Some solutions to problems in Manifolds, Sheaves and Cohomology

1 Chapter 3

3.1. Let $k \in \mathbb{N}$. Then, we have the function $x \mapsto \|x\| \in \mathcal{F}(B(0,k))$ where $B(0,k)$ is the open ball of radius k centered at 0. If $\mathcal{F}$ were a sheaf, then we could find $f \in \mathcal{F}(\mathbb{R}^n)$ such that $f_{|B(0,k)} : x \mapsto \|x\|$ for all k. Such a function cannot be bounded, so $\mathcal{F}$ is not a sheaf.
The inclusions $i_U : \mathcal{F}(U) \to \mathcal{C}_{\mathbb{R}^n,\mathbb{R}}(U)$ make up a morphism of sheaves $i : \mathcal{F} \to \mathcal{C}_{\mathbb{R}^n,\mathbb{R}}$. Let $x \in \mathbb{R}^n$. Then i_x is injective, and it is surjective because $\mathbb{R}^n$ is locally compact so a continuous function $f : \mathbb{R}^n \to \mathbb{R}$ is locally bounded. Since $\mathcal{C}_{\mathbb{R}^n,\mathbb{R}}$ is a sheaf, we get that it is the sheafification of $\mathcal{F}$.

3.2. For $U \subseteq V \subseteq X$ open, if $f : V \to \overline{\mathbb{R}}$ is measurable, then $f_{|U}$ also is (because U is open and we use the Borel σ-algebra of X). Il follows that $\mathcal{M}_X(U)$ is a presheaf of functions on X. Now assume X is Lindelöf. Let $U = \bigcup_i U_i$ be an open covering of an open set U and $s_i \in \mathcal{M}_X(U_i)$ be compatible on intersections. Then, the function s obtained by gluing together all the s_is is measurable, because there exists a countable subcover $U = \bigcup_{n\in\mathbb{N}} U_{i_n}$ and

$$s^{-1}(A) = \bigcup_{n\in\mathbb{N}} s_{i_n}^{-1}(A)$$

is a Borel subset of X by properties of σ-algebras. This shows $\mathcal{M}_X$ is a sheaf on X.

3.3.

1. For $U \subseteq V \subseteq \mathbb{R}^n$ open, if $f : V \to \mathbb{R}$ is Lebesgue integrable, then $f_{|U}$ is also Lebesgue integrable. Since a restriction of a function f such that $|f| = 0$ satisfies again $|f| = 0$, restriction morphisms pass to the quotient and $U \mapsto L^1(U)$ is a presheaf of $\mathbb{R}$-vector spaces. For $k \in \mathbb{N}$, the constant function with value 1 on $B(0,k)$ is Lebesgue integrable. However, the constant function with value 1 is not Lebesgue integrable on $\mathbb{R}^n$, which shows $U \mapsto L^1(U)$ is not a sheaf.

2. For $U \subseteq \mathbb{R}^n$ open, we have an inclusion map $i_U : L^1(U) \to L^1_{\text{loc}}(U)$. This inclusion map gives a morphism of presheaves $i : L^1 \to L^1_{\text{loc}}$. It is injective on stalks. Let $x \in \mathbb{R}^n$ and $(U,f) \in (L^1_{\text{loc}})_x$. Since $\mathbb{R}^n$ is locally compact, there exists a compact set $K \subseteq U$ and an open set $V \ni x$ contained in K. Then, $[(U,f)] = i([V, f_{|V}])$. This makes sense since $f_{|V}$ is Lebesgue integrable because f is Lebesgue integrable on $K \supseteq V$. This shows L^1_{loc} is the sheafification of L^1.

3.4. Let $\mathcal{G}$ be a subpresheaf of $\mathcal{F}$. Assume it is a sheaf. Let $U \subseteq X$ be open, $U = \bigcup_i U_i$ be an open covering of U, and $s \in \mathcal{F}(U)$ such that $s_{|U_i} \in \mathcal{G}(U_i)$ for all i. Set $s_i = s_{|U_i} \in \mathcal{G}(U_i)$ for all i. Then, $s_{i|U_i \cap U_j} = s_{j|U_i \cap U_j}$ because s_i and s_j are restrictions of s. Since $\mathcal{G}$ is a sheaf, there exists $s' \in \mathcal{G}(U)$ such that $s_i = s'_{|U_i}$. Since $\mathcal{F}$ is a sheaf, the uniqueness condition gives that $s = s'$, so $s \in \mathcal{G}(U)$. Conversely, assume that for every open set $U \subseteq X$, open covering $U = \bigcup_i U_i$ and $s \in \mathcal{F}(U)$ with $s_{|U_i} \in \mathcal{G}(U_i)$, we have $s \in \mathcal{G}(U)$. Let $U = \bigcup_i U_i$ be an open covering of an open set, and $s_i \in \mathcal{G}(U_i)$ be such that $s_{i U_i \cap U_j} = s_{j U_i \cap U_j}$ on all i, j. Since $\mathcal{F}$ is a sheaf, there exists a unique element $s \in \mathcal{F}(U)$ such that $s_i = s_{|U_i}$ for all i. Then, by assumption, $s \in \mathcal{G}(U)$. Uniqueness comes from the uniqueness of s. Therefore, $\mathcal{G}$ is a sheaf.

3.5.

1. It is clear that Ω is a presheaf. Let $U \subseteq X$ be open, $U = \bigcup_i U_i$ be an open covering and $V_i \subset U_i$ be open sets in X, such that $V_i \cap U_i \cap U_j = V_j \cap U_i \cap U_j$ for all i, j. Then, $\bigcup_i V_i$ is an open set in X contained in U, and

$$\left(\bigcup_i V_i\right) \cap U_j = \bigcup_i (V_i \cap U_j) = \bigcup_i (V_i \cap U_i \cap U_j) = \bigcup_i (V_j \cap U_i \cap U_j) = (V_j \cap U_j) \cap \left(\bigcup_i U_i\right) = V_j$$

   Moreover, if $W \subseteq U$ is another open set satisfying $W \cap U_j = V_j$ for all j, then $\bigcup_j V_j \subseteq W$ and any $x \in W$ is contained in one U_j and therefore in one V_j, so $W = \bigcup_j V_j$. Hence Ω is a sheaf.

2. We first check that $\mathcal{G}_\Phi$ is a subpresheaf of $\mathcal{F}$. Let $V \subseteq U \subseteq X$ be open sets and $s \in \mathcal{G}_\Phi(U)$. Then, $\Phi_V(s_{|V}) = \Phi_U(s) \cap V = U \cap V = V$, so $\mathcal{G}_\Phi$ is a subpresheaf of $\mathcal{F}$. Now, let $U \subseteq X$ be an open set, $U = \bigcup_i U_i$ be an open covering and $s \in \mathcal{F}(U)$ be such that $s_{|U_i} \in \mathcal{G}_\Phi(U_i)$. Then, $(\Phi_U(s)) \cap U_i = \Phi_{U_i}(s_{|U_i}) = U_i$ since s_i is in $\mathcal{G}_\Phi(U_i)$. Therefore,

$$\Phi_U(s) \cap U = \Phi_U(s) \cap \left(\bigcup_i U_i\right) = \bigcup_i \Phi_U(s) \cap U_i = \bigcup_i U_i = U$$

   So $\Phi_U(s) \supseteq U$. Since $\Phi_U(s) \in \Omega(U)$, we have $\Phi_U(s) \subseteq U$, so $\Phi_U(s) = U$ and $s \in \mathcal{G}_\Phi(U)$. The preceding problem allows us to conclude that $\mathcal{G}_\Phi$ is a subsheaf of $\mathcal{F}$.
   We have a map

$$\begin{array}{rcl} \mathrm{Hom}_{\mathrm{Sh}(X)}(\mathcal{F}, \Omega) & \to & \{\text{subsheaves of } \mathcal{F}\} \\ \Phi & \mapsto & \mathcal{G}_\Phi \end{array}$$

   Let $\mathcal{G}$ be a subsheaf of $\mathcal{F}$. For $U \subseteq X$ open, we define

$$\begin{array}{rrcl} \Phi_U^{\mathcal{G}}: & \mathcal{F}(U) & \to & \Omega(U) \\ & s & \mapsto & \displaystyle\bigcup_{\substack{V \subseteq U \text{ open} \\ s_{|V} \in \mathcal{G}(V)}} V \end{array}$$

   Let us check that $\Phi^{\mathcal{G}}$ is a morphism of sheaves $\mathcal{F} \to \Omega$. Let $V \subseteq U \subseteq X$ be open sets, and $s \in \mathcal{F}(U)$. Then,

$$\Phi_V^{\mathcal{G}}(s_{|V}) = \bigcup_{\substack{W \subseteq V \text{ open} \\ s_{|W} \in \mathcal{G}(W)}} W$$

and

$$\Phi_U^{\mathcal{G}}(s) \cap V = V \cap \bigcup_{\substack{W \subseteq U \text{ open} \\ s_{|W} \in \mathcal{G}(W)}} W = \bigcup_{\substack{W \subseteq U \text{ open} \\ s_{|W} \in \mathcal{G}(W)}} (V \cap W) = \bigcup_{\substack{W \subseteq V \text{ open} \\ s_{|W} \in \mathcal{G}(W)}} W$$

So $\Phi^{\mathcal{G}}$ is a morphism of sheaves $\mathcal{F} \to \Omega$.
Now, we check that $\Phi \mapsto \mathcal{G}_\Phi$ and $\mathcal{G} \mapsto \Phi^{\mathcal{G}}$ are inverse bijections of each other. Let $\Phi : \mathcal{F} \to \Omega$ be a morphism of sheaves. Then, for any open set $U \subseteq X$:

$$\Phi_U^{\mathcal{G}_\Phi}(s) = \bigcup_{\substack{V \subseteq U \text{ open} \\ s_{|V} \in \mathcal{G}_\Phi(V)}} V = \bigcup_{\substack{V \subseteq U \text{ open} \\ \Phi_V(s_{|V}) = V}} V = \bigcup_{\substack{V \subseteq U \text{ open} \\ \Phi_U(s) \cap V = V}} V = \bigcup_{\substack{V \subseteq U \text{ open} \\ V \subseteq \Phi_U(s)}} V = \Phi_U(s)$$

And the other way around,

$$\begin{aligned} \mathcal{G}_{\Phi^{\mathcal{G}}}(U) &= \{s \in \mathcal{F}(U) \mid \Phi_U^{\mathcal{G}}(s) = U\} \\ &= \left\{ s \in \mathcal{F}(U) \mid \bigcup_{\substack{V \subseteq U \text{ open} \\ s_{|V} \in \mathcal{G}(V)}} V = U \right\} \end{aligned}$$

If $s \in \mathcal{G}(U)$, then $s \in \mathcal{G}_{\Phi^{\mathcal{G}}}(U)$ (take $V = U$). If $s \in \mathcal{G}_{\Phi^{\mathcal{G}}}(U)$, since $U = \bigcup_{\substack{V \subseteq U \text{ open} \\ s_{|V} \in \mathcal{G}(V)}} V$ is an open covering of U, $\mathcal{G}$ is a subsheaf of $\mathcal{F}$, and $s_{|V} \in \mathcal{G}(V)$ for all V as in the union, we have $s \in \mathcal{G}(U)$. By double inclusion, $\mathcal{G}_{\Phi^{\mathcal{G}}}(U) = \mathcal{G}(U)$.
Functoriality is the only thing that remains to be checked. Let $\mathcal{H}$ be another sheaf on X and $\alpha : \mathcal{F} \to \mathcal{H}$ be a morphism of sheaves. We have an induced map $(- \circ \alpha) : \mathrm{Hom}(\mathcal{H}, \Omega) \to \mathrm{Hom}(\mathcal{F}, \Omega)$. We need to get a map $\{\text{subsheaves of } \mathcal{H}\} \to \{\text{subsheaves of } \mathcal{F}\}$ from α. Consider

$$\begin{array}{rcl} \{\text{subsheaves of } \mathcal{H}\} & \to & \{\text{subsheaves of } \mathcal{F}\} \\ \mathcal{G} & \mapsto & (\alpha^{-1}\mathcal{G} : U \mapsto \alpha_U^{-1}(\mathcal{G}(U))) \end{array}$$

We check that this map is well-defined. Let $\mathcal{G}$ be a subsheaf of $\mathcal{H}$. We have a commutative diagram

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\alpha_U} & \mathcal{H}(U) \\ \downarrow & & \downarrow \\ \mathcal{F}(V) & \xrightarrow{\alpha_V} & \mathcal{H}(V) \end{array}$$

so if $s \in \alpha_U^{-1}(\mathcal{G}(U))$, then $\alpha_U(s) \in \mathcal{G}(U)$, so $(\alpha_V(s_{|V})) = (\alpha_U(s))_{|V} \in \mathcal{G}(V)$, so $s_{|V} \in \mathcal{G}(V)$. Hence $\alpha^{-1}\mathcal{G}$ is a subpresheaf of $\mathcal{F}$. Now let $U \subseteq X$ be an open set, $U = \bigcup_i U_i$ and $s \in \mathcal{F}(U)$ with $s_{|U_i} \in (\alpha^{-1}\mathcal{G})(U_i)$ for all i. Then, $(\alpha_U(s))_{|U_i} = \alpha_{U_i}(s_{|U_i}) \in \mathcal{G}(U_i)$ and since $\mathcal{G}$ is a sheaf, $\alpha_U(s) \in \mathcal{G}(U)$, so $s \in (\alpha^{-1}\mathcal{G})(U)$, which shows $\alpha^{-1}\mathcal{G}$ is a subsheaf of $\mathcal{F}$ by the preceding problem.
Functoriality means that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{Sh}(X)}(\mathcal{H},\Omega) & \longrightarrow & \{\text{subsheaves of } \mathcal{H}\} \\ \Big\downarrow{\scriptstyle -\circ\alpha} & & \Big\downarrow{\scriptstyle \mathcal{G}\mapsto\alpha^{-1}\mathcal{G}} \\ \mathrm{Hom}_{\mathrm{Sh}(X)}(\mathcal{F},\Omega) & \longrightarrow & \{\text{subsheaves of } \mathcal{F}\} \end{array}$$

If $\Phi : \mathcal{H} \to \Omega$ is a morphism of sheaves, then we need to check that $\alpha^{-1}\mathcal{G}_\Phi = \mathcal{G}_{\Phi\circ\alpha}$. For $U \subseteq X$ open,

$$\begin{aligned} \alpha^{-1}\mathcal{G}_\Phi(U) &= \{s \in \mathcal{F}(U) \mid \alpha_U(s) \in \mathcal{G}_\Phi(U)\} \\ &= \{s \in \mathcal{F}(U) \mid \Phi_U(\alpha_U(s)) = U\} \\ &= \{s \in \mathcal{F}(U) \mid (\Phi \circ \alpha)_U(s) = U\} \\ &= \mathcal{G}_{\Phi\circ\alpha}(U) \end{aligned}$$

and this concludes the proof.

3.6.

1. The category **Set** is isomorphic to the category $\mathrm{Sh}(\{x\})$. Using this identification one obtains directly that $(i_x)_*$ is a functor $\mathrm{Sh}(\{x\}) \to \mathrm{Sh}(X)$. If E is set, so a sheaf on $\{x\}$, and $\mathcal{F}$ is a sheaf of X, we have a natural bijection

$$\mathrm{Hom}_{\mathrm{Sh}(\{x\})(i_x^{-1}\mathcal{F},E)} \leftrightarrow \mathrm{Hom}_{\mathrm{Sh}(X)}(\mathcal{F},(i_x)_*(E))$$

TODO

2.