$$\textcircled{1}\ \int_C f\,ds = \int_a^b f(t)\left|\frac{d\vec{r}}{dt}\right| dt\ \textcircled{2}$$

$$\underbrace{(x, y, z)} \Rightarrow (x(t), y(t), z(t))$$

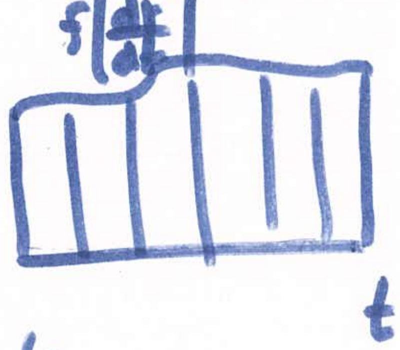
$$\vec{r} = (x, y, z) \Rightarrow \vec{r} = (x(t), y(t), z(t)) = \vec{r}(t)$$

$$\frac{d\vec{r}}{dt} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$$

$$f = f(x, y, z) \Rightarrow f(x(t), y(t), z(t)) = f(t)$$

1

2



Parameterisation options

a) $C: \ y = g(x) \Rightarrow \begin{cases} x = t \\ y = g(t) \end{cases} \quad x_0 \le t \le x_1$

$$\vec{r} = (t, g(t))$$

$$\frac{d\vec{r}}{dt} = \left(1, \frac{dg}{dt}\right)$$

$$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{1^2 + \left(\frac{dg}{dt}\right)^2}$$

b)

C

$$0 \le t \le 2\pi$$

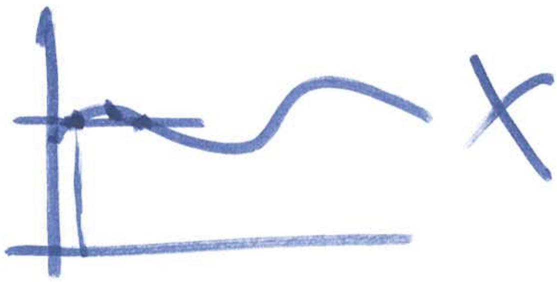
$$x = R\cos t$$

$$y = R\sin t$$

$$\vec{r} = (x, y) = (R\cos t, R\sin t)$$

$$\frac{d\vec{r}}{dt} = (-R\sin t, R\cos t)$$

$$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{R^2\sin^2 t + R^2\cos^2 t} = R$$

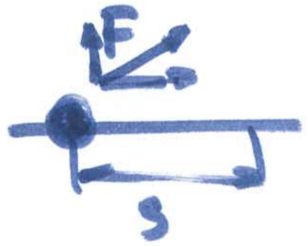


c) $\vec{r} = (x, y, z) = (g_1(t), g_2(t), g_3(t))$

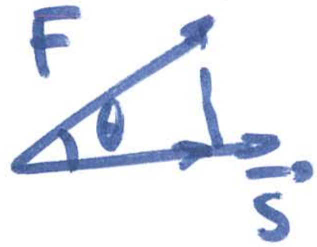
$$\frac{d\vec{r}}{dt} = \left(\frac{dg_1}{dt}, \frac{dg_2}{dt}, \frac{dg_3}{dt}\right)$$

$$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{\left(\frac{dg_1}{dt}\right)^2 + \left(\frac{dg_2}{dt}\right)^2 + \left(\frac{dg_3}{dt}\right)^2}$$

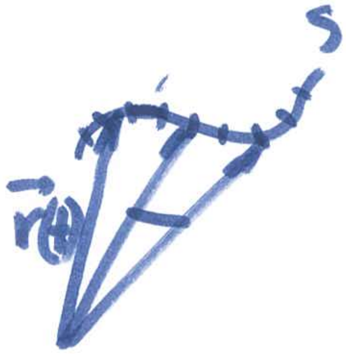
Work integral



$$Fs = W \text{ - work}$$



$$W = |\vec{F}| \cos\theta \cdot |\vec{s}|$$
$$= \vec{F} \cdot \vec{s}$$



$$d\vec{s} = \frac{d\vec{r}}{dt}\, dt$$

$$W = \int_C \vec{F} \cdot d\vec{s} = \boxed{\int_a^b \vec{F}(t) \cdot \frac{d\vec{r}}{dt}\, dt}$$

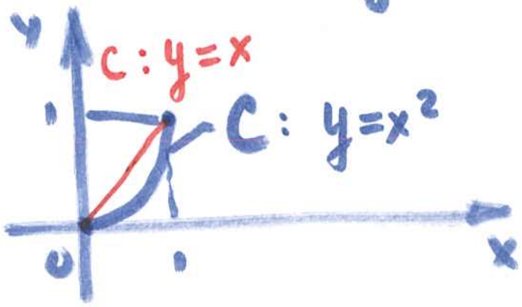
$$\vec{r} = (x(t), y(t), z(t)), \quad \frac{d\vec{r}}{dt} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$$

$$\vec{F} = \Big(F_1(x(t), y(t), z(t)), F_2(x(t), y(t), z(t)), F_3(x(t), y(t), z(t)) \Big)$$

$$\boxed{W = \int_a^b \left(F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt}\right) dt}$$

Example. $\vec{F} = (x^3, xy)$

a) $C: \quad y = x^2$ between $(0,0)$ and $(1,1)$



$$\begin{cases} x = t \\ y = t^2 \end{cases} \quad 0 \le t \le 1$$

$$\vec{r} = (x, y) = (t, t^2)$$

$$\frac{d\vec{r}}{dt} = (1, 2t)$$

$$\vec{F} = (t^3, t^3)$$

$$W = \int_0^1 (t^3, t^3) \cdot (1, 2t)\, dt$$

$$= \int_0^1 (t^3 + 2t^4)\, dt = \frac{t^4}{4} + \frac{2t^5}{5}\Big|_0^1$$

$$= \frac{1}{4} + \frac{2}{5} = \boxed{\frac{13}{20}}$$

b) $C: \quad y = x \qquad \begin{cases} x = t \\ y = t \end{cases} \quad 0 \le t \le 1$

$$\vec{r} = (x, y) = (t, t), \quad \frac{d\vec{r}}{dt} = (1, 1)$$

$$\vec{F} = (t^3, t^2)$$

$$W = \int_0^1 (t^3, t^2) \cdot (1, 1)\, dt = \int_0^1 (t^3 + t^2)\, dt$$

$$= \frac{t^4}{4} + \frac{t^3}{3}\Big|_0^1 = \frac{1}{4} + \frac{1}{3} = \boxed{\frac{7}{12}}$$

Exampl $\vec{v} = (6x^2 - 2y,\ x+z,\ 12yz)$

$C: \vec{r} = (x, y, z) = (t, t^2, t^3)$ between $(0,0,0)$ and $(1,1,1)$

$x = t$ $\quad x = 0 \Rightarrow t = 0$

$y = t^2$ $\quad y = 0^2 = 0$ ✓

$z = t^3$ $\quad z = 0^3 = 0$ ✓ $\quad \Longrightarrow 0 \le t \le 1$

$x = 1 \Rightarrow t = 1$

$y = 1^2 = 1$ ✓

$z = 1^3 = 1$ ✓

$$\frac{d\vec{r}}{dt} = (1, 2t, 3t^2)$$

$$\vec{v} = (6t^2 - 2t^2,\ t + t^3,\ 12t^2 \cdot t^3)$$

$$= (4t^2,\ t + t^3,\ 12t^5)$$

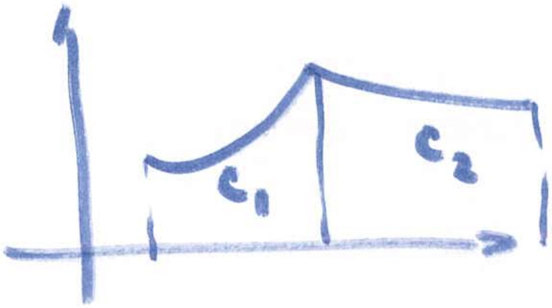
$$W = \int_0^1 (4t^2, t + t^3, 12t^5) \cdot (1, 2t, 3t^2)\, dt$$

$$= \int_0^1 (4t^2 + 2t^2 + 2t^4 + 12 \cdot 3t^7)\, dt$$

$$= \int_0^1 (6t^2 + 2t^4 + 36t^7)\, dt = 2t^3 + \frac{2t^5}{5} + \frac{36}{8}t^8 \Big|_0^1$$

$$= 2 + \frac{2}{5} + \frac{9}{2} = \frac{20}{10} + \frac{4}{10} + \frac{45}{10} = \boxed{\frac{69}{10}}$$

Combining paths



$C = C_1 + C_2$

$$\int_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r}$$

$-C$ - same path but travelled backwards

$C: \ \vec{r} = \vec{r}(t), \quad a \le t \le b$

$-C: \ \vec{r}^*(t) = \vec{r}(a+b-t)$

$$W_C = \int_a^b \vec{F}(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt} \, dt$$

$$W_{-C} = \int_a^b \vec{F}(\vec{r}(a+b-t)) \cdot \frac{d\vec{r}(a+b-t)}{dt} \, dt$$

$$\left\{ a+b-t = \tau \ : \ \frac{d\vec{r}(t)}{dt} = \frac{d\vec{r}}{d\tau} \cdot \frac{d\tau}{dt} = -\frac{d\vec{r}}{d\tau} \right\}$$

$$= -\int_a^b F(\vec{r}(\tau)) \cdot \frac{d\vec{r}}{d\tau} \, d\tau = \int_b^a F \vec{r}(\tau) \frac{d\vec{r}}{d\tau} \, d\tau$$

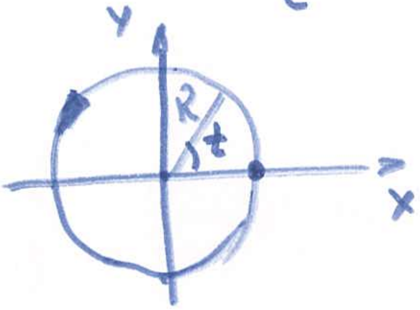
Circulation

$$W = \int_C \vec{F}\cdot d\vec{r} \Longrightarrow \Gamma = \oint_C \vec{F}\cdot d\vec{r}$$

work integral $\quad \int_a^b \vec{F}(t)\cdot \frac{d\vec{r}}{dt}\,dt \quad$ circulation

Example: $\quad \vec{V} = (-y, x)$

$\Gamma = \oint_C \vec{F}\cdot d\vec{r}$, where $C: x^2+y^2=R^2$



$$\begin{cases} x = R\cos t \\ y = R\sin t \end{cases} \quad 0 \le t \le 2\pi$$

$$\vec{r} = (x, y) = (R\cos t, R\sin t)$$

$$\frac{d\vec{r}}{dt} = (-R\sin t, R\cos t)$$

$$\vec{V} = (-R\sin t, R\cos t)$$

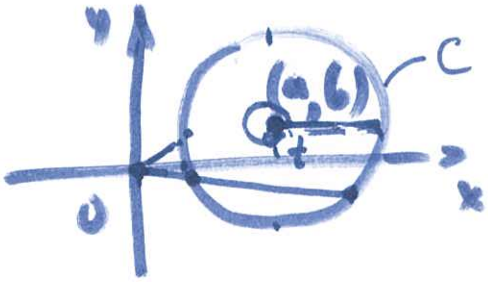
$$\Gamma = \oint_C \vec{V}\cdot d\vec{r} = \int_0^{2\pi} (-R\sin t, R\cos t)\cdot(-R\sin t, R\cos t)\,dt$$

$$= \int_0^{2\pi} (R^2\sin^2 t + R^2\cos^2 t)\,dt = R^2 t\Big|_0^{2\pi} = \boxed{2\pi R^2}$$

$S = \pi R^2$, what is 2?

$$\nabla\times\vec{V} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix} = \begin{matrix} \vec{i}\,[0] \\ -\vec{j}\,[0] \\ +\vec{k}\,[1+1] \end{matrix} = 2\vec{k} = (0,0,2)$$

$$|\nabla\times\vec{v}| = \sqrt{0^2+0^2+2^2} = 2$$



$C: \ (x-a)^2 + (y-b)^2 = R^2$ (35)

$$\begin{cases} x = a + R\cos t \\ y = b + R\sin t \end{cases} \quad 0 \le t \le 2\pi$$

$$\vec{r} = (x, y) = (a + R\cos t,\ b + R\sin t)$$

$$\frac{d\vec{r}}{dt} = (-R\sin t\ ,\ R\cos t)$$

$$\vec{V} = (-b - R\sin t,\ a + R\cos t)$$

$$\Gamma = \int_0^{2\pi} (-b - R\sin t, a + R\cos t)\cdot(-R\sin t, R\cos t)\,dt$$

$$= \int_0^{2\pi} (bR\sin t + \underbrace{R^2\sin^2 t + aR\cos t + R^2\cos^2 t}_{= R^2})\,dt$$

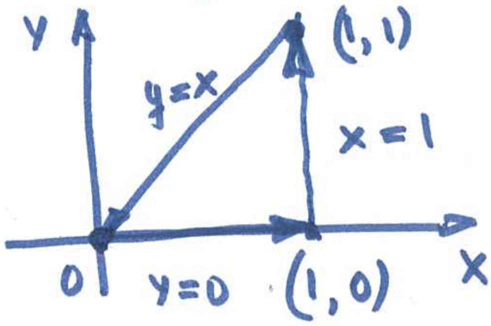
$$= \int_0^{2\pi} (bR(\sin t) + aR\cos t + R^2)\,dt = R^2\cdot 2\pi$$

$$\Gamma = 4\Delta^2 \left[\frac{v(x+\Delta, y) - v(x-\Delta, y)}{2\Delta} - \frac{u(x, y+\Delta) - u(x, y-\Delta)}{2\Delta} \right]$$

let $\Delta \to 0$ $\quad \Gamma \to \underbrace{4\Delta^2}_{\text{Area } S} \underbrace{\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)}_{\text{curl } \vec{v}}$

$$\to \quad |\nabla \times \vec{v}| \cdot S = \Gamma$$

Example 3.9



$\vec{F} = (x^2, -5xy)$

1) $\begin{cases} x = t \\ y = 0 \end{cases}, \quad 0 \le t \le 1$

$\vec{r} = (x, y) = (t, 0)$

$\frac{d\vec{r}}{dt} = (1, 0)$

$\vec{F} = (t^2, 0)$

$$\int_0^1 (t^2, 0) \cdot (1, 0)\, dt = \int_0^1 t^2\, dt = \frac{t^3}{3}\Big|_0^1 = \frac{1}{3}$$

2) - Exercise for you

3) $\begin{cases} x = t \\ y = t \end{cases}, \quad 1 \ge t \ge 0 \qquad F = (t^2, -5t^2)$

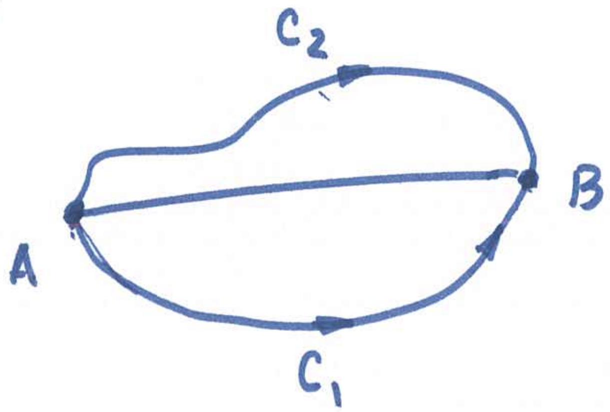
$\vec{r} = (x, y) = (t, t), \quad \frac{d\vec{r}}{dt} = (1, 1)$

$$\int_1^0 (t^2, -5t^2) \cdot (1, 1)\, dt = \int_1^0 (t^2 - 5t^2)\, dt$$

$$= -4 \int_1^0 t^2 dt = -4 \frac{t^3}{3}\Big|_1^0 = 0 - \left(-4 \cdot \frac{1}{3}\right)$$

$$= \frac{4}{3}$$

Path independence



$\vec{F}$

$$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_2} \vec{F}\cdot d\vec{r}$$

$$\int_{C_1} \vec{F}\cdot d\vec{r} - \int_{C_2} \vec{F}\cdot d\vec{r} = 0$$

$$\int_{AC_1B} \vec{F}\cdot d\vec{r} + \int_{BC_2A} \vec{F}\cdot d\vec{r} = 0 \implies \Gamma = 0$$

Path independence $\implies \Gamma = 0$

Example: $\vec{F} = (y+z,\ x+z,\ x+y)$

$P(1,2,3)$ & $Q(2,-1,0)$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y+z & x+z & x+y \end{vmatrix} = \begin{matrix} \vec{i}\,[1-1] \\ -\vec{j}\,[1-1] \\ +\vec{k}\,[1-1] \end{matrix} = \vec{0}$$

$$\frac{\partial \varphi}{\partial x} = y+z \implies \varphi = yx + zx + f(y,z)$$

$$\frac{\partial \varphi}{\partial y} = x+z \qquad \cancel{x} + \frac{\partial f}{\partial y} = \cancel{x} + z \implies f = yz + g(z)$$

$$\varphi = yx + zx + yz + g(z)$$

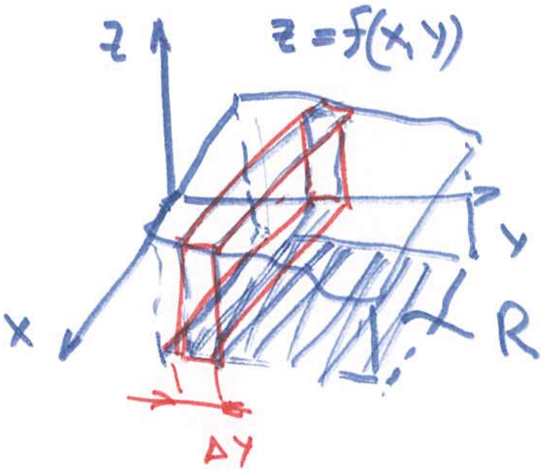
$$\frac{\partial \varphi}{\partial z} = x+y \qquad \cancel{x} + \cancel{y} + \frac{dg}{dz} = \cancel{x} + \cancel{y}, \quad g = C$$

$$\varphi = xy + xz + yz + C$$

$$\int_C \vec{F}\cdot d\vec{r} = \varphi(Q) - \varphi(P) = 2(-1) + 2\cdot 0 + (-1)\cdot 0 - [1\cdot 2 + 1\cdot 3 + 2\cdot 3]$$

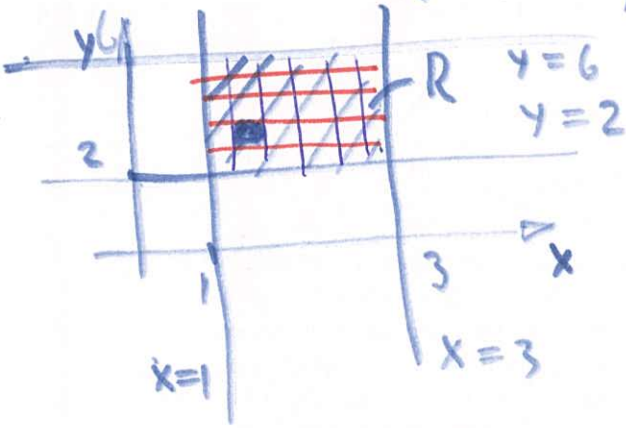
$$= -2 - 2 - 3 - 6 = -13$$

Repeated integrals



Example: $z = 2x + 4y$

$R: \ x = 1, \ x = 3, \ y = 2, \ y = 6$



$$V = \iint_R z(x,y)\, dR$$ - double integral

$$I_1 = \int_1^3 \int_2^6 z(x,y)\, dy\, dx$$ repeated integral

$$I_2 = \int_2^6 \int_1^3 z(x,y)\, dx\, dy$$

$$I_1 = \int_1^3 \int_2^6 (2x+4y)\, dy\, dx$$

$$= \int_1^3 (2xy + 2y^2)\Big|_{y=2}^{y=6} dx = \int_1^3 (12x + 72 - 4x - 8)\, dx$$

$$= \int_1^3 (8x + 64)\, dx = 4x^2 + 64x \Big|_{x=1}^{x=3} = 36 + 192 - 4 - 64 = 160$$

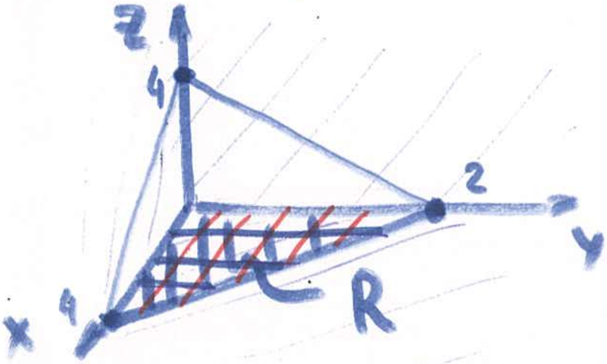
$$I_2 = \int_2^6 \int_1^3 (2x+4y)\,dx\,dy = \int_2^6 (x^2+4xy)\Big|_{x=1}^{x=3} dy$$

$$= \int_2^6 (9+12y-1-4y)\,dy = \int_2^6 (8+8y)\,dy$$

$$= 8y+4y^2 \Big|_{y=2}^{y=6} = 48+144-16-16$$
$$= 160$$

Exampl6. $z = 4-(x+2y)$

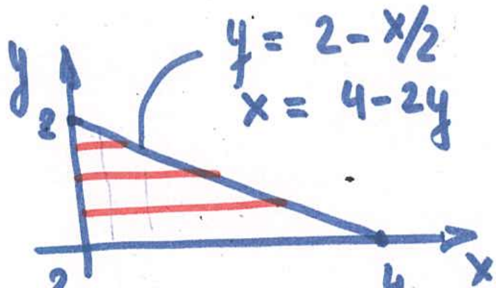
$z=0$, $x=0$, $y=0$



$x=0$: $z = 4-2y$
$y=0$: $z = 4-x$
$z=0$: $4-x-2y=0$
$y = 2-x/2$

Study guide : $V = \iint (\;)\,dy\,dx$

We will do : $V = \iint (\;)\,dx\,dy$



$$V = \int_0^2 \int_0^{4-2y} (4-x-2y)\,dx\,dy$$

$$= \int_0^2 \left(4x - \frac{x^2}{2} - 2xy\right)\Big|_{x=0}^{x=4-2y} dy = \int_0^2 \left(16-8y-2(2-y)^2-2(4-2y)y\right)dy$$

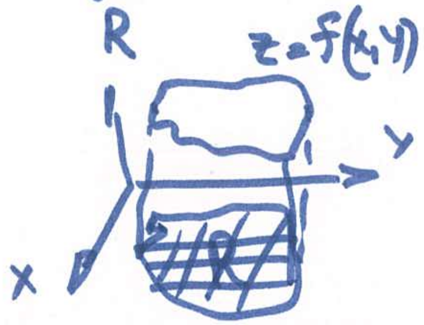
$$= \int_0^2 \left(16 - 8y - 8 + 8y - 2y^2 - 8y + 4y^2\right) dy$$

$$= \int_0^2 \left(8 - 8y + 2y^2\right) dy = \left(8y - 4y^2 + \frac{2y^3}{3}\right)\Big|_{y=0}^{y=2}$$

$$= 16 - 16 + \frac{2 \cdot 8}{3} = \frac{16}{3}$$

Double Integrals (continued)

$$\iint_R f \, dR = \int_a^b \int_{g(y)}^{h(y)} f \, dx\,dy$$



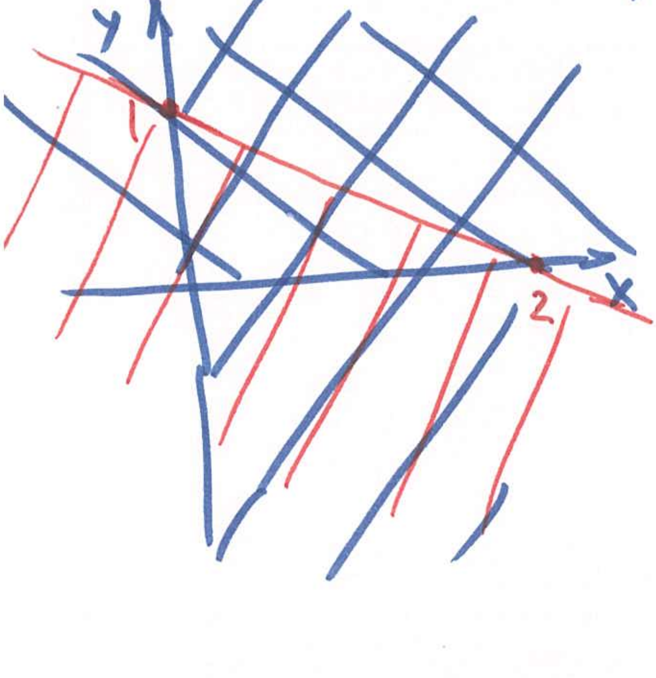
$dR = dx \times dy$

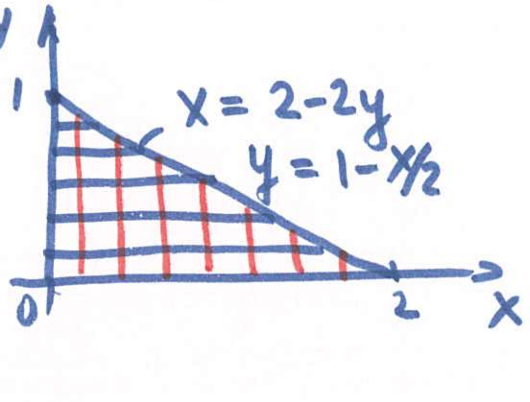
<u>Example</u>: $I = \iint_R (x^2+y^2)\, dR = \int_0^1 \int_0^{2-2y} (x^2+y^2)\, dx\,dy$

$x \geq 0$, $y \geq 0$, $x + 2y \leq 2$

$2y \leq 2 - x$, $y \leq 1 - x/2$

$y = 1 - x/2$, $x = 2 - 2y$





$$I = \int_0^1 \left(\frac{x^3}{3} + xy^2\right)\Bigg|_{x=0}^{x=2-2y} dy = \int_0^1 \left(\frac{8(1-y)^3}{3} + 2(1-y)y^2\right)dy$$

$$= \int_0^1 \frac{8}{3}(1-y)^3 dy + 2\int_0^1 (y^2 - y^3)\, dy$$

$$= \left\{ \begin{array}{l} 1-y = u \\ -1 = \frac{du}{dy} \\ dy = -du \\ y=0 \Rightarrow u=1 \\ y=1 \Rightarrow u=0 \end{array} \right\} = -\int_1^0 \frac{8}{3} u^3 du + 2\left[\frac{y^3}{3} - \frac{y^4}{4}\right]\Bigg|_{y=0}^{y=1}$$

$$= -\frac{2u^4}{3}\Bigg|_{u=1}^{u=0} + 2\left(\frac{1}{3} - \frac{1}{4}\right)$$

$$= \frac{2}{3} + \frac{2}{3} - \frac{1}{2} = \frac{4}{3} - \frac{1}{2} = \frac{8}{6} - \frac{3}{6} = \boxed{\frac{5}{6}}$$

$$I = \int_0^2 \int_0^{1-x/2} (x^2+y^2)\, dy\, dx = \int_0^2 \left(yx^2 + \frac{y^3}{3}\right)\Bigg|_{y=0}^{y=1-x/2} dx$$

$$= \int_0^2 \left(\left(1-\frac{x}{2}\right)x^2 + \frac{(1-x/2)^3}{3}\right) dx = \int_0^2 \left(x^2 - \frac{x^3}{2}\right) dx$$

$$+ \frac{1}{3}\int_0^2 (1-x/2)^3 dx = \left\{ \begin{array}{ll} u = 1-x/2 & x=0 \Rightarrow u=1 \\ \frac{du}{dx} = -\frac{1}{2} & x=2 \Rightarrow u=0 \\ dx = -2du & \end{array} \right\}$$

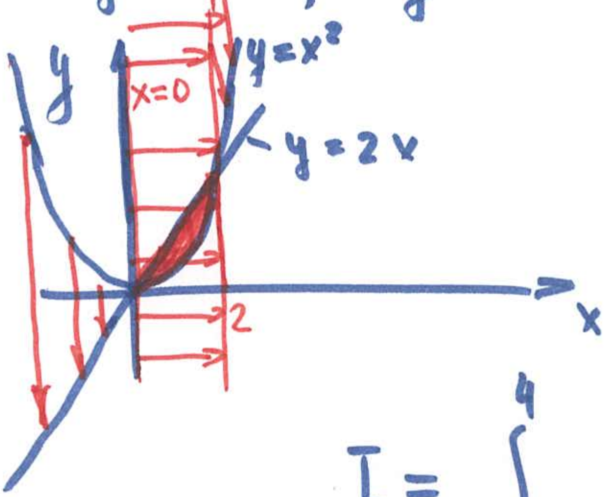
$$= \left(\frac{x^3}{3} - \frac{x^4}{8}\right)\Bigg|_{x=0}^{x=2} + \frac{1}{3}\int_1^0 u^3(-2)\, du$$

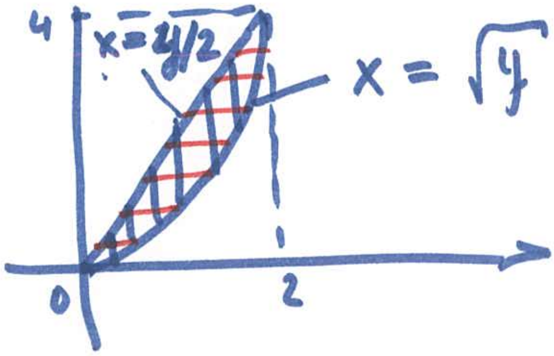
$$= \frac{8}{3} - \frac{16}{8} - \frac{2}{3} \cdot \frac{u^4}{4}\Bigg|_1^0 = \frac{8}{3} - \frac{16}{8} + \frac{2}{3\cdot 4} = \frac{8}{3} - 2 + \frac{1}{6}$$

$$= \frac{16}{6} + \frac{1}{6} - \frac{12}{6} = \boxed{\frac{5}{6}}$$

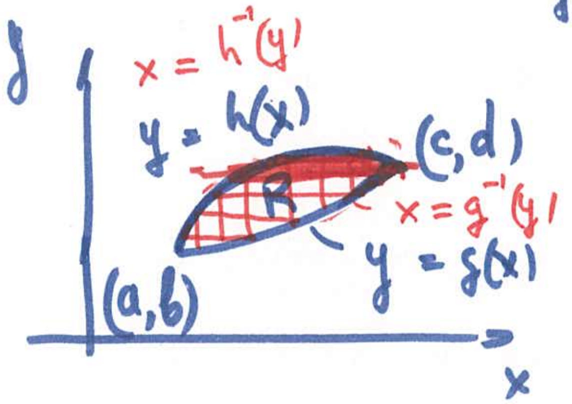
Example. $I = \int_0^2 \int_{x^2}^{2x} f \, dy \, dx$

$y = x^2, \quad y = 2x, \quad x = 0, \quad x = 2$





$$I = \int_0^4 \int_{y/2}^{\sqrt{y}} f \, dx \, dy$$

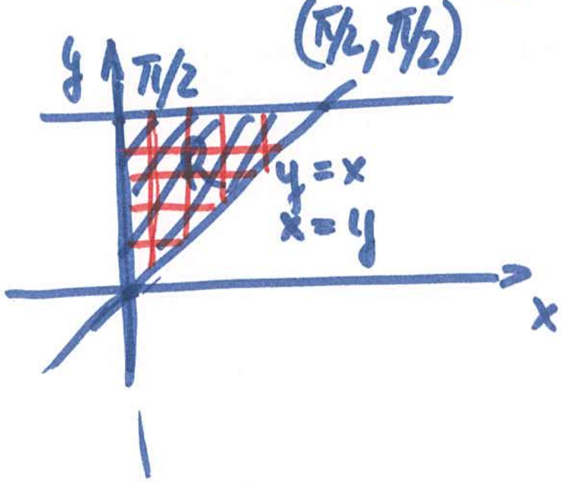


Why do we need to change the integration order?

$$\iint_R f \, dR = \int_a^c \int_{g(x)}^{h(x)} f \, dy \, dx$$

$$\int \int_{h^{-1}(y)}^{?} f \, dx \, dy$$

Example. $I = \iint_R \frac{\sin y}{y}\, dR \qquad R: \begin{cases} x = 0 \\ y = \pi/2 \\ y = x \end{cases}$



$$I = \int_0^{\pi/2} \int_x^{\pi/2} \frac{\sin y}{y}\, dy\, dx$$

$$= \int_0^{\pi/2} \int_0^{y} \frac{\sin y}{y}\, dx\, dy$$

$$= \int_0^{\pi/2} \frac{\sin y}{y}\, x \Big|_{x=0}^{x=y} dy = \int_0^{\pi/2} \sin y\, dy$$

$$= -\cos y \Big|_0^{\pi/2} = -\left[0 - 1\right] = 1$$