Ph.D. Qualifying Examination
Sem 2, 2000/2001
Algebra

1. Let G be a finite group with a unique maximal subgroup. Show that G is cyclic.

2. Let A be a subgroup of index n of a finite group G and let

$$\{g_1 A, g_2 A, \cdots, g_n A\}$$

be a set of coset representatives of G/A. For each $g \in G$, define

$$f_g : G/A \to G/A$$

by $f_g(g_i A) = g g_i A$. Prove that f_g is a bijection. Define $\chi : G \to S_n$ by

$$\chi(g) = f_g .$$

Prove that χ is a group homomorphism. Determine the kernel of χ.

3. Let R be a commutative ring with identity and let $\chi : R \to F$ be a nontrivial ring homomorphism, where F is an integral domain. Prove that kernel of χ is a prime ideal.

4. Let V be a vector space of finite dimension over a field F. Suppose that V is an integral domain. Prove that V is a field.

5. Let E/F be a field extension and let $a, b \in E$ be algebraic over F. Prove that every element in $F(a, b)$ is algebraic over F.

— END OF PAPER —

Ph.D. Qualifying Examination
Sem 1, 2001/2002
Algebra

1.(a) Show that if R is a commutative ring with identity, then every maximal ideal of R is a prime ideal. [15 marks]

(b) Show that if R is a Principal Ideal Domain, then every prime ideal of R is a maximal ideal. [15 marks]

(c) Give an example of a ring R which has a prime ideal that is not maximal. [10 marks]

2.(a) Let G and H be finite groups with relatively prime orders. Let $\theta : G \to H$ be a group homomorphism. What can conclude about θ and why? [10 marks]

(b) Let H be a subgroup of a group G with index 2. Prove that $H \triangleleft G$. [15 marks]

(c) Give an example to show that H may not be a normal subgroup of G if $|G : H| = 3$. [10 marks]

3. If L is a field extension of K such that $[L : K] = p$ where p is a prime number, show that $L = K(a)$ for every $a \in L$ that is not in K. [15 marks]

4. Give an example of two algebraic numbers a and b of degrees 2 and 3, respectively, such that ab is of degree less than 6 over $\mathbb{Q}$. [10 marks]

— END OF PAPER —

Ph.D. Qualifying Examination
Sem 2, 2001/2002
Algebra

1. Classify all groups of order n up to isomorphism, where

   (a) n is the square of a prime integer; [10 marks]

   (b) $n = pq$ where p, q are primes with $p > q$ and q does not divide $p - 1$. [10 marks]

2. Let R be a commutative ring with 1.

   (a) If $I, J_1, J_2, \ldots, J_n$ are ideals of R such that I is prime and $I \supseteq \bigcap_{r=1}^{n} J_r$, prove that $I \supseteq J_s$ for some s. [10 marks]

   (b) If the intersection of all maximal ideals of R is prime but not maximal, prove that R has infinitely many maximal ideals. [5 marks]

   (c) If $I, J_1, J_2, \ldots, J_n$ are ideals of R such that J_r's are prime for all r, and $I \subseteq \bigcup_{r=1}^{n} J_r$, prove that $I \subseteq J_s$ for some s. [15 marks]

3. Let R be a ring and M be a left R-module. Show that the following statements are equivalent: [30 marks]

   (a) Every submodule of M is finitely generated.

   (b) Every non-empty collection of submodules of M has a maximal element (with respect to inclusion).

   (c) Whenever $N_1 \subseteq N_2 \subseteq \cdots$ is an ascending chain of submodules of M, there is an integer k such that $N_l = N_k$ for all $l \geq k$.

4. A complex number is *algebraic* if and only if it satisfies a polynomial with rational coefficients.

   (a) Prove that a complex number α is algebraic if and only if $\alpha \in F$ for some finite field extension F of $\mathbb{Q}$. [8 marks]

   (b) Hence, or otherwise, show that the set K of algebraic numbers is a field. [6 marks]

   (c) Show further that K is algebraically closed, i.e. every polynomial with coefficients in K has a root in K. [6 marks]

— END OF PAPER —

Ph.D. Qualifying Examination
Sem 1, 2002/2003
Algebra

1. Let p be a prime number. Show that

$$\mathbb{Q}(\sqrt{p}) = \{a + b\sqrt{p} \mid a, b \in \mathbb{Q}\}$$

is a subfield of $\mathbb{R}$.
If p and q are distinct prime numbers, prove that

(i) $\mathbb{Q}(\sqrt{p})$ and $\mathbb{Q}(\sqrt{q})$ are isomorphic as additive groups;

(ii) $\mathbb{Q}(\sqrt{p})$ and $\mathbb{Q}(\sqrt{q})$ are not isomorphic as fields;

(iii) $\mathbb{Q}(\sqrt{p} + \sqrt{q})$ is the compositum of $\mathbb{Q}(\sqrt{p})$ and $\mathbb{Q}(\sqrt{q})$.

2.(i) Let $f(x) \in \mathbb{Z}_2[x]$. Prove that $(x - 1)$ divides $f(x)$ in $\mathbb{Z}_2[x]$ if and only if $f(x)$ has an even number of nonzero coefficients.

(ii) Prove that if $\deg f(x) > 1$ and $f(x)$ is irreducible in $\mathbb{Z}_2[x]$, then $f(x)$ has constant term 1 and an odd number of nonzero coefficients.

(iii) Determine all irreducible polynomials of degree 4 or less over $\mathbb{Z}_2$.

(iv) If p is a prime number, how many monic irreducible polynomials of degree 2 over $\mathbb{Z}_p$ are there? Justify your answer.

3.(a) Determine whether each of the following pairs of groups are isomorphic:

(i) $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2,\ \mathbb{Z}_8$;

(ii) $\mathbb{Z}, \mathbb{Q}$;

(iii) $\mathbb{R}^*, \mathbb{C}^*$;

(iv) $\mathbb{R}^*, \mathbb{Q}^*$;

(v) $\mathbb{Q}, \mathbb{Q} \times \mathbb{Q}$.

(b) $\mathbb{Z}[i] = \{a+bi \mid a, b \in \mathbb{Z}\}$ is a Euclidean domain with respect to the Euclidean distance d, where

$$d(a+bi) = a^2 + b^2.$$

(i) Find $\alpha, \beta \in \mathbb{Z}[i]$ such that

$$1 - 5i = (1+2i)\alpha + \beta,$$

where $|\beta| < 5$.

(ii) Decide, with reasons, which of the following elements are irreducible in $\mathbb{Z}[i]$:

$$1+i, 2+3i, 1+3i.$$

4.(a) If p is a prime number, show that the symmetric group S_p has exactly $(p-2)!$ Sylow p-subgroups. Deduce that $(p-1)!+1$ is divisible by p.

(b) Prove that a ring with a prime number of elements is either a field or a zero ring (i.e. a ring in which all products are zero).

5. If ϕ is an automorphism of a group G, show that the set $H = \{h \in G \mid \phi(h) = h\}$ is a subgroup of G.

Prove that ϕ commutes with the inner automorphism $\psi : G \to G$ given by $\psi(g) = a^{-1}ga$ if and only if $\phi(a)a^{-1}$ is in the centre Z of G.

If ϕ commutes with every inner automorphism of G, prove that the mapping

$$\theta : G \to G, \quad \theta(a) = \phi(a)a^{-1}$$

is a homomorphism of G onto a subgroup of Z. Hence or otherwise, show that then H is normal in G and G/H is abelian.

— **END OF PAPER** —

Ph.D. Qualifying Examination
Sem 2, 2002/2003
Algebra

1. Let G be a group of order $2p$, where p is an odd prime. Prove that either G is cyclic, or $G = \{1, a, a^2, \ldots, a^{p-1}, b, ab, a^2b, \ldots, a^{p-1}b\}$ where a has order p, b has order 2, and $ba = a^{-1}b$. [20 marks]

2. Let $\phi : R \to S$ be a homomorphism of commutative rings with 1. Prove or disprove the following statements:
   (a) If I is a prime ideal of S, then $\phi^{-1}(I)$ is a prime ideal of R. [10 marks]
   (b) If J is a maximal ideal of S, then $\phi^{-1}(J)$ is a maximal ideal of R. [10 marks]

3.(a) If R and S are simple rings with 1, find all ideals of $R \times S$. [8 marks]
(b) If M and N are simple left R-modules (where R is a ring with 1), find all submodules of $M \oplus N$. [12 marks]

4. Let R be a ring, M be a left R-module, and $B = \{m_1, m_2, \ldots, m_n\} \subseteq M$. Show that the following statements are equivalent: [20 marks]
   (a) Every element of M has a unique expression of the form $r_1m_1 + r_2m_2 + \ldots + r_nm_n$ ($r_i \in R$ for all i).
   (b) Every function from B to a left R-module N can be uniquely extended to a module homomorphism from M to N.

5. Let F be a finite field with p^n elements. Prove that
   (a) the multiplicative group $F^\times = F \setminus \{0\}$ is cyclic. [10 marks]
   (b) F contains a subfield with p^m elements if and only if $m \mid n$. [10 marks]

— END OF PAPER —

Ph.D. Qualifying Examination
Algebra

1. (a) Let G be the additive group $\mathbb{Q}/\mathbb{Z}$. Show that any finite subgroup of G is cyclic.

   (b) For the ring $R = \mathbb{Z} \times \mathbb{Z}$, give an example for each of the following:

   (i) a maximal ideal of R;

   (ii) a prime ideal of R that is not maximal.

2. (a) Let G be a finite group, and H be a subgroup of index 2. Show that $x^2 \in H$ for any $x \in G$ and hence deduce that H contains all elements of G of odd order.

   (b) Let $n > 3$ be an integer, and let G be a subgroup of S_n. Assume that G has an odd permutation. Show that G has a normal subgroup of index 2.

   (c) Let A_4 be the subgroup of even permutations in S_4. Show that A_4 has no subgroup of index 2.

3. Recall that an element p of an integral domain D is called irreducible if p is a non-zero, non-unit and in any factorization $p = rs$ with $r, s \in D$, one of r, s is a unit. Now let

$$D = \mathbb{Z}[\sqrt{-7}] = \{a + b\sqrt{-7} | a, b \in \mathbb{Z}\}.$$

   (i) By using the norm function $N(a + b\sqrt{-7}) = a^2 + 7b^2$, show that $2, 1 \pm \sqrt{-7}$ are irreducible elements of D.

   (ii) Is $2D$ a prime ideal? Is D a unique factorization domain? Justify your answers.

4. (a) Let R be a finite commutative ring with 1, such that $1 \neq 0$. Let $R^* = R\backslash\{0\}$ and put

$$k = \prod_{r \in R^*} r,$$

   is a field.

   (b) Let p be a positive prime number such that $p = 4k + 1$ for some $k \in \mathbb{Z}$. Show that there exists $a \in \mathbb{Z}_p$ such that $a^2 = -1$ in $\mathbb{Z}_p$.

5. Show that each of the following polynomials is irreducible over $\mathbb{Q}$: you may want to consider reduction modulo a prime number.

(i) $3x^4 - 2x^2 + 72x - 10$;

(ii) $x^3 + 1003x + 1002$.

END OF PAPER

PH.D. QUALIFYING EXAMINATION 2003/2004 (Sem 2)
ALGEBRA

1. Show that if a and b are elements in a group G, then ab and ba have the same order. [10 marks]

2. (a) Let H and K be subgroups of a group G with H normal in G. Show that
$$HK := \{hk : h \in H, k \in K\}$$
is a subgroup of G and show that H is normal in HK. [10 marks]

   (b) Show that $(H \cap K)$ is normal in K and that
$$K/(H \cap K) \simeq HK/H.$$
[10 marks]

   (c) Show that if H is a normal subgroup of G such that
$$\gcd(|H|, [G : H]) = 1,$$
then H is the unique subgroup of G of order $|H|$.
[15 marks]

3. (a) Show that if R is a finite integral domain with a unit element, then R is a field. [10 marks]

   (b) Show that if R is a finite commutative ring with a unit element, then every prime ideal of R is a maximal ideal. [10 marks]

4. Let R is a ring with a unit element, 1_R, in which
$$(ab)^2 = a^2b^2$$
for all $a, b \in R$. Prove that R must be commutative. [15 marks]

5. (a) Let K be a finite field of p elements, where p is a prime. Let $\gcd(n,p) = 1$ and F be the splitting field of $x^n - 1_K$ over K. Show that if $(F : K) = f$ then n divides $q^f - 1$.
[10 marks]

   (b) Show that f is the smallest integer m for which $q^m - 1$ is divisible by n.
[10 marks]

END OF PAPER

Ph.D. Qualifying Examination : Algebra

Ring Theory

(a) Let p be a prime. Find all the rings (up to ring isomorphism) of p elements. Justify your answers.
(b) Let $R = M_n(\mathbb{R})$ be the ring of all $n \times n$ matrices over the real numbers. Find all the ideals of R. Justify your answers.

Group Theory

(a) Let p be a prime. Find all the groups (up to group isomorphism) of order $2p$. Justify your answers.
(b) Let p be a prime and let G be a group of p^3 elements. Suppose that G is not abelian. Prove that $Z(G)$ is cyclic of order p.

Field Theory

(a) Let p be a prime. Find all the fields (up to field isomorphism) of p^2 elements Justify your answers.
(b) Let σ be a field automorphism from $\mathbb{R}$ to $\mathbb{R}$. Prove that $\sigma(x) = x$ for all $x \in \mathbb{R}$.

Ph.D. Qualifying Examination

Algebra

Semester 2, 2004/2005

Ring Theory

(a) Prove that every integral domain can be imbedded in a field.

(b) Let D be an integral domain and let $F = \{x \in D : xd = 1 \text{ for some } d \in D\}$. Suppose that D is a finite dimensional vector space over F. Prove that D is a field.

Group Theory

(a) Let G be a nonabelian finite group generated x and y, where $o(x) = o(y) = 2$. Prove that G is isomorphic to a dihedral group.

(b) Let G be a group of order 56. Suppose that G has 2 or more subgroups of order 7. Prove that G has a subgroup isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

Field Theory

(a) Let F be a finite field. Prove that $F - \{0\}$ under multiplication is a cyclic group.

(b) Prove that $\sqrt{2} + \sqrt{3} + \sqrt{5} + \sqrt{7}$ is irrational.

END OF PAPER

Algebra, 2005/2006, Sem 1

Answer all questions. Each question carries 25 marks.

(1) Classify all groups of order 8 up to isomorphism.

(2) Prove or disprove each of the following statements:

(a) A field is a Euclidean domain.

(b) If R is a Euclidean domain but not a field, and S is a subring of R with multiplicative identity, then S is the unique factorization domain.

(3) For each of the following polynomials $f(X)$, find the degree of K over $\mathbb{Q}$, where K is the splitting field of $f(X)$.

(a) $f(X) = X^4 - 1$;
(b) $f(X) = X^3 - 1$;
(c) $f(X) = X^4 - 2$;
(d) $f(X) = X^3 - 2$.

(4) Let R be a ring with 1. A *simple* left R-module M is a left R-module such that $|M| > 1$ and if N is a submodule of M, then either $N = M$ or $N = \{0\}$.

(a) Let I be a maximal left ideal of R. Show that R/I is a simple R-module.
(b) Let m be a nonzero element of a simple left R-module M. Prove that:
(i) $Rm := \{rm \mid r \in R\}$ equals M;
(ii) $\operatorname{Ann}(m) := \{r \in R \mid rm = 0\}$ is a maximal left ideal of R;
(iii) $R/\operatorname{Ann}(m) \cong M$ as left R-modules .

PhD Qualifying Exam
Algebra
Sem 2, 2005/2006

Answer all questions.
Each question carries 25 marks.

(1) (a) Prove that a group of order 12 either has a normal subgroup of order 3, or is isomorphic to A_4, the alternating group on 4 letters. [10 marks]

(b) Show that any simple group acting on a set of n elements is isomorphic to a subgroup of A_n, the alternating group on n letters. [15 marks]

(2) Let R be a ring, not necessarily commutative and not necessarily containing the multiplicative identity. Prove that if $R[X]$ is a principal ideal domain, then R is a field. [25 marks]

(3) Let K be the splitting field of $X^4 - 2$ over $\mathbb{Q}$. Find all intermediate fields between $\mathbb{Q}$ and K. [25 marks]

(4) Let R be a ring with multiplicative identity, and let M be a left R-module. Show that the following statements are equivalent: [25 marks]

(a) Every submodule of M is finitely generated.
(b) Whenever $N_1 \subseteq N_2 \subseteq \cdots$ is an ascending chain of submodules of M, there is an integer k such that $N_l = N_k$ for all $l \geq k$.
(c) Every non-empty collection of submodules of M has a maximal element (with respect to inclusion).

The End

PhD Qualifying Examination

Algebra

Sem 1, 2007/2008

Answer all questions. Each question carries 25 marks.

(1) Let R be a Euclidean domain, and denote the $n \times n$-matrix ring over R by $M_n(R)$. Let $M \in M_n(R)$. Prove that there exist units $P, Q \in M_n(R)$ such that $PMQ = \text{diag}(d_1, d_2, \ldots, d_n)$ with $d_1 \mid d_2 \mid \cdots \mid d_n$.

(2) Prove that a simple group of order 60 is isomorphic to A_5.

(3) Let $K \subseteq L \subseteq M$ be fields. Prove or disprove each of the following statements:

(a) If L is algebraic over K and M is algebraic over L, then M is algebraic over K.
(b) If L is separable over K and M is separable over L, then M is separable over K.
(c) If L is Galois over K and M is Galois over L, then M is Galois over K.
(d) If L is radical over K and M is radical over L, then M is radical over K.

(4) Let R be a ring with multiplicative identity, and let M be a left R-module. Let $k \in \mathbb{Z}^+$. Prove that the following statements are equivalent:

(a) M is isomorphic to $R^k := \{(r_1, r_2, \ldots, r_k) \mid r_i \in R \text{ for all } i\}$ as left R-modules.
(b) there exist $m_1, m_2, \ldots, m_k \in M$ such that for every $m \in M$, there exist unique $r_1, \ldots, r_k \in R$ such that $m = r_1 m_1 + r_2 m_2 + \cdots + r_k m_k$.
(c) there exist $m_1, m_2, \ldots, m_k \in M$ such that every function f from $\{m_1, m_2, \ldots, m_k\}$ to a left R-module N can be uniquely extended to a left module homomorphism $\tilde{f} : M \to N$.

National University of Singapore

Ph.D. Qualifying Examination
Year 2007–2008 Semester II

Algebra

Answer all questions. Each question carries 25 marks.

(1) Let R be a commutative ring with 1.

(a) Let I be an ideal of R. Explain briefly what is meant to say that (i) I is *prime*, (ii) I is *maximal*.

(b) Prove or disprove each of the following statements:

(i) If I is a maximal ideal of R, then I is prime.

(ii) If I is a nonzero prime ideal of R, then I is maximal.

(2) Let p and q be a prime integers with $p \leq q$.

(a) Show that any group of order pq has a normal subgroup of order q.

(b) Hence, or otherwise, classify all groups of order pq up to isomorphism.

(3) Let n be a fixed positive integer.

(a) Prove that $\mathbb{Q}(\cos \frac{2\pi i}{n})$ is an algebraic extension over $\mathbb{Q}$.

(b) Determine the degree $[\mathbb{Q}(\cos \frac{2\pi i}{n}) : \mathbb{Q}]$.

(4) Let R be a ring with 1, and let M be a left R-module. Prove that the following statements are equivalent:

(a) M is nonzero, and if N is a submodule of M, then $N = 0$ or $N = M$.

(b) For every $m \in M \setminus \{0\}$, $M = \{rm \mid r \in R\}$.

(c) There exists a maximal left ideal I of R such that $M \cong R/I$ as left R-modules.

NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

SEMESTER 1 2008-2009

Ph.D. QUALIFYING EXAMINATION

PAPER 1

ALGEBRA

Time allowed : 3 hours

INSTRUCTIONS TO CANDIDATES

1. Answer **ALL** questions.

Ph.D. Qualifying Examination
Year 2008/2009, Semester 1

ALGEBRA

Answer all questions. Each question carries 20 marks.

(1) (a) Let G be a finite simple group, and suppose that H is a proper subgroup of G of index k. Show that there exists an injective group homomorphism from G to the alternating group A_k of degree k.
(b) Show that a group of order 120 is not simple.

(2) Let V be a finite-dimensional vector space of an algebraically closed field F of positive characteristic p. Let $\alpha : V \to V$ be a linear operator on V, and suppose that there exists a positive integer n such that $\alpha^n(v) = v$ for all $v \in V$, while for each positive integer i less than n, there exists $v_i \in V$ such that $\alpha^i(v_i) \neq v_i$. Show that α is diagonalisable if and only if n is not divisible by p.

(3) (a) Let R and S be integral domains with $R \subseteq S$. Prove or disprove the following:
(i) If R is a Euclidean domain, then S is a unique factorisation domain.
(ii) If S is a Euclidean domain, then R is a unique factorisation domain.
(b) Let $\phi : T \to U$ be a surjective ring homomorphism between two integral domains T and U. Prove or disprove the following:
(i) If T is a principal ideal domain, then U is a principal ideal domain.
(ii) If T is a unique factorisation domain, then U is a unique factorisation domain.

(4) Let K be the splitting field of $X^4 - 2$ over the field $\mathbb{Q}$ of rational numbers.
(a) Show that there exist field automorphisms τ and σ of K satisfying the following properties:
- τ has order 2;
- σ has order 4;
- $\tau \circ \sigma = \sigma^{-1} \circ \tau$.

(b) Hence, or otherwise, find all intermediate fields between $\mathbb{Q}$ and K.

(5) Let R be a ring with multiplicative identity, and let M be a finitely generated left R-module.
(a) Let B be a non-empty finite subset of M. Show that M is a free R-module with basis B if and only if every function from B to any left R-module N can be uniquely extended to a left R-module homomorphism from M to N.
(b) Suppose further that R is a principal ideal domain. Prove that M is a free R-module if and only if M is a projective R-module.

END OF PAPER

Ph.D. Qualifying Examination
Sem 2, 2000/2001
Analysis

1. Show that a function $f : [0,1] \to \mathbb{R}$ is uniformly continuous on $[0,1]$ if and only if $(f(x_n))_{n=1}^{\infty}$ is a Cauchy sequence whenever $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence in $[0,1]$.

2. Let f be a continuous real-valued function such that
$$\int_0^1 x^n f(x)\,dx = 0 \text{ for } n = 0, 1, 2, \ldots$$
Show that f is the zero function.

3. Let $C[0,1]$ denote the set of all real-valued continuous functions on $[0,1]$. Show that there is a unique $f \in C[0,1]$ such that
$$f(x) = \int_0^{x/2} f(t)\,dt$$
for all $x \in [0,1]$.

4. Let (X,d) be a complete metric space. For any $x \in X$ and any $\epsilon > 0$, let $B(x,\epsilon)$ denote the open ball of radius ϵ centered at x. Suppose that A is a subset of X so that for any $\epsilon > 0$, there exists a compact subset A_ϵ of X satisfying
$$A \subseteq \cup_{x \in A_\epsilon} B(x,\epsilon).$$
Show that A is relatively compact, i.e., the closure of A is a compact set.

5. Let $(f_n)_{n=1}^{\infty}$ be a sequence of real-valued differentiable functions on $[0,1]$. Assume that there is a constant $C < \infty$ so that
   (i) $|f_n(0)| \le C$ for all $n \in \mathbb{N}$;
   (ii) $|f_n'(x)| \le C$ for all $x \in [0,1]$ and all $n \in \mathbb{N}$. Show that $(f_n)_{n=1}^{\infty}$ has a uniformly convergent subsequence.

— END OF PAPER —

Ph.D. Qualifying Examination
Sem 1, 2001/2002
Analysis

1. Let d be a metric on a nonempty set M. For each of the following, determine whether in general ρ defines a metric on M. Justify your answers.
   (i) $\rho(x,y) = (d(x,y))^2$, $x, y \in M$.
   (ii) $\rho(x,y) = \min\{2, d(x,y)\}$, $x, y \in M$.

2. Prove or disprove the following statements.
   (a) In a metric space, every closed subset of a compact set is compact.
   (b) In a metric space, every closed and bounded set is compact.

3. Let ℓ^∞ be the space of all bounded sequences of complex numbers endowed with the metric
$$d(\zeta,\eta) = \sup_{j\in\mathbb{N}} |\zeta_j - \eta_j|, \qquad \zeta = \{\zeta_j\}_{j\in\mathbb{N}},\ \eta = \{\eta_j\}_{j\in\mathbb{N}} \in \ell^\infty.$$
Suppose that $K : \mathbb{N}^2 \longrightarrow \mathbb{C}$ is a function for which there exists $\lambda \in (0,1)$ such that
$$\sum_{l\in\mathbb{N}} |K(j,l)| \leq \lambda, \qquad j \in \mathbb{N}.$$
Show that for every $\beta = \{\beta_j\}_{j\in\mathbb{N}} \in \ell^\infty$, there exists a unique $\alpha = \{\alpha_j\}_{j\in\mathbb{N}} \in \ell^\infty$ such that
$$\alpha_j = \sum_{l\in\mathbb{N}} K(j,l)\alpha_l + \beta_j, \qquad j \in \mathbb{N}.$$

4. Determine whether the function $g(x,y) = \displaystyle\sum_{k=1}^{\infty} \frac{(x-2y)^k \sin(kx+y)}{\sqrt{k!}\,(1+x^{2k}y^{4k})}$ is continuous on $\mathbb{R}^2$. Justify your answer.

5. Let $f_k : [0,1] \longrightarrow \mathbb{R}$, $k \geq 1$, be a sequence of continuous functions such that for every $k \geq 1$,
$$\int_0^1 (f_k(t))^2\, dt = 1.$$
Define a sequence of functions $F_k : [0,1] \longrightarrow \mathbb{R}$, $k \geq 1$, by
$$F_k(x) = \int_0^x t f_k(t)\, dt.$$
Prove that the sequence F_k, $k \geq 1$, has a uniformly convergent subsequence.

— END OF PAPER —

Ph.D. Qualifying Examination
Sem 2, 2001/2002
Analysis

1. Let $(a_n)_{n=1}^{\infty}$ be a bounded sequence of real numbers such that $a_n \neq 0$ for all $n \in \mathbb{N}$.

   (a) Show that
   $$\limsup_{n\to\infty} |a_n|^{1/n} \leq \limsup_{n\to\infty} |a_{n+1}/a_n|.$$

   (b) If $\lim_{n\to\infty} |a_{n+1}/a_n|$ exists, show that $\lim_{n\to\infty} |a_n|^{1/n}$ exists and the two limits are equal.

   (c) Give an example where equality does not hold in (a).

2. Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be sequences of real numbers.

   (a) Define $b_0 = 0$ and $c_n = b_n - b_{n-1}$ for all $n \in \mathbb{N}$. Show that if $p, q \in \mathbb{N}$, $p \leq q$, then
   $$\sum_{n=p}^{q} a_n b_n = \left(\sum_{n=p}^{q} a_n\right) b_{p-1} + \sum_{j=p}^{q} \left(\sum_{n=j}^{q} a_n\right) c_j.$$

   (b) Suppose that $(b_n)_{n=1}^{\infty}$ is increasing and converges to $b \in \mathbb{R}$, and that $\sum_{n=1}^{\infty} a_n$ converges. Let M and m be real numbers such that $m \leq \sum_{n=p}^{q} a_n \leq M$ for all p, $q \in \mathbb{N}$, $p \leq q$. Show that $\sum_{n=1}^{\infty} a_n b_n$ converges and that $mb \leq \sum_{n=1}^{\infty} a_n b_n \leq Mb$.

   (c) If $\sum_{n=1}^{\infty} a_n x^n$ converges for all $x \in [0,1]$, show that
   $$\lim_{x\to 1^-} \sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} a_n.$$

3. Let $f : X \to Y$ be a function mapping between metric spaces X and Y. Show that f is continuous on X if and only if $f(\overline{A}) \subseteq \overline{f(A)}$ for every subset A of X. (Here $\overline{S}$ denotes the closure of the set S.)

4. Let $(f_n)_{n=1}^{\infty}$ be a sequence of continuous functions from a metric space X into a metric space Y. If $(f_n)_{n=1}^{\infty}$ converges uniformly to a function f on X and $(x_n)_{n=1}^{\infty}$ is a sequence in X that converges to an element $x \in X$, show that $(f_n(x_n))_{n=1}^{\infty}$ converges to $f(x)$.

5. Let $f : (0,1] \to \mathbb{R}$ be a continuous function on $(0,1]$. Show that f is uniformly continuous on $(0,1]$ if and only if $\lim_{x\to 0^+} f(x)$ exists and has a real value.

— END OF PAPER —

Ph.D. Qualifying Examination
Sem 1, 2002/2003
Analysis

1.(a) Let $f : [0, \infty) \to \mathbb{R}$. Suppose that f is continuous on $[0, \infty)$ and differentiable on $[100, \infty)$ with bounded derivatives there. Prove that f is uniformly continuous on $[0, \infty)$.

(b) Let $f : (0, 1] \to \mathbb{R}$ be continuous. Is f uniformly continuous on $(0, 1]$? Justify your answer.

2.(a) State, without proof, the Heine-Borel Theorem.

(b) Let δ be a positive function defined on $[a, b]$. Prove that there exist a finite number of interval-point pairs $([u_i, v_i], x_i)$, with $x_i \in [u_i, v_i] \subset (x_i - \delta(x_i), x_i + \delta(x_i))$, $i = 1, 2, \ldots, n$, satisfying the following properties:

(i) $(u_i, v_i) \cap (u_j, v_j) = \phi$ for $i \neq j$;

(ii) $x_i \in [u_i, v_i]$ for each i; and

(iii) $\displaystyle\bigcup_{i=1}^{n} [u_i, v_i] = [a, b]$.

3.(a) Let $f : [a, b] \to \mathbb{R}$. Suppose that f is unbounded on $[a, b]$. Prove that there exists a convergent sequence $\{y_n\}$ in $[a, b]$ such that $|f(y_n)| > n$, for each n.

(b) Use (a) to prove that if f is continuous on $[a, b]$, then f is bounded on $[a, b]$.

4. Let $\{f_n\}$ and $\{g_n\}$ be two sequences of functions defined on $[a, b]$. Suppose that

(i) $\displaystyle\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on $[a, b]$;

(ii) $g_n(x) \leq g_{n+1}(x)$ for all $x \in [a, b]$ and all n; and

(iii) there exists a real number L such that $|g_n(x)| \leq L$ for all $x \in [a, b]$ and all n.

Prove that $\displaystyle\sum_{n=1}^{\infty} f_n(x) g_n(x)$ converges uniformly on $[a, b]$.

Hint: Use Cauchy Criterion and Abel's partial summation

$$\sum_{k=1}^{n} a_k b_k = \sum_{k=1}^{n-1} (a_k - a_{k+1}) B_k + a_n B_n$$

where $B_k = \displaystyle\sum_{i=1}^{k} b_i$.

5. Let $C^*[0,1]$ be the space of all functions $x : [0,1] \to [0,1]$, which are continuous and $x(0) = 0$. Let $f : [0,1] \times [0,1] \to \mathbb{R}$ be continuous. For each $x \in C^*[0,1]$, define $F(x) : [0,1] \to \mathbb{R}$ as follows:

$$F(x)(t) = \int_0^t f(s, x(s))ds \text{ for } t \in [0,1].$$

Let $G = \{F(x) : x \in C^*[0,1]\}$. Prove that

(i) G is sequentially compact i.e., every sequence in G has a subsequence which is uniformly convergent on $[0,1]$;

(ii) $F : C^*[0,1] \to C[0,1]$ is continuous under the uniform norm $\| \ \|$, where $C[0,1]$ is the space of all continuous functions on $[0,1]$ and $\|x\| = \sup\{x(t) : t \in [0,1]\}$.

— END OF PAPER —

Ph.D. Qualifying Examination
Sem 2, 2002/2003
Analysis

(1) [20 marks]

(a) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. If $f'(-1) < 2$ and $f'(1) > 2$, show that there exists $x_0 \in (-1, 1)$ such that $f'(x_0) = 2$. (Hint: consider the function $f(x) - 2x$ and recall the proof of Rolle's theorem)

(b) Let $f : (-1, 1) \to \mathbb{R}$ be a differentiable function on $(-1, 0) \cup (0, 1)$ such that $\lim_{x\to 0} f'(x) = l$. If f is continuous on $(-1, 1)$, show that f is indeed differentiable at 0 and $f'(0) = l$.

(2) Let $\mathbb{P}_n$ be the space of polynomials of degree $\leq n$ on $\mathbb{R}$ for each $n \in \mathbb{N}$. If $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$, define [20 marks]

$$||p||_M = \max\{|a_0|, |a_1|, \cdots, |a_n|\}$$

$$||p||_\infty = \max\{|p(x)| : x \in [0, 1]\}, \text{ and } ||p||_1 = \int_0^1 |p(x)|dx.$$

(i) Show that $|| \cdot ||_1$ is a norm of the space $\mathbb{P}_n$.

(ii) Use the fact that $|| \cdot ||_M$ and $|| \cdot ||_\infty$ are also norms of $\mathbb{P}_n$, or otherwise, to show that there exists a positive constant c_n such that

$$c_n||p||_\infty \leq ||p||_1 \leq (1/c_n)||p||_M$$

for all $p \in \mathbb{P}_n$. (Hint: note that $\mathbb{P}_n \equiv \mathbb{R}^{n+1}$)

(iii) With the help of the Weiestrass approximation theorem, show that there is no positive constant c such that $c_n > c$ for all n. (Note that for each $\varepsilon > 0$, there is a nonnegative continuous function f_ε on $[0, 1]$ such that $f_\varepsilon(0) = 1$ and $||f_\varepsilon||_1 < \varepsilon$.)

(3) Prove or disprove each of the following statements. [40 marks]

(a) If $f : [1,5] \to [1,5]$ is a continuous function, then there exists $x_0 \in [1,5]$ such that $f(x_0) = x_0$.

(b) Let $\{f_n\}$ be a sequence of uniformly continuous functions on an interval I. If $\{f_n\}$ converges uniformly to a function f on I, then f is also uniformly continuous on I.

(c) Let $\{f_n\}$ be a sequence of functions that converges uniformly to a function f on (0,2). If each of the f_n is differentiable on (0,2), then f is also differentiable on (0,2).

(d) If f is a continuous function on [-1,1], then there exists a constant $M > 0$ such that

$$|f(x_1) - f(x_2)| \leq M|x_1 - x_2| \quad \text{for all } x_1, x_2 \in [-1,1].$$

(e) If f is a uniformly continuous function on $(0,5)$, then there exists a positive number ε such that the function $g(x) = 1/(f(x)+\varepsilon)$ is also uniformly continuous on $(0,5)$.

(4) Let $g : [0,1] \times [0,1] \to [0,1]$ be a continuous function and let $\{f_n\}$ be a sequence of functions such that [20 marks]

$$f_n(x) = \begin{cases} 0, & 0 \leq x \leq 1/n, \\ \displaystyle\int_0^{x-\frac{1}{n}} g(t, f_n(t))dt, & 1/n \leq x \leq 1. \end{cases}$$

With the help of the Arzela-Ascoli theorem or otherwise, show that there exists a continuous function $f : [0,1] \to \mathbb{R}$ such that

$$f(x) = \int_0^x g(t, f(t))dt$$

for all $x \in [0,1]$. (Hint: first show that $|f_n(x_1) - f_n(x_2)| \leq |x_1 - x_2|$.)

— END OF PAPER —

Ph.D. Qualifying Examination

Analysis

1. In this question, the metric d used is the usual metric $d(x,y) = |x - y|$.

   (i) Let $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$. Prove that f is uniformly continuous on D if and only if whenever $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ in D with $d(x_n, y_n) \to 0$ as $n \to \infty$, we have $d(f(x_n), f(y_n)) \to 0$ as $n \to \infty$;

   (ii) Let $f : [0,1) \to \mathbb{R}$ be continuous. Is f uniformly continuous on $[0,1)$? Justify your answer; and

   (iii) Let $f : E \to \mathbb{R}$ be uniformly continuous. Is E closed and bounded? Justify your answer.

2. (a) Give four different kinds of metric defined on $\mathbb{R}^n$. (You do not have to justify your answer.);

   (b) Give a metric d defined on $\mathbb{R}^n$ such that $\|\alpha x\| \neq |\alpha| \; \|x\|$, where $\|y\| = d(y,0)$; and

   (c) Let $S \subseteq \mathbb{R}$. Then S is said to have the Bolzano-Weierstrass property if every sequence in S has a convergent subsequence <u>with limit in S</u>.

      (i) Prove that, under the usual metric $d(x,y) = |x - y|$, S has the Bolzano-Weierstrass property if and only if S is bounded and closed.
      (You may use the fact that every bounded sequence has a convergent subsequence.)

      (ii) Is (i) true for any metric defined on $\mathbb{R}$? Justify your answer.

3. (i) Let $\alpha, \beta \geq 0$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Prove that $\frac{\alpha^p}{p} + \frac{\beta^q}{q} \geq \alpha\beta$.

   (ii) Use (i) to prove the following Hölder inquality:

$$\sum_{i=1}^{n} |x_i y_i| \leq \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |y_i|^q \right)^{\frac{1}{q}}.$$

4. Let $B[0,1]$ be the space of all bounded functions defined on $[0,1]$. On $B[0,1]$, define a metric d_∞ as follows:

$$d_\infty(f,g) = \sup\{|f(x) - g(x)| : x \in [0,1]\}.$$

Let $f_k \in B[0,1]$, $M_k \in \mathbb{R}$, $k = 1, 2, \ldots$, and $|f_k(x)| \le M_k$ for all $x \in [0,1]$ and all k. Suppose that $\sum_{k=1}^{\infty} M_k < \infty$. Prove that

(i) $\left(\sum_{k=1}^{n} f_k(x)\right)_{n=1}^{\infty}$ converges in $(B[0,1], d_\infty)$.

(ii) if each f_n is continuous on $[0,1]$, then $\sum_{k=1}^{\infty} f_k(x)$ is continuous on $[0,1]$; and

(iii) if each f_n is Riemann integrable on $[0,1]$, then $\sum_{k=1}^{\infty} f_k(x)$ is Riemann integrable on $[0,1]$ and $\sum_{k=1}^{\infty} \int_0^1 f_k(x)dx = \int_0^1 \sum_{k=1}^{\infty} f_k(x)dx$.

END OF PAPER

NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

SEMESTER 1 2003-2004

Ph.D. QUALIFYING EXAMINATION

PAPER 2

Time allowed : 3 hours

INSTRUCTIONS TO CANDIDATES

1. Answer **ALL** questions from **BOTH** sections.

2. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

Ph.D. Qualifying Examination

Analysis

Notation

$(\mathbb{R}, d_1)$ denotes the metric space of real numbers with metric $d_1(x,y) = |x-y|$.

$(\mathbb{R}, d_p)$ denotes the n-dimensional Euclidean space with metric $d_p(x,y) = \left(\sum\limits_{k=1}^{n} |x_k - y_k|^p\right)^{\frac{1}{p}}$, where $p \geq 1$.

(ℓ_p, d_p) denotes the metric space of real sequences $x = (x_k)_{k=1}^{\infty}$ with $\sum\limits_{k=1}^{\infty} |x_k|^p < \infty$ and $d_p(x,y) = \left(\sum\limits_{k=1}^{\infty} |x_k - y_k|^p\right)^{\frac{1}{p}}$, where $p \geq 1$.

$S_r(x)$ and $S_r[x]$ denote an open sphere and a closed sphere in a metric space resepc-tively.

$\overline{A}$ denotes the closure of A in a metric space.

1. (a) (i) Let $(X, \|\cdot\|)$ be a normed space. Prove that $\overline{S_r(x)} = S_r[x]$.

   (ii) Let $(X, \hat{d})$ be a discrete metric space. Does $\overline{S_r(x)} = S_r[x]$ hold? Justify your answer.

   (b) For each $n = 1, 2, \ldots$, let $S_{\frac{1}{n}}(x_n)$ and $S_{\frac{1}{n}}[x_n]$ be an open sphere and a closed sphere respectively in a complete metric space (X, ρ). Suppose that for each n, $S_{\frac{1}{n+1}}(x_{n+1}) \subseteq S_{\frac{1}{n}}(x_n)$. Prove that $\bigcap\limits_{n=1}^{\infty} S_{\frac{1}{n}}[x_n]$ is not empty.

2. (a) Let $f : ([0,1], d_1) \to (\mathbb{R}, d_1)$. Suppose for each $x \in [0,1]$, there exists $S_{r_x}(x) = \{y : |x-y| < r_x\}$ such that f is bounded on $S_{r_x}(x)$. Prove that f is bounded on $[0,1]$.

   (b) (i) Let f and g be uniformly continuous on $A \subseteq \mathbb{R}$, under the standard distance d_1. Suppose that f and g are bounded on A. Prove that their product fg is uniformly continuous on A.

   (ii) Give an example to show that (i) does not hold if " f and g are bounded on A" is omitted.

3. (a) (i) Let A and B be subsets of $\mathbb{R}$. Suppose that A is compact and B is closed in $(\mathbb{R}, d_1)$. Prove that $A+B$ is closed, where $A+B = \{x+y : x \in A, y \in B\}$.

(ii) Give an example to show that (i) does not hold if "A is compact" is replaced by "A is closed".

(b) Let $(X, \hat{d})$ be a discrete metric space and $A \subseteq X$. Prove that if A is finite, then A is compact. Is the converse true? Justify your answer.

4. (a) Let $(x^{(n)})_{n=1}^{\infty}$ be a squence in $\mathbb{R}^m$. Prove that $d_p(x^{(n)}, x) \to 0$ as $n \to \infty$ if and only if $d_q(x^{(n)}, x) \to 0$ as $n \to \infty$, where p, $q \geq 1$.

(b) Let x, $y \in \mathbb{R}^n$ with $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$ define

$$d_\infty(x, y) = \max\{|x_i - y_i| : i = 1, 2, ..., n\}.$$

Prove that

$$d_\infty(x, y) = \lim_{p \to \infty} d_p(x, y).$$

(c) Let $(x^{(n)})_{n=1}^{\infty}$ be a sequence in (ℓ_p, d_p) with $x^{(n)} = (x_1^{(n)}, x_2^{(n)}, ...)$. Suppose that for each k, $\lim_{n \to \infty} x_k^{(n)} = x_k$ and there exists $y \in \ell_p$ such that $|x_k^{(n)}| \leq |y_k|$ for each k and n. Prove that $d_p(x^{(n)}, x) \to 0$ as $n \to \infty$.

END OF PAPER

NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

SEMESTER 2 2003-2004

Ph.D. QUALIFYING EXAMINATION

PAPER 2

Time allowed : 3 hours

INSTRUCTIONS TO CANDIDATES

1. Answer **ALL** questions.

2. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

Ph.D. Qualifying Examination

Analysis

1. (a) Discuss the convergence, both pointwise and uniform, of

$$S_n(x) = \frac{nx}{1+n^2x^2}, \quad n = 1, 2, \dots .$$

on

(i) $[0, 1]$; and

(ii) $[c, 1]$, where $c > 0$.

(b) Let $S_{m,n} : [a, b] \to \mathbb{R}$, $m = 1, 2, \dots, n = 1, 2, \dots$. Suppose that

(i) for each n, $|S_{m,n}(x)| \leq g_n(x)$ for all m and all $x \in [a, b]$;

(ii) $\sum_{n=1}^{\infty} g_n(x)$ converges uniformly on $[a, b]$; and

(iii) for each n, $S_{m,n}(x) \to S_n(x)$ on $[a, b]$ as $m \to \infty$.

Prove that $\sum_{n=1}^{\infty} S_n(x)$ converges uniformly on $[a, b]$.

2. (a) Let (X, ρ) be a metric space. Prove that (X, ρ) is compact if and only if every class of closed sets with finite intersection property has nonempty intersection. (A class of subsets of X is said to have the finite intersection property if every finite subclass has nonempty intersection.)

(b) Let $(\mathbb{R}^n, d_2)$ be the n-dimensional Euclidean space with the usual metric d_2 and $E = \prod_{i=1}^{n} [a_i, b_i]$ a compact subinterval in $\mathbb{R}^n$. Suppose δ is a positive function defined on E. Prove that there exists a finite collection $\{(I_i, x^{(i)})\}_{i=1}^{m}$ of interval-point pairs, where $x^{(i)} \in I_i \subseteq E$ for all i, such that for each i,

$$x^{(i)} \in I_i \subseteq B\big(x^{(i)}, \delta\big(x^{(i)}\big)\big),$$

where $B(x^{(i)}, \delta(x^{(i)})) = \{y \in \mathbb{R}^n : d_2(x^{(i)}, y) < \delta(x^{(i)})\}$.
Hint: Proof by contradiction.

3. (a) Let (X, ρ) be a metric space with $x_0 \in X$. Define $f : X \to \mathbb{R}$ by $f(x) = \rho(x, x_0)$. Prove that f is uniformly continuous on X.

   (b) Let (X, ρ) be a metric space and A a nonempty subset of X. Let $f(x) = dist(x, A) = \inf\{\rho(x, y) : y \in A\}$.

   Prove that $f : X \to \mathbb{R}$ is continuous. Is f uniformly continuous on X? Justify your answer.

   (c) Prove that in a separate metric space every uncountable set contains a convergent sequence of distinct points.

4. (a) A collection of continuous real-valued functions on a set $S \subseteq \mathbb{R}$ is said to be equicontinuous if for each $\varepsilon > 0$, there is a $\delta > 0$ so that $|f(x) - f(y)| \leq \varepsilon$ when f is in the collection, x, y are in S and $|x - y| \leq \delta$.

   (i) Is the collection $\{\cos nx, n = 1, 2, ...\}$ equicontinuous on $(-\infty, \infty)$? Justify your answer.

   (ii) Let $\{f_n\}$ be equicontinuous on $[a, b]$ and $f_n \to f$ uniformly on $[a, b] \cap \mathbb{Q}$. Prove that $\|f_n - f\|_\infty \to 0$ as $n \to \infty$, where $\|g\|_\infty = \sup\{|g(x)| : x \in [a, b]\}$.

   (b) Prove that every compact metric space is separable.

END OF PAPER

NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

SEMESTER 1 2004-2005

Ph.D. QUALIFYING EXAMINATION

PAPER 2

Time allowed : 3 hours

INSTRUCTIONS TO CANDIDATES

1. Answer **ALL** questions.
2. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

Ph.D. Qualifying Examination

Analysis

Sem 2, 2004/2005

Notation.

$(\mathbb{R}^n, \|\cdot\|_p)$ denotes the n-dimensional Euclidean space with norm $\|x\|_p = (\sum_{k=1}^{n} |x_k|^p)^{\frac{1}{p}}$, where $p \geq 1$.

$(R^n, \|\cdot\|_\infty)$ denotes the n-dimensional Euclidean space with norm $\|x\|_\infty = \max\{|x_k|; k = 1, 2, ..., n\}$.

$(\ell_p, \| \ \|_p)$ denotes the norm space of real sequences $x = (x_k)_{k=1}^{\infty}$ with norm $\|x\|_p = (\sum_{k=1}^{\infty} |x_k|^p)^{\frac{1}{p}} < \infty$, where $p \geq 1$.
$(\ell_\infty, \|\cdot\|_\infty)$ denotes the norm space of bounded real sequences $x = (x_k)_{k=1}^{\infty}$ with norm $\|x\|_\infty = \sup\{|x_k| : k = 1, 2, ...\}$.

1. (a) Let $K \subset \mathbb{R}$ consists of 0 and the numbers $\frac{1}{n}$, for $n = 1, 2, 3, \ldots$. Prove that every open cover of K contains a finite subcover.

   (b) A metric space is called separable if it contains a countable dense subset. Show that $\mathbb{R}^2$ is separable.

   (c) A collection $\{V_\alpha\}$ of open sets of a metric space X is said to be a base for X if the following is true: For every $x \in X$ and every open set $G \subset X$ such that $x \in G$, we have $x \in V_\alpha \subset G$ for some α.

   Prove that every separable metric space has a countable base.

2. If E is a nonempty subset of a metric space (X, d), define the distance from $x \in X$ to E by

$$\rho_E(x) = \inf_{z \in E} d(x, z).$$

(a) Prove that

$$\rho_E(x) \leq d(x, y) + d(y, z)$$

for all $z \in E$ and all x, $y \in X$.

(b) Prove that

$$|\rho_E(x) - \rho_E(y)| \leq d(x, y) \quad \text{for all } x, y \in X.$$

(c) Suppose K and F are disjoint sets in X, K is compact and F is closed. Prove that

(i) ρ_F is a continuous function on the compact set K; and

(ii) there exists $\delta > 0$ such that $d(p, q) > \delta$ for all $p \in K$ and all $q \in F$.

Show that the statement (ii) may fail for two disjoint closed sets if neither is compact.

3. (a) Discuss the convergence, both pointwise and uniform, of

$$S_n(x) = \frac{1 - x^n}{1 - x}, \ n = 1, 2, ...$$

on $(-1, 1)$.

(b) Suppose K is compact and

(α) $\{g_n\}$ is a sequence of continuous functions on K,

(β) $\{g_n\}$ converges to 0 on K,

(γ) $g_n(x) \geq g_{n+1}(x)$ for all $x \in K$, $n = 1, 2, 3, ...$

Let $\varepsilon > 0$ be given and for $n = 1, 2, 3, ...$, K_n the set of all $x \in K$ with $g_n(x) \geq \varepsilon$.

Prove that

(i) each K_n is closed and compact;

(ii) $\bigcap_{n=1}^{\infty} K_n$ is empty; and

(iii) there exists N such that $0 \leq g_n(x) < \varepsilon$ for all $x \in K$ and all $n \geq N$.

4. (a) Let $x \in \mathbb{R}^n$, $x^{(m)} \in \mathbb{R}^n$, $m = 1, 2, ...$, prove that

(i) $\|x^{(m)}\|_p \to 0$ as $m \to \infty$ iff for each k, $x_k^{(m)} \to 0$ as $m \to \infty$, where $x^{(m)} = (x_1^{(m)}, x_2^{(m)}, ..., x_n^{(m)})$; and

(ii) $\|x\|_\infty = \lim_{p \to \infty} \|x\|_p$.

(b) (i) Prove that $\ell_p \subset \ell_q$ if $1 \leq p \leq q \leq \infty$.

(ii) Is (a)(i) true for $x^{(m)} \in \ell_p$, $m = 1, 2, ...$, where $x^{(m)} = (x_1^{(m)}, x_2^{(m)}, ...)$? Justify your answer.

END OF PAPER

NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

SEMESTER 2 2004-2005

Ph.D. QUALIFYING EXAMINATION

PAPER 2

Time allowed : 3 hours

INSTRUCTIONS TO CANDIDATES

1. Answer **ALL** questions.

2. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

Ph.D. Qualifying Examination

Analysis

Sem 1, 2005/2006

1. (a) Give an example of an open cover of $(0,1)$ which has no finite subcover.

   (b) A subset K of a metric space X is said to be compact if every open cover of K contains a finite subcover. Prove that every closed subset (relative to X) of a compact set is compact.

   (c) Let E_n, $n = 1,2,3,...$ be a sequence of countable sets, and $S = \bigcup_{n=1}^{\infty} E_n$. Prove that S is countable.

2. (a) Suppose (X,d) is a complete metric space and $\emptyset \neq A_n \subseteq X$ is closed for $n = 1,2,3,...$ and $A_1 \supseteq A_2 \supseteq A_3 \supseteq ...$ with $\lim_{n\to\infty} d(A_n) = 0$. Prove that $\bigcap_{n=1}^{\infty} A_n$ is a singleton set.

   (b) Let (X,d) be a metric space. Prove that $f : X \to \mathbb{R}$ is continuous if and only if for each open set G in $\mathbb{R}$, $f^{-1}(G)$ is open in X.

3. (a) Prove that $\sum_{k=1}^{\infty} \frac{\sin kx}{k^2}$ is uniformly convergent on $(-\infty,\infty)$.

   (b) Consider the sequence $\{S_n(x)\}$ defined on $[0,1]$ by
   $$S_n(x) = \begin{cases} n - n^2x & \text{if } x \in (0,\frac{1}{n}) \\ 0 & \text{otherwise.} \end{cases}$$
   Does $\{S_n(x)\}$ converge uniformly on $[0,1]$? Justify your answer.

   (c) Is the uniform limit of a sequence of differentiable functions on $[-1,1]$ differentiable on $[-1,1]$? Justify your answer.

4. Let $f : [a,b] \to \mathbb{R}$. Then f is said to be regulated if for each $x \in [a,b]$, $\lim_{t\to x-} f(t)$ and $\lim_{t\to x+} f(t)$ exist.

   Use the Heine-Borel open covering theorem to prove that if f is regulated on $[a,b]$, then for each $\epsilon > 0$, there exists a finite sequence $a = t_0 < t_1 < t_2 < ... < t_n = b$ such that for each $i = 1,2,...,n$ and any two points t', t'' with $t_{i-1} < t' < t'' < t_i$, we have
   $$|f(t'') - f(t')| \leq \epsilon.$$

END OF PAPER

NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

SEMESTER 1 2005-2006

Ph.D. QUALIFYING EXAMINATION

PAPER 2

Time allowed : 3 hours

INSTRUCTIONS TO CANDIDATES

1. Answer **ALL** questions.
2. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

Ph.D. Qualifying Examination

Analysis

Sem 2, 2005/2006

Do All Questions

1. (i) Suppose $\{p_n\}$ and $\{q_n\}$ are Cauchy sequences in a metric space X. Show that the sequence $\{d(p_n, q_n)\}$ converges.

   (ii) Let $\{E_n\}$ be a sequence of closed and bounded sets in a complete metric space. If $E_n \supset E_{n+1}$ for all n and $\lim_{n\to\infty} \text{diam } E_n = 0$, prove that $\bigcap_{n=1}^{\infty} E_n$ consists of exactly one point. Can the condition "complete" be omitted? Justify your answer.

2. (i) Let d be a discrete metric defined on $\mathbb{R}$. What sets are open in $(\mathbb{R}, d)$? What functions are uniformly continuous on $(\mathbb{R}, d)$? Justify your answers.

   (ii) A metric space is called separable if it contains a countable dense subset. Let X be a metric space in which every infinite subset has a limit point. Prove that X is separable.

3. (i) Let $B[0,1]$ be the space of all bounded functions defined on $[0,1]$. Give a norm $\|\cdot\|$ defined on $B[0,1]$ such that $\|f_n - f\| \to 0$ as $n \to \infty$ if and only if $\{f_n\}$ converges to f uniformly on $[0,1]$.

   (ii) Suppose $\{f_n\}$ converges to f uniformly on $[0,1]$ and $\lim_{t\to x} f_n(t)$ exists for each n. Prove that

   (i) $\lim_{n\to\infty}\lim_{t\to x} f_n(t)$ exists, and

   (ii) $\lim_{t\to x}\lim_{n\to\infty} f_n(t) = \lim_{n\to\infty}\lim_{t\to x} f_n(t)$.

4. (a) Let f be an increasing function defined on (a, b). Prove that

   (i) $f(x+)$ and $f(x-)$ exist at every point of x in $(a.b)$,

   (ii) the set of points in (a, b) at which f is discontinuous is at most countable.

   (b) If f is continuous on $[0,1]$ and if

$$\int_0^1 f(x)x^n dx = 0, \quad n = 0, 1, 2, ...,$$

   prove that $f(x) = 0$ on $[0,1]$.

END OF PAPER

NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

SEMESTER 2 2005-2006

Ph.D. QUALIFYING EXAMINATION

PAPER 2

Time allowed : 4 hours

INSTRUCTIONS TO CANDIDATES

1. Answer **ALL** questions.
2. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

Ph.D. Qualifying Examination

Analysis

Sem 1, 2006/2007

Answer All Questions

1. A subset K of a metric space X is said to be compact if every open cover of K contains a finite subcover.

   (a) Prove that if K is a compact subset in a metric space X, the K is closed and bounded. Is the converse true? Justify your answer.

   (b) Prove that if $\{K_\alpha\}$ is a collection of compact subsets of a metric space X such that the intersection of every finite subcollection of $\{K_\alpha\}$ is nonempty, then $\cap K_\alpha$ is nonempty.

2. Let $\delta : [0,1] \times [0,1] \to (0,\infty)$. Use Q.1(b) to show that there exist finite collections of nonoverlapping rectangles $\{I_k\}_{k=1}^n$ and points $\{x_k\}_{k=1}^n$ such that $x_k \in I_k \subseteq B(x_k, \delta(x_k))$ and $\bigcup\limits_{k=1}^{n} I_k = [0,1] \times [0,1]$, where $B(x, \delta(x)) = \{y \in [0,1] \times [0,1]; d(x,y) < \delta(x)\}$ and d is the euclidean metric on $\mathbb{R}^2$.

3. (a) Prove that a function $f : (a,b) \to \mathbb{R}$ is uniformly continuous on (a,b) if and only if it can be extended to a function $\hat{f}$ that is continuous on $[a,b]$.

   (b) Let f and g be real-valued functions that are uniformly continuous on a compact set $D \subseteq \mathbb{R}$. Suppose that $g(x) \neq 0$ for all $x \in D$. Is $\dfrac{f}{g}$ uniformly continuous on D? Justify your answer.

4. The family $\mathcal{F}$ of functions from the metric space (S,d) to the metric space (T,ρ) is called *equicontinuous* on S if given any $\varepsilon > 0$ there is a $\delta > 0$ such that for every $f \in \mathcal{F}$, $\rho(f(x_1), f(x_2)) < \varepsilon$ whenever $d(x_1, x_2) < \delta$. Prove that if (S,d) is a compact metric space and the sequence $f_n : S \to T$ is equicontinuous on S and $f(x) = \lim\limits_{n\to\infty} f_n(x)$ for each $x \in S$ then the sequence $\{f_n\}$ converges uniformly to f on S.

END OF PAPER

NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

SEMESTER 1 2006-2007

Ph.D. QUALIFYING EXAMINATION

PAPER 2

Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

Answer **ALL** questions.

Ph.D. Qualifying Examination

Analysis

Sem 2, 2006/2007

1. (a) Let E be a nonempty subset of $\mathbb{R}$ and suppose that f_k, $g_k : E \to \mathbb{R}$, $k \in \mathbb{N}$, if

$$\left|\sum_{k=1}^{n} f_k(x)\right| \leq M < \infty$$

for $n \in \mathbb{N}$ and $x \in E$, and if $g_k \downarrow 0$ uniformly on E as $k \to \infty$, prove that $\sum_{k=1}^{\infty} f_k g_k$ converges uniformly on E.

(Abel's formula: $\sum_{k=m}^{n} a_k b_k = A_{n,m} b_n - \sum_{k=m}^{n-1} A_{k,m}(b_{k+1} - b_k)$, where $A_{n,m} = \sum_{k=m}^{n} a_k$).

(b) Prove that, for each $x \in (0, 2\pi)$,

$$\left|\sum_{k=1}^{n} \cos(kx)\right| \leq \frac{1}{|\sin\frac{x}{2}|}$$

(Formula: $2 \sin \alpha \cos \beta = \sin(\alpha + \beta) + \sin(\alpha - \beta)$).

(c) Prove that if $a_k \downarrow 0$ as $k \to \infty$, then $\sum_{k=d}^{\infty} a_k \cos(kx)$ converges uniformly on any closed subinterval $[a, b]$ of $(0, 2\pi)$.

2. A metric space is called separable if it contains a countable dense subset. A subset K of a metric space is said to be compact if every open cover of K contains a finite subcover.

Prove that every compact metric space is separable.

3. $f : [a, b] \to \mathbb{R}$ be bounded.

(i) The oscillation of f on an interval J that intersects $[a, b]$ is defined to be

$$\Omega_f(J) := \sup_{x,y \in J \cap [a,b]} (f(x) - f(y)).$$

(ii) The oscillation of f at a point $t \in [a, b]$ is defined to be

$$\omega_f(t) := \lim_{h \to 0_+} \Omega_f((t - h, t + h)).$$

Prove that

(a) f is continuous at $t \in [a,b]$ if and only if $\omega_f(t) = 0$.

(b) let E represent the set of points of discontinuity of f in $[a,b]$. Prove that

$$E = \bigcup_{j=1}^{\infty} \left\{ t \in [a,b] : \omega_f(t) \geq \frac{1}{j} \right\}.$$

(c) For each $\varepsilon > 0$, let

$$H = \{t \in [a,b] : \omega_f(t) \geq \varepsilon\}.$$

Prove that H is compact.

(Hint: H is compact if and only if H is bounded and closed.)

(d) Let I be a closed subinterval of $[a,b]$ and $\varepsilon > 0$. If $\omega_f(t) < \varepsilon$ for all $t \in I$, prove that there exists $\delta > 0$ such that $\Omega_f(J) < \varepsilon$ for all closed subintervals J of I that satisfy $|J| < \delta$.

4. Let $\{f_n\}$ be a uniformly bounded sequence of functions which are Riemann-intergrable on $[a,b]$, and put

$$F_n(x) = \int_a^x f_n(t)dt, \quad (a \leq x \leq b).$$

Prove that there exists a subsequence $\{F_{n_k}\}$ of $\{F_n\}$ which converges uniformly on $[a,b]$.

END OF PAPER

Ph.D. Qualifying Examination

Analysis

Sem 1, 2007/2008

1. (a) A subset K of a metric space X is said to be compact if every open conver of K contains a finite subcover.

   Prove that compact subsets of metric spaces are closed.

   (b) A subset E of a metric space X is said to be perfect if E is closed and if every point of E is a limit point of E.

   Prove that if E is a non-empty perfect set of $\mathbb{R}$. Then E is uncountable.

   (c) Prove that the open interval (a, b) is uncountable.

2. (a) Let $f_n : \mathbb{R} \to \mathbb{R}$ be continuous, $n = 1, 2, ...$ Suppose that

$$f(x) = \sum_{u=0}^{\infty} f_n(x) \quad \text{exists for every} \quad x \in \mathbb{R}.$$

   Is $f : \mathbb{R} \to \mathbb{R}$ continuous ? Justify your answer.

   (b) Suppose $f_n \to f$ uniformly on a set E in a metric space. Let x be a limit point of E and suppose that

$$\lim_{t \to x} f_n(t) = A_n, \quad n = 1, 2, \ldots .$$

   Prove that $\lim_{n \to \infty} A_n$ exists and

$$\lim_{t \to x} \lim_{n \to \infty} f_n(t) = \lim_{n \to \infty} \lim_{t \to x} f_n(t).$$

   (c) Let $\{f_n\}$ be a sequence of continuous functions on $(0, 1)$ such that $\{f_n\}$ converges pointwise to a continuous function on $(0, 1)$ and $f_n(x) \geq f_{n+1}(x)$ for all $x \in (0, 1)$, $n = 1, 2, \ldots$.

   Does $\{f_n\}$ converge uniformly to f on $(0, 1)$? Justify your answer.

3. Let $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$, let $\alpha > 0$. If there exists a constant $k > 0$ such that

$$|f(x) - f(y)| \leq k|x - y|^{\alpha}$$

for all x, $y \in A$, then f is said to be a Lipschitz function of order α on A.

(a) Suppose f is a Lipschitz function of order α on $(0, 1)$ where $\alpha > 1$. Prove that f is differentiable on $(0, 1)$ and find its derivative f'.

(b) Give an example of a Lipschitz function of order $\frac{1}{2}$ but not of order 1 on $[0, 1]$.

(c) Is every uniformly continuous function on $[0, 1]$ is a Lipschitz function of order 1? Justify your answer.

4. (a) Let $f_n : [0, 1] \to \mathbb{R}$ be continuous, $n = 1, 2, \ldots$. Suppose $\{f_n\}$ converges uniformly on $[0, 1]$. Prove that $\{f_n\}$ is equicontinuous on $[0, 1]$.

(b) Let $f_n : [0, 1] \to \mathbb{R}$ be continuous, $n = 1, 2, \ldots.$ Suppose $\{f_n\}$ is pointwise bounded and equicontinuous on $[0, 1]$. Prove that (i) $\{f_n\}$ is uniformly bounded on $[0, 1]$; (ii) $\{f_n\}$ contains a uniformly convergent subsequence.

END OF PAPER

NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

SEMESTER 1 2007-2008

Ph.D. QUALIFYING EXAMINATION

PAPER 2

Time allowed : 4 hours

INSTRUCTIONS TO CANDIDATES

Answer **ALL** questions.

NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

SEMESTER 1 2008-2009

Ph.D. QUALIFYING EXAMINATION

PAPER 2

ANALYSIS

Time allowed : 3 hours

INSTRUCTIONS TO CANDIDATES

1. Answer **ALL** questions.

Ph.D. Qualifying Examination

Year 2008–2009, Semester I

Analysis

Part 1. (65 marks)

1. **[5 points each]** Each of the following statements is either **TRUE** or **FALSE**. Prove the true statements and give counterexamples to the false statements.

   (a) Let X and Y be metric spaces. A function $f: X \to Y$ is uniformly continuous on X if and only if it maps Cauchy sequences in X onto Cauchy sequences in Y.

   (b) If f is a real-valued function defined on $\mathbb{R}^2$ such that f_x and f_y exist on $\mathbb{R}^2$ and are bounded there, then f is continuous on $\mathbb{R}^2$.

   (c) Let $(r_n)_{n=1}^{\infty}$ be an arbitrary sequence of numbers in $[0,1]$. The series
   $$\sum_{n=1}^{\infty} \frac{1}{n^2\sqrt{|x - r_n|}}$$
   converges for almost all x in $[0,1]$.

2. **[10 points]** Let $f : [a,b] \to \mathbb{R}$ be a function of bounded variation on $[a,b]$. For any $x \in [a,b]$, let $V(x)$ be the variation of f on $[a,x]$. Show that if V is absolutely continuous on $[a,b]$, then so is f.

3. **[10 points]** Let f_1 and f_2 be nonnegative Lebesgue measurable functions on $\mathbb{R}$. Suppose that the sets $\{x : f_1(x) > a\}$ and $\{x : f_2(x) > a\}$ are equal in measure for all $a > 0$. Prove that f_1 is Lebesgue integrable if and only if f_2 is Lebesgue integrable; in which case, show that $\int f_1 = \int f_2$.

4. **[15 points]** Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable on $\mathbb{R}$ and assume that f' is continuous on $\mathbb{R}$. Define $F : \mathbb{R} \to \mathbb{R}$ by
   $$F(x) = \int_0^x f(x+t)\,dt.$$
   Show that F is differentiable on $\mathbb{R}$ and that, for all $a \in \mathbb{R}$,
   $$F'(a) = f(2a) + \int_0^a f'(a+t)\,dt.$$

5. **[15 points]** Let $(f_n)_{n=1}^{\infty}$ be a sequence of Lebesgue integrable functions on $[0,1]$ such that

   (a) $\sup_n \int_0^1 |f_n| < \infty$,
   (b) For all $\epsilon > 0$, there exists $\delta > 0$ so that $\sup_n \int_E |f_n| < \epsilon$ for every measurable subset E of $[0,1]$ with $|E| < \delta$.

   Show that if $(f_n)_{n=1}^{\infty}$ converges almost everywhere on $[0,1]$ to a function f, then f is integrable on $[0,1]$ and $\int_0^1 f = \lim_{n\to\infty} \int_0^1 f_n$.
   (It may be helpful to consider the functions $\max(\min(f_n, N), -N)$ for $N \in \mathbb{N}$.)

Part 2 (35 marks)

6. **[13 marks]**

(a) Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function in the complex plane $\mathbb{C}$. Let z_o be a point in $\mathbb{C}$, and let C be the unit circle $|z - z_o| = 1$ centered at z_o and oriented in the counter-clockwise direction. It is given that $f(z) \neq f(z_o)$ for all $\underline{z \text{ inside or on } C \text{ except } z_o}$,

$$f'(z_o) = 2, \quad f''(z_o) = 3 \quad \text{and} \quad \int_C \frac{f'(z)}{f(z) - f(z_o)}\, dz = 2\pi i.$$

Evaluate the integral $\displaystyle\int_C \frac{1}{(f(z) - f(z_o))^2}\, dz$. Justify your answer.

(b) Solve the equation $4 + \cos z = 2\sinh(iz)$. Express your answers in Cartesian form.

7. **[12 marks]** Let $D := \{z \in \mathbb{C} : |z| < 2\}$ denote the disc of radius 2 and centered at the origin in the complex plane $\mathbb{C}$. Suppose the function $f : D \setminus \{\frac{i}{2}\} \to \mathbb{C}$ is analytic in $D \setminus \{\frac{i}{2}\}$, and f has a simple pole at the point $z = \frac{i}{2}$. Let $\sum_{n=0}^{\infty} a_n z^n$ denote the Maclaurin series of f. It is also given that $a_n \neq 0$ for all $n \geq 0$. Is it true that

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = -2i\,?$$

Justify your answer.

8. **[10 marks]** Let $f : \mathbb{C} \to \mathbb{C}$ be a <u>non-constant</u> entire function in the complex plane $\mathbb{C}$ such that $f(z + i) = f(z)$ for all $z \in \mathbb{C}$. Let U be an open subset of $\mathbb{C}$, and let $z_o \in U$. Suppose $g : U \setminus \{z_o\} \to \mathbb{C}$ is an analytic function on $U \setminus \{z_o\}$. It is given that z_o is not a removable singularity of g. Is it true that $f \circ g$ has an essential singularity at z_o? Justify your answer.

END OF PAPER