Singularities and residues.

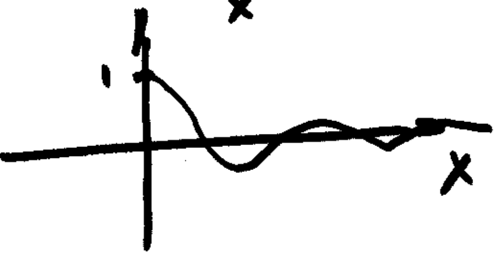
$$f(z) = \frac{\sin z}{z} = \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} \cdots\right)$$

$$= 1 - \frac{z^2}{6} + \frac{z^4}{120} - \cdots$$

$$f(z) = \begin{cases} \frac{\sin z}{z}, & z \neq 0 \\ 1, & z = 0 \end{cases}$$

Example of a <u>removeable</u> singularity.

$\frac{\sin x}{x}$



principal part residu

$$f(z) = \cdots + \boxed{\frac{c_{-n}}{(z-z_0)^n} + \frac{c_{-n+1}}{(z-z_0)^{n-1}} \cdots + \frac{c_{-1}}{z-z_0}} + c_0$$

$$+ c_1(z-z_0) + c_2(z-z_0)^2 + \cdots$$

$$\frac{\sin z}{z^3} = \frac{1}{z^2} - \frac{1}{6} + \frac{z^2}{120} \cdots$$

pole of degree 2

$$\frac{\sin z}{z^2} = \frac{1}{z} - \frac{z}{6} + \frac{z^3}{120} - \cdots$$

pole of degree 1 $\equiv$ simple pole

The coefficient divided by z (multiplying z^{-1} or $(z-z_0)^{-1}$ is residue.

How to find $c_{-1} = \text{Res}\,(f(z), z_0)$

a) Simple pole

$$f(z) = \frac{c_{-1}}{z - z_0} + c_0 + c_1 (z - z_0) + \ldots$$

$$c_{-1} = (z - z_0)\, f(z) - c_0 (z - z_0) - c_1 (z - z_0)^2 - \ldots$$

$$\boxed{c_{-1} = \lim_{z \to z_0} (z - z_0)\, f(z)} \quad \boxed{m = 1}$$

$$f(z) = \frac{\sin z}{z^2} \qquad z_0 = 0 \text{ - singular point}$$

$$c_{-1} = \lim_{z \to 0} \cancel{z} \cdot \frac{\sin z}{z^{\cancel{2}}} = \lim_{z \to 0} \frac{\sin z}{z} = 1$$

$$\frac{\sin z}{z^2} = \frac{\overset{c_{-1}}{1}}{z} + \ldots$$

$$\text{Res}\left(\frac{\sin z}{z^2}, 0\right) = c_{-1} = 1$$

<u>Example</u>: $f(z) = \dfrac{2z}{(z^2+1)(2z-1)}$

$z^2 + 1 = 0 \implies z^2 = -1 \implies z = \pm i$

$2z - 1 = 0 \implies 2z = 1 \implies z = 1/2$

$$f(z) = \frac{\cancel{2} z}{\cancel{2}(z+i)(z-i)(z - 1/2)} = \frac{z}{(z+i)(z-i)(z-1/2)}$$

$$= \frac{A}{z+i} + \frac{B}{z-i} + \frac{C}{z - 1/2}$$

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$f(z)$ has <u>simple</u> poles at $z = i$, $z = -i$
$z = 1/2$

$$\mathrm{Res}\left(f(z), i\right) = \lim_{z \to i} \cancel{(z-i)} \frac{z}{(z+i)\cancel{(z-i)}(z-1/2)}$$

$$= \frac{\cancel{i}}{2\cancel{i}\,(i - 1/2)} = \frac{1}{2i-1} = \frac{-1-2i}{5}$$

$$\mathrm{Res}\left(f(z), -i\right) = \lim_{z \to -i} \cancel{(z+i)} \frac{z}{\cancel{(z+i)}(z-i)(z-1/2)}$$

$$= \frac{+\cancel{i}}{+2\cancel{i}\,(-i-1/2)} = \frac{1}{-2i-1} = \frac{-1+2i}{5}$$

$$\mathrm{Res}\left(f(z), 1/2\right) = \lim_{z \to 1/2} \cancel{(z-1/2)} \frac{z}{(z+i)(z-i)\cancel{(z-1/2)}}$$

$$= \lim_{z \to 1/2} z \Big/ (z^2+1) = \frac{1/2}{\frac{1}{4}+1} = \frac{\frac{1}{2}\cdot 4}{5} = \frac{2}{5}$$

Example $f(z) = \dfrac{c_{-2}}{(z-z_0)^2} + \dfrac{c_{-1}}{z-z_0} + c_0 + \dots$

$$(z-z_0)^2 f(z) = c_{-2} + c_{-1}(z-z_0) + c_0(z-z_0)^2 + \dots$$

$\cancel{2(z-z_0)f(z)} -$

$$\frac{d}{dz}\left[(z-z_0)^2 f(z)\right] = c_{-1} + 2c_0(z-z_0) + \dots$$

$$\boxed{c_{-1} = \lim_{z \to z_0} \frac{d}{dz}(z-z_0)^2 f(z)}$$ - Pole of degree 2

What if pole is of degree m?

$$f(z) = \frac{c_{-m}}{(z-z_0)^m} + \frac{c_{-m+1}}{(z-z_0)^{m-1}} \dots + \frac{c_{-1}}{z-z_0} + \dots$$

$$c_{-1} = \lim_{z \to z_0} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right]$$

$$(z-z_0)^m f(z) = c_{-m} + c_{-m+1}(z-z_0) + \dots + c_{-1}(z-z_0)^{m-1} + c_0(z-z_0)^m$$

$$\frac{d}{dz}(z-z_0)^m f(z) = c_{-m+1} + \dots + (m-1)c_{-1}(z-z_0)^{m-2}$$

$$\frac{d^2}{dz^2}(z-z_0)^m f(z) = + \dots (m-2)(m-1)\, c_{-1}(z-z_0)^{m-3}$$

$$c_{-1} = \lim_{z \to z_0} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left[(z-z_0)^m f(z) \right]$$

<u>Example</u>: $f(z) = \dfrac{e^z}{(z+1)^2}$

$z+1=0 \Rightarrow z=-1$ is a pole of degree 2. $m=2$

$$c_{-1} = \operatorname{Res}(f(z),-1) = \lim_{z\to -1} \frac{1}{(2-1)!} \frac{d}{dz} \left[\cancel{(z+1)^2} \frac{e^z}{\cancel{(z+1)^2}} \right]$$

$$= \lim_{z\to -1} \frac{d}{dz} e^z = \lim_{z \to -1} e^z = e^{-1}$$

<u>Example</u>: $f(z) = \dfrac{z^5}{(z+1)^4}$, $z+1=0$, $z_0=-1$

Pole of deg. $m=4$

$$c_{-1} = \operatorname{Res}(f(z),-1) = \lim_{z \to z_0} \left[\frac{1}{3!} \frac{d^3}{dz^3} z^5 \right]$$

$$= \frac{1}{6}\left(5z^4\right)'' = \frac{5}{6}\left(4z^3\right)' = \frac{10}{\cancel{3}} \cancel{3} z^2 = 10z^2$$

as $z \to -1$

$$c_{-1} = 10 \cdot (-1)^2 = 10$$

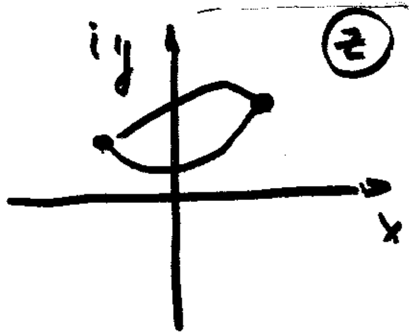
$f(z) = e^{1/z}$ $\qquad$ $z=0$ is a singular point

$= 1 + \dfrac{1}{z} + \dfrac{1}{2!z^2} + \dfrac{1}{3!z^3} + \cdots$

$z=0$ is essential singularity.

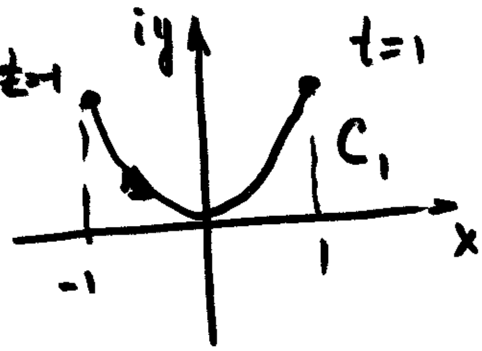
$\mathrm{Res}\left(\cancel{f(z)}, e^{1/z}, 0\right) = 1$

Contour integration in a complex plane



$$z = x(t) + i\,y(t)$$

Real plan $y = x^2$: $\vec{r}(t) = (x, y) = (t, t^2)$



$x = \text{Real}(z)$, $y = \text{Im}(z)$

$z = x(t) + i\,y(t)$

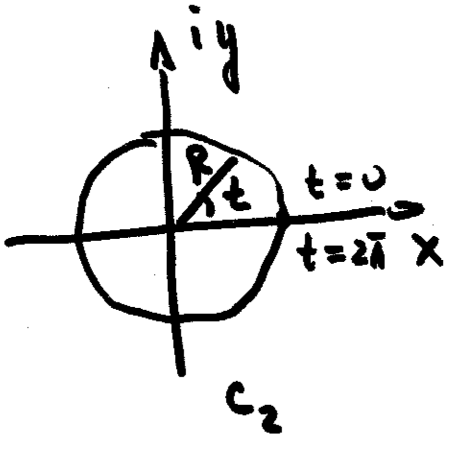
$x = t$

$y = f(t) = x^2 = t^2$

C_1: $z(t) = t + i t^2 \quad -1 \le t \le 1$

Connects: $z(-1) = -1 + i$ and

$z(1) = 1 + i$



Real plain $x = R\cos t$
$y = R\sin t$

complex plane:

$z(t) = x(t) + i\,y(t)$

$= R\cos t + i R\sin t$

$= R(\cos t + i\sin t) = R e^{it}$

$0 \le t \le 2\pi$

$$\boxed{\int_a^b f(z(t))\,\frac{dz}{dt}\,dt = \int_C f(z)\,dz}$$

Example: $f(z) = z^2 \qquad c_1 : z(t) = t + it^2$

$$f(z) = (t+it^2)^2 = t^2 + 2it^3 - t^4 = 2it^3 + t^2 - t^4$$

$$\frac{dz}{dt} = 1 + 2it$$

$$\int_{c_1} f(z)\,dz = \int_{-1}^{1} (t^2 - t^4 + 2it^3)(1+2it)\,dt$$

$$= \int_{-1}^{1} (t^2 - t^4 + 2it^3 + 2it^3 - 2it^5 - 4t^4)\,dt$$

$$= \int_{-1}^{1} (t^2 - 5t^4 + 4it^3 - 2it^5)\,dt$$

$$= \frac{t^3}{3} - t^5 + it^4 - \frac{\cancel{2}it^6}{\cancel{6}3} \Bigg|_{-1}^{1}$$

$$= \frac{1}{3} - 1 + \cancel{i} - \cancel{\frac{i}{3}} - \left(-\frac{1}{3} + 1 + \cancel{i} - \cancel{\frac{i}{3}}\right)$$

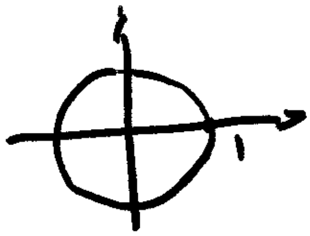
$$= \frac{2}{3} - 2 = \frac{2-6}{3} = -\frac{4}{3}$$

Example: $f(z) = z^2 = \cancel{t^2 - t^4 + 2it^3}\ R^2 e^{2it}$

$$z = Re^{it}, \quad \frac{dz}{dt} = iRe^{it}$$

$$\oint_{c_2} f(z)\,dz = \int_0^{2\pi} R^2 e^{2it} \cdot iRe^{it}\,dt = iR^3 \int_0^{2\pi} e^{3it}\,dt$$

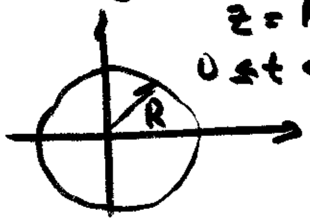
$$= \cancel{i}R^3 \frac{e^{3it}}{3\cancel{i}} \Bigg|_0^{2\pi} = \frac{R^3}{3}\left(\underbrace{e^{6i\pi}}_{1} - \underbrace{e^0}_{1}\right) = 0$$



Cauchy's Theorem

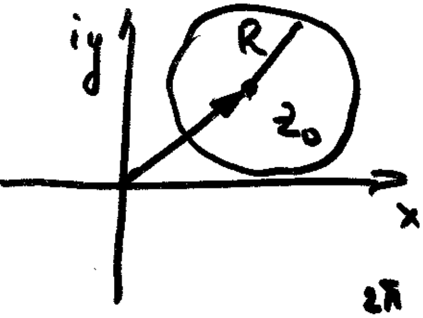
$$\oint_C z^2 \, dz = 0$$

Example. $\oint_C f(z)\,dz$, where



$f(z) = (z - z_0)^n$, $n \neq -1$, n integer

C: a circle of radius R centered at z_0.



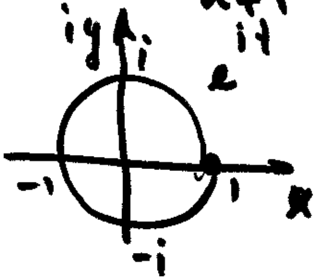
$$z(t) = z_0 + Re^{it}$$

$$u = z - z_0 = R e^{it} \qquad \frac{du}{dt} = \frac{dz}{dt} = iRe^{it}$$

$$\oint_C u^n \, du = \int_0^{2\pi} iR\,(Re^{it})^n e^{it} \, dt$$

$$= iR^{n+1} \int_0^{2\pi} e^{it(n+1)} dt = \frac{iR^{n+1}}{i(n+1)} e^{it(n+1)} \Big|_{t=0}^{t=2\pi}$$

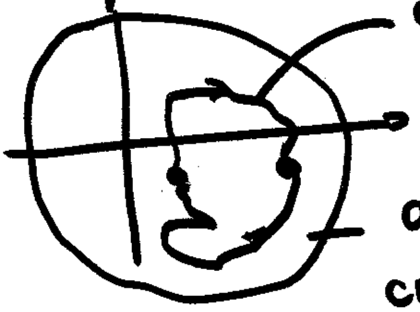
$$= \frac{R^{n+1}}{n+1}\left(e^{i(n+1)2\pi} - 1\right) = \frac{R^{n+1}}{n+1}(1-1) = 0.$$



Let $f(z)$ be analytic at z_0 $\Rightarrow$ CRC are satisfied $\Rightarrow$ $f(z)$ is differentiable $\Rightarrow$ infinitely many times $\Rightarrow$ $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$

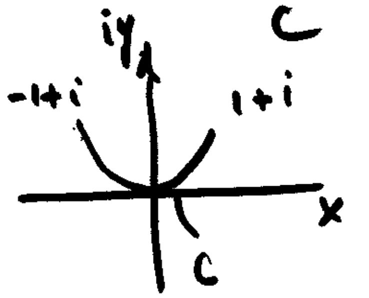
$$\oint_C f(z)\,dz = \int_0^{2\pi} f(t) \frac{dz}{dt} dt = \oint_C \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} \oint_C (z - z_0)^n dz = 0$$



analycity region (the region of convergence of Taylor series representation of $f(z)$)

$$\oint_C f(z)\, dz = 0$$



$$I = \int_C f(z)\, dz \quad , \text{ where } f(z) = z^2$$

$$I = -\frac{4}{3}$$

$f(z) = z^2$ - analytic $\Rightarrow$ sat. CRC

$$\int_{-1+i}^{1+i} z^2 dz = \frac{z^3}{3}\Bigg|_{-1+i}^{1+i} = \frac{(1+i)^3 - (-1+i)^3}{3}$$

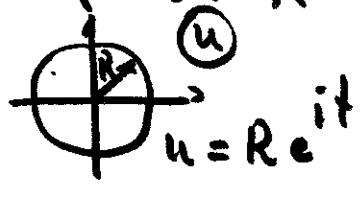
$$= \left[1 + 3i - 3 + i^3 - ((-1)^3 + 3i + 3 + i^3)\right] / 3$$

$$(a+b)^3 = a^3 + 3a^2 b + 3ab^2 + b^3$$

$$= \frac{1 + \cancel{3i} - 3 + \cancel{i^3} + 1 - \cancel{3i} - 3 - \cancel{i^3}}{3} = \boxed{\frac{-4}{3}}$$

Example. $\oint_C f(z) dz$, where $f(z) = \frac{1}{z - z_0}$

$$\oint_{|z-z_0|=R} \frac{dz}{z - z_0} = \left\{ \begin{array}{l} u = z - z_0 \\ z = z_0 + Re^{it} \\ 0 \le t < 2\pi \end{array} \right\} \int_{|u|=R} \frac{du}{u} = \int_0^{2\pi} \frac{i \cancel{R} \cancel{e^{it}} dt}{\cancel{R} \cancel{e^{it}}}$$



$$= i \int_0^{2\pi} dt = 2\pi i \neq 0$$

Cauchy's integral formula

$$f(z) = f(z_0) + f'(z_0)(z-z_0) + \frac{f''(z_0)}{2!}(z-z_0)^2 + \dots$$

$$\frac{f(z)}{z-z_0} = \frac{f(z_0)}{z-z_0} + f'(z_0) + \frac{f''(z_0)}{2!}(z-z_0) + \dots$$

$$\oint_C \frac{f(z)}{z-z_0}\,dz = \oint_C \frac{f(z_0)}{z-z_0}\,dz + \oint_C \dots\dots$$

$$\oint \frac{f(z)}{z-z_0}\,dz = f(z_0)\oint_C \frac{dz}{z-z_0} + \underbrace{\oint \dots}_{\Rightarrow 0}$$

$$= 2\pi i\, f(z_0)$$

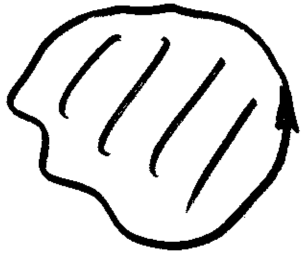
$$= 2\pi i\, C_{-1} = 2\pi i \,\mathrm{Res}\left(\frac{f(z)}{z-z_0}, z_0\right)$$

$$\boxed{\oint_C \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0)}$$

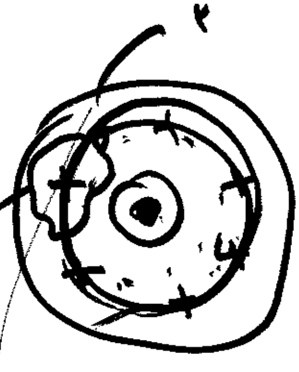
$$\frac{f(z)}{(z-z_0)^{n+1}} = \frac{f(z_0)}{(z-z_0)^{n+1}} + \frac{f'(z_0)}{(z-z_0)^{n}} + \dots + \frac{f^{(n)}(z_0)}{n!(z-z_0)} + \dots$$

$$\boxed{\oint_C \frac{f(z)\,dz}{(z-z_0)^{n+1}} = 2\pi i \frac{f^{(n)}(z_0)}{n!}}$$

CIF

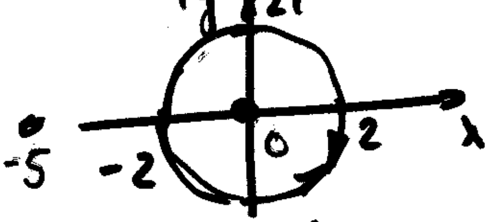


$\int_c \nabla \times F\, dc$



<u>Example</u>:

$$I = \oint_{|z|=2} \frac{\cos z}{(z+5)z}\, dz$$



a) $z_0 = 0$
$n+1 = 1 \Rightarrow n = 0$
$f(z) = \dfrac{\cos z}{z+5}$

$$I = \frac{2\pi i}{0!} \frac{\cos 0}{(0+5)} = \frac{2\pi i \cdot 1}{1 \cdot 5} = \frac{2\pi i}{5}$$

?

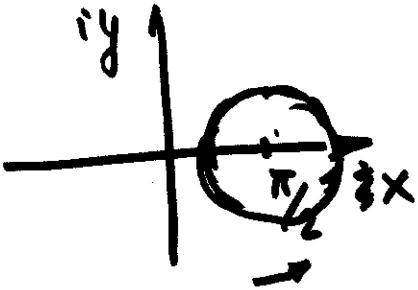
b) $\cancel{z_0 = -5}$
$\cancel{n+1 = 1 \Rightarrow n=0}$
$\cancel{f(z) = \dfrac{\cos z}{z}}$

$$\cancel{I = 2\pi i \frac{\cos(-5)}{-5}} = -\frac{2\pi i \cos 5}{5}$$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \quad \{ z = iy \}$$

$$= \frac{e^{iiy} + e^{-iiy}}{2} = \frac{e^{-y} + e^{y}}{2}$$

Example:
$$I = \oint_C \frac{\cos(2z)\, dz}{(z - \pi/2)^5}$$



$z_0 = \pi/2$

$n = 4$

$f(z) = \cos(2z)$

$$I = \frac{2\pi i}{4!} \cos^{(4)}(2 z_0) = \frac{2\pi i}{4!} 2^4 \cos(2z)\Big|_{z=\pi/2}$$

$$= \frac{2 \cdot 16 \pi i}{24} \cos(\pi) = \boxed{-\frac{4}{3}\pi i}$$