MATHESON I

Setting

$G = (X, \mathcal{F})$ (hypergraph)

- X finite (point) set
- $\mathcal{F} \subseteq 2^X \setminus \{\emptyset\}$ (hyper-) edges

Coverings

$C \subseteq \{C_1, \dots, C_k\} \subseteq \mathcal{F}$ is a covering, if $\forall x \in X \; \exists C' \in C : x \in C'$.

Let $\operatorname{Cov} G$ denote the size of a smallest covering.

Greedy Procedures pick a locally best solution in each step

BUBBLESORT

Greedy Blueprint

- Pick a best candidate f
- Add f to solution C
- Remove f from G

repeat until G empty

Q: Best edge for covering?

A: Covers most new points.

Incidence Matrices

Let $N := |X|$, $M := |\mathcal{F}|$ and
$A \in \{0,1\}^{N \times M}$ s.t.
$a_{x,F} = 1 \iff x \in F$

MATHESON II

Define
$a := \max |f_i|$
$v := \delta \mathcal{G}$, i.e. each vertex is
contained in at least v edges

Lovász-Stein Algorithm (LSA)
$i \leftarrow a$
$A'_i \leftarrow$ Largest set of A_i-columns,
each with i ones &
all pw. disjoint
$K_i \leftarrow |A'_i|$
Add columns in A'_i to C
From A_i remove
- all columns in A'_i
- all rows with a 1 in A'_i

$\Big\}$ call the result A_{i-1}

$i \leftarrow i-1$
repeat until C covers $\mathcal{G}$

Observe: Let $K := \sum_{i=1}^{a} K_i$ be the size of the covering found by LSA.
Then $\operatorname{Cov} \mathcal{G} \leq K \leq ?$.

Say we got k_i rows left in A_i
Thus $k_a = N - aK_a$,
$k_{a-1} = N - aK_a - (a-1)K_{a-1}, \dots$

Rewrite to $K_a = \frac{N-k_a}{a}$:

$$K_i = \frac{k_{i+1}+k_i}{i} \qquad (i=1,\dots,a).$$

Next, bound k_i:

- Rows of A_{i-1} contain $\geq v$ ones
- Columns of A_{i-1} contain $\leq i-1$ ones

Thus $v k_i \leq (i-1)(M-K_a-\dots-K_{i+1}) \leq (i-1)M$

$\Rightarrow k_i \leq \frac{(i-1)M}{v}$.

In conclusion

$$K = \Sigma_{i=1}^{a} K_i = \Sigma_{i=1}^{a} \frac{k_{i+1}-k_i}{i}$$

$$= \frac{k_{a+1}}{a} + \frac{k_a}{a(a-1)} + \frac{k_{a-1}}{(a-1)(a-2)} + \dots + \frac{k_2}{2\cdot 1} - k_1$$

$$\leq \frac{N}{a} + \frac{M}{v}\left(\frac{1}{a} + \frac{1}{a-1} + \dots + \frac{1}{2}\right)$$

$$\overset{(*)}{\leq} \frac{M}{v}\left(1 + \Sigma_{k=2}^{a} \frac{1}{k}\right)$$

Proof of (*). $X := \{(x,f) : f \in \mathcal{F}, x \in f\}$. Then

- $|X| = \Sigma_{x\in X} d(x) \geq Nv$
- $|X| = \Sigma_{f\in\mathcal{F}} |f| \leq Ma$

$\Rightarrow \frac{M}{v} \geq \frac{N}{a}$. $\circ$

LOVÁSZ-STEIN

Tightness

- LSA optimal, i.e. $K = \operatorname{Cov} \mathcal{G}$, if $\mathcal{F}$ is the only covering.
- LSA optimal $\not\Rightarrow$ bound thight:

  $\mathcal{F} := \{[n], \{1\}, \{2\}, \dots, \{n\}\}$

  $|\mathcal{F}| = n+1, \quad a = n, \quad v = 2, \quad$ thus

  $\operatorname{Cov} \mathcal{F} = 1 = K \leq \frac{n+1}{2}(1 + \ln n) \xrightarrow{n\to\infty} \infty$

- Bound tight if $a=1$, $v=1$: $\operatorname{cov} \mathcal{G} = K = |\mathcal{F}|$

HASH-FUNCTIONS

Proof-Sketch of Theorem 2:
Reduce to Lovasz-Stein by
(1) constructing A,
(2) showing that a covering of $\mathcal{G}_A$ corresponds to a PHF,
(3) estimating the size of A

(1). Correspondence between $Y^{|X|}$ and $\{f \mid f: X \to Y\} = Y^X$
$f_i \mapsto (f_i(x_1), \dots, f_i(x_{|X|}))^T$.
$\rightsquigarrow (f_i^T)_{1 \leq i \leq N} =: B \in Y^{|X| \times N}$

$\mathcal{H}$ perfect, if for all choices of ω many rows of B there is a column with distinct entries.

Construction of A:
- m^n columns, one for each f
- $\binom{n}{\omega}$ rows, one for each C
- $a_{C,f} = 1 \iff f_{|C}$ is injective, i.e. $(f(c_1), \dots, f(c_\omega))^T$ has distinct entries

(2) A covering of the ω-subsets corresponds to a set of functions which is a PHF.

(3) Counting ones in A we get

- $v = w! \binom{m}{w} m^{n-w}$, since
  - $\{f(c_1), \dots, f(c_w)\} \rightsquigarrow \binom{m}{w}$
  - $(f(c_1), \dots, f(c_w)) \rightsquigarrow w!$
  - there are $n-w$ elements of X left that can be mapped to $Y \rightsquigarrow m^{n-w}$
- $a = \binom{n}{w}$