ECE 438 - Laboratory 6a
Discrete Fourier Transform and Fast Fourier Transform Algorithms (Week 1)

Last updated on February 22, 2022

Date:2/23
Section:

Name	Signature	Time spent outside lab
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Student Name #2 [---%]		

	Below expectations	Lacks in some respect	Meets all expectations
Completeness of the report			
Organization of the report			
Quality of figures: *Correctly labeled with title, x-axis, y-axis, and name(s)*			
Understanding the effects of truncating the signal on its DTFT (20 pts): *Magnitude and phase plots, hamming/rect windows, questions*			
Implementation of DFT and inverse DFT (40 pts): *Python codes, frequency and time-domain plots, analytical expressions*			
Implementation of DFT and IDFT using matrix multiplication (30 pts): *Matrices A,B,C, matlab codes, plots, questions*			
Computation time comparison (10 pts): *Runtimes, questions*			

In [1]:
```
import numpy as np
import matplotlib.pyplot as plt
from helper import DTFT, hamming
import time
```

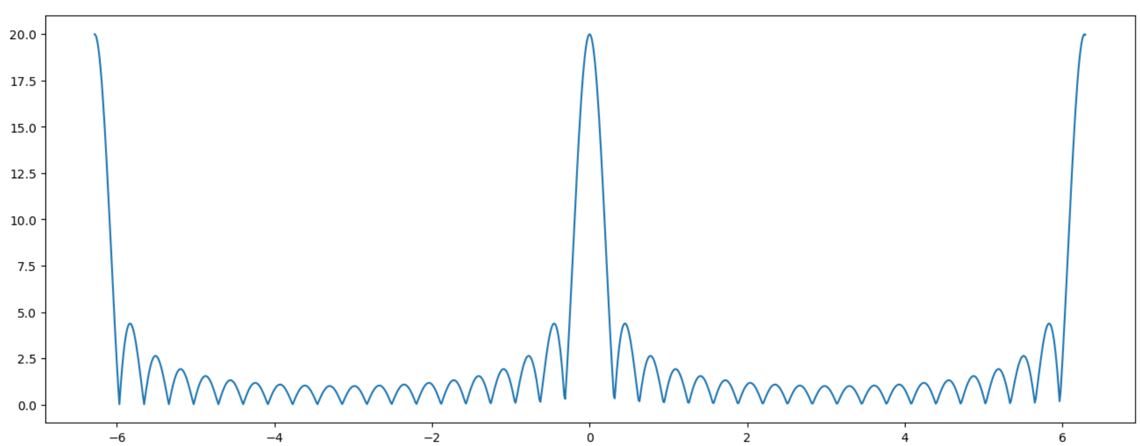
In [2]:
```
# make sure the plot is displayed in this notebook
%matplotlib inline
# specify the size of the plot
plt.rcParams['figure.figsize'] = (16, 6)

# for auto-reloading extenrnal modules
%load_ext autoreload
%autoreload 2
```

Exercise 2: Windowing Effects

1. Plot the magnitude of $W(e^{j\omega})$, using equations (10) and (11).

In [3]:
```
N = 20

w = np.linspace(-2*np.pi, 2*np.pi+0.01, 1001)
W = [0]*len(w)

W[0] = N
W[N] = N

for i in range(1, len(w), 1):
    W[i] = np.exp(-1j*w[i]*(N-1)/2)*np.sin(w[i]*N/2)/np.sin(w[i]/2)

plt.plot(w, np.abs(W))
```

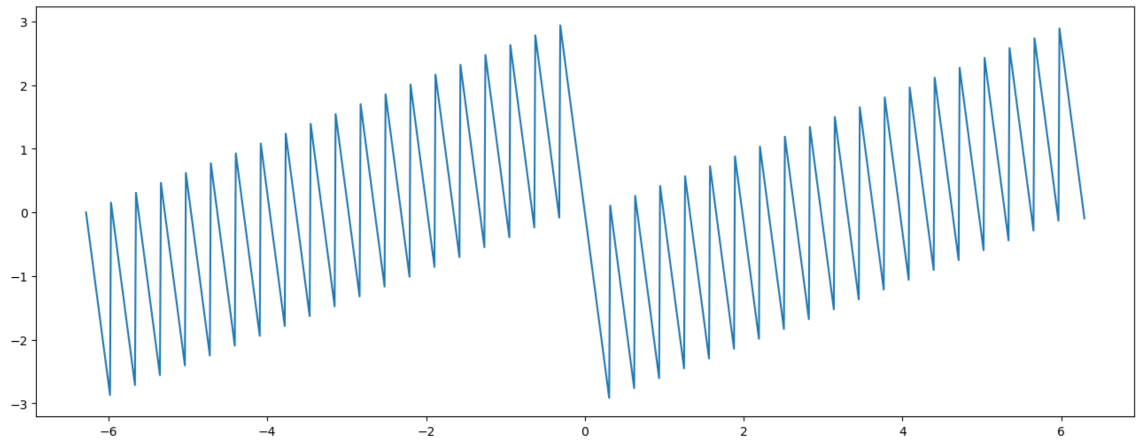
Out[3]: [<matplotlib.lines.Line2D at 0x1c74198e9d0>]



2. Plot the phase of $W(e^{j\omega})$, using equations (10) and (11).

In [4]:
```
plt.plot(w, np.angle(W))
```

Out[4]:
```
[<matplotlib.lines.Line2D at 0x1c74208c2e0>]
```



3. Determine an analytical expression for $X(e^{j\omega})$ (the DTFT of the non-truncated signal).

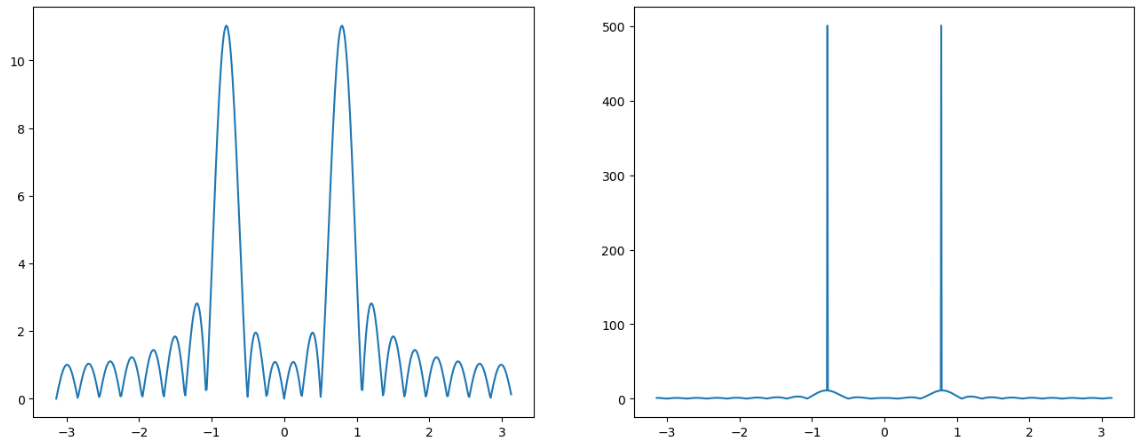
$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

4. Truncate the signal $x[n]$ using a window of size $N = 20$ and then use `DTFT` to compute $X_{\text{tr}}(e^{j\omega})$. Then plot the magnitude of $X_{\text{tr}}(e^{j\omega})$. Make sure that the plot contains a least 512 points.

Hint: Use the command `X, w = DTFT(x, 512)` .

In [5]:
```
n = np.linspace(0, N, N+1)
n1 = np.linspace(0, 1000, 1001)

x = np.cos(np.pi/4*n1)
xtr = np.cos(np.pi/4*n)

X, w1 = DTFT(x, 512)
Xtr, w = DTFT(xtr, 512)

plt.subplot(1,2,1)
plt.plot(w, np.abs(Xtr))

plt.subplot(1,2,2)
plt.plot(w1, np.abs(X))
```

Out[5]:
```
[<matplotlib.lines.Line2D at 0x1c7423d4130>]
```



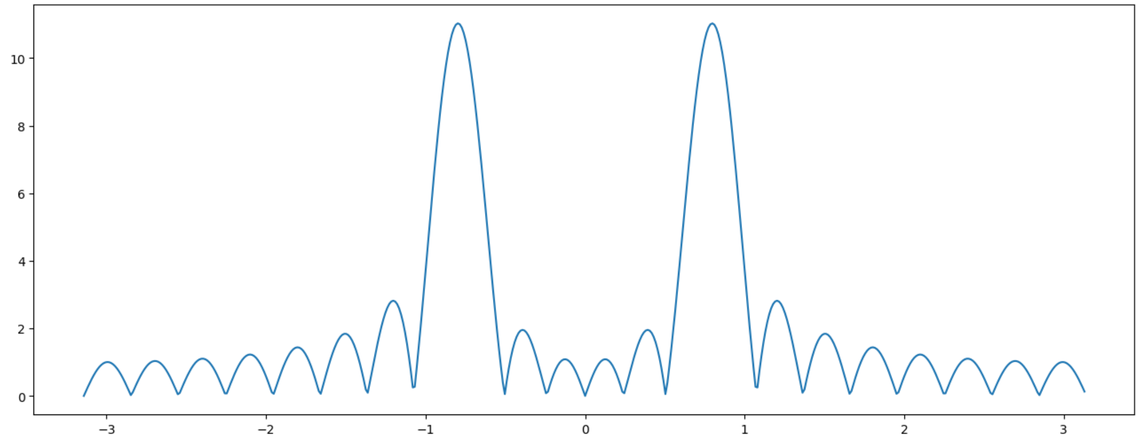
5. Describe the difference between $|X_{\text{tr}}(e^{j\omega})$ and $|X(e^{j\omega})|$. What is the reason for this difference?

The width of each lobe are bigger when X is truncated. It is bigger because it's convolved with a rectangular window.

6. What would you expect your plots to look like if you had used a Hamming window in place of the truncation (rectangular) window? (See Fig. 1 for a plot of a Hamming window of length 20 and its DTFT.) Submit the plot of the magnitude of the DTFT of the signal $x[n]$ windowed using a Hamming window. (Hint: The Python command for a Hamming window is `hamming(N)` .)

In [6]:
```
n = np.linspace(0, N, N+1)

h = hamming(N+1)
xtr_h = h*np.cos(np.pi/4*n)

Xtr_h, w = DTFT(xtr, 512)

plt.plot(w, np.abs(Xtr_h))
```

Out[6]: [<matplotlib.lines.Line2D at 0x1c742748ee0>]



7. Comment on the effects of using a different window for $w[n]$.

They are similar but sidelobe are reduced

Exercise 3.1: Computing the DFT

1. Write your own Python function to implement the DFT of equation (3). Your routine should implement the DFT exactly as specified by (3) using *for-loops* for n and k, and computing the exponentials as they appear.

Hint: initialize `X` as a vector of complex values by using `.astype(complex)` .

In [7]:
```
def DFTsum(x):
    """
    Parameters:
    ---
    x: the input signal, an N point vector contining the values x[0], ...,

    Returns:
    ---
    X: the DFT of x
    """

    X = [0]*len(x)
    for k in range(0, len(x)):
        for n in range(0, len(x)):
            X[k] += (x[n]*np.exp(-1j*2*np.pi*n*k/len(x))).astype(complex)


    return X
```

2. Test your routine `DFTsum` by computing $X_N(k)$ for each of the following cases:

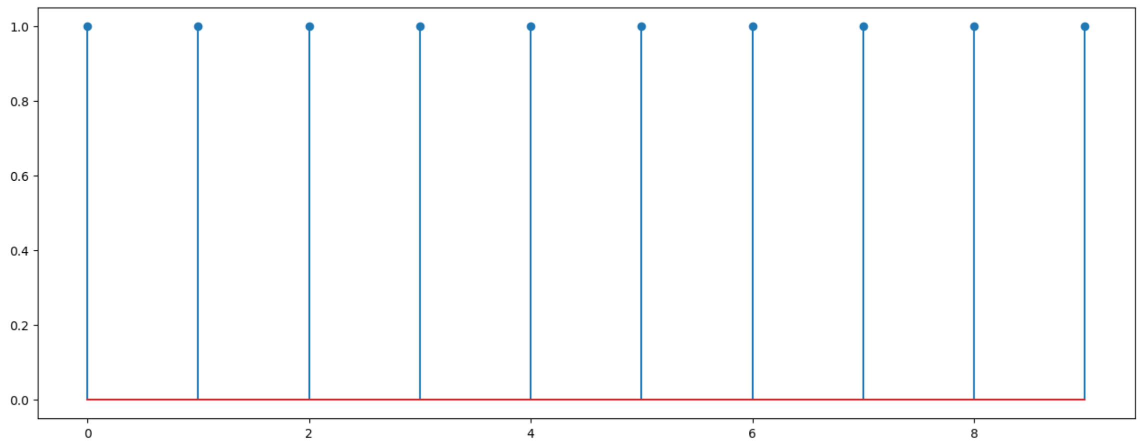
- $x(n) = \delta(n) \quad \text{for } N = 10$
- $x(n) = 1 \quad \text{for } N = 10$
- $x(n) = e^{j2\pi n/10} \quad \text{for } N = 10$
- $x(n) = \cos(2\pi n/10) \quad \text{for } N = 10$

and plot the magnitude of each of the DFT's.

In [8]:
```
N = 10
n = np.linspace(0, N-1, N)
print(n)
x1 = [0]*len(n)
x1[0] = 1
X1 = DFTsum(x1)
w = np.linspace(0, len(X1)-1, len(X1))

plt.stem(w, np.abs(X1))
```

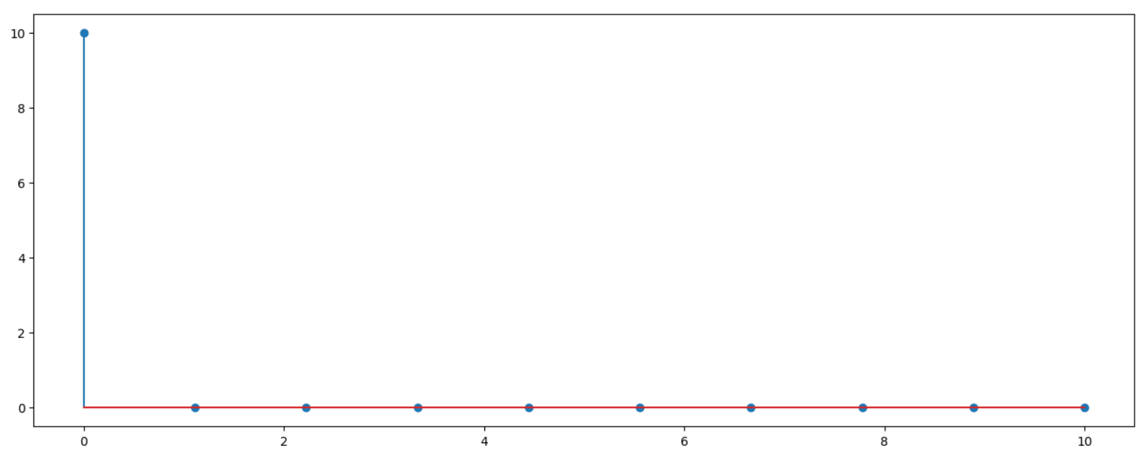
```
[0. 1. 2. 3. 4. 5. 6. 7. 8. 9.]
```

Out[8]:
```
<StemContainer object of 3 artists>
```



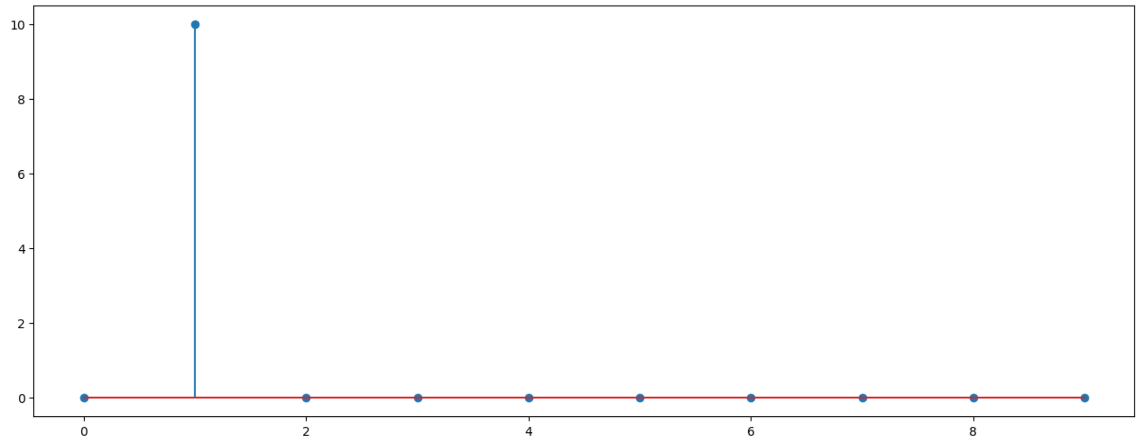
In [9]:
```
x2 = [1]*len(n)
X2 = DFTsum(x2)
w = np.linspace(0, len(X2), len(X2))

plt.stem(w, np.abs(X2))
```

Out[9]:
```
<StemContainer object of 3 artists>
```



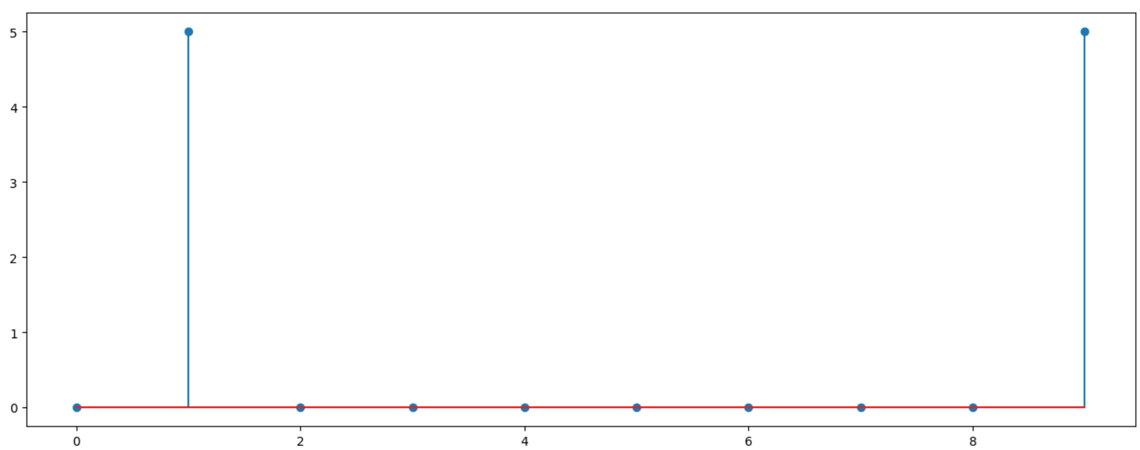
In [10]:
```
x3 = np.exp(1j*2*np.pi*n/10)
X3 = DFTsum(x3)
w = np.linspace(0, len(X3)-1, len(X3))
plt.stem(w, np.abs(X3))
```

Out[10]: <StemContainer object of 3 artists>



In [11]:
```
x4 = np.cos(2*np.pi*n/10)
X4 = DFTsum(x4)
w = np.linspace(0, len(X4)-1, len(X4))

plt.stem(w, np.abs(X4))
```

Out[11]: <StemContainer object of 3 artists>



3. Derive simple closed-form analytical expressions for the DFT (not the DTFT!) of each signal.

X1 = 1 X2 = sigma(k) [note:impulse at k] X3 = sigma(k-1) X4 = sigma(k-1) + sigma(k-10)

Exercise 3.2: Computing the Inverse DFT

1. Write a Python function for computing the inverse DFT of (4).

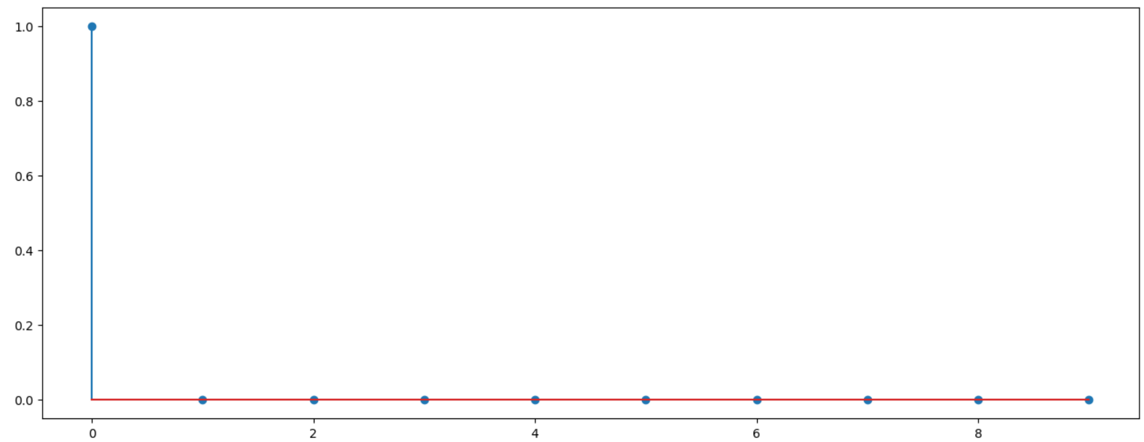
In [12]:
```
def IDFTsum(X):
    '''
    Parameters:
    ---
    X: the N point vector containing the DFT

    Returns:
    ---
    x: the corresponding time-domain signal
    '''

    x = [0]*len(X)
    for n in range(0, len(X)):
        for k in range(0, len(X)):
            x[n] += (X[k]*np.exp(1j*2*np.pi*n*k/len(x))/len(X)).astype(comp

    return x
```

2. Use `IDFTsum` to invert each of the DFT's computed in the previous problem. Plot the magnitudes of the inverted DFT's, and verify that those time-domain signals match the original ones. Use `np.real()` to eliminate any imaginary parts which roundoff error may produce.

In [13]:
```
x1 = IDFTsum(X1)
w = np.linspace(0, len(x1)-1, len(x1))
plt.stem(w,x1)
```

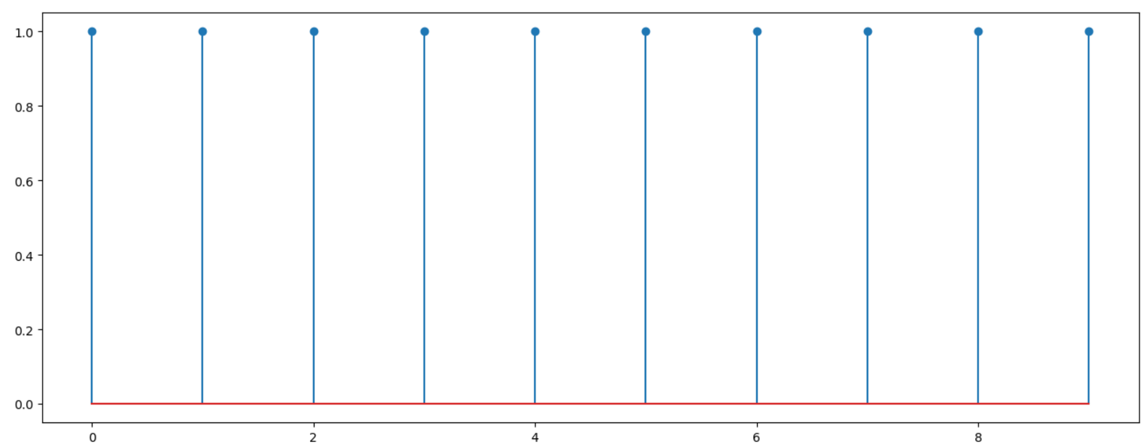
```
C:\Users\rxw14\anaconda3\lib\site-packages\numpy\ma\core.py:3379: ComplexW
arning: Casting complex values to real discards the imaginary part
  _data[indx] = dval
C:\Users\rxw14\anaconda3\lib\site-packages\matplotlib\cbook\__init__.py:12
98: ComplexWarning: Casting complex values to real discards the imaginary
part
  return np.asarray(x, float)
```

Out[13]: <StemContainer object of 3 artists>



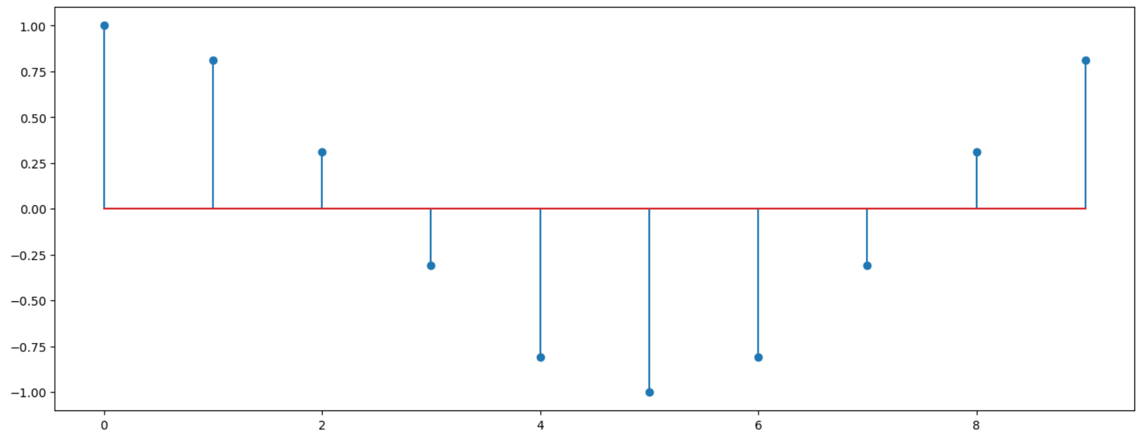
In [14]:
```
x2 = IDFTsum(X2)
w = np.linspace(0, len(x2)-1, len(x2))
plt.stem(w,x2)
```

Out[14]: <StemContainer object of 3 artists>



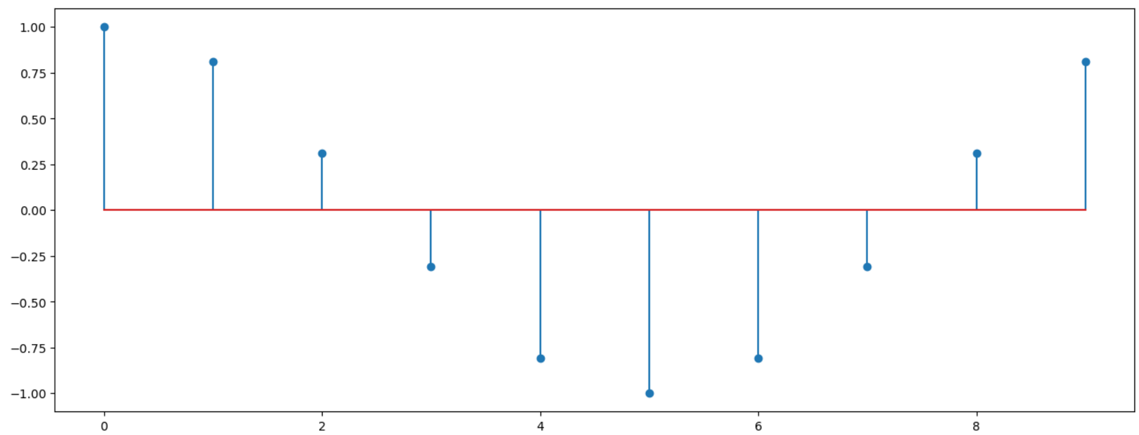
In [15]:
```
x3 = IDFTsum(X3)
w = np.linspace(0, len(x3)-1, len(x3))
plt.stem(w,x3)
```

Out[15]: <StemContainer object of 3 artists>



In [16]:
```
x4 = IDFTsum(X4)
w = np.linspace(0, len(x4)-1, len(x4))
plt.stem(w,x4)
```

Out[16]: <StemContainer object of 3 artists>



Exercise 3.3: Matrix Representation of the DFT

1. Write a Python function for computing the $N \times N$ DFT matrix A in equation (16).

In [17]:
```
def DFTmatrix(N):
    """
    Parameters:
    ---
    N: N point DFT

    Returns:
    ---
    A: an N x N DFT matrix
    """
    A = np.zeros((N,N)).astype(complex)

    for n in range(N):
        for k in range(N):
            A[k][n] = np.exp(-1j*2*np.pi*k*n/N)
    return A
```

2. Print out the matrix A for $N = 5$.

In [18]:
```
A1 = DFTmatrix(5)
print(A1)
```

```
[[ 1.        +0.j          1.        +0.j          1.        +0.j
   1.        +0.j          1.        +0.j        ]
 [ 1.        +0.j          0.30901699-0.95105652j -0.80901699-0.58778525j
  -0.80901699+0.58778525j  0.30901699+0.95105652j]
 [ 1.        +0.j         -0.80901699-0.58778525j  0.30901699+0.95105652j
   0.30901699-0.95105652j -0.80901699+0.58778525j]
 [ 1.        +0.j         -0.80901699+0.58778525j  0.30901699-0.95105652j
   0.30901699+0.95105652j -0.80901699-0.58778525j]
 [ 1.        +0.j          0.30901699+0.95105652j -0.80901699+0.58778525j
  -0.80901699-0.58778525j  0.30901699-0.95105652j]]
```

3. Use the matrix A to compute the DFT of the following signals.

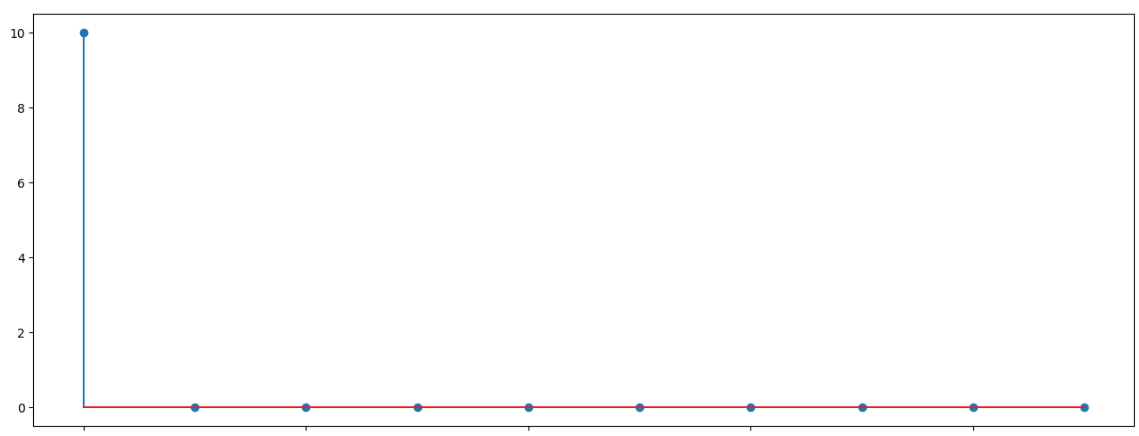
- $x(n) = \delta(n) \quad \text{for } N = 10$
- $x(n) = 1 \quad \text{for } N = 10$
- $x(n) = e^{j2\pi n/N} \quad \text{for } N = 10$

In [26]:
```
x1 = [0]*10

x1[0] = 1

A1 = DFTmatrix(10)
X1 = A1*x1


N = 10
x2 = [0]*N
for n in range(0,N-1,1):
    x2[n] = 1

A2 = DFTmatrix(N)
X2 = A2*x2


N = 10
x3 = [0]*N
for n in range(0,N-1,1):
    x3[n] = np.exp(1j*2*np.pi*n/N)

A3 = DFTmatrix(N)
X3 = A3*x3
```

4. Plot the magnitude plots of these 3 DFTs.

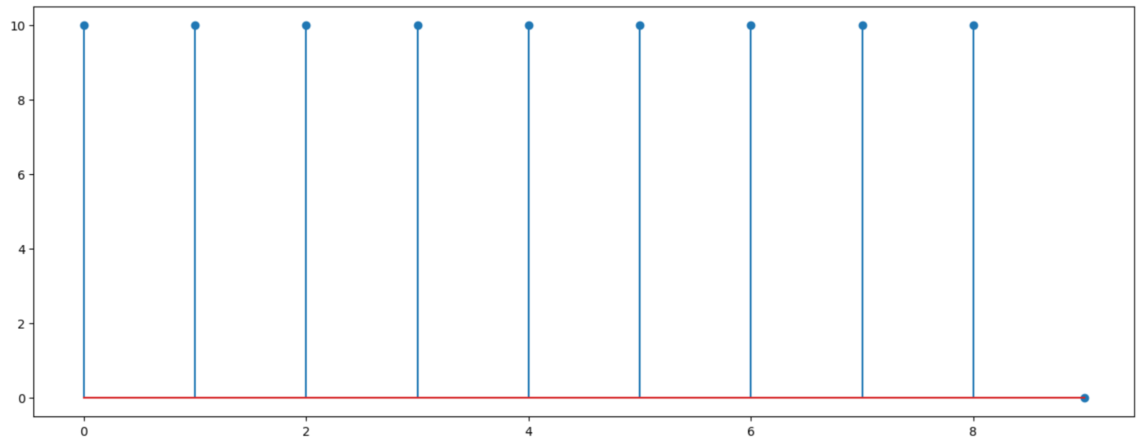
In [33]:
```
X1_mag = np.abs(X1)
X1_mag_arr = np.sum(X1_mag, axis=0)
n = np.linspace(0,9,10)

plt.stem(n, X1_mag_arr)
```

Out[33]: <StemContainer object of 3 artists>



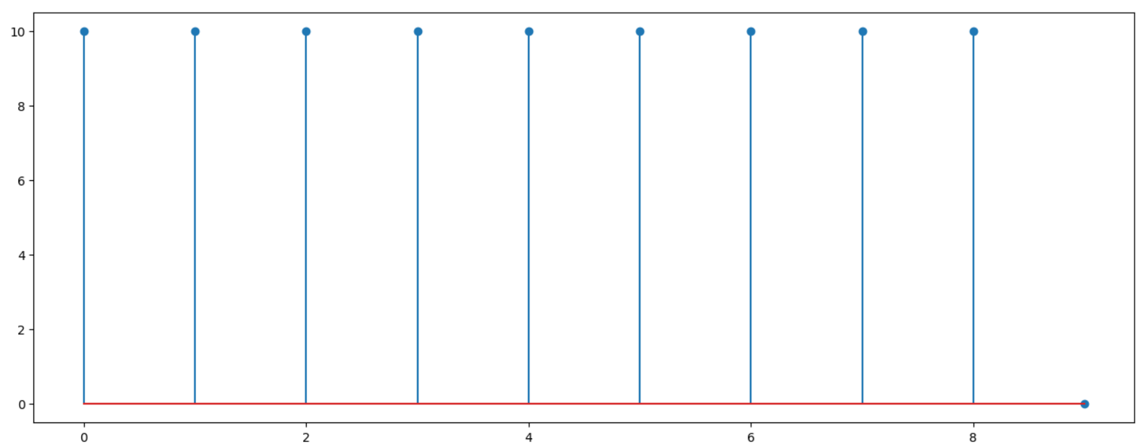
In [34]:
```
X2_mag = np.abs(X2)
X2_mag_arr = np.sum(X2_mag, axis=0)
n = np.linspace(0,9,10)

plt.stem(n, X2_mag_arr)
```

Out[34]: <StemContainer object of 3 artists>



In [35]:
```
X3_mag = np.abs(X3)
X3_mag_arr = np.sum(X3_mag, axis=0)
n = np.linspace(0,9,10)

plt.stem(n, X3_mag_arr)
```

Out[35]: <StemContainer object of 3 artists>



5. How many multiplies are required to compute an N point DFT using the matrix method (Consider a multiply as the multiplication of either complex or real numbers.)

N^2

Exercise 3.4: Matrix Representation of the Inverse DFT

1. Write an analytical expression for the elements of the inverse DFT matrix `B` , using the form of equation (16).

B = e^(j*2*pi*k*n/N)/N

2. Write a Python function for computing the $N \times N$ inverse DFT matrix B.

In [36]:
```
def IDFTmatrix(N):
    """
    Parameters:
    ---
    N: N-point IDFT

    Returns:
    ---
    B: the N x N inverse DFT matrix
    """
    B = np.zeros((N,N)).astype(complex)

    for n in range(N):
        for k in range(N):
            B[k][n] = np.exp(1j*2*np.pi*k*n/N)/N
    return B
```

3. Print out the matrix B for $N = 5$.

In [37]:
```
B1 = IDFTmatrix(5)
print(B1)
```

```
[[ 0.2       +0.j          0.2       +0.j          0.2       +0.j
   0.2       +0.j          0.2       +0.j        ]
 [ 0.2       +0.j          0.0618034+0.1902113j  -0.1618034+0.11755705j
  -0.1618034-0.11755705j  0.0618034-0.1902113j ]
 [ 0.2       +0.j         -0.1618034+0.11755705j  0.0618034-0.1902113j
   0.0618034+0.1902113j  -0.1618034-0.11755705j]
 [ 0.2       +0.j         -0.1618034-0.11755705j  0.0618034+0.1902113j
   0.0618034-0.1902113j  -0.1618034+0.11755705j]
 [ 0.2       +0.j          0.0618034-0.1902113j  -0.1618034-0.11755705j
  -0.1618034+0.11755705j  0.0618034+0.1902113j ]]
```

4. Compute the matrices A for $N = 5$. Then compute and print out the elements of $C = BA$.

In [40]:
```
A1 = DFTmatrix(5)

C = A1*B1

print(C)
```

```
[[0.2+0.0000000e+00j 0.2+0.0000000e+00j 0.2+0.0000000e+00j
  0.2+0.0000000e+00j 0.2+0.0000000e+00j]
 [0.2+0.0000000e+00j 0.2+0.0000000e+00j 0.2+0.0000000e+00j
  0.2+0.0000000e+00j 0.2+0.0000000e+00j]
 [0.2+0.0000000e+00j 0.2+0.0000000e+00j 0.2+0.0000000e+00j
  0.2+0.0000000e+00j 0.2+0.0000000e+00j]
 [0.2+0.0000000e+00j 0.2+0.0000000e+00j 0.2+0.0000000e+00j
  0.2-6.9388939e-18j 0.2+0.0000000e+00j]
 [0.2+0.0000000e+00j 0.2+0.0000000e+00j 0.2+0.0000000e+00j
  0.2+0.0000000e+00j 0.2+0.0000000e+00j]]
```

5. What form does C have? Why does it have this form?

C is a constant matrix of 1/N. Because they are inverse of each other.

Exercise 3.5: Computation Time Comparison

1. Compute the signal $x(n) = \cos(2\pi n/10)$ for $N = 512$.

In [43]:
```
n = np.linspace(0, 511, 512)

x = np.cos(2*np.pi*n/10)
```

2. Compute the matrix A for $N = 512$.

In [44]:
```
A = DFTmatrix(512)
```

3. Compare the computation time of `DFTsum(x)` with a matrix implementation `X = A.dot(x)` by using the *time* function from *time* library before and after the program execution (See the example below). Do not include the computation of A in your timing calculations.

Report the time required for each of the two implementations.

```
t1 = time.time()
# program execution
t2 = time.time()
print(f"time taken: {t2 - t1:.4f}")
```

In [45]:
```
t1 = time.time()
X = DFTsum(x)
t2 = time.time()
print(f"time taken: {t2 - t1:.4f}")

t1 = time.time()
X = A.dot(x)
t2 = time.time()
print(f"time taken: {t2 - t1:.4f}")
```

```
time taken: 1.7816
time taken: 0.0110
```

4. Which method is faster? Which method requires less storage?

The matrix method is faster, but requires extra storage for matrix A