

Cheetah MPC

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1 Conventions

Notation	
i	ith leg
\mathcal{B}	left subscript for body coordinate system
$[\mathbf{x}]_{\times} \in \mathbb{R}^{3 \times 3}$	skew-symmetric matrix, $[\mathbf{x}]_{\times} \mathbf{y} = \mathbf{x} \times \mathbf{y}$
$\Theta = [\phi, \theta, \psi]^T$	orientation as ZYX Euler angles
$\mathbf{J}_i \in \mathbb{R}^{3 \times 3}$	foot Jacobain
$\Lambda_i \in \mathbb{R}^{3 \times 3}$	operational space inertia matrix
$\mathbf{K}_p, \mathbf{K}_d \in \mathbb{R}^{3 \times 3}$	diagonal positive definite pd gain matrices

2 State Space Model

$$\frac{d}{dt} \hat{\Theta} = R_z(\psi) \hat{\Theta} \quad (1)$$

$$\frac{d}{dt} \hat{\mathbf{p}} = \hat{\mathbf{p}} \quad (2)$$

$$\frac{d}{dt} \hat{\omega} = \hat{\mathbf{I}}^{-1} \sum_{i=1}^n \mathbf{r}_i \times \mathbf{f}_i = \hat{\mathbf{I}}^{-1} ([\mathbf{r}_1]_{\times} \mathbf{f}_1 + \dots + [\mathbf{r}_n]_{\times} \mathbf{f}_n) \quad (3)$$

$$\frac{d}{dt} \hat{\mathbf{p}} = \frac{\sum_{i=1}^n \mathbf{f}_i}{m} - \mathbf{g} = \frac{\mathbf{f}_1 + \dots + \mathbf{f}_n}{m} - \mathbf{g} \quad (4)$$

$$\begin{aligned} \frac{d}{dt} \begin{bmatrix} \hat{\Theta} \\ \hat{\mathbf{p}} \\ \hat{\omega} \\ \hat{\mathbf{p}} \\ \mathbf{g} \end{bmatrix} &= \begin{bmatrix} \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{R}_z(\psi) & \mathbf{0}_3 & \mathbf{0}_{3 \times 1} \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_{3 \times 1} \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_{3 \times 1} \\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & [0 \ 0 \ 1]^T \\ \mathbf{0}_{1 \times 3} & \mathbf{0}_{1 \times 3} & \mathbf{0}_{1 \times 3} & \mathbf{0}_{1 \times 3} & 0 \end{bmatrix} \begin{bmatrix} \hat{\Theta} \\ \hat{\mathbf{p}} \\ \hat{\omega} \\ \hat{\mathbf{p}} \\ \mathbf{g} \end{bmatrix} \\ &\quad + \begin{bmatrix} \mathbf{0}_3 & \dots & \mathbf{0}_3 \\ \mathbf{0}_3 & \dots & \mathbf{0}_3 \\ \hat{\mathbf{I}}^{-1} [\mathbf{r}_1]_{\times} & \dots & \hat{\mathbf{I}}^{-1} [\mathbf{r}_4]_{\times} \\ \mathbf{I}_3/m & \dots & \mathbf{I}_3/m \\ \mathbf{0}_{1 \times 3} & \dots & \mathbf{0}_{1 \times 3} \end{bmatrix} \begin{bmatrix} \mathbf{f}_1 \\ \vdots \\ \mathbf{f}_4 \end{bmatrix} \end{aligned}$$

3 ZOOH Discretization

Given a linear system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$ and its solution

$$\mathbf{x}(t) = e^{\mathbf{A}(t-t_0)} \mathbf{x}(t_0) + \int_{t_0}^t e^{\mathbf{A}(t-\tau)} \mathbf{B}\mathbf{u}(\tau) d\tau$$

Let $t_0 = t_k$, $t = t_{k+1}$, and $t_{k+1} - t_k = T$. Therefore,

$$\begin{aligned} \mathbf{x}_{k+1} &= e^{\mathbf{A}T} \mathbf{x}_k + \int_{t_k}^{t_{k+1}} e^{\mathbf{A}(t_{k+1}-\tau)} \mathbf{B}\mathbf{u}(\tau) d\tau \\ &= e^{\mathbf{A}T} \mathbf{x}_k + \int_{t_k}^{t_{k+1}} e^{\mathbf{A}(t_{k+1}-\tau)} \mathbf{B} d\tau \mathbf{u}_k \end{aligned}$$

Let $\tau' = t_{k+1} - \tau$, we have

$$\int_{t_k}^{t_{k+1}} e^{\mathbf{A}(t_{k+1}-\tau)} \mathbf{B} d\tau = \int_T^0 e^{\mathbf{A}\tau'} (-d\tau') \mathbf{B} = \int_0^T e^{\mathbf{A}\tau} d\tau \mathbf{B}$$

Thus, $\mathbf{x}_{k+1} = \mathbf{A}_d \mathbf{x}_k + \mathbf{B}_d \mathbf{u}_k$, where $\mathbf{A}_d = e^{\mathbf{A}T}$ and $\mathbf{B}_d = \int_0^T e^{\mathbf{A}\tau} d\tau \mathbf{B}$

4 QP Formulation

$$\mathbf{x}_k = \mathbf{A}^k \mathbf{x}_0 + \sum_{i=0}^{k-1} \mathbf{A}^{k-1-i} \mathbf{B} \mathbf{u}_i$$

$$\begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \vdots \\ \mathbf{x}_k \end{bmatrix} = \begin{bmatrix} \mathbf{A}^1 \\ \mathbf{A}^2 \\ \vdots \\ \mathbf{A}^k \end{bmatrix} \mathbf{x}_0 + \begin{bmatrix} \mathbf{0} & & & \cdots & \mathbf{0} \\ \mathbf{B} & \mathbf{0} & & \cdots & \mathbf{0} \\ \mathbf{AB} & \mathbf{B} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & & & \ddots & \vdots \\ \mathbf{A}^{k-1}\mathbf{B} & \mathbf{A}^{k-2}\mathbf{B} & \mathbf{0} & \cdots & \mathbf{B} \end{bmatrix} \begin{bmatrix} \mathbf{u}_0 \\ \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_{k-1} \end{bmatrix}$$