Symmetric matrices

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} \qquad \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} \qquad A^T = A$$

symmetric

Properties

1. E.values of real symmetric matrices are real.
2. E.vectors of s.m. corresponding to distinct e.values are mutually orthogonal. Modal matrix P composed of <u>unit</u> e.vectors is orthogonal.
3. Any $n \times n$ s.m has n mutually orth. e.vectors (even if some e.values are repeated).

Proof: $\vec{u} = (a, b, c)$ where a, b, c are complex numbers

$\vec{v} = (d, e, f)$ is a complex vector

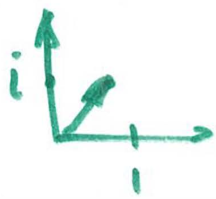
$$(\vec{u} \cdot \vec{v}) = a^* d + b^* e + c^* f$$

$i = \sqrt{-1}$

$$\vec{u} = (1, i) = \vec{v}$$

$$(\vec{u} \cdot \vec{v}) = (\vec{u} \cdot \vec{u}) = 1 \cdot 1 + i \cdot i = 1 - 1 = 0$$

$$(\vec{u} \cdot \vec{u}) = (1, -i) \cdot (1, i) = 1 + 1 = 2$$



$$(\vec{u}\cdot\vec{v}) = [a^* \ b^* \ c^*]\begin{bmatrix} d \\ e \\ f \end{bmatrix} = \vec{u}^{*T}\vec{v}$$

$$(\vec{x}\cdot\vec{x}) = \vec{x}^{*T}\vec{x} = \|\vec{x}\|^2 \geq 0$$

$$(\|x\|^2)^* = \|x\|^2$$

$$(\|x\|^2)^* = (\vec{x}\cdot\vec{x})^* = (\vec{x}^{*T}\vec{x})^* = \vec{x}^T\vec{x}^* =$$

$$= \|x\|^2 = (\vec{x}\cdot\vec{x}) = \vec{x}^{*T}\vec{x}$$

Assume that e. val λ, e. vec $\vec{x}$, is complex : $\cancel{\lambda^* \neq \lambda}$

$$\left.\begin{aligned} A\vec{x} &= \lambda\vec{x} \\ A^*\vec{x}^* &= \lambda^*\vec{x}^* \end{aligned}\right\}$$

A is symmetric, A is real

$A^* = A$, $A^T = A$

$$A\vec{x}^* = \lambda^*\vec{x}^*$$

$$\vec{x}^T A\vec{x}^* = \vec{x}^T(A\vec{x}^*) = \vec{x}^T(\lambda^*\vec{x}^*) = \lambda^*\vec{x}^T\vec{x}^*$$

$$= \lambda^*(\vec{x}\cdot\vec{x})$$

$$\vec{x}^T A\vec{x}^* = (\vec{x}^T A)\vec{x}^* = (A^T\vec{x})^T\vec{x}^* = (A\vec{x})^T\vec{x}^*$$

$$\boxed{[AB]^T = B^T A^T}$$

$$= (\lambda\vec{x})^T\vec{x}^* = \lambda\vec{x}^T\vec{x}^* = \lambda(\vec{x}\cdot\vec{x})$$

$$\lambda^*(\vec{x}\cdot\vec{x}) = \lambda(\vec{x}\cdot\vec{x})$$

$(\vec{x}\cdot\vec{x}) \neq 0$ - e.v.

$\lambda^* = \lambda$, λ is real.

Let $\vec{u}$ & $\vec{v}$ be e. vectors of a real sym. m
corresponding to λ_1 & λ_2 (e. values) $\lambda_1 \neq \lambda_2$

$$\vec{u}^{*T}(A\vec{v}) = u^{*T}\lambda_2\vec{v} = \lambda_2(\vec{u}\cdot\vec{v})$$

$$(\vec{u}^{*T}A)\vec{v} = \underset{A.s.m}{(A^T\vec{u}^*)^T\vec{v}} = (A\vec{u}^*)^T\vec{v} = \underset{A\text{ real}}{(A^*\vec{u}^*)^T\vec{v}}$$

$$= (A\vec{u})^{*T}\vec{v} = (\lambda_1\vec{u})^{*T}\vec{v} = \lambda_1\vec{u}^{*T}\vec{v} = \lambda_1(\vec{u}\cdot\vec{v})$$

$$\lambda_2(\vec{u}\cdot\vec{v}) = \lambda_1(\vec{u}\cdot\vec{v})$$

$$\underbrace{(\lambda_2-\lambda_1)}_{\neq 0}(u\cdot\vec{v}) = 0$$

$\vec{u}\cdot\vec{v} = 0$ $\vec{u}$ & $\vec{v}$ are $\perp$

$$\begin{bmatrix} \vec{u} & \vec{v} \end{bmatrix}$$ - orthogonal

If $\hat{\vec{u}}$, $\hat{\vec{v}}$ are unit vectors

$$P = \begin{bmatrix} \hat{\vec{u}} & \hat{\vec{v}} \end{bmatrix}$$ is orthogonal

$$P^{-1} = P^T$$

Qudratic Curves

$$a x_1^2 + b x_2^2 + 2c x_1 x_2 = 1$$

$$[x_1 \quad x_2] \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} =$$

$$= [x_1 \quad x_2] \begin{bmatrix} A x_1 + B x_2 \\ C x_1 + D x_2 \end{bmatrix} =$$

$$= A x_1^2 + B x_1 x_2 + C x_1 x_2 + D x_2^2$$

$$= A x_1^2 + D x_2^2 + (B+C) x_1 x_2$$

$a = A$, $b = D$, $2c = B + C$, $B = C = c$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} a & c \\ c & b \end{bmatrix}$$

Example: $2x_1^2 + 2x_2^2 - 6x_1 x_2 = 1$; $\begin{bmatrix} 2 & -3 \\ -3 & 2 \end{bmatrix}$

$$[x_1 \quad x_2] \begin{bmatrix} 2 & -3 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 1$$

1) $x_1^2 + x_2^2 = r^2$ - circle $\frac{x_1^2}{r^2} + \frac{x_2^2}{r^2} = 1$

2) $\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1$ - ellipse $a \neq b$

3) $\frac{x_1^2}{a^2} - \frac{x_2^2}{b^2} = 1$ - hyperbola

1) $\begin{bmatrix} \frac{1}{r^2} & 0 \\ 0 & \frac{1}{r^2} \end{bmatrix}$ 2) $\begin{bmatrix} \frac{1}{a^2} & 0 \\ 0 & \frac{1}{b^2} \end{bmatrix}$ 3) $\begin{bmatrix} \frac{1}{a^2} & 0 \\ 0 & -\frac{1}{b^2} \end{bmatrix}$

General form

$$a x_1^2 + b x_2^2 + 2c x_1 x_2 = 1$$

$$[x_1, x_2] \begin{bmatrix} a & c \\ c & b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 1$$

$$\vec{x}^T A \vec{x} = 1$$

Canonical form

$$\lambda_1 y_1^2 + \lambda_2 y_2^2 = 1$$

$$[y_1 \ y_2] \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 1$$

$$\vec{y}^T \Lambda \vec{y} = 1$$

$M^{-1} A M = \Lambda$ - matrix diagonalisation

$$\vec{y}^T M^{-1} A (M \vec{y}) = \vec{x}^T A \vec{x}$$

$$M\vec{y} = \vec{x} \Rightarrow \vec{x}^T = (M\vec{y})^T = \vec{y}^T M^T \stackrel{?}{=} \vec{y}^T M^{-1}$$

$M^T = M^{-1}$ - orthogonal matrix

A - is real-valued symmetric, its eigenvectors are orthogonal. M - modal matrix of A

$$M = \begin{bmatrix} \vec{e}_1 & \vec{e}_2 \end{bmatrix} \qquad \vec{e}_1 \perp \vec{e}_2$$

$$M^{-1}M = M^T M = I = \begin{bmatrix} \vec{e}_1 \\ \vec{e}_2 \end{bmatrix} \begin{bmatrix} \vec{e}_1 & \vec{e}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \vec{e}_1 \cdot \vec{e}_1 & \vec{e}_1 \cdot \vec{e}_2 \\ \vec{e}_2 \cdot \vec{e}_1 & \vec{e}_2 \cdot \vec{e}_2 \end{bmatrix} \Rightarrow \begin{array}{l} \vec{e}_1 \cdot \vec{e}_2 = \vec{e}_2 \cdot \vec{e}_1 = 0 \\ \vec{e}_1 \cdot \vec{e}_1 = |\vec{e}_1|^2 = 1 \\ \vec{e}_2 \cdot \vec{e}_2 = |\vec{e}_2|^2 = 1 \end{array}$$

$$(\vec{e}_1, \vec{e}_2) \Rightarrow (\hat{\vec{e}}_1, \hat{\vec{e}}_2)$$

$$M \Rightarrow P$$

$$\begin{bmatrix} \vec{e}_1 & \vec{e}_2 \end{bmatrix} \Rightarrow \begin{bmatrix} \hat{\vec{e}}_1 & \hat{\vec{e}}_2 \end{bmatrix}$$

$$\boxed{P\vec{y} = \vec{x} \iff \vec{y} = P^T \vec{x}}$$

Example: $\quad 5x_1^2 + 6x_1x_2 + 5x_2^2 = 8$

$$[x_1 \;\; x_2]\begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 8$$

$$\begin{vmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{vmatrix} = (5-\lambda)^2 - 9 = (5-\lambda-3)(5-\lambda+3)$$

$$= (2-\lambda)(8-\lambda) = 0$$

$$\lambda_1 = 2, \quad \lambda_2 = 8 \implies 2y_1^2 + 8y_2^2 = 8$$

$$\frac{1}{4}y_1^2 + y_2^2 = 1$$

$$\lambda_1 = 2 \qquad \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \implies [1 \;\; 1] \implies \vec{e}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} t$$

$$\hat{\vec{e}}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\lambda_2 = 8 \qquad \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} \implies [1 \;\; -1] \implies \vec{e}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} t$$

$$\hat{\vec{e}}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = P^T\vec{x} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} x_1 - x_2 \\ x_1 + x_2 \end{bmatrix}$$

$y_1 = \frac{1}{\sqrt{2}}(x_1 - x_2)$, $y_2 = \frac{1}{\sqrt{2}}(x_1 + x_2)$

$$\vec{x} = P\vec{y} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} y_1 + y_2 \\ -y_1 + y_2 \end{bmatrix}$$

$x_1 = \frac{1}{\sqrt{2}}(y_1 + y_2)$, $x_2 = \frac{1}{\sqrt{2}}(-y_1 + y_2)$

$$\frac{5}{(\sqrt{2})^2}(y_1 + y_2)^2 + \frac{6}{2}(y_1 + y_2)(-y_1 + y_2) +$$

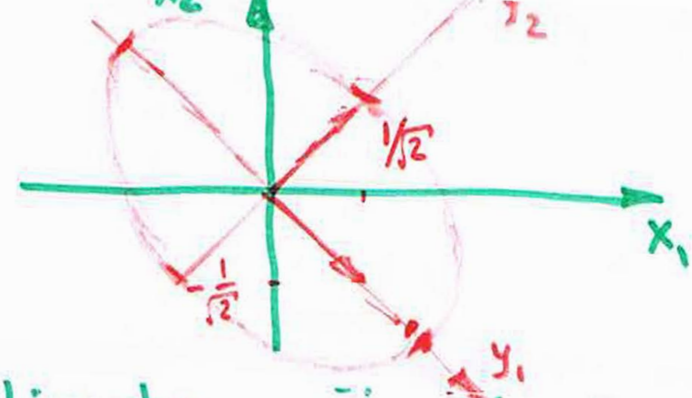
$$+ \frac{5}{2}(-y_1 + y_2)^2 = \frac{1}{2}\left[5(y_1^2 + 2y_1y_2 + y_2^2) + \right.$$

$$\left. + 6(y_2^2 - y_1^2) + 5(y_1^2 - 2y_1y_2 + y_2^2)\right]$$

$$= \frac{1}{2}\left[\underline{5y_1^2} + 10y_1y_2 + 5y_2^2 + \underline{6y_2^2} - 6y_1^2 + 5y_1^2 - 10y_1y_2 + 5y_2^2\right]$$

$$= \frac{1}{2}\left[16y_2^2 + 4y_1^2\right] = 8y_2^2 + 2y_1^2 = 8$$

$$\boxed{y_2^2 + \frac{1}{4}y_1^2 = 1}$$ - ellipse



Y-coordinates $\quad \bar{e}_{1y} = (1, 0)$ $\quad e_{2y} = (0, 1)$

$(y_1, y_2) \qquad (y_1, y_2)$

$\bar{e}_{1y} \Longrightarrow \quad x_1 = \frac{1}{\sqrt{2}}, \; x_2 = -\frac{1}{\sqrt{2}}$

$\bar{e}_{2y} \Longrightarrow \quad x_1 = \frac{1}{\sqrt{2}}, \; x_2 = \frac{1}{\sqrt{2}}$

Note: $$P = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \cos \pi/4 & \sin \pi/4 \\ -\sin \pi/4 & \cos \pi/4 \end{bmatrix}$$

$$= R[\Theta], \quad \Theta = \pi/4 \text{ - rotation by } \pi/4$$

Mystery of Example 1.2 solved!

Quadratic surfaces

(1) $a x_1^2 + b x_2^2 + c x_3^2 + 2d x_1 x_2 + 2e x_2 x_3 + 2f x_1 x_3 =$

$$[x_1 \quad x_2 \quad x_3] \underbrace{\begin{bmatrix} a & d & f \\ d & b & e \\ f & e & c \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 1$$

$$A \to (\lambda_1, \lambda_2, \lambda_3) \to (\vec{e}_1, \vec{e}_2, \vec{e}_3) \to (\hat{\vec{e}}_1, \hat{\vec{e}}_2, \hat{\vec{e}}_3)$$

$$\to P = \left[\begin{pmatrix} \hat{\vec{e}}_1 \end{pmatrix} \begin{pmatrix} \hat{\vec{e}}_2 \end{pmatrix} \begin{pmatrix} \hat{\vec{e}}_3 \end{pmatrix} \right] \to \vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = P^T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = P \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

(1) $\to \quad \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2 = 1$

a) $\lambda_1 > 0, \lambda_2 > 0, \lambda_3 > 0$ – ellipsoid
$\lambda_1 = \lambda_2 = \lambda_3 > 0$ – sphere

b) $\lambda_1 > 0, \lambda_2 > 0, \lambda_3 < 0$ one sheet hyperboloid

c) $\lambda_1 > 0, \lambda_2 < 0, \lambda_3 < 0$ two sheet hyperboloid

d) $\lambda_1 = 0, \lambda_2 > 0, \lambda_3 > 0$ – elliptic cylinder
$\lambda_1 = 0, \lambda_2 = \lambda_3 > 0$ – circular cylinder

e) $\lambda_1 = 0, \lambda_2 > 0, \lambda_3 < 0$ – hyperbolic cylinder

Example.

$$\underline{x_1^2} + \underline{6x_1}x_2 - \underline{2x_2^2} - 2x_2x_3 + \underline{x_3^2} = \underline{1}$$

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \\ 3 & -2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 1$$

$$\begin{vmatrix} 1-\lambda & 3 & 0 \\ 3 & -2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{vmatrix} = \begin{aligned} &(1-\lambda) \times [(2+\lambda)(1-\lambda)-1] \\ &- 3 \times [3(1-\lambda)] \\ &+ 0 \times [\quad] \end{aligned}$$

$$= (1-\lambda)[(2+\lambda)(\lambda-1)-1] - 9(1-\lambda)$$

$$= (1-\lambda)[(2+\lambda)(\lambda-1)-10] = (1-\lambda)[\lambda^2+\lambda-12] = 0$$

$$\lambda_1 = 1, \qquad \lambda_{2,3}^2 + \lambda_{2,3} - 12 = 0$$

$$\lambda_{2,3} = \frac{-1 \pm \sqrt{1+48}}{2} = 3, -4$$

$$\lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2 = 1$$

$$y_1^2 + 3y_2^2 - 4y_3^2 = 1$$ – hyperboloid of one sheet

$\lambda_1 = 1$:

$$\begin{bmatrix} 0 & 3 & 0 \\ 3 & -3 & -1 \\ 0 & -1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & -3 & -1 \\ 0 & 1 & 0 \\ & & \end{bmatrix} \Rightarrow \vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

$$\begin{cases} 3x - 3y - z = 0 \\ \quad\quad\; y = 0 \end{cases} \qquad 3x = z$$

$$\hat{\vec{e}}_1 = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

$$\lambda_2 = 3 \quad \begin{bmatrix} -2 & 3 & 0 \\ 3 & -5 & -1 \\ 0 & -1 & -2 \end{bmatrix} = \begin{bmatrix} 2 & -3 & 0 \\ 0 & 1 & 2 \\ & & \end{bmatrix} \Rightarrow \vec{e}_2 = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} t_2$$

$$\lambda_3 = -4 \quad \begin{bmatrix} 5 & 3 & 0 \\ 3 & 2 & -1 \\ 0 & -1 & 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 5 & 3 & 0 \\ 0 & 1 & -5 \\ & & \end{bmatrix} \qquad \hat{\vec{e}}_2 = \frac{1}{\sqrt{14}} \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$$

$$\vec{e}_3 = \begin{bmatrix} -3 \\ 5 \\ 1 \end{bmatrix} t_3$$

$$\hat{\vec{e}}_3 = \begin{bmatrix} -3 \\ 5 \\ 1 \end{bmatrix} \frac{1}{\sqrt{35}}$$

$$P = \begin{bmatrix} \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{14}} & \frac{-3}{\sqrt{35}} \\ 0 & \frac{2}{\sqrt{14}} & \frac{5}{\sqrt{35}} \\ \frac{3}{\sqrt{10}} & \frac{-1}{\sqrt{14}} & \frac{1}{\sqrt{35}} \end{bmatrix}$$

$$\vec{x} = P\vec{y}, \quad \vec{y} = P^T \vec{x}$$

(x_1, x_2, x_3)