Algorithmic Game Theory

BIT — 2022 Spring

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Homework 1

§ Nash Equilibrium §

Problem 1: Favourite Meal/Dish in School Canteen

东食堂二层的牛肉板面，学校少有的卖宽面条的地方，但对于不太能吃辣的我需要点微微微微辣的。

Problem 2: Equivalence of Nash Equilibrium's Definition

(1) Definition Clarification

In this section, we modify formulas of the definition to make preparations for further proof.

For **definition 1**, a pair of mixed strategies (x, y) is Nash Equilibrium iff neither of the player can increase their expected payoff by deviating from the initial strategy while the other's unchanged.

Let R of size $m \times n$ be the payoff matrix for the row player and $\Delta_m = \{\mathbf{x} = (x_1, \ldots, x_m) \in \mathbb{R}^m \mid \sum_{i\in[m]} x_i = 1,\ x_i \geq 0\}$ be the set of all mixed strategies over m actions. When the strategy y keeps constant, for $\forall \mathbf{x}^{'} \in \Delta_m$, the expected payoff for the row player $\mathbf{x}^T R\mathbf{y}$ is always greater than or equal to that of all other mixed strategies $\mathbf{x}^{'} R\mathbf{y}$, namely

$$\mathbf{x}^T R\mathbf{y} \geq \mathbf{x}^{'T} R\mathbf{y},\ \forall \mathbf{x}^{'} \in \Delta_m \tag{2.1}$$

Define $\mathbf{S} = R\mathbf{y} = \{s_1, s_2, \cdots, s_m\}$ to simplify representations and rearrange equation Eq. (2.1), i.e. $\mathbf{x}^T\mathbf{S} > \mathbf{x}^{'T}\mathbf{S}$,

$$x_1 s_1 + x_2 s_2 + \cdots + x_m s_m \geq x^{'}_1 s_1 + x^{'}_2 s_2 + \cdots + x^{'}_m s_m,\ \forall \mathbf{x}^{'} \in \Delta_m \tag{2.2}$$

Let C stand for the payoff matrix for the column player, similarly, for $\forall \mathbf{y}^{'} \in \Delta_n$,

$$\begin{gathered} \mathbf{x}^T C\mathbf{y} \geq \mathbf{x}^T C\mathbf{y}^{'} \\ \mathbf{T}\mathbf{y} \geq \mathbf{T}\mathbf{y}^{'} \\ y_1 t_1 + y_2 t_2 + \cdots + y_n t_n \geq y^{'}_1 t_1 + y^{'}_2 t_2 + \cdots + y^{'}_n t_n \end{gathered} \tag{2.3}$$

where $\mathbf{T} = \mathbf{x}^T C = \{t_1, t_2, \cdots, t_n\}$.

For **definition 2**, a pair of strategies (x, y) is Nash Equilibrium iff each action in the support of $\mathbf{x}$ (or $\mathbf{y}$) is the best response to the other. When action x_i is in favor of, namely the possibility for action i x_i is greater than zero, the expected payoff of the pure strategy for action i $e_i^T R\mathbf{y}$ is better than all other pure strategies, that is

$$x_i > 0 \Rightarrow \mathbf{e}_i^T R\mathbf{y} \geq \mathbf{e}_k^T R\mathbf{y}, \forall k \in [m] \tag{2.4}$$

Using notation $\mathbf{S}$ as defined above, we can draw a conclusion that

$$x_i > 0 \Rightarrow s_i \geq s_k, \forall k \in [m] \tag{2.5}$$

Therefore, if $x_i > 0$ and $x_k > 0$, then $s_i = s_k$, or it will contradict Eq. (2.5).
The same goes for the row player, for $\forall k \in n$

$$y_j > 0 \Rightarrow \mathbf{x}^T C\mathbf{e}_j \geq \mathbf{x}^T C\mathbf{e}_k \tag{2.6}$$

$$y_j > 0 \Rightarrow t_j \geq t_k \tag{2.7}$$

$$y_i > 0, y_j > 0 \Rightarrow t_i = t_j \tag{2.8}$$

To sum up, definition 1 can be written in the form of Eq. (2.3) and definition 2 in the form of equation Eq. (2.6) and Eq. (2.7), with its conclusion equation Eq. (2.8).

(2) Proof of Equivalence

In this section, using equations obtained previously, we aim to prove equivalence from the perspective of sufficiency and necessity.

a. Definition 1 $\Rightarrow$ Definition 2

Proof. By contradiction.

Assume that definition 2 does not hold. That is,

$$\text{for } \mathbf{x} = \{x_1, x_2, \cdots, x_i, \cdots, x_k, \cdots, x_m\},\ x_i > 0, \exists k \in [m],\ \text{s.t. } s_i < s_k \tag{2.9}$$

As $x_i > 0$, $x_i s_i < x_i s_k$. It is obvious that $x_i s_i + x_k s_k < x_i s_k + x_k s_k = 0 + (x_i + x_k)s_k$.
Thus, $\exists \mathbf{x}' = \{x_1, x_2 \cdots, 0, \cdots, x_i + x_k, \cdots, x_m\}$, s.t. $\mathbf{x}^T\mathbf{S} < \mathbf{x}'^T\mathbf{S}$.
However, this contradicts definition 1.
Therefore, definition 1 is the sufficient condition for definition 2. □

b. Definition 2 $\Rightarrow$ Definition 1

Proof. Let $\alpha = \{a, b, \cdots, d\}$ be the set of all subscripts that make x greater than 0, i.e., $x_a > 0, x_b > 0 \cdots, x_d > 0$ and $x_a + x_b \cdots + x_d = 1$. Then, there exists $\alpha' = \{o, p, \cdots, r\}$, which makes $x_o' > 0, x_p' > 0, \cdots, x_r' > 0$ and $x_o' + x_p' + \cdots + x_r' = 1$.

case 1: $\alpha' \subseteq \alpha$

Based on definition 2, $s_a = s_b = \cdots = s_d = s_o = s_p = \cdots = s_r$.
Thus, $x_1 s_1 + x_2 s_2 + \cdots + x_m s_m = (x_a + \cdots + x_c + x_d)s_a = s_a$.
Similarly, $x_1' s_1 + x_2' s_2 + \cdots + x_m' s_m = s_a$.

It can be inferred from above that $x_1s_1 + x_2s_2 + \cdots + x_ms_m = x_1^{'}s_1 + x_2^{'}s_2 + \cdots + x_m^{'}s_m$, which satisfies definition 1.

case 2: $\alpha^{'} \nsubseteq \alpha$

$x_p = 0, x_p^{'} > 0$, according to definition 2 in equation Eq. (2.5), for $x_a > 0, s_a \geq s_p$.

Thus, $x_1^{'}s_1 + x_2^{'}s_2 + \cdots + x_m^{'}s_m = (x_o^{'} + \cdots + x_r^{'})s_a + x_p^{'}s_p \leq (x_o^{'} + \cdots + x_r^{'} + x_p^{'})s_a = s_a$.

As $x_1s_1 + x_2s_2 + \cdots + x_ms_m = (x_a + \cdots + x_c + x_d)s_a = s_a$, $\mathbf{x}^T\mathbf{S} \geq \mathbf{x}^{'T}\mathbf{S}$, which satisfies definition 1.

Combining case 1 and case 2, definition 2 is the necessary condition for definition 1. □

Hence, two definitions are equivalent.

Problem 3: Proof of Symmetric Nash Equilibrium

We can solve the problem using **Brouwer's Fixed Point Theorem**: let D be a convex and compact subset of $\mathbb{R}^n$. If a function $f: D \to D$ is continuous, then exists $x \in D$ such that $f(x) = x$. Firstly, we will construct a function that satisfies the theorem's requirement. Then, we will prove that any fixed point of function f is an Nash Equilibrium of the game.

(1) Construction of Function f

For Nash Equilibrium, set $f : \Delta \to \Delta$, where Δ is the set containing mixed strategies of the row player (referred as player 1 in the following part) and the column player (referred as player 2 in the following part). Thus, Δ can be described as $[\mathbf{x}, \mathbf{x}]$, where $x \in \Delta_m$ as defined in Problem 2.

The expected payoff for player 1 and 2, similar to Problem 2, is as follows:

$$\begin{gathered} u(\mathbf{x}_1) = \mathbf{x}_1^T R \mathbf{x_2} \\ u(\mathbf{x}_2) = \mathbf{x}_1^T C \mathbf{x}_2 = \mathbf{x}_1^T R^T \mathbf{x}_2 = (\mathbf{x}_1^T R^T \mathbf{x}_2)^T = \mathbf{x}_2^T R \mathbf{x}_1 \end{gathered} \tag{3.1}$$

Define a gain function $G_{p,s_p} = \max(u_p(s_p; \mathbf{x}_{-p}) - u_p(\mathbf{x}), 0)$ to describe the utility player p can increase using pure strategy s_i, when the other's mixed strategy is fixed,

$$\begin{aligned} G_{1,s_1} &= \max(\mathbf{e}_{s_1}^T R \mathbf{x}_2 - \mathbf{x}_1^T R \mathbf{x_2}, 0) \\ G_{2,s_2} &= \max(\mathbf{e}_{s_2}^T R \mathbf{x_1} - \mathbf{x}_2^T R \mathbf{x_1}, 0) \end{aligned} \tag{3.2}$$

Let $\mathbf{y} = f(\mathbf{x})$, where

$$y_{p,s_p} = \frac{x_{p,s_p} + G_{p,s_p}(\mathbf{x})}{1 + \sum_{s_p^{'} \in S_P} G_{p,s_p^{'}}(\mathbf{x})} \tag{3.3}$$

If $\mathbf{x}_1 = \mathbf{x}_2$, according to equation Eq. (3.2), then $G_{1,s_1} = G_{2,s_2}$. Thus, $\mathbf{y}_1 = \mathbf{y}_2$. For $\Delta = [\mathbf{x}, \mathbf{x}]$, f satisfies the mapping $[\mathbf{x}, \mathbf{x}] \to [\mathbf{y}, \mathbf{y}]$. For each player $\sum y_{p,s_p} = 1$ and $y_{p,s_p} \geq 0$, meaning that $[\mathbf{y}, \mathbf{y}] \in \Delta$. Therefore, the mapping can be turned into $\Delta \to \Delta$. Moreover, f is continuous. According to Brouwer's Fixed Point Theorem, there exists a $\mathbf{X} = [\mathbf{x}, \mathbf{x}] \in \Delta$ such that $f(\mathbf{X}) = \mathbf{X}$.

(2) Proof of any Fixed Point being an NE of the Game

To show a fixed point $[\mathbf{x}, \mathbf{x}]$ is the symmetric Nash Equilibrium of the game, we only need to prove that $G_{p,s_p}(\mathbf{x}) = 0, \forall p \in [2]$.

Proof. By contradiction.

Assume that $G_{p,s_p}(\mathbf{x}) = 0$ does not hold. That is, there exists $p \in [2], s_p$ such that $G_{p,s_p} > 0$.

As $[\mathbf{x}, \mathbf{x}]$ is a fixed point, we can infer that $x_{p,s_p} > 0$. Otherwise, if $x_{p,s_p} = 0$, then $y_{p,s_p} > 0 \neq x_{p,s_p}$, which breaks the premise that $[\mathbf{x}, \mathbf{x}]$ is a fixed point.

For player p, the utility $u_p(\mathbf{x})$ equals the sum of $x_{p,s_p} \cdot u_p(s_p; \mathbf{x}_{-p}), \forall s \in S_p$. If $G_{p,s_p} > 0$, then according to the definition of the gain function, $u_p(s_p; \mathbf{x}_{-p}) > u_p(\mathbf{x})$. Moreover, as mentioned above, $x_{p,s_p} > 0$. Hence, to make the equation $u_p(\mathbf{x}) = \sum_{s \in S_p} x_{p,s_p} \cdot u_p(s_p; \mathbf{x}_{-p})$ hold, there exists some other pure strategy $s_p^{'}$ such that $x_{p,s_p^{'}} > 0$ and $u_p(s_p; \mathbf{x}_{-p}) > u_p(\mathbf{x})$. In this case, $G_{p,s_p} < 0$ and $y_{p,s_p^{'}} < x_{p,s_p^{'}}$. However, this contradicts the fact that $[\mathbf{x}, \mathbf{x}]$ is a fixed point.

Hence, there exists $G_{p,s_p}(\mathbf{x}) = 0, \forall p \in [2]$ and any symmetric game (R, C) where $R = C^T$ has a symmetric Nash Equilibrium $[\mathbf{x}, \mathbf{x}]$. □

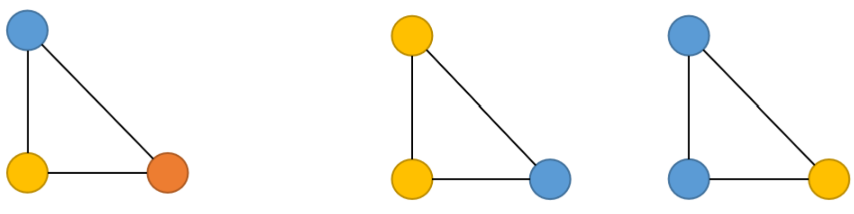
Problem 4: Proof of Sperner's Lemma

(1) Proof by double-counting

Proof. We can prove the lemma by counting the number of yellow-blue edges from two different perspectives.

a. Counting By Color

Assume that there are a triangles colored by all the three colors and b triangles colored by yellow and blue, as is shown in Figure 1.



(a) One Yellow-blue Edge (b) Two Yellow-blue Edges

Figure 1: Counting by color

Thus, the number of yellow-blue edges all triangles contain is $a + 2b$.

b. Counting by position

As can be seen in the figure of the problem, triangles inside the square share edges so that all yellow-blue edges can be counted twice in the last section. Meanwhile, when yellow-blue edges are the boundary of the square, they are counted only once. Assume there are c internal edges and d boundary edges, the number of yellow-blue edges all triangles contain can be double-counted as $2c + d$.

According to the way by which we are supposed to color the boundary, there are no yellow node on the top and right boundary, no blue on the left. Thus, yellow-blue edges can only appear at the bottom of the square. Moreover, the lower left node are supposed to be yellow, the lower right node blue. Therefore, there must be an odd number of yellow-blue edges on the bottom boundary, meaning that d is an odd number.

As two ways of counting are equivalent, $a + 2b = 2c + d$. As d is an odd number, a is also an odd number, which proves the existence of a tri-chromatic triangle. □

(2) Proof by path-following

Proof. Firstly, we will define directed edges, as is shown in Figure 2. The path start from the center of a triangle to the center of another. It can cross the adjacent edge of two triangles only when the edge is yellow-red with red on our left hand.

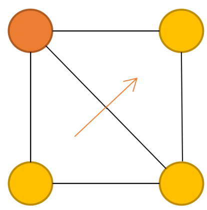


Figure 2: Definition of Directed Edges

Then, for convenience, we introduce an outer boundary for the square in Figure 3. It is obvious that this way of adding an outer boundary does not result in more tri-chromatic triangles. We set the initial source node in the bottom-left triangle, as is marked in the following figure. Following the rule of directed path as mentioned above, we wander around the square.

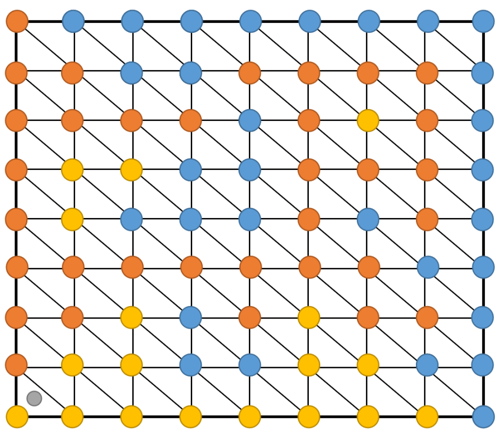
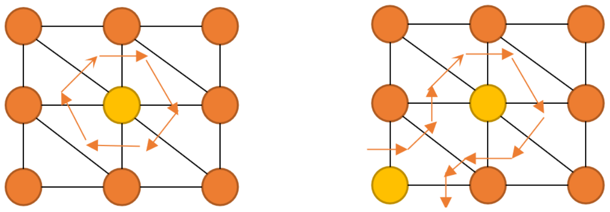


Figure 3: Introduction of an Outer Boundary

As we can only cross yellow-red edges with red on left hand, yet there's only one yellow-red edge with red on right hand in the outer boundary, as a result, the walk cannot exit the square. Neither could it loop inside. There will be a loop when there's a closed red square, according to Figure 4. But based on the rule we've designed, this could not happen. Hence, the path must stop somewhere inside the square.



(a) Loop (b) No Loop

Figure 4: Loop Conditions

The path will stop only in a triangle colored by all the three colors (see Figure 5). This proves the existence of a tri-chromatic triangle.

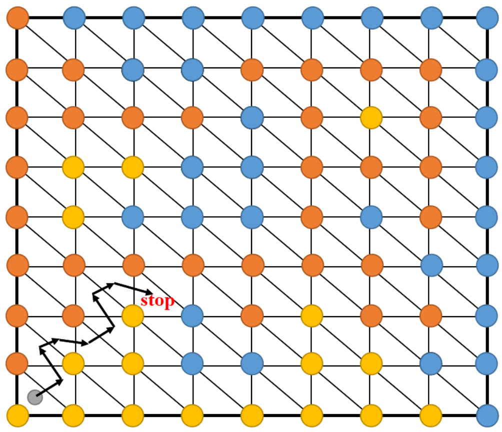


Figure 5: Stop Conditions

□