1 One-particle fermion Green's function for a Hamiltonian system

A thermal Green's function for a Hamiltonian system of fermions in the Matsubara representation:

$$g_{12}(\omega) \equiv -\langle c_{\omega 1}\bar{c}_{\omega 2}\rangle = -\int_0^\beta e^{i\omega\tau} d\tau \,\mathrm{Tr}[\hat{w}\mathbb{T}\hat{c}_1(\tau)\hat{c}_2^+(0)] =$$
$$= \sum_{mn} \frac{\langle n|\hat{c}_1|m\rangle\langle m|\hat{c}_2^+|n\rangle(w_m + w_n)}{i\omega - (E_m - E_n)}$$

where indices n, m denote Fock states of the system, E_n is an energy level, β is the inverse temperature and

$$w_n \equiv \frac{e^{-\beta E_n}}{Z}, \quad Z \equiv \sum_n e^{-\beta E_n}$$

2 Two-particle fermion Green's function and two-particle irreducible vertex part

For an energy-conserving combination of frequencies $(\omega_1, \omega_2; \omega_3, \omega_4 = \omega_1 + \omega_2 - \omega_3)$ a two-particle Green's function is defined as follows:

$$\chi_{1234}(\omega_1, \omega_2; \omega_3, \omega_4) \equiv \langle c_{\omega_1 1} c_{\omega_2 2}\bar{c}_{\omega_3 3}\bar{c}_{\omega_4 4}\rangle = \int_0^\beta e^{i\omega_1\tau_1 + i\omega_2\tau_2 - i\omega_3\tau_3} d\tau \,\mathrm{Tr}[\hat{w}\mathbb{T}\hat{c}_1(\tau_1)\hat{c}_2(\tau_2)\hat{c}_3^+(\tau_3)\hat{c}_4^+(0)]$$

$$\chi_{1234}(\omega_1, \omega_2; \omega_3, \omega_4) = \sum_\Pi \mathrm{sgn}(\Pi) \sum_{ijkl} \langle i|\hat{O}_{\Pi_1}|j\rangle\langle j|\hat{O}_{\Pi_2}|k\rangle\langle k|\hat{O}_{\Pi_3}|l\rangle\langle l|\hat{c}_4^+|i\rangle \phi_{ijkl}(z_{\Pi_1}, z_{\Pi_2}, z_{\Pi_3})$$

where a sum over Π is a sum over all permutations of 3 indices, $\hat{O} = \{\hat{c}_1, \hat{c}_2, \hat{c}_3^+\}$, $z = \{i\omega_1, i\omega_2, -i\omega_3\}$. The spectral kernel ϕ_{ijkl} has the following form:

$$\phi_{ijkl}(z_1, z_2, z_3) =$$
$$= \frac{w_i + w_l}{(z_1 + E_i - E_j)(z_1 + z_2 + z_3 + E_i - E_l)(z_3 + E_k - E_l)} - \frac{w_j + w_k}{(z_1 + E_i - E_j)(z_2 + E_j - E_k)(z_3 + E_k - E_l)} +$$
$$+ \frac{1}{(z_1 + E_i - E_j)(z_3 + E_k - E_l)} \left[\beta w_i \delta_{z_1+z_2}\delta_{E_i - E_k} + \frac{w_k - w_i}{z_1 + z_2 + E_i - E_k}(1 - \delta_{z_1+z_2}\delta_{E_i - E_k})\right] -$$
$$- \frac{1}{(z_1 + E_i - E_j)(z_3 + E_k - E_l)} \left[\beta w_j \delta_{z_2+z_3}\delta_{E_j - E_l} + \frac{w_l - w_j}{z_2 + z_3 + E_j - E_l}(1 - \delta_{z_2+z_3}\delta_{E_j - E_l})\right]$$

The Wick part of a two-particle Green's function:

$$\chi^0_{1234}(\omega_1, \omega_2; \omega_3, \omega_4) = \beta\delta_{\omega_1\omega_4}\delta_{\omega_2\omega_3} g_{14}(\omega_1) g_{23}(\omega_2) - \beta\delta_{\omega_1\omega_3}\delta_{\omega_2\omega_4} g_{13}(\omega_1) g_{24}(\omega_2)$$

An irreducible vertex part:

$$\Gamma_{1234}(\omega_1, \omega_2; \omega_3, \omega_4) \equiv \chi_{1234}(\omega_1, \omega_2; \omega_3, \omega_4) - \chi^0_{1234}(\omega_1, \omega_2; \omega_3, \omega_4)$$

An amputated irreducible vertex part:

$$\gamma_{1234}(\omega_1, \omega_2; \omega_3, \omega_4) \equiv \sum_{1'2'3'4'} (g^{-1}(\omega_1))_{11'}(g^{-1}(\omega_2))_{22'}\Gamma_{1'2'3'4'}(\omega_1, \omega_2; \omega_3, \omega_4)(g^{-1}(\omega_3))_{3'3}(g^{-1}(\omega_4))_{4'4}$$

3 Singular part of an irreducible vertex part.

The singular part of an irreducible vertex part is defined as a sum of all terms proportional to β. It can be expressed using a function F (Wick-like contribution) and a function R ("superconductive" contribution).

$$\Gamma^s_{1234} = \beta F_{1234}(\omega_1,\omega_2)\delta_{\omega_2-\omega_3} - \beta F_{2134}(\omega_2,\omega_1)\delta_{\omega_1-\omega_3} + \beta R_{1234}(\omega_1,\omega_3)\delta_{\omega_1+\omega_2}$$

$$\begin{aligned}
F_{1234}(\omega_1,\omega_2) = \sum_{ijkl} w_i \Bigg\{ & \frac{\delta_{E_i-E_k}\langle i|\hat{c}_2|l\rangle\langle l|\hat{c}_3^+|k\rangle\langle k|\hat{c}_4^+|j\rangle\langle j|\hat{c}_1|i\rangle - w_k\langle k|\hat{c}_2|l\rangle\langle l|\hat{c}_3^+|k\rangle\langle i|\hat{c}_4^+|j\rangle\langle j|\hat{c}_1|i\rangle}{(i\omega_1-(E_i-E_j))(i\omega_2-(E_l-E_k))} + \\
& + \frac{\delta_{E_i-E_k}\langle i|\hat{c}_3^+|l\rangle\langle l|\hat{c}_2|k\rangle\langle k|\hat{c}_4^+|j\rangle\langle j|\hat{c}_1|i\rangle - w_k\langle k|\hat{c}_3^+|l\rangle\langle l|\hat{c}_2|k\rangle\langle i|\hat{c}_4^+|j\rangle\langle j|\hat{c}_1|i\rangle}{(i\omega_1-(E_i-E_j))(i\omega_2-(E_k-E_l))} + \\
& + \frac{\delta_{E_i-E_k}\langle i|\hat{c}_1|j\rangle\langle j|\hat{c}_4^+|k\rangle\langle k|\hat{c}_2|l\rangle\langle l|\hat{c}_3^+|i\rangle - w_k\langle i|\hat{c}_1|j\rangle\langle j|\hat{c}_4^+|i\rangle\langle k|\hat{c}_2|l\rangle\langle l|\hat{c}_3^+|k\rangle}{(i\omega_1-(E_j-E_i))(i\omega_2-(E_l-E_k))} + \\
& + \frac{\delta_{E_i-E_k}\langle i|\hat{c}_1|j\rangle\langle j|\hat{c}_4^+|k\rangle\langle k|\hat{c}_3^+|l\rangle\langle l|\hat{c}_2|i\rangle - w_k\langle i|\hat{c}_1|j\rangle\langle j|\hat{c}_4^+|i\rangle\langle k|\hat{c}_3^+|l\rangle\langle l|\hat{c}_2|k\rangle}{(i\omega_1-(E_j-E_i))(i\omega_2-(E_k-E_l))} \Bigg\}
\end{aligned}$$

$$\begin{aligned}
R_{1234}(\omega_1,\omega_3) = -\sum_{ijkl}\delta_{E_i-E_k} w_i \Bigg\{ & \frac{\langle i|\hat{c}_1|j\rangle\langle j|\hat{c}_2|k\rangle\langle k|\hat{c}_3^+|l\rangle\langle l|\hat{c}_4^+|i\rangle}{(i\omega_1-(E_j-E_i))(i\omega_3-(E_k-E_l))} + \frac{\langle i|\hat{c}_2|j\rangle\langle j|\hat{c}_1|k\rangle\langle k|\hat{c}_3^+|l\rangle\langle l|\hat{c}_4^+|i\rangle}{(i\omega_1-(E_i-E_j))(i\omega_3-(E_k-E_l))} + \\
& + \frac{\langle k|\hat{c}_1|j\rangle\langle j|\hat{c}_2|i\rangle\langle i|\hat{c}_4^+|l\rangle\langle l|\hat{c}_3^+|k\rangle}{(i\omega_1-(E_j-E_i))(i\omega_3-(E_l-E_k))} + \frac{\langle k|\hat{c}_2|j\rangle\langle j|\hat{c}_1|i\rangle\langle i|\hat{c}_4^+|l\rangle\langle l|\hat{c}_3^+|k\rangle}{(i\omega_1-(E_i-E_j))(i\omega_3-(E_l-E_k))} \Bigg\}
\end{aligned}$$

$\beta\to\infty$ **limit.** Weight w_i goes to $1/g$ in the limit of low temperatures, if i is a component of a g-fold ground state, and vanishes for excited states. Thus a summation over i and k in the formulae above includes only components of the ground state:

$$\sum_{ijkl} w_i\delta_{E_i-E_k} \mapsto \frac{1}{g}\sum_{ik\in\{|gs\rangle\}}\sum_{jl}$$

$$\sum_{ijkl} w_i w_k \mapsto \frac{1}{g^2}\sum_{ik\in\{|gs\rangle\}}\sum_{jl}$$

If the ground state is not degenerate ($g=1$), then $\beta F_{1234}(\omega_1,\omega_2)\to 0$ as $\beta\to\infty$. It is easy to prove taking the limit for this special case:

$$w_i\to\delta_{i,gs};\qquad \sum_{ijkl} w_i\delta_{E_i,E_k},\ \sum_{ijkl} w_i w_k \mapsto \sum_{jl};\qquad E_i,E_k\mapsto E_{gs};\qquad |i\rangle,|k\rangle\mapsto|gs\rangle$$