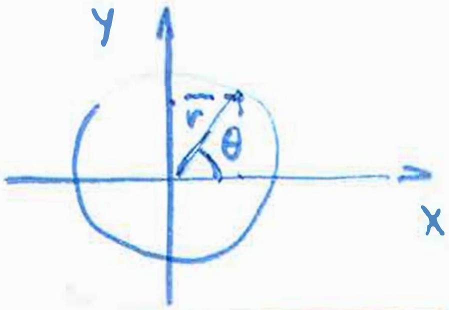
Double integrals in polar coordinates



$$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}}$$

$r \geq 0$

$0 \leq \theta < 2\pi$

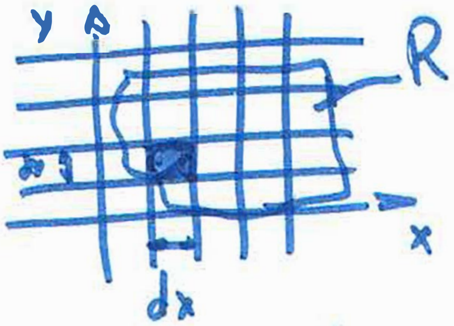
$$x^2+y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2$$

$$x^2+y^2 = r^2$$

$$\boxed{\begin{aligned} r &= \sqrt{x^2+y^2} \\ \theta &= \tan^{-1}(y/x) \end{aligned}}$$

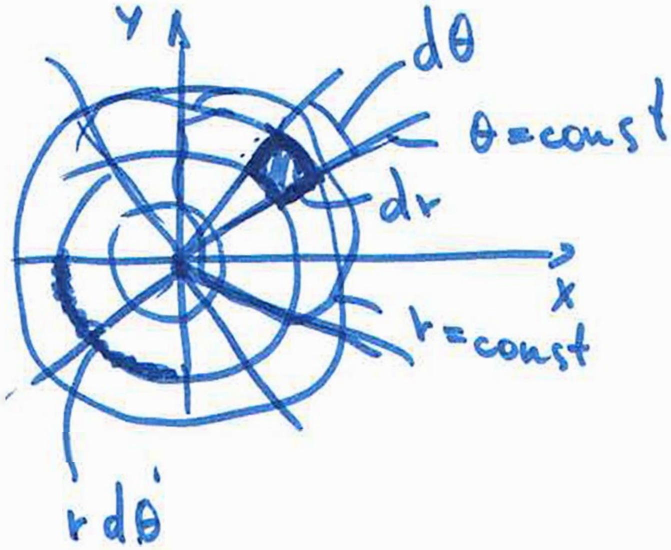
$$\frac{y}{x} = \tan\theta$$

$$dR = dx\,dy$$



$x = \text{const}$

$y = \text{const}$

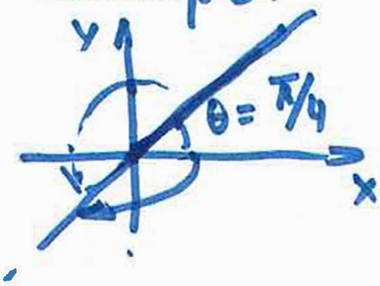


$$dR = r\,dr\,d\theta$$

$$\iint f(x,y)\,dx\,dy$$

$$\Rightarrow \iint f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

Example.



$y = x$

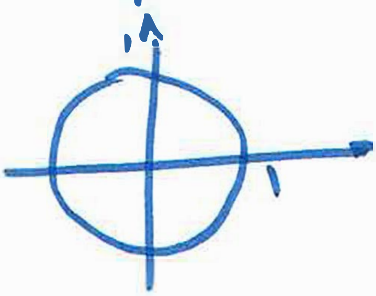
$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$r\cos\theta = r\sin\theta$$
$$\frac{\sin\theta}{\cos\theta} = 1 = \tan\theta$$

$\vee\ \theta = \pi/4$

$\theta = -\frac{3}{4}\pi$ X $\notin [0, 2\pi)$

$\vee\ \theta = \frac{5}{4}\pi$

Example.



$r = 1$

$$x^2 + y^2 = 1$$
$$y = \pm\sqrt{1 - x^2}$$

Example.

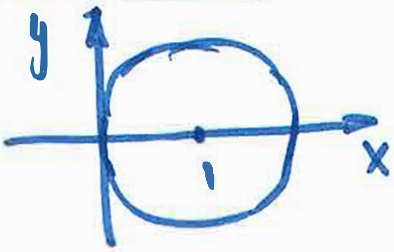
$r = 2\cos\theta$

$$x = r\cos\theta$$
$$y = r\sin\theta$$

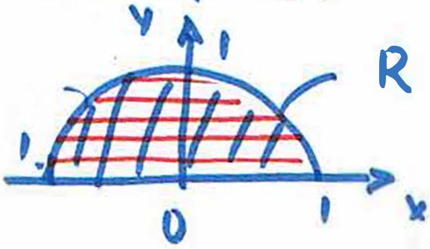
$\cos\theta = \frac{x}{r}$ $\Rightarrow$ $r = \frac{2x}{r}$ $\Rightarrow$ $r^2 = 2x$

$x^2 + y^2 = 2x$ $\Rightarrow$ $\underbrace{x^2 - 2x + 1} + y^2 = 1$

$\underline{(x-1)^2 + y^2 = 1}$ $\Rightarrow$ $y = \pm\sqrt{1 - (x-1)^2}$



Example:



$$I=\iint_R (x^2+y^2)\,dR$$

$$=\int_0^1\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}(x^2+y^2)\,dx\,dy$$

$y=\sqrt{1-x^2}$

$x^2+y^2=1$

$x=\pm\sqrt{1-y^2}$

$x\to r\cos\theta$

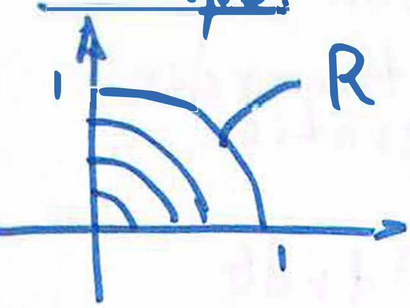
$y\to r\sin\theta$

$dx\,dy\to r\,dr\,d\theta$

$$I=\int_0^{\pi}\int_0^1 r^2\cdot r\,dr\,d\theta=\int_0^{\pi}\frac{r^4}{4}\Big|_0^1 d\theta=\int_0^{\pi}\frac{1}{4}\,d\theta$$

$$=\frac{1}{4}\theta\Big|_0^{\pi}=\frac{\pi}{4}$$

Example.



$$\iint_R y^2\,dR=\int_0^1\int_0^{\pi/2} r^2\sin^2\theta\; r\,d\theta\,dr$$

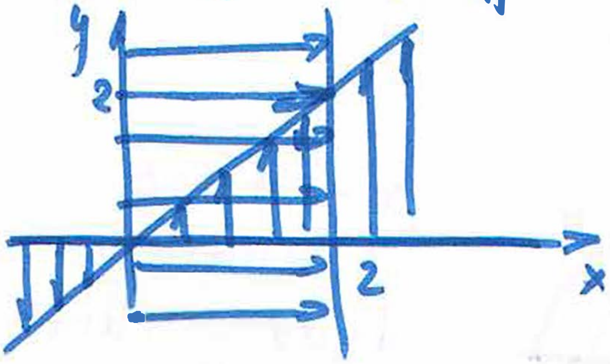
$$=\int_0^1\int_0^{\pi/2} r^3\,\frac{1-\cos(2\theta)}{2}\,d\theta\,dr$$

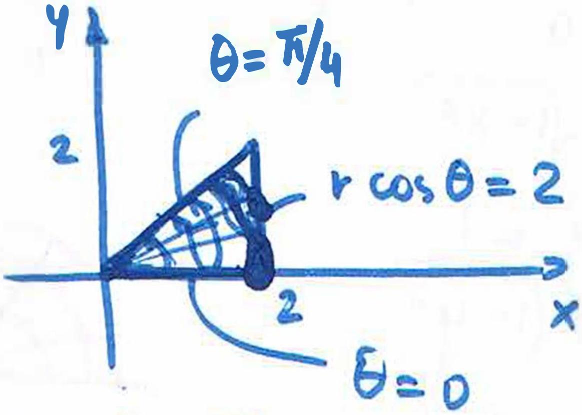
$$=\int_0^1 r^3\,\frac{\theta-\frac{1}{2}\sin(2\theta)}{2}\Bigg|_{\theta=0}^{\theta=\pi/2}dr$$

$$=\int_0^1 r^3\,\frac{\pi}{4}\,dr=\frac{\pi}{4}\,\frac{r^4}{4}\Big|_0^1=\frac{\pi}{16}$$

Exampb. $I = \int_0 \int_0 y \, dy \, dx$

$y = 0$, $y = x$, $\cancel{y}$ $x = 0$, $x = 2$





$x = r \cos\theta$

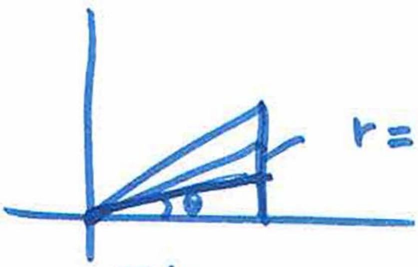
$\cancel{y} = r \sin\theta$

$y = x \Rightarrow \theta = \pi/4$

$$I = \int \int_{0 \times ?}^{\pi/4} r \sin\theta \; r \, d\theta \, dr$$

Geometrically too complicated, needs splitting the region $\Rightarrow$ change the order of integration.

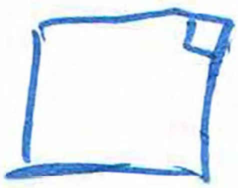
$$= \int_0^{\pi/4} \int_0^{2/\cos\theta} r \sin\theta \, r \, dr \, d\theta = \int_0^{\pi/4} \int_0^{2/\cos\theta} \sin\theta \, r^2 \, dr \, d\theta$$



$r = \frac{2}{\cos\theta}$

$$= \int_0^{\pi/4} \frac{r^3}{3} \sin\theta \Big|_{r=0}^{r = \frac{2}{\cos\theta}} d\theta$$

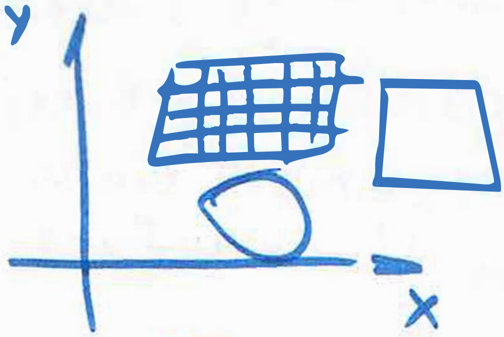
$$= \int_0^{\pi/4} \frac{8}{3} \frac{\sin\theta}{\cos^3\theta} \, d\theta = \left\{ \begin{array}{ll} \cos\theta = u & \theta = \pi/4 \Rightarrow u = \frac{\sqrt{2}}{2} \\ \frac{du}{d\theta} = -\sin\theta & \\ d\theta = \frac{du}{-\sin\theta} & \theta = 0 \Rightarrow u = 1 \end{array} \right\}$$



$$= \frac{8}{3}\int_1^{\sqrt{2}/2} \frac{\cancel{\sin\theta}}{u^3}\,\frac{du}{-\cancel{\sin\theta}} = -\frac{8}{3}\int_1^{\sqrt{2}/2}\frac{du}{u^3} = -\frac{8}{3}\,\frac{u^{-2}}{(-2)}$$

$$= \frac{4}{3}\left[\left(\frac{\sqrt{2}}{2}\right)^{-2} - 1\right] = \frac{4}{3}\left[\frac{4}{2} - 1\right] = \boxed{\frac{4}{3}}$$

Description of surfaces in space and surface integrals



$dA = dR = dx\,dy$

$dA = dR = r\,dr\,d\theta$

Description of surfaces

Elliptic cylinder : example of parameterisation

$$\vec{r}(u,v) = (x(u,v),\ y(u,v),\ z(u,v))$$

$$= (\overset{x}{4\cos u},\ \overset{y}{2\sin u},\ \overset{z}{v})$$

$$0 \le u \le 2\pi,\ -\infty < v < \infty$$

$v = \text{const} \Rightarrow z = \text{const}$

$x = 4\cos u, \quad y = 2\sin u$ $\Big\}$ $\sin^2 u + \cos^2 u = 1$

$\cos u = \dfrac{x}{4}, \quad \sin u = \dfrac{y}{2}$

$$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{2}\right)^2 = 1 \Rightarrow \boxed{\frac{x^2}{16} + \frac{y^2}{4} = 1}$$

ellipse

$u = \text{const}$ $\quad$ $\left.\begin{array}{l} x = 4\cos u = \text{const} = x_0 \\ y = 2\sin u = \text{const} = y_0 \end{array}\right\} \Rightarrow$ straight vertical line (parallel to z axis)

$\left.\begin{array}{l} \vec{r}(u, \text{const}) \\ \vec{r}(\text{const}, v) \end{array}\right\}$ - parameterisation lines that form the surface

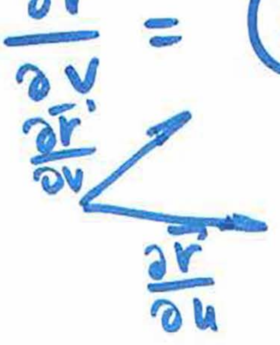
$\dfrac{\partial \vec{r}}{\partial u}$ - vector tangent to the u-line

$\dfrac{\partial \vec{r}}{\partial v}$ - vector tangent to the v-line

$$\frac{\partial \vec{r}}{\partial u} = \frac{\partial}{\partial u}(4\cos u, 2\sin u, v)$$
$$= (-4\sin u, 2\cos u, 0)$$

$$\frac{\partial \vec{r}}{\partial v} = (0, 0, 1)$$

The plane containing $\dfrac{\partial \vec{r}}{\partial u}$ & $\dfrac{\partial \vec{r}}{\partial v}$ is tangent to the surface



(52)

$\vec{N} = ?$

$$\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} = \vec{N}$$

$$\frac{\partial \vec{r}}{\partial v} \times \frac{\partial \vec{r}}{\partial u} = -\vec{N}$$

$$\vec{n} = \frac{\vec{N}}{|\vec{N}|}$$

$$\vec{N} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} i & j & k \\ -4\sin u & 2\cos u & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{matrix} \vec{i} & (2\cos u) \\ -\vec{j} & (-4\sin u) \\ +\vec{k} & 0 \end{matrix}$$

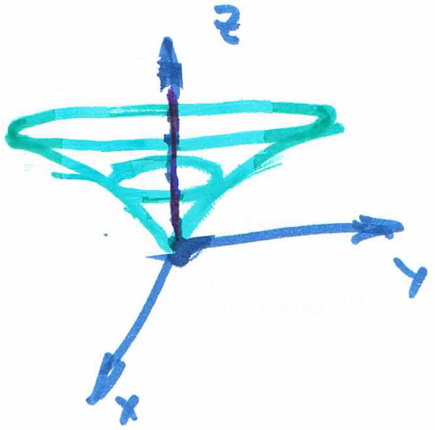
$$= (2\cos u,\ 4\sin u,\ 0)$$

$$|\vec{N}| = \sqrt{4\cos^2 u + 16\sin^2 u} = \sqrt{4 + 12\sin^2 u}$$

$$= 2\sqrt{1 + 3\sin^2 u}$$

$$\vec{n} = \frac{\vec{N}}{|\vec{N}|} = \frac{(\cos u,\ 2\sin u,\ 0)}{\sqrt{1 + 3\sin^2 u}}$$

Example : Find a suitable parametric representation of a surface in the form of a parabolic cup (paraboloid)



$$x^2+y^2=r^2=z^4$$

$$r=z^2$$

$$x=r\cos u\ ,\quad y=r\sin u$$

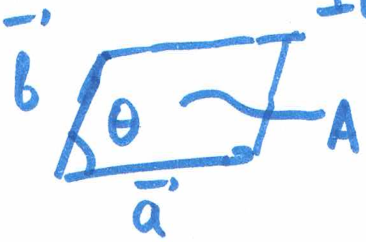
$$z=v \quad \Rightarrow \quad r=v^2$$

$$r^2\cos^2 u+r^2\sin^2 u=v^4$$

$$v^4\cos^2 u+v^4\sin^2 u=v^4 \quad \checkmark$$

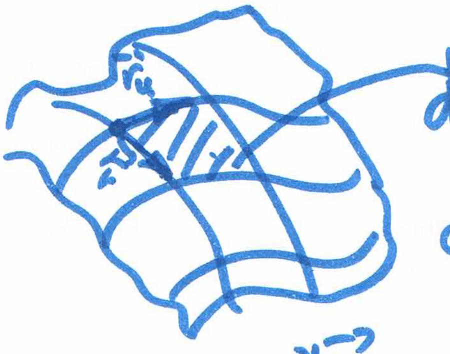
$$\vec{r}=(x,y,z)=(v^2\cos u,\ v^2\sin u,\ v)$$

Integration over a general surface



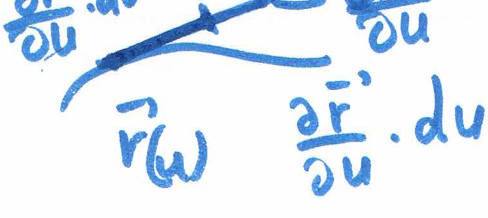
$$\vec{a}\times\vec{b}=|\vec{a}|\,|\vec{b}|\sin\theta=A$$

$$\vec{r}_u=\frac{\partial\vec{r}}{\partial u} \qquad \frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v}=\vec{N}$$

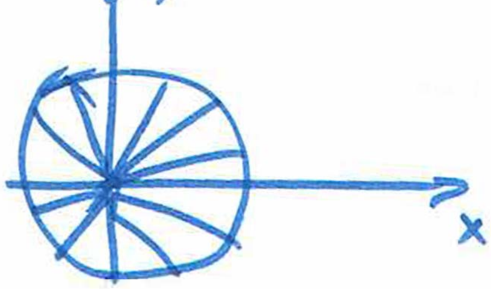


$$\vec{r}_v=\frac{\partial\vec{r}}{\partial v}$$

$$du\,dv\,|\vec{N}|=dA$$



$$\iint_S G\,dA=\iint_R G(\vec{r}(u,v))\,|\vec{r}_u\times\vec{r}_v|\,du\,dv$$



$(x, y) \longrightarrow (r, \theta) \equiv (u, v)$ (5

$x = u \cos v$

$y = u \sin v$

$\vec{r} = (u \cos v, \ u \sin v)$

$\vec{r}_u = (\cos v, \ \sin v)$

$\vec{r}_v = (-u \sin v, \ u \cos v)$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} i & j & k \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & 0 \end{vmatrix} = \vec{i}\ 0 - \vec{j}\ 0 + \vec{k}\,(u\cos^2 v + u\sin^2 v$$

$= (0, 0, u)$

$|\vec{r}_u \times \vec{r}_v| = u$

$$\iint_S G\, dA = \iint_R G(u,v)\, |\vec{r}_u \times \vec{r}_v|\, du\, dv$$

$$= \iint_R G(u,v)\, u\, du\, dv$$

$$\Longrightarrow \iint_R G(r,\theta)\, r\, dr\, d\theta$$

$$\vec{r} = (x(u,v),\ y(u,v),\ 0)$$

$$\vec{r}_u = (x_u,\ y_u,\ 0)$$

$$\vec{r}_v = (x_v,\ y_v,\ 0)$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_u & y_u & 0 \\ x_v & y_v & 0 \end{vmatrix} = \begin{matrix} \vec{i}\ 0 \\ \vec{j}\ 0 \\ \vec{k}\ (x_u y_v - x_v y_u) \end{matrix}$$

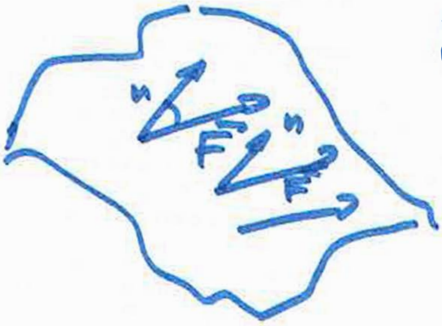
$$= (0,\ 0,\ x_u y_v - x_v y_u)$$

$$|\vec{r}_u \times \vec{r}_v| = |\underbrace{x_u y_v - x_v y_u}_{\text{Jacobian}}| = J$$

$$\iint\limits_{\substack{S \\ \text{flat}}} G\, dA = \iint\limits_{R} G(u,v)\, J\, du\, dv$$

$$\iint_S G\,dA = \iint_R G(\vec{r}(u,v))\,|\vec{r}_u \times \vec{r}_v|\,du\,dv$$

Flux integrals



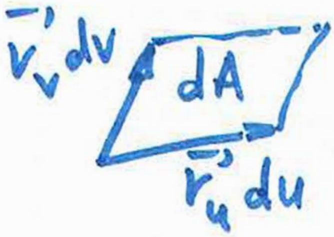
$F_n = \vec{F}\cdot\vec{n}$ – normal component of the vector $\vec{F}$

$F\cdot\vec{n}\cdot dA$ – flux through small area dA

$$|\vec{N}|\cdot\vec{n}$$

$$\Phi = \int_S F_n\,dA = \int_S \vec{F}\cdot\vec{n}\cdot dA$$

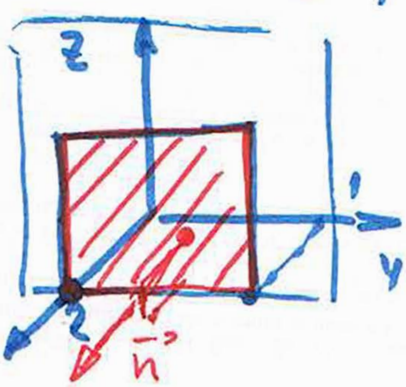
$$\vec{r}_u \times \vec{r}_v = \vec{N} \qquad |\vec{N}| = dA \quad , \quad \vec{n} = \frac{\vec{N}}{|\vec{N}|}$$



$$\Phi = \int_S F_n\,dA = \iint_R \vec{F}\cdot(\vec{r}_u \times \vec{r}_v)\,du\,dv$$

Example. $\vec{F} = (x^2y,\ yz,\ 3x^3z)$

S: $x=2$, $y=0$, $y=1$, $z=0$, $z=3$



$$\vec{n} = (1,0,0)$$

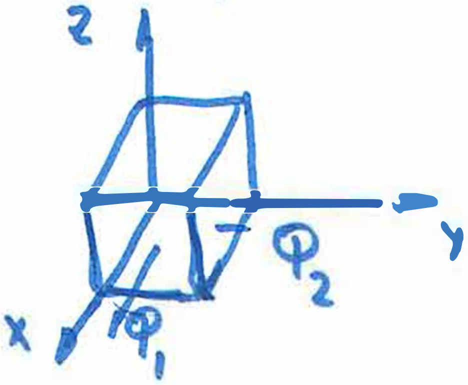
$$\Phi = \int_0^3\int_0^1 (x^2y,\ yz,\ 3x^3z)\cdot(1,0,0)\,dy\,dz$$

$$= \int_0^3\int_0^1 x^2y\,dy\,dz = \int_0^3 \frac{x^2y^2}{2}\Big|_{y=0}^{y=1} dz$$

$$= \int_0 \frac{x^2}{2}\,dz = \frac{x^2 z}{2}\Big|_{z=0}^{z=3} = \frac{3x^2}{2} = \{x=2\} =$$

$$= 6$$

Example:



$$\vec{F} = (x^2 y,\ yz,\ 3x^3 z)$$

$$\Phi_1 = \int_0^1\int_0^1 (x^2 y, yz, 3x^3 z)\cdot(1,0,0)\,dy$$

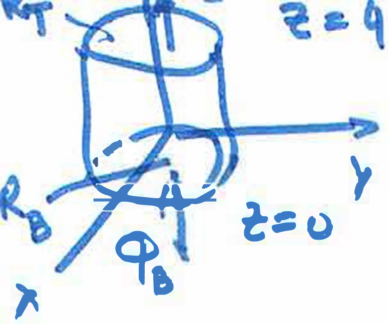
$$= \int_0^1\int_0^1 x^2 y\,dy\,dz = \int_0^1 \frac{x^2 y^2}{2}\Big|_{y=0}^{y=1} dz$$

$$= \int_0^1 \frac{x^2}{2}\,dz = \frac{x^2 z}{2}\Big|_{z=0}^{z=1} = \frac{x^2}{2} = \frac{1}{2}$$

$$\Phi_2 = \int_0^1\int_0^1 (x^2 y, yz, 3x^3 z)\cdot(0,1,0)\,dx\,dz$$

$$= \int_0^1\int_0^1 yz\,dx\,dz = \int_0^1 xyz\Big|_{x=0}^{x=1} dz = \int_0^1 yz\,dz$$

$$= y\int_0^1 z\,dz = y\frac{z^2}{2}\Big|_{z=0}^{z=1} = \frac{y}{2} = \frac{1}{2}$$



$$\vec{F} = (2x, 2y, z^2)$$

$$\vec{r}(u,v) = (\overset{x}{4\cos u}, \overset{y}{2\sin u}, \overset{z}{v})$$

$$0 \le u \le 2\pi, \quad 0 \le v \le 4$$

$$\vec{N} = (2\cos u, 4\sin u, 0)$$

$$\Phi_S = \int_0^4 \int_0^{2\pi} (8\cos u, 4\sin u, v^2) \cdot (2\cos u, 4\sin u, 0)\, du\, dv$$

$$= \int_0^4 \int_0^{2\pi} (16\cos^2 u + 16\sin^2 u)\, du\, dv$$

$$= 16 \int_0^4 \int_0^{2\pi} du\, dv = 16 \cdot 2\pi \cdot 4 = 128\pi$$

$$\Phi_B = \iint_R (2x, 2y, z^2) \cdot (0,0,-1)\, dR_B$$

$$= \iint -z^2\, dR_B = \iint 0\, dR_B = 0$$

$$\Phi_T = \iint_R (2x, 2y, z^2) \cdot (0,0,1)\, dR_T$$

$$= \iint_{R_T} z^2\, dR_T = 16 \iint_{R_T} dR_T = 16 A_{ellips}$$

$$16\pi ab = 16\pi \cdot 4 \cdot 2 = 128\pi$$

$$\left(\frac{x}{4}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$$

$$\Phi = \Phi_S + \Phi_B + \Phi_T = 256\pi$$

$$\iint_S \vec{F}\cdot\vec{n}\,dA = \iint_R \vec{F}\cdot(\vec{r}_u \times \vec{r}_v)\,du$$

$\vec{v}$: $\iint_S \vec{v}\cdot d\vec{A}$ - volumetric flow rate

$$[\vec{v}] = \frac{L}{T}, \quad [A] = L^2$$

$$[\vec{v}\cdot A] = \frac{L}{T}\cdot L^2 = \frac{L^3}{T}$$

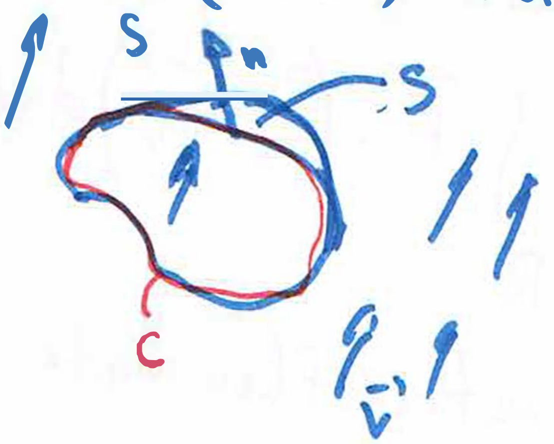
$\rho\vec{v}$: $\iint_S \rho\vec{v}\cdot d\vec{A}$ - mass flow rate

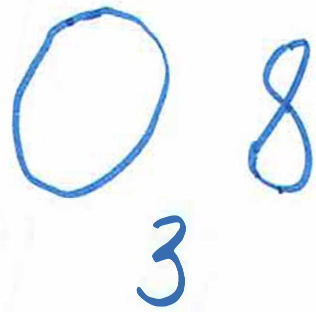
$$[\rho] = \frac{M}{L^3} \qquad [\rho\vec{v}A] = \frac{M}{\cancel{L^3}}\cdot\frac{\cancel{L^3}}{T} = \frac{M}{T}$$

$$\vec{n}\,dA = d\vec{A}$$

Stokes Theorem

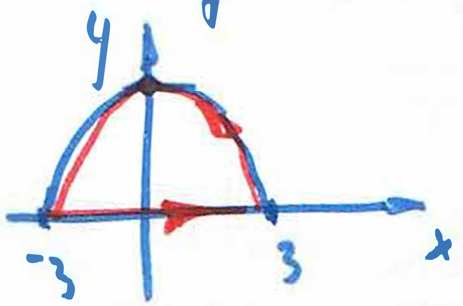
$$\iint_S (\nabla \times \vec{v}) \cdot \vec{n}\, dA = \oint_C \vec{v} \cdot d\vec{r}$$





Example. $\vec{V} = \left(x^2, \frac{1}{y}, z^2\right)$; Γ_c - ?

$C: \quad y = 9 - x^2, \quad -3 \le x \le 3$

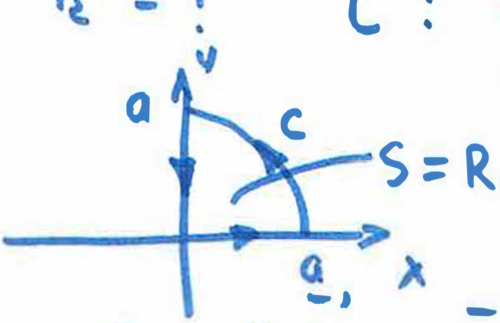


$$\boxed{\Gamma_c = 0}$$

Example. $\vec{V} = (z, xy, x)$ $\quad \Gamma_c$ - ?

$\Gamma_c = ?$ $\quad C: \; x^2 + y^2 = a^2, \quad 0 \le x \le a$

$0 \le y \le a$

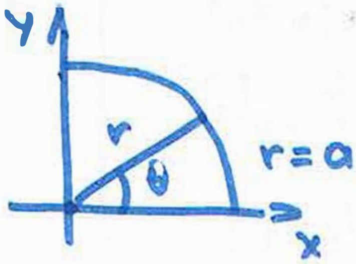


$$\Gamma_c = \oint_C \vec{v} \cdot d\vec{r} = \iint_S (\nabla \times \vec{v}) \cdot \vec{n}\, dA$$

$$\nabla \times \vec{V} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & xy & x \end{vmatrix} = \begin{array}{l} \vec{i}, \; 0 \\ -\vec{j}, \; (1-1) \\ +\vec{k} \; (y-0) \end{array} = (0, 0, y)$$

$\vec{n} = (0,0,1)$

$$\Gamma_c = \iint_R (0,0,y)\cdot(0,0,1)\,dR = \iint_R y\,dR$$



$0 \le r \le a$

$0 \le \theta \le \pi/2$

$y = r\sin\theta$

$dR = r\,dr\,d\theta$

$$\Gamma_c = \int_0^{\pi/2}\int_0^a r\sin\theta\, r\,dr\,d\theta = \int_0^{\pi/2}\int_0^a \sin\theta\, r^2\,dr\,d\theta$$

$$= \int_0^{\pi/2} \sin\theta \left.\frac{r^3}{3}\right|_{r=0}^{r=a} d\theta = \frac{a^3}{3}\int_0^{\pi/2} \sin\theta\,d\theta$$

$$= \frac{a^3}{3}\left(-\cos\theta\right)\Big|_{\theta=0}^{\theta=\pi/2} = \frac{a^3}{3}\left(-0-(-1)\right) = \frac{a^3}{3}$$

Existance of potential

If $\varphi : \vec{F} = \nabla\varphi$, exists $\rightarrow \nabla\times\vec{F} = \nabla\times\nabla\varphi = 0$

Potential Field is irrotacctial.

If $\nabla\times\vec{F}' = 0$, is $\vec{F}' = \nabla\varphi$?

$Q(x,y,z)$

$P(x_0,y_0,z_0)$

$$\varphi_1 = \int_{C_1} \vec{F}\cdot d\vec{r} \;;\quad \varphi_2 = \int_{C_2} \vec{F}\cdot d\vec{r}$$

$C_3 = C_1 + (-C_2)$

$= C_1 - C_2$

$$\oint_{C_3} \vec{F}\cdot d\vec{r} = \varphi_1 - \varphi_2 = \iint_S (\nabla\times\vec{F})\cdot\vec{n}\,d\,\ldots$$

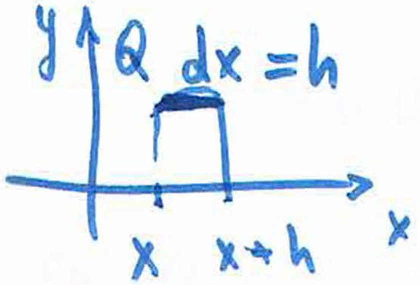
$\boxed{\varphi_1 = \varphi_2}$ $= 0$

$$\varphi_1 = \varphi_2 = \varphi$$

$$\frac{\partial \varphi}{\partial x} = \lim_{h \to 0} \frac{1}{h}\left[\varphi(x+h, y, z) - \varphi(x, y, z)\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\int_{(x_0, y_0, z_0)}^{(x+h, y, z)} \vec{F} \cdot d\vec{r} - \int_{(x_0, y_0, z_0)}^{(x, y, z)} \vec{F} \cdot d\vec{r}\right].$$

$$= \lim_{h \to 0} \frac{1}{h} \int_{(x, y, z)}^{(x+h, y, z)} \vec{F} \cdot d\vec{r} = \lim_{h \to 0} \frac{1}{h} \int_{(x, y, z)}^{(x+h, y, z)} \vec{F} \cdot d\vec{r}$$



$$d\vec{r} = (dx, \overset{0}{\cancel{dy}}, \overset{0}{\cancel{dz}}) = (dx, 0, 0)$$

$$= (h, 0, 0)$$

$$\vec{F} = (F_1, F_2, F_3) \Longrightarrow \vec{F} \cdot d\vec{r} = F_1 h$$

$$\frac{\partial \varphi}{\partial x} = \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} F_1\, dx = \lim_{h \to 0} \frac{1}{\cancel{h}} F_1 \cdot \cancel{h}$$

$$= F_1$$

$$\boxed{\frac{\partial \varphi}{\partial x} = F_1}$$

Green's theorem

Let $\vec{v}$ be 2D : $\vec{v} = (v_1, v_2)$

$v_1 = v_1(x, y)$, $v_2 = v_2(x, y)$

$$\nabla \times \vec{v} = \left(0, 0, \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)$$

The region is 2D $\Longrightarrow$ it is flat $\Longrightarrow$

$\vec{n} = (0, 0, 1)$

$$(\nabla \times \vec{v}) \cdot \vec{n} = \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}$$

$d\vec{r} = (dx, dy)$

$\vec{v} \cdot d\vec{r} = v_1 dx + v_2 dy$

$$\boxed{\iint_R \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right) dA = \oint_C (v_1 dx + v_2 dy)}$$