Algorithmic Game Theory, Spring 2022
Homework 2

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Instructions:

1. Feel free to discuss with fellow students, but write your own answers. If you do discuss a problem with someone then write their names at the starting of the answer for that problem.
2. Please type your solutions if possible in LaTeX or word whatever is suitable.
3. Even if you are not able to solve a problem completely, do submit whatever you have. Partial proofs, high-level ideas, examples, and so on.

Problem 1. (2pt) What's your favourite movie?

Problem 2. (3pt) Pick your favourite theorem in our class, and tell me the reason.

Problem 3. (5pt) Given a valuation function $v : 2^{[m]} \to \mathbb{R}$ which is *normalized* (i.e., $v(\emptyset) = 0$) and *monotone* (i.e., $v(S) \leq v(T)$ once $S \subseteq T \subseteq [m]$). v is *additive* iff $v(S) = \sum_{i \in S} v(i)$. v is *submodular* iff $v(S \cup T) + v(S \cap T) \leq v(S) + v(T)$ for any $S, T \subseteq [m]$. And v is *XOS* iff there exist t additive valuations $\{a_1, \ldots, a_t\}$ such that $v(S) = \max_{r \in [t]} a_r(S)$ for every $S \subseteq [m]$.

Prove that any monotone and normalized submodular function can be written as an XOS function.

Problem 4. This problem derives an interesting interpretation of a virtual valuation $\varphi(v) = v - \frac{1-F(v)}{f(v)}$ and the regularity condition. Consider a strictly increasing distribution function F with a strictly positive density f on the interval $[0, v_{\max}]$ (with $v_{\max} < +\infty$).

For $q \in [0, 1]$, define $V(q) = F^{-1}(1 - q)$ as the posted price resulting in a probability q of a sale (for a single bidder with valuation drawn from F). Define $R(q) = q \cdot V(q)$ as the expected revenue obtained when (for a single bidder) the probability of a sale is q. The function $R(q)$, for $q \in [0, 1]$, is often called the "revenue curve" of a distribution F. Note that $R(0) = R(1) = 0$.

1. (3pt) What is the revenue curve for the uniform distribution on $[0, 1]$?
2. (2pt) Prove that the slope of the revenue curve at q (i.e., $R'(q)$) is precisely $\varphi(V(q))$, where φ is the virtual valuation function of F.

3. (2pt) Prove that a distribution is regular if and only if its revenue curve is concave.

4. (3pt) Prove that, for a regular distribution, the median price $V(1/2)$ is a $(1/2)$-approximation of the optimal posted price. That is, prove that $R(1/2) \geq 1/2 \cdot \max_{q \in [0,1]} R(q)$.