MTH20014 Mathematics 3B. Tutorial 12 (including problems for independent preparation for the final examination)

1. Classify the singularities of the following functions:
   (a) $\dfrac{\sinh z}{z}$; (b) $\dfrac{z+1}{z^2}$; (c) $\dfrac{\cosh z}{z^3}$; (d) $e^{\frac{1}{z}}$.

2. Calculate the residues at the indicated poles
   (a) $\dfrac{\cos z}{z}$, $\quad z = 0$;
   (b) $\dfrac{z}{\sin z}$, $\quad z = \pi$;
   (c) $\dfrac{1}{(1+z^2)^2}$, $\quad z = \mathrm{i}$.

3. Evaluate $\oint_C \dfrac{2z}{z^2+4}\,\mathrm{d}z$, where C is the circle $|z| = 1$, using a suitable path parameterisation.

4. Evaluate $\oint_C \dfrac{z^2+3\mathrm{i}z-2}{z^3+9z}\,\mathrm{d}z$, where C is the circle $|z| = 1$, using Cauchy's integral formula.

5. Use Cauchy's residue theorem to evaluate the following integrals along the circle $|z| = 4$:
   (a) $\oint_C \dfrac{2z}{z^2+4}\,\mathrm{d}z$, (b) $\oint_C \dfrac{z^2+3\mathrm{i}z-2}{z^3+9z}\,\mathrm{d}z$.

6. Use closed contour integration to evaluate
   (a) $\displaystyle\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{x^2+6x+10}$,
   (b) $\displaystyle\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{x^2+4x+13}$.

7. Use integration along the unit circle $|z| = 1$ to evaluate
   (a) $\displaystyle\int_0^{2\pi} \frac{\mathrm{d}\theta}{3+2\cos\theta}$,
   (b) $\displaystyle\int_0^{2\pi} \frac{\mathrm{d}\theta}{4+\sin\theta}$.

Answers

2. (a) 1;
   (b) $-\pi$;
   (c) $-\dfrac{\mathrm{i}}{4}$.
3. 0.
4. $-\dfrac{4}{9}\pi\mathrm{i}$.
5. (a) $4\pi\mathrm{i}$,
   (b) $2\pi\mathrm{i}$.
6. (a) π,
   (b) $\dfrac{\pi}{3}$.
7. (a) $\dfrac{2\pi}{\sqrt{5}}$,
   (b) $\dfrac{2\pi}{\sqrt{15}}$.