Homological Algebra

Original lectures notes by Baptiste Rognerud, but now typeset in LaTeX

1 Introduction to category theory

References:

- Emily Riehl, Category Theory in Context (chapter I)
- Saunders Mac Lane, Categories for the Working Mathematician
- Ibrahim Assem, Introduction au langage catégorique (chapters I, II)

❯ Near 1945 Eilenberg and Mac Lane gave the good formalism for a “natural isomorphism” (the general theory of natural transformations). For instance, if V is a finite-dimensional vector space, $V \simeq V^*$ and $V \simeq V^{**}$, but the first isomorphism is not natural (“a choice needs to be made”), while the second is. But why?
It turns out solving this question gave a formalism, category theory, that unified already existing mathematical concepts, gave new links between different notions and also gave new questions!
⚠ Category theory is not a theory that trivializes mathematics.
It is used today by (almost) everyone: algebraic geometry, algebra, representation theory, topology, combinatorics, ...

1.1 Categories and functors

Definition 1.1. A *category* $\mathcal{C}$ is the data of

- A collection of *morphisms* $\mathrm{Mor}(\mathcal{C})$
- A collection of *objects* $\mathrm{Ob}(\mathcal{C})$

such that

1. Every morphism $f \in \mathrm{Mor}(\mathcal{C})$ has a specified domain $X \in \mathrm{Ob}(\mathcal{C})$ and codomain $Y \in \mathrm{Ob}(\mathcal{C})$. We write $f : X \to Y$.
2. For every object $X \in \mathrm{Ob}(\mathcal{C})$ there exists a morphism $1_X : X \to X$ (the *identity* of X), also written id_X
3. For any three objects $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$ and morphims $f : X \to Y$ and $g : Y \to Z$ there exists a morphism $g \circ f : X \to Z$ (we often omit $\circ$ and just write gf)

satisfying

(Identity) $\forall f : X \to Y, 1_Y f = f = f 1_X$

(Associativity) $\forall f : W \to X, g : X \to Y, h : Y \to Z,\ h(gf) = (hg)f$

Remark.

1. We use the term "collection" because we don't want to worry about set-theoretical issues
2. If $\mathrm{Mor}(\mathcal{C})$ is a set, we say that $\mathcal{C}$ is *small*
3. We denote by $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ (or $\mathcal{C}(X, Y)$) the collection of $f : X \to Y \in \mathrm{Mor}(\mathcal{C})$

Examples 1.2 (Concrete categories)**.**

1. The category **Set**, where objects are sets and morphisms are just maps.
2. **Top**, where objects are topological spaces and morphisms are continuous maps.
3. Groups together with group homomorphisms form a category called **Grp**. The same can be said about rings, fields...
4. k-vector spaces, or more generally left/right R-modules, together with linear maps, form a category denoted $R\mathbf{Mod}$ or $\mathbf{Mod}R$ (for left or right R-modules).

In these previous examples, objects are sets with additional structure, and morphisms between two objects are in particular maps between the two underlying sets. Such categories are called *concrete categories* (a rigorous definition will be given later). However, a category need not be concrete.

Examples 1.3 (Abstract categories)**.**

1. Let k be a field. There exists a category $\mathbf{Mat}_k$ where objects are the natural numbers $\mathbb{N}$ and morphisms are $\mathrm{Hom}(m, n) = \mathrm{Mat}_{n,m}(k)$, where composition is given by matrix multiplication.
2. If G is a group, there exists a category BG which has only one object $\bullet$, and morphisms $\mathrm{Hom}(\bullet, \bullet) = G$, where composition is multiplication in G.
3. If $(P, \leq)$ is a *poset* (a partially ordered set, that is a set P together with a reflexive, transitive relation $\leq$), then one can construct a category $\hat{P}$ by setting $\mathrm{Ob}(\hat{P}) = P$ and $|\mathrm{Hom}(x, y)| = \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{otherwise} \end{cases}$, where composition is defined in the only possible way.
4. The homotopy category of topological spaces: objects are topological spaces, and $\mathrm{Hom}(X, Y)$ is $\mathrm{Hom}_{\mathbf{Top}}(X, Y)/\sim$ where $\sim$ is homotopy of continuous maps.

Exercise. Check the categories defined above really are categories. In (2), what are the minimal hypotheses needed on G for BG to be a category? In (3), what are the minimal hypotheses needed on $\leq$ for $\hat{P}$ to be a category?

Examples 1.4 (Categories constructed from categories)**.**

1. If $\mathcal{C}$ is a category, one can construct its *opposite category* $\mathcal{C}^{\mathrm{op}}$, defined by $\mathrm{Ob}(\mathcal{C}^{\mathrm{op}}) = \mathrm{Ob}(\mathcal{C})$ and $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X)$, with composition described by the following diagram:

$$\begin{tikzcd} X \arrow[d, "f"] \arrow[dd, bend right, "gf"'] & X \\ Y \arrow[d, "g"] \arrow[r, squiggly] & Y \arrow[u, "f^{\mathrm{op}}"'] \\ Z & Z \arrow[u, "g^{\mathrm{op}}"'] \arrow[uu, bend right, "f^{\mathrm{op}}g^{\mathrm{op}}"'] \end{tikzcd}$$

2. Let $\mathcal{C}$ be a category. A *subcategory* $\mathcal{D}$ of $\mathcal{C}$ is another category such that $\mathrm{Ob}(\mathcal{D}) \subset \mathrm{Ob}(\mathcal{C})$ and $\mathrm{Mor}(\mathcal{D}) \subset \mathrm{Mor}(\mathcal{C})$ and the composition in $\mathcal{D}$ is induced by the one in $\mathcal{C}$. For instance, **Ab**, the category of abelian groups and group homomorphisms, is a subcategory of **Grp**.

3. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. The *product category* of $\mathcal{C}$ and $\mathcal{D}$ is the category $\mathcal{C} \times \mathcal{D}$ defined by $\mathrm{Ob}(\mathcal{C} \times \mathcal{D}) = \mathrm{Ob}(\mathcal{C}) \times \mathrm{Ob}(\mathcal{D})$ and $\mathrm{Mor}(\mathcal{C} \times \mathcal{D}) = \mathrm{Mor}(\mathcal{C}) \times \mathrm{Mor}(\mathcal{D})$, composition and identities being defined componentwise.

Exercise. Describe $(BG)^{\mathrm{op}}$ for G a group and $\hat{P}^{\mathrm{op}}$ for $(P, \leq)$ a poset.

$\mathbf{Set}^{\mathrm{op}}$ is not **Set**. **TODO**

Remark. In a category $\mathcal{C}$ the objects can be anything, so saying $x \in X$ for $X \in \mathrm{Ob}(\mathcal{C})$ doesn't make sense. Hence, categorical notions are defined using arrows and not elements.

Definition 1.5. Let $\mathcal{C}$ be a category.

1. $f : X \to Y$ is an *isomorphism* if there exists $g : Y \to X$ such that $gf = \mathrm{id}_X$ and $fg = \mathrm{id}_Y$.

2. $f : X \to Y$ is a *monomorphism* if for all $g, h : W \to X$ such that $fg = fh$, $g = h$ (f is left-cancellable).

3. $f : X \to Y$ is an *epimorphism* if for all $g, h : Y \to Z$ such that $gf = hf$, $g = h$ (f is right-cancellable).

Being both a mono and an epi doesn't imply being an iso. **TODO**

Definition 1.6. Let $\mathcal{C}, \mathcal{D}$ be two categories. A *(covariant) functor* $F : \mathcal{C} \to \mathcal{D}$ is the data of

- An object $F(X) \in \mathrm{Ob}(\mathcal{D})$ for all $X \in \mathrm{Ob}(\mathcal{C})$
- A morphism $F(f) \in \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ for all $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$

such that $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for all $X \in \mathrm{Ob}(\mathcal{C})$ and $F(gf) = F(g)F(f)$ whenever $f, g \in \mathrm{Mor}(\mathcal{C})$ are composable.

Definition 1.7. A *contravariant* functor from $\mathcal{C}$ to $\mathcal{D}$ is a functor from $\mathcal{C}^{\mathrm{op}}$ to $\mathcal{D}$ (so composition is reversed, i.e. $F(gf) = F(f)F(g)$).

Examples 1.8.

1. $U : \mathbf{Grp} \to \mathbf{Set}, U(G) = G, U(f) = f$ the functor that to a group assigns it its underlying set and to a homomorphism the underlying map. It is called the *forgetful functor* from groups to sets, because it forgets the group structure.

2. $U : \begin{array}{ccc} \mathbf{Ass} & \to & \mathbf{Lie} \\ (A,+,\cdot) & \mapsto & (A,+,[-,-]) \end{array}$ the forgetful functor from the category of associative algebras to the category of Lie algebras. It forgets the "associative structure" but remembers the underlying abelian group.
3. $F : \mathbf{Set} \to \mathbf{Ab}, X \mapsto \mathbb{Z}[X], f \mapsto \mathbb{Z}[f]$, which to a set assigns the free abelian group with basis X (the group of finite linear combinations of elements of X). A map $f : X \to Y$ can then be uniquely extended to a linear map $\mathbb{Z}[f] : \mathbb{Z}[X] \to \mathbb{Z}[Y]$ that agrees with f on the bases of $\mathbb{Z}[X]$ and $\mathbb{Z}[Y]$.
4. Suppose $\mathcal{C}$ is locally small (i.e. for any X, Y, $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ is a set). For all $X \in \mathcal{C}$, $\mathrm{Hom}(X,-)$ is a functor $\mathcal{C} \to \mathbf{Set}$. Similarly, $\mathrm{Hom}_{\mathcal{C}}(-,X)$ is a contravariant functor $\mathcal{C} \to \mathbf{Set}$. $\mathrm{Hom}_{\mathcal{C}}(-,-)$ is a functor $\mathcal{C} \times \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$.
5. Functors $\mathcal{C} \xrightarrow[F]{} \mathcal{D} \xrightarrow[G]{} \mathcal{E}$ can be composed in the obvious sense.

TODO: DRAW DIAGRAMS

Definition 1.9. Let $\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] & \mathcal{D} \end{tikzcd}$ be two functors. A *natural transformation* η from F to G is the data of morphisms $\eta_X : F(X) \to G(X) \in \mathrm{Mor}(\mathcal{D})$ for all $X \in \mathrm{Ob}(\mathcal{C})$ such that for all $f : X \to Y \in \mathrm{Mor}(\mathcal{C})$, the diagram

$$\begin{tikzcd} F(X) \arrow[r, "\eta_X"] \arrow[d, "F(f)"] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\eta_Y"] & G(Y) \end{tikzcd}$$

commutes, that is $G(f)\eta_X = \eta_Y F(f)$. We write $\eta : F \Rightarrow G$ or draw $\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathcal{D} \end{tikzcd}$

Example 1.10. Let V be a k-vector space. $\mathrm{id}_{\mathbf{Vect}_k}$ and $D^2 = \mathrm{Hom}_{\mathbf{Vect}_k}(\mathrm{Hom}_{\mathbf{Vect}_k}(-,k),k)$ are two endofunctors of $\mathbf{Vect}_k$. $\mathrm{ev}_- : \begin{array}{ccc} V & \to & V^{**} \\ v & \mapsto & \begin{array}{ccc} \mathrm{Hom}(V,k) & \to & k \\ \phi & \mapsto & \phi(v) \end{array} \end{array}$ defines a natural transformation between them:

$$\begin{tikzcd} V \arrow[r, "\mathrm{ev}"] \arrow[d, "f"'] & V^{**} \arrow[d, "D^2(f)"] \\ W \arrow[r, "\mathrm{ev}"'] & W^{**} \end{tikzcd}$$

For $a \in V$, $D^2(f) \circ \mathrm{ev}_a : \begin{array}{ccc} W^* & \to & k \\ \phi & \mapsto & \phi(f(a)) \end{array} \in W^{**}$ and in the other direction $(\mathrm{ev} \circ f)(a) = \mathrm{ev}_{f(a)}$.
However, there is no natural transformation from $\mathrm{id}_{\mathbf{Vect}_k}$ to D. For one, the first is covariant and the second is contravariant. To get a natural transformation from a covariant to a contravariant functor, we can modify the definition of naturality to be that $\begin{array}{ccc} V & \to & V^* \\ \downarrow & & \uparrow \\ W & \to & W^* \end{array}$ commutes, but even such natural transformations do not exist from $\mathrm{id}_{\mathbf{Vect}_k}$ to D.

Definition 1.11. A natural transformation $\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"{name=U}] \arrow[r, bend right, "G"'{name=D}] & \mathcal{D} \arrow[from=U, to=D, Rightarrow, "\eta"] \end{tikzcd}$ is a *natural isomorphism* if η_X is an isomorphism for all $X \in \mathrm{Ob}(\mathcal{C})$.

Remark. One can compose natural transformations in two ways, "vertical composition":

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=60, "F"{name=F}] \arrow[r, "G"{name=G}] \arrow[r, bend right=60, "H"'{name=H}] & \mathcal{D} \arrow[from=F, to=G, Rightarrow, "\alpha"] \arrow[from=G, to=H, Rightarrow, "\beta"] \end{tikzcd} \rightsquigarrow \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"{name=U}] \arrow[r, bend right, "H"'{name=D}] & \mathcal{D} \arrow[from=U, to=D, Rightarrow, "\beta\circ\alpha"] \end{tikzcd} \text{ where } (\beta \circ \alpha)_X = \beta_X \circ \alpha_X$$

or "horizontal composition":

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F_1"{name=A}] \arrow[r, bend right, "G_1"'{name=B}] & \mathcal{D} \arrow[from=A, to=B, Rightarrow, "\alpha_1"] \arrow[r, bend left, "F_2"{name=C}] \arrow[r, bend right, "G_2"'{name=E}] & \mathcal{E} \arrow[from=C, to=E, Rightarrow, "\alpha_2"] \end{tikzcd} \rightsquigarrow \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F_2\circ F_1"{name=U}] \arrow[r, bend right, "G_2\circ G_1"'{name=D}] & \mathcal{E} \arrow[from=U, to=D, Rightarrow, "\alpha_2 * \alpha_1"] \end{tikzcd} \text{ where } (\alpha_2 * \alpha_1)_X = G_2((\alpha_1)_X) \circ (\alpha_2)_{F_1(X)}$$

Horizontal composition can also be defined in another equivalent way using commutativity of

$$\begin{tikzcd} F_2F_1(X) \arrow[r, "(\alpha_2)_{F_1(X)}"] \arrow[d, "F_2((\alpha_1)_X)"'] & G_2F_1(X) \arrow[d, "G_2((\alpha_1)_X)"] \\ F_2G_1(X) \arrow[r, "(\alpha_2)_{G_1(X)}"] & G_2G_1(X) \end{tikzcd}$$

The diagram commutes by naturality of α_2, so $(\alpha_2 * \alpha_1) = (\alpha_2)_{G_1(X)} \circ F_2((\alpha_1)_X)$.

Definition 1.12. Let $\mathcal{C}, \mathcal{D}$ be two categories. Then the *functor category from $\mathcal{C}$ to $\mathcal{D}$* written $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ or $\mathcal{D}^{\mathcal{C}}$ is the category whose objects are functors from $\mathcal{C}$ to $\mathcal{D}$ and morphisms are natural transformations.

Remark. Categories, together with functors and natural transformations between them is the prototypal example of a 2-category.

1.2 Equivalences of categories

Definition 1.13. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. An *equivalence of categories* from $\mathcal{C}$ to $\mathcal{D}$ is the data of

1. $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ wo functors
2. Natural isomorphisms $\eta : 1_{\mathcal{C}} \Rightarrow GF$ and $\varepsilon : FG \Rightarrow 1_{\mathcal{D}}$ where $1_{\mathcal{C}}$ and $1_{\mathcal{D}}$ are the identity functors of $\mathcal{C}$ and $\mathcal{D}$.

Remark.

1. G is called a *quasi-inverse* of F.
2. Most of the time we say that F is an equivalence if there exists G such that (F, G) is an equivalence.

3. If F, G are contravariant, we speak of *duality* between $\mathcal{C}$ and $\mathcal{D}$.
4. If two categories are equivalent, every property that can be expressed "in terms of arrows" is preserved.

Definition 1.14. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then, we say

1. F is *faithful* if $\forall X, Y \in \mathrm{Ob}(\mathcal{C})$, $F : \begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(X, Y) & \to & \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y)) \\ f & \mapsto & F(f) \end{array}$ is injective.
2. F is *full* if the previous map is surjective.
3. F is *essentially surjective* if for all $Y \in \mathrm{Ob}(\mathcal{D})$ there is $X \in \mathrm{Ob}(\mathcal{C})$ such that $F(X) \simeq Y$ in $\mathcal{D}$.

Theorem 1.15. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then F is an equivalence of categories if and only if it is fully faithful and essentially surjective.*

Proof. ⚠ There is a little set-theoretic issue: an equivalence of categories is always fully faithful and essentially surjective, but the converse requires the axiom of choice for the class $\mathrm{Ob}(\mathcal{C})$.
Suppose $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories, and let $G : \mathcal{D} \to \mathcal{C}$ be a quasi-inverse of F, together with natural isomorphisms $\eta : 1_{\mathcal{C}} \to GF$ and $\varepsilon : 1_{\mathcal{D}} \to FG$. If Y is an object of $\mathcal{D}$, then $Y \simeq FG(Y)$, so F is essentially surjective. Let X, Y be objects of $\mathcal{C}$. To show F is fully faithful we will construct an inverse to $F : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$. For any $f \in \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$, we have a commutative diagram

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "f"'] & GF(X) \arrow[d, "GF(f)"] \\ Y \arrow[r, "\eta_Y"] & GF(Y) \end{tikzcd}$$

which prompts us to define $\phi : \begin{array}{ccc} \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y)) & \to & \mathrm{Hom}_{\mathcal{C}}(X, Y) \\ g & \mapsto & \eta_Y^{-1} \circ G(g) \circ \eta_X \end{array}$. We now check it is the map we're looking for. If $f : X \to Y$, since the above diagram commutes and η_Y is invertible, we get that $\phi(F(f)) = f$, so $\phi \circ F = \mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(X,Y)}$, which means F is faithful. We have two commutative diagrams, by definition of ϕ and by naturality of η:

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "\phi(g)"'] & GF(X) \arrow[d, "G(g)"] \\ Y \arrow[r, "\eta_Y"] & GF(Y) \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "\phi(g)"'] & GF(X) \arrow[d, "GF(\phi(g))"] \\ Y \arrow[r, "\eta_Y"] & GF(Y) \end{tikzcd}$$

therefore, $G(g) \circ \eta_X = GF(\phi(g)) \circ \eta_X$. Since η_X is invertible, $G(g) = GF(\phi(g))$. The previous point shows that G is faithful, so $g = F(\phi(g))$, hence F is full.
Now suppose F is fully faithful and essentially surjective. Our goal is to construct G. For any $Y \in \mathrm{Ob}(\mathcal{D})$, since F is essentially surjective, there exists $X_Y \in \mathrm{Ob}(\mathcal{C})$ and an isomorphism $\varepsilon_Y : Y \to F(X_Y)$. Therefore, for any $Y, Z \in \mathrm{Ob}(\mathcal{D})$ and $f : Y \to Z$, we have a commutative diagram

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "\varepsilon_Y"] & Z \arrow[d, "\varepsilon_Z"] \\ F(X_Y) \arrow[r, "\varepsilon_Z \circ f \circ \varepsilon_Y^{-1}"'] & F(X_Z) \end{tikzcd}$$

Which leads us to define $G(Y) = X_Y$ and $G(f)$ to be the unique morphism $m_f : X_Y \to X_Z$ such that $F(m_f) = \varepsilon_Z \circ f \circ \varepsilon_Y^{-1}$ (this works because F is fully faithful). We have $G(\mathrm{id}_Y) = \mathrm{id}_{X_Y}$ since $\varepsilon_Y \circ \mathrm{id}_Y \circ \varepsilon_Y^{-1} = \mathrm{id}_Y$ and $F(\mathrm{id}_{X_Y}) = \mathrm{id}_Y$. The next diagram shows $G(g \circ f) = G(g) \circ G(f)$:

$$\begin{tikzcd}
W \arrow[r, "f"] \arrow[rr, bend left, "g\circ f"] \arrow[d, "\varepsilon_W"] & Y \arrow[r, "g"] \arrow[d, "\varepsilon_Y"] & Z \arrow[d, "\varepsilon_Z"] \\
F(X_W) \arrow[r, "F(m_f)"'] \arrow[rr, bend right, "F(m_g\circ m_f)=F(m_g)\circ F(m_f)"'] & F(X_Y) \arrow[r, "F(m_g)"'] & F(X_Z)
\end{tikzcd}$$

By this construction, ε is a natural isomorphism $\mathrm{id}_{\mathcal{D}} \Rightarrow FG$ (look at the above diagrams). Now, pick $Y, Z \in \mathrm{Ob}(\mathcal{C})$ and $f : Y \to Z$. We have $GF(Y) = X_{F(Y)}$ and $\varepsilon_Y : F(Y) \xrightarrow{\sim} F(X_{F(Y)})$. Since F is fully faithful, there exists a unique $\eta_Y : Y \to X_{F(Y)} = GF(Y)$ such that $F(\eta_Y) = \varepsilon_Y$. Here, $\eta_Y = G(\varepsilon_Y)$, which means that η_Y is an isomorphism since functors preserve isomorphisms. We obtain the diagram

$$\begin{tikzcd}
Y \arrow[r, "\eta_Y"] \arrow[d, "f"] & GF(Y) \arrow[d, "GF(f)"] \\
Z \arrow[r, "\eta_Z"] & GF(Z)
\end{tikzcd}$$

The diagram commutes because $GF(f)$ is the unique morphism such that

$$F(GF(f)) = \varepsilon_Z \circ F(f) \circ \varepsilon_Y^{-1} = F(\eta_Z \circ f \circ \eta_Y^{-1})$$

and F is faithful. η is then a natural isomorphism $\mathrm{id}_{\mathcal{C}} \Rightarrow GF$. $\square$

Example 1.16. $\mathbf{Vect}_k \simeq \mathbf{Mat}_k$ through the functor $n \mapsto k^n$ and $(A : n \to m) \mapsto (f_A : k^n \to k^m)$.

2 Universal properties

References:

- Riehl (Chapters II, III, IV)
- Mac Lane (Chapters III, IV, V)
- Assem (Chapters III, IV, V)

❯ Let S be a set together with an equivalence relation $\sim$. Let ${}^{S}/_{\sim}$ be the quotient set, and $\pi : S \to {}^{S}/_{\sim}$ be the projection. For any $f : S \to X$ compatible with $\sim$, there exists a unique map $\bar{f} : {}^{S}/_{\sim} \to X$ such that $f = \bar{f} \circ \pi$. This is represented by the following commutative diagram :

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d, "\pi"'] & X \\ {}^{S}/_{\sim} \arrow[ur, dashed, "\exists! \bar{f}"'] & \end{tikzcd}$$

We say that $S \xrightarrow{\pi} {}^{S}/_{\sim}$ is a solution to the universal problem posed by the compatible maps. Such a solution is unique up to unique isomorphism: if $S \xrightarrow{p} S'$ is another solution, then we get the three commutative diagrams

$$\begin{tikzcd} S \arrow[r, "p"] \arrow[d, "\pi"'] & S' \\ {}^{S}/_{\sim} \arrow[ur, dashed, "\exists! a"'] & \end{tikzcd} \quad \begin{tikzcd} S \arrow[r, "\pi"] \arrow[d, "p"'] & {}^{S}/_{\sim} \\ S' \arrow[ur, dashed, "\exists! b"'] & \end{tikzcd} \quad \begin{tikzcd} S \arrow[r, "p"] \arrow[d, "p"'] & S' \\ S' \arrow[ur, "\mathrm{id}_{S'}"'] \arrow[r, "b"'] & {}^{S}/_{\sim} \arrow[u, "a"'] \end{tikzcd}$$

then $abp = a\pi = p$. The identity of S' also makes this diagram commute so by uniqueness $ab = \mathrm{id}_{S'}$ and similarly $ba = \mathrm{id}_{S/\sim}$.

2.1 Initial and final objects

Definition 2.1. Let $\mathcal{C}$ be a category. An object $c \in \mathrm{Ob}(\mathcal{C})$ is *initial* (*final*) if for all $d \in \mathrm{Ob}(\mathcal{C})$ there exists a unique morphism $c \to d$ (a unique morphism $d \to c$).

Proposition 2.2. *If an initial/final object exists, then it is unique up to unique isomorphism.*

Proof. Let c, c' be two initial objects. Then there exists a unique morphism $f : c \to c'$ and a unique morphism $g : c' \to c$. There also exists a unique morphism $c \to c$, that is id_c. Therefore, $gf = \mathrm{id}_c$. In the same way, $fg = \mathrm{id}_{c'}$. Therefore, c and c' are isomorphic and the isomorphism is unique. $\square$

Examples 2.3.

1. $\varnothing$ is initial in **Set** and any singleton is final.
2. $\{0\}$ is both initial and final in $\mathbf{Vect}_k$ (or $R\mathbf{Mod}$).
3. The category of fields does not have initial/final objects (reason on field characteristics).

We want to say a universal object is an initial or final object. A category has at most 2, so this may seem a little restrictive, but this is solved by thinking of a good category.

Definition 2.4. Let $F : \mathcal{C} \to \mathbf{Set}$ be a functor. Let $\int F$ be the category defined by

$$\begin{aligned} &\mathrm{Ob}(\textstyle\int F) = \{(c, x) \mid c \in \mathrm{Ob}(\mathcal{C}) \text{ and } x \in F(c)\} \\ &\mathrm{Hom}((c, x), (c', x')) = \{f \in \mathrm{Hom}(c, c') \mid F(f)(x) = x'\} \end{aligned}$$

where composition is composition in $\mathcal{C}$, and $\mathrm{id}_{(c,x)} = \mathrm{id}_c$ for all x. If F is contravariant, let $\int F$ have the same objects and morphisms $\mathrm{Hom}((c, x), (c', x')) = \{f \in \mathrm{Hom}(c, c') \mid F(f)(x') = x\}$.

Proposition 2.5. *There is a forgetful functor $\pi : \int F \to \mathcal{C}$ defined by $\pi(c, x) = c$ and $\pi(f : (c, x) \to (c', x')) = f : c \to c'$.*

Example 2.6. Let S be a set, and $\sim$ an equivalence relation on S. Let $F : \mathbf{Set} \to \mathbf{Set}$ be defined by $F(X) = \{f : S \to X \mid x \sim y \Rightarrow f(x) = f(y)\}$ and $F(\alpha : X \to Y) = \alpha \circ -$.

$\int F$ has for objects $(X, S \xrightarrow{f} X)$ where f is compatible with $\sim$, and for morphisms α that makes this diagram commute:

$$\begin{tikzcd} S \arrow[r, "f'"] \arrow[d, "f"'] & X' \\ X \arrow[ur, "\alpha"'] & \end{tikzcd}$$

$({}^{S}\!/\!\sim, S \xrightarrow{\pi} {}^{S}\!/\!\sim)$ is an initial object of $\int F$.

Definition 2.7. Let $F : \mathcal{C} \to \mathbf{Set}$ be a functor. A *universal element* for F is an initial object of $\int F$, that is a pair (c, x) with $c \in \mathrm{Ob}(\mathcal{C})$ and $x \in F(c)$ such that

$$\forall (d, y), d \in \mathrm{Ob}(\mathcal{C}), y \in F(d), \exists! \alpha : c \to d, y = F(\alpha)(x)$$

Definition 2.8. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor and $d \in \mathrm{Ob}(\mathcal{D})$. A *universal arrow from d to F* is a pair (c, f) where $c \in \mathrm{Ob}(\mathcal{C})$ and $f \in \mathrm{Hom}_{\mathcal{D}}(d, F(c))$, such that

$$\forall (c', f'), c' \in \mathrm{Ob}(\mathcal{C}), f' : d \to F(c'), \exists! \alpha \in \mathrm{Hom}_{\mathcal{C}}(c, c'), F(\alpha) \circ f = f'$$

$$\begin{tikzcd} & d \arrow[dl, "f"'] \arrow[dr, "\forall f'"] & \\ F(c) \arrow[rr, dashed, "F(\alpha)"'] & & F(c') \end{tikzcd}$$

$$\begin{tikzcd} c \arrow[r, "\exists! \alpha"] & c' \end{tikzcd}$$

Exercise. Define a category $d \downarrow F$ such that a universal arrow is an initial object of $d \downarrow F$.

Example 2.9. Let $U : \mathbf{Vect}_k \to \mathbf{Set}$ be the forgetful functor. Let $X \in \mathbf{Set}$. A universal arrow from X to U is the "best" k-vector space V_X with a map $X \to V_X$. Set $V_X = k[X]$ the k-vector space with basis X, and $i : X \to V_X$ that maps $x \in X$ to the corresponding basis element. Then, for any vector space V and map $f : X \to U(V)$, f can be extended by linearity into a linear map $\tilde{f} : k[X] \to V$, which makes this diagram commute:

$$\begin{tikzcd} & X \arrow[dl, "i"'] \arrow[dr, "f"] & \\ k[X] \arrow[rr, dashed, "\tilde{f}"'] & & U(V) \end{tikzcd}$$

If α is another map that makes the diagram commute then α and $\tilde{f}$ coincide on a basis of $k[X]$ and therefore are equal.

Proposition 2.10. *Universal elements and arrows are two equivalent notions.*

Proof. Let $F : \mathcal{C} \to \mathbf{Set}$ be a functor and (c, x) a universal element for F. Consider $f_x : \begin{array}{ccc} \{*\} & \to & F(c) \\ * & \mapsto & x \end{array}$. Then, (c, f_x) is a universal arrow $* \to F$, because $F(\alpha)(x) = y$ iff $F(\alpha) \circ f_x = f_y$.

$$\begin{tikzcd} & \{*\} \arrow[dl, "f_x"'] \arrow[dr, "f_y"] & \\ F(c) \arrow[rr, dashed, "F(\alpha)"'] & & F(c') \end{tikzcd}$$

Conversely, if $F : \mathcal{C} \to \mathcal{D}$ is a functor and (c, f) is a universal arrow $d \to F$, then consider the functor $\mathrm{Hom}_{\mathcal{D}}(d, F(-)) : \begin{array}{ccc} \mathcal{C} & \to & \mathbf{Set} \\ x & \mapsto & \mathrm{Hom}_{\mathcal{D}}(d, F(x)) \end{array}$ (we need to assume $\mathcal{D}$ is locally small so the functor is set-valued). Then, $f \in \mathrm{Hom}_{\mathcal{D}}(d, F(c))$ is a universal element for this functor. $\square$

2.2 Representable functors

Definition 2.11. Let $\mathcal{C}$ be a (locally small) category, and $F : \mathcal{C} \to \mathbf{Set}$ a functor.

1. We say that F is *representable* if there is some $c \in \mathrm{Ob}(\mathcal{C})$ such that F and $\mathrm{Hom}_{\mathcal{C}}(c, -)$ are naturally isomorphic (if F is contravariant, use $\mathrm{Hom}_{\mathcal{C}}(-, c)$ instead).
2. A *representation* of F is the data of $c \in \mathrm{Ob}(\mathcal{C})$ and a natural isomorphism $\eta : \mathrm{Hom}(c, -) \Rightarrow F$.

Example 2.12. The forgetful functor $U : \mathbf{Grp} \to \mathbf{Set}$ is representable since $\mathrm{Hom}_{\mathbf{Grp}}(\mathbb{Z}, -) \simeq U$. The natural isomorphism is given by $\alpha \in \mathrm{Hom}_{\mathbf{Grp}}(\mathbb{Z}, G) \mapsto \alpha(1) \in G$.

The following theorem explains how to find the natural isomorphism $\alpha : \mathrm{Hom}_{\mathcal{C}}(c, -) \Rightarrow F$ in general.

Theorem 2.13 (Yoneda lemma). *Let $F : \mathcal{C} \to \mathbf{Set}$ be a functor with $\mathcal{C}$ locally small, and $c \in \mathrm{Ob}(\mathcal{C})$. Then,*

$$\begin{array}{ccc} \mathrm{Nat}(\mathrm{Hom}(c, -), F) & \xrightarrow{\sim} & F(c) \\ \alpha & \mapsto & \alpha_c(\mathrm{id}_c) \end{array}$$

and this isomorphism is natural in c and in F.

Proof. Let $\alpha \in \mathrm{Nat}(\mathrm{Hom}(c, -), F)$. Let $d \in \mathcal{C}$ and $f : c \to d$. By naturality, we have a commutative diagram

$$\begin{tikzcd} \mathrm{Hom}(c, c) \arrow[r, "\alpha_c"] \arrow[d, "f \circ -"] & F(c) \arrow[d, "F(f)"] \\ \mathrm{Hom}(c, d) \arrow[r, "\alpha_d"] & F(d) \end{tikzcd}$$

This means that $F(f) \circ \alpha_c = \alpha_d \circ (f \circ -)$. Evaluating at id_c, we get $F(f) \circ \alpha_c(\mathrm{id}_c) = \alpha_d(f)$. This shows that the natural transformation α is entirely determined by the value of $\alpha_c(\mathrm{id}_c)$, which shows the map defined above is injective. Conversely, if $e \in F(c)$, then we define $\alpha^e : \mathrm{Hom}(c, -) \Rightarrow F$ by $\alpha^e_d : g \mapsto F(g)(e)$. We check it is a natural transformation:

$$\begin{tikzcd}
\mathrm{Hom}(c,c) \arrow[r, "g\mapsto F(g)(e)"] \arrow[d, "f\circ -"] & F(c) \arrow[d, "F(f)"] \\
\mathrm{Hom}(c,d) \arrow[r, "h\mapsto F(h)(e)"] & F(d)
\end{tikzcd}$$

and this diagram commutes since for $g : c \to c$ we have

$$F(f)(F(g)(e)) = F(f \circ g)(e) = F((f \circ -)(g))(e)$$

This shows the map in the theorem is surjective, therefore an isomorphism, and its inverse is given by $e \in F(c) \mapsto \alpha^e$. We now check naturality. We first need to understand what it means to say the isomorphism is natural in c. Let $f : c \to d$. $\mathrm{Nat}(\mathrm{Hom}(c,-), F)$ is functorial in c, as it is the composition of two contravariant hom-functors. More concretely:

$$c \xrightarrow{f} d \rightsquigarrow \mathrm{Hom}(d,-) \xrightarrow{-\circ f} \mathrm{Hom}(c,-) \rightsquigarrow \mathrm{Nat}(\mathrm{Hom}(c,-),F) \xrightarrow{-\circ(-\circ f)} \mathrm{Nat}(\mathrm{Hom}(d,-),F)$$

(Nat is the hom-functor of the functor category $\mathcal{C}^{\mathbf{Set}}$). One thing to note is that the morphism $f : c \to d$ induces a natural transformation $\mathrm{Hom}(d,-) \xrightarrow{-\circ f} \mathrm{Hom}(c,-)$, and this makes the whole thing work. This is in general a property of functors defined on a product category, see the remark below. Now, saying the isomorphism, which we'll write $\Phi_{d,F}$, is natural means that the square

$$\begin{tikzcd}
\mathrm{Nat}(\mathrm{Hom}(c,-),F) \arrow[r, "\Phi_{c,F}"] \arrow[d, "-\circ(-\circ f)"] & F(c) \arrow[d, "F(f)"] \\
\mathrm{Nat}(\mathrm{Hom}(d,-),F) \arrow[r, "\Phi_{d,F}"] & F(d)
\end{tikzcd}$$

commutes. And indeed, if $\alpha : \mathrm{Hom}(c,-) \Rightarrow F$ is a natural transformation,

$$\begin{aligned}
\Phi_{d,F}(\alpha \circ (- \circ f)) &= (\alpha \circ (- \circ f))_d(\mathrm{id}_d) = [\alpha_d \circ (- \circ f)](\mathrm{id}_d) = \alpha_d(f) \\
F(f)(\Phi_{c,F}(\alpha)) &= F(f)(\alpha_c(\mathrm{id}_c)) = \alpha_d(f \circ \mathrm{id}_c) = \alpha_d(f)
\end{aligned}$$

The second to last equality comes from the naturality of α.
We now turn to naturality in F. Let G be another functor $\mathcal{C} \to \mathbf{Set}$ and $\beta : F \Rightarrow G$ be a natural transformation. We check that

$$\begin{tikzcd}
\mathrm{Nat}(\mathrm{Hom}(c,-),F) \arrow[r, "\Phi_{c,F}"] \arrow[d, "\beta\circ -"] & F(c) \arrow[d, "\beta_c"] \\
\mathrm{Nat}(\mathrm{Hom}(c,-),G) \arrow[r, "\Phi_{c,G}"] & G(c)
\end{tikzcd}$$

commutes. For $\alpha : \mathrm{Hom}(c,-) \Rightarrow F$, we have

$$\beta_c(\Phi_{c,F}(\alpha)) = \beta_c(\alpha_c(\mathrm{id}_c)) = (\beta \circ \alpha)_c(\mathrm{id}_c) = \Phi_{c,G}(\beta \circ \alpha)$$

which completes the proof of naturality. $\square$

Remark.

1. If $F : \mathcal{C} \to \mathbf{Set}$, then (c, x) is a universal element for F if and only if the natural transformation $\alpha_x : \mathrm{Hom}(c, -) \Rightarrow F$ induced by x is an isomorphism. Indeed, α_x is an isomorphism iff $\forall c' \in \mathcal{C}$, $\begin{array}{rrcl} (\alpha_x)_{c'} : & \mathrm{Hom}(c, c') & \to & F(c') \\ & f & \mapsto & F(f)(x) \end{array}$ is bijective iff

$$\forall c' \in \mathcal{C}, \forall y \in F(c'), \exists! f : c \to c', F(f)(x) = y$$

2. For universal arrows, use $\mathrm{Hom}_{\mathcal{D}}(d, F(-))$ as before.

3. Let $\mathcal{C}, \mathcal{D}$ and $\mathcal{E}$ be categories, and $F : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ be a functor. Let $c, d \in \mathrm{Ob}(\mathcal{C})$, $x, y \in \mathrm{Ob}(\mathcal{D})$ and morphisms $f : c \to d$, $g : x \to y$. The morphism f induces a natural transformation $F(f, \mathrm{id}_-) : F(c, -) \Rightarrow F(d, -)$, see the commutative square:

$$\begin{tikzcd}
F(c,x) \arrow[r, "F(f,\mathrm{id}_x)"] \arrow[d, "F(\mathrm{id}_c,g)"] & F(d,x) \arrow[d, "F(\mathrm{id}_d,g)"] \\
F(c,y) \arrow[r, "F(f,\mathrm{id}_y)"'] & F(d,y)
\end{tikzcd}$$

2.3 Examples of objects defined by universal properties

2.3.1 Products, coproducts

Let $\mathcal{C}$ be a small category and $X, Y \in \mathrm{Ob}(\mathcal{C})$. We define a category $\mathcal{C}_{X,Y}$ whose objects are tuples (Z, f, g) where $Z \in \mathrm{Ob}(\mathcal{C})$ and $f : Z \to X$, $g : Z \to Y$ and morphisms are maps $\alpha : Z \to Z'$ in $\mathcal{C}$ such that the following diagram commutes:

$$\begin{tikzcd}
& Z \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[dd, "\alpha"] & \\
X & & Y \\
& Z' \arrow[ul, "f'"] \arrow[ur, "g'"'] &
\end{tikzcd}$$

Definition 2.14. A *product* of X and Y is a final object in $\mathcal{C}_{X,Y}$. Concretely, it is an object $X \times Y$ together with two maps $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ such that for any $(Z, f, g) \in \mathrm{Ob}(\mathcal{C}_{X,Y})$, we have a commutative diagram

$$\begin{tikzcd}
& Z \arrow[dl, "\forall f"'] \arrow[dr, "\forall g"] \arrow[d, dashed, "\exists! \alpha"] & \\
X & X \times Y \arrow[l, "\pi_X"] \arrow[r, "\pi_Y"'] & Y
\end{tikzcd}$$

Since it is defined as being a final object, if it exists, a product is unique up to unique isomorphism.

Examples 2.15. In **Set**, the product of X and Y is the cartesian product. In **Grp**, it is the product group. In **Top**, it is the cartesian product equipped with the product topology. In these examples, the maps in the definition are the canonical projections.

The notion dual to the one of a product is called a coproduct.

Definition 2.16. A *coproduct* of X and Y is a product in $\mathcal{C}^{\mathrm{op}}$. Concretely, it satisfies the universal property expressed by this commutative diagram:

$$\begin{tikzcd}
X \arrow[r, "i_X"] \arrow[dr, "\forall f"'] & X \sqcup Y \arrow[d, dashed, "\exists!\alpha"] & Y \arrow[l, "i_Y"'] \arrow[dl, "\forall g"] \\
& Z &
\end{tikzcd}$$

Examples 2.17. In **Set**, the coproduct of X and Y is the disjoint union together with canonical inclusion. In **Top**, the coproduct of X and Y is their disjoint union equipped with the disjoint union topology. However, in **Grp**, the underlying set of the coproduct of two groups is not the disjoint union.

2.3.2 Equalizers and coequalizers

Definition 2.18. Let $\mathcal{C}$ be a category and $X, Y \in \mathrm{Ob}(\mathcal{C})$, $f, g : X \to Y$. Consider the contravariant functor $F : \mathcal{C} \to \mathbf{Set}$ defined by $F(c) = \{\alpha : c \to X \mid f\alpha = g\alpha\}$ and $F(\beta) = - \circ \beta$. An *equalizer* in $\mathcal{C}$ is a representation of the contravariant functor F.

By the Yoneda lemma, a natural transformation $\mathrm{Hom}(-, c) \Rightarrow F$ is the same as an element of $F(c)$, so a representation of F is a pair (c, e) with $c \in \mathrm{Ob}(\mathcal{C})$ and $e \in F(c)$ such that the natural transformation given by the Yoneda lemma is an isomorphism. Concretely, we want $\begin{array}{rccc} \eta_e : & \mathrm{Hom}(d, c) & \to & F(d) \\ & h & \mapsto & F(h)(e) \end{array}$ to be an isomorphism for all $d \in \mathrm{Ob}(c)$. This translates into the follwing diagram:

$$\begin{tikzcd}
& d \arrow[dl, dashed, "\exists!\alpha"'] \arrow[d, "\forall h"] & \\
c \arrow[r, "e"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y
\end{tikzcd}$$

Example 2.19. In **Set**, $E = \{x \in X \mid f(x) = g(x)\} \hookrightarrow X$ is an equalizer.

The dual notion is that of a coequalizer.

Definition 2.20. A *coequalizer* of $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ is an object $Z \in \mathrm{Ob}(\mathcal{C})$ together with a morphism $\pi : Y \to Z$ such that $\pi f = \pi g$ and that universal to this property:

$$\begin{tikzcd}
X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "\pi"] \arrow[d, "\forall h"] & Z \arrow[dl, dashed, "\exists!\alpha"] \\
& Z' &
\end{tikzcd}$$

Example 2.21. In **Set**, consider the equivalence relation $\sim$ on Y generated by $f(x) \sim g(x)$ (the smallest equivalence relation on Y with this property). Then $y \xrightarrow{\pi} Y/{\sim}$ is a coequalizer.

2.4 Adjoint functors

This notion was introduced by Kan in 1958.

Definition 2.22. An *adjunction* (G, D) is a pair of functors $G : \mathcal{C} \to \mathcal{D}$ and $D : \mathcal{D} \to \mathcal{C}$ together with an isomorphism $\mathrm{Hom}_{\mathcal{D}}(G(c), d) \simeq \mathrm{Hom}_{\mathcal{C}}(c, D(d))$ which is natural in both c and d.
We write $G \dashv D$ and say G is left adjoint to D and D is right adjoint to G.

If $G \dashv D$ we have $\forall c, d \in \mathrm{Ob}(\mathcal{C}) \times \mathrm{Ob}(\mathcal{D})$,

$$\mathrm{Hom}_{\mathcal{D}}(G(c), d) \xrightarrow[\alpha_{c,d}]{\sim} \mathrm{Hom}_{\mathcal{C}}(c, D(d))$$

and in particular when $d = G(c)$ we get $\mathrm{Hom}_{\mathcal{D}}(G(c), G(c)) \xrightarrow[\alpha_{c,G(c)}]{\sim} \mathrm{Hom}_{\mathcal{C}}(c, DG(c))$.
Let $\eta_c : c \to DG(c)$ be the image of $\mathrm{id}_{G(c)}$. This gives a collection of morphisms $- \to DG(-)$.

Proposition 2.23. *These morphisms make up a natural transformation* $\mathrm{id}_{\mathcal{C}} \Rightarrow DG$.

Proof. Let $f : c \to d$. We want to show that

$$\begin{tikzcd}
c \arrow[r, "\eta_c = \alpha_{c,G(c)}(\mathrm{id}_{G(c)})"] \arrow[d, "f"] & DG(c) \arrow[d, "DG(f)"] \\
d \arrow[r, "\eta_d = \alpha_{d,G(d)}(\mathrm{id}_{G(d)})"] & DG(d)
\end{tikzcd}$$

commutes. By naturality of the isomorphism α given by the adjunction, we get the following commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}(G(c), G(c)) \arrow[r, "\sim", "\alpha_{c,G(c)}"'] \arrow[d, "G(f)\circ -"'] & \mathrm{Hom}(c, DG(c)) \arrow[d, "DG(f)\circ -"'] \\
\mathrm{Hom}(G(c), G(d)) \arrow[r, "\sim", "\alpha_{c,G(d)}"'] & \mathrm{Hom}(c, DG(d)) \\
\mathrm{Hom}(G(d), G(d)) \arrow[r, "\sim", "\alpha_{d,G(d)}"'] \arrow[u, "-\circ G(f)"] & \mathrm{Hom}(d, DG(d)) \arrow[u, "-\circ f"]
\end{tikzcd}$$

which gives us these equations:

$$DG(f) \circ \eta_c = DG(f) \circ \alpha_{c,G(c)}(\mathrm{id}_{G(c)}) = \alpha_{c,G(d)}(G(f) \circ \mathrm{id}_{G(c)}) = \alpha_{c,G(d)}(G(f))$$
$$\eta_d \circ f = \alpha_{d,G(d)}(\mathrm{id}_{G(d)}) \circ f = \alpha_{c,G(d)}(\mathrm{id}_{G(c)} \circ G(f)) = \alpha_{c,G(d)}(G(f))$$

which completes the proof. $\square$

We also get a natural transformation $\varepsilon : GD \Rightarrow \mathrm{id}_{\mathcal{D}}$ when $c = D(d)$ by setting $\varepsilon_d = \alpha^{-1}_{D(d),d}(\mathrm{id}_{D(d)})$.

Definition 2.24. The natural transformation $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow DG$ is called the *unit* of the adjunction. The natural transformation $\varepsilon : GD \Rightarrow \mathrm{id}_{\mathcal{D}}$ is called its *counit*.

Proposition 2.25. *Let* $\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "G"] & \mathcal{D} \arrow[l, bend left, "D"] \end{tikzcd}$ *be two functors. Then,* $G \dashv D$ *if and only if there are natural transformations* $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow DG$ *and* $\varepsilon : GD \Rightarrow \mathrm{id}_{\mathcal{D}}$ *such that the following diagrams commute:*

$$\begin{tikzcd} G \arrow[r, Rightarrow, "G\eta"] \arrow[dr, equal, "\mathrm{id}_G"'] & GDG \arrow[d, Rightarrow, "\varepsilon G"] \\ & G \end{tikzcd} \qquad \begin{tikzcd} D \arrow[r, Rightarrow, "\eta D"] \arrow[dr, equal, "\mathrm{id}_D"'] & DGD \arrow[d, Rightarrow, "D\varepsilon"] \\ & D \end{tikzcd}$$

where $G\eta$ is the natural transformation given by the morphisms $G(\eta_c)$ and εG is the one give by morphisms $\varepsilon_{G(c)}$ (and similarly for ηD and $D\varepsilon$).

Proof. Suppose $G \dashv D$. Let $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow DG$ and $\varepsilon : GD \Rightarrow \mathrm{id}_{\mathcal{D}}$ be the unit and counit of the adjunction. Let $c \in \mathcal{C}$. We have

$$(\varepsilon G)_c \circ (G\eta)_c = \varepsilon_{G(c)} \circ G(\eta_c) = \alpha^{-1}_{DG(c),G(c)}(\mathrm{id}_{DG(c)}) \circ G(\alpha_{c,G(c)}(\mathrm{id}_{G(c)}))$$

and the naturality of α gives the following commutative diagram

$$\begin{tikzcd} \mathrm{Hom}(G(c),G(c)) & \mathrm{Hom}(c,DG(c)) \arrow[l, "\sim"', "\alpha^{-1}_{c,G(c)}"] \\ \mathrm{Hom}(GDG(c),G(c)) \arrow[u, "-\circ G(\alpha_{c,G(c)}(\mathrm{id}_{G(c)}))"] & \mathrm{Hom}(DG(c),DG(c)) \arrow[l, "\sim"', "\alpha^{-1}_{DG(c),G(c)}"] \arrow[u, "-\circ \alpha_{c,G(c)}(\mathrm{id}_{G(c)})"'] \end{tikzcd}$$

which shows that $(\varepsilon G)_c \circ (G\eta)_c = \mathrm{id}_{G(c)}$, hence $\varepsilon G \circ G\eta = \mathrm{id}_G$. The commutativity of the other triangle is treated in a similar way.
Now assume that there are natural transformations η and ε that make both triangles commute. We define two maps

$$\begin{array}{rrcl} \alpha_{c,d} : & \mathrm{Hom}(G(c),d) & \to & \mathrm{Hom}(c,D(d)) \\ & f & \mapsto & D(f) \circ \eta_c \\ \beta_{c,d} : & \mathrm{Hom}(c,D(d)) & \to & \mathrm{Hom}(G(c),d) \\ & g & \mapsto & \varepsilon_d \circ G(g) \end{array}$$

and we show these are natural isomorphisms that give us the adjunction. First we check naturality of α. Let $f : c \to c' \in \mathrm{Mor}(\mathcal{C})$ and $g : d \to d' \in \mathrm{Mor}(\mathcal{D})$. We need to check that the diagrams

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{D}}(G(c),d) \arrow[r, "\alpha_{c,d}"] & \mathrm{Hom}_{\mathcal{C}}(c,D(d)) \\ \mathrm{Hom}_{\mathcal{D}}(G(c'),d) \arrow[r, "\alpha_{c',d}"] \arrow[u, "-\circ G(f)"] & \mathrm{Hom}_{\mathcal{C}}(c',D(d)) \arrow[u, "-\circ f"] \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Hom}_{\mathcal{D}}(G(c),d) \arrow[r, "\alpha_{c,d}"] \arrow[d, "g\circ -"] & \mathrm{Hom}_{\mathcal{C}}(c,D(d)) \arrow[d, "D(g)\circ -"] \\ \mathrm{Hom}_{\mathcal{D}}(G(c),d') \arrow[r, "\alpha_{c,d'}"] & \mathrm{Hom}_{\mathcal{C}}(c,D(d')) \end{tikzcd}$$

commute. We only check the left diagram and leave the right to the reader (sorry). We have

$$\alpha_{c,d} \circ (- \circ G(f)) = (D(-) \circ \eta_c) \circ (- \circ G(f)) = D(- \circ G(f)) \circ \eta_c = D(-) \circ DG(f) \circ \eta_c$$
$$(- \circ f) \circ \alpha_{c',d} = (- \circ f) \circ (D(-) \circ \eta_{c'}) = D(-) \circ \eta_{c'} \circ f = D(-) \circ DG(f) \circ \eta_c$$

One shows β is natural in c and d in a similar way. We leave it to the reader (sorry again). Now we need to check that α and β are inverses of each other, and that's where the triangle diagrams come into play.

$$\alpha_{c,d} \circ \beta_{c,d} = D(\varepsilon_d \circ G(-)) \circ \eta_c = D(\varepsilon_d) \circ DG(-) \circ \eta_c = D(\varepsilon_d) \circ \eta_{D(d)} \circ - = -$$

We used the definitions of α and β, the functoriality of D, the naturality of η and the second triangle diagram. We leave to the reader (sorry) to check that $\beta_{c,d} \circ \alpha_{c,d}$ is also the identity. $\square$

Examples 2.26.

1. The forgetful functor $\mathbf{Ab} \to \mathbf{Set}$ is right adjoint to the free abelian group functor $\mathbf{Set} \to \mathbf{Ab}$.
2. The forgetful functor $\mathbf{Ab} \to \mathbf{Grp}$ is right adjoint to the abelianization functor $\mathbf{Grp} \to \mathbf{Ab}$ that sends a group G to its abelianization $G^{ab} = G/[G,G]$ and a morphism $f : G \to H$ to the induced morphism $f^{ab} : G^{ab} \to H^{ab}$.
3. The forgetful functor $\mathbf{Top} \to \mathbf{Set}$ is right adjoint to the functor $\mathbf{Set} \to \mathbf{Top}$ that takes a set and equips it with the coarse topology. The forgetful functor $\mathbf{Top} \to \mathbf{Set}$ is also left adjoint to the functor $\mathbf{Set} \to \mathbf{Top}$ that equips a set with the discrete topology.
4. Let G be a group, H one of its subgroups and k be a field. We have a functor from the category $\mathbf{Rep}_k(G)$ of representations of G on k-vector spaces to the category $\mathbf{Rep}_k(H)$ of representations of H on k-vector spaces. It is the restriction functor Res_H^G. Its left adjoint is Ind_H^G, the induced representation functor.

Theorem 2.27. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. The following are equivalent:*

1. *F admits a left adjoint*
2. *For all $X \in \mathrm{Ob}(\mathcal{D})$, $\mathrm{Hom}_{\mathcal{D}}(X, F(-))$ is representable*
3. *For all $X \in \mathrm{Ob}(\mathcal{D})$, there exists a universal arrow $X \to F$*

Corollary. *If they exist, adjoints are unique up to isomorphism.*

Proof. $2 \iff 3$ was the subject of a previous remark right after the Yoneda lemma. We prove $1 \iff 2$. Suppose F admits a left adjoint G. Let $X \in \mathrm{Ob}(\mathcal{D})$. Then for all $Y \in \mathrm{Ob}(\mathcal{C})$ we have a bijection $\mathrm{Hom}_{\mathcal{D}}(X, F(Y)) \simeq \mathrm{Hom}_{\mathcal{C}}(G(X), Y)$ which is natural in Y, so $G(X)$ represents $\mathrm{Hom}_{\mathcal{D}}(X, F(-))$. For the converse, suppose all functors $\mathrm{Hom}_{\mathcal{D}}(X, F(-))$ are representable. We define $G(X)$ to be an object of $\mathcal{C}$ that represents $\mathrm{Hom}_{\mathcal{D}}(X, F(-))$. Now choose $X, Y \in \mathrm{Ob}(\mathcal{D})$ and $f : X \to Y$. We need to define $G(f)$. We wish to have a commuting square

$$\begin{tikzcd}
\mathrm{Hom}(G(X), -) \arrow[r, "\sim"] & \mathrm{Hom}(X, F(-)) \\
\mathrm{Hom}(G(Y), -) \arrow[r, "\sim"] \arrow[u, dashed, "\exists ! \gamma"] & \mathrm{Hom}(Y, F(-)) \arrow[u, "- \circ f"']
\end{tikzcd}$$

We need to recover a map $G(X) \to G(Y)$ such that composing with it gives us γ. This works by the Yoneda lemma, which tells us that the natural transformation γ comes from an element $\mathrm{Hom}(G(X), G(Y))$. Call it $G(f)$. Using this diagram with $X = Y$ and $f = \mathrm{id}_X$ shows that $G(\mathrm{id}_X) = \mathrm{id}_{G(X)}$. It remains to check this does define a functor. Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{C}$. Then we have the diagram

$$\begin{tikzcd}
\mathrm{Hom}(G(Z), -) \arrow[r, "- \circ G(g)"] \arrow[d, "\wr"] \arrow[rr, bend left, "- \circ G(g \circ f)"] & \mathrm{Hom}(G(Y), -) \arrow[r, "- \circ G(f)"] \arrow[d, "\wr"] & \mathrm{Hom}(G(X), -) \arrow[d, "\wr"] \\
\mathrm{Hom}(Z, F(-)) \arrow[r, "- \circ g"] \arrow[rr, bend right, "- \circ (g \circ f)"'] & \mathrm{Hom}(Y, F(-)) \arrow[r, "- \circ f"] & \mathrm{Hom}(X, F(-))
\end{tikzcd}$$

Which shows $G(g \circ f) = G(g) \circ G(f)$ (because the map γ above is unique). $\square$

This theorem shows there is a deep link between universal properties and adjoint functors.

2.5 Limits and colimits

(This subsection may be skipped on a first reading.)

Definition 2.28.