Harmonic functions

$$f(z) = u(x,y) + i\, v(x,y) \qquad \text{CRC:}$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad ; \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \quad ; \quad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x} = 0$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{- Laplace's eqn}$$

$$\nabla^2 u = 0$$

Harmonic conjugate:

$u = x^3 - 3xy^2 + 4xy$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 4y, \quad \frac{\partial^2 u}{\partial x^2} = 6x$$

$$\frac{\partial u}{\partial y} = -6xy + 4x, \quad \frac{\partial^2 u}{\partial y^2} = -6x$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x - 6x = 0$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow \frac{\partial v}{\partial y} = 3x^2 - 3y^2 + 4y \Rightarrow v = 3x^2y - y^3 + 2y^2 + f(x)$$

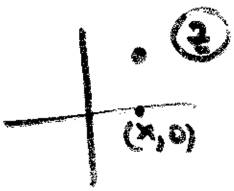
$$-\frac{\partial u}{\partial y} = +\frac{\partial v}{\partial x} \Rightarrow \frac{\partial v}{\partial x} = 6xy - 4x \Rightarrow \cancel{6xy} + \frac{df}{dx} = \cancel{6xy} - 4x$$

$$\frac{df}{dx} = -4x \Longrightarrow f = -2x^2 + C$$

$$v = 3x^2y - y^3 + 2y^2 - 2x^2 + C$$

$$W(x+iy) = x^3 - 3xy^2 + 4xy + i(3x^2y - y^3 + 2y^2 - 2x^2 + c)$$

$W(z) - ?$: Set $y = 0$ and then replace $x \to z$



$$W(x) = x^3 + i(-2x^2 + c)$$

$$W(z) = z^3 + i(-2z^2 + c)$$

$$= z^3 - 2iz^2 + iC, \quad C \in R$$

$$= z^3 - 2iz^2 + C, \quad C \in Z$$

Complex Taylor series

$$f(x) = \frac{f(x_0)}{0!} + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \dots$$

$$f(z) = f(z_0) + f'(z_0)(z-z_0) + \frac{f''(z_0)}{2}(z-z_0)^2 + \dots + \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n + \dots = \sum_{n=0}^{\infty} \frac{f^{n}(z_0)}{n!}(z-z_0)^n$$

$f(z) = u(x,y) + i\,v(x,y)$, $f'(z)$ exists if

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \text{CRC}$$

If CRC are satisfied then $f'(z)$ exists and does not depend on the direction of differentiation.

$$f'(z) = f'(x+iy) = \underbrace{\frac{\partial u}{\partial x}}_{u_1} + i\underbrace{\frac{\partial v}{\partial x}}_{v_1} = u_1(x,y) + i\,v_1(x,y)$$

$$\frac{\partial u_1}{\partial x} \overset{?}{=} \frac{\partial v_1}{\partial y} \Rightarrow \frac{\partial^2 u}{\partial x^2} \overset{?}{=} \frac{\partial^2 v}{\partial y \partial x} \Rightarrow \frac{\partial}{\partial x}\Big(\underbrace{\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}}_{=0}\Big) \overset{?}{=} 0 \checkmark$$

$$\frac{\partial u_1}{\partial y} \overset{?}{=} -\frac{\partial v_1}{\partial x} \Rightarrow \frac{\partial^2 u}{\partial y \partial x} \overset{?}{=} -\frac{\partial^2 v}{\partial x^2} \Rightarrow \frac{\partial}{\partial x}\Big(\underbrace{\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}}_{=0}\Big) \overset{?}{=} 0 \checkmark$$

If ~~the~~ a function satisfies CRC it can be differentiated infinitely many times!

Example. $f(z) = e^z = \underbrace{e^x \cos y}_{u} + i \underbrace{e^x \sin y}_{v}$ ⑨

$\frac{\partial u}{\partial x} = e^x \cos y = \frac{\partial v}{\partial y}$ ✓

CRC are satisfied

$\frac{\partial u}{\partial y} = -e^x \sin y = -\frac{\partial v}{\partial x}$ ✓

for $z_0 = 0$:

$f(z) = e^z$, $f(0) = 1$

$f'(z) = e^z$, $f'(0) = 1$

$f''(z) = e^z$, $f''(0) = 1$

$\vdots$

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \dots$$

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots \frac{z^n}{n!} + \dots$$

$z = x + iy$. Let $x = 0 \Rightarrow z = iy$

$$e^{iy} = 1 + iy - \frac{y^2}{2!} - i\frac{y^3}{3!} + \frac{y^4}{4!} + i\frac{y^5}{5!} \dots$$

$$= u + iv$$

$$u = \underbrace{1 - \frac{y^2}{2!} + \frac{y^4}{4!} + \dots}_{\cos y}, \quad v = \underbrace{y - \frac{y^3}{3!} + \frac{y^5}{5!} - \dots}_{\sin y}$$

$$\boxed{e^{iy} = \cos y + i \sin y}$$ Euler's formula

For $y = \pi$

$$e^{i\pi} = -1 \quad \Rightarrow \quad \boxed{e^{i\pi} + 1 = 0}$$

Example. $f(z)=\sqrt{z}$ at $z=1$

$$z = x+iy = re^{i\theta}, \quad r = |z| = \sqrt{x^2+y^2}$$

$$\theta = \tan^{-1}\frac{y}{x}$$

$$\sqrt{z} = \sqrt{re^{i\theta}} = \sqrt{r}\, e^{i\theta/2} = \underbrace{\sqrt[4]{x^2+y^2}\cos\theta/2}_{u} + i\underbrace{\sqrt[4]{x^2+y^2}\sin\theta/2}_{V}$$

$$\frac{\partial u}{\partial x} = \frac{2x}{4}\left(x^2+y^2\right)^{-3/4}\cos\theta/2$$

$$+\sqrt[4]{x^2+y^2}\left(-\frac{1}{2}\sin\theta\right)\cdot\frac{d}{dx}\left(\tan^{-1}\frac{y}{x}\right)$$

$$= \frac{1}{2}\,x\left(x^2+y^2\right)^{-3/4}\cos\theta/2 + \frac{1}{2}\sqrt[4]{x^2+y^2}\sin\theta\,\frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right)$$

$$= \frac{1}{2}\,x\left(x^2+y^2\right)^{-3/4}\cos\theta/2 + \frac{1}{2}\sqrt[4]{x^2+y^2}\,\frac{y}{x^2+y^2}\sin\theta$$

$$= \underline{\frac{1}{2}\left(x^2+y^2\right)^{-3/4}\left(x\cos\theta/2 + y\sin\theta/2\right)}$$

$$\frac{1}{\frac{x^2+y^2}{x^2}}\cdot\frac{1}{x}$$

$$\frac{\partial v}{\partial y} = \frac{1}{4}\,2y\left(x^2+y^2\right)^{-3/4}\sin\theta/2 + \frac{1}{2}\sqrt[4]{x^2+y^2}\cos\theta/2\left(\frac{1}{1+\frac{y^2}{x^2}}\cdot\frac{1}{x}\right)$$

$$= \underline{\frac{1}{2}y\left(x^2+y^2\right)^{-3/4}\sin\theta/2 + \frac{1}{2}\,\frac{x\sqrt[4]{x^2+y^2}}{x^2+y^2}\cos\theta/2}$$

$$f(z) = z^{1/2} \qquad f(1) = 1$$

$$f'(z) = \frac{1}{2} z^{-1/2} \qquad f'(1) = \frac{1}{2} = 1 \cdot \frac{1}{2}$$

$$f''(z) = -\frac{1}{4} z^{-3/2} \qquad f''(1) = -\frac{1}{4} = -1 \cdot \frac{1}{2} \cdot \frac{1}{2}$$

$$f'''(z) = \frac{3}{8} z^{-5/2} \qquad f'''(1) = \frac{3}{8} = 1 \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{3}{2}$$

$$f^{(iv)}(z) = -\frac{15}{16} z^{-7/2} \qquad f^{(iv)}(1) = -\frac{15}{16} = (-1)^5 \frac{5!!}{2^4}$$

$$f^{(n)}(1) = \frac{(n+1)!!}{2^n} (-1)^{n+1}$$

$$f(z) = 1 + \frac{1}{2}(z-1) - \frac{1}{8}(z-1)^2 + \frac{3}{8 \cdot 6}(z-1)^3$$

$$- \frac{15}{16} \frac{1}{4!} (z-1)^4 + \dots$$

$$= 1 + \frac{1}{2}(z-1) - \frac{1}{8}(z-1)^2 + \frac{1}{16}(z-1)^3 - \frac{5}{128}(z-1)^4 + \dots$$

Power series

$$f(z) = f(z_0) + f'(z_0)(z-z_0) + \frac{f''(z_0)}{2}(z-z_0)^2 + \dots$$

$$= a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots = \sum_{n=0} a_n (z-z_0)^n$$

$n \geq 0$ – non-negative powers

$$z - z_0 = \frac{1}{u}$$

$$f(z) = a_0 + a_1 \frac{1}{u} + a_2 \frac{1}{u^2} + \dots + = \sum_{n=0}^{\infty} a_n u^{-n}$$

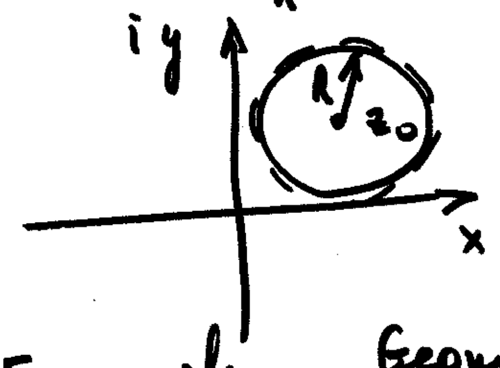
$-n \leq 0$ – non-positive powers $= \sum_{n=0}^{\infty} \frac{a_n}{u^n}$

Series containing only non-negative or only non-positive ~~term~~ powers are called power series.

Series is convergent within the disk of convergence with radius R

$$R = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| \quad , \quad |z - z_0| < R$$

disk of convergence



Example. Geometric series

$$1 + x + x^2 + x^3 + \dots = \frac{1}{1-x}$$

$$1 + z + z^2 + z^3 + \dots = \frac{1}{1-z}$$

$$S = 1 + x + x^2 + x^3 + \ldots$$
$$xS = x + x^2 + x^3 + x^4 + \ldots$$
$$xS - S = -1 \implies (x-1)S = -1$$
$$S = \frac{1}{1-x}$$

$$\frac{1}{1-x} = 1 + x + x^2 + \ldots$$

$$a_n = 1, \quad a_{n+1} = 1$$

$$\lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \left| \frac{1}{1} \right| = 1 = R$$

$$x = \frac{1}{2} < 1 = R$$
$$\frac{1}{1-\frac{1}{2}} = 2 = 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \ldots$$
$$x = 2 > 1 = R$$

$$\frac{1}{1-2} = -1 \neq 1 + 2 + 2^2 + 2^3 + \ldots$$

$$\boxed{\frac{1}{1-z} = 1 + z + z^2 + z^3 + z^4 + \ldots}$$

$$\frac{1}{1+z} = \left\{ \begin{matrix} -z = u \\ z = -u \end{matrix} \right\} = \frac{1}{1-u} = 1 + u + u^2 + u^3 + \ldots$$
$$= 1 - z + z^2 - z^3 + z^4 - z^5 \ldots \qquad |z| < 1 = R$$

$$\int \frac{dz}{1+z} = \ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \ldots, \quad |z| < 1$$

$$\frac{-1}{(1+z)^2} = -1 + 2z - 3z^2 + 4z^3 - 5z^4 + \ldots$$

$$\frac{1}{1+z^n} = \left\{ \begin{matrix} z^n = -u \\ u = -z^n \end{matrix} \right\} = \frac{1}{1-u} = 1 + u + u^2 + u^3 + \cdots$$

$$= 1 - z^n + z^{2n} - z^{3n} \cdots\cdots$$

$$\frac{1}{1+z^2} = 1 - z^2 + z^4 - z^6 + \cdots \qquad |z| < 1$$

$$\int \frac{dz}{1+z^2} = \tan^{-1} z = z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7} + \cdots$$

Binomial series

$$(1+z)^n = \underbrace{1}_{m=0} + \underbrace{nz}_{m=1} + \underbrace{\frac{n(n-1)}{2!} z^2}_{m=2} + \frac{n(n-1)(n-2)}{3!} z^3 + \cdots$$

$$a_m = \frac{n(n-1)(n-2)\ldots(n-m+1)}{m!}$$

$$a_{m+1} = \frac{n(n-1)(n-2)\ldots(n-m)}{(m+1)!}$$

$$R = \lim_{m \to \infty} \left| \frac{a_m}{a_{m+1}} \right| = \lim_{m \to \infty} \left| \frac{\cancel{n}\,\cancel{(n-1)}\,\cancel{(n-2)}\ldots\cancel{(n-m+1)}}{\cancel{n}\,\cancel{(n-1)}\,\cancel{(n-2)}\ldots\cancel{(\quad)}\,(n-m)} \, \frac{(m+1)!}{m!} \right|$$

$$= \lim_{m \to \infty} \left| \frac{m+1}{n-m} \right| = \left| \frac{m}{-m} \right| = |-1| = 1$$

$$|z| < 1$$

$$\frac{1}{(1+z)^2} = \left\{ n=-2 \right\} = 1 - 2z + 3z^2 - 4z^3 + \dots$$

If $n \geq 0$ Binomial series is finite if n is integer

$(1+z)^n$

$n=0$ $\quad 1$

$n=1$ $\quad 1+z$

$n=2$ $\quad (1+z)^2 = 1+2z+z^2$

$n=3$ $\quad (1+z)^3 = 1+3z+3z^2+z^3$

Example: $\dfrac{1}{2+z}$

a) $$\frac{1}{2+z} = \frac{1}{2}\,\frac{1}{1+z/2} = \left\{ \begin{matrix} z/2 = -u \\ u = -z/2 \end{matrix} \right\}$$

$$= \frac{1}{2}\,\frac{1}{1-u} = \frac{1}{2}\left(1+u+u^2+u^3+\dots\right)$$

$$= \frac{1}{2}\left(1 - \frac{z}{2} + \left(\frac{z}{2}\right)^2 - \left(\frac{z}{2}\right)^3 + \dots\right)$$

$R_u = 1$, $R_{-z/2} = 1$

$$|u| < 1 \implies \left|-\frac{z}{2}\right| < 1 \Rightarrow \left|\frac{z}{2}\right| < 1$$

$|z| < 2$

b) $$\frac{1}{2+z} = \frac{1}{1+(1+z)} = \left\{ \begin{matrix} 1+z = -u \\ u = -(1+z) \end{matrix} \right\} = \frac{1}{1-u}$$

$$= 1 + u + u^2 + \dots$$

$$= 1 - (1+z) + (1+z)^2 - (1+z)^3 + \dots$$

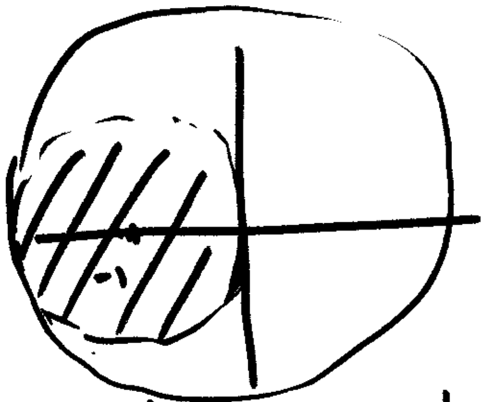
$|u| < 1$, $|-(1+z)| < 1$, $|1+z| < 1$

$R = 1 \qquad |1+z|$

$|z - z_0| < R \quad , \quad z_0 = -1$

$|1+z| < 1 \iff |z - z_0| < R$

$z_0 = -1 \ , \ R = 1$



c) $\dfrac{1}{2+z} = \dfrac{1}{z} \cdot \dfrac{1}{1+\frac{2}{z}} = \left\{ \begin{array}{l} \frac{2}{z} = -u \\ u = -\frac{2}{z} \end{array} \right\} =$

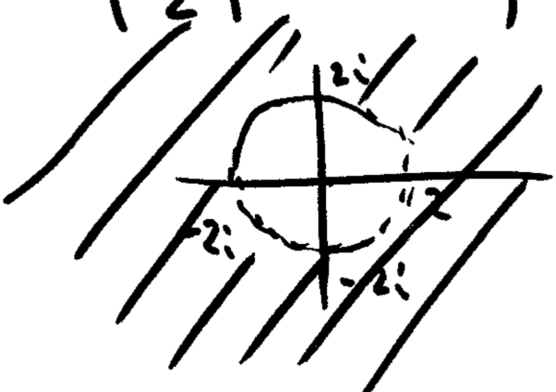
$= \dfrac{1}{z} \cdot \dfrac{1}{1-u} = \dfrac{1}{z}\left(1 + u + u^2 + \dots\right)$

$= \dfrac{1}{z}\left(1 - \dfrac{2}{z} + \left(\dfrac{2}{z}\right)^2 - \left(\dfrac{2}{z}\right)^3 + \dots\right)$

$= \dfrac{1}{z} - \dfrac{2}{z^2} + \dfrac{4}{z^3} - \dfrac{8}{z^4} + \dots$

$|u| < 1 \quad , \quad \left|-\dfrac{2}{z}\right| < 1 \ , \ \left|\dfrac{2}{z}\right| < 1$

$\left|\dfrac{z}{2}\right| > 1 \quad , \quad |z| > 2$



Laurent Series

$$f(z) = \cdots + \frac{c_{-3}}{(z-z_0)^3} + \frac{c_{-2}}{(z-z_0)^2} + \frac{c_{-1}}{z-z_0} + c_0 +$$

$$c_1(z-z_0) + c_2(z-z_0)^2 + \cdots$$

$$= \sum_{-\infty}^{\infty} c_n (z-z_0)^n$$

<u>Example</u>: $\quad f(z) = \dfrac{1}{z^2(1+z)} = \dfrac{1}{z^2} \; \dfrac{1}{1+z}$

$$\frac{1}{1-u} = 1 + u + u^2 + \cdots, \quad |u| < 1$$

$$\left\{ \begin{array}{l} u = -z \\ z = -u \end{array} \right\} \qquad f(z) = f(u) = \frac{1}{u^2} \; \frac{1}{1-u} =$$

$$= \frac{1}{u^2} \left(1 + u + u^2 + \cdots \right) =$$

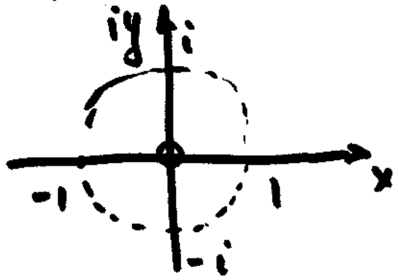
$$= \frac{1}{u^2} + \frac{1}{u} + 1 + u + \cdots$$

$$= \frac{1}{z^2} - \frac{1}{z} + 1 - z + z^2 + \cdots$$

$$c_{-3} = c_{-4} = \cdots = c_{-n} = \cdots = 0$$

Laurent series.

$$|u| < 1 \implies |-z| < 1 \implies |z| < 1$$



b) Expand about $z = -1$

$$f(z) = \frac{1}{z^2}\,\frac{1}{1+z} = \left\{\begin{matrix} 1+z = u \\ z = u-1 \end{matrix}\right\} = \frac{1}{(u-1)^2}\,\frac{1}{u} = \frac{1}{u}\,\frac{1}{(1-u)^2}$$

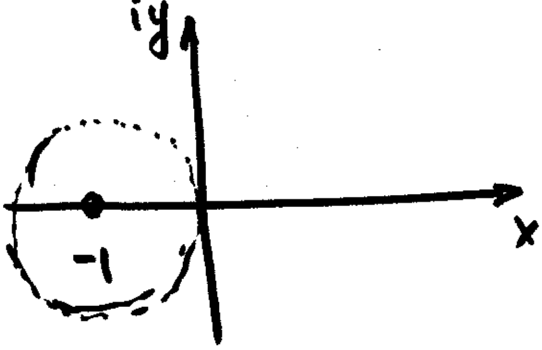
$$\frac{1}{(1-u)^2} = (1-u)^{-2} = 1 + 2u + 3u^2 + 4u^3 + \dots, \quad |u|<1$$

$$f(u) = \frac{1}{u}\left(1 + 2u + 3u^2 + 4u^3 + \dots\right) = \frac{1}{u} + 2 + 3u + 4u^2 + \dots$$

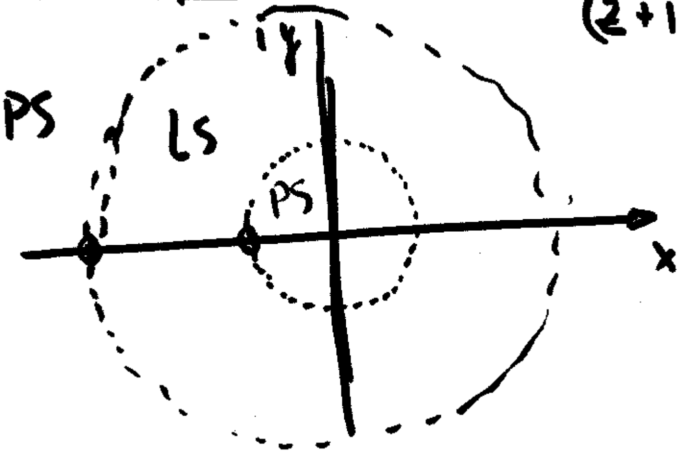
$$= \frac{1}{1+z} + 2 + 3(1+z) + 4(1+z)^2 + \dots$$

Laurent series

$$|u|<1 \Rightarrow |1+z| < 1$$



<u>Example</u>: $f(z) = \dfrac{1}{(z+1)(z+3)}$



a) $|z| < 1$

$$f(z) = \frac{1}{1+z}\cdot\frac{1}{3+z}$$

$$= \left(1 - z + z^2 - z^3 + \dots\right) \times$$

$$\times \frac{1}{3}\,\frac{1}{1+z/3} = \left\{\begin{matrix} z/3 = -u \\ u = -z/3 \end{matrix}\right\}$$

$$= \frac{1}{3}\left(1 - z + z^2 - z^3 + \dots\right)\left(1 - \frac{z}{3} + \left(\frac{z}{3}\right)^2 - \left(\frac{z}{3}\right)^3 + \dots\right)$$

$$= \frac{1}{3} - \frac{z}{3} - \frac{z}{9} + \dots = \frac{1}{3} - \frac{4z}{9} + \dots$$

$$f(z) = \frac{1}{(z+1)(z+3)} = \frac{A}{z+1} + \frac{B}{z+3}$$

$$= \frac{A(z+3) + B(z+1)}{(z+1)(z+3)} = \frac{(A+B)z + 3A + B}{(z+1)(z+3)}$$

$$1 = \underbrace{(A+B)}_{=0} z + 3A + B$$

$$A = -B \quad \Rightarrow \quad 1 = 3A - A = 2A, \quad A = \frac{1}{2} = -B$$

$$f(z) = \frac{1}{2}\frac{1}{(1+z)} - \frac{1}{2}\frac{1}{3+z} = \frac{1}{2}\frac{1}{1+z} - \frac{1}{6}\frac{1}{1+z/3}$$

$$= \frac{1}{2}\left(1 - z + z^2 - z^3 + \ldots\right) - \frac{1}{6}\left(1 - \frac{z}{3} + \frac{z^2}{9} - \frac{z^3}{27} + \ldots\right)$$

$|z| < 1$ $\qquad\qquad$ $\left|\frac{z}{3}\right| < 1$

$|z| < 3$

$$= \frac{1}{3} - \frac{4}{9}z + \frac{13}{27}z^2 - \ldots$$ Power series

b) $$f(z) = \frac{1}{2}\frac{1}{1+z} - \frac{1}{2}\frac{1}{3+z} =$$

$$= \frac{1}{2z}\frac{1}{1+\frac{1}{z}} - \frac{1}{6}\frac{1}{1+z/3}$$

$$= \frac{1}{2z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3}\cdots\right) - \frac{1}{6}\left(1 - \frac{z}{3} + \frac{z^2}{9} - \frac{z^3}{27} + \ldots\right)$$

$$= \frac{1}{2z} - \frac{1}{2z^2} + \frac{1}{2z^3} - \frac{1}{2z^4}\cdots - \frac{1}{6} + \frac{1}{18}z - \frac{1}{54}z^2 + \frac{1}{6\cdot 27}z^3 \ldots$$

$$= \cdots - \frac{1}{2z^4} + \frac{1}{2z^3} - \frac{1}{2z^2} + \frac{1}{2z} - \frac{1}{6} + \frac{1}{18}z - \frac{1}{54}z^2 + \cdots$$

Laurent series

c) $|z| > 3$

$$f(z) = \frac{1}{2}\,\frac{1}{1+z} - \frac{1}{2}\,\frac{1}{3+z}$$

$$= \frac{1}{2z}\,\frac{1}{1+\frac{1}{z}} - \frac{1}{2z}\,\frac{1}{1+3/z}$$

$$= \frac{1}{2z}\left(1 - \frac{1}{z} + \frac{1}{z^2}\cdots\right) - \frac{1}{2z}\left(1 - \frac{3}{z} + \frac{9}{z^2} - \frac{27}{z^3} + \cdots\right.$$

$$= \cancel{\frac{1}{2z}} - \frac{1}{2z^2} + \frac{1}{2z^3}\cdots - \cancel{\frac{1}{2z}} + \frac{3}{2z^2} - \frac{9}{2z^3} + \cdots$$

$$= \frac{1}{z^2} - \frac{4}{z^3} + \cdots$$

Power series