MTH20014 Mathematics 3B. Tutorial 2

1. Find the eigenvalues and corresponding eigenvectors of the following matrices

$$\text{(a)} \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}; \qquad \text{(b)} \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}; \qquad \text{(c)} \begin{bmatrix} 1 & 0 & -4 \\ 0 & 5 & 4 \\ -4 & 4 & 3 \end{bmatrix}; \quad \text{(d)} \begin{bmatrix} 2 & 0 & 1 \\ -1 & 4 & -1 \\ -1 & 2 & 0 \end{bmatrix};$$

$$\text{(e)} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}; \quad \text{(f)} \begin{bmatrix} 1 & 2 & -1 \\ 1 & 2 & -1 \\ 2 & 2 & -1 \end{bmatrix}; \quad \text{(g)} \begin{bmatrix} 3 & 0 & -1 \\ 0 & 1 & 0 \\ 2 & 0 & 0 \end{bmatrix}; \quad \text{(h)} \begin{bmatrix} -2 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 6 \end{bmatrix}.$$

   Hint: it is often easier to solve cubic characteristic equations resulting from 3×3 matrices by factorisation rather than by expansion.

2. Given that $\lambda = 1$ is the three-times repeated eigenvalue of the matrix $\begin{bmatrix} -3 & -7 & -5 \\ 2 & 4 & 3 \\ 1 & 2 & 2 \end{bmatrix}$, determine a corresponding set of linearly independent eigenvectors.

3. Compute squares of matrices (a) and (b) in Question 1 above and find the eigenvalues and eigenvectors of the resulting matrices. Determine how these eigenvalues and eigenvectors are related to the eigenvalues and eigenvectors of the original matrices and use this observation to determine the eigenvalues and eigenvectors of squares of matrices (c)–(h) in Question 1 without actually computing these squares.

Answers

1. (a) $\lambda_1 = 2$, $\mathbf{e}_1 = \beta_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$; $\lambda_2 = 3$, $\mathbf{e}_2 = \beta_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$;

   (b) $\lambda_1 = 1$, $\mathbf{e}_1 = \beta_1 \begin{bmatrix} 1 \\ -3 \end{bmatrix}$; $\lambda_2 = 5$, $\mathbf{e}_2 = \beta_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$;

   (c) $\lambda_1 = 9$, $\mathbf{e}_1 = \beta_1 \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$; $\lambda_2 = 3$, $\mathbf{e}_2 = \beta_2 \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$; $\lambda_3 = -3$, $\mathbf{e}_3 = \beta_3 \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$;

   (d) $\lambda_1 = 1$, $\mathbf{e}_1 = \beta_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$; $\lambda_2 = 2$, $\mathbf{e}_2 = \beta_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$; $\lambda_3 = 3$, $\mathbf{e}_3 = \beta_3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$;

   (e) $\lambda_1 = 4$, $\mathbf{e}_1 = \beta_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$; $\lambda_2 = 1$, $\mathbf{e}_2 = \beta_2 \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$; $\lambda_3 = -1$, $\mathbf{e}_3 = \beta_3 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$;

   (f) $\lambda_1 = 0$, $\mathbf{e}_1 = \beta_1 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$; $\lambda_{2,3} = 1$, $\mathbf{e}_2 = \beta_2 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$;

   (g) $\lambda_1 = 2$, $\mathbf{e}_1 = \beta_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$; $\lambda_2 = 1$, $\mathbf{e}_2 = \beta_2 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$; $\lambda_3 = 1$, $\mathbf{e}_3 = \beta_3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$;

   (h) $\lambda_1 = 1$, $\mathbf{e}_1 = \beta_1 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$; $\lambda_2 = -3$, $\mathbf{e}_2 = \beta_2 \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$; $\lambda_3 = 7$, $\mathbf{e}_3 = \beta_3 \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$.

2. $\mathbf{e}_1 = \beta_1 \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$.

3.

$$(a) \begin{bmatrix} 14 & -5 \\ 10 & -1 \end{bmatrix}; \quad (b) \begin{bmatrix} 19 & 6 \\ 18 & 7 \end{bmatrix}.$$