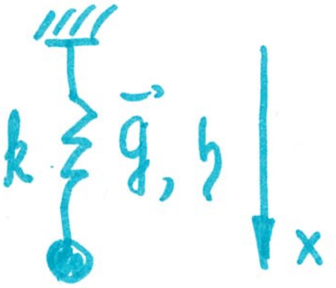
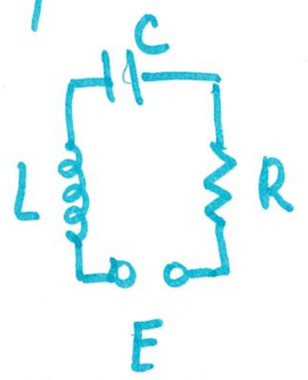
Differential equations (ODE)



$$m\ddot{x} = -kx - \eta\dot{x} \; (+mg)$$

$$\frac{dx}{dt} = \dot{x}$$

$$m\ddot{x} + kx + \eta\dot{x} = \begin{cases} 0 \\ mg \end{cases}$$



$$E = V_L + V_C + V_R$$

$V_R = RI$ - Ohm

$$V_C = \frac{Q}{C}$$

$$V_L = L\dot{I}$$

$$I = \dot{Q}$$

$$L\ddot{Q} + R\dot{Q} + \frac{1}{C}Q = \begin{cases} E \\ 0 \end{cases}$$

$$\ddot{x} + \frac{\eta}{m}\dot{x} + \frac{k}{m}x = \begin{cases} 0 \\ g \end{cases}$$

$$x \to y, \quad \frac{\eta}{m} \to a, \quad \frac{k}{m} \to b$$

$$\begin{Bmatrix} 0 \\ g \end{Bmatrix} \to r$$

$$\ddot{Q} + \frac{R}{L}\dot{Q} + \frac{1}{CL}Q = \begin{cases} E/L \\ 0 \end{cases}$$

$$Q \to y, \quad \frac{R}{L} \to a, \quad \frac{1}{CL} \to b$$

$$\begin{Bmatrix} E/L \\ 0 \end{Bmatrix} \to r$$

$$\ddot{y} + a\dot{y} + by = r$$

$y = ce^{\lambda t}$ - general solution

$$x_1 = y \Rightarrow \boxed{\dot{x}_1 = \dot{y} = x_2}$$

$$\dot{x}_2 = \ddot{x}_1 = \ddot{y}$$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -a x_2 - b x_1 \end{cases}$$

$$\sin t = \frac{e^{it} - e^{-it}}{2i}$$

$$\cos t = \frac{e^{it} + e^{-it}}{2}$$

Euler's formula

$r = 0$ for the time being.

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \dot{\vec{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}$$

$$\dot{\vec{x}} = \begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix} \vec{x} \Rightarrow \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} e^{\lambda t} \Rightarrow \frac{d}{dt}\left(\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} e^{\lambda t}\right) = \lambda \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} e^{\lambda t}$$

$$\lambda \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \cancel{e^{\lambda t}} = \begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \cancel{e^{\lambda t}}$$

$$\begin{bmatrix} -\lambda & 1 \\ -b & -a-\lambda \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} \frac{dx_1}{dt} &= ax_1 + bx_2 \\ \frac{dx_2}{dt} &= cx_1 + dx_2 \end{aligned}\right\} \qquad \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} e^{\lambda_1 t} + \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} e^{\lambda_2 t}$$

where λ_1 & λ_2 are e.val. and $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ & $\begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$ are the corresponding e.vectors of A

Solving systems of ODE

$$\begin{cases} \dot{x}_1 = 3x_1 + 4x_2 \\ \dot{x}_2 = 5x_1 + 2x_2 \\ \dot{x}_2 = 2x_2 + 5x_1 \end{cases}$$

$x_1(0) = 10$

$x_2(0) = 1$

$\dot{\vec{x}} = A\vec{x}$

Method 1

$$(*) \begin{cases} A = \begin{bmatrix} 3 & 4 \\ 5 & 2 \end{bmatrix} ; \quad \begin{vmatrix} 3-\lambda & 4 \\ 5 & 2-\lambda \end{vmatrix} = (3-\lambda)(2-\lambda) - 20 = \lambda^2 - 5\lambda - 14 = 0 \\ \lambda_{1,2} = \dfrac{5 \pm \sqrt{25 + 4 \cdot 14}}{2} = 7, -2 \end{cases}$$

$$\lambda_1 = 7 : \begin{bmatrix} -4 & 4 \\ 5 & -5 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 \end{bmatrix} \Rightarrow \vec{e}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} c_1$$

$$\lambda_2 = -2 \quad \begin{bmatrix} 5 & 4 \\ 5 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 5 & 4 \end{bmatrix} \Rightarrow \vec{e}_2 = \begin{bmatrix} 4 \\ -5 \end{bmatrix} c_2$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{7t} + c_2 \begin{bmatrix} 4 \\ -5 \end{bmatrix} e^{-2t}$$ – general solution

$$x_1 = c_1 e^{7t} + 4c_2 e^{-2t} \qquad \begin{cases} 10 = c_1 + 4c_2 \\ 1 = c_1 - 5c_2 \end{cases}$$

$$x_2 = c_1 e^{7t} - 5c_2 e^{-2t}$$

$9 = 9c_2 \Rightarrow c_2 = 1 \quad , \quad c_1 = 10 - 4 \cdot 1 = 6$

$x_1 = 6e^{7t} + 4e^{-2t}$; $x_2 = 6e^{7t} - 5e^{-2t}$ – specific solution

Method 2. Rationale:

$\dot{x} = ax \Rightarrow x = ce^{at}$

(*) $\dot{\vec{x}} = A\vec{x} \Rightarrow \vec{x} = e^{At}\vec{c}$

$$e^{At} = \alpha_1 A + \alpha_0 I$$

$$\begin{cases} e^{7t} = 7\alpha_1 + \alpha_0 \\ e^{-2t} = -2\alpha_1 + \alpha_0 \end{cases} \qquad \begin{array}{l} a = e^{7t} \\ b = e^{-2t} \end{array} \qquad \begin{cases} a = 7\alpha_1 + \alpha_0 \\ b = -2\alpha_1 + \alpha_0 \end{cases}$$

$$a - b = 9\alpha_1 \Rightarrow \alpha_1 = \frac{1}{9}(a-b)$$

$$a = \frac{7}{9}(a-b) + \alpha_0 \Rightarrow \alpha_0 = a - \frac{7}{9}a + \frac{7}{9}b$$

$$= \frac{2}{9}a + \frac{7}{9}b$$

$$e^{At} = \frac{1}{9}(a-b)\begin{bmatrix} 3 & 4 \\ 5 & 2 \end{bmatrix} + \frac{1}{9}(2a+7b)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 3a-3b+2a+7b & 4(a-b) \\ 5(a-b) & 2a-2b+2a+7b \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 5a+4b & 4a-4b \\ 5a-5b & 4a+5b \end{bmatrix}$$

$$\vec{x} = e^{At}\vec{c} \qquad t=0 \Rightarrow \vec{x}(0) = I \cdot \vec{c} = \vec{c}$$

$$\vec{c} = \begin{bmatrix} 10 \\ 1 \end{bmatrix}$$

$$\vec{x} = e^{At}\vec{c} = \frac{1}{9}\begin{bmatrix} 5a+4b & 4a-4b \\ 5a-5b & 4a+5b \end{bmatrix}\begin{bmatrix} 10 \\ 1 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 50a+40b+4a-4b \\ 50a-50b+4a+5b \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 54a+36b \\ 54a-45b \end{bmatrix}$$

$$= \begin{bmatrix} 6a+4b \\ 6a-5b \end{bmatrix} = \begin{bmatrix} 6e^{7t}+4e^{-2t} \\ 6e^{7t}-5e^{-2t} \end{bmatrix}$$

Another example of solving systems of linear first-order ODEs

Example
$$\begin{cases} \dot{x}_1 = x_1 \\ \dot{x}_2 = x_1 + x_2 \end{cases} \qquad A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$\begin{vmatrix} 1-\lambda & 0 \\ 1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 = 0 \;, \quad \lambda_1 = \lambda_2 = 1$$

Method 1.

$$\lambda_1 = 1: \quad \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \Rightarrow [1 \;\; 0] \,, \quad \vec{e}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} c_1$$

$\lambda_2 = 1 \qquad \vec{e}_2$ does not exist

Method 1 does not work!

Method 2.

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = e^{At} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} ; \quad e^{At} = \alpha_1 A + \alpha_0 I \qquad \begin{cases} e^{\lambda t} = \alpha_1 \lambda + \alpha_0 \\ \alpha_1 = t e^{\lambda t} \end{cases}$$

$$\alpha_1 = t e^{t} \,, \quad \alpha_0 = e^{\lambda t} - \alpha_1 \lambda = e^{t} - t e^{t} = e^{t}(1-t)$$

$$e^{At} = \alpha_1 \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + \alpha_0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \alpha_1 + \alpha_0 & 0 \\ \alpha_1 & \alpha_1 + \alpha_0 \end{bmatrix} = \begin{bmatrix} \cancel{te^t} + e^t - \cancel{te^t} & 0 \\ te^t & e^t \end{bmatrix}$$

$$= \begin{bmatrix} e^t & 0 \\ te^t & e^t \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} e^t & 0 \\ te^t & e^t \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_1 e^t \\ c_1 te^t + c_2 e^t \end{bmatrix}$$

$$\begin{cases} x_1 = c_1 e^t \\ x_2 = (c_1 t + c_2) e^t \end{cases}$$ General solution ■

Solve one eqn. at a time

$x_1 = c_1 e^t$

$\dot{x}_2 = x_2 + c_1 e^t \Rightarrow \dot{x}_2 - x_2 = c_1 e^t$

$\dot{x}_2 - x_2 = 0 \Rightarrow x_2 = c_2 e^t$