MHF4U - Rates of Change in Rational Functions
Assessment Answers

Part 1: Multiple Choice Questions (Conceptual Understanding)

1. Which of the following is true about the average rate of change of a function between two points?

Answer: A) It is the same as the slope of a secant line.

Explanation: The **average rate of change** of a function between two points is equivalent to the **slope** of the secant line connecting these points.

2. If a graph has a horizontal tangent line, what does this indicate about the instantaneous rate of change at that point?

Answer: C) The instantaneous rate of change is zero.

Explanation: A **horizontal tangent line** implies that the **slope** at that point is zero, which means the instantaneous rate of change is zero.

3. In which of the following cases would the average rate of change be zero?

Answer: D) A function that decreases and then increases.

Explanation: The **average rate of change** between two points is zero when the function starts and ends at the same value, even if it increases and decreases in between.

4. What is the average rate of change for the function $f(x) = x^2$ between $x = 1$ and $x = 3$?

Answer: B) 4

Explanation: The average rate of change is calculated using the formula:

$$\frac{f(3) - f(1)}{3 - 1} = \frac{9 - 1}{2} = 4$$

Part 2: Short Answer (Conceptual and Problem Solving)

1. Define the instantaneous rate of change and explain how it differs from the average rate of change.

Answer:

- **Instantaneous Rate of Change**: It refers to the rate of change at a specific point on the curve. It is the slope of the tangent line at that point.
- **Average Rate of Change**: It is the change in the value of the function over an interval divided by the length of that interval, representing the slope of the secant line.

2. Given the following graph of a function $f(x)$, estimate the instantaneous rate of change at $x = 2$ using secant lines.

Answer: To estimate the instantaneous rate of change, draw secant lines near $x = 2$ (for example, from $x = 1.9$ to $x = 2.1$), calculate the slope of these secant lines, and use this slope to approximate the instantaneous rate of change.

3. For the function $f(x) = x^2$, calculate the average rate of change between $x = -2$ and $x = 2$. How does this differ from the instantaneous rate of change at $x = 2$?

Answer:

- **Average Rate of Change**:

$$\frac{f(2) - f(-2)}{2 - (-2)} = \frac{4 - 4}{4} = 0$$

- **Instantaneous Rate of Change at $x = 2$**: The derivative of $f(x) = x^2$ is $f'(x) = 2x$. At $x = 2$:

$$f'(2) = 2(2) = 4$$

4. You are driving a car and the speedometer shows a changing speed. Describe how you would interpret the instantaneous rate of change in this context.

Answer: The **instantaneous rate of change** in this context represents the **car's speed** at a specific moment in time. It shows how fast the car is traveling at that exact point, as opposed to the **average speed** over a longer period.

Part 3: Graphing and Operations with Functions

1. Sketch the graph of a function that increases at a constant rate, becomes constant, and then decreases at a constant rate. Indicate the rate of change at different intervals.

Answer: The graph is a piecewise linear function with three sections:

- **Increasing section**: A straight line with a positive slope.
- **Constant section**: A horizontal line (rate of change = 0).
- **Decreasing section**: A straight line with a negative slope.

2. Given $f(x) = x^3$, graph the function and its rate of change (first derivative).
Answer:

- The graph of $f(x) = x^3$ is a cubic function with a turning point at $(0, 0)$.
- The rate of change (first derivative) is:

$$f'(x) = 3x^2$$

- The graph of $f'(x) = 3x^2$ is a parabola that opens upwards, with a vertex at the origin.

3. Estimate the instantaneous rate of change at $x = 1$ for the function $f(x) = 2x^2 - 3x + 4$ using a secant line.

Answer: To estimate the instantaneous rate of change, calculate the slope of secant lines near $x = 1$. For example, calculate the slope of the secant line from $x = 0.9$ to $x = 1.1$, then find the average of these slopes.

4. Graph the function $y = 3x + 2$ and the secant line between the points $(1, f(1))$ and $(3, f(3))$. Calculate the average rate of change.

Answer:

$$\text{Average rate of change} = \frac{f(3) - f(1)}{3 - 1}$$

For the function $y = 3x + 2$:

$$f(3) = 3(3) + 2 = 11 \quad \text{and} \quad f(1) = 3(1) + 2 = 5$$

$$\frac{11 - 5}{3 - 1} = \frac{6}{2} = 3$$

So, the average rate of change is **3**.

Part 4: Word Problems (Real-World Applications)

1. A car accelerates from 0 to 60 km/h over a period of 5 seconds. What is the average rate of change of speed? What is the instantaneous rate of change at $t = 5$ seconds?

Answer:

- Average rate of change:
$$\frac{60-0}{5-0} = 12\,\text{km/h per second}$$
- Instantaneous rate of change at $t = 5$ seconds depends on the function describing the car's acceleration. If the acceleration is constant, the instantaneous rate of change is the same as the average rate of change.

2. A tank is being filled with water. The rate of change of water volume is modeled by $V(t) = 4t^2$, where t is time in hours. Find the instantaneous rate of change at $t = 3$ hours.

Answer:
$$V'(t) = 8t$$
At $t = 3$:
$$V'(3) = 8(3) = 24\,\text{cubic units per hour}$$

3. A company's profit $P(x)$ is modeled by the function $P(x) = 100x - 2x^2$, where x is the number of units sold. Calculate the instantaneous rate of change of profit when 10 units are sold.

Answer:
$$P'(x) = 100 - 4x$$
At $x = 10$:
$$P'(10) = 100 - 4(10) = 60$$
The instantaneous rate of change of profit when 10 units are sold is **60 units of profit per unit sold**.