Lovász-Stein Theorem

Seminar Extremal Graph Theory

Robin Link

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Advisors:
Prof. Dr. Maria Axenovich & Dr. Alexander Neal Riasanovsky

Lovász-Stein Theorem

Motivation

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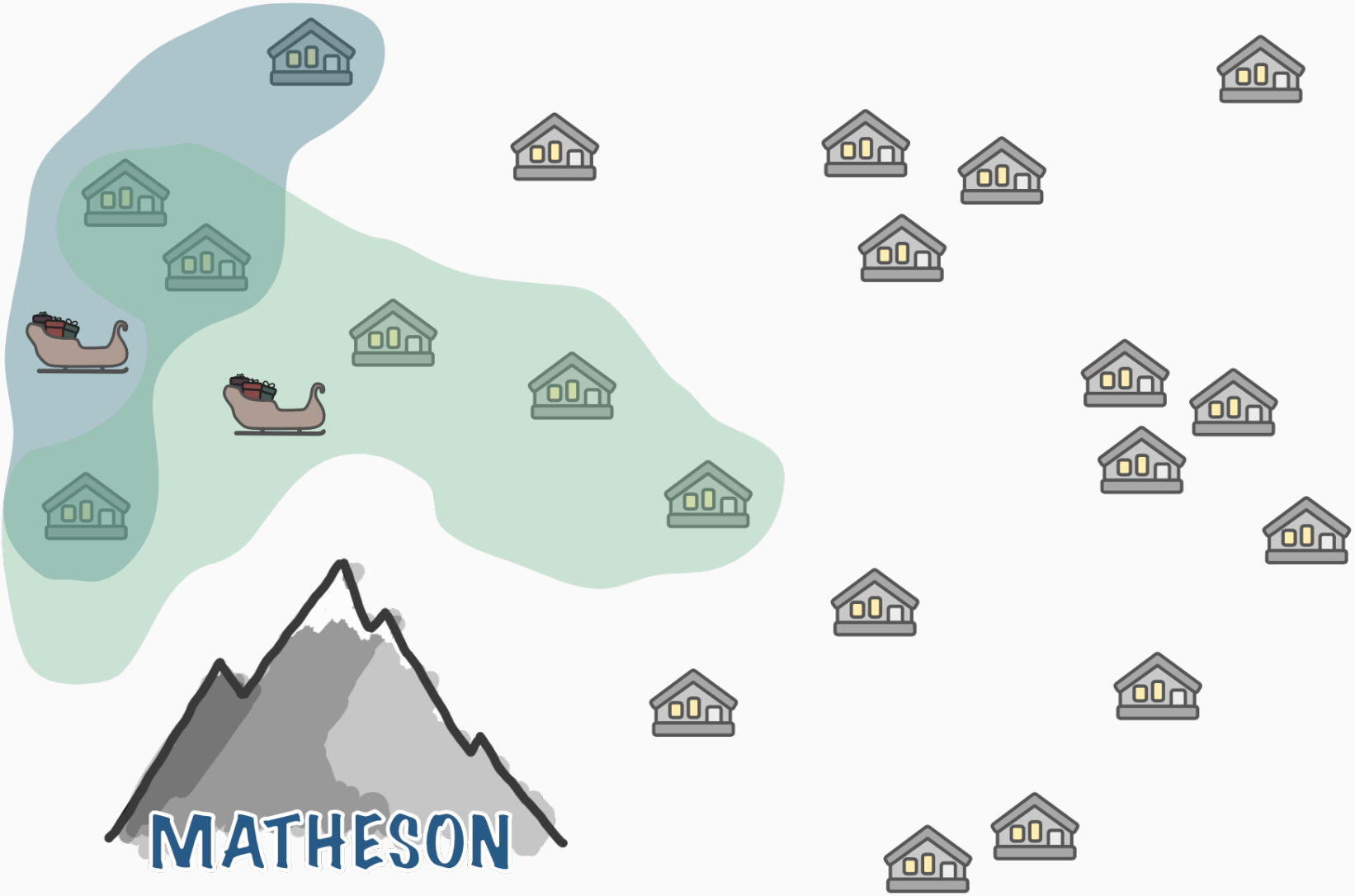

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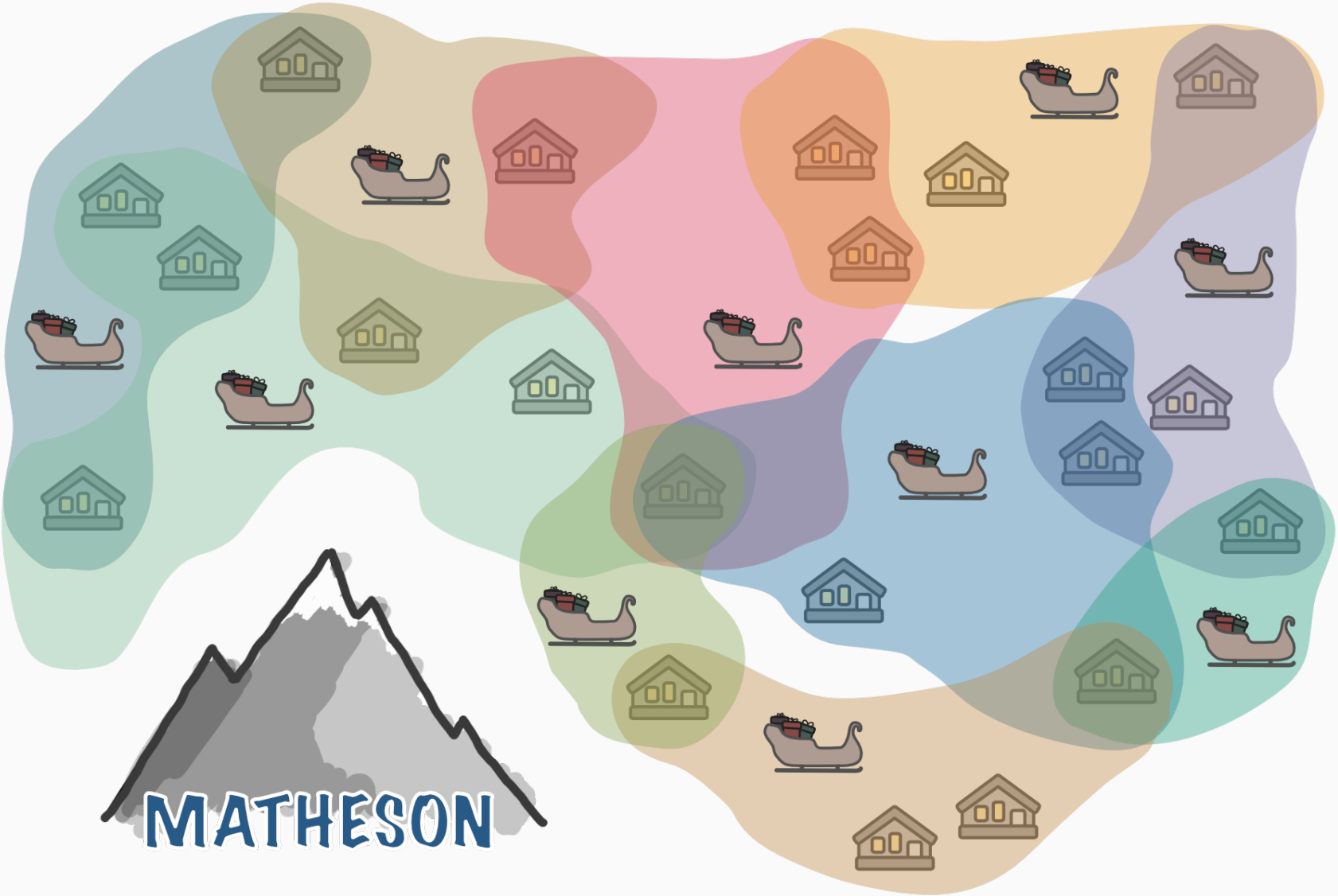

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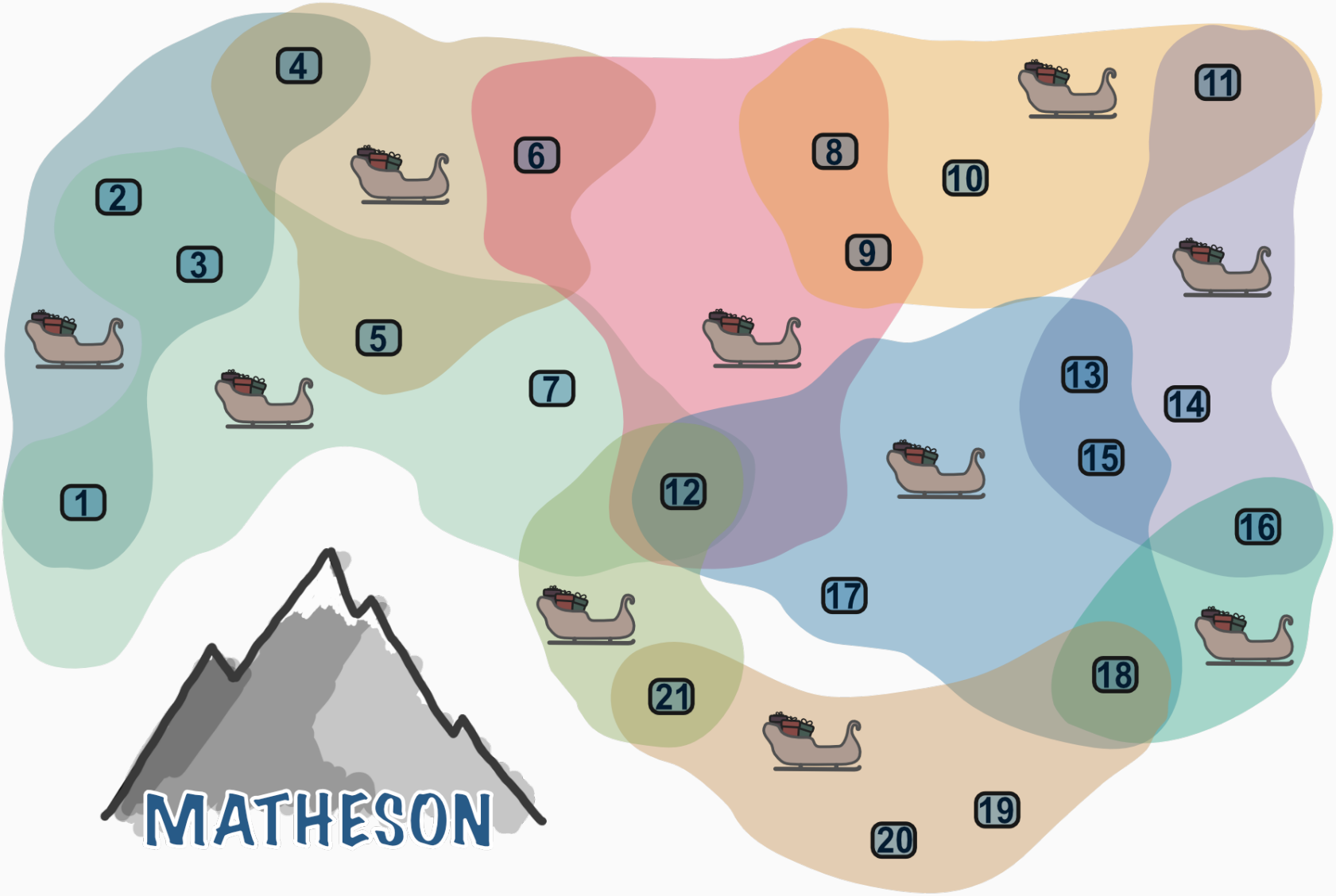

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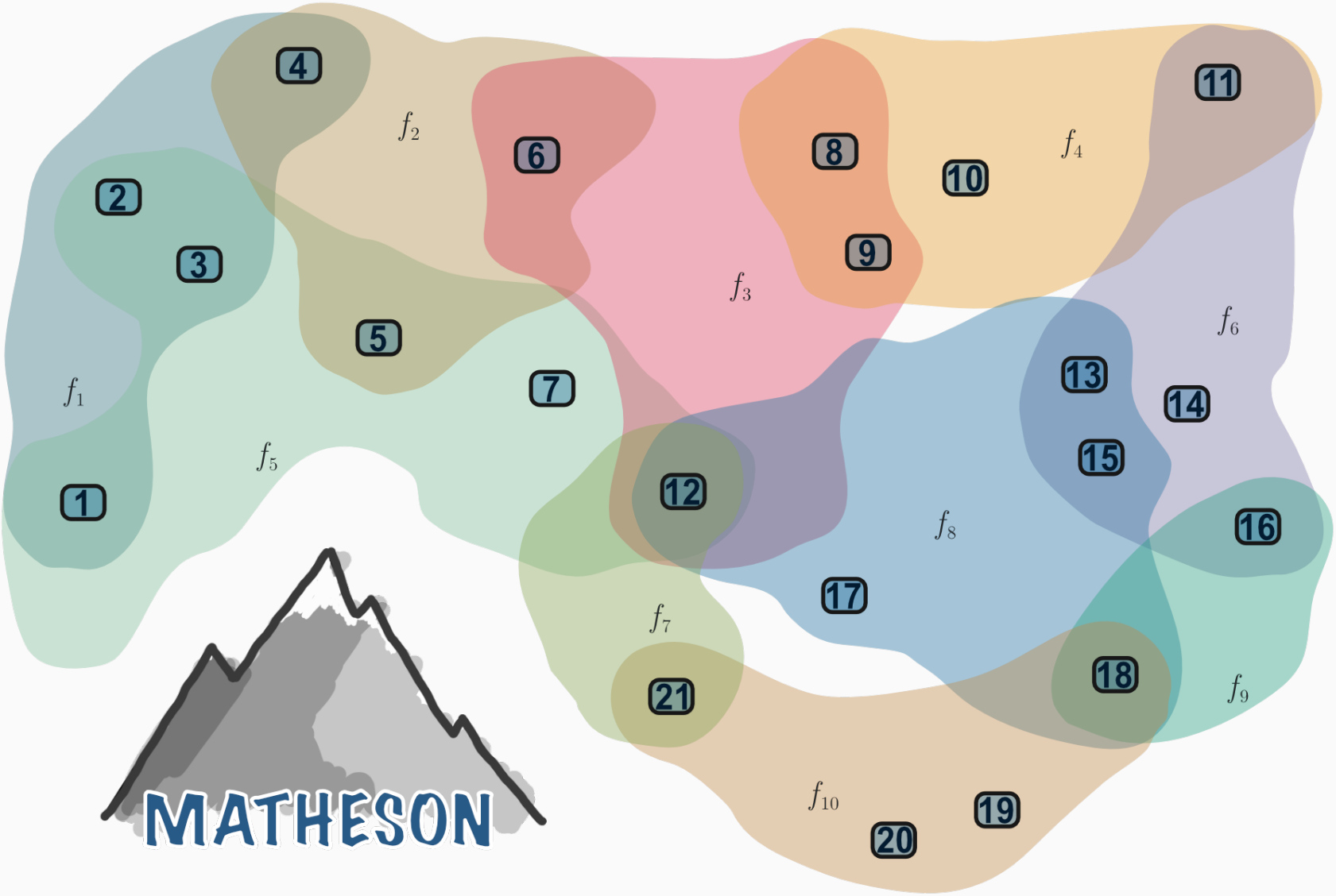
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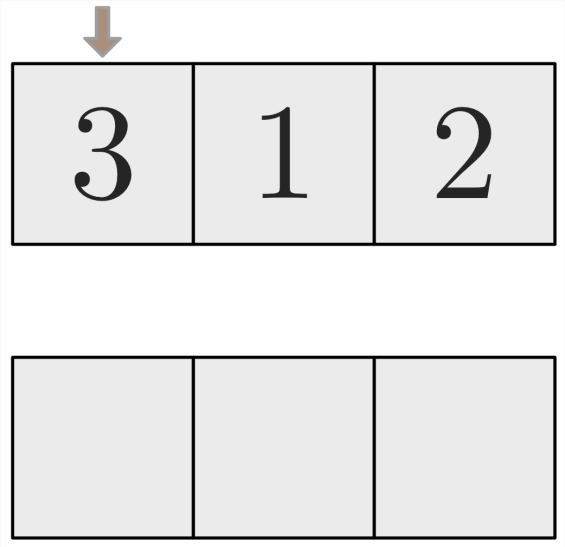
MATHESON

Greedy Sorting



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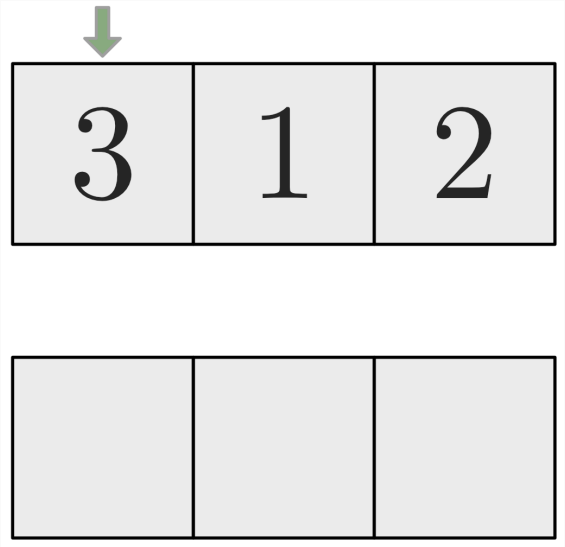

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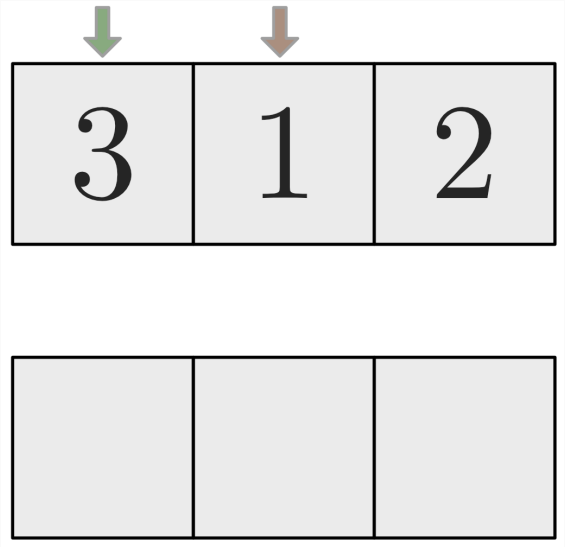

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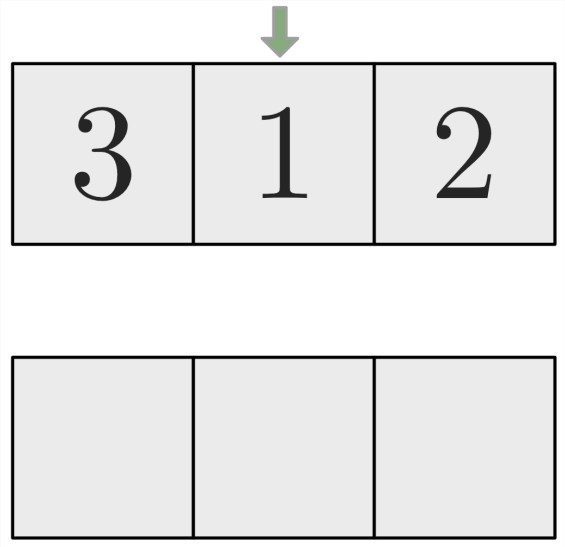

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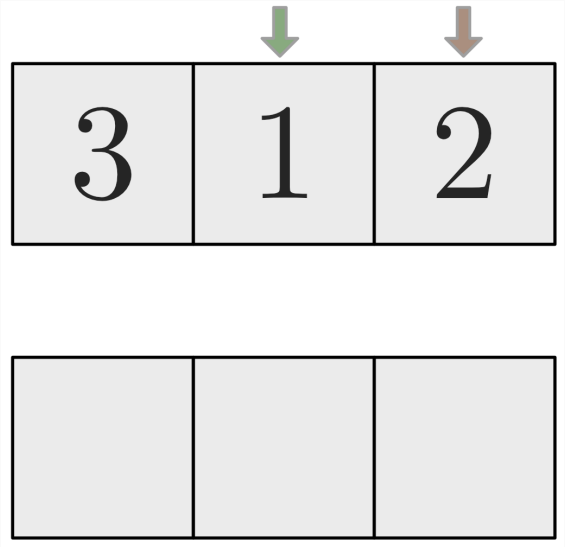

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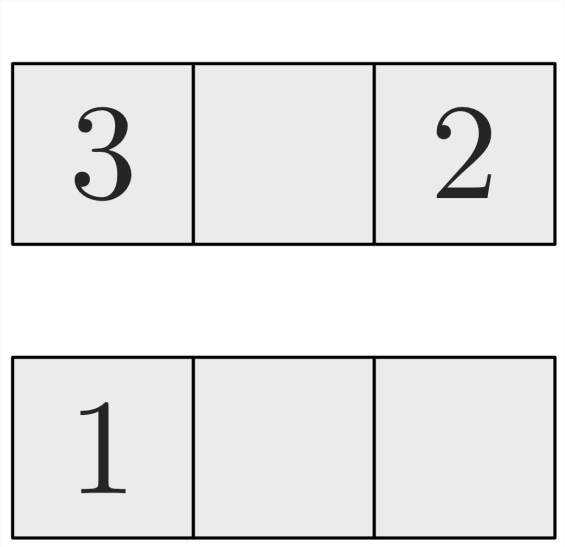

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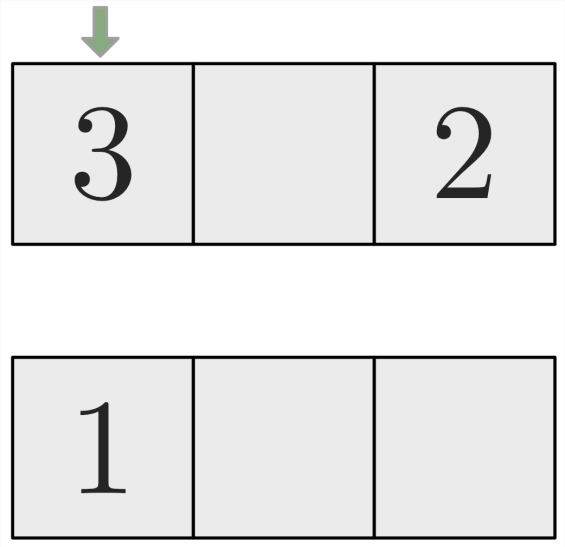

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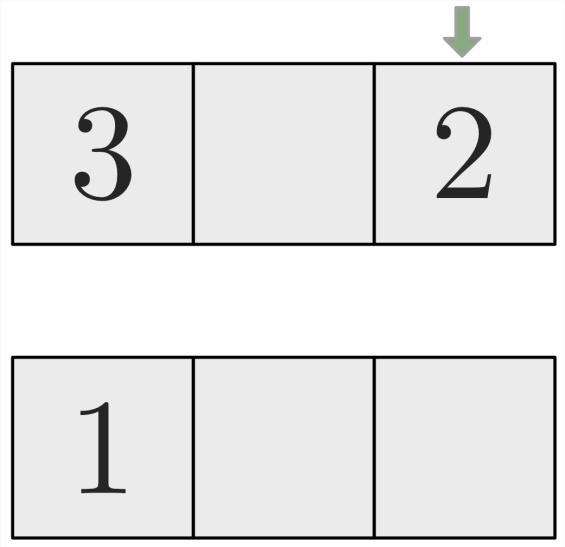

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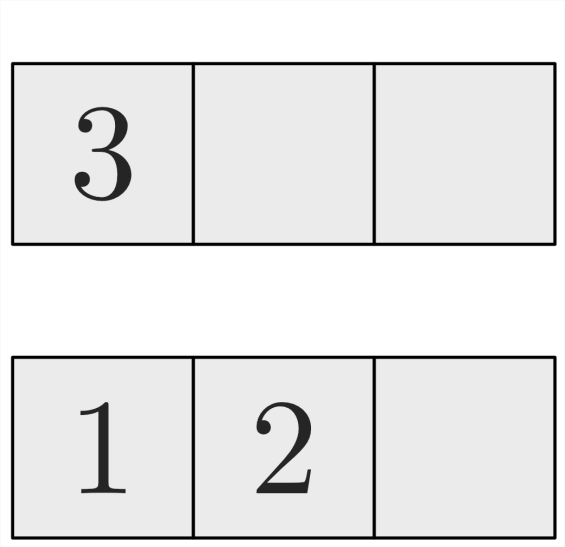

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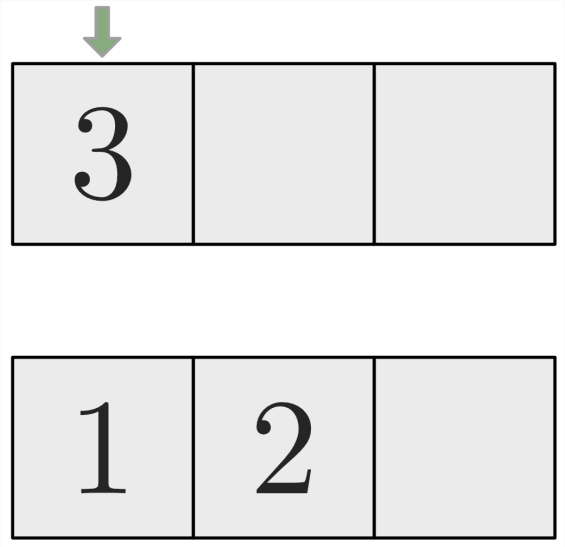

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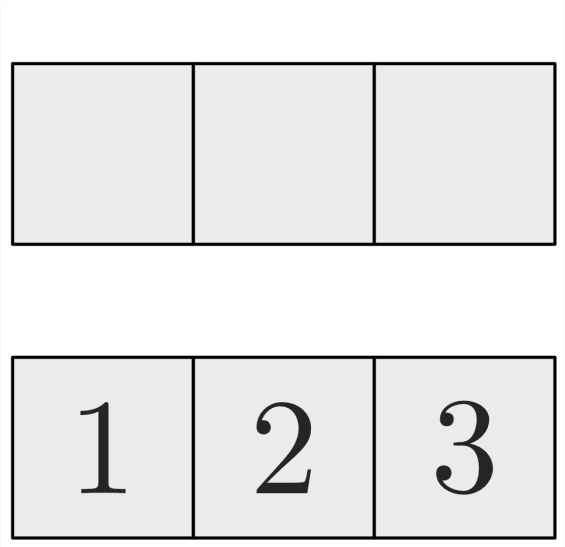
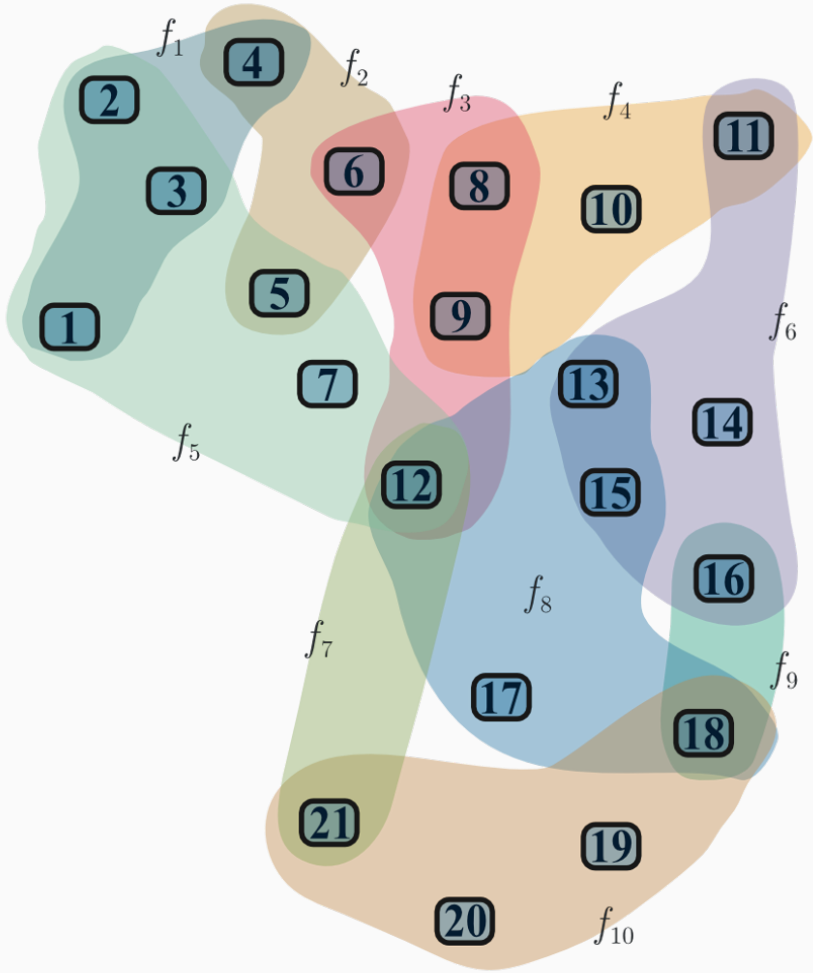


Lovász-Stein Algorithm



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	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}
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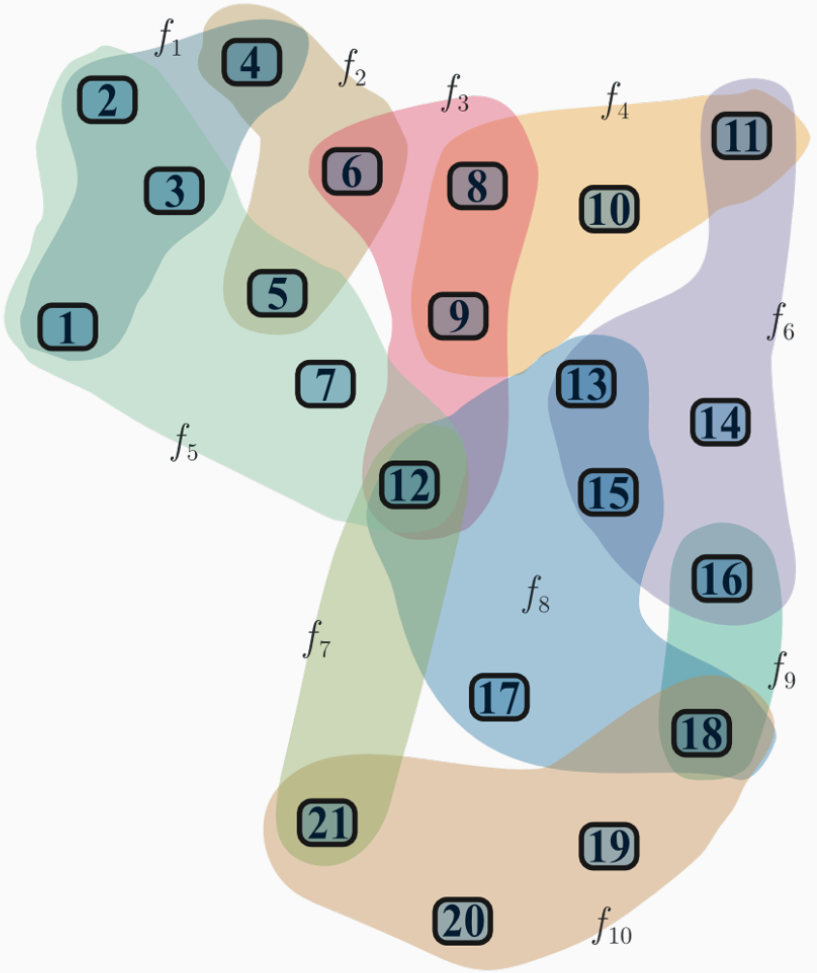

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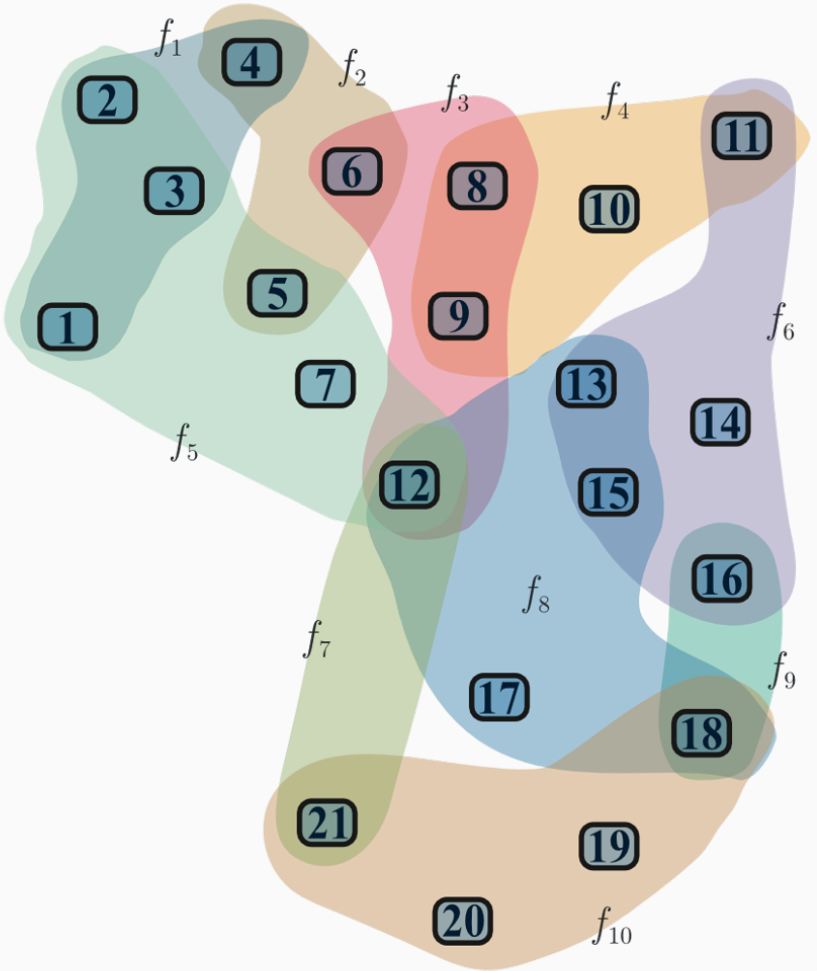

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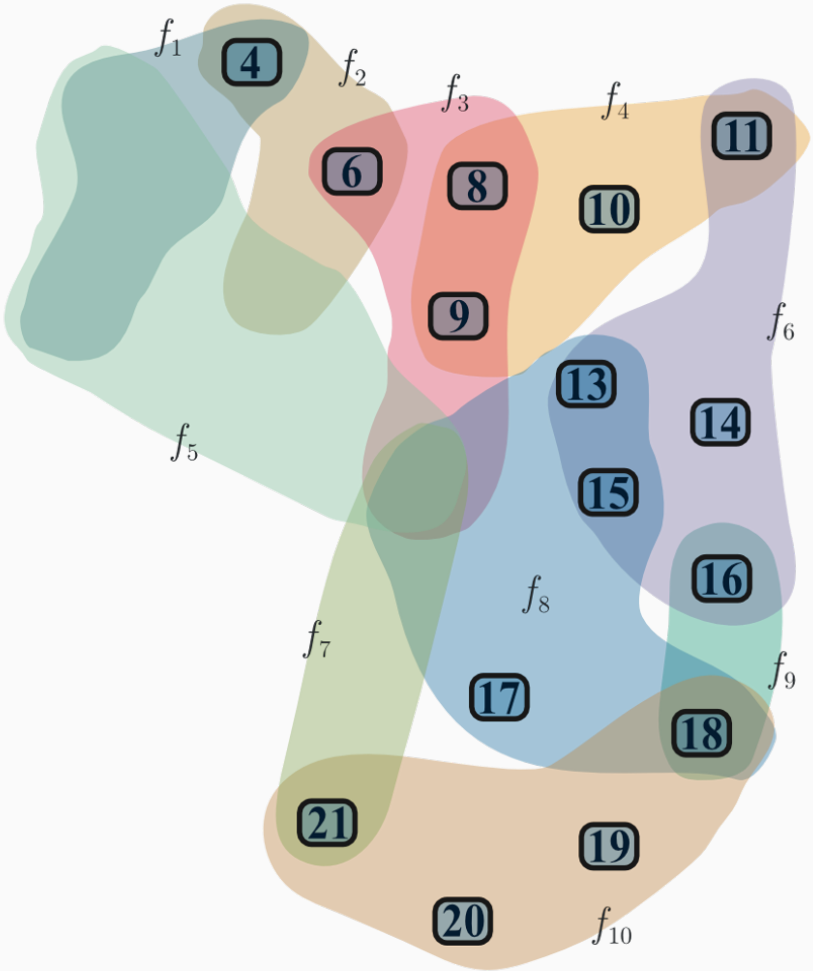

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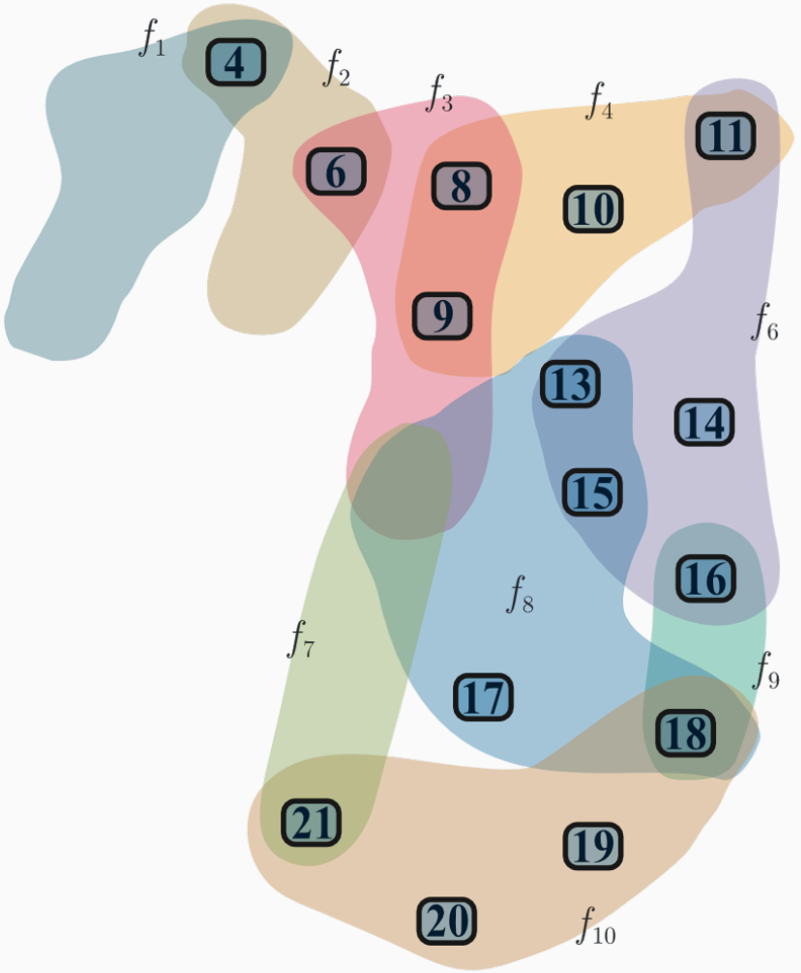

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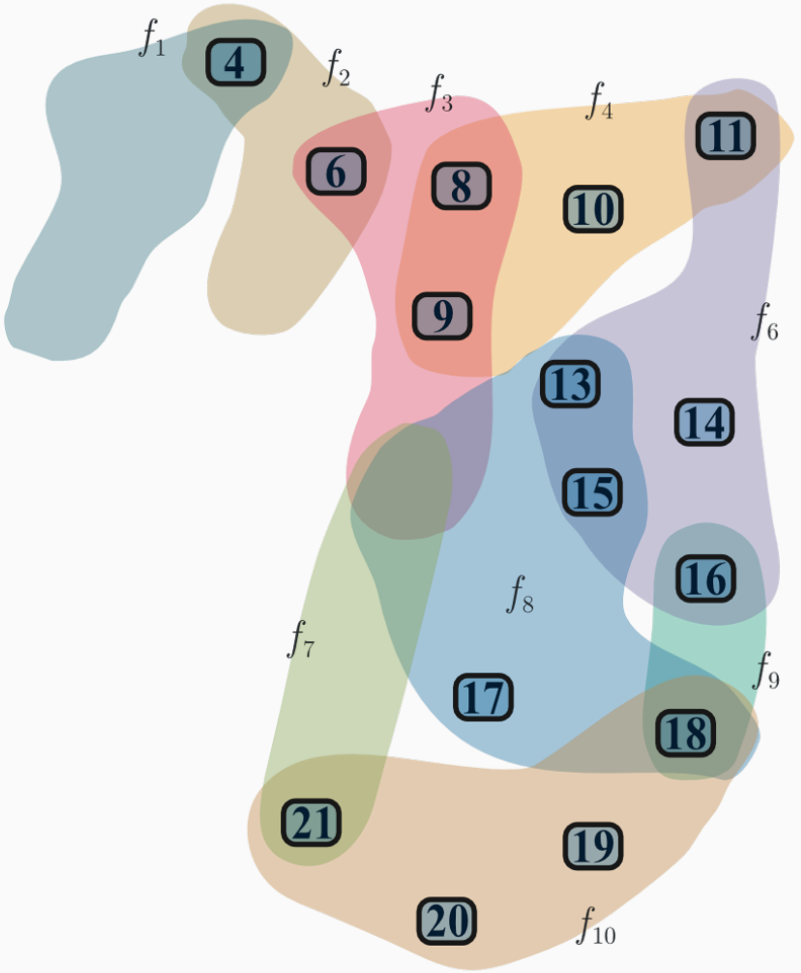

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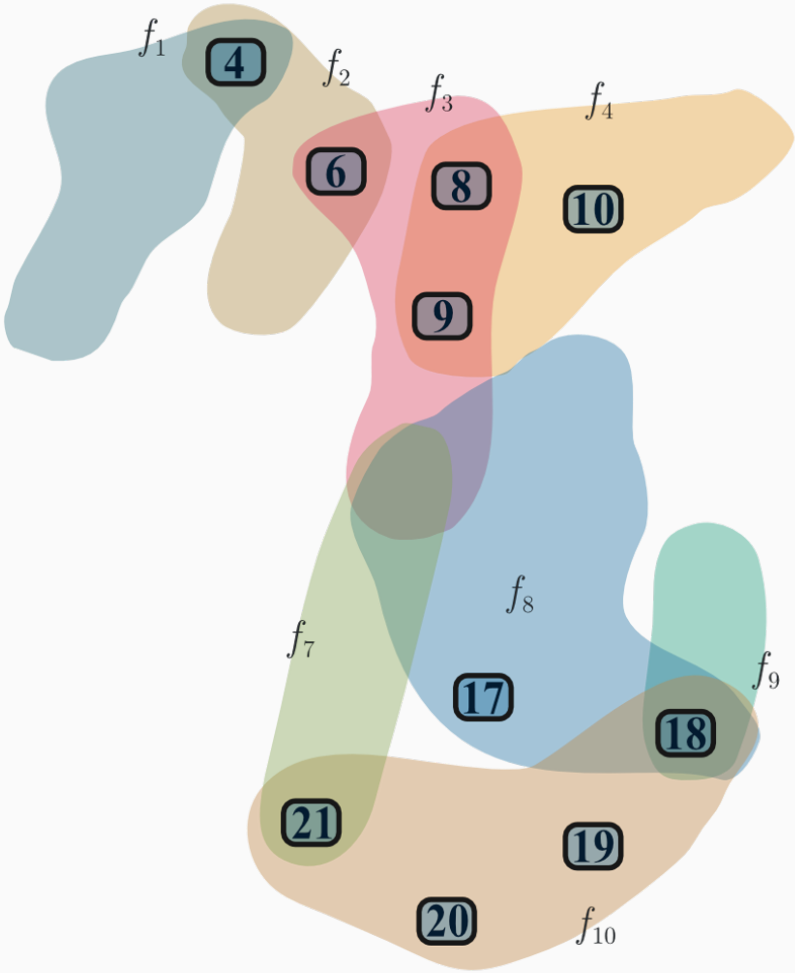

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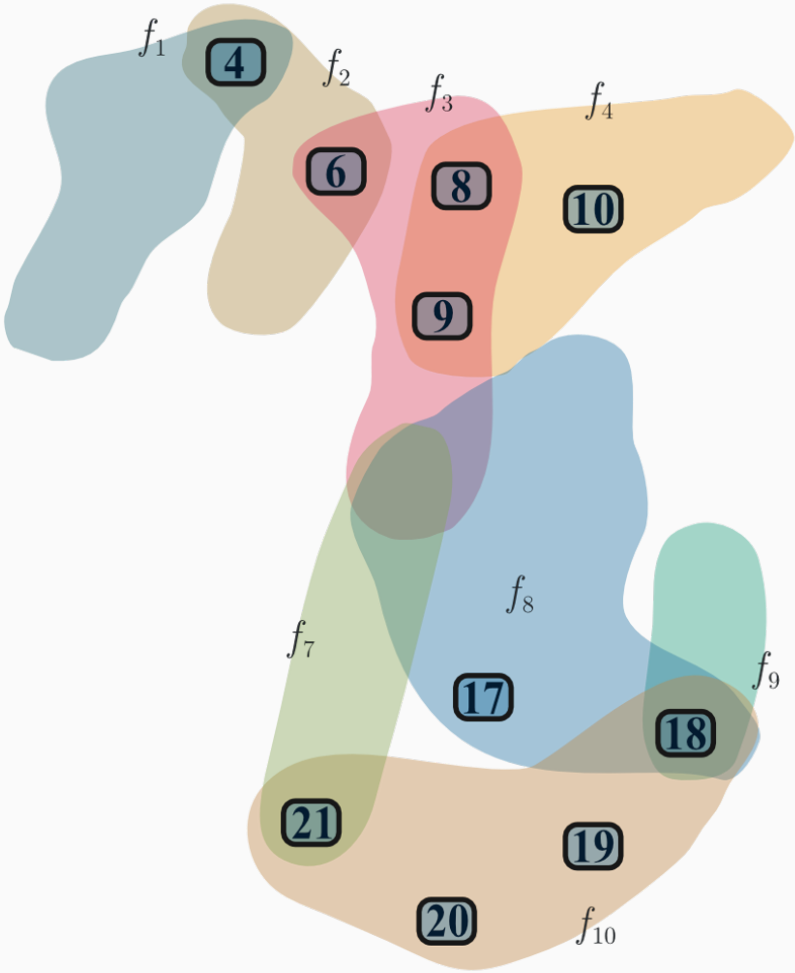

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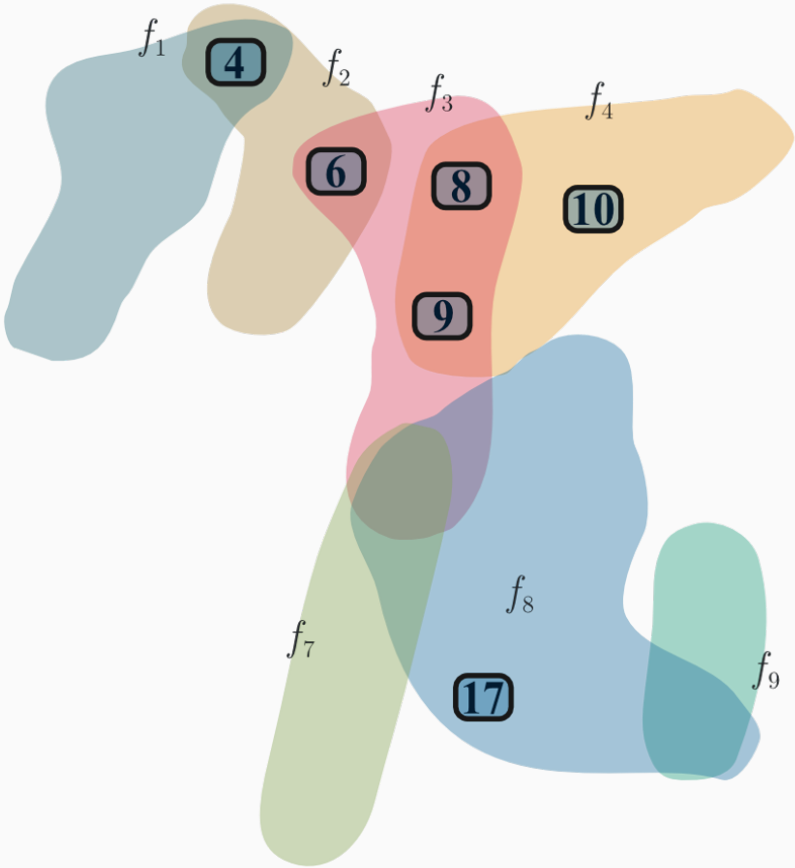

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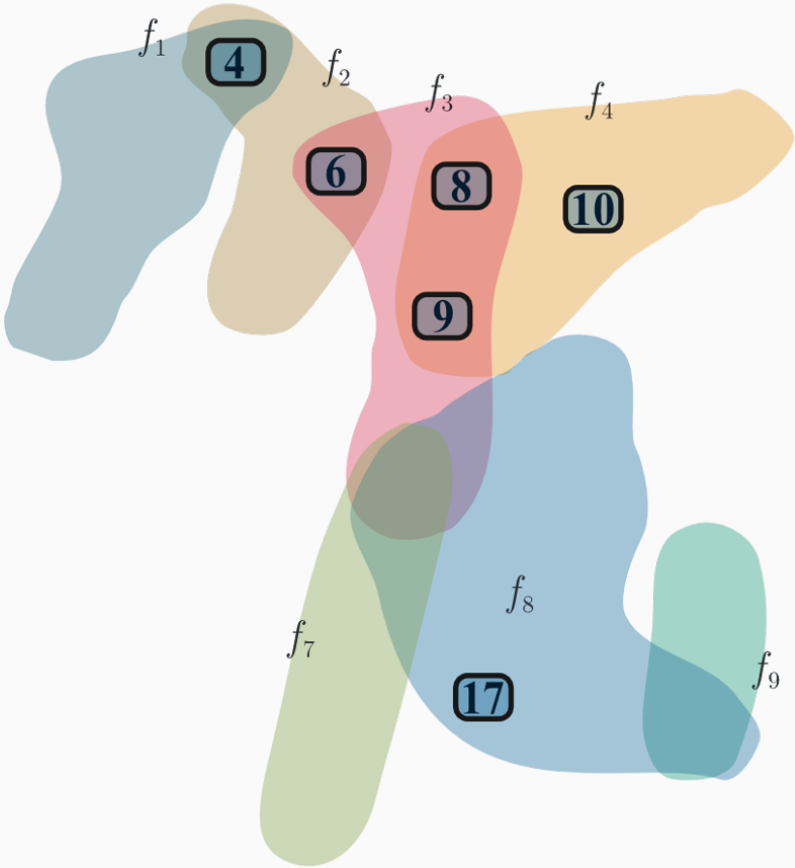

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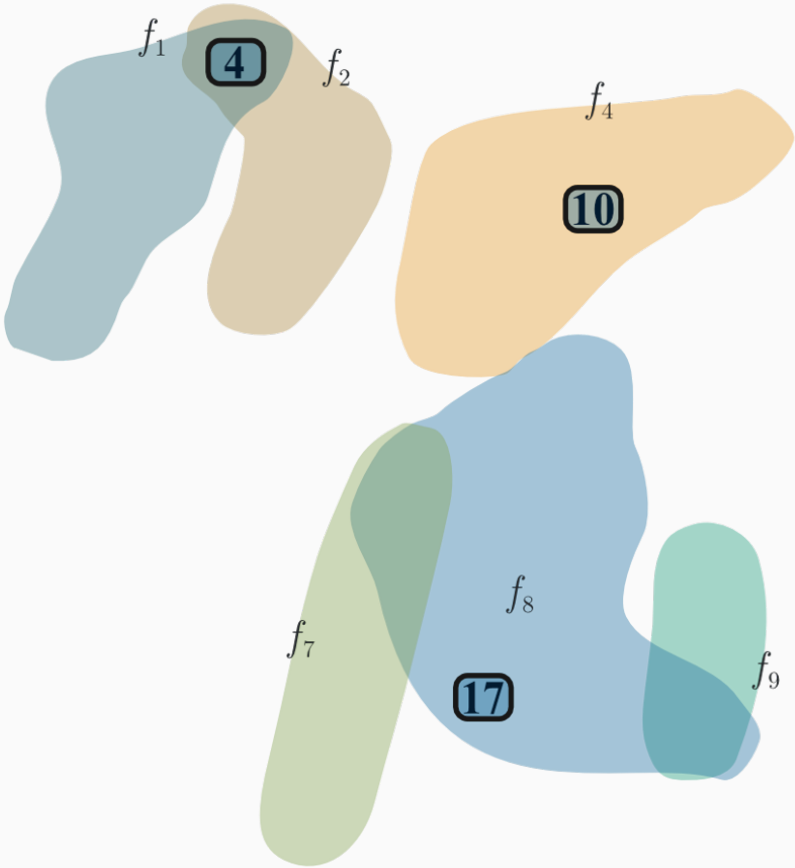

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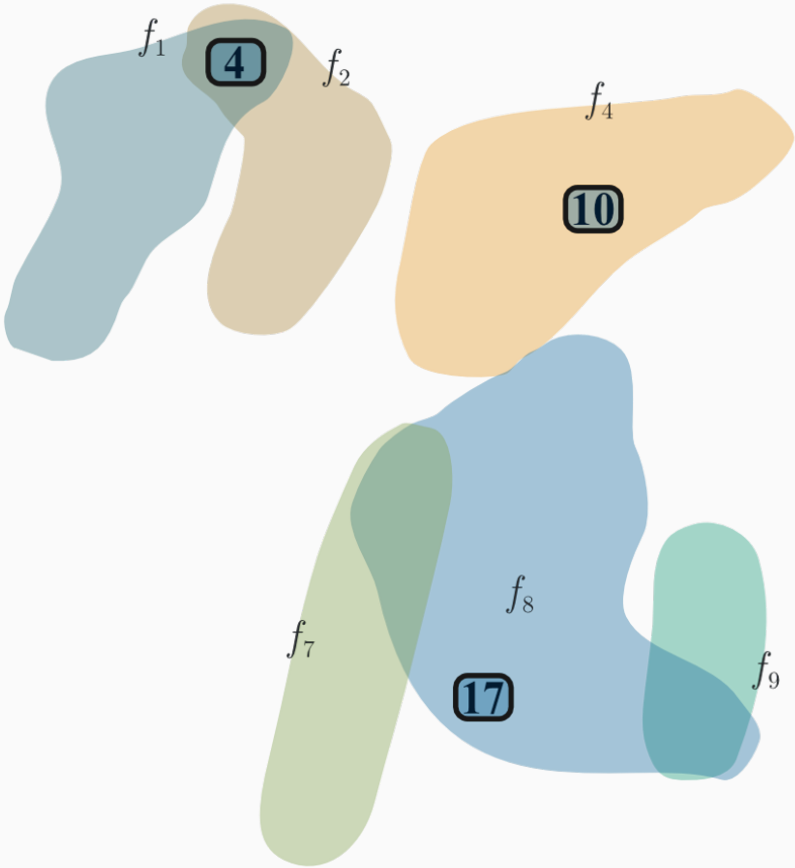

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Memos

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(3) $k_i \leq \frac{(i-1)M}{v}$

Lovász-Stein-Theorem

Lovász-Stein Theorem (1974)

If each member of $\mathcal{F}$ has at most a elements, and each point belongs to at least v of the sets in $\mathcal{F}$, then

$$Cov(\mathcal{G}) \leq \frac{|\mathcal{F}|}{v}(1 + \ln(a)).$$

Hashing

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How fast can $x \overset{?}{\in} M$ be decided by an algorithm?

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(1) For each $m \in M$ check $x \stackrel{?}{=} m$.

(2) If M is sortable, all elements can be stored in a binary tree of height $\log(|M|)$. Traverse the tree, checking $x \stackrel{?}{=} m$ for all visited elements m.

Hash Functions Motivation

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Some ideas

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(2) If M is sortable, all elements can be stored in a binary tree of height $\log(|M|)$. Traverse the tree, checking $x \stackrel{?}{=} m$ for all visited elements m.

We get a linear-time algorithm for (1) and a logarithmic-time algorithm for (2).

Hash Functions Motivation

How fast can $x \stackrel{?}{\in} M$ be decided by an algorithm?

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Question: How close can we get to constant-time?

Hash Functions

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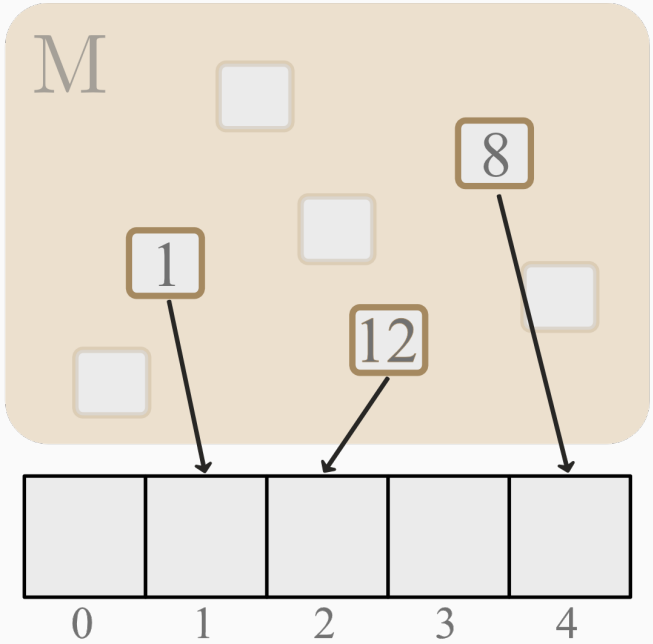
Example ($h(x) = x \bmod 5$):



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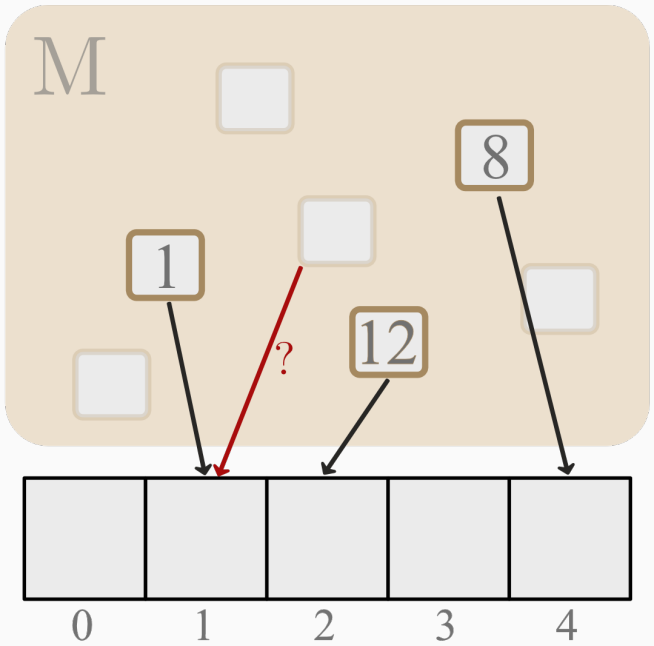

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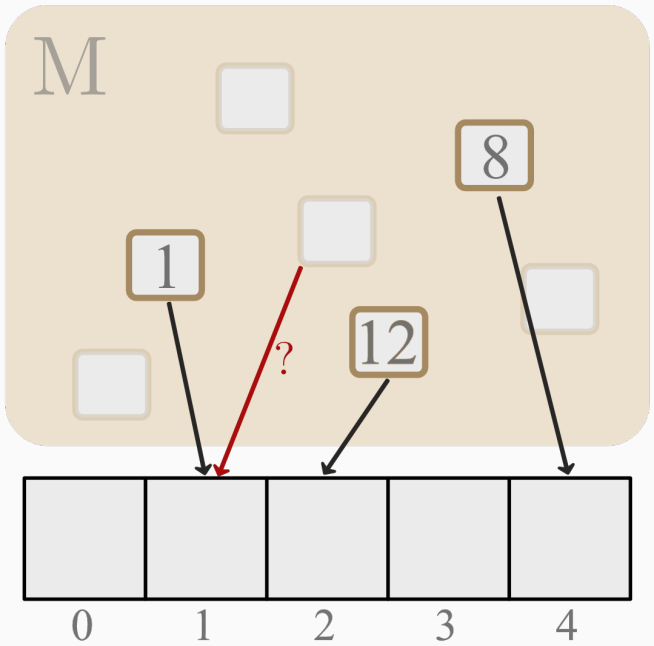

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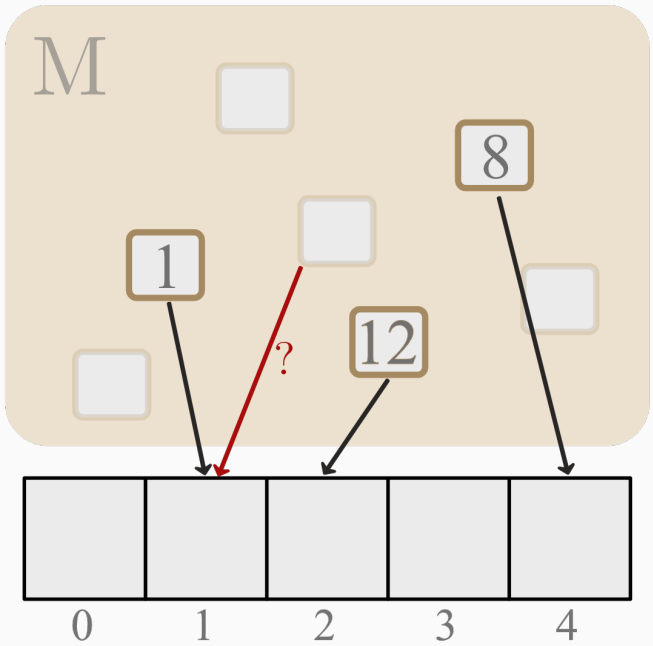

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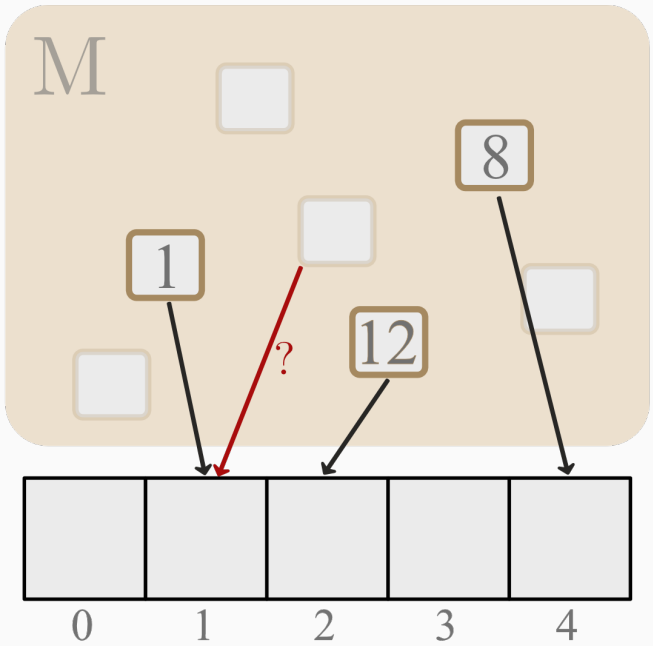

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Question: Can we get to constant-time?

Perfect Hash Functions

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Theorem 2 (2011)

There exists a perfect hashing family with

$$M \leq \frac{n^w}{w!\binom{n}{w}}\left(1 + \ln\binom{m}{w}\right).$$

Thank You!

And that's a good place to stop.

Questions?

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