MTH20014 Mathematics 3B. Tutorial 10

1. For $z = 2 - 3\mathrm{i}$ find
   (i) $\mathrm{i}z$, (ii) z^*, (iii) $\dfrac{1}{z}$, (iv) $(z^*)^*$.

2. Plot the points $z_1 = -1 - \sqrt{3}\mathrm{i}$, $z_2 = -1 + \sqrt{3}\mathrm{i}$, $z_3 = -1$, $z_4 = 2 + 2\mathrm{i}$ on an Argand diagram.

3. Express $z = \dfrac{(2-\mathrm{i})(3+2\mathrm{i})}{3-4\mathrm{i}}$ in (i) Cartesian form, (ii) complex exponential form.

4. For $z_1 = 2e^{\frac{\mathrm{i}\pi}{3}}$ and $z_2 = 4e^{-\frac{2\mathrm{i}\pi}{3}}$ find
   (i) $\left|\dfrac{z_1^2}{z_2^3}\right|$, (ii) $\arg \dfrac{z_1^2}{z_2^3}$.

5. For $z = -1 - \mathrm{i}\sqrt{3}$ find z^3.

6. Show that the following functions are analytic, i.e. that their real and imaginary parts satisfy the Cauchy-Riemann conditions, and find an expression for $f'(z) = \frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial v}{\partial x}$

   (a) $w = (1+3\mathrm{i})z^2$,

   (b) $w = (1+\mathrm{i})z^3$,

   (c) $w = \dfrac{z^2+\mathrm{i}}{z}$.

7. Show that the following functions $u(x,y)$ are harmonic, find their conjugate harmonic functions $v(x,y)$ and determine $f(z) = u(x,y) + \mathrm{i}v(x,y)$ as functions of z.

   (a) $u(x,y) = 2x^2 - 2xy - 5x - 2y^2$,

   (b) $u(x,y) = 3x^2 - 8xy - 3y^2 + 2y$,

   (c) $u(x,y) = 2e^x \cos y - 3e^x \sin y$,

   (d) $u(x,y) = 3e^{-x} \cos y + 5e^{-x} \sin y$,

   (e) $u(x,y) = e^x x \cos y - e^x y \sin y$,

   (f) $u(x,y) = 2x^3 - 3yx^2 - 5x^2 - 6y^2x + y^3 + 5y^2$.

Answers

1. (i) $3+2\mathrm{i}$, (ii) $2+3\mathrm{i}$, (iii) $\dfrac{2+3\mathrm{i}}{13}$, (iv) $2-3\mathrm{i}$.

3. (ii) $\dfrac{4}{5}+\dfrac{7}{5}\mathrm{i}$, (ii) $\dfrac{\sqrt{65}}{5}e^{\mathrm{i}\theta}$, where $\theta = \tan^{-1}\dfrac{7}{4}$.

4. (i) $\dfrac{1}{16}$, (ii) $\dfrac{2\pi}{3}$.

5. 8.

6. (a) $2x-6y+\mathrm{i}(6x+2y)$,
   (b) $3x^2-3y^2-6xy+\mathrm{i}(3x^2-3y^2+6xy)$,
   (c) $1-\dfrac{2xy}{(x^2+y^2)^2}+\mathrm{i}\dfrac{y^2-x^2}{(x^2+y^2)^2}$.

7. (a) $f(z)=(2+\mathrm{i})z^2-5z$,
   (b) $f(z)=(3+4\mathrm{i})z^2-2jz$,
   (c) $f(z)=(2+3\mathrm{i})e^z$,
   (d) $f(z)=(3+5\mathrm{i})e^{-z}$,
   (e) $f(z)=ze^z$,
   (f) $f(z)=(2+\mathrm{i})z^3-5z^2$.