Cohomological induction and Theta correspondence

and the unitarizability of the constructions

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Notations

We follow Wallach's approach [Wal84] [Wal88]

G a real reductive Lie group of inner type

θ Cartan involution of G, with compact subalgebra $\mathfrak{k}$.

$\mathfrak{h}$ fundamental Cartan subalgebra of $\mathfrak{g}$

$\mathfrak{t}$ $\mathfrak{k} \cap \mathfrak{h}$ is maximal abelian in $\mathfrak{k}$

H a element in $i\mathfrak{t}$.

$\mathfrak{l}$ $= \{ X \in \mathfrak{g} \mid [H, X] = 0 \}$

$\mathfrak{u}$ $= \{ X \in \mathfrak{g} \mid [H, X] = \lambda X, \lambda > 0 \}$

$\mathfrak{u}_k$ $= \mathfrak{u} \cap \mathfrak{k}_{\mathbb{C}}$

$\mathfrak{q}$ $= \mathfrak{l}_{\mathbb{C}} + \mathfrak{u}$, θ-stable parabolic subalgebra.

$\mathfrak{m}$ $= \mathfrak{k} \cap \mathfrak{l}$.

M $= \{ g \in K \mid \mathrm{Ad}(g)H = H \} = K \cap L$.

Cohomology induction

Fix a $(\mathfrak{l}, M)$-module W.

(I) Regard W as a $(\mathfrak{q}, M)$-module on which $\mathfrak{u}$ act trivially.

(II) Define $M(\mathfrak{q}, W) = U(\mathfrak{g}_{\mathbb{C}}) \otimes_{U(\mathfrak{q})} W$, which is a $(\mathfrak{g}, M)$-module.

(III) Apply the Zuckerman functor, $\Gamma^j(M(\mathfrak{q}, W))$.

$\Gamma^j(M(\mathfrak{q}, W))$ is trivial for all $j < n$. ($n = \dim \mathfrak{u}_k$).

Zuckerman functor

The Zuckerman functor is defined by ($H(K)$:left K-finite smooth functions on K)

$$\Gamma^j(V) = \mathrm{H}^j(\mathfrak{k}, M; V \otimes H(K)).$$

$\mathrm{H}^j(V)$ is the j-th cohomology group of the cochian complex

$$C^j(\mathfrak{k}, M; V) = \mathrm{Hom}_M(\bigwedge^j(\mathfrak{k}/\mathfrak{m}), V)$$

Γ^j is a functor from $\mathscr{C}(\mathfrak{g}, M)$ to $\mathscr{C}(\mathfrak{g}, K)$, where K action given by right transform of $H(K)$ and $\mathfrak{g}$ action given by the commutative relation

$$\begin{tikzcd}
\Gamma^j(U(\mathfrak{g}) \otimes V) \arrow[r, "T_{U(\mathfrak{g})}(V)"] \arrow[d, "\Gamma^j(m)"'] & U(\mathfrak{g}) \otimes \Gamma^j(V) \arrow[d, "m"] \\
\Gamma^j(V) \arrow[r, "\mathrm{Identity}"] & \Gamma^j(V)
\end{tikzcd}$$

Hermitian Form

Now assume that W is an irreducible $(\mathfrak{l}, M)$-module admitting a positive definite Hermitian form $\langle \cdot, \cdot \rangle$. We construct a hermitian form on $\Gamma^n(M(\mathfrak{q}, W))$.

(I) Shapovalov Form on $M(\mathfrak{q}, W) = U(\mathfrak{g}_{\mathbb{C}}) \otimes_{U(\mathfrak{q})} W$.

$$(x \otimes w, y \otimes v) = \langle p(y^* x) w, v \rangle ,$$

p — projection from $U(\mathfrak{g}_{\mathbb{C}})$ to $U(\mathfrak{l}_{\mathbb{C}})$ by decomposition

$$U(\mathfrak{g}_{\mathbb{C}}) = U(\mathfrak{l}_{\mathbb{C}}) \oplus (\overline{\mathfrak{u}} U(\mathfrak{g}_{\mathbb{C}}) + U(\mathfrak{g}_{\mathbb{C}}) \mathfrak{u}).$$

(II) Hermitian form on $\Gamma^n(M(\mathfrak{q}, W))$ — by the nature pairing between complexes $C^j(M(\mathfrak{q}, W)) \subset \bigwedge^j (\mathfrak{t}_{\mathbb{C}}/\mathfrak{m}_{\mathbb{C}})^* \otimes M(\mathfrak{q}, W)$ and $C^{(2n-j)}(M(\mathfrak{q}, W))$:

$$\langle \alpha \otimes w, \beta \otimes v \rangle = (\alpha, \beta)(w, v)$$

If $M(\mathfrak{q}, W)$ is also irreducible, $(\cdot, \cdot)$ non-degenerate and only $\Gamma^n(M(\mathfrak{q}, W))$ nontrivial.

Signature character

For a $(\mathfrak{g}, K)$ module V define the *character* of it:

$$\mathrm{ch}_M(V) = \sum_{\gamma \in \widehat{M}} \dim \mathrm{Hom}_M(F_\gamma, V)\gamma.$$

V admits Hermitian form $\langle \cdot, \cdot \rangle$, then define *signature character*:

$$\mathrm{ch}_s(V, \langle \cdot, \cdot \rangle) = \sum_{\gamma \in \widehat{M}} \frac{1}{d(\gamma)}(p_\gamma - q_\gamma)\gamma$$

(p_γ, q_γ) is the signature of the Hermitian form $\langle \cdot, \cdot \rangle$ restricted on isotropic component $V(\gamma)$.
Note that $\langle \cdot, \cdot \rangle$ is definite if and only if

$$\mathrm{ch}_s(V, \langle \cdot, \cdot \rangle) = \pm \mathrm{ch}_M(V)$$

Unitarizablity I

By vanishing theorems and the construction of the Hermition form on $\Gamma^j(V)$ we have:

$$\begin{aligned}\mathrm{ch}_s(\Gamma^n(V)) &= \sum_{\gamma\in\widehat{K}} \mathrm{sgn}(H^n(\mathfrak{t},M;V\otimes F_\gamma^*)\otimes F_\gamma) \\ &= \sum_{\gamma\in\widehat{K}} \mathrm{sgn}(\bigoplus_j C^j(\mathfrak{t},M;V\otimes F_\gamma^*))\mathrm{ch}_M\gamma\end{aligned}$$

Hence reduce computing $\mathrm{ch}_s(\Gamma^n(M(\mathfrak{q},W)))$ to computing

$$\mathrm{ch}_s(\bigwedge^j(\mathfrak{t}_{\mathbb{C}}/\mathfrak{m}_{\mathbb{C}})^*\otimes M(\mathfrak{q},W)\otimes F_\gamma^*)$$

The key is to compute

$$\mathrm{ch}_s(M(\mathfrak{q},W)).$$

Unitarizablity II

Now we use a "continous argument".
Assume there exists $\mu \in i\mathfrak{l}^*$ such that $(\mu, \alpha) > 0$ for $\alpha \in \Delta(\mathfrak{u}_k, \mathfrak{t}_{\mathbb{C}})$ and some technical condition. $M(\mathfrak{q}, W \otimes C_{-t\mu})$ is irreducible for $t \geq 0$.

(I) the decomposition of $M(\mathfrak{q}, W \otimes C_{-t\mu})$ into eigenspaces is independent of t;

(II) signature takes integer value, then is constant for all $t \geq 0$;

(III) Key formula:

$$\begin{aligned} &\left\langle X^I Y^J \otimes w, X^{I'} Y^{J'} \otimes w' \right\rangle_t \\ =&t^{|I|+|J|}\delta_{I,I'}\delta_{J,J'}(-1)^{|J|}\prod_a (\mu, \alpha_a)^{i_a} \prod_b (\mu, \beta_b)^{j_b} \langle w, w' \rangle \\ &+ \text{lower degree terms.} \end{aligned}$$

This gives an expression for $\mathrm{ch}_s(M(\mathfrak{q}, W))$ by $\mathrm{ch}_M(W)$.

For simplectic group Sp (except complex case), for each central character χ there is an *oscillator representation* $\omega = \omega_\chi$ of the double cover $\widetilde{\mathrm{Sp}}$ of Sp.
For any subgroup G of Sp, let $\widetilde{G}$ be the perimage of projecetion $\varphi\colon \widetilde{\mathrm{Sp}} \to \mathrm{Sp}$. Suppose ω is realized on $\mathscr{Y}$, with smooth vector $\mathscr{Y}^\infty$. Let

$$\mathscr{R}(\widetilde{G}, \omega) = \left\{ \begin{array}{c} \text{countinous} \\ \text{irreducible} \\ \text{admissible} \end{array} \ \rho \text{ of } \widetilde{G} \ \middle| \ \rho \simeq \mathscr{Y}^\infty / \mathscr{N}_\rho \right\} \Big/ \begin{array}{c} infinitesimal \\ equivalence \end{array}$$

Reductive dual pair in Symplectic group

(G, G') called a reductive dual pair if G, G' are reductive subgroups of some symplectic group Sp and are mutual centralizers.
We can only consider irreducible dual pairs. Let $\mathrm{Sp} = \mathrm{Sp}(W)$

Type I $W = V \otimes_D V'$, D with involution τ. G and G' are group of isometries of sesqui-linear hermitian form $(,)$ and skew-hermitian $(,)'$.

$$\langle v_1 \otimes v_1', v_2 \otimes v_2' \rangle = \mathrm{tr}((v_1, v_2)(v_2', v_1')')$$

E.g. $(O_{p,q}, \mathrm{Sp}_{2n}(\mathbb{R}))) \subset Sp_{2(p+q)n}(\mathbb{R})$.

Type II $W = U \oplus U^*$, $U = V \otimes_D V'$, $U^* = V^* \otimes_D V'^*$
$G = \mathrm{GL}_D(V)$, $G' = \mathrm{GL}_D(V')$.
E.g. $(\mathrm{GL}_m(\mathbb{R}), \mathrm{GL}_n(\mathbb{R})) \subset \mathrm{Sp}_{2nm}(\mathbb{R})$

Theta correspondence

We follow [How89]. There is a one-one correspondence between $\mathscr{R}(\widetilde{G}, \omega)$ and $\mathscr{R}(\widetilde{G}', \omega)$. $\mathscr{R}(\widetilde{G}\widetilde{G}', \omega)$ is the graph of this bijection.
Steps to give this correspondence:

(1) Given $\rho' \in \mathscr{R}(\widetilde{G}', \omega)$,

(2) Define

$$\mathscr{Y}_{\rho'} = \bigcap_{\rho \cong \mathscr{Y}^\infty / \mathscr{Y}_1} \mathscr{Y}_1.$$

(3) Consider $\mathscr{Y}^\infty / \mathscr{Y}_{\rho'}$ be a $\widetilde{G}\widetilde{G}'$ module, then

$$\mathscr{Y}^\infty / \mathscr{Y}_{\rho'} = \rho_1 \otimes \rho'$$

ρ_1 is a $\widetilde{G}$ module

(4) ρ_1 has a unique irreducible quotient ρ.

(5) $\rho' \leftrightarrow \rho$ gives the correspondence.

The proof is purely algebraic by classical invariant theory.

Unitary version

Want to know when this correspondence preserves unitarity.
Li [Li89] gives results in *stable range* for Type I reductive dual pair.
$\widehat{\widetilde{G}}(\varepsilon)$ — unitary dual of $\widetilde{G}$ with nontrivial action on $\mathbb{Z}_2 = \pm 1$.

(1) Fix realizaitions $\mathscr{Y}^\infty$ and H_σ^∞ of ω and $\sigma \in \widehat{\widetilde{G}'}(\varepsilon)$

(2) Consider $\mathscr{Y}^\infty \otimes H_\sigma^\infty$.

(3)

$$(\Phi, \Phi')_\sigma = \int_{G'} (\Phi, (\omega \otimes \sigma)(g)\Phi')dg$$

(4) Make sense in *stable range*: Witt index of $V \geq \dim V'$.

(5)

$$R = \left\{ \Phi \in \mathscr{Y}^\infty \otimes H_\sigma^\infty \;\middle|\; (\Phi, \Phi')_\sigma = 0 \text{ for all} \Phi' \right\}$$

(6) $H(\sigma) \triangleq (\mathscr{Y}^\infty \otimes H_\sigma^\infty)/R$ gives irreducible unitary representation denoted by $\pi(\sigma)$.

Howe's duality correspondence gives an injection (in the stable range)

$$\widehat{\widetilde{G'}}(\varepsilon) \hookrightarrow \widehat{\widetilde{G}}(\varepsilon).$$

This correspondence is the same as Howe's by $\sigma \to \pi(\sigma^*)$. The proof is by analysis.

Unitary dual

A central problem in represnetaion theory is to understand the unitary dual of group G.

For real reductive group, we can construct unitary representations by parabolic induction. This gives $\mathrm{principle\ series}$.

But they are not all. It may also have $\mathrm{discrete\ series}$ (can be constructed by cohomological induction or other methods) and other types.

Theta-correspondence gives another way to construct.

There seems to be some relation between cohomological induction and theta-correspondence [WZ04]. We are trying to find some kind of such relations.

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