MHF4U - Rates of Change in Rational Functions Assessment Answers

Part 1: Multiple Choice Questions (Conceptual Understanding)

1. Which of the following is a valid operation on two functions, $f(x)$ and $g(x)$?
Answer: D) All of the above
Explanation: All options represent valid operations on functions. You can add, multiply, and divide functions, and $f(x) \cdot g(x)$ represents the product of the two functions.

2. Which of the following is the correct domain for the function $f(x) = \frac{1}{x-2}$?
Answer: A) $x \in (-\infty, 2) \cup (2, \infty)$
Explanation: The domain of the function is all real numbers except $x = 2$, since division by zero is undefined.

3. Which transformation occurs when the absolute value function $f(x) = |x|$ is modified to $f(x) = |x - 3|$?
Answer: D) Horizontal shift right by 3 units
Explanation: The modification $|x - 3|$ shifts the function 3 units to the right along the x-axis.

4. What is the range of the function $f(x) = |x - 2|$?
Answer: D) $[0, \infty)$
Explanation: The range of an absolute value function is always non-negative. Since the minimum value of $|x - 2|$ is 0, the range is $[0, \infty)$.

Part 2: Short Answer (Conceptual and Problem Solving)

1. Define inverse functions. How can you verify if two functions are inverses of each other?
Answer:

- **Inverse Functions**: Two functions $f(x)$ and $g(x)$ are said to be inverses of each other if $f(g(x)) = x$ and $g(f(x)) = x$.
- **Verification**: To verify if two functions are inverses, we compose them. If both compositions result in the identity function x, then they are inverses.

2. Given the piecewise function:

$$f(x) = \begin{cases} 2x + 3 & \text{if } x < 0 \\ x^2 - 1 & \text{if } x \geq 0 \end{cases}$$

Domain: All real numbers ($\mathbb{R}$), because the function is defined for all x.
Range:

- For $x < 0$, the output of $f(x) = 2x + 3$ is all real numbers.
- For $x \geq 0$, the output of $f(x) = x^2 - 1$ is $[-1, \infty)$.

Therefore, the range is $(-\infty, \infty)$.

3. Sketch the graph of this piecewise function.
Explanation: The graph consists of a line for $x < 0$ with slope 2 and y-intercept 3, and a parabola for $x \geq 0$ opening upwards starting from $y = -1$.

4. Find the inverse of the function $f(x) = 3x - 4$.
Answer: To find the inverse:

$$y = 3x - 4$$

Solve for x:

$$x = \frac{y + 4}{3}$$

Thus, the inverse function is:

$$f^{-1}(x) = \frac{x + 4}{3}$$

5. Solve the equation $|2x - 3| = 7$. Show all steps.
Answer:

$$|2x - 3| = 7$$

This gives two cases:

- Case 1: $2x - 3 = 7 \rightarrow 2x = 10 \rightarrow x = 5$
- Case 2: $2x - 3 = -7 \rightarrow 2x = -4 \rightarrow x = -2$

Therefore, the solutions are $x = 5$ and $x = -2$.

Part 3: Graphing and Operations with Functions

1. Sketch the graph of the following functions and indicate key features:

- $f(x) = |x - 1|$: This is a V-shaped graph, with the vertex at $(1, 0)$. The graph opens upwards.
- $g(x) = 2x + 3$: This is a straight line with slope 2 and y-intercept 3.

2. Perform the following operations and graph the resulting functions:

$$h(x) = f(x) + g(x), \text{ where } f(x) = 2x - 1 \text{ and } g(x) = x^2 + 3x$$

Answer:

$$h(x) = (2x - 1) + (x^2 + 3x) = x^2 + 5x - 1$$

Domain: All real numbers ($\mathbb{R}$)
Range: $(-\infty, \infty)$, since the function is a parabola opening upwards.

3. Given $f(x) = x^2 - 4$ and $g(x) = x + 1$, find $(f \circ g)(x)$.
Answer:

$$(f \circ g)(x) = f(g(x)) = f(x + 1)$$

Substituting $x + 1$ into $f(x)$:

$$f(x + 1) = (x + 1)^2 - 4 = x^2 + 2x + 1 - 4 = x^2 + 2x - 3$$

Thus, $(f \circ g)(x) = x^2 + 2x - 3$.

Part 4: Word Problems (Real-World Applications)

1. A population of rabbits grows exponentially according to the function $P(t) = 50e^{0.05t}$**, where** t **is time in years and** $P(t)$ **is the population size.**

How many rabbits are in the population after 10 years?

Answer:

$$P(10) = 50e^{0.05\times 10} = 50e^{0.5} \approx 50 \times 1.6487 \approx 82.44$$

The population after 10 years is approximately 82 rabbits.

What is the rate of change of the population at $t = 10$**?**

Answer:

$$P'(t) = 50e^{0.05t} \times 0.05$$

At $t = 10$:

$$P'(10) = 82.44 \times 0.05 \approx 4.12$$

The rate of change at $t = 10$ is approximately 4.12 rabbits per year.

2. A company's profit is modeled by the function $P(x) = 100x - 5x^2$**, where** x **is the number of units sold.**

Find the number of units that maximizes profit.

Answer:

$$P'(x) = 100 - 10x = 0 \quad \Rightarrow \quad x = 10$$

The number of units that maximizes profit is 10 units.

What is the maximum profit?

Answer:

$$P(10) = 100(10) - 5(10)^2 = 1000 - 500 = 500$$

The maximum profit is $500.

3. The cost to produce x **units of a product is given by the function** $C(x) = 100 + 10x$**, and the revenue function is** $R(x) = 50x$**.**

Determine the break-even point by solving $C(x) = R(x)$**.**

Answer:

$$100 + 10x = 50x \quad \Rightarrow \quad 100 = 40x \quad \Rightarrow \quad x = 2.5$$

The break-even point is at 2.5 units.