ALGORITHMIC GAME THEORY

BIT — 2022 Spring

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Homework 2

§ Mechanism Design §

Problem 1: Favourite Movie

Interstellar directed by Christopher Nolan. It's a grand and thoughtful journey into time and space and gravity, into the beautiful and dazzling universe. But it goes beyond that, as "love is the one thing that transcends time and space." It's a movie that awes me and inspires me into the known.

Of course, *A Beautiful Mind* is a movie worth watching, too.

Problem 2: Favourite Theorem in Class

My favorite one is Myerson's Lemma. In the first place, the theorem itself is powerful and concise: it turns the elusive "implementable" goal into an explicit one with uniqueness and offers a simple formula to achieve it. Furthermore, the lemma is useful in many cases to attain DSIC mechanism, for example, revenue-maximization auction design. It lays foundation for subsequent theories and thus is of significant importance. Most importantly, it sparks my interest and inspires to me to learn more about mechanism design from that class on.

Problem 3: XOS Function

(Discussed with Jingyi Gang, Sitian Shen)

To prove this, we will first construct 2^m additive valuations for the mapping of $a_T(S), T \in [m]$ such that $v(S) = \max_{T\subseteq[m]} a_T(S)$. Then, using given information, we will prove that any monotone and normalized submodular function $v(S)$ can be written as an XOS function using this way of construction.

(1) Construction of Additive Valuations

In this section, we construct 2^m additive valuations such that $v(S) = a_S(S)$.

For $S = \emptyset$, define $a_\emptyset(\emptyset) \equiv 0$. For $i \in [m]$, define $a_\emptyset(i) \equiv 0$

For $S \notin \emptyset, i \in [m]$, taking random selection of an element j from the set S, we define $a_S(i)$ as follows:

$$a_S(i) = \begin{cases} -\infty, & i \notin S \\ a_{S-\{j\}}(i), & i \neq j \\ v(S) - v(S - \{j\}), & i = j \end{cases} \tag{3.1}$$

As a_S is an additive function, using information that v is normalized(i.e., $v(\emptyset) = 0$), we can infer that

$$\begin{aligned} a_S(S) &= \sum_{i \in S} a_S(i) = a_S(j) + \sum_{i \in S - \{j\}} a_S(i) \\ &= v(S) - v(S - \{j\}) + a_{S-\{j\}}(S - \{j\}) = \cdots \\ &= v(S) - v(S - \{j\}) + v(S - \{j\}) \cdots - v(\{k\}) + v(\{k\}) - v(\emptyset) + a_\emptyset(\emptyset) \\ &= v(S) \end{aligned} \tag{3.2}$$

For $T \in [m]$, we can obtain that

$$a_T(S) = \begin{cases} -\infty, & S \not\subseteq T \\ a_{T-\{i\}}(S), & S \subseteq T - \{i\} \\ a_{T-\{i\}}(S) + v(T) - v(T - \{i\}), & \text{other} \end{cases} \tag{3.3}$$

(2) Proof of XOS

In this section, using the construction above, we aim to prove that any monotone and normalized submodular function $v(S)$ can be written as an XOS function. That is, $v(S) = a_S(S) = max_{T \in [m]}(S)$.

case 1 $S \not\subseteq T$

From Eq. (3.3), it is obvious that $a_T(S) = -\infty$. As v is monotone and normalized, we can infer that $v(S) \geq v(\emptyset) = 0$. Thus, $a_T(S) = -\infty < a_S(S) = v(S)$.

case 2 $S \subseteq S + \{i\} \subseteq \cdots \subseteq T - \{j\} \subseteq T$ when $a_T(S)$ is inherited from $a_S(S)$

From Eq. (3.3), we can know that $a_T(S) = a_{T-\{j\}}(S) = \cdots = a_{S+\{i\}}(S) = a_S(S)$.

case 3 $S \subseteq T$, but not in case 2.

Suppose $T = S + \{i\}$ and T is not inherited from S, we can show that $a_T(S) \leq v(S)$.

Prove by mathematical induction.

For $S = \{j\}, |S| = 1$ and T is inherited from $\{i\}$

$$\begin{aligned} a_T(i) &= a_{\{i\}}(i) = v(\{i\}) - v(\emptyset) = v(\{i\}) \\ a_T(j) &= v(T) - v(\{i\}) \\ a_T(T) &= v(T) \end{aligned} \tag{3.4}$$

Recall that v is submodular, that is,

$$v(s \cup t) + v(s \cap t) \leq v(s) + v(t) \tag{3.5}$$

Thus, $v(T) = v(\{i\} + \{j\}) + v(\emptyset) < v(\{i\}) + v(\{j\})$,

$$a_T(S) = a_T(j) = v(T) - v(\{i\}) < v(j) = v(S) \tag{3.6}$$

For $|S| \neq 1$ and T is inherited from $S - \{j\}$, assume that $a_{T-\{j\}}(S - \{j\}) \leq v(S - \{j\})$, we can infer that

$$\begin{aligned} a_T(j) &= v(T) - v(T - \{j\}) \\ a_T(S) = a_T(T - \{i\}) &= a_T(S - \{j\}) + a_T(\{j\}) \\ &\leq v(S - \{j\}) + v(S + \{i\}) - v(S + \{i\} - \{j\}) \end{aligned} \tag{3.7}$$

Use the submodular identity again, according to Eq. (3.5). Set $s = S$, $t = S + \{i\} - \{j\}$ and substitute the formula

$$\begin{aligned} v(S + \{i\}) + v(S - \{j\}) &\leq v(S) + v(S + \{i\} - \{j\}) \\ a_T(S) \leq v(S + \{i\}) + v(S - \{j\}) - v(S + \{i\} - \{j\}) &\leq v(S) \end{aligned} \tag{3.8}$$

Thus, for $T = S + \{i\}$, we can show that $a_T(S) \leq v(S)$.

For all the situations in case 3, we can prove that $a_T(S) \leq v(S)$ always exists using mathematical induction similar to the previous one.

Hence, combining case 1,2 and 3, we can prove that any monotone and normalized submodular function $v(S)$ can be written as an XOS function.

Problem 4: Virtual Valuation and Regularity Condition

(1) For a uniform distribution, the distribution function is as follows:

$$F(x) = x, \text{where } x \in [0, 1] \tag{4.1}$$

As $F^{-1}(x) = x$, $V(q) = F^{-1}(1 - q) = 1 - q$, which denotes the posted price resulting in a probability q of a sale.

Hence, the revenue curve $R(q) = q \cdot V(q) = q(1 - q)$.

(2) The revenue curve at q

$$R(q) = q \cdot V(q) = q \cdot F^{-1}(1 - q), q \in [0, 1] \tag{4.2}$$

Take the inverse function of F^{-1}, we can obtain

$$F(\frac{R(q)}{q}) = 1 - q \tag{4.3}$$

Then, take the derivative of both sides

$$f(\frac{R(q)}{q}) \cdot \frac{R^{'}(q) \cdot q - R(q) \cdot 1}{q^2} = -1 \tag{4.4}$$

Based on equation Eq. (4.2), we simplify Eq. (4.4)

$$\begin{aligned} f(V(q)) \cdot \frac{R^{'}(q) \cdot q - q \cdot V(q)}{q^2} &= -1 \\ f(V(q)) \cdot \frac{R^{'}(q) - V(q)}{q} &= -1 \end{aligned} \tag{4.5}$$

Thus, we get the slope of the revenue curve at q

$$R^{'}(q) = V(q) - \frac{q}{f(V(q))} \tag{4.6}$$

According to equation Eq. (4.3), $q = 1 - F(\frac{R(q)}{q}) = 1 - F(V(q))$. Hence, $R^{'}(q) = \varphi(V(q))$, since the virtual valuation $\varphi(v) = v - \frac{1-F(v)}{f(v)}$ and $V(q)$ can get all possible values of v for $q \in [0,1]$.

(3) We have proved in Sec. (4.2) that

$$R^{'}(q) = \varphi(V(q)) \tag{4.7}$$

Take the derivative of both sides

$$R^{''}(q) = [\varphi(V(q))]^{'} = \varphi^{'}(V(q)) \cdot V^{'}(q) \tag{4.8}$$

Then, take the derivative of equation Eq. (4.2), we can get the derivative of $V(q)$

$$\begin{aligned} f(V(q)) \cdot V^{'}(q) &= -1 \\ V^{'}(q) &= -\frac{1}{f(V(q))} \end{aligned} \tag{4.9}$$

Since the distribution density is strictly positive, $V^{'}(q) < 0$. Thus,

$$R^{''}(q) < 0 \text{ iff } \varphi^{'}(V(q)) > 0, q \in [0,1] \tag{4.10}$$

Furthermore, for $q \in [0,1]$, $V(q)$ can get all possible values of v. Hence, equation 4.10 can be turned into

$$R^{''}(q) < 0 \text{ iff } \varphi^{'}(v) > 0, v \in [0, v_{max}] \tag{4.11}$$

Thus, the revenue curve is concave if and only if $\varphi(v)$ is a monotone non-decreasing function for $v \in [0, v_{max}]$, which is equivalent to the definition of a regular distribution.

(4) As proven in section Sec. (4.3), if a valuation distribution is regular, its revenue curve is concave. That is to say, for $q \in [0,1]$, $R^{''}(q) < 0$.

Let $q^{'}$ be the the maximum point of R, namely $R(q^{'}) = max_{q\in[0,1]}R(q)$. According to Jensen's inequality, the secant line of a convex function lies above the graph of the function, we can know that

$$\begin{aligned} R\left(\frac{0+q^{'}}{2}\right) &\geq \frac{R(0)+R(q^{'})}{2} \\ R\left(\frac{q^{'}+1}{2}\right) &\geq \frac{R(q^{'})+R(1)}{2} \end{aligned} \tag{4.12}$$

Since $R(0) = R(1) = 0$, Eq. (4.12) can be simplified as follows:

$$\begin{aligned} R\left(\frac{q^{'}}{2}\right) &\geq \frac{1}{2}R(q^{'}) \\ R\left(\frac{q^{'}+1}{2}\right) &\geq \frac{1}{2}R(q^{'}) \end{aligned} \tag{4.13}$$

As $0 \leq q^{'} \leq 1$, $\frac{q^{'}}{2} \leq \frac{1}{2} \leq \frac{q^{'}+1}{2}$. According to the properties of convex functions

$$\begin{aligned} R\left(\frac{1}{2}\right) \geq R\left(\frac{q^{'}}{2}\right) &\geq \frac{1}{2}R(q^{'}) \\ R\left(\frac{1}{2}\right) \geq R\left(\frac{q^{'}+1}{2}\right) &\geq \frac{1}{2}R(q^{'}) \\ \Rightarrow R\left(\frac{1}{2}\right) &\geq \frac{1}{2}max_{q\in[0,1]}R(q) \end{aligned} \tag{4.14}$$

Thus, we can prove that the median price $V\left(\frac{1}{2}\right)$ is a $\frac{1}{2}$-approximation of the optimal posted price for a regular distribution.