Systems of equations: review

$$\begin{cases} 2x + 3y = 8 \\ 4x + y = 6 \end{cases} \qquad \begin{cases} ax + by = e \\ cx + dy = f \end{cases}$$

$$\begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = ax + by$$

$$\begin{bmatrix} c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = cx + dy$$

$$\underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\vec{z}} = \underbrace{\begin{bmatrix} e \\ f \end{bmatrix}}_{\vec{r}} \qquad A\vec{z} = \vec{r}$$

$$x = \frac{e - by}{a} \Rightarrow c\,\frac{e - by}{a} + dy = f$$

$$\boxed{y = \frac{af - ce}{ad - bc}, \quad x = \frac{de - bf}{ad - bc}}$$

$$A|\vec{r} = \left[\begin{array}{cc|c} a & b & e \\ c & d & f \end{array}\right] \Longrightarrow$$

②

$$\Longrightarrow \left[\begin{array}{cc|c} ac & bc & ec \\ ac & ad & fa \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} ac & bc & ec \\ 0 & bc-ad & ec-fa \end{array}\right]$$

$$(bc-ad)\,y = ec-fa$$

$$y = \frac{(-1)\,ec-fa}{(-1)\,bc-ad} = \frac{af-ec}{ad-bc}$$

$$ad-bc = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \det A = \Delta$$

$$af-ec = \begin{vmatrix} a & e \\ c & f \end{vmatrix} = \Delta_y$$

$$de-bf = \begin{vmatrix} e & b \\ f & d \end{vmatrix} = \Delta_x$$

$$x = \frac{\Delta_x}{\Delta}, \quad y = \frac{\Delta_y}{\Delta}$$ - Cramer's rule

$$kx = l \Longrightarrow \quad x = \frac{l}{k} = \left(\underbrace{k^{-1}k}_{1}x = k^{-1}l\right)$$

$$A\vec{z} = \vec{r} \Longrightarrow \quad A^{-1}(A\vec{z}) = A^{-1}\vec{r}$$

$$\Longrightarrow \underline{\vec{z} = A^{-1}\vec{r}}$$

$$A^{-1}A = I \quad \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)$$

$$A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\vec{z} = \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} e \\ f \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} de-bf \\ af-ce \end{bmatrix}$$

$\det A = 0$?

$$ad = bc \Longrightarrow \frac{a}{c} = \frac{b}{d}$$

<u>Geometric meaning</u> of L.E.

$$\underset{A}{\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}}, \quad \det A = 2\cdot 1 - 3\cdot 4 = -10 \neq 0$$

$y = \frac{8}{3} - \frac{2x}{3}$; $y = 6 - 4x$ - straight lines

$$* \begin{cases} 2x + 3y = 8 \\ 6x + 9y = 24 \end{cases}$$

$\det A = 2 \cdot 9 - 3 \cdot 6 = 0$

$$\begin{cases} y = \frac{8}{3} - \frac{2x}{3} \\ y = \frac{24}{9} - \frac{6x}{9} \end{cases}$$

$$\begin{cases} 2x + 3y = 8 \\ 6x + 9y = 18 \end{cases} \qquad \begin{cases} y = \frac{8}{3} - \frac{2}{3}x \\ y = 2 - \frac{2}{3}x \end{cases}$$

$$\left[\begin{array}{cc|c} 2 & 3 & 8 \\ 6 & 9 & 18 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 2 & 3 & 8 \\ 0 & 0 & 6 \end{array}\right]$$

$$\begin{cases} 2x + 3y = 8 \\ 0x + 0 \cdot y = 6 \end{cases} \Rightarrow \cancel{0 = 6}$$

$$* \left[\begin{array}{cc|c} 2 & 3 & 8 \\ 6 & 9 & 24 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 2 & 3 & 8 \\ 0 & 0 & 0 \end{array}\right]$$

$0x + 0y = 0$

$$2x + 3y = 8$$

$x = t$ - parameter

$$y = \frac{8}{3} - \frac{2t}{3}, \quad t \in \mathbb{R}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t \\ \frac{8}{3} - \frac{2t}{3} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{8}{3} \end{bmatrix} + t \begin{bmatrix} 1 \\ -\frac{2}{3} \end{bmatrix}$$

$$\{t = 3p\} = \begin{bmatrix} 0 \\ \frac{8}{3} \end{bmatrix} + p \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

$$\left\{p = p_1 + \frac{1}{3}\right\} = \begin{bmatrix} 0 \\ \frac{8}{3} \end{bmatrix} + \left(p_1 + \frac{1}{3}\right) \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ \frac{8}{3} \end{bmatrix} + \begin{bmatrix} 1 \\ -\frac{2}{3} \end{bmatrix} + p_1 \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

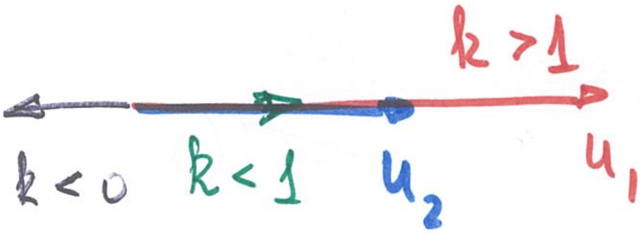
$$= \begin{bmatrix} 1 \\ 2 \end{bmatrix} + p_1 \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

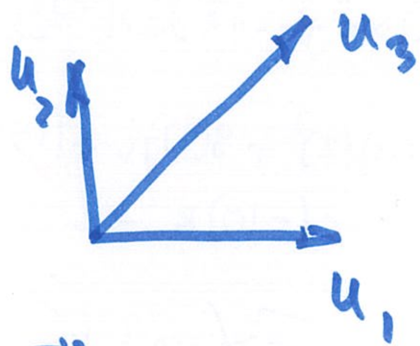
$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\vec{u}_1 = \begin{bmatrix} a \\ c \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} b \\ d \end{bmatrix}$

$\det A = 0 \implies \dfrac{a}{c} = \dfrac{b}{d}$

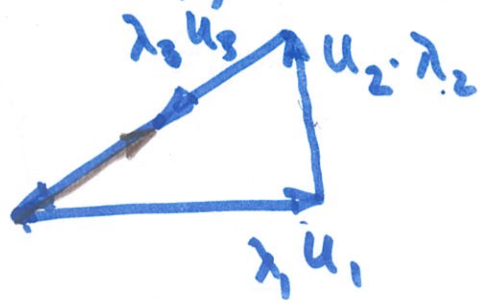
$ad - bc = 0 \implies \dfrac{d}{b} = \dfrac{c}{d} = k$

$\vec{u}_1 = k\,\vec{u}_2$





$\lambda_1 \vec{u}_1 + \lambda_2 \vec{u}_2 + \lambda_3 \vec{u}_3 = \vec{0}$



$\left.\begin{array}{l} \lambda_1 \neq 0 \\ \lambda_2 \neq 0 \\ \lambda_3 \neq 0 \end{array}\right\}$ $\vec{u}_1, \vec{u}_2, \vec{u}_3$ are linearly dependent

$\vec{u}_1 = -\dfrac{\lambda_2 \vec{u}_2 - \lambda_2 \vec{u}_3}{\lambda_1}$

$\lambda_1 = \lambda_2 = 0$

$\vec{u}_1$ & $\vec{u}_2$ are linearly independent.

Rotation Matrix

$$\begin{cases} 2x + 3y = 8 \\ 4x + y = 6 \end{cases}, \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}, \quad \vec{z} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \vec{r} = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$$

$$A\vec{z} = \vec{r}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}}_{R[\theta]} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}; \quad R[\theta] = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

rotation matrix.

$$\det R = \cos^2\theta + \sin^2\theta = 1$$

$$R[-\theta] = \begin{bmatrix} \cos(-\theta) & \sin(-\theta) \\ -\sin(-\theta) & \cos(-\theta) \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = R^T[\theta]$$

$$R[-\theta]\left(R[\theta]\vec{z}\right) = R[-\theta]\vec{w} = I\vec{z}$$

$$\boxed{R[-\theta] \cdot R[\theta] = I}$$

$$R^T[\theta] \cdot R[\theta] = I$$

$$R^{-1}[\theta] \cdot R[\theta] = I$$

$$\boxed{R^{-1}[\theta] = R^T[\theta]}$$ – orthogonal Matrix

If A is orthogonal $\Rightarrow A^{-1} = A^T$

$$AA^T = I$$

$$\begin{array}{cccc} a_{11} & a_{12} & a_{13} & a_{14} \end{array}$$

$$\begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{pmatrix} \begin{pmatrix} c_1 & c_2 & c_3 & c_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$c_1 = r_1^T, \quad c_2 = r_2^T \ \ldots$$

$$\vec{r}_1 \cdot c_1 = 1, \quad \vec{r}_2 \cdot c_1 = 0 \ \ldots\ldots$$

$$\vec{r}_i \cdot \vec{c}_i = 1, \quad \vec{r}_i \cdot \vec{c}_j = 0, \quad i \neq j$$

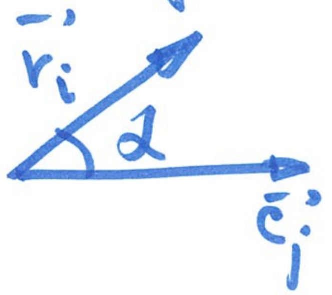
$$\vec{r}_1 c_1 = [a_{11}, a_{12}, a_{13}, a_{14}] \cdot \begin{bmatrix} a_{11} \\ a_{12} \\ a_{13} \\ a_{14} \end{bmatrix} =$$

$$= a_{11} \cdot a_{11} + a_{12} \cdot a_{12} + a_{13} a_{13} + a_{14} a_{14}$$

$= \vec{r}_1 \cdot \vec{c}_1$ – dot product.

$$= a_{11}^2 + a_{12}^2 + a_{13}^2 + a_{14}^2 = |\vec{r}_1|^2 = |\vec{c}_1|^2 = 1$$

$r_i \cdot c_j$ – is a dot product



$$\vec{r_i} \cdot \vec{c_j} = |r_i||c_j| \cdot \cos\alpha = 0$$

$$\Rightarrow \cos\alpha = 0, \quad \alpha = \pi/2$$

$\vec{r_i}, \vec{c_j}$ are orthogonal

Example. What is the shape of the curve given by $5x_1^2 + 6x_1x_2 + 5x_2^2 = 8$

Hint: Consider rotation of coord syst. by $-\pi/4$ (45° clockwise).

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

orig. $\vec{x}$ rotated $\vec{y}$

$$\vec{y} = R\left[-\frac{\pi}{4}\right]\vec{x}$$

$$= \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \vec{x}$$

$$= \frac{\sqrt{2}}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \vec{x}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{\sqrt{2}}{2} \begin{bmatrix} x_1 - x_2 \\ x_1 + x_2 \end{bmatrix}$$

$$y_1 = \frac{\sqrt{2}}{2}(x_1 - x_2) \qquad \vec{y} = R\left[-\frac{\pi}{4}\right] \vec{x}$$

$$y_2 = \frac{\sqrt{2}}{2}(x_1 + x_2) \qquad \vec{x} = R^{-1}\left[-\frac{\pi}{4}\right] \vec{y}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = R\left[\frac{\pi}{4}\right] \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$x_1 = \frac{\sqrt{2}}{2}(y_1 + y_2)$$

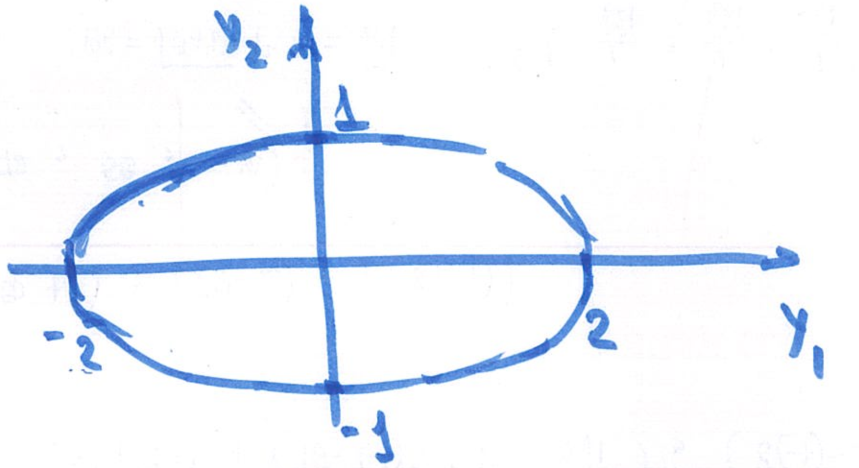
$$x_2 = \frac{\sqrt{2}}{2}(-y_1 + y_2)$$

$$5 \frac{2}{4}(y_1 + y_2)^2 + 6 \frac{2}{4}(y_1 + y_2)(y_2 - y_1) + 5 \cdot \frac{2}{4}(y_2 - y_1)^2 =$$

$$= \frac{1}{2}\left(5(y_1^2 + 2y_1y_2 + y_2^2) + 6(y_2^2 - y_1^2) + 5(y_2^2 - 2y_1y_2 + y_1^2)\right)$$

$$= \frac{1}{2}(4y_1^2 + 16y_2^2) = 2y_1^2 + 8y_2^2 = 8$$

$$\frac{y_1^2}{4} + y_2^2 = 1 \quad - \text{ellipse}$$



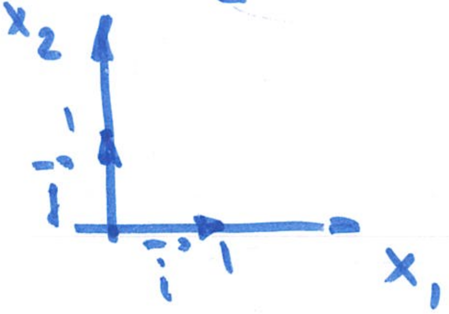
Stretching without rotation

$$\lambda \vec{x} = \lambda I \vec{x} = \Lambda \vec{x}$$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \lambda I = \Lambda = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$A = \begin{bmatrix} 4 & 5 \\ 3 & 2 \end{bmatrix} \quad ; \quad A\vec{i} = \begin{bmatrix} 4 & 5 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

$$A\vec{j} = \begin{bmatrix} 4 & 5 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$



$A\vec{x} = \lambda I \vec{x} \Rightarrow A\vec{x} - \lambda I \vec{x} = \vec{0}$

$(A - \lambda I)\vec{x} = \vec{0}$

$ax = 0 \Rightarrow x = 0/a = 0$

$a = 0 \Rightarrow 0 \cdot x = 0$, x is arbitrary

Assume that $\vec{x} \neq \vec{0}$

We must have $\det(A - \lambda I) = 0$ to get non-trivial solution $\vec{x} \neq \vec{0}$

$$\det(A - \lambda I) = \begin{vmatrix} 4-\lambda & 5 \\ 3 & 2-\lambda \end{vmatrix} = (4-\lambda)(2-\lambda) - 15$$

$$= \lambda^2 - 6\lambda - 7 = 0$$

$$\lambda_{1,2} = \frac{6 \pm \sqrt{6^2 + 28}}{2} = 7, -1$$

$$\lambda_1 = 7: \quad \begin{bmatrix} -3 & 5 \\ 3 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left\{ \begin{matrix} -3x_1 + 5x_2 = 0 \\ 3x_1 - 5x_2 = 0 \end{matrix} \right\} \Rightarrow -3x_1 + 5x_2 = 0$$

$$x_1 = t \quad , \quad x_2 = \frac{3x_1}{5} = \frac{3}{5}t$$

$$\vec{e}_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ \frac{3}{5}t \end{bmatrix} = t \begin{bmatrix} 1 \\ \frac{3}{5} \end{bmatrix} = t \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

$$\vec{e}_1 = t \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

$t \neq 0$

λ eigenvalue

$\vec{e}_1$ eigenvector

$\vec{e}_1 \neq \vec{0}$

Unit eigenvector

$$\hat{\vec{e}}_1 = \frac{1}{\sqrt{34}} \begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 5/\sqrt{34} \\ 3/\sqrt{34} \end{bmatrix}$$

$\lambda_2 = -1 \quad \begin{bmatrix} 5 & 5 \\ 3 & 3 \end{bmatrix}$

$x_1 + x_2 = 0 \qquad \vec{e}_2 = t \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad t \neq 0$

$\hat{\vec{e}}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Example.

$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} : \quad \begin{vmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 = 0$

$= (1-\lambda)(1-\lambda)$

$\lambda_{1,2} = 1$ – repeated eigenvalue.

$\lambda_1 = 1 \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \vec{e}_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\lambda_2 = 1 \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \vec{e}_2 = ?$

e_2 does not exist!

Example:

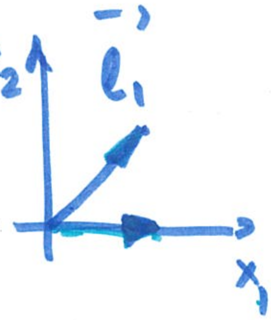
$$\begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix} \Rightarrow \begin{vmatrix} -\lambda & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{vmatrix} = \begin{matrix} -0\,[\qquad] \\ +(2-\lambda)[-\lambda(3-\lambda)+2] \\ -0\,[\qquad] \end{matrix}$$

$$= (2-\lambda)(\lambda^2 - 3\lambda + 2) = 0$$

$$\lambda_1 = 2 \quad , \quad \lambda_{2,3}^2 - 3\lambda_{2,3} + 2 = 0$$

$$\lambda_{2,3} = \frac{3 \pm \sqrt{9-8}}{2} = 2,\ 1$$

$$\begin{bmatrix} -2 & 0 & -2 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



$$\lambda_1 = 2 : x_1 + x_3 = 0 \qquad \vec{e}_1 = t\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\lambda_2 = 2 \qquad \vec{e}_2 = t\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\lambda_3 = 1 : \begin{vmatrix} -1 & 0 & -2 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{vmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \Rightarrow \vec{e}_3 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} t_3$$

Properties of eigenvalues:

1. A^T has the same spectrum as A.

Proof: $\det\left(A^T - \lambda I\right) = \det\left(A^T - \lambda I^T\right) =$ $\boxed{\text{Direct Proof}}$

$= \det\left(\left(A - \lambda I\right)^T\right) = \det\left(A - \lambda I\right) = 0.$

2. A^{-1} exists only if $\lambda_i \neq 0$, $i = 1, 2, \ldots n$

Proof what if $\lambda_i = 0$?

$\det\left(A - \lambda_i I\right) = \det(A) = 0$ $\boxed{\text{Proof by contrapositive}}$

3. Eigenvalues of $A^{-1} = \dfrac{1}{\lambda_i}$, λ_i are e.v. of A

$A\vec{x} = \lambda_i \vec{x}$, $A^{-1} A \vec{x} = \lambda_i A^{-1} \vec{x}$

$\vec{x} = \lambda_i A^{-1} \vec{x}$, $\underline{A^{-1}\vec{x} = \dfrac{1}{\lambda_i}\vec{x}} \Rightarrow \dfrac{1}{\lambda_i}$ is e.v. of A^{-1}!

4. E.v. of A^m are λ_i^m. , $m = 1, 2, 3, \ldots$

$\underbrace{A \cdot A \cdot A \cdot \ldots \cdot A}_{m}$

$\boxed{\text{Proof by Mathematical induction.}}$

1. Assume that it works fom some m
2. Prove algebraically that then it works fom $m+1$.
3. Demonstrate that it works fon $m=1$
4. State logically that then it works for any m

1. Assume $A^m \vec{x} = \lambda_i^m \vec{x}$

2. $A \cdot A^m \vec{x} = A^{m+1} \vec{x} = \lambda_i^m A \vec{x}$

By definition of e.v.p $A\vec{x} = \lambda_i \vec{x}$

Therefore $A^{m+1} \vec{x} = \lambda_i^m \cdot \lambda_i \vec{x} = \lambda_i^{m+1} \vec{x}$

3. $A\vec{x} = \lambda_i \vec{x}$ is true fon $m=1$

4. Proved by MMI.

5 + 6 : $Tr(A) = \sum_{i=1}^{n} \lambda_i$ "Magic" Properties

$\det(A) = \prod_{i=1}^{n} \lambda_i$

$$P(\lambda) = \det(A - \lambda \underline{I}) = (-1)^n \lambda^n + (Tr(A))\lambda^{n-1} + \ldots + C$$

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = (a-\lambda)(d-\lambda) - bc = 0$$

$$= ad - d\lambda - a\lambda + \lambda^2 - bc$$

$$= \lambda^2 - (a+d)\lambda + ad - bc$$

$$\begin{vmatrix} (a-\lambda) & b & c \\ d & (e-\lambda) & f \\ g & h & (i-\lambda) \end{vmatrix} = \begin{matrix} (a-\lambda)\,[(e-\lambda)(i-\lambda) - fh] \\ -b\,[d(i-\lambda) - fg] \\ +c\,[dh - (e-\lambda)\cdot g] \end{matrix}$$

$$P(0) = C = \det(A)$$

$$P(\lambda) = (-1)^n \lambda^n + (-1)^{n-1} Tr(A)\lambda^{n-1} + \ldots + \det A = 0$$

$$= (\lambda_1 - \lambda)(\lambda_2 - \lambda)(\lambda_3 - \lambda)\ldots(\lambda_n - \lambda) = 0$$

$$= (-\lambda)^n + \lambda_1 (-\lambda)^{n-1} + \lambda_2 (-\lambda)^{n-1} + \lambda_3 (-\lambda)^{n-1} + \ldots$$

$$= (-\lambda)^n + (-\lambda) \sum_{i=1}^{n-1\infty} \lambda_i + \ldots + \prod_{i=1}^{n} \lambda_i = 0$$

Theorem 1.7 (p 18.)

$\boxed{\text{Proof by contradiction.}}$

Assume e.v. are linearly dependent, but eigenvalues are still distinct

$$\vec{v}_0 = c_1 \vec{e}_1 + c_2 \vec{e}_2 + \ldots + c_m \vec{e}_m = \vec{0}$$

$(c_1, c_2, \ldots) \neq 0$, $1 \leq m \leq n$, where n is the total number of e.v.

$$\vec{v}_1 = A \vec{v}_0 = c_1 \lambda_1 \vec{e}_1 + c_2 \lambda_2 \vec{e}_2 + \ldots + c_m \lambda_m \vec{e}_m = \vec{0}$$

$$\vec{V}_1 - \lambda_1 \vec{V}_0 = \left(c_2 \lambda_2 \vec{e}_2 - c_2 \lambda_1 \vec{e}_2\right) + \dots =$$

$$\vec{V}_2 = c_2 (\lambda_2 - \lambda_1) \vec{e}_2 + c_3 (\lambda_3 - \lambda_1) \vec{e}_3 + \dots$$

$$AV_2 = c_2 (\lambda_2 - \lambda_1) \lambda_2 e_2 + c_3 (\lambda_3 - \lambda_1) \lambda_3 \vec{e}_3 + \dots$$

$$AV_2 - \lambda_2 V_2 = c_3 (\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2) \vec{e}_3 + \dots$$

$$\vdots$$

$$c_m (\lambda_m - \lambda_{m-1})(\lambda_m - \lambda_{m-2}) \cdots (\lambda_m - \lambda_1) \vec{e}_m = \vec{0}$$