# Homological Algebra

Original lectures notes by Baptiste Rognerud, but now typeset in LaTeX

## 1 Introduction to category theory

References:

- Emily Riehl, Category Theory in Context (chapter I)
- Saunders Mac Lane, Categories for the Working Mathematician
- Ibrahim Assem, Introduction au langage catégorique (chapters I, II)

❯ Near 1945 Eilenberg and Mac Lane gave the good formalism for a "natural isomorphism" (the general theory of natural transformations). For instance, if $V$ is a finite-dimensional vector space, $V \simeq V^*$ and $V \simeq V^{**}$, but the first isomorphism is not natural ("a choice needs to be made"), while the second is. But why?
It turns out solving this question gave a formalism, category theory, that unified already existing mathematical concepts, gave new links between different notions and also gave new questions!
⚠ Category theory is not a theory that trivializes mathematics.
It is used today by (almost) everyone: algebraic geometry, algebra, representation theory, topology, combinatorics, ...

### 1.1 Categories and functors

**Definition 1.1.** A *category* $\mathcal{C}$ is the data of

- A collection of *morphisms* $\mathrm{Mor}(\mathcal{C})$
- A collection of *objects* $\mathrm{Ob}(\mathcal{C})$

such that

1. Every morphism $f \in \mathrm{Mor}(\mathcal{C})$ has a specified domain $X \in \mathrm{Ob}(\mathcal{C})$ and codomain $Y \in \mathrm{Ob}(\mathcal{C})$. We write $f : X \to Y$.
2. For every object $X \in \mathrm{Ob}(\mathcal{C})$ there exists a morphism $1_X : X \to X$ (the *identity* of $X$), also written $\mathrm{id}_X$
3. For any three objects $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$ and morphism $f : X \to Y$ and $g : Y \to Z$ there exists a morphism $g \circ f : X \to Z$ (we often omit $\circ$ and just write $gf$)

satisfying

**(Identity)** $\forall f : X \to Y, 1_Y f = f = f 1_X$

**(Associativity)** $\forall f : W \to X, g : X \to Y, h : Y \to Z$, $h(gf) = (hg)f$

*Remark.*

1. We use the term "collection" because we don't want to worry about set-theoretical issues
2. If $\mathrm{Mor}(\mathcal{C})$ is a set, we say that $\mathcal{C}$ is *small*
3. We denote by $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ (or $\mathcal{C}(X, Y)$) the collection of $f : X \to Y \in \mathrm{Mor}(\mathcal{C})$

**Examples 1.2** (Concrete categories)**.**

1. The category **Set**, where objects are sets and morphisms are just maps.
2. **Top**, where objects are topological spaces and morphisms are continuous maps.
3. Groups together with group homomorphisms form a category called **Grp**. The same can be said about rings, fields...
4. $k$-vector spaces, or more generally left/right $R$-modules, together with linear maps, form a category denoted $R\mathbf{Mod}$ or $\mathbf{Mod}R$ (for left or right $R$-modules).

In these previous examples, objects are sets with additional structure, and morphisms between two objects are in particular maps between the two underlying sets. Such categories are called *concrete categories* (a rigorous definition will be given later). However, a category need not be concrete.

**Examples 1.3** (Abstract categories)**.**

1. Let $k$ be a field. There exists a category $\mathbf{Mat}_k$ where objects are the natural numbers $\mathbb{N}$ and morphisms are $\mathrm{Hom}(m, n) = \mathrm{Mat}_{n,m}(k)$, where composition is given by matrix multiplication.
2. If $G$ is a group, there exists a category $BG$ which has only one object $\bullet$, and morphisms $\mathrm{Hom}(\bullet, \bullet) = G$, where composition is multiplication in $G$.
3. If $(P, \leq)$ is a *poset* (a partially ordered set, that is a set $P$ together with a reflexive, transitive relation $\leq$), then one can construct a category $\hat{P}$ by setting $\mathrm{Ob}(\hat{P}) = P$ and $|\mathrm{Hom}(x, y)| = \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{otherwise} \end{cases}$, where composition is defined in the only possible way.
4. The homotopy category of topological spaces: objects are topological spaces, and $\mathrm{Hom}(X, Y)$ is $\mathrm{Hom}_{\mathbf{Top}}(X, Y)/\sim$ where $\sim$ is homotopy of continuous maps.

*Exercise.* Check the categories defined above really are categories. In (2), what are the minimal hypotheses needed on $G$ for $BG$ to be a category? In (3), what are the minimal hypotheses needed on $\leq$ for $\hat{P}$ to be a category?

**Examples 1.4** (Categories constructed from categories)**.**

1. If $\mathcal{C}$ is a category, one can construct its *opposite category* $\mathcal{C}^{\mathrm{op}}$, defined by $\mathrm{Ob}(\mathcal{C}^{\mathrm{op}}) = \mathrm{Ob}(\mathcal{C})$ and $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X)$, with composition described by the following diagram:

$$
\begin{tikzcd}
X \arrow[d, "f"] \arrow[dd, bend right, "gf"'] & X \\
Y \arrow[d, "g"] \arrow[r, rightsquigarrow] & Y \arrow[u, "f^{\mathrm{op}}"'] \\
Z & Z \arrow[u, "g^{\mathrm{op}}"'] \arrow[uu, bend right, "f^{\mathrm{op}}g^{\mathrm{op}}"']
\end{tikzcd}
$$

2. Let $\mathcal{C}$ be a category. A *subcategory* $\mathcal{D}$ of $\mathcal{C}$ is another category such that $\mathrm{Ob}(\mathcal{D}) \subset \mathrm{Ob}(\mathcal{C})$ and $\mathrm{Mor}(\mathcal{D}) \subset \mathrm{Mor}(\mathcal{C})$ and the composition in $\mathcal{D}$ is induced by the one in $\mathcal{C}$. For instance, **Ab**, the category of abelian groups and group homomorphisms, is a subcategory of **Grp**.

3. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. The *product category* of $\mathcal{C}$ and $\mathcal{D}$ is the category $\mathcal{C} \times \mathcal{D}$ defined by $\mathrm{Ob}(\mathcal{C} \times \mathcal{D}) = \mathrm{Ob}(\mathcal{C}) \times \mathrm{Ob}(\mathcal{D})$ and $\mathrm{Mor}(\mathcal{C} \times \mathcal{D}) = \mathrm{Mor}(\mathcal{C}) \times \mathrm{Mor}(\mathcal{D})$, composition and identities being defined componentwise.

*Exercise.* Describe $(BG)^{\mathrm{op}}$ for $G$ a group and $\hat{P}^{\mathrm{op}}$ for $(P, \leq)$ a poset.

⚠ $\mathbf{Set}^{\mathrm{op}}$ is not **Set**. **TODO**

*Remark.* In a category $\mathcal{C}$ the objects can be anything, so saying $x \in X$ for $X \in \mathrm{Ob}(\mathcal{C})$ doesn't make sense. Hence, categorical notions are defined using arrows and not elements.

**Definition 1.5.** Let $\mathcal{C}$ be a category.

1. $f : X \to Y$ is an *isomorphism* if there exists $g : Y \to X$ such that $gf = \mathrm{id}_X$ and $fg = \mathrm{id}_Y$.

2. $f : X \to Y$ is a *monomorphism* if for all $g, h : W \to X$ such that $fg = fh$, $g = h$ ($f$ is left-cancellable).

3. $f : X \to Y$ is an *epimorphism* if for all $g, h : Y \to Z$ such that $gf = hf$, $g = h$ ($f$ is right-cancellable).

⚠ Being both a mono and an epi doesn't imply being an iso. **TODO**

**Definition 1.6.** Let $\mathcal{C}, \mathcal{D}$ be two categories. A *(covariant) functor* $F : \mathcal{C} \to \mathcal{D}$ is the data of

- An object $F(X) \in \mathrm{Ob}(\mathcal{D})$ for all $X \in \mathrm{Ob}(\mathcal{C})$
- A morphism $F(f) \in \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ for all $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$

such that $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for all $X \in \mathrm{Ob}(\mathcal{C})$ and $F(gf) = F(g)F(f)$ whenever $f, g \in \mathrm{Mor}(\mathcal{C})$ are composable.

**Definition 1.7.** A *contravariant* functor from $\mathcal{C}$ to $\mathcal{D}$ is a functor from $\mathcal{C}^{\mathrm{op}}$ to $\mathcal{D}$ (so composition is reversed, i.e. $F(gf) = F(f)F(g)$).

**Examples 1.8.**

1. $U : \mathbf{Grp} \to \mathbf{Set}, U(G) = G, U(f) = f$ the functor that to a group assigns it its underlying set and to a homomorphism the underlying map. It is called the *forgetful functor* from groups to sets, because it forgets the group structure.

2. $U: \begin{array}{ccc} \mathbf{Ass} & \to & \mathbf{Lie} \\ (A,+,\cdot) & \mapsto & (A,+,[-,-]) \end{array}$ the forgetful functor from the category of associative algebras to the category of Lie algebras. It forgets the "associative structure" but remembers the underlying abelian group.

3. $F : \mathbf{Set} \to \mathbf{Ab}, X \mapsto \mathbb{Z}[X], f \mapsto \mathbb{Z}[f]$, which to a set assigns the free abelian group with basis $X$ (the group of finite linear combinations of elements of $X$). A map $f : X \to Y$ can then be uniquely extended to a linear map $\mathbb{Z}[f] : \mathbb{Z}[X] \to \mathbb{Z}[Y]$ that agrees with $f$ on the bases of $\mathbb{Z}[X]$ and $\mathbb{Z}[Y]$.

4. Suppose $\mathcal{C}$ is locally small (i.e. for any $X, Y$, $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ is a set). For all $X \in \mathcal{C}$, $\mathrm{Hom}(X,-)$ is a functor $\mathcal{C} \to \mathbf{Set}$. Similarly, $\mathrm{Hom}_{\mathcal{C}}(-,X)$ is a contravariant functor $\mathcal{C} \to \mathbf{Set}$. $\mathrm{Hom}_{\mathcal{C}}(-,-)$ is a functor $\mathcal{C} \times \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$.

5. Functors $\mathcal{C} \underset{F}{\to} \mathcal{D} \underset{G}{\to} \mathcal{E}$ can be composed in the obvious sense.

**TODO**: DRAW DIAGRAMS

**Definition 1.9.** Let $\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] & \mathcal{D} \end{tikzcd}$ be two functors. A *natural transformation* $\eta$ from $F$ to $G$ is the data of morphisms $\eta_X : F(X) \to G(X) \in \mathrm{Mor}(\mathcal{D})$ for all $X \in \mathrm{Ob}(\mathcal{C})$ such that for all $f : X \to Y \in \mathrm{Mor}(\mathcal{C})$, the diagram

$$\begin{tikzcd} F(X) \arrow[r, "\eta_X"] \arrow[d, "F(f)"] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\eta_Y"] & G(Y) \end{tikzcd}$$

commutes, that is $G(f)\eta_X = \eta_Y F(f)$. We write $\eta : F \Rightarrow G$ or draw $\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathcal{D} \end{tikzcd}$

**Example 1.10.** Let $V$ be a $k$-vector space. $\mathrm{id}_{\mathbf{Vect}_k}$ and $D^2 = \mathrm{Hom}_{\mathbf{Vect}_k}(\mathrm{Hom}_{\mathbf{Vect}_k}(-,k),k)$ are two endofunctors of $\mathbf{Vect}_k$. $\mathrm{ev}_- : \begin{array}{ccc} V & \to & V^{**} \\ v & \mapsto & \begin{array}{ccc} \mathrm{Hom}(V,k) & \to & k \\ \phi & \mapsto & \phi(v) \end{array} \end{array}$ defines a natural transformation between them:

$$\begin{tikzcd} V \arrow[r, "\mathrm{ev}"] \arrow[d, "f"'] & V^{**} \arrow[d, "D^2(f)"] \\ W \arrow[r, "\mathrm{ev}"'] & W^{**} \end{tikzcd}$$

For $a \in V$, $D^2(f) \circ \mathrm{ev}_a : \begin{array}{ccc} W^* & \to & k \\ \phi & \mapsto & \phi(f(a)) \end{array} \in W^{**}$ and in the other direction $(\mathrm{ev} \circ f)(a) = \mathrm{ev}_{f(a)}$.
However, there is no natural transformation from $\mathrm{id}_{\mathbf{Vect}_k}$ to $D$. For one, the first is covariant and the second is contravariant. To get a natural transformation from a covariant to a contravariant functor, we can modify the definition of naturality to be that $\begin{array}{ccc} V & \to & V^* \\ \downarrow & & \uparrow \\ W & \to & W^* \end{array}$ commutes, but even such natural transformations do not exist from $\mathrm{id}_{\mathbf{Vect}_k}$ to $D$.

**Definition 1.11.** A natural transformation $\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"{name=U}] \arrow[r, bend right, "G"'{name=L}] & \mathcal{D} \arrow[from=U, to=L, phantom, "\Downarrow\eta"] \end{tikzcd}$ is a *natural isomorphism* if $\eta_X$ is an isomorphism for all $X \in \mathrm{Ob}(\mathcal{C})$.

*Remark.* One can compose natural transformations in two ways, "vertical composition":

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=60, "F"{name=F}] \arrow[r, "G"{name=G, description}] \arrow[r, bend right=60, "H"'{name=H}] & \mathcal{D} \arrow[from=F, to=G, phantom, "\Downarrow\alpha"] \arrow[from=G, to=H, phantom, "\Downarrow\beta"] \end{tikzcd} \rightsquigarrow \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"{name=U}] \arrow[r, bend right, "H"'{name=L}] & \mathcal{D} \arrow[from=U, to=L, phantom, "\Downarrow\beta\circ\alpha"] \end{tikzcd} \text{ where } (\beta \circ \alpha)_X = \beta_X \circ \alpha_X$$

or "horizontal composition":

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F_1"{name=U1}] \arrow[r, bend right, "G_1"'{name=L1}] & \mathcal{D} \arrow[from=U1, to=L1, phantom, "\Downarrow\alpha_1"] \arrow[r, bend left, "F_2"{name=U2}] \arrow[r, bend right, "G_2"'{name=L2}] & \mathcal{E} \arrow[from=U2, to=L2, phantom, "\Downarrow\alpha_2"] \end{tikzcd} \rightsquigarrow \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F_2\circ F_1"{name=U}] \arrow[r, bend right, "G_2\circ G_1"'{name=L}] & \mathcal{E} \arrow[from=U, to=L, phantom, "\Downarrow\alpha_2 * \alpha_1"] \end{tikzcd} \text{ where } (\alpha_2 * \alpha_1)_X = G_2((\alpha_1)_X) \circ (\alpha_2)_{F_1(X)}$$

Horizontal composition can also be defined in another equivalent way using commutativity of

$$\begin{tikzcd} F_2F_1(X) \arrow[r, "(\alpha_2)_{F_1(X)}"] \arrow[d, "F_2((\alpha_1)_X)"'] & G_2F_1(X) \arrow[d, "G_2((\alpha_1)_X)"] \\ F_2G_1(X) \arrow[r, "(\alpha_2)_{G_1(X)}"] & G_2G_1(X) \end{tikzcd}$$

The diagram commutes by naturality of $\alpha_2$, so $(\alpha_2 * \alpha_1) = (\alpha_2)_{G_1(X)} \circ F_2((\alpha_1)_X)$.

**Definition 1.12.** Let $\mathcal{C}, \mathcal{D}$ be two categories. Then the *functor category from $\mathcal{C}$ to $\mathcal{D}$* written $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ or $\mathcal{D}^{\mathcal{C}}$ is the category whose objects are functors from $\mathcal{C}$ to $\mathcal{D}$ and morphisms are natural transformations.

*Remark.* Categories, together with functors and natural transformations between them is the prototypal example of a 2-category.

## 1.2 Equivalences of categories

**Definition 1.13.** Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. An *equivalence of categories* from $\mathcal{C}$ to $\mathcal{D}$ is the data of

1. $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ wo functors

2. Natural isomorphisms $\eta : 1_{\mathcal{C}} \Rightarrow GF$ and $\varepsilon : FG \Rightarrow 1_{\mathcal{D}}$ where $1_{\mathcal{C}}$ and $1_{\mathcal{D}}$ are the identity functors of $\mathcal{C}$ and $\mathcal{D}$.

*Remark.*

1. $G$ is called a *quasi-inverse* of $F$.

2. Most of the time we say that $F$ is an equivalence if there exists $G$ such that $(F, G)$ is an equivalence.

3. If $F, G$ are contravariant, we speak of *duality* between $\mathcal{C}$ and $\mathcal{D}$.
4. If two categories are equivalent, every property that can be expressed "in terms of arrows" is preserved.

**Definition 1.14.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then, we say

1. $F$ is *faithful* if $\forall X, Y \in \mathrm{Ob}(\mathcal{C})$, $F : \begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(X,Y) & \to & \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y)) \\ f & \mapsto & F(f) \end{array}$ is injective.
2. $F$ is *full* if the previous map is surjective.
3. $F$ is *essentially surjective* if for all $Y \in \mathrm{Ob}(\mathcal{D})$ there is $X \in \mathrm{Ob}(\mathcal{C})$ such that $F(X) \simeq Y$ in $\mathcal{D}$.

**Theorem 1.15.** *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then $F$ is an equivalence of categories if and only if it is fully faithful and essentially surjective.*

*Proof.* ⚠ There is a little set-theoretic issue: an equivalence of categories is always fully faithful and essentially surjective, but the converse requires the axiom of choice for the class $\mathrm{Ob}(\mathcal{C})$.
Suppose $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories, and let $G : \mathcal{D} \to \mathcal{C}$ be a quasi-inverse of $F$, together with natural isomorphisms $\eta : 1_{\mathcal{C}} \to GF$ and $\varepsilon : 1_{\mathcal{D}} \to FG$. If $Y$ is an object of $\mathcal{D}$, then $Y \simeq FG(Y)$, so $F$ is essentially surjective. Let $X, Y$ be objects of $\mathcal{C}$. To show $F$ is fully faithful we will construct an inverse to $F : \mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$. For any $f \in \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$, we have a commutative diagram

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "f"'] & GF(X) \arrow[d, "GF(f)"] \\ Y \arrow[r, "\eta_Y"] & GF(Y) \end{tikzcd}$$

which prompts us to define $\phi : \begin{array}{ccc} \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y)) & \to & \mathrm{Hom}_{\mathcal{C}}(X,Y) \\ g & \mapsto & \eta_Y^{-1} \circ G(g) \circ \eta_X \end{array}$. We now check it is the map we're looking for. If $f : X \to Y$, since the above diagram commutes and $\eta_Y$ is invertible, we get that $\phi(F(f)) = f$, so $\phi \circ F = \mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(X,Y)}$, which means $F$ is faithful. We have two commutative diagrams, by definition of $\phi$ and by naturality of $\eta$:

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "\phi(g)"'] & GF(X) \arrow[d, "G(g)"] \\ Y \arrow[r, "\eta_Y"] & GF(Y) \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "\phi(g)"'] & GF(X) \arrow[d, "GF(\phi(g))"] \\ Y \arrow[r, "\eta_Y"] & GF(Y) \end{tikzcd}$$

therefore, $G(g) \circ \eta_X = GF(\phi(g)) \circ \eta_X$. Since $\eta_X$ is invertible, $G(g) = GF(\phi(g))$. The previous point shows that $G$ is faithful, so $g = F(\phi(g))$, hence $F$ is full.
Now suppose $F$ is fully faithful and essentially surjective. Our goal is to construct $G$. For any $Y \in \mathrm{Ob}(\mathcal{D})$, since $F$ is essentially surjective, there exists $X_Y \in \mathrm{Ob}(\mathcal{C})$ and an isomorphism $\varepsilon_Y : Y \to F(X_Y)$. Therefore, for any $Y, Z \in \mathrm{Ob}(\mathcal{D})$ and $f : Y \to Z$, we have a commutative diagram

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "\varepsilon_Y"] & Z \arrow[d, "\varepsilon_Z"] \\ F(X_Y) \arrow[r, "\varepsilon_Z \circ f \circ \varepsilon_Y^{-1}"'] & F(X_Z) \end{tikzcd}$$

Which leads us to define $G(Y) = X_Y$ and $G(f)$ to be the unique morphism $m_f : X_Y \to X_Z$ such that $F(m_f) = \varepsilon_Z \circ f \circ \varepsilon_Y^{-1}$ (this works because $F$ is fully faithful). We have $G(\mathrm{id}_Y) = \mathrm{id}_{X_Y}$ since $\varepsilon_Y \circ \mathrm{id}_Y \circ \varepsilon_Y^{-1} = \mathrm{id}_Y$ and $F(\mathrm{id}_{X_Y}) = \mathrm{id}_Y$. The next diagram shows $G(g \circ f) = G(g) \circ G(f)$:

$$\begin{tikzcd}
W \arrow[r, "f"] \arrow[d, "\varepsilon_W"] \arrow[rr, bend left, "g\circ f"] & Y \arrow[r, "g"] \arrow[d, "\varepsilon_Y"] & Z \arrow[d, "\varepsilon_Z"] \\
F(X_W) \arrow[r, "F(m_f)"'] \arrow[rr, bend right, "F(m_g\circ m_f)=F(m_g)\circ F(m_f)"'] & F(X_Y) \arrow[r, "F(m_g)"'] & F(X_Z)
\end{tikzcd}$$

By this construction, $\varepsilon$ is a natural isomorphism $\mathrm{id}_{\mathcal{D}} \Rightarrow FG$ (look at the above diagrams). Now, pick $Y, Z \in \mathrm{Ob}(\mathcal{C})$ and $f : Y \to Z$. We have $GF(Y) = X_{F(Y)}$ and $\varepsilon_Y : F(Y) \xrightarrow{\sim} F(X_{F(Y)})$. Since $F$ is fully faithful, there exists a unique $\eta_Y : Y \to X_{F(Y)} = GF(Y)$ such that $F(\eta_Y) = \varepsilon_Y$. Here, $\eta_Y = G(\varepsilon_Y)$, which means that $\eta_Y$ is an isomorphism since functors preserve isomorphisms. We obtain the diagram

$$\begin{tikzcd}
Y \arrow[r, "\eta_Y"] \arrow[d, "f"] & GF(Y) \arrow[d, "GF(f)"] \\
Z \arrow[r, "\eta_Z"] & GF(Z)
\end{tikzcd}$$

The diagram commutes because $GF(f)$ is the unique morphism such that

$$F(GF(f)) = \varepsilon_Z \circ F(f) \circ \varepsilon_Y^{-1} = F(\eta_Z \circ f \circ \eta_Y^{-1})$$

and $F$ is faithful. $\eta$ is then a natural isomorphism $\mathrm{id}_{\mathcal{C}} \Rightarrow GF$. $\square$

**Example 1.16.** $\mathbf{Vect}_k \simeq \mathbf{Mat}_k$ through the functor $n \mapsto k^n$ and $(A : n \to m) \mapsto (f_A : k^n \to k^m)$.

# 2 Universal properties

References:

- Riehl (Chapters II, III, IV)
- Mac Lane (Chapters III, IV, V)
- Assem (Chapters III, IV, V)

❯ Let $S$ be a set together with an equivalence relation $\sim$. Let ${}^{S}/_{\sim}$ be the quotient set, and $\pi : S \to {}^{S}/_{\sim}$ be the projection. For any $f : S \to X$ compatible with $\sim$, there exists a unique map $\bar{f} : {}^{S}/_{\sim} \to X$ such that $f = \bar{f} \circ \pi$. This is represented by the following commutative diagram :

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d, "\pi"'] & X \\ {}^{S}/_{\sim} \arrow[ur, dashed, "\exists!\bar{f}"'] & \end{tikzcd}$$

We say that $S \xrightarrow{\pi} {}^{S}/_{\sim}$ is a solution to the universal problem posed by the compatible maps. Such a solution is unique up to unique isomorphism: if $S \xrightarrow{p} S'$ is another solution, then we get the three commutative diagrams

$$\begin{tikzcd} S \arrow[r, "p"] \arrow[d, "\pi"'] & S' \\ {}^{S}/_{\sim} \arrow[ur, dashed, "\exists!a"'] & \end{tikzcd} \quad \begin{tikzcd} S \arrow[r, "\pi"] \arrow[d, "p"'] & {}^{S}/_{\sim} \\ S' \arrow[ur, dashed, "\exists!b"'] & \end{tikzcd} \quad \begin{tikzcd} S \arrow[r, "p"] \arrow[d, "p"'] & S' \\ S' \arrow[ur, "\mathrm{id}_{S'}"'] \arrow[r, "b"'] & {}^{S}/_{\sim} \arrow[u, "a"'] \end{tikzcd}$$

then $abp = a\pi = p$. The identity of $S'$ also makes this diagram commute so by uniqueness $ab = \mathrm{id}_{S'}$ and similarly $ba = \mathrm{id}_{S/\sim}$.

## 2.1 Initial and final objects

**Definition 2.1.** Let $\mathcal{C}$ be a category. An object $c \in \mathrm{Ob}(\mathcal{C})$ is *initial* (*final*) if for all $d \in \mathrm{Ob}(\mathcal{C})$ there exists a unique morphism $c \to d$ (a unique morphism $d \to c$).

**Proposition 2.2.** *If an initial/final object exists, then it is unique up to unique isomorphism.*

*Proof.* Let $c, c'$ be two initial objects. Then there exists a unique morphism $f : c \to c'$ and a unique morphism $g : c' \to c$. There also exists a unique morphism $c \to c$, that is $\mathrm{id}_c$. Therefore, $gf = \mathrm{id}_c$. In the same way, $fg = \mathrm{id}_{c'}$. Therefore, $c$ and $c'$ are isomorphic and the isomorphism is unique. □

**Examples 2.3.**

1. $\varnothing$ is initial in **Set** and any singleton is final.
2. $\{0\}$ is both initial and final in $\mathbf{Vect}_k$ (or $R\mathbf{Mod}$).
3. The category of fields does not have initial/final objects (reason on field characteristics).

We want to say a universal object is an initial or final object. A category has at most 2, so this may seem a little restrictive, but this is solved by thinking of a good category.

**Definition 2.4.** Let $F : \mathcal{C} \to \mathbf{Set}$ be a functor. Let $\int F$ be the category defined by

$$\begin{aligned}
&\mathrm{Ob}(\textstyle\int F) = \{(c,x) \mid c \in \mathrm{Ob}(\mathcal{C}) \text{ and } x \in F(c)\}\\
&\mathrm{Hom}((c,x),(c',x')) = \{f \in \mathrm{Hom}(c,c') \mid F(f)(x) = x'\}
\end{aligned}$$

where composition is composition in $\mathcal{C}$, and $\mathrm{id}_{(c,x)} = \mathrm{id}_c$ for all $x$. If $F$ is contravariant, let $\int F$ have the same objects and morphisms $\mathrm{Hom}((c,x),(c',x')) = \{f \in \mathrm{Hom}(c,c') \mid F(f)(x') = x\}$.

**Proposition 2.5.** *There is a forgetful functor $\pi : \int F \to \mathcal{C}$ defined by $\pi(c,x) = c$ and $\pi(f : (c,x) \to (c',x')) = f : c \to c'$.*

**Example 2.6.** Let $S$ be a set, and $\sim$ an equivalence relation on $S$. Let $F : \mathbf{Set} \to \mathbf{Set}$ be defined by $F(X) = \{f : S \to X \mid x \sim y \Rightarrow f(x) = f(y)\}$ and $F(\alpha : X \to Y) = \alpha \circ -$.
$\int F$ has for objects $(X, S \xrightarrow{f} X)$ where $f$ is compatible with $\sim$, and for morphisms $\alpha$ that makes this diagram commute:
$$\begin{tikzcd} S \arrow[r, "f'"] \arrow[d, "f"'] & X' \\ X \arrow[ur, "\alpha"'] & \end{tikzcd}$$
$(S/{\sim}, S \xrightarrow{\pi} S/{\sim})$ is an initial object of $\int F$.

**Definition 2.7.** Let $F : \mathcal{C} \to \mathbf{Set}$ be a functor. A *universal element* for $F$ is an initial object of $\int F$, that is a pair $(c,x)$ with $c \in \mathrm{Ob}(\mathcal{C})$ and $x \in F(c)$ such that

$$\forall (d,y), d \in \mathrm{Ob}(\mathcal{C}), y \in F(d), \exists! \alpha : c \to d, y = F(\alpha)(x)$$

**Definition 2.8.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor and $d \in \mathrm{Ob}(\mathcal{D})$. A *universal arrow from $d$ to $F$* is a pair $(c,f)$ where $c \in \mathrm{Ob}(\mathcal{C})$ and $f \in \mathrm{Hom}_{\mathcal{D}}(d, F(c))$, such that

$$\forall (c',f'), c' \in \mathrm{Ob}(\mathcal{C}), f' : d \to F(c'), \exists! \alpha \in \mathrm{Hom}_{\mathcal{C}}(c,c'), F(\alpha) \circ f = f'$$

$$\begin{tikzcd} & d \arrow[dl, "f"'] \arrow[dr, "\forall f'"] & \\ F(c) \arrow[rr, dashed, "F(\alpha)"'] & & F(c') \end{tikzcd}$$

$$c \xrightarrow{\exists! \alpha} c'$$

*Exercise.* Define a category $d \downarrow F$ such that a universal arrow is an initial object of $d \downarrow F$.

**Example 2.9.** Let $U : \mathbf{Vect}_k \to \mathbf{Set}$ be the forgetful functor. Let $X \in \mathbf{Set}$. A universal arrow from $X$ to $U$ is the "best" $k$-vector space $V_X$ with a map $X \to V_X$. Set $V_X = k[X]$ the $k$-vector space with basis $X$, and $i : X \to V_X$ that maps $x \in X$ to the corresponding basis element. Then, for any vector space $V$ and map $f : X \to U(V)$, $f$ can be extended by linearity into a linear map $\tilde{f} : k[X] \to V$, which makes this diagram commute:

$$\begin{tikzcd} & X \arrow[dl, "i"'] \arrow[dr, "f"] & \\ k[X] \arrow[rr, dashed, "\tilde{f}"'] & & U(V) \end{tikzcd}$$

If $\alpha$ is another map that makes the diagram commute then $\alpha$ and $\tilde{f}$ coincide on a basis of $k[X]$ and therefore are equal.

**Proposition 2.10.** *Universal elements and arrows are two equivalent notions.*

*Proof.* Let $F : \mathcal{C} \to \mathbf{Set}$ be a functor and $(c, x)$ a universal element for $F$. Consider $f_x : \begin{array}{ccc} \{*\} & \to & F(c) \\ * & \mapsto & x \end{array}$. Then, $(c, f_x)$ is a universal arrow $* \to F$, because $F(\alpha)(x) = y$ iff $F(\alpha) \circ f_x = f_y$.

$$\begin{tikzcd} & \{*\} \arrow[dl, "f_x"'] \arrow[dr, "f_y"] & \\ F(c) \arrow[rr, dashed, "F(\alpha)"'] & & F(c') \end{tikzcd}$$

Conversely, if $F : \mathcal{C} \to \mathcal{D}$ is a functor and $(c, f)$ is a universal arrow $d \to F$, then consider the functor $\mathrm{Hom}_{\mathcal{D}}(d, F(-)) : \begin{array}{ccc} \mathcal{C} & \to & \mathbf{Set} \\ x & \mapsto & \mathrm{Hom}_{\mathcal{D}}(d, F(x)) \end{array}$ (we need to assume $\mathcal{D}$ is locally small so the functor is set-valued). Then, $f \in \mathrm{Hom}_{\mathcal{D}}(d, F(c))$ is a universal element for this functor. $\square$

## 2.2 Representable functors

**Definition 2.11.** Let $\mathcal{C}$ be a (locally small) category, and $F : \mathcal{C} \to \mathbf{Set}$ a functor.

1. We say that $F$ is *representable* if there is some $c \in \mathrm{Ob}(\mathcal{C})$ such that $F$ and $\mathrm{Hom}_{\mathcal{C}}(c, -)$ are naturally isomorphic (if $F$ is contravariant, use $\mathrm{Hom}_{\mathcal{C}}(-, c)$ instead).

2. A *representation* of $F$ is the data of $c \in \mathrm{Ob}(\mathcal{C})$ and a natural isomorphism $\eta : \mathrm{Hom}(c, -) \Rightarrow F$.

**Example 2.12.** The forgetful functor $U : \mathbf{Grp} \to \mathbf{Set}$ is representable since $\mathrm{Hom}_{\mathbf{Grp}}(\mathbb{Z}, -) \simeq U$. The natural isomorphism is given by $\alpha \in \mathrm{Hom}_{\mathbf{Grp}}(\mathbb{Z}, G) \mapsto \alpha(1) \in G$.

The following theorem explains how to find the natural isomorphism $\alpha : \mathrm{Hom}_{\mathcal{C}}(c, -) \Rightarrow F$ in general.

**Theorem 2.13** (Yoneda lemma)**.** *Let $F : \mathcal{C} \to \mathbf{Set}$ be a functor with $\mathcal{C}$ locally small, and $c \in \mathrm{Ob}(\mathcal{C})$. Then,*

$$\begin{array}{ccc} \mathrm{Nat}(\mathrm{Hom}(c, -), F) & \xrightarrow{\sim} & F(c) \\ \alpha & \mapsto & \alpha_c(\mathrm{id}_c) \end{array}$$

*and this isomorphism is natural in $c$ and in $F$.*

*Proof.* Let $\alpha \in \mathrm{Nat}(\mathrm{Hom}(c, -), F)$. Let $d \in \mathcal{C}$ and $f : c \to d$. By naturality, we have a commutative diagram

$$\begin{tikzcd} \mathrm{Hom}(c, c) \arrow[r, "\alpha_c"] \arrow[d, "f \circ -"] & F(c) \arrow[d, "F(f)"] \\ \mathrm{Hom}(c, d) \arrow[r, "\alpha_d"] & F(d) \end{tikzcd}$$

This means that $F(f) \circ \alpha_c = \alpha_d \circ (f \circ -)$. Evaluating at $\mathrm{id}_c$, we get $F(f) \circ \alpha_c(\mathrm{id}_c) = \alpha_d(f)$. This shows that the natural transformation $\alpha$ is entirely determined by the value of $\alpha_c(\mathrm{id}_c)$, which shows the map defined above is injective. Conversely, if $e \in F(c)$, then we define $\alpha^e : \mathrm{Hom}(c, -) \Rightarrow F$ by $\alpha^e_d : g \mapsto F(g)(e)$. We check it is a natural transformation:

$$\begin{array}{ccc} \mathrm{Hom}(c,c) & \xrightarrow{g \mapsto F(g)(e)} & F(c) \\ \Big\downarrow{\scriptstyle f\circ -} & & \Big\downarrow{\scriptstyle F(f)} \\ \mathrm{Hom}(c,d) & \xrightarrow{h \mapsto F(h)(e)} & F(d) \end{array}$$

and this diagram commutes since for $g : c \to c$ we have

$$F(f)(F(g)(e)) = F(f \circ g)(e) = F((f \circ -)(g))(e)$$

This shows the map in the theorem is surjective, therefore an isomorphism, and its inverse is given by $e \in F(c) \mapsto \alpha^e$. We now check naturality. We first need to understand what it means to say the isomorphism is natural in $c$. Let $f : c \to d$. $\mathrm{Nat}(\mathrm{Hom}(c, -), F)$ is functorial in $c$, as it is the composition of two contravariant hom-functors. More concretely:

$$c \xrightarrow{f} d \rightsquigarrow \mathrm{Hom}(d, -) \xrightarrow{-\circ f} \mathrm{Hom}(c, -) \rightsquigarrow \mathrm{Nat}(\mathrm{Hom}(c, -), F) \xrightarrow{-\circ(-\circ f)} \mathrm{Nat}(\mathrm{Hom}(d, -), F)$$

(Nat is the hom-functor of the functor category $\mathcal{C}^{\mathbf{Set}}$). One thing to note is that the morphism $f : c \to d$ induces a natural transformation $\mathrm{Hom}(d, -) \xrightarrow{-\circ f} \mathrm{Hom}(c, -)$, and this makes the whole thing work. This is in general a property of functors defined on a product category, see the remark below. Now, saying the isomorphism, which we'll write $\Phi_{d,F}$, is natural means that the square

$$\begin{array}{ccc} \mathrm{Nat}(\mathrm{Hom}(c,-),F) & \xrightarrow{\Phi_{c,F}} & F(c) \\ \Big\downarrow{\scriptstyle -\circ(-\circ f)} & & \Big\downarrow{\scriptstyle F(f)} \\ \mathrm{Nat}(\mathrm{Hom}(d,-),F) & \xrightarrow{\Phi_{d,F}} & F(d) \end{array}$$

commutes. And indeed, if $\alpha : \mathrm{Hom}(c, -) \Rightarrow F$ is a natural transformation,

$$\begin{aligned} \Phi_{d,F}(\alpha \circ (- \circ f)) &= (\alpha \circ (- \circ f))_d(\mathrm{id}_d) = [\alpha_d \circ (- \circ f)](\mathrm{id}_d) = \alpha_d(f) \\ F(f)(\Phi_{c,F}(\alpha)) &= F(f)(\alpha_c(\mathrm{id}_c)) = \alpha_d(f \circ \mathrm{id}_c) = \alpha_d(f) \end{aligned}$$

The second to last equality comes from the naturality of $\alpha$.
We now turn to naturality in $F$. Let $G$ be another functor $\mathcal{C} \to \mathbf{Set}$ and $\beta : F \Rightarrow G$ be a natural transformation. We check that

$$\begin{array}{ccc} \mathrm{Nat}(\mathrm{Hom}(c,-),F) & \xrightarrow{\Phi_{c,F}} & F(c) \\ \Big\downarrow{\scriptstyle \beta\circ -} & & \Big\downarrow{\scriptstyle \beta_c} \\ \mathrm{Nat}(\mathrm{Hom}(c,-),G) & \xrightarrow{\Phi_{c,G}} & G(c) \end{array}$$

commutes. For $\alpha : \mathrm{Hom}(c, -) \Rightarrow F$, we have

$$\beta_c(\Phi_{c,F}(\alpha)) = \beta_c(\alpha_c(\mathrm{id}_c)) = (\beta \circ \alpha)_c(\mathrm{id}_c) = \Phi_{c,G}(\beta \circ \alpha)$$

which completes the proof of naturality. $\square$

*Remark.*

1. If $F : \mathcal{C} \to \mathbf{Set}$, then $(c, x)$ is a universal element for $F$ if and only if the natural transformation $\alpha_x : \mathrm{Hom}(c, -) \Rightarrow F$ induced by $x$ is an isomorphism. Indeed, $\alpha_x$ is an isomorphism iff $\forall c' \in \mathcal{C}$, $(\alpha_x)_{c'} : \begin{array}{rcl} \mathrm{Hom}(c, c') & \to & F(c') \\ f & \mapsto & F(f)(x) \end{array}$ is bijective iff
$$\forall c' \in \mathcal{C}, \forall y \in F(c'), \exists! f : c \to c', F(f)(x) = y$$

2. For universal arrows, use $\mathrm{Hom}_{\mathcal{D}}(d, F(-))$ as before.

3. Let $\mathcal{C}, \mathcal{D}$ and $\mathcal{E}$ be categories, and $F : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ be a functor. Let $c, d \in \mathrm{Ob}(\mathcal{C})$, $x, y \in \mathrm{Ob}(\mathcal{D})$ and morphisms $f : c \to d$, $g : x \to y$. The morphism $f$ induces a natural transformation $F(f, \mathrm{id}_-) : F(c, -) \Rightarrow F(d, -)$, see the commutative square:
$$\begin{tikzcd} F(c,x) \arrow[r, "F(f{,}\mathrm{id}_x)"] \arrow[d, "F(\mathrm{id}_c{,}g)"] & F(d,x) \arrow[d, "F(\mathrm{id}_d{,}g)"] \\ F(c,y) \arrow[r, "F(f{,}\mathrm{id}_y)"'] & F(d,y) \end{tikzcd}$$

## 2.3 Examples of objects defined by universal properties

### 2.3.1 Products, coproducts

Let $\mathcal{C}$ be a small category and $X, Y \in \mathrm{Ob}(\mathcal{C})$. We define a category $\mathcal{C}_{X,Y}$ whose objects are tuples $(Z, f, g)$ where $Z \in \mathrm{Ob}(\mathcal{C})$ and $f : Z \to X$, $g : Z \to Y$ and morphisms are maps $\alpha : Z \to Z'$ in $\mathcal{C}$ such that the following diagram commutes:

$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[dd, "\alpha"] & \\ X & & Y \\ & Z' \arrow[ul, "f'"] \arrow[ur, "g'"'] & \end{tikzcd}$$

**Definition 2.14.** A *product* of $X$ and $Y$ is a final object in $\mathcal{C}_{X,Y}$. Concretely, it is an object $X \times Y$ together with two maps $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ such that for any $(Z, f, g) \in \mathrm{Ob}(\mathcal{C}_{X,Y})$, we have a commutative diagram

$$\begin{tikzcd} & Z \arrow[dl, "\forall f"'] \arrow[dr, "\forall g"] \arrow[d, dashed, "\exists ! \alpha"] & \\ X & X \times Y \arrow[l, "\pi_X"] \arrow[r, "\pi_Y"'] & Y \end{tikzcd}$$

Since it is defined as being a final object, if it exists, a product is unique up to unique isomorphism.

**Examples 2.15.** In **Set**, the product of $X$ and $Y$ is the cartesian product. In **Grp**, it is the product group. In **Top**, it is the cartesian product equipped with the product topology. In these examples, the maps in the definition are the canonical projections.

The notion dual to the one of a product is called a coproduct.

**Definition 2.16.** A *coproduct* of $X$ and $Y$ is a product in $\mathcal{C}^{\mathrm{op}}$. Concretely, it satisfies the universal property expressed by this commutative diagram:

$$\begin{tikzcd} X \arrow[r, "i_X"] \arrow[dr, "\forall f"'] & X \sqcup Y \arrow[d, dashed, "\exists!\alpha"] & Y \arrow[l, "i_Y"'] \arrow[dl, "\forall g"] \\ & Z & \end{tikzcd}$$

**Examples 2.17.** In **Set**, the coproduct of $X$ and $Y$ is the disjoint union together with canonical inclusion. In **Top**, the coproduct of $X$ and $Y$ is their disjoint union equipped with the disjoint union topology. However, in **Grp**, the underlying set of the coproduct of two groups is not the disjoint union.

### 2.3.2 Equalizers and coequalizers

**Definition 2.18.** Let $\mathcal{C}$ be a category and $X, Y \in \mathrm{Ob}(\mathcal{C})$, $f, g : X \to Y$. Consider the contravariant functor $F : \mathcal{C} \to \mathbf{Set}$ defined by $F(c) = \{\alpha : c \to X \mid f\alpha = g\alpha\}$ and $F(\beta) = - \circ \beta$. An *equalizer* in $\mathcal{C}$ is a representation of the contravariant functor $F$.

By the Yoneda lemma, a natural transformation $\mathrm{Hom}(-, c) \Rightarrow F$ is the same as an element of $F(c)$, so a representation of $F$ is a pair $(c, e)$ with $c \in \mathrm{Ob}(\mathcal{C})$ and $e \in F(c)$ such that the natural transformation given by the Yoneda lemma is an isomorphism. Concretely, we want $\begin{array}{rccc} \eta_e : & \mathrm{Hom}(d, c) & \to & F(d) \\ & h & \mapsto & F(h)(e) \end{array}$ to be an isomorphism for all $d \in \mathrm{Ob}(c)$. This translates into the follwing diagram:

$$\begin{tikzcd} & d \arrow[dl, dashed, "\exists!\alpha"'] \arrow[d, "\forall h"] & \\ c \arrow[r, "e"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \end{tikzcd}$$

**Example 2.19.** In **Set**, $E = \{x \in X \mid f(x) = g(x)\} \hookrightarrow X$ is an equalizer.

The dual notion is that of a coequalizer.

**Definition 2.20.** A *coequalizer* of $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ is an object $Z \in \mathrm{Ob}(\mathcal{C})$ together with a morphism $\pi : Y \to Z$ such that $\pi f = \pi g$ and that universal to this property:

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "\pi"] \arrow[d, "\forall h"] & Z \arrow[dl, dashed, "\exists!\alpha"] \\ & Z' & \end{tikzcd}$$

**Example 2.21.** In **Set**, consider the equivalence relation $\sim$ on $Y$ generated by $f(x) \sim g(x)$ (the smallest equivalence relation on $Y$ with this property). Then $y \xrightarrow{\pi} Y/\sim$ is a coequalizer.

## 2.4 Adjoint functors

This notion was introduced by Kan in 1958.

**Definition 2.22.** An *adjunction* $(G, D)$ is a pair of functors $G : \mathcal{C} \to \mathcal{D}$ and $D : \mathcal{D} \to \mathcal{C}$ together with an isomorphism $\mathrm{Hom}_{\mathcal{D}}(G(c), d) \simeq \mathrm{Hom}_{\mathcal{C}}(c, D(d))$ which is natural in both $c$ and $d$.
We write $G \dashv D$ and say $G$ is left adjoint to $D$ and $D$ is right adjoint to $G$.

If $G \dashv D$ we have $\forall c, d \in \mathrm{Ob}(\mathcal{C}) \times \mathrm{Ob}(\mathcal{D})$,

$$\mathrm{Hom}_{\mathcal{D}}(G(c), d) \xrightarrow[\alpha_{c,d}]{\sim} \mathrm{Hom}_{\mathcal{C}}(c, D(d))$$

and in particular when $d = G(c)$ we get $\mathrm{Hom}_{\mathcal{D}}(G(c), G(c)) \xrightarrow[\alpha_{c,G(c)}]{\sim} \mathrm{Hom}_{\mathcal{C}}(c, DG(c))$.
Let $\eta_c : c \to DG(c)$ be the image of $\mathrm{id}_{G(c)}$. This gives a collection of morphisms $- \to DG(-)$.

**Proposition 2.23.** *These morphisms make up a natural transformation* $\mathrm{id}_{\mathcal{C}} \Rightarrow DG$.

*Proof.* Let $f : c \to d$. We want to show that

$$\begin{tikzcd} c \arrow[r, "\eta_c = \alpha_{c,G(c)}(\mathrm{id}_{G(c)})"] \arrow[d, "f"] & DG(c) \arrow[d, "DG(f)"] \\ d \arrow[r, "\eta_d = \alpha_{d,G(d)}(\mathrm{id}_{G(d)})"] & DG(d) \end{tikzcd}$$

commutes. By naturality of the isomorphism $\alpha$ given by the adjunction, we get the following commutative diagram

$$\begin{tikzcd} \mathrm{Hom}(G(c), G(c)) \arrow[r, "\sim", "\alpha_{c,G(c)}"'] \arrow[d, "G(f)\circ -"'] & \mathrm{Hom}(c, DG(c)) \arrow[d, "DG(f)\circ -"'] \\ \mathrm{Hom}(G(c), G(d)) \arrow[r, "\sim", "\alpha_{c,G(d)}"'] & \mathrm{Hom}(c, DG(d)) \\ \mathrm{Hom}(G(d), G(d)) \arrow[r, "\sim", "\alpha_{d,G(d)}"'] \arrow[u, "-\circ G(f)"] & \mathrm{Hom}(d, DG(d)) \arrow[u, "-\circ f"'] \end{tikzcd}$$

which gives us these equations:

$$DG(f) \circ \eta_c = DG(f) \circ \alpha_{c,G(c)}(\mathrm{id}_{G(c)}) = \alpha_{c,G(d)}(G(f) \circ \mathrm{id}_{G(c)}) = \alpha_{c,G(d)}(G(f))$$
$$\eta_d \circ f = \alpha_{d,G(d)}(\mathrm{id}_{G(d)}) \circ f = \alpha_{c,G(d)}(\mathrm{id}_{G(c)} \circ G(f)) = \alpha_{c,G(d)}(G(f))$$

which completes the proof. $\square$

We also get a natural transformation $\varepsilon : GD \Rightarrow \mathrm{id}_{\mathcal{D}}$ when $c = D(d)$ by setting $\varepsilon_d = \alpha^{-1}_{D(d),d}(\mathrm{id}_{D(d)})$.

**Definition 2.24.** The natural transformation $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow DG$ is called the *unit* of the adjunction. The natural transformation $\varepsilon : GD \Rightarrow \mathrm{id}_{\mathcal{D}}$ is called its *counit*.

**Proposition 2.25.** *Let* $\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "G"] & \mathcal{D} \arrow[l, bend left, "D"] \end{tikzcd}$ *be two functors. Then,* $G \dashv D$ *if and only if there are natural transformations* $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow DG$ *and* $\varepsilon : GD \Rightarrow \mathrm{id}_{\mathcal{D}}$ *such that the following diagrams commute:*

$$\begin{tikzcd} G \arrow[r, Rightarrow, "G\eta"] \arrow[dr, equal, "\mathrm{id}_G"'] & GDG \arrow[d, Rightarrow, "\varepsilon G"] \\ & G \end{tikzcd} \qquad \begin{tikzcd} D \arrow[r, Rightarrow, "\eta D"] \arrow[dr, equal, "\mathrm{id}_D"'] & DGD \arrow[d, Rightarrow, "D\varepsilon"] \\ & D \end{tikzcd}$$

*where $G\eta$ is the natural transformation given by the morphisms $G(\eta_c)$ and $\varepsilon G$ is the one give by morphisms $\varepsilon_{G(c)}$ (and similarly for $\eta D$ and $D\varepsilon$).*

*Proof.* Suppose $G \dashv D$. Let $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow DG$ and $\varepsilon : GD \Rightarrow \mathrm{id}_{\mathcal{D}}$ be the unit and counit of the adjunction. Let $c \in \mathcal{C}$. We have

$$(\varepsilon G)_c \circ (G\eta)_c = \varepsilon_{G(c)} \circ G(\eta_c) = \alpha^{-1}_{DG(c),G(c)}(\mathrm{id}_{DG(c)}) \circ G(\alpha_{c,G(c)}(\mathrm{id}_{G(c)}))$$

and the naturality of $\alpha$ gives the following commutative diagram

$$\begin{tikzcd} \mathrm{Hom}(G(c),G(c)) & \mathrm{Hom}(c,DG(c)) \arrow[l, "\sim", "\alpha^{-1}_{c,G(c)}"'] \\ \mathrm{Hom}(GDG(c),G(c)) \arrow[u, "-\circ G(\alpha_{c,G(c)}(\mathrm{id}_{G(c)}))"] & \mathrm{Hom}(DG(c),DG(c)) \arrow[l, "\sim", "\alpha^{-1}_{DG(c),G(c)}"'] \arrow[u, "-\circ\alpha_{c,G(c)}(\mathrm{id}_{G(c)})"'] \end{tikzcd}$$

which shows that $(\varepsilon G)_c \circ (G\eta)_c = \mathrm{id}_{G(c)}$, hence $\varepsilon G \circ G\eta = \mathrm{id}_G$. The commutativity of the other triangle is treated in a similar way.
Now assume that there are natural transformations $\eta$ and $\varepsilon$ that make both triangles commute. We define two maps

$$\begin{array}{rrcl} \alpha_{c,d} : & \mathrm{Hom}(G(c),d) & \to & \mathrm{Hom}(c,D(d)) \\ & f & \mapsto & D(f) \circ \eta_c \\ \beta_{c,d} : & \mathrm{Hom}(c,D(d)) & \to & \mathrm{Hom}(G(c),d) \\ & g & \mapsto & \varepsilon_d \circ G(g) \end{array}$$

and we show these are natural isomorphisms that give us the adjunction. First we check naturality of $\alpha$. Let $f : c \to c' \in \mathrm{Mor}(\mathcal{C})$ and $g : d \to d' \in \mathrm{Mor}(\mathcal{D})$. We need to check that the diagrams

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{D}}(G(c),d) \arrow[r, "\alpha_{c,d}"] & \mathrm{Hom}_{\mathcal{C}}(c,D(d)) \\ \mathrm{Hom}_{\mathcal{D}}(G(c'),d) \arrow[u, "-\circ G(f)"] \arrow[r, "\alpha_{c',d}"] & \mathrm{Hom}_{\mathcal{C}}(c',D(d)) \arrow[u, "-\circ f"] \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Hom}_{\mathcal{D}}(G(c),d) \arrow[r, "\alpha_{c,d}"] \arrow[d, "g\circ-"] & \mathrm{Hom}_{\mathcal{C}}(c,D(d)) \arrow[d, "D(g)\circ-"] \\ \mathrm{Hom}_{\mathcal{D}}(G(c),d') \arrow[r, "\alpha_{c,d'}"] & \mathrm{Hom}_{\mathcal{C}}(c,D(d')) \end{tikzcd}$$

commute. We only check the left diagram and leave the right to the reader (sorry). We have

$$\alpha_{c,d} \circ (- \circ G(f)) = (D(-) \circ \eta_c) \circ (- \circ G(f)) = D(- \circ G(f)) \circ \eta_c = D(-) \circ DG(f) \circ \eta_c$$
$$(- \circ f) \circ \alpha_{c',d} = (- \circ f) \circ (D(-) \circ \eta_{c'}) = D(-) \circ \eta_{c'} \circ f = D(-) \circ DG(f) \circ \eta_c$$

One shows $\beta$ is natural in $c$ and $d$ in a similar way. We leave it to the reader (sorry again). Now we need to check that $\alpha$ and $\beta$ are inverses of each other, and that's where the triangle diagrams come into play.

$$\alpha_{c,d} \circ \beta_{c,d} = D(\varepsilon_d \circ G(-)) \circ \eta_c = D(\varepsilon_d) \circ DG(-) \circ \eta_c = D(\varepsilon_d) \circ \eta_{D(d)} \circ - = -$$

We used the definitions of $\alpha$ and $\beta$, the functoriality of $D$, the naturality of $\eta$ and the second triangle diagram. We leave to the reader (sorry) to check that $\beta_{c,d} \circ \alpha_{c,d}$ is also the identity. $\square$

**Examples 2.26.**

1. The forgetful functor $\mathbf{Ab} \to \mathbf{Set}$ is right adjoint to the free abelian group functor $\mathbf{Set} \to \mathbf{Ab}$.

2. The forgetful functor $\mathbf{Ab} \to \mathbf{Grp}$ is right adjoint to the abelianization functor $\mathbf{Grp} \to \mathbf{Ab}$ that sends a group $G$ to its abelianization $G^{ab} = G/[G,G]$ and a morphism $f : G \to H$ to the induced morphism $f^{ab} : G^{ab} \to H^{ab}$.

3. The forgetful functor $\mathbf{Top} \to \mathbf{Set}$ is right adjoint to the functor $\mathbf{Set} \to \mathbf{Top}$ that takes a set and equips it with the coarse topology. The forgetful functor $\mathbf{Top} \to \mathbf{Set}$ is also left adjoint to the functor $\mathbf{Set} \to \mathbf{Top}$ that equips a set with the discrete topology.

4. Let $G$ be a group, $H$ one of its subgroups and $k$ be a field. We have a functor from the category $\mathbf{Rep}_k(G)$ of representations of $G$ on $k$-vector spaces to the category $\mathbf{Rep}_k(H)$ of representations of $H$ on $k$-vector spaces. It is the restriction functor $\mathrm{Res}_H^G$. Its left adjoint is $\mathrm{Ind}_H^G$, the induced representation functor.

**Theorem 2.27.** *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. The following are equivalent:*

1. *$F$ admits a left adjoint*

2. *For all $X \in \mathrm{Ob}(\mathcal{D})$, $\mathrm{Hom}_{\mathcal{D}}(X, F(-))$ is representable*

3. *For all $X \in \mathrm{Ob}(\mathcal{D})$, there exists a universal arrow $X \to F$*

**Corollary 2.28.** *If they exist, adjoints are unique up to isomorphism.*

*Proof.* $2 \iff 3$ was the subject of a previous remark right after the Yoneda lemma. We prove $1 \iff 2$. Suppose $F$ admits a left adjoint $G$. Let $X \in \mathrm{Ob}(\mathcal{D})$. Then for all $Y \in \mathrm{Ob}(\mathcal{C})$ we have a bijection $\mathrm{Hom}_{\mathcal{D}}(X, F(Y)) \simeq \mathrm{Hom}_{\mathcal{C}}(G(X), Y)$ which is natural in $Y$, so $G(X)$ represents $\mathrm{Hom}_{\mathcal{D}}(X, F(-))$. For the converse, suppose all functors $\mathrm{Hom}_{\mathcal{D}}(X, F(-))$ are representable. We define $G(X)$ to be an object of $\mathcal{C}$ that represents $\mathrm{Hom}_{\mathcal{D}}(X, F(-))$. Now choose $X, Y \in \mathrm{Ob}(\mathcal{D})$ and $f : X \to Y$. We need to define $G(f)$. We wish to have a commuting square

$$\begin{tikzcd}
\mathrm{Hom}(G(X), -) \arrow[r, "\sim"] & \mathrm{Hom}(X, F(-)) \\
\mathrm{Hom}(G(Y), -) \arrow[r, "\sim"] \arrow[u, dashed, "\exists! \gamma"] & \mathrm{Hom}(Y, F(-)) \arrow[u, "- \circ f"']
\end{tikzcd}$$

We need to recover a map $G(X) \to G(Y)$ such that composing with it gives us $\gamma$. This works by the Yoneda lemma, which tells us that the natural transformation $\gamma$ comes from an element $\mathrm{Hom}(G(X), G(Y))$. Call it $G(f)$. It remains to check this does define a functor. Using this diagram with $X = Y$ and $f = \mathrm{id}_X$ shows that $G(\mathrm{id}_X) = \mathrm{id}_{G(X)}$. Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{C}$. Then we draw

$$\begin{tikzcd}
\mathrm{Hom}(G(Z), -) \arrow[r, "- \circ G(g)"'] \arrow[rr, bend left, "- \circ G(g \circ f)"] \arrow[d, "\wr"] & \mathrm{Hom}(G(Y), -) \arrow[r, "- \circ G(f)"'] \arrow[d, "\wr"] & \mathrm{Hom}(G(X), -) \arrow[d, "\wr"] \\
\mathrm{Hom}(Z, F(-)) \arrow[r, "- \circ g"] \arrow[rr, bend right, "- \circ (g \circ f)"'] & \mathrm{Hom}(Y, F(-)) \arrow[r, "- \circ f"] & \mathrm{Hom}(X, F(-))
\end{tikzcd}$$

and this diagram shows that $G(g \circ f) = G(g) \circ G(f)$ (because the map $\gamma$ above is unique). $\square$

This theorem shows there is a deep link between universal properties and adjoint functors.

## 2.5 Limits and colimits

(This subsection may be skipped on a first reading.)
Let us recall the definition of a functor category.

**Definition 2.29.** Let $\mathcal{C}, \mathcal{D}$ be two categories. Then $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$, also written $\mathcal{D}^{\mathcal{C}}$, is the category whose objects are functors $\mathcal{C} \to \mathcal{D}$ and morphisms are natural transformations between such functors, with composition given by vertical composition. It is called the *functor category category from $\mathcal{C}$ to $\mathcal{D}$*. When $\mathcal{J}$ is a small category we also say that $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is the category of *diagrams of shape $\mathcal{J}$ in $\mathcal{C}$*.

**Examples 2.30.**

1. Let $\mathbf{2}$ be the category $\bullet \longrightarrow \bullet$ which has two objects 1 and 2 and three morphisms (two of them being identities).
   Then $\mathbf{2} \times \mathbf{2}$ is the category
   $$\begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$
   which has 4 objects and 9 morphisms (4 of them being identities). Then, a functor from $\mathbf{2} \times \mathbf{2}$ to $\mathcal{C}$ is a commutative diagram of this shape in $\mathcal{C}$.

2. If $\mathcal{J}$ is a small category, there is a functor $\Delta : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Fun}(\mathcal{J}, \mathcal{C})$ where $\Delta(c)$ is the constant functor at $c$, that is the functor that sends all objects to $c$ and all morphisms to $\mathrm{id}_c$, and $\Delta(f) = f$, which works since a natural transformation $\Delta(c) \Rightarrow \Delta(d)$ is just the data of one morphism $c \to d$.

**Definition 2.31.** A *cone above a diagram $F : \mathcal{J} \to \mathcal{C}$ with summit $c \in \mathcal{C}$* is a natural transformation $\lambda : \Delta(c) \Rightarrow F$. Dually, a cone *under* $F$ with summit $c$, also called a *cocone*, is a natural transformation $\lambda : F \Rightarrow \Delta(c)$.

Let us unwrap this definition. A cone is a collection of maps $\lambda_j : c \to F(j)$ for all $j \in \mathrm{Ob}(\mathcal{J})$, such that for any morphism $f : i \to j \in \mathrm{Mor}(\mathcal{J})$, this diagram commutes:

$$\begin{tikzcd} & c \arrow[dl, "\lambda_i"'] \arrow[dr, "\lambda_j"] & \\ F(i) \arrow[rr, "F(f)"'] & & F(j) \end{tikzcd}$$

**Definition 2.32.** Let $F : \mathcal{J} \to \mathcal{C}$ be a diagram. A *limit* (or *projective limit* or *inverse limit*) of $F$ is a universal cone above $F$, in the sense that it is a final object in the category of cones above $F$. Dually, a *colimit* (or *inductive limite* or *direct limit*) is a universal cocone, that is an initial object in the category of cones under $F$.

Concretely, a limit of $F : \mathcal{J} \to \mathcal{C}$ is a pair $(\lim F, \phi)$ with $\lim F \in \mathrm{Ob}(\mathcal{C})$ and $\phi : \Delta(\lim F) \Rightarrow F$ is such that for any cone $\lambda : \Delta(c) \Rightarrow F$, there exists a unique morphism $f : X \to \lim F \in \mathrm{Mor}(\mathcal{C})$, such that the diagram on the left commutes:

$$\begin{tikzcd} \Delta(c) \arrow[rr, dashed, "\Delta(f)"] \arrow[dr, Rightarrow, "\lambda"'] & & \Delta(\lim F) \arrow[dl, Rightarrow, "\phi"] \\ & F & \end{tikzcd} \quad \text{which is equivalent to} \quad \forall j \in \mathcal{J}, \quad \begin{tikzcd} c \arrow[rr, dashed, "f"] \arrow[dr, Rightarrow, "\lambda_j"'] & & \lim F \arrow[dl, Rightarrow, "\phi_j"] \\ & F(j) & \end{tikzcd}$$

In compact form, $\mathrm{Hom}_{\mathcal{C}}(-, \lim F) \simeq \mathrm{Hom}_{\mathrm{Fun}(\mathcal{J},\mathcal{C})}(\Delta(-), F)$.

*Exercise.* Do the same for colimits.

*Remark.*

1. If a limit exists it is unique up to isomorphism (unique isomorphism that commutes with the legs of the cone)

2. If all limits exist, then lim becomes a functor $\lim : \mathrm{Fun}(\mathcal{J},\mathcal{C}) \to \mathcal{C}$ in the following way. Recall that theorem 2.27 says a functor $D$ admits a left adjoint iff for all objects $X$ in its codomain, $\mathrm{Hom}(X, D(-))$ is representable. The compact form of the definition of a limit says that the functor $\mathrm{Hom}(\Delta(-), F)$ is representable for all $F$ (since we assume all limits exist). A dual version of the theorem gives that $\Delta$ admits a right adjoint, which is lim since $\mathrm{Hom}(c, \lim F) \simeq \mathrm{Hom}(\Delta(c), F)$. If $\eta : F \Rightarrow G$ is a natural transformation, then $\lim(\eta)$ can be constructed in the following way: $\lim F \Rightarrow F \underset{\eta}{\Rightarrow} G$ is a cone above $G$, and $\lim(\eta) : \lim F \to \lim G$ comes from the universality of $\lim G$.

**Corollary 2.33.**

1. *If* $\mathcal{C}$ *has all* $\mathcal{J}$*-limits, then* $\lim : \mathrm{Fun}(\mathcal{J},\mathcal{C}) \to \mathcal{C}$ *is a right adjoint to* $\Delta$.

2. *If* $\mathcal{C}$ *has all* $\mathcal{J}$*-colimits, then* $\mathrm{colim} : \mathrm{Fun}(\mathcal{J},\mathcal{C}) \to \mathcal{C}$ *is a left adjoint to* $\Delta$.

**Example 2.34.**

1. If $\mathcal{J}$ is discrete, that is has no morphisms other than identities, then a functor $F : \mathcal{J} \to \mathcal{C}$ is the same as a collection $(X_i)_{i \in \mathcal{J}}$ of objects of $\mathcal{C}$. Then, a limit of $F$ is an object $\lim F \in \mathrm{Ob}(\mathcal{C})$ with morphisms $f_i : \lim F \to X_i$ such that for all objects $X \in \mathrm{Ob}(\mathcal{C})$ with morphisms $p_i : X \to X_i$, we have a unique map $\alpha : X \to \lim F$ that makes this diagram commute for all $i, j \in \mathcal{J}$:

$$\begin{tikzcd} & X \arrow[dl, "p_i"'] \arrow[d, dashed, "\alpha"] \arrow[dr, "p_j"] & \\ X_i & \lim F \arrow[l, "f_i"] \arrow[r, "f_j"'] & X_j \end{tikzcd}$$

   We write $\lim F = \prod_{j \in \mathcal{J}} F(j)$ and call it the *product of the* $F(j)$*s*. Morphisms $f_i$ are written $\pi_i$ and called *canonical projections*.
   Dually, the colimit of $F$ is called a coproduct and written $\coprod_{j \in \mathcal{J}} F(j)$.

2. If $\mathcal{J} = \bullet \rightrightarrows \bullet$, then a functor $F : \mathcal{J} \to \mathcal{C}$ is the data of two parallel morphisms in $\mathcal{C}$. A limit is an equalizer and a colimit is a coequalizer.

3. If $\mathcal{J} = \begin{tikzcd} & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$ then $F : \mathcal{J} \to \mathcal{C}$ is the data of $A, B, C \in \mathrm{Ob}(\mathcal{C})$ with two morphisms $f : A \to C$ and $g : B \to C$. The limit $\lim F$ is called a *pullback* of $f$ and $g$, with universal property depicted here:

$$\begin{tikzcd} X \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & A \times_C B \arrow[r, "\pi_A"] \arrow[d, "\pi_B"'] & A \arrow[d, "f"] \\ & B \arrow[r, "g"'] & C \end{tikzcd}$$

4. If $\mathcal{J} = \omega^{\mathrm{op}}$, that is $\mathcal{J} = \cdots \to 2 \to 1 \to 0$, then $\lim F$ is often called the "inverse limit" of $F$. Concretely, $F$ is the data of $\cdots \to F(2) \xrightarrow[\alpha_2]{} F(1) \xrightarrow[\alpha_1]{} F(0)$, and a cone above $F$ looks like

$$\begin{tikzcd} & & c \arrow[dl, "\lambda_2"'] \arrow[d, "\lambda_1"] \arrow[dr, "\lambda_0"] & \\ \cdots \arrow[r] & F(2) \arrow[r, "\alpha_2"'] & F(1) \arrow[r, "\alpha_1"'] & F(0) \end{tikzcd}$$

we have $(\alpha_i \circ \cdots \circ \alpha_n) \circ \lambda_n = \lambda_i$.

The typical example of an inverse limit is the one given by $F(n) = \mathbb{Z}/p^n\mathbb{Z}$ in **Ring** with morphisms $\mathbb{Z}/p^{n+1}\mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z}$ being reduction mod $p^n$. The inverse limit $\varprojlim \mathbb{Z}/p^n\mathbb{Z}$ is the ring of $p$-adic integers. Concretely, $a \in \varprojlim \mathbb{Z}/p^n\mathbb{Z}$ iff $a = (a_i)_{i \in \mathbb{N}}$ such that $a_i \equiv a_j \mod p^i \forall i \leq j$.

5. The dual notion, given by $\mathcal{J} = 0 \to 1 \to 2 \to \cdots$, is obtained by taking the colimit. It is called a *direct limit*. The typical example here is the Prüfer $p$-group $\varinjlim \mathbb{Z}/p^n\mathbb{Z} = \mathbb{Z}(p^\infty)$.

**Definition 2.35.** Let $\mathcal{C}$ be a category. We say $\mathcal{C}$ is *(co)complete* if it has all small (co)limits i.e. if for all diagrams $F : \mathcal{J} \to \mathcal{C}$ with $\mathcal{J}$ small, $F$ has a (co)limit.

**Theorem 2.36.** *A category $\mathcal{C}$ is (co)complete if and only if it has all small (co)products and (co)equalizers.*

*Proof.* Let $\mathcal{J}$ be a small category and $D : \mathcal{J} \to \mathcal{C}$ be a diagram. We have the products $\prod_{k \in \mathrm{Ob}(\mathcal{J})} D(k)$ and $\prod_{g \in \mathrm{Mor}(\mathcal{J})} D(\mathrm{cod}(g))$ where $\mathrm{cod}(g)$ is the codomain of $g$. We have two morphisms

$$\prod_{k \in \mathrm{Ob}(\mathcal{J})} D(k) \underset{t}{\overset{s}{\rightrightarrows}} \prod_{g \in \mathrm{Mor}(\mathcal{J})} D(\mathrm{cod}(g))$$

given by $s = \prod_{f:i \to j} D(f)\pi_i$ and $t = \prod_{f:i \to j} \pi_j$, or with diagrams, for any $f : i \to j \in \mathrm{Mor}(\mathcal{J})$ :

$$\begin{tikzcd} \prod_{k \in \mathrm{Ob}(\mathcal{J})} D(k) \arrow[r, dashed, "\exists! s"] \arrow[d, "\pi_i"] & \prod_{g \in \mathrm{Mor}(\mathcal{J})} D(\mathrm{cod}(g)) \arrow[d, "\pi_f"] \\ D(i) \arrow[r, "D(f)"] & D(j) \end{tikzcd} \qquad \begin{tikzcd} \prod_{k \in \mathrm{Ob}(\mathcal{J})} D(k) \arrow[rr, dashed, "\exists! t"] \arrow[dr, "\pi_j"'] & & \prod_{g \in \mathrm{Mor}(\mathcal{J})} D(\mathrm{cod}(g)) \arrow[dl, "\pi_f"] \\ & D(j) & \end{tikzcd}$$

We call $\lim D$ an equalizer of $s$ and $t$. A cone above $D$ is given by compositions

$$\lim D \xrightarrow{\alpha} \prod_{k \in \mathrm{Ob}(\mathcal{J})} D(k) \xrightarrow{\pi_i} D(i)$$

Indeed, for any morphism $f : i \to j \in \mathrm{Mor}(\mathcal{J})$, $D(f)\pi_i\alpha = \pi_f s\alpha = \pi_f t\alpha = \pi_j\alpha$.
Now let $\Delta(c) \underset{\lambda}{\Rightarrow} D$ be another cone above $D$. For any $k \in \mathrm{Ob}(\mathcal{J})$, we have $\lambda_k : c \to D(k)$, which gives a unique morphism $\lambda_* : c \to \prod_{k \in \mathrm{Ob}(\mathcal{J})} D(k)$ such that $\pi_i\lambda_* = \lambda_i$. Then, for any $f : i \to j \in \mathrm{Mor}(\mathcal{J})$, we have

$$\begin{aligned} \pi_f s\lambda_* &= D(f)\pi_i\lambda_* = D(f)\lambda_i = \lambda_j \\ \pi_f t\lambda_* &= \pi_j\lambda_* = \lambda_j \end{aligned}$$

and applying the universal property of the product shows that $s\lambda_* = t\lambda_*$. By the universal property of equalizers this gives the existence of a unique morphism $c \to \lim D$ and completes the proof. $\square$

**Definition 2.37.** $F : \mathcal{C} \to \mathcal{D}$ *preserves (co)limits* if for every diagram $D : \mathcal{J} \to \mathcal{C}$ and any (co)limit cone $(c, \phi)$ of $D$, the image $(F(c), F\phi)$ is a (co)limit cone over $FD : \mathcal{J} \to \mathcal{D}$.

*Remark.* Preserving limits is like having $F(\lim D) \simeq \lim FD$, but stronger:

$$\begin{tikzcd} \lim D \arrow[d, "\phi_i"] \\ D(i) \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} F(\lim D) \arrow[r, dashed, "\exists!\alpha"] \arrow[d, "FD(\phi_i)"'] & \lim FD \arrow[dl, "\lambda_i"] \\ FD(i) & \end{tikzcd}$$

and $\alpha$ is an isomorphism since $(F(\lim D), F\phi)$ is a limit cone.

**Proposition 2.38.** *Let $\mathcal{C}$ be a locally small category and $X \in \mathrm{Ob}(\mathcal{C})$. Then*

1. $\mathrm{Hom}_{\mathcal{C}}(X, -)$ *preserves all limits that exist in $\mathcal{C}$*
2. *The contravariant functor* $\mathrm{Hom}_{\mathcal{C}}(-, X)$ *transforms colimits in $\mathcal{C}$ into limits in* **Set***.*

*Proof.* Let $\mathcal{J}$ be a small category and $D : \mathcal{J} \to \mathcal{C}$ be a diagram. Let $F : \mathcal{C} \to$ **Set** be te hom-functor $\mathrm{Hom}_{\mathcal{C}}(X, -)$. Let $(L, \lambda)$ be a limit cone for $D$. Then, $(F(L), F(\lambda))$ is a cone in **Set** over $FD$, since for any $\alpha : i \to j \in \mathrm{Mor}(\mathcal{J})$ we have the commutative diagram

$$\begin{tikzcd} & F(L) \arrow[dl, "F(\lambda)"'] \arrow[dr, "F(\lambda_i)"] & \\ \mathrm{Hom}_{\mathcal{C}}(X, D(i)) \arrow[rr, "D(\alpha)\circ-"'] & & \mathrm{Hom}_{\mathcal{C}}(X, D(j)) \end{tikzcd}$$

It remains to show that $(F(L), F(\lambda))$ is a limit cone for $FD$. Let $S \underset{f}{\Rightarrow} FD$ be another cone. We have $f(i) : \begin{array}{ccc} S & \to & \mathrm{Hom}(X, D(i)) \\ s & \mapsto & f_i(s) \end{array}$ (we work in **Set** so morphisms are actual maps here). Fixing $S$, we get commutative diagrams:

$$\begin{tikzcd} & X \arrow[dl, "f_i(s)"'] \arrow[dr, "f_j(s)"] & \\ D(i) \arrow[rr, "D(\alpha)\circ -"'] & & D(j) \end{tikzcd}$$

so $(X, f_i(s))$ is a cone over $D$ hence there exists a unique morphism $u_s : X \to L$ such that $\lambda_i \circ u_s = f_i(s)$ for all $i \in \mathrm{Ob}(\mathcal{J})$. Now set $u : \begin{array}{ccc} S & \to & \mathrm{Hom}(X, L) \\ s & \mapsto & u_s \end{array}$ and we have $(F\lambda \circ u)(s) = (F\lambda)(u_s) = f$ so $u : S \to F(L)$ is a morphism of cones. We need to check it is unique. If $v$ is another one then $\lambda_i \circ v(s) = f_i(s)$ so $v(s) = u_s$ by uniqueness of $u_s$, which shows $v = u$.
Another proof is given here:

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}(X, \lim D) &\simeq \mathrm{Hom}_{\mathrm{Fun}(\mathcal{J},\mathcal{C})}(\Delta X, D) \\ &\simeq \mathrm{Hom}_{\mathrm{Fun}(\mathcal{J},\mathbf{Set})}(\Delta 1, \mathrm{Hom}_{\mathcal{C}}(X, D(-)) \\ &\simeq \mathrm{Hom}_{\mathbf{Set}}(1, \lim \mathrm{Hom}_{\mathcal{C}}(X, D(-))) \\ &\simeq \lim \mathrm{Hom}_{\mathcal{C}}(X, D(-)) \end{aligned}$$

(1 is a singleton.) The first and third isomorphisms are by definition of a limit. The last isomorphism comes from the fact that for any set $A$, maps $1 \to A$ correspond to elements of $A$. The second isomorphism works since a natural transformation $\Delta X \Rightarrow D$ is the same as a collection of morphisms $f_i : X \to D(i)$ indexed by $\mathrm{Ob}(\mathcal{J})$. $\square$

**Theorem 2.39.** *Right adjoints preserve limits. Left adjoints preserve colimits.*

*Proof.* We only need to prove the statement about right adjoints and then use opposite categories for left adjoints. Let $\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] & \mathcal{D} \arrow[l, bend left, "G"] \end{tikzcd}$ be two functors with $F \dashv G$ and $D : \mathcal{J} \to \mathcal{D}$ be a diagram, with $\eta : \Delta(\lim D) \Rightarrow D$ its limit cone. Our goal is to show that $(G \lim D, G\eta)$ is a limit cone for $G \circ D$. The fact that it is a cone above $G \circ D$ is clear. Now let $\mu : \Delta(c) \Rightarrow GD$ be another cone. For any $j \in \mathrm{Ob}(\mathcal{J})$, we have $\mu_j \in \mathrm{Hom}(c, GD(j))$. By adjunction, it corresponds to a morphism $\mu_j^* \in \mathrm{Hom}(F(c), D(j))$. We claim these morphisms make up a natural transformation $\mu^* : \Delta(F(c)) \Rightarrow D$. Indeed, for any morphism $f : i \to j \in \mathrm{Mor}(\mathcal{J})$, we have by naturality of the adjunction a commutative diagram

$$\begin{tikzcd} \mathrm{Hom}(F(c), D(i)) \arrow[r, "\sim"] \arrow[d, "D(f)\circ -"] & \mathrm{Hom}(c, GD(i)) \arrow[d, "GD(f)\circ -"] \\ \mathrm{Hom}(F(c), D(j)) \arrow[r, "\sim"] & \mathrm{Hom}(c, GD(j)) \end{tikzcd}$$

so $D(f) \circ \mu_i^* = (GD(f) \circ \mu_i)^* = \mu_j^*$. By universality of $\lim D$, there exists a unique morphism $\tau : F(c) \to \lim D$ that makes the appropriate diagram commute. Using the adjunction, we get a morphism $\tau^* : c \to G(\lim D)$, which is the morphism we are looking for. The commutativity of the appropriate diagram comes from naturality of the adjunction. Uniqueness comes from the uniqueness of $\tau$.

In compact form:

$$
\begin{aligned}
\mathrm{Hom}_{\mathcal{C}}(c, \lim GD) &\simeq \mathrm{Hom}_{\mathrm{Fun}(\mathcal{J},\mathcal{C})}(\Delta c, GD) \\
&\simeq \mathrm{Hom}_{\mathrm{Fun}(\mathcal{J},\mathcal{D})}(F\Delta c, D) \\
&\simeq \mathrm{Hom}_{\mathrm{Fun}(\mathcal{J},\mathcal{D})}(\Delta Fc, D) \\
&\simeq \mathrm{Hom}_{\mathcal{D}}(Fc, \lim D) \\
&\simeq \mathrm{Hom}_{\mathcal{C}}(C, G\lim D)
\end{aligned}
$$

$\square$

# 3 Tensor products

All rings considered here are assumed to be associative and to have a multiplicative unit 1. Let $A$ be a ring.

**Definition 3.1.**

- A *right $A$-module* is an abelian group $(M,+)$ with a map $\begin{array}{rcl} M \times A & \to & M \\ (m,a) & \mapsto & m \cdot a \end{array}$ such that

$$\begin{array}{llll} (1) & (m+n)\cdot a = m\cdot a + n\cdot a & (3) & m\cdot(ab) = (m\cdot a)b \\ (2) & m\cdot(a+b) = m\cdot a + m\cdot b & (4) & m\cdot 1_A = m \end{array}$$

  by symmetry one gets the notion of a *left $A$-module* (which is the equivalent of a vector space, but with a ring in place of the field).

- If $A, B$ are two rings, an *$A$-$B$-bimodule* is an abelian group $M$ with a left $A$-module and a right $B$-module structure such that for $(a,b) \in A \times B$ and $m \in M$, $a \cdot (m \cdot b) = (a \cdot m) \cdot b$.

- Let $M$ be a right $A$-module, $N$ be a left $A$-module and $G$ be an abelian group.
  A *bilinear* (or *balanced*) map $f : M \times N \to G$ is a map $f$ such that

$$\begin{array}{ll} (1) & f(m_1+m_2,n) = f(m_1,n) + f(m_2,n) \\ (2) & f(m,n_1+n_2) = f(m,n_1) + f(m,n_2) \\ (3) & f(ma,n) = f(m,an) \end{array}$$

The following theorem shows that there exists an abelian group $M \otimes_A N$ that is "universal" with respect to bilinear maps.

**Theorem 3.2.** *Let $M$ be a right $A$-module and $N$ be a left $A$-module. There exists an abelian group $M \otimes_A N$ together with a bilinear map $t : M \times N \to M \otimes_A N$ such that for any abelian group $G$ and bilinear map $b : M \times N \to G$, there exists a unique group homomorphism $\tilde{b}$ that makes this diagram commute:*

$$\begin{tikzcd} M \times N \arrow[r, "\forall b"] \arrow[d, "t"'] & G \\ M \otimes_A N \arrow[ur, dashed, "\exists! \tilde{b}"'] & \end{tikzcd}$$

*Proof.* Let $L = \mathbb{Z}[M \times N]$ be the free abelian group on $M \times N$. It has a basis, namely $\{(m,n) \mid m \in M, n \in N\}$. Now consider the subgroup

$$I = \langle (m_1+m_2,n) - (m_1,n) - (m_2,n), (m,n_1+n_2) - (m,n_1) - (m,n_2), (ma,n) - (m,an) \rangle$$

It is chosen so the relations we want hold in ${}^{L}/_{I}$, for instance $(ma,n) = (m,an)$ in the quotient group. Set $M \otimes_A N = {}^{L}/_{I}$ and $t : \begin{array}{rcl} M \times N & \to & {}^{L}/_{I} \\ (m,n) & \mapsto & [(m,n)] \end{array}$. By construction $M \otimes_A N$ is an abelian group and $t$ is bilinear. We need to check the universal property. Pick a bilinear map $b : M \times N \to G$. We have a diagram

$$\begin{tikzcd}
M \times N \arrow[r, "b"] \arrow[d, "i"'] & G \\
\mathbb{Z}[M \times N] \arrow[ur, dashed, "\exists!\tilde{b}"'] \arrow[d, "\pi"'] & \\
M \otimes_A N \arrow[uur, dashed, bend right, "\exists!\overline{b}"'] &
\end{tikzcd}$$

where $i : (m, n) \mapsto (m, n)$ is the inclusion map and $\pi : L \to {}^{L}/_{I}$ is the canonical projection. The map $\tilde{b}$ exists by universal property of the free abelian group. Moreover it passes to the quotient ($I \subset \ker(\tilde{b})$), so we get the map $\overline{b}$. We now check uniqueness. Let $f : M \otimes_A N \to G$ be another linear map that makes the diagram commute. Then, $f \circ \pi$ makes the top triangle commute, so by the universal property of the free abelian group, $f \circ \pi = \tilde{b}$. Applying the universal property of the quotient allows us to conclude $f = \overline{b}$. □

*Remark.*

1. The abelian group $M \otimes_A N$ is a unique up to unique isomorphism.

2. The class $[(m, n)] \in M \otimes_A N$ is written $m \otimes n$. It is called a "*pure tensor*". Pure tensors generate the tensor product:

$$x \in M \otimes_A N \iff \exists (m_i, n_i) \in M^n \times N^n, x = \sum_{i=1}^{n} m_i \otimes n_i$$

❯ The tensor product is a functor. Precisely, it is a bifunctor $- \otimes_A - : \mathbf{Mod}A \times A\mathbf{Mod} \to \mathbf{Ab}$. If $M, M'$ are two right $A$-modules, $N, N'$ are two left $A$-modules and $f : M \to M', g : N \to N'$ are linear maps, then writing $(f \otimes g)(m \otimes n) = f(m) \otimes g(n)$ gives a commutative diagram

$$\begin{tikzcd}
M \otimes_A N \arrow[r, "\mathrm{id}_M \otimes g"] \arrow[d, "f \otimes \mathrm{id}_N"'] \arrow[dr, dashed, "f \otimes g"] & M \otimes_A N' \arrow[d, "f \otimes \mathrm{id}_{N'}"] \\
M' \otimes_A N \arrow[r, "\mathrm{id}_{M'} \otimes g"'] & M' \otimes_A N'
\end{tikzcd}$$

One needs to be careful as $M \otimes_A N$ can be defined using a quotient or a universal property. Obtaining the arrow $f \otimes g$ is easier with the universal property:

$$\begin{tikzcd}
M \times N \arrow[r, "(f,g)"] \arrow[d, "t"] & M' \times N' \arrow[d, "t'"] \\
M \otimes_A N \arrow[r, dashed, "f \otimes g"] & M' \otimes_A N'
\end{tikzcd}$$

Since $t' \circ (f, g)$ is bilinear, we obtain the unique map $f \otimes g$ using the universal property of $M \otimes_A N$. Hence we obtain the lemma:

**Lemma 3.3.** *$- \otimes_A -$ is a bifunctor.*

**Corollary 3.4.** *1. If $M$ is a $B$-$A$-bimodule, then $M \otimes_A N$ is a left $B$-module*

2. *If $N$ is an $A$-$C$-bimodule, then $M \otimes_A N$ is a right $C$-module*

3. *If $M$ is a $B$-$A$-bimodule and $N$ is a $A$-$C$-bimodule then $M \otimes_A N$ is a $B$-$C$-bimodule.*

*Proof.* We do the proof of 1. We set $b \bullet (m \otimes n) = (bm) \otimes n$ and now we need to check that it is well defined. A good way is to fix $b \in B$ and let $\ell_b : \begin{array}{ccc} M & \to & M \\ m & \mapsto & b \cdot m \end{array}$ and notice that $\ell_b \in \mathrm{End}_A(M)$. By functoriality, we get a map $\ell_b \otimes \mathrm{id}_N : \begin{array}{ccc} M \otimes_A N & \to & M \otimes_A N \\ m \otimes n & \mapsto & (bm) \otimes n \end{array}$ so our action is well defined and this is a $B$-module structure on the tensor product. The proof of 2. is similar. The proof of 3. comes from the fact that $\ell_b \otimes \mathrm{id}_N$ and $\mathrm{id}_M \otimes r_c$ commutes. □

**Examples 3.5.**

1. $A \otimes_A N \simeq N$ as left $A$-modules. Isomorphisms are given by $a \otimes n \mapsto a \cdot n$ and $n \mapsto 1 \otimes n$. The well-definition of these maps comes from the universal property.

2. If $R$ is commutative then an $R$-module $M$ is an $R$-$R$-bimodule $\begin{array}{ccc} R \times M \times R & \to & (x, m, y) \\ M & \mapsto & mxy = myx \end{array}$ so $M \otimes_R N$ is always an $R$-module.

⚠ Over a field, $\dim(V \otimes W) = \dim(V)\dim(W)$ but this is false in general for a ring.

*Exercise.* Show that $\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z} = \{0\}$ when $\gcd(m, n) = 1$.

**Theorem 3.6** (Tensor-hom adjunction)**.** *Let $A, B$ be two rings and $M$ be an $A$-$B$-bimodule. We have a functor $- \otimes_A M : \mathbf{Mod}A \to \mathbf{Mod}B$ and a functor $\mathrm{Hom}_B(M, -) : \mathbf{Mod}B \to \mathbf{Mod}A$. Then $- \otimes_A M$ is left adjoint to $\mathrm{Hom}_B(M, -)$.*

The $A$-module structure on $\mathrm{Hom}_B(M, Y)$ for $Y$ a $B$-module is given by

$$\begin{array}{ccl} \mathrm{Hom}_B(M, Y) \times A & \to & \mathrm{Hom}_B(M, Y) \\ (f, a) & \mapsto & f \cdot a : \begin{array}{ccc} M & \to & Y \\ m & \mapsto & f(am) \end{array} \end{array}$$

*Proof.* **TODO** □

# 4 Additive categories

## 4.1 Preadditive and additive categories

**Definition 4.1.** A *zero object* in a category $\mathcal{C}$ is an object that is both final and initial.

**Example 4.2.** $\{0\}$ is a zero objet in $\mathbf{Mod}A$ for $A$ a ring.

**Definition 4.3.** Let $k$ be a commutative ring. A *$k$-category* is a category $\mathcal{C}$ such that all hom-sets are $k$-modules and composition is bilinear. When $k = \mathbb{Z}$ we say that $\mathcal{C}$ is *preadditive.*

*Remark.* One says that $\mathcal{C}$ is “enriched” over $\mathbf{Mod}k$.

**Lemma 4.4.** *Let $\mathcal{C}$ be a $k$-category. For $X, Y \in \mathrm{Ob}(\mathcal{C})$, the product $X \times Y$ exists iff the coproduct $X \sqcup Y$ exists. If so, they are isomorphic.*

*Proof.* Suppose $X \times Y$ exists. Define $i_X = (\mathrm{id}_X, 0) : X \to X \times Y$ and $i_Y = (0, \mathrm{id}_Y) : Y \to X \times Y$. We claim these maps together with the product are the coproduct of $X$ and $Y$. Let $Z \in \mathrm{Ob}(\mathcal{C})$ and $f : X \to Z$, $g : Y \to Z$. Then, define $f \sqcup g : X \times Y \to Z$ by $f \sqcup g = f\pi_X + g\pi_Y$. This makes this diagram commute:

$$
\begin{tikzcd}
X \arrow[r, "i_X"] \arrow[dr, bend right, "f"'] & X \times Y \arrow[d, "f \sqcup g"] & Y \arrow[l, "i_Y"'] \arrow[dl, bend left, "g"] \\
& Z &
\end{tikzcd}
$$

Now let $h : X \times Y \to Z$ be another arrow that makes the diagram commute. Then

$$h \circ (i_X\pi_X + i_Y\pi_Y) = hi_X\pi_X + hi_Y\pi_Y = f\pi_X + g\pi_Y = f \sqcup g$$

And uniqueness follows since $\mathrm{id}_{X \times Y} = i_X\pi_X + i_Y\pi_Y$. This comes from the universal property of the product and the diagram

$$
\begin{tikzcd}
& X \times Y \arrow[dl, bend right, "\pi_X"'] \arrow[d, dashed, "i_X\pi_X + i_Y\pi_Y"] \arrow[dr, bend left, "\pi_Y"] & \\
X & X \times Y \arrow[l, "\pi_X"] \arrow[r, "\pi_Y"'] & Y
\end{tikzcd}
$$

$\square$

**Definition 4.5.** Let $\mathcal{C}$ be a $k$-category. A *biproduct* of $X$ and $Y$ is an object $X \oplus Y \in \mathcal{C}$ with morphisms $X \underset{\pi_X}{\overset{i_X}{\rightleftarrows}} X \oplus Y \underset{i_Y}{\overset{\pi_Y}{\rightleftarrows}} Y$ such that

1. $i_X\pi_X + i_Y\pi_Y = \mathrm{id}_{X \oplus Y}$
2. $\pi_X i_Y = 0$, $\pi_Y i_X = 0$, $\pi_X i_X = \mathrm{id}_X$, $\pi_Y i_Y = \mathrm{id}_Y$

**Definition 4.6.** Let $k$ be a commutative ring. A *$k$-additive* (or *$k$-linear*) category is a $k$-category with finite products and finite coproducts.

*Remark.*

1. When $k = \mathbb{Z}$, we simply say the category is *additive.*
2. As seen above, finite products are finite coproducts and vice versa. Both are finite biproducts.
3. For $\mathcal{C}$ a $k$-category, the following are equivalent:
   (a) $\mathcal{C}$ is $k$-additive
   (b) $\mathcal{C}$ has a zero object and every pair of objects has a product
   (c) $\mathcal{C}$ has a zero object and every pair of objects has a coproduct
   (d) $\mathcal{C}$ has a zero object and every pair of objects has a biproduct

   Moreover $(b) \iff (c) \iff (d)$, and for $(a)$ we are just missing the empty product (or coproduct), which is the zero object.

4. If $A$ is additive there is a canonical interpretation of the group structure on $\mathrm{Hom}(-,-)$ using $-\oplus-$. See exercise sheets.

**Examples 4.7.**

0. The category **Ab** of abelian groups is additive.
1. If $A$ is a ring (or $k$-algebra) then $\mathbf{Mod}A$, $A\mathbf{Mod}$ and finitely generated versions are $k$-additive.
2. If $\mathcal{C}$ is additive, then $\mathcal{C}^{\mathrm{op}}$ is additive.
3. If $\mathcal{C}$ is additive and $I$ is a category then $\mathrm{Fun}(I,\mathcal{C})$ is additive.
4. If $A$ is a ring, then the category $BA$ with one object $\bullet$ and $\mathrm{Hom}(\bullet,\bullet)=A$ is preadditive but not additive.

**Definition 4.8.** Let $F:\mathcal{C}\to\mathcal{D}$ be a functor between two $k$-linear categories. The functor $F$ is said to be *$k$-linear* (or *additive* when $k=\mathbb{Z}$) if for any $X,Y\in\mathrm{Ob}(\mathcal{C})$, $\begin{array}{ccc}\mathrm{Hom}(X,Y) & \to & \mathrm{Hom}(FX,FY)\\ f & \mapsto & F(f)\end{array}$ is a $k$-linear map.

**Proposition 4.9.** *A functor $F:\mathcal{C}\to\mathcal{D}$ is additive if and only if $F(0)\simeq 0$ and $F(X\oplus Y)\simeq F(X)\oplus F(Y)$.*

*Proof.* Suppose $F$ is additive. $\mathrm{id}_0$ is the zero morphism of $\mathrm{Hom}_{\mathcal{C}}(0,0)$. Therefore $F(\mathrm{id}_0)=\mathrm{id}_{F(0)}$ is the zero morphism of $\mathrm{Hom}_{\mathcal{D}}(F(0),F(0))$. For any $Y\in\mathrm{Ob}(\mathcal{D})$ and $f:F(0)\to Y$, $f=f\mathrm{id}_{F(0)}=0$. This shows $F(0)$ is initial. A similar reasoning shows it is final. Therefore $F(0)$ is isomorphic to the zero object of $\mathcal{D}$. Now let $X \underset{\pi_X}{\overset{i_X}{\rightleftarrows}} X\oplus Y \underset{i_Y}{\overset{\pi_Y}{\rightleftarrows}} Y$ be a biproduct in $\mathcal{C}$. Then we have a diagram

$$F(X) \underset{F(\pi_X)}{\overset{F(i_X)}{\rightleftarrows}} F(X\oplus Y) \underset{F(i_Y)}{\overset{F(\pi_Y)}{\rightleftarrows}} F(Y)$$

And the relations we require for this diagram to be a biproduct are satisfied since $F$ is additive and $X\oplus Y$ is a biproduct.
Now assume $F(0)\simeq 0$ and $F(X\oplus Y)\simeq F(X)\oplus F(Y)$ for all $X,Y\in\mathrm{Ob}(\mathcal{C})$. Let $X,Y\in\mathrm{Ob}(\mathcal{C})$. **TODO** $\square$

**Example 4.10.** Let $A,B$ be two rings and $M$ be an $A$-$B$-bimodule. Then, $-\otimes_A M_B:\mathbf{Mod}A\to\mathbf{Mod}B$ is additive. This can be quickly proven using the proposition above: the functor is a left adjoint so it preserves coproducts!

## 4.2 Chain complexes in an additive category

In this subsection, all categories are assumed to be additive.

**Definition 4.11.** A *chain complex* in $\mathcal{C}$ is a collection $C_\bullet=\{C_n\mid n\in\mathbb{Z}\}$ of objects of $\mathcal{C}$ together with morphisms $\partial_n:C_n\to C_{n-1}$ of $\mathcal{C}$ such that $\partial_{n-1}\circ\partial_n=0$. The morphisms $\partial_n$ are called the *differentials* of the complex.
Dually, a *cochain complex* in $\mathcal{C}$ is a collection $C^\bullet=\{C^n\mid n\in\mathbb{Z}\}$ of objects of $\mathcal{C}$ together with morphisms $\delta_n:C^n\to C^{n+1}$ of $\mathcal{C}$ such that $\delta^{n+1}\circ\delta^n=0$.

*Remark.* If $C_\bullet$ is a chain complex, then $(C')^\bullet = C_{-n}$ together with $\delta^n = \partial_{-n}$ is a cochain complex, so both notions are mathematically the same. However in practice chain and cochain complexes represent different objects so it is good to distinguish the two.

**Definition 4.12.** Let $C_\bullet$ and $D_\bullet$ be two chain complexes in $\mathcal{C}$. A *morphism of chain complexes* $f_\bullet : C_\bullet \to D_\bullet$ is a collection of morphisms $f_n : C_n \to D_n$ such that all diagrams

$$\begin{tikzcd}
{} \arrow[r] & C_n \arrow[r, "\partial_n^C"] \arrow[d, "f_n"] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & {} \\
{} \arrow[r] & D_n \arrow[r, "\partial_n^D"] & D_{n-1} \arrow[r] & {}
\end{tikzcd}$$

commute ("$\partial f = f \partial$").

**Definition 4.13.** If $\mathcal{C}$ is an additive category, then the category $\mathrm{Ch}(\mathcal{C})$ is the category whose objects are chain complexes in $\mathcal{C}$ and morphisms are morphisms of chain complexes. We also write $\mathrm{Ch}_\bullet(\mathcal{C})$.

*Remark.* One can check that $\mathrm{Ch}(\mathcal{C})$ is an additive category.

**Example 4.14.** Let $\Delta_n = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n} x_i = 1 \text{ and } x_i \geq 0\} = \mathrm{Conv}(e_0, \dots, e_n)$ be the standard $n$-simplex. $\Delta_n$ appears $n+1$ times as a face of the standard $n+1$-simplex, and

$$\begin{array}{rrcl} d^i : & \Delta_n & \to & \Delta_{n+1} \\ & (x_0, \dots, x_n) & \mapsto & (x_0, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n) \end{array}$$

is the $i$-th face map. $\Delta_n$ is a topological space, so when $X$ is a topological space we can consider

$$\mathrm{Hom}_{\mathbf{Top}}(\Delta_n, X) = \{f : \Delta_n \to X \mid f \text{ continuous}\}$$

and we get

$$\begin{array}{rrcl} d_i : & \mathrm{Hom}_{\mathbf{Top}}(\Delta_{n+1}, X) & \to & \mathrm{Hom}_{\mathbf{Top}}(\Delta_n, X) \\ & \sigma & \mapsto & (\Delta_{n+1} \xrightarrow{d^i} \Delta_n \xrightarrow{\sigma} X) \end{array}$$

for $0 \leq i \leq n+1$.
Singular Chain Complex **TODO**

**Definition 4.15.** Singular simplices **TODO**

**Example 4.16.** Singular chain complex. **TODO**

**Proposition 4.17.** $C^{\text{sing}} : \mathbf{Top} \to \mathrm{Ch}_\bullet(\mathbf{Ab})$ *is a functor.*

*Proof.* **TODO** □

**Simplicial methods**

**Definition 4.18.**

- We define the *simplicial category* (or *simplex category*) $\Delta$ whose objects are $[n] = \{0, 1, \dots, n\}$ for $n \in \mathbb{N}$, and $\mathrm{Hom}([n], [m]) = \{f : [n] \to [m] \mid f \text{ increasing}\}$. This category is equivalent to the category of non-empty, finite, totally ordered sets with increasing maps as morphisms.

- A *simplicial set* is a contravariant functor $\Delta \to \mathbf{Set}$. More generally, if $\mathcal{C}$ is a category, a *simplicial object* in $\mathcal{C}$ is a contravariant functor from $\Delta$ to $\mathcal{C}$.
- Simplicial objects in a category $\mathcal{C}$ are objects of the category $\mathcal{C}^{\Delta^{\mathrm{op}}}$. We write $s\mathcal{C} = \mathcal{C}^{\Delta^{\mathrm{op}}}$ (so $s\mathbf{Set}$ is the category of simplicial sets).
- If $X : \Delta^{\mathrm{op}} \to \mathcal{C}$ is a simplicial object, we define $X_n = X([n])$ the *$n$-simplices* of $X$.
- In $\Delta$, we have $d^i : [n-1] \to [n]$ the injective map that "misses $i$", defined by

$$d^i(k) = \begin{cases} k & \text{if } k < i \\ k+1 & \text{if } k \geq i \end{cases}$$

**Proposition 4.19.** *We have $d^i \circ d^j = d^{j+1} \circ d^i$ when $i \leq j$.*

*Proof.* You can do it. I believe in you. (**TODO**) □

If $X : \Delta^{\mathrm{op}} \to \mathbf{Ab}$ is a simplicial abelian group, then we can define $(X_\bullet, d)$ with $X_n = X([n])$ and $\begin{array}{cccc} d_n : & X_n & \to & X_{n-1} \\ & x & \mapsto & \sum_{i=0}^n (-1)^i X(d^i)(x) \end{array}$.

**Proposition 4.20.** *If $X \in s\mathbf{Ab}$, then $(X_\bullet, d)$ is a chain complex of abelian groups. Moreover, $X \mapsto X_\bullet$ is a functor $s\mathbf{Ab} \to \mathrm{Ch}_\bullet(\mathbf{Ab})$.*

*Proof.* **TODO** □

Let $\begin{array}{cccc} s^i : & [n+1] & \to & [n] \\ & k & \mapsto & \begin{cases} k & \text{if } k \leq i \\ k-1 & \text{if } k > i \end{cases} \end{array}$ be the map that "hits $i$ twice".

**Theorem 4.21.** *Every morphism in $\Delta$ is a composition of maps of the form $d^i$ and $s^i$. These maps are subject to the so-called* simplicial *relations*

$$\begin{cases} d^j \circ d^i = d^i \circ d^{j-1} & i < j \quad (1) \\ s^i \circ s^j = s^j \circ s^{i-1} & i > j \quad (2) \\ d^i \circ s^j = \begin{cases} s^{j-1} \circ d^i & i < j \\ \mathrm{id} & i \in \{j, j+1\} \\ s^j \circ d^{i-1} & i > j+1 \end{cases} & (3) \end{cases} \tag{$*$}$$

***TODO****better typography*
*and this is a presentation of $\Delta$ by generators and relations*

This theorem says that to define a functor $F$ from $\Delta$ to $\mathcal{C}$ it is enough to define $F(d^i), F(s^i)$ and show that $(*)$ holds.

*Proof.* "Voir annexe." **TODO** □

The maps $d^i$s generate $\Delta_{\mathrm{inj}}$ so to construct $F : \Delta_{\mathrm{inj}}^{\mathrm{op}} \to \mathcal{C}$ and use proposition 4.20 we only need to define $F(d^i)$ and check (1).

**Theorem 4.22.** *If* $F : \Delta_{\text{inj}}^{\text{op}} \to \mathbf{Ab}$ *is a (semisimplicial abelian group) functor then* $(F([n]), d_\bullet)$ *with* $\begin{array}{rccc} d_n: & F([n]) & \to & F[n-1] \\ & x & \mapsto & \sum_i (-1)^i F(d^i)(x) \end{array}$ *is a chain complex of abelian groups. This also works if* $\mathbf{Ab}$ *is replaced by any additive category* $\mathcal{C}$*.*

**Examples 4.23.**

1. Writing

$$\begin{array}{ccccc} \mathbf{Top} & \longrightarrow & s\mathbf{Set} & \longrightarrow & s\mathbf{Ab} \\ X & \longmapsto & \mathrm{Hom}_{\mathbf{Top}}(\Delta(-), X) & \longmapsto & \mathbb{Z}[\mathrm{Hom}_{\mathbf{Top}}(\Delta(-), X)] \end{array}$$

   allows us to use the theorem to recover what we said about the singular chain complex before.

2. Let $G$ be a finite group, and $F_n$ be the free abelian group on $G^{n+1} = \{(g_0, \dots, g_n) \mid g_i \in G\}$. $F_n$ is a $\mathbb{Z}[G]$-module for $g \bullet (g_0, \dots, g_n) = (gg_0, \dots, gg_n)$, so $F_n \in \mathbb{Z}[G]\mathbf{Mod}$. We define maps

$$\begin{array}{rccc} \partial_i : & F_n & \to & F_{n-1} \\ & (g_0, \dots, g_n) & \mapsto & (g_0, \dots, g_{i-1}, g_{i+1}, \dots, g_n) \end{array}$$

   (the map removes $g_i$). For $i < j$, we have

$$\begin{aligned} \partial_i \circ \partial_j(g_0, \dots, g_n) &= \partial_i(-, \cancel{g_j}, -) = (-, \cancel{g_i}, -, \cancel{g_j}, -) \\ \partial_{j-1} \circ \partial_i(g_0, \dots, g_n) &= \partial_{j-1}(-, \cancel{g_i}, -) = (-, \cancel{g_i}, -, \cancel{g_j}, -) \end{aligned}$$

   so setting $F([n]) = F_n$ and $F(d^i) = \partial_i$ defines a functor $F : \Delta_{\text{inj}}^{\text{op}} \to \mathbb{Z}[G]\mathbf{Mod}$. Applying theorem 4.22 we have $(F_n, \partial_\bullet) \in \mathrm{Ch}(\mathbb{Z}[G]\mathbf{Mod})$ called the *bar resolution* of $G$.

3. Koszul complex, Hochschild complex...

**Definition 4.24.** Let $\mathcal{C}$ be an additive category, $C_\bullet, D_\bullet \in \mathrm{Ch}_\bullet(\mathcal{C})$ and $f, g \in \mathrm{Hom}(C_\bullet, D_\bullet)$.
A *homotopy* $H$ *from* $f$ *to* $g$ is the data of maps $h_i : C_i \to D_{i+1}$ in $\mathcal{C}$ such that the following diagram commutes

$$\begin{tikzcd} C_{n+1} \arrow[r, "d^C_{n+1}"] \arrow[d, "f_{n+1}-g_{n+1}"'] & C_n \arrow[r, "d^C_n"] \arrow[dl, "h_n"'] \arrow[d, "f_n-g_n"] & C_{n-1} \arrow[dl, "h_{n-1}"'] \arrow[d, "f_{n-1}-g_{n-1}"] \\ D_{n+1} \arrow[r, "d^D_{n+1}"'] & D_n \arrow[r, "d^D_n"'] & D_{n-1} \end{tikzcd}$$

which means $f_n - g_n = h_{n-1}d^C_n + d^D_{n+1}h_n$ ("$f - g = hd + dh$"). We write $f \sim g$ if $f$ and $g$ are homotopic. $f$ and $g$ are *homotopy equivalences* if $fg \sim \mathrm{id}_D$ and $gf \sim \mathrm{id}_C$.

The motivation for this definition comes from topology. Let $f, g : X \to Y$ be continuous maps between topological spaces. We say $f$ and $g$ are homotopic if there exists a continuous map $H : X \times I \to Y$ (here $I$ is the unit interval $[0, 1]$) such that $H(-, 0) = f$ and $H(-, 1) = g$. The map $H$ is a called a homotopy from $f$ to $g$.

**Theorem 4.25.** *Let $X, Y$ be two topological spaces and $f, g : X \to Y$ be two homotopic continuous maps. Then the induced maps $C^{\text{sing}}(f), C^{\text{sing}}(g) : C^{\text{sing}}(X) \to C^{\text{sing}}(Y)$ are homotopic as morphisms of chain complexes.*

*Proof.* **TODO** ☐

**Lemma 4.26.** *Let $X, Y, Z \in \mathrm{Ch}(\mathcal{C})$ and $f : X_\bullet \to Y_\bullet, g : Y_\bullet \to Z_\bullet$ be morphisms of chain complexes. Then $f \sim 0$ implies $g \circ f \sim 0$.*

*Proof.* Let $h_\bullet$ be a homotopy between $f$ and 0. Then $g_\bullet \circ h_\bullet$ is a homotopy between $g \circ f$ and 0. ☐

**Definition 4.27.** Let $\mathcal{C}$ be a category. The *homotopy category $K(\mathcal{C})$ of chain complexes in $\mathcal{C}$* is the category defined by $\mathrm{Ob}(K(\mathcal{C})) = \mathrm{Ob}(\mathrm{Ch}(\mathcal{C}))$ and $\mathrm{Hom}_{K(\mathcal{C})}(X, Y) = \mathrm{Hom}_{\mathrm{Ch}(\mathcal{C})}(X, Y)/\sim$.

Lemma 4.26 above shows that composition in $K(\mathcal{C})$ is well-defined: if $f \sim g$, then $f - g \sim 0$ so $h(f - g) \sim 0$, so $hf \sim hg$. In the same vein, if $f - g \sim 0$, $(f - g)h \sim 0$, so $fh \sim fg$. This shows composition in $K(\mathcal{C})$ is well-defined.

*Remark.*

1. $K(\mathcal{C})$ is an additive category.
2. ⚠ In general, $K(\mathcal{C})$ is a complicated object: it is a triangulated category.

# 5 Abelian categories

**Definition 5.1.** Let $\mathcal{C}$ be an additive category. A *kernel* of $f \in \mathrm{Mor}(\mathcal{C})$ is an equalizer of $(f, 0)$. Dually, a *cokernel* of $f$ is a coequalizer of $(f, 0)$.

Concretely, we have universal arrows for any $\alpha$ such that $f\alpha = 0$ (or $\alpha f = 0$ for a cokernel)

$$\begin{tikzcd} & Z \arrow[dl, dashed] \arrow[d, "\alpha"] & \\ \ker f \arrow[r, "\iota"] & X \arrow[r, "f"] & Y \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "f"] & Y \arrow[r, "\pi"] \arrow[d, "\alpha"] & \operatorname{coker} f \arrow[dl, dashed] \\ & Z & \end{tikzcd}$$

If we assume that every morphism in $\mathcal{C}$ has a kernel and a cokernel, then

$$\begin{tikzcd} \ker f \arrow[r, "\iota"] & X \arrow[r, "f"] \arrow[d, "p"] & Y \arrow[r, "\pi"] & \operatorname{coker} f \\ & \operatorname{coker}(\ker f) & \ker(\operatorname{coker} f) \arrow[u] & \end{tikzcd}$$

where $\operatorname{coker}(\ker f)$ is notation for $\operatorname{coker}(\iota)$ and $\ker(\operatorname{coker} f)$ is notation for $\ker(\pi)$.
Since $f \circ \iota = 0$, we have a unique map $\tilde{f} : \operatorname{coker}(\ker f) \to Y$ by the universal property of the cokernel.

$$\begin{tikzcd}
\ker f \arrow[r, "\iota"] & X \arrow[r, "f"] \arrow[d, "p"] & Y \arrow[r, "\pi"] & \operatorname{coker} f \\
& \operatorname{coker}(\ker f) \arrow[ur, dashed, "\tilde{f}"'] & &
\end{tikzcd}$$

And we have $\pi \circ \tilde{f} \circ p = \pi \circ f = 0$.

**Lemma 5.2.** *Kernels are monomorphisms and cokernels are epimorphisms.*

*Proof.* Draw a diagram

$$\begin{tikzcd}
W \arrow[d, shift right, "b"'] \arrow[d, shift left, "a"] & & \\
\ker f \arrow[r, "\iota"] & X \arrow[r, "f"] & Y
\end{tikzcd}$$

such that $\iota a = \iota b$. Then $f\iota(a-b) = 0$, so there is a unique map $c : W \to \ker f$ such that we have a commutative diagram

$$\begin{tikzcd}
W \arrow[d, dashed, "c"] \arrow[dr, "\iota(a-b)"] & & \\
\ker f \arrow[r, "\iota"] & X \arrow[r, "f"] & Y
\end{tikzcd}$$

However $a - b$ and $0$ already make the diagram commute, so $a - b = 0$, so $a = b$.
The proof that a cokernel is an epimorphism is similar. □

Hence, $\pi \circ \tilde{f} \circ p = \pi \circ f = 0 = 0 \circ p$ means that $\pi \circ \tilde{f} = 0$ since $p$ is an epimorphism. This means that $\tilde{f}$ factorizes through $\ker(\operatorname{coker} f)$. Setting $\operatorname{coim} f = \operatorname{coker}(\ker f)$ and $\operatorname{im} f = \ker(\operatorname{coker} f)$, we obtain the following commutative diagram

$$\begin{tikzcd}
\ker f \arrow[r, "\iota"] & X \arrow[r, "f"] \arrow[d] & Y \arrow[r, "\pi"] & \operatorname{coker} f \\
& \operatorname{coim} f \arrow[r, dashed, "\overline{f}"'] & \operatorname{im} f \arrow[u] &
\end{tikzcd}$$

**Example 5.3.** In $\mathcal{C} = A\mathbf{Mod}$, we have the canonical factorization

$$\begin{tikzcd}
\ker f \arrow[r, "\iota"] & X \arrow[r, "f"] \arrow[d] & Y \arrow[r, "\pi"] & Y/\operatorname{im} f \\
& X/\ker f \arrow[r, dashed, "\overline{f}"'] & \operatorname{im} f \arrow[u] &
\end{tikzcd}$$

and $\overline{f}$ is an isomorphism by the first isomorphism theorem.

**Definition 5.4.** Let $\mathcal{C}$ be an additive category. Then $\mathcal{C}$ is *abelian* if

1. Every morphism has a kernel and a cokernel in $\mathcal{C}$.
2. $\forall f : X \to Y$, the canonical morphism $\overline{f} : \operatorname{coim} f \to \operatorname{im} f$ is an isomorphism.

**Examples 5.5.**

1. If $A$ is a ring, $\mathbf{Mod}A$ is abelian. If $A$ is noetherian, then the full subcategory $\mathbf{mod}A$ of finitely generated modules is abelian.

2. If $\mathcal{C}$ is abelian, then so is $\mathcal{C}^{\mathrm{op}}$.

3. There are examples of categories that satisfy 1 but not 2. For instance, Hausdorff topological abelian groups, where kernels are given by the usual kernel and cokernels are the quotients by the closure of the image. We have

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Q} \arrow[r, hook] \arrow[d] & \mathbb{R} \arrow[r] & 0 \\ & \mathbb{Q} \arrow[r, "\not\cong"] & \mathbb{R} \arrow[u, hook] & \end{tikzcd}$$

**Proposition 5.6.** *Let $\mathcal{A}$ be an abelian category and $\mathcal{J}$ a small category. Then*

1. $\mathrm{Fun}(\mathcal{J},\mathcal{A})$ *is an abelian category.*

2. $\mathrm{Ch}_\bullet(\mathcal{A})$ *is an abelian category.*

*Sketch of proof.* Let $F, G \in \mathrm{Fun}(\mathcal{J},\mathcal{A})$ and $\eta : F \Rightarrow G$. We want to construct $\ker\eta$. For any morphism $f : i \to j \in \mathrm{Mor}(\mathcal{J})$, we have a diagram

$$\begin{tikzcd} \ker(\eta_i) \arrow[r, "\iota_i"] \arrow[d, dashed, "\alpha_f"] & F(i) \arrow[r, "\eta_i"] \arrow[d, "F(f)"] & G(i) \arrow[d, "G(f)"] \\ \ker(\eta_j) \arrow[r, "\iota_j"] & F(j) \arrow[r, "\eta_j"] & G(j) \end{tikzcd}$$

We have $0 = G(f)\eta_i\iota = \eta_j F(f)\iota$ so $F(f)\iota$ factorizes through $\ker(\eta_j)$, which gives the morphism $\alpha_f$. One can check $\ker(\eta)$, defined by $\ker(\eta)(i) = \ker(\eta_i)$ and $\ker(\eta)(f) = \alpha_f$ is a functor (this is proved using uniqueness of $\alpha_f$). One can check that $\iota : \ker(\eta) \Rightarrow F$ is a kernel of $\eta$ by drawing the adequate diagrams. Constructing cokernels is done similarly. The canonical factorization is an isomorphism since its evaluation at every object is an isomorphism because $\mathcal{A}$ is abelian.
$\mathrm{Ch}_\bullet(\mathcal{A})$ is a subcategory of $\mathrm{Fun}(\mathbb{Z},\mathcal{A})$ so kernels and cokernels exist in $\mathrm{Fun}(\mathbb{Z},\mathcal{A})$. There is a commutative diagram

$$\begin{tikzcd} \ker f_{i+1} \arrow[r, dashed, "\alpha_{i+1}"] \arrow[d] & \ker f_i \arrow[r, dashed, "\alpha_i"] \arrow[d] & \ker f_{i-1} \arrow[d] \\ C_{i+1} \arrow[r, "d_{i+1}"] \arrow[d, "f_{i+1}"] & C_i \arrow[r, "d_i"] \arrow[d, "f_i"] & C_{i-1} \arrow[d, "f_{i-1}"] \\ D_{i+1} \arrow[r, "d_{i+1}"] \arrow[d] & D_i \arrow[r, "d_i"] \arrow[d] & D_{i-1} \arrow[d] \\ \mathrm{coker}\, f_{i+1} \arrow[r, dashed, "\beta_{i+1}"'] & \mathrm{coker}\, f_i \arrow[r, dashed, "\beta_i"'] & \mathrm{coker}\, f_{i-1} \end{tikzcd}$$

And the universal property of $\ker(f_{i-1})$ means that $\alpha_i\alpha_{i-1}$ is the unique morphism induced by $d_{i+1}d_i = 0$, so $\alpha_i\alpha i - 1 = 0$ and kernels, cokernels of chain complexes are again chain complexes. $\square$

*Remark.*

1. There is another equivalent definition of abelian categories: a category is abelian iff it is preabelian (additive, and all kernels/cokernels exist) and every monomorphism is a kernel of some morphism and every epimorphism is a cokernel of some morphism.

2. Abelian categories have finite limits and colimits.

3. If $f \in \mathrm{Mor}(\mathcal{A})$ with $\mathcal{A}$ abelian, then $f$ is a monomorphism if and only if $\ker f = 0$ and $f$ is an epimorphism if and only if $\operatorname{coker} f = 0$. Moreover, a monomorphism that is also an epimorphism is an isomorphism.

Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be two composable morphisms in an abelian category such that $gf = 0$. The left diagram below shows that $0 = gf = g\alpha\overline{f}\pi = 0$, however $\overline{f}\pi$ is an epi so $g\alpha = 0$. Therefore, $\alpha$ factorizes into a map $\operatorname{im} f \to \ker g$ as shown in the right diagram.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, two heads, "\pi"] & Y \arrow[r, "g"] & Z \\ \operatorname{coim} f \arrow[r, "\overline{f}"'] & \operatorname{im} f \arrow[u, hook, "\alpha"'] & \end{tikzcd} \qquad \begin{tikzcd} \ker g \arrow[r, hook, "\iota"] & Y \arrow[r, "g"] & Z \\ & \operatorname{im} f \arrow[u, hook, "\alpha"'] \arrow[ul, dashed, "\exists!\overline{\alpha}"] & \end{tikzcd}$$

**Definition 5.7.** Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ such that $gf = 0$.

- We say it is *exact* if the canonical map $\operatorname{im} f \to \ker g$ is an isomorphism.
- A chain complex $(C_\bullet, d_\bullet)$ is *exact* if the canonical maps $\mathrm{Im}(d_i) \simeq \ker(d_i)$ are isomorphisms for all $i \in \mathbb{Z}$.
- A *short exact sequence* if an exact complex of the form $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$.

**Example 5.8.** In $\mathbf{Mod}A$, $gf = 0$ means that $\operatorname{im} f \subset \ker g$, so exactness is equivalent to $\operatorname{im} f = \ker g$.

**Proposition 5.9.** *The sequence $0 \to X \xrightarrow{f} Y$ is exact if and only if $f$ is a monomorphism. The sequence $X \xrightarrow{f} Y \to 0$ is exact if and only if $f$ is an epimorphism.*

*Proof.* We have $\mathrm{im}(0 \to X) = \ker(\mathrm{coker}(0 \to X))$. One shows that the cokernel of $0 \to X$ is $X \xrightarrow{\mathrm{id}} X$ since it satisfies the required universal property. Similarly, one can prove the kernel of $X \xrightarrow{\mathrm{id}} X$ is $0 \to X$ by checking the universal property. Therefore, $\mathrm{im}(0 \to X) = 0$. Exactness is therefore equivalent to asking $\ker f = 0$. Let $i$ be the universal morphism $\ker f \xrightarrow{i} X$. If $f$ is a mono, we have $\ker f = 0$ since $fi$ is a mono and $fi0 = fi\mathrm{id}_{kerf} = 0$. Conversely, if $\ker f = 0$ and $fg = fh$, then $f(g - h) = 0$ and the factorization shows that $g = h$.
A similar "dual proof" shows the second part of the proposition is true. $\square$

**Proposition 5.10.** *The sequence $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z$ is exact if and only if $f = \ker g$.*

*Proof.* Assume $f = \ker g$. Since kernels are monomorphisms, we have exactness at $X$. Now we need to show the canonical map $\operatorname{im} f \to \ker g$ is an isomorphism. Draw the diagram

$$\begin{tikzcd}
 & & & \operatorname{coker} f \\
0 \arrow[r] & X \arrow[r, "f"] \arrow[d] & Y \arrow[r, "g"] \arrow[ur, "\pi_f"] & Z \\
 & \operatorname{coim} f \arrow[r, "\sim"] & \operatorname{im} f \arrow[r] \arrow[u] & \ker g \arrow[ul, "i_g"']
\end{tikzcd}$$

By $f = \ker g$ we mean $X \simeq \ker g$ as kernels. This means that there is an isomorphism $\ker g \underset{\sim}{\xrightarrow{\phi}} X$ such that $i_g = f\phi$. Then, $\pi_f i_g = \pi_f f\phi = 0$, so $i_g$ factorizes through $\ker(\operatorname{coker} f) = \operatorname{im} f$ in a way that makes the whole diagram commute which shows the canonical map $\operatorname{im} f \to \ker g$ is an isomorphism, so we have exactness at $Y$.

Conversely, assume the sequence is exact. We just need to check $X \xrightarrow{f} Y$ satisfies the universal property of $\ker g$. Exactness tells us we have a diagram

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "f"] \arrow[d, two heads, "\pi"] & Y \arrow[r, "g"] & Z \\
 & \operatorname{coim} f \arrow[r, "\sim", "\overline{f}"'] & \operatorname{im} f \arrow[r, "\sim", "t"'] \arrow[u, hook, "\alpha"'] & \ker g \arrow[ul, "i_g"']
\end{tikzcd}$$

We have $\operatorname{coim} f = \operatorname{coker}(\ker f)$. The proof above shows that exactness at $X$ implies $\ker f \simeq 0$. One can then check that $\operatorname{coim} f = X$ and $\pi = \mathrm{id}$. Therefore we obtain an isomorphism $\phi : X \to \ker g$ such that $i_g \circ \phi = f$ or equivalently $f \circ \phi^{-1} = i_g$. Let $h : T \to Y$ be a morphism such that $gh = 0$, then we have a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \\
\ker g \arrow[u, "\phi^{-1}"] \arrow[ur, "i_g"] & T \arrow[u, "h"'] \arrow[l, dashed, "\overline{h}"]
\end{tikzcd}$$

So $\phi^{-1}\overline{h}$ is a factorization of $h$ through $f$. If we have another factorization $\psi$ then

$$\begin{tikzcd}
\ker g \arrow[r, "i_g"] & Y \arrow[r] & Z \\
X \arrow[u, "\phi"] \arrow[ur, "f"] & T \arrow[u, "h"'] \arrow[l, "\psi"]
\end{tikzcd}$$

so $i_g\phi\psi = f\psi = h$ and $\phi\psi = \overline{h}$, so $\psi = \phi^{-1}\overline{h}$. $\square$

*Remark.* The sequence $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ is exact if and only if $f$ is a monomorphism, $g$ is an epimorphism and $\operatorname{im} f \xrightarrow{\sim} \ker g$ is an isomorphism, which is equivalent to $g = \operatorname{coker} f$ and $f = \ker g$.

*Remark.* There is a difficult theorem of Freyd and Mitchell that says any abelian category can be seen as a full subcategory of $\mathbf{Mod}A$ for some ring $A$ in such a way that the abelian structure is induced by the usual one in $\mathbf{Mod}A$.

**Definition 5.11.** Let $\mathcal{A}$ be an abelian category and $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$, $0 \to D \xrightarrow{h} E \xrightarrow{k} F \to 0$ be two short exact sequences. A *morphism of short exact sequences* between them is the data of three morphisms $\alpha : A \to D$, $\beta : B \to E$ and $\gamma : C \to F$ such that the following diagram commutes:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & D \arrow[r, "h"] & E \arrow[r, "k"] & F \arrow[r] & 0 \end{tikzcd}$$

**Lemma 5.12** (Short five lemma)**.** *Using the same notations as in the definition above:*

- *If $\alpha$ and $\gamma$ are monomorphisms, so is $\beta$.*
- *If $\alpha$ and $\gamma$ are epimorphisms, so is $\beta$.*
- *If $\alpha$ and $\gamma$ are isomorphisms, so is $\beta$.*

We give two proofs of this result.

*Proof by diagram chase.* Assume we work in a category of modules $\mathbf{Mod}A$. Assume $\alpha, \gamma$ are monos. Let $x \in \ker \beta$. Then $\gamma g(x) = k\beta(x) = 0$ and $\gamma$ is a mono so $g(x) = 0$. By exactness at $B$, there exists $y \in A$ such that $f(y) = x$. Then $0 = \beta f(y) = h\alpha(y)$. By exactness at $D$, $h$ is a mono, so $\alpha(y) = 0$. since $\alpha$ is a mono, $y = 0$, so $x = 0$, which means $\beta$ is a mono.
Now assume $\alpha, \gamma$ are epis. Let $x \in E$. Since $\gamma, g$ are epis, there exists $y \in B$ such that $\gamma(g(y)) = k(x)$. Then, $k(\beta(y) - x) = 0$. By exactness at $E$ and since $\alpha$ is epi, there exists $z \in A$ such that $h(\alpha(z)) = \beta(y) - x$. Therefore $\beta f(z) = \beta(y) - x$, so $\beta(y - f(z)) = x$ and $\beta$ is epi. $\square$

*Categorical proof in any abelian category.* Assume $\alpha, \gamma$ are monos. Let us add $\ker \beta$ to the diagram.

$$\begin{tikzcd} & & \ker \beta \arrow[dl, dashed, "\exists j"'] \arrow[d, "i"] & & \\ 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & D \arrow[r, "h"] & E \arrow[r, "k"] & F \arrow[r] & 0 \end{tikzcd}$$

We have $\beta i = 0$, so $\gamma g i = k\beta i = 0$. Since $\gamma$ is a mono, $gi = 0$. Exactness tells us $f = \ker g$, so we obtain the map $j : \ker \beta \to A$ with the universal property of $\ker g$. Since the diagram commutes, $0 = \beta i = \beta f j = h\alpha j$. Since $h$ and $\alpha$ are both monos, $j = 0$, so $i = 0$, so $\beta$ is a mono.
Now assume $\alpha, \gamma$ are epis and consider the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & D \arrow[r, "h"] & E \arrow[r, "k"] \arrow[d, "\pi"] & F \arrow[r] \arrow[dl, dashed, "\exists \eta"] & 0 \\ & & \operatorname{coker} \beta & & \end{tikzcd}$$

We have $\pi\beta = 0$, so $\pi\beta f = \pi h \alpha = 0$. Since $\alpha$ is an epi, $\pi h = 0$. Exactness tells us $k = \operatorname{coker} h$, which gives us $\eta$. Then, $\eta k \beta = 0$, so $\eta\gamma g = 0$. Since $\gamma, g$ are epis, $\eta = 0$, so $\pi = 0$, so $\beta$ is an epi. $\square$

**Theorem 5.13** (Splitting lemma)**.** *Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a short exact sequence in an abelian category $\mathcal{A}$. The following are equivalent:*

*(1)* $\exists r : B \to A, rf = \mathrm{id}_A$

*(2)* $\exists s : C \to B, gs = \mathrm{id}_C$

*(3)* $\exists h : B \xrightarrow{\sim} A \oplus C$ *such that*

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, equal] & B \arrow[r, "g"] \arrow[d, "h"] & C \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & A \arrow[r, "i_A"] & A \oplus C \arrow[r, "\pi_C"] & C \arrow[r] & 0 \end{tikzcd}$$

*is an isomorphism of short exact sequences.*

*When these conditions are satisfied, we say the short exact sequence* splits*.*

*Proof.* Assume we have (3). Then we have the projection $\pi_A : A \oplus C \to A$. Letting $r = \pi_A h$, we have $rf = \pi_A hf = \pi_A i_A = \mathrm{id}_A$. Similarly, setting $s = h^{-1} i_C$ gives $gs = \pi_C h h^{-1} i_C = \mathrm{id}_C$.
Now assume (1). We have $r : B \to A$ and $g : B \to C$. This gives a morphism $r \oplus g : B \to A \oplus C$ defined by $r \oplus g = i_A r + i_C g$. Then, $(r \oplus g)f = i_A$ since $gf = 0$ and $\pi_C(r \oplus g) = g$ by properties of the biproduct. This means that $r \oplus g$ makes the diagram above commute. The short five lemma then tells us $r \oplus g$ is an isomorphism.
Assume (2). Then $f : A \to B$ and $s : C \to B$ induce a morphism $f \oplus s : A \oplus C \to B$ defined by $f \oplus s = f\pi_A + s\pi_C$. This morphism satisfies

$$(f \oplus s)i_A = f \quad \text{and} \quad g(f \oplus s) = \pi_C$$

so again we get an isomorphism of short exact sequences by the short five lemma. □

**Definition 5.14.** Let $\mathcal{C}$ and $\mathcal{D}$ be two abelian categories. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor.

1. We say $F$ is *left exact* if $F$ preserves finite limits.
2. We say $F$ is *right exact* if $F$ preserves finite colimits.
3. We say $F$ is *exact* if it preserves finite limits and finite colimits.

**Lemma 5.15.** *Let $F : \mathcal{C} \to \mathcal{D}$ be an additive functor between abelian categories. The following are equivalent;*

*(1) The functor $F$ is left exact.*

*(2) The functor $F$ preserves kernels i.e.* $F(\ker f) \simeq \ker(F(f))$.

*(3) If $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z$ is an exact sequence in $\mathcal{C}$, the sequence $0 \to F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z)$ is also exact.*

*(4) If $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ is a short exact sequence in $\mathcal{C}$, the sequence $0 \to F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z)$ is also exact.*

*Proof.*

(1) ⇒ (2) This is clear since a kernel is a limit (an equalizer).

(2) ⇒ (3) Assume we have (2). Then $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z$ is an exact sequence in $\mathcal{C}$ if and only if $f = \ker g$, so $F(f)$ is a kernel of $F(g)$, so $0 \to F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z)$ is exact.

(3) ⇒ (4) This is clear.

(2) ⇒ (1) The functor $F$ is additive so it preserves products. The equalizer of $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ is the kernel of $f - g$, so $F$ preserving kernels means it also preserves equalizers. Since any finite limit can be built out of products and equalizers, $F$ is left-exact.

(4) ⇒ (3) Let $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z$ be an exact sequence. Consider $0 \to X \xrightarrow{f} Y \to \mathrm{coker}(f) \to 0$. Applying $F$ shows that $F(f)$ is a monomorphism, so $F$ preserves monos. Moreover we have the exact sequence $0 \to X \xrightarrow{f} Y \xrightarrow{\overline{g}} \mathrm{Im}\, g \to 0$ so $0 \to F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(\overline{g})} F(\mathrm{Im}\, g)$ is also exact. Since $i : \mathrm{im}\, g \hookrightarrow Z$ is a mono and $F$ preserves monos, we know that $F(i) : \mathrm{im}\, g \to F(Z)$ is a mono so $F$ does not change the kernel. This means that $0 \to F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z)$ is exact.

□

**Corollary 5.16.** *For an additive functor $F : \mathcal{C} \to \mathcal{D}$ between abelian categories, the following are equivalent:*

1. *$F$ is exact.*
2. *For any short exact sequence $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ in $\mathcal{C}$, the sequence $0 \to F(A) \xrightarrow{F(\alpha)} F(B) \xrightarrow{F(\beta)} F(C) \to 0$ is exact.*

**Proposition 5.17.** *Let $\mathcal{C}$ be an abelian category.*

1. $\mathrm{Hom}_{\mathcal{C}}(-,-) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Mod}\mathbb{Z}$ *is left exact in each variable.*
2. $- \otimes_A - : \mathbf{Mod}A \times A\mathbf{Mod} \to \mathbb{Z}\mathbf{Mod}$ *is right exact in each variable.*
3. *If $F \dashv G$, then $F$ is right exact and $G$ is left exact.*

*Proof.* Since left adjoints preserve colimits, they are right exact, and dually for right adjoints. □

## 5.1 Chain complexes in abelian categories

**Definition 5.18.** Let $\mathcal{A}$ be an abelian category and $(X_\bullet, d_\bullet) \in \mathrm{Ch}_\bullet(\mathcal{A})$. Pour $n \in \mathbb{Z}$, we define:

- $Z_n(X) = \ker d_n$ ("$n$-cycles")
- $B_n(X) = \mathrm{im}\, d_{n+1}$ ("$n$-boundaries")
- $H_n(X) = Z_n(X)/B_n(X)$ ("$n$-th homology of $X$")

The definition $Z_n(X)/B_n(X)$ works in a category of modules ; in an arbitrary abelian category, one sets $H_n(X) = \operatorname{coker}(B_n \hookrightarrow Z_n)$. If we are working with a cochain complex, we speak of cocycles, coboundaries and cohomology.

Let $(X_\bullet, d_\bullet^X)$ and $(Y_\bullet, d_\bullet^Y)$ be two chain complexes in an abelian category $\mathcal{A}$, and $f : X_\bullet \to Y_\bullet$ be a chain morphism between them. Then we have the diagram

$$\begin{tikzcd}
\ker d_n^X \arrow[r, "i"] \arrow[d, dashed, "Z_n(f)"] & X_n \arrow[r, "d_n^X"] \arrow[d, "f_n"] & X_{n-1} \arrow[d, "f_{n-1}"] \\
\ker d_n^Y \arrow[r] & Y_n \arrow[r, "d_n^Y"'] & Y_{n-1}
\end{tikzcd}$$

Since $d_n^Y f_n i = f_{n-1} d_n^X i = 0$, we obtain the dashed arrow $Z_n(f)$ by the universal property of $\ker d_n^Y$. Similarly, $f_{n-1}$ induces a morphism $\overline{f_{n-1}} : \operatorname{coker}(d_n^X) \to \operatorname{coker}(d_n^Y)$ and we have the diagram:

$$\begin{tikzcd}
\operatorname{im} d_n^X \arrow[r, "i"] \arrow[d, dashed, "B(f_n)"] & X_{n-1} \arrow[r, "\pi"] \arrow[d, "f_{n-1}"] & \operatorname{coker}(d_n^X) \arrow[d, "\overline{f_{n-1}}"] \\
\operatorname{im} d_n^Y \arrow[r, "i"'] & Y_{n-1} \arrow[r, "\pi"'] & \operatorname{coker}(d_n^Y)
\end{tikzcd}$$

Since $\pi f_{n-1} i = \overline{f_{n-1}} \pi i = 0$, $f_{n-1} i$ induces a morphism $B_n(f) : \operatorname{im} d_n^X \to \operatorname{im} d_n^Y$. Therefore, we have the diagram

$$\begin{tikzcd}
\operatorname{im} d_{n+1}^X \arrow[r, hook, "i"] \arrow[d, "B_n(f)"] & \ker d_n \arrow[r] \arrow[d, "Z_n(f)"] & \operatorname{coker} i \arrow[d, dashed, "H_n(f)"] \\
\operatorname{im} d_{n+1}^Y \arrow[r, hook] & \ker d_n \arrow[r, "\pi"] & \operatorname{coker} i
\end{tikzcd}$$

In $\mathbf{Mod} - A$, we have $H_n(X) = {}^{\ker d_n}/_{\operatorname{im} d_{n+1}}$ and $H_n(f)([x]) = [f_n(x)]$. We get a functor $H_n : \mathrm{Ch}_\bullet(\mathcal{A}) \to \mathcal{A}$ called the $n$-th homology functor. Moreover, it is an additive functor.

**Definition 5.19.** Let $f : X_\bullet \to Y_\bullet$ be a morphism of chain complexes. We say $f$ is a *quasi-isomorphism* if $H_n(f)$ is an isomorphism for all $n \in \mathbb{Z}$.

**Proposition 5.20.** *Let $f : C_\bullet \to D_\bullet \in \operatorname{Mor}(\mathrm{Ch}_\bullet(\mathcal{A}))$ with $\mathcal{A}$ an abelian category.*

1. *If $f \sim g$ then $H_n(f) = H_n(g)$ for all $n$.*
2. *If $f$ is a homotopy equivalence, then it is a quasi isomorphism.*

*Proof.* Assume $f \sim 0$. Then we have a collection of morphisms $s_n : C_n \to D_{n+1}$ and a diagram

$$\begin{tikzcd}
C_{n+1} \arrow[r] \arrow[d, "f_{n+1}"'] & C_n \arrow[r] \arrow[d, "f_n"'] \arrow[dl, "s_n"] & C_{n-1} \arrow[d, "f_{n-1}"] \arrow[dl, "s_{n-1}"] \\
D_{n+1} \arrow[r] & D_n \arrow[r] & D_{n-1}
\end{tikzcd}$$

with $f_n = s_{n-1} d_n^C + d_{n+1}^D s_n$. Then, **TODO** □

**Definition 5.21.** Let $C_\bullet$ be a chain complex.

1. We say $C_\bullet$ is *contractible* if $C$ is homotopy equivalent to 0.
2. We say $C_\bullet$ is *acyclic* if $C$ is quasi-isomorphic to 0.

Of course, contractibility implies acyclicity.

**Theorem 5.22** (Long exact sequence). *A short exact sequence of chain complexes*

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$

*gives rise to a long exact sequence in homology*

$$\cdots \to H_n(A) \xrightarrow{H_n(\alpha)} H_n(B) \xrightarrow{H_n(\beta)} C \xrightarrow{\delta_n} H_{n-1}(A) \to \cdots$$

*The morphisms $\delta_n$ (which are defined in the proof) are called the* connecting *homomorphisms.*

*Proof.* **TODO**(snake lemma in abelian cat) □

*Remark.* The connecting homomorphism is natural in the following way. If we have a commutative diagram of chain complexes where the rows are short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\alpha"] \arrow[d, "f"'] & B \arrow[r, "\beta"] \arrow[d, "g"'] & C \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & A' \arrow[r, "\alpha'"'] & B' \arrow[r, "\beta'"'] & C' \arrow[r] & 0 \end{tikzcd}$$

we obtain a morphism of long exact sequences

$$\begin{tikzcd} \cdots \arrow[r] & H_n(A) \arrow[r, "H_n(\alpha)"] \arrow[d, "H_n(f)"'] & H_n(B) \arrow[r, "H_n(\beta)"] \arrow[d, "H_n(g)"'] & H_n(C) \arrow[r, "\delta_n"] \arrow[d, "H_n(h)"'] & H_{n-1}(A) \arrow[r, "H_{n-1}(\alpha)"] \arrow[d, "H_{n-1}(f)"] & H_{n-1}(B) \arrow[r] \arrow[d, "H_{n-1}(g)"] & \cdots \\ \cdots \arrow[r] & H_n(A') \arrow[r, "H_n(\alpha')"'] & H_n(B') \arrow[r, "H_n(\beta')"'] & H_n(C') \arrow[r, "\delta'_n"'] & H_{n-1}(A') \arrow[r, "H_{n-1}(\alpha')"'] & H_{n-1}(B') \arrow[r] & \cdots \end{tikzcd}$$

The commutativity of the first two squares comes from the fact that $H_n$ is a functor. The commutativity of the middle square, the one that involves connecting homomorphisms, needs a proof.

## 5.2 Projective and injective objects

**Definition 5.23.** Let $\mathcal{A}$ be an abelian category.

1. An object $I \in \mathrm{Ob}(\mathcal{A})$ is *injective* if $\mathrm{Hom}_{\mathcal{A}}(-, I) : \mathcal{A}^{\mathrm{op}} \to \mathbf{Ab}$ is exact
2. An object $P \in \mathrm{Ob}(\mathcal{A})$ is *projective* if $\mathrm{Hom}_{\mathcal{A}}(P, -) : \mathcal{A} \to \mathbf{Ab}$ is exact
3. The category $\mathcal{A}$ has *enough projectives* if for any $X \in \mathrm{Ob}(\mathcal{A})$ there exists an epimorphism $f : P \to X$ with $P$ projective.
4. The category $\mathcal{A}$ has *enough injectives* if for any $X \in \mathrm{Ob}(\mathcal{A})$ there exists a monomorphism $g : X \to I$ with $I$ injective.

**Proposition 5.24.** ***TODO****prop 5.18*

*Proof.* **TODO** $\square$

**Definition 5.25.** Let $\mathcal{A}$ be an abelian category. A *pullback* of $\alpha : A \to C$ and $\beta : B \to C$ is a limit of the diagram $\begin{array}{cc} & B \\ & \downarrow\beta \\ A \xrightarrow[\alpha]{} & C \end{array}$. Diagramatically, the pullback is the object $P$ together with the two morphisms $p_1, p_2$:

$$\begin{tikzcd}
Q \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left, "q_2"] \arrow[ddr, bend right, "q_1"'] & & \\
& P \arrow[r, "p_2"] \arrow[d, "p_1"] & B \arrow[d, "\beta"] \\
& A \arrow[r, "\alpha"'] & C
\end{tikzcd}$$

Dually, we have the notion of a pushout:

$$\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[d, "\beta"'] & B \arrow[d] \arrow[ddr, bend left] & \\
C \arrow[r] \arrow[drr, bend right] & P \arrow[dr, dashed, "\exists!"] & \\
& & Q
\end{tikzcd}$$

**Proposition 5.26.** *Let $\mathcal{A}$ be an abelian category.*

1. *Any pair of morphisms has a pullback and a pushout in $\mathcal{A}$.*

2. *In any category, if $\alpha$ is a monomorphism, then $p_2$ also is a monomorphism: pullbacks preserve monos.*

3. *Dually, pushouts preserve epis.*

4. *In an abelian category, pullbacks preserve kernels :*

$$\begin{tikzcd}
\ker p_2 \arrow[r] \arrow[d, dashed, "\simeq"] & P \arrow[r, "p_2"] \arrow[d, "p_1"] & Y \arrow[d, "g"] \\
\ker f \arrow[r] & X \arrow[r, "f"] & Z
\end{tikzcd}$$

5. *In an abelian category, pullbacks preserve epimorphisms.*

*Proof.* 1. is clear since pullback and pushouts are small (co)limits. Let's prove 2. We use the same notations as in the definition above. Assume $\alpha$ is a monomorphism. Consider a pair of parallel morphisms $W \overset{a}{\underset{b}{\rightrightarrows}} P$ such that $p_2 a = p_2 b$. Then we have a diagram

$$\begin{tikzcd}
W \arrow[dr, shift left, "b"] \arrow[dr, shift right, "a"'] \arrow[drr, bend left, "p_2 b"] \arrow[ddr, bend right, "p_1 a"'] \\
& P \arrow[r, "p_2"] \arrow[d, "p_1"] & B \arrow[d, "\beta"] \\
& A \arrow[r, "\alpha"'] & C
\end{tikzcd}$$

By commutativity, $\alpha p_1 a = \beta p_2 a = \beta p_2 b = \alpha p_1 b$. Since $\alpha$ is a mono, $p_1 a = p_1 b$. This means that the diagram above commutes. Therefore, $a$ and $b$ are two solutions to the same universal problem and $a = b$. This shows $p_2$ is a mono. The proof for 3. is dual. They both work in any category. Now we work in an abelian category so we can talk about kernels to prove 4. Consider the diagram

$$\begin{tikzcd}
\ker p_2 \arrow[r, "i_2"] \arrow[d, dashed, "\exists! p"] & P \arrow[r, "p_2"] \arrow[d, "p_1"] & Y \arrow[d, "g"] \\
\ker f \arrow[r, "i_1"] & X \arrow[r, "f"] & Z
\end{tikzcd}$$

We have $f p_1 i_2 = g p_2 i_2 = 0$, so we obtain the map $p$ by the universal property of $\ker f$. The universal property of the pullback gives us a map $t : \ker f \to P$ in the following way:

$$\begin{tikzcd}
& \ker f \arrow[dl, dashed, "\exists! q"'] \arrow[dr, dashed, "\exists! t"] \arrow[drr, bend left, "0"] \arrow[ddr, bend right, "i_1"'] \\
\ker p_2 \arrow[rr] & & P \arrow[r, "p_2"] \arrow[d, "p_1"] & Y \arrow[d, "g"] \\
& & X \arrow[r, "f"'] & Z
\end{tikzcd}$$

the map $q$ is induced by $t$ since $p_2 t = 0$. Now,

$$\begin{tikzcd}
\ker f \arrow[r, "i_1"] & X \arrow[r, "f"] & Z \\
\ker p_2 \arrow[u, "p"] & \ker f \arrow[l, "q"] \arrow[u, "i_1"']
\end{tikzcd}$$

and we have $i_1 p q = p_1 i_2 q = p_1 t = i_1$. Therefore, $pq$ and $\mathrm{id}_{\ker f}$ are two solutions to the same universal problem, so they are equal: $pq = \mathrm{id}_{\ker f}$. Now consider the diagram

$$\begin{tikzcd}
\ker p_2 \arrow[dr, "i_2 q p"] \arrow[drr, bend left, "0"] \arrow[ddr, bend right, "p_1 i_2"'] \\
& P \arrow[r, "p_2"] \arrow[d, "p_1"] & Y \arrow[d, "g"] \\
& X \arrow[r, "f"] & Z
\end{tikzcd}$$

This diagram commutes: indeed, $p_2 i_2 q p = 0$ and $p_1 i_2 q p = i_1 p = p_1 i_2$. Now, replacing $i_2 q p$ by $i_2 : \ker p_2 \to P$ also makes the diagram commute, so by universal property of the pullback,

$i_2 qp = i_2$. Since $i_2$ is a mono (it's a kernel), $qp = \mathrm{id}_{\ker p_2}$. This concludes the proof of 4.
Let

$$\begin{tikzcd} P \arrow[r, "p_2"] \arrow[d, "p_1"] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$

be a pullback. Then, we have an exact sequence $P \xrightarrow{(p_1,p_2)} X \oplus Y \xrightarrow{-f \oplus g} Z$ (**TODO**understand why and finish proof). $\square$

**Theorem 5.27.** *Let $\mathcal{A}$ be an abelian category.*

1. *Let $(P_\lambda)_{\lambda\in\Lambda}$ be a family of objects of $\mathcal{A}$. Then $\bigcup_{\lambda\in\Lambda} P_\lambda$ is projective if and only if $P_\lambda$ is projective for all $\lambda \in \Lambda$.*
2. *An object $P$ is projective if and only if for any epi $f : X \twoheadrightarrow P$, there exists $s : P \to X$ such that $fs = \mathrm{id}_P$.*
3. *An object $I$ is injective if and only if for any mono $f : I \hookrightarrow X$, there exists $r : X \to I$ such that $rf = \mathrm{id}_I$.*

*Proof.* **TODO** $\square$