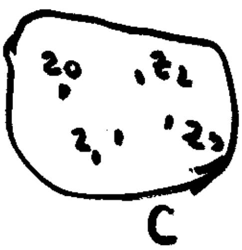
Cauchy's residue theorem

$$\oint_C \frac{f(z)}{z-z_0}\,dz = 2\pi i\, \underbrace{f(z_0)}_{c_{-1}}$$

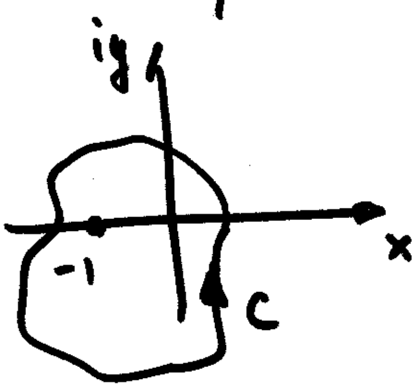
$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz = 2\pi i\, \underbrace{\frac{f^{(n)}(z_0)}{n!}}_{c_{-1}}$$

→ Cauchy's integral formula



$$\oint_C f(z)\,dz = 2\pi i \sum_{k=0}^{n} \underbrace{\text{Res}\left(f(z), z_k\right)}_{c_{-1\,k}}$$

Example.



$$\oint_C \frac{e^z}{(z+1)^2}\,dz$$

$$(z+1)^2 \overset{?}{=} 0, \quad z = -1$$

$$f(z) = \frac{e^z}{(z+1)^2}, \quad z_0 = -1$$

$$\oint_C \frac{e^z}{(z+1)^2}\,dz = 2\pi i\, \underbrace{\text{Res}\left(\frac{e^z}{(z+1)^2}, -1\right)}_{c_{-1}} = 2\pi i\, e^{-1}$$

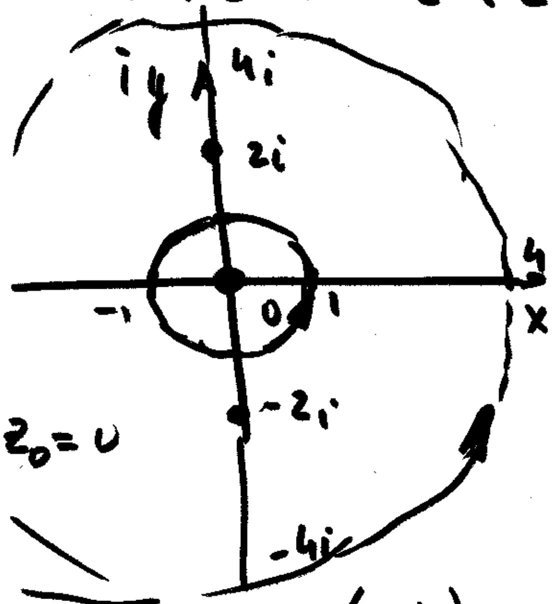
$$\boxed{c_{-1} = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} (z-z_0)^m f(z)}$$

$m = 2$

$$c_{-1} = \lim_{z \to -1} \frac{d}{dz}\left(e^z\right) = e^z\Big|_{z=-1} = e^{-1}$$

Example. $I = \oint_C \frac{z^3 - z^2 + z - 1}{z^3 + 4z}\, dz$

$$z^3 + 4z = z(z^2+4) = 0 \Rightarrow z = 0$$
$$z = \pm 2i$$



$z_0 = 0$

a) $|z| = 1 : C$

$$\mathrm{Res}(f(z), 0) = \lim_{z \to 0} \frac{z^3 - z^2 + z - 1}{z^2+4} =$$

$m = 1$

$$= -\frac{1}{4}$$

$$I = 2\pi i \left(-\frac{1}{4}\right) = \boxed{-\frac{\pi i}{2}}$$

b) $C : |z| = 4$

$$\mathrm{Res}(f(z), 2i) = \lim_{z \to 2i} \frac{z^3 - z^2 + z - 1}{z(z+2i)} =$$

$$\left\{ \begin{array}{l} z(z^2+4) \\ = z^1 (z+2i)^1 (z-2i)^1 \\ m = 1 \end{array} \right\}$$

$$= \frac{-8i + 4 + 2i - 1}{2i \cdot 4i}$$

$$= -\frac{3 - 6i}{8} = -\frac{3}{8} + \frac{3}{4} i$$

$$\mathrm{Res}(f(z), -2i) = \lim_{z \to -2i} \frac{z^3 - z^2 + z - 1}{z(z-2i)} = \frac{+8i + 4 - 2i - 1}{-2i(-4i)}$$

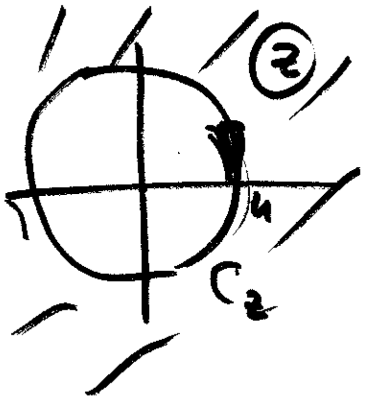
$$= \frac{6i + 3}{-8} = -\frac{3}{8} - \frac{3}{4} i$$

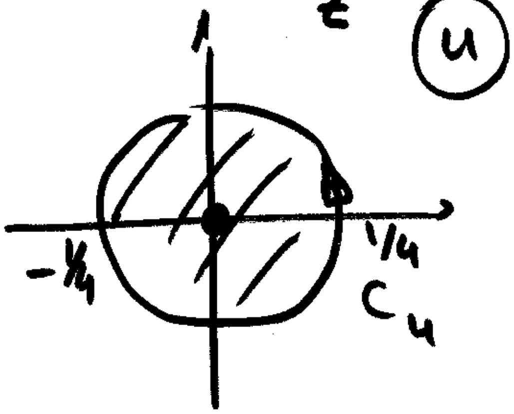
$$I = 2\pi i \left(-\frac{1}{4} - \frac{3}{8} + \cancel{\frac{3}{4}} i - \frac{3}{8} - \cancel{\frac{3}{4}} i\right)$$

$$\boxed{= -2\pi i}$$

$|z| = 4$ but traverse C in the clockwise direction: I_1 - find it.

$$I_1 \stackrel{?}{=} -I \qquad u = \frac{1}{z} \qquad \frac{du}{dz} = -\frac{1}{z^2} = -u^2$$





$$dz = -\frac{du}{u^2}$$

$$I_1 = \oint_{C_u} \left(- \frac{\frac{1}{u^3} - \frac{1}{u^2} + \frac{1}{u} - 1}{\frac{1}{u^3} + 4\frac{1}{u}} \right) \frac{du}{u^2}$$

$$= \oint_{C_u} \frac{u^3 - u^2 + u - 1}{(1+4u^2)u^2}\, du$$

$$\frac{u^3 - u^2 + u - 1}{(1+4u^2)u^2} = \frac{Au + B}{1 + 4u^2} + \frac{Cu + D}{u^2}$$

$$= \frac{Au^3 + Bu^2 + Cu + 4Cu^3 + D + 4Du^2}{(1+4u^2)\, u^2} =$$

$1 = A + 4C \qquad A = 1 - 4C = 1 - 4 = -3$

$-1 = B + 4D \qquad B = -4D - 1 = 4 - 1 = 3$

$1 = C$

$-1 = D$

$$= \frac{-3u + 3}{1 + 4u^2} + \frac{1}{u} - \frac{1}{u^2}$$

$$= -\frac{1}{u^2} + \frac{1}{u} + 3(1-u)\left(1 - 4u^2 + \dots\right)$$

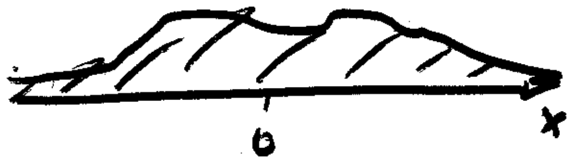
c_{-1}/u

$c_{-1} = 1 = \mathrm{Res}\,(f(u), 0)$ (37)

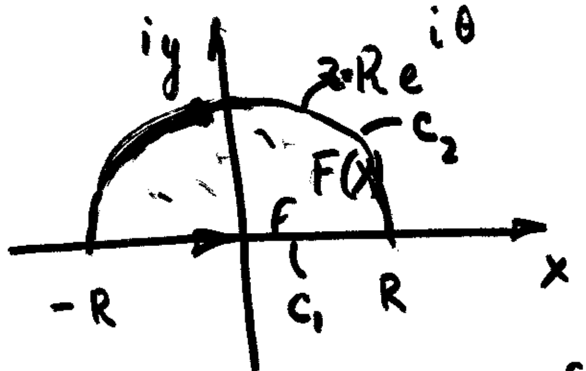
$\underline{I_1 = 2\pi i \cdot c_{-1} = 2\pi i \cdot 1 = 2\pi i = -I}$

$$\int_{-\infty}^{\infty} F(x)\,dx$$

$$I = \int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \tan^{-1} x \Big|_{-\infty}^{\infty} = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$$



$$I = \int_{-\infty}^{\infty} \frac{x^2}{1+x^4}\,dx = ?$$



$$I = \lim_{R\to\infty} \int_{-R}^{R} F(x)\,dx$$

$x \to z$

$F(x) \longrightarrow f(z)$

$0 \le \theta \le \pi$

$$\oint_{c_1+c_2} f(z)\,dz = \int_{-R}^{R} F(x)\,dx + \int_0^{\pi} f(Re^{i\theta}) \times Rie^{i\theta}\,d\theta$$

$$= 2\pi i \sum \mathrm{Res}$$

$z = Re^{i\theta}$

$dz = Ri\,e^{i\theta}\,d\theta$

$R \to \infty$

$$\int_0^{\pi} f(Re^{i\theta}) \cdot iRe^{i\theta}\,d\theta \xrightarrow{?} 0$$

For some functions f this is true

Example: rational functions

$$f(z) = \frac{a_k z^k + a_{k-1} z^{k-1} + \dots a_0}{b_\ell z^\ell + b_{\ell-1} z^{\ell-1} + \dots b_0}$$

$$\lim_{|z|\to\infty} f(z) = \frac{a_k z^k}{b_l z^l} = \frac{a_k}{b_l} z^{k-l} = \frac{c}{z^n}$$

$$\frac{a_k}{b_l} = c \qquad k-l=-n \;, \quad k<l$$

$$\left\{ f(z) = \frac{10z^3+10z}{5z^5+4} \;, \quad \lim_{|z|\to\infty} = \frac{10z^3}{5z^5} = \frac{2}{z^2} \to 0 \right\}$$

$$f(z)\,dz = \underbrace{f(Re^{i\theta})\, iRe^{i\theta}}\, d\theta$$

$$\lim_{|z|\to\infty} \left(f(z)dz\right) = \frac{c}{z^n}\, d\cancel{z} = \frac{icRe^{i\theta}}{(Re^{i\theta})^n}\, d\theta$$

$$= icR^{1-n} e^{i(1-n)\theta} d\theta \longrightarrow 0 \quad \text{as } R\to\infty$$

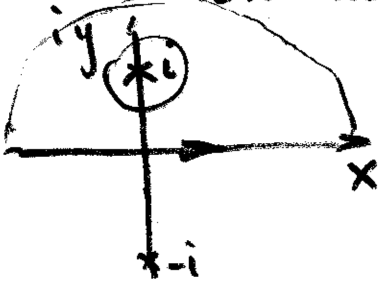
$$1-n<0 \qquad 1<n \;, \quad n=2,3,4,\dots$$

$n = l-k > 1$, $\Rightarrow$ The highest power in the numerator has to be at least 2 units smalle then that of the denominator.

Example: $I = \displaystyle\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$, $F(x) = \dfrac{1}{1+x^2}$

Numerator's degree is 0 $\qquad 2-0=2>1$ V

Denominator's degree is 2



$$f(z) = \frac{1}{1+z^2} = \frac{1}{(z-i)(z+i)}$$

$$I = \int_{-\infty}^{\infty} \frac{dx}{x^2+1} = 2\pi i \; \mathrm{Res}\,(f(z), i)$$

$$\operatorname{Res}\left(f(z), i\right) = \lim_{z \to i} \cancel{(z-i)} \frac{1}{(z+i)\cancel{(z-i)}} = \frac{1}{2i} = -\frac{i}{2}$$

$$I = \cancel{2}\pi i \left(-\frac{i}{\cancel{2}}\right) = \pi$$

<u>Example</u> $$I = \int_{-\infty}^{\infty} \frac{x^2}{x^4+1}\, dx$$

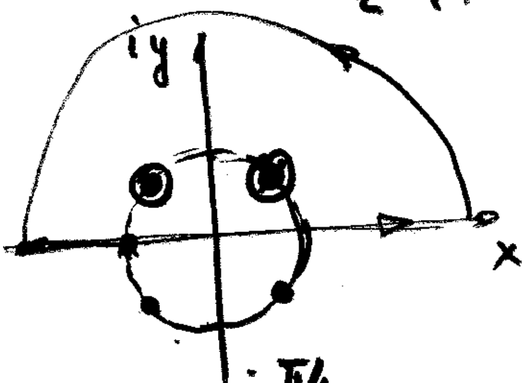
Num 2

Den 4

$4-2 = 2 > 1$ ✓

$$f(z) = \frac{z^2}{z^4+1} \quad ; \quad z^4+1 = 0 \;, \quad z^4 = -1 = e^{i\pi}$$

$$= e^{i(\pi+2\pi)} = e^{i(\pi+4\pi)} = e^{i(\pi+6\pi)} = e^{i(\pi+8\pi)}$$



$z^4 = e^{i\pi}$

$z = e^{i\pi/4}$

$z^4 = e^{i3\pi}$

$z = e^{i\frac{3\pi}{4}}$

$z^4 = e^{i5\pi}$

$z = e^{i5\pi/4}$

$z^4 = e^{i7\pi}$

$z = e^{i\frac{7\pi}{4}}$

$z = e^{i9\pi/4} = e^{i(\pi/4 + 2\pi)}$

$$z_1 = e^{i\pi/4} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$$

$$z_2 = e^{i3\pi/4} = -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$$

$$f(z) = \frac{z^2}{\left(z - e^{i\frac{3\pi}{4}}\right)\left(z - e^{i\pi/4}\right)\left(z - e^{i5\pi/4}\right)\left(z - e^{i7\pi/4}\right)}$$

$$\operatorname{Res}\left(f(z), e^{i\pi/4}\right) = \frac{e^{i\pi/2}}{\left(e^{i\pi/4} - e^{i3\pi/4}\right)\left(e^{i\pi/4} - e^{i\frac{5\pi}{4}}\right)\left(e^{i\pi/4} - e^{i7\pi/4}\right)}$$

$$= \frac{i}{\left(\frac{\sqrt{2}}{2} + \cancel{i\frac{\sqrt{2}}{2}} + \frac{\sqrt{2}}{2} - \cancel{i\frac{\sqrt{2}}{2}}\right)\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)\left(\cancel{\frac{\sqrt{2}}{2}} + i\frac{\sqrt{2}}{2} - \cancel{\frac{\sqrt{2}}{2}} + i\frac{\sqrt{2}}{2}\right)}$$

$$= \frac{\cancel{i}}{\sqrt{2}\,\left(\sqrt{2} + i\sqrt{2}\right)\cancel{i}\sqrt{2}} = \frac{1}{2}\,\frac{\sqrt{2} - i\sqrt{2}}{2+2} = \frac{1}{8}\left(\sqrt{2} - i\sqrt{2}\right)$$

$$= \frac{\sqrt{2}}{8}(1-i)$$

$$\operatorname{Res}\left(f(z), e^{i\frac{3\pi}{4}}\right) = -\frac{\sqrt{2}}{8}(1+i)$$

$$I = 2\pi i\left(\frac{\sqrt{2}}{8}(1-i) - \frac{\sqrt{2}}{8}(1+i)\right)$$

$$= \frac{\sqrt{2}\pi i}{4}(\cancel{1} - i - \cancel{1} - i) = \frac{\sqrt{2}}{4}\pi i\,(-2i) =$$

$$= \frac{\sqrt{2}}{2}\pi$$

Example. $I = \int_{-\infty}^{\infty} \frac{dx}{(x^2+4)^2}$, $F(x) = \frac{1}{(x^2+4)^2} = \frac{1}{x^4+8x^2+16}$

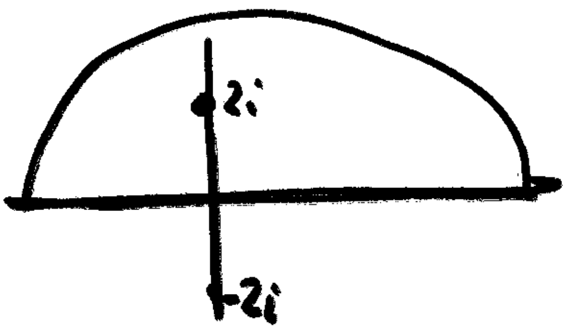
Num 0

Den 4

$4 - 0 = 4 > 1$.

$$f(z) = \frac{1}{(z^2+4)^2}$$

$$z^2+4 = 0 \Rightarrow z = \pm 2i$$



$\operatorname{Res}(f(z), 2i)$ - ?

$$f(z) = \frac{1}{(z-2i)^2(z+2i)^2}$$

Pole of degree 2.

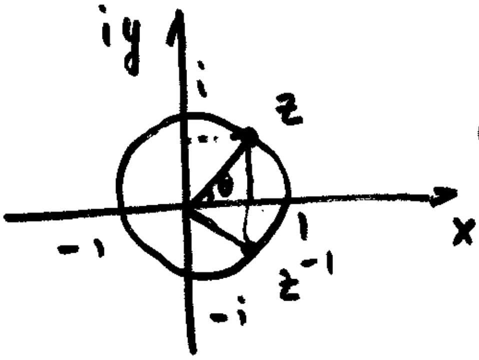
$$\operatorname{Res}(f(z), 2i) = \frac{1}{(2-1)!}\lim_{z\to 2i}\frac{d}{dz}\frac{1}{(z+2i)^2}$$

$$= -\frac{2}{(z+2i)^3}\bigg|_{z=2i} = -\frac{2}{(4i)^3} = \frac{+2i}{64\, i\cdot i}$$

$$= \frac{-i}{32}$$

$$I = 2\pi i \frac{-i}{32} = \boxed{\frac{\pi}{16}}$$

$$\int_0^{2\pi} g(\cos\theta, \sin\theta)\, d\theta$$



$$|z| = 1$$

(1) $z = e^{i\theta} = \cos\theta + i\sin\theta$

$$z^{-1} = \frac{1}{z} \Rightarrow |z^{-1}| = \frac{1}{1} = 1$$

$$z^{-1} = \left(e^{i\theta}\right)^{-1} = e^{-i\theta} = \bar{z}$$

(2) $z^{-1} = e^{-i\theta} = \cos\theta - i\sin\theta$

$$z + z^{-1} = 2\cos\theta \qquad z - z^{-1} = 2i\sin\theta$$

$$\boxed{\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)} \qquad \boxed{\sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right) = \frac{i}{2}\left(\frac{1}{z} - z\right)}$$

$$\boxed{dz = d\left(e^{i\theta}\right) = i e^{i\theta} d\theta \Rightarrow \frac{dz}{d\theta} = iz}$$

$$\boxed{d\theta = \frac{dz}{iz} = -i\frac{dz}{z}}$$

<u>Example</u>: $I = \displaystyle\int_0^{2\pi} \frac{d\theta}{3 + \cos\theta}$

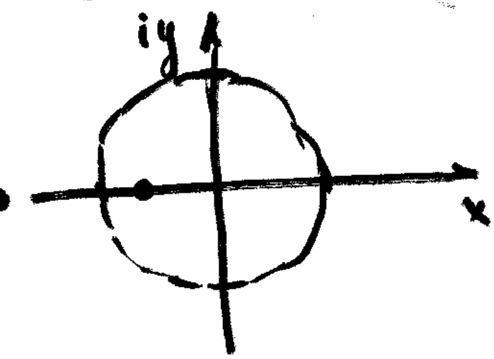
$$3 + \cos\theta = 3 + \frac{1}{2}\left(z + \frac{1}{z}\right) = 3 + \frac{z^2 + 1}{2z} = \frac{z^2 + 6z + 1}{2z}$$

$$d\theta = -i\frac{dz}{z}$$

$$I = -i \oint_{|z|=1} \frac{2z}{z^2 + 6z + 1} \frac{dz}{z} = -2i \oint_{|z|=1} \frac{dz}{z^2 + 6z + 1}$$

$z^2+6z+1=0 \qquad z=\frac{-6\pm\sqrt{36-4}}{2}=-3\pm 2\sqrt{2}$

$z^2+6z+1=(z+3+2\sqrt{2})(z+3-2\sqrt{2})$



$z=-3-2\sqrt{2}$ X

$z=-3+2\sqrt{2}$

$|-3+2\sqrt{2}| \overset{?}{<} 1$

$-3+2\sqrt{2}<0$

$9 \qquad 8$

$-3+2\sqrt{2} \overset{?}{>} -1$ V

$2\sqrt{2} \overset{?}{>} 2$ V

$\sqrt{2}>1$ V

$I=-2\pi i\, 2i\, \mathrm{Res}\left(f(z), -3+2\sqrt{2}\right)$

$f(z)=\frac{1}{(z+3+2\sqrt{2})(z+3-2\sqrt{2})}$

$\mathrm{Res}\left(f(z), -3+2\sqrt{2}\right)=\frac{1}{\cancel{-3}+2\sqrt{2}+\cancel{3}+2\sqrt{2}}=\frac{1}{4\sqrt{2}}$

$I=\cancel{4}\pi\frac{1}{\cancel{4}\sqrt{2}}=\frac{\pi\sqrt{2}}{2}$