

# Homological Algebra

Original lectures notes by Baptiste Rognerud, but now typeset in L<sup>A</sup>T<sub>E</sub>X

## 1 Introduction to category theory

References:

- Emily Riehl, Category Theory in Context (chapter I)
- Saunders Mac Lane, Categories for the Working Mathematician
- Ibrahim Assem, Introduction au langage catégorique (chapters I, II)

➤ Near 1945 Eilenberg and Mac Lane gave the good formalism for a “natural isomorphism” (the general theory of natural transformations). For instance, if  $V$  is a finite-dimensional vector space,  $V \simeq V^*$  and  $V \simeq V^{**}$ , but the first isomorphism is not natural (“a choice needs to be made”), while the second is. But why?

It turns out solving this question gave a formalism, category theory, that unified already existing mathematical concepts, gave new links between different notions and also gave new questions!

⚠ Category theory is not a theory that trivializes mathematics.

It is used today by (almost) everyone: algebraic geometry, algebra, representation theory, topology, combinatorics, ...

### 1.1 Categories and functors

**Definition 1.1.** A *category*  $\mathcal{C}$  is the data of

- A collection of *morphisms*  $\text{Mor}(\mathcal{C})$
- A collection of *objects*  $\text{Ob}(\mathcal{C})$

such that

1. Every morphism  $f \in \text{Mor}(\mathcal{C})$  has a specified domain  $X \in \text{Ob}(\mathcal{C})$  and codomain  $Y \in \text{Ob}(\mathcal{C})$ . We write  $f : X \rightarrow Y$ .
2. For every object  $X \in \text{Ob}(\mathcal{C})$  there exists a morphism  $1_X : X \rightarrow X$  (the *identity* of  $X$ ), also written  $\text{id}_X$
3. For any three objects  $X, Y, Z \in \text{Ob}(\mathcal{C})$  and morphism  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  there exists a morphism  $g \circ f : X \rightarrow Z$  (we often omit  $\circ$  and just write  $gf$ )

satisfying

**(Identity)**  $\forall f : X \rightarrow Y, 1_Y f = f = f 1_X$

**(Associativity)**  $\forall f : W \rightarrow X, g : X \rightarrow Y, h : Y \rightarrow Z, h(gf) = (hg)f$

*Remark.*

1. We use the term “collection” because we don’t want to worry about set-theoretical issues
2. If  $\text{Mor}(\mathcal{C})$  is a set, we say that  $\mathcal{C}$  is *small*
3. We denote by  $\text{Hom}_{\mathcal{C}}(X, Y)$  (or  $\mathcal{C}(X, Y)$ ) the collection of  $f : X \rightarrow Y \in \text{Mor}(\mathcal{C})$

**Examples 1.2** (Concrete categories).

1. The category **Set**, where objects are sets and morphisms are just maps.
2. **Top**, where objects are topological spaces and morphisms are continuous maps.
3. Groups together with group homomorphisms form a category called **Grp**. The same can be said about rings, fields...
4.  $k$ -vector spaces, or more generally left/right  $R$ -modules, together with linear maps, form a category denoted **RMod** or **ModR** (for left or right  $R$ -modules).

In these previous examples, objects are sets with additional structure, and morphisms between two objects are in particular maps between the two underlying sets. Such categories are called *concrete categories* (a rigorous definition will be given later). However, a category need not be concrete.

**Examples 1.3** (Abstract categories).

1. Let  $k$  be a field. There exists a category **Mat** $_k$  where objects are the natural numbers  $\mathbb{N}$  and morphisms are  $\text{Hom}(m, n) = \text{Mat}_{n,m}(k)$ , where composition is given by matrix multiplication.
2. If  $G$  is a group, there exists a category  $BG$  which has only one object  $\bullet$ , and morphisms  $\text{Hom}(\bullet, \bullet) = G$ , where composition is multiplication in  $G$ .
3. If  $(P, \leq)$  is a *poset* (a partially ordered set, that is a set  $P$  together with a reflexive, transitive relation  $\leq$ ), then one can construct a category  $\hat{P}$  by setting  $\text{Ob}(\hat{P}) = P$  and  $|\text{Hom}(x, y)| = \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{otherwise} \end{cases}$ , where composition is defined in the only possible way.
4. The homotopy category of topological spaces: objects are topological spaces, and  $\text{Hom}(X, Y)$  is  $\text{Hom}_{\text{Top}}(X, Y) / \sim$  where  $\sim$  is homotopy of continuous maps.

*Exercise.* Check the categories defined above really are categories. In (2), what are the minimal hypotheses needed on  $G$  for  $BG$  to be a category? In (3), what are the minimal hypotheses needed on  $\leq$  for  $\hat{P}$  to be a category?

**Examples 1.4** (Categories constructed from categories).

1. If  $\mathcal{C}$  is a category, one can construct its *opposite category*  $\mathcal{C}^{\text{op}}$ , defined by  $\text{Ob}(\mathcal{C}^{\text{op}}) = \text{Ob}(\mathcal{C})$  and  $\text{Hom}_{\mathcal{C}^{\text{op}}}(X, Y) = \text{Hom}_{\mathcal{C}}(Y, X)$ , with composition described by the following diagram:

$$\begin{array}{ccc}
X & & X \\
\downarrow f & & \uparrow f^{\text{op}} \\
Y & \rightsquigarrow & Y \\
\downarrow g & & \uparrow g^{\text{op}} \\
Z & & Z
\end{array}
\quad
\begin{array}{c}
gf \\
\downarrow \\
Z
\end{array}
\quad
\begin{array}{c}
f^{\text{op}}g^{\text{op}} \\
\downarrow \\
Z
\end{array}$$

2. Let  $\mathcal{C}$  be a category. A *subcategory*  $\mathcal{D}$  of  $\mathcal{C}$  is another category such that  $\text{Ob}(\mathcal{D}) \subset \text{Ob}(\mathcal{C})$  and  $\text{Mor}(\mathcal{D}) \subset \text{Mor}(\mathcal{C})$  and the composition in  $\mathcal{D}$  is induced by the one in  $\mathcal{C}$ . For instance, **Ab**, the category of abelian groups and group homomorphisms, is a subcategory of **Grp**.
3. Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories. The *product category* of  $\mathcal{C}$  and  $\mathcal{D}$  is the category  $\mathcal{C} \times \mathcal{D}$  defined by  $\text{Ob}(\mathcal{C} \times \mathcal{D}) = \text{Ob}(\mathcal{C}) \times \text{Ob}(\mathcal{D})$  and  $\text{Mor}(\mathcal{C} \times \mathcal{D}) = \text{Mor}(\mathcal{C}) \times \text{Mor}(\mathcal{D})$ , composition and identities being defined componentwise.

*Exercise.* Describe  $(BG)^{\text{op}}$  for  $G$  a group and  $\hat{P}^{\text{op}}$  for  $(P, \leq)$  a poset.

**⚠ Set<sup>op</sup> is not Set. TODO**

*Remark.* In a category  $\mathcal{C}$  the objects can be anything, so saying  $x \in X$  for  $X \in \text{Ob}(\mathcal{C})$  doesn't make sense. Hence, categorical notions are defined using arrows and not elements.

**Definition 1.5.** Let  $\mathcal{C}$  be a category.

1.  $f : X \rightarrow Y$  is an *isomorphism* if there exists  $g : Y \rightarrow X$  such that  $gf = \text{id}_X$  and  $fg = \text{id}_Y$ .
2.  $f : X \rightarrow Y$  is a *monomorphism* if for all  $g, h : W \rightarrow X$  such that  $fg = fh$ ,  $g = h$  ( $f$  is left-cancellable).
3.  $f : X \rightarrow Y$  is an *epimorphism* if for all  $g, h : Y \rightarrow Z$  such that  $gf = hf$ ,  $g = h$  ( $f$  is right-cancellable).

**⚠ Being both a mono and an epi doesn't imply being an iso. TODO**

**Definition 1.6.** Let  $\mathcal{C}, \mathcal{D}$  be two categories. A (*covariant*) *functor*  $F : \mathcal{C} \rightarrow \mathcal{D}$  is the data of

- An object  $F(X) \in \text{Ob}(\mathcal{D})$  for all  $X \in \text{Ob}(\mathcal{C})$
- A morphism  $F(f) \in \text{Hom}_{\mathcal{D}}(F(X), F(Y))$  for all  $f \in \text{Hom}_{\mathcal{C}}(X, Y)$

such that  $F(\text{id}_X) = \text{id}_{F(X)}$  for all  $X \in \text{Ob}(\mathcal{C})$  and  $F(gf) = F(g)F(f)$  whenever  $f, g \in \text{Mor}(\mathcal{C})$  are composable.

**Definition 1.7.** A *contravariant* functor from  $\mathcal{C}$  to  $\mathcal{D}$  is a functor from  $\mathcal{C}^{\text{op}}$  to  $\mathcal{D}$  (so composition is reversed, i.e.  $F(gf) = F(f)F(g)$ ).

**Examples 1.8.**

1.  $U : \mathbf{Grp} \rightarrow \mathbf{Set}, U(G) = G, U(f) = f$  the functor that to a group assigns it its underlying set and to a homomorphism the underlying map. It is called the *forgetful functor* from groups to sets, because it forgets the group structure.

2.  $U : \mathbf{Ass} \rightarrow \mathbf{Lie}$  the forgetful functor from the category of associative algebras to the category of Lie algebras. It forgets the “associative structure” but remembers the underlying abelian group.
 
$$(A, +, \cdot) \mapsto (A, +, [-, -])$$
3.  $F : \mathbf{Set} \rightarrow \mathbf{Ab}, X \mapsto \mathbb{Z}[X], f \mapsto \mathbb{Z}[f]$ , which to a set assigns the free abelian group with basis  $X$  (the group of finite linear combinations of elements of  $X$ ). A map  $f : X \rightarrow Y$  can then be uniquely extended to a linear map  $\mathbb{Z}[f] : \mathbb{Z}[X] \rightarrow \mathbb{Z}[Y]$  that agrees with  $f$  on the bases of  $\mathbb{Z}[X]$  and  $\mathbb{Z}[Y]$ .
4. Suppose  $\mathcal{C}$  is locally small (i.e. for any  $X, Y$ ,  $\text{Hom}_{\mathcal{C}}(X, Y)$  is a set). For all  $X \in \mathcal{C}$ ,  $\text{Hom}(X, -)$  is a functor  $\mathcal{C} \rightarrow \mathbf{Set}$ . Similarly,  $\text{Hom}_{\mathcal{C}}(-, X)$  is a contravariant functor  $\mathcal{C} \rightarrow \mathbf{Set}$ .  $\text{Hom}_{\mathcal{C}}(-, -)$  is a functor  $\mathcal{C} \times \mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$ .
5. Functors  $\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{E}$  can be composed in the obvious sense.

**TODO: DRAW DIAGRAMS**

**Definition 1.9.** Let  $\mathcal{C} \begin{array}{c} \xrightarrow{F} \\ \xrightarrow{G} \end{array} \mathcal{D}$  be two functors. A *natural transformation*  $\eta$  from  $F$  to  $G$  is the data of morphisms  $\eta_X : F(X) \rightarrow G(X) \in \text{Mor}(\mathcal{D})$  for all  $X \in \text{Ob}(\mathcal{C})$  such that for all  $f : X \rightarrow Y \in \text{Mor}(\mathcal{C})$ , the diagram

$$\begin{array}{ccc} F(X) & \xrightarrow{\eta_X} & G(X) \\ \downarrow F(f) & & \downarrow G(f) \\ F(Y) & \xrightarrow{\eta_Y} & G(Y) \end{array}$$

commutes, that is  $G(f)\eta_X = \eta_Y F(f)$ . We write  $\eta : F \Rightarrow G$  or draw  $\mathcal{C} \begin{array}{c} \xrightarrow{F} \\ \Downarrow \eta \\ \xrightarrow{G} \end{array} \mathcal{D}$

**Example 1.10.** Let  $V$  be a  $k$ -vector space.  $\text{id}_{\mathbf{Vect}_k}$  and  $D^2 = \text{Hom}_{\mathbf{Vect}_k}(\text{Hom}_{\mathbf{Vect}_k}(-, k), k)$  are two endofunctors of  $\mathbf{Vect}_k$ .  $\text{ev}_- : V \rightarrow V^{**}$  defines a natural transformation

$$\begin{array}{ccccc} v & \mapsto & \text{Hom}(V, k) & \rightarrow & k \\ & & \phi & \mapsto & \phi(v) \end{array}$$

between them:

$$\begin{array}{ccc} V & \xrightarrow{\text{ev}} & V^{**} \\ f \downarrow & & \downarrow D^2(f) \\ W & \xrightarrow{\text{ev}} & W^{**} \end{array}$$

For  $a \in V$ ,  $D^2(f) \circ \text{ev}_a : W^* \rightarrow k \in W^{**}$  and in the other direction  $(\text{ev} \circ f)(a) = \text{ev}_{f(a)}$ .  
 $\phi \mapsto \phi(f(a))$

However, there is no natural transformation from  $\text{id}_{\mathbf{Vect}_k}$  to  $D$ . For one, the first is covariant and the second is contravariant. To get a natural transformation from a covariant to a contravariant

functor, we can modify the definition of naturality to be that  $\begin{array}{ccc} V & \rightarrow & V^* \\ \downarrow & & \uparrow \\ W & \rightarrow & W^* \end{array}$  commutes, but even such

natural transformations do not exist from  $\text{id}_{\mathbf{Vect}_k}$  to  $D$ .

**Definition 1.11.** A natural transformation  $\mathcal{C} \begin{array}{c} \xrightarrow{F} \\ \Downarrow \eta \\ \xrightarrow{G} \end{array} \mathcal{D}$  is a *natural isomorphism* if  $\eta_X$  is an isomorphism for all  $X \in \text{Ob}(\mathcal{C})$ .

*Remark.* One can compose natural transformations in two ways, “vertical composition”:

$$\mathcal{C} \begin{array}{c} \xrightarrow{F} \\ \Downarrow \alpha \\ \xrightarrow{G} \\ \Downarrow \beta \\ \xrightarrow{H} \end{array} \mathcal{D} \rightsquigarrow \mathcal{C} \begin{array}{c} \xrightarrow{F} \\ \Downarrow \beta \circ \alpha \\ \xrightarrow{H} \end{array} \mathcal{D} \quad \text{where } (\beta \circ \alpha)_X = \beta_X \circ \alpha_X$$

or “horizontal composition”:

$$\mathcal{C} \begin{array}{c} \xrightarrow{F_1} \\ \Downarrow \alpha_1 \\ \xrightarrow{G_1} \end{array} \mathcal{D} \begin{array}{c} \xrightarrow{F_2} \\ \Downarrow \alpha_2 \\ \xrightarrow{G_2} \end{array} \mathcal{E} \rightsquigarrow \mathcal{C} \begin{array}{c} \xrightarrow{F_2 \circ F_1} \\ \Downarrow \alpha_2 * \alpha_1 \\ \xrightarrow{G_2 \circ G_1} \end{array} \mathcal{E} \quad \text{where } (\alpha_2 * \alpha_1)_X = G_2((\alpha_1)_X) \circ (\alpha_2)_{F_1(X)}$$

Horizontal composition can also be defined in another equivalent way using commutativity of

$$\begin{array}{ccc} F_2 F_1(X) & \xrightarrow{(\alpha_2)_{F_1(X)}} & G_2 F_1(X) \\ F_2((\alpha_1)_X) \downarrow & & \downarrow G_2((\alpha_1)_X) \\ F_2 G_1(X) & \xrightarrow{(\alpha_2)_{G_1(X)}} & G_2 G_1(X) \end{array}$$

The diagram commutes by naturality of  $\alpha_2$ , so  $(\alpha_2 * \alpha_1) = (\alpha_2)_{G_1(X)} \circ F_2((\alpha_1)_X)$ .

**Definition 1.12.** Let  $\mathcal{C}, \mathcal{D}$  be two categories. Then the *functor category from  $\mathcal{C}$  to  $\mathcal{D}$*  written  $\text{Fun}(\mathcal{C}, \mathcal{D})$  or  $\mathcal{D}^{\mathcal{C}}$  is the category whose objects are functors from  $\mathcal{C}$  to  $\mathcal{D}$  and morphisms are natural transformations.

*Remark.* Categories, together with functors and natural transformations between them is the prototypical example of a 2-category.

## 1.2 Equivalences of categories

**Definition 1.13.** Let  $\mathcal{C}$  and  $\mathcal{D}$  be two categories. An *equivalence of categories* from  $\mathcal{C}$  to  $\mathcal{D}$  is the data of

1.  $F : \mathcal{C} \rightarrow \mathcal{D}$  and  $G : \mathcal{D} \rightarrow \mathcal{C}$  two functors
2. Natural isomorphisms  $\eta : 1_{\mathcal{C}} \Rightarrow GF$  and  $\varepsilon : FG \Rightarrow 1_{\mathcal{D}}$  where  $1_{\mathcal{C}}$  and  $1_{\mathcal{D}}$  are the identity functors of  $\mathcal{C}$  and  $\mathcal{D}$ .

*Remark.*

1.  $G$  is called a *quasi-inverse* of  $F$ .
2. Most of the time we say that  $F$  is an equivalence if there exists  $G$  such that  $(F, G)$  is an equivalence.

3. If  $F, G$  are contravariant, we speak of *duality* between  $\mathcal{C}$  and  $\mathcal{D}$ .
4. If two categories are equivalent, every property that can be expressed “in terms of arrows” is preserved.

**Definition 1.14.** Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be a functor. Then, we say

1.  $F$  is *faithful* if  $\forall X, Y \in \text{Ob}(\mathcal{C}), F : \text{Hom}_{\mathcal{C}}(X, Y) \rightarrow \text{Hom}_{\mathcal{D}}(F(X), F(Y))$  is injective.  

$$f \mapsto F(f)$$
2.  $F$  is *full* if the previous map is surjective.
3.  $F$  is *essentially surjective* if for all  $Y \in \text{Ob}(\mathcal{D})$  there is  $X \in \text{Ob}(\mathcal{C})$  such that  $F(X) \simeq Y$  in  $\mathcal{D}$ .

**Theorem 1.15.** Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be a functor. Then  $F$  is an equivalence of categories if and only if it is fully faithful and essentially surjective.

*Proof.* **▲** There is a little set-theoretic issue: an equivalence of categories is always fully faithful and essentially surjective, but the converse requires the axiom of choice for the class  $\text{Ob}(\mathcal{C})$ .

Suppose  $F : \mathcal{C} \rightarrow \mathcal{D}$  is an equivalence of categories, and let  $G : \mathcal{D} \rightarrow \mathcal{C}$  be a quasi-inverse of  $F$ , together with natural isomorphisms  $\eta : 1_{\mathcal{C}} \rightarrow GF$  and  $\varepsilon : 1_{\mathcal{D}} \rightarrow FG$ . If  $Y$  is an object of  $\mathcal{D}$ , then  $Y \simeq FG(Y)$ , so  $F$  is essentially surjective. Let  $X, Y$  be objects of  $\mathcal{C}$ . To show  $F$  is fully faithful we will construct an inverse to  $F : \text{Hom}_{\mathcal{C}}(X, Y) \rightarrow \text{Hom}_{\mathcal{D}}(F(X), F(Y))$ . For any  $f \in \text{Hom}_{\mathcal{D}}(F(X), F(Y))$ , we have a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & GF(X) \\ f \downarrow & & \downarrow GF(f) \\ Y & \xrightarrow{\eta_Y} & GF(Y) \end{array}$$

which prompts us to define  $\phi : \text{Hom}_{\mathcal{D}}(F(X), F(Y)) \rightarrow \text{Hom}_{\mathcal{C}}(X, Y)$ . We now check it is  

$$g \mapsto \eta_Y^{-1} \circ G(g) \circ \eta_X$$

the map we’re looking for. If  $f : X \rightarrow Y$ , since the above diagram commutes and  $\eta_Y$  is invertible, we get that  $\phi(F(f)) = f$ , so  $\phi \circ F = \text{id}_{\text{Hom}_{\mathcal{C}}(X, Y)}$ , which means  $F$  is faithful. We have two commutative diagrams, by definition of  $\phi$  and by naturality of  $\eta$ :

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & GF(X) \\ \phi(g) \downarrow & & \downarrow G(g) \\ Y & \xrightarrow{\eta_Y} & GF(Y) \end{array} \quad \begin{array}{ccc} X & \xrightarrow{\eta_X} & GF(X) \\ \phi(g) \downarrow & & \downarrow GF(\phi(g)) \\ Y & \xrightarrow{\eta_Y} & GF(Y) \end{array}$$

therefore,  $G(g) \circ \eta_X = GF(\phi(g)) \circ \eta_X$ . Since  $\eta_X$  is invertible,  $G(g) = GF(\phi(g))$ . The previous point shows that  $G$  is faithful, so  $g = F(\phi(g))$ , hence  $F$  is full.

Now suppose  $F$  is fully faithful and essentially surjective. Our goal is to construct  $G$ . For any  $Y \in \text{Ob}(\mathcal{D})$ , since  $F$  is essentially surjective, there exists  $X_Y \in \text{Ob}(\mathcal{C})$  and an isomorphism  $\varepsilon_Y : Y \rightarrow F(X_Y)$ . Therefore, for any  $Y, Z \in \text{Ob}(\mathcal{D})$  and  $f : Y \rightarrow Z$ , we have a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{f} & Z \\ \varepsilon_Y \downarrow & & \downarrow \varepsilon_Z \\ F(X_Y) & \xrightarrow[\varepsilon_Z \circ f \circ \varepsilon_Y^{-1}]{} & F(X_Z) \end{array}$$

Which leads us to define  $G(Y) = X_Y$  and  $G(f)$  to be the unique morphism  $m_f : X_Y \rightarrow X_Z$  such that  $F(m_f) = \varepsilon_Z \circ f \circ \varepsilon_Y^{-1}$  (this works because  $F$  is fully faithful). We have  $G(\text{id}_Y) = \text{id}_{X_Y}$  since  $\varepsilon_Y \circ \text{id}_Y \circ \varepsilon_Y^{-1} = \text{id}_Y$  and  $F(\text{id}_{X_Y}) = \text{id}_Y$ . The next diagram shows  $G(g \circ f) = G(g) \circ G(f)$ :

$$\begin{array}{ccccc}
 & & g \circ f & & \\
 & \curvearrowright & & \curvearrowright & \\
 W & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\
 \downarrow \varepsilon_W & & \downarrow \varepsilon_Y & & \downarrow \varepsilon_Z \\
 F(X_W) & \xrightarrow{F(m_f)} & F(X_Y) & \xrightarrow{F(m_g)} & F(X_Z) \\
 & \curvearrowright & & \curvearrowright & \\
 & F(m_g \circ m_f) = F(m_g) \circ F(m_f) & & & 
 \end{array}$$

By this construction,  $\varepsilon$  is a natural isomorphism  $\text{id}_{\mathcal{D}} \Rightarrow FG$  (look at the above diagrams). Now, pick  $Y, Z \in \text{Ob}(\mathcal{C})$  and  $f : Y \rightarrow Z$ . We have  $GF(Y) = X_{F(Y)}$  and  $\varepsilon_Y : F(Y) \xrightarrow{\sim} F(X_{F(Y)})$ . Since  $F$  is fully faithful, there exists a unique  $\eta_Y : Y \rightarrow X_{F(Y)} = GF(Y)$  such that  $F(\eta_Y) = \varepsilon_Y$ . Here,  $\eta_Y = G(\varepsilon_Y)$ , which means that  $\eta_Y$  is an isomorphism since functors preserve isomorphisms. We obtain the diagram

$$\begin{array}{ccc}
 Y & \xrightarrow{\eta_Y} & GF(Y) \\
 \downarrow f & & \downarrow GF(f) \\
 Z & \xrightarrow{\eta_Z} & GF(Z)
 \end{array}$$

The diagram commutes because  $GF(f)$  is the unique morphism such that

$$F(GF(f)) = \varepsilon_Z \circ F(f) \circ \varepsilon_Y^{-1} = F(\eta_Z \circ f \circ \eta_Y^{-1})$$

and  $F$  is faithful.  $\eta$  is then a natural isomorphism  $\text{id}_{\mathcal{C}} \Rightarrow GF$ . □

**Example 1.16.**  $\mathbf{Vect}_k \simeq \mathbf{Mat}_k$  through the functor  $n \mapsto k^n$  and  $(A : n \rightarrow m) \mapsto (f_A : k^n \rightarrow k^m)$ .

## 2 Universal properties

References:

- Riehl (Chapters II, III, IV)
- Mac Lane (Chapters III, IV, V)
- Assem (Chapters III, IV, V)

► Let  $S$  be a set together with an equivalence relation  $\sim$ . Let  $S/\sim$  be the quotient set, and  $\pi : S \rightarrow S/\sim$  be the projection. For any  $f : S \rightarrow X$  compatible with  $\sim$ , there exists a unique map  $\bar{f} : S/\sim \rightarrow X$  such that  $f = \bar{f} \circ \pi$ . This is represented by the following commutative diagram :

$$\begin{array}{ccc} S & \xrightarrow{f} & X \\ \pi \downarrow & \nearrow \exists! \bar{f} & \\ S/\sim & & \end{array}$$

We say that  $S \xrightarrow{\pi} S/\sim$  is a solution to the universal problem posed by the compatible maps. Such a solution is unique up to unique isomorphism: if  $S \xrightarrow{p} S'$  is another solution, then we get the three commutative diagrams

$$\begin{array}{ccc} S & \xrightarrow{p} & S' \\ \pi \downarrow & \nearrow \exists! a & \\ S/\sim & & \end{array} \quad \begin{array}{ccc} S & \xrightarrow{\pi} & S/\sim \\ p \downarrow & \nearrow \exists! b & \\ S' & & \end{array} \quad \begin{array}{ccc} S & \xrightarrow{p} & S' \\ p \downarrow & \nearrow \text{id}_{S'} & \uparrow a \\ S' & \xrightarrow{b} & S/\sim \end{array}$$

then  $abp = a\pi = p$ . The identity of  $S'$  also makes this diagram commute so by uniqueness  $ab = \text{id}_{S'}$  and similarly  $ba = \text{id}_{S/\sim}$ .

### 2.1 Initial and final objects

**Definition 2.1.** Let  $\mathcal{C}$  be a category. An object  $c \in \text{Ob}(\mathcal{C})$  is *initial* (*final*) if for all  $d \in \text{Ob}(\mathcal{C})$  there exists a unique morphism  $c \rightarrow d$  (a unique morphism  $d \rightarrow c$ ).

**Proposition 2.2.** *If an initial/final object exists, then it is unique up to unique isomorphism.*

*Proof.* Let  $c, c'$  be two initial objects. Then there exists a unique morphism  $f : c \rightarrow c'$  and a unique morphism  $g : c' \rightarrow c$ . There also exists a unique morphism  $c \rightarrow c$ , that is  $\text{id}_c$ . Therefore,  $gf = \text{id}_c$ . In the same way,  $fg = \text{id}_{c'}$ . Therefore,  $c$  and  $c'$  are isomorphic and the isomorphism is unique.  $\square$

**Examples 2.3.**

1.  $\emptyset$  is initial in **Set** and any singleton is final.
2.  $\{0\}$  is both initial and final in **Vect<sub>k</sub>** (or **RMod**).
3. The category of fields does not have initial/final objects (reason on field characteristics).

We want to say a universal object is an initial or final object. A category has at most 2, so this may seem a little restrictive, but this is solved by thinking of a good category.



**Definition 2.4.** Let  $F : \mathcal{C} \rightarrow \mathbf{Set}$  be a functor. Let  $\int F$  be the category defined by

$$\begin{aligned}\mathrm{Ob}(\int F) &= \{(c, x) \mid c \in \mathrm{Ob}(\mathcal{C}) \text{ and } x \in F(c)\} \\ \mathrm{Hom}((c, x), (c', x')) &= \{f \in \mathrm{Hom}(c, c') \mid F(f)(x) = x'\}\end{aligned}$$

where composition is composition in  $\mathcal{C}$ , and  $\mathrm{id}_{(c, x)} = \mathrm{id}_c$  for all  $x$ . If  $F$  is contravariant, let  $\int F$  have the same objects and morphisms  $\mathrm{Hom}((c, x), (c', x')) = \{f \in \mathrm{Hom}(c, c') \mid F(f)(x') = x\}$ .

**Proposition 2.5.** *There is a forgetful functor  $\pi : \int F \rightarrow \mathcal{C}$  defined by  $\pi(c, x) = c$  and  $\pi(f : (c, x) \rightarrow (c', x')) = f : c \rightarrow c'$ .*

**Example 2.6.** Let  $S$  be a set, and  $\sim$  an equivalence relation on  $S$ . Let  $F : \mathbf{Set} \rightarrow \mathbf{Set}$  be defined by  $F(X) = \{f : S \rightarrow X \mid x \sim y \Rightarrow f(x) = f(y)\}$  and  $F(\alpha : X \rightarrow Y) = \alpha \circ -$ .

$\int F$  has for objects  $(X, S \xrightarrow{f} X)$  where  $f$  is compatible with  $\sim$ , and for morphisms  $\alpha$  that makes

this diagram commute:

$$\begin{array}{ccc} S & \xrightarrow{f'} & X' \\ f \downarrow & \nearrow \alpha & \\ X & & \end{array}$$

$(S/\sim, S \xrightarrow{\pi} S/\sim)$  is an initial object of  $\int F$ .

**Definition 2.7.** Let  $F : \mathcal{C} \rightarrow \mathbf{Set}$  be a functor. A *universal element* for  $F$  is an initial object of  $\int F$ , that is a pair  $(c, x)$  with  $c \in \mathrm{Ob}(\mathcal{C})$  and  $x \in F(c)$  such that

$$\forall (d, y), d \in \mathrm{Ob}(\mathcal{C}), y \in F(d), \exists! \alpha : c \rightarrow d, y = F(\alpha)(x)$$

**Definition 2.8.** Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be a functor and  $d \in \mathrm{Ob}(\mathcal{D})$ . A *universal arrow from  $d$  to  $F$*  is a pair  $(c, f)$  where  $c \in \mathrm{Ob}(\mathcal{C})$  and  $f \in \mathrm{Hom}_{\mathcal{D}}(d, F(c))$ , such that

$$\forall (c', f'), c' \in \mathrm{Ob}(\mathcal{C}), f' : d \rightarrow F(c'), \exists! \alpha \in \mathrm{Hom}_{\mathcal{C}}(c, c'), F(\alpha) \circ f = f'$$

$$\begin{array}{ccc} & d & \\ f \swarrow & & \searrow \forall f' \\ F(c) & \xrightarrow{F(\alpha)} & F(c') \\ c & \xrightarrow{\exists! \alpha} & c' \end{array}$$

*Exercise.* Define a category  $d \downarrow F$  such that a universal arrow is an initial object of  $d \downarrow F$ .

**Example 2.9.** Let  $U : \mathbf{Vect}_k \rightarrow \mathbf{Set}$  be the forgetful functor. Let  $X \in \mathbf{Set}$ . A universal arrow from  $X$  to  $U$  is the “best”  $k$ -vector space  $V_X$  with a map  $X \rightarrow V_X$ . Set  $V_X = k[X]$  the  $k$ -vector space with basis  $X$ , and  $i : X \rightarrow V_X$  that maps  $x \in X$  to the corresponding basis element. Then, for any vector space  $V$  and map  $f : X \rightarrow U(V)$ ,  $f$  can be extended by linearity into a linear map  $\tilde{f} : k[X] \rightarrow V$ , which makes this diagram commute:

$$\begin{array}{ccc} & X & \\ i \swarrow & & \searrow f \\ k[X] & \xrightarrow{\tilde{f}} & U(V) \end{array}$$

If  $\alpha$  is another map that makes the diagram commute then  $\alpha$  and  $\tilde{f}$  coincide on a basis of  $k[X]$  and therefore are equal.

**Proposition 2.10.** *Universal elements and arrows are two equivalent notions.*

*Proof.* Let  $F : \mathcal{C} \rightarrow \mathbf{Set}$  be a functor and  $(c, x)$  a universal element for  $F$ . Consider  $f_x : \{*\} \rightarrow F(c)$ . Then,  $(c, f_x)$  is a universal arrow  $* \rightarrow F$ , because  $F(\alpha)(x) = y$  iff  $F(\alpha) \circ f_x = f_y$ .

$$\begin{array}{ccc} & \{*\} & \\ f_x \swarrow & & \searrow f_y \\ F(c) & \xrightarrow{F(\alpha)} & F(c') \end{array}$$

Conversely, if  $F : \mathcal{C} \rightarrow \mathcal{D}$  is a functor and  $(c, f)$  is a universal arrow  $d \rightarrow F$ , then consider the functor  $\text{Hom}_{\mathcal{D}}(d, F(-)) : \mathcal{C} \rightarrow \mathbf{Set}$  (we need to assume  $\mathcal{D}$  is locally small so the functor is set-valued). Then,  $f \in \text{Hom}_{\mathcal{D}}(d, F(c))$  is a universal element for this functor.  $\square$

## 2.2 Representable functors

**Definition 2.11.** Let  $\mathcal{C}$  be a (locally small) category, and  $F : \mathcal{C} \rightarrow \mathbf{Set}$  a functor.

1. We say that  $F$  is *representable* if there is some  $c \in \text{Ob}(\mathcal{C})$  such that  $F$  and  $\text{Hom}_{\mathcal{C}}(c, -)$  are naturally isomorphic (if  $F$  is contravariant, use  $\text{Hom}_{\mathcal{C}}(-, c)$  instead).
2. A *representation* of  $F$  is the data of  $c \in \text{Ob}(\mathcal{C})$  and a natural isomorphism  $\eta : \text{Hom}(c, -) \Rightarrow F$ .

**Example 2.12.** The forgetful functor  $U : \mathbf{Grp} \rightarrow \mathbf{Set}$  is representable since  $\text{Hom}_{\mathbf{Grp}}(\mathbb{Z}, -) \simeq U$ . The natural isomorphism is given by  $\alpha \in \text{Hom}_{\mathbf{Grp}}(\mathbb{Z}, G) \mapsto \alpha(1) \in G$ .

The following theorem explains how to find the natural isomorphism  $\alpha : \text{Hom}_{\mathcal{C}}(c, -) \Rightarrow F$  in general.

**Theorem 2.13** (Yoneda lemma). *Let  $F : \mathcal{C} \rightarrow \mathbf{Set}$  be a functor with  $\mathcal{C}$  locally small, and  $c \in \text{Ob}(\mathcal{C})$ . Then,*

$$\begin{array}{ccc} \text{Nat}(\text{Hom}(c, -), F) & \xrightarrow{\sim} & F(c) \\ \alpha & \mapsto & \alpha_c(\text{id}_c) \end{array}$$

and this isomorphism is natural in  $c$  and in  $F$ .

*Proof.* Let  $\alpha \in \text{Nat}(\text{Hom}(c, -), F)$ . Let  $d \in \mathcal{C}$  and  $f : c \rightarrow d$ . By naturality, we have a commutative diagram

$$\begin{array}{ccc} \text{Hom}(c, c) & \xrightarrow{\alpha_c} & F(c) \\ \downarrow f \circ - & & \downarrow F(f) \\ \text{Hom}(c, d) & \xrightarrow{\alpha_d} & F(d) \end{array}$$

This means that  $F(f) \circ \alpha_c = \alpha_d \circ (f \circ -)$ . Evaluating at  $\text{id}_c$ , we get  $F(f) \circ \alpha_c(\text{id}_c) = \alpha_d(f)$ . This shows that the natural transformation  $\alpha$  is entirely determined by the value of  $\alpha_c(\text{id}_c)$ , which shows the map defined above is injective. Conversely, if  $e \in F(c)$ , then we define  $\alpha^e : \text{Hom}(c, -) \Rightarrow F$  by  $\alpha_d^e : g \mapsto F(g)(e)$ . We check it is a natural transformation:

$$\begin{array}{ccc}
\mathrm{Hom}(c, c) & \xrightarrow{g \mapsto F(g)(e)} & F(c) \\
\downarrow f \circ - & & \downarrow F(f) \\
\mathrm{Hom}(c, d) & \xrightarrow{h \mapsto F(h)(e)} & F(d)
\end{array}$$

and this diagram commutes since for  $g : c \rightarrow c$  we have

$$F(f)(F(g)(e)) = F(f \circ g)(e) = F((f \circ -)(g))(e)$$

This shows the map in the theorem is surjective, therefore an isomorphism, and its inverse is given by  $e \in F(c) \mapsto \alpha^e$ . We now check naturality. We first need to understand what it means to say the isomorphism is natural in  $c$ . Let  $f : c \rightarrow d$ .  $\mathrm{Nat}(\mathrm{Hom}(c, -), F)$  is functorial in  $c$ , as it is the composition of two contravariant hom-functors. More concretely:

$$c \xrightarrow{f} d \rightsquigarrow \mathrm{Hom}(d, -) \xrightarrow{- \circ f} \mathrm{Hom}(c, -) \rightsquigarrow \mathrm{Nat}(\mathrm{Hom}(c, -), F) \xrightarrow{- \circ (- \circ f)} \mathrm{Nat}(\mathrm{Hom}(d, -), F)$$

( $\mathrm{Nat}$  is the hom-functor of the functor category  $\mathcal{C}^{\mathbf{Set}}$ ). One thing to note is that the morphism  $f : c \rightarrow d$  induces a natural transformation  $\mathrm{Hom}(d, -) \xrightarrow{- \circ f} \mathrm{Hom}(c, -)$ , and this makes the whole thing work. This is in general a property of functors defined on a product category, see the remark below. Now, saying the isomorphism, which we'll write  $\Phi_{d,F}$ , is natural means that the square

$$\begin{array}{ccc}
\mathrm{Nat}(\mathrm{Hom}(c, -), F) & \xrightarrow{\Phi_{c,F}} & F(c) \\
\downarrow - \circ (- \circ f) & & \downarrow F(f) \\
\mathrm{Nat}(\mathrm{Hom}(d, -), F) & \xrightarrow{\Phi_{d,F}} & F(d)
\end{array}$$

commutes. And indeed, if  $\alpha : \mathrm{Hom}(c, -) \Rightarrow F$  is a natural transformation,

$$\begin{aligned}
\Phi_{d,F}(\alpha \circ (- \circ f)) &= (\alpha \circ (- \circ f))_d(\mathrm{id}_d) = [\alpha_d \circ (- \circ f)](\mathrm{id}_d) = \alpha_d(f) \\
F(f)(\Phi_{c,F}(\alpha)) &= F(f)(\alpha_c(\mathrm{id}_c)) = \alpha_d(f \circ \mathrm{id}_c) = \alpha_d(f)
\end{aligned}$$

The second to last equality comes from the naturality of  $\alpha$ .

We now turn to naturality in  $F$ . Let  $G$  be another functor  $\mathcal{C} \rightarrow \mathbf{Set}$  and  $\beta : F \Rightarrow G$  be a natural transformation. We check that

$$\begin{array}{ccc}
\mathrm{Nat}(\mathrm{Hom}(c, -), F) & \xrightarrow{\Phi_{c,F}} & F(c) \\
\downarrow \beta \circ - & & \downarrow \beta_c \\
\mathrm{Nat}(\mathrm{Hom}(c, -), G) & \xrightarrow{\Phi_{c,G}} & G(c)
\end{array}$$

commutes. For  $\alpha : \mathrm{Hom}(c, -) \Rightarrow F$ , we have

$$\beta_c(\Phi_{c,F}(\alpha)) = \beta_c(\alpha_c(\mathrm{id}_c)) = (\beta \circ \alpha)_c(\mathrm{id}_c) = \Phi_{c,G}(\beta \circ \alpha)$$

which completes the proof of naturality. □

*Remark.*

1. If  $F : \mathcal{C} \rightarrow \mathbf{Set}$ , then  $(c, x)$  is a universal element for  $F$  if and only if the natural transformation  $\alpha_x : \text{Hom}(c, -) \Rightarrow F$  induced by  $x$  is an isomorphism. Indeed,  $\alpha_x$  is an isomorphism iff  $\forall c' \in \mathcal{C}$ ,  $(\alpha_x)_{c'} : \text{Hom}(c, c') \rightarrow F(c')$  is bijective iff
 
$$f \mapsto F(f)(x)$$

$$\forall c' \in \mathcal{C}, \forall y \in F(c'), \exists ! f : c \rightarrow c', F(f)(x) = y$$

2. For universal arrows, use  $\text{Hom}_{\mathcal{D}}(d, F(-))$  as before.
3. Let  $\mathcal{C}, \mathcal{D}$  and  $\mathcal{E}$  be categories, and  $F : \mathcal{C} \times \mathcal{D} \rightarrow \mathcal{E}$  be a functor. Let  $c, d \in \text{Ob}(\mathcal{C})$ ,  $x, y \in \text{Ob}(\mathcal{D})$  and morphisms  $f : c \rightarrow d$ ,  $g : x \rightarrow y$ . The morphism  $f$  induces a natural transformation  $F(f, \text{id}_-) : F(c, -) \Rightarrow F(d, -)$ , see the commutative square:

$$\begin{array}{ccc} F(c, x) & \xrightarrow{F(f, \text{id}_x)} & F(d, x) \\ \downarrow F(\text{id}_c, g) & & \downarrow F(\text{id}_d, g) \\ F(c, y) & \xrightarrow{F(f, \text{id}_y)} & F(d, y) \end{array}$$

## 2.3 Examples of objects defined by universal properties

### 2.3.1 Products, coproducts

Let  $\mathcal{C}$  be a small category and  $X, Y \in \text{Ob}(\mathcal{C})$ . We define a category  $\mathcal{C}_{X,Y}$  whose objects are tuples  $(Z, f, g)$  where  $Z \in \text{Ob}(\mathcal{C})$  and  $f : Z \rightarrow X$ ,  $g : Z \rightarrow Y$  and morphisms are maps  $\alpha : Z \rightarrow Z'$  in  $\mathcal{C}$  such that the following diagram commutes:

$$\begin{array}{ccccc} & & Z & & \\ & f \swarrow & \downarrow \alpha & \searrow g & \\ X & & & & Y \\ & f' \swarrow & \downarrow & \searrow g' & \\ & & Z' & & \end{array}$$

**Definition 2.14.** A *product* of  $X$  and  $Y$  is a final object in  $\mathcal{C}_{X,Y}$ . Concretely, it is an object  $X \times Y$  together with two maps  $\pi_X : X \times Y \rightarrow X$  and  $\pi_Y : X \times Y \rightarrow Y$  such that for any  $(Z, f, g) \in \text{Ob}(\mathcal{C}_{X,Y})$ , we have a commutative diagram

$$\begin{array}{ccccc} & & Z & & \\ & \swarrow \forall f & \downarrow \exists ! \alpha & \searrow \forall g & \\ X & & X \times Y & & Y \\ & \xleftarrow{\pi_X} & & \xrightarrow{\pi_Y} & \end{array}$$

Since it is defined as being a final object, if it exists, a product is unique up to unique isomorphism.

**Examples 2.15.** In **Set**, the product of  $X$  and  $Y$  is the cartesian product. In **Grp**, it is the product group. In **Top**, it is the cartesian product equipped with the product topology. In these examples, the maps in the definition are the canonical projections.

The notion dual to the one of a product is called a coproduct.

**Definition 2.16.** A *coproduct* of  $X$  and  $Y$  is a product in  $\mathcal{C}^{\text{op}}$ . Concretely, it satisfies the universal property expressed by this commutative diagram:

$$\begin{array}{ccccc} X & \xrightarrow{i_X} & X \sqcup Y & \xleftarrow{i_Y} & Y \\ & \searrow \forall f & \downarrow \exists! \alpha & \swarrow \forall g & \\ & & Z & & \end{array}$$

**Examples 2.17.** In **Set**, the coproduct of  $X$  and  $Y$  is the disjoint union together with canonical inclusion. In **Top**, the coproduct of  $X$  and  $Y$  is their disjoint union equipped with the disjoint union topology. However, in **Grp**, the underlying set of the coproduct of two groups is not the disjoint union.

### 2.3.2 Equalizers and coequalizers

**Definition 2.18.** Let  $\mathcal{C}$  be a category and  $X, Y \in \text{Ob}(\mathcal{C})$ ,  $f, g : X \rightarrow Y$ . Consider the contravariant functor  $F : \mathcal{C} \rightarrow \mathbf{Set}$  defined by  $F(c) = \{\alpha : c \rightarrow X \mid f\alpha = g\alpha\}$  and  $F(\beta) = - \circ \beta$ . An *equalizer* in  $\mathcal{C}$  is a representation of the contravariant functor  $F$ .

By the Yoneda lemma, a natural transformation  $\text{Hom}(-, c) \Rightarrow F$  is the same as an element of  $F(c)$ , so a representation of  $F$  is a pair  $(c, e)$  with  $c \in \text{Ob}(\mathcal{C})$  and  $e \in F(c)$  such that the natural transformation given by the Yoneda lemma is an isomorphism. Concretely, we want  $\eta_e : \text{Hom}(d, c) \rightarrow F(d)$  to be an isomorphism for all  $d \in \text{Ob}(\mathcal{C})$ . This translates into

$$\begin{array}{ccc} & & d \\ & \swarrow \exists! \alpha & \downarrow \forall h \\ c & \xrightarrow{e} & X \end{array} \quad \begin{array}{ccc} & & Y \\ & \xleftarrow{f} & \\ & \xrightarrow{g} & \end{array}$$

$h \mapsto F(h)(e)$

the following diagram:

$$\begin{array}{ccc} & d & \\ \exists! \alpha \swarrow & \downarrow \forall h & \\ c & \xrightarrow{e} & X \end{array} \quad \begin{array}{ccc} & & Y \\ & \xleftarrow{f} & \\ & \xrightarrow{g} & \end{array}$$

**Example 2.19.** In **Set**,  $E = \{x \in X \mid f(x) = g(x)\} \hookrightarrow X$  is an equalizer.

The dual notion is that of a coequalizer.

**Definition 2.20.** A *coequalizer* of  $X \rightrightarrows Y$  is an object  $Z \in \text{Ob}(\mathcal{C})$  together with a morphism  $\pi : Y \rightarrow Z$  such that  $\pi f = \pi g$  and that universal to this property:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow g & \downarrow \forall h \\ & & Z' \end{array} \quad \begin{array}{ccc} & & Z \\ & \xleftarrow{\pi} & \\ & \swarrow \exists! \alpha & \end{array}$$

**Example 2.21.** In **Set**, consider the equivalence relation  $\sim$  on  $Y$  generated by  $f(x) \sim g(x)$  (the smallest equivalence relation on  $Y$  with this property). Then  $y \xrightarrow{\pi} Y/\sim$  is a coequalizer.

## 2.4 Adjoint functors

This notion was introduced by Kan in 1958.

**Definition 2.22.** An *adjunction*  $(G, D)$  is a pair of functors  $G : \mathcal{C} \rightarrow \mathcal{D}$  and  $D : \mathcal{D} \rightarrow \mathcal{C}$  together with an isomorphism  $\text{Hom}_{\mathcal{D}}(G(c), d) \simeq \text{Hom}_{\mathcal{C}}(c, D(d))$  which is natural in both  $c$  and  $d$ . We write  $G \dashv D$  and say  $G$  is left adjoint to  $D$  and  $D$  is right adjoint to  $G$ .

If  $G \dashv D$  we have  $\forall c, d \in \text{Ob}(\mathcal{C}) \times \text{Ob}(\mathcal{D})$ ,

$$\text{Hom}_{\mathcal{D}}(G(c), d) \xrightarrow[\alpha_{c,d}]{\sim} \text{Hom}_{\mathcal{C}}(c, D(d))$$

and in particular when  $d = G(c)$  we get  $\text{Hom}_{\mathcal{D}}(G(c), G(c)) \xrightarrow[\alpha_{c,G(c)}]{\sim} \text{Hom}_{\mathcal{C}}(c, DG(c))$ .

Let  $\eta_c : c \rightarrow DG(c)$  be the image of  $\text{id}_{G(c)}$ . This gives a collection of morphisms  $- \rightarrow DG(-)$ .

**Proposition 2.23.** *These morphisms make up a natural transformation  $\text{id}_{\mathcal{C}} \Rightarrow DG$ .*

*Proof.* Let  $f : c \rightarrow d$ . We want to show that

$$\begin{array}{ccc} c & \xrightarrow{\eta_c = \alpha_{c,G(c)}(\text{id}_{G(c)})} & DG(c) \\ \downarrow f & & \downarrow DG(f) \\ d & \xrightarrow{\eta_d = \alpha_{d,G(d)}(\text{id}_{G(d)})} & DG(d) \end{array}$$

commutes. By naturality of the isomorphism  $\alpha$  given by the adjunction, we get the following commutative diagram

$$\begin{array}{ccc} \text{Hom}(G(c), G(c)) & \xrightarrow[\alpha_{c,G(c)}]{\sim} & \text{Hom}(c, DG(c)) \\ G(f) \circ - \downarrow & & DG(f) \circ - \downarrow \\ \text{Hom}(G(c), G(d)) & \xrightarrow[\alpha_{c,G(d)}]{\sim} & \text{Hom}(c, DG(d)) \\ - \circ G(f) \uparrow & & - \circ f \uparrow \\ \text{Hom}(G(d), G(d)) & \xrightarrow[\alpha_{d,G(d)}]{\sim} & \text{Hom}(d, DG(d)) \end{array}$$

which gives us these equations:

$$\begin{aligned} DG(f) \circ \eta_c &= DG(f) \circ \alpha_{c,G(c)}(\text{id}_{G(c)}) = \alpha_{c,G(d)}(G(f) \circ \text{id}_{G(c)}) = \alpha_{c,G(d)}(G(f)) \\ \eta_d \circ f &= \alpha_{d,G(d)}(\text{id}_{G(d)}) \circ f = \alpha_{c,G(d)}(\text{id}_{G(c)} \circ G(f)) = \alpha_{c,G(d)}(G(f)) \end{aligned}$$

which completes the proof.  $\square$

We also get a natural transformation  $\varepsilon : GD \Rightarrow \text{id}_{\mathcal{D}}$  when  $c = D(d)$  by setting  $\varepsilon_d = \alpha_{D(d),d}^{-1}(\text{id}_{D(d)})$ .

**Definition 2.24.** The natural transformation  $\eta : \text{id}_{\mathcal{C}} \Rightarrow DG$  is called the *unit* of the adjunction. The natural transformation  $\varepsilon : GD \Rightarrow \text{id}_{\mathcal{D}}$  is called its *counit*.

**Proposition 2.25.** *Let  $\mathcal{C} \begin{array}{c} \xrightarrow{G} \\ \xleftarrow{D} \end{array} \mathcal{D}$  be two functors. Then,  $G \dashv D$  if and only if there are natural transformations  $\eta : \text{id}_{\mathcal{C}} \Rightarrow DG$  and  $\varepsilon : GD \Rightarrow \text{id}_{\mathcal{D}}$  such that the following diagrams commute:*

$$\begin{array}{ccc}
G & \xrightarrow{G\eta} & GDG \\
& \searrow \text{id}_G & \downarrow \varepsilon G \\
& & G
\end{array}
\qquad
\begin{array}{ccc}
D & \xrightarrow{\eta D} & DGD \\
& \searrow \text{id}_D & \downarrow D\varepsilon \\
& & D
\end{array}$$

where  $G\eta$  is the natural transformation given by the morphisms  $G(\eta_c)$  and  $\varepsilon G$  is the one give by morphisms  $\varepsilon_{G(c)}$  (and similarly for  $\eta D$  and  $D\varepsilon$ ).

*Proof.* Suppose  $G \dashv D$ . Let  $\eta : \text{id}_C \Rightarrow DG$  and  $\varepsilon : GD \Rightarrow \text{id}_D$  be the unit and counit of the adjunction. Let  $c \in \mathcal{C}$ . We have

$$(\varepsilon G)_c \circ (G\eta)_c = \varepsilon_{G(c)} \circ G(\eta_c) = \alpha_{DG(c), G(c)}^{-1}(\text{id}_{DG(c)}) \circ G(\alpha_{c, G(c)}(\text{id}_{G(c)}))$$

and the naturality of  $\alpha$  gives the following commutative diagram

$$\begin{array}{ccc}
\text{Hom}(G(c), G(c)) & \xleftarrow[\alpha_{c, G(c)}^{-1}]{\sim} & \text{Hom}(c, DG(c)) \\
\uparrow - \circ G(\alpha_{c, G(c)}(\text{id}_{G(c)})) & & \uparrow - \circ \alpha_{c, G(c)}(\text{id}_{G(c)}) \\
\text{Hom}(GDG(c), G(c)) & \xleftarrow[\alpha_{DG(c), G(c)}^{-1}]{\sim} & \text{Hom}(DG(c), DG(c))
\end{array}$$

which shows that  $(\varepsilon G)_c \circ (G\eta)_c = \text{id}_{G(c)}$ , hence  $\varepsilon G \circ G\eta = \text{id}_G$ . The commutativity of the other triangle is treated in a similar way.

Now assume that there are natural transformations  $\eta$  and  $\varepsilon$  that make both triangles commute. We define two maps

$$\begin{aligned}
\alpha_{c,d} : \text{Hom}(G(c), d) &\rightarrow \text{Hom}(c, D(d)) \\
f &\mapsto D(f) \circ \eta_c \\
\beta_{c,d} : \text{Hom}(c, D(d)) &\rightarrow \text{Hom}(G(c), d) \\
g &\mapsto \varepsilon_d \circ G(g)
\end{aligned}$$

and we show these are natural isomorphisms that give us the adjunction. First we check naturality of  $\alpha$ . Let  $f : c \rightarrow c' \in \text{Mor}(\mathcal{C})$  and  $g : d \rightarrow d' \in \text{Mor}(\mathcal{D})$ . We need to check that the diagrams

$$\begin{array}{ccc}
\text{Hom}_{\mathcal{D}}(G(c), d) & \xrightarrow{\alpha_{c,d}} & \text{Hom}_{\mathcal{C}}(c, D(d)) \\
\uparrow - \circ G(f) & & \uparrow - \circ f \\
\text{Hom}_{\mathcal{D}}(G(c'), d) & \xrightarrow{\alpha_{c',d}} & \text{Hom}_{\mathcal{C}}(c', D(d))
\end{array}
\qquad
\begin{array}{ccc}
\text{Hom}_{\mathcal{D}}(G(c), d) & \xrightarrow{\alpha_{c,d}} & \text{Hom}_{\mathcal{C}}(c, D(d)) \\
\downarrow g \circ - & & \downarrow D(g) \circ - \\
\text{Hom}_{\mathcal{D}}(G(c), d') & \xrightarrow{\alpha_{c,d'}} & \text{Hom}_{\mathcal{C}}(c, D(d'))
\end{array}$$

commute. We only check the left diagram and leave the right to the reader (sorry). We have

$$\begin{aligned}
\alpha_{c,d} \circ (- \circ G(f)) &= (D(-) \circ \eta_c) \circ (- \circ G(f)) = D(- \circ G(f)) \circ \eta_c = D(-) \circ DG(f) \circ \eta_c \\
(- \circ f) \circ \alpha_{c',d} &= (- \circ f) \circ (D(-) \circ \eta_{c'}) = D(-) \circ \eta_{c'} \circ f = D(-) \circ DG(f) \circ \eta_c
\end{aligned}$$

One shows  $\beta$  is natural in  $c$  and  $d$  in a similar way. We leave it to the reader (sorry again). Now we need to check that  $\alpha$  and  $\beta$  are inverses of each other, and that's where the triangle diagrams come into play.

$$\alpha_{c,d} \circ \beta_{c,d} = D(\varepsilon_d \circ G(-)) \circ \eta_c = D(\varepsilon_d) \circ DG(-) \circ \eta_c = D(\varepsilon_d) \circ \eta_{D(d)} \circ - = -$$

We used the definitions of  $\alpha$  and  $\beta$ , the functoriality of  $D$ , the naturality of  $\eta$  and the second triangle diagram. We leave to the reader (sorry) to check that  $\beta_{c,d} \circ \alpha_{c,d}$  is also the identity.  $\square$

**Examples 2.26.**

1. The forgetful functor  $\mathbf{Ab} \rightarrow \mathbf{Set}$  is right adjoint to the free abelian group functor  $\mathbf{Set} \rightarrow \mathbf{Ab}$ .
2. The forgetful functor  $\mathbf{Ab} \rightarrow \mathbf{Grp}$  is right adjoint to the abelianization functor  $\mathbf{Grp} \rightarrow \mathbf{Ab}$  that sends a group  $G$  to its abelianization  $G^{ab} = G/[G, G]$  and a morphism  $f : G \rightarrow H$  to the induced morphism  $f^{ab} : G^{ab} \rightarrow H^{ab}$ .
3. The forgetful functor  $\mathbf{Top} \rightarrow \mathbf{Set}$  is right adjoint to the functor  $\mathbf{Set} \rightarrow \mathbf{Top}$  that takes a set and equips it with the coarse topology. The forgetful functor  $\mathbf{Top} \rightarrow \mathbf{Set}$  is also left adjoint to the functor  $\mathbf{Set} \rightarrow \mathbf{Top}$  that equips a set with the discrete topology.
4. Let  $G$  be a group,  $H$  one of its subgroups and  $k$  be a field. We have a functor from the category  $\mathbf{Rep}_k(G)$  of representations of  $G$  on  $k$ -vector spaces to the category  $\mathbf{Rep}_k(H)$  of representations of  $H$  on  $k$ -vector spaces. It is the restriction functor  $\text{Res}_H^G$ . Its left adjoint is  $\text{Ind}_H^G$ , the induced representation functor.

**Theorem 2.27.** *Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be a functor. The following are equivalent:*

1.  $F$  admits a left adjoint
2. For all  $X \in \text{Ob}(\mathcal{D})$ ,  $\text{Hom}_{\mathcal{D}}(X, F(-))$  is representable
3. For all  $X \in \text{Ob}(\mathcal{D})$ , there exists a universal arrow  $X \rightarrow F$

**Corollary 2.28.** *If they exist, adjoints are unique up to isomorphism.*

*Proof.*  $2 \iff 3$  was the subject of a previous remark right after the Yoneda lemma. We prove  $1 \iff 2$ . Suppose  $F$  admits a left adjoint  $G$ . Let  $X \in \text{Ob}(\mathcal{D})$ . Then for all  $Y \in \text{Ob}(\mathcal{C})$  we have a bijection  $\text{Hom}_{\mathcal{D}}(X, F(Y)) \simeq \text{Hom}_{\mathcal{C}}(G(X), Y)$  which is natural in  $Y$ , so  $G(X)$  represents  $\text{Hom}_{\mathcal{D}}(X, F(-))$ . For the converse, suppose all functors  $\text{Hom}_{\mathcal{D}}(X, F(-))$  are representable. We define  $G(X)$  to be an object of  $\mathcal{C}$  that represents  $\text{Hom}_{\mathcal{D}}(X, F(-))$ . Now choose  $X, Y \in \text{Ob}(\mathcal{D})$  and  $f : X \rightarrow Y$ . We need to define  $G(f)$ . We wish to have a commuting square

$$\begin{array}{ccc} \text{Hom}(G(X), -) & \xrightarrow{\sim} & \text{Hom}(X, F(-)) \\ \exists! \gamma \uparrow & & \uparrow - \circ f \\ \text{Hom}(G(Y), -) & \xrightarrow{\sim} & \text{Hom}(Y, F(-)) \end{array}$$

We need to recover a map  $G(X) \rightarrow G(Y)$  such that composing with it gives us  $\gamma$ . This works by the Yoneda lemma, which tells us that the natural transformation  $\gamma$  comes from an element  $\text{Hom}(G(X), G(Y))$ . Call it  $G(f)$ . It remains to check this does define a functor. Using this diagram with  $X = Y$  and  $f = \text{id}_X$  shows that  $G(\text{id}_X) = \text{id}_{G(X)}$ . Let  $X \xrightarrow{f} Y \xrightarrow{g} Z$  in  $\mathcal{C}$ . Then we draw

$$\begin{array}{ccccc} & & \xrightarrow{- \circ G(g \circ f)} & & \\ \text{Hom}(G(Z), -) & \xrightarrow{- \circ G(g)} & \text{Hom}(G(Y), -) & \xrightarrow{- \circ G(f)} & \text{Hom}(G(X), -) \\ \downarrow \wr & & \downarrow \wr & & \downarrow \wr \\ \text{Hom}(Z, F(-)) & \xrightarrow{- \circ g} & \text{Hom}(Y, F(-)) & \xrightarrow{- \circ f} & \text{Hom}(X, F(-)) \\ & & \xleftarrow{- \circ (g \circ f)} & & \end{array}$$



and this diagram shows that  $G(g \circ f) = G(g) \circ G(f)$  (because the map  $\gamma$  above is unique).  $\square$

This theorem shows there is a deep link between universal properties and adjoint functors.

## 2.5 Limits and colimits

(This subsection may be skipped on a first reading.)

Let us recall the definition of a functor category.

**Definition 2.29.** Let  $\mathcal{C}, \mathcal{D}$  be two categories. Then  $\text{Fun}(\mathcal{C}, \mathcal{D})$ , also written  $\mathcal{D}^{\mathcal{C}}$ , is the category whose objects are functors  $\mathcal{C} \rightarrow \mathcal{D}$  and morphisms are natural transformations between such functors, with composition given by vertical composition. It is called the *functor category category from  $\mathcal{C}$  to  $\mathcal{D}$* . When  $\mathcal{J}$  is a small category we also say that  $\text{Fun}(\mathcal{C}, \mathcal{D})$  is the category of *diagrams of shape  $\mathcal{J}$  in  $\mathcal{C}$* .

**Examples 2.30.**

1. Let  $\mathbf{2}$  be the category  $\bullet \longrightarrow \bullet$  which has two objects 1 and 2 and three morphisms (two of them being identities).

Then  $\mathbf{2} \times \mathbf{2}$  is the category  $\begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ & \searrow & \downarrow \\ \bullet & \longrightarrow & \bullet \end{array}$  which has 4 objects and 9 morphisms (4 of them being identities). Then, a functor from  $\mathbf{2} \times \mathbf{2}$  to  $\mathcal{C}$  is a commutative diagram of this shape in  $\mathcal{C}$ .

2. If  $\mathcal{J}$  is a small category, there is a functor  $\Delta : \text{Ob}(\mathcal{C}) \rightarrow \text{Fun}(\mathcal{J}, \mathcal{C})$  where  $\Delta(c)$  is the constant functor at  $c$ , that is the functor that sends all objects to  $c$  and all morphisms to  $\text{id}_c$ , and  $\Delta(f) = f$ , which works since a natural transformation  $\Delta(c) \Rightarrow \Delta(d)$  is just the data of one morphism  $c \rightarrow d$ .

**Definition 2.31.** A *cone above a diagram  $F : \mathcal{J} \rightarrow \mathcal{C}$  with summit  $c \in \mathcal{C}$*  is a natural transformation  $\lambda : \Delta(c) \Rightarrow F$ . Dually, a *cone under  $F$  with summit  $c$* , also called a *cocone*, is a natural transformation  $\lambda : F \Rightarrow \Delta(c)$ .

Let us unwrap this definition. A cone is a collection of maps  $\lambda_j : c \rightarrow F(j)$  for all  $j \in \text{Ob}(\mathcal{J})$ , such that for any morphism  $f : i \rightarrow j \in \text{Mor}(\mathcal{J})$ , this diagram commutes:

$$\begin{array}{ccc} & c & \\ \lambda_i \swarrow & & \searrow \lambda_j \\ F(i) & \xrightarrow{F(f)} & F(j) \end{array}$$

**Definition 2.32.** Let  $F : \mathcal{J} \rightarrow \mathcal{C}$  be a diagram. A *limit* (or *projective limit* or *inverse limit*) of  $F$  is a universal cone above  $F$ , in the sense that it is a final object in the category of cones above  $F$ . Dually, a *colimit* (or *inductive limit* or *direct limit*) is a universal cocone, that is an initial object in the category of cones under  $F$ .

Concretely, a limit of  $F : \mathcal{J} \rightarrow \mathcal{C}$  is a pair  $(\lim F, \phi)$  with  $\lim F \in \text{Ob}(\mathcal{C})$  and  $\phi : \Delta(\lim F) \Rightarrow F$  is such that for any cone  $\lambda : \Delta(c) \Rightarrow F$ , there exists a unique morphism  $f : X \rightarrow \lim F \in \text{Mor}(\mathcal{C})$ , such that the diagram on the left commutes:

$$\begin{array}{ccc}
\Delta(c) & \xrightarrow{\Delta(f)} & \Delta(\lim F) \\
\searrow \lambda & & \swarrow \phi \\
& F &
\end{array}
\quad \text{which is equivalent to} \quad
\forall j \in \mathcal{J}, \quad
\begin{array}{ccc}
c & \xrightarrow{f} & \lim F \\
\searrow \lambda_j & & \swarrow \phi_j \\
& F(j) &
\end{array}$$

In compact form,  $\text{Hom}_{\mathcal{C}}(-, \lim F) \simeq \text{Hom}_{\text{Fun}(\mathcal{J}, \mathcal{C})}(\Delta(-), F)$ .

*Exercise.* Do the same for colimits.

*Remark.*

1. If a limit exists it is unique up to isomorphism (unique isomorphism that commutes with the legs of the cone)
2. If all limits exist, then  $\lim$  becomes a functor  $\lim : \text{Fun}(\mathcal{J}, \mathcal{C}) \rightarrow \mathcal{C}$  in the following way. Recall that theorem 2.27 says a functor  $D$  admits a left adjoint iff for all objects  $X$  in its codomain,  $\text{Hom}(X, D(-))$  is representable. The compact form of the definition of a limit says that the functor  $\text{Hom}(\Delta(-), F)$  is representable for all  $F$  (since we assume all limits exist). A dual version of the theorem gives that  $\Delta$  admits a right adjoint, which is  $\lim$  since  $\text{Hom}(c, \lim F) \simeq \text{Hom}(\Delta(c), F)$ . If  $\eta : F \Rightarrow G$  is a natural transformation, then  $\lim(\eta)$  can be constructed in the following way:  $\lim F \Rightarrow F \xRightarrow{\eta} G$  is a cone above  $G$ , and  $\lim(\eta) : \lim F \rightarrow \lim G$  comes from the universality of  $\lim G$ .

**Corollary 2.33.**

1. If  $\mathcal{C}$  has all  $\mathcal{J}$ -limits, then  $\lim : \text{Fun}(\mathcal{J}, \mathcal{C}) \rightarrow \mathcal{C}$  is a right adjoint to  $\Delta$ .
2. If  $\mathcal{C}$  has all  $\mathcal{J}$ -colimits, then  $\text{colim} : \text{Fun}(\mathcal{J}, \mathcal{C}) \rightarrow \mathcal{C}$  is a left adjoint to  $\Delta$ .

**Example 2.34.**

1. If  $\mathcal{J}$  is discrete, that is has no morphisms other than identities, then a functor  $F : \mathcal{J} \rightarrow \mathcal{C}$  is the same as a collection  $(X_i)_{i \in \mathcal{J}}$  of objects of  $\mathcal{C}$ . Then, a limit of  $F$  is an object  $\lim F \in \text{Ob}(\mathcal{C})$  with morphisms  $f_i : \lim F \rightarrow X_i$  such that for all objects  $X \in \text{Ob}(\mathcal{C})$  with morphisms  $p_i : X \rightarrow X_i$ , we have a unique map  $\alpha : X \rightarrow \lim F$  that makes this diagram commute for all  $i, j \in \mathcal{J}$ :

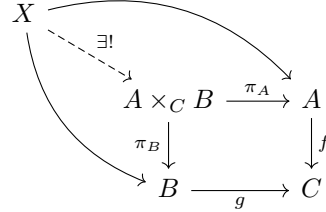
$$\begin{array}{ccccc}
& & X & & \\
& \swarrow p_i & \downarrow \alpha & \searrow p_j & \\
X_i & \xleftarrow{f_i} & \lim F & \xrightarrow{f_j} & X_j
\end{array}$$

We write  $\lim F = \prod_{j \in \mathcal{J}} F(j)$  and call it the *product of the  $F(j)$ s*. Morphisms  $f_i$  are written  $\pi_i$  and called *canonical projections*.

Dually, the colimit of  $F$  is called a coproduct and written  $\bigsqcup_{j \in \mathcal{J}} F(j)$ .

2. If  $\mathcal{J} = \bullet \rightrightarrows \bullet$ , then a functor  $F : \mathcal{J} \rightarrow \mathcal{C}$  is the data of two parallel morphisms in  $\mathcal{C}$ . A limit is an equalizer and a colimit is a coequalizer.

3. If  $\mathcal{J} = \begin{array}{c} \bullet \\ \downarrow \\ \bullet \rightarrow \bullet \end{array}$  then  $F : \mathcal{J} \rightarrow \mathcal{C}$  is the data of  $A, B, C \in \text{Ob}(\mathcal{C})$  with two morphisms  $f : A \rightarrow C$  and  $g : B \rightarrow C$ . The limit  $\lim F$  is called a *pullback* of  $f$  and  $g$ , with universal property depicted here:



4. If  $\mathcal{J} = \omega^{\text{op}}$ , that is  $\mathcal{J} = \cdots \rightarrow 2 \rightarrow 1 \rightarrow 0$ , then  $\lim F$  is often called the “inverse limit” of  $F$ . Concretely,  $F$  is the data of  $\cdots \rightarrow F(2) \xrightarrow{\alpha_2} F(1) \xrightarrow{\alpha_1} F(0)$ , and a cone above  $F$  looks like

$$\begin{array}{ccccc} & & c & & \\ & \swarrow \lambda_2 & \downarrow \lambda_1 & \searrow \lambda_0 & \\ \cdots & \longrightarrow & F(2) & \xrightarrow{\alpha_2} & F(1) & \xrightarrow{\alpha_1} & F(0) \end{array} \quad \text{we have } (\alpha_i \circ \cdots \circ \alpha_n) \circ \lambda_n = \lambda_i.$$

The typical example of an inverse limit is the one given by  $F(n) = \mathbb{Z}/p^n\mathbb{Z}$  in **Ring** with morphisms  $\mathbb{Z}/p^{n+1}\mathbb{Z} \rightarrow \mathbb{Z}/p^n\mathbb{Z}$  being reduction mod  $p^n$ . The inverse limit  $\varprojlim \mathbb{Z}/p^n\mathbb{Z}$  is the ring of  $p$ -adic integers. Concretely,  $a \in \varprojlim \mathbb{Z}/p^n\mathbb{Z}$  iff  $a = (a_i)_{i \in \mathbb{N}}$  such that  $a_i \equiv a_j \pmod{p^i} \forall i \leq j$ .

5. The dual notion, given by  $\mathcal{J} = 0 \rightarrow 1 \rightarrow 2 \rightarrow \cdots$ , is obtained by taking the colimit. It is called a *direct limit*. The typical example here is the Prüfer  $p$ -group  $\varinjlim \mathbb{Z}/p^n\mathbb{Z} = \mathbb{Z}(p^\infty)$ .

**Definition 2.35.** Let  $\mathcal{C}$  be a category. We say  $\mathcal{C}$  is (co)complete if it has all small (co)limits i.e. if for all diagrams  $F : \mathcal{J} \rightarrow \mathcal{C}$  with  $\mathcal{J}$  small,  $F$  has a (co)limit.

**Theorem 2.36.** A category  $\mathcal{C}$  is (co)complete if and only if it has all small (co)products and (co)equalizers.

*Proof.* Let  $\mathcal{J}$  be a small category and  $D : \mathcal{J} \rightarrow \mathcal{C}$  be a diagram. We have the products  $\prod_{k \in \text{Ob}(\mathcal{J})} D(k)$  and  $\prod_{g \in \text{Mor}(\mathcal{J})} D(\text{cod}(g))$  where  $\text{cod}(g)$  is the codomain of  $g$ . We have two morphisms

$$\prod_{k \in \text{Ob}(\mathcal{J})} D(k) \xrightleftharpoons[t]{s} \prod_{g \in \text{Mor}(\mathcal{J})} D(\text{cod}(g))$$

given by  $s = \prod_{f:i \rightarrow j} D(f)\pi_i$  and  $t = \prod_{f:i \rightarrow j} \pi_j$ , or with diagrams, for any  $f : i \rightarrow j \in \text{Mor}(\mathcal{J})$  :

$$\begin{array}{ccc} \prod_{k \in \text{Ob}(\mathcal{J})} D(k) & \xrightarrow{\exists! s} & \prod_{g \in \text{Mor}(\mathcal{J})} D(\text{cod}(g)) \\ \downarrow \pi_i & & \downarrow \pi_f \\ D(i) & \xrightarrow{D(f)} & D(j) \end{array} \quad \begin{array}{ccc} \prod_{k \in \text{Ob}(\mathcal{J})} D(k) & \xrightarrow{\exists! t} & \prod_{g \in \text{Mor}(\mathcal{J})} D(\text{cod}(g)) \\ \swarrow \pi_j & & \searrow \pi_f \\ & D(j) & \end{array}$$

We call  $\lim D$  an equalizer of  $s$  and  $t$ . A cone above  $D$  is given by compositions

$$\lim D \xrightarrow{\alpha} \prod_{k \in \text{Ob}(\mathcal{J})} D(k) \xrightarrow{\pi_i} D(i)$$

Indeed, for any morphism  $f : i \rightarrow j \in \text{Mor}(\mathcal{J})$ ,  $D(f)\pi_i\alpha = \pi_f s\alpha = \pi_f t\alpha = \pi_j\alpha$ . Now let  $\Delta(c) \Rightarrow_{\lambda} D$  be another cone above  $D$ . For any  $k \in \text{Ob}(\mathcal{J})$ , we have  $\lambda_k : c \rightarrow D(k)$ , which gives a unique morphism  $\lambda_* : c \rightarrow \prod_{k \in \text{Ob}(\mathcal{J})} D(k)$  such that  $\pi_i\lambda_* = \lambda_i$ . Then, for any  $f : i \rightarrow j \in \text{Mor}(\mathcal{J})$ , we have

$$\begin{aligned} \pi_f s\lambda_* &= D(f)\pi_i\lambda_* = D(f)\lambda_i = \lambda_j \\ \pi_f t\lambda_* &= \pi_j\lambda_* = \lambda_j \end{aligned}$$

and applying the universal property of the product shows that  $s\lambda_* = t\lambda_*$ . By the universal property of equalizers this gives the existence of a unique morphism  $c \rightarrow \lim D$  and completes the proof.  $\square$

**Definition 2.37.**  $F : \mathcal{C} \rightarrow \mathcal{D}$  preserves (co)limits if for every diagram  $D : \mathcal{J} \rightarrow \mathcal{C}$  and any (co)limit cone  $(c, \phi)$  of  $D$ , the image  $(F(c), F\phi)$  is a (co)limit cone over  $FD : \mathcal{J} \rightarrow \mathcal{D}$ .

*Remark.* Preserving limits is like having  $F(\lim D) \simeq \lim FD$ , but stronger:

$$\begin{array}{ccc} \lim D & & F(\lim D) \xrightarrow{\exists! \alpha} \lim FD \\ \downarrow \phi_i & \rightsquigarrow & \downarrow FD(\phi_i) \swarrow \lambda_i \\ D(i) & & FD(i) \end{array}$$

and  $\alpha$  is an isomorphism since  $(F(\lim D), F\phi)$  is a limit cone.

**Proposition 2.38.** Let  $\mathcal{C}$  be a locally small category and  $X \in \text{Ob}(\mathcal{C})$ . Then

1.  $\text{Hom}_{\mathcal{C}}(X, -)$  preserves all limits that exist in  $\mathcal{C}$
2. The contravariant functor  $\text{Hom}_{\mathcal{C}}(-, X)$  transforms colimits in  $\mathcal{C}$  into limits in **Set**.

*Proof.* Let  $\mathcal{J}$  be a small category and  $D : \mathcal{J} \rightarrow \mathcal{C}$  be a diagram. Let  $F : \mathcal{C} \rightarrow \mathbf{Set}$  be the hom-functor  $\text{Hom}_{\mathcal{C}}(X, -)$ . Let  $(L, \lambda)$  be a limit cone for  $D$ . Then,  $(F(L), F(\lambda))$  is a cone in **Set** over  $FD$ , since for any  $\alpha : i \rightarrow j \in \text{Mor}(\mathcal{J})$  we have the commutative diagram

$$\begin{array}{ccc} & F(L) & \\ F(\lambda) \swarrow & & \searrow F(\lambda_i) \\ \text{Hom}_{\mathcal{C}}(X, D(i)) & \xrightarrow{D(\alpha) \circ -} & \text{Hom}_{\mathcal{C}}(X, D(j)) \end{array}$$

It remains to show that  $(F(L), F(\lambda))$  is a limit cone for  $FD$ . Let  $S \xRightarrow{f} FD$  be another cone. We have  $f(i) : S \rightarrow \text{Hom}(X, D(i))$  (we work in **Set** so morphisms are actual maps here). Fixing  $s \mapsto f_i(s)$   $S$ , we get commutative diagrams:

$$\begin{array}{ccc}
& X & \\
f_i(s) \swarrow & & \searrow f_j(s) \\
D(i) & \xrightarrow{D(\alpha) \circ -} & D(j)
\end{array}$$

so  $(X, f_i(s))$  is a cone over  $D$  hence there exists a unique morphism  $u_s : X \rightarrow L$  such that  $\lambda_i \circ u_s = f_i(s)$  for all  $i \in \text{Ob}(\mathcal{J})$ . Now set  $u : S \rightarrow \text{Hom}(X, L)$  and we have  $(F\lambda \circ u)(s) = (F\lambda)(u_s) = f$   
 $s \mapsto u_s$

so  $u : S \rightarrow F(L)$  is a morphism of cones. We need to check it is unique. If  $v$  is another one then  $\lambda_i \circ v(s) = f_i(s)$  so  $v(s) = u_s$  by uniqueness of  $u_s$ , which shows  $v = u$ .

Another proof is given here:

$$\begin{aligned}
\text{Hom}_{\mathcal{C}}(X, \lim D) &\simeq \text{Hom}_{\text{Fun}(\mathcal{J}, \mathcal{C})}(\Delta X, D) \\
&\simeq \text{Hom}_{\text{Fun}(\mathcal{J}, \mathbf{Set})}(\Delta 1, \text{Hom}_{\mathcal{C}}(X, D(-))) \\
&\simeq \text{Hom}_{\mathbf{Set}}(1, \lim \text{Hom}_{\mathcal{C}}(X, D(-))) \\
&\simeq \lim \text{Hom}_{\mathcal{C}}(X, D(-))
\end{aligned}$$

(1 is a singleton.) The first and third isomorphisms are by definition of a limit. The last isomorphism comes from the fact that for any set  $A$ , maps  $1 \rightarrow A$  correspond to elements of  $A$ . The second isomorphism works since a natural transformation  $\Delta X \Rightarrow D$  is the same as a collection of morphisms  $f_i : X \rightarrow D(i)$  indexed by  $\text{Ob}(\mathcal{J})$ .  $\square$

**Theorem 2.39.** *Right adjoints preserve limits. Left adjoints preserve colimits.*

*Proof.* We only need to prove the statement about right adjoints and then use opposite categories

for left adjoints. Let  $\mathcal{C} \begin{array}{c} \xrightarrow{F} \\ \xleftarrow{G} \end{array} \mathcal{D}$  be two functors with  $F \dashv G$  and  $D : \mathcal{J} \rightarrow \mathcal{D}$  be a diagram,

with  $\eta : \Delta(\lim D) \Rightarrow D$  its limit cone. Our goal is to show that  $(G \lim D, G\eta)$  is a limit cone for  $G \circ D$ . The fact that it is a cone above  $G \circ D$  is clear. Now let  $\mu : \Delta(c) \Rightarrow GD$  be another cone. For any  $j \in \text{Ob}(\mathcal{J})$ , we have  $\mu_j \in \text{Hom}(c, GD(j))$ . By adjunction, it corresponds to a morphism  $\mu_j^* \in \text{Hom}(F(c), D(j))$ . We claim these morphisms make up a natural transformation  $\mu^* : \Delta(F(c)) \Rightarrow D$ . Indeed, for any morphism  $f : i \rightarrow j \in \text{Mor}(\mathcal{J})$ , we have by naturality of the adjunction a commutative diagram

$$\begin{array}{ccc}
\text{Hom}(F(c), D(i)) & \xrightarrow{\sim} & \text{Hom}(c, GD(i)) \\
\downarrow D(f) \circ - & & \downarrow GD(f) \circ - \\
\text{Hom}(F(c), D(j)) & \xrightarrow{\sim} & \text{Hom}(c, GD(j))
\end{array}$$

so  $D(f) \circ \mu_i^* = (GD(f) \circ \mu_i)^* = \mu_j^*$ . By universality of  $\lim D$ , there exists a unique morphism  $\tau : F(c) \rightarrow \lim D$  that makes the appropriate diagram commute. Using the adjunction, we get a morphism  $\tau^* : c \rightarrow G(\lim D)$ , which is the morphism we are looking for. The commutativity of the appropriate diagram comes from naturality of the adjunction. Uniqueness comes from the uniqueness of  $\tau$ .

In compact form:

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}}(c, \lim GD) &\simeq \mathrm{Hom}_{\mathrm{Fun}(\mathcal{J}, \mathcal{C})}(\Delta c, GD) \\ &\simeq \mathrm{Hom}_{\mathrm{Fun}(\mathcal{J}, \mathcal{D})}(F\Delta c, D) \\ &\simeq \mathrm{Hom}_{\mathrm{Fun}(\mathcal{J}, \mathcal{D})}(\Delta Fc, D) \\ &\simeq \mathrm{Hom}_{\mathcal{D}}(Fc, \lim D) \\ &\simeq \mathrm{Hom}_{\mathcal{C}}(C, G \lim D)\end{aligned}$$

□

### 3 Tensor products

All rings considered here are assumed to be associative and to have a multiplicative unit 1. Let  $A$  be a ring.

**Definition 3.1.**

- A *right  $A$ -module* is an abelian group  $(M, +)$  with a map  $M \times A \rightarrow M$  such that

$$\begin{array}{ll} (1) & (m+n) \cdot a = m \cdot a + n \cdot a \\ (2) & m \cdot (a+b) = m \cdot a + m \cdot b \\ (3) & m \cdot (ab) = (m \cdot a)b \\ (4) & m \cdot 1_A = m \end{array}$$

by symmetry one gets the notion of a *left  $A$ -module* (which is the equivalent of a vector space, but with a ring in place of the field).

- If  $A, B$  are two rings, an  *$A$ - $B$ -bimodule* is an abelian group  $M$  with a left  $A$ -module and a right  $B$ -module structure such that for  $(a, b) \in A \times B$  and  $m \in M$ ,  $a \cdot (m \cdot b) = (a \cdot m) \cdot b$ .
- Let  $M$  be a right  $A$ -module,  $N$  be a left  $A$ -module and  $G$  be an abelian group. A *bilinear* (or *balanced*) map  $f : M \times N \rightarrow G$  is a map  $f$  such that

$$\begin{array}{ll} (1) & f(m_1 + m_2, n) = f(m_1, n) + f(m_2, n) \\ (2) & f(m, n_1 + n_2) = f(m, n_1) + f(m, n_2) \\ (3) & f(ma, n) = f(m, an) \end{array}$$

The following theorem shows that there exists an abelian group  $M \otimes_A N$  that is “universal” with respect to bilinear maps.

**Theorem 3.2.** *Let  $M$  be a right  $A$ -module and  $N$  be a left  $A$ -module. There exists an abelian group  $M \otimes_A N$  together with a bilinear map  $t : M \times N \rightarrow M \otimes_A N$  such that for any abelian group  $G$  and bilinear map  $b : M \times N \rightarrow G$ , there exists a unique group homomorphism  $\tilde{b}$  that makes this diagram commute:*

$$\begin{array}{ccc} M \times N & \xrightarrow{\forall b} & G \\ t \downarrow & \nearrow \exists \tilde{b} & \\ M \otimes_A N & & \end{array}$$

*Proof.* Let  $L = \mathbb{Z}[M \times N]$  be the free abelian group on  $M \times N$ . It has a basis, namely  $\{(m, n) \mid m \in M, n \in N\}$ . Now consider the subgroup

$$I = \langle (m_1 + m_2, n) - (m_1, n) - (m_2, n), (m, n_1 + n_2) - (m, n_1) - (m, n_2), (ma, n) - (m, an) \rangle$$

It is chosen so the relations we want hold in  $L/I$ , for instance  $(ma, n) = (m, an)$  in the quotient group. Set  $M \otimes_A N = L/I$  and  $t : M \times N \rightarrow L/I$ . By construction  $M \otimes_A N$  is an abelian

group and  $t$  is bilinear. We need to check the universal property. Pick a bilinear map  $b : M \times N \rightarrow G$ . We have a diagram

$$\begin{array}{ccc}
M \times N & \xrightarrow{b} & G \\
\downarrow i & \nearrow \exists! \bar{b} & \uparrow \\
\mathbb{Z}[M \times N] & & \\
\downarrow \pi & \nearrow \exists! \bar{b} & \\
M \otimes_A N & & 
\end{array}$$

where  $i : (m, n) \mapsto (m, n)$  is the inclusion map and  $\pi : L \rightarrow L/I$  is the canonical projection. The map  $\bar{b}$  exists by universal property of the free abelian group. Moreover it passes to the quotient ( $I \subset \ker(\bar{b})$ ), so we get the map  $\bar{b}$ . We now check uniqueness. Let  $f : M \otimes_A N \rightarrow G$  be another linear map that makes the diagram commute. Then,  $f \circ \pi$  makes the top triangle commute, so by the universal property of the free abelian group,  $f \circ \pi = \bar{b}$ . Applying the universal property of the quotient allows us to conclude  $f = \bar{b}$ .  $\square$

*Remark.*

1. The abelian group  $M \otimes_A N$  is a unique up to unique isomorphism.
2. The class  $[(m, n)] \in M \otimes_A N$  is written  $m \otimes n$ . It is called a “*pure tensor*”. Pure tensors generate the tensor product:

$$x \in M \otimes_A N \iff \exists (m_i, n_i) \in M^n \times N^n, x = \sum_{i=1}^n m_i \otimes n_i$$

► The tensor product is a functor. Precisely, it is a bifunctor  $-\otimes_A - : \mathbf{Mod} A \times A \mathbf{Mod} \rightarrow \mathbf{Ab}$ . If  $M, M'$  are two right  $A$ -modules,  $N, N'$  are two left  $A$ -modules and  $f : M \rightarrow M', g : N \rightarrow N'$  are linear maps, then writing  $(f \otimes g)(m \otimes n) = f(m) \otimes g(n)$  gives a commutative diagram

$$\begin{array}{ccc}
M \otimes_A N & \xrightarrow{\text{id}_M \otimes g} & M \otimes_A N' \\
f \otimes \text{id}_N \downarrow & \dashrightarrow f \otimes g & \downarrow f \otimes \text{id}_{N'} \\
M' \otimes_A N & \xrightarrow{\text{id}_{M'} \otimes g} & M' \otimes_A N'
\end{array}$$

One needs to be careful as  $M \otimes_A N$  can be defined using a quotient or a universal property. Obtaining the arrow  $f \otimes g$  is easier with the universal property:

$$\begin{array}{ccc}
M \times N & \xrightarrow{(f, g)} & M' \times N' \\
\downarrow t & & \downarrow t' \\
M \otimes_A N & \dashrightarrow f \otimes g & M' \otimes_A N'
\end{array}$$

Since  $t' \circ (f, g)$  is bilinear, we obtain the unique map  $f \otimes g$  using the universal property of  $M \otimes_A N$ . Hence we obtain the lemma:

**Lemma 3.3.**  $-\otimes_A -$  is a bifunctor.

**Corollary 3.4.** 1. If  $M$  is a  $B$ - $A$ -bimodule, then  $M \otimes_A N$  is a left  $B$ -module



2. If  $N$  is an  $A$ - $C$ -bimodule, then  $M \otimes_A N$  is a right  $C$ -module

3. If  $M$  is a  $B$ - $A$ -bimodule and  $N$  is a  $A$ - $C$ -bimodule then  $M \otimes_A N$  is a  $B$ - $C$ -bimodule.

*Proof.* We do the proof of 1. We set  $b \bullet (m \otimes n) = (bm) \otimes n$  and now we need to check that it is well defined. A good way is to fix  $b \in B$  and let  $\ell_b : M \rightarrow M$  and notice that  $\ell_b \in \text{End}_A(M)$ .

$$m \mapsto b \cdot m$$

By functoriality, we get a map  $\ell_b \otimes \text{id}_N : M \otimes_A N \rightarrow M \otimes_A N$  so our action is well defined

$$m \otimes n \mapsto (bm) \otimes n$$

and this is a  $B$ -module structure on the tensor product. The proof of 2. is similar. The proof of 3. comes from the fact that  $\ell_b \otimes \text{id}_N$  and  $\text{id}_M \otimes r_c$  commutes.  $\square$

### Examples 3.5.

1.  $A \otimes_A N \simeq N$  as left  $A$ -modules. Isomorphisms are given by  $a \otimes n \mapsto a \cdot n$  and  $n \mapsto 1 \otimes n$ . The well-definition of these maps comes from the universal property.

2. If  $R$  is commutative then an  $R$ -module  $M$  is an  $R$ - $R$ -bimodule

$$\begin{aligned} R \times M \times R &\rightarrow (x, m, y) \\ M &\mapsto mxy = myx \end{aligned}$$

so  $M \otimes_R N$  is always an  $R$ -module.

**!** Over a field,  $\dim(V \otimes W) = \dim(V) \dim(W)$  but this is false in general for a ring.

*Exercise.* Show that  $\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z} = \{0\}$  when  $\gcd(m, n) = 1$ .

**Theorem 3.6** (Tensor-hom adjunction). *Let  $A, B$  be two rings and  $M$  be an  $A$ - $B$ -bimodule. We have a functor  $- \otimes_A M : \mathbf{Mod} A \rightarrow \mathbf{Mod} B$  and a functor  $\text{Hom}_B(M, -) : \mathbf{Mod} B \rightarrow \mathbf{Mod} A$ . Then  $- \otimes_A M$  is left adjoint to  $\text{Hom}_B(M, -)$ .*

The  $A$ -module structure on  $\text{Hom}_B(M, Y)$  for  $Y$  a  $B$ -module is given by

$$\begin{aligned} \text{Hom}_B(M, Y) \times A &\rightarrow \text{Hom}_B(M, Y) \\ (f, a) &\mapsto f \cdot a : M \rightarrow Y \\ m &\mapsto f(am) \end{aligned}$$

*Proof.* **TODO**  $\square$

## 4 Additive categories

### 4.1 Preadditive and additive categories

**Definition 4.1.** A *zero object* in a category  $\mathcal{C}$  is an object that is both final and initial.

**Example 4.2.**  $\{0\}$  is a zero object in  $\mathbf{Mod} A$  for  $A$  a ring.

**Definition 4.3.** Let  $k$  be a commutative ring. A  *$k$ -category* is a category  $\mathcal{C}$  such that all hom-sets are  $k$ -modules and composition is bilinear. When  $k = \mathbb{Z}$  we say that  $\mathcal{C}$  is *preadditive*.

*Remark.* One says that  $\mathcal{C}$  is “enriched” over  $\mathbf{Mod} k$ .

**Lemma 4.4.** *Let  $\mathcal{C}$  be a  $k$ -category. For  $X, Y \in \text{Ob}(\mathcal{C})$ , the product  $X \times Y$  exists iff the coproduct  $X \sqcup Y$  exists. If so, they are isomorphic.*

*Proof.* Suppose  $X \times Y$  exists. Define  $i_X = (\text{id}_X, 0) : X \rightarrow X \times Y$  and  $i_Y = (0, \text{id}_Y) : Y \rightarrow X \times Y$ . We claim these maps together with the product are the coproduct of  $X$  and  $Y$ . Let  $Z \in \text{Ob}(\mathcal{C})$  and  $f : X \rightarrow Z$ ,  $g : Y \rightarrow Z$ . Then, define  $f \sqcup g : X \times Y \rightarrow Z$  by  $f \sqcup g = f\pi_X + g\pi_Y$ . This makes this diagram commute:

$$\begin{array}{ccccc} X & \xrightarrow{i_X} & X \times Y & \xleftarrow{i_Y} & Y \\ & \searrow f & \downarrow f \sqcup g & \swarrow g & \\ & & Z & & \end{array}$$

Now let  $h : X \times Y \rightarrow Z$  be another arrow that makes the diagram commute. Then

$$h \circ (i_X\pi_X + i_Y\pi_Y) = hi_X\pi_X + hi_Y\pi_Y = f\pi_X + g\pi_Y = f \sqcup g$$

And uniqueness follows since  $\text{id}_{X \times Y} = i_X\pi_X + i_Y\pi_Y$ . This comes from the universal property of the product and the diagram

$$\begin{array}{ccccc} & & X \times Y & & \\ \pi_X \swarrow & & \downarrow i_X\pi_X + i_Y\pi_Y & & \searrow \pi_Y \\ X & \xleftarrow{\pi_X} & X \times Y & \xrightarrow{\pi_Y} & Y \end{array}$$

□

**Definition 4.5.** Let  $\mathcal{C}$  be a  $k$ -category. A *biproduct* of  $X$  and  $Y$  is an object  $X \oplus Y \in \mathcal{C}$  with morphisms  $X \xrightleftharpoons[\pi_X]{i_X} X \oplus Y \xrightleftharpoons[i_Y]{\pi_Y} Y$  such that

1.  $i_X\pi_X + i_Y\pi_Y = \text{id}_{X \oplus Y}$
2.  $\pi_X i_Y = 0$ ,  $\pi_Y i_X = 0$ ,  $\pi_X i_X = \text{id}_X$ ,  $\pi_Y i_Y = \text{id}_Y$

**Definition 4.6.** Let  $k$  be a commutative ring. A  $k$ -*additive* (or  $k$ -*linear*) category is a  $k$ -category with finite products and finite coproducts.

*Remark.*

1. When  $k = \mathbb{Z}$ , we simply say the category is *additive*.
2. As seen above, finite products are finite coproducts and vice versa. Both are finite biproducts.
3. For  $\mathcal{C}$  a  $k$ -category, the following are equivalent:
  - (a)  $\mathcal{C}$  is  $k$ -additive
  - (b)  $\mathcal{C}$  has a zero object and every pair of objects has a product
  - (c)  $\mathcal{C}$  has a zero object and every pair of objects has a coproduct
  - (d)  $\mathcal{C}$  has a zero object and every pair of objects has a biproduct

Moreover (b)  $\iff$  (c)  $\iff$  (d), and for (a) we are just missing the empty product (or coproduct), which is the zero object.

4. If  $A$  is additive there is a canonical interpretation of the group structure on  $\text{Hom}(-, -)$  using  $- \oplus -$ . See exercise sheets.

**Examples 4.7.**

0. The category **Ab** of abelian groups is additive.
1. If  $A$  is a ring (or  $k$ -algebra) then **Mod** $A$ , **A****Mod** and finitely generated versions are  $k$ -additive.
2. If  $\mathcal{C}$  is additive, then  $\mathcal{C}^{\text{op}}$  is additive.
3. If  $\mathcal{C}$  is additive and  $I$  is a category then  $\text{Fun}(I, \mathcal{C})$  is additive.
4. If  $A$  is a ring, then the category  $BA$  with one object  $\bullet$  and  $\text{Hom}(\bullet, \bullet) = A$  is preadditive but not additive.

**Definition 4.8.** Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be a functor between two  $k$ -linear categories. The functor  $F$  is said to be  $k$ -linear (or *additive* when  $k = \mathbb{Z}$ ) if for any  $X, Y \in \text{Ob}(\mathcal{C})$ ,

$$\begin{array}{ccc} \text{Hom}_{\mathcal{C}}(X, Y) & \rightarrow & \text{Hom}_{\mathcal{D}}(FX, FY) \\ f & \mapsto & F(f) \end{array}$$

is a  $k$ -linear map.

**Proposition 4.9.** A functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  is additive if and only if  $F(0) \simeq 0$  and  $F(X \oplus Y) \simeq F(X) \oplus F(Y)$ .

*Proof.* Suppose  $F$  is additive.  $\text{id}_0$  is the zero morphism of  $\text{Hom}_{\mathcal{C}}(0, 0)$ . Therefore  $F(\text{id}_0) = \text{id}_{F(0)}$  is the zero morphism of  $\text{Hom}_{\mathcal{D}}(F(0), F(0))$ . For any  $Y \in \text{Ob}(\mathcal{D})$  and  $f : F(0) \rightarrow Y$ ,  $f = f \text{id}_{F(0)} = 0$ . This shows  $F(0)$  initial. A similar reasoning shows it is final. Therefore  $F(0)$  is isomorphic to the zero object of  $\mathcal{D}$ . **TODO** □

**Example 4.10.** Let  $A, B$  be two rings and  $M$  be an  $A$ - $B$ -bimodule. Then,  $- \otimes_A M_B : \mathbf{Mod} A \rightarrow \mathbf{Mod} B$  is additive. This can be quickly proven using the proposition above: the functor is a left adjoint so it preserves coproducts!