Volume integrals

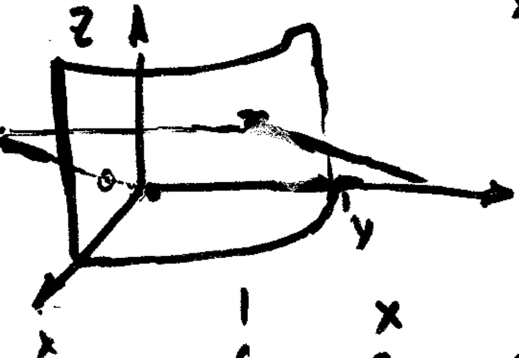
$$\iiint_T f(x,y,z)\,dV = \int \int \int f\,dx\,dy\,dz$$

$$= \int_{a_1}^{a_2} \int_{b_1(z)}^{b_2(z)} \int_{c_1(y,z)}^{c_2(y,z)} f\,dx\,dy\,dz$$

Example. Find the mass of T:

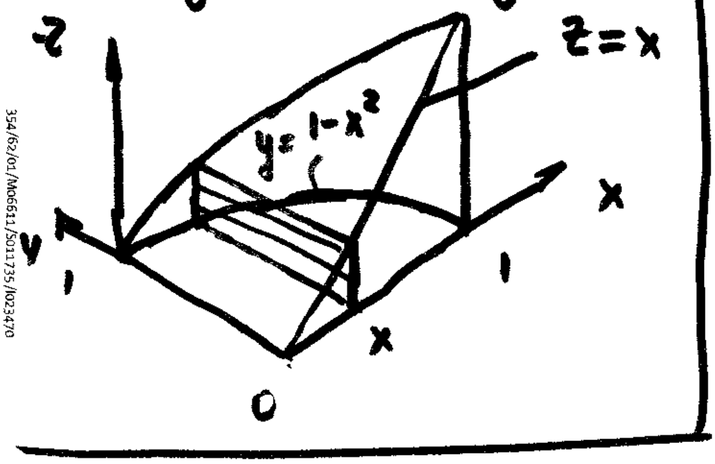
$T = \{ (x,y,z) :\ y = 1-x^2,\ z = x\}$ if $\rho = 4z$

$x \geq 0,\ y \geq 0,\ z \geq 0$



$$\iiint_T 4z\,dV =$$

$$= \int_0^1 \int_0^x \int_0^{1-x^2} 4z\,dy\,dz\,dx =$$



$$= \int_0^1 \int_0^x 4zy \Big|_{y=0}^{y=1-x^2} dz\,dx$$

$$= \int_0^1 \int_0^x 4z(1-x^2)\,dz\,dx$$

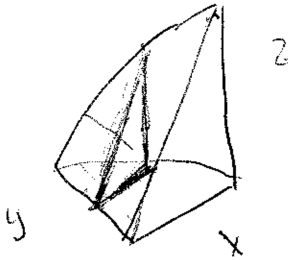
$$= \int_0^1 2z^2(1-x^2) \Big|_{z=0}^{z=x} dx$$

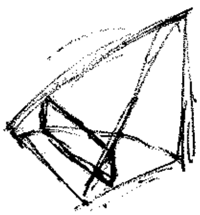
$$= \int_0^1 2x^2(1-x^2)\,dx = \int_0^1 (2x^2 - 2x^4)\,dx = \frac{2x^3}{3} - \frac{2x^5}{5}\Big|_{x=0}^{x=1}$$

$$= \frac{2}{3} - \frac{2}{5} = \frac{2\cdot 5 - 2\cdot 3}{3\cdot 5} = \frac{4}{15}$$

64a

dydz dx





$$\int_0^1\int_0^x\int_0^{1-x^2} 4\cdot z\,dy\,dz\,dx = \int_0^1\int_0^x 4yz\Big|_{y=0}^{y=1-x^2} dz\,dx$$

$$=\int_0^1\int_0^x 4(1-x^2)z\,dz\,dx = \int_0^1 2(1-x^2)z^2\Big|_{z=0}^{z=x} dx$$

$$=\int_0^1 2(1-x^2)x^2\,dx = \int_0^1 (2x^2-2x^4)\,dx$$

$$=\frac{2x^3}{3}-\frac{2x^5}{5}\Big|_0^1 = 2\left(\frac{1}{3}-\frac{1}{5}\right)$$

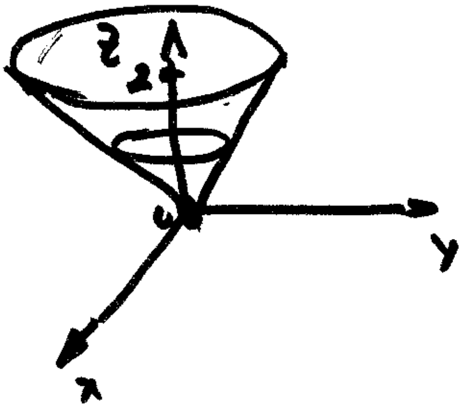
$$=\frac{2(5-3)}{15}=\frac{4}{15}$$

Ostrogradsky – Gauss' divergence theorem

$$\iiint_T \nabla \cdot \vec{F}\, dv = \iint_S \vec{F} \cdot \vec{n}\, dA$$

Example. $\iint_S \vec{F} \cdot \vec{n}\, dA$, $\vec{F} = (x^2, (1-2x)y, 4z)$

S: $x^2 + y^2 \le z^2$, $0 \le z \le 2$



$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(1-2x)y + \frac{\partial}{\partial z}(4z)$$

$$= \cancel{2x} - \cancel{2x} + 1 + 4 = 5$$

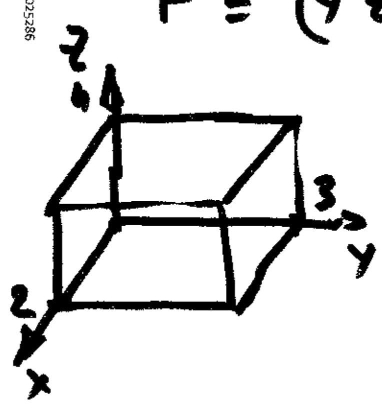
$$\iint_S \vec{F} \cdot \vec{n}\, dA = \iiint_T 5\, dV = 5 \iiint_T dV$$

$$= 5 \cdot V = 5 \cdot \frac{1}{3} \cdot 2 \cdot \pi 2^2 = \frac{40\pi}{3}$$

$$\left\{ \frac{1}{3} \cdot h \cdot \pi r^2 \right\}$$

Example:

$\vec{F} = (yz, xz, xy)$ $\Phi = \iint_S \vec{F} \cdot \vec{n}\, dA$ – ?



$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(yz) + \frac{\partial}{\partial y}(xz) + \frac{\partial}{\partial z}(xy)$$

$$= 0 + 0 + 0 = 0$$

$$\Phi = 0.$$

Example. $\vec{F} = \vec{r} = (x, y, z)$

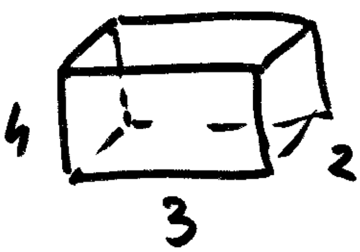
$$\iint_S \vec{F}\cdot\vec{n}\,dA = \iint_S \vec{r}\cdot\vec{n}\,dA$$

$$\nabla\cdot\vec{F} = \nabla\cdot\vec{r} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1+1+1 = 3$$

$$\iint_S \vec{r}\cdot\vec{n}\,dA = 3\iiint_T dV = 3V$$

$$V = \frac{1}{3}\iint_S \vec{r}\cdot\vec{n}\,dA$$, where S is the surface enclosing Volume of T

Example. $\iint_S \vec{r}\cdot\vec{n}\,dA$ for

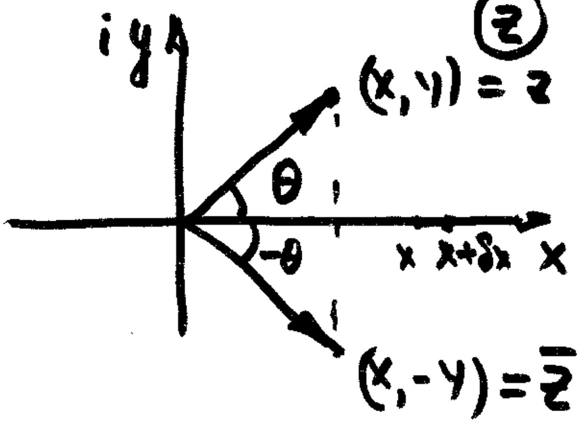


$$V = 2\cdot 3\cdot 4 = 24$$

$$\iint_S \vec{r}\cdot\vec{n}\,dA = 3\cdot V = 3\cdot 24 = 72$$

Functions of a complex variable

$z = x + iy$ $\qquad i = \sqrt{-1}\ ,\ i^2 = -1$



$|z|^2 = x^2 + y^2$

$x = z\cos\theta$

$y = z\sin\theta$

$z = |z|\, e^{i\theta}$

$e^{i\theta} = \cos\theta + i\sin\theta$ - Euler formula

$w = f(z)$

$w = (1+2i)\, z \qquad , \qquad z = (x, y)$

$w = z^2$

$w = \frac{1}{z}$

$w = e^{z}$

Example : $w = (1+2i)\,z = (1+2i)(x+iy)$

$= x + 2ix + iy + 2i^2 y = \underbrace{x - 2y}_{u} + i\,\underbrace{(2x+y)}_{v}$

$u = Re\{w\}\ ,\ v = Im\{w\}$

$z^2 = (x+iy)^2 = x^2 + 2xyi - y^2 = x^2 - y^2 + 2xyi$

$\underset{RP}{u = x^2 - y^2}\ ,\ \underset{IP}{v = 2xy}$

$w = \frac{1}{z}\frac{\bar{z}}{\bar{z}} = \frac{1}{x+iy} = \frac{(x-iy)}{(x+iy)(x-iy)} = \frac{x-iy}{x^2 + \cancel{ixy} - \cancel{ixy} + y^2}$

$= \frac{x-iy}{x^2+y^2} = \underbrace{\frac{x}{x^2+y^2}}_{u} + i\,\underbrace{\frac{-y}{x^2+y^2}}_{v}$

$$z \cdot z = x^2 + y^2 = |z|^2$$

$$z \cdot \bar{z} = |z|^2$$

$$w = e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y)$$

$$= \underbrace{e^x \cos y}_{u} + i\, \underbrace{e^x \sin y}_{v}$$

Differentiation of a function of a complex variable.

$$\frac{df(x)}{dx} = f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$\frac{df(z)}{dz} = f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$$

Example. $f(z) = z^2$

$$\frac{df}{dz} = \lim_{\delta z \to 0} \frac{(z+\delta z)^2 - z^2}{\delta z} = \lim_{\delta z \to 0} \frac{z^2 + 2z\delta z + (\delta z)^2 - z^2}{\delta z}$$

$$= \lim_{\delta z \to 0} (2z + \delta z) = 2z$$

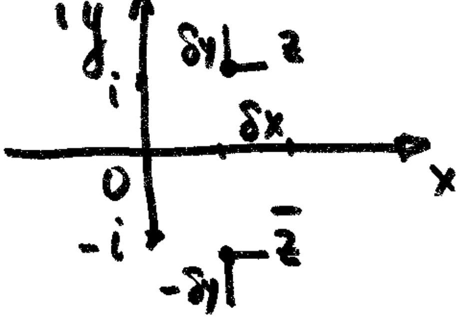
$$f(x) = x^2 \Rightarrow f'(x) = 2x$$

$$f(z) = z^2 \Rightarrow f'(z) = 2z$$

$$f(z) = \bar{z} \qquad f'(z) - ?$$

$$f'(z) = \lim_{\delta z \to 0} \frac{\overline{z+\delta z} - \bar{z}}{\delta z} = \lim_{\delta z \to 0} \frac{\cancel{\bar{z}} + \overline{\delta z} - \cancel{\bar{z}}}{\delta z}$$

$$= \lim_{\delta z \to 0} \frac{\overline{\delta z}}{\delta z}$$



$$\delta z = \delta x \quad , \quad \overline{\delta z} = \delta x$$

$$\frac{\overline{\delta z}}{\delta z} = \frac{\delta x}{\delta x} = 1$$

$$\delta z = i\delta y \quad , \quad \overline{\delta z} = \overline{i\delta y} = -i\delta y$$

$$\frac{\overline{\delta z}}{\delta z} = \frac{-i\delta y}{i\delta y} = -1$$

Cauchy-Riemann conditions.

$$w = f(z) = z^2 = \underbrace{x^2 - y^2}_{u} + \underbrace{2xy}_{v}\, i$$

$$f = w = u(x,y) + i\, v(x,y)$$

δz : a) $\delta z = \delta x$ b) $\delta z = i\,\delta y$

a) $$f'(z) = \lim_{\delta x \to 0} \frac{f(z+\delta x) - f(z)}{\delta x} = \lim_{\delta x \to 0} \Big(u(x+\delta x, y) + i\, v(x+\delta x, y) - u(x,y) - i\, v(x,y)\Big) \Big/ \delta x$$

$$= \lim_{\delta x \to 0} \frac{u(x+\delta x, y) - u(x,y) + i\,(v(x+\delta x, y) - v(x,y))}{\delta x}$$

$$= \boxed{\frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x}}$$

b) $$f'(z) = \lim_{\delta y \to 0} \frac{u(x, y+\delta y) - u(x,y) + i(v(x,y+\delta y) - v(x,y))}{i\delta y}$$

$$= \frac{1}{i}\frac{\partial u}{\partial y} + \frac{i}{i}\frac{\partial v}{\partial y} = \frac{i}{i^2}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = \boxed{-i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}}$$

For $f'(z)$ not to depend on the direction in a complex plane

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}}$$

$u = x^2 - y^2$, $v = 2xy$ for $f(z) = z^2$

V $\begin{cases} \frac{\partial u}{\partial x} = 2x \\ \frac{\partial v}{\partial y} = 2x \end{cases}$ V $\begin{cases} \frac{\partial v}{\partial x} = 2y \\ \frac{\partial u}{\partial y} = -2y \end{cases}$ V

Example: you can show that $f = \frac{1}{z}$ satisfies the Cauchy-Riemann (CR) conditions

Thu $f'(z)$ can be found

$$f(z) \longrightarrow f(x) \longrightarrow f'(x) \longrightarrow f'(z)$$

$$f(z) = \frac{1}{z} \rightarrow f(x) = \frac{1}{x} \rightarrow f'(x) = -\frac{1}{x^2} \rightarrow f'(z) = -\frac{1}{z^2}$$

Orthogonality of u & v

Let $f(z) = u(x,y) + i\, v(x,y)$ satisfy CRC.

$f(z)$ - analytic

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad -\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$$

$$\nabla u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right), \quad \nabla v = \left(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}\right)$$

$$\nabla u \cdot \nabla v = \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}$$

$$= \frac{\partial v}{\partial y}\frac{\partial v}{\partial x} - \frac{\partial v}{\partial x}\cdot\frac{\partial v}{\partial y} = 0 \implies \underline{\nabla u \perp \nabla v}$$

$$\left.\begin{array}{l} \nabla u \perp u = \text{const} \\ \nabla v \perp v = \text{const} \end{array}\right\} \implies u = \text{const} \perp v = \text{const}$$