Matrix diagonalisation

$A^{n\times n}$, $\lambda_1, \lambda_2, \lambda_3, \dots \lambda_n$ distinct eigenvalus

$\Rightarrow$ n linearly independent eigenvector

$$M = \begin{bmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 & \vec{e}_n \end{bmatrix}$$ - modal Matrix of A

$$AM = \begin{bmatrix} \vec{r}_1 \\ \vec{r}_2 \\ \\ \vec{r}_n \end{bmatrix} \begin{bmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_n \end{bmatrix}$$

$$= \begin{bmatrix} A\vec{e}_1 & A\vec{e}_2 & A\vec{e}_n \end{bmatrix} = \begin{bmatrix} \lambda_1 e_1 & \lambda_2 \vec{e}_2 & \dots & \lambda_n \vec{e}_n \end{bmatrix}$$

$$M^{-1}M = I = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdot & 1 \end{bmatrix} = \begin{bmatrix} M^{-1}\vec{e}_1 & M^{-1}\vec{e}_2 & M^{-1}\vec{e}_n \end{bmatrix}$$

$$M^{-1}\vec{e}_1 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}; \quad M^{-1}\vec{e}_2 = \begin{bmatrix}0\\1\\0\\0\\0\\0\end{bmatrix} \;\cdots\; ; M^{-1}\vec{e}_n = \begin{bmatrix}0\\0\\0\\ \\1\end{bmatrix}$$

$$M^{-1}AM = M^{-1}\begin{bmatrix}\lambda_1\vec{e}_1 & \cdots & \lambda_n\vec{e}_n\end{bmatrix} = \begin{bmatrix}\lambda_1\begin{bmatrix}1\\0\\0\\ \vdots\end{bmatrix} & \lambda_2\begin{bmatrix}0\\1\\0\\0\\0\end{bmatrix} & \cdots & \lambda_n\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\end{bmatrix}$$

$$= \begin{bmatrix}\lambda_1 & 0 & & 0\\ 0 & \lambda_2 & & 0\\ 0 & 0 & & 0\\ 0 & 0 & & \lambda_n\end{bmatrix} = \Lambda$$ - spectral matrix of A

$$\boxed{M^{-1}AM = \Lambda}$$ - similarity transformation

A & Λ - are similar matrices

Example: $A = \begin{bmatrix}4 & 5\\ 3 & 2\end{bmatrix}$

$$\begin{vmatrix}4-\lambda & 5\\ 3 & 2-\lambda\end{vmatrix} = (4-\lambda)(2-\lambda) - 15 = \lambda^2 - 6\lambda - 7$$

$$\lambda_{1,2} = \frac{6 \pm \sqrt{36+28}}{2} = 7, -1$$

$\lambda_1 = 7$: $\quad -3x + 5y = 0 \qquad \vec{e}_1 = \begin{bmatrix}x\\y\end{bmatrix}$

$\qquad 3x - 5y = 0$

$$\vec{e}_1 = t\begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

$\lambda_2 = -1$: $\quad 5x + 5y = 0 \qquad \vec{e}_2 = t\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$M = \begin{bmatrix} 5 & 1 \\ 3 & -1 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 7 & 0 \\ 0 & -1 \end{bmatrix}$$

$$M^{-1} A M = \Lambda.$$

$$M = \begin{bmatrix} 1 & 5 \\ -1 & 3 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} -1 & 0 \\ 0 & 7 \end{bmatrix}$$

Example of diagonalising a 3 × 3 matrix

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$$

$$\begin{vmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{vmatrix} = \begin{matrix} -0 \times [\qquad] \\ +(2-\lambda)\left[(1-\lambda)(3-\lambda)+2\right] \\ -2\left[\underbrace{1-\lambda+1}_{2-\lambda}\right] \end{matrix}$$

$$= (2-\lambda)\left[(1-\lambda)(3-\lambda)\right] = 0$$

$\lambda_1 = 2, \quad \lambda_2 = 1, \quad \lambda_3 = 3$

$\lambda_1 = 2:$

$$\begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

$$\vec{e}_1 = t \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$$

$\lambda_2 = 1$

$$\begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \vec{e}_2 = t \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$\lambda_3 = 3: \begin{bmatrix} -2 & 0 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ & & \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & -1 \end{bmatrix}$$

$$\vec{e}_3 = t \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

$$M = \begin{bmatrix} -2 & 1 & -1 \\ 1 & -1 & 1 \\ 2 & 0 & 2 \end{bmatrix} - \text{Modal Matrix}$$

$$M^{-1} A M = \Lambda = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Matrix Powers

$$A^m = \underbrace{A \cdot A \cdot A \dots A}_{m} \quad ; \quad M M^{-1} A M M^{-1} = M \Lambda M^{-1}$$

$$A = M \Lambda M^{-1}$$

$$A^m = \underbrace{M \Lambda M^{-1}}_{A} \underbrace{M}_{I} \underbrace{\Lambda M^{-1}}_{A} \underbrace{M}_{I} \Lambda \dots \underbrace{M^{-1} M}_{I} \underbrace{\Lambda M^{-1}}_{A}$$

$$= M \Lambda^m M^{-1}$$

$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}^2 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{bmatrix}$$

$$\Lambda^m = \begin{bmatrix} \lambda_1^m & & \\ & \lambda_2^m & \\ & & \lambda_n^m \end{bmatrix}$$

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} \lambda_1^m & 0 \\ 0 & \lambda_2^m \end{bmatrix} = \begin{bmatrix} m_{11}\lambda_1^m & m_{12}\lambda_2^m \\ m_{21}\lambda_1^m & m_{22}\lambda_2^m \end{bmatrix}$$

Example

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}; \quad M = \begin{bmatrix} -2 & 1 & -1 \\ 1 & -1 & 1 \\ 2 & 0 & 2 \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \qquad A^4 = ?$$

Solution: Need M^{-1}

1. $\det M = \begin{vmatrix} -2 & 1 & -1 \\ 1 & -1 & 1 \\ 2 & 0 & 2 \end{vmatrix} = \begin{matrix} 2 \times [1-1] \\ -0 \times [\quad] \\ +2 \times [2-1] \end{matrix} = 2$

2. Compute minors

$M_{11} = -2, \quad M_{12} = 2-2=0, \quad M_{13} = 2$

$M_{21} = 2, \quad M_{22} = -4+2=-2, \quad M_{23} = -2$

$M_{31} = 0, \quad M_{32} = -2+1=-1, \quad M_{33} = 2-1 = 1$

3. Cofactors

$$C = \begin{bmatrix} -2 & 0 & 2 \\ -2 & -2 & +2 \\ 0 & +1 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 2 \\ -2 & -2 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

4. $M^{-1} = \frac{C^T}{\det M} = \frac{1}{2}\begin{bmatrix} -2 & -2 & 0 \\ 0 & -2 & 1 \\ 2 & 2 & 1 \end{bmatrix}$

$A^4 = M \Lambda^4 M^{-1}$

$= \frac{1}{2}\begin{bmatrix} -2 & 1 & -1 \\ 1 & -1 & 1 \\ 2 & 0 & 2 \end{bmatrix}\begin{bmatrix} 16 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 81 \end{bmatrix}\begin{bmatrix} -2 & -2 & 0 \\ 0 & -2 & 1 \\ 2 & 2 & 1 \end{bmatrix}$

$= \frac{1}{2}\begin{bmatrix} -32 & 1 & -81 \\ 16 & -1 & 81 \\ 32 & 0 & 162 \end{bmatrix}\begin{bmatrix} -2 & -2 & 0 \\ 0 & -2 & 1 \\ 2 & 2 & 1 \end{bmatrix}$

$= \frac{1}{2}\begin{bmatrix} 64-162 & 64-2-162 & 1-81 \\ & & \\ & & \end{bmatrix} = \begin{bmatrix} -49 & -50 & -40 \\ & & \\ & & \end{bmatrix}$

$m \geq 3$

The Cayley-Hamilton Theorem

Every matrix satisfies its own characteristic equation.

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}; \quad \det(A) = a_{11}a_{22} - a_{12}a_{21}$$

$$Tr(A) = a_{11} + a_{22}$$

$$\begin{vmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{vmatrix} = (a_{11}-\lambda)(a_{22}-\lambda) - a_{12}a_{21} =$$

$$= \lambda^2 \underbrace{-(a_{11}+a_{22})}_{b}\lambda + \underbrace{a_{11}a_{22} - a_{12}a_{21}}_{c} =$$

$c \equiv \det(A)$

$b \equiv -Tr(A)$

$$= \lambda^2 + b\lambda + c = 0$$

$$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$$

$$\lambda_1 + \lambda_2 = -b = +Tr(A) \Longrightarrow b = -Tr(A)$$

$$\lambda_1 \cdot \lambda_2 = \frac{b^2 - \left(\sqrt{b^2 - 4c}\right)^2}{4} = \frac{b^2 - b^2 + 4c}{4} = c = \det(A)$$

Start with $\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

$$\Lambda^2 + b\Lambda + cI = 0$$

$$\begin{bmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{bmatrix} + \begin{bmatrix} b\lambda_1 & 0 \\ 0 & b\lambda_2 \end{bmatrix} + \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix} =$$

$$= \begin{bmatrix} \lambda_1^2 + b\lambda_1 + c & 0 \\ 0 & \lambda_2^2 + b\lambda_2 + c \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbb{O}$$

$$M^{-1} A M = \Lambda \quad \Rightarrow \quad A = M \Lambda M^{-1}$$

$$\Lambda^2 + b\Lambda + cI = \Lambda\Lambda + b\Lambda + cI$$

$$= M^{-1} A \underbrace{M M^{-1}}_{=I} A M + b M^{-1} A M + c M^{-1} I M$$

$$= M^{-1} A^2 M + M^{-1} b A M + M^{-1} c I M =$$

$$= \underbrace{M M^{-1}}_{I} (A^2 + bA + cI) \underbrace{M M^{-1}}_{I} = \underbrace{M \mathbb{O} M^{-1}}_{\mathbb{O}}$$

$$A^2 + bA + cI = \mathbb{O}$$

Corollary: $A^2 - Tr(A)A + det(A)I = \mathbb{O}$

$$A^2 = Tr(A)A - det(A)I$$

$$A^3 = Tr(A)A^2 - det(A)A =$$

$$= Tr(A)\left[Tr(A)A - det(A)I\right] - det(A)A$$

$$= Tr(A)^2 A - det(A)A - Tr(A)det(A)I$$

$$= \left(T(A)^2 - det(A)\right)A - Tr(A)det(A)I$$

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$A^7 - ?\quad A = \begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix}$

$$\begin{vmatrix} -1-\lambda & 2 \\ -3 & 4-\lambda \end{vmatrix} = -(1+\lambda)(4-\lambda) + 6 = +\lambda^2 - 3\lambda + 2 = 0$$

$$\lambda^2 - 3\lambda + 2 = 0$$

$$\lambda_{1,2} = \frac{3 \pm \sqrt{9-8}}{2} = 2, 1$$

$$A^7 = \alpha_1 A + \alpha_2 I \Rightarrow \begin{cases} 2^7 = 2\alpha_1 + \alpha_2 \\ 1 = \alpha_1 + \alpha_2 \end{cases} \qquad 2^7 = a$$

$$2^7 - 1 = \alpha_1, \quad \alpha_2 = 2^7 - 2\alpha_1 = 2^7 - 2 \cdot 2^7 + 2$$

$$= 2^7(1-2) + 2 = 2 - 2^7$$

$$A^7 = (2^7 - 1)\begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix} + (2 - 2^7)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 - 2^7 + 2 - 2^7 & 2(2^7 - 1) \\ -3(2^7 - 1) & 4(2^7 - 1) + (2 - 2^7) \end{bmatrix} =$$

$$\cancel{\alpha_1 = 2^7 - 1}$$

$$\cancel{\alpha_2 = 2 - 2^7}$$

$$\left.\begin{aligned} A^2 &= \alpha_1 A + \alpha_2 I \\ A^3 &= \alpha_1 A + \alpha_2 I \end{aligned}\right\} \text{ structurally}$$

$$\boxed{A^m = \alpha_1 A + \alpha_2 I} \qquad \left\{\begin{aligned} \lambda_1^m &= \alpha_1 \lambda_1 + \alpha_2 \\ \lambda_2^m &= \alpha_1 \lambda_2 + \alpha_2 \end{aligned}\right.$$

In particular :

$$\left(A^2 - Tr(A)\,A + det(A)\,I\right)A^{-1} = 0$$

$$A - Tr(A)\,I + det(A)\,A^{-1} = 0$$

$$A^{-1} = \frac{Tr(A)\,I - A}{det(A)}$$

$$\boxed{\alpha_1 = \frac{\lambda_2^m - \lambda_1^m}{\lambda_2 - \lambda_1}} \Rightarrow \alpha_2 = \lambda_1^m - \alpha_1 \lambda_1$$

$$\boxed{\alpha_2 = \frac{\lambda_1^m \lambda_2 - \lambda_2^m \lambda_1}{\lambda_2 - \lambda_1}}$$

If $\lambda_1 = \lambda_2 = \lambda$ then

$$\frac{d}{d\lambda}\left(\lambda^m = \alpha_1 \lambda + \alpha_2\right) \Rightarrow \boxed{m\lambda^{m-1} = \alpha_1}$$

$$\lambda^m = m\,\lambda^{m-1}\cdot\lambda + \alpha_2 \Rightarrow \alpha_2 = \lambda^m - m\lambda^{m-1+1}$$

$$\boxed{\alpha_2 = \lambda^m(1-m)}$$

$$\begin{cases} a = 2\alpha_1 + \alpha_2 \\ 1 = \alpha_1 + \alpha_2 \end{cases} \qquad \begin{aligned} a - 1 &= \alpha_1 \\ 1 - \alpha_1 &= 1 - a + 1 = 2 - a = \alpha_2 \end{aligned}$$

$$A^7 = (a-1)\begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix} + (2-a)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-a+2-a & 2(a-1) \\ -3(a-1) & 4(a-1)+2-a \end{bmatrix} = \begin{bmatrix} 3-2a & 2(a-1) \\ 3(1-a) & 3a-2 \end{bmatrix}$$

$$= \begin{bmatrix} 3-2^8 & 2(2^7-1) \\ 3(1-2^7) & 3\cdot 2^7 - 2 \end{bmatrix}$$

A^{15} -?, $\quad A = \begin{bmatrix} 2 & 0 \\ 4 & 2 \end{bmatrix} \Rightarrow \lambda_1 = \lambda_2 = \lambda = 2$

$$A^{15} = \alpha_1 A + \alpha_2 I \implies \lambda^{15} = \alpha_1 \lambda + \alpha_2$$

$$\begin{cases} 2^{15} = 2\alpha_1 + \alpha_2 \\ 15 \cdot 2^{14} = \alpha_1 \end{cases} \qquad 15\lambda^{14} = \alpha_1$$

$$a = 2^{14}, \quad 2^{15} = 2a \Rightarrow \begin{cases} 2a = 2\alpha_1 + \alpha_2 \\ \alpha_1 = 15a \\ \alpha_2 = 2a - 2\cdot(15a) \\ \quad = -28a \end{cases}$$

$$A^{15} = 15a\begin{bmatrix} 2 & 0 \\ 4 & 2 \end{bmatrix} - 28a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 30a - 28a & 0 \\ 60a & 30a - 28a \end{bmatrix} = \begin{bmatrix} 2a & 0 \\ 60a & 2a \end{bmatrix}$$

$$= 2a\begin{bmatrix} 1 & 0 \\ 30 & 1 \end{bmatrix} = 2^{15}\begin{bmatrix} 1 & 0 \\ 30 & 1 \end{bmatrix}$$

Matrix Functions

Taylor Series: (Mclaurent)

$$f(x) = f(0) + f'(0)\,x + f''(0)\,\frac{x^2}{2} + \dots$$

$$+ f^{(n)}(0)\,\frac{x^n}{n!} + \dots = \sum_{n=0}^{\infty} f^{(n)}(0)\,\frac{x^n}{n!}$$

$x = \lambda t$

$$f(\lambda t) = \sum_{n=0}^{\infty} f^{(n)}(0)\,\frac{\lambda^n t^n}{n!} = \underbrace{\sum_{n=0}^{\infty} a_n(t)\,\lambda^n}_{\text{Power series}}$$

λ is eigenvalue of A

$$f(At) = \sum_{n=0}^{\infty} a_n(t)\,A^n$$

$A^n = \tilde{\alpha}_{1n} A + \tilde{\alpha}_{0n} I$ (for 2×2 matrix)

$$\sum_{n=0}^{\infty} a_n(t)\left[\tilde{\alpha}_{1n} A + \tilde{\alpha}_{0n} I\right] =$$

$$= A \underbrace{\sum_{n=0}^{\infty}\left(a_n(t)\tilde{\alpha}_{1n}\right)}_{\alpha_1(t)} + I \cdot \underbrace{\sum_{n=0}^{\infty}\left(a_n(t) \cdot \tilde{\alpha}_{0n}\right)}_{\alpha_0(t)}$$

$$= \alpha_1(t) A + \alpha_0(t) I = f(At)$$

<u>Example.</u>

$$e^{At} = \alpha_1(t) A + \alpha_0(t) I \quad , \quad \lambda \text{ is e.v. of } A$$

$$e^{\lambda t} = \alpha_1(t)\lambda + \alpha_0(t)$$

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \Rightarrow \begin{vmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) - 8$$

$$= \lambda^2 - 4\lambda - 5 = 0$$

$$\lambda_{1,2} = \frac{4 \pm \sqrt{16+20}}{2} = 5, -1$$

$$\begin{cases} e^{5t} = 5\alpha_1 + \alpha_0 \\ e^{-t} = -\alpha_1 + \alpha_0 \end{cases} \qquad e^{5t} = a \qquad e^{-t} = b$$

$$\begin{cases} a = 5\alpha_1 + \alpha_0 \\ b = -\alpha_1 + \alpha_0 \end{cases} \Rightarrow a - b = 6\alpha_1 \Rightarrow \frac{a-b}{6} = \alpha_1$$

$$a - 5\alpha_1 = a - \frac{5}{6}(a-b) = $$

$$= \frac{a}{6} + \frac{5b}{6} = \frac{a+5b}{6} = \alpha_0$$

$$e^{At} = \frac{a-b}{6}\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} + \frac{a+5b}{6}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \frac{1}{6}\left\{\begin{bmatrix} a-b & 2a-2b \\ 4a-4b & 3a-3b \end{bmatrix} + \begin{bmatrix} a+5b & 0 \\ 0 & a+5b \end{bmatrix}\right\}$$

$$= \frac{1}{6}\begin{bmatrix} 2a+4b & 2a-2b \\ 4a-4b & 4a+2b \end{bmatrix} = \frac{1}{3}\begin{bmatrix} a+2b & a-b \\ 2a-2b & 2a+b \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} e^{5t}+2e^{-t} & e^{5t}-e^{-t} \\ 2e^{5t}-2e^{-t} & 2e^{5t}+e^{-t} \end{bmatrix}$$

$$t=0: \quad e^{A\cdot 0} = e^{0} = \frac{1}{3}\begin{bmatrix} 1+2 & 1-1 \\ 2-2 & 2+1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Example: $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $\sin(At)$

$$\sin(At) = \alpha_1(t)A + \alpha_0(t)I$$

$$\begin{vmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 = (1-\lambda)(1-\lambda) = 0$$

$$\lambda_{1,2} = 1$$

$$\sin(\lambda t) = \alpha_1(t)\lambda + \alpha_0(t) \quad ; \quad t\cos(\lambda t) = \alpha_1(t)$$

$$\sin t = \alpha_1(t) + \alpha_0(t) \qquad t\cos(t) = \alpha_1(t)$$

$$\sin t = t\cos(t) + \alpha_0(t)$$

$$\alpha_0(t) = \sin(t) - t\cos(t)$$

$$\sin(At) = t\cos t \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + (\sin(t) - t\cos(t))\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} t\cos t & t\cos t \\ 0 & t\cos t \end{bmatrix} + \begin{bmatrix} \sin(t) - t\cos(t) & 0 \\ 0 & \sin(t) - t\cos(t) \end{bmatrix}$$

$$\begin{bmatrix} \sin(t) & t\cos(t) \\ 0 & \sin(t) \end{bmatrix} = \sin(At)$$

1. If $\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, then $f(\Lambda t) = \begin{bmatrix} f(\lambda_1 t) & 0 \\ 0 & f(\lambda_2 t) \end{bmatrix}$

2. If two matrices have identical eigenvalues, then expansion of their powers have identical coefficients.

Example: $\cos(At)$, $A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$

$$\begin{vmatrix} 1-\lambda & 3 \\ 2 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda) - 6 = \lambda^2 - 3\lambda - 4 = 0$$

$$\lambda_{1,2} = \frac{3 \pm \sqrt{9+16}}{2} = 4, -1$$

$\lambda_1 = 4$: $\begin{bmatrix} -3 & 3 \\ 2 & -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -1 \end{bmatrix} \Rightarrow \vec{e}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} t$

$\lambda_2 = -1$ $\begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 3 \end{bmatrix} \Rightarrow \vec{e}_2 = \begin{bmatrix} 3 \\ -2 \end{bmatrix} t$

$$M = \begin{bmatrix} 1 & 3 \\ 1 & -2 \end{bmatrix}, \quad M^{-1} = \begin{bmatrix} -2 & -3 \\ -1 & 1 \end{bmatrix} \frac{1}{(-5)} =$$

$$= \frac{1}{5} \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}$$

$$\underline{\Lambda} = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}$$

$$M^{-1} A M = \Lambda \Rightarrow A = M \Lambda M^{-1}$$

$$\cos(At) = \alpha_1 A + \alpha_0 I = \alpha_1 M \Lambda M^{-1} + \alpha_0 M I M^{-1}$$

$$= M (\alpha_1 \Lambda + \alpha_0 I) M^{-1} = M \cos(\underline{\Lambda} t) M^{-1}$$

$$= M \begin{bmatrix} \cos(4t) & 0 \\ 0 & \cos(t) \end{bmatrix} M^{-1}$$

$$= \frac{1}{5} \begin{bmatrix} 1 & 3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} \cos(4t) & 0 \\ 0 & \cos t \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} \cos(4t) & 3\cos t \\ \cos(4t) & -2\cos t \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} 2\cos(4t) + 3\cos t & 3\cos(4t) - 3\cos t \\ 2\cos(4t) - 2\cos t & 3\cos(4t) + 2\cos t \end{bmatrix}$$

Generalisation to larger matrices.

2×2:
$$\lambda^2 + b\lambda + c = 0$$
$$A^2 + bA + c\underline{I} = 0$$
$$A^2 = -bA - c\underline{I} = \alpha_1 A + \alpha_0 \underline{I}$$
$$\alpha_1 = -b, \quad \alpha_0 = c$$

3×3:
$$\lambda^3 + b\lambda^2 + c\lambda + d = 0$$
$$A^3 + bA^2 + cA + d\underline{I} = 0$$
$$A^3 = -bA^2 - cA - d\underline{I}$$
$$A^3 = \alpha_2 A^2 + \alpha_1 A + \alpha_0 \underline{I}$$
$$A\cdot A = [A][A]$$

$m\times m$: $A^m = \alpha_{m-1}A^{m-1} + \alpha_{m-2}A^{m-2} + \dots + \alpha_0 \underline{I}$