Differential Equations

Overview:

1. First Order Equation : $\underbrace{\frac{dy}{dx} = ay + q(t)}_{\text{linear eq}^n}$ ↗ i/p $\frac{dy}{dt} = f(y)$ ↙ could be anything (Non-linear)

2. Second Order Equation : $\frac{d^2y}{dx^2} = -ky \rightarrow$ linear eq^n

* $\frac{dy}{dx}$ = Slope of the curve ; $\frac{d^2y}{dx^2}$ = Bending of the curve.

$$\text{Eg. } my'' + by' + ky = f(t)$$

GOOD EQUATION

Linear Constant Coefficients

(If m, b = linear const) ⇒ Nice fun^n $f(t)$ → Nice fun^n $y(t)$

3. System of n equation , $\frac{dy}{dt} = Ay$, $\frac{d^2y}{dt^2} = -Sy$

$y \to [\ \ ]_{n\times 1}$, $A \to [\ \ ]_{n\times n}$

* THE CALCULUS WE NEED

1. Basic derivatives. Eg x^n, $\sin x$, $\cos x$, e^x, $\ln x$.
2. Rules for $f+g$, $f(x)\cdot g(x)$, $\frac{f(x)}{g(x)}$, $f(g(x))$
3. Fundamental. Theoram : $\frac{d}{dx}\int_0^x y(t)\,dt = y(x)$

* Charts attached.

Derivative Rules

Exponential Functions

$$\frac{d}{dx}(e^x) = e^x$$
$$\frac{d}{dx}(a^x) = a^x \ln a$$
$$\frac{d}{dx}(e^{g(x)}) = e^{g(x)} g'(x)$$
$$\frac{d}{dx}(a^{g(x)}) = \ln(a) a^{g(x)} g'(x)$$

Logarithmic Functions

$$\frac{d}{dx}(\ln x) = \frac{1}{x}, x > 0$$
$$\frac{d}{dx}\ln(g(x)) = \frac{g'(x)}{g(x)}$$
$$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}, x > 0$$
$$\frac{d}{dx}(\log_a g(x)) = \frac{g'(x)}{g(x) \ln a}$$

Trigonometric Functions

$$\frac{d}{dx}(\sin x) = \cos x$$
$$\frac{d}{dx}(\cos x) = -\sin x$$
$$\frac{d}{dx}(\tan x) = \sec^2 x$$
$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$
$$\frac{d}{dx}(\sec x) = \sec x \tan x$$
$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

Inverse Trigonometric Functions

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}, x \neq \pm 1$$
$$\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}, x \neq \pm 1$$
$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$
$$\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$$
$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}, x \neq \pm 1, 0$$
$$\frac{d}{dx}(\csc^{-1} x) = \frac{-1}{x\sqrt{x^2-1}}, x \neq \pm 1, 0$$

Hyperbolic Functions

$$\frac{d}{dx}(\sinh x) = \cosh x$$
$$\frac{d}{dx}(\cosh x) = \sinh x$$
$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$$
$$\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$$
$$\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$
$$\frac{d}{dx}(\coth x) = -\operatorname{csch} x$$

Inverse Hyperbolic Functions

$$\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}}$$
$$\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}}, x > 1$$
$$\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2}, |x| < 1$$
$$\frac{d}{dx}(\operatorname{csch}^{-1} x) = \frac{-1}{|x|\sqrt{1-x^2}}, x \neq 0$$
$$\frac{d}{dx}(\operatorname{sech}^{-1} x) = \frac{-1}{x\sqrt{1-x^2}}, 0 < x < 1$$
$$\frac{d}{dx}(\coth^{-1} x) = \frac{1}{1-x^2}, |x| > 1$$

Basic Derivatives Rules

Constant Rule: $\frac{d}{dx}(c) = 0$

Constant Multiple Rule: $\frac{d}{dx}[cf(x)] = cf'(x)$

Power Rule: $\frac{d}{dx}(x^n) = nx^{n-1}$

Sum Rule: $\frac{d}{dx}[f(x)+g(x)] = f'(x)+g'(x)$

Difference Rule: $\frac{d}{dx}[f(x)-g(x)] = f'(x)-g'(x)$

Product Rule: $\frac{d}{dx}[f(x)g(x)] = f(x)g'(x)+g(x)f'(x)$

Quotient Rule: $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x)-f(x)g'(x)}{[g(x)]^2}$

Chain Rule: $\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$

9 $y(t) = \int_0^t e^{t-s} q(s)\, ds$ Solves $\frac{dy}{dt} = y + q(t)$

$$\Rightarrow y(t) = e^t \int_0^t e^{-s} q(s)\, ds$$

$$\frac{dy}{dt} = \underbrace{e^t \int_0^t e^{-s} q(s)\, ds}_{} + e^t \cdot [e^{-t} q(t)]$$

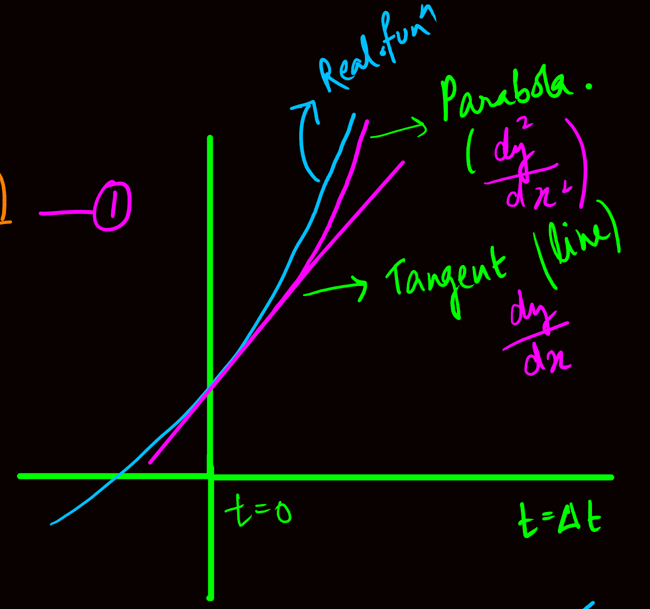
$$= y + q(t)$$

// Proved.

④ Tangent Line to the graph

$$f(t+\Delta t) \approx f(t) + \Delta t \frac{d f(t)}{dt} \quad \text{— ①}$$

$$\therefore \frac{\Delta f}{\Delta t} \approx \frac{df}{dt}$$



Taylor's Series

$$f(t+\Delta t) \overset{*}{=} f(t) + \Delta t \frac{d f(t)}{dt} + \frac{1}{2!}(\Delta t)^2 \frac{d^2 f}{dt^2} \ldots$$

$$\ldots\ldots + \frac{1}{n!}(\Delta t)^n \frac{d^n f}{dt^n} + \ldots \quad \text{— ②}$$

As we go towards higher derivatives we can recreate the fun^n.

$\{\therefore \approx$ in ① can be $=$ in eq^n ② $*\}$

RESPONSE TO EXPONENTIAL INPUT.

$$\frac{dy}{dt} = ay + e^{st} \qquad y(t) = \text{" Exponential Response"}$$

$$y = y(0) \text{ at } t=0.$$

In this case the particular solution $y_p = Ye^{st}$ — ①

$$\frac{d(Ye^{st})}{dt} = aYe^{st} + e^{st}$$

$$\Rightarrow Ys\cancel{e^{st}} = aY\cancel{e^{st}} + \cancel{e^{st}}$$

$$\Rightarrow (s-a)Y = 1$$

$$\Rightarrow Y = \frac{1}{s-a}.$$

↙ from ① (Particular Sol^n)

$$y(t) = \frac{e^{st}}{s-a} + \text{Null Sol}^n.$$

Null sol^n i.e $q(t) = e^{st} = 0$

$$\therefore \frac{dy}{dt} = ay$$

Sol^n: $y(t) = Ce^{at}$

$$y(t) = \underbrace{\frac{e^{st}}{s-a}}_{y_p} + \underbrace{Ce^{at}}_{y_h} \quad \text{— ②}$$

and. $y(0) = \frac{1}{s-a} + C$

$$C = y(0) - \frac{1}{(s-a)} \quad \text{— ③}$$

$\therefore$ From ② & ③ we get,

$$y(t) = \frac{e^{st}}{(s-a)} + \left[y(0) - \frac{1}{(s-a)}\right]e^{at}$$

$$y(t) = \boxed{\frac{e^{st} - e^{at}}{(s-a)}} + y(0)e^{at} = y_{vp} + y_N$$

= 0/0 form at s = a

another particular sol^n

(G.S calls this a very particular solution)

* Point of Resonance at s=a

$$y(t)_{s=a} = te^{at} + y(0)e^{at}$$

*

* L'Hospital Rule (0/0). $= \dfrac{\frac{d}{ds}(e^{st} - e^{at})}{\frac{d}{ds}(s-a)} = \dfrac{te^{st} - 0}{1} = te^{at}$

RESPONSE TO OSCILLATING INPUT

$$\frac{dy}{dt} = ay + \cos \omega t \quad , \quad y = y(0) \text{ at } t = 0$$

Here Particular Solⁿ is given by,

$$\boxed{y_p(t) = M\cos \omega t + N\sin \omega t} \rightarrow \text{Form 1}$$

$$\Rightarrow -M\omega \sin \omega t + N\omega \cos \omega t = aM\cos \omega t + aN\sin \omega t + \cos \omega t$$

$$\Rightarrow \underbrace{-aM + \omega N = 1}_{\textcircled{1}} \quad \& \quad -\omega M = aN \text{ --- } \textcircled{2}$$

From ① & ② we get,

$$M = \frac{-a}{\omega^2 + a^2} \quad \& \quad N = \frac{\omega}{\omega^2 + a^2}.$$

Also,

Another form of $\boxed{y(t) = G\cos(\omega t - \alpha)} \leftarrow$ Particular Solⁿ.

$$= G(\cos \omega t \cos \alpha + \sin \omega t \sin \alpha) \text{ --- } \textcircled{3}$$

Here $M = G\cos\alpha$, $N = G\sin\alpha$ — ④

$$\therefore M^2 + N^2 = G^2 \overbrace{(\cos^2\alpha + \sin^2\alpha)}^{=1}$$

$$G = \sqrt{M^2 + N^2}$$

Now, To find α

from eqⁿ ④ $\quad \dfrac{M}{N} = \dfrac{G\cos\alpha}{G\sin\alpha}$

$\Rightarrow \tan\alpha = \frac{N}{M}$

SOLUTION FOR ANY INPUT.

$$\frac{dy}{dt} = ay + q(t) \quad . \quad \text{Start at } y(0)$$

{starting deposit}

{interest added} {new deposits}

$$y(t) = \underbrace{y(0)\,e^{at}}_{\text{Null sol}^n} + \underbrace{\int_0^t e^{a(t-s)} q(s)\,ds}_{\text{Particular Sol}^n}$$

{any instant deposit}

Null soln — only initial deposit, $q(t) = 0$

{interest or growth in the remaining time t}.

* Special $q(t) = c$
$= e^{st}$
$= \cos\omega t$ or $\sin\omega t$
$=$ Step funn
$=$ Delta funn

} To be covered.

* $$y(t) = y(0)e^{at} + \int_0^t e^{a(t-s)} q(s)\,ds$$

Check if the formula is correct;

Now,

$$y_p(t) = e^{at} \int_{s=0}^{s=t} e^{-as} q(s)\,ds$$

$$\frac{dy_p(t)}{dt} = \int_{s=0}^{s=t} e^{-as} q(s)\,ds \,.\, e^{at}.a - \cancel{e^{at}} . \cancel{e^{-at}} q(t)$$

{By Fundamental Theoram of Calculas}

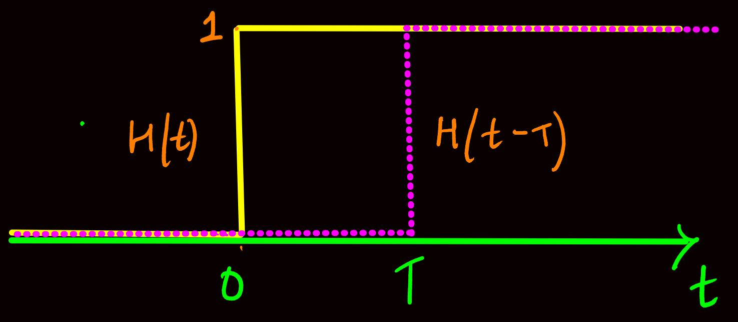
$$\Rightarrow \frac{dy_p}{dt} = ay + q(t)$$

STEP FUNCTION & DELTA FUNCTION

① Step Function $\{H(t)\}$

Step Funⁿ

$$H(t) = \begin{cases} 0 & t<0 \\ 1 & t \geq 0. \end{cases}$$



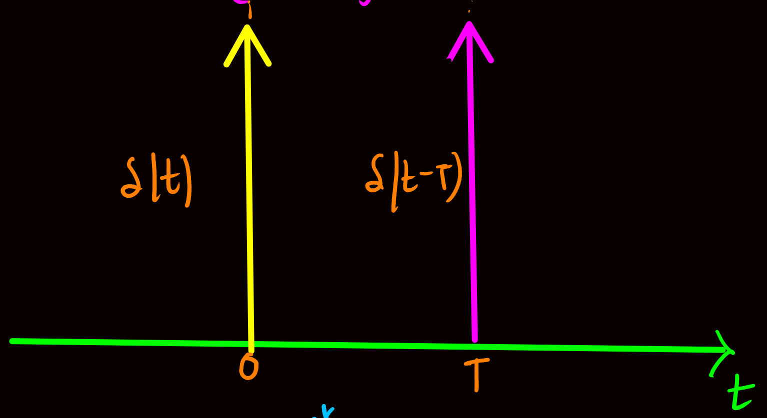
$\frac{dH}{dt} = 0$ at $t=0$ $\frac{dH}{dt} = 0$

$$\frac{dH}{dt} = \delta(t) = \infty$$

Similarly for $H(t-T)$.

$$\left.\frac{d(H)}{dt}\right|_{at\ t=T} = \delta(t-T) = \infty$$

② Delta Function $\{\delta(t)\}$



$$\int_{-\infty}^{\infty} \delta(t)\,dt = H(t)\Big|_{t=-\infty}^{t=\infty} = 1 - 0$$

$$\boxed{\int_{-\infty}^{\infty} \delta(t)\,dt = 1}$$

only exists at $t=0$

$$* \int_{-\infty}^{\infty} \delta(t) f(t)\,dt = f(0)$$

$$* \int_{-\infty}^{\infty} \delta(t) \sin t\,dt = \sin 0 = 0$$

* $\int_{-\infty}^{\infty} \delta(t) e^{t} dt = e^{0} = 1$ * $\int_{-\infty}^{\infty} \delta(t-T) e^{t} dt = e^{T}$

9. $\frac{dy}{dt} = ay + \delta(t-T)$; $y(0) = 0$

$$y(t) = \begin{cases} 0 & \text{upto } t = T \\ e^{a(t-T)} & t \geq T \end{cases}$$

RESPONSE TO COMPLEX EXPONENTIAL

$$\frac{dy}{dt} = ay_c + \cos\omega t + i\sin\omega t$$

$$= ay_c + e^{i\omega t} \quad \{\text{Euler's formula}\}$$

$\therefore y_c(t) = Ye^{i\omega t}$

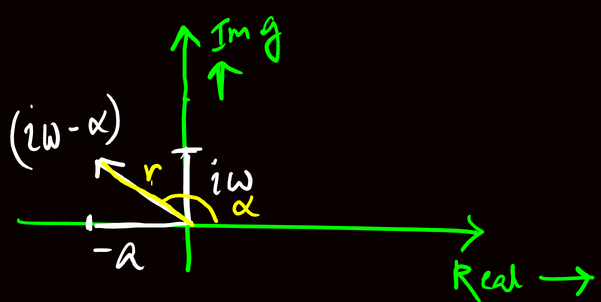
$$\therefore \frac{d(Ye^{i\omega t})}{dt} = aYe^{i\omega t} + e^{i\omega t}$$

$$\Rightarrow i\omega Y\cancel{e^{i\omega t}} = aY\cancel{e^{i\omega t}} + \cancel{e^{i\omega t}}$$

$$\Rightarrow (i\omega - a)Y = 1$$

$$\Rightarrow Y = \frac{1}{(i\omega - a)}$$

Now, $i\omega - a = re^{i\alpha}$



$$i\omega - a = \sqrt{a^2+\omega^2}\, e^{i\alpha}$$

$$\therefore\ y_c(t) = \frac{1}{i\omega - a} e^{i\omega t} = \boxed{\frac{1}{\sqrt{a^2+\omega^2}} e^{-i\alpha}} e^{i\omega t}$$

Y

$$= \frac{1}{\sqrt{a^2+\omega^2}} e^{i(\omega t - \alpha)}$$

Real Part $y(t) = \dfrac{1}{\sqrt{a^2+\omega^2}} \cos(\omega t - \alpha) = G\cos(\omega t - \alpha)$

{ Done Previously }

INTEGRATING FACTOR FOR A CONSTANT RATE

$\dfrac{dy}{dt} = a(t)y + q(t)$. Solve by integrating factor M

$\dfrac{dM}{dt} = -a(t)M$ $\quad a = \text{const} \longrightarrow M = e^{-at}$

$\quad a = \text{varying} \longrightarrow M = e^{-\int_0^t a(T)\,dT}$

$$\frac{d}{dt}(M.y) = \frac{dy}{dt}.M + y.\frac{dM}{dt} = \frac{dy}{dt}.M - a(t)My = M\left[\frac{dy}{dt} - a(t)y\right]$$

$$= Mq$$

Integrating both sides we get

$$M(t)y(t) - \overset{1}{M(0)}\,y(0) = \int_0^t M(s)q(s)\,ds$$

$$\Rightarrow M(t)y(t) = y(0) + \int_0^t M(s)q(s)\,ds$$

$$\Rightarrow y(t) = e^{\int_0^t a(T)\,dT} y(0) + \int_0^t e^{t^2-s^2} q(s)\,ds.$$

$$\frac{M(s)}{M(t)} = e^{\int_0^t a} \cdot e^{-\int_0^s a} = e^{\int_s^t a}$$

$*e^{t^2}$

when $a = 2t$ *

$$\frac{dy}{dt} = 2ty + q(t)$$ Solve by integrating factor $I(t)$

Sol^n Here $a(t) = 2t$

$$I(t) = e^{-t^2} \quad \because \frac{dI}{dt} = -2t\,I(t) \quad \{ I(0) = e^0 = 1 \}$$

$$\frac{d(I.y)}{dt} = I.\frac{dy}{dt} + y.\frac{dI}{dt} = I(\cancel{2ty} + q(t)) + \cancel{-2tI(t).y}$$

$$\frac{d(Iy)}{dt} = I q(t)$$

$$\Rightarrow I(t).y(t) - \underbrace{I(0)}_{=1} y(0) = \int_{s=0}^{s=t} e^{-s^2} q(s)\,ds.$$

$$\Rightarrow I(t).y(t) - y(0) = \int_0^t e^{-s^2} q(s)\,ds \qquad \{ I(t) = e^{-t^2}$$

$$\Rightarrow y(t) = y(0).e^{t^2} + \int_0^t e^{t^2-s^2} q(s)\,ds$$

General Rule for choosing Integrating factor is:

$$I(t) = e^{-\int_0^t a(T)\,dT} \qquad \{ a(T) = \text{const or variable}$$