MTH20014 Engineering Mathematics 3B. Tutorial 5

1. Write the following quadratic forms as $V(\mathbf{x}) = \mathbf{x}^T\mathbf{A}\mathbf{x} = 1$, determine their canonical forms, find the modal matrices (i.e. the matrices of unit eigenvectors) of the corresponding transformations and write down explicite expressions for canonical coordinates (y_1, y_2, y_3) in terms of the original coordinates (x_1, x_2, x_3). State what surfaces these quadratic forms correspond to:

   (a) $-x_1^2 + x_2^2 + x_3^2 + 4x_1x_2 + 4x_1x_3 - 8x_2x_3 = 1$;

   (b) $x_1^2 - 3x_2^2 - 3x_3^2 + 4x_1x_2 + 4x_1x_3 + 12x_2x_3 = 1$;

   (c) $4x_1^2 + 2x_2^2 + 2x_3^2 + 2x_1x_2 + 2x_1x_3 + 6x_2x_3 = 1$.

2. Solve the following systems of differential equations using the eigenvector and matrix exponential techniques:

   (a) $\begin{array}{rcl} \dot{x} &=& 3x + 4y \\ \dot{y} &=& 4x - 3y \end{array}$ $x(0) = 5,\ y(0) = 1$;

   (b) $\begin{array}{rcl} \dot{x} &=& 3x + y \\ \dot{y} &=& -2x \end{array}$ $x(0) = 9,\ y(0) = 3$;

   (c) $\begin{array}{rcl} \dot{x} &=& 6x + 2y \\ \dot{y} &=& 2x + 3y \end{array}$.

Answers

1. (a) $-5y_1^2 + 5y_2^2 + y_3^2$, one-sheet hyperboloid, $\mathbf{P} = \begin{bmatrix} \frac{-1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}$,

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}}(-x_1 + x_2 + x_3) \\ \frac{1}{\sqrt{2}}(-x_2 + x_3) \\ \frac{1}{\sqrt{6}}(2x_1 + x_2 + x_3) \end{bmatrix};$$

(b) $-9y_1^2 + 5y_2^2 - y_3^2$, two-sheet hyperboloid, $\mathbf{P} = \begin{bmatrix} 0 & \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}$,

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}}(-x_2 + x_3) \\ \frac{1}{\sqrt{3}}(x_1 + x_2 + x_3) \\ \frac{1}{\sqrt{6}}(-2x_1 + x_2 + x_3) \end{bmatrix};$$

(c) $6y_1^2 + 3y_2^2 - y_3^2$, one-sheet hyperboloid, $\mathbf{P} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{bmatrix}$,

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}}(x_1 + x_2 + x_3) \\ \frac{1}{\sqrt{6}}(-2x_1 + x_2 + x_3) \\ \frac{1}{\sqrt{2}}(-x_2 + x_3) \end{bmatrix}.$$

2. (a) $x = \dfrac{3}{5}e^{-5t} + \dfrac{22}{5}e^{5t}$, $y = -\dfrac{6}{5}e^{-5t} + \dfrac{11}{5}e^{5t}$;

(b) $x = -12e^t + 21e^{2t}$, $y = 24e^t - 21e^{2t}$;

(c) $x = c_1e^{2t} + 2c_2e^{7t}$, $y = -2c_1e^{2t} + c_2e^{7t}$.