Cheetah MPC

Zhuang Yulun

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1 Conventions

Notation	
i	ith leg
$\mathcal{B}$	left subscript for body coordinate system
$[\mathbf{x}]_\times \in \mathbb{R}^{3\times 3}$	skew-symmetric matrix, $[\mathbf{x}]_\times \mathbf{y} = \mathbf{x} \times \mathbf{y}$
$\boldsymbol{\Theta} = [\phi, \theta, \psi]^T$	orientation as ZYX Euler angles
$\mathbf{J}_i \in \mathbb{R}^{3\times 3}$	foot Jacobain
$\boldsymbol{\Lambda}_i \in \mathbb{R}^{3\times 3}$	operational space inertia matrix
$\mathbf{K}_p, \mathbf{K}_d \in \mathbb{R}^{3\times 3}$	diagonal positive definite pd gain matrices

2 Dynamics

$$\ddot{\mathbf{p}} = \frac{\sum_{i=1}^n \mathbf{f}_i}{m} - \mathbf{g} \tag{1}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{I}\boldsymbol{\omega}) = \sum_{i=1}^n \mathbf{r}_i \times \mathbf{f}_i \tag{2}$$

$$\dot{\mathbf{R}} = [\boldsymbol{\omega}]_\times \mathbf{R} \tag{3}$$

$$\mathbf{R} = \mathbf{R}_z(\psi)\mathbf{R}_y(\theta)\mathbf{R}_x(\phi) \tag{4}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{I}\boldsymbol{\omega}) = \mathbf{I}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times (\mathbf{I}\boldsymbol{\omega}) \approx \mathbf{I}\dot{\boldsymbol{\omega}} \tag{5}$$

$$\boldsymbol{\omega} = \begin{bmatrix} \cos(\theta)\cos(\psi) & -\sin(\psi) & 0 \\ \cos(\theta)\sin(\psi) & \cos(\psi) & 0 \\ -\sin(\theta) & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}$$

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} \cos(\psi)/\cos(\theta) & \sin(\psi)/\cos(\theta) & 0 \\ -\sin(\psi) & \cos(\psi) & 0 \\ \cos(\psi)\tan(\theta) & \sin(\psi)\tan(\theta) & 1 \end{bmatrix} \boldsymbol{\omega} \tag{6}$$

$$\approx \begin{bmatrix} \cos(\psi) & \sin(\psi) & 0 \\ -\sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \boldsymbol{\omega} \approx \mathbf{R}_z^\top(\psi)\boldsymbol{\omega} \tag{7}$$

3 State Space Model

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{\Theta}} = R_z(\psi)\hat{\mathbf{\Theta}} \tag{8}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{p}} = \hat{\dot{\mathbf{p}}} \tag{9}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\omega} = \hat{\mathbf{I}}^{-1}\sum_{i=1}^{n}\mathbf{r}_i \times \mathbf{f}_i = \hat{\mathbf{I}}^{-1}\left([\mathbf{r}_1]_\times \mathbf{f}_1 + \cdots + [\mathbf{r}_n]_\times \mathbf{f}_n\right) \tag{10}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\dot{\mathbf{p}}} = \frac{\sum_{i=1}^{n}\mathbf{f}_i}{m} - \mathbf{g} = \frac{\mathbf{f}_1 + \cdots + \mathbf{f}_n}{m} - \mathbf{g} \tag{11}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\hat{\mathbf{\Theta}}\\ \hat{\mathbf{p}}\\ \hat{\omega}\\ \hat{\dot{\mathbf{p}}}\\ \mathbf{g}\end{bmatrix} = \begin{bmatrix}\mathbf{0}_3 & \mathbf{0}_3 & \mathbf{R}_z(\psi) & \mathbf{0}_3 & \mathbf{0}_{3\times1}\\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{I}_3 & \mathbf{0}_{3\times1}\\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_{3\times1}\\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & [0\ 0\ 1]^T\\ \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & 0\end{bmatrix}\begin{bmatrix}\hat{\mathbf{\Theta}}\\ \hat{\mathbf{p}}\\ \hat{\omega}\\ \hat{\dot{\mathbf{p}}}\\ \mathbf{g}\end{bmatrix} \tag{12}$$

$$+\begin{bmatrix}\mathbf{0}_3 & \cdots & \mathbf{0}_3\\ \mathbf{0}_3 & \cdots & \mathbf{0}_3\\ \hat{\mathbf{I}}^{-1}[\mathbf{r}_1]_\times & \cdots & \hat{\mathbf{I}}^{-1}[\mathbf{r}_4]_\times\\ \mathbf{I}_3/m & \cdots & \mathbf{I}_3/m\\ \mathbf{0}_{1\times3} & \cdots & \mathbf{0}_{1\times3}\end{bmatrix}\begin{bmatrix}\mathbf{f}_1\\ \vdots\\ \mathbf{f}_4\end{bmatrix} \tag{13}$$

$$\dot{\mathbf{x}}(t) = \mathbf{A}_c(\psi)\mathbf{x}(t) + \mathbf{B}_c(\mathbf{r}_1, \ldots, \mathbf{r}_n, \psi)\,\mathbf{u}(t)$$

where $\mathbf{A}_c \in \mathbb{R}^{13\times13}$ and $\mathbf{B}_c \in \mathbb{R}^{13\times3n}$

4 ZOH Discretization

Given a linear system $\dot{x} = Ax + Bu$ and its solution

$$\mathbf{x}(t) = e^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0) + \int_{t_0}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau$$

Let $t_0 = t_k, t = t_{k+1}$, and $t_{k+1} - t_k = T$. Therefore,

$$\begin{aligned}\mathbf{x}_{k+1} &= e^{\mathbf{A}T}\mathbf{x}_k + \int_{t_k}^{t_{k+1}} e^{\mathbf{A}(t_{k+1}-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau\\ &= e^{\mathbf{A}T}\mathbf{x}_k + \int_{t_k}^{t_{k+1}} e^{\mathbf{A}(t_{k+1}-\tau)}\mathbf{B}d\tau\mathbf{u}_k\end{aligned}$$

Let $\tau' = t_{k+1} - \tau$, we have

$$\int_{t_k}^{t_{k+1}} e^{\mathbf{A}(t_{k+1}-\tau)}\mathbf{B}d\tau = \int_{T}^{0} e^{\mathbf{A}\tau'}(-d\tau')\,\mathbf{B} = \int_{0}^{T} e^{\mathbf{A}\tau}d\tau\mathbf{B}$$

Thus, $\mathbf{x}_{k+1} = \mathbf{A}_d\mathbf{x}_k + \mathbf{B}_d\mathbf{u}_k$, where $\mathbf{A}_d = e^{\mathbf{A}T}$ and $\mathbf{B}_d = \int_0^T e^{\mathbf{A}\tau}d\tau\mathbf{B}$

$$\mathbf{x}_{k+1} = \mathbf{A}_d\mathbf{x}_k + \mathbf{B}_d\mathbf{u}_k \tag{14}$$

$$\mathbf{A}_d = e^{\mathbf{A}T} \tag{15}$$

$$\mathbf{B}_d = \int_0^T e^{\mathbf{A}\tau}d\tau\mathbf{B} \tag{16}$$

5 MPC Formulation

$$\min_{\mathbf{x},\mathbf{u}} \sum_{i=0}^{k-1} \left\|\mathbf{x}_{i+1} - \mathbf{x}_{i+1,\mathrm{ref}}\right\|_{\mathbf{Q}_i} + \left\|\mathbf{u}_i\right\|_{\mathbf{R}_i}$$

$$\textit{subject to} \quad \mathbf{x}_{i+1} = \mathbf{A}_i\mathbf{x}_i + \mathbf{B}_i\mathbf{u}_i, i = 0 \ldots k-1$$

$$\underline{\mathbf{c}}_i \le \mathbf{C}_i\mathbf{u}_i \le \overline{\mathbf{c}}_i, i = 0 \ldots k-1$$

$$\mathbf{D}_i\mathbf{u}_i = 0, i = 0 \ldots k-1$$

$$\begin{array}{r}
f_{\min} \le f_z \le f_{\max} \\
-\mu f_z \le \pm f_x \le \mu f_z \\
-\mu f_z \le \pm f_y \le \mu f_z
\end{array}$$

6 QP Formulation

$$\boldsymbol{x_k} = \mathbf{A}^k \boldsymbol{x}_0 + \sum_{i=0}^{k-1} \mathbf{A}^{k-1-i}\mathbf{B}\mathbf{u}_i$$

$$\begin{bmatrix} \boldsymbol{x}_1 \\ \boldsymbol{x}_2 \\ \vdots \\ \boldsymbol{x}_k \end{bmatrix} = \begin{bmatrix} \mathbf{A}^1 \\ \mathbf{A}^2 \\ \vdots \\ \mathbf{A}^k \end{bmatrix} \boldsymbol{x}_0 + \begin{bmatrix} \mathbf{0} & & & \cdots & \mathbf{0} \\ \mathbf{B} & \mathbf{0} & & \cdots & \mathbf{0} \\ \mathbf{AB} & \mathbf{B} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & & & \ddots & \vdots \\ \mathbf{A}^{k-1}\mathbf{B} & \mathbf{A}^{k-1}\mathbf{B} & \mathbf{0} & \cdots & \mathbf{B} \end{bmatrix} \begin{bmatrix} \mathbf{u}_0 \\ \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_{k-1} \end{bmatrix}$$

$$\begin{array}{ll}
\min_{\mathbf{U}} & \frac{1}{2}\mathbf{U}^\top\mathbf{H}\mathbf{U} + \mathbf{U}^\top\mathbf{g} \\
\text{s. t.} & \underline{\mathbf{c}} \le \mathbf{C}\mathbf{U} \le \overline{\mathbf{c}}
\end{array}$$

$$\mathbf{H} = 2\left(\mathbf{B}_{\mathrm{qp}}^{\boldsymbol{L}}\mathbf{B}_{\mathrm{qp}} + \mathbf{K}\right) \tag{17}$$

$$\mathbf{g} = 2\mathbf{B}_{\mathrm{qp}}^{\top}\mathbf{L}\left(\mathbf{A}_{\mathrm{qp}}\mathbf{x}_0 - \mathbf{y}\right) \tag{18}$$

7 Ground Normal Estimation

To enable the ability to traverse stairs and sloped terrain without vision, we use measurements of each footstep location

$$\boldsymbol{p}_i = (p_i^x, p_i^y, p_i^z)$$

to approximate the local slope of the walking surface.
The walking surface is modeled as a plane:

$$z(x, y) = a_0 + a_1 x + a_2 y$$

Coefficients $\boldsymbol{a} = (a_0, a_1, a_2)^T$ are obtained through the solution of the least squares problem

$$\boldsymbol{a} = \left(\boldsymbol{W}^T\boldsymbol{W}\right)^{\dagger}\boldsymbol{W}^T\boldsymbol{p}^z$$

$$\boldsymbol{W} = \begin{bmatrix} \mathbf{1} & \boldsymbol{p}^x & \boldsymbol{p}^y \end{bmatrix}_{4\times 3}$$

where $\boldsymbol{p}^x = (p_1^x, p_2^x, p_3^x, p_4^x)$