Graph Colorama: Finding the Chromatic Number
Interactive Play and Challenges

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Objective

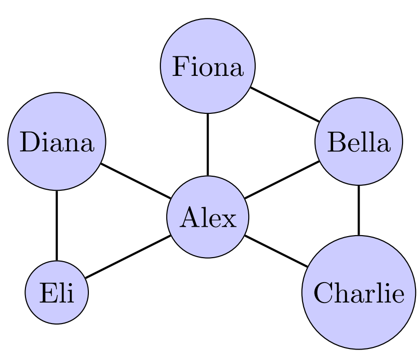
Learn and understanding of graph coloring through hands-on play and structured problem-solving.

Part 1: Getting Started with the Graph Coloring Game

Let's start with some basic definitions to help guide you through graph coloring:

- **Graph:** A collection of vertices (points) connected by edges (lines).
- **Graph Coloring:** A way of coloring the vertices of a graph so that no two adjacent vertices (connected by an edge) have the same color.
- **Chromatic Number (χ):** The minimum number of colors needed to achieve a proper coloring of a graph.
- **Greedy Algorithm:** The game uses a Greedy Algorithm to approximate the chromatic number. This algorithm colors each vertex in sequence, assigning the smallest possible color that hasn't been used by its neighbors.

Friendship Networks: Visualizing Relationships as a Graph



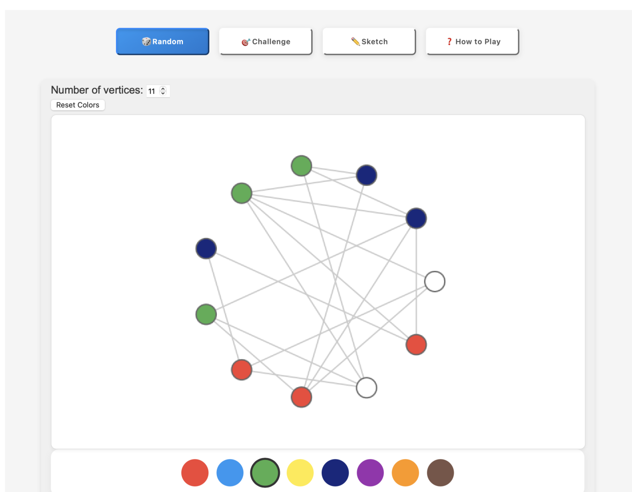
Instructions

Begin by exploring the graph coloring game using the link below:

`https://reben80.github.io/chromatic-number-new-/`

Alternatively, scan the QR code below to access the game:

Below is an example screenshot of the app interface:



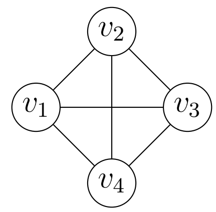
- Try to color each graph with the fewest colors possible.
- Work through the different levels and challenges in the game.
- Observe how the Greedy Algorithm helps approximate the chromatic number for different graphs.

This interactive experience introduces key graph coloring concepts. After playing the game, proceed to the challenges below.

Challenge 1: Exploring Complete Graphs and Their Chromatic Numbers

Introduction

Complete Graph (K_n): A graph where each vertex is connected to every other vertex.
Example: In a complete graph K_4, there are 4 vertices, and each vertex is connected to the other 3 vertices, forming a fully connected structure.



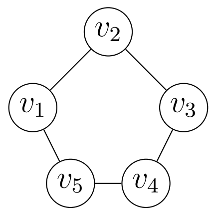
Questions

Answer the following questions about the chromatic numbers of complete graphs:

1. What is the chromatic number of the following complete graphs:
   - K_1 (1 vertex)
   - K_2 (2 vertices)
   - K_3 (3 vertices)
   - K_{40} (40 vertices)
2. In general, what is the chromatic number of a complete graph K_n, where it has n vertices and each vertex is connected to all other vertices?
3. For removing one vertex, think about how the number of vertices and edges is affected and what that means for coloring.
4. For removing two edges, consider whether the graph remains complete or becomes a different type of graph. How does this influence the chromatic number?

Challenge 2: Cycle Graphs and Their Chromatic Numbers

Cycle Graph: A graph where vertices form a closed loop, with each vertex connected to two others. **Example:** In a cycle graph C_5, there are 5 vertices $(v_1, v_2, v_3, v_4, v_5)$, and they are connected in a circular manner, such that v_1 is connected to v_2 and v_5, v_2 is connected to v_1 and v_3, and so on, forming a closed loop.



Questions

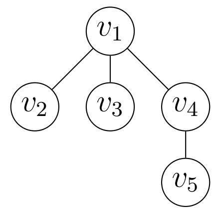
Answer the following questions about the chromatic numbers of cycle graphs:

1. What is the chromatic number of the following cycle graphs:
   - C_2 (2 vertices)
   - C_3 (3 vertices)
   - C_4 (4 vertices)
   - C_5 (5 vertices)
   - C_6 (4 vertices)
   - C_7 (5 vertices)
   - C_{100} (100 vertices)
   - C_{101} (101 vertices)
2. In general, what is the chromatic number of a cycle graph C_n? explain your reasoning.

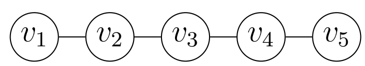
Challenge 3: Trees and Their Chromatic Numbers

Tree: A graph with no cycles, meaning it has no closed loops and connected

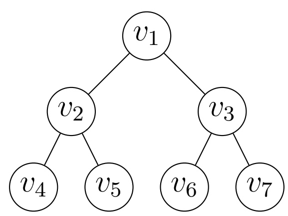
Example 1: A tree with 5 vertices could have v_1 as the root vertex, connected to v_2, v_3, and v_4, with v_4 further connected to v_5.



Example 2: A tree with 5 vertices can also have all the vertices aligned in a straight line, where v_1 is connected to v_2, v_2 is connected to v_3, v_3 is connected to v_4, and v_4 is connected to v_5.



Example 3: A tree with 7 vertices can be represented as a binary tree with v_1 as the root vertex, connected to v_2 and v_3. Vertex v_2 is further connected to v_4 and v_5, and vertex v_3 is connected to v_6 and v_7.



Questions

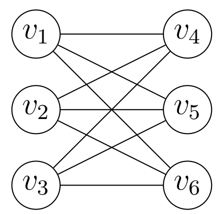
Answer the following questions about the chromatic numbers of trees:

1. What is the chromatic number of the following trees:
   - A tree with 1 vertex
   - A tree with 2 vertices
   - A tree with 3 vertices (connected in a straight line)
   - Example 1, Example 2 and Example 3 above.
2. In general, what is the chromatic number of any tree with n vertices? Explain your reasoning.
3. What happens to the chromatic number if the tree is transformed into a graph by adding one edge to form a cycle? Provide an example and explanation.

Challenge 4: Bipartite Graphs and Their Chromatic Numbers

Bipartite Graph: A graph whose vertices can be divided into two distinct sets, where edges only exist between (not within) the two sets.

Example: In a bipartite graph with 6 vertices, the vertices can be divided into two sets of 3 each. For instance, set $A = \{v_1, v_2, v_3\}$ and set $B = \{v_4, v_5, v_6\}$, and edges exist only between vertices in A and B.



Questions

Answer the following questions about the chromatic numbers of bipartite graphs:

1. What is the chromatic number of the following bipartite graphs:
   - A bipartite graph with 2 vertices (1 in each set).
   - A bipartite graph with 4 vertices (2 in each set).
   - A bipartite graph with 6 vertices (3 in each set), as shown in the example above.
   - A bipartite graph with $m = 5$ vertices in set A and $n = 7$ vertices in set B.
2. If you remove one edge from the bipartite graph in the example above, what is the chromatic number of the resulting graph? Draw the new graph.
3. If you add one edge within set A or B, what happens to the chromatic number? Explain and draw the new graph.
4. In general, what is the chromatic number of any bipartite graph, and why?

Challenge 5: Greedy Coloring on a Same Graph but Different Orders

Below are two graph diagrams from a 4x4 grid of numbered vertices. You are asked to use the **greedy coloring algorithm** to color each diagram, following the specified order.

Diagram 1: Column-wise Order

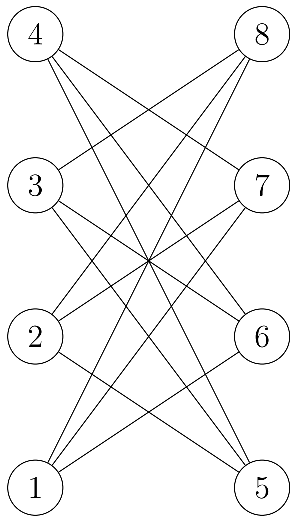
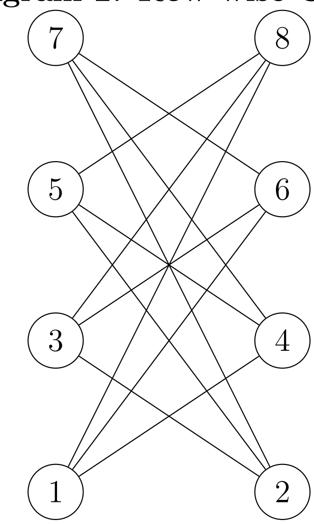


Diagram 2: Row-wise Order



Use the **greedy coloring algorithm**

1. Apply the greedy algorithm to each diagram and assign colors to each vertex in the specified order. How many colors are used in each order?
2. Compare the two orders: Did the number of colors used change depending on the order? If so, explain why.
3. Based on your results, discuss how the choice of coloring order can affect the total number of colors needed.

Acknowledgment

I would like to thank Alexandra Veremeychik for her support and valuable ideas that made this workshop better. Her efforts in the SK Day activity at Montgomery College brought excitement and engagement to the event. Thank you for your hard work and dedication!