Laplacian

1. differentiable manifold: locally similar enough to a linear space manifold to allow calculus.

2. Laplacian $L: C^\infty(M) \to C^\infty(M)$ unknown manifold.

   spectrum $spec(L) = \{\lambda_0, \lambda_1, \lambda_2, \cdots \lambda_k \cdots\} \to \infty$

3. Given $spec(L)$. we can infer M: its dim, volume and its <u>total scalar curvature</u>.

4. Data: $\{x_i\}$. $x_i \in \mathbb{R}^N$. submanifold: $M \subset \mathbb{R}^N$

5. <u>Eigen functions</u> of L on M can be used to define lower dim embeddings.

6. Idea: ① Model M by $G = (V, E)$. close data points are connected.

   ② Construct <u>graph Laplacian</u> L on G

   ③ Compute $spec(L)$ and corresponding eigen functions.

   ④ Construct an embedding $F: V \to \mathbb{R}^m$ for $m < N$.

7. Spectral Theorem: There exists an orthogonal basis of $\mathbb{R}^n$ consisting of eigen vectors of A. and eigen values are real

8. Min (max) imizing Property:

   $$\underset{\|f\|=1}{\operatorname{argmax}} \langle Af, f \rangle = f_n \;/\; f_0 \text{ when asked minimization.}$$

9. W: $w_{ij} = \begin{cases} 1, & \text{if node } i \text{ and } j \text{ connected.} \\ 0 & \text{o.w.} \end{cases}$ D: d_{ii} = degree of node i.

$L \triangleq D - W$

10. generalized eigen value problem: $Lf = \lambda D f$. or $D^{-1} L f = \lambda f$.

11 eigen vector $y := f_1 = \underline{(0,-3,1,2)}$ coordinates for y_1, y_2, y_3, y_4

12. Alg. ①. Construct. $G = (V, E)$. (You can connect the points within a sphere, or connect n nearest)

② Choose Weights for edge $\rightarrow$ $W_{ij} = 1$

$\rightarrow$ $W_{ij} = \exp\{-\frac{\|x_i - x_j\|^2}{t}\}$

③ Solve $Lf = \lambda D f$. where. $D_{ii} = \sum_{j=1}^{k} W_{ij}$, $L = D - W$

④ $f_0, f_1, \cdots f_{k-1}$ be the eigen vectors, then.

$F(i) \triangleq (f_1(i), \cdots f_m(i))$

The ith point's coordinate is the ith element of $f_1, \cdots f_m$.

13. Why does it work?

Objective: Construct an embedding $F: V \rightarrow \mathbb{R}^m$: $Y = \begin{pmatrix} Y_1^T \\ \vdots \\ Y_k^T \end{pmatrix} \in M_{k \times m}(\mathbb{R})$.

min $\sum_{i,j=1}^{k} \|Y_i - Y_j\|^2 W_{ij}$, or $\underline{\min \operatorname{tr}(Y^T L Y)}$. ☆

$\Rightarrow$ Problem reduced to find

$\underset{\operatorname{tr}(Y^T D Y = I)}{\operatorname{argmin}} \operatorname{tr}(Y^T L Y) \Rightarrow \underset{\operatorname{tr}(Z^T Z = I)}{\operatorname{argmin}} \operatorname{tr}(Z^T D^{-1/2} L D^{-1/2} Z)$

$\Rightarrow$ find minimum non-zero eigen values of

$D^{-1/2} L D^{-1/2} f = f\lambda$ or $\underline{Lf = Df\lambda}$. generalized eigen value problem.

Prove: $\min \sum_{i,j=1}^{k} \|Y_i - Y_j\|^2 W_{ij}$ equals to $\min tr(Y^T L Y)$

$$\text{min left} = \min \sum_{i,j=1}^{k} (\|Y_i\|^2 + \|Y_j\|^2 - \underline{2 Y_i^T Y_j) W_{ij}}$$

$$= \min \left[\sum_{i,j=1}^{k} (-2 Y_i^T Y_j) W_{ij} + 2 \times \sum_{i,j=1}^{k} \|Y_i\|^2 W_{ij} \right]$$

$$= \min \left[\sum_{i,j=1}^{k} (-2 Y_i^T Y_j) W_{ij} + \sum_{i=1}^{k} \|Y_i\|^2 D_{ii} \right]$$

$$\text{min right} = \min tr\left[(Y_1 \cdots Y_k) \begin{bmatrix} L_{11} & \cdots & L_{1k} \\ \vdots & \cdots & \vdots \\ L_{k1} & \cdots & L_{kk} \end{bmatrix} \begin{bmatrix} Y_1^T \\ \vdots \\ Y_k^T \end{bmatrix} \right]$$

$$= \min tr\left(\sum_{i,j=1}^{k} Y_i L_{ij} Y_j^T \right)$$

$$= \cancel{\#} \min \sum_{i,j=1}^{k} tr(Y_i L_{ij} Y_j^T)$$

$$= \min \sum_{i,j=1}^{k} L_{ij}\, tr(Y_i^T Y_j)$$

$$= \min \sum_{i,j=1}^{k} L_{ij} Y_i^T Y_j$$

$$= \min \sum_{i,j=1}^{k} (D_{ij} Y_i^T Y_j - W_{ij} Y_i^T Y_j)$$

$$= \min \left[\sum_{i,j=1}^{k} -W_{ij} Y_i^T Y_j + \sum_{i=1}^{k} D_{ii} \|Y_i\|^2 \right]$$

$\therefore$ min left $\Leftrightarrow$ min right $\square$.

Complement

1. The Laplacian for (M,g) Riemann Manifold.

$$L = -\sum_{i,j=1}^{n} g^{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \text{lower order terms.}$$

2. Spectral Theorem.

L is symmetric with respect to the inner product in $C_c^\infty(M)$

$$(f,g)_{L^2} = (g,f)_{L^2} = \int_M f(x) g(x)\, dx$$

if M is compact, there exists an orthogonal basis of $L^2(M)$ consisting of eigen vectors of L, each eigen value is real.

3. Let (M,g) be a compact Riemann manifold. $f: M \to \mathbb{R}$.

① For close x, z.

$$|f(x) - f(z)| \le \text{dist}_M(x,z) \|\nabla f\| + o(\text{dist}_M(x,z))$$

② Preserve locality? minimizing the upper bound $\Rightarrow$

$$\underset{\|f\|_{L^2(M)}=1}{\text{argmin}} \int_M \|\nabla f\|^2 dx \quad (1)$$

(1) must be the eigen vector of Laplacian.

③ By Stokes Theorem.

$$\int_M \|\nabla f\|^2 dx = \int_M (Lf) f\, dx = (Lf, f)_{L^2}.$$

Double A