

Homological Algebra

Original lectures notes by Baptiste Rognerud, but now typeset in L^AT_EX

1 Introduction to category theory

References:

- Emily Riehl, Category Theory in Context (chapter I)
- Saunders Mac Lane, Categories for the Working Mathematician
- Ibrahim Assem, Introduction au langage catégorique (chapters I, II)

➤ Near 1945 Eilenberg and Mac Lane gave the good formalism for a “natural isomorphism” (the general theory of natural transformations). For instance, if V is a finite-dimensional vector space, $V \simeq V^*$ and $V \simeq V^{**}$, but the first isomorphism is not natural (“a choice needs to be made”), while the second is. But why?

It turns out solving this question gave a formalism, category theory, that unified already existing mathematical concepts, gave new links between different notions and also gave new questions!

⚠ Category theory is not a theory that trivializes mathematics.

It is used today by (almost) everyone: algebraic geometry, algebra, representation theory, topology, combinatorics, ...

1.1 Categories and functors

Definition 1.1. A *category* \mathcal{C} is the data of

- A collection of *morphisms* $\text{Mor}(\mathcal{C})$
- A collection of *objects* $\text{Ob}(\mathcal{C})$

such that

1. Every morphism $f \in \text{Mor}(\mathcal{C})$ has a specified domain $X \in \text{Ob}(\mathcal{C})$ and codomain $Y \in \text{Ob}(\mathcal{C})$. We write $f : X \rightarrow Y$.
2. For every object $X \in \text{Ob}(\mathcal{C})$ there exists a morphism $1_X : X \rightarrow X$ (the *identity* of X), also written id_X
3. For any three objects $X, Y, Z \in \text{Ob}(\mathcal{C})$ and morphism $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ there exists a morphism $g \circ f : X \rightarrow Z$ (we often omit \circ and just write gf)

satisfying

(Identity) $\forall f : X \rightarrow Y, 1_Y f = f = f 1_X$

(Associativity) $\forall f : W \rightarrow X, g : X \rightarrow Y, h : Y \rightarrow Z, h(gf) = (hg)f$

Remark.

1. We use the term “collection” because we don’t want to worry about set-theoretical issues
2. If $\text{Mor}(\mathcal{C})$ is a set, we say that \mathcal{C} is *small*
3. We denote by $\text{Hom}_{\mathcal{C}}(X, Y)$ (or $\mathcal{C}(X, Y)$) the collection of $f : X \rightarrow Y \in \text{Mor}(\mathcal{C})$

Examples 1.2 (Concrete categories).

1. The category **Set**, where objects are sets and morphisms are just maps.
2. **Top**, where objects are topological spaces and morphisms are continuous maps.
3. Groups together with group homomorphisms form a category called **Grp**. The same can be said about rings, fields...
4. k -vector spaces, or more generally left/right R -modules, together with linear maps, form a category denoted **RMod** or **ModR** (for left or right R -modules).

In these previous examples, objects are sets with additional structure, and morphisms between two objects are in particular maps between the two underlying sets. Such categories are called *concrete categories* (a rigorous definition will be given later). However, a category need not be concrete.

Examples 1.3 (Abstract categories).

1. Let k be a field. There exists a category **Mat** $_k$ where objects are the natural numbers \mathbb{N} and morphisms are $\text{Hom}(m, n) = \text{Mat}_{n,m}(k)$, where composition is given by matrix multiplication.
2. If G is a group, there exists a category BG which has only one object \bullet , and morphisms $\text{Hom}(\bullet, \bullet) = G$, where composition is multiplication in G .
3. If (P, \leq) is a *poset* (a partially ordered set, that is a set P together with a reflexive, transitive relation \leq), then one can construct a category \hat{P} by setting $\text{Ob}(\hat{P}) = P$ and $|\text{Hom}(x, y)| = \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{otherwise} \end{cases}$, where composition is defined in the only possible way.
4. The homotopy category of topological spaces: objects are topological spaces, and $\text{Hom}(X, Y)$ is $\text{Hom}_{\text{Top}}(X, Y) / \sim$ where \sim is homotopy of continuous maps.

Exercise. Check the categories defined above really are categories. In (2), what are the minimal hypotheses needed on G for BG to be a category? In (3), what are the minimal hypotheses needed on \leq for \hat{P} to be a category?

Examples 1.4 (Categories constructed from categories).

1. If \mathcal{C} is a category, one can construct its *opposite category* \mathcal{C}^{op} , defined by $\text{Ob}(\mathcal{C}^{\text{op}}) = \text{Ob}(\mathcal{C})$ and $\text{Hom}_{\mathcal{C}^{\text{op}}}(X, Y) = \text{Hom}_{\mathcal{C}}(Y, X)$, with composition described by the following diagram:

$$\begin{array}{ccc}
X & & X \\
\downarrow f & & \uparrow f^{\text{op}} \\
Y & \rightsquigarrow & Y \\
\downarrow g & & \uparrow g^{\text{op}} \\
Z & & Z
\end{array}
\quad
\begin{array}{c}
gf \\
\downarrow \\
Z
\end{array}
\quad
\begin{array}{c}
f^{\text{op}}g^{\text{op}} \\
\downarrow \\
Z
\end{array}$$

- Let \mathcal{C} be a category. A *subcategory* \mathcal{D} of \mathcal{C} is another category such that $\text{Ob}(\mathcal{D}) \subset \text{Ob}(\mathcal{C})$ and $\text{Mor}(\mathcal{D}) \subset \text{Mor}(\mathcal{C})$ and the composition in \mathcal{D} is induced by the one in \mathcal{C} . For instance, **Ab**, the category of abelian groups and group homomorphisms, is a subcategory of **Grp**.
- Let \mathcal{C} and \mathcal{D} be categories. The *product category* of \mathcal{C} and \mathcal{D} is the category $\mathcal{C} \times \mathcal{D}$ defined by $\text{Ob}(\mathcal{C} \times \mathcal{D}) = \text{Ob}(\mathcal{C}) \times \text{Ob}(\mathcal{D})$ and $\text{Mor}(\mathcal{C} \times \mathcal{D}) = \text{Mor}(\mathcal{C}) \times \text{Mor}(\mathcal{D})$, composition and identities being defined componentwise.

Exercise. Describe $(BG)^{\text{op}}$ for G a group and \hat{P}^{op} for (P, \leq) a poset.

▲ Set^{op} is not Set. TODO

Remark. In a category \mathcal{C} the objects can be anything, so saying $x \in X$ for $X \in \text{Ob}(\mathcal{C})$ doesn't make sense. Hence, categorical notions are defined using arrows and not elements.

Definition 1.5. Let \mathcal{C} be a category.

- $f : X \rightarrow Y$ is an *isomorphism* if there exists $g : Y \rightarrow X$ such that $gf = \text{id}_X$ and $fg = \text{id}_Y$.
- $f : X \rightarrow Y$ is a *monomorphism* if for all $g, h : W \rightarrow X$ such that $fg = fh$, $g = h$ (f is left-cancellable).
- $f : X \rightarrow Y$ is an *epimorphism* if for all $g, h : Y \rightarrow Z$ such that $gf = hf$, $g = h$ (f is right-cancellable).

▲ Being both a mono and an epi doesn't imply being an iso. TODO

Definition 1.6. Let \mathcal{C}, \mathcal{D} be two categories. A (*covariant*) *functor* $F : \mathcal{C} \rightarrow \mathcal{D}$ is the data of

- An object $F(X) \in \text{Ob}(\mathcal{D})$ for all $X \in \text{Ob}(\mathcal{C})$
- A morphism $F(f) \in \text{Hom}_{\mathcal{D}}(F(X), F(Y))$ for all $f \in \text{Hom}_{\mathcal{C}}(X, Y)$

such that $F(\text{id}_X) = \text{id}_{F(X)}$ for all $X \in \text{Ob}(\mathcal{C})$ and $F(gf) = F(g)F(f)$ whenever $f, g \in \text{Mor}(\mathcal{C})$ are composable.

Definition 1.7. A *contravariant* functor from \mathcal{C} to \mathcal{D} is a functor from \mathcal{C}^{op} to \mathcal{D} (so composition is reversed, i.e. $F(gf) = F(f)F(g)$).

Examples 1.8.

- $U : \mathbf{Grp} \rightarrow \mathbf{Set}, U(G) = G, U(f) = f$ the functor that to a group assigns it its underlying set and to a homomorphism the underlying map. It is called the *forgetful functor* from groups to sets, because it forgets the group structure.

2. $U : \mathbf{Ass} \rightarrow \mathbf{Lie}$ the forgetful functor from the category of associative algebras to the category of Lie algebras. It forgets the “associative structure” but remembers the underlying abelian group.

$$(A, +, \cdot) \mapsto (A, +, [-, -])$$
3. $F : \mathbf{Set} \rightarrow \mathbf{Ab}, X \mapsto \mathbb{Z}[X], f \mapsto \mathbb{Z}[f]$, which to a set assigns the free abelian group with basis X (the group of finite linear combinations of elements of X). A map $f : X \rightarrow Y$ can then be uniquely extended to a linear map $\mathbb{Z}[f] : \mathbb{Z}[X] \rightarrow \mathbb{Z}[Y]$ that agrees with f on the bases of $\mathbb{Z}[X]$ and $\mathbb{Z}[Y]$.
4. Suppose \mathcal{C} is locally small (i.e. for any X, Y , $\text{Hom}_{\mathcal{C}}(X, Y)$ is a set). For all $X \in \mathcal{C}$, $\text{Hom}(X, -)$ is a functor $\mathcal{C} \rightarrow \mathbf{Set}$. Similarly, $\text{Hom}_{\mathcal{C}}(-, X)$ is a contravariant functor $\mathcal{C} \rightarrow \mathbf{Set}$. $\text{Hom}_{\mathcal{C}}(-, -)$ is a functor $\mathcal{C} \times \mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$.
5. Functors $\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{E}$ can be composed in the obvious sense.

TODO: DRAW DIAGRAMS

Definition 1.9. Let $\mathcal{C} \begin{array}{c} \xrightarrow{F} \\ \xrightarrow{G} \end{array} \mathcal{D}$ be two functors. A *natural transformation* η from F to G is the data of morphisms $\eta_X : F(X) \rightarrow G(X) \in \text{Mor}(\mathcal{D})$ for all $X \in \text{Ob}(\mathcal{C})$ such that for all $f : X \rightarrow Y \in \text{Mor}(\mathcal{C})$, the diagram

$$\begin{array}{ccc} F(X) & \xrightarrow{\eta_X} & G(X) \\ \downarrow F(f) & & \downarrow G(f) \\ F(Y) & \xrightarrow{\eta_Y} & G(Y) \end{array}$$

commutes, that is $G(f)\eta_X = \eta_Y F(f)$. We write $\eta : F \Rightarrow G$ or draw $\mathcal{C} \begin{array}{c} \xrightarrow{F} \\ \Downarrow \eta \\ \xrightarrow{G} \end{array} \mathcal{D}$

Example 1.10. Let V be a k -vector space. $\text{id}_{\mathbf{Vect}_k}$ and $D^2 = \text{Hom}_{\mathbf{Vect}_k}(\text{Hom}_{\mathbf{Vect}_k}(-, k), k)$ are two endofunctors of \mathbf{Vect}_k . $\text{ev}_- : V \rightarrow V^{**}$ defines a natural transformation

$$\begin{array}{ccccc} v & \mapsto & \text{Hom}(V, k) & \rightarrow & k \\ & & \phi & \mapsto & \phi(v) \end{array}$$

between them:

$$\begin{array}{ccc} V & \xrightarrow{\text{ev}} & V^{**} \\ f \downarrow & & \downarrow D^2(f) \\ W & \xrightarrow{\text{ev}} & W^{**} \end{array}$$

For $a \in V$, $D^2(f) \circ \text{ev}_a : W^* \rightarrow k \in W^{**}$ and in the other direction $(\text{ev} \circ f)(a) = \text{ev}_{f(a)}$.
 $\phi \mapsto \phi(f(a))$

However, there is no natural transformation from $\text{id}_{\mathbf{Vect}_k}$ to D . For one, the first is covariant and the second is contravariant. To get a natural transformation from a covariant to a contravariant

functor, we can modify the definition of naturality to be that $\begin{array}{ccc} V & \rightarrow & V^* \\ \downarrow & & \uparrow \\ W & \rightarrow & W^* \end{array}$ commutes, but even such

natural transformations do not exist from $\text{id}_{\mathbf{Vect}_k}$ to D .

Definition 1.11. A natural transformation $\mathcal{C} \begin{array}{c} \xrightarrow{F} \\ \Downarrow \eta \\ \xrightarrow{G} \end{array} \mathcal{D}$ is a *natural isomorphism* if η_X is an isomorphism for all $X \in \text{Ob}(\mathcal{C})$.

Remark. One can compose natural transformations in two ways, “vertical composition”:

$$\mathcal{C} \begin{array}{c} \xrightarrow{F} \\ \Downarrow \alpha \\ \xrightarrow{G} \\ \Downarrow \beta \\ \xrightarrow{H} \end{array} \mathcal{D} \rightsquigarrow \mathcal{C} \begin{array}{c} \xrightarrow{F} \\ \Downarrow \beta \circ \alpha \\ \xrightarrow{H} \end{array} \mathcal{D} \quad \text{where } (\beta \circ \alpha)_X = \beta_X \circ \alpha_X$$

or “horizontal composition”:

$$\mathcal{C} \begin{array}{c} \xrightarrow{F_1} \\ \Downarrow \alpha_1 \\ \xrightarrow{G_1} \end{array} \mathcal{D} \begin{array}{c} \xrightarrow{F_2} \\ \Downarrow \alpha_2 \\ \xrightarrow{G_2} \end{array} \mathcal{E} \rightsquigarrow \mathcal{C} \begin{array}{c} \xrightarrow{F_2 \circ F_1} \\ \Downarrow \alpha_2 * \alpha_1 \\ \xrightarrow{G_2 \circ G_1} \end{array} \mathcal{E} \quad \text{where } (\alpha_2 * \alpha_1)_X = G_2((\alpha_1)_X) \circ (\alpha_2)_{F_1(X)}$$

Horizontal composition can also be defined in another equivalent way using commutativity of

$$\begin{array}{ccc} F_2 F_1(X) & \xrightarrow{(\alpha_2)_{F_1(X)}} & G_2 F_1(X) \\ F_2((\alpha_1)_X) \downarrow & & \downarrow G_2((\alpha_1)_X) \\ F_2 G_1(X) & \xrightarrow{(\alpha_2)_{G_1(X)}} & G_2 G_1(X) \end{array}$$

The diagram commutes by naturality of α_2 , so $(\alpha_2 * \alpha_1) = (\alpha_2)_{G_1(X)} \circ F_2((\alpha_1)_X)$.

Definition 1.12. Let \mathcal{C}, \mathcal{D} be two categories. Then the *functor category from \mathcal{C} to \mathcal{D}* written $\text{Fun}(\mathcal{C}, \mathcal{D})$ or $\mathcal{D}^{\mathcal{C}}$ is the category whose objects are functors from \mathcal{C} to \mathcal{D} and morphisms are natural transformations.

Remark. Categories, together with functors and natural transformations between them is the prototypical example of a 2-category.

1.2 Equivalences of categories

Definition 1.13. Let \mathcal{C} and \mathcal{D} be two categories. An *equivalence of categories* from \mathcal{C} to \mathcal{D} is the data of

1. $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{C}$ two functors
2. Natural isomorphisms $\eta : 1_{\mathcal{C}} \Rightarrow GF$ and $\varepsilon : FG \Rightarrow 1_{\mathcal{D}}$ where $1_{\mathcal{C}}$ and $1_{\mathcal{D}}$ are the identity functors of \mathcal{C} and \mathcal{D} .

Remark.

1. G is called a *quasi-inverse* of F .
2. Most of the time we say that F is an equivalence if there exists G such that (F, G) is an equivalence.

3. If F, G are contravariant, we speak of *duality* between \mathcal{C} and \mathcal{D} .
4. If two categories are equivalent, every property that can be expressed “in terms of arrows” is preserved.

Definition 1.14. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor. Then, we say

1. F is *faithful* if $\forall X, Y \in \text{Ob}(\mathcal{C}), F : \text{Hom}_{\mathcal{C}}(X, Y) \rightarrow \text{Hom}_{\mathcal{D}}(F(X), F(Y))$ is injective.

$$f \mapsto F(f)$$
2. F is *full* if the previous map is surjective.
3. F is *essentially surjective* if for all $Y \in \text{Ob}(\mathcal{D})$ there is $X \in \text{Ob}(\mathcal{C})$ such that $F(X) \simeq Y$ in \mathcal{D} .

Theorem 1.15. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor. Then F is an equivalence of categories if and only if it is fully faithful and essentially surjective.

Proof. **▲** There is a little set-theoretic issue: an equivalence of categories is always fully faithful and essentially surjective, but the converse requires the axiom of choice for the class $\text{Ob}(\mathcal{C})$.

Suppose $F : \mathcal{C} \rightarrow \mathcal{D}$ is an equivalence of categories, and let $G : \mathcal{D} \rightarrow \mathcal{C}$ be a quasi-inverse of F , together with natural isomorphisms $\eta : 1_{\mathcal{C}} \rightarrow GF$ and $\varepsilon : 1_{\mathcal{D}} \rightarrow FG$. If Y is an object of \mathcal{D} , then $Y \simeq FG(Y)$, so F is essentially surjective. Let X, Y be objects of \mathcal{C} . To show F is fully faithful we will construct an inverse to $F : \text{Hom}_{\mathcal{C}}(X, Y) \rightarrow \text{Hom}_{\mathcal{D}}(F(X), F(Y))$. For any $f \in \text{Hom}_{\mathcal{D}}(F(X), F(Y))$, we have a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & GF(X) \\ f \downarrow & & \downarrow GF(f) \\ Y & \xrightarrow{\eta_Y} & GF(Y) \end{array}$$

which prompts us to define $\phi : \text{Hom}_{\mathcal{D}}(F(X), F(Y)) \rightarrow \text{Hom}_{\mathcal{C}}(X, Y)$. We now check it is

$$g \mapsto \eta_Y^{-1} \circ G(g) \circ \eta_X$$

the map we’re looking for. If $f : X \rightarrow Y$, since the above diagram commutes and η_Y is invertible, we get that $\phi(F(f)) = f$, so $\phi \circ F = \text{id}_{\text{Hom}_{\mathcal{C}}(X, Y)}$, which means F is faithful. We have two commutative diagrams, by definition of ϕ and by naturality of η :

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & GF(X) \\ \phi(g) \downarrow & & \downarrow G(g) \\ Y & \xrightarrow{\eta_Y} & GF(Y) \end{array} \quad \begin{array}{ccc} X & \xrightarrow{\eta_X} & GF(X) \\ \phi(g) \downarrow & & \downarrow GF(\phi(g)) \\ Y & \xrightarrow{\eta_Y} & GF(Y) \end{array}$$

therefore, $G(g) \circ \eta_X = GF(\phi(g)) \circ \eta_X$. Since η_X is invertible, $G(g) = GF(\phi(g))$. The previous point shows that G is faithful, so $g = F(\phi(g))$, hence F is full.

Now suppose F is fully faithful and essentially surjective. Our goal is to construct G . For any $Y \in \text{Ob}(\mathcal{D})$, since F is essentially surjective, there exists $X_Y \in \text{Ob}(\mathcal{C})$ and an isomorphism $\varepsilon_Y : Y \rightarrow F(X_Y)$. Therefore, for any $Y, Z \in \text{Ob}(\mathcal{D})$ and $f : Y \rightarrow Z$, we have a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{f} & Z \\ \varepsilon_Y \downarrow & & \downarrow \varepsilon_Z \\ F(X_Y) & \xrightarrow[\varepsilon_Z \circ f \circ \varepsilon_Y^{-1}]{} & F(X_Z) \end{array}$$

Which leads us to define $G(Y) = X_Y$ and $G(f)$ to be the unique morphism $m_f : X_Y \rightarrow X_Z$ such that $F(m_f) = \varepsilon_Z \circ f \circ \varepsilon_Y^{-1}$ (this works because F is fully faithful). We have $G(\text{id}_Y) = \text{id}_{X_Y}$ since $\varepsilon_Y \circ \text{id}_Y \circ \varepsilon_Y^{-1} = \text{id}_Y$ and $F(\text{id}_{X_Y}) = \text{id}_Y$. The next diagram shows $G(g \circ f) = G(g) \circ G(f)$:

$$\begin{array}{ccccc}
 & & g \circ f & & \\
 & \nearrow & & \searrow & \\
 W & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\
 \downarrow \varepsilon_W & & \downarrow \varepsilon_Y & & \downarrow \varepsilon_Z \\
 F(X_W) & \xrightarrow{F(m_f)} & F(X_Y) & \xrightarrow{F(m_g)} & F(X_Z) \\
 & \searrow & & \nearrow & \\
 & & F(m_g \circ m_f) = F(m_g) \circ F(m_f) & &
 \end{array}$$

By this construction, ε is a natural isomorphism $\text{id}_{\mathcal{D}} \Rightarrow FG$ (look at the above diagrams). Now, pick $Y, Z \in \text{Ob}(\mathcal{C})$ and $f : Y \rightarrow Z$. We have $GF(Y) = X_{F(Y)}$ and $\varepsilon_Y : F(Y) \xrightarrow{\sim} F(X_{F(Y)})$. Since F is fully faithful, there exists a unique $\eta_Y : Y \rightarrow X_{F(Y)} = GF(Y)$ such that $F(\eta_Y) = \varepsilon_Y$. Here, $\eta_Y = G(\varepsilon_Y)$, which means that η_Y is an isomorphism since functors preserve isomorphisms. We obtain the diagram

$$\begin{array}{ccc}
 Y & \xrightarrow{\eta_Y} & GF(Y) \\
 \downarrow f & & \downarrow GF(f) \\
 Z & \xrightarrow{\eta_Z} & GF(Z)
 \end{array}$$

The diagram commutes because $GF(f)$ is the unique morphism such that

$$F(GF(f)) = \varepsilon_Z \circ F(f) \circ \varepsilon_Y^{-1} = F(\eta_Z \circ f \circ \eta_Y^{-1})$$

and F is faithful. η is then a natural isomorphism $\text{id}_{\mathcal{C}} \Rightarrow GF$. □

Example 1.16. $\mathbf{Vect}_k \simeq \mathbf{Mat}_k$ through the functor $n \mapsto k^n$ and $(A : n \rightarrow m) \mapsto (f_A : k^n \rightarrow k^m)$.

2 Universal properties

References:

- Riehl (Chapters II, III, IV)
- Mac Lane (Chapters III, IV, V)
- Assem (Chapters III, IV, V)

► Let S be a set together with an equivalence relation \sim . Let S/\sim be the quotient set, and $\pi : S \rightarrow S/\sim$ be the projection. For any $f : S \rightarrow X$ compatible with \sim , there exists a unique map $\bar{f} : S/\sim \rightarrow X$ such that $f = \bar{f} \circ \pi$. This is represented by the following commutative diagram :

$$\begin{array}{ccc} S & \xrightarrow{f} & X \\ \pi \downarrow & \nearrow \exists! \bar{f} & \\ S/\sim & & \end{array}$$

We say that $S \xrightarrow{\pi} S/\sim$ is a solution to the universal problem posed by the compatible maps. Such a solution is unique up to unique isomorphism: if $S \xrightarrow{p} S'$ is another solution, then we get the three commutative diagrams

$$\begin{array}{ccc} S & \xrightarrow{p} & S' \\ \pi \downarrow & \nearrow \exists! a & \\ S/\sim & & \end{array} \quad \begin{array}{ccc} S & \xrightarrow{\pi} & S/\sim \\ p \downarrow & \nearrow \exists! b & \\ S' & & \end{array} \quad \begin{array}{ccc} S & \xrightarrow{p} & S' \\ p \downarrow & \nearrow \text{id}_{S'} & \uparrow a \\ S' & \xrightarrow{b} & S/\sim \end{array}$$

then $abp = a\pi = p$. The identity of S' also makes this diagram commute so by uniqueness $ab = \text{id}_{S'}$ and similarly $ba = \text{id}_{S/\sim}$.

2.1 Initial and final objects

Definition 2.1. Let \mathcal{C} be a category. An object $c \in \text{Ob}(\mathcal{C})$ is *initial* (*final*) if for all $d \in \text{Ob}(\mathcal{C})$ there exists a unique morphism $c \rightarrow d$ (a unique morphism $d \rightarrow c$).

Proposition 2.2. *If an initial/final object exists, then it is unique up to unique isomorphism.*

Proof. Let c, c' be two initial objects. Then there exists a unique morphism $f : c \rightarrow c'$ and a unique morphism $g : c' \rightarrow c$. There also exists a unique morphism $c \rightarrow c$, that is id_c . Therefore, $gf = \text{id}_c$. In the same way, $fg = \text{id}_{c'}$. Therefore, c and c' are isomorphic and the isomorphism is unique. \square

Examples 2.3.

1. \emptyset is initial in **Set** and any singleton is final.
2. $\{0\}$ is both initial and final in **Vect**_k (or **RMod**).
3. The category of fields does not have initial/final objects (reason on field characteristics).

We want to say a universal object is an initial or final object. A category has at most 2, so this may seem a little restrictive, but this is solved by thinking of a good category.

Definition 2.4. Let $F : \mathcal{C} \rightarrow \mathbf{Set}$ be a functor. Let $\int F$ be the category defined by

$$\begin{aligned}\mathrm{Ob}(\int F) &= \{(c, x) \mid c \in \mathrm{Ob}(\mathcal{C}) \text{ and } x \in F(c)\} \\ \mathrm{Hom}((c, x), (c', x')) &= \{f \in \mathrm{Hom}(c, c') \mid F(f)(x) = x'\}\end{aligned}$$

where composition is composition in \mathcal{C} , and $\mathrm{id}_{(c, x)} = \mathrm{id}_c$ for all x . If F is contravariant, let $\int F$ have the same objects and morphisms $\mathrm{Hom}((c, x), (c', x')) = \{f \in \mathrm{Hom}(c, c') \mid F(f)(x') = x\}$.

Proposition 2.5. *There is a forgetful functor $\pi : \int F \rightarrow \mathcal{C}$ defined by $\pi(c, x) = c$ and $\pi(f : (c, x) \rightarrow (c', x')) = f : c \rightarrow c'$.*

Example 2.6. Let S be a set, and \sim an equivalence relation on S . Let $F : \mathbf{Set} \rightarrow \mathbf{Set}$ be defined by $F(X) = \{f : S \rightarrow X \mid x \sim y \Rightarrow f(x) = f(y)\}$ and $F(\alpha : X \rightarrow Y) = \alpha \circ -$.

$\int F$ has for objects $(X, S \xrightarrow{f} X)$ where f is compatible with \sim , and for morphisms α that makes

this diagram commute:

$$\begin{array}{ccc} S & \xrightarrow{f'} & X' \\ f \downarrow & \nearrow \alpha & \\ X & & \end{array}$$

$(S/\sim, S \xrightarrow{\pi} S/\sim)$ is an initial object of $\int F$.

Definition 2.7. Let $F : \mathcal{C} \rightarrow \mathbf{Set}$ be a functor. A *universal element* for F is an initial object of $\int F$, that is a pair (c, x) with $c \in \mathrm{Ob}(\mathcal{C})$ and $x \in F(c)$ such that

$$\forall (d, y), d \in \mathrm{Ob}(\mathcal{C}), y \in F(d), \exists! \alpha : c \rightarrow d, y = F(\alpha)(x)$$

Definition 2.8. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor and $d \in \mathrm{Ob}(\mathcal{D})$. A *universal arrow from d to F* is a pair (c, f) where $c \in \mathrm{Ob}(\mathcal{C})$ and $f \in \mathrm{Hom}_{\mathcal{D}}(d, F(c))$, such that

$$\forall (c', f'), c' \in \mathrm{Ob}(\mathcal{C}), f' : d \rightarrow F(c'), \exists! \alpha \in \mathrm{Hom}_{\mathcal{C}}(c, c'), F(\alpha) \circ f = f'$$

$$\begin{array}{ccc} & d & \\ f \swarrow & & \searrow \forall f' \\ F(c) & \xrightarrow{F(\alpha)} & F(c') \\ c & \xrightarrow{\exists! \alpha} & c' \end{array}$$

Exercise. Define a category $d \downarrow F$ such that a universal arrow is an initial object of $d \downarrow F$.

Example 2.9. Let $U : \mathbf{Vect}_k \rightarrow \mathbf{Set}$ be the forgetful functor. Let $X \in \mathbf{Set}$. A universal arrow from X to U is the “best” k -vector space V_X with a map $X \rightarrow V_X$. Set $V_X = k[X]$ the k -vector space with basis X , and $i : X \rightarrow V_X$ that maps $x \in X$ to the corresponding basis element. Then, for any vector space V and map $f : X \rightarrow U(V)$, f can be extended by linearity into a linear map $\tilde{f} : k[X] \rightarrow V$, which makes this diagram commute:

$$\begin{array}{ccc} & X & \\ i \swarrow & & \searrow f \\ k[X] & \xrightarrow{\tilde{f}} & U(V) \end{array}$$

If α is another map that makes the diagram commute then α and \tilde{f} coincide on a basis of $k[X]$ and therefore are equal.

Proposition 2.10. *Universal elements and arrows are two equivalent notions.*

Proof. Let $F : \mathcal{C} \rightarrow \mathbf{Set}$ be a functor and (c, x) a universal element for F . Consider $f_x : \{*\} \rightarrow F(c)$. Then, (c, f_x) is a universal arrow $* \rightarrow F$, because $F(\alpha)(x) = y$ iff $F(\alpha) \circ f_x = f_y$.

$$\begin{array}{ccc} & \{*\} & \\ f_x \swarrow & & \searrow f_y \\ F(c) & \xrightarrow{F(\alpha)} & F(c') \end{array}$$

Conversely, if $F : \mathcal{C} \rightarrow \mathcal{D}$ is a functor and (c, f) is a universal arrow $d \rightarrow F$, then consider the functor $\text{Hom}_{\mathcal{D}}(d, F(-)) : \mathcal{C} \rightarrow \mathbf{Set}$ (we need to assume \mathcal{D} is locally small so the functor is set-valued). Then, $f \in \text{Hom}_{\mathcal{D}}(d, F(c))$ is a universal element for this functor. \square

2.2 Representable functors

Definition 2.11. Let \mathcal{C} be a (locally small) category, and $F : \mathcal{C} \rightarrow \mathbf{Set}$ a functor.

1. We say that F is *representable* if there is some $c \in \text{Ob}(\mathcal{C})$ such that F and $\text{Hom}_{\mathcal{C}}(c, -)$ are naturally isomorphic (if F is contravariant, use $\text{Hom}_{\mathcal{C}}(-, c)$ instead).
2. A *representation* of F is the data of $c \in \text{Ob}(\mathcal{C})$ and a natural isomorphism $\eta : \text{Hom}(c, -) \Rightarrow F$.

Example 2.12. The forgetful functor $U : \mathbf{Grp} \rightarrow \mathbf{Set}$ is representable since $\text{Hom}_{\mathbf{Grp}}(\mathbb{Z}, -) \simeq U$. The natural isomorphism is given by $\alpha \in \text{Hom}_{\mathbf{Grp}}(\mathbb{Z}, G) \mapsto \alpha(1) \in G$.

The following theorem explains how to find the natural isomorphism $\alpha : \text{Hom}_{\mathcal{C}}(c, -) \Rightarrow F$ in general.

Theorem 2.13 (Yoneda lemma). *Let $F : \mathcal{C} \rightarrow \mathbf{Set}$ be a functor with \mathcal{C} locally small, and $c \in \text{Ob}(\mathcal{C})$. Then,*

$$\begin{array}{ccc} \text{Nat}(\text{Hom}(c, -), F) & \xrightarrow{\sim} & F(c) \\ \alpha & \mapsto & \alpha_c(\text{id}_c) \end{array}$$

and this isomorphism is natural in c and in F .

Proof. Let $\alpha \in \text{Nat}(\text{Hom}(c, -), F)$. Let $d \in \mathcal{C}$ and $f : c \rightarrow d$. By naturality, we have a commutative diagram

$$\begin{array}{ccc} \text{Hom}(c, c) & \xrightarrow{\alpha_c} & F(c) \\ \downarrow f \circ - & & \downarrow F(f) \\ \text{Hom}(c, d) & \xrightarrow{\alpha_d} & F(d) \end{array}$$

This means that $F(f) \circ \alpha_c = \alpha_d \circ (f \circ -)$. Evaluating at id_c , we get $F(f) \circ \alpha_c(\text{id}_c) = \alpha_d(f)$. This shows that the natural transformation α is entirely determined by the value of $\alpha_c(\text{id}_c)$, which shows the map defined above is injective. Conversely, if $e \in F(c)$, then we define $\alpha^e : \text{Hom}(c, -) \Rightarrow F$ by $\alpha_d^e : g \mapsto F(g)(e)$. We check it is a natural transformation:

$$\begin{array}{ccc}
\mathrm{Hom}(c, c) & \xrightarrow{g \mapsto F(g)(e)} & F(c) \\
\downarrow f \circ - & & \downarrow F(f) \\
\mathrm{Hom}(c, d) & \xrightarrow{h \mapsto F(h)(e)} & F(d)
\end{array}$$

and this diagram commutes since for $g : c \rightarrow c$ we have

$$F(f)(F(g)(e)) = F(f \circ g)(e) = F((f \circ -)(g))(e)$$

This shows the map in the theorem is surjective, therefore an isomorphism, and its inverse is given by $e \in F(c) \mapsto \alpha^e$. We now check naturality. We first need to understand what it means to say the isomorphism is natural in c . Let $f : c \rightarrow d$. $\mathrm{Nat}(\mathrm{Hom}(c, -), F)$ is functorial in c , as it is the composition of two contravariant hom-functors. More concretely:

$$c \xrightarrow{f} d \rightsquigarrow \mathrm{Hom}(d, -) \xrightarrow{- \circ f} \mathrm{Hom}(c, -) \rightsquigarrow \mathrm{Nat}(\mathrm{Hom}(c, -), F) \xrightarrow{- \circ (- \circ f)} \mathrm{Nat}(\mathrm{Hom}(d, -), F)$$

(Nat is the hom-functor of the functor category $\mathcal{C}^{\mathbf{Set}}$). One thing to note is that the morphism $f : c \rightarrow d$ induces a natural transformation $\mathrm{Hom}(d, -) \xrightarrow{- \circ f} \mathrm{Hom}(c, -)$, and this makes the whole thing work. This is in general a property of functors defined on a product category, see the remark below. Now, saying the isomorphism, which we'll write $\Phi_{d,F}$, is natural means that the square

$$\begin{array}{ccc}
\mathrm{Nat}(\mathrm{Hom}(c, -), F) & \xrightarrow{\Phi_{c,F}} & F(c) \\
\downarrow - \circ (- \circ f) & & \downarrow F(f) \\
\mathrm{Nat}(\mathrm{Hom}(d, -), F) & \xrightarrow{\Phi_{d,F}} & F(d)
\end{array}$$

commutes. And indeed, if $\alpha : \mathrm{Hom}(c, -) \Rightarrow F$ is a natural transformation,

$$\begin{aligned}
\Phi_{d,F}(\alpha \circ (- \circ f)) &= (\alpha \circ (- \circ f))_d(\mathrm{id}_d) = [\alpha_d \circ (- \circ f)](\mathrm{id}_d) = \alpha_d(f) \\
F(f)(\Phi_{c,F}(\alpha)) &= F(f)(\alpha_c(\mathrm{id}_c)) = \alpha_d(f \circ \mathrm{id}_c) = \alpha_d(f)
\end{aligned}$$

The second to last equality comes from the naturality of α .

We now turn to naturality in F . Let G be another functor $\mathcal{C} \rightarrow \mathbf{Set}$ and $\beta : F \Rightarrow G$ be a natural transformation. We check that

$$\begin{array}{ccc}
\mathrm{Nat}(\mathrm{Hom}(c, -), F) & \xrightarrow{\Phi_{c,F}} & F(c) \\
\downarrow \beta \circ - & & \downarrow \beta_c \\
\mathrm{Nat}(\mathrm{Hom}(c, -), G) & \xrightarrow{\Phi_{c,G}} & G(c)
\end{array}$$

commutes. For $\alpha : \mathrm{Hom}(c, -) \Rightarrow F$, we have

$$\beta_c(\Phi_{c,F}(\alpha)) = \beta_c(\alpha_c(\mathrm{id}_c)) = (\beta \circ \alpha)_c(\mathrm{id}_c) = \Phi_{c,G}(\beta \circ \alpha)$$

which completes the proof of naturality. □

Remark.

1. If $F : \mathcal{C} \rightarrow \mathbf{Set}$, then (c, x) is a universal element for F if and only if the natural transformation $\alpha_x : \text{Hom}(c, -) \Rightarrow F$ induced by x is an isomorphism. Indeed, α_x is an isomorphism iff $\forall c' \in \mathcal{C}$, $(\alpha_x)_{c'} : \text{Hom}(c, c') \rightarrow F(c')$ is bijective iff

$$f \mapsto F(f)(x)$$

$$\forall c' \in \mathcal{C}, \forall y \in F(c'), \exists ! f : c \rightarrow c', F(f)(x) = y$$

2. For universal arrows, use $\text{Hom}_{\mathcal{D}}(d, F(-))$ as before.
3. Let \mathcal{C}, \mathcal{D} and \mathcal{E} be categories, and $F : \mathcal{C} \times \mathcal{D} \rightarrow \mathcal{E}$ be a functor. Let $c, d \in \text{Ob}(\mathcal{C})$, $x, y \in \text{Ob}(\mathcal{D})$ and morphisms $f : c \rightarrow d$, $g : x \rightarrow y$. The morphism f induces a natural transformation $F(f, \text{id}_-) : F(c, -) \Rightarrow F(d, -)$, see the commutative square:

$$\begin{array}{ccc} F(c, x) & \xrightarrow{F(f, \text{id}_x)} & F(d, x) \\ \downarrow F(\text{id}_c, g) & & \downarrow F(\text{id}_d, g) \\ F(c, y) & \xrightarrow{F(f, \text{id}_y)} & F(d, y) \end{array}$$

2.3 Examples of objects defined by universal properties

2.3.1 Products, coproducts

Let \mathcal{C} be a small category and $X, Y \in \text{Ob}(\mathcal{C})$. We define a category $\mathcal{C}_{X,Y}$ whose objects are tuples (Z, f, g) where $Z \in \text{Ob}(\mathcal{C})$ and $f : Z \rightarrow X$, $g : Z \rightarrow Y$ and morphisms are maps $\alpha : Z \rightarrow Z'$ in \mathcal{C} such that the following diagram commutes:

$$\begin{array}{ccccc} & & Z & & \\ & f \swarrow & \downarrow \alpha & \searrow g & \\ X & & & & Y \\ & f' \swarrow & \downarrow & \searrow g' & \\ & & Z' & & \end{array}$$

Definition 2.14. A *product* of X and Y is a final object in $\mathcal{C}_{X,Y}$. Concretely, it is an object $X \times Y$ together with two maps $\pi_X : X \times Y \rightarrow X$ and $\pi_Y : X \times Y \rightarrow Y$ such that for any $(Z, f, g) \in \text{Ob}(\mathcal{C}_{X,Y})$, we have a commutative diagram

$$\begin{array}{ccccc} & & Z & & \\ & \swarrow \forall f & \downarrow \exists ! \alpha & \searrow \forall g & \\ X & \xleftarrow{\pi_X} & X \times Y & \xrightarrow{\pi_Y} & Y \end{array}$$

Since it is defined as being a final object, if it exists, a product is unique up to unique isomorphism.

Examples 2.15. In **Set**, the product of X and Y is the cartesian product. In **Grp**, it is the product group. In **Top**, it is the cartesian product equipped with the product topology. In these examples, the maps in the definition are the canonical projections.

The notion dual to the one of a product is called a coproduct.

Definition 2.16. A *coproduct* of X and Y is a product in \mathcal{C}^{op} . Concretely, it satisfies the universal property expressed by this commutative diagram:

$$\begin{array}{ccccc} X & \xrightarrow{i_X} & X \sqcup Y & \xleftarrow{i_Y} & Y \\ & \searrow \forall f & \downarrow \exists! \alpha & \swarrow \forall g & \\ & & Z & & \end{array}$$

Examples 2.17. In **Set**, the coproduct of X and Y is the disjoint union together with canonical inclusion. In **Top**, the coproduct of X and Y is their disjoint union equipped with the disjoint union topology. However, in **Grp**, the underlying set of the coproduct of two groups is not the disjoint union.

2.3.2 Equalizers and coequalizers

Definition 2.18. Let \mathcal{C} be a category and $X, Y \in \text{Ob}(\mathcal{C})$, $f, g : X \rightarrow Y$. Consider the contravariant functor $F : \mathcal{C} \rightarrow \mathbf{Set}$ defined by $F(c) = \{\alpha : c \rightarrow X \mid f\alpha = g\alpha\}$ and $F(\beta) = - \circ \beta$. An *equalizer* in \mathcal{C} is a representation of the contravariant functor F .

By the Yoneda lemma, a natural transformation $\text{Hom}(-, c) \Rightarrow F$ is the same as an element of $F(c)$, so a representation of F is a pair (c, e) with $c \in \text{Ob}(\mathcal{C})$ and $e \in F(c)$ such that the natural transformation given by the Yoneda lemma is an isomorphism. Concretely, we want $\eta_e : \text{Hom}(d, c) \rightarrow F(d)$ to be an isomorphism for all $d \in \text{Ob}(\mathcal{C})$. This translates into

$$\begin{array}{ccc} & & d \\ & \swarrow \exists! \alpha & \downarrow \forall h \\ c & \xrightarrow{e} & X \end{array} \quad \begin{array}{ccc} & & Y \\ & \xleftarrow{f} & \\ & \xrightarrow{g} & \end{array}$$

$h \mapsto F(h)(e)$

the following diagram:

$$\begin{array}{ccc} & d & \\ \exists! \alpha \swarrow & \downarrow \forall h & \\ c & \xrightarrow{e} & X \end{array} \quad \begin{array}{ccc} & & Y \\ & \xleftarrow{f} & \\ & \xrightarrow{g} & \end{array}$$

Example 2.19. In **Set**, $E = \{x \in X \mid f(x) = g(x)\} \hookrightarrow X$ is an equalizer.

The dual notion is that of a coequalizer.

Definition 2.20. A *coequalizer* of $X \rightrightarrows Y$ is an object $Z \in \text{Ob}(\mathcal{C})$ together with a morphism $\pi : Y \rightarrow Z$ such that $\pi f = \pi g$ and that universal to this property:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow g & \downarrow \forall h \\ & & Z' \end{array} \quad \begin{array}{ccc} & & Z \\ & \xleftarrow{\pi} & \\ & \swarrow \exists! \alpha & \end{array}$$

Example 2.21. In **Set**, consider the equivalence relation \sim on Y generated by $f(x) \sim g(x)$ (the smallest equivalence relation on Y with this property). Then $y \xrightarrow{\pi} Y/\sim$ is a coequalizer.

2.4 Adjoint functors

This notion was introduced by Kan in 1958.

Definition 2.22. An *adjunction* (G, D) is a pair of functors $G : \mathcal{C} \rightarrow \mathcal{D}$ and $D : \mathcal{D} \rightarrow \mathcal{C}$ together with an isomorphism $\text{Hom}_{\mathcal{D}}(G(c), d) \simeq \text{Hom}_{\mathcal{C}}(c, D(d))$ which is natural in both c and d . We write $G \dashv D$ and say G is left adjoint to D and D is right adjoint to G .

If $G \dashv D$ we have $\forall c, d \in \text{Ob}(\mathcal{C}) \times \text{Ob}(\mathcal{D})$,

$$\text{Hom}_{\mathcal{D}}(G(c), d) \xrightarrow[\alpha_{c,d}]{\sim} \text{Hom}_{\mathcal{C}}(c, D(d))$$

and in particular when $d = G(c)$ we get $\text{Hom}_{\mathcal{D}}(G(c), G(c)) \xrightarrow[\alpha_{c,G(c)}]{\sim} \text{Hom}_{\mathcal{C}}(c, DG(c))$.

Let $\eta_c : c \rightarrow DG(c)$ be the image of $\text{id}_{G(c)}$. This gives a collection of morphisms $- \rightarrow DG(-)$.

Proposition 2.23. *These morphisms make up a natural transformation $\text{id}_{\mathcal{C}} \Rightarrow DG$.*

Proof. Let $f : c \rightarrow d$. We want to show that

$$\begin{array}{ccc} c & \xrightarrow{\eta_c = \alpha_{c,G(c)}(\text{id}_{G(c)})} & DG(c) \\ \downarrow f & & \downarrow DG(f) \\ d & \xrightarrow{\eta_d = \alpha_{d,G(d)}(\text{id}_{G(d)})} & DG(d) \end{array}$$

commutes. By naturality of the isomorphism α given by the adjunction, we get the following commutative diagram

$$\begin{array}{ccc} \text{Hom}(G(c), G(c)) & \xrightarrow[\alpha_{c,G(c)}]{\sim} & \text{Hom}(c, DG(c)) \\ G(f) \circ - \downarrow & & DG(f) \circ - \downarrow \\ \text{Hom}(G(c), G(d)) & \xrightarrow[\alpha_{c,G(d)}]{\sim} & \text{Hom}(c, DG(d)) \\ - \circ G(f) \uparrow & & - \circ f \uparrow \\ \text{Hom}(G(d), G(d)) & \xrightarrow[\alpha_{d,G(d)}]{\sim} & \text{Hom}(d, DG(d)) \end{array}$$

which gives us these equations:

$$\begin{aligned} DG(f) \circ \eta_c &= DG(f) \circ \alpha_{c,G(c)}(\text{id}_{G(c)}) = \alpha_{c,G(d)}(G(f) \circ \text{id}_{G(c)}) = \alpha_{c,G(d)}(G(f)) \\ \eta_d \circ f &= \alpha_{d,G(d)}(\text{id}_{G(d)}) \circ f = \alpha_{c,G(d)}(\text{id}_{G(c)} \circ G(f)) = \alpha_{c,G(d)}(G(f)) \end{aligned}$$

which completes the proof. \square

We also get a natural transformation $\varepsilon : GD \Rightarrow \text{id}_{\mathcal{D}}$ when $c = D(d)$ by setting $\varepsilon_d = \alpha_{D(d),d}^{-1}(\text{id}_{D(d)})$.

Definition 2.24. The natural transformation $\eta : \text{id}_{\mathcal{C}} \Rightarrow DG$ is called the *unit* of the adjunction. The natural transformation $\varepsilon : GD \Rightarrow \text{id}_{\mathcal{D}}$ is called its *counit*.

Proposition 2.25. *Let $\mathcal{C} \begin{array}{c} \xrightarrow{G} \\ \xleftarrow{D} \end{array} \mathcal{D}$ be two functors. Then, $G \dashv D$ if and only if there are natural transformations $\eta : \text{id}_{\mathcal{C}} \Rightarrow DG$ and $\varepsilon : GD \Rightarrow \text{id}_{\mathcal{D}}$ such that the following diagrams commute:*

$$\begin{array}{ccc}
G & \xrightarrow{G\eta} & GDG \\
& \searrow \text{id}_G & \downarrow \varepsilon G \\
& & G
\end{array}
\qquad
\begin{array}{ccc}
D & \xrightarrow{\eta D} & DGD \\
& \searrow \text{id}_D & \downarrow D\varepsilon \\
& & D
\end{array}$$

where $G\eta$ is the natural transformation given by the morphisms $G(\eta_c)$ and εG is the one give by morphisms $\varepsilon_{G(c)}$ (and similarly for ηD and $D\varepsilon$).

Proof. Suppose $G \dashv D$. Let $\eta : \text{id}_C \Rightarrow DG$ and $\varepsilon : GD \Rightarrow \text{id}_D$ be the unit and counit of the adjunction. Let $c \in \mathcal{C}$. We have

$$(\varepsilon G)_c \circ (G\eta)_c = \varepsilon_{G(c)} \circ G(\eta_c) = \alpha_{DG(c), G(c)}^{-1}(\text{id}_{DG(c)}) \circ G(\alpha_{c, G(c)}(\text{id}_{G(c)}))$$

and the naturality of α gives the following commutative diagram

$$\begin{array}{ccc}
\text{Hom}(G(c), G(c)) & \xleftarrow[\alpha_{c, G(c)}^{-1}]{\sim} & \text{Hom}(c, DG(c)) \\
\uparrow - \circ G(\alpha_{c, G(c)}(\text{id}_{G(c)})) & & \uparrow - \circ \alpha_{c, G(c)}(\text{id}_{G(c)}) \\
\text{Hom}(GDG(c), G(c)) & \xleftarrow[\alpha_{DG(c), G(c)}^{-1}]{\sim} & \text{Hom}(DG(c), DG(c))
\end{array}$$

which shows that $(\varepsilon G)_c \circ (G\eta)_c = \text{id}_{G(c)}$, hence $\varepsilon G \circ G\eta = \text{id}_G$. The commutativity of the other triangle is treated in a similar way.

Now assume that there are natural transformations η and ε that make both triangles commute. We define two maps

$$\begin{aligned}
\alpha_{c,d} : \text{Hom}(G(c), d) &\rightarrow \text{Hom}(c, D(d)) \\
f &\mapsto D(f) \circ \eta_c \\
\beta_{c,d} : \text{Hom}(c, D(d)) &\rightarrow \text{Hom}(G(c), d) \\
g &\mapsto \varepsilon_d \circ G(g)
\end{aligned}$$

and we show these are natural isomorphisms that give us the adjunction. First we check naturality of α . Let $f : c \rightarrow c' \in \text{Mor}(\mathcal{C})$ and $g : d \rightarrow d' \in \text{Mor}(\mathcal{D})$. We need to check that the diagrams

$$\begin{array}{ccc}
\text{Hom}_{\mathcal{D}}(G(c), d) & \xrightarrow{\alpha_{c,d}} & \text{Hom}_{\mathcal{C}}(c, D(d)) \\
\uparrow - \circ G(f) & & \uparrow - \circ f \\
\text{Hom}_{\mathcal{D}}(G(c'), d) & \xrightarrow{\alpha_{c',d}} & \text{Hom}_{\mathcal{C}}(c', D(d))
\end{array}
\qquad
\begin{array}{ccc}
\text{Hom}_{\mathcal{D}}(G(c), d) & \xrightarrow{\alpha_{c,d}} & \text{Hom}_{\mathcal{C}}(c, D(d)) \\
\downarrow g \circ - & & \downarrow D(g) \circ - \\
\text{Hom}_{\mathcal{D}}(G(c), d') & \xrightarrow{\alpha_{c,d'}} & \text{Hom}_{\mathcal{C}}(c, D(d'))
\end{array}$$

commute. We only check the left diagram and leave the right to the reader (sorry). We have

$$\begin{aligned}
\alpha_{c,d} \circ (- \circ G(f)) &= (D(-) \circ \eta_c) \circ (- \circ G(f)) = D(- \circ G(f)) \circ \eta_c = D(-) \circ DG(f) \circ \eta_c \\
(- \circ f) \circ \alpha_{c',d} &= (- \circ f) \circ (D(-) \circ \eta_{c'}) = D(-) \circ \eta_{c'} \circ f = D(-) \circ DG(f) \circ \eta_c
\end{aligned}$$

One shows β is natural in c and d in a similar way. We leave it to the reader (sorry again). Now we need to check that α and β are inverses of each other, and that's where the triangle diagrams come into play.

$$\alpha_{c,d} \circ \beta_{c,d} = D(\varepsilon_d \circ G(-)) \circ \eta_c = D(\varepsilon_d) \circ DG(-) \circ \eta_c = D(\varepsilon_d) \circ \eta_{D(d)} \circ - = -$$

We used the definitions of α and β , the functoriality of D , the naturality of η and the second triangle diagram. We leave to the reader (sorry) to check that $\beta_{c,d} \circ \alpha_{c,d}$ is also the identity. \square

Examples 2.26.

1. The forgetful functor $\mathbf{Ab} \rightarrow \mathbf{Set}$ is right adjoint to the free abelian group functor $\mathbf{Set} \rightarrow \mathbf{Ab}$.
2. The forgetful functor $\mathbf{Ab} \rightarrow \mathbf{Grp}$ is right adjoint to the abelianization functor $\mathbf{Grp} \rightarrow \mathbf{Ab}$ that sends a group G to its abelianization $G^{ab} = G/[G, G]$ and a morphism $f : G \rightarrow H$ to the induced morphism $f^{ab} : G^{ab} \rightarrow H^{ab}$.
3. The forgetful functor $\mathbf{Top} \rightarrow \mathbf{Set}$ is right adjoint to the functor $\mathbf{Set} \rightarrow \mathbf{Top}$ that takes a set and equips it with the coarse topology. The forgetful functor $\mathbf{Top} \rightarrow \mathbf{Set}$ is also left adjoint to the functor $\mathbf{Set} \rightarrow \mathbf{Top}$ that equips a set with the discrete topology.
4. Let G be a group, H one of its subgroups and k be a field. We have a functor from the category $\mathbf{Rep}_k(G)$ of representations of G on k -vector spaces to the category $\mathbf{Rep}_k(H)$ of representations of H on k -vector spaces. It is the restriction functor Res_H^G . Its left adjoint is Ind_H^G , the induced representation functor.

Theorem 2.27. *Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor. The following are equivalent:*

1. F admits a left adjoint
2. For all $X \in \text{Ob}(\mathcal{D})$, $\text{Hom}_{\mathcal{D}}(X, F(-))$ is representable
3. For all $X \in \text{Ob}(\mathcal{D})$, there exists a universal arrow $X \rightarrow F$

Corollary. *If they exist, adjoints are unique up to isomorphism.*

Proof. $2 \iff 3$ was the subject of a previous remark right after the Yoneda lemma. We prove $1 \iff 2$. Suppose F admits a left adjoint G . Let $X \in \text{Ob}(\mathcal{D})$. Then for all $Y \in \text{Ob}(\mathcal{C})$ we have a bijection $\text{Hom}_{\mathcal{D}}(X, F(Y)) \simeq \text{Hom}_{\mathcal{C}}(G(X), Y)$ which is natural in Y , so $G(X)$ represents $\text{Hom}_{\mathcal{D}}(X, F(-))$. For the converse, suppose all functors $\text{Hom}_{\mathcal{D}}(X, F(-))$ are representable. We define $G(X)$ to be an object of \mathcal{C} that represents $\text{Hom}_{\mathcal{D}}(X, F(-))$. Now choose $X, Y \in \text{Ob}(\mathcal{D})$ and $f : X \rightarrow Y$. We need to define $G(f)$. We wish to have a commuting square

$$\begin{array}{ccc} \text{Hom}(G(X), -) & \xrightarrow{\sim} & \text{Hom}(X, F(-)) \\ \exists! \gamma \uparrow & & \uparrow - \circ f \\ \text{Hom}(G(Y), -) & \xrightarrow{\sim} & \text{Hom}(Y, F(-)) \end{array}$$

We need to recover a map $G(X) \rightarrow G(Y)$ such that composing with it gives us γ . This works by the Yoneda lemma, which tells us that the natural transformation γ comes from an element $\text{Hom}(G(X), G(Y))$. Call it $G(f)$. It remains to check this does define a functor. Using this diagram with $X = Y$ and $f = \text{id}_X$ shows that $G(\text{id}_X) = \text{id}_{G(X)}$. Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ in \mathcal{C} . Then we draw

$$\begin{array}{ccccc} & & \xrightarrow{- \circ G(g \circ f)} & & \\ \text{Hom}(G(Z), -) & \xrightarrow{- \circ G(g)} & \text{Hom}(G(Y), -) & \xrightarrow{- \circ G(f)} & \text{Hom}(G(X), -) \\ \downarrow \wr & & \downarrow \wr & & \downarrow \wr \\ \text{Hom}(Z, F(-)) & \xrightarrow{- \circ g} & \text{Hom}(Y, F(-)) & \xrightarrow{- \circ f} & \text{Hom}(X, F(-)) \\ & & \xleftarrow{- \circ (g \circ f)} & & \end{array}$$

and this diagram shows that $G(g \circ f) = G(g) \circ G(f)$ (because the map γ above is unique). \square

This theorem shows there is a deep link between universal properties and adjoint functors.

2.5 Limits and colimits

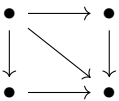
(This subsection may be skipped on a first reading.)

Let us recall the definition of a functor category.

Definition 2.28. Let \mathcal{C}, \mathcal{D} be two categories. Then $\text{Fun}(\mathcal{C}, \mathcal{D})$, also written $\mathcal{D}^{\mathcal{C}}$, is the category whose objects are functors $\mathcal{C} \rightarrow \mathcal{D}$ and morphisms are natural transformations between such functors, with composition given by vertical composition. It is called the *functor category category from \mathcal{C} to \mathcal{D}* . When \mathcal{J} is a small category we also say that $\text{Fun}(\mathcal{C}, \mathcal{D})$ is the category of *diagrams of shape \mathcal{J} in \mathcal{C}* .

Examples 2.29.

1. Let $\mathbf{2}$ be the category $\bullet \longrightarrow \bullet$ which has two objects 1 and 2 and three morphisms (two of them being identities).

Then $\mathbf{2} \times \mathbf{2}$ is the category  which has 4 objects and 9 morphisms (4 of them being identities). Then, a functor from $\mathbf{2} \times \mathbf{2}$ to \mathcal{C} is a commutative diagram of this shape in \mathcal{C} .

2. If \mathcal{J} is a small category, there is a functor $\Delta : \text{Ob}(\mathcal{C}) \rightarrow \text{Fun}(\mathcal{J}, \mathcal{C})$ where $\Delta(c)$ is the constant functor at c , that is the functor that sends all objects to c and all morphisms to id_c , and $\Delta(f) = f$, which works since a natural transformation $\Delta(c) \Rightarrow \Delta(d)$ is just the data of one morphism $c \rightarrow d$.

Definition 2.30. A *cone above a diagram $F : \mathcal{J} \rightarrow \mathcal{C}$ with summit $c \in \mathcal{C}$* is a natural transformation $\lambda : \Delta(c) \Rightarrow F$. Dually, a *cone under F with summit c* , also called a *cocone*, is a natural transformation $\lambda : F \Rightarrow \Delta(c)$.

Let us unwrap this definition. A cone is a collection of maps $\lambda_j : c \rightarrow F(j)$ for all $j \in \text{Ob}(\mathcal{J})$, such that for any morphism $f : i \rightarrow j \in \text{Mor}(\mathcal{J})$, this diagram commutes:

$$\begin{array}{ccc} & c & \\ \lambda_i \swarrow & & \searrow \lambda_j \\ F(i) & \xrightarrow{F(f)} & F(j) \end{array}$$

Definition 2.31. Let $F : \mathcal{J} \rightarrow \mathcal{C}$ be a diagram. A *limit* (or *projective limit* or *inverse limit*) of F is a universal cone above F , in the sense that it is a final object in the category of cones above F . Dually, a *colimit* (or *inductive limit* or *direct limit*) is a universal cocone, that is an initial object in the category of cones under F .

Concretely, a limit of $F : \mathcal{J} \rightarrow \mathcal{C}$ is a pair $(\lim F, \phi)$ with $\lim F \in \text{Ob}(\mathcal{C})$ and $\phi : \Delta(\lim F) \Rightarrow F$ is such that for any cone $\lambda : \Delta(c) \Rightarrow F$, there exists a unique morphism $f : X \rightarrow \lim F \in \text{Mor}(\mathcal{C})$, such that the diagram on the left commutes:

$$\begin{array}{ccc}
\Delta(c) & \xrightarrow{\Delta(f)} & \Delta(\lim F) \\
\searrow \lambda & & \swarrow \phi \\
& F &
\end{array}
\quad \text{which is equivalent to} \quad
\forall j \in \mathcal{J}, \quad
\begin{array}{ccc}
c & \xrightarrow{f} & \lim F \\
\searrow \lambda_j & & \swarrow \phi_j \\
& F(j) &
\end{array}$$

In compact form, $\text{Hom}_{\mathcal{C}}(-, \lim F) \simeq \text{Hom}_{\text{Fun}(\mathcal{J}, \mathcal{C})}(\Delta(-), F)$.

Exercise. Do the same for colimits.

Remark.

1. If a limit exists it is unique up to isomorphism (unique isomorphism that commutes with the legs of the cone)
2. If all limits exist, then \lim becomes a functor $\lim : \text{Fun}(\mathcal{J}, \mathcal{C}) \rightarrow \mathcal{C}$ in the following way. Recall that theorem 2.27 says a functor D admits a left adjoint iff for all objects X in its codomain, $\text{Hom}(X, D(-))$ is representable. The compact form of the definition of a limit says that the functor $\text{Hom}(\Delta(-), F)$ is representable for all F (since we assume all limits exist). A dual version of the theorem gives that Δ admits a right adjoint, which is \lim since $\text{Hom}(c, \lim F) \simeq \text{Hom}(\Delta(c), F)$. If $\eta : F \Rightarrow G$ is a natural transformation, then $\lim(\eta)$ can be constructed in the following way: $\lim F \Rightarrow F \xRightarrow{\eta} G$ is a cone above G , and $\lim(\eta) : \lim F \rightarrow \lim G$ comes from the universality of $\lim G$.

Corollary.

1. If \mathcal{C} has all \mathcal{J} -limits, then $\lim : \text{Fun}(\mathcal{J}, \mathcal{C}) \rightarrow \mathcal{C}$ is a right adjoint to Δ .
2. If \mathcal{C} has all \mathcal{J} -colimits, then $\text{colim} : \text{Fun}(\mathcal{J}, \mathcal{C}) \rightarrow \mathcal{C}$ is a left adjoint to Δ .

Example 2.32.

1. If \mathcal{J} is discrete, that is has no morphisms other than identities, then a functor $F : \mathcal{J} \rightarrow \mathcal{C}$ is the same as a collection $(X_i)_{i \in \mathcal{J}}$ of objects of \mathcal{C} . Then, a limit of F is an object $\lim F \in \text{Ob}(\mathcal{C})$ with morphisms $f_i : \lim F \rightarrow X_i$ such that for all objects $X \in \text{Ob}(\mathcal{C})$ with morphisms $p_i : X \rightarrow X_i$, we have a unique map $\alpha : X \rightarrow \lim F$ that makes this diagram commute for all $i, j \in \mathcal{J}$:

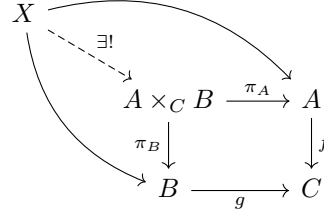
$$\begin{array}{ccccc}
& & X & & \\
& \swarrow p_i & \vdots \alpha & \searrow p_j & \\
X_i & \xleftarrow{f_i} & \lim F & \xrightarrow{f_j} & X_j
\end{array}$$

We write $\lim F = \prod_{j \in \mathcal{J}} F(j)$ and call it the *product of the $F(j)$ s*. Morphisms f_i are written π_i and called *canonical projections*.

Dually, the colimit of F is called a coproduct and written $\bigsqcup_{j \in \mathcal{J}} F(j)$.

2. If $\mathcal{J} = \bullet \rightrightarrows \bullet$, then a functor $F : \mathcal{J} \rightarrow \mathcal{C}$ is the data of two parallel morphisms in \mathcal{C} . A limit is an equalizer and a colimit is a coequalizer.

3. If $\mathcal{J} = \begin{array}{c} \bullet \\ \downarrow \\ \bullet \rightarrow \bullet \end{array}$ then $F : \mathcal{J} \rightarrow \mathcal{C}$ is the data of $A, B, C \in \text{Ob}(\mathcal{C})$ with two morphisms $f : A \rightarrow C$ and $g : B \rightarrow C$. The limit $\lim F$ is called a *pullback* of f and g , with universal property depicted here:



4. If $\mathcal{J} = \omega^{\text{op}}$, that is $\mathcal{J} = \cdots \rightarrow 2 \rightarrow 1 \rightarrow 0$, then $\lim F$ is often called the “inverse limit” of F . Concretely, F is the data of $\cdots \rightarrow F(2) \xrightarrow{\alpha_2} F(1) \xrightarrow{\alpha_1} F(0)$, and a cone above F looks like

$$\begin{array}{c} & c & \\ \lambda_2 \swarrow & \downarrow \lambda_1 & \searrow \lambda_0 \\ \cdots \longrightarrow F(2) & \xrightarrow{\alpha_2} F(1) & \xrightarrow{\alpha_1} F(0) \end{array} \quad \text{we have } (\alpha_i \circ \cdots \circ \alpha_n) \circ \lambda_n = \lambda_i.$$

The typical example of an inverse limit is the one given by $F(n) = \mathbb{Z}/p^n\mathbb{Z}$ in **Ring** with morphisms $\mathbb{Z}/p^{n+1}\mathbb{Z} \rightarrow \mathbb{Z}/p^n\mathbb{Z}$ being reduction mod p^n . The inverse limit $\varprojlim \mathbb{Z}/p^n\mathbb{Z}$ is the ring of p -adic integers. Concretely, $a \in \varprojlim \mathbb{Z}/p^n\mathbb{Z}$ iff $a = (a_i)_{i \in \mathbb{N}}$ such that $a_i \equiv a_j \pmod{p^i} \forall i \leq j$.

5. The dual notion, given by $\mathcal{J} = 0 \rightarrow 1 \rightarrow 2 \rightarrow \cdots$, is obtained by taking the colimit. It is called a *direct limit*. The typical example here is the Prüfer p -group $\varinjlim \mathbb{Z}/p^n\mathbb{Z} = \mathbb{Z}(p^\infty)$.