Homological Algebra

Original lectures notes by Baptiste Rognerud, but now typeset in LaTeX

1 Introduction to category theory

References:

- Emily Riehl, Category Theory in Context (chapter I)
- Saunders Mac Lane, Categories for the Working Mathematician
- Ibrahim Assem, Introduction au langage catégorique (chapters I, II)

❯ Near 1945 Eilenberg and Mac Lane gave the good formalism for a "natural isomorphism" (the general theory of natural transformations). For instance, if V is a finite-dimensional vector space, $V \simeq V^*$ and $V \simeq V^{**}$, but the first isomorphism is not natural ("a choice needs to be made"), while the second is. But why?
It turns out solving this question gave a formalism, category theory, that unified already existing mathematical concepts, gave new links between different notions and also gave new questions!
⚠ Category theory is not a theory that trivializes mathematics.
It is used today by (almost) everyone: algebraic geometry, algebra, representation theory, topology, combinatorics, . . .

1.1 Categories and functors

Definition 1.1. A *category* $\mathcal{C}$ is the data of

- A collection of *morphisms* $\mathrm{Mor}(\mathcal{C})$
- A collection of *objects* $\mathrm{Ob}(\mathcal{C})$

such that

1. Every morphism $f \in \mathrm{Mor}(\mathcal{C})$ has a specified domain $X \in \mathrm{Ob}(\mathcal{C})$ and codomain $Y \in \mathrm{Ob}(\mathcal{C})$. We write $f : X \to Y$.
2. For every object $X \in \mathrm{Ob}(\mathcal{C})$ there exists a morphism $1_X : X \to X$ (the *identity* of X), also written id_X
3. For any three objects $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$ and morphism $f : X \to Y$ and $g : Y \to Z$ there exists a morphism $g \circ f : X \to Z$ (we often omit $\circ$ and just write gf)

satisfying

(Identity) $\forall f : X \to Y, 1_Y f = f = f 1_X$

(Associativity) $\forall f : W \to X, g : X \to Y, h : Y \to Z$, $h(gf) = (hg)f$

Remark.

1. We use the term "collection" because we don't want to worry about set-theoretical issues
2. If $\mathrm{Mor}(\mathcal{C})$ is a set, we say that $\mathcal{C}$ is *small*
3. We denote by $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ (or $\mathcal{C}(X, Y)$) the collection of $f : X \to Y \in \mathrm{Mor}(\mathcal{C})$

Examples 1.2 (Concrete categories)**.**

1. The category **Set**, where objects are sets and morphisms are just maps.
2. **Top**, where objects are topological spaces and morphisms are continuous maps.
3. Groups together with group homomorphisms form a category called **Grp**. The same can be said about rings, fields...
4. k-vector spaces, or more generally left/right R-modules, together with linear maps, form a category denoted $R\mathbf{Mod}$ or $\mathbf{Mod}R$ (for left or right R-modules).

In these previous examples, objects are sets with additional structure, and morphisms between two objects are in particular maps between the two underlying sets. Such categories are called *concrete categories* (a rigorous definition will be given later). However, a category need not be concrete.

Examples 1.3 (Abstract categories)**.**

1. Let k be a field. There exists a category $\mathbf{Mat}_k$ where objects are the natural numbers $\mathbb{N}$ and morphisms are $\mathrm{Hom}(m, n) = \mathrm{Mat}_{n,m}(k)$, where composition is given by matrix multiplication.
2. If G is a group, there exists a category BG which has only one object $\bullet$, and morphisms $\mathrm{Hom}(\bullet, \bullet) = G$, where composition is multiplication in G.
3. If $(P, \leq)$ is a *poset* (a partially ordered set, that is a set P together with a reflexive, transitive relation $\leq$), then one can construct a category $\hat{P}$ by setting $\mathrm{Ob}(\hat{P}) = P$ and $|\mathrm{Hom}(x, y)| = \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{otherwise} \end{cases}$, where composition is defined in the only possible way.
4. The homotopy category of topological spaces: objects are topological spaces, and $\mathrm{Hom}(X, Y)$ is $\mathrm{Hom}_{\mathbf{Top}}(X, Y)/\sim$ where $\sim$ is homotopy of continuous maps.

Exercise. Check the categories defined above really are categories. In (2), what are the minimal hypotheses needed on G for BG to be a category? In (3), what are the minimal hypotheses needed on $\leq$ for $\hat{P}$ to be a category?

Examples 1.4 (Categories constructed from categories)**.**

1. If $\mathcal{C}$ is a category, one can construct its *opposite category* $\mathcal{C}^{\mathrm{op}}$, defined by $\mathrm{Ob}(\mathcal{C}^{\mathrm{op}}) = \mathrm{Ob}(\mathcal{C})$ and $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X)$, with composition described by the following diagram:

$$
\begin{tikzcd}
X \arrow[d, "f"] \arrow[dd, bend right, "gf"'] & X \\
Y \arrow[d, "g"] \arrow[r, rightsquigarrow] & Y \arrow[u, "f^{\mathrm{op}}"] \\
Z & Z \arrow[u, "g^{\mathrm{op}}"] \arrow[uu, bend right, "f^{\mathrm{op}}g^{\mathrm{op}}"']
\end{tikzcd}
$$

2. Let $\mathcal{C}$ be a category. A *subcategory* $\mathcal{D}$ of $\mathcal{C}$ is another category such that $\mathrm{Ob}(\mathcal{D}) \subset \mathrm{Ob}(\mathcal{C})$ and $\mathrm{Mor}(\mathcal{D}) \subset \mathrm{Mor}(\mathcal{C})$ and the composition in $\mathcal{D}$ is induced by the one in $\mathcal{C}$. For instance, **Ab**, the category of abelian groups and group homomorphisms, is a subcategory of **Grp**.

3. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. The *product category* of $\mathcal{C}$ and $\mathcal{D}$ is the category $\mathcal{C} \times \mathcal{D}$ defined by $\mathrm{Ob}(\mathcal{C} \times \mathcal{D}) = \mathrm{Ob}(\mathcal{C}) \times \mathrm{Ob}(\mathcal{D})$ and $\mathrm{Mor}(\mathcal{C} \times \mathcal{D}) = \mathrm{Mor}(\mathcal{C}) \times \mathrm{Mor}(\mathcal{D})$, composition and identities being defined componentwise.

Exercise. Describe $(BG)^{\mathrm{op}}$ for G a group and $\hat{P}^{\mathrm{op}}$ for $(P, \leq)$ a poset.

⚠ $\mathbf{Set}^{\mathrm{op}}$ is not **Set**. **TODO**

Remark. In a category $\mathcal{C}$ the objects can be anything, so saying $x \in X$ for $X \in \mathrm{Ob}(\mathcal{C})$ doesn't make sense. Hence, categorical notions are defined using arrows and not elements.

Definition 1.5. Let $\mathcal{C}$ be a category.

1. $f : X \to Y$ is an *isomorphism* if there exists $g : Y \to X$ such that $gf = \mathrm{id}_X$ and $fg = \mathrm{id}_Y$.

2. $f : X \to Y$ is a *monomorphism* if for all $g, h : W \to X$ such that $fg = fh$, $g = h$ (f is left-cancellable).

3. $f : X \to Y$ is an *epimorphism* if for all $g, h : Y \to Z$ such that $gf = hf$, $g = h$ (f is right-cancellable).

⚠ Being both a mono and an epi doesn't imply being an iso. **TODO**

Definition 1.6. Let $\mathcal{C}, \mathcal{D}$ be two categories. A *(covariant) functor* $F : \mathcal{C} \to \mathcal{D}$ is the data of

- An object $F(X) \in \mathrm{Ob}(\mathcal{D})$ for all $X \in \mathrm{Ob}(\mathcal{C})$

- A morphism $F(f) \in \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ for all $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$

such that $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for all $X \in \mathrm{Ob}(\mathcal{C})$ and $F(gf) = F(g)F(f)$ whenever $f, g \in \mathrm{Mor}(\mathcal{C})$ are composable.

Definition 1.7. A *contravariant* functor from $\mathcal{C}$ to $\mathcal{D}$ is a functor from $\mathcal{C}^{\mathrm{op}}$ to $\mathcal{D}$ (so composition is reversed, i.e. $F(gf) = F(f)F(g)$).

Examples 1.8.

1. $U : \mathbf{Grp} \to \mathbf{Set}, U(G) = G, U(f) = f$ the functor that to a group assigns it its underlying set and to a homomorphism the underlying map. It is called the *forgetful functor* from groups to sets, because it forgets the group structure.

2. $U : \begin{array}{ccc} \mathbf{Ass} & \to & \mathbf{Lie} \\ (A,+,\cdot) & \mapsto & (A,+,[-,-]) \end{array}$ the forgetful functor from the category of associative algebras to the category of Lie algebras. It forgets the "associative structure" but remembers the underlying abelian group.

3. $F : \mathbf{Set} \to \mathbf{Ab}, X \mapsto \mathbb{Z}[X], f \mapsto \mathbb{Z}[f]$, which to a set assigns the free abelian group with basis X (the group of finite linear combinations of elements of X). A map $f : X \to Y$ can then be uniquely extended to a linear map $\mathbb{Z}[f] : \mathbb{Z}[X] \to \mathbb{Z}[Y]$ that agrees with f on the bases of $\mathbb{Z}[X]$ and $\mathbb{Z}[Y]$.

4. Suppose $\mathcal{C}$ is locally small (i.e. for any X, Y, $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ is a set). For all $X \in \mathcal{C}$, $\mathrm{Hom}(X,-)$ is a functor $\mathcal{C} \to \mathbf{Set}$. Similarly, $\mathrm{Hom}_{\mathcal{C}}(-,X)$ is a contravariant functor $\mathcal{C} \to \mathbf{Set}$. $\mathrm{Hom}_{\mathcal{C}}(-,-)$ is a functor $\mathcal{C} \times \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$.

5. Functors $\mathcal{C} \xrightarrow[F]{} \mathcal{D} \xrightarrow[G]{} \mathcal{E}$ can be composed in the obvious sense.

TODO: DRAW DIAGRAMS

Definition 1.9. Let $\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] & \mathcal{D} \end{tikzcd}$ be two functors. A *natural transformation* η from F to G is the data of morphisms $\eta_X : F(X) \to G(X) \in \mathrm{Mor}(\mathcal{D})$ for all $X \in \mathrm{Ob}(\mathcal{C})$ such that for all $f : X \to Y \in \mathrm{Mor}(\mathcal{C})$, the diagram

$$\begin{tikzcd} F(X) \arrow[r, "\eta_X"] \arrow[d, "F(f)"] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\eta_Y"] & G(Y) \end{tikzcd}$$

commutes, that is $G(f)\eta_X = \eta_Y F(f)$. We write $\eta : F \Rightarrow G$ or draw $\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathcal{D} \end{tikzcd}$

Example 1.10. Let V be a k-vector space. $\mathrm{id}_{\mathbf{Vect}_k}$ and $D^2 = \mathrm{Hom}_{\mathbf{Vect}_k}(\mathrm{Hom}_{\mathbf{Vect}_k}(-,k),k)$ are two endofunctors of $\mathbf{Vect}_k$. $\mathrm{ev}_- : \begin{array}{ccc} V & \to & V^{**} \\ v & \mapsto & \begin{array}{ccc} \mathrm{Hom}(V,k) & \to & k \\ \phi & \mapsto & \phi(v) \end{array} \end{array}$ defines a natural transformation between them:

$$\begin{tikzcd} V \arrow[r, "\mathrm{ev}"] \arrow[d, "f"'] & V^{**} \arrow[d, "D^2(f)"] \\ W \arrow[r, "\mathrm{ev}"'] & W^{**} \end{tikzcd}$$

For $a \in V$, $D^2(f) \circ \mathrm{ev}_a : \begin{array}{ccc} W^* & \to & k \\ \phi & \mapsto & \phi(f(a)) \end{array} \in W^{**}$ and in the other direction $(\mathrm{ev} \circ f)(a) = \mathrm{ev}_{f(a)}$.
However, there is no natural transformation from $\mathrm{id}_{\mathbf{Vect}_k}$ to D. For one, the first is covariant and the second is contravariant. To get a natural transformation from a covariant to a contravariant functor, we can modify the definition of naturality to be that $\begin{array}{ccc} V & \to & V^* \\ \downarrow & & \uparrow \\ W & \to & W^* \end{array}$ commutes, but even such natural transformations do not exist from $\mathrm{id}_{\mathbf{Vect}_k}$ to D.

Definition 1.11. A natural transformation $\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"{name=U}] \arrow[r, bend right, "G"'{name=D}] & \mathcal{D} \arrow[from=U, to=D, Rightarrow, "\eta"] \end{tikzcd}$ is a *natural isomorphism* if η_X is an isomorphism for all $X \in \mathrm{Ob}(\mathcal{C})$.

Remark. One can compose natural transformations in two ways, "vertical composition":

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=60, "F"{name=U}] \arrow[r, "G"{name=M, description}] \arrow[r, bend right=60, "H"'{name=D}] & \mathcal{D} \arrow[from=U, to=M, Rightarrow, "\alpha"] \arrow[from=M, to=D, Rightarrow, "\beta"] \end{tikzcd} \rightsquigarrow \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"{name=U}] \arrow[r, bend right, "H"'{name=D}] & \mathcal{D} \arrow[from=U, to=D, Rightarrow, "\beta\circ\alpha"] \end{tikzcd} \text{ where } (\beta \circ \alpha)_X = \beta_X \circ \alpha_X$$

or "horizontal composition":

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F_1"{name=U1}] \arrow[r, bend right, "G_1"'{name=D1}] & \mathcal{D} \arrow[r, bend left, "F_2"{name=U2}] \arrow[r, bend right, "G_2"'{name=D2}] & \mathcal{E} \arrow[from=U1, to=D1, Rightarrow, "\alpha_1"] \arrow[from=U2, to=D2, Rightarrow, "\alpha_2"] \end{tikzcd} \rightsquigarrow \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F_2\circ F_1"{name=U}] \arrow[r, bend right, "G_2\circ G_1"'{name=D}] & \mathcal{E} \arrow[from=U, to=D, Rightarrow, "\alpha_2*\alpha_1"] \end{tikzcd} \text{ where } (\alpha_2 * \alpha_1)_X = G_2((\alpha_1)_X) \circ (\alpha_2)_{F_1(X)}$$

Horizontal composition can also be defined in another equivalent way using commutativity of

$$\begin{tikzcd} F_2F_1(X) \arrow[r, "(\alpha_2)_{F_1(X)}"] \arrow[d, "F_2((\alpha_1)_X)"'] & G_2F_1(X) \arrow[d, "G_2((\alpha_1)_X)"] \\ F_2G_1(X) \arrow[r, "(\alpha_2)_{G_1(X)}"] & G_2G_1(X) \end{tikzcd}$$

The diagram commutes by naturality of α_2, so $(\alpha_2 * \alpha_1) = (\alpha_2)_{G_1(X)} \circ F_2((\alpha_1)_X)$.

Definition 1.12. Let $\mathcal{C}, \mathcal{D}$ be two categories. Then the *functor category from $\mathcal{C}$ to $\mathcal{D}$* written $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ or $\mathcal{D}^{\mathcal{C}}$ is the category whose objects are functors from $\mathcal{C}$ to $\mathcal{D}$ and morphisms are natural transformations.

Remark. Categories, together with functors and natural transformations between them is the prototypal example of a 2-category.

1.2 Equivalences of categories

Definition 1.13. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. An *equivalence of categories* from $\mathcal{C}$ to $\mathcal{D}$ is the data of

1. $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ wo functors
2. Natural isomorphisms $\eta : 1_{\mathcal{C}} \Rightarrow GF$ and $\varepsilon : FG \Rightarrow 1_{\mathcal{D}}$ where $1_{\mathcal{C}}$ and $1_{\mathcal{D}}$ are the identity functors of $\mathcal{C}$ and $\mathcal{D}$.

Remark.

1. G is called a *quasi-inverse* of F.
2. Most of the time we say that F is an equivalence if there exists G such that (F, G) is an equivalence.

3. If F, G are contravariant, we speak of *duality* between $\mathcal{C}$ and $\mathcal{D}$.

4. If two categories are equivalent, every property that can be expressed "in terms of arrows" is preserved.

Definition 1.14. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then, we say

1. F is *faithful* if $\forall X, Y \in \mathrm{Ob}(\mathcal{C})$, $F : \begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(X,Y) & \to & \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y)) \\ f & \mapsto & F(f) \end{array}$ is injective.

2. F is *full* if the previous map is surjective.

3. F is *essentially surjective* if for all $Y \in \mathrm{Ob}(\mathcal{D})$ there is $X \in \mathrm{Ob}(\mathcal{C})$ such that $F(X) \simeq Y$ in $\mathcal{D}$.

Theorem 1.15. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then F is an equivalence of categories if and only if it is fully faithful and essentially surjective.*

Proof. ⚠ There is a little set-theoretic issue: an equivalence of categories is always fully faithful and essentially surjective, but the converse requires the axiom of choice for the class $\mathrm{Ob}(\mathcal{C})$.
Suppose $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories, and let $G : \mathcal{D} \to \mathcal{C}$ be a quasi-inverse of F, together with natural isomorphisms $\eta : 1_{\mathcal{C}} \to GF$ and $\varepsilon : 1_{\mathcal{D}} \to FG$. If Y is an object of $\mathcal{D}$, then $Y \simeq FG(Y)$, so F is essentially surjective. Let X, Y be objects of $\mathcal{C}$. To show F is fully faithful we will construct an inverse to $F : \mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))$. For any $f \in \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))$, we have a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & GF(X) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle GF(f)} \\ Y & \xrightarrow{\eta_Y} & GF(Y) \end{array}$$

which prompts us to define $\phi : \begin{array}{ccc} \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y)) & \to & \mathrm{Hom}_{\mathcal{C}}(X,Y) \\ g & \mapsto & \eta_Y^{-1} \circ G(g) \circ \eta_X \end{array}$. We now check it is the map we're looking for. If $f : X \to Y$, since the above diagram commutes and η_Y is invertible, we get that $\phi(F(f)) = f$, so $\phi \circ F = \mathrm{id}_{\mathrm{Hom}_{\mathcal{C}}(X,Y)}$, which means F is faithful. We have two commutative diagrams, by definition of ϕ and by naturality of η:

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & GF(X) \\ {\scriptstyle \phi(g)}\downarrow & & \downarrow{\scriptstyle G(g)} \\ Y & \xrightarrow{\eta_Y} & GF(Y) \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{\eta_X} & GF(X) \\ {\scriptstyle \phi(g)}\downarrow & & \downarrow{\scriptstyle GF(\phi(g))} \\ Y & \xrightarrow{\eta_Y} & GF(Y) \end{array}$$

therefore, $G(g) \circ \eta_X = GF(\phi(g)) \circ \eta_X$. Since η_X is invertible, $G(g) = GF(\phi(g))$. The previous point shows that G is faithful, so $g = F(\phi(g))$, hence F is full.
Now suppose F is fully faithful and essentially surjective. Our goal is to construct G. For any $Y \in \mathrm{Ob}(\mathcal{D})$, since F is essentially surjective, there exists $X_Y \in \mathrm{Ob}(\mathcal{C})$ and an isomorphism $\varepsilon_Y : Y \to F(X_Y)$. Therefore, for any $Y, Z \in \mathrm{Ob}(\mathcal{D})$ and $f : Y \to Z$, we have a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{f} & Z \\ \downarrow{\scriptstyle \varepsilon_Y} & & \downarrow{\scriptstyle \varepsilon_Z} \\ F(X_Y) & \xrightarrow[\varepsilon_Z \circ f \circ \varepsilon_Y^{-1}]{} & F(X_Z) \end{array}$$

Which leads us to define $G(Y) = X_Y$ and $G(f)$ to be the unique morphism $m_f : X_Y \to X_Z$ such that $F(m_f) = \varepsilon_Z \circ f \circ \varepsilon_Y^{-1}$ (this works because F is fully faithful). We have $G(\mathrm{id}_Y) = \mathrm{id}_{X_Y}$ since $\varepsilon_Y \circ \mathrm{id}_Y \circ \varepsilon_Y^{-1} = \mathrm{id}_Y$ and $F(\mathrm{id}_{X_Y}) = \mathrm{id}_Y$. The next diagram shows $G(g \circ f) = G(g) \circ G(f)$:

$$\begin{tikzcd}
W \arrow[r, "f"] \arrow[rr, bend left, "g\circ f"] \arrow[d, "\varepsilon_W"] & Y \arrow[r, "g"] \arrow[d, "\varepsilon_Y"] & Z \arrow[d, "\varepsilon_Z"] \\
F(X_W) \arrow[r, "F(m_f)"'] \arrow[rr, bend right, "F(m_g\circ m_f)=F(m_g)\circ F(m_f)"'] & F(X_Y) \arrow[r, "F(m_g)"'] & F(X_Z)
\end{tikzcd}$$

By this construction, ε is a natural isomorphism $\mathrm{id}_{\mathcal{D}} \Rightarrow FG$ (look at the above diagrams). Now, pick $Y, Z \in \mathrm{Ob}(\mathcal{C})$ and $f : Y \to Z$. We have $GF(Y) = X_{F(Y)}$ and $\varepsilon_Y : F(Y) \xrightarrow{\sim} F(X_{F(Y)})$. Since F is fully faithful, there exists a unique $\eta_Y : Y \to X_{F(Y)} = GF(Y)$ such that $F(\eta_Y) = \varepsilon_Y$. Here, $\eta_Y = G(\varepsilon_Y)$, which means that η_Y is an isomorphism since functors preserve isomorphisms. We obtain the diagram

$$\begin{tikzcd}
Y \arrow[r, "\eta_Y"] \arrow[d, "f"] & GF(Y) \arrow[d, "GF(f)"] \\
Z \arrow[r, "\eta_Z"] & GF(Z)
\end{tikzcd}$$

The diagram commutes because $GF(f)$ is the unique morphism such that

$$F(GF(f)) = \varepsilon_Z \circ F(f) \circ \varepsilon_Y^{-1} = F(\eta_Z \circ f \circ \eta_Y^{-1})$$

and F is faithful. η is then a natural isomorphism $\mathrm{id}_{\mathcal{C}} \Rightarrow GF$. $\square$

Example 1.16. $\mathbf{Vect}_k \simeq \mathbf{Mat}_k$ through the functor $n \mapsto k^n$ and $(A : n \to m) \mapsto (f_A : k^n \to k^m)$.

2 Universal properties

References:

- Riehl (Chapters II, III, IV)
- Mac Lane (Chapters III, IV, V)
- Assem (Chapters III, IV, V)

❯ Let S be a set together with an equivalence relation $\sim$. Let ${}^{S}\!/\!_{\sim}$ be the quotient set, and $\pi : S \to {}^{S}\!/\!_{\sim}$ be the projection. For any $f : S \to X$ compatible with $\sim$, there exists a unique map $\bar{f} : {}^{S}\!/\!_{\sim} \to X$ such that $f = \bar{f} \circ \pi$. This is represented by the following commutative diagram :

$$\begin{tikzcd}
S \arrow[r, "f"] \arrow[d, "\pi"'] & X \\
{}^{S}\!/\!_{\sim} \arrow[ur, dashed, "\exists ! \bar{f}"'] &
\end{tikzcd}$$

We say that $S \xrightarrow{\pi} {}^{S}\!/\!_{\sim}$ is a solution to the universal problem posed by the compatible maps. Such a solution is unique up to unique isomorphism: if $S \xrightarrow{p} S'$ is another solution, then we get the three commutative diagrams

$$\begin{tikzcd}
S \arrow[r, "p"] \arrow[d, "\pi"'] & S' \\
{}^{S}\!/\!_{\sim} \arrow[ur, dashed, "\exists ! a"'] &
\end{tikzcd}
\quad
\begin{tikzcd}
S \arrow[r, "\pi"] \arrow[d, "p"'] & {}^{S}\!/\!_{\sim} \\
S' \arrow[ur, dashed, "\exists ! b"'] &
\end{tikzcd}
\quad
\begin{tikzcd}
S \arrow[r, "p"] \arrow[d, "p"'] & S' \\
S' \arrow[ur, "\mathrm{id}_{S'}"'] \arrow[r, "b"'] & {}^{S}\!/\!_{\sim} \arrow[u, "a"']
\end{tikzcd}$$

then $abp = a\pi = p$. The identity of S' also makes this diagram commute so by uniqueness $ab = \mathrm{id}_{S'}$ and similarly $ba = \mathrm{id}_{S/\sim}$.

2.1 Initial and final objects

Definition 2.1. Let $\mathcal{C}$ be a category. An object $c \in \mathrm{Ob}(\mathcal{C})$ is *initial* (*final*) if for all $d \in \mathrm{Ob}(\mathcal{C})$ there exists a unique morphism $c \to d$ (a unique morphism $d \to c$).

Proposition 2.2. *If an initial/final object exists, then it is unique up to unique isomorphism.*

Proof. Let c, c' be two initial objects. Then there exists a unique morphism $f : c \to c'$ and a unique morphism $g : c' \to c$. There also exists a unique morphism $c \to c$, that is id_c. Therefore, $gf = \mathrm{id}_c$. In the same way, $fg = \mathrm{id}_{c'}$. Therefore, c and c' are isomorphic and the isomorphism is unique. $\square$

Examples 2.3.

1. $\varnothing$ is initial in **Set** and any singleton is final.
2. $\{0\}$ is both initial and final in $\mathbf{Vect}_k$ (or R**Mod**).
3. The category of fields does not have initial/final objects (reason on field characteristics).

We want to say a universal object is an initial or final object. A category has at most 2, so this may seem a little restrictive, but this is solved by thinking of a good category.

Definition 2.4. Let $F : \mathcal{C} \to \mathbf{Set}$ be a functor. Let $\int F$ be the category defined by

$$\begin{aligned} &\mathrm{Ob}(\textstyle\int F) = \{(c, x) \mid c \in \mathrm{Ob}(\mathcal{C}) \text{ and } x \in F(c)\} \\ &\mathrm{Hom}((c, x), (c', x')) = \{f \in \mathrm{Hom}(c, c') \mid F(f)(x) = x'\} \end{aligned}$$

where composition is composition in $\mathcal{C}$, and $\mathrm{id}_{(c,x)} = \mathrm{id}_c$ for all x. If F is contravariant, let $\int F$ have the same objects and morphisms $\mathrm{Hom}((c, x), (c', x')) = \{f \in \mathrm{Hom}(c, c') \mid F(f)(x') = x\}$.

Proposition 2.5. *There is a forgetful functor $\pi : \int F \to \mathcal{C}$ defined by $\pi(c, x) = c$ and $\pi(f : (c, x) \to (c', x')) = f : c \to c'$.*

Example 2.6. Let S be a set, and $\sim$ an equivalence relation on S. Let $F : \mathbf{Set} \to \mathbf{Set}$ be defined by $F(X) = \{f : S \to X \mid x \sim y \Rightarrow f(x) = f(y)\}$ and $F(\alpha : X \to Y) = \alpha \circ -$.

$\int F$ has for objects $(X, S \xrightarrow{f} X)$ where f is compatible with $\sim$, and for morphisms α that makes this diagram commute:

$$\begin{tikzcd} S \arrow[r, "f'"] \arrow[d, "f"'] & X' \\ X \arrow[ur, "\alpha"'] & \end{tikzcd}$$

$({}^{S}\!/\!_{\sim}, S \xrightarrow{\pi} {}^{S}\!/\!_{\sim})$ is an initial object of $\int F$.

Definition 2.7. Let $F : \mathcal{C} \to \mathbf{Set}$ be a functor. A *universal element* for F is an initial object of $\int F$, that is a pair (c, x) with $c \in \mathrm{Ob}(\mathcal{C})$ and $x \in F(c)$ such that

$$\forall (d, y), d \in \mathrm{Ob}(\mathcal{C}), y \in F(d), \exists! \alpha : c \to d, y = F(\alpha)(x)$$

Definition 2.8. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor and $d \in \mathrm{Ob}(\mathcal{D})$. A *universal arrow from d to F* is a pair (c, f) where $c \in \mathrm{Ob}(\mathcal{C})$ and $f \in \mathrm{Hom}_{\mathcal{D}}(d, F(c))$, such that

$$\forall (c', f'), c' \in \mathrm{Ob}(\mathcal{C}), f' : d \to F(c'), \exists! \alpha \in \mathrm{Hom}_{\mathcal{C}}(c, c'), F(\alpha) \circ f = f'$$

$$\begin{tikzcd} & d \arrow[dl, "f"'] \arrow[dr, "\forall f'"] & \\ F(c) \arrow[rr, dashed, "F(\alpha)"'] & & F(c') \end{tikzcd}$$

$$\begin{tikzcd} c \arrow[r, "\exists! \alpha"] & c' \end{tikzcd}$$

Exercise. Define a category $d \downarrow F$ such that a universal arrow is an initial object of $d \downarrow F$.

Example 2.9. Let $U : \mathbf{Vect}_k \to \mathbf{Set}$ be the forgetful functor. Let $X \in \mathbf{Set}$. A universal arrow from X to U is the "best" k-vector space V_X with a map $X \to V_X$. Set $V_X = k[X]$ the k-vector space with basis X, and $i : X \to V_X$ that maps $x \in X$ to the corresponding basis element. Then, for any vector space V and map $f : X \to U(V)$, f can be extended by linearity into a linear map $\tilde{f} : k[X] \to V$, which makes this diagram commute:

$$\begin{tikzcd} & X \arrow[dl, "i"'] \arrow[dr, "f"] & \\ k[X] \arrow[rr, dashed, "\tilde{f}"'] & & U(V) \end{tikzcd}$$

If α is another map that makes the diagram commute then α and $\tilde{f}$ coincide on a basis of $k[X]$ and therefore are equal.

Proposition 2.10. *Universal elements and arrows are two equivalent notions.*

Proof. Let $F : \mathcal{C} \to \mathbf{Set}$ be a functor and (c, x) a universal element for F. Consider $f_x : \begin{array}{ccc} \{*\} & \to & F(c) \\ * & \mapsto & x \end{array}$. Then, (c, f_x) is a universal arrow $* \to F$, because $F(\alpha)(x) = y$ iff $F(\alpha) \circ f_x = f_y$.

$$\begin{tikzcd} & \{*\} \arrow[dl, "f_x"'] \arrow[dr, "f_y"] & \\ F(c) \arrow[rr, dashed, "F(\alpha)"'] & & F(c') \end{tikzcd}$$

Conversely, if $F : \mathcal{C} \to \mathcal{D}$ is a functor and (c, f) is a universal arrow $d \to F$, then consider the functor $\mathrm{Hom}_{\mathcal{D}}(d, F(-)) : \begin{array}{ccc} \mathcal{C} & \to & \mathbf{Set} \\ x & \mapsto & \mathrm{Hom}_{\mathcal{D}}(d, F(x)) \end{array}$ (we need to assume $\mathcal{D}$ is locally small so the functor is set-valued). Then, $f \in \mathrm{Hom}_{\mathcal{D}}(d, F(c))$ is a universal element for this functor. $\square$

2.2 Representable functors

Definition 2.11. Let $\mathcal{C}$ be a (locally small) category, and $F : \mathcal{C} \to \mathbf{Set}$ a functor.

1. We say that F is *representable* if there is some $c \in \mathrm{Ob}(\mathcal{C})$ such that F and $\mathrm{Hom}_{\mathcal{C}}(c, -)$ are naturally isomorphic (if F is contravariant, use $\mathrm{Hom}_{\mathcal{C}}(-, c)$ instead).
2. A *representation* of F is the data of $c \in \mathrm{Ob}(\mathcal{C})$ and a natural isomorphism $\eta : \mathrm{Hom}(c, -) \Rightarrow F$.

Example 2.12. The forgetful functor $U : \mathbf{Grp} \to \mathbf{Set}$ is representable since $\mathrm{Hom}_{\mathbf{Grp}}(\mathbb{Z}, -) \simeq U$. The natural isomorphism is given by $\alpha \in \mathrm{Hom}_{\mathbf{Grp}}(\mathbb{Z}, G) \mapsto \alpha(1) \in G$.

The following theorem explains how to find the natural isomorphism $\alpha : \mathrm{Hom}_{\mathcal{C}}(c, -) \Rightarrow F$ in general.

Theorem 2.13 (Yoneda lemma)**.** *Let $F : \mathcal{C} \to \mathbf{Set}$ be a functor with $\mathcal{C}$ locally small, and $c \in \mathrm{Ob}(\mathcal{C})$. Then,*

$$\begin{array}{rcl} \mathrm{Nat}(\mathrm{Hom}(c, -), F) & \xrightarrow{\sim} & F(c) \\ \alpha & \mapsto & \alpha_c(\mathrm{id}_c) \end{array}$$

and this isomorphism is natural in c and in F.

Proof. Let $\alpha \in \mathrm{Nat}(\mathrm{Hom}(c, -), F)$. Let $d \in \mathcal{C}$ and $f : c \to d$. By naturality, we have a commutative diagram

$$\begin{tikzcd} \mathrm{Hom}(c, c) \arrow[r, "\alpha_c"] \arrow[d, "f \circ -"] & F(c) \arrow[d, "F(f)"] \\ \mathrm{Hom}(c, d) \arrow[r, "\alpha_d"] & F(d) \end{tikzcd}$$

This means that $F(f) \circ \alpha_c = \alpha_d \circ (f \circ -)$. Evaluating at id_c, we get $F(f) \circ \alpha_c(\mathrm{id}_c) = \alpha_d(f)$. This shows that the natural transformation α is entirely determined by the value of $\alpha_c(\mathrm{id}_c)$, which shows the map defined above is injective. Conversely, if $e \in F(c)$, then we define $\alpha^e : \mathrm{Hom}(c, -) \Rightarrow F$ by $\alpha^e_d : g \mapsto F(g)(e)$. We check it is a natural transformation:

$$\begin{array}{ccc} \mathrm{Hom}(c,c) & \xrightarrow{g\mapsto F(g)(e)} & F(c) \\ \Big\downarrow{\scriptstyle f\circ -} & & \Big\downarrow{\scriptstyle F(f)} \\ \mathrm{Hom}(c,d) & \xrightarrow{h\mapsto F(h)(e)} & F(d) \end{array}$$

and this diagram commutes since for $g : c \to c$ we have

$$F(f)(F(g)(e)) = F(f \circ g)(e) = F((f \circ -)(g))(e)$$

This shows the map in the theorem is surjective, therefore an isomorphism, and its inverse is given by $e \in F(c) \mapsto \alpha^e$. We now check naturality. We first need to understand what it means to say the isomorphism is natural in c. Let $f : c \to d$. $\mathrm{Nat}(\mathrm{Hom}(c, -), F)$ is functorial in c, as it is the composition of two contravariant hom-functors. More concretely:

$$c \xrightarrow{f} d \rightsquigarrow \mathrm{Hom}(d, -) \xrightarrow{-\circ f} \mathrm{Hom}(c, -) \rightsquigarrow \mathrm{Nat}(\mathrm{Hom}(c, -), F) \xrightarrow{-\circ(-\circ f)} \mathrm{Nat}(\mathrm{Hom}(d, -), F)$$

(Nat is the hom-functor of the functor category $\mathcal{C}^{\mathbf{Set}}$). One thing to note is that the morphism $f : c \to d$ induces a natural transformation $\mathrm{Hom}(d, -) \xrightarrow{-\circ f} \mathrm{Hom}(c, -)$, and this makes the whole thing work. This is in general a property of functors defined on a product category, see the remark below. Now, saying the isomorphism, which we'll write $\Phi_{d,F}$, is natural means that the square

$$\begin{array}{ccc} \mathrm{Nat}(\mathrm{Hom}(c,-),F) & \xrightarrow{\Phi_{c,F}} & F(c) \\ \Big\downarrow{\scriptstyle -\circ(-\circ f)} & & \Big\downarrow{\scriptstyle F(f)} \\ \mathrm{Nat}(\mathrm{Hom}(d,-),F) & \xrightarrow{\Phi_{d,F}} & F(d) \end{array}$$

commutes. And indeed, if $\alpha : \mathrm{Hom}(c, -) \Rightarrow F$ is a natural transformation,

$$\Phi_{d,F}(\alpha \circ (- \circ f)) = (\alpha \circ (- \circ f))_d(\mathrm{id}_d) = [\alpha_d \circ (- \circ f)](\mathrm{id}_d) = \alpha_d(f)$$
$$F(f)(\Phi_{c,F}(\alpha)) = F(f)(\alpha_c(\mathrm{id}_c)) = \alpha_d(f \circ \mathrm{id}_c) = \alpha_d(f)$$

The second to last equality comes from the naturality of α.
We now turn to naturality in F. Let G be another functor $\mathcal{C} \to \mathbf{Set}$ and $\beta : F \Rightarrow G$ be a natural transformation. We check that

$$\begin{array}{ccc} \mathrm{Nat}(\mathrm{Hom}(c,-),F) & \xrightarrow{\Phi_{c,F}} & F(c) \\ \Big\downarrow{\scriptstyle \beta\circ -} & & \Big\downarrow{\scriptstyle \beta_c} \\ \mathrm{Nat}(\mathrm{Hom}(c,-),G) & \xrightarrow{\Phi_{c,G}} & G(c) \end{array}$$

commutes. For $\alpha : \mathrm{Hom}(c, -) \Rightarrow F$, we have

$$\beta_c(\Phi_{c,F}(\alpha)) = \beta_c(\alpha_c(\mathrm{id}_c)) = (\beta \circ \alpha)_c(\mathrm{id}_c) = \Phi_{c,G}(\beta \circ \alpha)$$

which completes the proof of naturality. $\square$

Remark.

1. If $F : \mathcal{C} \to \mathbf{Set}$, then (c, x) is a universal element for F if and only if the natural transformation $\alpha_x : \mathrm{Hom}(c, -) \Rightarrow F$ induced by x is an isomorphism. Indeed, α_x is an isomorphism iff $\forall c' \in \mathcal{C}$, $(\alpha_x)_{c'} : \begin{array}{rcl} \mathrm{Hom}(c, c') & \to & F(c') \\ f & \mapsto & F(f)(x) \end{array}$ is bijective iff

$$\forall c' \in \mathcal{C}, \forall y \in F(c'), \exists! f : c \to c', F(f)(x) = y$$

2. For universal arrows, use $\mathrm{Hom}_{\mathcal{D}}(d, F(-))$ as before.

3. Let $\mathcal{C}, \mathcal{D}$ and $\mathcal{E}$ be categories, and $F : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ be a functor. Let $c, d \in \mathrm{Ob}(\mathcal{C})$, $x, y \in \mathrm{Ob}(\mathcal{D})$ and morphisms $f : c \to d$, $g : x \to y$. The morphism f induces a natural transformation $F(f, \mathrm{id}_-) : F(c, -) \Rightarrow F(d, -)$, see the commutative square:

$$\begin{tikzcd} F(c,x) \arrow[r, "F(f,\mathrm{id}_x)"] \arrow[d, "F(\mathrm{id}_c,g)"] & F(d,x) \arrow[d, "F(\mathrm{id}_d,g)"] \\ F(c,y) \arrow[r, "F(f,\mathrm{id}_y)"'] & F(d,y) \end{tikzcd}$$

2.3 Examples of objects defined by universal properties

2.3.1 Products, coproducts

Let $\mathcal{C}$ be a small category and $X, Y \in \mathrm{Ob}(\mathcal{C})$. We define a category $\mathcal{C}_{X,Y}$ whose objects are tuples (Z, f, g) where $Z \in \mathrm{Ob}(\mathcal{C})$ and $f : Z \to X$, $g : Z \to Y$ and morphisms are maps $\alpha : Z \to Z'$ in $\mathcal{C}$ such that the following diagram commutes:

$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[dd, "\alpha"] & \\ X & & Y \\ & Z' \arrow[ul, "f'"] \arrow[ur, "g'"'] & \end{tikzcd}$$

Definition 2.14. A *product* of X and Y is a final object in $\mathcal{C}_{X,Y}$. Concretely, it is an object $X \times Y$ together with two maps $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ such that for any $(Z, f, g) \in \mathrm{Ob}(\mathcal{C}_{X,Y})$, we have a commutative diagram

$$\begin{tikzcd} & Z \arrow[dl, "\forall f"'] \arrow[dr, "\forall g"] \arrow[d, dashed, "\exists! \alpha"] & \\ X & X \times Y \arrow[l, "\pi_X"] \arrow[r, "\pi_Y"'] & Y \end{tikzcd}$$

Since it is defined as being a final object, if it exists, a product is unique up to unique isomorphism.

Examples 2.15. In **Set**, the product of X and Y is the cartesian product. In **Grp**, it is the product group. In **Top**, it is the cartesian product equipped with the product topology. In these examples, the maps in the definition are the canonical projections.

The notion dual to the one of a product is called a coproduct.

Definition 2.16. A *coproduct* of X and Y is a product in $\mathcal{C}^{\mathrm{op}}$. Concretely, it satisfies the universal property expressed by this commutative diagram:

$$\begin{tikzcd} X \arrow[r, "i_X"] \arrow[dr, "\forall f"'] & X \sqcup Y \arrow[d, dashed, "\exists ! \alpha"] & Y \arrow[l, "i_Y"'] \arrow[dl, "\forall g"] \\ & Z & \end{tikzcd}$$

Examples 2.17. In **Set**, the coproduct of X and Y is the disjoint union together with canonical inclusion. In **Top**, the coproduct of X and Y is their disjoint union equipped with the disjoint union topology. However, in **Grp**, the underlying set of the coproduct of two groups is not the disjoint union.

2.3.2 Equalizers and coequalizers

Definition 2.18. Let $\mathcal{C}$ be a category and $X, Y \in \mathrm{Ob}(\mathcal{C})$, $f, g : X \to Y$. Consider the contravariant functor $F : \mathcal{C} \to \mathbf{Set}$ defined by $F(c) = \{\alpha : c \to X \mid f\alpha = g\alpha\}$ and $F(\beta) = - \circ \beta$. An *equalizer* in $\mathcal{C}$ is a representation of the contravariant functor F.

By the Yoneda lemma, a natural transformation $\mathrm{Hom}(-, c) \Rightarrow F$ is the same as an element of $F(c)$, so a representation of F is a pair (c, e) with $c \in \mathrm{Ob}(\mathcal{C})$ and $e \in F(c)$ such that the natural transformation given by the Yoneda lemma is an isomorphism. Concretely, we want $\begin{array}{cccc} \eta_e : & \mathrm{Hom}(d, c) & \to & F(d) \\ & h & \mapsto & F(h)(e) \end{array}$ to be an isomorphism for all $d \in \mathrm{Ob}(c)$. This translates into the follwing diagram:

$$\begin{tikzcd} & d \arrow[dl, dashed, "\exists ! \alpha"'] \arrow[d, "\forall h"] & \\ c \arrow[r, "e"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \end{tikzcd}$$

Example 2.19. In **Set**, $E = \{x \in X \mid f(x) = g(x)\} \hookrightarrow X$ is an equalizer.

The dual notion is that of a coequalizer.

Definition 2.20. A *coequalizer* of $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ is an object $Z \in \mathrm{Ob}(\mathcal{C})$ together with a morphism $\pi : Y \to Z$ such that $\pi f = \pi g$ and that universal to this property:

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "\pi"] \arrow[d, "\forall h"] & Z \arrow[dl, dashed, "\exists ! \alpha"] \\ & Z' & \end{tikzcd}$$

Example 2.21. In **Set**, consider the equivalence relation $\sim$ on Y generated by $f(x) \sim g(x)$ (the smallest equivalence relation on Y with this property). Then $y \xrightarrow{\pi} Y/\sim$ is a coequalizer.

2.4 Adjoint functors

This notion was introduced by Kan in 1958.

Definition 2.22. An *adjunction* (G, D) is a pair of functors $G : \mathcal{C} \to \mathcal{D}$ and $D : \mathcal{D} \to \mathcal{C}$ together with an isomorphism $\mathrm{Hom}_{\mathcal{D}}(G(c), d) \simeq \mathrm{Hom}_{\mathcal{C}}(c, D(d))$ which is natural in both c and d.
We write $G \dashv D$ and say G is left adjoint to D and D is right adjoint to G.

If $G \dashv D$ we have $\forall c, d \in \mathrm{Ob}(\mathcal{C}) \times \mathrm{Ob}(\mathcal{D})$,

$$\mathrm{Hom}_{\mathcal{D}}(G(c), d) \xrightarrow[\alpha_{c,d}]{\sim} \mathrm{Hom}_{\mathcal{C}}(c, D(d))$$

and in particular when $d = G(c)$ we get $\mathrm{Hom}_{\mathcal{D}}(G(c), G(c)) \xrightarrow[\alpha_{c,G(c)}]{\sim} \mathrm{Hom}_{\mathcal{C}}(c, DG(c))$.
Let $\eta_c : c \to DG(c)$ be the image of $\mathrm{id}_{G(c)}$. This gives a collection of morphisms $- \to DG(-)$.

Proposition 2.23. *These morphisms make up a natural transformation* $\mathrm{id}_{\mathcal{C}} \Rightarrow DG$*.*

Proof. Let $f : c \to d$. We want to show that

$$\begin{tikzcd}
c \arrow[r, "\eta_c = \alpha_{c,G(c)}(\mathrm{id}_{G(c)})"] \arrow[d, "f"] & DG(c) \arrow[d, "DG(f)"] \\
d \arrow[r, "\eta_d = \alpha_{d,G(d)}(\mathrm{id}_{G(d)})"] & DG(d)
\end{tikzcd}$$

commutes. By naturality of the isomorphism α given by the adjunction, we get the following commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}(G(c), G(c)) \arrow[r, "\sim", "\alpha_{c,G(c)}"'] \arrow[d, "G(f)\circ -"'] & \mathrm{Hom}(c, DG(c)) \arrow[d, "DG(f)\circ -"'] \\
\mathrm{Hom}(G(c), G(d)) \arrow[r, "\sim", "\alpha_{c,G(d)}"'] & \mathrm{Hom}(c, DG(d)) \\
\mathrm{Hom}(G(d), G(d)) \arrow[r, "\sim", "\alpha_{d,G(d)}"'] \arrow[u, "-\circ G(f)"] & \mathrm{Hom}(d, DG(d)) \arrow[u, "-\circ f"]
\end{tikzcd}$$

which gives us these equations:

$$DG(f) \circ \eta_c = DG(f) \circ \alpha_{c,G(c)}(\mathrm{id}_{G(c)}) = \alpha_{c,G(d)}(G(f) \circ \mathrm{id}_{G(c)}) = \alpha_{c,G(d)}(G(f))$$
$$\eta_d \circ f = \alpha_{d,G(d)}(\mathrm{id}_{G(d)}) \circ f = \alpha_{c,G(d)}(\mathrm{id}_{G(c)} \circ G(f)) = \alpha_{c,G(d)}(G(f))$$

which completes the proof. ☐

We also get a natural transformation $\varepsilon : GD \Rightarrow \mathrm{id}_{\mathcal{D}}$ when $c = D(d)$ by setting $\varepsilon_d = \alpha^{-1}_{D(d),d}(\mathrm{id}_{D(d)})$.

Definition 2.24. The natural transformation $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow DG$ is called the *unit* of the adjunction. The natural transformation $\varepsilon : GD \Rightarrow \mathrm{id}_{\mathcal{D}}$ is called its *counit*.

Proposition 2.25. *Let* $\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "G"] & \mathcal{D} \arrow[l, bend left, "D"] \end{tikzcd}$ *be two functors. Then,* $G \dashv D$ *if and only if there are natural transformations* $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow DG$ *and* $\varepsilon : GD \Rightarrow \mathrm{id}_{\mathcal{D}}$ *such that the following diagrams commute:*

$$\begin{tikzcd} G \arrow[r, Rightarrow, "G\eta"] \arrow[dr, equal, "\mathrm{id}_G"'] & GDG \arrow[d, Rightarrow, "\varepsilon G"] \\ & G \end{tikzcd} \qquad \begin{tikzcd} D \arrow[r, Rightarrow, "\eta D"] \arrow[dr, equal, "\mathrm{id}_D"'] & DGD \arrow[d, Rightarrow, "D\varepsilon"] \\ & D \end{tikzcd}$$

where $G\eta$ is the natural transformation given by the morphisms $G(\eta_c)$ and εG is the one give by morphisms $\varepsilon_{G(c)}$ (and similarly for ηD and $D\varepsilon$).

Proof. Suppose $G \dashv D$. Let $\eta : \mathrm{id}_{\mathcal{C}} \Rightarrow DG$ and $\varepsilon : GD \Rightarrow \mathrm{id}_{\mathcal{D}}$ be the unit and counit of the adjunction. Let $c \in \mathcal{C}$. We have

$$(\varepsilon G)_c \circ (G\eta)_c = \varepsilon_{G(c)} \circ G(\eta_c) = \alpha^{-1}_{DG(c),G(c)}(\mathrm{id}_{DG(c)}) \circ G(\alpha_{c,G(c)}(\mathrm{id}_{G(c)}))$$

and the naturality of α gives the following commutative diagram

$$\begin{tikzcd} \mathrm{Hom}(G(c),G(c)) & \mathrm{Hom}(c,DG(c)) \arrow[l, "\sim", "\alpha^{-1}_{c,G(c)}"'] \\ \mathrm{Hom}(GDG(c),G(c)) \arrow[u, "-\circ G(\alpha_{c,G(c)}(\mathrm{id}_{G(c)}))"] & \mathrm{Hom}(DG(c),DG(c)) \arrow[l, "\sim", "\alpha^{-1}_{DG(c),G(c)}"'] \arrow[u, "-\circ\alpha_{c,G(c)}(\mathrm{id}_{G(c)})"'] \end{tikzcd}$$

which shows that $(\varepsilon G)_c \circ (G\eta)_c = \mathrm{id}_{G(c)}$, hence $\varepsilon G \circ G\eta = \mathrm{id}_G$. The commutativity of the other triangle is treated in a similar way.
Now assume that there are natural transformations η and ε that make both triangles commute. We define two maps

$$\begin{array}{rrcl} \alpha_{c,d}: & \mathrm{Hom}(G(c),d) & \to & \mathrm{Hom}(c,D(d)) \\ & f & \mapsto & D(f)\circ\eta_c \\ \beta_{c,d}: & \mathrm{Hom}(c,D(d)) & \to & \mathrm{Hom}(G(c),d) \\ & g & \mapsto & \varepsilon_d \circ G(g) \end{array}$$

and we show these are natural isomorphisms that give us the adjunction. First we check naturality of α. Let $f : c \to c' \in \mathrm{Mor}(\mathcal{C})$ and $g : d \to d' \in \mathrm{Mor}(\mathcal{D})$. We need to check that the diagrams

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{D}}(G(c),d) \arrow[r, "\alpha_{c,d}"] & \mathrm{Hom}_{\mathcal{C}}(c,D(d)) \\ \mathrm{Hom}_{\mathcal{D}}(G(c'),d) \arrow[u, "-\circ G(f)"] \arrow[r, "\alpha_{c',d}"] & \mathrm{Hom}_{\mathcal{C}}(c',D(d)) \arrow[u, "-\circ f"] \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Hom}_{\mathcal{D}}(G(c),d) \arrow[r, "\alpha_{c,d}"] \arrow[d, "g\circ-"] & \mathrm{Hom}_{\mathcal{C}}(c,D(d)) \arrow[d, "D(g)\circ-"] \\ \mathrm{Hom}_{\mathcal{D}}(G(c),d') \arrow[r, "\alpha_{c,d'}"] & \mathrm{Hom}_{\mathcal{C}}(c,D(d')) \end{tikzcd}$$

commute. We only check the left diagram and leave the right to the reader (sorry). We have

$$\alpha_{c,d} \circ (- \circ G(f)) = (D(-) \circ \eta_c) \circ (- \circ G(f)) = D(- \circ G(f)) \circ \eta_c = D(-) \circ DG(f) \circ \eta_c$$
$$(- \circ f) \circ \alpha_{c',d} = (- \circ f) \circ (D(-) \circ \eta_{c'}) = D(-) \circ \eta_{c'} \circ f = D(-) \circ DG(f) \circ \eta_c$$

One shows β is natural in c and d in a similar way. We leave it to the reader (sorry again). Now we need to check that α and β are inverses of each other, and that's where the triangle diagrams come into play.

$$\alpha_{c,d} \circ \beta_{c,d} = D(\varepsilon_d \circ G(-)) \circ \eta_c = D(\varepsilon_d) \circ DG(-) \circ \eta_c = D(\varepsilon_d) \circ \eta_{D(d)} \circ - = -$$

We used the definitions of α and β, the functoriality of D, the naturality of η and the second triangle diagram. We leave to the reader (sorry) to check that $\beta_{c,d} \circ \alpha_{c,d}$ is also the identity. $\square$

Examples 2.26.

1. The forgetful functor $\mathbf{Ab} \to \mathbf{Set}$ is right adjoint to the free abelian group functor $\mathbf{Set} \to \mathbf{Ab}$.
2. The forgetful functor $\mathbf{Ab} \to \mathbf{Grp}$ is right adjoint to the abelianization functor $\mathbf{Grp} \to \mathbf{Ab}$ that sends a group G to its abelianization $G^{ab} = G/[G,G]$ and a morphism $f : G \to H$ to the induced morphism $f^{ab} : G^{ab} \to H^{ab}$.
3. The forgetful functor $\mathbf{Top} \to \mathbf{Set}$ is right adjoint to the functor $\mathbf{Set} \to \mathbf{Top}$ that takes a set and equips it with the coarse topology. The forgetful functor $\mathbf{Top} \to \mathbf{Set}$ is also left adjoint to the functor $\mathbf{Set} \to \mathbf{Top}$ that equips a set with the discrete topology.
4. Let G be a group, H one of its subgroups and k be a field. We have a functor from the category $\mathbf{Rep}_k(G)$ of representations of G on k-vector spaces to the category $\mathbf{Rep}_k(H)$ of representations of H on k-vector spaces. It is the restriction functor Res_H^G. Its left adjoint is Ind_H^G, the induced representation functor.

Theorem 2.27. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. The following are equivalent:*

1. *F admits a left adjoint*
2. *For all $X \in \mathrm{Ob}(\mathcal{D})$, $\mathrm{Hom}_{\mathcal{D}}(X, F(-))$ is representable*
3. *For all $X \in \mathrm{Ob}(\mathcal{D})$, there exists a universal arrow $X \to F$*

Corollary 2.28. *If they exist, adjoints are unique up to isomorphism.*

Proof. $2 \iff 3$ was the subject of a previous remark right after the Yoneda lemma. We prove $1 \iff 2$. Suppose F admits a left adjoint G. Let $X \in \mathrm{Ob}(\mathcal{D})$. Then for all $Y \in \mathrm{Ob}(\mathcal{C})$ we have a bijection $\mathrm{Hom}_{\mathcal{D}}(X, F(Y)) \simeq \mathrm{Hom}_{\mathcal{C}}(G(X), Y)$ which is natural in Y, so $G(X)$ represents $\mathrm{Hom}_{\mathcal{D}}(X, F(-))$. For the converse, suppose all functors $\mathrm{Hom}_{\mathcal{D}}(X, F(-))$ are representable. We define $G(X)$ to be an object of $\mathcal{C}$ that represents $\mathrm{Hom}_{\mathcal{D}}(X, F(-))$. Now choose $X, Y \in \mathrm{Ob}(\mathcal{D})$ and $f : X \to Y$. We need to define $G(f)$. We wish to have a commuting square

$$\begin{tikzcd}
\mathrm{Hom}(G(X), -) \arrow[r, "\sim"] & \mathrm{Hom}(X, F(-)) \\
\mathrm{Hom}(G(Y), -) \arrow[r, "\sim"] \arrow[u, dashed, "\exists! \gamma"] & \mathrm{Hom}(Y, F(-)) \arrow[u, "- \circ f"']
\end{tikzcd}$$

We need to recover a map $G(X) \to G(Y)$ such that composing with it gives us γ. This works by the Yoneda lemma, which tells us that the natural transformation γ comes from an element $\mathrm{Hom}(G(X), G(Y))$. Call it $G(f)$. It remains to check this does define a functor. Using this diagram with $X = Y$ and $f = \mathrm{id}_X$ shows that $G(\mathrm{id}_X) = \mathrm{id}_{G(X)}$. Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{C}$. Then we draw

$$\begin{tikzcd}
\mathrm{Hom}(G(Z), -) \arrow[r, "- \circ G(g)"'] \arrow[rr, bend left, "- \circ G(g \circ f)"] \arrow[d, "\wr"] & \mathrm{Hom}(G(Y), -) \arrow[r, "- \circ G(f)"'] \arrow[d, "\wr"] & \mathrm{Hom}(G(X), -) \arrow[d, "\wr"] \\
\mathrm{Hom}(Z, F(-)) \arrow[r, "- \circ g"] \arrow[rr, bend right, "- \circ (g \circ f)"'] & \mathrm{Hom}(Y, F(-)) \arrow[r, "- \circ f"] & \mathrm{Hom}(X, F(-))
\end{tikzcd}$$

and this diagram shows that $G(g \circ f) = G(g) \circ G(f)$ (because the map γ above is unique). $\square$

This theorem shows there is a deep link between universal properties and adjoint functors.

2.5 Limits and colimits

(This subsection may be skipped on a first reading.)
Let us recall the definition of a functor category.

Definition 2.29. Let $\mathcal{C}, \mathcal{D}$ be two categories. Then $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$, also written $\mathcal{D}^{\mathcal{C}}$, is the category whose objects are functors $\mathcal{C} \to \mathcal{D}$ and morphisms are natural transformations between such functors, with composition given by vertical composition. It is called the *functor category category from $\mathcal{C}$ to $\mathcal{D}$*. When $\mathcal{J}$ is a small category we also say that $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is the category of *diagrams of shape $\mathcal{J}$ in $\mathcal{C}$*.

Examples 2.30.

1. Let $\mathbf{2}$ be the category $\bullet \longrightarrow \bullet$ which has two objects 1 and 2 and three morphisms (two of them being identities).
   Then $\mathbf{2} \times \mathbf{2}$ is the category $\begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$ which has 4 objects and 9 morphisms (4 of them being identities). Then, a functor from $\mathbf{2} \times \mathbf{2}$ to $\mathcal{C}$ is a commutative diagram of this shape in $\mathcal{C}$.

2. If $\mathcal{J}$ is a small category, there is a functor $\Delta : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Fun}(\mathcal{J}, \mathcal{C})$ where $\Delta(c)$ is the constant functor at c, that is the functor that sends all objects to c and all morphisms to id_c, and $\Delta(f) = f$, which works since a natural transformation $\Delta(c) \Rightarrow \Delta(d)$ is just the data of one morphism $c \to d$.

Definition 2.31. A *cone above a diagram $F : \mathcal{J} \to \mathcal{C}$ with summit $c \in \mathcal{C}$* is a natural transformation $\lambda : \Delta(c) \Rightarrow F$. Dually, a cone *under* F with summit c, also called a *cocone*, is a natural transformation $\lambda : F \Rightarrow \Delta(c)$.

Let us unwrap this definition. A cone is a collection of maps $\lambda_j : c \to F(j)$ for all $j \in \mathrm{Ob}(\mathcal{J})$, such that for any morphism $f : i \to j \in \mathrm{Mor}(\mathcal{J})$, this diagram commutes:

$$\begin{tikzcd} & c \arrow[dl, "\lambda_i"'] \arrow[dr, "\lambda_j"] & \\ F(i) \arrow[rr, "F(f)"'] & & F(j) \end{tikzcd}$$

Definition 2.32. Let $F : \mathcal{J} \to \mathcal{C}$ be a diagram. A *limit* (or *projective limit* or *inverse limit*) of F is a universal cone above F, in the sense that it is a final object in the category of cones above F. Dually, a *colimit* (or *inductive limite* or *direct limit*) is a universal cocone, that is an initial object in the category of cones under F.

Concretely, a limit of $F : \mathcal{J} \to \mathcal{C}$ is a pair $(\lim F, \phi)$ with $\lim F \in \mathrm{Ob}(\mathcal{C})$ and $\phi : \Delta(\lim F) \Rightarrow F$ is such that for any cone $\lambda : \Delta(c) \Rightarrow F$, there exists a unique morphism $f : X \to \lim F \in \mathrm{Mor}(\mathcal{C})$, such that the diagram on the left commutes:

$$\begin{tikzcd} \Delta(c) \arrow[rr, dashed, "\Delta(f)"] \arrow[dr, Rightarrow, "\lambda"'] & & \Delta(\lim F) \arrow[dl, Rightarrow, "\phi"] \\ & F & \end{tikzcd} \quad \text{which is equivalent to} \quad \forall j \in \mathcal{J}, \quad \begin{tikzcd} c \arrow[rr, dashed, "f"] \arrow[dr, Rightarrow, "\lambda_j"'] & & \lim F \arrow[dl, Rightarrow, "\phi_j"] \\ & F(j) & \end{tikzcd}$$

In compact form, $\mathrm{Hom}_{\mathcal{C}}(-, \lim F) \simeq \mathrm{Hom}_{\mathrm{Fun}(\mathcal{J},\mathcal{C})}(\Delta(-), F)$.

Exercise. Do the same for colimits.

Remark.

1. If a limit exists it is unique up to isomorphism (unique isomorphism that commutes with the legs of the cone)

2. If all limits exist, then lim becomes a functor $\lim : \mathrm{Fun}(\mathcal{J},\mathcal{C}) \to \mathcal{C}$ in the following way. Recall that theorem 2.27 says a functor D admits a left adjoint iff for all objects X in its codomain, $\mathrm{Hom}(X, D(-))$ is representable. The compact form of the definition of a limit says that the functor $\mathrm{Hom}(\Delta(-), F)$ is representable for all F (since we assume all limits exist). A dual version of the theorem gives that Δ admits a right adjoint, which is lim since $\mathrm{Hom}(c, \lim F) \simeq \mathrm{Hom}(\Delta(c), F)$. If $\eta : F \Rightarrow G$ is a natural transformation, then $\lim(\eta)$ can be constructed in the following way: $\lim F \Rightarrow F \underset{\eta}{\Rightarrow} G$ is a cone above G, and $\lim(\eta) : \lim F \to \lim G$ comes from the universality of $\lim G$.

Corollary 2.33.

1. *If $\mathcal{C}$ has all $\mathcal{J}$-limits, then* $\lim : \mathrm{Fun}(\mathcal{J},\mathcal{C}) \to \mathcal{C}$ *is a right adjoint to* Δ.

2. *If $\mathcal{C}$ has all $\mathcal{J}$-colimits, then* $\mathrm{colim} : \mathrm{Fun}(\mathcal{J},\mathcal{C}) \to \mathcal{C}$ *is a left adjoint to* Δ.

Example 2.34.

1. If $\mathcal{J}$ is discrete, that is has no morphisms other than identities, then a functor $F : \mathcal{J} \to \mathcal{C}$ is the same as a collection $(X_i)_{i \in \mathcal{J}}$ of objects of $\mathcal{C}$. Then, a limit of F is an object $\lim F \in \mathrm{Ob}(\mathcal{C})$ with morphisms $f_i : \lim F \to X_i$ such that for all objects $X \in \mathrm{Ob}(\mathcal{C})$ with morphisms $p_i : X \to X_i$, we have a unique map $\alpha : X \to \lim F$ that makes this diagram commute for all $i, j \in \mathcal{J}$:

   $$\begin{tikzcd} & X \arrow[dl, "p_i"'] \arrow[d, dashed, "\alpha"] \arrow[dr, "p_j"] & \\ X_i & \lim F \arrow[l, "f_i"] \arrow[r, "f_j"'] & X_j \end{tikzcd}$$

   We write $\lim F = \prod_{j \in \mathcal{J}} F(j)$ and call it the *product of the $F(j)$s.* Morphisms f_i are written π_i and called *canonical projections.*
   Dually, the colimit of F is called a coproduct and written $\coprod_{j \in \mathcal{J}} F(j)$.

2. If $\mathcal{J} = \bullet \rightrightarrows \bullet$, then a functor $F : \mathcal{J} \to \mathcal{C}$ is the data of two parallel morphisms in $\mathcal{C}$. A limit is an equalizer and a colimit is a coequalizer.

3. If $\mathcal{J} = \begin{tikzcd} & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$ then $F : \mathcal{J} \to \mathcal{C}$ is the data of $A, B, C \in \mathrm{Ob}(\mathcal{C})$ with two morphisms $f : A \to C$ and $g : B \to C$. The limit $\lim F$ is called a *pullback* of f and g, with universal property depicted here:

$$\begin{tikzcd} X \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & A \times_C B \arrow[r, "\pi_A"] \arrow[d, "\pi_B"'] & A \arrow[d, "f"] \\ & B \arrow[r, "g"'] & C \end{tikzcd}$$

4. If $\mathcal{J} = \omega^{\mathrm{op}}$, that is $\mathcal{J} = \cdots \to 2 \to 1 \to 0$, then $\lim F$ is often called the "inverse limit" of F. Concretely, F is the data of $\cdots \to F(2) \xrightarrow[\alpha_2]{} F(1) \xrightarrow[\alpha_1]{} F(0)$, and a cone above F looks like

$$\begin{tikzcd} & & c \arrow[dl, "\lambda_2"'] \arrow[d, "\lambda_1"] \arrow[dr, "\lambda_0"] & \\ \cdots \arrow[r] & F(2) \arrow[r, "\alpha_2"'] & F(1) \arrow[r, "\alpha_1"'] & F(0) \end{tikzcd}$$

we have $(\alpha_i \circ \cdots \circ \alpha_n) \circ \lambda_n = \lambda_i$.

The typical example of an inverse limit is the one given by $F(n) = \mathbb{Z}/p^n\mathbb{Z}$ in **Ring** with morphisms $\mathbb{Z}/p^{n+1}\mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z}$ being reduction mod p^n. The inverse limit $\varprojlim \mathbb{Z}/p^n\mathbb{Z}$ is the ring of p-adic integers. Concretely, $a \in \varprojlim \mathbb{Z}/p^n\mathbb{Z}$ iff $a = (a_i)_{i \in \mathbb{N}}$ such that $a_i \equiv a_j \mod p^i \forall i \leq j$.

5. The dual notion, given by $\mathcal{J} = 0 \to 1 \to 2 \to \cdots$, is obtained by taking the colimit. It is called a *direct limit*. The typical example here is the Prüfer p-group $\varinjlim \mathbb{Z}/p^n\mathbb{Z} = \mathbb{Z}(p^\infty)$.

Definition 2.35. Let $\mathcal{C}$ be a category. We say $\mathcal{C}$ is *(co)complete* if it has all small (co)limits i.e. if for all diagrams $F : \mathcal{J} \to \mathcal{C}$ with $\mathcal{J}$ small, F has a (co)limit.

Theorem 2.36. *A category $\mathcal{C}$ is (co)complete if and only if it has all small (co)products and (co)equalizers.*

Proof. Let $\mathcal{J}$ be a small category and $D : \mathcal{J} \to \mathcal{C}$ be a diagram. We have the products $\prod_{k \in \mathrm{Ob}(\mathcal{J})} D(k)$ and $\prod_{g \in \mathrm{Mor}(\mathcal{J})} D(\mathrm{cod}(g))$ where $\mathrm{cod}(g)$ is the codomain of g. We have two morphisms

$$\prod_{k \in \mathrm{Ob}(\mathcal{J})} D(k) \underset{t}{\overset{s}{\rightrightarrows}} \prod_{g \in \mathrm{Mor}(\mathcal{J})} D(\mathrm{cod}(g))$$

given by $s = \prod_{f:i \to j} D(f)\pi_i$ and $t = \prod_{f:i \to j} \pi_j$, or with diagrams, for any $f : i \to j \in \mathrm{Mor}(\mathcal{J})$:

$$\begin{tikzcd} \prod_{k \in \mathrm{Ob}(\mathcal{J})} D(k) \arrow[r, dashed, "\exists! s"] \arrow[d, "\pi_i"] & \prod_{g \in \mathrm{Mor}(\mathcal{J})} D(\mathrm{cod}(g)) \arrow[d, "\pi_f"] \\ D(i) \arrow[r, "D(f)"] & D(j) \end{tikzcd} \qquad \begin{tikzcd} \prod_{k \in \mathrm{Ob}(\mathcal{J})} D(k) \arrow[rr, dashed, "\exists! t"] \arrow[dr, "\pi_j"'] & & \prod_{g \in \mathrm{Mor}(\mathcal{J})} D(\mathrm{cod}(g)) \arrow[dl, "\pi_f"] \\ & D(j) & \end{tikzcd}$$

We call $\lim D$ an equalizer of s and t. A cone above D is given by compositions

$$\lim D \xrightarrow{\alpha} \prod_{k\in\mathrm{Ob}(\mathcal{J})} D(k) \xrightarrow{\pi_i} D(i)$$

Indeed, for any morphism $f : i \to j \in \mathrm{Mor}(\mathcal{J})$, $D(f)\pi_i\alpha = \pi_f s\alpha = \pi_f t\alpha = \pi_j\alpha$.
Now let $\Delta(c) \underset{\lambda}{\Rightarrow} D$ be another cone above D. For any $k \in \mathrm{Ob}(\mathcal{J})$, we have $\lambda_k : c \to D(k)$, which gives a unique morphism $\lambda_* : c \to \prod_{k\in\mathrm{Ob}(\mathcal{J})} D(k)$ such that $\pi_i\lambda_* = \lambda_i$. Then, for any $f : i \to j \in \mathrm{Mor}(\mathcal{J})$, we have

$$\begin{aligned}
&\pi_f s\lambda_* = D(f)\pi_i\lambda_* = D(f)\lambda_i = \lambda_j \\
&\pi_f t\lambda_* = \pi_j\lambda_* = \lambda_j
\end{aligned}$$

and applying the universal property of the product shows that $s\lambda_* = t\lambda_*$. By the universal property of equalizers this gives the existence of a unique morphism $c \to \lim D$ and completes the proof. $\square$

Definition 2.37. $F : \mathcal{C} \to \mathcal{D}$ *preserves (co)limits* if for every diagram $D : \mathcal{J} \to \mathcal{C}$ and any (co)limit cone (c, ϕ) of D, the image $(F(c), F\phi)$ is a (co)limit cone over $FD : \mathcal{J} \to \mathcal{D}$.

Remark. Preserving limits is like having $F(\lim D) \simeq \lim FD$, but stronger:

$$\begin{tikzcd}
\lim D \arrow[d, "\phi_i"] & \rightsquigarrow & F(\lim D) \arrow[r, dashed, "\exists!\alpha"] \arrow[d, "FD(\phi_i)"'] & \lim FD \arrow[dl, "\lambda_i"] \\
D(i) & & FD(i) &
\end{tikzcd}$$

and α is an isomorphism since $(F(\lim D), F\phi)$ is a limit cone.

Proposition 2.38. *Let $\mathcal{C}$ be a locally small category and $X \in \mathrm{Ob}(\mathcal{C})$. Then*

1. $\mathrm{Hom}_{\mathcal{C}}(X, -)$ *preserves all limits that exist in* $\mathcal{C}$
2. *The contravariant functor* $\mathrm{Hom}_{\mathcal{C}}(-, X)$ *transforms colimits in* $\mathcal{C}$ *into limits in* **Set**.

Proof. Let $\mathcal{J}$ be a small category and $D : \mathcal{J} \to \mathcal{C}$ be a diagram. Let $F : \mathcal{C} \to \mathbf{Set}$ be te hom-functor $\mathrm{Hom}_{\mathcal{C}}(X, -)$. Let (L, λ) be a limit cone for D. Then, $(F(L), F(\lambda))$ is a cone in **Set** over FD, since for any $\alpha : i \to j \in \mathrm{Mor}(\mathcal{J})$ we have the commutative diagram

$$\begin{tikzcd}
& F(L) \arrow[dl, "F(\lambda)"'] \arrow[dr, "F(\lambda_i)"] & \\
\mathrm{Hom}_{\mathcal{C}}(X, D(i)) \arrow[rr, "D(\alpha)\circ-"'] & & \mathrm{Hom}_{\mathcal{C}}(X, D(j))
\end{tikzcd}$$

It remains to show that $(F(L), F(\lambda))$ is a limit cone for FD. Let $S \underset{f}{\Rightarrow} FD$ be another cone. We have $f(i) : \begin{array}{ccc} S & \to & \mathrm{Hom}(X, D(i)) \\ s & \mapsto & f_i(s) \end{array}$ (we work in **Set** so morphisms are actual maps here). Fixing S, we get commutative diagrams:

$$
\begin{tikzcd}
& X \arrow[dl, "f_i(s)"'] \arrow[dr, "f_j(s)"] & \\
D(i) \arrow[rr, "D(\alpha)\circ -"'] & & D(j)
\end{tikzcd}
$$

so $(X, f_i(s))$ is a cone over D hence there exists a unique morphism $u_s : X \to L$ such that $\lambda_i \circ u_s = f_i(s)$ for all $i \in \mathrm{Ob}(\mathcal{J})$. Now set $u : \begin{array}{ccc} S & \to & \mathrm{Hom}(X, L) \\ s & \mapsto & u_s \end{array}$ and we have $(F\lambda \circ u)(s) = (F\lambda)(u_s) = f$ so $u : S \to F(L)$ is a morphism of cones. We need to check it is unique. If v is another one then $\lambda_i \circ v(s) = f_i(s)$ so $v(s) = u_s$ by uniqueness of u_s, which shows $v = u$.

Another proof is given here:

$$
\begin{aligned}
\mathrm{Hom}_{\mathcal{C}}(X, \lim D) &\simeq \mathrm{Hom}_{\mathrm{Fun}(\mathcal{J},\mathcal{C})}(\Delta X, D) \\
&\simeq \mathrm{Hom}_{\mathrm{Fun}(\mathcal{J},\mathbf{Set})}(\Delta 1, \mathrm{Hom}_{\mathcal{C}}(X, D(-)) \\
&\simeq \mathrm{Hom}_{\mathbf{Set}}(1, \lim \mathrm{Hom}_{\mathcal{C}}(X, D(-))) \\
&\simeq \lim \mathrm{Hom}_{\mathcal{C}}(X, D(-))
\end{aligned}
$$

(1 is a singleton.) The first and third isomorphisms are by definition of a limit. The last isomorphism comes from the fact that for any set A, maps $1 \to A$ correspond to elements of A. The second isomorphism works since a natural transformation $\Delta X \Rightarrow D$ is the same as a collection of morphisms $f_i : X \to D(i)$ indexed by $\mathrm{Ob}(\mathcal{J})$. □

Theorem 2.39. *Right adjoints preserve limits. Left adjoints preserve colimits.*

Proof. We only need to prove the statement about right adjoints and then use opposite categories for left adjoints. Let $\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] & \mathcal{D} \arrow[l, bend left, "G"] \end{tikzcd}$ be two functors with $F \dashv G$ and $D : \mathcal{J} \to \mathcal{D}$ be a diagram, with $\eta : \Delta(\lim D) \Rightarrow D$ its limit cone. Our goal is to show that $(G \lim D, G\eta)$ is a limit cone for $G \circ D$. The fact that it is a cone above $G \circ D$ is clear. Now let $\mu : \Delta(c) \Rightarrow GD$ be another cone. For any $j \in \mathrm{Ob}(\mathcal{J})$, we have $\mu_j \in \mathrm{Hom}(c, GD(j))$. By adjunction, it corresponds to a morphism $\mu_j^* \in \mathrm{Hom}(F(c), D(j))$. We claim these morphisms make up a natural transformation $\mu^* : \Delta(F(c)) \Rightarrow D$. Indeed, for any morphism $f : i \to j \in \mathrm{Mor}(\mathcal{J})$, we have by naturality of the adjunction a commutative diagram

$$
\begin{tikzcd}
\mathrm{Hom}(F(c), D(i)) \arrow[r, "\sim"] \arrow[d, "D(f)\circ -"] & \mathrm{Hom}(c, GD(i)) \arrow[d, "GD(f)\circ -"] \\
\mathrm{Hom}(F(c), D(j)) \arrow[r, "\sim"] & \mathrm{Hom}(c, GD(j))
\end{tikzcd}
$$

so $D(f) \circ \mu_i^* = (GD(f) \circ \mu_i)^* = \mu_j^*$. By universality of $\lim D$, there exists a unique morphism $\tau : F(c) \to \lim D$ that makes the appropriate diagram commute. Using the adjunction, we get a morphism $\tau^* : c \to G(\lim D)$, which is the morphism we are looking for. The commutativity of the appropriate diagram comes from naturality of the adjunction. Uniqueness comes from the uniqueness of τ.

In compact form:

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{C}}(c, \lim GD) &\simeq \mathrm{Hom}_{\mathrm{Fun}(\mathcal{J},\mathcal{C})}(\Delta c, GD) \\
&\simeq \mathrm{Hom}_{\mathrm{Fun}(\mathcal{J},\mathcal{D})}(F\Delta c, D) \\
&\simeq \mathrm{Hom}_{\mathrm{Fun}(\mathcal{J},\mathcal{D})}(\Delta Fc, D) \\
&\simeq \mathrm{Hom}_{\mathcal{D}}(Fc, \lim D) \\
&\simeq \mathrm{Hom}_{\mathcal{C}}(C, G\lim D)
\end{aligned}$$

$\square$

3 Tensor products

All rings considered here are assumed to be associative and to have a multiplicative unit 1. Let A be a ring.

Definition 3.1.

- A *right A-module* is an abelian group $(M, +)$ with a map $\begin{array}{rcl} M \times A & \to & M \\ (m, a) & \mapsto & m \cdot a \end{array}$ such that

$$\begin{array}{llll} (1) & (m+n) \cdot a = m \cdot a + n \cdot a & (3) & m \cdot (ab) = (m \cdot a)b \\ (2) & m \cdot (a+b) = m \cdot a + m \cdot b & (4) & m \cdot 1_A = m \end{array}$$

  by symmetry one gets the notion of a *left A-module* (which is the equivalent of a vector space, but with a ring in place of the field).

- If A, B are two rings, an *A-B-bimodule* is an abelian group M with a left A-module and a right B-module structure such that for $(a, b) \in A \times B$ and $m \in M$, $a \cdot (m \cdot b) = (a \cdot m) \cdot b$.

- Let M be a right A-module, N be a left A-module and G be an abelian group.
  A *bilinear* (or *balanced*) map $f : M \times N \to G$ is a map f such that

$$\begin{array}{ll} (1) & f(m_1 + m_2, n) = f(m_1, n) + f(m_2, n) \\ (2) & f(m, n_1 + n_2) = f(m, n_1) + f(m, n_2) \\ (3) & f(ma, n) = f(m, an) \end{array}$$

The following theorem shows that there exists an abelian group $M \otimes_A N$ that is "universal" with respect to bilinear maps.

Theorem 3.2. *Let M be a right A-module and N be a left A-module. There exists an abelian group $M \otimes_A N$ together with a bilinear map $t : M \times N \to M \otimes_A N$ such that for any abelian group G and bilinear map $b : M \times N \to G$, there exists a unique group homomorphism $\tilde{b}$ that makes this diagram commute:*

$$\begin{tikzcd} M \times N \arrow[r, "\forall b"] \arrow[d, "t"'] & G \\ M \otimes_A N \arrow[ur, dashed, "\exists! \tilde{b}"'] & \end{tikzcd}$$

Proof. Let $L = \mathbb{Z}[M \times N]$ be the free abelian group on $M \times N$. It has a basis, namely $\{(m, n) \mid m \in M, n \in N\}$. Now consider the subgroup

$$I = \langle (m_1 + m_2, n) - (m_1, n) - (m_2, n), (m, n_1 + n_2) - (m, n_1) - (m, n_2), (ma, n) - (m, an) \rangle$$

It is chosen so the relations we want hold in ${}^{L}/_{I}$, for instance $(ma, n) = (m, an)$ in the quotient group. Set $M \otimes_A N = {}^{L}/_{I}$ and $t : \begin{array}{rcl} M \times N & \to & {}^{L}/_{I} \\ (m, n) & \mapsto & [(m, n)] \end{array}$. By construction $M \otimes_A N$ is an abelian group and t is bilinear. We need to check the universal property. Pick a bilinear map $b : M \times N \to G$. We have a diagram

$$\begin{tikzcd}
M \times N \arrow[r, "b"] \arrow[d, "i"'] & G \\
\mathbb{Z}[M \times N] \arrow[ur, dashed, "\exists! \tilde{b}"'] \arrow[d, "\pi"'] & \\
M \otimes_A N \arrow[uur, dashed, bend right, "\exists! \overline{b}"'] &
\end{tikzcd}$$

where $i : (m, n) \mapsto (m, n)$ is the inclusion map and $\pi : L \to {}^{L}/_{I}$ is the canonical projection. The map $\tilde{b}$ exists by universal property of the free abelian group. Moreover it passes to the quotient ($I \subset \ker(\tilde{b})$), so we get the map $\overline{b}$. We now check uniqueness. Let $f : M \otimes_A N \to G$ be another linear map that makes the diagram commute. Then, $f \circ \pi$ makes the top triangle commute, so by the universal property of the free abelian group, $f \circ \pi = \tilde{b}$. Applying the universal property of the quotient allows us to conclude $f = \overline{b}$. $\square$

Remark.

1. The abelian group $M \otimes_A N$ is a unique up to unique isomorphism.

2. The class $[(m, n)] \in M \otimes_A N$ is written $m \otimes n$. It is called a "*pure tensor*". Pure tensors generate the tensor product:

$$x \in M \otimes_A N \iff \exists (m_i, n_i) \in M^n \times N^n, x = \sum_{i=1}^{n} m_i \otimes n_i$$

❯ The tensor product is a functor. Precisely, it is a bifunctor $- \otimes_A - : \mathbf{Mod}A \times A\mathbf{Mod} \to \mathbf{Ab}$. If M, M' are two right A-modules, N, N' are two left A-modules and $f : M \to M', g : N \to N'$ are linear maps, then writing $(f \otimes g)(m \otimes n) = f(m) \otimes g(n)$ gives a commutative diagram

$$\begin{tikzcd}
M \otimes_A N \arrow[r, "\mathrm{id}_M \otimes g"] \arrow[d, "f \otimes \mathrm{id}_N"'] \arrow[dr, dashed, "f \otimes g"] & M \otimes_A N' \arrow[d, "f \otimes \mathrm{id}_{N'}"] \\
M' \otimes_A N \arrow[r, "\mathrm{id}_{M'} \otimes g"'] & M' \otimes_A N'
\end{tikzcd}$$

One needs to be careful as $M \otimes_A N$ can be defined using a quotient or a universal property. Obtaining the arrow $f \otimes g$ is easier with the universal property:

$$\begin{tikzcd}
M \times N \arrow[r, "(f,g)"] \arrow[d, "t"] & M' \times N' \arrow[d, "t'"] \\
M \otimes_A N \arrow[r, dashed, "f \otimes g"] & M' \otimes_A N'
\end{tikzcd}$$

Since $t' \circ (f, g)$ is bilinear, we obtain the unique map $f \otimes g$ using the universal property of $M \otimes_A N$. Hence we obtain the lemma:

Lemma 3.3. *$- \otimes_A -$ is a bifunctor.*

Corollary 3.4. *1. If M is a B-A-bimodule, then $M \otimes_A N$ is a left B-module*

2. *If N is an A-C-bimodule, then $M \otimes_A N$ is a right C-module*

3. *If M is a B-A-bimodule and N is a A-C-bimodule then $M \otimes_A N$ is a B-C-bimodule.*

Proof. We do the proof of 1. We set $b \bullet (m \otimes n) = (bm) \otimes n$ and now we need to check that it is well defined. A good way is to fix $b \in B$ and let $\ell_b : \begin{array}{ccc} M & \to & M \\ m & \mapsto & b \cdot m \end{array}$ and notice that $\ell_b \in \mathrm{End}_A(M)$. By functoriality, we get a map $\ell_b \otimes \mathrm{id}_N : \begin{array}{ccc} M \otimes_A N & \to & M \otimes_A N \\ m \otimes n & \mapsto & (bm) \otimes n \end{array}$ so our action is well defined and this is a B-module structure on the tensor product. The proof of 2. is similar. The proof of 3. comes from the fact that $\ell_b \otimes \mathrm{id}_N$ and $\mathrm{id}_M \otimes r_c$ commutes. $\square$

Examples 3.5.

1. $A \otimes_A N \simeq N$ as left A-modules. Isomorphisms are given by $a \otimes n \mapsto a \cdot n$ and $n \mapsto 1 \otimes n$. The well-definition of these maps comes from the universal property.

2. If R is commutative then an R-module M is an R-R-bimodule $\begin{array}{ccc} R \times M \times R & \to & (x, m, y) \\ M & \mapsto & mxy = myx \end{array}$ so $M \otimes_R N$ is always an R-module.

⚠ Over a field, $\dim(V \otimes W) = \dim(V)\dim(W)$ but this is false in general for a ring.

Exercise. Show that $\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z} = \{0\}$ when $\gcd(m, n) = 1$.

Theorem 3.6 (Tensor-hom adjunction)**.** *Let A, B be two rings and M be an A-B-bimodule. We have a functor $- \otimes_A M : \mathbf{Mod}A \to \mathbf{Mod}B$ and a functor $\mathrm{Hom}_B(M, -) : \mathbf{Mod}B \to \mathbf{Mod}A$. Then $- \otimes_A M$ is left adjoint to $\mathrm{Hom}_B(M, -)$.*

The A-module structure on $\mathrm{Hom}_B(M, Y)$ for Y a B-module is given by

$$\begin{array}{ccl} \mathrm{Hom}_B(M, Y) \times A & \to & \mathrm{Hom}_B(M, Y) \\ (f, a) & \mapsto & f \cdot a : \begin{array}{ccc} M & \to & Y \\ m & \mapsto & f(am) \end{array} \end{array}$$

Proof. **TODO** $\square$

4 Additive categories

4.1 Preadditive and additive categories

Definition 4.1. A *zero object* in a category $\mathcal{C}$ is an object that is both final and initial.

Example 4.2. $\{0\}$ is a zero objet in $\mathbf{Mod}A$ for A a ring.

Definition 4.3. Let k be a commutative ring. A *k-category* is a category $\mathcal{C}$ such that all hom-sets are k-modules and composition is bilinear. When $k = \mathbb{Z}$ we say that $\mathcal{C}$ is *preadditive*.

Remark. One says that $\mathcal{C}$ is "enriched" over $\mathbf{Mod}k$.

Lemma 4.4. *Let $\mathcal{C}$ be a k-category. For $X, Y \in \mathrm{Ob}(\mathcal{C})$, the product $X \times Y$ exists iff the coproduct $X \sqcup Y$ exists. If so, they are isomorphic.*

Proof. Suppose $X \times Y$ exists. Define $i_X = (\mathrm{id}_X, 0) : X \to X \times Y$ and $i_Y = (0, \mathrm{id}_Y) : Y \to X \times Y$. We claim these maps together with the product are the coproduct of X and Y. Let $Z \in \mathrm{Ob}(\mathcal{C})$ and $f : X \to Z$, $g : Y \to Z$. Then, define $f \sqcup g : X \times Y \to Z$ by $f \sqcup g = f\pi_X + g\pi_Y$. This makes this diagram commute:

$$\begin{tikzcd} X \arrow[r, "i_X"] \arrow[dr, bend right, "f"'] & X \times Y \arrow[d, "f \sqcup g"] & Y \arrow[l, "i_Y"'] \arrow[dl, bend left, "g"] \\ & Z & \end{tikzcd}$$

Now let $h : X \times Y \to Z$ be another arrow that makes the diagram commute. Then

$$h \circ (i_X\pi_X + i_Y\pi_Y) = hi_X\pi_X + hi_Y\pi_Y = f\pi_X + g\pi_Y = f \sqcup g$$

And uniqueness follows since $\mathrm{id}_{X\times Y} = i_X\pi_X + i_Y\pi_Y$. This comes from the universal property of the product and the diagram

$$\begin{tikzcd} & X \times Y \arrow[dl, bend right, "\pi_X"'] \arrow[d, dashed, "i_X\pi_X + i_Y\pi_Y"] \arrow[dr, bend left, "\pi_Y"] & \\ X & X \times Y \arrow[l, "\pi_X"] \arrow[r, "\pi_Y"'] & Y \end{tikzcd}$$

$\square$

Definition 4.5. Let $\mathcal{C}$ be a k-category. A *biproduct* of X and Y is an object $X \oplus Y \in \mathcal{C}$ with morphisms $X \underset{\pi_X}{\overset{i_X}{\rightleftarrows}} X \oplus Y \underset{i_Y}{\overset{\pi_Y}{\rightleftarrows}} Y$ such that

1. $i_X\pi_X + i_Y\pi_Y = \mathrm{id}_{X\oplus Y}$
2. $\pi_X i_Y = 0$, $\pi_Y i_X = 0$, $\pi_X i_X = \mathrm{id}_X$, $\pi_Y i_Y = \mathrm{id}_Y$

Definition 4.6. Let k be a commutative ring. A *k-additive* (or *k-linear*) category is a k-category with finite products and finite coproducts.

Remark.

1. When $k = \mathbb{Z}$, we simply say the category is *additive.*
2. As seen above, finite products are finite coproducts and vice versa. Both are finite biproducts.
3. For $\mathcal{C}$ a k-category, the following are equivalent:
   (a) $\mathcal{C}$ is k-additive
   (b) $\mathcal{C}$ has a zero object and every pair of objects has a product
   (c) $\mathcal{C}$ has a zero object and every pair of objects has a coproduct
   (d) $\mathcal{C}$ has a zero object and every pair of objects has a biproduct

   Moreover $(b) \iff (c) \iff (d)$, and for (a) we are just missing the empty product (or coproduct), which is the zero object.

4. If A is additive there is a canonical interpretation of the group structure on $\mathrm{Hom}(-,-)$ using $-\oplus-$. See exercise sheets.

Examples 4.7.

0. The category **Ab** of abelian groups is additive.
1. If A is a ring (or k-algebra) then $\mathbf{Mod}A$, $A\mathbf{Mod}$ and finitely generated versions are k-additive.
2. If $\mathcal{C}$ is additive, then $\mathcal{C}^{\mathrm{op}}$ is additive.
3. If $\mathcal{C}$ is additive and I is a category then $\mathrm{Fun}(I,\mathcal{C})$ is additive.
4. If A is a ring, then the category BA with one object $\bullet$ and $\mathrm{Hom}(\bullet,\bullet)=A$ is preadditive but not additive.

Definition 4.8. Let $F:\mathcal{C}\to\mathcal{D}$ be a functor between two k-linear categories. The functor F is said to be *k-linear* (or *additive* when $k=\mathbb{Z}$) if for any $X,Y\in\mathrm{Ob}(\mathcal{C})$, $\begin{array}{ccc}\mathrm{Hom}(X,Y) & \to & \mathrm{Hom}(FX,FY)\\ f & \mapsto & F(f)\end{array}$ is a k-linear map.

Proposition 4.9. *A functor $F:\mathcal{C}\to\mathcal{D}$ is additive if and only if $F(0)\simeq 0$ and $F(X\oplus Y)\simeq F(X)\oplus F(Y)$.*

Proof. Suppose F is additive. id_0 is the zero morphism of $\mathrm{Hom}_{\mathcal{C}}(0,0)$. Therefore $F(\mathrm{id}_0)=\mathrm{id}_{F(0)}$ is the zero morphism of $\mathrm{Hom}_{\mathcal{D}}(F(0),F(0))$. For any $Y\in\mathrm{Ob}(\mathcal{D})$ and $f:F(0)\to Y$, $f=f\mathrm{id}_{F(0)}=0$. This shows $F(0)$ is initial. A similar reasoning shows it is final. Therefore $F(0)$ is isomorphic to the zero object of $\mathcal{D}$. Now let $X \underset{\pi_X}{\overset{i_X}{\rightleftarrows}} X\oplus Y \underset{i_Y}{\overset{\pi_Y}{\rightleftarrows}} Y$ be a biproduct in $\mathcal{C}$. Then we have a diagram

$$F(X) \underset{F(\pi_X)}{\overset{F(i_X)}{\rightleftarrows}} F(X\oplus Y) \underset{F(i_Y)}{\overset{F(\pi_Y)}{\rightleftarrows}} F(Y)$$

And the relations we require for this diagram to be a biproduct are satisfied since F is additive and $X\oplus Y$ is a biproduct.
Now assume $F(0)\simeq 0$ and $F(X\oplus Y)\simeq F(X)\oplus F(Y)$ for all $X,Y\in\mathrm{Ob}(\mathcal{C})$. Let $X,Y\in\mathrm{Ob}(\mathcal{C})$.
TODO $\square$

Example 4.10. Let A,B be two rings and M be an A-B-bimodule. Then, $-\otimes_A M_B:\mathbf{Mod}A\to\mathbf{Mod}B$ is additive. This can be quickly proven using the proposition above: the functor is a left adjoint so it preserves coproducts!

4.2 Chain complexes in an additive category

In this subsection, all categories are assumed to be additive.

Definition 4.11. A *chain complex* in $\mathcal{C}$ is a collection $C_\bullet=\{C_n\mid n\in\mathbb{Z}\}$ of objects of $\mathcal{C}$ together with morphisms $\partial_n:C_n\to C_{n-1}$ of $\mathcal{C}$ such that $\partial_{n-1}\circ\partial_n=0$. The morphisms ∂_n are called the *differentials* of the complex.
Dually, a *cochain complex* in $\mathcal{C}$ is a collection $C^\bullet=\{C^n\mid n\in\mathbb{Z}\}$ of objects of $\mathcal{C}$ together with morphisms $\delta_n:C^n\to C^{n+1}$ of $\mathcal{C}$ such that $\delta^{n+1}\circ\delta^n=0$.

Remark. If $C_\bullet$ is a chain complex, then $(C')^\bullet = C_{-n}$ together with $\delta^n = \partial_{-n}$ is a cochain complex, so both notions are mathematically the same. However in practice chain and cochain complexes represent different objects so it is good to distinguish the two.

Definition 4.12. Let $C_\bullet$ and $D_\bullet$ be two chain complexes in $\mathcal{C}$. A *morphism of chain complexes* $f_\bullet : C_\bullet \to D_\bullet$ is a collection of morphisms $f_n : C_n \to D_n$ such that all diagrams

$$\begin{tikzcd} {} \arrow[r] & C_n \arrow[r, "\partial_n^C"] \arrow[d, "f_n"] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & {} \\ {} \arrow[r] & D_n \arrow[r, "\partial_n^D"] & D_{n-1} \arrow[r] & {} \end{tikzcd}$$

commute ("$\partial f = f\partial$").

Definition 4.13. If $\mathcal{C}$ is an additive category, then the category $\mathrm{Ch}(\mathcal{C})$ is the category whose objects are chain complexes in $\mathcal{C}$ and morphisms are morphisms of chain complexes. We also write $\mathrm{Ch}_\bullet(\mathcal{C})$.

Remark. One can check that $\mathrm{Ch}(\mathcal{C})$ is an additive category.

Example 4.14. Let $\Delta_n = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid \sum\limits_{i=1}^{n} x_i = 1 \text{ and } x_i \geq 0\} = \mathrm{Conv}(e_0, \ldots, e_n)$ be the standard n-simplex. Δ_n appears $n+1$ times as a face of the standard $n+1$-simplex, and

$$\begin{array}{rrcl} d^i: & \Delta_n & \to & \Delta_{n+1} \\ & (x_0, \ldots, x_n) & \mapsto & (x_0, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) \end{array}$$

is the i-th face map. Δ_n is a topological space, so when X is a topological space we can consider

$$\mathrm{Hom}_{\mathbf{Top}}(\Delta_n, X) = \{f : \Delta_n \to X \mid f \text{ continuous}\}$$

and we get

$$\begin{array}{rrcl} d_i: & \mathrm{Hom}_{\mathbf{Top}}(\Delta_{n+1}, X) & \to & \mathrm{Hom}_{\mathbf{Top}}(\Delta_n, X) \\ & \sigma & \mapsto & (\Delta_{n+1} \xrightarrow{d^i} \Delta_n \xrightarrow{\sigma} X) \end{array}$$

for $0 \leq i \leq n+1$.
Singular Chain Complex **TODO**

Definition 4.15. Singular simplices **TODO**

Example 4.16. Singular chain complex. **TODO**

Proposition 4.17. $C^{\mathrm{sing}} : \mathbf{Top} \to \mathrm{Ch}_\bullet(\mathbf{Ab})$ *is a functor.*

Proof. **TODO** □

Simplicial methods

Definition 4.18.

- We define the *simplicial category* (or *simplex category*) Δ whose objects are $[n] = \{0, 1, \ldots, n\}$ for $n \in \mathbb{N}$, and $\mathrm{Hom}([n], [m]) = \{f : [n] \to [m] \mid f \text{ increasing}\}$. This category is equivalent to the category of non-empty, finite, totally ordered sets with increasing maps as morphisms.

- A *simplicial set* is a contravariant functor $\Delta \to \mathbf{Set}$. More generally, if $\mathcal{C}$ is a category, a *simplicial object* in $\mathcal{C}$ is a contravariant functor from Δ to $\mathcal{C}$.
- Simplicial objects in a category $\mathcal{C}$ are objects of the category $\mathcal{C}^{\Delta^{\mathrm{op}}}$. We write $s\mathcal{C} = \mathcal{C}^{\Delta^{\mathrm{op}}}$ (so $s\mathbf{Set}$ is the category of simplicial sets).
- If $X : \Delta^{\mathrm{op}} \to \mathcal{C}$ is a simplicial object, we define $X_n = X([n])$ the *n-simplices* of X.
- In Δ, we have $d^i : [n-1] \to [n]$ the injective map that "misses i", defined by

$$d^i(k) = \begin{cases} k & \text{if } k < i \\ k+1 & \text{if } k \geq i \end{cases}$$

Proposition 4.19. *We have $d^i \circ d^j = d^{j+1} \circ d^i$ when $i \leq j$.*

Proof. You can do it. I believe in you. (**TODO**) □

If $X : \Delta^{\mathrm{op}} \to \mathbf{Ab}$ is a simplicial abelian group, then we can define $(X_\bullet, d)$ with $X_n = X([n])$ and $\begin{array}{rccc} d_n : & X_n & \to & X_{n-1} \\ & x & \mapsto & \sum_{i=0}^{n} (-1)^i X(d^i)(x) \end{array}$.

Proposition 4.20. *If $X \in s\mathbf{Ab}$, then $(X_\bullet, d)$ is a chain complex of abelian groups. Moreover, $X \mapsto X_\bullet$ is a functor $s\mathbf{Ab} \to \mathrm{Ch}_\bullet(\mathbf{Ab})$.*

Proof. **TODO** □

Let $\begin{array}{rccc} s^i : & [n+1] & \to & [n] \\ & k & \mapsto & \begin{cases} k & \text{if } k \leq i \\ k-1 & \text{if } k > i \end{cases} \end{array}$ be the map that "hits i twice".

Theorem 4.21. *Every morphism in Δ is a composition of maps of the form d^i and s^i. These maps are subject to the so-called* simplicial *relations*

$$\begin{cases} d^j \circ d^i = d^i \circ d^{j-1} & i < j \quad (1) \\ s^i \circ s^j = s^j \circ s^{i-1} & i > j \quad (2) \\ d^i \circ s^j = \begin{cases} s^{j-1} \circ d^i & i < j \\ \mathrm{id} & i \in \{j, j+1\} \\ s^j \circ d^{i-1} & i > j+1 \end{cases} \quad (3) & \end{cases} \qquad (*)$$

TODO*better typography*
and this is a presentation of Δ by generators and relations

This theorem says that to define a functor F from Δ to $\mathcal{C}$ it is enough to define $F(d^i), F(s^i)$ and show that $(*)$ holds.

Proof. "Voir annexe." **TODO** □

The maps d^is generate Δ_{inj} so to construct $F : \Delta_{\mathrm{inj}}^{\mathrm{op}} \to \mathcal{C}$ and use proposition 4.20 we only need to define $F(d^i)$ and check (1).

Theorem 4.22. *If $F : \Delta_{\text{inj}}^{\text{op}} \to$ **Ab** is a (semisimplicial abelian group) functor then $(F([n]), d_\bullet)$ with* $d_n : \begin{array}{rcl} F([n]) & \to & F[n-1] \\ x & \mapsto & \sum_i (-1)^i F(d^i)(x) \end{array}$ *is a chain complex of abelian groups. This also works if* **Ab** *is replaced by any additive category $\mathcal{C}$.*

Examples 4.23.

1. Writing

$$\begin{tikzcd} \mathbf{Top} \arrow[r] & s\mathbf{Set} \arrow[r] & s\mathbf{Ab} \\ X \arrow[r, mapsto] & \mathrm{Hom}_{\mathbf{Top}}(\Delta(-), X) \arrow[r, mapsto] & \mathbb{Z}[\mathrm{Hom}_{\mathbf{Top}}(\Delta(-), X)] \end{tikzcd}$$

   allows us to use the theorem to recover what we said about the singular chain complex before.

2. Let G be a finite group, and F_n be the free abelian group on $G^{n+1} = \{(g_0, \ldots, g_n) \mid g_i \in G\}$. F_n is a $\mathbb{Z}[G]$-module for $g \bullet (g_0, \ldots, g_n) = (gg_0, \ldots, gg_n)$, so $F_n \in \mathbb{Z}[G]\mathbf{Mod}$. We define maps

$$\begin{array}{rrcl} \partial_i : & F_n & \to & F_{n-1} \\ & (g_0, \ldots, g_n) & \mapsto & (g_0, \ldots, g_{i-1}, g_{i+1}, \ldots, g_n) \end{array}$$

   (the map removes g_i). For $i < j$, we have

$$\begin{aligned} \partial_i \circ \partial_j(g_0, \ldots, g_n) &= \partial_i(-, \cancel{g_j}, -) = (-, \cancel{g_i}, -, \cancel{g_j}, -) \\ \partial_{j-1} \circ \partial_i(g_0, \ldots, g_n) &= \partial_{j-1}(-, \cancel{g_i}, -) = (-, \cancel{g_i}, -, \cancel{g_j}, -) \end{aligned}$$

   so setting $F([n]) = F_n$ and $F(d^i) = \partial_i$ defines a functor $F : \Delta_{\text{inj}}^{\text{op}} \to \mathbb{Z}[G]\mathbf{Mod}$. Applying theorem 4.22 we have $(F_n, \partial_\bullet) \in \mathrm{Ch}(\mathbb{Z}[G]\mathbf{Mod})$ called the *bar resolution* of G.

3. Koszul complex, Hochschild complex...

Definition 4.24. Let $\mathcal{C}$ be an additive category, $C_\bullet, D_\bullet \in \mathrm{Ch}_\bullet(\mathcal{C})$ and $f, g \in \mathrm{Hom}(C_\bullet, D_\bullet)$.
A *homotopy H from f to g* is the data of maps $h_i : C_i \to D_{i+1}$ in $\mathcal{C}$ such that the following diagram commutes

$$\begin{tikzcd} C_{n+1} \arrow[r, "d_{n+1}^C"] \arrow[d, "f_{n+1}-g_{n+1}"'] & C_n \arrow[r, "d_n^C"] \arrow[d, "f_n-g_n"] \arrow[dl, "h_n"'] & C_{n-1} \arrow[d, "f_{n-1}-g_{n-1}"] \arrow[dl, "h_{n-1}"'] \\ D_{n+1} \arrow[r, "d_{n+1}^D"'] & D_n \arrow[r, "d_n^D"'] & D_{n-1} \end{tikzcd}$$

which means $f_n - g_n = h_{n-1}d_n^C + d_{n+1}^D h_n$ ("$f - g = hd + dh$"). We write $f \sim g$ if f and g are homotopic. f and g are *homotopy equivalences* if $fg \sim \mathrm{id}_D$ and $gf \sim \mathrm{id}_C$.

The motivation for this definition comes from topology. Let $f, g : X \to Y$ be continuous maps between topological spaces. We say f and g are homotopic if there exists a continuous map $H : X \times I \to Y$ (here I is the unit interval $[0, 1]$) such that $H(-, 0) = f$ and $H(-, 1) = g$. The map H is a called a homotopy from f to g.

Theorem 4.25. *Let X, Y be two topological spaces and $f, g : X \to Y$ be two homotopic continuous maps. Then the induced maps $C^{\text{sing}}(f), C^{\text{sing}}(g) : C^{\text{sing}}(X) \to C^{\text{sing}}(Y)$ are homotopic as morphisms of chain complexes.*

Proof. **TODO** ☐

Lemma 4.26. *Let $X, Y, Z \in \mathrm{Ch}(\mathcal{C})$ and $f : X_\bullet \to Y_\bullet, g : Y_\bullet \to Z_\bullet$ be morphisms of chain complexes. Then $f \sim 0$ implies $g \circ f \sim 0$.*

Proof. Let $h_\bullet$ be a homotopy between f and 0. Then $g_\bullet \circ h_\bullet$ is a homotopy between $g \circ f$ and 0. ☐

Definition 4.27. Let $\mathcal{C}$ be a category. The *homotopy category $K(\mathcal{C})$ of chain complexes in $\mathcal{C}$* is the category defined by $\mathrm{Ob}(K(\mathcal{C})) = \mathrm{Ob}(\mathrm{Ch}(\mathcal{C}))$ and $\mathrm{Hom}_{K(\mathcal{C})}(X, Y) = \mathrm{Hom}_{\mathrm{Ch}(\mathcal{C})}(X, Y)/\sim$.

Lemma 4.26 above shows that composition in $K(\mathcal{C})$ is well-defined: if $f \sim g$, then $f - g \sim 0$ so $h(f - g) \sim 0$, so $hf \sim hg$. In the same vein, if $f - g \sim 0$, $(f - g)h \sim 0$, so $fh \sim fg$. This shows composition in $K(\mathcal{C})$ is well-defined.

Remark.

1. $K(\mathcal{C})$ is an additive category.
2. ⚠ In general, $K(\mathcal{C})$ is a complicated object: it is a triangulated category.

5 Abelian categories

Definition 5.1. Let $\mathcal{C}$ be an additive category. A *kernel* of $f \in \mathrm{Mor}(\mathcal{C})$ is an equalizer of $(f, 0)$. Dually, a *cokernel* of f is a coequalizer of $(f, 0)$.

Concretely, we have universal arrows for any α such that $f\alpha = 0$ (or $\alpha f = 0$ for a cokernel)

$$\begin{tikzcd} & Z \arrow[dl, dashed] \arrow[d, "\alpha"] & \\ \ker f \arrow[r, "\iota"] & X \arrow[r, "f"] & Y \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "f"] & Y \arrow[r, "\pi"] \arrow[d, "\alpha"] & \operatorname{coker} f \arrow[dl, dashed] \\ & Z & \end{tikzcd}$$

If we assume that every morphism in $\mathcal{C}$ has a kernel and a cokernel, then

$$\begin{tikzcd} \ker f \arrow[r, "\iota"] & X \arrow[r, "f"] \arrow[d, "p"] & Y \arrow[r, "\pi"] & \operatorname{coker} f \\ & \operatorname{coker}(\ker f) & \ker(\operatorname{coker} f) \arrow[u] & \end{tikzcd}$$

where $\operatorname{coker}(\ker f)$ is notation for $\operatorname{coker}(\iota)$ and $\ker(\operatorname{coker} f)$ is notation for $\ker(\pi)$.
Since $f \circ \iota = 0$, we have a unique map $\tilde{f} : \operatorname{coker}(\ker f) \to Y$ by the universal property of the cokernel.

$$\begin{tikzcd}
\ker f \arrow[r, "\iota"] & X \arrow[r, "f"] \arrow[d, "p"] & Y \arrow[r, "\pi"] & \operatorname{coker} f \\
& \operatorname{coker}(\ker f) \arrow[ur, dashed, "\tilde{f}"'] & &
\end{tikzcd}$$

And we have $\pi \circ \tilde{f} \circ p = \pi \circ f = 0$.

Lemma 5.2. *Kernels are monomorphisms and cokernels are epimorphisms.*

Proof. Draw a diagram

$$\begin{tikzcd}
W \arrow[d, shift right, "b"'] \arrow[d, shift left, "a"] & & \\
\ker f \arrow[r, "\iota"] & X \arrow[r, "f"] & Y
\end{tikzcd}$$

such that $\iota a = \iota b$. Then $f\iota(a-b) = 0$, so there is a unique map $c : W \to \ker f$ such that we have a commutative diagram

$$\begin{tikzcd}
W \arrow[d, dashed, "c"] \arrow[dr, "\iota(a-b)"] & & \\
\ker f \arrow[r, "\iota"] & X \arrow[r, "f"] & Y
\end{tikzcd}$$

However $a-b$ and 0 already make the diagram commute, so $a-b=0$, so $a=b$.
The proof that a cokernel is an epimorphism is similar. □

Hence, $\pi \circ \tilde{f} \circ p = \pi \circ f = 0 = 0 \circ p$ means that $\pi \circ \tilde{f} = 0$ since p is an epimorphism. This means that $\tilde{f}$ factorizes through $\ker(\operatorname{coker} f)$. Setting $\operatorname{coim} f = \operatorname{coker}(\ker f)$ and $\operatorname{im} f = \ker(\operatorname{coker} f)$, we obtain the following commutative diagram

$$\begin{tikzcd}
\ker f \arrow[r, "\iota"] & X \arrow[r, "f"] \arrow[d] & Y \arrow[r, "\pi"] & \operatorname{coker} f \\
& \operatorname{coim} f \arrow[r, dashed, "\overline{f}"'] & \operatorname{im} f \arrow[u] &
\end{tikzcd}$$

Example 5.3. In $\mathcal{C} = A\mathbf{Mod}$, we have the canonical factorization

$$\begin{tikzcd}
\ker f \arrow[r, "\iota"] & X \arrow[r, "f"] \arrow[d] & Y \arrow[r, "\pi"] & Y/\operatorname{im} f \\
& X/\ker f \arrow[r, dashed, "\overline{f}"'] & \operatorname{im} f \arrow[u] &
\end{tikzcd}$$

and $\overline{f}$ is an isomorphism by the first isomorphism theorem.

Definition 5.4. Let $\mathcal{C}$ be an additive category. Then $\mathcal{C}$ is *abelian* if

1. Every morphism has a kernel and a cokernel in $\mathcal{C}$.
2. $\forall f : X \to Y$, the canonical morphism $\overline{f} : \operatorname{coim} f \to \operatorname{im} f$ is an isomorphism.

Examples 5.5.

1. If A is a ring, $\mathbf{Mod}A$ is abelian. If A is noetherian, then the full subcategory $\mathbf{mod}A$ of finitely generated modules is abelian.

2. If $\mathcal{C}$ is abelian, then so is $\mathcal{C}^{\mathrm{op}}$.

3. There are examples of categories that satisfy 1 but not 2. For instance, Hausdorff topological abelian groups, where kernels are given by the usual kernel and cokernels are the quotients by the closure of the image. We have

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Q} \arrow[r, hook] \arrow[d] & \mathbb{R} \arrow[r] & 0 \\ & \mathbb{Q} \arrow[r, "\not\cong"] & \mathbb{R} \arrow[u, hook] & \end{tikzcd}$$

Proposition 5.6. *Let $\mathcal{A}$ be an abelian category and $\mathcal{J}$ a small category. Then*

1. $\mathrm{Fun}(\mathcal{J}, \mathcal{A})$ *is an abelian category.*

2. $\mathrm{Ch}_\bullet(\mathcal{A})$ *is an abelian category.*

Sketch of proof. Let $F, G \in \mathrm{Fun}(\mathcal{J}, \mathcal{A})$ and $\eta : F \Rightarrow G$. We want to construct $\ker \eta$. For any morphism $f : i \to j \in \mathrm{Mor}(\mathcal{J})$, we have a diagram

$$\begin{tikzcd} \ker(\eta_i) \arrow[r, "\iota_i"] \arrow[d, dashed, "\alpha_f"] & F(i) \arrow[r, "\eta_i"] \arrow[d, "F(f)"] & G(i) \arrow[d, "G(f)"] \\ \ker(\eta_j) \arrow[r, "\iota_j"] & F(j) \arrow[r, "\eta_j"] & G(j) \end{tikzcd}$$

We have $0 = G(f)\eta_i\iota = \eta_j F(f)\iota$ so $F(f)\iota$ factorizes through $\ker(\eta_j)$, which gives the morphism α_f. One can check $\ker(\eta)$, defined by $\ker(\eta)(i) = \ker(\eta_i)$ and $\ker(\eta)(f) = \alpha_f$ is a functor (this is proved using uniqueness of α_f). One can check that $\iota : \ker(\eta) \Rightarrow F$ is a kernel of η by drawing the adequate diagrams. Constructing cokernels is done similarly. The canonical factorization is an isomorphism since its evaluation at every object is an isomorphism because $\mathcal{A}$ is abelian.
$\mathrm{Ch}_\bullet(\mathcal{A})$ is a subcategory of $\mathrm{Fun}(\mathbb{Z}, \mathcal{A})$ so kernels and cokernels exist in $\mathrm{Fun}(\mathbb{Z}, \mathcal{A})$. There is a commutative diagram

$$\begin{tikzcd} \ker f_{i+1} \arrow[r, dashed, "\alpha_{i+1}"] \arrow[d] & \ker f_i \arrow[r, dashed, "\alpha_i"] \arrow[d] & \ker f_{i-1} \arrow[d] \\ C_{i+1} \arrow[r, "d_{i+1}"] \arrow[d, "f_{i+1}"] & C_i \arrow[r, "d_i"] \arrow[d, "f_i"] & C_{i-1} \arrow[d, "f_{i-1}"] \\ D_{i+1} \arrow[r, "d_{i+1}"] \arrow[d] & D_i \arrow[r, "d_i"] \arrow[d] & D_{i-1} \arrow[d] \\ \operatorname{coker} f_{i+1} \arrow[r, dashed, "\beta_{i+1}"'] & \operatorname{coker} f_i \arrow[r, dashed, "\beta_i"'] & \operatorname{coker} f_{i-1} \end{tikzcd}$$

And the universal property of $\ker(f_{i-1})$ means that $\alpha_i\alpha_{i-1}$ is the unique morphism induced by $d_{i+1}d_i = 0$, so $\alpha_i\alpha i - 1 = 0$ and kernels, cokernels of chain complexes are again chain complexes. $\square$

Remark.

1. There is another equivalent definition of abelian categories: a category is abelian iff it is preabelian (additive, and all kernels/cokernels exist) and every monomorphism is a kernel of some morphism and every epimorphism is a cokernel of some morphism.

2. Abelian categories have finite limits and colimits.

3. If $f \in \mathrm{Mor}(\mathcal{A})$ with $\mathcal{A}$ abelian, then f is a monomorphism if and only if $\ker f = 0$ and f is an epimorphism if and only if $\operatorname{coker} f = 0$. Moreover, a monomorphism that is also an epimorphism is an isomorphism.

Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be two composable morphisms in an abelian category such that $gf = 0$. The left diagram below shows that $0 = gf = g\alpha\overline{f}\pi = 0$, however $\overline{f}\pi$ is an epi so $g\alpha = 0$. Therefore, α factorizes into a map $\operatorname{im} f \to \ker g$ as shown in the right diagram.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, two heads, "\pi"] & Y \arrow[r, "g"] & Z \\ \operatorname{coim} f \arrow[r, "\overline{f}"'] & \operatorname{im} f \arrow[u, hook, "\alpha"'] & \end{tikzcd} \qquad \begin{tikzcd} \ker g \arrow[r, hook, "\iota"] & Y \arrow[r, "g"] & Z \\ & \operatorname{im} f \arrow[u, hook, "\alpha"'] \arrow[ul, dashed, "\exists!\overline{\alpha}"] & \end{tikzcd}$$

Definition 5.7. Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ such that $gf = 0$.

- We say it is *exact* if the canonical map $\operatorname{im} f \to \ker g$ is an isomorphism.

- A chain complex $(C_\bullet, d_\bullet)$ is *exact* if the canonical maps $\mathrm{Im}(d_i) \simeq \ker(d_i)$ are isomorphisms for all $i \in \mathbb{Z}$.

- A *short exact sequence* if an exact complex of the form $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$.

Example 5.8. In $\mathbf{Mod}A$, $gf = 0$ means that $\operatorname{im} f \subset \ker g$, so exactness is equivalent to $\operatorname{im} f = \ker g$.

Proposition 5.9. *The sequence $0 \to X \xrightarrow{f} Y$ is exact if and only if f is a monomorphism. The sequence $X \xrightarrow{f} Y \to 0$ is exact if and only if f is an epimorphism.*

Proof. We have $\mathrm{im}(0 \to X) = \ker(\operatorname{coker}(0 \to X))$. One shows that the cokernel of $0 \to X$ is $X \xrightarrow{\mathrm{id}} X$ since it satisfies the required universal property. Similarly, one can prove the kernel of $X \xrightarrow{\mathrm{id}} X$ is $0 \to X$ by checking the universal property. Therefore, $\mathrm{im}(0 \to X) = 0$. Exactness is therefore equivalent to asking $\ker f = 0$. Let i be the universal morphism $\ker f \xrightarrow{i} X$. If f is a mono, we have $\ker f = 0$ since fi is a mono and $fi0 = fi\mathrm{id}_{kerf} = 0$. Conversely, if $\ker f = 0$ and $fg = fh$, then $f(g - h) = 0$ and the factorization shows that $g = h$.
A similar "dual proof" shows the second part of the proposition is true. $\square$

Proposition 5.10. *The sequence $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z$ is exact if and only if $f = \ker g$.*

Proof. Assume $f = \ker g$. Since kernels are monomorphisms, we have exactness at X. Now we need to show the canonical map $\operatorname{im} f \to \ker g$ is an isomorphism. Draw the diagram

$$\begin{tikzcd}
 & & & \operatorname{coker} f \\
0 \arrow[r] & X \arrow[r, "f"] \arrow[d] & Y \arrow[r, "g"] \arrow[ur, "\pi_f"] & Z \\
 & \operatorname{coim} f \arrow[r, "\sim"] & \operatorname{im} f \arrow[u] \arrow[r] & \ker g \arrow[ul, "i_g"]
\end{tikzcd}$$

By $f = \ker g$ we mean $X \simeq \ker g$ as kernels. This means that there is an isomorphism $\ker g \underset{\sim}{\xrightarrow{\phi}} X$ such that $i_g = f\phi$. Then, $\pi_f i_g = \pi_f f \phi = 0$, so i_g factorizes through $\ker(\operatorname{coker} f) = \operatorname{im} f$ in a way that makes the whole diagram commute which shows the canonical map $\operatorname{im} f \to \ker g$ is an isomorphism, so we have exactness at Y.

Conversely, assume the sequence is exact. We just need to check $X \xrightarrow{f} Y$ satisfies the universal property of $\ker g$. Exactness tells us we have a diagram

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "f"] \arrow[d, two heads, "\pi"] & Y \arrow[r, "g"] & Z \\
 & \operatorname{coim} f \arrow[r, "\sim", "\overline{f}"'] & \operatorname{im} f \arrow[u, hook, "\alpha"] \arrow[r, "\sim", "t"'] & \ker g \arrow[ul, "i_g"]
\end{tikzcd}$$

We have $\operatorname{coim} f = \operatorname{coker}(\ker f)$. The proof above shows that exactness at X implies $\ker f \simeq 0$. One can then check that $\operatorname{coim} f = X$ and $\pi = \mathrm{id}$. Therefore we obtain an isomorphism $\phi : X \to \ker g$ such that $i_g \circ \phi = f$ or equivalently $f \circ \phi^{-1} = i_g$. Let $h : T \to Y$ be a morphism such that $gh = 0$, then we have a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \\
\ker g \arrow[u, "\phi^{-1}"] \arrow[ur, "i_g"] & T \arrow[u, "h"'] \arrow[l, dashed, "\overline{h}"]
\end{tikzcd}$$

So $\phi^{-1}\overline{h}$ is a factorization of h through f. If we have another factorization ψ then

$$\begin{tikzcd}
\ker g \arrow[r, "i_g"] & Y \arrow[r] & Z \\
X \arrow[u, "\phi"] \arrow[ur, "f"] & T \arrow[u, "h"'] \arrow[l, "\psi"]
\end{tikzcd}$$

so $i_g \phi \psi = f\psi = h$ and $\phi\psi = \overline{h}$, so $\psi = \phi^{-1}\overline{h}$. $\square$

Remark. The sequence $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ is exact if and only if f is a monomorphism, g is an epimorphism and $\operatorname{im} f \xrightarrow{\sim} \ker g$ is an isomorphism, which is equivalent to $g = \operatorname{coker} f$ and $f = \ker g$.

Remark. There is a difficult theorem of Freyd and Mitchell that says any abelian category can be seen as a full subcategory of $\mathbf{Mod}A$ for some ring A in such a way that the abelian structure is induced by the usual one in $\mathbf{Mod}A$.

Definition 5.11. Let $\mathcal{A}$ be an abelian category and $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$, $0 \to D \xrightarrow{h} E \xrightarrow{k} F \to 0$ be two short exact sequences. A *morphism of short exact sequences* between them is the data of three morphisms $\alpha : A \to D$, $\beta : B \to E$ and $\gamma : C \to F$ such that the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & D \arrow[r, "h"] & E \arrow[r, "k"] & F \arrow[r] & 0
\end{tikzcd}$$

Lemma 5.12 (Short five lemma)**.** *Using the same notations as in the definition above:*

- *If α and γ are monomorphisms, so is β.*
- *If α and γ are epimorphisms, so is β.*
- *If α and γ are isomorphisms, so is β.*

We give two proofs of this result.

Proof by diagram chase. Assume we work in a category of modules $\mathbf{Mod}A$. Assume α, γ are monos. Let $x \in \ker \beta$. Then $\gamma g(x) = k\beta(x) = 0$ and γ is a mono so $g(x) = 0$. By exactness at B, there exists $y \in A$ such that $f(y) = x$. Then $0 = \beta f(y) = h\alpha(y)$. By exactness at D, h is a mono, so $\alpha(y) = 0$. since α is a mono, $y = 0$, so $x = 0$, which means β is a mono.
Now assume α, γ are epis. Let $x \in E$. Since γ, g are epis, there exists $y \in B$ such that $\gamma(g(y)) = k(x)$. Then, $k(\beta(y) - x) = 0$. By exactness at E and since α is epi, there exists $z \in A$ such that $h(\alpha(z)) = \beta(y) - x$. Therefore $\beta f(z) = \beta(y) - x$, so $\beta(y - f(z)) = x$ and β is epi. $\square$

Categorical proof in any abelian category. Assume α, γ are monos. Let us add $\ker \beta$ to the diagram.

$$\begin{tikzcd}
 & & \ker \beta \arrow[dl, dashed, "\exists j"'] \arrow[d, "i"] & & \\
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & D \arrow[r, "h"] & E \arrow[r, "k"] & F \arrow[r] & 0
\end{tikzcd}$$

We have $\beta i = 0$, so $\gamma g i = k\beta i = 0$. Since γ is a mono, $gi = 0$. Exactness tells us $f = \ker g$, so we obtain the map $j : \ker \beta \to A$ with the universal property of $\ker g$. Since the diagram commutes, $0 = \beta i = \beta f j = h\alpha j$. Since h and α are both monos, $j = 0$, so $i = 0$, so β is a mono.
Now assume α, γ are epis and consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & D \arrow[r, "h"] & E \arrow[r, "k"] \arrow[d, "\pi"] & F \arrow[r] \arrow[dl, dashed, "\exists \eta"] & 0 \\
 & & \operatorname{coker} \beta & &
\end{tikzcd}$$

We have $\pi\beta = 0$, so $\pi\beta f = \pi h\alpha = 0$. Since α is an epi, $\pi h = 0$. Exactness tells us $k = \operatorname{coker} h$, which gives us η. Then, $\eta k\beta = 0$, so $\eta\gamma g = 0$. Since γ, g are epis, $\eta = 0$, so $\pi = 0$, so β is an epi. $\square$

Theorem 5.13 (Splitting lemma)**.** *Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a short exact sequence in an abelian category $\mathcal{A}$. The following are equivalent:*

(1) $\exists r : B \to A, rf = \mathrm{id}_A$

(2) $\exists s : C \to B, gs = \mathrm{id}_C$

(3) $\exists h : B \xrightarrow{\sim} A \oplus C$ *such that*

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, equal] & B \arrow[r, "g"] \arrow[d, "h"] & C \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & A \arrow[r, "i_A"] & A \oplus C \arrow[r, "\pi_C"] & C \arrow[r] & 0 \end{tikzcd}$$

is an isomorphism of short exact sequences.

When these conditions are satisfied, we say the short exact sequence splits*.*

Proof. Assume we have (3). Then we have the projection $\pi_A : A \oplus C \to A$. Letting $r = \pi_A h$, we have $rf = \pi_A hf = \pi_A i_A = \mathrm{id}_A$. Similarly, setting $s = h^{-1} i_C$ gives $gs = \pi_C h h^{-1} i_C = \mathrm{id}_C$.
Now assume (1). We have $r : B \to A$ and $g : B \to C$. This gives a morphism $r \oplus g : B \to A \oplus C$ defined by $r \oplus g = i_A r + i_C g$. Then, $(r \oplus g)f = i_A$ since $gf = 0$ and $\pi_C(r \oplus g) = g$ by properties of the biproduct. This means that $r \oplus g$ makes the diagram above commute. The short five lemma then tells us $r \oplus g$ is an isomorphism.
Assume (2). Then $f : A \to B$ and $s : C \to B$ induce a morphism $f \oplus s : A \oplus C \to B$ defined by $f \oplus s = f\pi_A + s\pi_C$. This morphism satisfies

$$(f \oplus s)i_A = f \quad \text{and} \quad g(f \oplus s) = \pi_C$$

so again we get an isomorphism of short exact sequences by the short five lemma. $\square$

Definition 5.14. Let $\mathcal{C}$ and $\mathcal{D}$ be two abelian categories. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor.

1. We say F is *left exact* if F preserves finite limits.
2. We say F is *right exact* if F preserves finite colimits.
3. We say F is *exact* if it preserves finite limits and finite colimits.

Lemma 5.15. *Let $F : \mathcal{C} \to \mathcal{D}$ be an additive functor between abelian categories. The following are equivalent;*

(1) The functor F is left exact.

(2) The functor F preserves kernels i.e. $F(\ker f) \simeq \ker(F(f))$.

(3) If $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z$ *is an exact sequence in* $\mathcal{C}$*, the sequence* $0 \to F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z)$ *is also exact.*

(4) If $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ *is a short exact sequence in* $\mathcal{C}$*, the sequence* $0 \to F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z)$ *is also exact.*

Proof.

(1) $\Rightarrow$ (2) This is clear since a kernel is a limit (an equalizer).

(2) $\Rightarrow$ (3) Assume we have (2). Then $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z$ is an exact sequence in $\mathcal{C}$ if and only if $f = \ker g$, so $F(f)$ is a kernel of $F(g)$, so $0 \to F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z)$ is exact.

(3) $\Rightarrow$ (4) This is clear.

(2) $\Rightarrow$ (1) The functor F is additive so it preserves products. The equalizer of $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ is the kernel of $f - g$, so F preserving kernels means it also preserves equalizers. Since any finite limit can be built out of products and equalizers, F is left-exact.

(4) $\Rightarrow$ (3) Let $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z$ be an exact sequence. Consider $0 \to X \xrightarrow{f} Y \to \operatorname{coker}(f) \to 0$. Applying F shows that $F(f)$ is a monomorphism, so F preserves monos. Moreover we have the exact sequence $0 \to X \xrightarrow{f} Y \xrightarrow{\overline{g}} \operatorname{Im} g \to 0$ so $0 \to F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(\overline{g})} F(\operatorname{Im} g)$ is also exact. Since $i : \operatorname{im} g \hookrightarrow Z$ is a mono and F preserves monos, we know that $F(i) : \operatorname{im} g \to F(Z)$ is a mono so F does not change the kernel. This means that $0 \to F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z)$ is exact.

□

Corollary 5.16. *For an additive functor $F : \mathcal{C} \to \mathcal{D}$ between abelian categories, the following are equivalent:*

1. *F is exact.*
2. *For any short exact sequence $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ in $\mathcal{C}$, the sequence $0 \to F(A) \xrightarrow{F(\alpha)} F(B) \xrightarrow{F(\beta)} F(C) \to 0$ is exact.*

Proposition 5.17. *Let $\mathcal{C}$ be an abelian category.*

1. $\operatorname{Hom}_{\mathcal{C}}(-,-) : \mathcal{C}^{\text{op}} \times \mathcal{C} \to \mathbf{Mod}\mathbb{Z}$ *is left exact in each variable.*
2. $- \otimes_A - : \mathbf{Mod}A \times A\mathbf{Mod} \to \mathbb{Z}\mathbf{Mod}$ *is right exact in each variable.*
3. *If $F \dashv G$, then F is right exact and G is left exact.*

Proof. Since left adjoints preserve colimits, they are right exact, and dually for right adjoints. □