ECE 438: Digital Signal Processing with Applications
Lab 3: Frequency Analysis

Date:

Section:

Name	Signature	Time spent outside lab
Student Name #1 [Ruixiang Wang]		
Student Name #2 [---%]		

	Below expectations	Lacks in some respect	Meets all expectations
Completeness of the report			
Organization of the report			
Quality of figures: *Correctly labeled with title, x-axis, y-axis, and name(s)*			
Ability to compute Fourier series expansion and synthesize periodic signals using the expansion in Simulink (26 pts): *Derivation and sketch, plots of synthesized signals, questions*			
Understanding of modulator and CT system analysis (26 pts): *Output plots, questions*			
Implementation of DTFT (21 pts): *Python function, DTFT's magnitude and phase plots*			
DT system analysis (27 pts): *Exercises in 5.2, completed block diagram, table of measurements, impulse and frequency response*			

In [2]:
```
import numpy as np
import matplotlib.pyplot as plt
```

In [3]:
```
# make sure the plot is displayed in this notebook
%matplotlib inline
# specify the size of the plot
plt.rcParams['figure.figsize'] = (16, 6)

# for auto-reloading externnal modules
%load_ext autoreload
%autoreload 2
```

Exercise 2.1

1. For each of these two signals, do the following on a blank sheet of paper (or type the equations in the Markdown cell if you are familiar with LaTex):

- **Compute the Fourier series expansion in the form**

$$s(t) = a_0 + \sum_{k=1}^{\infty} A_k \sin(2\pi k f_0 t + \theta_k)$$

where $f_0 = \frac{1}{T_0}$.

Hint : You may want to use one of the following references:

Sec. 4.1 of “Digital Signal Processing”, by Proakis and Manolakis, 1996;

Sec. 4.2 of “Signals and Systems”, by A. Oppenheim and A. Willsky, 1983;

Sec. 3.3 of “Signals and Systems”, A. Oppenheim and A. Willsky, 1997.

Note that in the expression above, the function in the summation is $\sin(2\pi k f_0 t + \theta k)$, rather than a complex sinusoid. The formulas in the above references must be modified to accommodate this. You can compute the cos/sin version of the Fourier series, then convert the coefficients.

(1)

$T_0 = 2 \qquad w_0 = \frac{2\pi}{T_0} = \pi$

$e^{j\theta} = \cos\theta + j\sin\theta$

$e^{-j\theta} = \cos\theta - j\sin\theta$

$$a_0 = \frac{1}{T_0}\int_0^1 u(t) - u(t-1)\, dt = \frac{1}{T_0} t\Big|_0^1 = \frac{1}{2}$$

$$a_k = \frac{1}{T_0}\int_0^1 (u(t)-u(t-1))\, e^{-jk\pi t}\, dt = \frac{1}{2}\int_0^1 e^{-jk\pi t}\, dt = \frac{1}{-j2k\pi}\left(e^{-jk\pi} - 1\right) = \frac{1-e^{-jk\pi}}{2jk\pi}$$

$e^{-jk\pi} \Downarrow \cos(k\pi) - j\sin(k\pi)$

$\cos(k\pi) \to (-1)^k$, $\sin(k\pi) \to 0$ for k is integer

$$= \begin{cases} 0, & k = \text{even} \\ \frac{1}{jk\pi}, & k = \text{odd} \end{cases}$$

$$a_k^* = \begin{cases} 0, & k = \text{even} \\ \frac{1}{-jk\pi}, & k = \text{even} \end{cases}$$

$$s(t) = a_0 + \sum_{k=1}^{\infty} a_k e^{jkw_0 t} + \sum_{k=1}^{\infty} a_k^* e^{-jkw_0 t}$$

$$= \frac{1}{2} + \sum_{k=1,3,5}^{\infty} \frac{1}{jk\pi}\left(e^{jk\pi t} - e^{-jk\pi t}\right)$$

$$= \frac{1}{2} + \sum_{k=1,3,5}^{\infty} \frac{1}{k\pi}\, 2\sin(k\pi t)$$

(2) Note: a0 is 0, there is a mistake here

$$-\frac{1}{2}$$

$$a_0 = \frac{1}{T_0}\int_{-\frac{1}{4}}^{\frac{1}{4}} \mathrm{rect}(2t)-\frac{1}{2}\,dt + \frac{1}{T_0}\int_{\frac{1}{4}}^{\frac{1}{2}} \mathrm{rect}(2t)-\frac{1}{2}\,dt + \frac{1}{T_0}\int_{-\frac{1}{2}}^{-\frac{1}{4}} \mathrm{rect}(2t)-\frac{1}{2}\,dt$$

$$= \frac{1}{8}$$

$$a_k = \frac{1}{T_0}\int_{-\frac{1}{4}}^{\frac{1}{4}} \frac{1}{2} e^{-jk2\pi t} + \frac{1}{T_0}\int_{\frac{1}{4}}^{\frac{1}{2}} -\frac{1}{2} e^{-jk2\pi t} + \frac{1}{T_0}\int_{-\frac{1}{2}}^{-\frac{1}{4}} -\frac{1}{2} e^{-jk2\pi t}$$

$$= \frac{1}{-4jk\pi}\left(e^{-jk\frac{\pi}{2}} - e^{jk\frac{\pi}{2}}\right) + \frac{1}{4jk\pi}\left(e^{-jk\pi} - e^{-jk\frac{\pi}{2}}\right) + \frac{1}{4jk\pi}\left(e^{jk\frac{\pi}{2}} - e^{jk\pi}\right)$$

$$= \frac{1}{4jk\pi}\Big(e^{-jk\pi} - e^{jk\pi} + \underbrace{e^{jk\frac{\pi}{2}} - e^{-jk\frac{\pi}{2}} - e^{-jk\frac{\pi}{2}} + e^{jk\frac{\pi}{2}}}_{2e^{jk\frac{\pi}{2}} - 2e^{-jk\frac{\pi}{2}}}\Big)$$

$$= \frac{1}{4jk\pi}\Big(\underbrace{-2j\sin(k\pi)}_{0} + 4j\sin\left(\frac{\pi}{2}k\right)\Big)$$

$$= \frac{1}{k\pi}\sin\left(k\frac{\pi}{2}\right) \qquad \begin{cases} 0, & k \text{ even} \\ \frac{(-1)^{k+1}}{k\pi}, & k \text{ odd} \end{cases}$$

$$a_k^* = \begin{cases} 0, & k \text{ even} \\ \frac{(-1)^{k+1}}{k\pi}, & k \text{ odd} \end{cases}$$

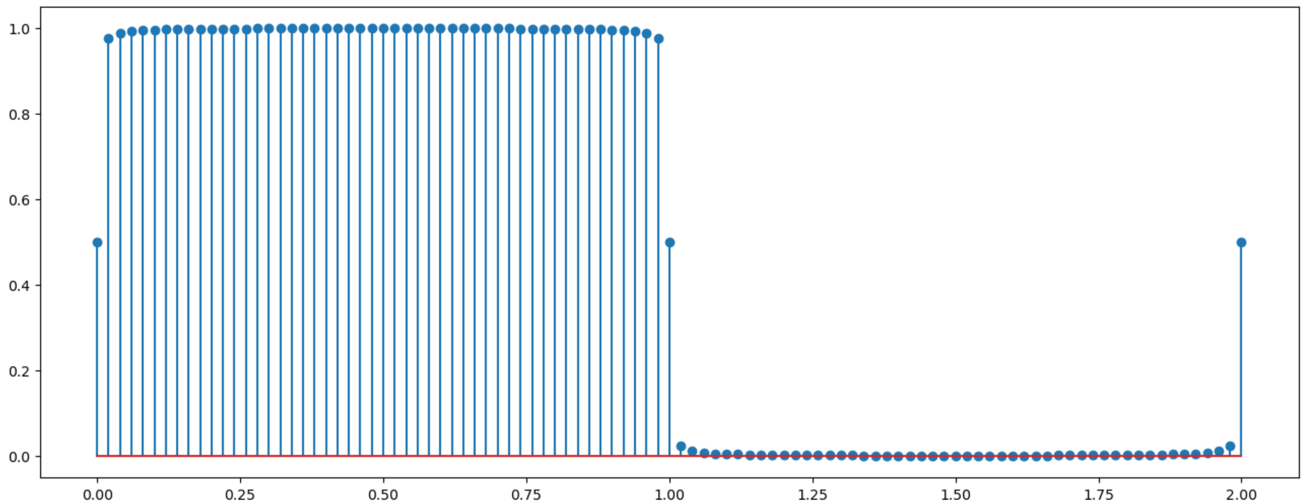
$$s(t) = \frac{1}{8} + \sum_{k=1,3,5}^{\infty} \frac{1}{k\pi} e^{jk2\pi t} + \sum_{k=1,3,5}^{\infty} \frac{1}{k\pi} e^{-jk2\pi t}$$

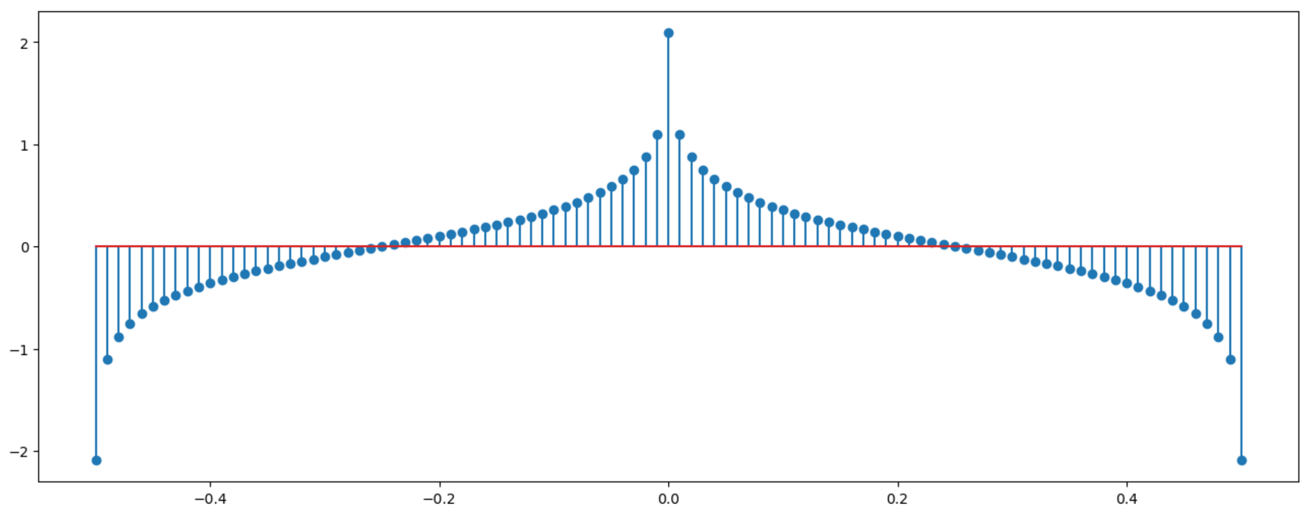
$$= \frac{1}{8} + \sum_{k=1,3,5}^{\infty} \frac{2j}{k\pi} \sin(k2\pi t)$$

2. Write code to approximate the two signals using the Fourier series expansion above. Use 200 (instead of infinite number of) Sine waves. Then, plot these two signals.

In [17]:
```
import math
t1 = np.linspace(0, 2, 101)
t2 = np.linspace(-0.5, 0.5, 101)

s1 = 0.5 + sum(2/(k*math.pi) * np.sin(k*math.pi*t1) for k in range(1, 200, 2))
s2 = sum(2/(k*math.pi) * np.cos(k*2*math.pi*t2) for k in range(1, 200, 2))

plt.figure(1)
plt.stem(t1, s1)

plt.figure(2)
plt.stem(t2, s2)
```

Out[17]:
```
<StemContainer object of 3 artists>
```

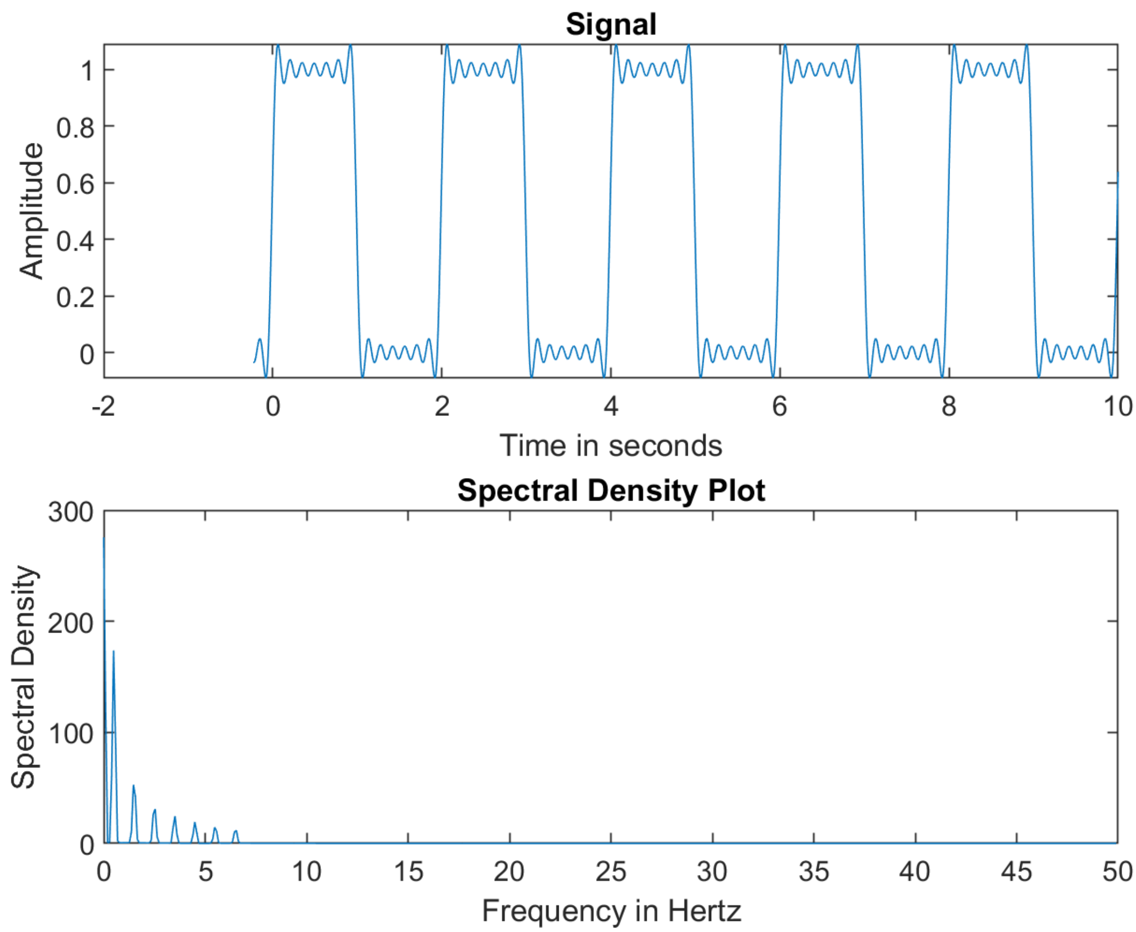


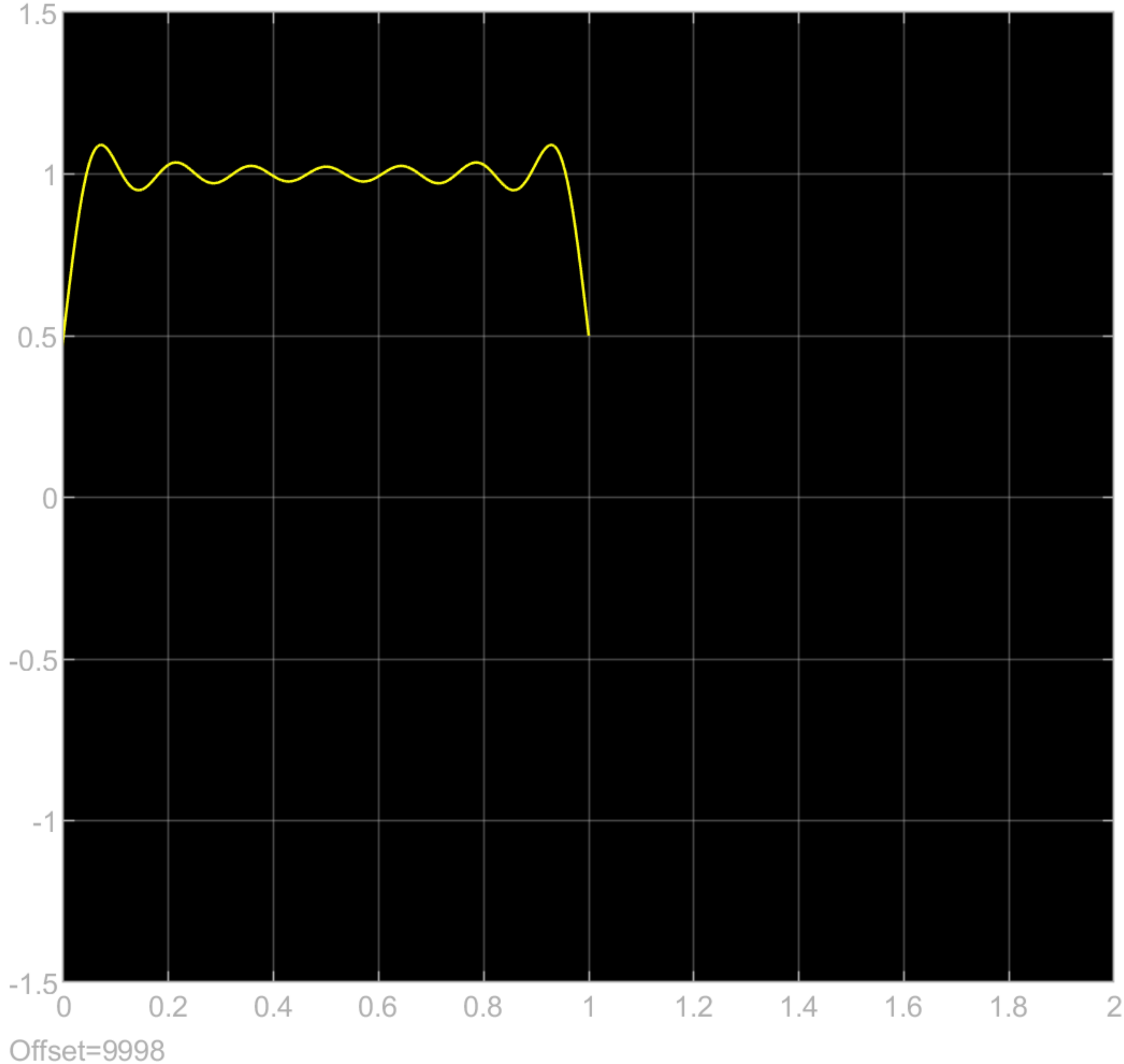


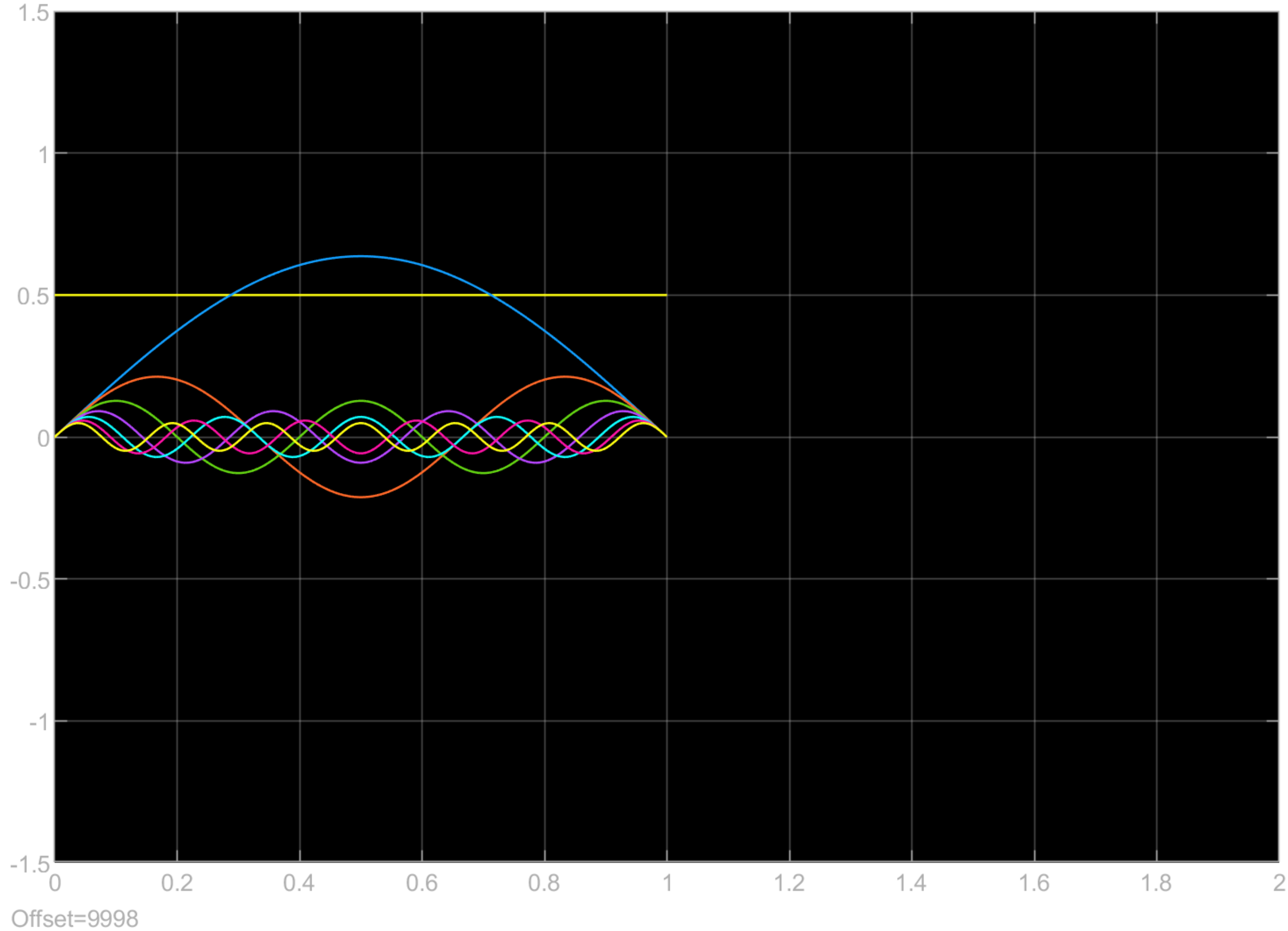
Exercise 4.1

1. Hand in plots of the Spectrum Analyzer output for each of the three synthesized waveforms.

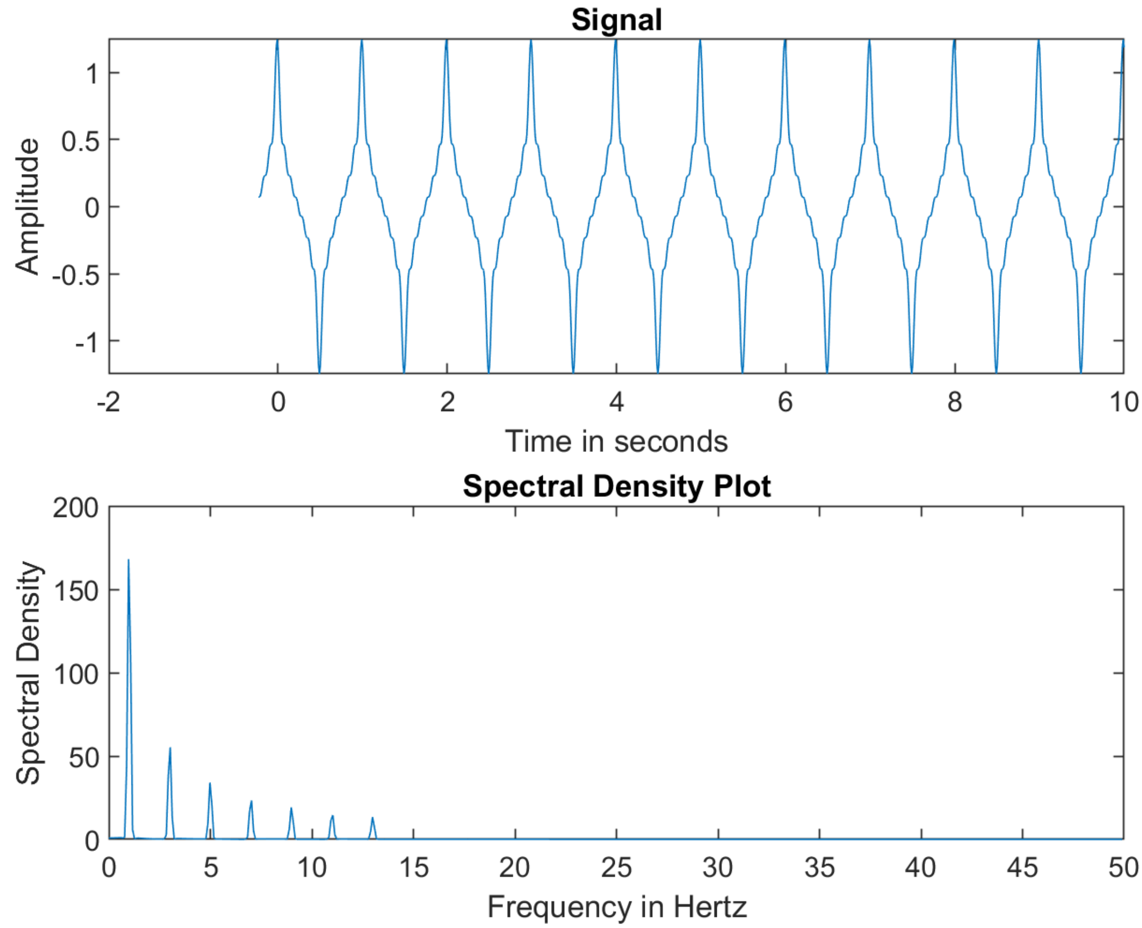
(1)

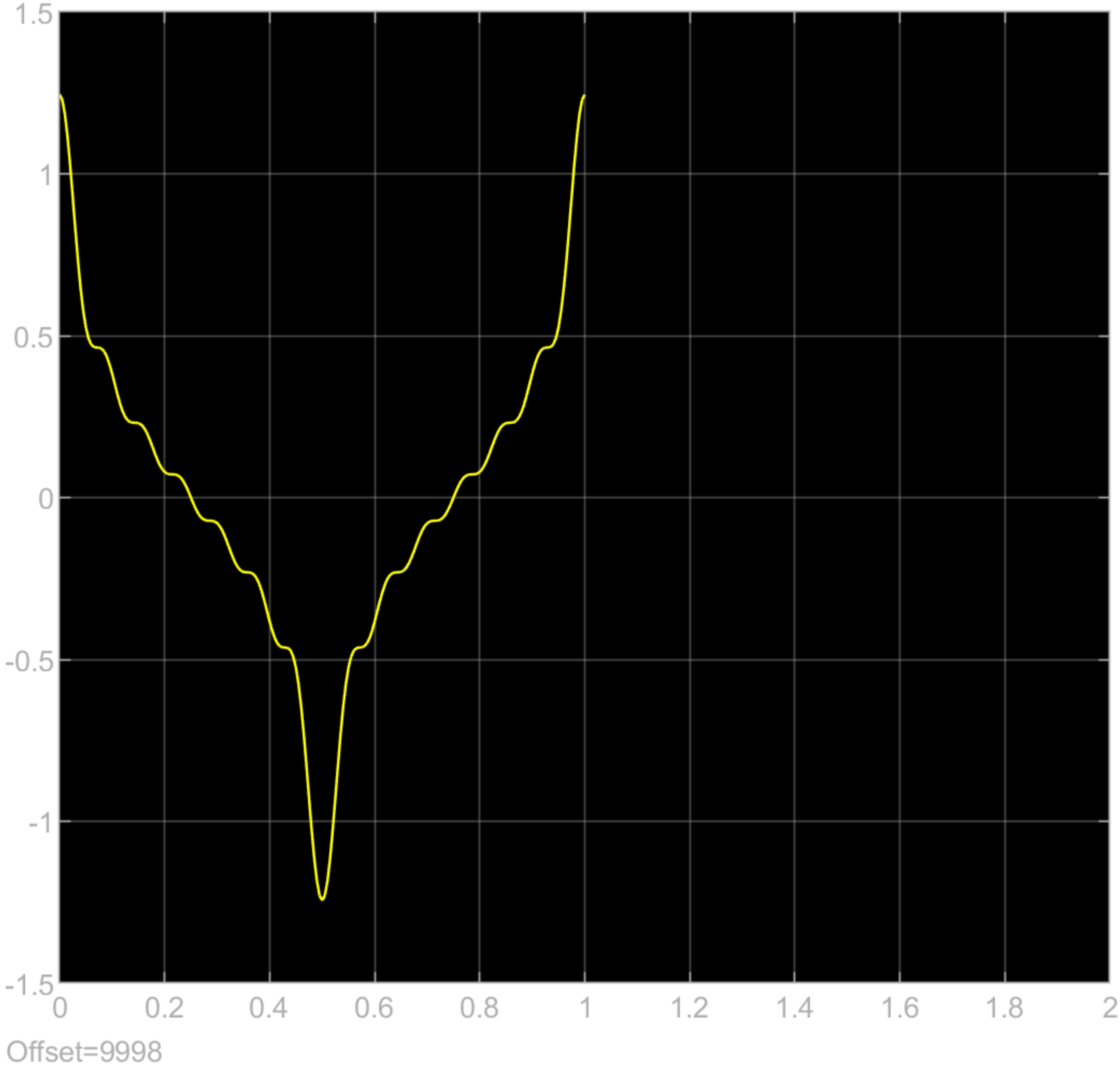

Signal
Amplitude
1
0.8
0.6
0.4
0.2
0
-2
0
2
4
6
8
10
Time in seconds
Spectral Density Plot
Spectral Density
300
200
100
0
0
5
10
15
20
25
30
35
40
45
50
Frequency in Hertz

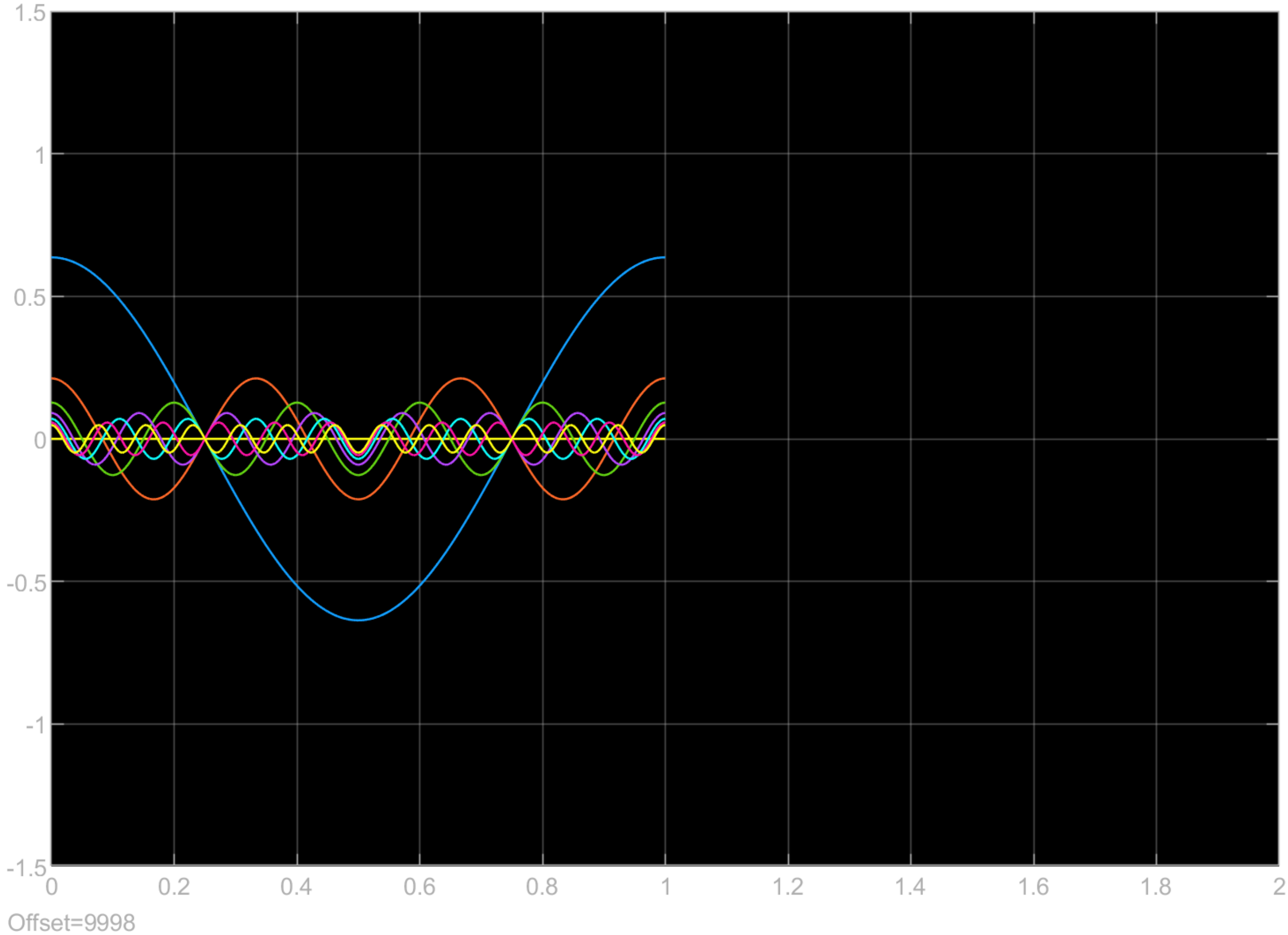

1.5
1
0.5
0
-0.5
-1
-1.5
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
Offset=9998




```
(2)
```


Signal
Amplitude
Time in seconds
Spectral Density Plot
Spectral Density
Frequency in Hertz


1.5
1
0.5
0
-0.5
-1
-1.5
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
Offset=9998

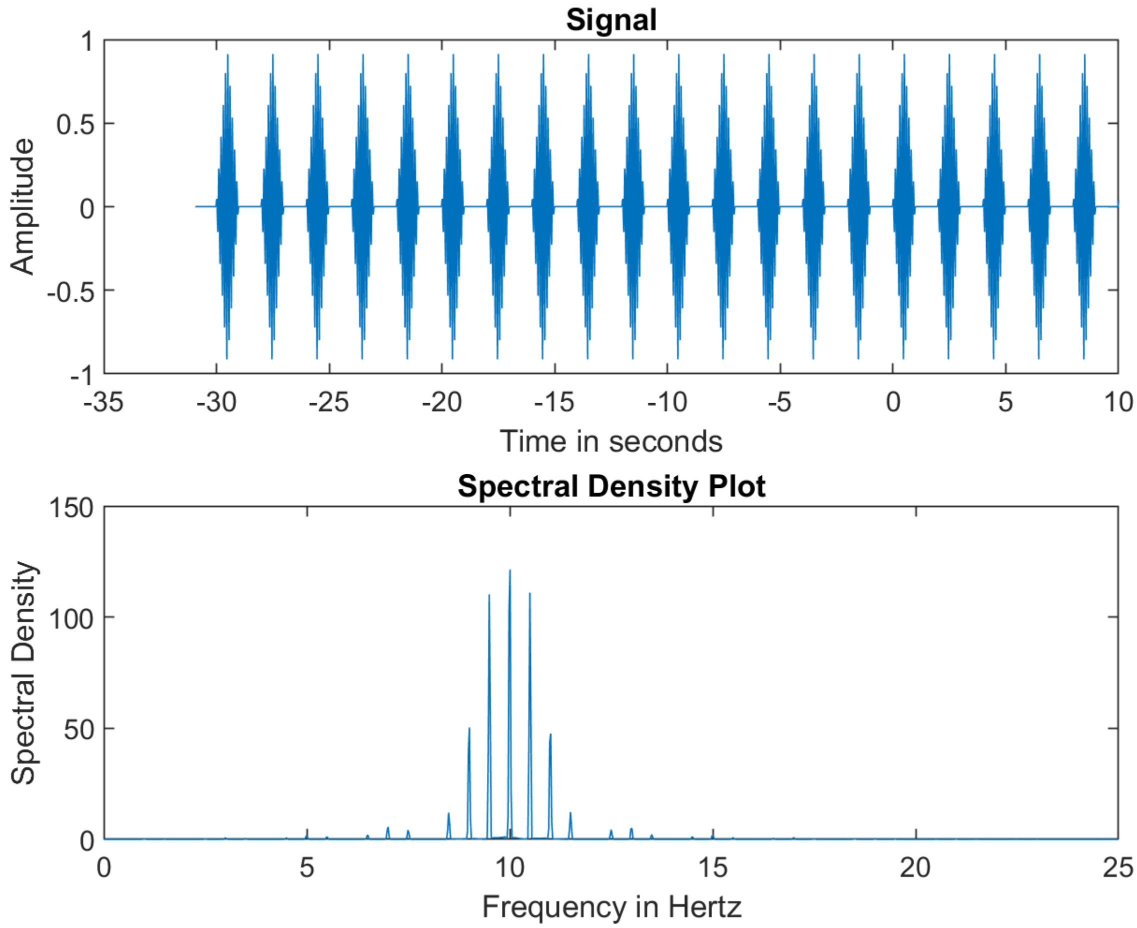


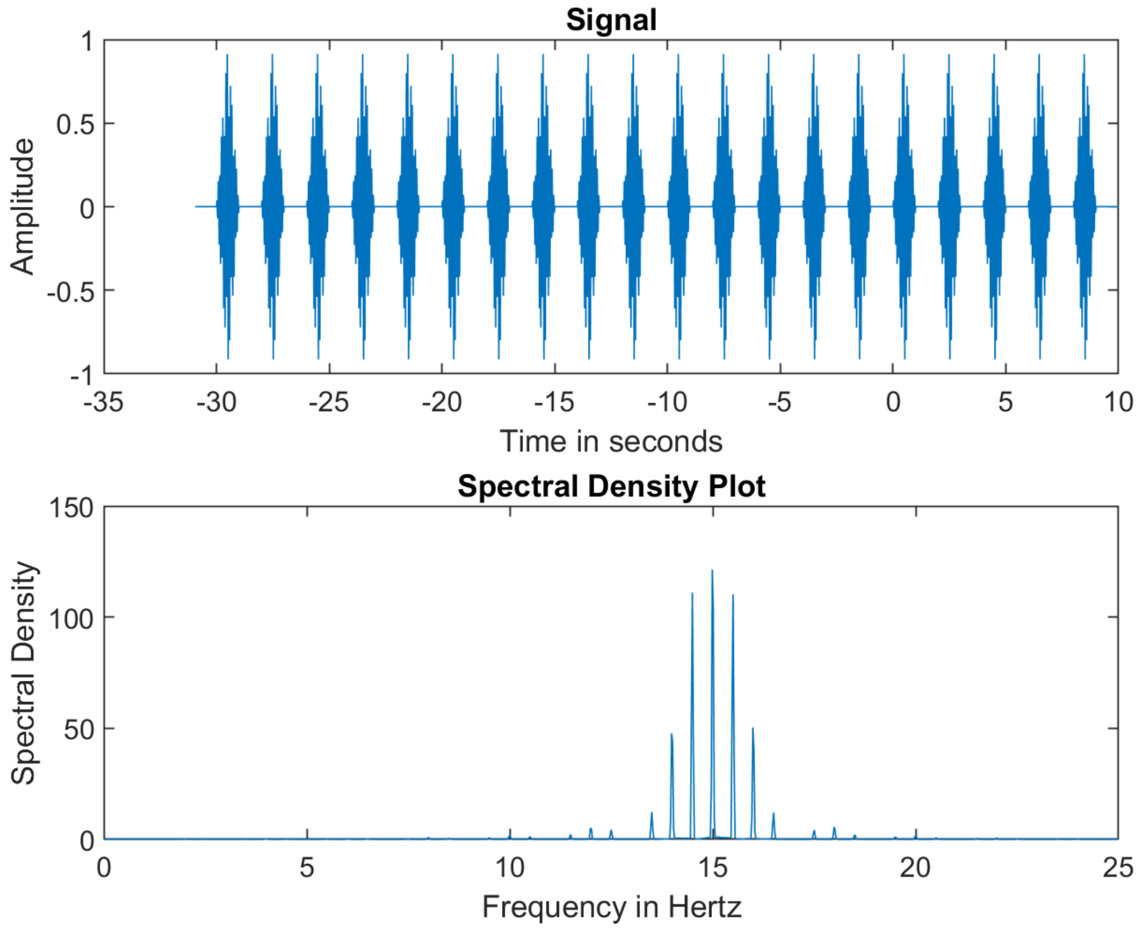
2. For each case in Q1, comment on how the synthesized waveform differs from the desired signal,and on the structure of the spectral density.

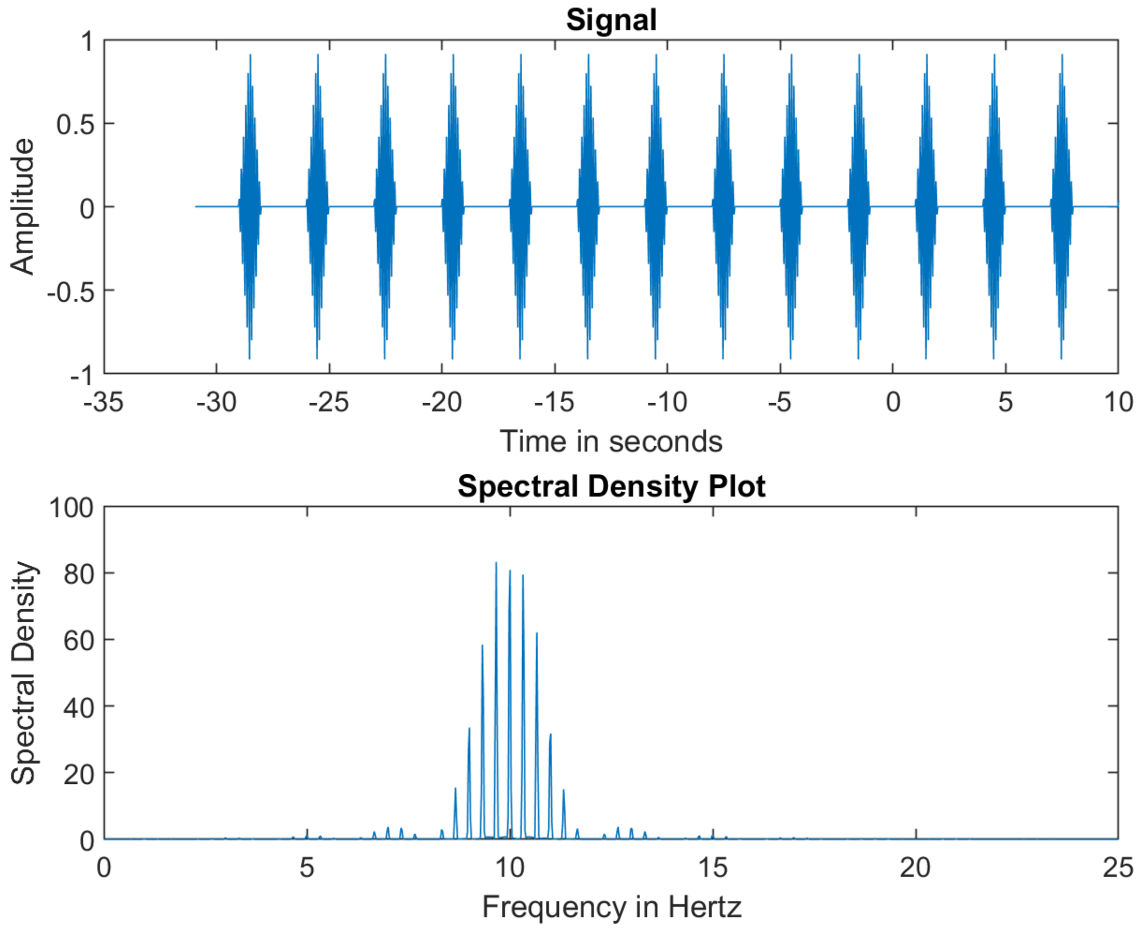
For signal (1), the result is similar to its original signal. For signal (2), the result does not describe original signal very well

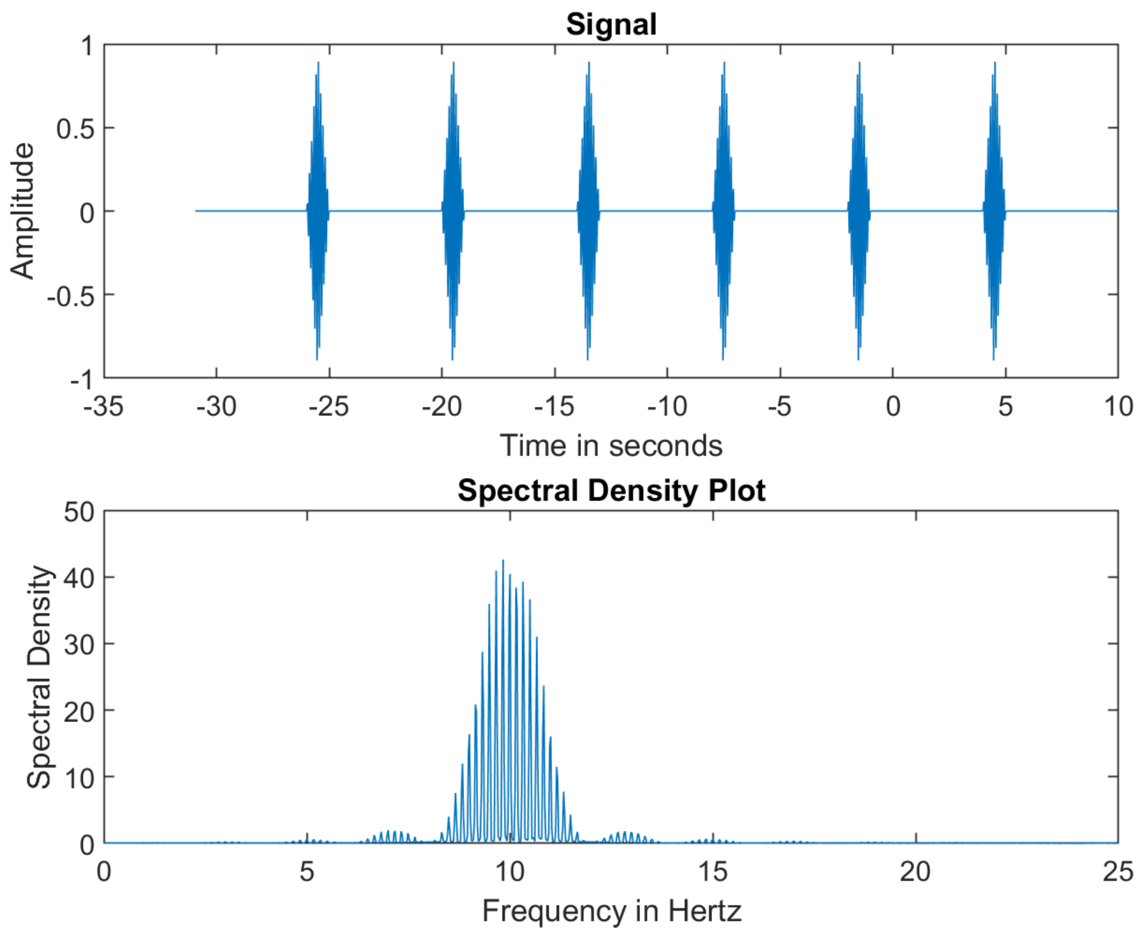
Exercise 4.2

1. Hand in plots of the output of the *Spectrum Analyzer* for each signal.


Signal
Amplitude
1
0.5
0
-0.5
-1
-35
-30
-25
-20
-15
-10
-5
0
5
10
Time in seconds
Spectral Density Plot
Spectral Density
150
100
50
0
0
5
10
15
20
25
Frequency in Hertz


Signal
Amplitude
1
0.5
0
-0.5
-1
-35
-30
-25
-20
-15
-10
-5
0
5
10
Time in seconds
Spectral Density Plot
Spectral Density
150
100
50
0
0
5
10
15
20
25
Frequency in Hertz


Signal
Amplitude
Time in seconds
1
0.5
0
-0.5
-1
-35
-30
-25
-20
-15
-10
-5
0
5
10
Spectral Density Plot
Spectral Density
Frequency in Hertz
100
80
60
40
20
0
0
5
10
15
20
25



2. What effect does changing the modulating frequency have on the spectral density?

the spectral density follow the modulating frequency and center at that value.

3. Why does the spectrum have a comb structure and what is the spectral distance between impulses? Why?

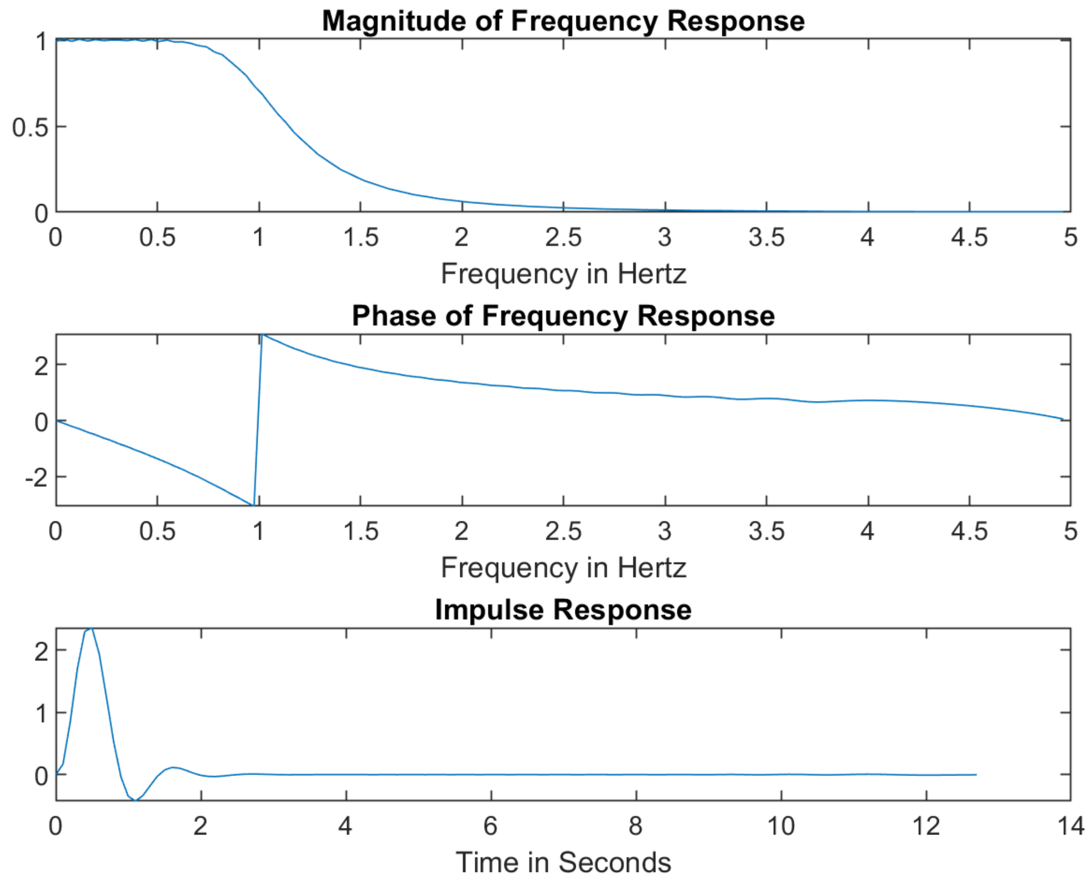
Because in time domain it has rep structure for our triangle signal. The distance is 0.5 for period of 2 because it is inverse of 2.

4. What would happen to the spectral density if the period of the triangle pulse were to increase toward infinity? (in the limit)

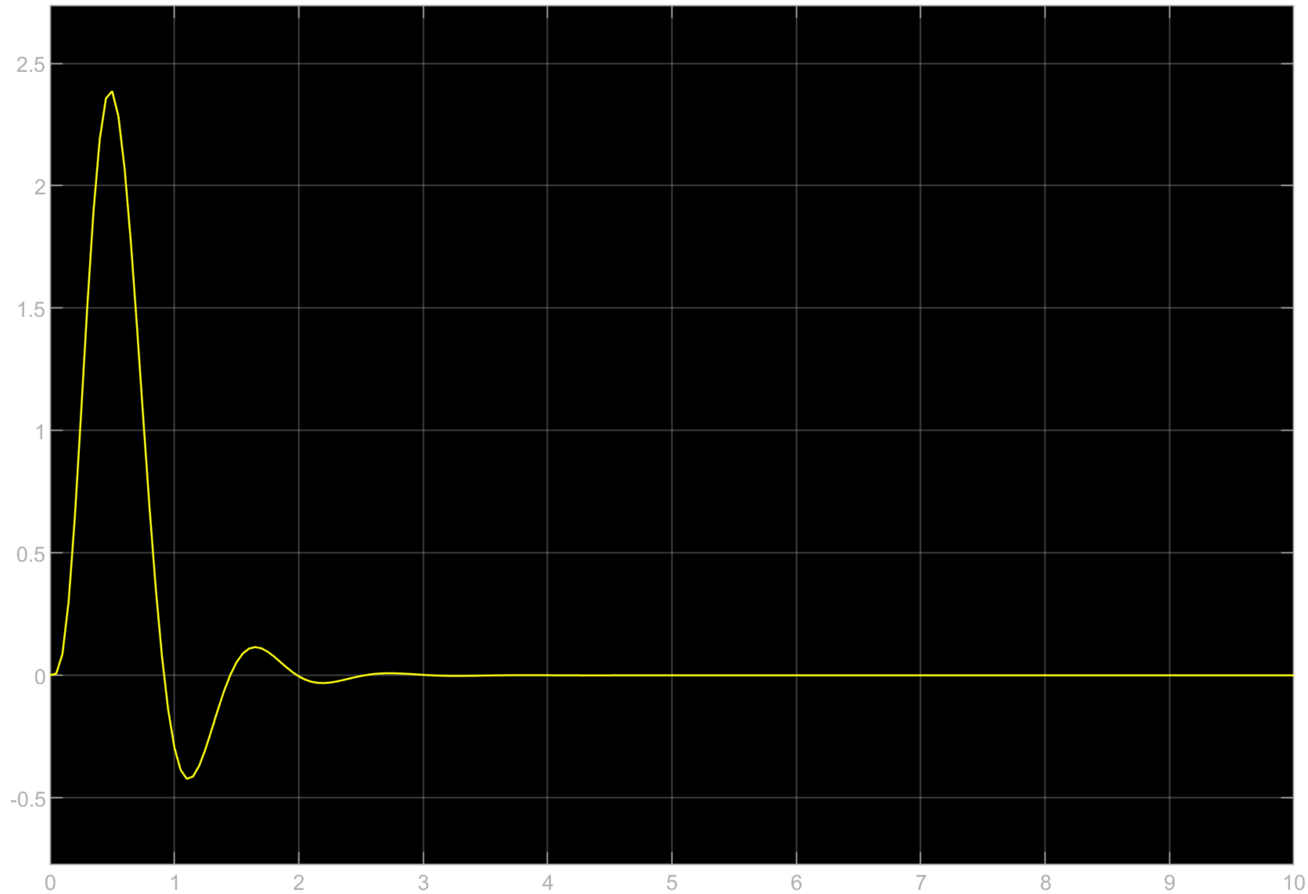
The distance between impulses will approach zero.

Exercise 4.3

1. Hand in the printout of the output of the Network Analyzer (magnitude and phase of the frequency response, and the impulse response).



2. Hand in the plot of the impulse response obtained using a unit step.



3. What are the advantages and disadvantages of each method?

The first method gives more information regarding phase and magnitude, but the frequency response is distorted by a little. The second method is the opposite.

Exercise 5.1

1. Complete the follwing function that computes the DTFT of a discrete-time signal.

```
def DTFT(x,n0,w):
    """
    This function computes the DTFT of a discrete-time signal.

    Parameters
    ---
    x: the discrete-time signal
    n0: time index corresponding to the 1st element of the x vector
    w: frequencies

    Returns
    ---
    X: the computed DTFT
    """
    pass
```

Note that if `x` is a vector of length N, then its DTFT is computed by

$$X(\omega) = \sum_{n=0}^{N-1} x[n]e^{-jw(n+n0)}$$

where w is a vector that contains the frequencies from $-\pi$ to π.

Hint: In Python, `1j` is defined as $\sqrt{-1}$. Use `np.exp(x)` to calculate e^x.

In [20]:
```
def DTFT(x,n0,w):
    X = sum(x[n]*np.exp(-1j*w*(n+n0)) for n in range(n0, len(x)-1, 1))
    return X
```

2. For the following signals

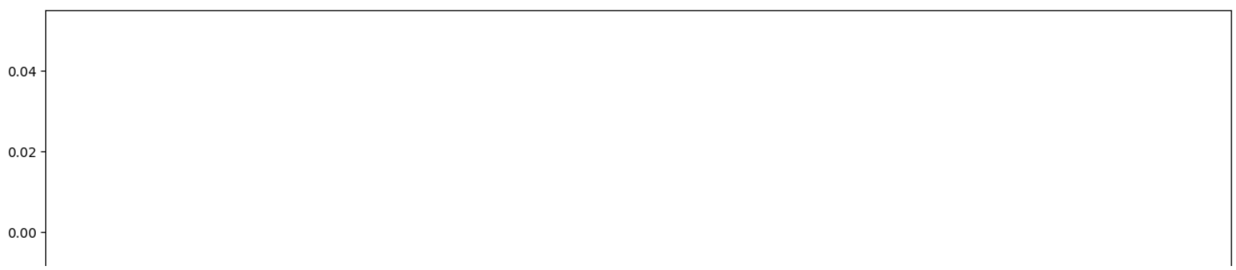
- $x[n] = \delta[n]$
- $x[n] = \delta[n-5]$
- $x[n] = (0.5)^n u[n]$

use your DTFT function to compute $X(\omega)$, and plots its magnitude and phase.

Hint: Use `np.power(a,b)` to calculate a^b. Use `np.abs()` and `np.angle()` to compute the magnitude and phase.

In [24]:
```
t = np.linspace(0, 50, 51)
x1 = [0]*50
x1[0] = 1
x1_mag = np.abs(DTFT(x1,0,2*math.pi))
x1_pha = np.angle(DTFT(x1,0,2*math.pi))

plt.figure(1)
plt.stem(t, x1_pha)
```

```
    497         if x.shape[0] != y.shape[0]:
--> 498             raise ValueError(f"x and y must have same first dimension, but "
    499                              f"have shapes {x.shape} and {y.shape}")
    500         if x.ndim > 2 or y.ndim > 2:

ValueError: x and y must have same first dimension, but have shapes (51,) and (1,)
```



Exercise 5.2: Magnitude and Phase of the Frequency Response of a Discrete-Time Systems

Consider the discrete-time system described by the following difference equation:

$$y[n] = 0.9y[n-1] + 0.3x[n] + 0.24x[n-1]$$

Assume that the system is **causal**.

1. Draw a system diagram.

insert your diagram here

2. Obtain the impulse response of the system by replacing $x[n]$ with $\delta[n]$ in the above equation. (Use causality to set up the initial conditions.)

write your answer here

3. Use your answer in Q2 to obtain the frequency response of the system.

write your answer here

4. Find the frequency response of the system using another method. Specifically, take the DTFT of the left-hand-side and right-hand-side of the difference equation, and then use linearity and the time-shifting property of the DTFT along with the fact that $H(\omega) = \frac{Y(\omega)}{X(\omega)}$

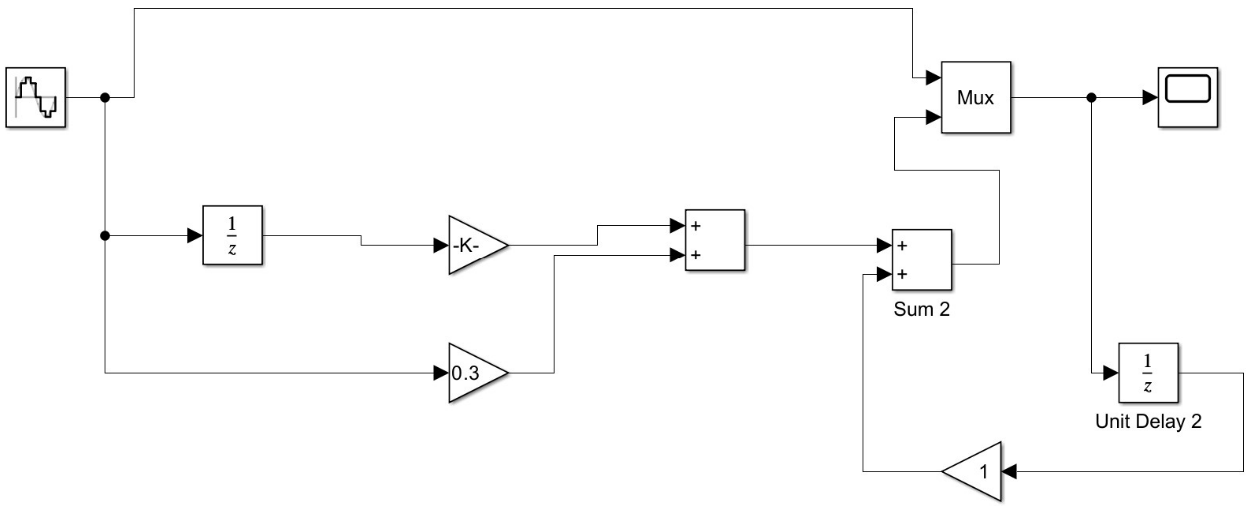
write your answer here

5. Write Python code to compute and plot the magnitude and phase responses, $|H(\omega)|$ and $\angle H(\omega)$, for $-\pi < \omega < \pi$.

In [6]:
```
# write your code here
```

Exercise 5.3

1. Insert the printout of your completed block diagram.



2. Enter both the amplitude measurements you made and their theoretical values.

ω	Measurements	Theoretical Values
$\pi/16$	6	
$\pi/8$		
$\pi/4$		

3. Plot the impulse response, and the magnitude and phase of the frequency response by using your DTFT function.

In [7]:
```
# write your code here
```