Diagnostic Medical Image Processing
Reconstruction – Parallel Beam Reconstruction: Practical Aspects

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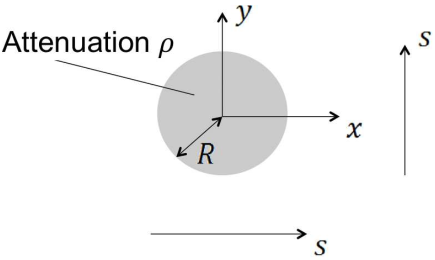
Topics

How to Implement a Parallel Beam Algorithm

On Noise, Filtering and Window Functions

Sinograms

Example: Homogeneous Cylinder (My First Phantom)



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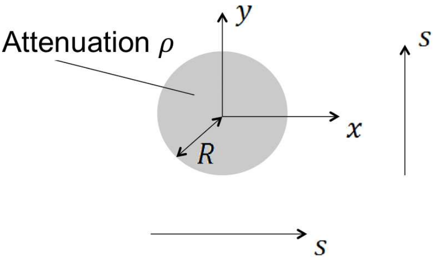
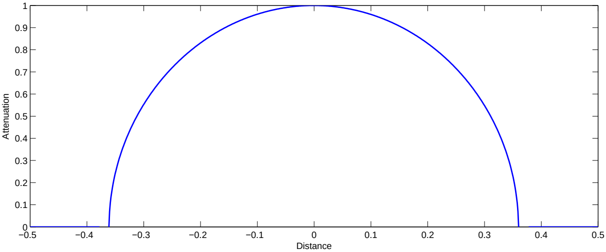


- Disc is in the center $\rightarrow$ Projection is the same in all views:

$$p(s) = \begin{cases} 2\rho\sqrt{R^2 - s^2} & s \leq R \\ 0 & s > R \end{cases}$$

Example: Homogeneous Cylinder (2)



Filtered Backprojection – Practical Algorithm

- Apply Filter on the detector row:

$$q(s,\theta) = h(s) * p(s,\theta)$$

$$h(s) = \int_{-\infty}^{\infty} |\omega| e^{2\pi i\omega s} \mathrm{d}\omega$$

- Backproject $q(s,\theta)$:

$$f(x,y) = \int_{0}^{\pi} q(s,\theta)|_{s=x\cos\theta+y\sin\theta} \mathrm{d}\theta$$

Discrete Spatial Form of the Ramp Filter

- Find the inverse Fourier transform of $|\omega|$
- Set cut-off frequency of the ramp filter at $\omega = \frac{1}{2}$

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$$h(s) = \frac{1}{2} \quad \frac{\sin \pi s}{\pi s} - \frac{1}{4} \left[\frac{\sin\left(\frac{\pi s}{2}\right)}{\frac{\pi s}{2}} \right]^2$$

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- Convert to discrete form: Let $s = n$ (integer)

$$h(n) = \begin{cases} \frac{1}{4} & n = 0 \\ 0 & n \text{ even} \\ -\frac{1}{n^2\pi^2} & n \text{ odd} \end{cases}$$

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- Also known as the “Ramachandran-Lakshminarayanan” convolver or “Ram-Lak” convolver

Discrete Spatial Form of the Ramp Filter (3)

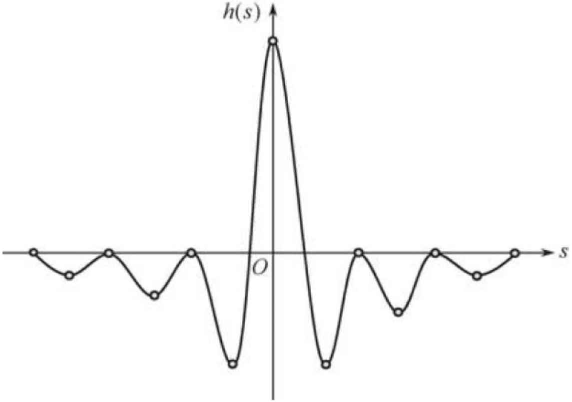


Image: Zeng, 2009

Discrete Spatial vs. Continuous Frequency Version

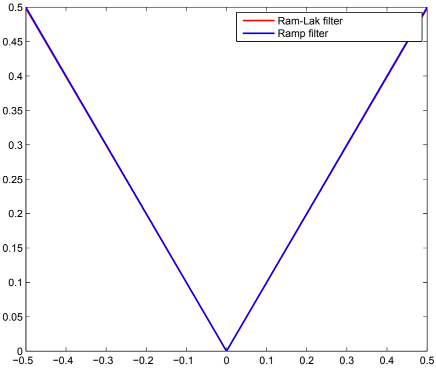
- Continuous frequency representation of the ramp filter:

$$H(\omega) = |\omega|$$

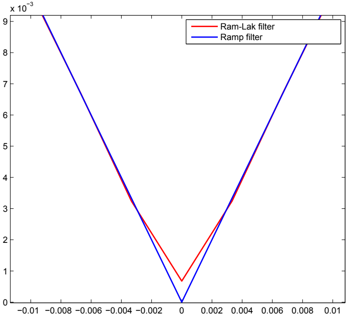
- Discrete spatial form:

$$h(n) = \begin{cases} \frac{1}{4} & n = 0 \\ 0 & n \text{ even} \\ -\frac{1}{n^2\pi^2} & n \text{ odd} \end{cases}$$

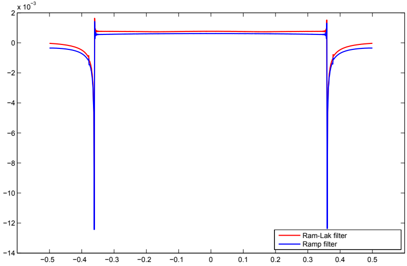
Discrete Spatial vs. Continuous Frequency Version (2)



Discrete Spatial vs. Continuous Frequency Version (3)



Example: Homogeneous Cylinder after Filter



Practical Algorithm - Filtering

- Precompute filter h(s) in spatial domain - $O(N)$

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  - Apply filter $P(\omega, \theta) \cdot H(\omega) - O(N)$

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  - Apply filter $P(\omega, \theta) \cdot H(\omega) - O(N)$
  - Compute filtered projection $q(s)$ via iFFT - $O(N \log N)$
- Total complexity:

$$O(N + N \log N + \#P(N + 2N \log N)) = O(\#P\ N \log N)$$

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- For each of $N \times N$ pixels:
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    - Compute $s = x \cos\theta + y \sin\theta - O(1)$

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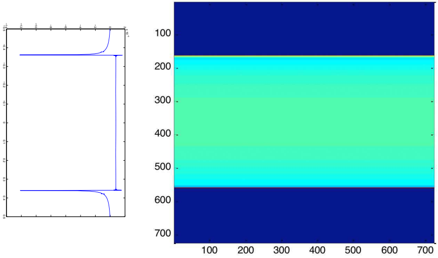
    - Update $f(x, y) + = q(s, \theta) - O(1)$

Practical Algorithm - Backprojection

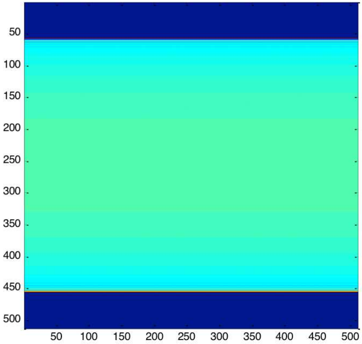
- Initialize $f(x, y) = 0 - O(N^2)$
- For each of $N \times N$ pixels:
  - For each of $\#P$ projections:
    - Compute $s = x\cos\theta + y\sin\theta - O(1)$
    - Update $f(x, y) += q(s, \theta) - O(1)$
- Total complexity:

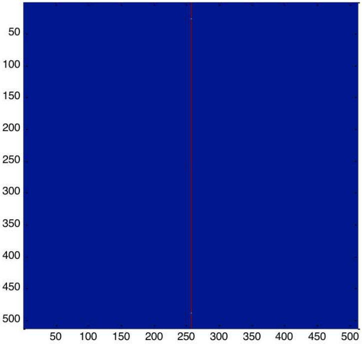
$$O(N^2 + N^2\#P(1 + 1)) = O(N^2\#P)$$

Backprojection Example

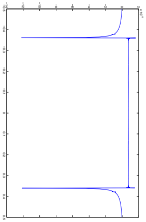


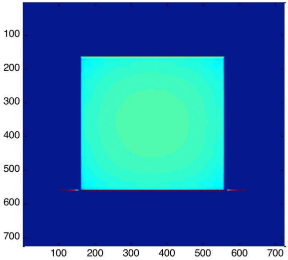
Backprojection and Fourier Slice Theorem

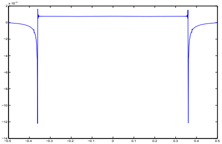




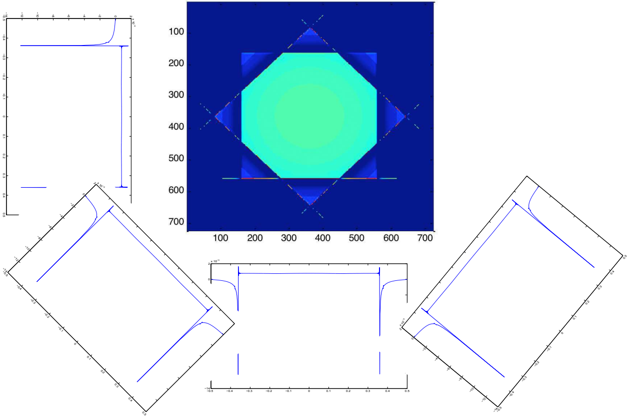
Backprojection Example







Backprojection Example



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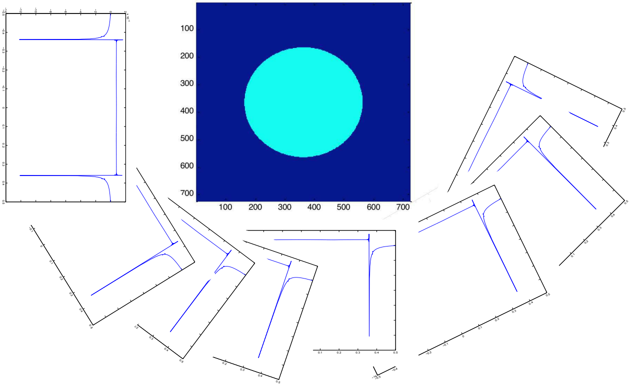

Filtered Backprojection – Practical Algorithm

- Apply Filter on the detector row:

$$O(\#P\ N \log N)$$

- Backproject:

$$O(\#P\ N^2)$$

Topics

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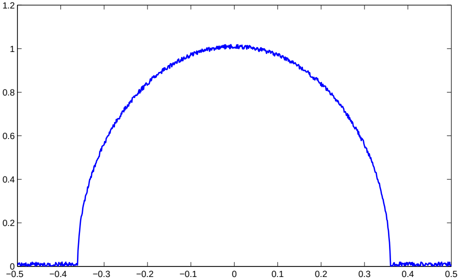
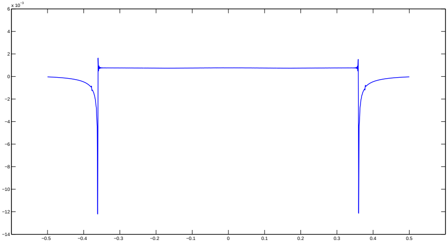
Sinograms

Additive Noise (+2%)



Additive Noise – After Filtering



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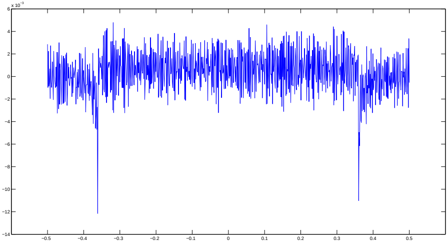

Window Functions

- Window functions are used to improve signals
  - High frequencies are reduced
    - $\rightarrow$ Noise reduction
    - $\rightarrow$ Reduces high frequencies caused by cutting
  - Many window functions are known:
    - Cosine Window
    - Shepp-Logan Window
    - ...

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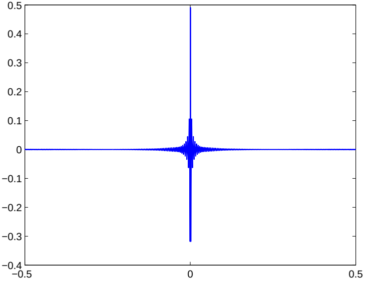
- Rewrite filtering equation to adjusted filter:

$$Q'(\omega, \theta) = H'(\omega) \cdot P(\omega, \theta)$$

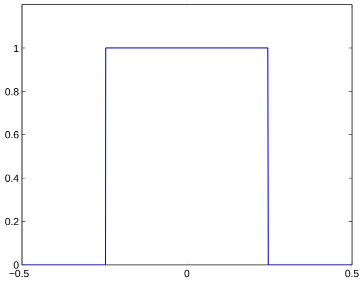
$$H'(\omega) = H(\omega) \cdot W(\omega)$$

Rectangular Window

Spatial Domain

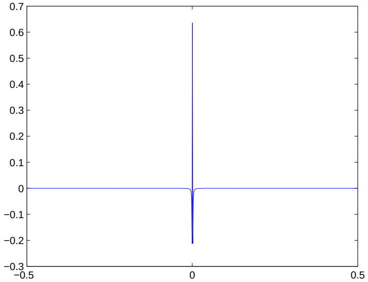


Frequency Domain

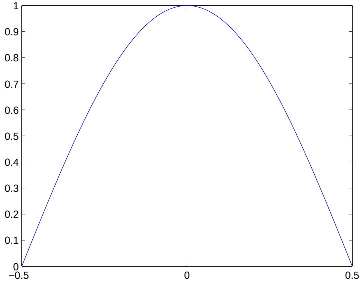


Cosine Window – $\cos(\pi \cdot x)$

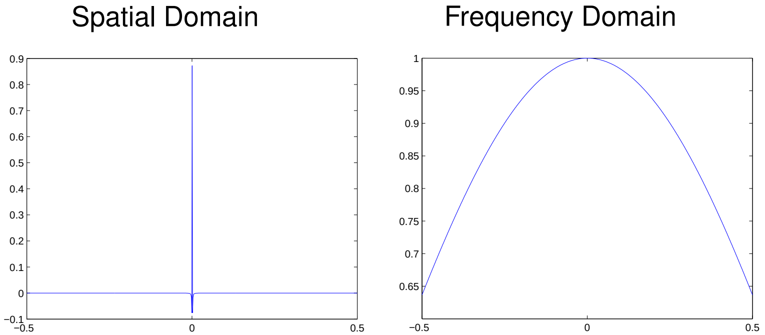
Spatial Domain



Frequency Domain

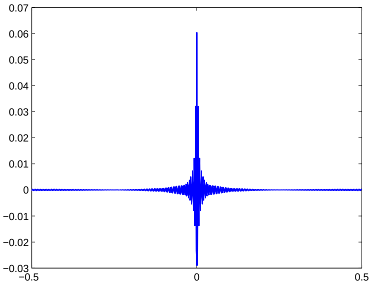


Shepp-Logan Window – $\dfrac{\sin(\pi \cdot x)}{(\pi \cdot x)}$

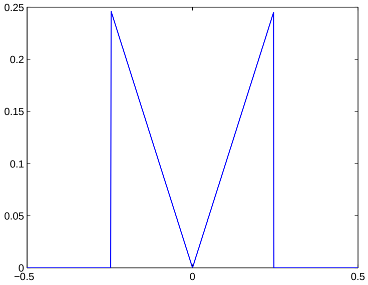


Rectangular Filter

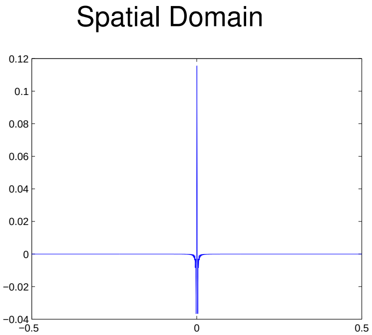
Spatial Domain

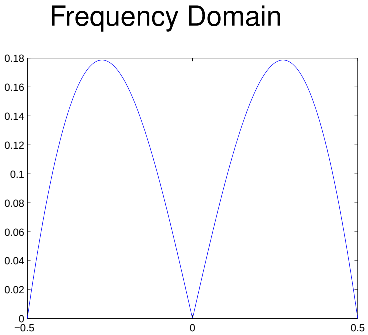


Frequency Domain



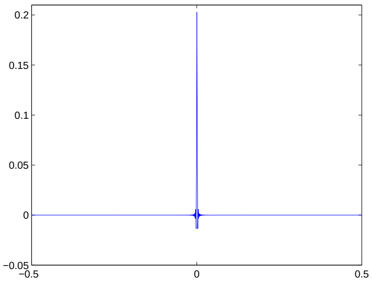
Cosine Filter



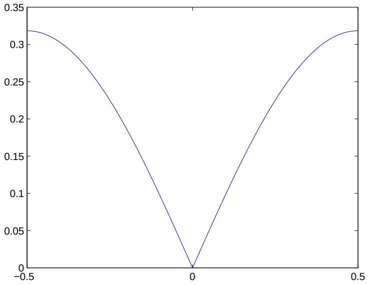


Shepp-Logan Filter

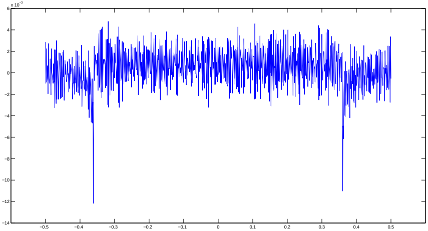
Spatial Domain



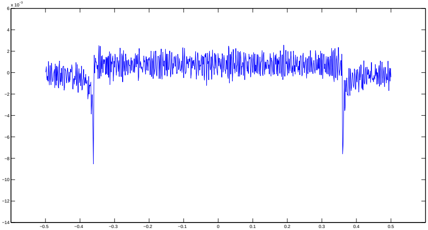
Frequency Domain



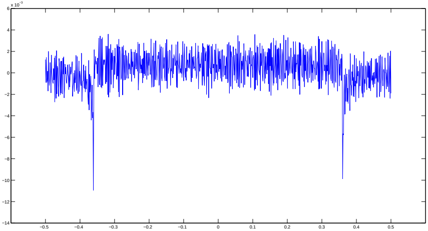
Ramp Filter Result



Shepp-Logan Filter Result



Cosine Filter Result



Noise

- Is amplified by ramp filter
- Has to be taken care of in an appropriate manner
- Is indirectly proportional to the applied dose
- Affects different reconstruction methods differently

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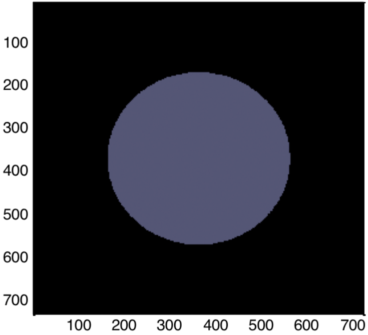
Sinograms

Sinogram

- Method to visualize all projections in one image
- Contains all information to reconstruct one slice
- Also called ‘Fanogram” in fan beam geometry
- Popular method for visualization with narrow detectors

Sinogram (2)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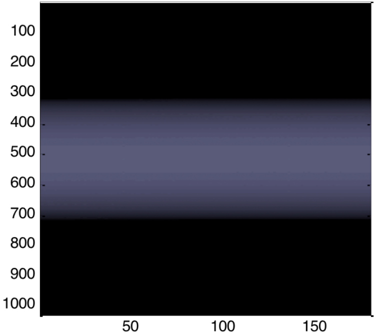
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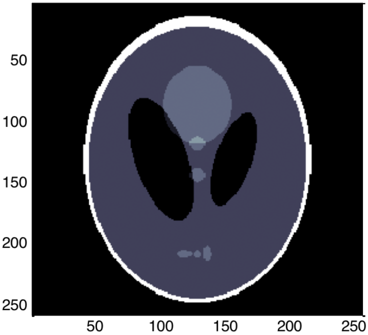
Sinogram (3)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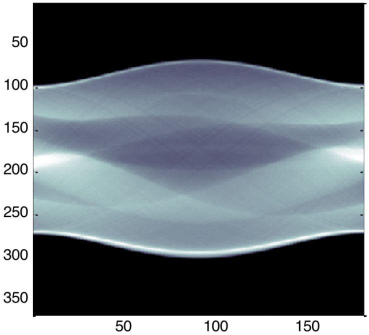
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Further Readings

- Gengsheng Lawrence “Larry” Zeng. "Medical Image Reconstruction – A Conceptual Tutorial". Springer 2009
- Ronald N. Bracewell. "The Fourier Transform and Its Applications". McGraw-Hill Publishing Company. 1999
- Thorsten M. Buzug. "Computed Tomography: From Photon Statistics to Modern Cone-Beam CT". Springer 2008

Questions?