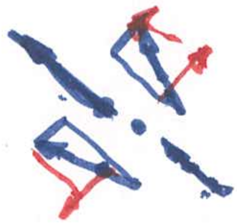


$\nabla \vec{V} > 0$

$\nabla \vec{V} < 0$?

Laplacian

$$\nabla \cdot \nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

$$= \frac{\partial}{\partial x}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\frac{\partial}{\partial y} + \frac{\partial}{\partial z}\frac{\partial}{\partial z}$$

$$= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \nabla^2$$

$\nabla^2 \varphi$, where $\varphi = 2x^2 y - 2y^2 x + 4z^2$

$$\frac{\partial \varphi}{\partial x} = 4xy - 2y^2, \quad \frac{\partial \varphi}{\partial y} = 2x^2 - 4xy, \quad \frac{\partial \varphi}{\partial z} = 8z$$

$$\frac{\partial^2 \varphi}{\partial x^2} = 4y, \quad \frac{\partial^2 \varphi}{\partial y^2} = -4x, \quad \frac{\partial^2 \varphi}{\partial z^2} = 8$$

$$\boxed{\nabla^2 \varphi = 4y - 4x + 8}$$

Laplacian in Fluids

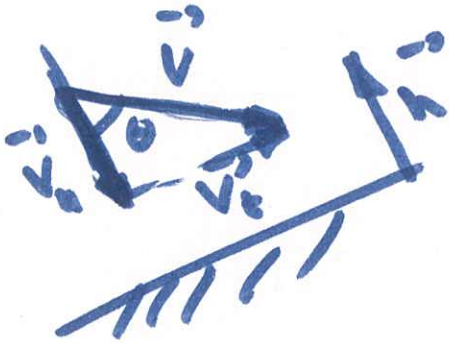
$\vec{V} = \nabla \varphi$, $\quad \nabla \cdot \vec{V} = 0 \Rightarrow \nabla \cdot \nabla \varphi = \nabla^2 \varphi = 0$

Laplace's eqn. $\boxed{\nabla^2 \varphi = 0}$

$\vec{V}_n = 0$?

$\vec{V} = \nabla \varphi$

$\vec{V} = \vec{V}_t + \vec{V}_n$



$$\vec{v}\cdot\vec{n} = |\vec{v}|\cdot|\vec{n}|\cdot\cos\theta = |\vec{v}|\cdot\cos\theta = |\vec{v}_n|$$

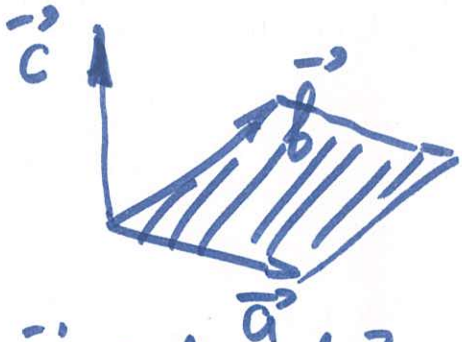
$$\nabla\varphi\cdot\vec{n} \Rightarrow D_{\vec{n}}\varphi$$

$$\boxed{D_{\vec{n}}\varphi = 0}$$

The Curl

$$\vec{a} = (a_1, a_2, a_3), \quad \vec{b} = (b_1, b_2, b_3)$$

$$\vec{c} = \vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$



$$a \rightarrow \nabla$$

$$\nabla\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ b_1 & b_2 & b_3 \end{vmatrix} = \vec{i}\left[\frac{\partial b_3}{\partial y} - \frac{\partial b_2}{\partial z}\right] - \vec{j}\left[\frac{\partial b_3}{\partial x} - \frac{\partial b_1}{\partial z}\right] + \vec{k}\left[\frac{\partial b_2}{\partial x} - \frac{\partial b_1}{\partial y}\right]$$

Example: $\vec{V} = (-y, x, z)$

$$\nabla\times\vec{V} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & z \end{vmatrix} = \vec{i}\,[0] - \vec{j}\,[0] + \vec{k}\,[1+1] = 2\vec{k} = (0,0,2)$$

$\nabla \times \vec{v}$, where $\vec{v} = (u, v, 0)$, $(u, v) \neq F(z)$

$$\nabla \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & 0 \end{vmatrix}$$

$$= \vec{i}\ 0 - \vec{j} \cdot 0 + \vec{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

$$= \left(0, 0, \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

<u>Curl measures rotation of a fluid</u>

$\vec{v} = (\underbrace{-y}_{u}, \underbrace{x}_{v})$, $\nabla \times \vec{v} = (0, 0, 1+1) = (0, 0, 2)$

$|\vec{v}| = \sqrt{(-y)^2 + x^2} = \sqrt{x^2+y^2} = r$

solid body rotation

$\vec{v} = \frac{(-y, x)}{x^2+y^2}$ Kitchen sink vortex

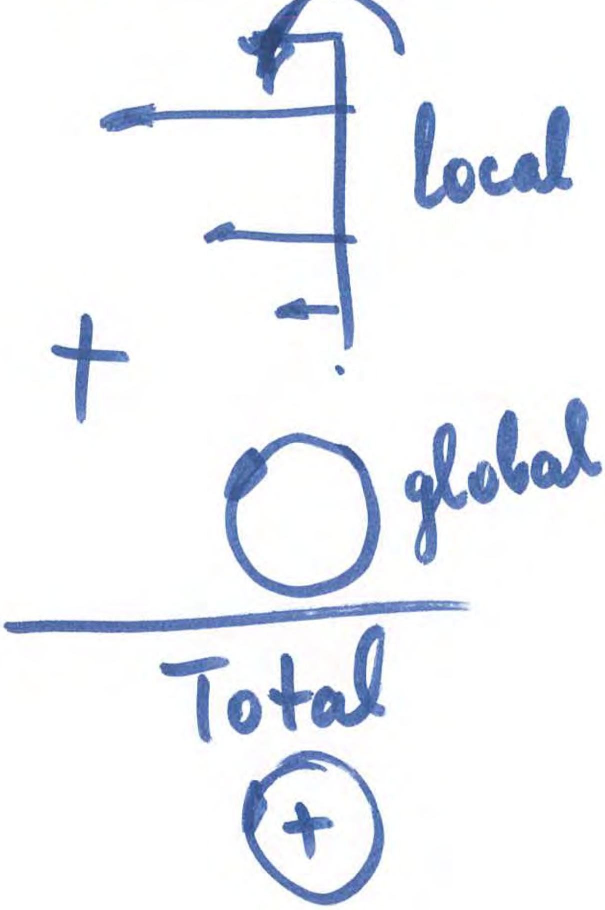
$$\nabla \times \vec{v} = \left(0, 0, \frac{\partial}{\partial x}\frac{x}{x^2+y^2} + \frac{\partial}{\partial y}\frac{y}{x^2+y^2}\right)$$

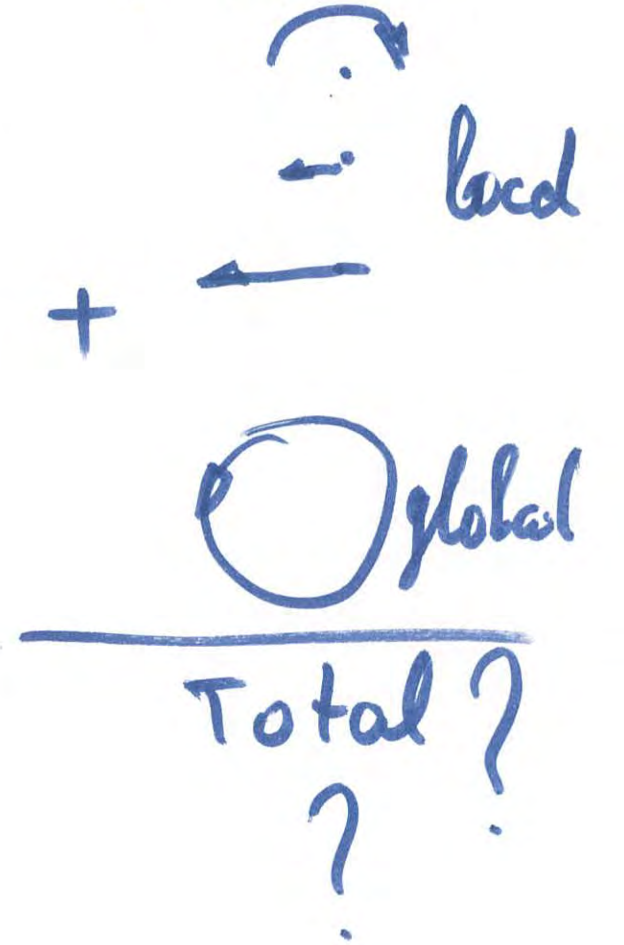
$$\frac{\partial}{\partial x}\frac{x}{x^2+y^2} = \frac{1}{x^2+y^2} + x \cdot \frac{(-1)}{(x^2+y^2)^2} \cdot 2x$$

$$= \frac{x^2+y^2 - 2x^2}{x^2+y^2} = \frac{y^2-x^2}{x^2+y^2}$$

$$\frac{\partial}{\partial y}\frac{y}{x^2+y^2} = \frac{x^2-y^2}{x^2+y^2}$$

$$\frac{\partial}{\partial x}(\;)+\frac{\partial}{\partial y}(\;) = \frac{\cancel{y^2}-\cancel{x^2}+\cancel{x^2}-\cancel{y^2}}{x^2+y^2} \overset{?}{=} 0$$

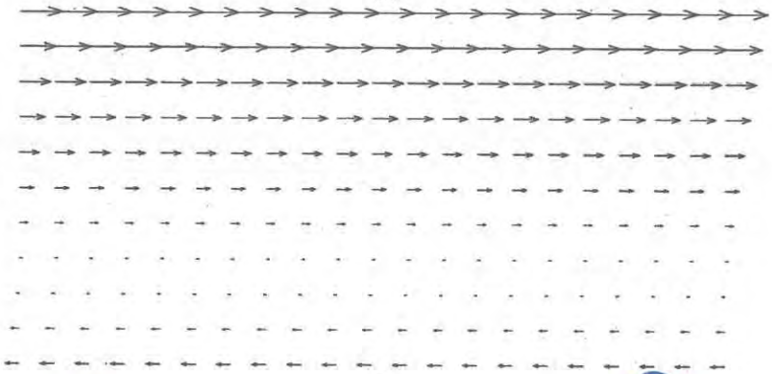




Exercises

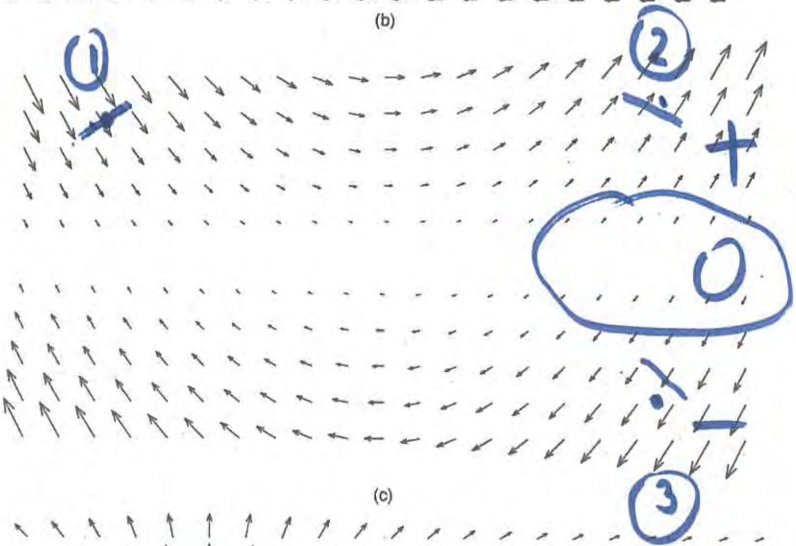
Ex. 2.3. For the following vector fields, sketch regions where you think the divergence is positive, negative, and approximately zero. Consider a variety of points, and for any one point look at the net effect of vectors in its immediate neighbourhood.

(a)



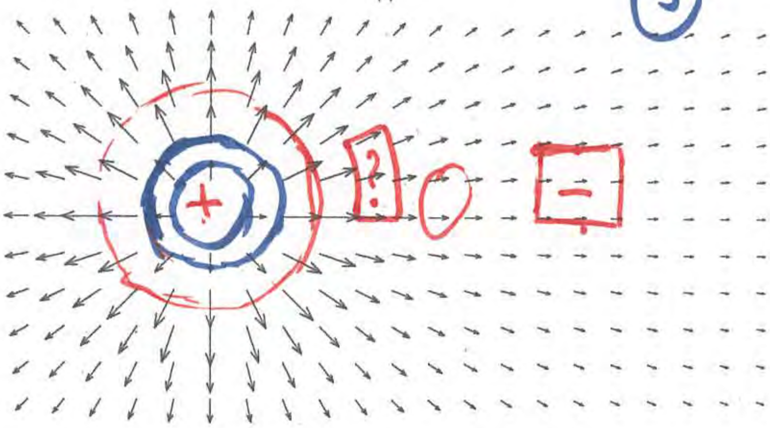
(b)

?



	1	2	3
g	+	+	−
e	−	+	−
T	?	+	−

(c)



Example. $\vec{V} = (y+z, x+z, x+y)$ determine if it has a potential. If so, find it.

$$\nabla \times \vec{V} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y+z & x+z & x+y \end{vmatrix} = \begin{matrix} \vec{i}\,[1-1] \\ -\vec{j}\,[1-1] \\ +\vec{k}\,[1-1] \end{matrix} = \vec{0}$$

$$\nabla \varphi = \vec{V}$$

$$\frac{\partial \varphi}{\partial x} = y+z \implies \varphi = xy + xz + f(y,z) = xy + xz + yz + g(z)$$

$$\frac{\partial \varphi}{\partial y} = x+z \implies \cancel{x} + \frac{\partial f}{\partial y} = \cancel{x} + z \,;\; \frac{\partial f}{\partial y} = z$$

$$f = yz + g(z)$$

$$\frac{\partial \varphi}{\partial z} = x+y \implies \cancel{x+y} + \frac{dg}{dz} = \cancel{x+y}$$

$$\frac{dg}{dz} = 0 \implies g = c$$

$$\boxed{\varphi = xy + xz + yz + c}$$

$\nabla \times \nabla f$

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

$$\nabla \times \nabla f = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = \vec{i}\left[\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right] - \vec{j}\left[\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial z \partial x}\right] + \vec{k}\left[\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right]$$

$$= (0, 0, 0) = \vec{0}$$

$$\boxed{\nabla \times \nabla f = \vec{0}}$$ - identity

<u>Theorem</u>: A vector field has a potential function <u>if and only if</u> it is irrotational.

$$\varphi = \varphi(x, y), \quad \nabla \varphi = \left(\frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y}\right) = (v_1, v_2)$$

$$\nabla \times \nabla \varphi = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 \\ v_1 & v_2 & 0 \end{vmatrix} = \vec{i} \cdot 0 + \vec{j} \cdot 0 + \vec{k}\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right) = \vec{0}$$

$$\boxed{\frac{\partial v_2}{\partial x} = \frac{\partial v_1}{\partial y}}$$

Line integrals

$$= \int_a^b f\,dx$$

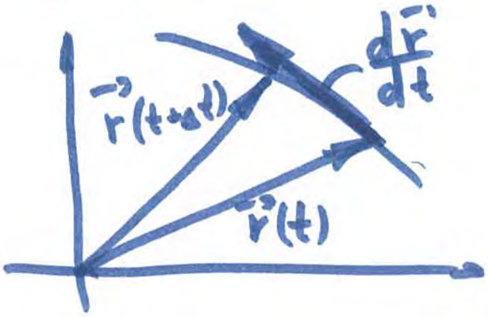
$$\Delta S = \overset{\Delta x}{\Delta x}\, y_i = f_i\,\Delta x \implies A = \sum_{i=1}^{N} f_i \Delta x$$

$$A = \lim_{\Delta x \to 0} \sum_{i=1}^{N} f_i \Delta x \quad , \quad N = L/_{\Delta x}$$

For curvilinear regions $\Delta X \longrightarrow \Delta S$.

$y = f(x)$

$$\Delta S = \sqrt{\Delta x^2 + \Delta y^2}$$



$$\frac{d\vec{r}}{dt}$$

$$\vec{r} = (x, y)$$

$$\vec{r}(t) = (x(t), y(t))$$

$$\Delta S = \left|\frac{d\vec{r}}{dt}\right| \cdot \Delta t$$

$$y = g(x) \qquad \begin{cases} x = t \\ y = g(t) \end{cases} \text{ - parametrisation}$$

$$\vec{r} = (x, f(x))$$

$$\vec{r}(t) = (t, g(t)) \implies \frac{d\vec{r}}{dt} = \left(\frac{dx}{dt}, \frac{dy}{dt}\right) = \left(1, \frac{dg}{dt}\right)$$

$$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{1^2 + \left(\frac{dg}{dt}\right)^2}$$

$$\boxed{\boxed{\int_C f \cdot ds} = \int_a^b f(t) \left|\frac{d\vec{r}}{dt}\right| dt}$$

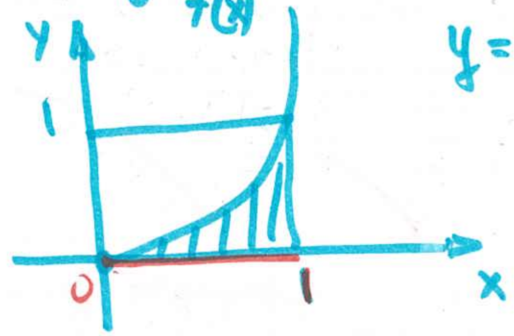
line integral along path C

$$(x, y) = \big(x(t), y(t)\big) \implies \begin{aligned} (x_0, y_0) &= (x(a), y(a)) \\ (x_1, y_1) &= (x(b), y(b)) \end{aligned}$$

Examples of line integration

Example 1. (comparison with standard integrals)

$$I = \int_0^1 \underbrace{x^2}_{f(x)} dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}$$



$y = x^2$

$$I = \int_C f\,ds = \int_{t_0}^{t_1} f(t) \left|\frac{d\vec{r}}{dt}\right| dt$$

$$\begin{cases} 0 \le x \le 1 \\ y = 0 \end{cases} \quad \begin{cases} x = t \\ y = 0 \end{cases} \Rightarrow \begin{cases} \vec{r} = (x, y) = (t, 0) \\ 0 \le t \le 1, \; t_0 = 0, t_1 = 1 \end{cases}$$

$$\frac{d\vec{r}}{dt} = (1, 0), \quad \left|\frac{d\vec{r}}{dt}\right| = \sqrt{1^2 + 0^2} = 1$$

$$f(x) = x^2 \Rightarrow f(t) = t^2$$

$$I = \int_0^1 t^2 \cdot 1 \cdot dt = \int_0^1 t^2 dt = \frac{t^3}{3}\Big|_0^1 = \frac{1}{3}$$

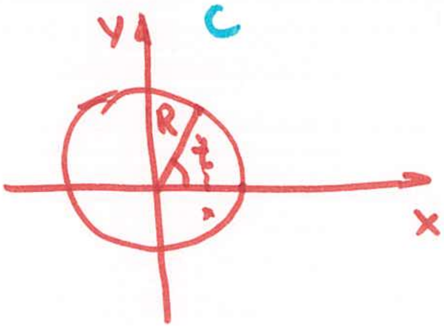
Generalisation: if the curve is given by

$$y = g(x) \Rightarrow \vec{r} = (t, g(t)), \quad \frac{d\vec{r}}{dt} = \left(1, \frac{dg}{dt}\right)$$

$$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{1 + \left(\frac{dg}{dt}\right)^2} \qquad \boxed{\begin{cases} x = t \\ y = g(t) \end{cases}}$$

Example 2 (geometrical description of the curve)

$$I = \int_C f\,ds = \int_{t_0}^{t_1} f(t) \left| \frac{d\vec{r}}{dt} \right| dt$$



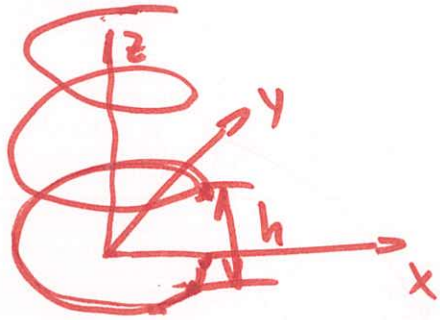
$$\boxed{\begin{cases} x = R\cos t & z = \frac{h}{2\pi} t \\ y = R\sin t & 0 \le t \le 2\pi \end{cases}}$$

$0 \le t \le 2\pi$

$z = at$

$t = 0 \quad z = 0$

$t = 2\pi \quad z = h$



$$h = a(2\pi) \Rightarrow a = \frac{h}{2\pi}$$

$$\vec{r} = (x, y, z) = \left(R\cos t,\ R\sin t,\ \frac{h}{2\pi} t\right)$$

$$\frac{d\vec{r}}{dt} = \left(-R\sin t,\ R\cos t,\ \frac{h}{2\pi}\right)$$

$$\left| \frac{d\vec{r}}{dt} \right| = \sqrt{R^2\sin^2 t + R^2\cos^2 t + \frac{h^2}{4\pi^2}} = \sqrt{R^2 + \frac{h^2}{4\pi^2}}$$

$$f(t) = f(x(t), y(t), z(t)) = (x^2 + y^2 + z^2)^2$$

$$= \left(R^2\cos^2 t + R^2\sin^2 t + \frac{h^2}{4\pi^2} t^2\right)^2$$

$$= \left(R^2 + \frac{h^2}{4\pi^2} t^2\right)^2$$

Choose for simplicity $R = 2$, $h = 6\pi$

$$f(t) = (4 + 9t^2)^2 = 16 + 72t^2 + 81t^4$$

$$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{4+9} = \sqrt{13}$$

$$I = \sqrt{13}\int_0^{2\pi} (16 + 72t^2 + 81t^4)\,dt = \sqrt{13}\left(16t + 24t^3 + \frac{81}{5}t^5\right)\Big|_0^{2\pi}$$

$$6 \cdot 4 \cdot 8$$

$$= \sqrt{13}\left(16 \cdot 2\pi + 24 \cdot 8\pi^3 + \frac{81}{5}32 \cdot \pi^5\right)$$

$$= 32\pi\sqrt{13}\left(1 + 6\pi^2 + \frac{81}{5}\pi^4\right).$$

Example 3 (problems with given curve parameterisation)

Find the length of the curve given by

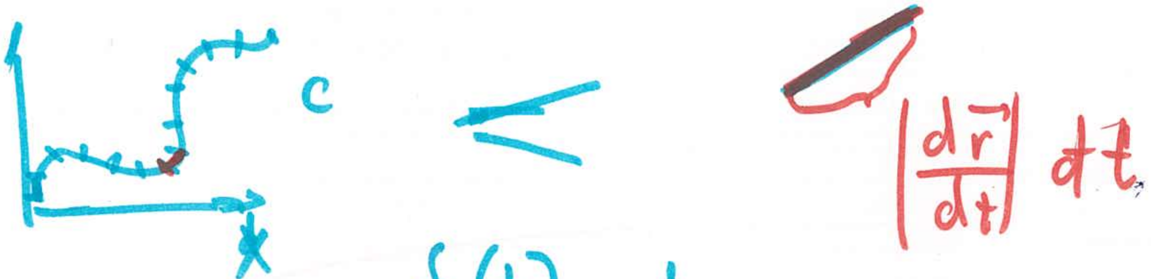
$\vec{r}(t) = (t, t^2, \frac{2}{3}t^3)$, between $A(0,0,0)$ & $B(1, 1, 2/3)$.

$\vec{r} = (x, y, z)$

$x = t$
$y = t^2$
$z = \frac{2}{3}t^3$

$A\begin{pmatrix}0\\0\\0\end{pmatrix} \Rightarrow t_0 = 0$; $B\begin{pmatrix}1\\1\\\frac{2}{3}\end{pmatrix}$ $t_1 = 1$

$$I = \int_{t_0}^{t_1} f(t) \left| \frac{d\vec{r}}{dt} \right| dt$$



$$f(t) = 1$$

$$\frac{d\vec{r}}{dt} = (1, 2t, 2t^2), \quad \left| \frac{d\vec{r}}{dt} \right| = \sqrt{1 + 4t^2 + 4t^4} =$$

$$= \sqrt{(1+2t^2)^2} = 1 + 2t^2$$

$$I = \int_0^1 1 \cdot (1+2t^2)\, dt = \left(t + \frac{2t^3}{3} \right) \Big|_0^1 = 1 + \frac{2}{3}$$

$$= 5/3$$