Part 2

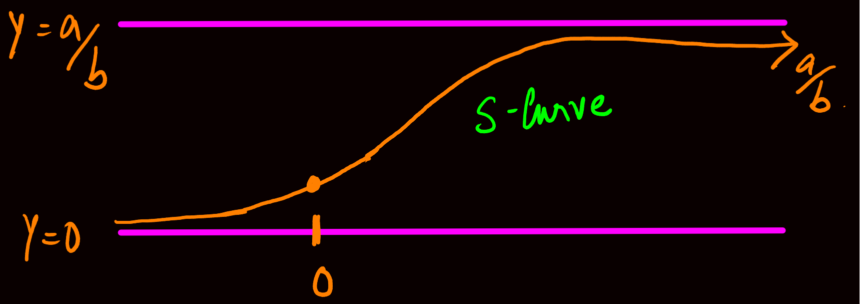
Logistic Equation (Population Growth)

$$\frac{dy}{dt} = ay - by^2$$

$$z = \frac{1}{y}$$

$$\frac{dz}{dt} = -1\, y^{-2} \frac{dy}{dt} = \frac{-1}{y^2}\left(ay - by^2\right) = -az + b$$

$$y = \frac{1}{z} = \frac{a}{de^{-at} + b}$$



at $t \to \infty$ $\quad y(t) = \frac{a}{b}$

at $t \to 0$ $\quad y(t) = 0$

Steady States

$$\frac{dy}{dt} = ay - by^2 = f(y)$$

Value at which $\frac{dy}{dt} = 0$ $\quad$ { If I start at 0 I stay at zero }
{ If I " " a/b " " a/b }

$$aY - bY^2 = 0$$

$$Y(a - bY) = 0$$

$$Y = 0, \; Y = a/b$$

$Y = 0$: unstable { We leave that steady state }

$Y = a/b$: stable { We approach that steady state }

Stability & Instability of Steady States

$\frac{dy}{dt} = f(y) \rightarrow$ Linear or Non Linear.

$f(Y) = 0 \rightarrow$ at steady state $\left[\frac{dy}{dt} = 0 \right.$ ∴ slope is 0 & we don't go anywhere$)$

$\ast\; y = Y$ forward time

eg Examples

① $\frac{dy}{dt} = ay$

For steady state, $Y = 0$

② $\frac{dy}{dt} = y - y^2$ (Logistic eq^n)

For steady state

$y - y^2 = 0 \Rightarrow y(1-y) = 0$

$\Rightarrow y = 0, 1$

$\therefore Y = 0$ or 1

③ $\frac{dy}{dt} = y - y^3$

For Steady State,

$y - y^3 = 0$

$Y = 0, 1, -1$

STABLE OR NOT

eg $\frac{dy}{dt} = at$

Sol^n to this eq^n $y(t) = e^{at}$

For sol^n to approach steady state $a < 0$

But for $y - y^2$ or $y - y^3$ how do we find the stable state?

Look at the derivative

$\frac{df^*}{dy}$ at $y = Y$ $\begin{cases} \text{If } \frac{df}{dy} < 0 \text{ then stable} \\ \text{otherwise, unstable} \end{cases}$

Ex. $\frac{dy}{dt} = y - y^2 = f(y)$

$\therefore \frac{df}{dy} = 1 - 2y \Big|_{Y=0,1}$

at $Y = 0 \Rightarrow \frac{df}{dy} = 1 > 0$

$Y = 1 \Rightarrow \frac{df}{dy} = -1 < 0$

$\therefore Y = 0 \longrightarrow$ unstable

$Y = 1 \longrightarrow$ stable

Ex. $\frac{dy}{dt} = y - y^3 = f(y)$

$\frac{df}{dy} = 1 - 3y^2$

$Y = 0 \Rightarrow \frac{df}{dy} = 1 > 0$

$Y = 1 \Rightarrow \frac{df}{dy} = -2 < 0$

$Y = -1 \Rightarrow \frac{df}{dy} = -2 < 0$

$\therefore Y = 0 \longrightarrow$ unstable

$Y = 1$ or $-1 \longrightarrow$ Stable

Reasoning behind the steady state

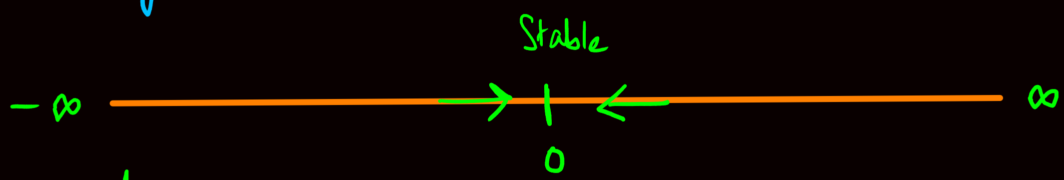
$$\frac{d}{dt}(y - Y) = f(y) - f(Y) \approx \left(\frac{df}{dy}\right)(y - Y)$$

$\rightarrow$ By mean value theoram

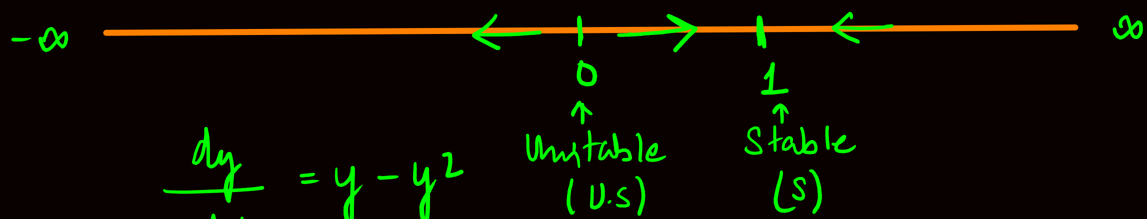
Here is "a" $\rightarrow$ if $a < 0$ Stable

$\rightarrow$ if $a > 0$ Unstable

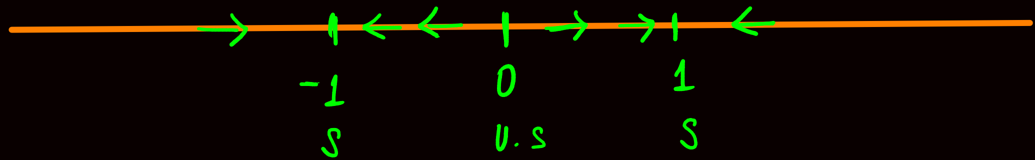
Stability lines



$$\frac{dy}{dt} = -y \quad \text{(Solution approaches 0)}$$



$$\frac{dy}{dt} = y - y^2$$



$$\frac{dy}{dt} = y - y^3$$

SEPERABLE EQUATIONS

$$\boxed{\frac{dy}{dt} = \frac{g(t)}{f(t)}}$$

$$\boxed{\int_{y(0)}^{y(t)} f(y)\,dy = \int_0^t g(s)\,ds} \rightarrow \text{Then solve for } y$$

① If $f(t) = 1$

$$\frac{dy}{dt} = g(t) \quad \text{then} \quad y = \int^t g(s)\,ds$$

② If $g(t) = 1$

$$\Rightarrow \frac{dy}{dt} = \frac{1}{f(y)}$$

then $\int f(y)\,dy = \int dt = t$

Example

① $\frac{dy}{dt} = \frac{t}{y}$ $\Rightarrow y\,dy = t\,dt$

$$\Rightarrow \int_{y(0)}^{y(t)} y\,dy = \int_0^t t\,dt$$

$$\Rightarrow \frac{1}{2}\left[y^2\right]_{y(0)}^{y(t)} = \frac{1}{2}t^2$$

$$\Rightarrow \frac{1}{2}y(t)^2 - \frac{1}{2}y(0)^2 = \frac{1}{2}t^2$$

$$\Rightarrow y(t) = \sqrt{t^2 + y(0)^2}$$ ← Solution

* At $y(0) = 0 \longrightarrow y(t) = t,\ y(t) = -t$

{Two Sol^n}

* At $t=0$, $y(t) = y(0) \longrightarrow$ gives 0/0 form

② $\frac{dy}{dt} = y - y^2$

$$\Rightarrow \int \frac{dy}{y-y^2} = \int_0^t dt$$

Integrating with Partial Fractions

$$\frac{1}{y-y^2} = \frac{A}{y} + \frac{B}{(1-y)}$$

where $A = 1$, $B = 1$

$$\Rightarrow \int_{y(0)}^{y(t)} \frac{1}{y}\,dy + \int_{y(0)}^{y(t)} \frac{1}{(1-y)}\,dy = \int_0^t dt$$

$$\Rightarrow \left[\log y\right]_{y(0)}^{y(t)} + \left(-\log(1-y)\right)_{y(0)}^{y(t)} = t$$

$$\Rightarrow \log(y(t)) - \log(y(0)) - \log(1-y(t)) + \log(1-y(0)) = t$$

$$\Rightarrow \log(y(t)\cdot y(0)) - \log(\{1-y(t)\}\cdot\{1-y(0)\}) = t$$

$$\Rightarrow \log(y(t)\cdot y(0)) - \log\{1 - y(0) - y(t) + y(t)\cdot y(0)\} = \log e^t$$

$$\Rightarrow \frac{y(t)\cdot y(0)}{1-y(0)-y(t)-y(t)\cdot y(0)} = e^t$$

$$\Rightarrow y(t)(y(0) - e^t + e^t y(0) + e^t y(t) - e^t y(t)\, y(0) = 0$$

$$\Rightarrow y(t)\left[y(0) + e^t - e^t y(0)\right] = e^t(1-y(0))$$

$$\Rightarrow y(t) = \frac{e^t(1-y(0))}{y(0)+e^t-e^t y(0)}$$