ECE 438 - Laboratory 7a
Discrete-Time Random Process (Week 1)

Last updated on March 1, 2022

Date:3/9
Section:

Name	Signature	Time spent outside lab
Student Name #1 [Ruixiang Wang]		
Student Name #2 [---%]		

	Below expectations	Lacks in some respect	Meets all expectations
Completeness of the report			
Organization of the report			
Quality of figures: *Correctly labeled with title, x-axis, y-axis, and name(s)*			
Understanding of random variables and linear transformations (35 pts): *Plots, sample means and variances of X and Y, derivation of mean and variance of Y, transformation and pdf of Y, Python code, questions*			
Understanding of CDF estimation (20 pts): *Python code and plots*			
Understanding of generating samples from a given distribution (20 pts): *Derivation of transformation, Python code, plots*			
Understanding of PDF estimation (25 pts): *Plots, questions*			

In [2]:
```
import numpy as np
import matplotlib.pyplot as plt
```

In [3]:
```
# make sure the plot is displayed in this notebook
%matplotlib inline
# specify the size of the plot
plt.rcParams['figure.figsize'] = (16, 6)

# for auto-reloading externnal modules
%load_ext autoreload
%autoreload 2
```

Exercise 2.1

1. Use the Python function [`np.random.normal(loc=0, scale=1, size=1000)` (https://numpy.org/doc/stable/reference/random/generated/numpy.random.normal.html)](https://numpy.org/doc/stable/reference/random/generated/numpy.random.normal.html) to generate 1000 samples of X, denoted as $X_1, X_2, \ldots, X_{1000}$. We will assume our generated samples are *i.i.d.*.

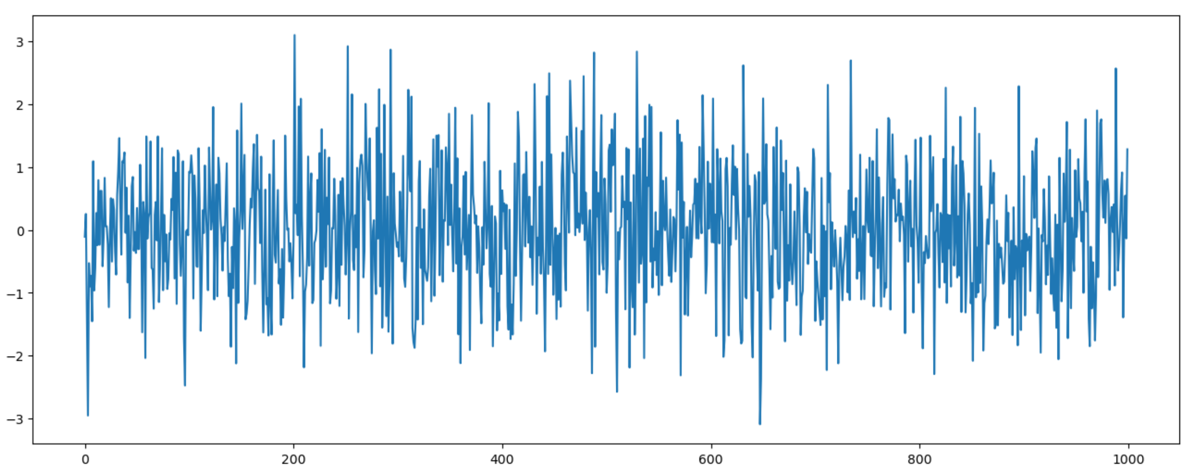
Note: `loc` is the mean ("centre") of the distribution, while `scale` is the **standard deviation** (spread or "width") of the distribution.

In [3]:
```
X = np.random.normal(loc=0, scale=1, size=1000)
```

2. Plot them using the function `plt.plot()` .

In [4]:
```
plt.plot(X)
```

Out[4]: [<matplotlib.lines.Line2D at 0x1ef20251610>]



3. Write Python functions to compute the sample mean and sample variance of equations (6) and (7) without using the predefined `mean()` , `variance()` , `np.mean()` and `np.var()` functions.

In [10]:
```
def get_sample_mean(X):
    """
    Parameters
    ---
    X: the samples of the random variable

    Returns
    ---
    mean_X: the sample mean of the random variable
    """
    mean_X = 0
    for i in range(len(X)):
        mean_X += X[i]

    mean_X /= len(X)
    return mean_X
```

In [11]:
```
def get_sample_var(X):
    """
    Parameters
    ---
    X: the samples of the random variable

    Returns:
    ---
    var_X: the sample variance of the random variable
    """
    var_X = 0
    mean_X = get_sample_mean(X)
    for i in range(len(X)):
        var_X += (X[i] - mean_X)**2

    var_X /= (len(X) - 1)
    return var_X
```

4. Use these functions to compute the sample mean and sample variance of the samples you just generated.

Hint: the following functions may be useful: `np.sum()` and `np.square`

In [17]:
```
mean_X = get_sample_mean(X)
var_X = get_sample_var(X)
print(mean_X)
print(var_X)
```

```
-0.003378020679146177
1.0106247274107099
```

Exercise 2.2

1. Using the linearity property of expectation, find the mean μ_Y and variance σ_Y^2 of Y in terms of a, b, μ_X and σ_X^2. Show your derivation in detail.

Hint: First find the mean, then substitute the result when finding the variance.

μY = aμX + b
$\sigma^{2}Y$ = $a^2\sigma^{2}X$

2. Consider a linear transformation of a Gaussian random variable X with mean 0 and variance 1. Calculate the constants a and b which make the mean and the variance of Y 3 and 9, respectively.

3 = b
+-3 = a

3. Use equation (5) to find the probability density function (PDF) for Y.

fY(y) = exp(-(y-3)^2/18)/sqrt(18pi)

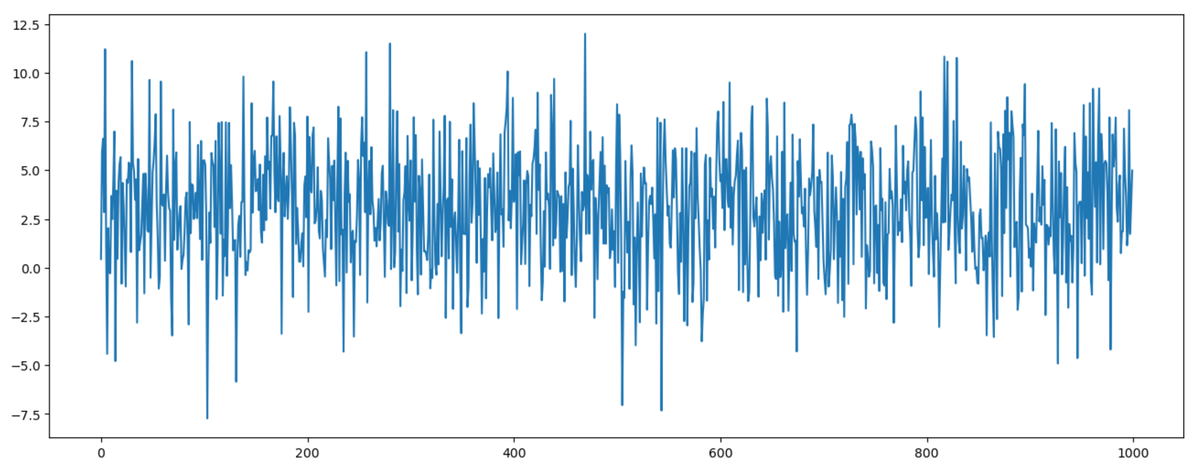
4. Generate 1000 samples of X, and then calculate 1000 samples of Y by applying the linear transformation in equation (10), using the a and b that you just determined.

In [69]:
```
X = np.random.normal(loc=0, scale=1, size=1000)
a = -3
b = 3
Y = a*X + b
```

5. Plot the resulting samples of Y.

In [70]:
```
plt.plot(Y)
```

Out[70]:
```
[<matplotlib.lines.Line2D at 0x1ef233d55b0>]
```



6. Use your functions to calculate the sample mean and sample variance of the samples of Y.

In [71]:
```
mean_Y = get_sample_mean(Y)
var_Y = get_sample_var(Y)
print(mean_Y)
print(var_Y)
```

```
3.02268540329822
8.939229651364979
```

Exercise 3.1

1. Write a function to compute the empirical CDF $\hat{F}_X(t)$ from the sample vector X at the points specified in the vector t.

Hint: The expression `np.sum(X <= s)` will return the number of elements in the vector `X` which are less than or equal to `s` .**

In [83]:
```
def empcdf(X,t):
    """
    Parameter
    ---
    X: the samples of the random variable
    t: the samples of time

    Return
    ---
    F: the empirical CDF
    """
    N = len(X)
    F = [0]*len(t)
    i = 0
    for s in t:
        F[i] = np.sum(X <= s)/N
        i += 1

    return F
```

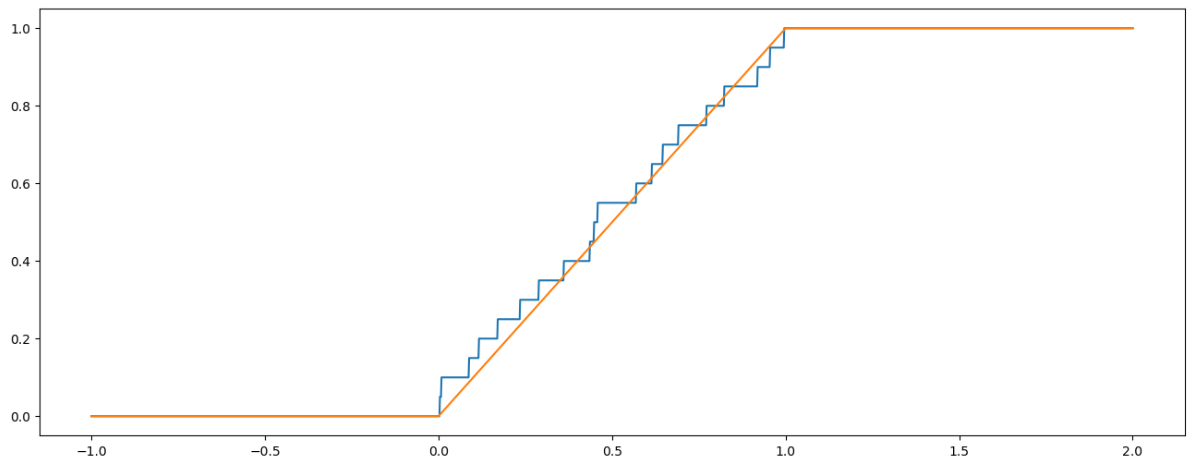
2. For $N = 20$ and $N = 200$,

- **Generate a sample of $\text{Uniform}[0, 1]$ random variables using the function `X = np.random.uniform(0, 1, N)` .**
- **Plot the CDF estimate in the range `t = np.linspace(-1, 2, 2000)` , and superimpose the true distribution for a $\text{Uniform}[0, 1]$ random variable.**

Note: make sure the figures for $N = 20$ and $N = 200$ are plotted in separate cells.

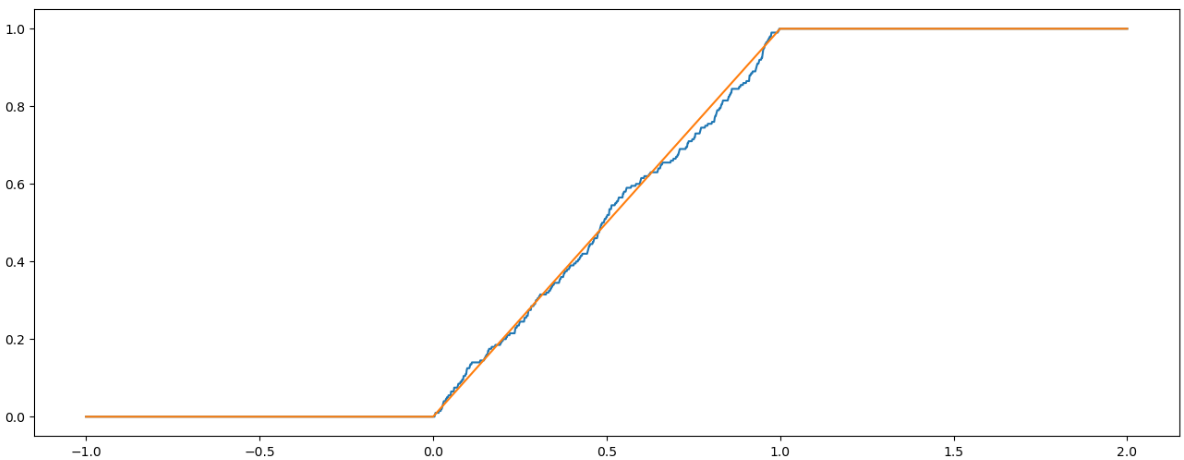
In [106]:
```
# N = 20
X_20 = np.random.uniform(0,1,20)
t = np.linspace(-1,2,2000)
n = [0]*len(t)
i = 0
for s in t:
    if s<0:
        n[i] = 0
        i+=1
    elif s<1:
        n[i] = s
        i+=1
    else:
        n[i] = 1
        i+=1



F_20 = empcdf(X_20, t)
plt.plot(t,F_20)
plt.plot(t,n)
```

Out[106]: [<matplotlib.lines.Line2D at 0x1ef25fb18b0>]



In [108]:
```
# N = 200
X_200 = np.random.uniform(0,1,200)
t = np.linspace(-1,2,2000)
n = [0]*len(t)
i = 0
for s in t:
    if s<0:
        n[i] = 0
        i+=1
    elif s<1:
        n[i] = s
        i+=1
    else:
        n[i] = 1
        i+=1

F_200 = empcdf(X_200, t)
plt.plot(t,F_200)
plt.plot(t,n)
```

Out[108]:
```
[<matplotlib.lines.Line2D at 0x1ef263136d0>]
```



Exercise 4.1

1. Derive the required transformation.

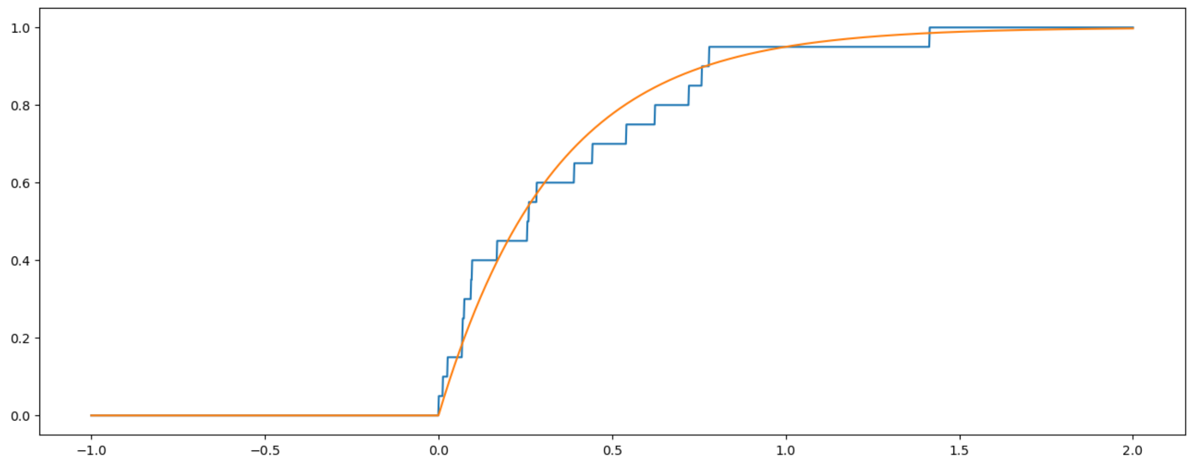
Y = (1-exp(-3X)) X = ln(1-Y)/-3

2. Generate samples of X when $N = 20$ and $N = 200$.

In [87]:
```
x = np.linspace(-1,2,2000)
```

In [113]:
```
Y = np.random.uniform(0,1,20)
X1 = empcdf(np.log(1-Y)/-3,x)

n = [0]*len(x)
i = 0
for s in x:
    if s<0:
        n[i] = 0
        i+=1
    else:
        n[i] = (1-np.exp(-3*s))
        i+=1

plt.plot(x,X1)
plt.plot(t,n)
```

Out[113]: [<matplotlib.lines.Line2D at 0x1ef28119b80>]

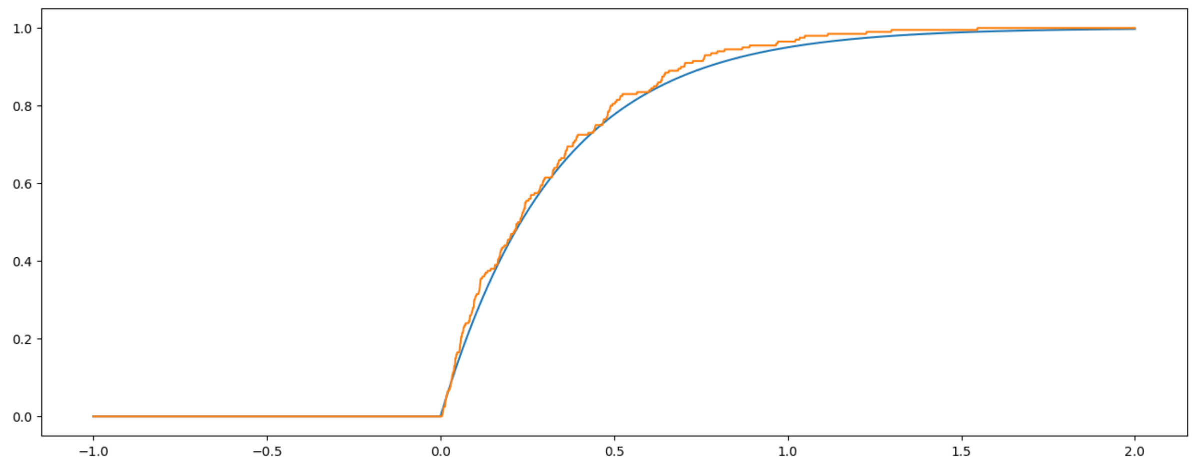


In [114]:

```
Y = np.random.uniform(0,1,200)
X2 = empcdf(np.log(1-Y)/-3,x)
n = [0]*len(x)
i = 0
for s in x:
    if s<0:
        n[i] = 0
        i+=1
    else:
        n[i] = (1-np.exp(-3*s))
        i+=1

plt.plot(t,n)
plt.plot(x,X2)
```

Out[114]:

```
[<matplotlib.lines.Line2D at 0x1ef2818b520>]
```

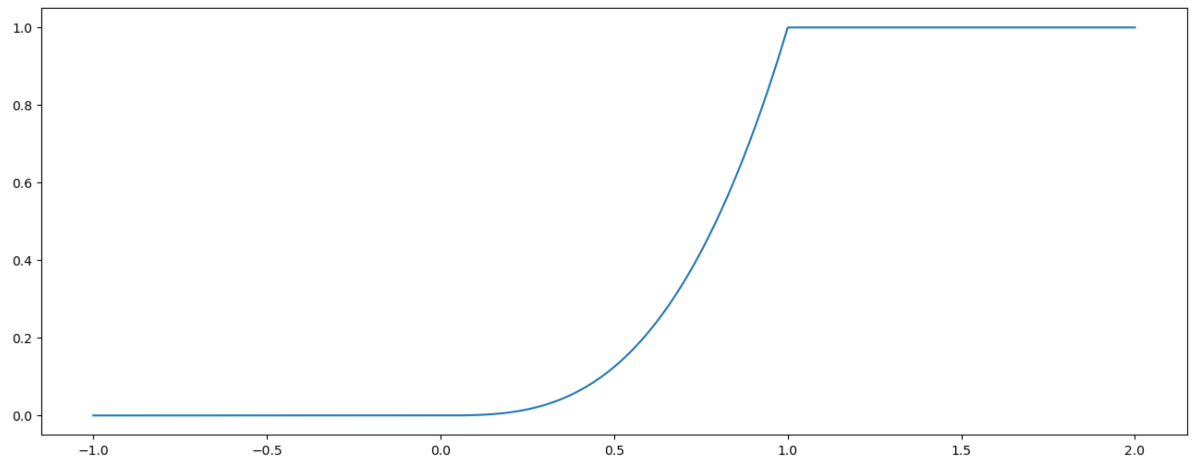


Exercise 5.1

1. Plot $F_X(x)$ for $x \in [0, 1]$.

In [117]:
```
x = np.linspace(-1,2,2000)
n = [0]*len(x)
i = 0
for s in x:
    if s<0:
        n[i] = 0
        i+=1
    elif s<=1:
        n[i] = s**3
        i+=1
    else:
        n[i] = 1
        i+=1

plt.plot(x,n)
```

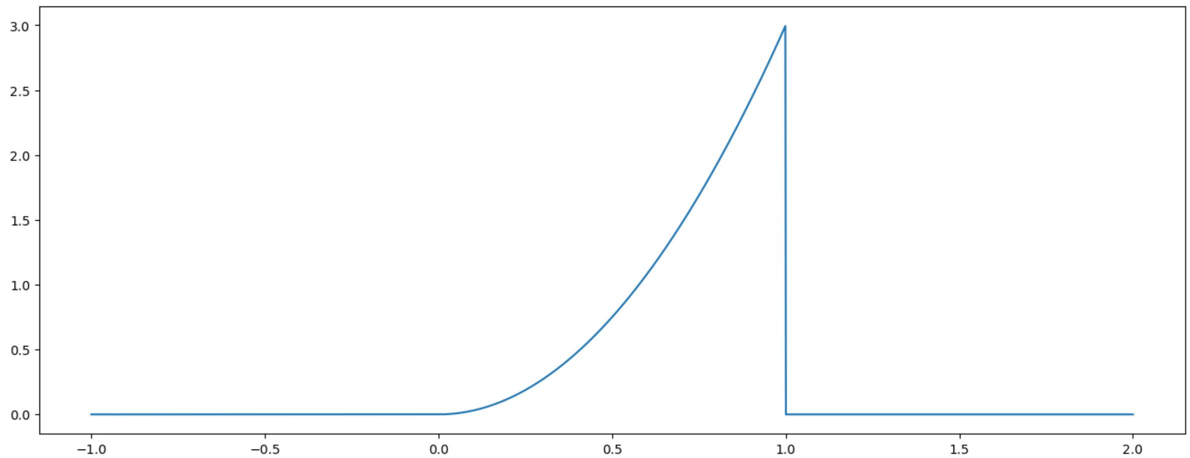
Out[117]:
```
[<matplotlib.lines.Line2D at 0x1ef28797ee0>]
```



2. Analytically calculate the probability density $f_X(x)$, and plot it for $x \in [0, 1]$.

In [4]:
```
x = np.linspace(-1,2,2000)
n = [0]*len(x)
i = 0
for s in x:
    if s<0:
        n[i] = 0
        i+=1
    elif s<=1:
        n[i] = 3*s**2
        i+=1
    else:
        n[i] = 0
        i+=1

plt.plot(x,n)
```

Out[4]:
```
[<matplotlib.lines.Line2D at 0x260e33d8340>]
```

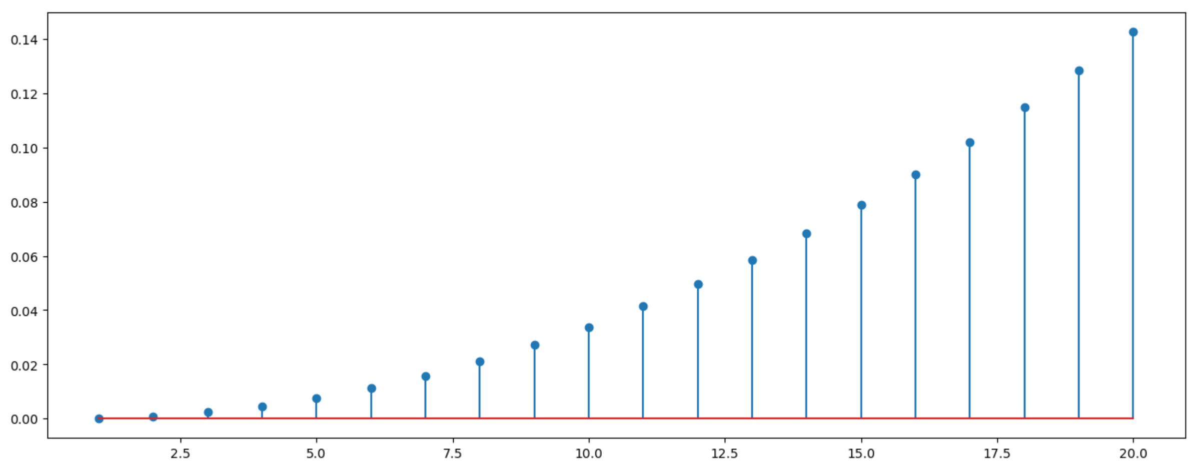


3. Using $L = 20$, $x_0 = 0$ and $x_L = 1$, write code to compute $\tilde{f}(k)$, the probability of X falling into $bin(k)$. Plot $\tilde{f}(k)$ for $k = 1, \ldots, L$ using the `plt.stem()` function.

Hint: Use the fact that $\tilde{f}(k) = F_X(x_k) - F_X(x_{k-1})$.

In [24]:
```
L = 20
delta = 1 / L
k = np.linspace(0, L, 21)

f_telda = [0]*len(k)


x = np.linspace(0,1,21)
n = [0]*len(x)

i = 0
for s in x:
    if s<0:
        n[i] = 0
        i+=1
    elif s<=1:
        n[i] = s**3
        i+=1
    else:
        n[i] = 1
        i+=1
ctr = 0
for s in k:
    f_telda[ctr] = n[ctr] - n[ctr-1]
    ctr+=1

k_1 = k[1:]
f_telda_1 = f_telda[1:]

plt.stem(k_1,f_telda_1)
```

Out[24]: <StemContainer object of 3 artists>



4. Show (mathematically) how $f_X(x)$ and $\tilde{f}(k)$ are related.

f_telda is the probability between xk and xk-1 in distribution function, thus f_telda is the f scaled down by delta

5. Generate 1000 samples of a random variable U that is uniformly distributed between 0 and 1 (using the `np.random.uniform(0, 1, 1000)`). Then form the random vector X by computing $X = U^{1/3}$.

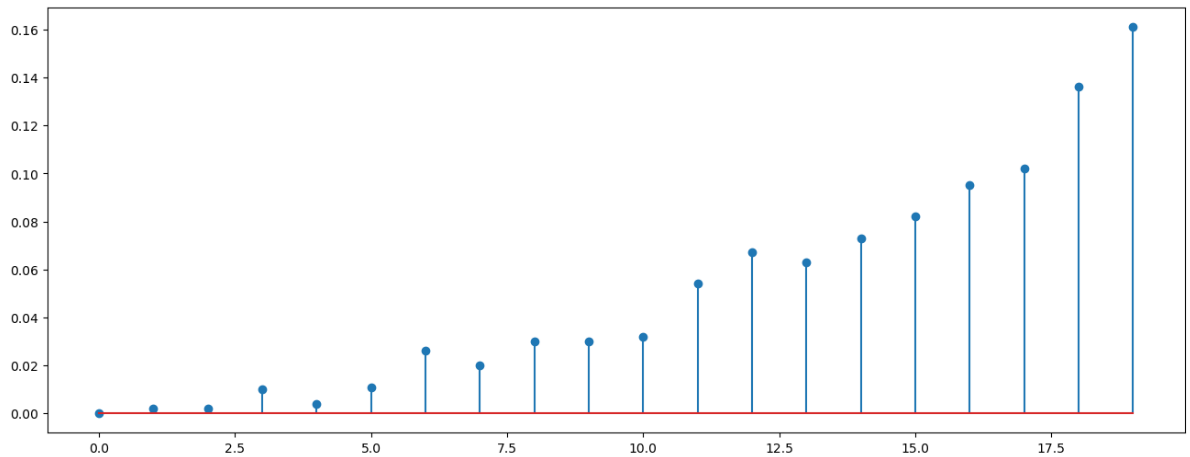
In [25]:
```
U = np.random.uniform(0, 1, 1000)
X = U**(1/3)
```

6. Use the Python function [`np.histogram()` (https://numpy.org/doc/stable/reference/generated/numpy.histogram.html)](https://numpy.org/doc/stable/reference/generated/numpy.histogram.html) to plot a *normalized* histogram for your samples of X, using 20 bins uniformly spaced on the interval $[0, 1]$. Use the `plt.stem()` command to plot the normalized histogram $H(k)/N$.

Hint: Use the Python command `H, _ = np.histogram(X, bins=20, range=(0, 1))` to obtain the normalized histogram. The underscore `_` means that whatever the second argument the function returns, I don't care and don't bother assigning it to a variable.

In [26]:
```
H, _ = np.histogram(X, bins=20, range=(0,1))
plt.stem(H/len(X))
```

Out[26]: <StemContainer object of 3 artists>



7. How do these plots ($H(k)/N$ and $\tilde{f}(k)$) compare?

Very similar to each other

8. Discuss the tradeoffs (advantages and the disadvantages) between selecting a very large or very small bin-width.

A very small bin-width means a more accurate representation of f, but f_telda is scaled down even more.