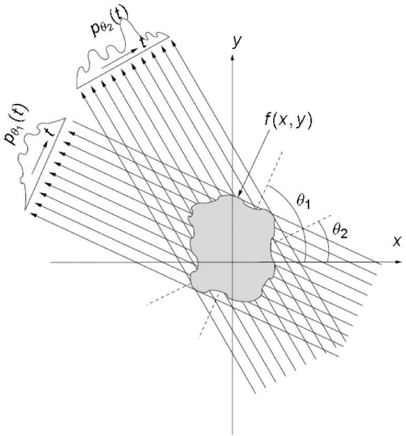
Diagnostic Medical Image Processing Reconstruction – Fan Beam Reconstruction

WS 2015/2016
Andreas Maier, Joachim Hornegger, Markus Kowarschik
Pattern Recognition Lab (CS 5)

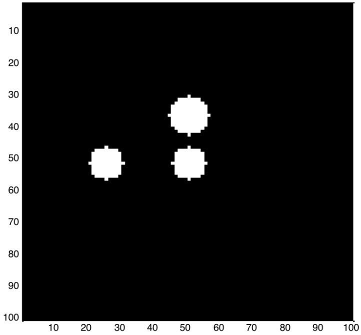
FRIEDRICH-ALEXANDER
UNIVERSITÄT
ERLANGEN-NÜRNBERG
TECHNISCHE FAKULTÄT

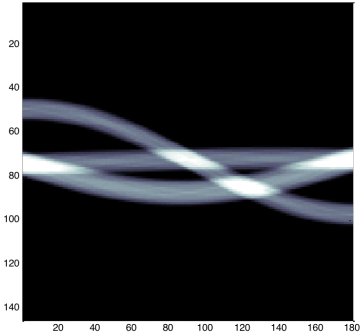
Parallel Beam Geometry



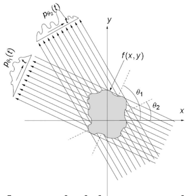
- Earliest Acquisition Geometry
- Principle: Rotate & Translate

Parallel Beam Geometry – Sinogram





Parallel Beam Geometry

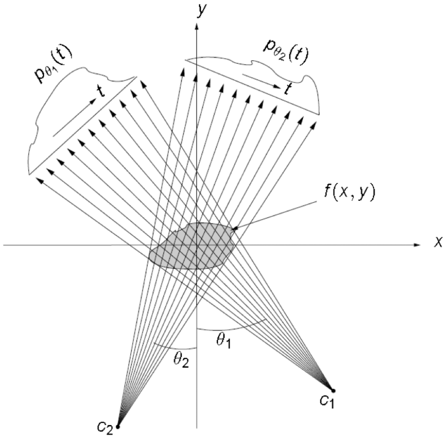


- Acquisition took 5 Minutes
- Reconstruction took 30 Minutes
- Slice resolution was 80 x 80 pixels

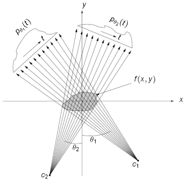
First CT Scanner: EMI (1971)

Image: Wikipedia

Fan Beam Geometry



Fan Beam Geometry



- Fan beam Scanners became available in 1975 (20s / slice)
- Fast rotations became possible 1987 with slip rings (300ms / slice)

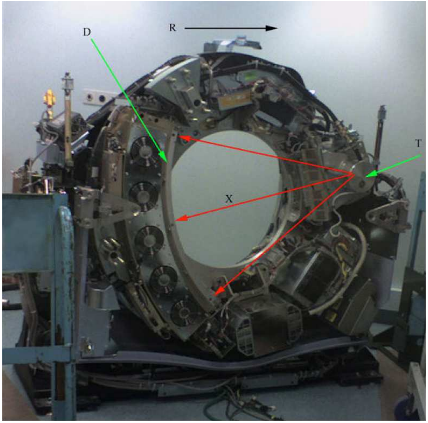


Image: Wikipedia

Topics

Fan Beam Geometry

Parallel Beam to Fan Beam Conversion

Short Scan

Fan Beam vs Parallel Beam

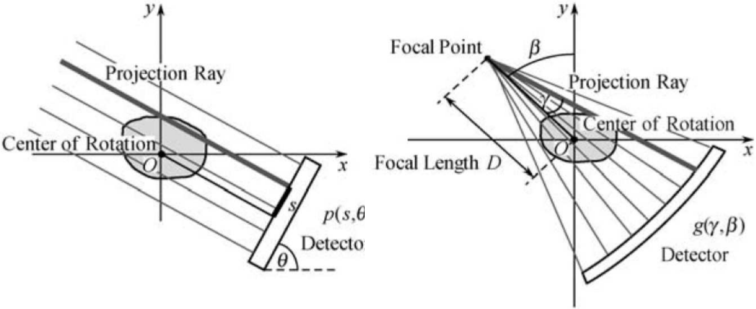
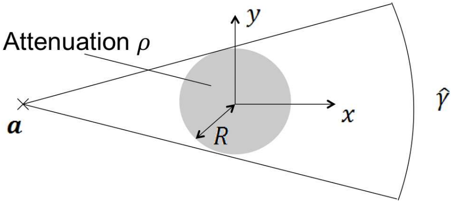


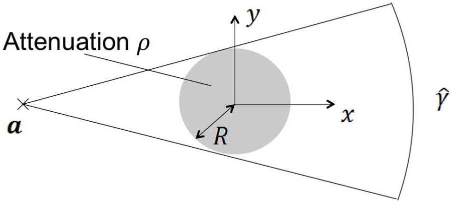
Image: Zeng, 2009

- Parallel beam algorithms cannot be applied directly anymore
- We do not have a central slice theorem anymore

Example: Homogeneous Cylinder



Example: Homogeneous Cylinder

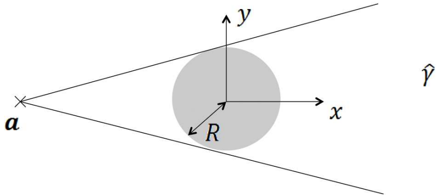


- Source is at position $\boldsymbol{a} = (a_x, a_y)^\top$
- Detector detects rays

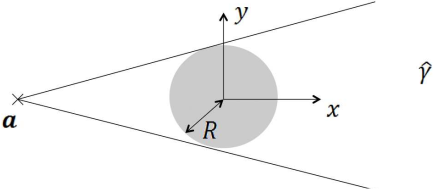
$$g(\boldsymbol{a}, \hat{\gamma}) = \int_{-\infty}^{\infty} f(a_x + t\cos\hat{\gamma}, a_y + t\sin\hat{\gamma})dt$$

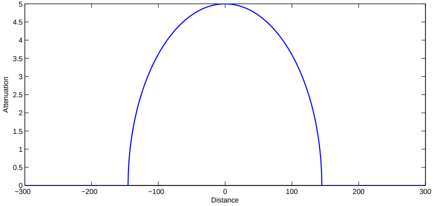
- Object is bounded by $R^2 = (x^2 + y^2)$

Example: Homogeneous Cylinder (2)



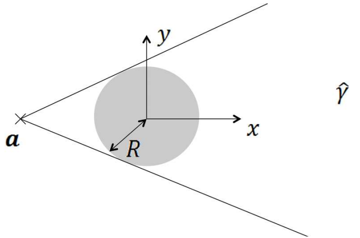
Example: Homogeneous Cylinder (2)

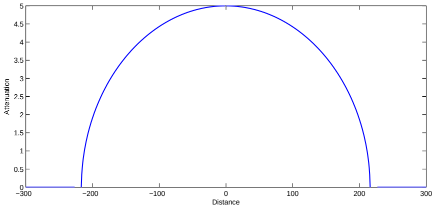




Projection $g(\boldsymbol{a}, \hat{\gamma})$

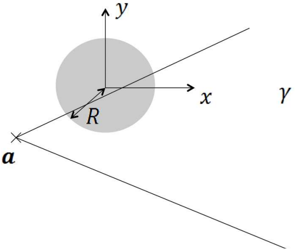
Example: Homogeneous Cylinder (3)

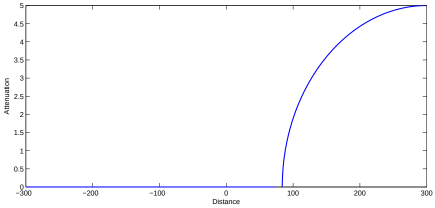




Projection $g(\boldsymbol{a}, \hat{\gamma})$

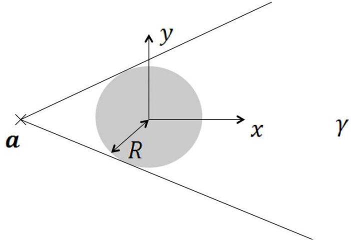
Example: Homogeneous Cylinder (4)

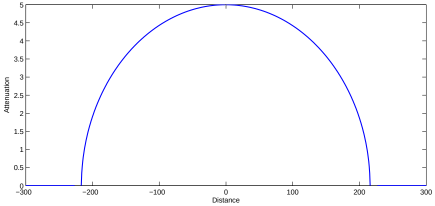




Projection $g(\boldsymbol{a}, \hat{\gamma})$

Example: Homogeneous Cylinder (5)





Projection $g(\boldsymbol{a}, \hat{\gamma})$

Point Spread Function – Parallel Beam

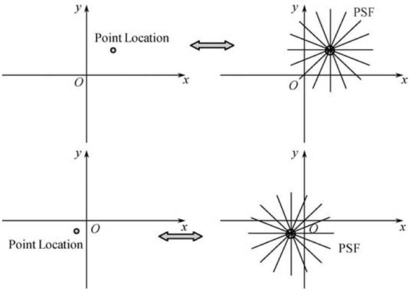


Image: Zeng, 2009

- Draw a line through the reconstructed point that is perpendicular to the detector
- Repeat for every detector position

Point Spread Function – Parallel Beam (2)

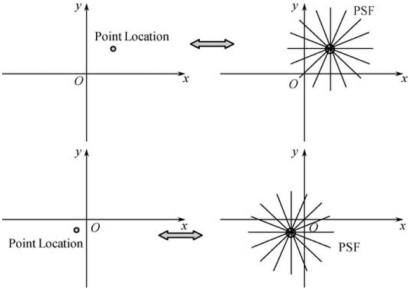


Image: Zeng, 2009

- In this case, the point spread function is shift-invariant, i.e., every reconstructed point shows the same pattern

Point Spread Function – Fan Beam

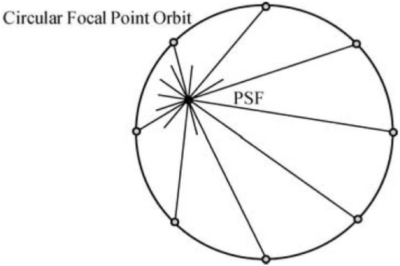


Image: Zeng, 2009

- Draw a line through the reconstructed point and the source position
- Repeat for every source position

Point Spread Function – Fan Beam (2)

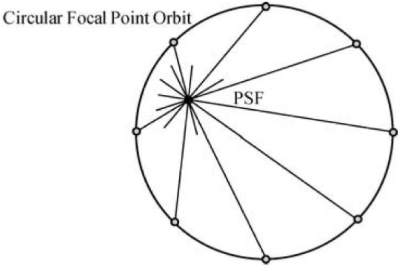


Image: Zeng, 2009

- For a complete circle, the pattern is also shift-invariant
- It can be shown that the full circle PSF is equivalent to the parallel beam PSF

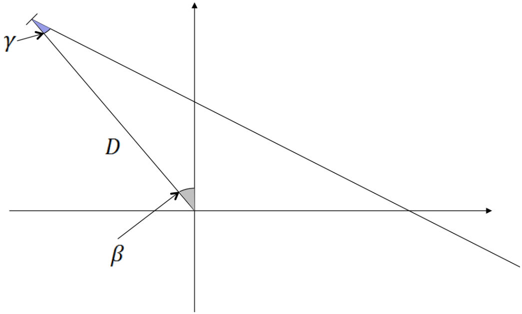
Topics

Fan Beam Geometry

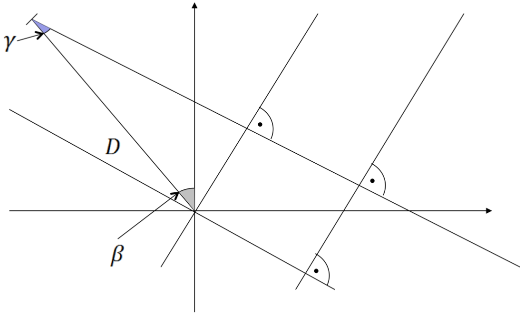
Parallel Beam to Fan Beam Conversion

Short Scan

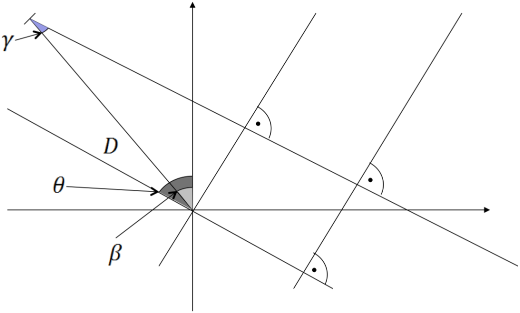
Parallel Beam to Fan Beam Conversion



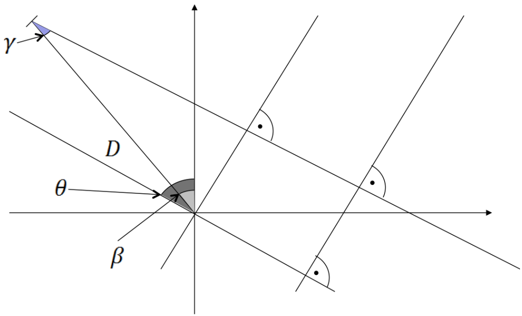
Parallel Beam to Fan Beam Conversion



Parallel Beam to Fan Beam Conversion

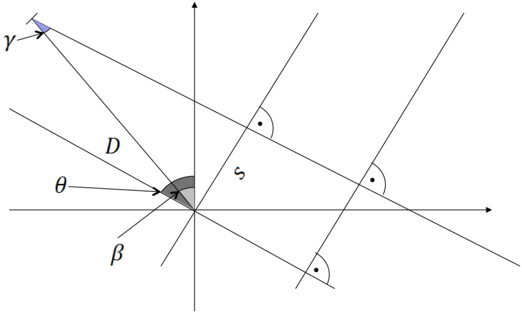


Parallel Beam to Fan Beam Conversion



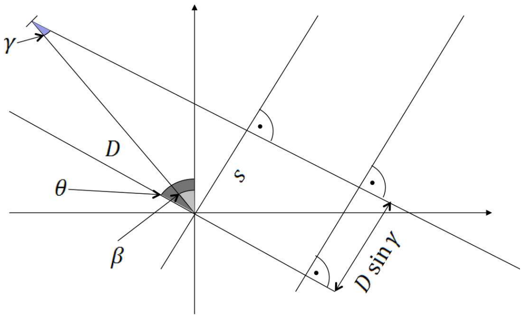
$$\theta = \gamma + \beta$$

Parallel Beam to Fan Beam Conversion



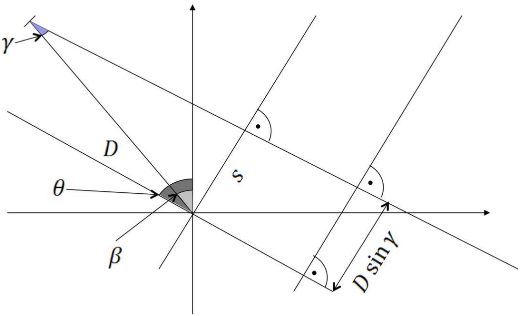
$$\theta = \gamma + \beta$$

Parallel Beam to Fan Beam Conversion



$$\theta = \gamma + \beta$$

Parallel Beam to Fan Beam Conversion



$$\theta = \gamma + \beta$$
$$s = D \sin \gamma$$

Parallel Beam to Fan Beam Conversion (2)

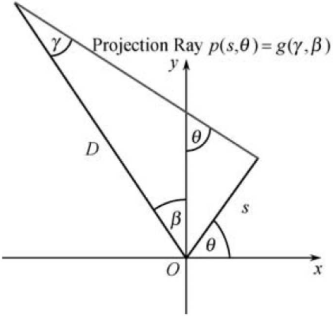


Image: Zeng, 2009

- Idea: Find equal rays in both geometries:

$$
\begin{aligned}
\theta &= \gamma + \beta \\
s &= D \sin \gamma
\end{aligned}
$$

- Then set

$$
p(s, \theta) = g(\gamma, \beta)
$$

- This process is called "Rebinning"

Parallel Beam to Fan Beam Conversion (2)

- Rebinning is a feasible solution
- Change of coordinate systems requires interpolation which may introduce inaccuracies
- Hence, rebinning may not be the method of choice

$\Rightarrow$ Derive reconstruction method for fan beam data by conversion of the reconstruction algorithm

Parallel Beam to Fan Beam Conversion (3)

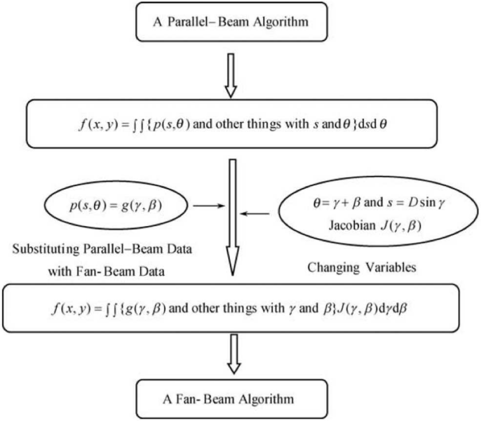


Image: Zeng, 2009

Equally-spaced and Equiangular Detectors

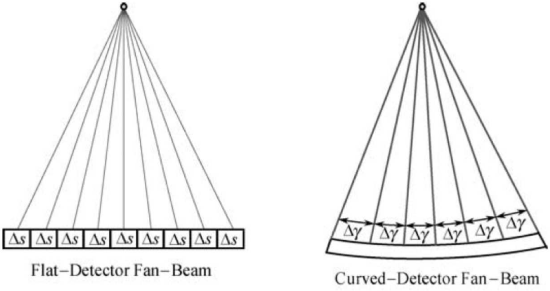


Image: Zeng, 2009

- Sampling is different in both geometries.
- Hence, different reconstruction formulas are obtained.

FBP for the Equiangular Case

- We start with a parallel beam backprojection.

$$f(x,y) = \frac{1}{2}\int_0^{2\pi}\int_{-\infty}^{\infty} p(s,\theta)h(x\cos\theta + y\sin\theta - s)\mathrm{d}s\mathrm{d}\theta$$

FBP for the Equiangular Case (2)

- Perform cosine weighting

$$g_1(\gamma, \beta) = g(\gamma, \beta) \cos \gamma$$

- Apply fan beam filter:

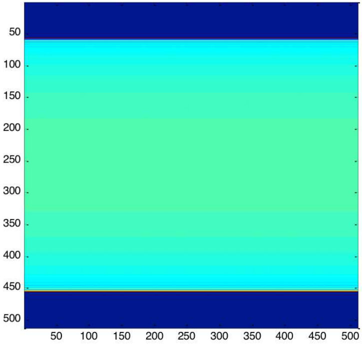
$$g_2(\gamma, \beta) = g_1(\gamma, \beta) * h_{\text{fan}}(\gamma)$$

$$h_{\text{fan}}(\gamma) = \frac{D}{2} \left(\frac{\gamma}{\sin \gamma} \right)^2 h(\gamma)$$

- Backproject with distance weight:

$$f(r, \varphi) = \int_0^{2\pi} \frac{1}{D'^2} g_2(\gamma', \beta) \mathrm{d}\beta$$

Backprojection and Fourier Slice Theorem



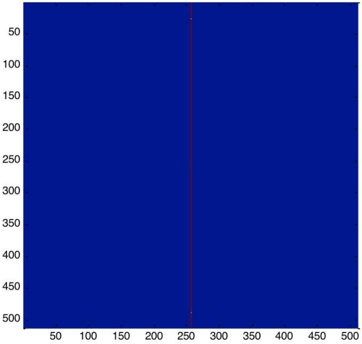


Figure: Backprojection of a single view and its Fourier Transform.

Backprojection and Fourier Slice Theorem

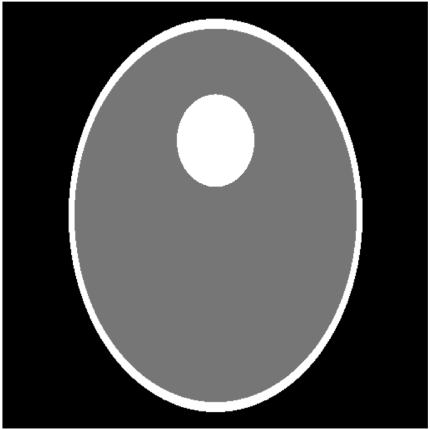


Figure: Slice view

Backprojection and Fourier Slice Theorem

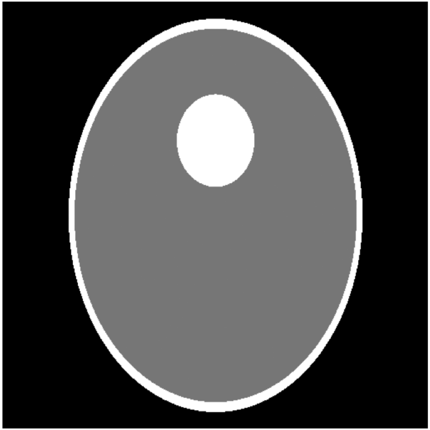


Figure: Slice view

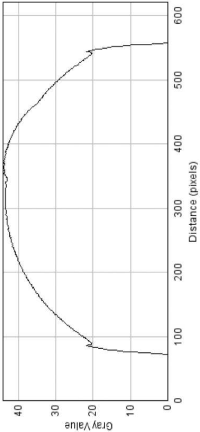


Figure: Projection

Backprojection and Fourier Slice Theorem (2)

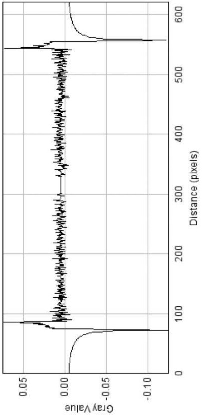


Figure: Filtered projection

Backprojection and Fourier Slice Theorem (2)

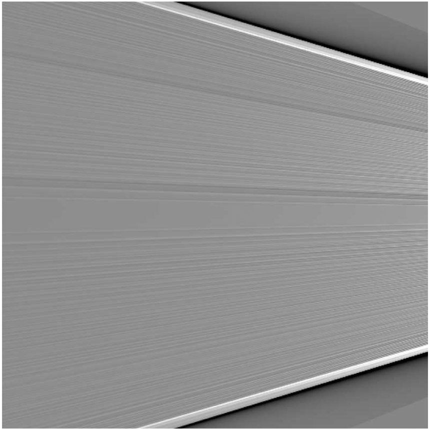


Figure: Fan-beam backprojection

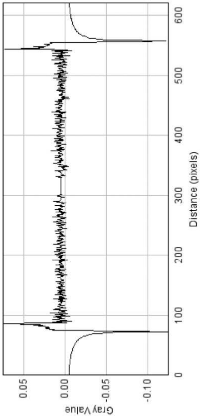


Figure: Filtered projection

Backprojection and Fourier Slice Theorem (3)

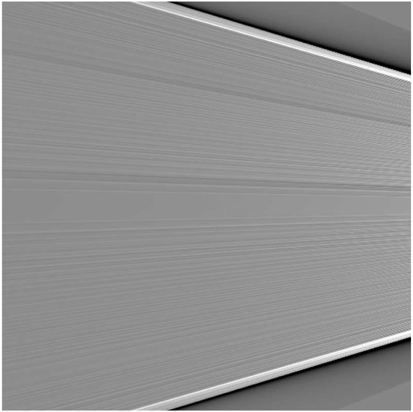


Figure: Fan-beam backprojection

Backprojection and Fourier Slice Theorem (3)

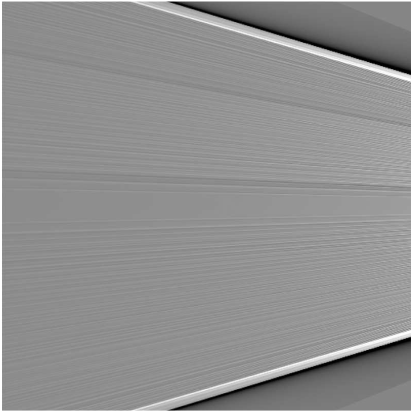


Figure: Fan-beam backprojection

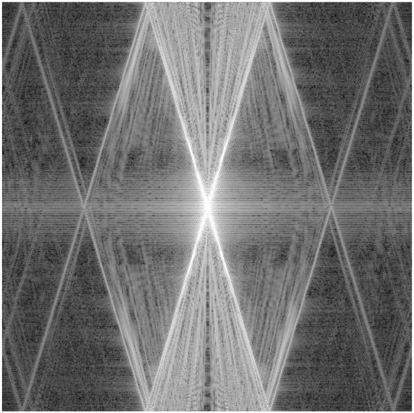


Figure: 2D Fourier Transform

Backprojection and Fourier Slice Theorem (4)

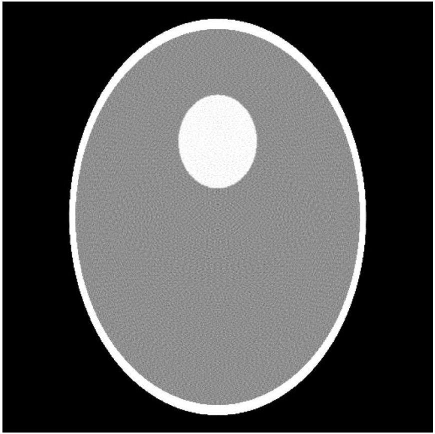


Figure: Reconstruction from fan-beam data.

FBP for the Equally-spaced Case

- We start with a parallel beam backprojection with polar coordinates (r, φ) with $x = r\cos\varphi$, $y = r\sin\varphi$ and $x\cos\theta + y\sin\theta = r\cos(\theta - \varphi)$.

$$f(r, \varphi) = \frac{1}{2}\int_0^{2\pi}\int_{-\infty}^{\infty} p(s, \theta)h(r\cos(\theta - \varphi) - s)\mathrm{d}s\mathrm{d}\theta$$

Parallel Beam to Fan Beam Conversion

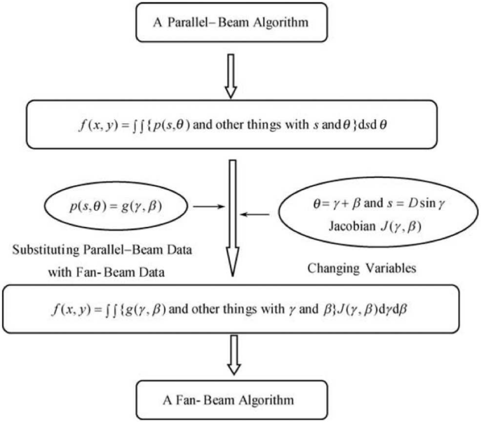


Image: Zeng, 2009

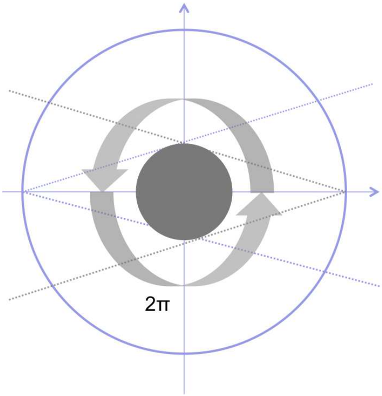
Topics

Fan Beam Geometry

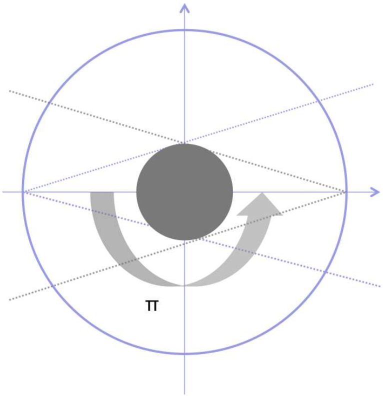
Parallel Beam to Fan Beam Conversion

Short Scan

Full Scan vs Half Scan

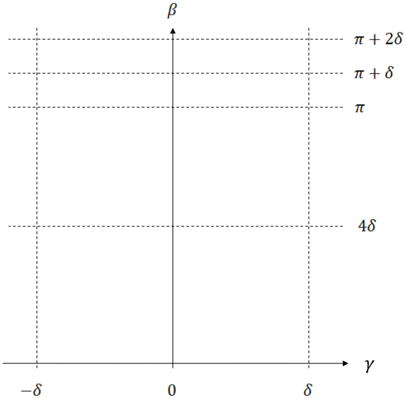


Full Scan



Half Scan

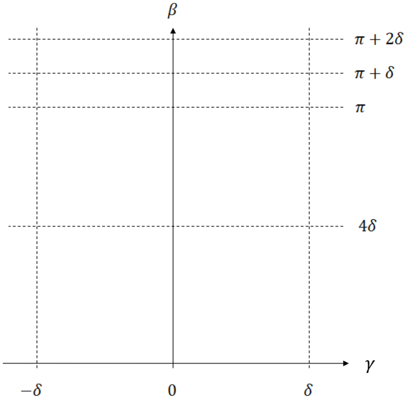
Redundant Areas – Sinogram



Identical rays:

$$
\begin{aligned}
\gamma_1 &= -\gamma_2 \\
\beta_2 &= \beta_1 - 2\gamma_1 + \pi
\end{aligned}
$$

Redundant Areas – Sinogram



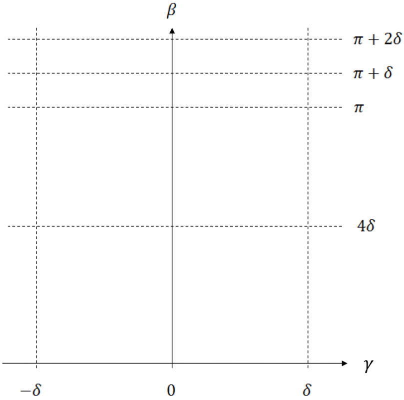
Identical rays:

$$\begin{aligned} \gamma_1 &= -\gamma_2 \\ \beta_2 &= \beta_1 - 2\gamma_1 + \pi \end{aligned}$$

Upper triangle:

$$\pi + 2\gamma_1 \leq \beta_1 \leq \pi + 2\delta$$

Redundant Areas – Sinogram



Identical rays:

$$
\begin{aligned}
\gamma_1 &= -\gamma_2 \\
\beta_2 &= \beta_1 - 2\gamma_1 + \pi
\end{aligned}
$$

Upper triangle:

$$\pi + 2\gamma_1 \leq \beta_1 \leq \pi + 2\delta$$

Lower triangle:

$$0 \leq \beta_2 \leq 2\gamma_2 + 2\delta$$

Parker Redundancy Weighting

Idea: Weight identical rays to reduce redundancy

Containts (upper triangle):

$$(1')f_1(\pi + 2\delta) = 0$$
$$(2')f_1(\pi + 2\gamma) = 1$$

Containts (lower triangle):

$$(1)f_2(0) = 0$$
$$(2)f_2(2\delta + 2\gamma) = 1$$

Solve redundancy:

$$(3)f_1(\beta_1) + f_2(\beta_2) = 1$$

Identical rays:

$$\begin{aligned} \gamma_1 &= -\gamma_2 \\ \beta_2 &= \beta_1 - 2\gamma_1 + \pi \end{aligned}$$

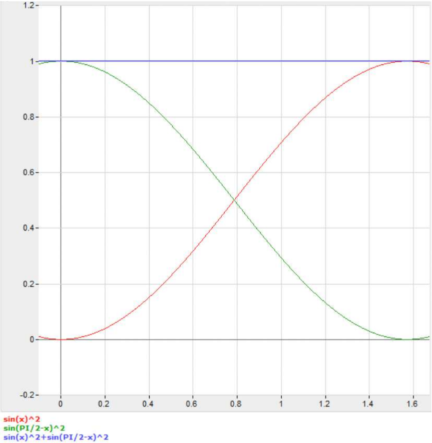
Upper triangle:

$$\pi + 2\gamma_1 \leq \beta_1 \leq \pi + 2\delta$$

Lower triangle:

$$0 \leq \beta_2 \leq 2\gamma_2 + 2\delta$$

Parker Redundancy Weighting (2)



Parker Redundancy Weighting (3)

Parker's trick:

$$\sin^2(\gamma) + \cos^2(\gamma) = 1$$
$$\sin(\frac{\pi}{2} - \gamma) = \cos(\gamma)$$

New containts:

$$(1 + 1')f(x) = 0$$
$$(2 + 2')f(x) = \frac{\pi}{2}$$
$$(3)f_1(\beta_1) + f_2(\beta_2) = 1$$

Weighting functions:

$$f_1(\beta_1) = \frac{\pi}{2}\frac{\pi + 2\delta - \beta_1}{(\pi + 2\delta) - (\pi + 2\gamma)} = \frac{\pi}{4}\frac{\pi + 2\delta - \beta_1}{\delta - \gamma}$$
$$f_2(\beta_2) = \frac{\pi}{2}\frac{\beta_2}{2\delta + 2\gamma} = \frac{\pi}{4}\frac{\beta_2}{\delta + \gamma}$$

Identical rays:

$$\begin{aligned} \gamma_1 &= -\gamma_2 \\ \beta_2 &= \beta_1 - 2\gamma_1 + \pi \end{aligned}$$

Upper triangle:

$$\pi + 2\gamma_1 \leq \beta_1 \leq \pi + 2\delta$$

Lower triangle:

$$0 \leq \beta_2 \leq 2\gamma_2 + 2\delta$$

Polynomial Parker Weighting

Parker Weighting – Example

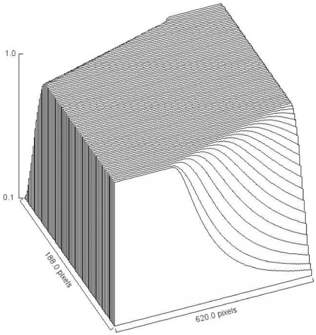


Figure: Parker Weights for a super short scan trajectory.

FBP for the Equiangular Case and Parker Weight

- Perform Parker weighting with $w_p(t, \beta)$:

$$g_1(\gamma, \beta) = g(\gamma, \beta) w_p(\gamma, \beta)$$

- Perform cosine weighting:

$$g_2(\gamma, \beta) = g_1(\gamma, \beta) \cos \gamma$$

- Apply fan beam filter:

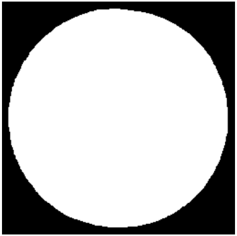
$$g_3(\gamma, \beta) = g_2(\gamma, \beta) * h_{\text{fan}}(\gamma)$$

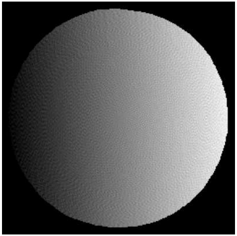
$$h_{\text{fan}}(\gamma) = \frac{D}{2} \left(\frac{\gamma}{\sin \gamma} \right)^2 h(\gamma)$$

- Backproject with distance weight:

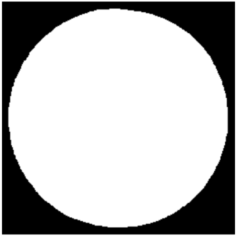
$$f(r, \varphi) = \int_0^{2\pi} \frac{1}{D'^2} g_3(\gamma', \beta) \mathrm{d}\beta$$

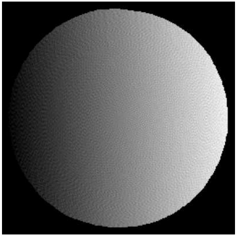
No Redundancy Weights – Example

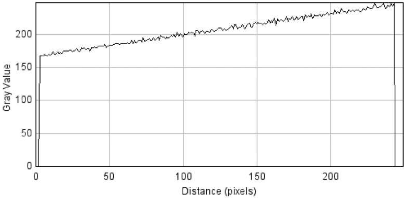




No Redundancy Weights – Example







Short Scan – Point Spread Function

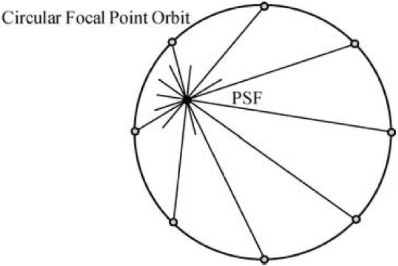
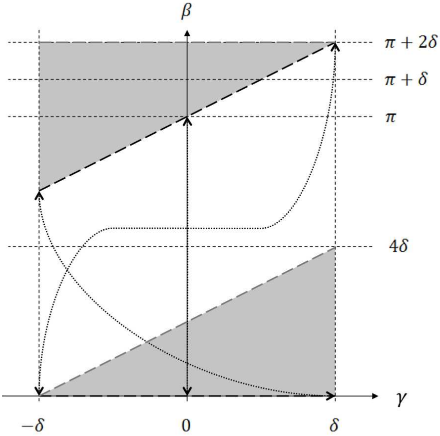


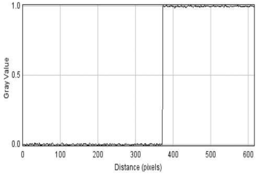
Image: Zeng, 2009

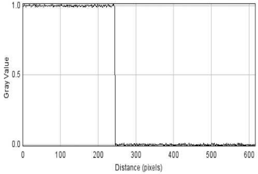
- Point spread function is no longer uniform
- Reconstruction resolution changes over the image

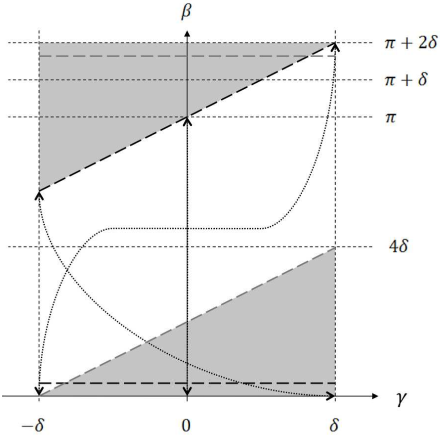
Less than a short scan?



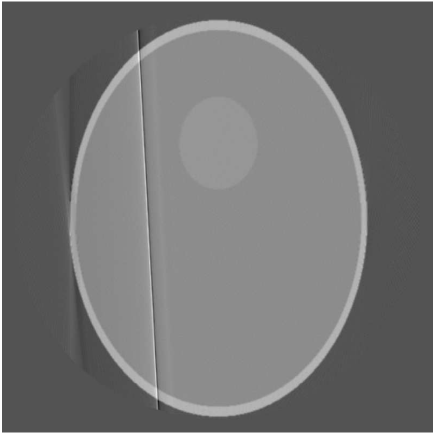
Less than a short scan? (2)







Less than a short scan? (3)



Less than a short scan? (4)

- Apply Parker weight $w_p(t, \beta)$ in reconstruction formular:

$$f(r, \varphi) = \frac{1}{2} \int_0^{2\pi} \frac{1}{U^2} \int_{-\infty}^{\infty} \frac{D}{\sqrt{D^2 + t^2}} (w_p(t, \beta) g(t, \beta)) h(\hat{t} - t) \mathrm{d}t \mathrm{d}\beta$$

Less than a short scan? (4)

- Apply Parker weight $w_p(t, \beta)$ in reconstruction formular:

$$f(r, \varphi) = \frac{1}{2} \int_0^{2\pi} \frac{1}{U^2} \int_{-\infty}^{\infty} \frac{D}{\sqrt{D^2 + t^2}} (w_p(t, \beta) g(t, \beta)) h(\hat{t} - t) \mathrm{d}t \mathrm{d}\beta$$

- Not possible to pull $w_p(t, \beta)$ after the convolution without introducing artifacts

Less than a short scan? (4)

- Apply Parker weight $w_p(t, \beta)$ in reconstruction formular:

$$f(r, \varphi) = \frac{1}{2} \int_0^{2\pi} \frac{1}{U^2} \int_{-\infty}^{\infty} \frac{D}{\sqrt{D^2 + t^2}} (w_p(t, \beta) g(t, \beta)) h(\hat{t} - t) \mathrm{d}t \mathrm{d}\beta$$

- Not possible to pull $w_p(t, \beta)$ after the convolution without introducing artifacts
- There is a solution that can solve this problem, but it won't be covered in this class

Less than a short scan? (4)

- Apply Parker weight $w_p(t, \beta)$ in reconstruction formular:

$$f(r, \varphi) = \frac{1}{2} \int_0^{2\pi} \frac{1}{U^2} \int_{-\infty}^{\infty} \frac{D}{\sqrt{D^2 + t^2}} (w_p(t, \beta) g(t, \beta)) h(\hat{t} - t) \mathrm{d}t \mathrm{d}\beta$$

- Not possible to pull $w_p(t, \beta)$ after the convolution without introducing artifacts
- There is a solution that can solve this problem, but it won't be covered in this class
- This reconstruction method is known as "super-short-scan"

Further Readings

- Gengsheng Lawrence “Larry” Zeng.“Medical Image Reconstruction – A Conceptual Tutorial”. Springer 2009
- Ronald N. Bracewell.“The Fourier Transform and Its Applications”. McGraw-Hill Publishing Company. 1999
- Dennis Parker. “Optimal short scan convolution reconstruction for fanbeam CT”. Medical Physics. 9(2): 254-257. 1982
- Frederic Noo, Michel Defrise, Rolf Clackdoyle, Hiroyuki Kudo. “Image reconstruction from fan-beam projections on less than a short scan”. Physics in Medicine and Biology 47: 2525-2546. 2002

Questions?