Diagnostic Medical Image Processing Reconstruction – Parallel Beam Reconstruction

WS 2015/2016

Andreas Maier, Joachim Hornegger, Markus Kowarschik
Pattern Recognition Lab (CS 5)

FRIEDRICH-ALEXANDER UNIVERSITÄT ERLANGEN-NÜRNBERG
TECHNISCHE FAKULTÄT

FRIEDRICH-ALEXANDER UNIVERSITÄT ERLANGEN-NÜRNBERG
TECHNISCHE FAKULTÄT

Topics

Important Methods

Central Slice Theorem

Filtered Backprojection

Filtering Revisited

Differentiated Backprojection

Fourier Transform

- 1D Fourier transform:

$$P(\omega) = \int_{-\infty}^{\infty} p(s) e^{-2\pi i s \omega} \mathrm{d}s$$

- 1D inverse Fourier transform:

$$p(s) = \int_{-\infty}^{\infty} P(\omega) e^{2\pi i s \omega} \mathrm{d}\omega$$

- Fourier pairs

Convolution

- Convolution:

$$(f * g)(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)\mathrm{d}\tau = \int_{-\infty}^{\infty} f(t-\tau)g(\tau)\mathrm{d}\tau$$

- Convolution theorem:

$$q(s) = f(s) * g(s)$$

$$Q(\omega) = F(\omega) \cdot G(\omega)$$

Hilbert Transform

- Spatial representation:

$$(f * h)(t) \quad = \quad \text{p.v.} \int_{-\infty}^{\infty} f(t-\tau)h(\tau)\mathrm{d}\tau$$

$$h(\tau) \quad = \quad \frac{1}{\pi\tau}$$

- Fourier representation:

$$H(\omega) = -i \,\mathrm{sgn}(\omega)$$

Topics

Important Methods

Central Slice Theorem

Filtered Backprojection

Filtering Revisited

Differentiated Backprojection

Central Slice Theorem

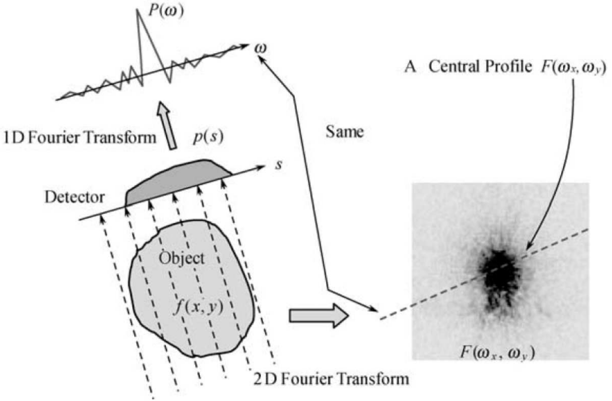

Image: Zeng, 2009

Central Slice Theorem (2)

$$P(\omega, \theta) = F(\omega \cos\theta, \omega \sin\theta)$$

Idea for Reconstruction

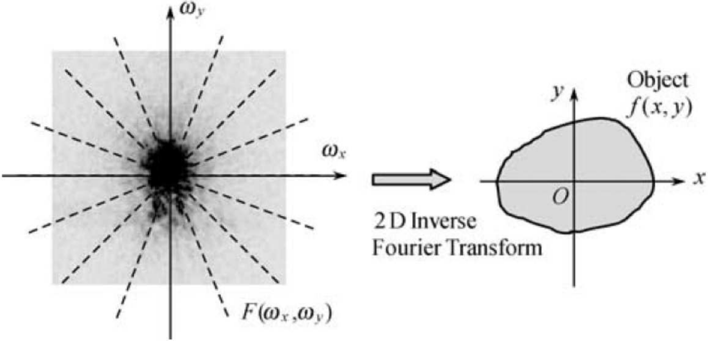

Image: Zeng, 2009

Topics

Important Methods

Central Slice Theorem

Filtered Backprojection

Filtering Revisited

Differentiated Backprojection

Filtered Backprojection

- Fourier transform in polar coordinates:

$$f(x, y) = \int_0^{2\pi} \int_0^{\infty} F_{\text{polar}}(\omega, \theta) e^{2\pi i\omega(x\cos\theta + y\sin\theta)} \omega \mathrm{d}\omega \mathrm{d}\theta$$

Filtered Backprojection – Practical Algorithm

- Apply Filter on the detector row:

$$q(s,\theta) = h(s) * p(s,\theta)$$

- Backproject $q(s,\theta)$:

$$f(x,y) = \int_0^\pi q(s,\theta)|_{s=x\cos\theta+y\sin\theta}\mathrm{d}\theta$$

Topics

Important Methods

Central Slice Theorem

Filtered Backprojection

Filtering Revisited

Differentiated Backprojection

Filtering Revisited

- Rewrite $|\omega|$:

$$|\omega| = (2\pi i\omega) \cdot \left\{ \frac{1}{2\pi} [-i \operatorname{sgn}(\omega)] \right\}$$

Filtering Revisited

- Rewrite $|\omega|$:

$$|\omega| = (2\pi i\omega) \cdot \left\{ \frac{1}{2\pi} [-i\,\mathrm{sgn}(\omega)] \right\}$$

- Note that multiplication in frequency space with $-i\,\mathrm{sgn}(\omega)$ is a Hilbert transform, i.e. equivalent to a convolution with $\frac{1}{\pi s}$

Filtering Revisited

- Rewrite $|\omega|$:

$$|\omega| = (2\pi i\omega) \cdot \left\{ \frac{1}{2\pi} [-i \operatorname{sgn}(\omega)] \right\}$$

- Note that multiplication in frequency space with $-i \operatorname{sgn}(\omega)$ is a Hilbert transform, i.e. equivalent to a convolution with $\frac{1}{\pi s}$
- Note that the inverse Fourier of $2\pi i\omega$ is the derivative operator

Differentiation Hilbert Backprojection Algorithm

- Compute first derivative of the detector row:

$$q_1(s,\theta) = \frac{\partial p(s,\theta)}{\partial s}$$

- Apply Hilbert Transform:

$$q_2(s,\theta) = \frac{1}{2\pi^2 s} * q_1(s,\theta)$$

- Backproject $q_2(s,\theta)$:

$$f(x,y) = \int_0^\pi q_2(s,\theta)|_{s=x\cos\theta+y\sin\theta} \mathrm{d}\theta$$

Topics

Important Methods

Central Slice Theorem

Filtered Backprojection

Filtering Revisited

Differentiated Backprojection

Differentiated Backprojection

- Definition of the backprojection:

$$b(x, y) = \int_0^{\pi} \mathbf{H}p(s, \theta)|_{s=x\cos\theta + y\sin\theta} \mathrm{d}\theta$$

- **H** is the Hilbert transform with respect to *s*

Differentiated Backprojection (2)

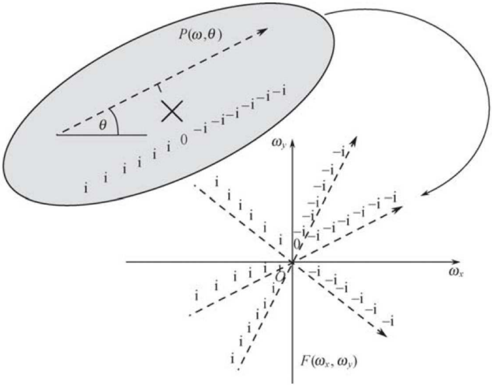

Image: Zeng, 2009

Many reconstruction algorithms are possible ...

Method	Step 1	Step 2	Step 3
	1D Ramp filter with Fourier transform	Backprojection	
	1D Ramp filter with convolution	Backprojection	
	Backprojection	2D Ramp filter with Fourier transform	
	Backprojection	2D Ramp filter with 2D convolution	
	Derivative	Hilbert transform	Backprojection
	Derivative	Backprojection	Hilbert transform
	Backprojection	Derivative	Hilbert transform
	Hilbert transform	Derivative	Backprojection
	Hilbert transform	Backprojection	Derivative
	Backprojection	Hilbert transform	Derivative

Image: Zeng, 2009

Further Readings

- Gengsheng Lawrence “Larry” Zeng. "Medical Image Reconstruction – A Conceptual Tutorial". Springer 2009
- Ronald N. Bracewell. "The Fourier Transform and Its Applications". McGraw-Hill Publishing Company. 1999

Questions?