MTH20014 Mathematics 3B. Tutorial 6

1. Find the vector $\boldsymbol{\nabla}\phi$ for each of the following functions. In each case sketch some of the curves $\phi = const.$ and show $\boldsymbol{\nabla}\phi$. Endeavour to get the direction and the relative magnitude of the gradient vectors right.

   (a) $\phi = 2x - y$, (b) $\phi = x^2 - y$, (c) $\phi = 2x - 5y$,
   (d) $\phi = x^2 + y^2$, (e) $\phi = 4x^2 - y$, (f) $\phi = xy$,
   (g) $\phi = 9x^2 + y^2$, (h) $\phi = \frac{x}{y}$, (i) $\phi = \frac{1}{xy}$.

2. Find the derivative of f in the direction of $\mathbf{u}$ and the maximum rate of change of f at point P and specify its direction.

   (a) $f(x, y) = 2x^2 - xy$, $P(3, 1)$, $\mathbf{u} = (1, 1)$;
   (b) $f(x, y) = x^2 - y$, $P(1, -2)$, $\mathbf{u} = (2, 1)$;
   (c) $f(x, y) = 4x + y$, $P(3, 2)$, $\mathbf{u} = (1, 2)$;
   (d) $f(x, y) = x^2 + y^2$, $P(1, 2)$, $\mathbf{u} = (3, 4)$;
   (e) $f(x, y) = e^y \sin x$, $P\left(\frac{\pi}{4}, 2\right)$, $\mathbf{u} = (2, 2)$;
   (f) $f(x, y) = y \ln x$, $P(2, 4)$, $\mathbf{u} = (1, -2)$.

3. If $\phi(x, y, z) = 3x^2y - y^3z^2$, find $\boldsymbol{\nabla}\phi$ at the point $(x, y, z) = (1, -2, 1)$.

4. Find a unit normal vector to the surface $x^2y + 2xz = 4$ at the point $(x, y, z) = (2, -2, 3)$.

5. Find the maximum rate of change of ϕ at the point $(x, y, z) = (2, 1, 1)$ for each of the following functions (note: $r^2 = x^2 + y^2 + z^2$).

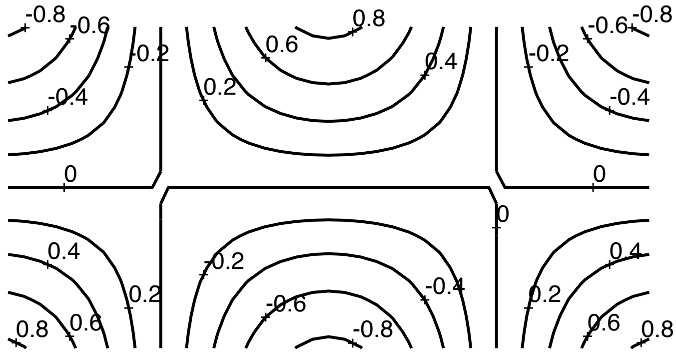
   (a) $\phi = 2x + y - z$, (b) $\phi = x^2 - zy$, (c) $\phi = x^2y - 5yz^2$,
   (d) $\phi = x^2 + y^2 + z^2$, (e) $\phi = \frac{1}{r}$, (f) $\phi = r^3$,
   (g) $\phi = xyz$, (h) $\phi = \ln(x^2 + y^2 + z^2)$, (i) $\phi = e^{-r}$.

6. Calculate $\boldsymbol{\nabla}\cdot\mathbf{F}$ for the following vector fields:

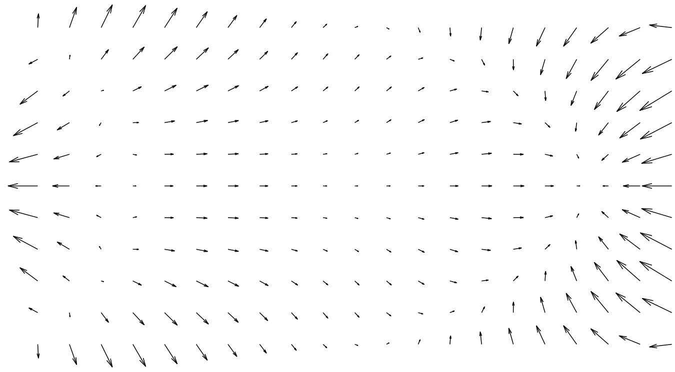
   (a) $\mathbf{F} = (x, 2y, 3z)$, (b) $\mathbf{F} = (3x^2 - y^2, x^2 - 2xy, 3z)$,
   (c) $\mathbf{F} = (3x^2, y^2, z^2)$, (d) $\mathbf{F} = (xy, yz, xz)$,
   (e) $\mathbf{F} = (yz, -zx, xy$, (f) $\mathbf{F} = (e^{-x}, xe^{-y}, xy)$,
   (g) $\mathbf{F} = (2xy + 3x^2z, x^2 - 3z^2, x^3 - 6yz)$,
   (h) $\mathbf{F} = (3x^2y + z^2, x^3 + 6y^2z, 2xz + 2y^3)$.

7. Given that the vector field $\mathbf{F} = (x + 3y, y - 2z, x + az)$ has zero divergence, find a.

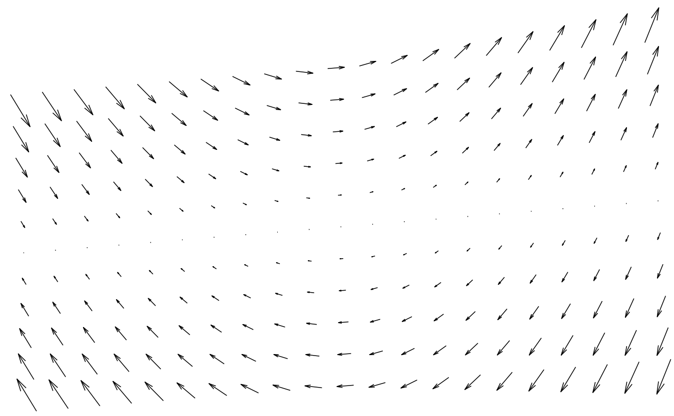
8. Sketch the approximate gradient field for a scalar function contour plot of which is shown below. Endeavour to get the directions and relative magnitudes of vectors correctly.



9. Sketch the approximate level curves (contours) of the scalar function if its gradient field is shown below. Endeavour to get the shape and the density of contour lines correctly. Mark the approximate locations of maxima, minima and saddle points if they exist.



10. Sketch approximate regions, where the divergence of the vector field shown below is positive, negative or zero.



Answers

1.

(a) $\boldsymbol{\nabla}\phi = (2,-1)$, (b) $\boldsymbol{\nabla}\phi = (2x,-1)$, (c) $\boldsymbol{\nabla}\phi = (2,-5)$,
(d) $\boldsymbol{\nabla}\phi = (2x,2y)$, (e) $\boldsymbol{\nabla}\phi = (8x,-1)$, (f) $\boldsymbol{\nabla}\phi = (y,x)$,
(g) $\boldsymbol{\nabla}\phi = (18x,2y)$, (h) $\boldsymbol{\nabla}\phi = \left(\frac{1}{y},-\frac{x}{y^2}\right)$, (i) $\boldsymbol{\nabla}\phi = \left(-\frac{1}{x^2y},-\frac{1}{xy^2}\right)$.

2. (a) $4\sqrt{2}$, $\sqrt{130}$ in the direction of $(11,-3)$;
   (b) $\frac{3\sqrt{5}}{5}$, $\sqrt{5}$ in the direction of $(2,-1)$;
   (c) $\frac{6\sqrt{5}}{5}$, $\sqrt{17}$ in the direction of $(4,1)$;
   (d) $\frac{22}{5}$, $2\sqrt{5}$ in the direction of $(1,2)$;
   (e) e^2, e^2 in the direction of $(1,1)$;
   (f) $\frac{2\sqrt{5}}{5}(1-\ln 2)$, $\sqrt{4+\ln^2 2}$ in the direction of $(2,\ln 2)$.

3. $\boldsymbol{\nabla}\phi = (-12,-9,16)$.

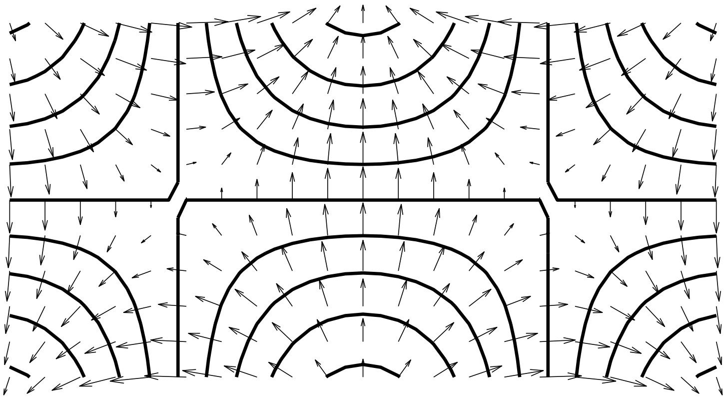
4. $\mathbf{n} = \left(-\frac{1}{3},\frac{2}{3},\frac{2}{3}\right)$.

5. (a) $\sqrt{6}$, (b) $3\sqrt{2}$, (c) $3\sqrt{13}$, (d) $2\sqrt{6}$, (e) $\frac{1}{6}$, (f) 18, (g) 3, (h) $\frac{\sqrt{6}}{3}$, (i) $e^{-\sqrt{6}}$.

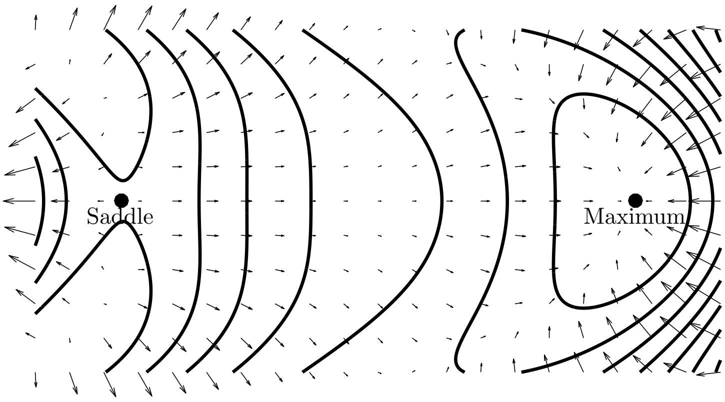
6.

	$\boldsymbol{\nabla}\cdot\mathbf{F}$
(a)	6
(b)	$4x+3$
(c)	$6x+2y+2z$
(d)	$y+z+x$
(e)	0
(f)	$-e^{-x}-xe^{-y}$
(g)	$6xz-4y$
(h)	$6xy+12yz+2x$

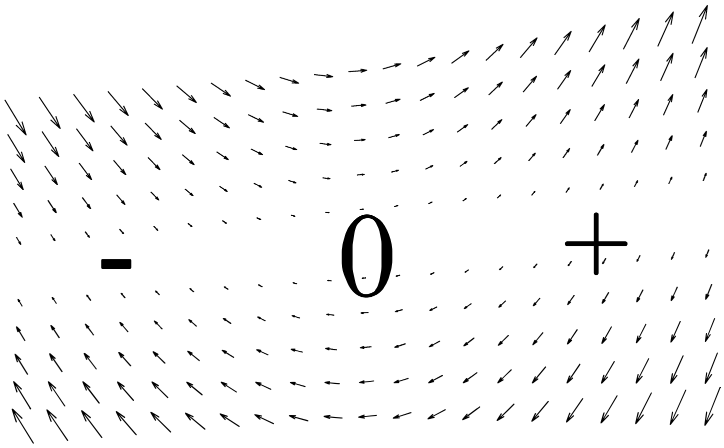
7. $a=-2$.



8.



9.



10.