

# Cohomological induction and Theta correspondence and the unitarizability of the constructions

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April 30, 2011



## Notations

We follow Wallach's approach [?] [?]

$G$  a real reductive Lie group of inner type

$\theta$  Cartan involution of  $G$ , with compact subalgebra  $\mathfrak{k}$ .

$\mathfrak{h}$  fundamental Cartan subalgebra of  $\mathfrak{g}$

$\mathfrak{t}$   $\mathfrak{k} \cap \mathfrak{h}$  is maximal abelian in  $\mathfrak{k}$

$H$  a element in  $\mathfrak{it}$ .

$$\mathfrak{l} = \{ X \in \mathfrak{g} \mid [H, X] = 0 \}$$

$$\mathfrak{u} = \{ X \in \mathfrak{g} \mid [H, X] = \lambda X, \lambda > 0 \}$$

$$\mathfrak{u}_k = \mathfrak{u} \cap \mathfrak{k}_{\mathbb{C}}$$

$\mathfrak{q} = \mathfrak{l}_{\mathbb{C}} + \mathfrak{u}$ ,  $\theta$ -stable parabolic subalgebra.

$$\mathfrak{m} = \mathfrak{k} \cap \mathfrak{l}.$$

$$M = \{ g \in K \mid \text{Ad}(g)H = H \} = K \cap L.$$



## Cohomology induction

Fix a  $(\mathfrak{l}, M)$ -module  $W$ .

- (I) Regard  $W$  as a  $(\mathfrak{q}, M)$ -module on which  $\mathfrak{u}$  act trivially.
  - (II) Define  $M(\mathfrak{q}, W) = U(\mathfrak{g}_{\mathbb{C}}) \otimes_{U(\mathfrak{q})} W$ , which is a  $(\mathfrak{g}, M)$ -module.
  - (III) Apply the Zuckerman functor,  $\Gamma^j(M(\mathfrak{q}, W))$ .
- $\Gamma^j(M(\mathfrak{q}, W))$  is trivial for all  $j < n$ . ( $n = \dim \mathfrak{u}_k$ ).

## Zuckerman functor

The Zuckerman functor is defined by  $(H(K): \text{left } K\text{-finite smooth functions on } K)$

$$\Gamma^j(V) = H^j(\mathfrak{k}, M; V \otimes H(K)).$$

$H^j(V)$  is the  $j$ -th cohomology group of the cochain complex

$$C^j(\mathfrak{k}, M; V) = \text{Hom}_M\left(\bigwedge^j(\mathfrak{k}/\mathfrak{m}), V\right)$$

$\Gamma^j$  is a functor from  $\mathcal{C}(\mathfrak{g}, M)$  to  $\mathcal{C}(\mathfrak{g}, K)$ , where  $K$  action given by right transform of  $H(K)$  and  $\mathfrak{g}$  action given by the commutative relation

$$\begin{array}{ccc} \Gamma^j(U(\mathfrak{g}) \otimes V) & \xrightarrow{T_{U(\mathfrak{g})}(V)} & U(\mathfrak{g}) \otimes \Gamma^j(V) \\ \Gamma^j(m) \downarrow & & \downarrow m \\ \Gamma^j(V) & \xrightarrow{\text{Identity}} & \Gamma^j(V) \end{array}$$

## Hermitian Form

Now assume that  $W$  is an irreducible  $(\mathfrak{l}, M)$ -module admitting a positive definite Hermitian form  $\langle \cdot, \cdot \rangle$ . We construct a hermitian form on  $\Gamma^n(M(\mathfrak{q}, W))$ .

(I) Shapovalov Form on  $M(\mathfrak{q}, W) = U(\mathfrak{g}_{\mathbb{C}}) \otimes_{U(\mathfrak{q})} W$ .

$$(x \otimes w, y \otimes v) = \langle p(y^* x)w, v \rangle,$$

$p$  — projection from  $U(\mathfrak{g}_{\mathbb{C}})$  to  $U(\mathfrak{l}_{\mathbb{C}})$  by decomposition

$$U(\mathfrak{g}_{\mathbb{C}}) = U(\mathfrak{l}_{\mathbb{C}}) \oplus (\bar{\mathfrak{u}}U(\mathfrak{g}_{\mathbb{C}}) + U(\mathfrak{g}_{\mathbb{C}})\mathfrak{u}).$$

(II) Hermitian form on  $\Gamma^n(M(\mathfrak{q}, W))$  — by the nature pairing between complexes  $C^j(M(\mathfrak{q}, W)) \subset \bigwedge^j(\mathfrak{t}_{\mathbb{C}}/\mathfrak{m}_{\mathbb{C}})^* \otimes M(\mathfrak{q}, W)$  and  $C^{(2n-j)}(M(\mathfrak{q}, W))$ :

$$\langle \alpha \otimes w, \beta \otimes v \rangle = (\alpha, \beta)(w, v)$$

If  $M(\mathfrak{q}, W)$  is also irreducible,  $(\cdot, \cdot)$  non-degenerate and only  $\Gamma^n(M(\mathfrak{q}, W))$  nontrivial

## Signature character

For a  $(\mathfrak{g}, K)$  module  $V$  define the *character* of it:

$$\mathrm{ch}_M(V) = \sum_{\gamma \in \widehat{M}} \dim \mathrm{Hom}_M(F_\gamma, V) \gamma.$$

$V$  admits Hermitian form  $\langle \cdot, \cdot \rangle$ , then define *signature character*:

$$\mathrm{ch}_s(V, \langle \cdot, \cdot \rangle) = \sum_{\gamma \in \widehat{M}} \frac{1}{d(\gamma)} (p_\gamma - q_\gamma) \gamma$$

$(p_\gamma, q_\gamma)$  is the signature of the Hermitian form  $\langle \cdot, \cdot \rangle$  restricted on isotropic component  $V(\gamma)$ .

Note that  $\langle \cdot, \cdot \rangle$  is definite if and only if

$$\mathrm{ch}_s(V, \langle \cdot, \cdot \rangle) = \pm \mathrm{ch}_M(V)$$

## Unitarizability I

By vanishing theorems and the construction of the Hermitian form on  $\Gamma^j(V)$  we have:

$$\begin{aligned} \mathrm{ch}_s(\Gamma^n(V)) &= \sum_{\gamma \in \widehat{K}} \mathrm{sgn}(H^n(\mathfrak{t}, M; V \otimes F_\gamma^*) \otimes F_\gamma) \\ &= \sum_{\gamma \in \widehat{K}} \mathrm{sgn}\left(\bigoplus_j C^j(\mathfrak{t}, M; V \otimes F_\gamma^*)\right) \mathrm{ch}_M \gamma \end{aligned}$$

Hence reduce computing  $\mathrm{ch}_s(\Gamma^n(M(\mathfrak{q}, W)))$  to computing

$$\mathrm{ch}_s\left(\bigwedge^j (\mathfrak{t}_{\mathbb{C}}/\mathfrak{m}_{\mathbb{C}})^* \otimes M(\mathfrak{q}, W) \otimes F_\gamma^*\right)$$

The key is to compute

$$\mathrm{ch}_s(M(\mathfrak{q}, W)).$$



## Unitarizability II

Now we use a “continuous argument”.

Assume there exists  $\mu \in i\mathfrak{l}^*$  such that  $(\mu, \alpha) > 0$  for  $\alpha \in \Delta(\mathfrak{u}_k, \mathfrak{t}_{\mathbb{C}})$  and some technical condition.  $M(\mathfrak{q}, W \otimes C_{-t\mu})$  is irreducible for  $t \geq 0$ .

- (I) the decomposition of  $M(\mathfrak{q}, W \otimes C_{-t\mu})$  into eigenspaces is independent of  $t$ ;
- (II) signature takes integer value, then is constant for all  $t \geq 0$ ;
- (III) Key formula:

$$\begin{aligned} & \left\langle X^I Y^J \otimes w, X^{I'} Y^{J'} \otimes w' \right\rangle_t \\ &= t^{|I|+|J|} \delta_{I,I'} \delta_{J,J'} (-1)^{|J|} \prod_a (\mu, \alpha_a)^{i_a} \prod_b (\mu, \beta_b)^{j_b} \langle w, w' \rangle \end{aligned}$$

+ lower degree terms.

This gives an expression for  $\text{ch}_s(M(\mathfrak{q}, W))$  by  $\text{ch}_M(W)$ .





## Representations of double cover of Symplectic group

For symplectic group  $\mathrm{Sp}$  (except complex case), for each central character  $\chi$  there is an *oscillator representation*  $\omega = \omega_\chi$  of the double cover  $\widetilde{\mathrm{Sp}}$  of  $\mathrm{Sp}$ .

For any subgroup  $G$  of  $\mathrm{Sp}$ , let  $\widetilde{G}$  be the perimage of projection  $\varphi: \widetilde{\mathrm{Sp}} \rightarrow \mathrm{Sp}$ . Suppose  $\omega$  is realized on  $\mathcal{Y}$ , with smooth vector  $\mathcal{Y}^\infty$ . Let

$$\mathcal{R}(\widetilde{G}, \omega) = \left\{ \begin{array}{l} \text{continuous} \\ \text{irreducible} \\ \text{admissible} \end{array} \rho \text{ of } \widetilde{G} \mid \rho \simeq \mathcal{Y}^\infty / \mathcal{N}_\rho \right\} / \text{infinitesimal equivalence}$$

## Reductive dual pair in Symplectic group

$(G, G')$  called a reductive dual pair if  $G, G'$  are reductive subgroups of some symplectic group  $\mathrm{Sp}$  and are mutual centralizers.

We can only consider irreducible dual pairs. Let  $\mathrm{Sp} = \mathrm{Sp}(W)$

**Type I**  $W = V \otimes_D V'$ ,  $D$  with involution  $\tau$ .  $G$  and  $G'$  are group of isometries of sesqui-linear hermitian form  $(,)$  and skew-hermitian  $(,)'$ .

$$\langle v_1 \otimes v'_1, v_2 \otimes v'_2 \rangle = \mathrm{tr}((v_1, v_2)(v'_2, v'_1)')$$

E.g.  $(O_{p,q}, \mathrm{Sp}_{2n}(\mathbb{R})) \subset \mathrm{Sp}_{2(p+q)n}(\mathbb{R})$ .

**Type II**  $W = U \oplus U^*$ ,  $U = V \otimes_D V'$ ,  $U^* = V^* \otimes_D V'^*$   
 $G = \mathrm{GL}_D(V)$ ,  $G' = \mathrm{GL}_D(V')$ .

E.g.  $(\mathrm{GL}_m(\mathbb{R}), \mathrm{GL}_n(\mathbb{R})) \subset \mathrm{Sp}_{2nm}(\mathbb{R})$



## Theta correspondence

We follow [?]. There is a one-one correspondence between  $\mathcal{R}(\tilde{G}, \omega)$  and  $\mathcal{R}(\tilde{G}', \omega)$ .  $\mathcal{R}(\tilde{G}\tilde{G}', \omega)$  is the graph of this bijection.

Steps to give this correspondence:

- (1) Given  $\rho' \in \mathcal{R}(\tilde{G}', \omega)$ ,
- (2) Define

$$\mathcal{Y}_{\rho'} = \bigcap_{\rho \cong \mathcal{Y}^\infty / \mathcal{Y}_1} \mathcal{Y}_1.$$

- (3) Consider  $\mathcal{Y}^\infty / \mathcal{Y}_{\rho'}$  be a  $\tilde{G}\tilde{G}'$  module, then

$$\mathcal{Y}^\infty / \mathcal{Y}_{\rho'} = \rho_1 \otimes \rho'$$

$\rho_1$  is a  $\tilde{G}$  module

- (4)  $\rho_1$  has a unique irreducible quotient  $\rho$ .
- (5)  $\rho' \leftrightarrow \rho$  gives the correspondence.

The proof is purely algebraic by classical invariant theory.



## Unitary version

Want to know when this correspondence preserves unitarity. Li [?] gives results in *stable range* for Type I reductive dual pair.  $\widehat{\widetilde{G}}(\varepsilon)$  — unitary dual of  $\widetilde{G}$  with nontrivial action on  $\mathbb{Z}_2 = \pm 1$ .

- (1) Fix realizations  $\mathscr{Y}^\infty$  and  $H_\sigma^\infty$  of  $\omega$  and  $\sigma \in \widehat{\widetilde{G}'}(\varepsilon)$
- (2) Consider  $\mathscr{Y}^\infty \otimes H_\sigma^\infty$ .
- (3)

$$(\Phi, \Phi')_\sigma = \int_{G'} (\Phi, (\omega \otimes \sigma)(g)\Phi') dg$$

- (4) Make sense in *stable range*: Witt index of  $V \geq \dim V'$ .
- (5)

$$R = \{ \Phi \in \mathscr{Y}^\infty \otimes H_\sigma^\infty \mid (\Phi, \Phi')_\sigma = 0 \text{ for all } \Phi' \}$$

- (6)  $H(\sigma) \triangleq (\mathscr{Y}^\infty \otimes H_\sigma^\infty)/R$  gives irreducible unitary representation denoted by  $\pi(\sigma)$ .



Howe's duality correspondence gives an injection (in the stable range)

$$\widehat{\widehat{G'}}(\varepsilon) \hookrightarrow \widehat{\widehat{G}}(\varepsilon).$$

This correspondence is the same as Howe's by  $\sigma \rightarrow \pi(\sigma^*)$ .  
The proof is by analysis.



## Unitary dual

A central problem in representation theory is to understand the unitary dual of group  $G$ .

For real reductive group, we can construct unitary representations by parabolic induction. This gives principle series.

But they are not all. It may also have discrete series (can be constructed by cohomological induction or other methods) and other types.

Theta-correspondence gives another way to construct.

There seems to be some relation between cohomological induction and theta-correspondence [?]. We are trying to find some kind of such relations.



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