MTH20014 Mathematics 3B. Tutorial 4

1. Find matrix functions $\exp(\mathbf{A}t)$, $\sin(\mathbf{A}t)$ and $\cos(\mathbf{A}t)$ if $\mathbf{A}$ is

$$\text{(a)} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \; ; \quad \text{(b)} \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} \; ; \quad \text{(c)} \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix} \, .$$

   Observe the pattern in your answers.

2. Determine the type of quadratic curves given by the following equations by converting them to a canonical form:

   (a) $11x_1^2+6x_1x_2+3x_2^2 = 1\,;$ (b) $x_1^2+8x_1x_2+7x_2^2 = 1\,;$ (c) $x_1^2-4x_1x_2-2x_2^2 = 1\,.$

   Give expressions for the canonical coordinates (y_1, y_2) in terms of the original coordinates (x_1, x_2) in each case.

Answers

1.

(a) $$\exp(\mathbf{A}t) = \frac{1}{2}\begin{bmatrix} e^{3t}+e^{-t} & e^{3t}-e^{-t} \\ e^{3t}-e^{-t} & e^{3t}+e^{-t} \end{bmatrix},$$

$$\sin(\mathbf{A}t) = \frac{1}{2}\begin{bmatrix} \sin(3t)-\sin(t) & \sin(3t)+\sin(t) \\ \sin(3t)+\sin(t) & \sin(3t)-\sin(t) \end{bmatrix},$$

$$\cos(\mathbf{A}t) = \frac{1}{2}\begin{bmatrix} \cos(3t)+\cos(t) & \cos(3t)-\cos(t) \\ \cos(3t)-\cos(t) & \cos(3t)+\cos(t) \end{bmatrix};$$

(b) $$\exp(\mathbf{A}t) = \begin{bmatrix} 2e^{3t}-e^{2t} & e^{2t}-e^{3t} \\ 2e^{3t}-2e^{2t} & 2e^{2t}-e^{3t} \end{bmatrix},$$

$$\sin(\mathbf{A}t) = \begin{bmatrix} 2\sin(3t)-\sin(2t) & \sin(2t)-\sin(3t) \\ 2\sin(3t)-2\sin(2t) & 2\sin(2t)-\sin(3t) \end{bmatrix},$$

$$\cos(\mathbf{A}t) = \begin{bmatrix} 2\cos(3t)-\cos(2t) & \cos(2t)-\cos(3t) \\ 2\cos(3t)-2\cos(2t) & 2\cos(2t)-\cos(3t) \end{bmatrix};$$

(c) $$\exp(\mathbf{A}t) = \frac{1}{4}\begin{bmatrix} 3e^{5t}+e^{t} & e^{5t}-e^{t} \\ 3e^{5t}-3e^{t} & e^{5t}+3e^{t} \end{bmatrix},$$

$$\sin(\mathbf{A}t) = \frac{1}{4}\begin{bmatrix} 3\sin(5t)+\sin(t) & \sin(5t)-\sin(t) \\ 3\sin(5t)-3\sin(t) & \sin(5t)+3\sin(t) \end{bmatrix},$$

$$\cos(\mathbf{A}t) = \frac{1}{4}\begin{bmatrix} 3\cos(5t)+\cos(t) & \cos(5t)-\cos(t) \\ 3\cos(5t)-3\cos(t) & \cos(5t)+3\cos(t) \end{bmatrix};$$

2.

(a) ellipse $2y_1^2 + 12y_2^2 = 1$, $$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{10}}(x_1 - 3x_2) \\ \frac{1}{\sqrt{10}}(3x_1 + x_2) \end{bmatrix};$$

(b) hyperbola $9y_1^2 - y_2^2 = 1$, $$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{5}}(x_1 + 2x_2) \\ \frac{1}{\sqrt{5}}(2x_1 - x_2) \end{bmatrix};$$

(c) hyperbola $2y_1^2 - 3y_2^2 = 1$, $$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{5}}(2x_1 - x_2) \\ \frac{1}{\sqrt{5}}(x_1 + 2x_2) \end{bmatrix}.$$