Diagnostic Medical Image Processing Reconstruction – 3D Reconstruction

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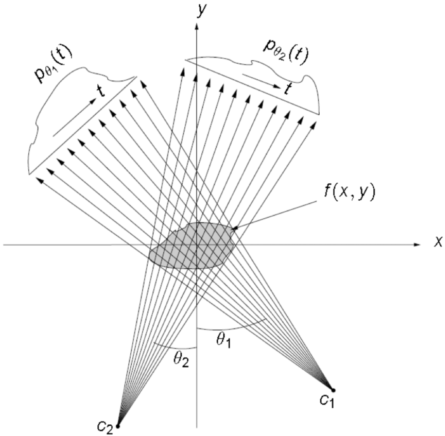
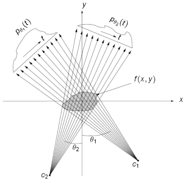
Reprise: Fan Beam Reconstruction

3D Reconstruction

Fan Beam Geometry



Fan Beam Geometry



- Fan beam Scanners became available in 1975 (20s / slice)
- Fast rotations became possible 1987 with slip rings (300ms / slice)

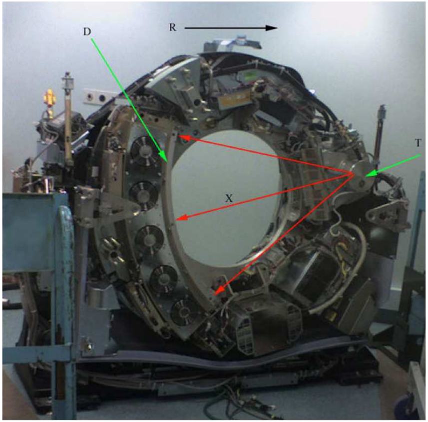


Image: Wikipedia

Reprise: Fan Beam Reconstruction

3D Reconstruction

- Parallel Line-Integral Data
- Parallel Plane-Integral Data
- Cone Beam Data
- Feldkamp's Algorithm
- Grangeat's Algorithm
- Katsevich's Algorithm

Topics

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Parallel Line-Integral Data

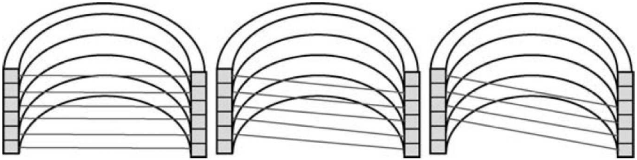

Image: Zeng, 2009

- Line-integrals may be along rays that are not perpendicular to the axis of rotation
- How does this affect our reconstruction?

Central Slice Theorem for Line-Integral Data

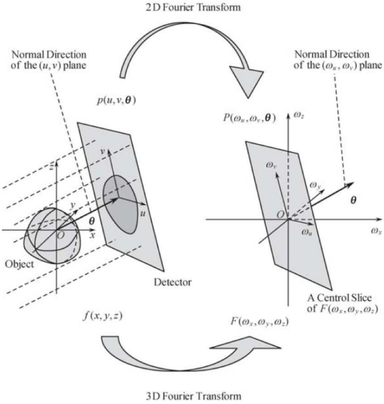

Image: Zeng, 2009

Parallel Line-Integral Data (2)

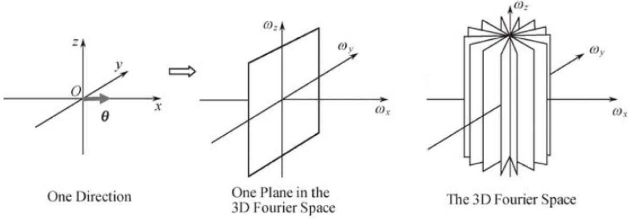

Image: Zeng, 2009

- Parallel projections on rays that are perpendicular to the rotation axis can fill the complete Fourier space

Parallel Line-Integral Data (3)

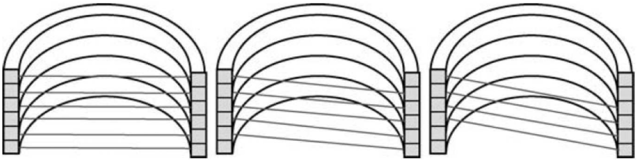

Image: Zeng, 2009

Fourier Space:

Parallel Line-Integral Data (4)

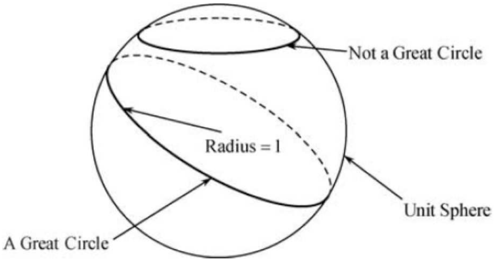

Image: Zeng, 2009

- Orlov's condition:
  A complete data set can be obtained, if every great circle intersects the trajectory

Parallel Line-Integral Data (5)

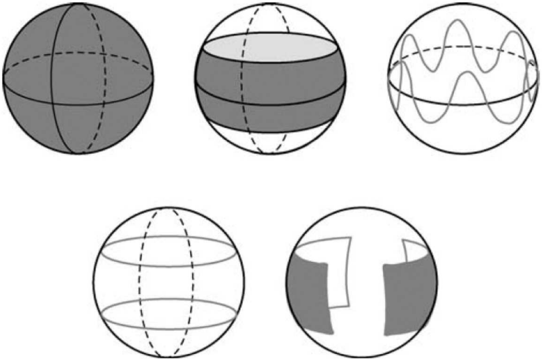

Image: Zeng, 2009

Parallel Line-Integral Data (6)

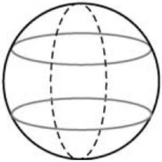

Image: Zeng, 2009

Fourier Space:

Parallel Line-Integral Data (7)

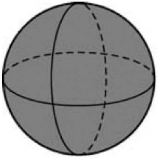

Image: Zeng, 2009

Fourier Space:

Parallel Line-Integral Data (8)

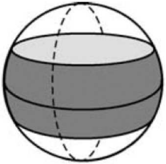

Image: Zeng, 2009

Fourier Space:

Parallel Line-Integral Data – BPF Algorithm

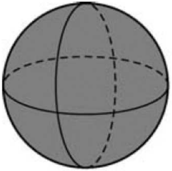

Image: Zeng, 2009

- For the unit sphere the PSF can be shown to be

$$h(x, y, z) = \frac{1}{x^2 + y^2 + z^2} = \frac{1}{r^2}$$

Parallel Line-Integral Data – BPF Algorithm

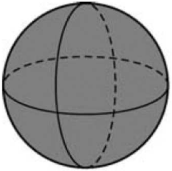

Image: Zeng, 2009

- For the unit sphere the PSF can be shown to be

$$h(x, y, z) = \frac{1}{x^2 + y^2 + z^2} = \frac{1}{r^2}$$

- The back projection of the data is computed by

$$b = f *** h$$

$$B(\omega_x, \omega_y, \omega_z) = F(\omega_x, \omega_y, \omega_z) \cdot H(\omega_x, \omega_y, \omega_z)$$

Parallel Line-Integral Data – BPF Algorithm

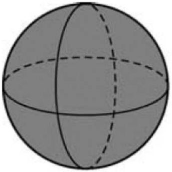

Image: Zeng, 2009

- With

$$H(\omega_x, \omega_y, \omega_z) = \frac{\pi}{\sqrt{(\omega_x^2, \omega_y^2, \omega_z^2)}}$$

$$F(\omega_x, \omega_y, \omega_z) = B(\omega_x, \omega_y, \omega_z) \frac{\sqrt{\omega_x^2, \omega_y^2, \omega_z^2}}{\pi}$$

Parallel Line-Integral Data – BPF Algorithm (2)

$$F(\omega_x, \omega_y, \omega_z) = B(\omega_x, \omega_y, \omega_z) \frac{\sqrt{\omega_x^2, \omega_y^2, \omega_z^2}}{\pi}$$

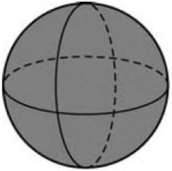

Image: Zeng, 2009

- Another version of a Backprojection-Then-Filtering-Algorithm
- This “trajectory” has heavy redundancy
- Depending on how the redundancy is weighted, different reconstructions emerge
- PSF and therewith the filter kernel is dependent on the imaging geometry

Parallel Line-Integral Data – FBP Algorithm

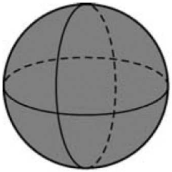

Image: Zeng, 2009

- When filtering is performed in projection domain, a 2D filter is applied:

$$g(\boldsymbol{\theta}, u, v) = p(\boldsymbol{\theta}, u, v) ** h(\boldsymbol{\theta}, u, v)$$
$$Q(\boldsymbol{\theta}, \omega_u, \omega_v) = P(\boldsymbol{\theta}, \omega_u, \omega_v) \cdot H(\boldsymbol{\theta}, \omega_u, \omega_v)$$

- This filter $H_{\boldsymbol{\theta}}(\omega_u, \omega_v)$ is often view-dependent
- For the unit sphere the filter is view-independent:

$$H(\omega_u, \omega_v) = \frac{\sqrt{\omega_u^2 + \omega_v^2}}{\pi}$$

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Parallel Plane-Integral Data

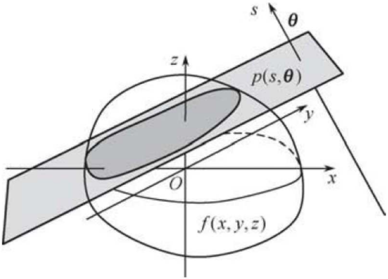

Image: Zeng, 2009

- 3D Radon transform
- Is not equal to the x-ray transform!

Parallel Plane-Integral Data (2)

- We do not have direct detectors for this transform
- We can compute the value of such a "detector" as a line-integral on a 2D parallel beam detector
- There is a central slice theorem for this transform
- Backprojection is different, as a point has to be backprojected as a plane in 3D

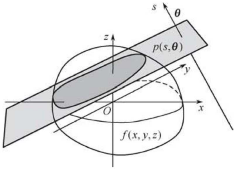

Image: Zeng, 2009

Parallel Plane-Integral Data (3)

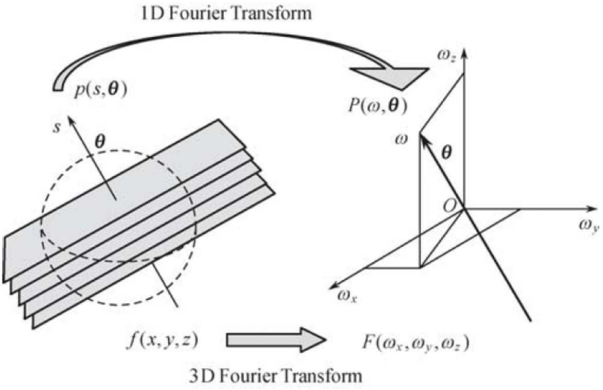

Image: Zeng, 2009

Parallel Plane-Integral Data – Backprojection

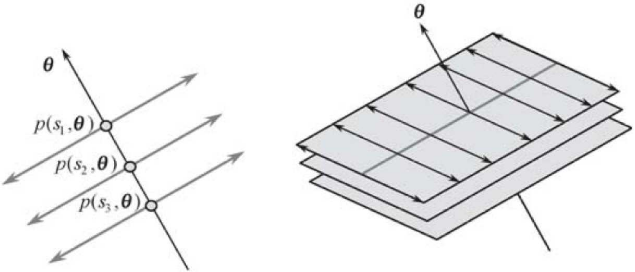

Image: Zeng, 2009

3D Radon Inversion Formula

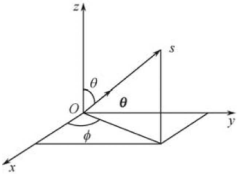

Image: Zeng, 2009

$$f(x, y, z) = -\frac{1}{8\pi^2} \iint\limits_{4\pi} \frac{\partial^2 p(s, \boldsymbol{\theta})}{\partial s^2}\Big|_{s=\boldsymbol{x} \cdot \boldsymbol{\theta}} \sin\theta \mathrm{d}\theta \mathrm{d}\phi$$

$$\boldsymbol{\theta} = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}$$

Backprojection-Then-Filtering Algorithm

- The backprojection of the image is obtained as

$$b(x, y, z) = \iint\limits_{2\pi} p(s, \boldsymbol{\theta})\big|_{s=\boldsymbol{x}\cdot\boldsymbol{\theta}} \sin\theta \mathrm{d}\theta \mathrm{d}\phi$$

Backprojection-Then-Filtering Algorithm

- The backprojection of the image is obtained as

$$b(x, y, z) = \iint_{2\pi} p(s, \boldsymbol{\theta})\big|_{s=\boldsymbol{x} \cdot \boldsymbol{\theta}} \sin\theta \mathrm{d}\theta \mathrm{d}\phi$$

- For the 3D Radon case, the PSF is given as

$$H(\omega_x, \omega_y, \omega_z) = \frac{1}{\omega_x^2 + \omega_y^2 + \omega_z^2}$$

Backprojection-Then-Filtering Algorithm

- The backprojection of the image is obtained as

$$b(x,y,z) = \iint_{2\pi} p(s,\boldsymbol{\theta})\big|_{s=\boldsymbol{x}\cdot\boldsymbol{\theta}} \sin\theta \mathrm{d}\theta \mathrm{d}\phi$$

- For the 3D Radon case, the PSF is given as

$$H(\omega_x,\omega_y,\omega_z) = \frac{1}{\omega_x^2+\omega_y^2+\omega_z^2}$$

- Hence the image can be obtained by

$$F(\omega_x,\omega_y,\omega_z) = B(\omega_x,\omega_y,\omega_z)\cdot(\omega_x^2+\omega_y^2+\omega_z^2)$$

- In spatial domain this can be written as

$$f(x,y,z) = \triangle b(x,y,z) = \frac{\partial^2 b(x,y,z)}{\partial x^2} + \frac{\partial^2 b(x,y,z)}{\partial y^2} + \frac{\partial^2 b(x,y,z)}{\partial z^2}$$

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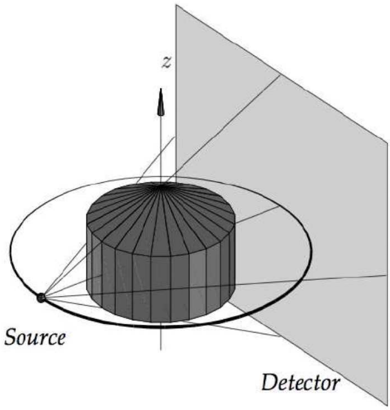
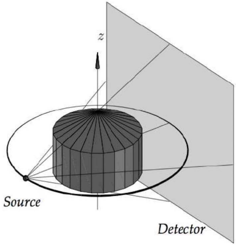
Cone Beam Geometry



Cone Beam Geometry (2)

- Flat panel or multi-row detectors can detect considerably more data at a time
- This geometry is very popular
- Projection and backprojection can be described using projection matrices
- As in fan-beam geometry, we do not have a central slice theorem
- Data sufficiency conditions are different than in 3D parallel beam geometry



Cone Beam Geometry – Data Sufficiency

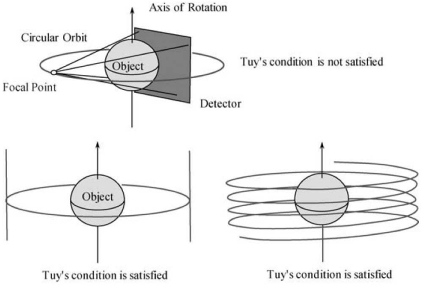


Image: Zeng, 2009

- Tuy's condition:
  Every plane that intersects the object of interest must contain a cone beam focal point

Cone Beam Geometry – Data Sufficiency (2)

- How to obtain a helical trajectory?

- How to make a circular scan complete?

Cone Beam Geometry – Data Sufficiency (3)

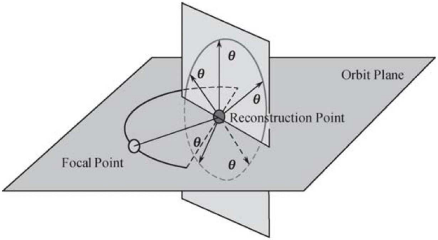


Image: Zeng, 2009

- Consider plane-integrals along θ for a reconstruction point in the plane of rotation
- All angles are observed within the plane that is perpendicular to the viewing direction

Cone Beam Geometry – Data Sufficiency (4)

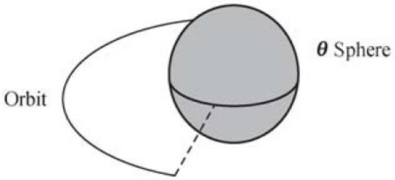


Image: Zeng, 2009

- If this is repeated for every point on the orbit, a full sphere will be sampled
- Hence, data for this point is complete
- We refer to this as a π-segment

Cone Beam Geometry – Data Sufficiency (5)

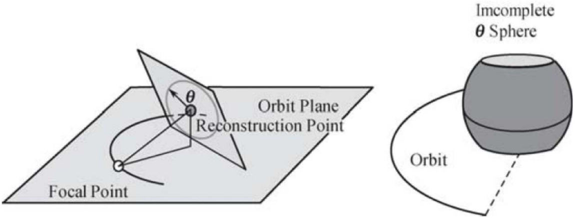


Image: Zeng, 2009

- On points above and below the orbit plane, there is missing data
- The reconstruction will contain artifacts
- The higher the angle to the reconstruction point, the more artifact will appear

Cone Beam Geometry – Data Redundancy

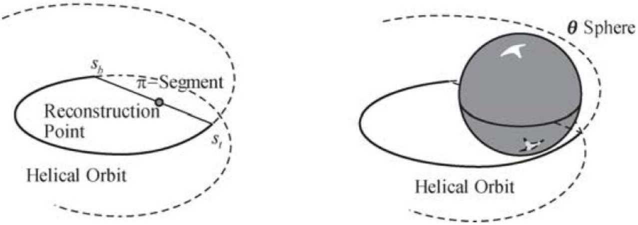


Image: Zeng, 2009

- A helical orbit will contain redundant observations

Cone Beam Geometry – Data Redundancy (2)

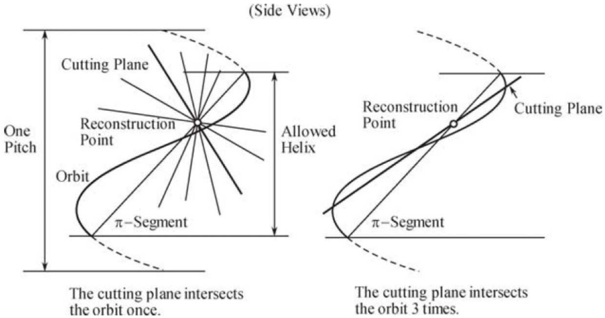


Image: Zeng, 2009

- Redundant observations will occur on cutting planes that hit the helix more than once

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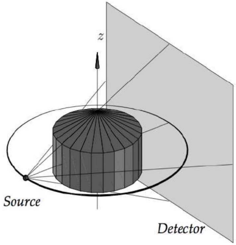
Feldkamp's Algorithm

- Is also known as the Feldkamp, Davis, Kress (FDK) algorithm named after the author of the original publication
- Is designed for circular trajectories and is thus approximate
- Is a commonly used cone beam algorithm as it is fast and robust
- Is based on a fan beam algorithm with appropriate cosine weights
- Is exact for objects that do not vary in z-direction



Feldkamp's Algorithm – Geometry

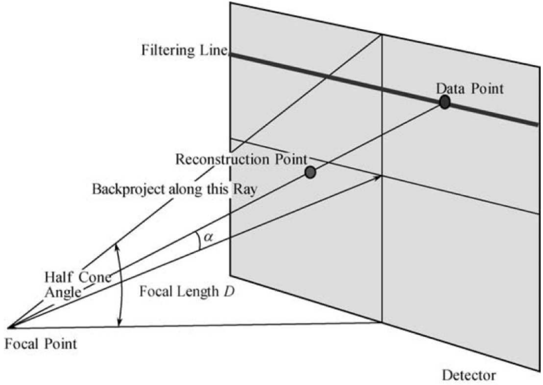


Image: Zeng, 2009

Fan Beam vs. Cone Beam

- FBP for fan beam data:

$$f(r,\varphi) = \frac{1}{2}\int_0^{2\pi} \frac{1}{U^2}\int_{-\infty}^{\infty} \frac{D}{\sqrt{D^2+t^2}} g(t,\beta)\cdot h(\hat{t}-t)\,\mathrm{d}t\,\mathrm{d}\beta$$

Fan Beam vs. Cone Beam

- FBP for fan beam data:

$$f(r,\varphi) = \frac{1}{2}\int_0^{2\pi} \frac{1}{U^2} \int_{-\infty}^{\infty} \frac{D}{\sqrt{D^2+t^2}} g(t,\beta) \cdot h(\hat{t}-t)\mathrm{d}t\mathrm{d}\beta$$

- Feldkamp's algorithm for cone beam data:

$$f(r,\varphi,z) = \frac{1}{2}\int_0^{2\pi} \frac{1}{U^2} \int_{-\infty}^{\infty} \frac{D}{\sqrt{D^2+t^2+u^2}} g(t,\beta) \cdot h(\hat{t}-t)\mathrm{d}t\mathrm{d}\beta$$

Feldkamp's Algorithm (2)

- Perform adjusted cosine weighting:

$$g_1(t,u,\beta) = g(t,u,\beta)\frac{D}{\sqrt{D^2+t^2+u^2}}$$

- Apply ramp filter for each detector row:

$$g_2(t,u,\beta) = g_1(t,u,\beta) * h(t)$$

- Backproject with distance weight:

$$f(r,\varphi,z) = \frac{1}{2}\int_0^{2\pi} \frac{1}{U^2} g_2(\hat{t},\hat{u},\beta)\mathrm{d}\beta$$

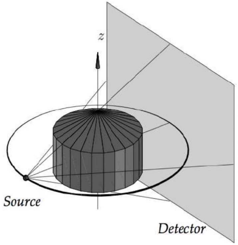
Feldkamp's Algorithm (3)

- Filtering is row-wise, hence its complexity is $O(N * N * \log N)$
- Backprojection is in 3D $\Rightarrow O(N * N * N)$
- In C-arm CT, projection is performed using projection matrices
- This processing speed can be improved if Horner's scheme is applied
- Backprojection is often implemented on special hardware

Topics

Grangeat's Algorithm

- Converts the cone-beam problem to a 3D Radon inversion problem
- Can provide exact reconstructions, if Tuy's condition is met
- Uses the idea to convert cone beam ray sums to plane-integrals



Grangeat's Algorithm (2)

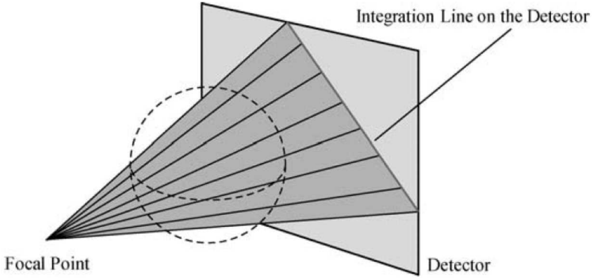


Image: Zeng, 2009

- Line-integral on a cone beam detector is a weighted plane-integral
- The line integral hast to be weighted with $\frac{1}{r}$ to get the regular unweighted plane-integral

Grangeat's Algorithm (3)

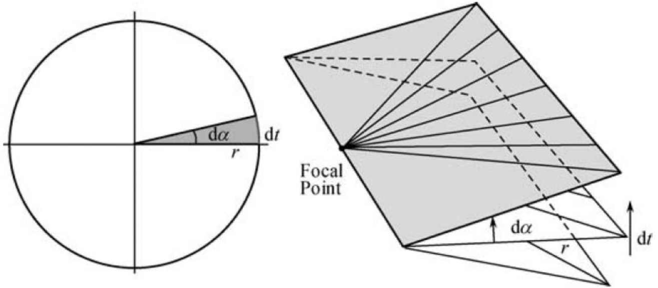
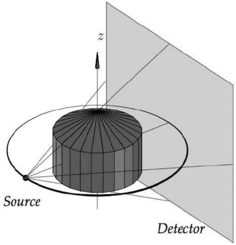


Image: Zeng, 2009

- the derivative along the tangential direction $\mathrm{d}t$ is equal to the derivative of a $\frac{1}{r}$ weighted plane integral along $\mathrm{d}\alpha$

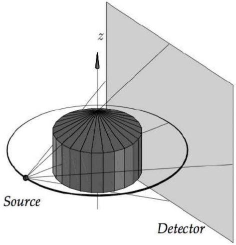
Grangeat's Algorithm (4)

- Form all possible line-integrals on each detector plane (all locations and orientations)
- Perform the angular derivative
- Rebin the data to Radon space
- Take the derivative with respect to *t*
- Perform the 3D Radon backprojection



Grangeat's Algorithm (5)

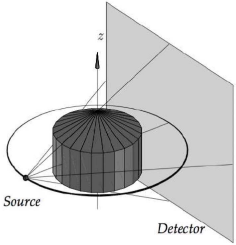
- Is not a filtered backprojection algorithm
- Requires interpolation on non-uniformly sampled data
- Needs to handle redundancy correctly (Divide by the number of redundant observations)
- Not commonly used for reconstruction
- Very useful to look at reconstruction problems



Topics

Katsevich's Algorithm

- First developed for helical trajectories
- Later expanded to more general orbits
- Is in the form FBP
- Filtering can be made shift-invariant, i.e. independent on the reconstruction location



Katsevich's Algorithm (2)

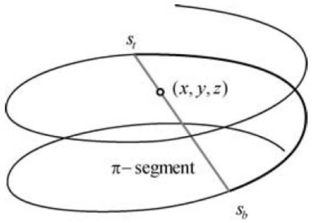


Image: Zeng, 2009

- For all points inside the helix there is one line that passes the point and hits the helix at two points that are separated by less than one pitch
- This line is called a π-line or π-segment

Katsevich's Algorithm (3)

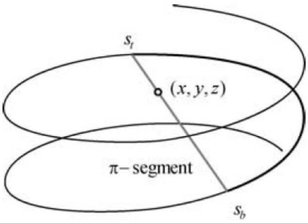


Image: Zeng, 2009

- If redundant data occurs, it occurs three times
- Redundancy is solve by assigning the weights:

$$1, -1, 1$$

Katsevich's Algorithm (4)

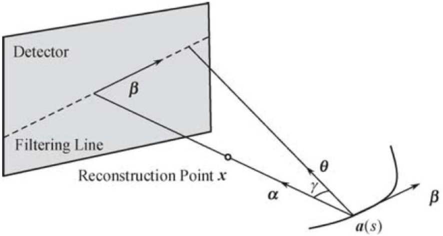


Image: Zeng, 2009

- Perform derivative along the teajectory $\boldsymbol{a}(s)$
- Filter along direction β

Katsevich's Algorithm (5)

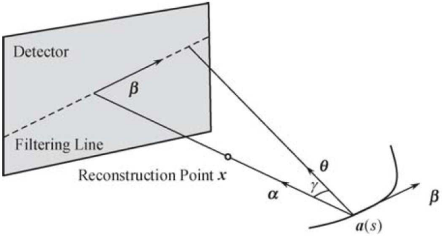


Image: Zeng, 2009

- Katsevich reconstruction formula:

$$f(\boldsymbol{x}) = \frac{-1}{2\pi^2} \int_{I_\pi(x)} \frac{1}{||\boldsymbol{x} - \boldsymbol{a}(s)||} \int_{-\pi/2}^{\pi/2} \frac{\partial g(\boldsymbol{\theta}(\gamma), \boldsymbol{a}(q))}{\partial q} |_{q=s} \frac{1}{\sin\gamma} \mathrm{d}\gamma \mathrm{d}s$$

Katsevich's Algorithm (6)

- Katsevich reconstruction formula:

$$f(\boldsymbol{x}) = \frac{-1}{2\pi^2} \int_{I_\pi(x)} \frac{1}{||\boldsymbol{x} - \boldsymbol{a}(s)||} \int_{-\pi/2}^{\pi/2} \frac{\partial g(\boldsymbol{\theta}(\gamma), \boldsymbol{a}(q))}{\partial q}\Big|_{q=s} \frac{1}{\sin\gamma} \mathrm{d}\gamma \mathrm{d}s$$

- Algorithm
  - Compute derivative along $\boldsymbol{a}(q)$
  - Perform weighting
  - Perform Hilbert transform along β
  - Perform cone beam backprojection with distance weighting

Katsevich's Algorithm (7)

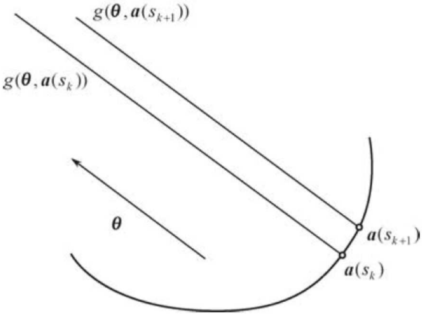


Image: Zeng, 2009

- Compute derivative along the trajectory as discrete difference between two neighboring projections

Katsevich's Algorithm (8)

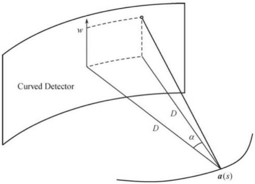


Image: Zeng, 2009

- Perform weighting of the projection data with

$$\frac{D}{\sqrt{D^2 + w^2}}$$

where D is the source detector distance and w the axis of the detector that points along the rotation axis

Katsevich's Algorithm (9)

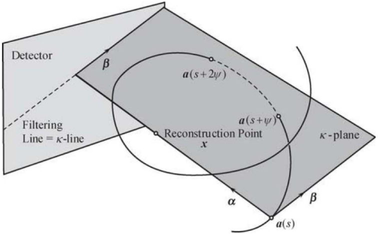


Image: Zeng, 2009

- Perform derivative along the trajectory $\boldsymbol{a}(s)$
- Choose β as the angle of a plane κ that contains the points $\boldsymbol{x}, \boldsymbol{a}(s), \boldsymbol{a}(s+\psi)$, and $\boldsymbol{a}(s+2\psi)$
- This is not unique. Hence, choose $|\psi|$ to be minimal

Katsevich's Algorithm (10)

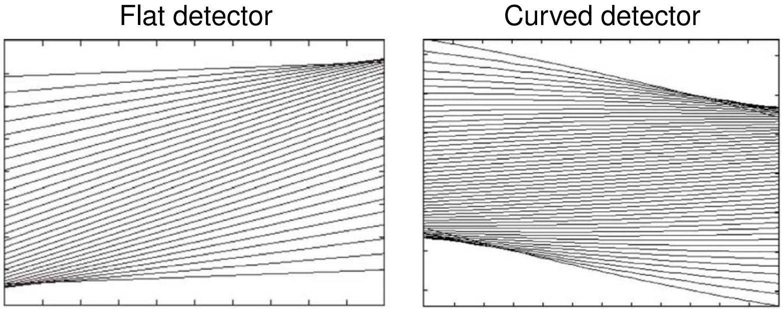


Image: Zeng, 2009

- The projection of a κ-plane onto the detector is called a κ-line
- The orientation changes with the reconstruction point

Katsevich's Algorithm (11)

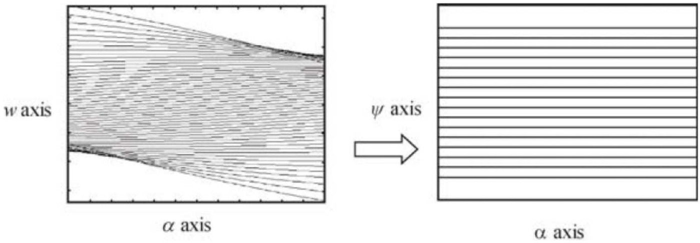


Image: Zeng, 2009

- Rebin the data to parallel lines to prepare for the Hilbert transform
- Perform the 1D Hilbert transform along each line

Katsevich's Algorithm (12)

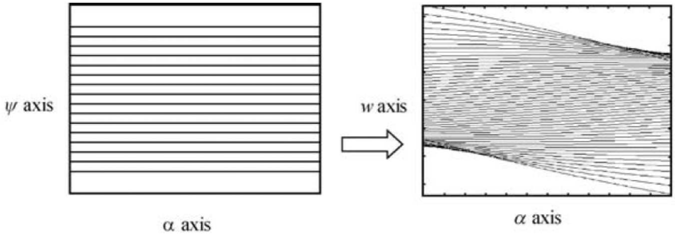


Image: Zeng, 2009

- Invert the rebinding to go back to detector coordinates
- Perform remaining weighting with cos α
- Perform cone beam backprojection with distance weighting for all voxels $\boldsymbol{x}$

Katsevich's Algorithm (13)

- Katsevich's algorithm performs an exact reconstruction
- The algorithms causes some additional computational overhead compared to other algorithms as the FDK algorithm
- Katsevich's algorithm does not use all available data
- Katsevich's algorithm deletes redundant data ($1, -1, 1$ weighting)

Exact vs. Approximate Reconstruction

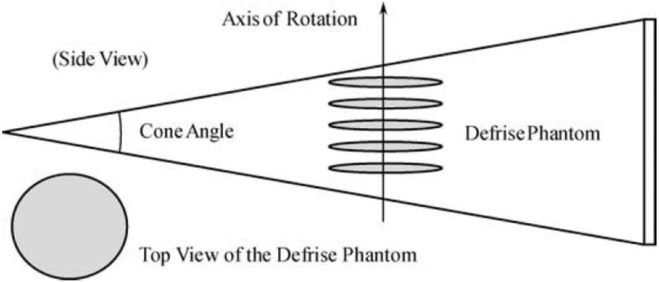


Image: Zeng, 2009

- The Defrise phantom is often used to investigate cone beam artifact

Exact vs. Approximate Reconstruction (2)

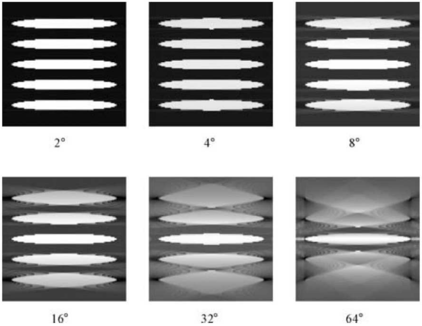


Image: Zeng, 2009

- The higher the cone angel the more cone artifact will appear

Exact vs. Approximate Reconstruction (3)

- The higher the cone angle, the more exact methods benefit
- Helical CT Scanners usually have rather small cone angles (due to image artifacts)
- Flat panel scanners usually have circular trajectories that allow only approximate reconstruction

$\Rightarrow$ Only few exact methods are used in clinical practice

3D Reconstruction

- Cone beam geometry allows a much faster data acquisition
- Approximate methods allow robust reconstruction
- Exact reconstruction provides artifact-free reconstruction, if the data is complete
- Cone beam geometry suffers from physical effects such as scatter much more than fan beam geometries

4D Reconstruction?

- There are methods to model even more dimensions such as time
- This makes the reconstruction problem even more difficult
- Approaches are:
  - Fast scanning
  - Motion gating (regular motion)
  - Motion estimation
- All these methods are subject to current research

Further Readings

- Gengsheng Lawrence “Larry” Zeng.“Medical Image Reconstruction – A Conceptual Tutorial”. Springer 2009

Questions?