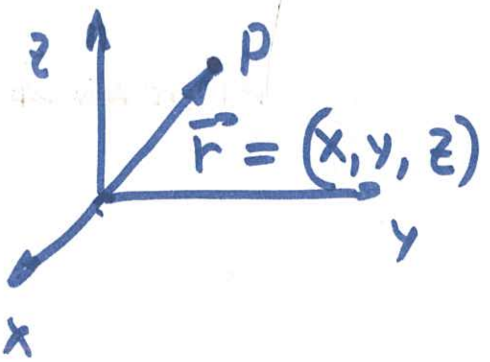
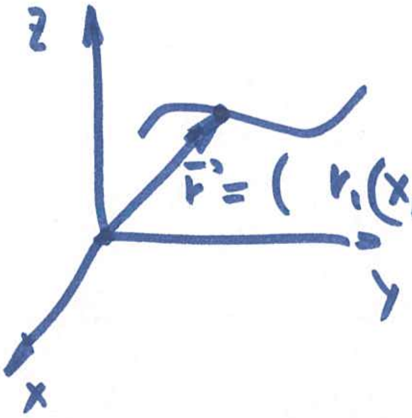
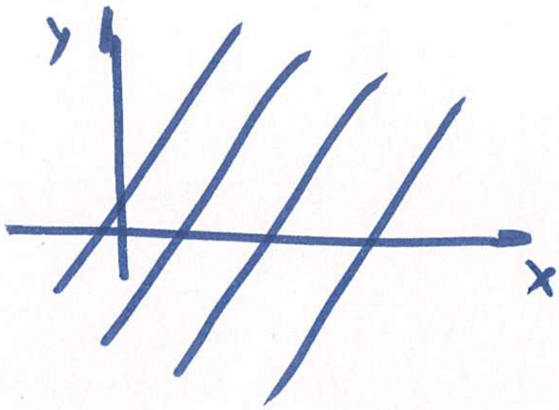
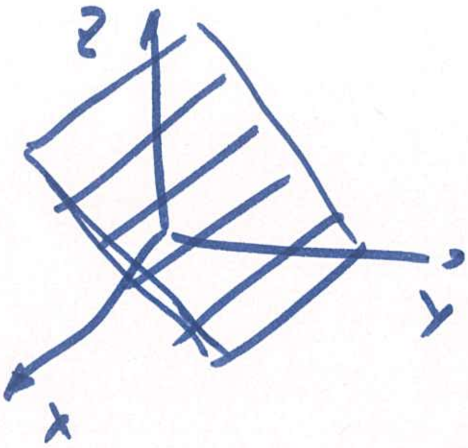
$P(x, y, z)$





$\vec{r} = (r_1(x,y,z),\ r_2(x,y,z),\ r_3(x,y,z))$

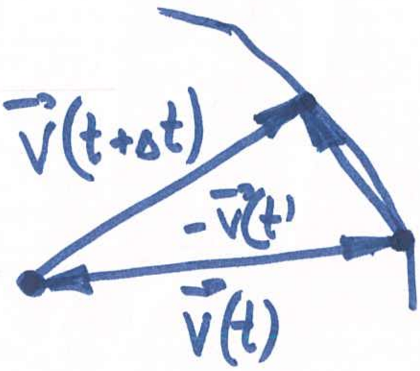




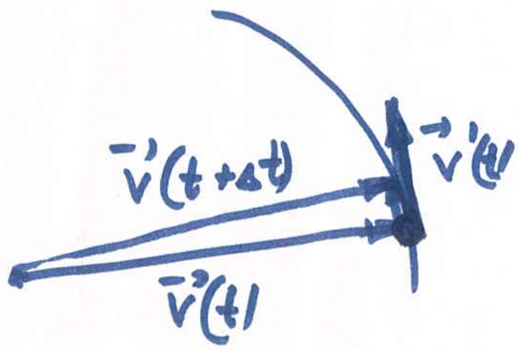
$\vec{v}(t) \longrightarrow \vec{a}$

Derivative of of a vector field

$$\vec{v}\,'(t) = \lim_{\Delta t \to 0} \frac{\vec{v}(t+\Delta t) - \vec{v}(t)}{\Delta t}$$



$\Delta t \to 0$



$\vec{v}\,'(t)$ - is tangent to the curve or trajectory described by a vector function $\vec{v}(t)$.

$$\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

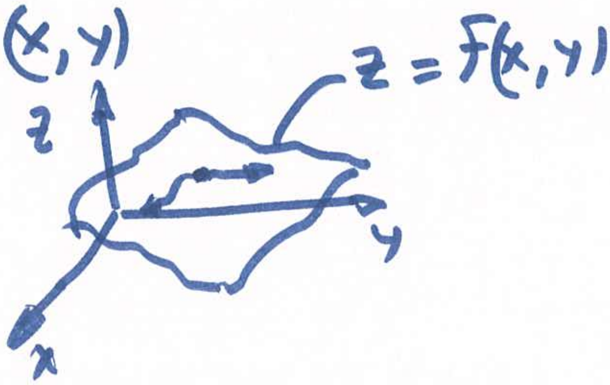
$$f(x) \longrightarrow f(x,y,z)$$

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y, z) - f(x,y,z)}{\Delta x}$$

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y, z) - f(x,y,z)}{\Delta y}$$

$$\frac{\partial f}{\partial z} = \lim_{\Delta z \to 0} \frac{f(x, y, z+\Delta z) - f(x,y,z)}{\Delta z}$$

$z = f(x,y)$



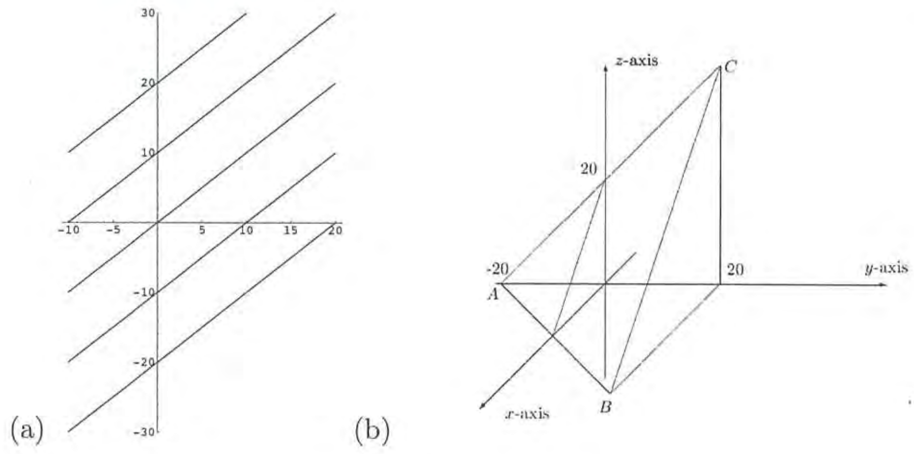


Figure 2.3: Contours of $z = 20 - x + y$.

the contour has equation $y = x + c - 20$ that describes straight lines parallel to the line $y = x$ shown in Figure 2.3 (a) for $z = 0$, 10, 20, 30 and 40. Figure 2.3 (b) shows part of the graph of the function as the triangular plane region ABC. ■

Vector fields

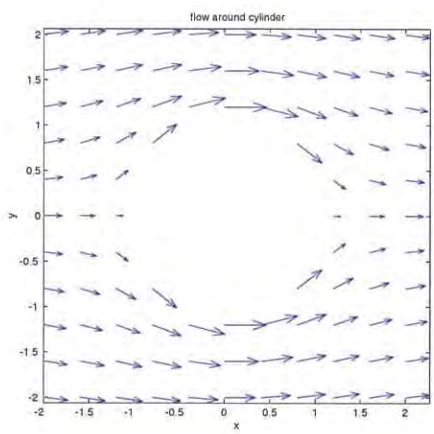
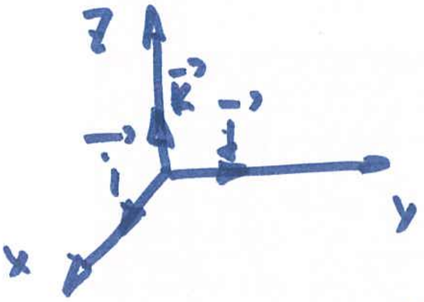


Figure 2.4: Flow velocity vector field around a circular cylinder.

Visualizing a two-dimensional vector field requires plotting vectors at each point in a plane in such a way that their lengths are proportional to their computed magnitudes. For example, a vector field of velocities around a

Operator nabla ∇

$$\nabla = \vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z} = \underbrace{\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)}_{\text{vector operator}}$$



Gradient of a scalar field.

$$\nabla f(x,y,z) = \underbrace{\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)}_{\nabla} f = \underline{\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)}$$

$$\vec{u} = (x_0, y_0, z_0)$$

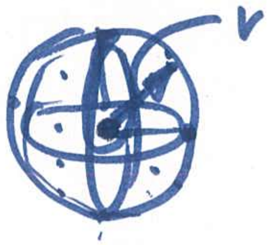
$$k\vec{u} = \vec{u}\cdot k = (kx_0, ky_0, kz_0) = (x_0 k, y_0 k, z_0 k)$$

$$\nabla f \overset{?}{\neq} f\,\nabla = \cancel{\left(f\frac{\partial}{\partial x}, f\frac{\partial}{\partial y}, f\frac{\partial}{\partial z}\right)}$$

<u>Example</u>: $f = x^2 + y^3 + z^4$

$$\nabla f\Big|_{(x_0,y_0,z_0)} = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)\Bigg|_{(x_0,y_0,z_0)} = (2x_0, 3y_0^2, 4z_0^3)$$

Gradient of spherically symmetric functions.



$$x^2+y^2+z^2=r^2$$

$$\vec{r}=(x,y,z) \implies r=|\vec{r}|=\sqrt{x^2+y^2+z^2}$$

$$f=f(r)$$

$$\frac{\partial f}{\partial x}=\frac{df}{dr}\cdot\frac{\partial r}{\partial x} \implies \frac{\partial r}{\partial x}=\frac{\frac{1}{2}\cdot 2x}{\sqrt{x^2+y^2+z^2}}=\frac{x}{r}$$

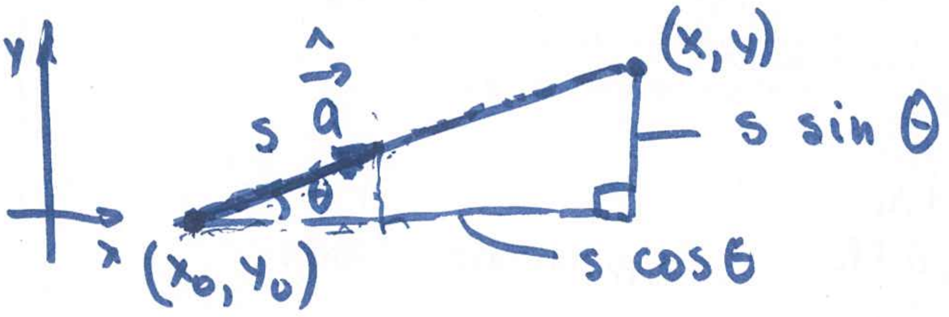
$$\frac{\partial r}{\partial y}=\frac{y}{r}, \quad \frac{\partial r}{\partial z}=\frac{z}{r}$$

$$\boxed{\nabla f}=\left(\frac{df}{dr}\frac{x}{r}, \frac{df}{dr}\frac{y}{r}, \frac{df}{dr}\frac{z}{r}\right)$$

$$=\frac{df}{dr}\frac{(x,y,z)}{r}=\frac{df}{dr}\cdot\frac{\vec{r}}{r}=\boxed{\frac{df}{dr}\hat{\vec{r}}}$$

Opertor	Name	Input	Output
∇	gradient	scalar function	vector function

Directional derivative



$$\hat{\vec{a}} = (\cos\theta, \sin\theta)$$

$$\vec{r} = s\,\hat{\vec{a}} + (x_0, y_0) = (\overbrace{x_0 + s\cos\theta}^{x}, \overbrace{y_0 + s\sin\theta}^{y})$$

$f(s)$ $$\frac{df}{ds} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial x}{\partial s} = \cos\theta \quad , \quad \frac{\partial y}{\partial s} = \sin\theta$$

$$\frac{df}{ds} = \frac{\partial f}{\partial x}\cdot\cos\theta + \frac{\partial f}{\partial y}\cdot\sin\theta$$

$$\vec{u} = (x_1, y_1) \quad , \quad \vec{v} = (x_2, y_2)$$

$$\boxed{\vec{u}\cdot\vec{v} = x_1 x_2 + y_1 y_2}$$

$$\frac{df}{ds} = \underbrace{\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)\cdot\left(\cos\theta, \sin\theta\right)}_{\nabla f\,\cdot\,\hat{\vec{a}}}$$

$$\times\ \boxed{\frac{df}{ds} = \nabla f\cdot\hat{\vec{a}}}$$ - directional derivative

$$\boxed{\frac{df}{d\vec{a}} = \nabla f\cdot\frac{\vec{a}}{|\vec{a}|}} \qquad \boxed{D_{\vec{a}}f = \nabla f\cdot\frac{\vec{a}}{|\vec{a}|}}$$

Example: $f = x^2 + xyz$. Find directional derivative in the direction of $\vec{u} = (1, 2, -3)$ at $P(1, 2, -1)$
(x, y, z)

$$\vec{D}_{\vec{u}} = \nabla f \cdot \frac{\vec{u}}{|\vec{u}|}$$

$$\nabla f = (2x + yz,\ xz,\ xy) = (2 \cdot 1 + 2(-1),\ 1 \cdot (-1),\ 1 \cdot 2)$$

$$= (0, -1, 2)$$

$$|\vec{u}| = \sqrt{1^2 + 2^2 + (-3)^2} = \sqrt{14}$$

$$D_{\vec{u}} = (0, -1, 2) \cdot \frac{(1, 2, -3)}{\sqrt{14}} = \frac{0 - 2 - 6}{\sqrt{14}}$$

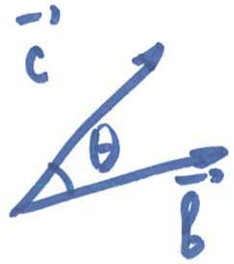
$$= \frac{-8}{\sqrt{14}}$$

$D_{\vec{a}}$	Directional derivative	Scalar Function	scalar

Meaning of the gradient

$$D_{\vec{a}} f = \nabla f \cdot \frac{\vec{a}}{|\vec{a}|} = \nabla f \cdot \hat{\vec{a}}$$

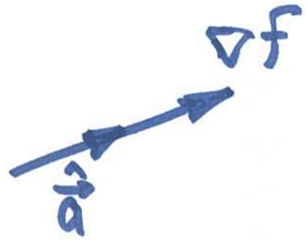
$\vec{b}, \vec{c} \Rightarrow \vec{b} \cdot \vec{c} = |\vec{b}| \cdot |\vec{c}| \cdot \cos\theta$



$$D_{\vec{a}} f = |\nabla f| \cdot |\hat{\vec{a}}| \cos\theta$$
$$= |\nabla f| \cos\theta$$

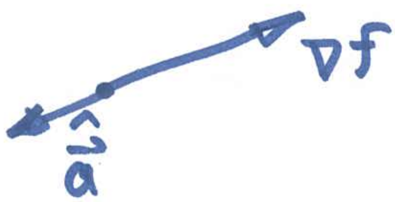
1) $\cos\theta = 1 \Rightarrow \theta = 0 \Rightarrow D_{\vec{a}} f = |\nabla f|$

Maximum value of $D_{\vec{a}} f$ is in the direction of the gradient vector



2) $\cos\theta = -1 \Rightarrow \theta = \pi \Rightarrow D_{\vec{a}} f = -|\nabla f|$

Minimum value of $D_{\vec{a}} f$ is in the direction opposite to the gradient.



3) $\cos\theta = 0 \Rightarrow \theta = \pi/2 \Rightarrow D_{\vec{a}} f = 0$

This is the direction of a level curve.

Example. $ax + by = d$ - equ for a straight line

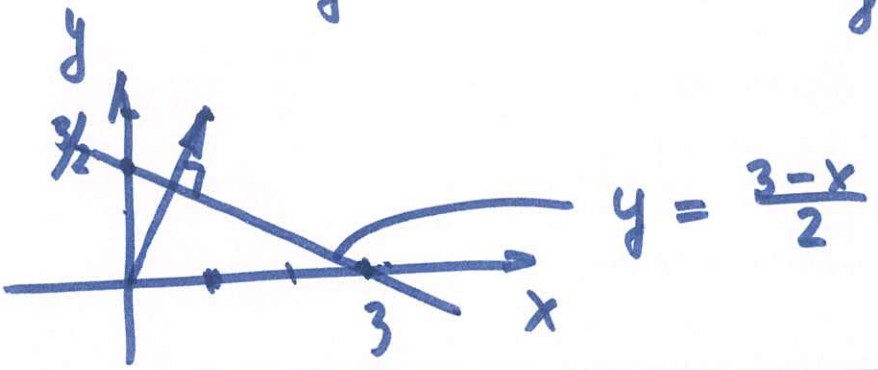
$f(x,y) = ax + by \Rightarrow f(x,y) = d$

$\nabla f = (a, b) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$

$\vec{c} \perp \nabla f \Rightarrow \vec{c}$ - shows the direction of a level

$\Rightarrow (a, b)$ is $\perp$ to the line.

$x + 2y = 3 \Rightarrow y = \frac{3-x}{2}$



$(a, b) = (1, 2)$

Example. $ax + by + cz = d \Rightarrow f(x,y,z) = d$

$f(x,y,z) = ax + by + cz$

$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = (a, b, c)$

$(a, b, c) \perp$ to the plane.

Consider $x^2+y^2=1+z^2$, $(x_0, y_0, z_0) = (1,-1,1)$

Normal vector to this surface at (x_0, y_0, z_0)?

$f(x,y,z) = x^2+y^2-z^2$

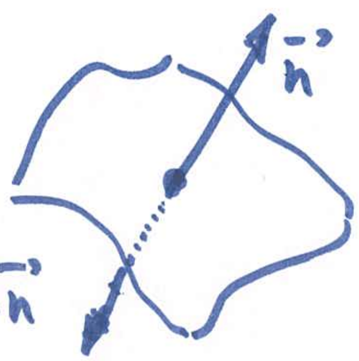
$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = (2x, 2y, -2z)$$

$$\nabla f \Big|_{(1,-1,1)} = (2,-2,-2) = \vec{N}$$

Unit normal is $\vec{n} = \dfrac{\vec{N}}{|\vec{N}|} = \dfrac{(2,-2,-2)}{\sqrt{12}}$

$$= \frac{\cancel{2}\cdot(1,-1,-1)}{\cancel{2}\cdot\sqrt{3}}$$

$$= \frac{(1,-1,-1)}{\sqrt{3}}$$

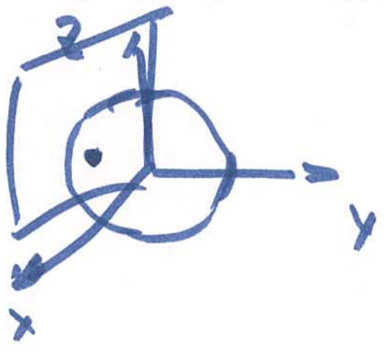


$-\vec{n} = \dfrac{(-1,1,1)}{\sqrt{3}}$ is also a normal to that surface

---

Example. Find tangent plane to

$x^2+y^2+z^2 = 3/4$ at $(x_0, y_0, z_0) = \left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right)$



$ax + by + cz = d$ - plane

(a, b, c) is $\perp$ to the plane and to the surface

$f(x,y,z) = x^2+y^2+z^2$

$\nabla f = (2x, 2y, 2z)$

$(2x_0, 2y_0, 2z_0) = (a, b, c) = (1, -1, 1)$

$x - y + z = d$

$d = \frac{1}{2} - \left(\frac{-1}{2}\right) + \frac{1}{2} = \frac{3}{2}$

$x - y + z = 3/2$ - tangent plane to the sphere at (x_0, y_0, z_0)

Potential functions.

$\varphi = x^2 y \implies \nabla\varphi = \left(\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}\right) = (2xy, x^2)$

Is converse true?

Given $\vec{F}$, can you find φ ?

$\vec{F} = (2xy, x^2 - 2y)$, φ - ?

$\nabla\varphi = \left(\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}\right) = \vec{F} = (2xy, x^2 - 2y)$

$$\begin{cases} \frac{\partial\varphi}{\partial x} = 2xy \\ \frac{\partial\varphi}{\partial y} = x^2 - 2y \end{cases} \implies \begin{array}{l} \varphi = x^2 y + f(y) \\ \cancel{x^2} + \frac{df}{dy} = \cancel{x^2} - 2y \end{array}$$

$\frac{df}{dy} = -2y \implies f = -y^2 + C$

$\varphi = x^2 y - y^2 + c$

Example : $\vec{V} = (y^2, x^2)$, $\cancel{\varphi ?}$

$$\vec{V} = \left(\frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y}\right) = (y^2, x^2)$$

$$\begin{cases} \dfrac{\partial \varphi}{\partial x} = y^2 \\ \dfrac{\partial \varphi}{\partial y} = x^2 \end{cases} \Longrightarrow \quad \varphi = x y^2 + f(y)$$

$$2xy + \frac{df}{dy} = x^2$$

$$\frac{df}{dy} = x^2 - 2xy .$$

Test for the existence of the potential.

$$\nabla \varphi = \vec{V} = (v_1, v_2) = \left(\frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y}\right)$$

$$\frac{\partial^2 \varphi}{\partial x \partial y} = \frac{\partial^2 \varphi}{\partial y \partial x}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial \varphi}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial \varphi}{\partial x}\right)$$

$$\boxed{\frac{\partial}{\partial x} v_2 = \frac{\partial}{\partial y} v_1}$$

Ex. 1 : $\vec{F} = (\underbrace{2xy}_{V_1}, \underbrace{x^2 - 2y}_{V_2})$

$$\frac{\partial V_1}{\partial y} = 2x = \frac{\partial V_2}{\partial x} = 2x \quad \checkmark$$

Ex. 2 : $\vec{V} = (\underbrace{y^2}_{V_1}, \underbrace{x^2}_{V_2})$

$$\frac{\partial V_2}{\partial x} = 2x, \quad \frac{\partial V_1}{\partial y} = 2y$$

$$2x \neq 2y \quad \times$$

Potential Functions (continued)

$$\vec{F} = (6xy,\ 3x^2 - 5z,\ 2z - 5y)$$

$$\nabla \varphi = \left(\frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y}, \frac{\partial \varphi}{\partial z}\right) = \vec{F}$$

$$\frac{\partial \varphi}{\partial x} = 6xy \implies \varphi = 3x^2 y + f(y,z)$$

$$\frac{\partial \varphi}{\partial y} = 3x^2 - 5z \implies \cancel{3x^2} + \frac{\partial f}{\partial y}(y,z) = \cancel{3x^2} - 5z$$

$$f(y,z) = -5yz + g(z)$$

$$\frac{\partial \varphi}{\partial z} = 2z - 5y$$

$$\varphi = 3x^2 y - 5yz + g(z)$$

$$\implies \cancel{-5y} + \frac{dg}{dz} = 2z \cancel{-5y}$$

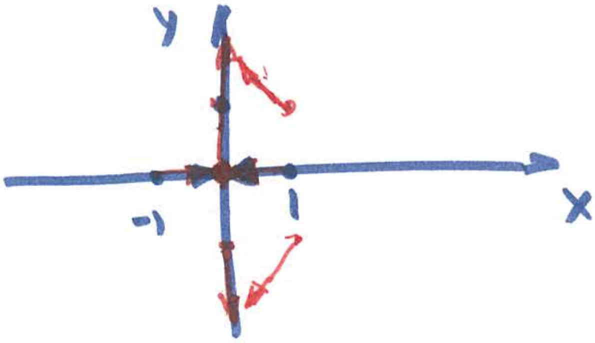
$$g = z^2 + c$$

$$\boxed{\varphi = 3x^2 y - 5yz + z^2 + c}$$

Velocity potential

$$\varphi = -\frac{1}{2}x^2 + \frac{1}{2}y^2, \quad \vec{v} = \left(\frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y}\right)$$

$$\vec{v} = (-x, y)$$



$$\varphi = \underline{-\frac{1}{2}x^2 + \frac{1}{2}y^2} = c/2$$

$$y^2 = x^2 + c$$

$$y = \pm\sqrt{x^2 + c}$$

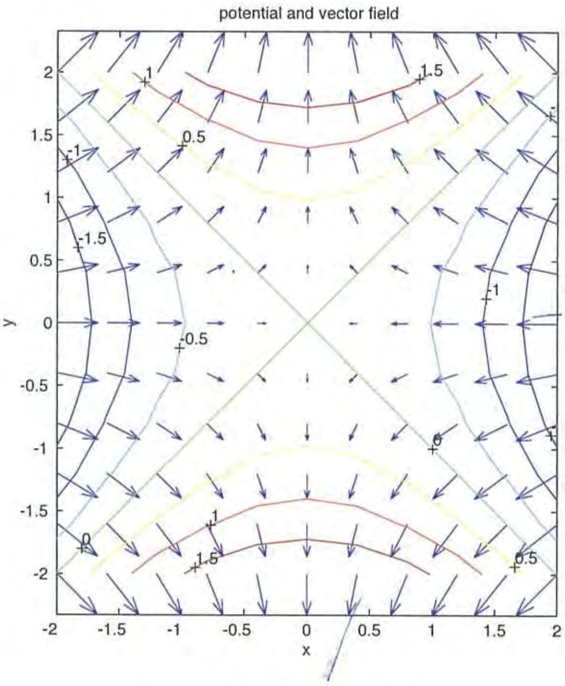
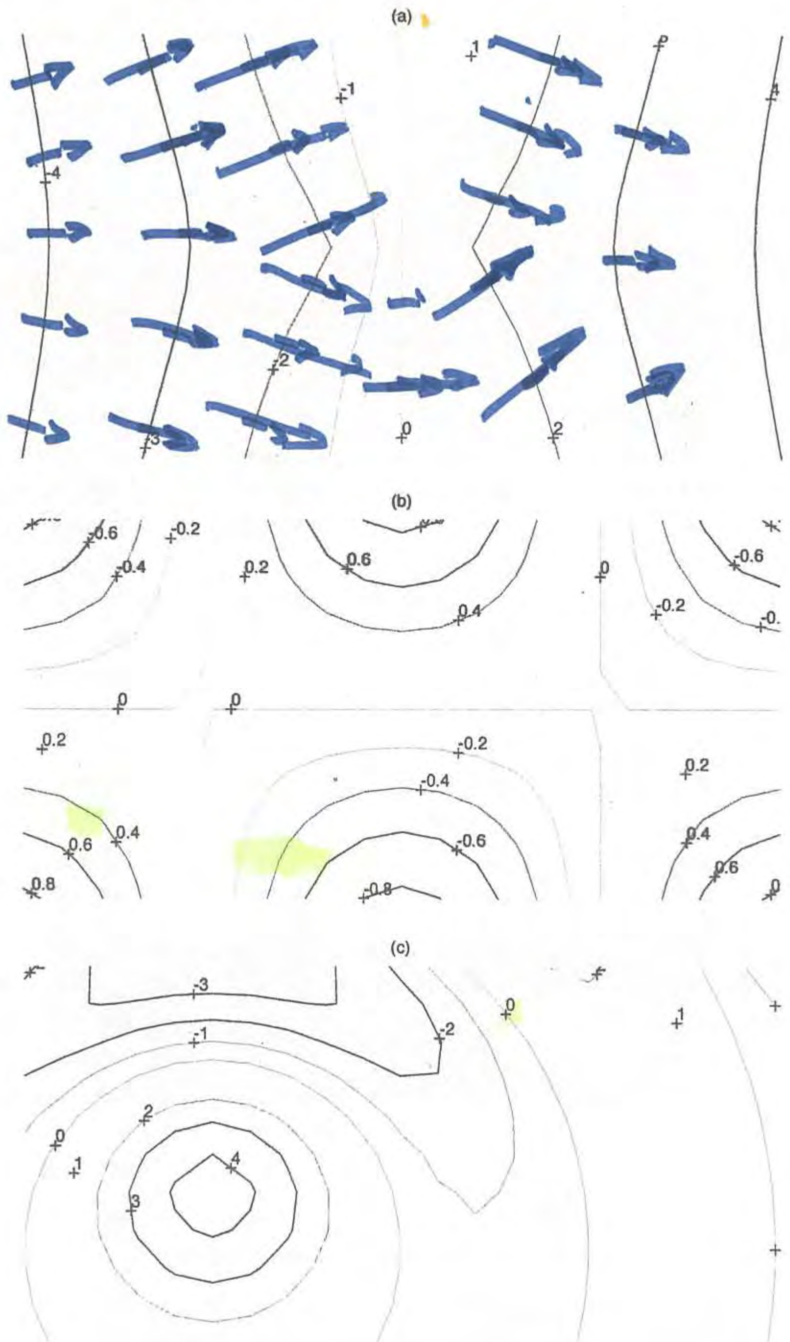


Figure 2.7: Corner flow.

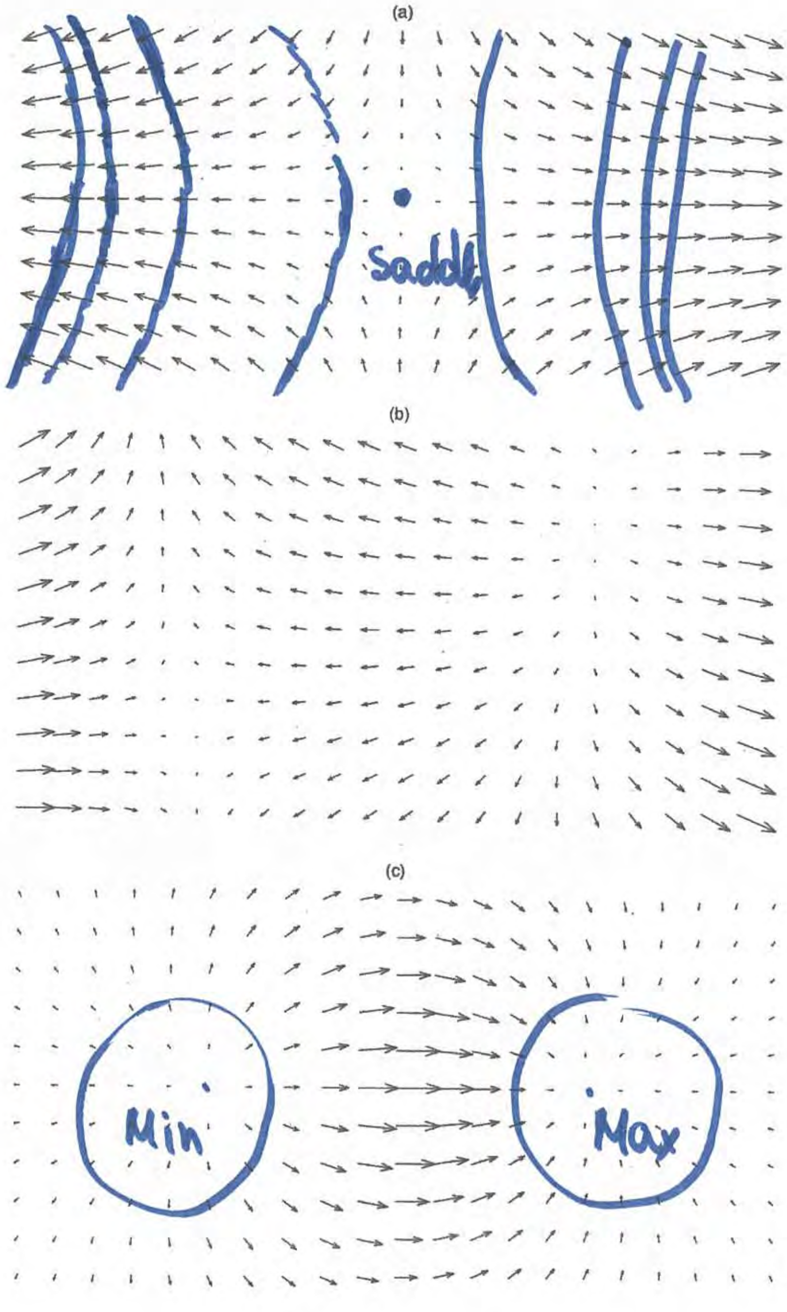
As a point of interest note that the flow depicted in Figure 2.7 is known as a *corner flow*. The name is derived from the fact that the velocity field is such that the fluid particles initially located in one of the quadrants will never penetrate the other quadrants. Therefore the flow would not change if the $x = 0$ and $y = 0$ lines were replaced with solid walls forming four corners with a common vertex at the origin. A flow of this kind also arises when two water jets of equal strength collide at the origin.

Exercises

Ex. 2.1. For each of the scalar functions with level curves plotted below sketch the vector field of its gradient. Endeavour to get the directions and relative magnitudes of vectors correctly.

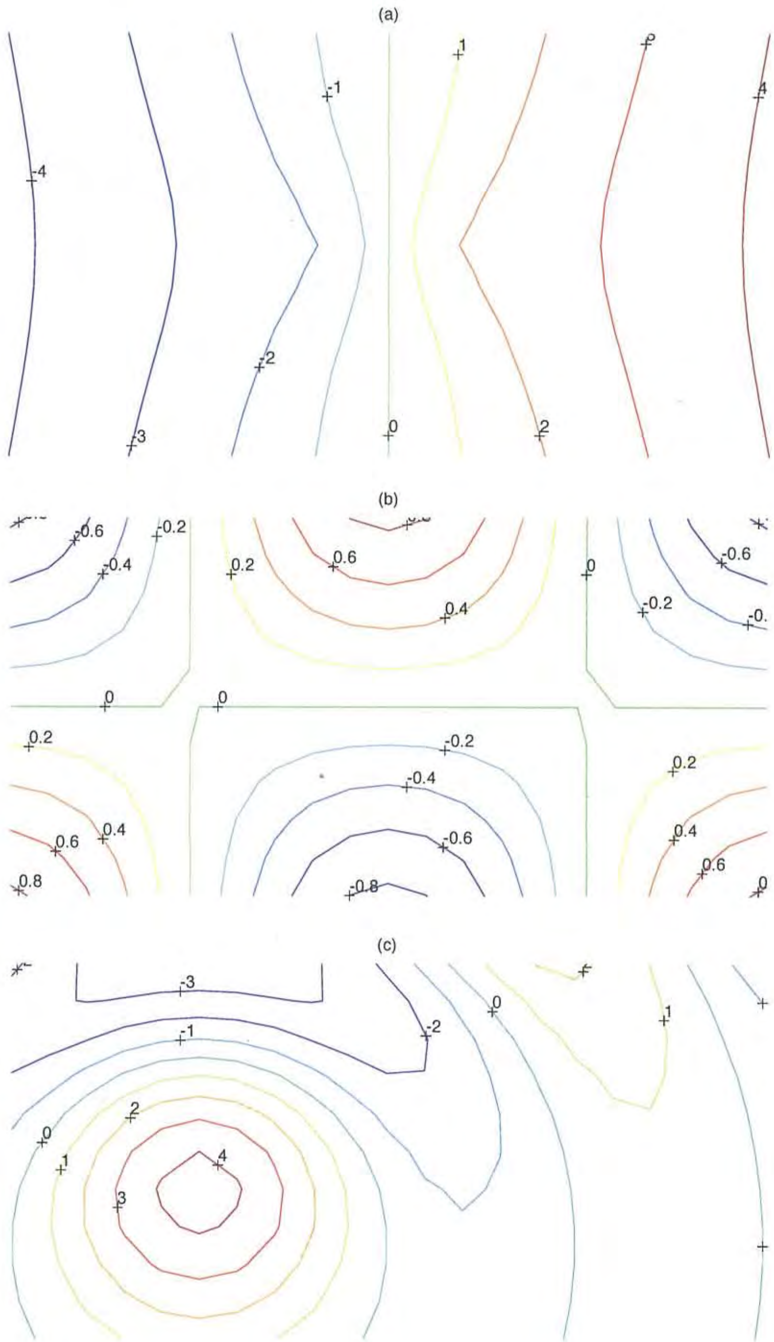


Ex. 2.2. Given that the following vector fields are the gradient of a potential function, sketch level curves of the potential and indicate where the potential has a local maximum, minimum or a saddle point.



Exercises

Ex. 2.1. For each of the scalar functions with level curves plotted below sketch the vector field of its gradient. Endeavour to get the directions and relative magnitudes of vectors correctly.



The Divergence

$$\vec{V} \Longrightarrow \boxed{\nabla \cdot \vec{V}} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (u, v, w)$$

$$\vec{V} = (u, v, w)$$

$$\boxed{= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}}$$

Example: $\vec{V} = (x^2 y,\ x^2 + yz,\ xyz)$

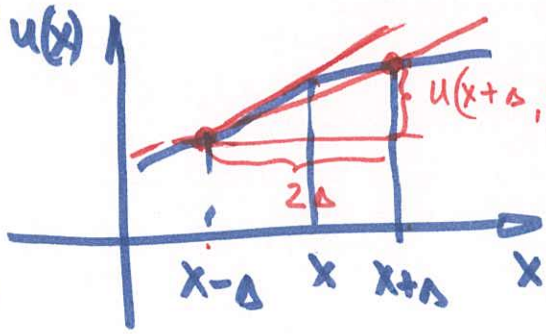
$$\nabla \cdot \vec{V} = \frac{\partial}{\partial x}(x^2 y) + \frac{\partial}{\partial y}(x^2 + yz) + \frac{\partial}{\partial z}(xyz)$$

$$= 2xy + z + xy = 3xy + z$$

Geometrical meaning of divergence:

$$\frac{1}{4}\left[u(x+\Delta, y) + v(x, y+\Delta) - u(x-\Delta, y) - v(x, y-\Delta)\right]$$

$$= \frac{\Delta}{2}\left[\frac{u(x+\Delta, y) - u(x-\Delta, y)}{2\Delta} + \frac{v(x, y+\Delta) - v(x, y-\Delta)}{2\Delta}\right]$$



$u(x+\Delta, y) - u(x-\Delta, y)$ $\quad \Delta \to 0$

$$\lim_{\Delta \to 0} \frac{u(x+\Delta, y) - u(x-\Delta, y)}{2\Delta} = \frac{\partial u}{\partial x}$$

$$\approx \frac{\Delta}{2}\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right] = \frac{\Delta}{2}\, \nabla \cdot \vec{V}$$

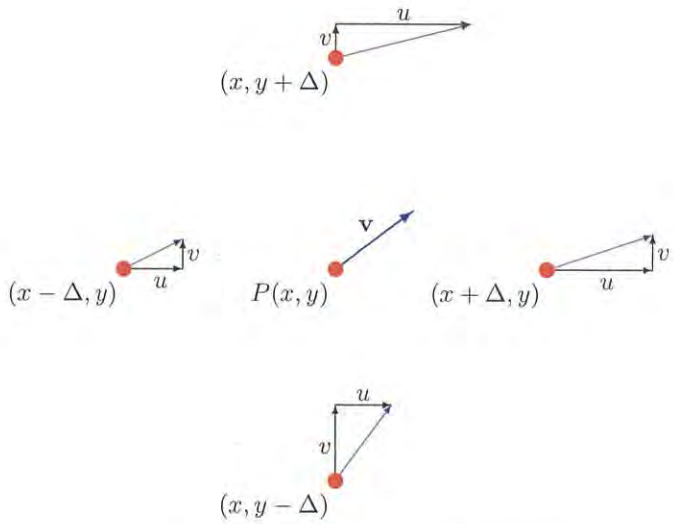


Figure 2.8: The meaning of divergence.

- to the right of P at $(x+\Delta, y)$ the component is $u(x+\Delta, y)$ as the v component is directed neither towards nor away from P;
- above P at $(x, y+\Delta)$ the component is $v(x, y+\Delta)$ as here it is the u component that is neither towards nor away from P;
- to the left of P at $(x-\Delta, y)$ the component is $-u(x-\Delta, y)$ (minus because positive u points towards P);
- below P at $(x, y-\Delta)$ the component is $-v(x, y-\Delta)$ (minus because positive v points towards P).

When the distance Δ between the neighbouring points tends to zero the average of these "pointing away" components is

$$\begin{aligned}
& \frac{1}{4}\left[u(x+\Delta, y)+v(x, y+\Delta)-u(x-\Delta, y)-v(x, y-\Delta)\right] \\
= \; & \frac{\Delta}{2}\left[\frac{u(x+\Delta, y)-u(x-\Delta x, y)}{2\Delta}+\frac{v(x, y+\Delta)-v(x, y-\Delta)}{2\Delta}\right] \\
\approx \; & \frac{\Delta}{2}\left[\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right] \quad \text{by a limit definition of the derivatives} \\
= \; & \frac{\Delta}{2}(\boldsymbol{\nabla}\cdot\mathbf{v}) \quad \text{by Cartesian formula for divergence.}
\end{aligned}$$

This demonstrates that the divergence of a vector field at any point is proportional to the average of how much the vector field "points away" from the point.

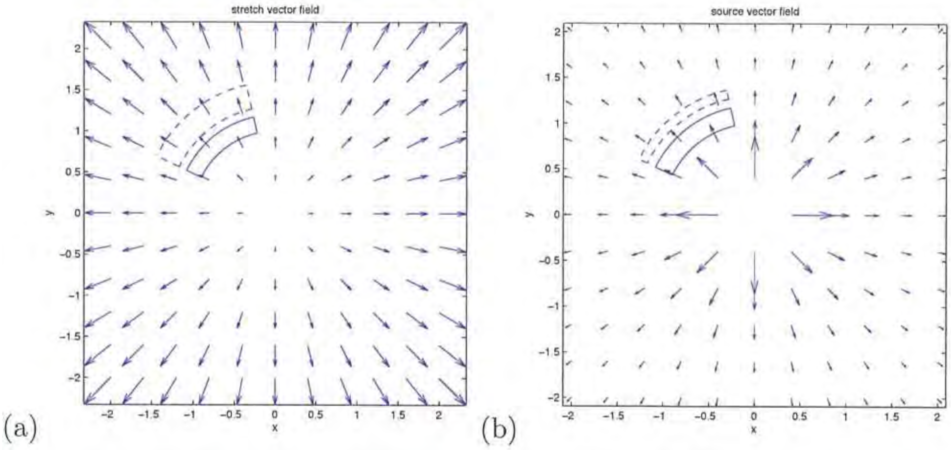


Figure 2.9: Examples of (a) non-zero and (b) zero divergence fields.

2.3.1 Divergence in fluids

In the fluids context a non-zero divergence of a velocity field at a point measures the rate, per unit volume, at which the fluid is flowing away from that point:

$$\nabla\cdot\mathbf{v} = \lim_{\Delta V\to 0} \frac{\text{flow out of } \Delta V}{\Delta V},$$

where ΔV is a volume enclosing the point. For example, contrast the vector fields $\mathbf{v} = (x, y)$ and $\mathbf{v} = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)$ shown in Figure 2.9 (a) and (b), respectively. In the first case the divergence is $\nabla\cdot\mathbf{v} = 2$ everywhere, while in the second the divergence is $\nabla\cdot\mathbf{v} = 0$ except at the singularity at the origin. The vector field shown in Figure 2.9 (a) corresponds to a uniform expansion field such as that of a rubber sheet that is being stretched. Examine any point P and the neighbouring arrows: some point towards P, but the arrows pointing away from P are bigger. Thus there is a net "pointing away" and the divergence is everywhere positive. The vector field shown in Figure 2.9 (b) corresponds to the sort of flow obtained when you turn on a tap and pour water down onto a flat surface: immediately under the tap there is a source of water (the singularity), but everywhere else it just spreads with a velocity that slows further away from the source. Even though the calculations show that the divergence is zero, being a borderline case, it is hard to establish this for sure by visual inspection.

Definition 2.11 *If the divergence of a vector field is zero everywhere then this vector field is called* solenoidal.

Fluids whose velocity fields are solenoidal are called *incompressible*. The amount of such fluid entering any region on one side is exactly equal to its