**Assignment-I**

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**Q1 Consider the following cases, form finite difference matrix with the governing equations and boundary conditions (with at least 2 internal points) and check the singularity of the matrix. Also, solve the problems analytically to find uniqueness and existence of the solution. Relate it with the matrix property.**

And boundary conditions of the form:

**Solution**

To find the solution of this second order (1D) heat diffusion equation. By integrating the equation twice, the solution is of the form:

The boundary condition at x = 0:

And similarly using the boundary condition at x = 1:

Unique solutions for and exist if:

Assuming that unique solution for and exist, they can be written as:

And the analytical solution to the problem:

Using finite differences, a second order 3-point stencil-based discretization can be done for the second derivative:

The first order derivatives (in the boundary conditions) are discretized using first order forward and backward difference operations:

Where . Starting from using a uniform grid,

Thus, if where are the number of points in the grid, the differential equation can be discretized as:

@ i = 0

@ i = 1

@ i = 2

@i = N-1

@ i = N

Which gives the system of linear equations:

With the following structure:

This system of linear equations can be solved easily using simple methods such as Jacobi or Gauss-Seidel iteration. This linear system is sparse with a total of non-zero entries.

The solution to the linear system exists if and that is strictly diagonally dominant.

A matrix is said to be diagonally dominant if

Jacobi or Gauss-seidel iterations will converge if there is at least one row such that

The solution converges under Jacobi or Gauss-Seidel iterations if:

Or,

The matrix has a unique solution if

For e.g., N = 2

Simplifying:

For a unique solution, ,

For N = 2, ,

Solving,

Or,

Which is the same as the analytical condition posed before. Thus, the Jacobi and the Gauss Seidel iterations will converge to the analytical solution given that the matrices are diagonally dominant.

**Cases with different boundary conditions:**

The analytical solution is:

The determinant:

Thus, a unique solution exists. The discretization matrix is diagonally dominant.

The determinant:

A unique solution does not exist. The discretization matrix is singular.

The determinant:

The matrix is non-singular, unique solution should exist.

The unique solution is:

However, the numerical solution can’t be computed as the discretization matrix is diagonally undominant. Here’s a simple proof:

Looking at the first and last rows of the discretization matrix, the following conditions were derived:

Plugging in values,

The matrix is diagonally undominant if number of discretization points

This suggests that this problem cannot be solved using Gauss-Seidel or Jacobi Iteration methods for finer grids, since for finer grids, the coefficient matrices are not diagonally dominant.

This can be solved by Gauss Seidel or Jacobi using matrix relaxation and a deferred correction-based approach. Without matrix relaxation, other techniques such as GMRES should be applied to solve the problem.

In the code, diagonal (Jacobi) preconditioning is performed to reduce the condition number of the matrix .

**Q2**