ECE 438 - Laboratory 6b
Discrete Fourier Transform and Fast Fourier Transform Algorithms (Week 2)

Last updated on February 27, 2022

Date:3/2
Section:

Name	Signature	Time spent outside lab
Student Name #1 [Ruixiang Wang]		
Student Name #2 [---%]		

	Below expectations	Lacks in some respect	Meets all expectations
Completeness of the report			
Organization of the report			
Quality of figures: *Correctly labeled with title, x-axis, y-axis, and name(s)*			
Understanding of the frequency range of DFT and effects of zero-padding (50 pts): *DFT and DTFT plots, Python code (DTFTsamples), questions*			
Implementation of Divide-and-Conquer DFT and FFT (50 pts): *Python codes (dcDFT, fft2, fft4, fft8, fft_stages), questions*			

In [1]:
```
import numpy as np
import matplotlib.pyplot as plt
from helper import DTFT, DFTsum, hamming
```

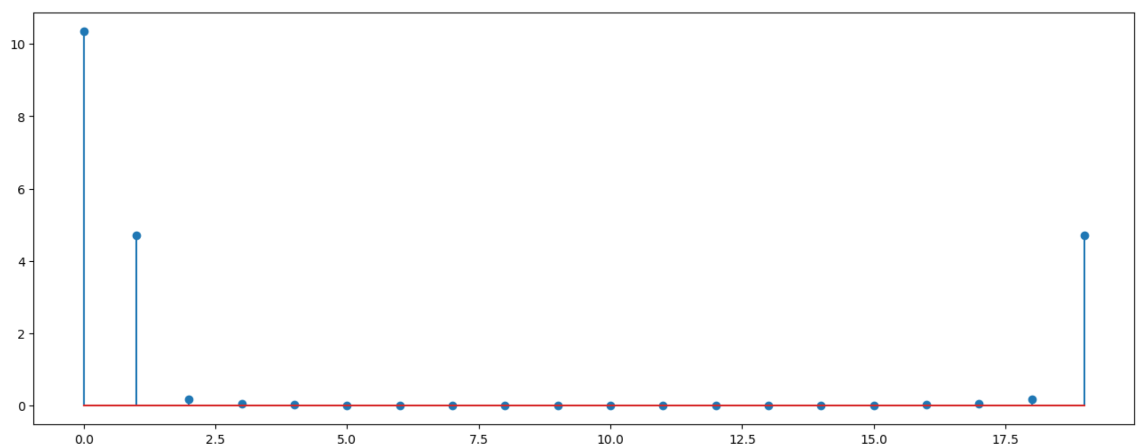
In [2]:
```
# make sure the plot is displayed in this notebook
%matplotlib inline
# specify the size of the plot
plt.rcParams['figure.figsize'] = (16, 6)

# for auto-reloading extenrnal modules
%load_ext autoreload
%autoreload 2
```

Exercise 2.1: Shifting the Frequency Range

1. Create a Hamming window `x` of length `N = 20`, using the provided function `hamming()`, then compute the 20 point DFT of `x` using the provided function `DFTsum()`, and finally, plot the magnitude of the DFT, $|X_{20}(k)|$, versus the index k.

In [24]:
```
x = hamming(20)
X_20 = DFTsum(x)
k = np.linspace(0,19,20)
plt.stem(k, np.abs(X_20))
```

Out[24]: `<StemContainer object of 3 artists>`



2. Complete the function `DTFTsamples` below to compute the samples of the DTFT and their corresponding frequencies.

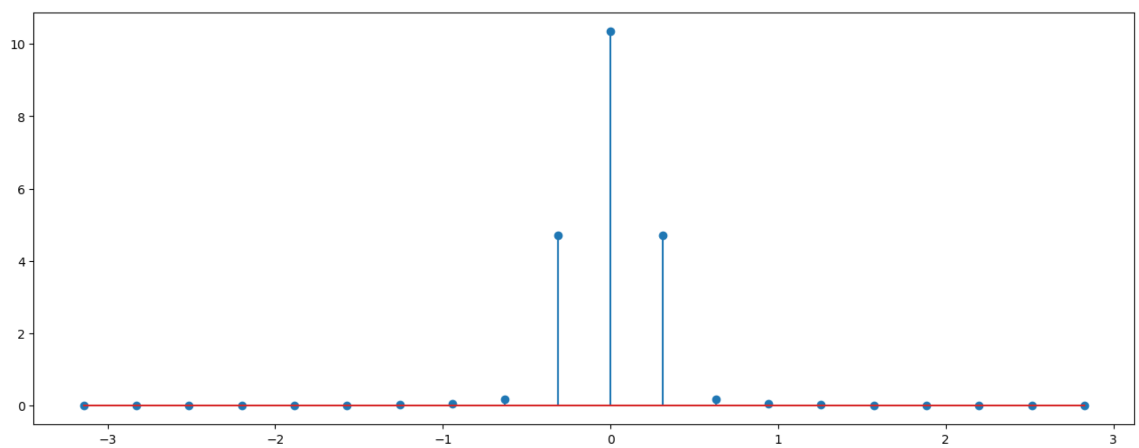
Note: Your function `DTFTsamples(x)` should call your function `DFTsum()` and use the function `np.fft.fftshift()`.

In [30]:
```
def DTFTsamples(x):
    """
    Compute samples of the DTFT and their corresponding frequencies in the

    Parameters:
    ---
    x: an N-point vector

    Returns:
    ---
    X: the length N vector of DTFT samples
    w: he length N vector of corresponding radial frequencies
    """
    k = np.linspace(0, len(x)-1, len(x))
    w = 2*np.pi*k/len(x)
    w[w>=np.pi] = w[w>=np.pi] - 2*np.pi
    N = len(x)
    X1 = DFTsum(x)
#     print(w)
    X = np.fft.fftshift(X1)
    w = np.fft.fftshift(w)
    return X, w
```

3. Use your function `DTFTsamples` to compute DTFT samples of the Hamming window of length $N = 20$. Plot the magnitude of these DTFT samples versus frequency in rad/sample.

In [31]:
```
X_20,w = DTFTsamples(x)
# print(w)
plt.stem(w, np.abs(X_20))
```

Out[31]: <StemContainer object of 3 artists>

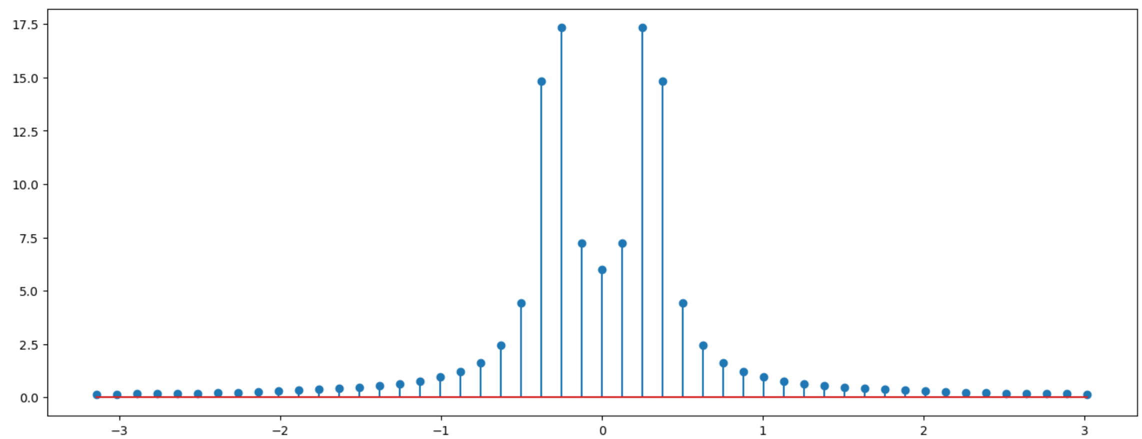


Exercise 2.2: Zero Padding

1. For $N = 50$, compute the vector x containing the values $x[0], \ldots, x[N-1]$, then compute the samples of $X[k]$ using your function `DTFTsamples()` , and finally plot the magnitude of the DTFT samples versus frequency in rad/sample.

In [32]:
```
N = 50
x = [0]*N

for n in range(0, N-1, 1):
    x[n] = np.sin(0.1*np.pi*n)

X,w = DTFTsamples(x)

plt.stem(w, np.abs(X))
```

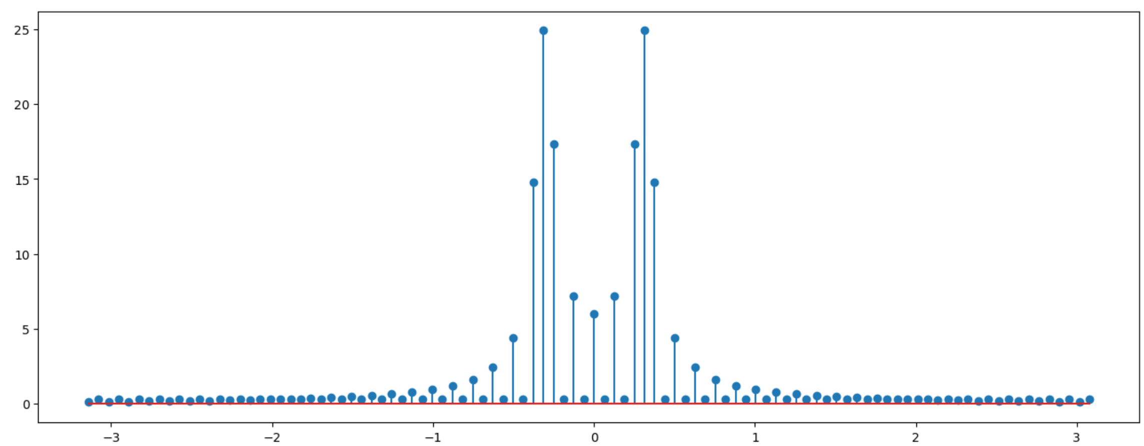
Out[32]: <StemContainer object of 3 artists>



2. For $N = 100$, compute the vector x containing the values $x[0], \ldots, x[N-1]$, then compute the samples of $X[k]$ using your function `DTFTsamples()` , and finally plot the magnitude of the DTFT samples versus frequency in rad/sample.

In [33]:
```
N = 100
x = [0]*N

for n in range(0, 49, 1):
    x[n] = np.sin(0.1*np.pi*n)


for n in range(50, N-1, 1):
    x[n] = 0

X,w = DTFTsamples(x)

plt.stem(w, np.abs(X))
```

Out[33]: `<StemContainer object of 3 artists>`



3. Which plot looks more like the true DTFT?

N = 100 looks more like the true DTFT because it has more sample point

3. Explain why the plots look so different.

The zeros pedding increase sampling point by increase N -> increase k

Exercise 3.1: Implementation of Divide-and-Conquer DFT

1. Complete the function `dcDFT` below.

In [54]:
```
def dcDFT(x):
    """
    Parameters:
    ---
    x: a vector of even length N

    Returns: the DFT of x
    """


    # Step 1
    # Separate the samples of x into even and odd points.
    # Hint: The Python function x0 = x[0:N:2] can be used to obtain the “ev
    N = len(x)
    x0 = x[0:N:2]
    x1 = x[1:N:2]

    # Step 2
    # Use your function DFTsum to compute the two N/2 point DFT’s.
    X0 = DFTsum(x0)
    X1 = DFTsum(x1)

    # Step 3
    # Multiply by the twiddle factors$
    N_div_2 = int(N/2)
    W = [0]*N_div_2
    for k in range(N_div_2):
        W[k] = np.exp(-1j*2*np.pi*k/N)
    X_k = [0]*N

    # Step 4
    # Combine the two DFT’s to form X
    for k in range(N_div_2):
        X_k[k] = X0[k] + W[k]*X1[k]

    for k in range(N_div_2):
        X_k[k+N_div_2] = X0[k] - W[k]*X1[k]

    X = X_k
    return X
```

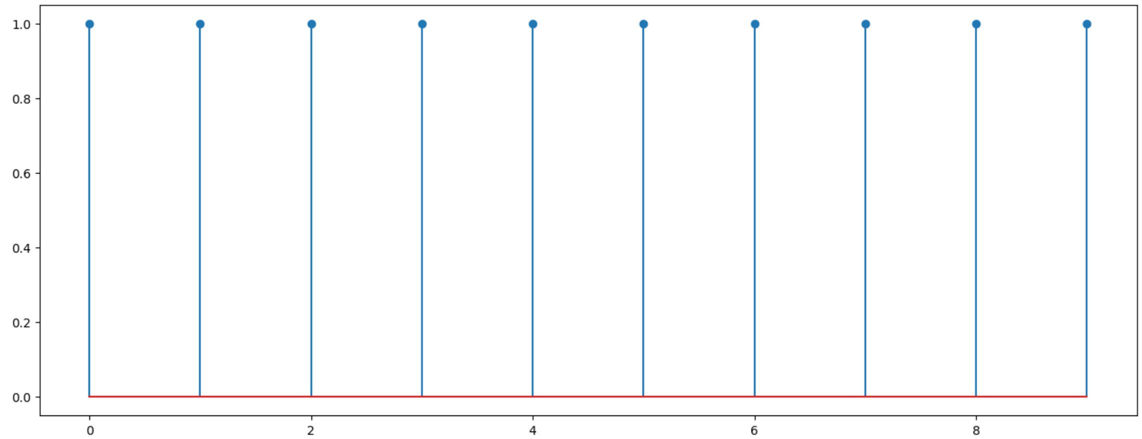
2. Test your function `dcDFT` by using it to compute and plot the DFT’s of the following signals:

- $x[n] = \delta[n]$ for $N = 10$
- $x[n] = 1$ for $N = 10$
- $x[n] = e^{j2\pi n/10}$ for $N = 10$

Make sure you plot the results in separate cells.

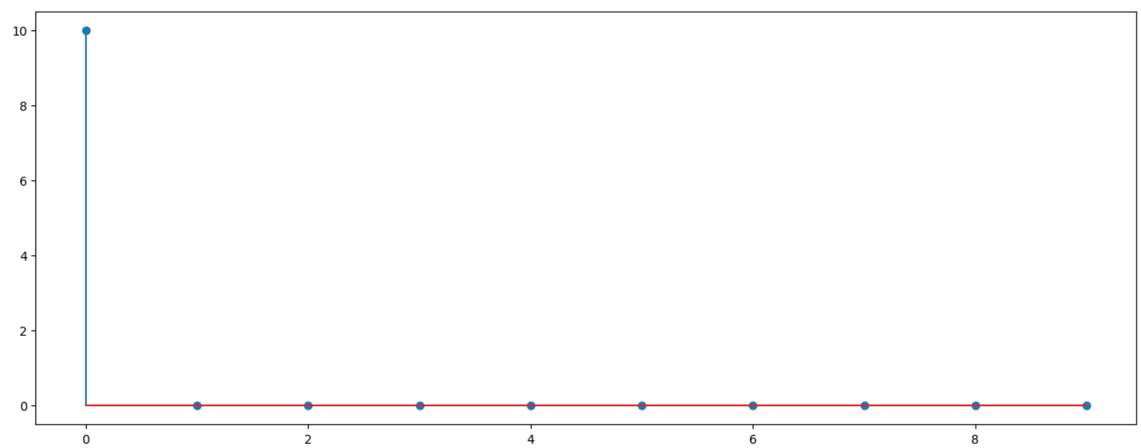
In [55]:
```
n = np.linspace(0, 9, 10)
x = [0]*len(n)
x[0] = 1
X1 = dcDFT(x)
plt.stem(n,X1)
```

Out[55]: <StemContainer object of 3 artists>



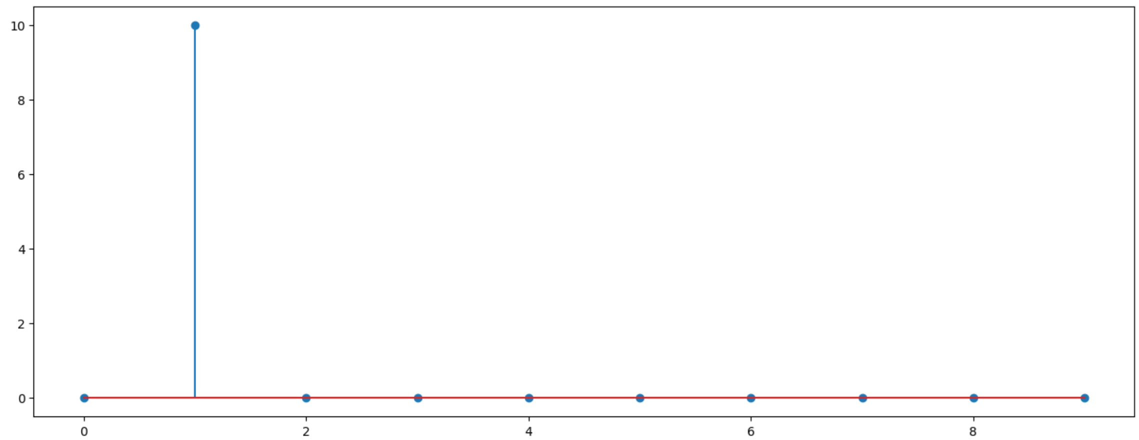
In [56]:
```
x = [1]*len(n)
X2 = dcDFT(x)
plt.stem(n,X2)
```

Out[56]: <StemContainer object of 3 artists>



In [57]:
```
x = [0]*len(n)
for i in range(len(n)):
    x[i] = np.exp(1j*2*np.pi*i/10)
X3 = dcDFT(x)
plt.stem(n,X3)
```

Out[57]: <StemContainer object of 3 artists>



3. Determine the number of multiplies that are required in this approach to computing an N point DFT. (Consider a multiply to be one multiplication of real or complex numbers.)

Hint: Refer to the diagram of Figure 1, and remember to consider the $N/2$ point DFTs.

N/2+(N/2)^2

Exercise 3.2: Recursive Divide and Conquer (Part 1)

1. Complete the Python functions below to compute the 2, 4, and 8-point FFT's.

Note: The function `FFT2` should directly compute the 2-point DFT using (13), but the functions `FFT4` and `FFT8` should compute their respective FFT's using the divide and conquer strategy. This means that `FFT8` should call `FFT4` , and `FFT4` should call `FFT2` .

In [78]:
```
def FFT2(x):
    """
    Parameters:
    ---
    x: the input signal

    Returns:
    ---
    X: the 2-point DFT of x
    """
    X = [0]*2
    X[0] = x[0]+x[1]
    X[1] = x[0]-x[1]
    return X
```

In [79]:
```
def FFT4(x):
    """
    Parameters:
    ---
    x: the input signal

    Returns:
    ---
    X: the 4-point DFT of x
    """
    X = [0]*4
    X0 = FFT2(x[0:4:2])
    X1 = FFT2(x[1:4:2])
    N=4

    N_div_2 = int(N/2)
    W = [0]*N_div_2
    for k in range(N_div_2):
        W[k] = np.exp(-1j*2*np.pi*k/N)
    X_k = [0]*N
    for k in range(N_div_2):
        X_k[k] = X0[k] + W[k]*X1[k]

    for k in range(N_div_2):
        X_k[k+N_div_2] = X0[k] - W[k]*X1[k]

    return X_k
```

In [80]:
```
def FFT8(x):
    """
    Parameters:
    ---
    x: the input signal

    Returns:
    ---
    X: the 8-point DFT of x
    """


    X = [0]*8
    X0 = FFT4(x[0:8:2])
    X1 = FFT4(x[1:8:2])
    N=8

    N_div_2 = int(N/2)
    W = [0]*N_div_2
    for k in range(N_div_2):
        W[k] = np.exp(-1j*2*np.pi*k/N)
    X_k = [0]*N
    for k in range(N_div_2):
        X_k[k] = X0[k] + W[k]*X1[k]

    for k in range(N_div_2):
        X_k[k+N_div_2] = X0[k] - W[k]*X1[k]

    return X_k
```

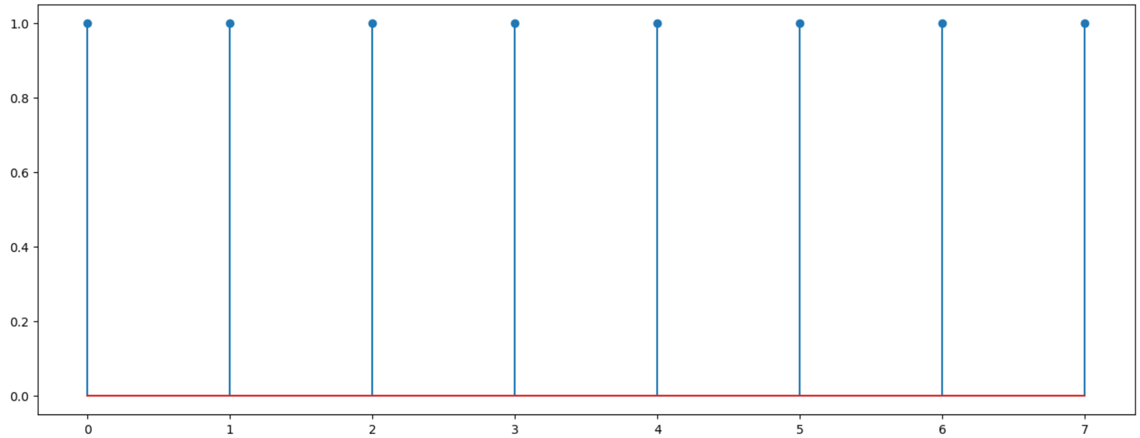
2. Test your function `FFT8` by using it to compute the DFT's of the following signals. Compare these results to the previous ones.

- $x[n] = \delta[n]$ for $N = 8$
- $x[n] = 1$ for $N = 8$
- $x[n] = e^{j2\pi n/8}$ for $N = 8$

Make sure you plot the results in separate cells.

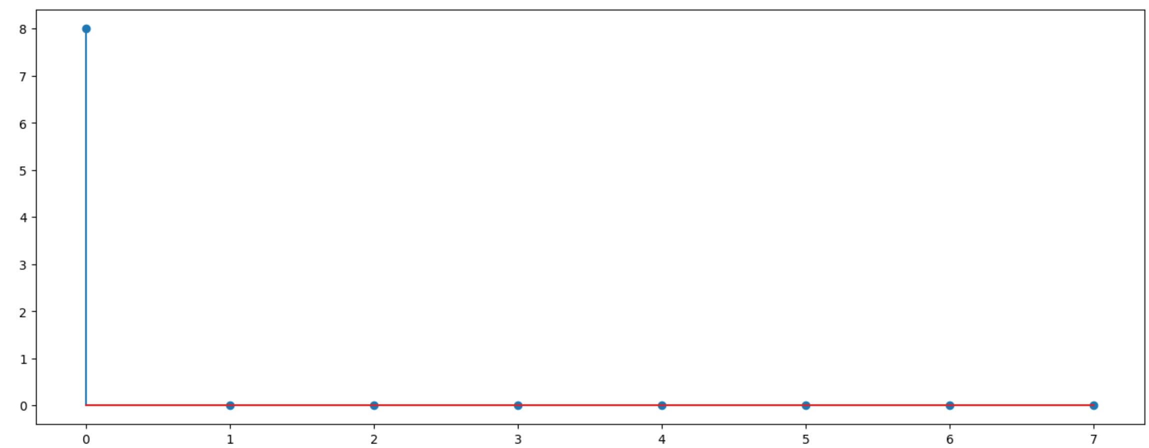
In [82]:
```
n = np.linspace(0,7,8)
x = [0]*len(n)
x[0] = 1
X1 = FFT8(x)
plt.stem(n,X1)
```

Out[82]: <StemContainer object of 3 artists>



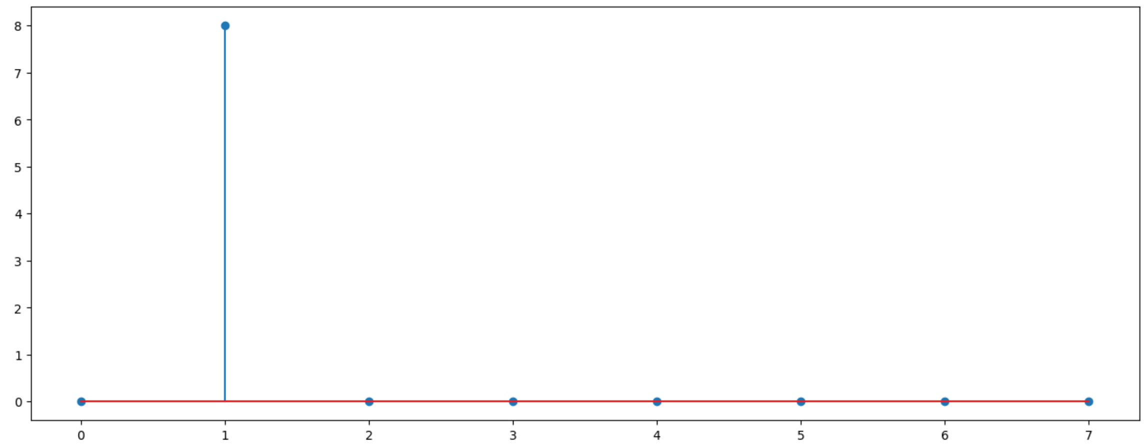
In [83]:
```
x = [1]*len(n)
X2 = FFT8(x)
plt.stem(n,X2)
```

Out[83]: <StemContainer object of 3 artists>



In [91]:
```
x = [0]*len(n)
for i in range(len(n)):
    x[i] = np.exp(1j*2*np.pi*i/8)
X3 = FFT8(x)
plt.stem(n,X3)
```

Out[91]: <StemContainer object of 3 artists>



3. List the output (not plot) of `FFT8` for the case $x[n] = 1$ for $N = 8$.

In [85]:
```
print(X2)
```

```
[(8+0j), 0j, 0j, 0j, 0j, 0j, 0j, 0j]
```

4. Calculate the total number of multiplies by twiddle factors required for your 8-point FFT. (A multiply is a multiplication by a real or complex number.)

8

5. Determine a formula for the number of multiplies required for an $N = 2^p$ point FFT. Leave the expression in terms of N and p. How does this compare to the number of multiplies required for direct implementation when $p = 10$?

That would be N multiplies. For direct implementation when p = 10, number of multiplies is 1024/2+(1024/2)^2 = 262656. Significantly reduced.

Exercise 3.3: Recursive Divide and Conquer (Part 2)

1. Complete the recursive function `fft_stage` below to perform one stage of the FFT algorithm for a power-of-2 length signal.

Note: the body of this function should look very similar to previous functions written in this lab.

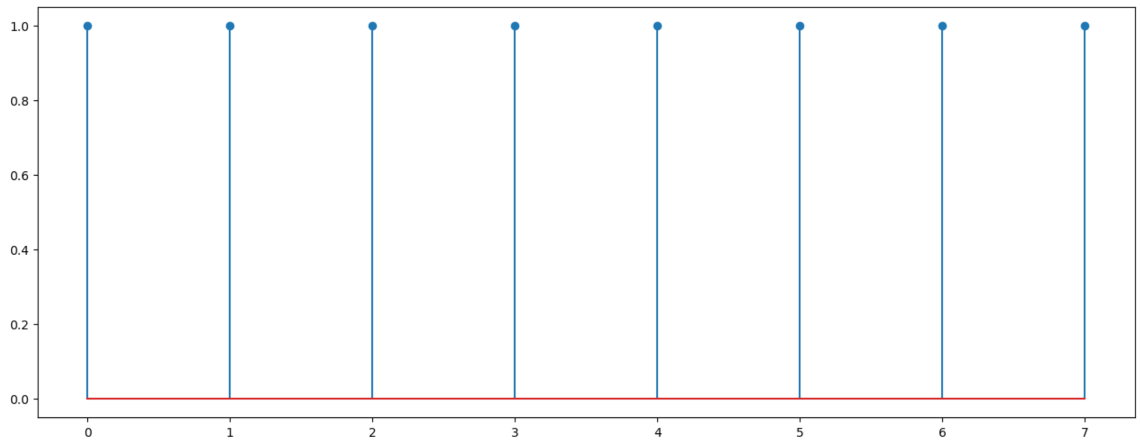
In [86]:
```
def fft_stage(x):
    """
    Performs one stage of the FFT algorithm for a power-of-2 length signal

    Parameters:
    ---
    x: a power-of-2 length signal

    Returns:
    ---
    X: the DFT of the inpu signal
    """

    # Step 1
    # Determine the length of the input signal.
    N = len(x)

    # Step 2
    # If N == 2, then the function should just compute the 2-pt DFT as in e
    if N == 2:
        X = [0]*2
        X[0] = x[0]+x[1]
        X[1] = x[0]-x[1]
        return X

    # Step 3
    # If N > 2, then the function should perform the FFT steps described pr
    # (i.e. decimate, compute (N/2)-pt DFTs, re-combine),
    # calling fft_stage(x) to compute the (N/2)-pt DFTs.
    elif N > 2:
        X = [0]*N
        X0 = fft_stage(x[0:N:2])
        X1 = fft_stage(x[1:N:2])

        N_div_2 = int(N/2)
        W = [0]*N_div_2
        for k in range(N_div_2):
            W[k] = np.exp(-1j*2*np.pi*k/N)
        X = [0]*N
        for k in range(N_div_2):
            X[k] = X0[k] + W[k]*X1[k]

        for k in range(N_div_2):
            X[k+N_div_2] = X0[k] - W[k]*X1[k]

    return X
```

2. Test `fft_stage(x)` on the three 8-point signals given above, and verify that it returns the same results as `FFT8(x)`.

Make sure you plot the results in separate cells.

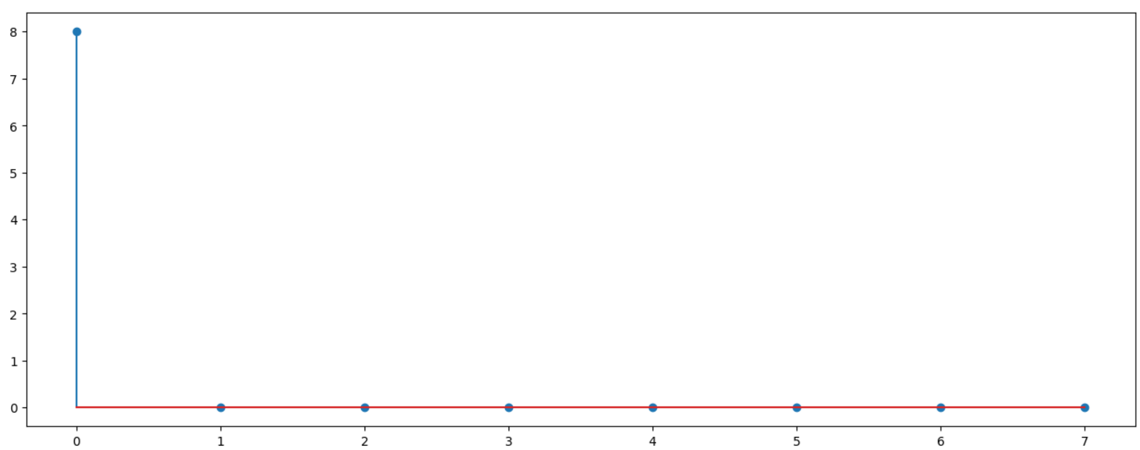
In [87]:
```
n = np.linspace(0,7,8)
x = [0]*len(n)
x[0] = 1
X1 = fft_stage(x)
plt.stem(n,X1)
```

Out[87]: <StemContainer object of 3 artists>



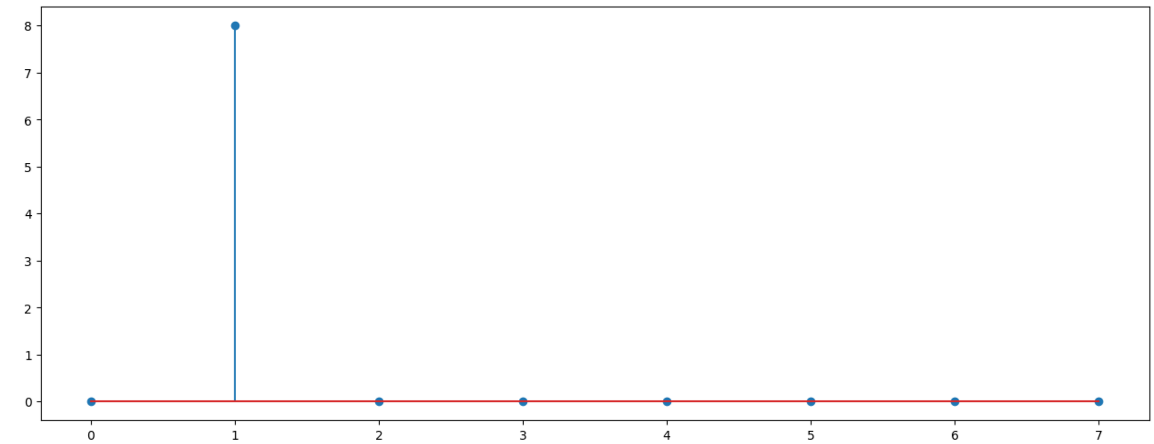
In [88]:
```
x = [1]*len(n)
X2 = fft_stage(x)
plt.stem(n,X2)
```

Out[88]: <StemContainer object of 3 artists>



In [90]:
```
x = [0]*len(n)
for i in range(len(n)):
    x[i] = np.exp(1j*2*np.pi*i/8)
X3 = fft_stage(x)
plt.stem(n,X3)
```

Out[90]:
```
<StemContainer object of 3 artists>
```



4. References

[1] J. W. Cooley and J. W. Tukey, “An algorithm for the machine calculation of complex Fourier series,” Mathematics of Computation, vol. 19, no. 90, p. 297-301, April 1965.