Solutions to Exercise 3A Foundation questions

1a $(\cos\theta + i\sin\theta)^5$

$= (\text{cis}\,\theta)^5$

$= \text{cis}\,5\theta$

1b $(\cos\theta + i\sin\theta)^{-3}$

$= (\text{cis}\,\theta)^{-3}$

$= \text{cis}(-3\theta)$

1c $(\cos 2\theta + i\sin 2\theta)^4$

$= (\text{cis}\,2\theta)^4$

$= \text{cis}(4 \times 2\theta)$

$= \text{cis}\,8\theta$

1d $\cos\theta - i\sin\theta$

$= \cos(-\theta) + i\sin(-\theta)$

$= \text{cis}(-\theta)$

1e $(\cos\theta - i\sin\theta)^{-7}$

$= \left(\text{cis}(-\theta)\right)^{-7}$

$= \text{cis}(-7 \times -\theta)$

$= \text{cis}\,7\theta$

1f $(\cos 3\theta - i\sin 3\theta)^2$

$= \left(\text{cis}(-3\theta)\right)^2$

$= \text{cis}(2 \times -3\theta)$

$= \text{cis}(-6\theta)$

2a

$$\begin{aligned}
&\frac{(\cos\theta + i\sin\theta)^6(\cos\theta + i\sin\theta)^{-3}}{(\cos\theta - i\sin\theta)^4}\\
&= \frac{(\operatorname{cis}\theta)^6(\operatorname{cis}\theta)^{-3}}{\left(\operatorname{cis}(-\theta)\right)^4}\\
&= \frac{\operatorname{cis}6\theta \times \operatorname{cis}(-3\theta)}{\operatorname{cis}(-4\theta)}\\
&= \frac{\operatorname{cis}(6\theta - 3\theta)}{\operatorname{cis}(-4\theta)}\\
&= \frac{\operatorname{cis}3\theta}{\operatorname{cis}(-4\theta)}\\
&= \operatorname{cis}\left(3\theta - (-4\theta)\right)\\
&= \operatorname{cis}7\theta
\end{aligned}$$

2b

$$\begin{aligned}
&\frac{(\cos 3\theta + i\sin 3\theta)^5(\cos 2\theta - i\sin 2\theta)^{-4}}{(\cos 4\theta - i\sin 4\theta)^{-7}}\\
&= \frac{(\operatorname{cis}3\theta)^5\left(\operatorname{cis}(-2\theta)\right)^{-4}}{\left(\operatorname{cis}(-4\theta)\right)^{-7}}\\
&= \frac{\operatorname{cis}15\theta \times \operatorname{cis}8\theta}{\operatorname{cis}28\theta}\\
&= \frac{\operatorname{cis}(15\theta + 8\theta)}{\operatorname{cis}28\theta}\\
&= \frac{\operatorname{cis}23\theta}{\operatorname{cis}28\theta}\\
&= \operatorname{cis}(23\theta - 28\theta)\\
&= \operatorname{cis}(-5\theta)
\end{aligned}$$

3a

$$\begin{aligned}
&\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^4\\
&= \left(\operatorname{cis}\frac{\pi}{4}\right)^4\\
&= \operatorname{cis}\left(4 \times \frac{\pi}{4}\right)\\
&= \operatorname{cis}\pi\\
&= \cos\pi + i\sin\pi\\
&= -1 + 0i\\
&= -1
\end{aligned}$$

3b

$$\begin{aligned}
&\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)^3\\
&=\left(\operatorname{cis}\frac{\pi}{2}\right)^3\\
&=\operatorname{cis}\left(3\times\frac{\pi}{2}\right)\\
&=\operatorname{cis}\frac{3\pi}{2}\\
&=\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}\\
&=0-1i\\
&=-i
\end{aligned}$$

3c

$$\begin{aligned}
&\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)^5\\
&=\left(\operatorname{cis}\frac{\pi}{6}\right)^5\\
&=\operatorname{cis}\left(5\times\frac{\pi}{6}\right)\\
&=\operatorname{cis}\frac{5\pi}{6}\\
&=\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\\
&=-\frac{\sqrt{3}}{2}+\frac{1}{2}i
\end{aligned}$$

3d

$$\begin{aligned}
&\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)^{-2}\\
&=\left(\operatorname{cis}\frac{2\pi}{3}\right)^{-2}\\
&=\operatorname{cis}\left(-2\times\frac{2\pi}{3}\right)\\
&=\operatorname{cis}\left(\frac{-4\pi}{3}\right)\\
&=\cos\frac{4\pi}{3}-i\sin\frac{4\pi}{3}\\
&=-\frac{1}{2}+\frac{\sqrt{3}}{2}i
\end{aligned}$$

3e

$$\left(\cos\frac{3\pi}{8} - i\sin\frac{3\pi}{8}\right)^{-6}$$
$$= \left(\operatorname{cis}\left(\frac{-3\pi}{8}\right)\right)^{-6}$$
$$= \operatorname{cis}\left(-6\times\frac{-3\pi}{8}\right)$$
$$= \operatorname{cis}\frac{9\pi}{4}$$
$$= \cos\frac{9\pi}{4} + i\sin\frac{9\pi}{4}$$
$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$$

3f

$$\left(\cos\frac{5\pi}{12} - i\sin\frac{5\pi}{12}\right)^{4}$$
$$= \left(\operatorname{cis}\left(-\frac{5\pi}{12}\right)\right)^{4}$$
$$= \operatorname{cis}\left(-\frac{5\pi}{3}\right)$$
$$= \cos\frac{5\pi}{3} - i\sin\frac{5\pi}{3}$$
$$= \frac{1}{2} + \frac{\sqrt{3}}{2}i$$

4a $1 + i$

$$= \sqrt{1^2 + 1^2}\operatorname{cis}\left(\tan^{-1}\left(\frac{1}{1}\right)\right)$$
$$= \sqrt{2}\operatorname{cis}\frac{\pi}{4}$$

4b $(1+i)^{17}$

$$= \left(\sqrt{2}\operatorname{cis}\frac{\pi}{4}\right)^{17}$$
$$= \left(\sqrt{2}\right)^{17}\operatorname{cis}\left(\frac{\pi}{4}\times 17\right)$$
$$= 256\sqrt{2}\operatorname{cis}\left(\frac{17\pi}{4}\right)$$

$$= 256\sqrt{2}\operatorname{cis}\frac{\pi}{4}$$
$$= 256\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$
$$= 256\sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$$
$$= 256 + 256i$$

5a $z = 1 + i\sqrt{3}$

$$= \sqrt{1^2 + \left(\sqrt{3}\right)^2}\operatorname{cis}\left(\tan^{-1}\frac{\sqrt{3}}{1}\right)$$
$$= 2\operatorname{cis}\frac{\pi}{3}$$

5b z^{11}

$$= \left(2\operatorname{cis}\frac{\pi}{3}\right)^{11}$$
$$= 2^{11}\operatorname{cis}\left(\frac{\pi}{3} \times 11\right)$$
$$= 2048\operatorname{cis}\frac{11\pi}{3}$$
$$= 2048\operatorname{cis}\frac{5\pi}{3}$$
$$= 2048\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$$
$$= 2048\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$$
$$= 1024 - 1024\sqrt{3}i$$

6a $z = -\sqrt{3} + i$

$|z|$
$= \sqrt{\left(-\sqrt{3}\right)^2 + 1^2}$
$= \sqrt{3 + 1}$
$= \sqrt{4}$
$= 2$

$\text{Arg}(z)$
$= \pi - \tan^{-1}\dfrac{1}{\sqrt{3}}$
$= \pi - \dfrac{\pi}{6}$
$= \dfrac{5\pi}{6}$

6b $z^7 + 64z$

$= \left(2\text{cis}\dfrac{5\pi}{6}\right)^7 + 64\left(2\text{cis}\dfrac{5\pi}{6}\right)$

$= 2^7\text{cis}\left(\dfrac{5\pi}{6} \times 7\right) + 128\text{cis}\left(\dfrac{5\pi}{6}\right)$

$= 128\text{cis}\dfrac{35\pi}{6} + 128\text{cis}\dfrac{5\pi}{6}$

$= 128\left(\text{cis}\dfrac{35\pi}{6} + \text{cis}\dfrac{5\pi}{6}\right)$
$= 128\left(\cos\dfrac{35\pi}{6} + i\sin\dfrac{35\pi}{6} + \cos\dfrac{5\pi}{6} + i\sin\dfrac{5\pi}{6}\right)$

$= 128\left(\dfrac{\sqrt{3}}{2} - \dfrac{1}{2}i - \dfrac{\sqrt{3}}{2} + \dfrac{1}{2}i\right)$

$= 0$

7a $\sqrt{3} - i$
$= \sqrt{\left(\sqrt{3}\right)^2 + 1^2}\ \text{cis}\left(\tan^{-1}(-\dfrac{1}{\sqrt{3}})\right)$
$= 2\text{cis}\left(-\dfrac{\pi}{6}\right)$

7b $\left(\sqrt{3}-i\right)^7$

$$= \left(2\text{cis}\left(-\frac{\pi}{6}\right)\right)^7$$

$$= 2^7\text{cis}\left(-\frac{\pi}{6}\times 7\right)$$

$$= 128\text{cis}\left(-\frac{7\pi}{6}\right)$$

$$= 128\text{cis}\frac{5\pi}{6}$$

7c $\left(\sqrt{3}-i\right)^7$

$$= 128\text{cis}\frac{5\pi}{6}$$

$$= 128\left(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right)$$

$$= 128\left(-\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)$$

$$= -64\sqrt{3}+64i$$

8a $\left(-1-i\sqrt{3}\right)$

$$= \sqrt{1^2+\left(\sqrt{3}\right)^2}\ \text{cis}\left(-\pi+\tan^{-1}\frac{\sqrt{3}}{1}\right)$$

$$= 2\text{cis}\left(-\frac{2\pi}{3}\right)$$

8b $\left(-1-i\sqrt{3}\right)^5$

$$= \left(2\text{cis}\left(-\frac{2\pi}{3}\right)\right)^5$$

$$= 2^5\ \text{cis}\left(-\frac{2\pi}{3}\times 5\right)$$

$$= 32\ \text{cis}\left(-\frac{10\pi}{3}\right)$$

$$= 32\ \text{cis}\frac{2\pi}{3}$$

8c $\left(-1 - i\sqrt{3}\right)^5$

$= 32 \operatorname{cis} \dfrac{2\pi}{3}$

$= 32\left(\cos\dfrac{2\pi}{3} + i\sin\dfrac{2\pi}{3}\right)$

$= 32\left(-\dfrac{1}{2} + \dfrac{i\sqrt{3}}{2}\right)$

$= -16 + 16i\sqrt{3}$

9a $\sqrt{2} - i\sqrt{2}$

$= \sqrt{\left(\sqrt{2}\right)^2 + \left(-\sqrt{2}\right)^2} \operatorname{cis}\left(\tan^{-1}\left(-\dfrac{\sqrt{2}}{\sqrt{2}}\right)\right)$

$= \sqrt{2+2} \operatorname{cis}\left(-\dfrac{\pi}{4}\right)$

$= 2\operatorname{cis}\left(-\dfrac{\pi}{4}\right)$

9b z^{22}

$= \left(2\operatorname{cis}\left(-\dfrac{\pi}{4}\right)\right)^{22}$

$= 2^{22} \operatorname{cis}\left(-\dfrac{\pi}{4} \times 22\right)$

$= 2^{22} \operatorname{cis}\left(-\dfrac{11\pi}{2}\right)$

$= 2^{22} \operatorname{cis}\dfrac{\pi}{2}$

$= 2^{22} i$

Solutions to Exercise 3A Development questions

10a $(1+i)^{10}$ (1st quadrant)

$= \left(\sqrt{2}\,\text{cis}\left(\frac{\pi}{4}\right)\right)^{10}$

$= \left(\sqrt{2}\right)^{10}\,\text{cis}\left(\frac{\pi}{4}\times 10\right)$

$= 2^5\,\text{cis}\left(\frac{5\pi}{2}\right)$

$= 2^5\,\text{cis}\left(\frac{5\pi}{2} - 2\pi\right)$

$= 2^5\,\text{cis}\left(\frac{\pi}{2}\right)$

$= 2^5 i$

which is purely imaginary

10b $\left(1 - i\sqrt{3}\right)^9$ (4th quadrant)

$= \left(2\,\text{cis}\left(-\frac{\pi}{3}\right)\right)^9$

$= 2^9\,\text{cis}\left(-\frac{\pi}{3}\times 9\right)$

$= 2^9\,\text{cis}(-3\pi)$

$= 2^9\,\text{cis}(-\pi)$

$= -2^9$

which is real

10c $(-1+i)^4$ (2nd quadrant)

$= \left(\sqrt{2}\,\text{cis}\left(\frac{3\pi}{4}\right)\right)^4$

$= \left(\sqrt{2}\right)^4 \text{cis}(3\pi)$

$= 2^2\,\text{cis}(\pi)$

$= 4\,\text{cis}(\pi)$

$= -4$

Hence $-1+i$ is a fourth root of -4.

10d $\left(-\sqrt{3}-i\right)^6$ (3rd quadrant)

$= \left(2\,\text{cis}\left(-\frac{5\pi}{6}\right)\right)^6$

$= 2^6\,\text{cis}(-5\pi)$

$= 2^6\,\text{cis}(-\pi)$

$= -2^6$

$= -64$

Hence $-\sqrt{3}-i$ is a sixth root of -64.

11 If k is a multiple of 4 then $k = 4n$ where n is an integer. Thus,

$(-1+i)^k$

$= (-1+i)^{4n}$ (2nd quadrant)

$= \left(\sqrt{2}\,\text{cis}\left(\frac{3\pi}{4}\right)\right)^{4n}$

$= \left(\left(\sqrt{2}\right)^4 \text{cis}(3\pi)\right)^n$

$= (-2^2)^n$

$= (-4)^n$

which is real as required

12a i $\left(\sqrt{3}+i\right)^m$ (1st quadrant)

$$= \left(2 \operatorname{cis}\left(\frac{\pi}{6}\right)\right)^m$$

$$= 2^m \operatorname{cis}\left(\frac{m\pi}{6}\right)$$

which is real when $\frac{m\pi}{6}$ is a multiple of π. The lowest positive integer for which this is true is when $m = 6$.

12a ii $\left(\sqrt{3}+i\right)^m$

$$= \left(2 \operatorname{cis}\left(\frac{\pi}{6}\right)\right)^m$$

$$= 2^m \operatorname{cis}\left(\frac{m\pi}{6}\right)$$

which is imaginary when $\frac{m\pi}{6}$ is of the form $n\pi \pm \frac{\pi}{2}$ where n is an integer. The lowest positive integer for which this is true is when $m = 3$.

12b i $\left(\sqrt{3}+i\right)^6$

$$= 2^6 \operatorname{cis}\left(\frac{6\pi}{6}\right)$$

$$= -2^6$$

$$= -64$$

12b ii $\left(\sqrt{3}+i\right)^3$

$$= 2^3 \operatorname{cis}\left(\frac{3\pi}{6}\right)$$

$$= 2^3 i$$

$$= 8i$$

13a $(1+i)^n + (1-i)^n$ (1st and 4th quadrants)

$$= \left(\sqrt{2}\operatorname{cis}\left(\tfrac{\pi}{4}\right)\right)^n + \left(\sqrt{2}\operatorname{cis}\left(-\tfrac{\pi}{4}\right)\right)^n$$

$$= \left(\sqrt{2}\right)^n \operatorname{cis}\left(\frac{n\pi}{4}\right) + \left(\sqrt{2}\right)^n \operatorname{cis}\left(-\frac{n\pi}{4}\right)$$

$$= \left(\sqrt{2}\right)^n \left[\operatorname{cis}\left(\frac{n\pi}{4}\right) + \operatorname{cis}\left(-\frac{n\pi}{4}\right)\right]$$

$$= \left(\sqrt{2}\right)^n \left[\cos\left(\frac{n\pi}{4}\right) + i\sin\left(\frac{n\pi}{4}\right) + \cos\left(\frac{n\pi}{4}\right) - i\sin\left(\frac{n\pi}{4}\right)\right]$$

$$= \left(\sqrt{2}\right)^n \left(2\cos\left(\frac{n\pi}{4}\right)\right)$$

$$= 2\left(\sqrt{2}\right)^n \cos\left(\frac{n\pi}{4}\right)$$

which is real

13b $2\left(\sqrt{2}\right)^n \cos\left(\frac{n\pi}{4}\right) = 0$

This expression will be 0 when,

$\frac{n\pi}{4} = 2m\pi \pm \frac{\pi}{2}$ (where m in an integer)

This gives,

$n = 8m \pm 2$

Since n is a positive integer, $n = 2, 6, 10, 14, 18 \ldots$

14 $\left(-\sqrt{3}+i\right)^n-\left(-\sqrt{3}-i\right)^n$ (2^{nd} and 3^{rd} quadrants)

$$=\left(2\text{ cis}\left(\frac{5\pi}{6}\right)\right)^n-\left(2\text{ cis}\left(-\frac{5\pi}{6}\right)\right)^n$$

$$=2^n\text{cis}\left(\frac{5n\pi}{6}\right)-2^n\text{cis}\left(-\frac{5n\pi}{6}\right)$$

$$=2^n\left[\text{cis}\left(\frac{5n\pi}{6}\right)-\text{cis}\left(-\frac{5n\pi}{6}\right)\right]$$

$$=2^n\left[\text{cis}\left(\frac{5n\pi}{6}\right)-\text{cis}\left(-\frac{5n\pi}{6}\right)\right]$$

$$=2^n\left[\cos\left(\frac{5n\pi}{6}\right)+i\sin\left(\frac{5n\pi}{6}\right)-\left(\cos\left(\frac{5n\pi}{6}\right)-i\sin\left(\frac{5n\pi}{6}\right)\right)\right]$$

$$=2^n\left[\cos\left(\frac{5n\pi}{6}\right)+i\sin\left(\frac{5n\pi}{6}\right)-\cos\left(\frac{5n\pi}{6}\right)+i\sin\left(\frac{5n\pi}{6}\right)\right]$$

$$=2^n\left[2i\sin\left(\frac{5n\pi}{6}\right)\right]$$

$$=2^{n+1}i\sin\left(\frac{5n\pi}{6}\right)$$

$$=2^{n+1}\sin\left(\frac{5n\pi}{6}\right)i$$

15a $\left(1+\sqrt{3}i\right)^{2n}+\left(1-\sqrt{3}i\right)^{2n}$ (1st and 4th quadrants)

$$=\left(2\text{ cis}\left(\frac{\pi}{3}\right)\right)^{2n}+\left(2\text{ cis}\left(-\frac{\pi}{3}\right)\right)^{2n}$$

$$=2^{2n}\text{ cis}\left(\frac{2n\pi}{3}\right)+2^{2n}\text{ cis}\left(-\frac{2n\pi}{3}\right)$$

$$=2^{2n}\left[\text{cis}\left(\frac{2n\pi}{3}\right)+\text{cis}\left(-\frac{2n\pi}{3}\right)\right]$$

$$=2^{2n}\left[\cos\left(\frac{2n\pi}{3}\right)+i\sin\left(\frac{2n\pi}{3}\right)+\cos\left(\frac{2n\pi}{3}\right)-i\sin\left(\frac{2n\pi}{3}\right)\right]$$

$$=2^{2n}\left[2\cos\left(\frac{2n\pi}{3}\right)\right]$$

$$=2^{2n+1}\cos\left(\frac{2n\pi}{3}\right)$$

If n is divisible by 3 then $n=3m$ where m is an integer. Hence

$$\left(1+\sqrt{3}i\right)^{2n}+\left(1-\sqrt{3}i\right)^{2n}$$

$$=2^{2n+1}\cos\left(\frac{2(3m)\pi}{3}\right)$$

$$=2^{2n+1}\cos(2\pi m)$$

$$=2^{2n+1}$$

15b $\left(1+\sqrt{3}i\right)^{2n}+\left(1-\sqrt{3}i\right)^{2n}=2^{2n+1}\cos\left(\frac{2n\pi}{3}\right)$ from part **a.**

Since n is not divisible by 3,

$$2^{2n+1}\cos\left(\frac{2n\pi}{3}\right)$$

$$=2^{2n+1}\cos\left(\pm\frac{2\pi}{3}\right),2^{2n+1}\cos\left(\pm\frac{4\pi}{3}\right),2^{2n+1}\cos\left(\pm\frac{8\pi}{3}\right)\dots$$

Note that the terms above will always be in the first and third quadrants, since as n increases, we always just add an extra $\frac{2\pi}{3}$ and exclude the origin. Hence,

$$=2^{2n+1}\left(-\frac{1}{2}\right)$$

$$=-2^{2n}$$

16

$$\left(\frac{1+\cos 2\theta + i\sin 2\theta}{1+\cos 2\theta - i\sin 2\theta}\right)^n$$

$$= \left(\frac{1+\cos^2\theta - \sin^2\theta + i\sin 2\theta}{1+\cos^2\theta - \sin^2\theta - i\sin 2\theta}\right)^n$$

$$= \left(\frac{\cos^2\theta + \cos^2\theta + i\sin 2\theta}{\cos^2\theta + \cos^2\theta - i\sin 2\theta}\right)^n$$

$$= \left(\frac{2\cos^2\theta + i\sin 2\theta}{2\cos^2\theta - i\sin 2\theta}\right)^n$$

$$= \left(\frac{2\cos^2\theta + 2i\cos\theta\sin\theta}{2\cos^2\theta - 2i\cos\theta\sin\theta}\right)^n$$

$$= \left(\frac{2\cos\theta\,(\cos\theta + i\sin\theta)}{2\cos\theta\,(\cos\theta - i\sin\theta)}\right)^n$$

$$= \left(\frac{(\cos\theta + i\sin\theta)}{(\cos\theta - i\sin\theta)}\right)^n$$

$$= \left(\frac{\text{cis}(\theta)}{\text{cis}(-\theta)}\right)^n$$

$$= \left(\text{cis}(2\theta)\right)^n$$

$$= \text{cis}(2n\theta)$$

17 $(1+\cos\alpha+i\sin\alpha)^k+(1+\cos\alpha-i\sin\alpha)^k$

$$= \left(1+\cos^2\frac{1}{2}\alpha-\sin^2\frac{1}{2}\alpha+i\sin\alpha\right)^k+\left(1+\cos^2\frac{1}{2}\alpha-\sin^2\frac{1}{2}\alpha-i\sin\alpha\right)^k$$

$$= \left(\cos^2\frac{1}{2}\alpha+1-\sin^2\frac{1}{2}\alpha+i\sin\alpha\right)^k+\left(\cos^2\frac{1}{2}\alpha+1-\sin^2\frac{1}{2}\alpha-i\sin\alpha\right)^k$$

$$= \left(\cos^2\frac{1}{2}\alpha+\cos^2\frac{1}{2}\alpha+i\sin\alpha\right)^k+\left(\cos^2\frac{1}{2}\alpha+\cos^2\frac{1}{2}\alpha-i\sin\alpha\right)^k$$

$$= \left(2\cos^2\frac{1}{2}\alpha+i\sin\alpha\right)^k+\left(2\cos^2\frac{1}{2}\alpha-i\sin\alpha\right)^k$$

$$= \left(2\cos^2\frac{1}{2}\alpha+2i\sin\frac{1}{2}\alpha\cos\frac{1}{2}\alpha\right)^k+\left(2\cos^2\frac{1}{2}\alpha-2i\sin\frac{1}{2}\alpha\cos\frac{1}{2}\alpha\right)^k$$

$$= \left(2\cos\left(\frac{1}{2}\alpha\right)\right)^k\left(\cos\frac{1}{2}\alpha+i\sin\frac{1}{2}\alpha\right)^k+\left(2\cos\left(\frac{1}{2}\alpha\right)\right)^k\left(\cos\frac{1}{2}\alpha-i\sin\frac{1}{2}\alpha\right)^k$$

$$= \left(2\cos\left(\frac{1}{2}\alpha\right)\right)^k\left(\operatorname{cis}\frac{1}{2}\alpha\right)^k+\left(2\cos\left(\frac{1}{2}\alpha\right)\right)^k\left(\operatorname{cis}\left(-\frac{1}{2}\alpha\right)\right)^k$$

$$= \left(2\cos\left(\frac{1}{2}\alpha\right)\right)^k\left[\left(\operatorname{cis}\frac{1}{2}\alpha\right)^k+\left(\operatorname{cis}\left(-\frac{1}{2}\alpha\right)\right)^k\right]$$

$$= \left(2\cos\left(\frac{1}{2}\alpha\right)\right)^k\left[\operatorname{cis}\left(\frac{1}{2}k\alpha\right)+\operatorname{cis}\left(-\frac{1}{2}k\alpha\right)\right]$$

$$= \left(2\cos\left(\frac{1}{2}\alpha\right)\right)^k\left[\cos\left(\frac{1}{2}k\alpha\right)+i\sin\left(\frac{1}{2}k\alpha\right)+\cos\left(\frac{1}{2}k\alpha\right)-i\sin\left(\frac{1}{2}k\alpha\right)\right]$$

$$= \left(2\cos\left(\frac{1}{2}\alpha\right)\right)^k\left[2\cos\left(\frac{1}{2}k\alpha\right)\right]$$

$$= 2^{k+1}\cos^k\left(\frac{1}{2}\alpha\right)\cos\left(\frac{1}{2}k\alpha\right)$$

$$= 2^{k+1}\cos\left(\frac{1}{2}k\alpha\right)\cos^k\left(\frac{1}{2}\alpha\right)$$

Solutions to Exercise 3A Enrichment questions

18a $1 + z + z^2 + \ldots + z^{2n-1}$

$= \dfrac{z^{2n} - 1}{z - 1}$ (Geometric sum noting z $\neq 1$, by definition)

$= \dfrac{\text{cis } 2\pi - 1}{z - 1}$ (de Moivre $\text{cis}^n\theta = \cos(n\theta)$)

$= \dfrac{1 - 1}{z - 1}$

$= 0$

Alternatively,

$1 + z + z^2 + \ldots z^{2n-1}$

$= 1 + z + z^2 + \ldots + z^{n-1} + z^n + z^{n+1} + z^{n+2} + \cdots + z^{2n-1}$

$= (1 + z + z^2 + \cdots + z^{n-1}) + z^n(1 + z + z^2 + \cdots + z^{n-1})$

$= (1 + z + z^2 + \cdots + z^{n-1})(1 + z^n)$

But,

$(1 + z^n)$

$= 1 + \left(\text{cis}\dfrac{\pi}{n}\right)^n$

$= 1 + \text{cis } \pi$ (By de Moivre)

$= 1 - 1$

$= 0$

Hence,

$1 + z + z^2 + \ldots z^{2n-1}$

$(= 1 + z + z^2 + \cdots + z^{n-1}) \times 0$

$= 0$

18b LHS

$$= \frac{z^n - 1}{z - 1} \text{ (Geometric sum noting z } \neq 1\text{, by definition)}$$

$$= \frac{-1 - 1}{z - 1} \text{ (By De Moivre)}$$

$$= \frac{2}{1 - z}$$

$$= \frac{2}{1 - \operatorname{cis}\left(\frac{\pi}{n}\right)} \times \frac{\operatorname{cis}\frac{-\pi}{2n}}{\operatorname{cis}\frac{-\pi}{2n}}$$

$$= \frac{2\operatorname{cis}\frac{-\pi}{2n}}{\operatorname{cis}\frac{-\pi}{2n} - cis\frac{\pi}{2n}}$$

$$= \frac{2\left(\cos\frac{\pi}{2n} - i\sin\frac{\pi}{2n}\right)}{-2i\sin\frac{\pi}{2n}}$$

$$= -\frac{1}{i}\cot\frac{\pi}{2n} + 1$$

$$= 1 + i\cot\frac{\pi}{2n}$$

Solutions to Exercise 3B Foundation questions

1a $\cos 3\theta + i\sin 3\theta$

$= (\cos\theta + i\sin\theta)^3$

$= \cos^3\theta + 3\cos^2\theta\,(i\sin\theta) + 3\cos\theta\,(i\sin\theta)^2 + (i\sin\theta)^3$

$= \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$

$= \cos^3\theta - 3\cos\theta\sin^2\theta + i(3\cos^2\theta\sin\theta - \sin^3\theta)$

1a i Equating the real components:

$$\begin{aligned}\cos 3\theta &= \cos^3\theta - 3\cos\theta\sin^2\theta\\ &= \cos^3\theta - 3\cos\theta\,(1-\cos^2\theta)\\ &= \cos^3\theta - 3\cos\theta + 3\cos^3\theta\\ &= 4\cos^3\theta - 3\cos\theta\end{aligned}$$

1aii Equating imaginary components:

$$\begin{aligned}\sin 3\theta &= 3\cos^2\theta\sin\theta - \sin^3\theta\\ &= 3(1-\sin^2\theta)\sin\theta - \sin^3\theta\\ &= 3\sin\theta - 3\sin^3\theta - \sin^3\theta\\ &= 3\sin\theta - 4\sin^3\theta\end{aligned}$$

1b $\tan 3\theta$

$$\begin{aligned}&= \frac{\sin 3\theta}{\cos 3\theta}\\ &= \frac{3\sin\theta - 4\sin^3\theta}{4\cos^3\theta - 3\cos\theta}\\ &= \frac{3\sin\theta - 4\sin^3\theta}{4\cos^3\theta - 3\cos\theta} \div \frac{\cos^3\theta}{\cos^3\theta}\\ &= \frac{3\tan\theta\sec^2\theta - 4\tan^3\theta}{4 - 3\sec^2\theta}\\ &= \frac{3\tan\theta\left(\tan^2\theta + 1\right) - 4\tan^3\theta}{4 - 3\left(\tan^2\theta + 1\right)}\\ &= \frac{3\tan^3\theta + 3\tan\theta - 4\tan^3\theta}{4 - 3\tan^2\theta - 3}\\ &= \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}\end{aligned}$$

2 $\cos 4\theta + i\sin 4\theta$

$$\begin{aligned}&= (\cos\theta + i\sin\theta)^4\\ &= \cos^4\theta + 4\cos^3\theta\,(i\sin\theta) + 6\cos^2\theta\,(i\sin\theta)^2 + 4\cos\theta\,(i\sin\theta)^3 + (i\sin\theta)^4\\ &= \cos^4\theta + 4i\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta - 4i\cos\theta\sin^3\theta + \sin^4\theta\end{aligned}$$

2a Equating the real components of the above equation:

$$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$$

2b Equating the imaginary components of the above equation:

$$\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$$

2c $\tan 4\theta$

$$\begin{aligned}&= \frac{\sin 4\theta}{\cos 4\theta}\\ &= \frac{4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta}{\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta}\\ &= \frac{4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta}{\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta} \div \frac{\cos^4\theta}{\cos^4\theta}\\ &= \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}\end{aligned}$$

3a $z^n + z^{-n}$

$$\begin{aligned}&= (\cos\theta + i\sin\theta)^n + (\cos\theta + i\sin\theta)^{-n}\\ &= \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta\\ &= 2\cos n\theta\end{aligned}$$

3b $(z + z^{-1})^4$

$$\begin{aligned}&= z^4 + 4z^3z^{-1} + 6z^2z^{-2} + 4z^1z^{-3} + z^{-4}\\ &= z^4 + 4z^2 + 6 + 4z^{-2} + z^{-4}\\ &= (z^4 + z^{-4}) + 4(z^2 + z^{-2}) + 6\end{aligned}$$

3c $\cos^4\theta$

$$= \left(\frac{1}{2} \times 2\cos\theta\right)^4$$
$$= \left(\frac{1}{2}(z + z^{-1})\right)^4$$
$$= \frac{1}{16}(z + z^{-1})^4$$
$$= \frac{1}{16}\left((z^4 + z^{-4}) + 4(z^2 + z^{-2}) + 16\right)$$
$$= \frac{1}{16}(2\cos 4\theta + 8\cos 2\theta + 16)$$
$$= \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + 1$$

4 $z^n - z^{-n}$

$$= (\cos\theta + i\sin\theta)^n - (\cos\theta + i\sin\theta)^{-n}$$
$$= \cos n\theta + i\sin n\theta - (\cos n\theta - i\sin n\theta)$$
$$= \cos n\theta + i\sin n\theta - \cos n\theta + i\sin n\theta$$
$$= 2i\sin n\theta$$

$\sin^4\theta$

$$= \left(\frac{1}{2} \times 2\sin\theta\right)^4$$
$$= \left(\frac{1}{2i}(z - z^{-1})\right)^4$$
$$= \frac{1}{(2i)^4}(z - z^{-1})^4$$
$$= \frac{1}{2^4 i^4}(z^4 - 4z^3z^{-1} + 6z^2z^{-2} - 4zz^{-3} + z^{-4})$$
$$= \frac{1}{16}\left((z^4 + z^{-4}) - 4(z^2 + z^{-2}) + 6\right)$$
$$= \frac{1}{16}(2\cos 4\theta - 8\cos 2\theta + 6)$$
$$= \frac{\cos 4\theta}{8} - \frac{\cos 2\theta}{2} + \frac{3}{8}$$

Solutions to Exercise 3B Development questions

5a Let $z = \cos\theta + i\sin\theta$

From question 3,

$z^n + z^{-n} = 2\cos n\theta$

Now following question 3 expanding $(z + z^{-1})^5$ gives,

$(z + z^{-1})^5$

$= z^5 + 5z^3 + 10z + 10z^{-1} + 5z^{-3} + z^{-5}$

$= (z^5 + z^{-5}) + 5(z^3 + z^{-3}) + 10(z + z^{-1})$

$= 2\cos(5\theta) + 5(2\cos(3\theta)) + 10(2\cos(\theta))$ (using the result above from Q3)

$= 2\cos 5\theta + 10\cos 3\theta + 20\cos\theta$

Hence, using the result of 3 question again for the LHS, we have,

$2^5\cos^5\theta = 2\cos 5\theta + 10\cos 3\theta + 20\cos\theta$

$$\cos^5\theta = \frac{1}{2^5}(2\cos 5\theta + 10\cos 3\theta + 20\cos\theta)$$

$$\cos^5\theta = \frac{1}{16}(\cos 5\theta + 5\cos 3\theta + 10\cos\theta)$$

5b

$$\int_0^{\frac{\pi}{2}} \cos^5\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{16}(\cos 5\theta + 5\cos 3\theta + 10\cos\theta)\, d\theta$$

$$= \left[\frac{1}{16}\left(\frac{1}{5}\sin 5\theta + \frac{5}{3}\sin 3\theta + 10\sin\theta\right)\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{16}\left(\frac{1}{5}\sin\frac{5\pi}{2} + \frac{5}{3}\sin\frac{3\pi}{2} + 10\sin\frac{\pi}{2}\right)$$

$$= \frac{1}{16}\left(\frac{1}{5}(1) + \frac{5}{3}(-1) + 10(1)\right)$$

$$= \frac{8}{15}$$

6a $\cos 6\alpha + \mathrm{i}\sin 6\alpha$

$= \operatorname{cis}(6\alpha)$

$= \left(\operatorname{cis}(\alpha)\right)^6$

$= (\cos\alpha + i\sin\alpha)^6$

$= \cos^6\alpha + 6i\cos^5\alpha\sin\alpha + 15i^2\cos^4\alpha\sin^2\alpha + 20i^3\cos^3\alpha\sin^3\alpha$

$+15i^4\cos^2\alpha\sin^4\alpha + 6i^5\cos\alpha\sin^5\alpha + i^6\sin^6\alpha$

$= \cos^6\alpha + 6i\cos^5\alpha\sin\alpha - 15\cos^4\alpha\sin^2\alpha - 20i\cos^3\alpha\sin^3\alpha$

$+15\cos^2\alpha\sin^4\alpha + 6i\cos\alpha\sin^5\alpha - \sin^6\alpha$

$= (\cos^6\alpha - 15\cos^4\alpha\sin^2\alpha + 15\cos^2\alpha\sin^4\alpha - \sin^6\alpha)$

$+i(6\cos^5\alpha\sin\alpha - 20\cos^3\alpha\sin^3\alpha + 6\cos\alpha\sin^5\alpha)$

Equating real and imaginary components

$\cos 6\alpha$

$= \cos^6\alpha - 15\cos^4\alpha\sin^2\alpha + 15\cos^2\alpha\sin^4\alpha - \sin^6\alpha$

$= \cos^6\alpha - 15\cos^4\alpha\,(1-\cos^2\alpha) + 15\cos^2\alpha\,(1-\cos^2\alpha)^2 - (1-\cos^2\alpha)^3$

$= \cos^6\alpha - 15\cos^4\alpha\,(1-\cos^2\alpha) + 15\cos^2\alpha\,(1-2\cos^2\alpha+\cos^4\alpha)$

$-(1-3\cos^2\alpha+3\cos^4\alpha-\cos^6\alpha)$

$= 32\cos^6\alpha - 48\cos^4\alpha + 18\cos^2\alpha - 1$

6b $32\cos^6\alpha - 48\cos^4\alpha + 18\cos^2\alpha - 1 = 0$

is solved when

$\cos 6\alpha = 0$ (from part a)

which is when, $6\alpha = \frac{n\pi}{2}$, for n not divisible by 2. Hence,

$\alpha = \frac{n\pi}{12}$ for $n = 1, 3, 4, 5, 9, 11$ (i.e. n is not divisible by 2)

and, letting $x = \cos\alpha$, the equation becomes,

$32x^6 + 4x^3 - 6x^2 - 4x + 1 = 0$

Which has solutions when,

$x = \cos\frac{n\pi}{12}$ for $n = 1, 3, 5, 7, 9, 11$

6c The product of the roots is

$$\cos\frac{\pi}{12}\cos\frac{3\pi}{12}\cos\frac{5\pi}{12}\cos\frac{7\pi}{12}\cos\frac{9\pi}{12}\cos\frac{11\pi}{12}=\frac{-1}{32}$$

$$\left(\cos\frac{\pi}{12}\right)\left(\cos\frac{\pi}{4}\right)\left(\cos\frac{5\pi}{12}\right)\left(-\cos\frac{5\pi}{12}\right)\left(\cos\frac{3\pi}{4}\right)\left(-\cos\frac{\pi}{12}\right)=\frac{-1}{32}$$

$$\left(\cos\frac{\pi}{12}\right)\left(\frac{1}{\sqrt{2}}\right)\left(\cos\frac{5\pi}{12}\right)\left(-\cos\frac{5\pi}{12}\right)\left(\frac{-1}{\sqrt{2}}\right)\left(-\cos\frac{\pi}{12}\right)=\frac{-1}{32}$$

$$\left(\cos\frac{\pi}{12}\right)\left(\frac{1}{\sqrt{2}}\right)\left(\cos\frac{5\pi}{12}\right)\left(\cos\frac{5\pi}{12}\right)\left(\frac{1}{\sqrt{2}}\right)\left(\cos\frac{\pi}{12}\right)=\frac{1}{32}$$

$$\frac{1}{2}\left(\cos\frac{\pi}{12}\cos\frac{5\pi}{12}\right)^2=\frac{1}{32}$$

$$\left(\cos\frac{\pi}{12}\cos\frac{5\pi}{12}\right)^2=\frac{1}{16}$$

$$\cos\frac{\pi}{12}\cos\frac{5\pi}{12}=\pm\frac{1}{4}$$

Since $\cos\frac{\pi}{12}>0$ and $\cos\frac{5\pi}{12}>0$ it follows that

$$\cos\frac{\pi}{12}\cos\frac{5\pi}{12}=\frac{1}{4}$$

7a Let $x=\tan\theta$

$$x^4+4x^3-6x^2-4x+1=0$$

$$\tan^4\theta+4\tan^3\theta-6\tan^2\theta-4\tan\theta+1=0$$

$$\tan^4\theta-6\tan^2\theta+1=4\tan\theta-4\tan^3\theta$$

$$1=\frac{4\tan^3\theta-4\tan\theta}{\tan^4\theta-6\tan^2\theta+1}$$

$$1=\tan 4\theta$$

$$4\theta=\frac{\pi}{4},\frac{5\pi}{4},\frac{9\pi}{4},\frac{13\pi}{4}$$

$$\theta=\frac{\pi}{16},\frac{5\pi}{16},\frac{9\pi}{16},\frac{13\pi}{16}$$

Hence the equation is solved when $x=\tan\frac{\pi}{16},\tan\frac{5\pi}{16},\tan\frac{9\pi}{16},\tan\frac{13\pi}{16}$

7b The sum of the roots is

$$\tan\frac{\pi}{16}+\tan\frac{5\pi}{16}+\tan\frac{9\pi}{16}+\tan\frac{13\pi}{16}=-4$$

Hence

$$\left(\tan\frac{\pi}{16}+\tan\frac{5\pi}{16}+\tan\frac{9\pi}{16}+\tan\frac{13\pi}{16}\right)^2=16$$

$$\tan^2\frac{\pi}{16}+\tan^2\frac{5\pi}{16}+\tan^2\frac{9\pi}{16}+\tan^2\frac{13\pi}{16}$$

$$+2\left(\tan\frac{\pi}{16}\tan\frac{5\pi}{16}+\tan\frac{\pi}{16}\tan\frac{9\pi}{16}+\tan\frac{\pi}{16}\tan\frac{13\pi}{16}+\tan\frac{5\pi}{16}\tan\frac{9\pi}{16}\right.$$
$$\left.+\tan\frac{5\pi}{16}\tan\frac{13\pi}{16}+\tan\frac{9\pi}{16}\tan\frac{13\pi}{16}\right)=16$$

However, the term in the brackets is just the sum of the products of the roots, hence

$$\tan^2\frac{\pi}{16}+\tan^2\frac{5\pi}{16}+\tan^2\frac{9\pi}{16}+\tan^2\frac{13\pi}{16}+2(-6)=16$$

$$\tan^2\frac{\pi}{16}+\tan^2\frac{5\pi}{16}+\tan^2\frac{9\pi}{16}+\tan^2\frac{13\pi}{16}=28$$

$$\tan^2\frac{\pi}{16}+\tan^2\frac{5\pi}{16}+\left(\tan\left(\pi-\frac{7\pi}{16}\right)\right)^2+\left(\tan\left(\pi-\frac{3\pi}{16}\right)\right)^2=28$$

$$\tan^2\frac{\pi}{16}+\tan^2\frac{5\pi}{16}+\left(-\tan\frac{7\pi}{16}\right)^2+\left(-\tan\frac{3\pi}{16}\right)^2=28$$

$$\tan^2\frac{\pi}{16}+\tan^2\frac{3\pi}{16}+\tan^2\frac{5\pi}{16}+\tan^2\frac{7\pi}{16}=28$$

8a $\cos 5\theta + i\sin 5\theta$

$= \mathrm{cis}(5\theta)$

$= \left(\mathrm{cis}(\theta)\right)^5$

$= (\cos\theta + i\sin\theta)^5$

$= \cos^5\theta + 5i\cos^4\theta\sin\theta + 10i^2\cos^3\theta\sin^2\theta + 10i^3\cos^2\theta\sin^3\theta$

$\quad +5i^4\cos\theta\sin^4\theta + i^5\sin^5\theta$

$= \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$

$\quad +i(5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta)$

Equating the imaginary parts of this equation gives:

$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$

$= 5(1-\sin^2\theta)^2\sin\theta - 10(1-\sin^2\theta)\sin^3\theta + \sin^5\theta$

$= 5(1-2\sin^2\theta+\sin^4\theta)\sin\theta - 10(1-\sin^2\theta)\sin^3\theta + \sin^5\theta$

$= 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$

8b Let $x = \sin\theta$

$16x^5 - 20x^3 + 5x - 1 = 0$

$16\sin^5\theta - 20\sin^3\theta + 5\sin\theta - 1 = 0$

$\sin 5\theta - 1 = 0$ (from part a)

$\sin 5\theta = 1$

$5\theta = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \frac{13\pi}{2}, \frac{17\pi}{2}$

$\theta = \frac{\pi}{10}, \frac{\pi}{2}, \frac{9\pi}{10}, \frac{13\pi}{10}, \frac{17\pi}{10}$

$x = \sin\frac{\pi}{10}, \sin\frac{\pi}{2}, \sin\frac{9\pi}{10}, \sin\frac{13\pi}{10}, \sin\frac{17\pi}{10}$

$x = 1, \sin\frac{\pi}{10}, \sin\frac{9\pi}{10}, \sin\frac{13\pi}{10}, \sin\frac{17\pi}{10}$

as required

8c $(4x^2 + bx + c)^2$

$= 16x^4 + 4bx^3 + 4cx^2 + 4bx^3 + b^2x^2 + bcx + 4cx^2 + bcx + c^2$

$= 16x^4 + (4b + 4b)x^3 + (4c + b^2 + 4c)x^2 + (bc + bc)x + c^2$

$= 16x^4 + 8bx^3 + (8c + b^2)x^2 + 2bcx + c^2$

Since

$16x^4 + 16x^3 - 4x^2 - 4x + 1 = (4x^2 + bx + c)^2$

$16x^4 + 16x^3 - 4x^2 - 4x + 1 = 16x^4 + 8bx^3 + (8c + b^2)x^2 + 2bcx + c^2$

Equating coefficients of x gives

$16 = 8b$ (1)

$-4 = 8c + b^2$ (2)

From (1), $b = 2$.

Substituting this into (2) gives $c = -1$.

Any root of $16x^4 + 16x^3 - 4x^2 - 4x + 1$ is necessarily a root of $(4x^2 + bx + c)$. Since there are two factors of $(4x^2 + bx + c)$ in the original equation, any root will be a root of the quadratic and will be a double root of the initial equations. Since quadratics have two roots, it follows that $16x^4 + 16x^3 - 4x^2 - 4x + 1 = 0$ must have two double roots (or one quadruple root).

8d Dividing $16x^5 - 20x^3 + 5x - 1$ by $(x - 1)$ yields $16x^4 + 16x^3 - 4x^2 + 1$.

Hence it follows that since $x = 1, \sin\frac{\pi}{10}, \sin\frac{9\pi}{10}, \sin\frac{13\pi}{10}, \sin\frac{17\pi}{10}$ are roots of the former equation, and since $x = 1$ is a root of the divisor,

$x = \sin\frac{\pi}{10}, \sin\frac{9\pi}{10}, \sin\frac{13\pi}{10}, \sin\frac{17\pi}{10}$ must be roots of the quotient $16x^4 + 16x^3 - 4x^2 + 1$ as required.

There is no contradiction to part c as $\sin\frac{9\pi}{10} = \sin\left(\pi - \frac{\pi}{10}\right) = \sin\frac{\pi}{10}$ and

$\sin\frac{13\pi}{10} = \sin\left(3\pi - \frac{13\pi}{10}\right) = \sin\frac{17\pi}{10}$ and so indeed there are two double roots.

8e The sum of the roots is

$$\sin\frac{\pi}{10}+\sin\frac{\pi}{10}+\sin\frac{13\pi}{10}+\sin\frac{13\pi}{10}+1=0$$

$$\sin\frac{\pi}{10}+\sin\frac{\pi}{10}-\sin\frac{3\pi}{10}-\sin\frac{3\pi}{10}+1=0$$

$$2\sin\frac{\pi}{10}-2\sin\frac{3\pi}{10}+1=0$$

$$\sin\frac{\pi}{10}-\sin\frac{3\pi}{10}+\frac{1}{2}=0$$

Hence

$$\sin\frac{\pi}{10}=\sin\frac{3\pi}{10}-\frac{1}{2} \qquad (1)$$

The product of the roots is

$$\sin\frac{\pi}{10}\sin\frac{\pi}{10}\sin\frac{13\pi}{10}\sin\frac{13\pi}{10}(1)=\frac{1}{16}$$

$$\sin\frac{\pi}{10}\sin\frac{\pi}{10}\left(-\sin\frac{3\pi}{10}\right)\left(-\sin\frac{3\pi}{10}\right)=\frac{1}{16}$$

$$\sin^2\frac{\pi}{10}\sin^2\frac{3\pi}{10}=\frac{1}{16} \qquad (2)$$

Hence,

$\sin\frac{\pi}{10}\sin\frac{3\pi}{10}=\frac{1}{4}$ (we take the positive solution as $\sin\frac{\pi}{10}>0$ and $\sin\frac{3\pi}{10}>0$)

Substituting (1) in (2) gives:

$$\left(\sin\frac{3\pi}{10}-\frac{1}{2}\right)\sin\frac{3\pi}{10}=\frac{1}{4}$$

$$\sin^2\frac{3\pi}{10}-\frac{1}{2}\sin\frac{3\pi}{10}-\frac{1}{4}=0$$

So,

$$\sin\frac{3\pi}{10}=\frac{-\left(-\frac{1}{2}\right)\pm\sqrt{\left(-\frac{1}{2}\right)^2-4(1)\left(-\frac{1}{4}\right)}}{2}$$

$$=\frac{1\pm\sqrt{5}}{4}$$

Since $\sin\dfrac{3\pi}{10} > 0$,

$$\sin\frac{3\pi}{10} = \frac{1+\sqrt{5}}{4}$$

$$\sin\frac{\pi}{10} = \frac{1+\sqrt{5}}{4} - \frac{1}{2}$$

$$= \frac{-1+\sqrt{5}}{4}$$

$$= \frac{\sqrt{5}-1}{4}$$

9a $\cos 7\theta + i\sin 7\theta$

$= \text{cis}(7\theta)$

$= \left(\text{cis}(\theta)\right)^7$

$= (\cos\theta + i\sin\theta)^7$

$= \cos^7\theta + 7i\cos^6\theta\sin\theta + 21i^2\cos^5\theta\sin^2\theta + 35i^3\cos^4\theta\sin^3\theta$

$+35i^4\cos^3\theta\sin^4\theta + 21i^5\cos^2\theta\sin^5\theta + 7i^6\cos\theta\sin^6\theta + i^7\sin^7\theta$

$= \cos^7\theta + 7i\cos^6\theta\sin\theta - 21\cos^5\theta\sin^2\theta - 35i\cos^4\theta\sin^3\theta + 35\cos^3\theta\sin^4\theta$

$+21i\cos^2\theta\sin^5\theta - 7\cos\theta\sin^6\theta - i\sin^7\theta$

Equating the real components on both sides of the equation

$\cos 7\theta = \cos^7\theta - 21\cos^5\theta\sin^2\theta + 35\cos^3\theta\sin^4\theta - 7\cos\theta\sin^6\theta$

$= \cos^7\theta - 21\cos^5\theta(1-\cos^2\theta) + 35\cos^3\theta(1-\cos^2\theta)^2$

$-7\cos\theta(1-\cos^2\theta)^3$

$= \cos^7\theta - 21\cos^5\theta(1-\cos^2\theta) + 35\cos^3\theta(1-2\cos^2\theta+\cos^4\theta)$

$-7\cos\theta(1-3\cos^2\theta+3\cos^4\theta-\cos^6\theta)$

$= 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta$

9b Let $x = 4\cos^2\theta$, then the polynomial becomes,

$(4\cos^2\theta)^3 - 7(4\cos^2\theta)^2 + 14(4\cos^2\theta) - 7 = 0$

$64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta = 0$ (multiply by $\cos\theta$)

Noting that roots in the form $\cos\theta = 0$, won't be solutions of the original polynomial. Now using part a we have,

$\cos 7\theta = 0$

Which is true whenever we have,

$7\theta = n\pi \pm \frac{\pi}{2}$ where n is an integer, Hence,

$$7\theta = \left(\frac{2n \pm 1}{2}\right)\pi$$

$$\theta = \left(\frac{2n \pm 1}{14}\right)\pi$$

So

$$x = 4\cos^2\left(\frac{2n \pm 1}{14}\right)\pi$$ where n is an integer, hence some unique roots are:

$$x = 4\cos^2\frac{\pi}{14}, 4\cos^2\frac{3\pi}{14}, 4\cos^2\frac{5\pi}{14}, n = 0, 1, 2 \text{ and taking } + 1$$

9c i The sum of the roots in the above equation is

$$4\cos^2\frac{\pi}{14} + 4\cos^2\frac{3\pi}{14} + 4\cos^2\frac{5\pi}{14} = \frac{-(-7)}{1} = 7$$

Hence

$$\cos^2\frac{\pi}{14} + \cos^2\frac{3\pi}{14} + \cos^2\frac{5\pi}{14} = \frac{7}{4}$$

9c ii

$$\left(\cos^2\frac{\pi}{14}+\cos^2\frac{3\pi}{14}+\cos^2\frac{5\pi}{14}\right)^2=\left(\frac{7}{4}\right)^2 \quad \text{(using 9c i)}$$

Expanding gives,

$$\cos^4\frac{\pi}{14}+\cos^4\frac{3\pi}{14}+\cos^4\frac{5\pi}{14}$$
$$+2\left(\cos^2\frac{\pi}{14}\cos^2\frac{3\pi}{14}+\cos^2\frac{3\pi}{14}\cos^2\frac{5\pi}{14}+\cos^2\frac{\pi}{14}\cos^2\frac{5\pi}{14}\right)$$
$$=\left(\frac{7}{4}\right)^2$$

$$\cos^4\frac{\pi}{14}+\cos^4\frac{3\pi}{14}+\cos^4\frac{5\pi}{14}$$
$$+\frac{2}{16}\left(4\cos^2\frac{\pi}{14}4\cos^2\frac{3\pi}{14}+4\cos^2\frac{3\pi}{14}4\cos^2\frac{5\pi}{14}+4\cos^2\frac{\pi}{14}4\cos^2\frac{5\pi}{14}\right)$$
$$=\left(\frac{7}{4}\right)^2$$

Now using the sum of root products, we have,

$$4\cos^2\frac{\pi}{14}4\cos^2\frac{3\pi}{14}+4\cos^2\frac{3\pi}{14}4\cos^2\frac{5\pi}{14}+4\cos^2\frac{\pi}{14}4\cos^2\frac{5\pi}{14}=\frac{14}{1}=14$$

Hence, we have,

$$\cos^4\frac{\pi}{14}+\cos^4\frac{3\pi}{14}+\cos^4\frac{5\pi}{14}+\frac{2}{16}(14)=\left(\frac{7}{4}\right)^2$$

$$\cos^4\frac{\pi}{14}+\cos^4\frac{3\pi}{14}+\cos^4\frac{5\pi}{14}=\left(\frac{7}{4}\right)^2-\frac{28}{16}$$

$$\cos^4\frac{\pi}{14}+\cos^4\frac{3\pi}{14}+\cos^4\frac{5\pi}{14}=\frac{21}{16}$$

10a Let $z = \cos\theta + i\sin\theta = \text{cis}(\theta)$

Using the result from question 3 we have,

$z^n - z^{-n} = 2i\sin n\theta$

We also have that,

$(z - z^{-1})^5$

$= z^5 - 5z^3 + 10z - 10z^{-1} + 5z^{-3} - z^{-5}$

$= (z^5 - z^{-5}) - 5(z^3 - z^{-3}) + 10(z - z^{-1})$

Using the result above we get,

$(2i\sin\theta)^5 = 2i\sin 5\theta - 5(2i\sin 3\theta) + 10(2i\sin\theta)$

$2^5 i^5 \sin^5\theta = 2i\sin 5\theta - 5(2i\sin 3\theta) + 10(2i\sin\theta)$

$2^5 i\sin^5\theta = 2i\sin 5\theta - 5(2i\sin 3\theta) + 10(2i\sin\theta)$

$2^4\sin^5\theta = \sin 5\theta - 5\sin 3\theta + 10\sin\theta$

$\sin^5\theta = \dfrac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$

10b From above we know that

$16\sin^5\theta = \sin 5\theta - 5\sin 3\theta + 10\sin\theta$

Hence, $16\sin^5\theta = \sin 5\theta$ if and only if

$-5\sin 3\theta + 10\sin\theta = 0$

$10\sin\theta = 5\sin 3\theta$

$\sin\theta = \frac{1}{2}\sin 3\theta$

$\sin\theta = \frac{1}{2}(\sin\theta\cos 2\theta + \sin 2\theta\cos\theta)$ (using angle sum identity)

$\sin\theta = \frac{1}{2}(\sin\theta\,(1 - 2\sin^2\theta) + (2\sin\theta\cos\theta)\cos\theta)$ (double angle identity)

$\sin\theta = \frac{1}{2}(\sin\theta\,(1 - 2\sin^2\theta) + 2\sin\theta\cos^2\theta)$

$\sin\theta = \frac{1}{2}\left(\sin\theta\,(1 - 2\sin^2\theta) + 2\sin\theta\,(1 - \sin^2\theta)\right)$

$\sin\theta = \frac{1}{2}\sin\theta\left((1 - 2\sin^2\theta) + 2(1 - \sin^2\theta)\right)$

$\sin\theta = \frac{1}{2}\sin\theta\,(3 - 4\sin^2\theta)$

$\frac{1}{2}\sin\theta\,(3 - 4\sin^2\theta) - \sin\theta = 0$

$\sin\theta\,(3 - 4\sin^2\theta) - 2\sin\theta = 0$

$\sin\theta\,(1 - 4\sin^2\theta) = 0$

$\sin\theta\,(1 - 2\sin\theta)(1 + 2\sin\theta) = 0$

Hence the solution occurs whenever $\sin\theta = 0, \pm\frac{1}{2}$

Thus, $\theta = 0, \frac{\pi}{6}, \frac{5\pi}{6}, \pi, \frac{7\pi}{6}, \frac{11\pi}{6}, \ldots$

11a Let $z = \cos\theta + i\sin\theta = \text{cis}(\theta)$

So,

$\cos 5\theta + i\sin 5\theta$

$= \text{cis}(5\theta)$

$= \left(\text{cis}(\theta)\right)^5$

$= (\cos\theta + i\sin\theta)^5$

$= \cos^5\theta + 5i\cos^4\theta\sin\theta + 10i^2\cos^3\theta\sin^2\theta + 10i^3\cos^2\theta\sin^3\theta$
$\quad +5i^4\cos\theta\sin^4\theta + i^5\sin^5\theta$

$= (\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta)$
$\quad +i(5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta)$

Equating real and imaginary components,

$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$

$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$

$$\begin{aligned}\tan 5\theta &= \frac{\sin 5\theta}{\cos 5\theta}\\ &= \frac{5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta}{\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta}\\ &= \frac{5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta}{\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta} \div \frac{\cos^5\theta}{\cos^5\theta}\\ &= \frac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}\end{aligned}$$

11b Let $x = \tan\theta$, then $x^4 - 10x^2 + 5 = 0$ becomes,

$\tan^4\theta - 10\tan^2\theta + 5 = 0$

$\tan\theta\,(\tan^4\theta - 10\tan^2\theta + 5) = \tan\theta \times 0$

$\tan^5\theta - 10\tan^3\theta + 5\tan\theta = 0$

$$\frac{\tan^5\theta - 10\tan^3\theta + 5\tan\theta}{1 - 10\tan^2\theta + 5\tan^4\theta} = 0$$

$\tan 5\theta = 0$ (from part a)

$5\theta = \pm\pi, \pm 2\pi$

$\theta = \pm\frac{\pi}{5}, \pm\frac{2\pi}{5}$

Hence the roots are

$x = \tan\left(\pm\frac{\pi}{5}\right), \tan\left(\pm\frac{2\pi}{5}\right)$

$x = \pm\tan\left(\frac{\pi}{5}\right), \pm\tan\left(\frac{2\pi}{5}\right)$

11c The product of roots is

$$\left(\tan\left(\frac{\pi}{5}\right)\right)\left(-\tan\left(\frac{\pi}{5}\right)\right)\left(\tan\left(\frac{2\pi}{5}\right)\right)\left(-\tan\left(\frac{2\pi}{5}\right)\right) = 5$$

$\tan^2\frac{\pi}{5}\tan^2\frac{2\pi}{5} = 5$

$\tan\frac{\pi}{5}\tan\frac{2\pi}{5} = \sqrt{5}$ $\left(\text{taking the positive solution as } \tan\frac{\pi}{5} > 0 \text{ and } \tan\frac{2\pi}{5} > 0\right)$

The product of the pairs of roots is

$$\left(\tan\left(\frac{\pi}{5}\right)\right)\left(-\tan\left(\frac{\pi}{5}\right)\right) + \left(\tan\left(\frac{2\pi}{5}\right)\right)\left(-\tan\left(\frac{2\pi}{5}\right)\right) + \left(\tan\left(\frac{\pi}{5}\right)\right)\left(\tan\left(\frac{2\pi}{5}\right)\right)$$
$$+ \left(-\tan\left(\frac{\pi}{5}\right)\right)\left(\tan\left(\frac{2\pi}{5}\right)\right) + \left(\tan\left(\frac{\pi}{5}\right)\right)\left(-\tan\left(\frac{2\pi}{5}\right)\right) + \left(-\tan\left(\frac{\pi}{5}\right)\right)\left(-\tan\left(\frac{2\pi}{5}\right)\right)$$

$= -10$

Which simplifies to,

$\tan^2\frac{\pi}{5} + \tan^2\frac{2\pi}{5} = 10$

12a

$$z^n + \frac{1}{z^n}$$

$$= \left(\text{cis}(\theta)\right)^n + \frac{1}{\left(\text{cis}(\theta)\right)^n}$$

$$= \text{cis}(n\theta) + \frac{1}{\text{cis}(n\theta)}$$

$$= \text{cis}(n\theta) + \text{cis}(-n\theta)$$

$$= \cos n\theta + i \sin n\theta + \cos n\theta - i \sin n\theta$$

$$= 2 \cos n\theta$$

$$z^n - \frac{1}{z^n}$$

$$= \left(\text{cis}(\theta)\right)^n - \frac{1}{\left(\text{cis}(\theta)\right)^n}$$

$$= \text{cis}(n\theta) - \frac{1}{\text{cis}(n\theta)}$$

$$= \text{cis}(n\theta) - \text{cis}(-n\theta)$$

$$= \cos n\theta + i \sin n\theta - (\cos n\theta - i \sin n\theta)$$

$$= 2i \sin n\theta$$

12b $128 \cos^3 \theta \sin^4 \theta$

$$= 8\cos^3 \theta \, 16 \sin^4 \theta$$

$$= (2 \cos \theta)^3 (2i \sin \theta)^4$$

$$= \left(z + \frac{1}{z}\right)^3 \left(z - \frac{1}{z}\right)^4 \text{ (from part a)}$$

$$= \left(z^3 + 3z + \frac{3}{z} + \frac{1}{z^3}\right)\left(z^4 - 4z^2 + 6 - \frac{4}{z^2} + \frac{1}{z^4}\right)$$

$$= z^7 - 4z^5 + 6z^3 - 4z + \frac{1}{z} + 3z^5 - 12z^3 + 18z - \frac{12}{z} + \frac{3}{z^3} + 3z^3 - 12z + \frac{18}{z} - \frac{12}{z^3} + \frac{3}{z^5} + z - \frac{4}{z} + \frac{6}{z^3} - \frac{4}{z^5} + \frac{1}{z^7}$$

$$= \left(z^7 + \frac{1}{z^7}\right) - \left(z^5 + \frac{1}{z^5}\right) - 3\left(z^3 + \frac{1}{z^3}\right) + 3\left(z + \frac{1}{z}\right)$$

12c From part b,

$$128\cos^3\theta\sin^4\theta = \left(z^7+\frac{1}{z^7}\right)-\left(z^5+\frac{1}{z^5}\right)-3\left(z^3+\frac{1}{z^3}\right)+3\left(z+\frac{1}{z}\right)$$

Using the result from part a,

$$128\cos^3\theta\sin^4\theta = 2\cos 7\theta - 2\cos 5\theta - 3\times 2\cos 3\theta + 3\times 2\cos\theta$$

Hence

$$\cos^3\theta\sin^4\theta = \frac{1}{64}(\cos 7\theta - \cos 5\theta - 3\cos 3\theta + 3\cos\theta)$$

as required

13a $5z^4 - 11z^3 + 16z^2 - 11z + 5 = 0$

$$5z^2 - 11z + 16 - \frac{11}{z} + \frac{5}{z^2} = 0$$

$$5\left(z^2+\frac{1}{z^2}\right) - 11\left(z+\frac{1}{z}\right) + 16 = 0$$

$$5(2\cos 2\theta) - 11(2\cos\theta) + 16 = 0$$

$$5\cos 2\theta - 11\cos\theta + 8 = 0$$

13b $5(2\cos^2\theta - 1) - 11\cos\theta + 8 = 0$ (using the result in a and double angle id)

$$10\cos^2\theta - 11\cos\theta + 3 = 0$$

$$\cos\theta = \frac{-(-11)\pm\sqrt{(-11)^2-4(10)(3)}}{2(10)}$$

$$= \frac{11\pm\sqrt{1}}{20}$$

$$= \frac{1}{2} \text{ or } \frac{3}{5}$$

When $\cos\theta = \frac{1}{2}, \sin\theta = \pm\frac{\sqrt{2^2-1}}{2} = \pm\frac{\sqrt{3}}{2}$

When $\cos\theta = \frac{3}{5}, \sin\theta = \pm\frac{\sqrt{5^2-3^2}}{5} = \pm\frac{\sqrt{16}}{5} = \pm\frac{4}{5}$

$$z = \frac{1}{2}\pm\frac{\sqrt{3}}{2}i$$

$$z = \frac{3}{5}\pm\frac{4}{5}i$$

14a

$$\frac{\sin 8\theta}{\sin\theta\cos\theta}$$

$$=\frac{2\sin 4\theta\cos 4\theta}{\sin\theta\cos\theta}$$

$$=\frac{2(2\sin 2\theta\cos 2\theta)(2\cos^2 2\theta-1)}{\sin\theta\cos\theta}$$

$$=\frac{2\big(2(2\sin\theta\cos\theta)(1-2\sin^2\theta)\big)(2(1-2\sin^2\theta)^2-1)}{\sin\theta\cos\theta}$$

$$=8(1-2\sin^2\theta)(2(1-2\sin^2\theta)^2-1)$$

Let $s=\sin\theta$

$$=8(1-2s^2)(2(1-2s^2)^2-1)$$

$$=8(1-2s^2)(2(1-4s^2+4s^4)-1)$$

$$=8(1-2s^2)(1-8s^2+8s^4)$$

$$=8(1-8s^2+8s^4-2s^2+16s^4-16s^6)$$

$$=8(1-10s^2+24s^4-16s^6)$$

14b $x^6-6x^4+10x^2-4=0$

Let $x=2\sin\theta=2s$

$$(2s)^6-6(2s)^4+10(2s)^2-4=0$$

$$64s^6-6\times 16s^4+40s^2-4=0$$

$$4(16s^6-24s^4+10s^2-1)=0$$

$$-\frac{1}{2}\big(8(1-10s^2+24s^4-16s^6)\big)=0$$

$$-\frac{1}{2}\left(\frac{\sin 8\theta}{\sin\theta\cos\theta}\right)=0$$

Which is 0 when, $\sin 8\theta=0$. Hence,

$8\theta=n\pi$ for $n=\pm1,\pm2,\pm3$

Thus,

$$x=2\sin\frac{n\pi}{8}\ \text{ for } n=\pm1,\pm2,\pm3$$

Solutions to Exercise 3B Enrichment questions

15a $\text{cis}\,(2n+1)\theta$

$= (\text{cis}\,\theta)^{2n+1}$ (By de Moivre)

$= {}^{2n+1}C_0 \cos^{2n+1}\theta + i \times {}^{2n+1}C_1 \cos^{2n}\theta \sin\theta - {}^{2n+1}C_2 \cos^{2n-1}\theta \sin^2\theta -$

$i \times {}^{2n+1}C_3 \cos^{2n-2}\theta \sin^3\theta + {}^{2n+1}C_4 \cos^{2n-3}\theta \sin^4\theta + i \times {}^{2n+1}C_5 \cos^{2n-4}\theta \sin^5\theta$

$+ \cdots + i^{2n+1} \times {}^{2n+1}C_{2n+1} \sin^{2n+1}\theta$

Take imaginary points, and note $i^{2n} = (-1)^n$, to get:

$\sin(2n+1)\theta$

$= {}^{2n+1}C_1 \cos^{2n}\theta \sin\theta - {}^{2n+1}C_3 \cos^{2n-2}\theta \sin^3\theta + {}^{2n+1}C_5 \cos^{2n-4}\theta \sin^5\theta$

$+ \cdots + (-1)^n \sin^{2n+1}\theta$

15b Divide through by $\sin^{2n+1}\theta$ for $\sin\theta \neq 0$.

$$\frac{\sin(2n+1)\theta}{\sin^{2n+1}\theta} = {}^{2n+1}C_1 \cot^{2n}\theta - {}^{2n+1}C_3 \cot^{2n+1}\theta + {}^{2n+1}C_5 \cot^{2n-4}\theta + \cdots + (-1)^n$$

Let $x = \cot^2\theta$, so that,

$$\frac{\sin(2n+1)\theta}{\sin^{2n+1}\theta} = {}^{2n+1}C_1 x^n - {}^{2n+1}C_3\, x^{n-1}\,\theta + {}^{2n+1}C_5 x^{n-2} + \cdots + (-1)^n = P(x)$$

Now $P(x) = 0$ when $\sin\,(2n+1)\,\theta = 0$ with $\sin\theta \neq 0$, which has solutions

$(2n+1)\theta = k\pi$, for integer k with $k \neq 0$ ($\sin\theta \neq 0$).

For principal values we have $-\pi < \frac{k\pi}{2n+1} \leq \pi$ and so for distinct solutions of x,

$k = 1, 2, 3, \dots, n$ (degree n polynomial)

Hence,

$$\theta = \frac{k\pi}{2n+1}, k = 1, 2, 3, \dots, n$$

Hence, $P(x) = 0$ for,

$$x = \cot^2\frac{k\pi}{2n+1}, k = 1, 2, 3, \dots, n$$

15c Summing the roots of the polynomial:

$$\sum_{k=1}^{n} \cot^2\left(\frac{k\pi}{2n+1}\right)$$

$$= \frac{{}^{2n+1}C_3}{{}^{2n+1}C_1} \qquad \left(-\frac{\text{coefficient of } x^{n-1}}{\text{coefficient of } x^n}\right)$$

$$= \frac{\cancel{(2n+1)!}}{3!\,(2n-2)!} \cdot \frac{1!\,2n!}{\cancel{(2n+1)!}}$$

$$= \frac{2n(2n-1)\cancel{(2n-2)!}}{3\times 2\times \cancel{(2n-2)!}}$$

$$= \frac{n(2n-1)}{3}$$

15d

$\cot\theta < \frac{1}{\theta}$ for $0 < \theta < \frac{\pi}{2}$, so for $\theta = \frac{k\pi}{2n+1}$,

$$\sum_{k=1}^{n} \cot^2\left(\frac{k\pi}{2n+1}\right) < \sum_{k=1}^{n} \left(\frac{2n+1}{k\pi}\right)^2$$

Hence,

$$\left(\frac{2n+1}{\pi}\right)^2 \sum_{k=1}^{n} \frac{1}{k^2} > \frac{2n(2n-1)}{6} \quad \text{(using part c)}$$

$$\frac{(2n+1)^2}{2n(2n-1)} \sum_{k=1}^{n} \frac{1}{k^2} > \frac{\pi^2}{6}$$

This is,

$$\frac{(2n+1)^2}{2n(2n-1)}\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{n^2}\right) > \frac{\pi^2}{6}$$

Solutions to Exercise 3C Foundation questions

1a $z^3 = 1$

$(r\text{cis }\theta)^3 = 1$

$r^3\text{cis }3\theta = 1$

$r^3\text{cis }3\theta = 1 \times \text{cis }2k\pi$

$r = \sqrt[3]{1} = 1$ and $3\theta = 2k\pi$ so $\theta = \frac{2k\pi}{3}$

$$z = \text{cis}\left(\frac{\pi + 2k\pi}{3}\right)$$

$$= \text{cis}\frac{2\pi}{3}, \text{cis}\left(-\frac{2\pi}{3}\right), \text{cis }0$$

$$= -\frac{1}{2} + \frac{\sqrt{3}}{2}i, -\frac{1}{2} - \frac{\sqrt{3}}{2}i, 1$$

1b

$$\left|\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)-\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)\right|$$
$$= \left|\sqrt{3}i\right|$$
$$= \sqrt{3}$$

$$\left|\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)-1\right|$$
$$= \left|-\frac{3}{2}+\frac{\sqrt{3}}{2}i\right|$$
$$= \sqrt{\left(-\frac{3}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$
$$= \sqrt{\frac{9}{4}+\frac{3}{4}}$$
$$= \sqrt{3}$$

$$\left|\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)-1\right|$$
$$= \left|-\frac{3}{2}-\frac{\sqrt{3}}{2}i\right|$$
$$= \sqrt{\left(-\frac{3}{2}\right)^2+\left(-\frac{\sqrt{3}}{2}\right)^2}$$
$$= \sqrt{\frac{9}{4}+\frac{3}{4}}$$
$$= \sqrt{3}$$

This shows that all sides of the triangle have the same length and thus it is equilateral.

1c In the case that $\operatorname{cis}\frac{2\pi}{3}$ is the root,

$$\left(\operatorname{cis}\frac{2\pi}{3}\right)^2 = \operatorname{cis}\frac{4\pi}{3} = \operatorname{cis}\left(\frac{4\pi}{3} - 2\pi\right) = \operatorname{cis}\left(-\frac{2\pi}{3}\right) \text{ is the other root.}$$

In the case that $\operatorname{cis}\left(-\frac{2\pi}{3}\right)$ is the root,

$$\left(\operatorname{cis}\left(-\frac{2\pi}{3}\right)\right)^2 = \operatorname{cis}\left(-\frac{4\pi}{3}\right) = \operatorname{cis}\left(2\pi - \frac{4\pi}{3}\right) = \operatorname{cis}\frac{2\pi}{3} \text{ is the other root.}$$

1d i

$$\left(\operatorname{cis}\frac{2\pi}{3}\right)^3 = \operatorname{cis}\frac{6\pi}{3} = \operatorname{cis}2\pi = 1$$

$$\left(\operatorname{cis}\frac{4\pi}{3}\right)^3 = \operatorname{cis}\frac{12\pi}{3} = \operatorname{cis}4\pi = 1$$

so in either case the answer is one.

Alternately, covering both cases at once:

$$\omega = \operatorname{cis}\frac{2k\pi}{3}$$

$$\begin{aligned}\omega^3 &= \operatorname{cis}\left(3 \times \frac{2k\pi}{3}\right)\\ &= \operatorname{cis}2k\pi\\ &= 1\end{aligned}$$

1d ii If $\omega = \text{cis}\frac{2\pi}{3}$,

$1 + \omega + \omega^2$

$$= 1 + \text{cis}\frac{2\pi}{3} + \left(\text{cis}\frac{2\pi}{3}\right)^2$$
$$= 1 + \text{cis}\frac{2\pi}{3} + \text{cis}\frac{4\pi}{3}$$
$$= 1 + \text{cis}\frac{2\pi}{3} + \text{cis}\left(-\frac{2\pi}{3}\right)$$
$$= 1 + \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} + \cos\frac{2\pi}{3} - i\sin\frac{2\pi}{3}$$
$$= 1 + 2\cos\frac{2\pi}{3}$$
$$= 1 + 2\left(-\frac{1}{2}\right)$$
$$= 0$$

If $\omega = \text{cis}\frac{4\pi}{3}$,

$1 + \omega + \omega^2$

$$= 1 + \text{cis}\frac{4\pi}{3} + \left(\text{cis}\frac{4\pi}{3}\right)^2$$
$$= 1 + \text{cis}\frac{4\pi}{3} + \text{cis}\frac{8\pi}{3}$$
$$= 1 + \text{cis}\left(-\frac{2\pi}{3}\right) + \text{cis}\frac{2\pi}{3}$$
$$= 1 + \cos\frac{2\pi}{3} - i\sin\frac{2\pi}{3} + \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$$
$$= 1 + 2\cos\frac{2\pi}{3}$$
$$= 1 + 2\left(-\frac{1}{2}\right)$$
$$= 0$$

1e i $(1+\omega^2)^3$
$= (-\omega)^3$
$= -\omega^3$
$= -1$

1e ii $(1-\omega-\omega^2)(1-\omega+\omega^2)(1+\omega-\omega^2)$
$= \big(1-(\omega+\omega^2)\big)(1+\omega^2-\omega)(1+\omega-\omega^2)$
$= \big(1-(-1)\big)(-\omega-\omega)(-\omega^2-\omega^2)$
$= 2(-2\omega)(-2\omega^2)$
$= 8\omega^3$
$= 8(1)$
$= 8$

1e iii $(1-\omega)(1-\omega^2)(1-\omega^4)(1-\omega^5)$
$= (1-\omega)(1-\omega^2)(1-\omega^3\omega)(1-\omega^3\omega^2)$
$= (1-\omega)(1-\omega^2)(1-(1)\omega)(1-(1)\omega^2)$
$= (1-\omega)(1-\omega^2)(1-\omega)(1-\omega^2)$
$= \big((1-\omega)(1-\omega^2)\big)^2$
$= (1-\omega-\omega^2+\omega^3)^2$
$= (1-\omega-\omega^2+1)^2$
$= (1+1+1)^2$
$= 3^2$
$= 9$

2a $z^6 = 1$

Let $z = r\text{cis}\,\theta$

$(r\text{cis}\,\theta)^6 = 1$

$r^6\,\text{cis}\,6\theta = 1$

$r = 1$

$\text{cis}\,6\theta = \text{cis}\,2k\pi$

$6\theta = 2k\pi$

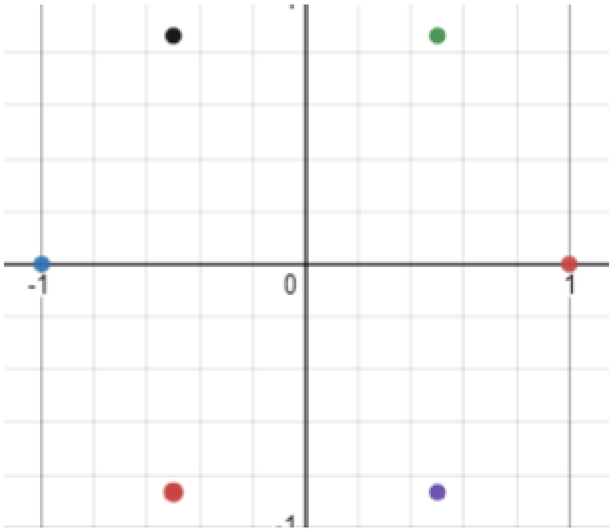
$\theta = \dfrac{k\pi}{3}$

$z = \text{cis}\,\dfrac{k\pi}{3}$

$z = \text{cis}\,0, \text{cis}\left(\pm\dfrac{\pi}{3}\right), \text{cis}\left(\pm\dfrac{2\pi}{3}\right), \text{cis}\,\pi$

$z = 1, \dfrac{1}{2} \pm \dfrac{\sqrt{3}}{2}i, -\dfrac{1}{2} \pm \dfrac{\sqrt{3}}{2}i, -1$

2b



All points are the same distance from the origin as,

$|1| = 1$

$$\left|\frac{1}{2} \pm \frac{\sqrt{3}}{2}i\right| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = \sqrt{1} = 1$$

$$\left|-\frac{1}{2} \pm \frac{\sqrt{3}}{2}i\right| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = \sqrt{1} = 1$$

$|-1| = 1$

Since $z = \text{cis}\,\frac{k\pi}{3}$, each root has an argument of $\frac{\pi}{3}$ between it and the adjacent roots, hence all roots are the same distance from the origin with the same argument between them relative to the origin so they form the corners of a regular hexagon.

2c

$$\alpha = \text{cis}\frac{\pi}{3}$$

$$\alpha^2 = \left(\text{cis}\frac{\pi}{3}\right)^2 = \text{cis}\left(\frac{2\pi}{3}\right) \text{ which is a root}$$

$$\alpha^{-2} = \left(\text{cis}\frac{\pi}{3}\right)^{-2} = \text{cis}\left(-\frac{2\pi}{3}\right) \text{ which is a root}$$

$$\alpha^{-1} = \left(\text{cis}\frac{\pi}{3}\right)^{-1} = \text{cis}\left(-\frac{\pi}{3}\right) \text{ which is a root}$$

2d

$$\begin{aligned}&(z^4 + z^2 + 1)(z^2 - 1)\\ &= z^6 + z^4 + z^2 - (z^4 + z^2 + 1)\\ &= z^6 - 1\end{aligned}$$

2e The roots of $z^2 - 1$ are $z = \pm 1$, which are the real roots of $z^6 - 1$. So the roots of $z^4 + z^2 + 1$ must be the complex roots of $z^6 - 1$. Thus

$$\begin{aligned}&z^4 + z^2 + 1\\ &= \left(z - \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\right)\left(z - \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\right)\left(z - \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\right)\left(z - \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\right)\\ &= \left(z^2 - \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i + \frac{1}{2} + \frac{\sqrt{3}}{2}i\right)z + \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\right) \times\\ &\qquad \left(z^2 - \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i - \frac{1}{2} + \frac{\sqrt{3}}{2}i\right)z + \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\right)\\ &= \left(z^2 - z + \left(\frac{1}{4} + \frac{3}{4}\right)\right)\left(z^2 - (-1)z + \left(\frac{1}{4} + \frac{3}{4}\right)\right)\\ &= (z^2 - z + 1)(z^2 + z + 1)\end{aligned}$$

3a $z^4 = -1$

Let $z = r\text{cis}\,\theta$

$(r\text{cis}\,\theta)^4 = -1$

$r^4\text{cis}\,4\theta = -1$

$r = 1$

$\text{cis}\,4\theta = -1$

$4\theta = \pi \pm 2k\pi$

$$\theta = \frac{\pi \pm 2k\pi}{4}$$

$$z = \text{cis}\left(\frac{\pi \pm 2k\pi}{4}\right)$$

$$z = \text{cis}\left(\pm\frac{\pi}{4}\right), \text{cis}\left(\pm\frac{3\pi}{4}\right)$$

$$z = \frac{1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}}i, -\frac{1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}}i$$

3b $$\left(z - \left(\tfrac{1}{\sqrt{2}} - \tfrac{1}{\sqrt{2}}i\right)\right)\left(z - \left(\tfrac{1}{\sqrt{2}} + \tfrac{1}{\sqrt{2}}i\right)\right)\left(z - \left(-\tfrac{1}{\sqrt{2}} - \tfrac{1}{\sqrt{2}}i\right)\right)\left(z - \left(-\tfrac{1}{\sqrt{2}} + \tfrac{1}{\sqrt{2}}i\right)\right)$$
$$= \left(z^2 - \sqrt{2}z + 1\right)\left(z^2 + \sqrt{2}z + 1\right)$$

4a $z^6 + 1 = 0$

$z^6 = -1$

$z = r\text{cis}\,\theta$

$(r\text{cis}\,\theta)^6 = -1$

$r^6\text{cis}\,6\theta = -1$

$r = 1$

$\text{cis}\,6\theta = -1$

$6\theta = \pi \pm 2k\pi$

$\theta = \dfrac{\pi \pm 2k\pi}{6}$

$z = \text{cis}\left(\dfrac{\pi \pm 2k\pi}{6}\right)$

$z = \text{cis}\left(\pm\dfrac{\pi}{6}\right), \text{cis}\left(\pm\dfrac{3\pi}{6}\right), \text{cis}\left(\pm\dfrac{5\pi}{6}\right)$

$z = \dfrac{\sqrt{3}}{2} \pm \dfrac{1}{2}i, \pm i, -\dfrac{\sqrt{3}}{2} \pm \dfrac{1}{2}i$

4b $(z^6 + 1)$

$= (z - i)(z + i)\left(z - \left(\dfrac{\sqrt{3}}{2} + \dfrac{1}{2}i\right)\right)\left(z - \left(\dfrac{\sqrt{3}}{2} - \dfrac{1}{2}i\right)\right)\left(z - \left(-\dfrac{\sqrt{3}}{2} + \dfrac{1}{2}i\right)\right) \times$

$\left(z - \left(-\dfrac{\sqrt{3}}{2} - \dfrac{1}{2}i\right)\right)$

$= (z^2 + 1)(z^2 - \sqrt{3}z + 1)(z^2 + \sqrt{3}z + 1)$

4c $z^6 + 1 = (z^2 + 1)\left(z^2 - \sqrt{3}z + 1\right)\left(z^2 + \sqrt{3}z + 1\right)$

$$\frac{z^6 + 1}{z^3} = \frac{(z^2 + 1)\left(z^2 - \sqrt{3}z + 1\right)\left(z^2 + \sqrt{3}z + 1\right)}{z^3}$$

$$z^3 + z^{-3} = (z + z^{-1})\left(z - \sqrt{3} + z^{-1}\right)\left(z + \sqrt{3} + z^{-1}\right)$$

Using the result from question 3 in Exercise 3B:

$$2\cos 3\theta = (2\cos\theta)\left(2\cos\theta - \sqrt{3}\right)\left(2\cos\theta + \sqrt{3}\right)$$

$$\cos 3\theta = (\cos\theta)\left(2\cos\theta - \sqrt{3}\right)\left(2\cos\theta + \sqrt{3}\right)$$

$$\cos 3\theta = 4\cos\theta\left(\cos\theta - \frac{\sqrt{3}}{2}\right)\left(\cos\theta + \frac{\sqrt{3}}{2}\right)$$

$$\cos 3\theta = 4\cos\theta\left(\cos\theta - \cos\frac{\pi}{6}\right)\left(\cos\theta - \cos\frac{5\pi}{6}\right)$$

5a $z^5 = i$

$$(r\operatorname{cis}\theta)^5 = \operatorname{cis}\frac{\pi}{2}$$

$$r^5\operatorname{cis}5\theta = \operatorname{cis}\left(\frac{\pi}{2} + 2k\pi\right)$$

$r^5 = 1$ and hence $r = 1$

$$5\theta = \frac{\pi}{2} + 2k\pi$$

$$\theta = \frac{1}{5}\left(\frac{\pi}{2} + 2k\pi\right)$$

$$z = \operatorname{cis}\left(\frac{1}{5}\left(\frac{\pi}{2} + 2k\pi\right)\right)$$

$$= \operatorname{cis}\left(-\frac{7\pi}{10}\right), \operatorname{cis}\left(-\frac{3\pi}{10}\right), \operatorname{cis}\frac{\pi}{10}, \operatorname{cis}\frac{\pi}{2}, \operatorname{cis}\frac{9\pi}{10}$$

5b $z^4 = -i$

Let $z = (r\text{cis}\,\theta)^4$
$= r^4\text{cis}\,4\theta$

$r^4\text{cis}\,4\theta = -i$

$r = 1$

$\text{cis}\,4\theta = -i$

$4\theta = \dfrac{3\pi}{2} \pm 2k\pi$

$\theta = \dfrac{3\pi}{8} \pm \dfrac{k\pi}{2}$

$z = \text{cis}\left(\dfrac{3\pi}{8} \pm \dfrac{k\pi}{2}\right)$

$z = \text{cis}\left(-\dfrac{5\pi}{8}\right), \text{cis}\left(-\dfrac{\pi}{8}\right), \text{cis}\,\dfrac{3\pi}{8}, \text{cis}\,\dfrac{7\pi}{8}$

5c $z^4 = -8 - 8\sqrt{3}i$

$(r\text{cis}\,\theta)^4 = -8 - 8\sqrt{3}i$

$r^4\text{cis}\,4\theta = -8 - 8\sqrt{3}i$

$r^4 = \sqrt{8^2 + \left(8\sqrt{3}\right)^2}$
$= 16$

$r = 2$

$4\theta = 2k\pi + \left(-\pi + \tan^{-1}\left(\dfrac{8\sqrt{3}}{8}\right)\right)$

$4\theta = 2k\pi - \dfrac{2\pi}{3}$

$\theta = \dfrac{1}{2}\left(k\pi - \dfrac{\pi}{3}\right)$

$z = 2\text{cis}\left(\dfrac{1}{2}\left(k\pi - \dfrac{\pi}{3}\right)\right)$
$= 2\text{cis}\left(-\dfrac{\pi}{6}\right), 2\text{cis}\left(-\dfrac{2\pi}{3}\right), 2\text{cis}\,\dfrac{\pi}{3}, 2\text{cis}\,\dfrac{5\pi}{6}$
$= \sqrt{3} - i, -1 - i\sqrt{3}, 1 + i\sqrt{3}, -\sqrt{3} + i$

5d $z^5 = 16\sqrt{2} - 16\sqrt{2}i$

$(r\text{cis}\,\theta)^5 = 16\sqrt{2} - 16\sqrt{2}i$

$r^5\text{cis}\,5\theta = 16\sqrt{2} - 16\sqrt{2}i$

$$r^5 = \sqrt{\left(16\sqrt{2}\right)^2 + \left(16\sqrt{2}\right)^2}$$
$$= 32$$

$r = 2$

$$5\theta = 2k\pi + \left(\tan^{-1}\left(\frac{-16\sqrt{2}}{16\sqrt{2}}\right)\right)$$

$$5\theta = 2k\pi - \frac{\pi}{4}$$

$$\theta = \frac{2k\pi}{5} - \frac{\pi}{20}$$

$$z = 2\text{ cis}\left(\frac{2k\pi}{5} - \frac{\pi}{20}\right)$$

$$= 2\text{ cis} -\frac{17\pi}{20}, 2\text{ cis} -\frac{9\pi}{20}, 2\text{ cis} -\frac{\pi}{20}, 2\text{ cis}\frac{7\pi}{20}, 2\text{ cis}\frac{3\pi}{4}$$

Solutions to Exercise 3C Development questions

6a Let $z = r\text{cis}(\theta)$ be a fifth root of -1.

$r^5\text{cis}^5(\theta) = -1$

$r^5\text{cis}(5\theta) = \cos(\pi) = \text{cis}(\pi)$

Hence $r = 1$ and $5\theta = 2\lambda\pi + \pi$ where λ is an integer.

This means that $\theta = \left(\frac{2\lambda+1}{5}\right)\pi$ and so

$$\theta = \pm\frac{\pi}{5}, \pm\frac{3\pi}{5}, \pi$$

$$z = \text{cis}\left(\pm\frac{\pi}{5}\right), \text{cis}\left(\pm\frac{3\pi}{5}\right), \text{cis}(\pi)$$

$$z = \text{cis}\left(\pm\frac{\pi}{5}\right), \text{cis}\left(\pm\frac{3\pi}{5}\right), -1$$

6b The root with least positive principle argument is $\alpha = \text{cis}\left(\frac{\pi}{5}\right)$. Now we have,

$$\alpha^3 = \left(\text{cis}\left(\frac{\pi}{5}\right)\right)^3 = \text{cis}\left(\frac{3\pi}{5}\right)$$

$$\alpha^7 = \left(\text{cis}\left(\frac{\pi}{5}\right)\right)^7 = \text{cis}\left(\frac{7\pi}{5}\right) = \text{cis}\left(\frac{7\pi}{5} - 2\pi\right) = \text{cis}\left(-\frac{3\pi}{5}\right)$$

$$\alpha^9 = \left(\text{cis}\left(\frac{\pi}{5}\right)\right)^9 = \text{cis}\left(\frac{9\pi}{5}\right) = \text{cis}\left(\frac{9\pi}{5} - 2\pi\right) = \text{cis}\left(-\frac{\pi}{5}\right)$$

Hence these are the three other complex roots.

6c α^7

$= \left(\text{cis}\left(\frac{\pi}{5}\right)\right)^7$

$= \text{cis}\left(\frac{7\pi}{5}\right)$

$= \text{cis}\left(\pi + \frac{2\pi}{5}\right)$

$= \text{cis}(\pi)\text{cis}\left(\frac{2\pi}{5}\right)$

$= \text{cis}(\pi)\left(\text{cis}\left(\frac{\pi}{5}\right)\right)^2$

$= -\alpha^2$

α^9

$= \left(\text{cis}\left(\frac{\pi}{5}\right)\right)^9$

$= \text{cis}\left(\frac{9\pi}{5}\right)$

$= \text{cis}\left(\pi + \frac{4\pi}{5}\right)$

$= \text{cis}(\pi)\text{cis}\left(\frac{4\pi}{5}\right)$

$= -\left(\text{cis}\left(\frac{\pi}{5}\right)\right)^4$

$= -\alpha^4$

6d $(1 + \alpha^2 + \alpha^4)$

$= 1 - \alpha^7 - \alpha^9$ (from part c)

$= -(-1 + \alpha^7 + \alpha^9)$

$= -(-1 + \alpha + \alpha^3 + \alpha^7 + \alpha^9) + (\alpha + \alpha^3)$

$= -0 + \alpha + \alpha^3$ (the sum of roots is 0, since the polynomial is $z^5 + 1 = 0$)

$= \alpha + \alpha^3$

7a Let $z = r\,\text{cis}(\theta)$, be a seventh root of unity then we have,

$z^7 = r^7\text{cis}(7\theta) = \text{cis}(0) = 1$

Hence $r = 1$ and $7\theta = 2n\pi$, where n is an integer. So we have,

$$\theta = \frac{2n\pi}{7}$$

$$z = \text{cis}(0), \text{cis}\left(\pm\frac{2\pi}{7}\right), \text{cis}\left(\pm\frac{4\pi}{7}\right), \text{cis}\left(\pm\frac{6\pi}{7}\right)$$

$$z = 1, \text{cis}\left(\pm\frac{2\pi}{7}\right), \text{cis}\left(\pm\frac{4\pi}{7}\right), \text{cis}\left(\pm\frac{6\pi}{7}\right)$$

7b The sum of the roots is

$$1 + \text{cis}\left(\frac{2\pi}{7}\right) + \text{cis}\left(-\frac{2\pi}{7}\right) + \text{cis}\left(\frac{4\pi}{7}\right) + \text{cis}\left(-\frac{4\pi}{7}\right) + \text{cis}\left(\frac{6\pi}{7}\right) + \text{cis}\left(-\frac{6\pi}{7}\right) = 0$$

(as the coefficient of z^6 in the equation $z^7 - 1 = 0$ is 0)

Expanding we have,

$$1 + \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7} + \cos\frac{2\pi}{7} - i\sin\frac{2\pi}{7} + \cos\frac{4\pi}{7} + i\sin\frac{4\pi}{7} + \cos\frac{4\pi}{7} - i\sin\frac{4\pi}{7}$$
$$+ \cos\frac{6\pi}{7} + i\sin\frac{6\pi}{7} + \cos\frac{6\pi}{7} - i\sin\frac{6\pi}{7} = 0$$

$$1 + 2\cos\frac{2\pi}{7} + 2\cos\frac{4\pi}{7} + 2\cos\frac{6\pi}{7} = 0$$

$$2\cos\frac{2\pi}{7} + 2\cos\frac{4\pi}{7} + 2\cos\frac{6\pi}{7} = -1$$

$$\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7} = -\frac{1}{2}$$

7c Writing the equation as a product of factors gives

$$(z-1)\left(z-\text{cis}\left(\frac{2\pi}{7}\right)\right)\left(z-\text{cis}\left(-\frac{2\pi}{7}\right)\right)\left(z-\text{cis}\left(\frac{4\pi}{7}\right)\right)\left(z-\text{cis}\left(-\frac{4\pi}{7}\right)\right)$$

$$\left(z-\text{cis}\left(\frac{6\pi}{7}\right)\right)\left(z-\text{cis}\left(-\frac{6\pi}{7}\right)\right)$$

$$=(z-1)\left(z^2-z\left(\text{cis}\left(\frac{2\pi}{7}\right)+\text{cis}\left(-\frac{2\pi}{7}\right)\right)+\left(\text{cis}\left(\frac{2\pi}{7}\right)\text{cis}\left(-\frac{2\pi}{7}\right)\right)\right)$$

$$\left(z^2-z\left(\text{cis}\left(\frac{4\pi}{7}\right)+\text{cis}\left(-\frac{4\pi}{7}\right)\right)+\left(\text{cis}\left(\frac{4\pi}{7}\right)\text{cis}\left(-\frac{4\pi}{7}\right)\right)\right)$$

$$\left(z^2-z\left(\text{cis}\left(\frac{6\pi}{7}\right)+\text{cis}\left(-\frac{6\pi}{7}\right)\right)+\left(\text{cis}\left(\frac{6\pi}{7}\right)\text{cis}\left(-\frac{6\pi}{7}\right)\right)\right)$$

$$=(z-1)\left(z^2-z\left(\text{cis}\left(\frac{2\pi}{7}\right)+\text{cis}\left(-\frac{2\pi}{7}\right)\right)+(\text{cis}(0))\right)$$

$$\left(z^2-z\left(\cos\left(\frac{4\pi}{7}\right)+\text{cis}\left(-\frac{4\pi}{7}\right)\right)+(\text{cis}(0))\right)$$

$$\left(z^2-z\left(\text{cis}\left(\frac{6\pi}{7}\right)+\text{cis}\left(-\frac{6\pi}{7}\right)\right)+(\text{cis}(0))\right)$$

Now using the fact that $\text{cis}\,x+\text{cis}(-x)=2\cos x$. We have,

$$=(z-1)\left(z^2-z\left(2\cos\frac{2\pi}{7}\right)+(\text{cis}(0))\right)\left(z^2-z\left(2\cos\frac{4\pi}{7}\right)+(\text{cis}(0))\right)$$

$$\left(z^2-z\left(2\cos\frac{6\pi}{7}\right)+(\text{cis}(0))\right)$$

$$=(z-1)\left(z^2-2z\cos\frac{2\pi}{7}+1\right)\left(z^2-2z\cos\frac{4\pi}{7}+1\right)\left(z^2-2z\cos\frac{6\pi}{7}+1\right)$$

7d The least positive principal argument is $\frac{2\pi}{7}$. Hence,

$$\alpha = \operatorname{cis}\left(\frac{2\pi}{7}\right)$$

$$\alpha^2 = \left(\operatorname{cis}\left(\frac{2\pi}{7}\right)\right)^2$$

$$= \operatorname{cis}\left(\frac{4\pi}{7}\right)$$

$$\alpha^3 = \left(\operatorname{cis}\left(\frac{2\pi}{7}\right)\right)^3$$

$$= \operatorname{cis}\left(\frac{6\pi}{7}\right)$$

$$\alpha^4 = \left(\operatorname{cis}\left(\frac{2\pi}{7}\right)\right)^4$$

$$= \operatorname{cis}\left(\frac{8\pi}{7}\right)$$

$$= \operatorname{cis}\left(\frac{8\pi}{7} - 2\pi\right)$$

$$= \operatorname{cis}\left(-\frac{6\pi}{7}\right)$$

$$\alpha^5 = \left(\operatorname{cis}\left(\frac{2\pi}{7}\right)\right)^5$$

$$= \operatorname{cis}\left(\frac{10\pi}{7}\right)$$

$$= \operatorname{cis}\left(\frac{10\pi}{7} - 2\pi\right)$$

$$= \operatorname{cis}\left(-\frac{4\pi}{7}\right)$$

$$\alpha^6 = \left(\operatorname{cis}\left(\frac{2\pi}{7}\right)\right)^6$$

$$= \operatorname{cis}\left(\frac{12\pi}{7}\right)$$

$$= \operatorname{cis}\left(-\frac{2\pi}{7}\right)$$

These are the other complex roots that we have previously found.

7e Since we have the roots of the polynomial, we can write it in factorised form as,

$$0 = \big(x - (\alpha + \alpha^6)\big)\big(x - (\alpha^2 + \alpha^5)\big)\big(x - (\alpha^3 + \alpha^4)\big)$$

Now, expanding we find that,

$$\begin{aligned}0 = x^3 &- ((\alpha + \alpha^6) + (\alpha^2 + \alpha^5) + (\alpha^3 + \alpha^4))x^2 \\ &+ \big((\alpha + \alpha^6)(\alpha^2 + \alpha^5) + (\alpha + \alpha^6)(\alpha^3 + \alpha^4) \\ &+ (\alpha^2 + \alpha^5)(\alpha^3 + \alpha^4)\big)x - (\alpha + \alpha^6)(\alpha^2 + \alpha^5)(\alpha^3 + \alpha^4)\end{aligned}$$

$$\begin{aligned}0 = x^3 &- (\alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5 + \alpha^6)x^2 \\ &+ \big((\alpha^3 + \alpha^6 + \alpha^8 + \alpha^{11}) + (\alpha^4 + \alpha^5 + \alpha^9 + \alpha^{10}) \\ &+ (\alpha^5 + \alpha^6 + \alpha^8 + \alpha^9)\big)x - (\alpha^6 + \alpha^7 + \alpha^9 + \alpha^{10} + \alpha^{11} + \alpha^{12} \\ &+ \alpha^{14} + \alpha^{15})\end{aligned}$$

Also note that we have,

$$\alpha^7 = \left(\operatorname{cis}\left(\frac{2\pi}{7}\right)\right)^7 = \operatorname{cis}(2\pi) = 1$$

and because the α are roots of the equation $z^7 - 1 = 0$, we also have,

$$0 = 1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5 + \alpha^6 \text{ (sum of roots)}$$

Now using the fact that $\alpha^7 = 1$, we can simplify the terms in the above expression to get,

$$\begin{aligned}0 = x^3 &- (\alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5 + \alpha^6)x^2 \\ &+ \big((\alpha^3 + \alpha^6 + \alpha^1 + \alpha^4) + (\alpha^4 + \alpha^5 + \alpha^2 + \alpha^3) \\ &+ (\alpha^5 + \alpha^6 + \alpha^1 + \alpha^2)\big)x \\ &- (\alpha^6 + 1 + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5 + 1 + \alpha^1)\end{aligned}$$

$$\begin{aligned}0 = x^3 &- (\alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5 + \alpha^6)x^2 \\ &+ \big((\alpha^3 + \alpha^6 + \alpha^1 + \alpha^4) + (\alpha^4 + \alpha^5 + \alpha^2 + \alpha^3) \\ &+ (\alpha^5 + \alpha^6 + \alpha^1 + \alpha^2)\big)x \\ &- (\alpha^6 + 1 + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5 + 1 + \alpha^1)\end{aligned}$$

$$\begin{aligned}0 = x^3 &- (\alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5 + \alpha^6)x^2 + 2(\alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5 + \alpha^6)x \\ &- \big((1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5 + \alpha^6) + 1\big)\end{aligned}$$

Which using the sum of roots result above becomes,

$$0 = x^3 - (-1)x^2 + 2(-1)x - (0 + 1)$$

$$0 = x^3 + x^2 - 2x - 1$$

8a i Let $z = r\text{cis}(\theta)$ be a fifth root of 1. Then,

$r^5\text{cis}^5(\theta) = 1$

$r^5\text{cis}(5\theta) = \cos(0) = \text{cis}(0)$

Hence $r = 1$ and $5\theta = 2n\pi$ where n is an integer.

This means that $\theta = \left(\frac{2n}{5}\right)\pi$ and so

$$\theta = 0, \pm\frac{2\pi}{5}, \pm\frac{4\pi}{5}$$

$$z = \text{cis}(0), \text{cis}\left(\pm\frac{2\pi}{5}\right), \text{cis}\left(\pm\frac{4\pi}{5}\right)$$

$$z = 1, \text{cis}\left(\pm\frac{2\pi}{5}\right), \text{cis}\left(\pm\frac{4\pi}{5}\right)$$

8a ii Note that all roots have a modulus of 1 from the origin, and that the angle between each of the roots in consecutive order is $\frac{2\pi}{5}$ radians. For example, $\frac{\text{cis}\left(\frac{4\pi}{5}\right)}{\text{cis}\left(\frac{2\pi}{5}\right)} = \text{cis}(\frac{2\pi}{5})$. Hence, the sections between consecutive roots equally divide 2π into 5 parts, and because each root has modulus 1 the distance between each root is equal. Thus, the roots form the 5 vertices of a regular pentagon.

8a iii Noting that the coefficient of z^4 in the equation $z^5 - 1 = 0$ is 0, the sum of the roots is,

$$\text{cis}\left(\frac{2\pi}{5}\right) + \text{cis}\left(-\frac{2\pi}{5}\right) + \text{cis}\left(\frac{4\pi}{5}\right) + \text{cis}\left(-\frac{4\pi}{5}\right) + 1 = 0$$

$$\cos\left(\frac{2\pi}{5}\right) + i\sin\left(\frac{2\pi}{5}\right) + \cos\left(\frac{2\pi}{5}\right) - i\sin\left(\frac{2\pi}{5}\right) + \cos\left(\frac{4\pi}{5}\right) + i\sin\left(\frac{4\pi}{5}\right) + \cos\left(\frac{4\pi}{5}\right)$$
$$- i\sin\left(\frac{4\pi}{5}\right) + 1 = 0$$

$$2\cos\frac{2\pi}{5} + 2\cos\frac{4\pi}{5} + 1 = 0$$

$$\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = -\frac{1}{2}$$

8b i $(z-1)(z^4+z^3+z^2+z+1)$

$= z^5+z^4+z^3+z^2+z-(z^4+z^3+z^2+z+1)$

$= z^5-1$

8b ii Since the roots of $z^5-1=(z-1)(z^4+z^3+z^2+z+1)$ are
$z=\text{cis}\left(\pm\frac{2\pi}{5}\right), \text{cis}\left(\pm\frac{4\pi}{5}\right), 1$

it follows that the roots of $(z^4+z^3+z^2+z+1)$ are

$z=\text{cis}\left(\pm\frac{2\pi}{5}\right), \text{cis}\left(\pm\frac{4\pi}{5}\right)$

Factorising $(z^4+z^3+z^2+z+1)$

$$\left(z-\text{cis}\left(\frac{2\pi}{5}\right)\right)\left(z-\text{cis}\left(-\frac{2\pi}{5}\right)\right)\left(z-\text{cis}\left(-\frac{4\pi}{5}\right)\right)\left(z-\text{cis}\left(-\frac{4\pi}{5}\right)\right)$$

$$=\left(z^2-z\left(\text{cis}\left(\frac{2\pi}{5}\right)+\text{cis}\left(-\frac{2\pi}{5}\right)\right)+\text{cis}\left(\frac{2\pi}{5}\right)\text{cis}\left(-\frac{2\pi}{5}\right)\right)$$

$$\left(z^2-z\left(\text{cis}\left(\frac{4\pi}{5}\right)+\text{cis}\left(-\frac{4\pi}{5}\right)\right)+\text{cis}\left(\frac{4\pi}{5}\right)\text{cis}\left(-\frac{4\pi}{5}\right)\right)$$

$$=\left(z^2-z\left(\text{cis}\left(\frac{2\pi}{5}\right)+\text{cis}\left(-\frac{2\pi}{5}\right)\right)+\text{cis}(0)\right)$$

$$\left(z^2-z\left(\text{cis}\left(\frac{4\pi}{5}\right)+\text{cis}\left(-\frac{4\pi}{5}\right)\right)+\text{cis}(0)\right)$$

$$=\left(z^2-2z\cos\frac{2\pi}{5}+1\right)\left(z^2-2z\cos\frac{4\pi}{5}+1\right) \text{ (using cis(x) + cis(−x) = 2}\cos x\text{)}$$

as required

8b iii Expanding the result from above we have,

$$\left(z^2 - 2z\cos\frac{2\pi}{5} + 1\right)\left(z^2 - 2z\cos\frac{4\pi}{5} + 1\right)$$

$$= z^4 + z^3\left(-2\cos\frac{2\pi}{5} - 2\cos\frac{4\pi}{5}\right) + z^2\left(2 + 4\cos\frac{2\pi}{5}\cos\frac{4\pi}{5}\right)$$
$$+ z\left(-2\cos\frac{2\pi}{5} - 2\cos\frac{4\pi}{5}\right) + 1$$

Equating the coefficients of z^2 gives,

$$1 = 2 + 4\cos\frac{2\pi}{5}\cos\frac{4\pi}{5}$$

$$-\frac{1}{4} = \cos\frac{2\pi}{5}\cos\frac{4\pi}{5}$$

Now from part a iii we also have the identity $\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = -\frac{1}{2}$. Subbing this in above we get,

$$-\frac{1}{4} = -\left(\cos\frac{4\pi}{5} + \frac{1}{2}\right)\cos\frac{4\pi}{5}$$

$$\left(\cos\frac{4\pi}{5}\right)^2 + \frac{1}{2}\cos\frac{4\pi}{5} - \frac{1}{4} = 0$$

Solving this quadratic equation, we find,

$$\cos\frac{4\pi}{5} = \frac{-\frac{1}{2} \pm \sqrt{\frac{1}{4} + 1}}{2}$$

$$\cos\frac{4\pi}{5} = \frac{-1 \pm \sqrt{5}}{4}$$

Now, $\cos\frac{4\pi}{5} < 0$, and so we have,

$$\cos\frac{4\pi}{5} = \frac{-1 - \sqrt{5}}{4}$$

However,

$$\cos\frac{4\pi}{5} = -\cos\left(\pi - \frac{4\pi}{5}\right) = -\cos\left(\frac{\pi}{5}\right)$$

Thus, we have,

$$\cos\left(\frac{\pi}{5}\right) = -\cos\frac{4\pi}{5} = \frac{1 + \sqrt{5}}{4}$$

8c i

Let $x = u + \dfrac{1}{u}$, then $x^2 + x - 1 = 0$ becomes,

$$\left(u + \frac{1}{u}\right)^2 + \left(u + \frac{1}{u}\right) - 1 = 0$$

$$u^2 + 2 + \frac{1}{u^2} + u + \frac{1}{u} - 1 = 0$$

$$u^2 + \frac{1}{u^2} + u + \frac{1}{u} + 1 = 0$$

$$u^4 + 1 + u^3 + u + u^2 = 0$$

$$u^4 + u^3 + u^2 + u + 1 = 0$$

Which has roots $u = \text{cis}\left(\pm\frac{2\pi}{5}\right), \text{cis}\left(\pm\frac{4\pi}{5}\right)$ from part b. Hence,

For $u = \text{cis}\left(\dfrac{2\pi}{5}\right)$,

$$x = \text{cis}\left(\frac{2\pi}{5}\right) + \frac{1}{\text{cis}\left(\frac{2\pi}{5}\right)}$$

$$= \text{cis}\left(\frac{2\pi}{5}\right) + \text{cis}\left(-\frac{2\pi}{5}\right)$$

$$= \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5} + \cos\frac{2\pi}{5} - i\sin\frac{2\pi}{5}$$

$$= 2\cos\frac{2\pi}{5}$$

For $u = \text{cis}\left(-\dfrac{2\pi}{5}\right)$,

$$x = \text{cis}\left(-\frac{2\pi}{5}\right) + \frac{1}{\text{cis}\left(-\frac{2\pi}{5}\right)}$$

$$= \text{cis}\left(-\frac{2\pi}{5}\right) + \text{cis}\left(\frac{2\pi}{5}\right)$$

$$= \cos\frac{2\pi}{5} - i\sin\frac{2\pi}{5} + \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$$

$$= 2\cos\frac{2\pi}{5}$$

For $u = \text{cis}\left(\frac{4\pi}{5}\right)$,

$$x = \text{cis}\left(\frac{4\pi}{5}\right) + \frac{1}{\text{cis}\left(\frac{4\pi}{5}\right)}$$

$$= \text{cis}\left(\frac{4\pi}{5}\right) + \text{cis}\left(-\frac{4\pi}{5}\right)$$

$$= \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5} + \cos\frac{4\pi}{5} - i\sin\frac{4\pi}{5}$$

$$= 2\cos\frac{4\pi}{5}$$

For $u = \text{cis}\left(-\frac{4\pi}{5}\right)$,

$$x = \text{cis}\left(-\frac{4\pi}{5}\right) + \frac{1}{\text{cis}\left(-\frac{4\pi}{5}\right)}$$

$$= \text{cis}\left(-\frac{4\pi}{5}\right) + \text{cis}\left(\frac{4\pi}{5}\right)$$

$$= \cos\frac{4\pi}{5} - i\sin\frac{4\pi}{5} + \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}$$

$$= 2\cos\frac{4\pi}{5}$$

Hence, the polynomial has roots $2\cos\frac{2\pi}{5}$ and $2\cos\frac{4\pi}{5}$.

8c ii Using the above result to factorise the polynomial we have,

$$\left(x - 2\cos\frac{2\pi}{5}\right)\left(x - 2\cos\frac{4\pi}{5}\right) = 0$$

$$x^2 - 2x\left(\cos\frac{4\pi}{5} + \cos\frac{2\pi}{5}\right) + 4\cos\frac{2\pi}{5}\cos\frac{4\pi}{5} = 0$$

$$x^2 - 2x\left(-\cos\left(\pi - \frac{4\pi}{5}\right) + \cos\frac{2\pi}{5}\right) - 4\cos\frac{2\pi}{5}\cos\left(\pi - \frac{4\pi}{5}\right) = 0$$

$$x^2 + 2x\left(\cos\frac{\pi}{5} - \cos\frac{2\pi}{5}\right) - 4\cos\frac{2\pi}{5}\cos\frac{\pi}{5} = 0$$

Comparing coefficients with $x^2 + x - 1 = 0$ we find that,

$$4\cos\frac{2\pi}{5}\cos\frac{\pi}{5} = 1$$

$$\cos\frac{2\pi}{5}\cos\frac{\pi}{5} = \frac{1}{4}$$

9a Let $z = r\,\text{cis}(\theta)$ be a ninth root of unity, so,

$$z^9 = r^9\,\text{cis}(9\theta) = 1 = \text{cis}(0)$$

Hence, $r = 1$ and $9\theta = 2n\pi$, where n is an integer. This gives,

$$\theta = \frac{2n\pi}{9} \text{ so}$$

$$z = 1, \text{cis}\left(\pm\frac{2\pi}{9}\right), \text{cis}\left(\pm\frac{4\pi}{9}\right), \text{cis}\left(\pm\frac{6\pi}{9}\right), \text{cis}\left(\pm\frac{8\pi}{9}\right)$$

9b $z^9 - 1$

$= (z^9 + z^6 + z^3) - (z^6 + z^3 + 1)$

$= (z^3 - 1)(z^6 + z^3 + 1)$

Writing $z^9 - 1$ as a product of factors gives

$z^9 - 1$

$$= (z-1)\left(z - \operatorname{cis}\left(-\frac{2\pi}{9}\right)\right)\left(z - \operatorname{cis}\left(\frac{2\pi}{9}\right)\right)\left(z - \operatorname{cis}\left(-\frac{4\pi}{9}\right)\right)\left(z - \operatorname{cis}\left(\frac{4\pi}{9}\right)\right)$$

$$\left(z - \operatorname{cis}\left(-\frac{6\pi}{9}\right)\right)\left(z - \operatorname{cis}\left(\frac{6\pi}{9}\right)\right)\left(z - \operatorname{cis}\left(-\frac{8\pi}{9}\right)\right)\left(z - \operatorname{cis}\left(\frac{8\pi}{9}\right)\right)$$

Note that

$\big(z - \operatorname{cis}(x)\big)\big(z - \operatorname{cis}(-x)\big)$

$= \big(z^2 - z\big(\operatorname{cis}(x) + \operatorname{cis}(-x)\big) + (\operatorname{cis}(0)\big)$

$= z^2 - z(\cos x + i\sin x + \cos x - i\sin x) + 1$

$= z^2 - 2z\cos x + 1$

Thus

$z^9 - 1$

$$= (z-1)\left(z^2 - 2z\cos\frac{2\pi}{9} + 1\right)\left(z^2 - 2z\cos\frac{4\pi}{9} + 1\right)$$

$$\left(z^2 - 2z\cos\frac{6\pi}{9} + 1\right)\left(z^2 - 2z\cos\frac{8\pi}{9} + 1\right)$$

$$= (z-1)\left(z^2 - 2z\cos\frac{2\pi}{9} + 1\right)\left(z^2 - 2z\cos\frac{4\pi}{9} + 1\right)$$

$$\left(z^2 - 2z\cos\frac{2\pi}{3} + 1\right)\left(z^2 - 2z\cos\frac{8\pi}{9} + 1\right)$$

$$= (z-1)\left(z^2 - 2z\cos\frac{2\pi}{9} + 1\right)\left(z^2 - 2z\cos\frac{4\pi}{9} + 1\right)$$

$$(z^2 + z + 1)\left(z^2 - 2z\cos\frac{8\pi}{9} + 1\right)$$

$= (z-1)(z^2+z+1)\left(z^2-2z\cos\frac{2\pi}{9}+1\right)$

$\left(z^2-2z\cos\frac{4\pi}{9}+1\right)\left(z^2-2z\cos\frac{8\pi}{9}+1\right)$

$= (z^3-1)\left(z^2-2z\cos\frac{2\pi}{9}+1\right)\left(z^2-2z\cos\frac{4\pi}{9}+1\right)\left(z^2-2z\cos\frac{8\pi}{9}+1\right)$

Hence using the result at the start of this question we have,

$z^6+z^3+1=\left(z^2-2z\cos\frac{2\pi}{9}+1\right)\left(z^2-2z\cos\frac{4\pi}{9}+1\right)\left(z^2-2z\cos\frac{8\pi}{9}+1\right)$

9c $z^6+z^3+1=\left(z^2-2z\cos\frac{2\pi}{9}+1\right)\left(z^2-2z\cos\frac{4\pi}{9}+1\right)\left(z^2-2z\cos\frac{8\pi}{9}+1\right)$

Dividing both sides by z^3 gives,

$z^3+1+z^{-3}=\left(z-2\cos\frac{2\pi}{9}+z^{-1}\right)\left(z-2\cos\frac{4\pi}{9}+z^{-1}\right)\left(z-2\cos\frac{8\pi}{9}+z^{-1}\right)$

$z^3+z^{-3}+1=\left(z+z^{-1}-2\cos\frac{2\pi}{9}\right)\left(z+z^{-1}-2\cos\frac{4\pi}{9}\right)\left(z+z^{-1}-2\cos\frac{8\pi}{9}\right)$

Now

z^n+z^{-n}

$=\operatorname{cis}(n\theta)+\operatorname{cis}(-n\theta)$

$=\cos n\theta+i\sin n\theta+\cos n\theta-i\sin n\theta$

$=2\cos n\theta$

Hence, the equation above becomes

$2\cos 3\theta+1=8\left(\cos\theta-\cos\frac{2\pi}{9}\right)\left(\cos\theta-\cos\frac{4\pi}{9}\right)\left(\cos\theta-\cos\frac{8\pi}{9}\right)$

10a $\omega = \text{cis}\left(\frac{2\pi}{9}\right)$

Let $z = \omega^k$ then

$$z^9 = (\omega^k)^9$$
$$= (\omega^9)^k$$
$$= \left(\left(\text{cis}\left(\frac{2\pi}{9}\right)\right)^9\right)^k$$
$$= \left(\text{cis}(2\pi)\right)^k$$
$$= (1)^k$$
$$= 1$$

10b $(\omega - 1)(1 + \omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 + \omega^7 + \omega^8)$

$$= \omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 + \omega^7 + \omega^8 + \omega^9$$
$$-(1 + \omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 + \omega^7 + \omega^8)$$
$$= \omega^9 - 1$$

Hence the equation
$(\omega - 1)(1 + \omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 + \omega^7 + \omega^8) = 0$

has the same roots as $\omega^9 - 1 = 0$ which are the ninth roots of unity.

Hence, since $\omega = \text{cis}\left(\frac{2\pi}{9}\right) \neq 1$ is a ninth root of unity we have,

$$1 + \omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 + \omega^7 + \omega^8 = 0$$

and so

$$\omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 + \omega^7 + \omega^8 = -1$$

10c Now using the result of part b we have,

$$-1 = \left(\operatorname{cis}\frac{2\pi}{9}\right) + \left(\operatorname{cis}\frac{2\pi}{9}\right)^2 + \left(\operatorname{cis}\frac{2\pi}{9}\right)^3 + \left(\operatorname{cis}\frac{2\pi}{9}\right)^4 + \left(\operatorname{cis}\frac{2\pi}{9}\right)^5$$

$$+ \left(\operatorname{cis}\frac{2\pi}{9}\right)^6 + \left(\operatorname{cis}\frac{2\pi}{9}\right)^7 + \left(\operatorname{cis}\frac{2\pi}{9}\right)^8$$

$$-1 = \operatorname{cis}\frac{2\pi}{9} + \operatorname{cis}\frac{4\pi}{9} + \operatorname{cis}\frac{6\pi}{9} + \operatorname{cis}\frac{8\pi}{9} + \operatorname{cis}\frac{10\pi}{9} + \operatorname{cis}\frac{12\pi}{9} + \operatorname{cis}\frac{14\pi}{9} + \operatorname{cis}\frac{16\pi}{9}$$

$$-1 = \operatorname{cis}\frac{2\pi}{9} + \operatorname{cis}\frac{4\pi}{9} + \operatorname{cis}\frac{6\pi}{9} + \operatorname{cis}\frac{8\pi}{9} + \operatorname{cis}\left(\frac{10\pi}{9} - 2\pi\right) + \operatorname{cis}\left(\frac{12\pi}{9} - 2\pi\right)$$

$$+ \operatorname{cis}\left(\frac{14\pi}{9} - 2\pi\right) + \operatorname{cis}\left(\frac{16\pi}{9} - 2\pi\right)$$

$$-1 = \operatorname{cis}\frac{2\pi}{9} + \operatorname{cis}\frac{4\pi}{9} + \operatorname{cis}\frac{6\pi}{9} + \operatorname{cis}\frac{8\pi}{9} + \operatorname{cis}\left(\frac{-8\pi}{9}\right) + \operatorname{cis}\left(-\frac{6\pi}{9}\right)$$

$$+ \operatorname{cis}\left(-\frac{4\pi}{9}\right) + \operatorname{cis}\left(-\frac{2\pi}{9}\right)$$

$$-1 = \operatorname{cis}\frac{2\pi}{9} + \operatorname{cis}\left(-\frac{2\pi}{9}\right) + \operatorname{cis}\frac{4\pi}{9} + \operatorname{cis}\left(-\frac{4\pi}{9}\right) + \operatorname{cis}\frac{6\pi}{9}$$

$$+ \operatorname{cis}\left(-\frac{6\pi}{9}\right) + \operatorname{cis}\frac{8\pi}{9} + \operatorname{cis}\left(\frac{-8\pi}{9}\right)$$

Using the result $\operatorname{cis}(x) + \operatorname{cis}(-x) = 2\cos x$, the above equation becomes

$$2\cos\frac{2\pi}{9} + 2\cos\frac{4\pi}{9} - 1 + 2\cos\frac{8\pi}{9} = -1$$

$$2\cos\frac{2\pi}{9} + 2\cos\frac{4\pi}{9} + 2\cos\frac{8\pi}{9} = 0$$

$$\cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} - \cos\left(\pi - \frac{8\pi}{9}\right) = 0$$

$$\cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} - \cos\frac{\pi}{9} = 0$$

$$\cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} = \cos\frac{\pi}{9}$$

10d Using the result of part c

$$\cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}$$

$$= \left(\cos\frac{2\pi}{9} + \cos\frac{4\pi}{9}\right)\left(\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}\right)$$

Using the result $\text{cis}(x) + \text{cis}(-x) = 2\cos x$, we can rewrite this in terms of ω as

$$= \frac{1}{8}\left(\omega + \frac{1}{\omega} + \omega^2 + \frac{1}{\omega^2}\right)\left(\left(\omega + \frac{1}{\omega}\right)\left(\omega^2 + \frac{1}{\omega^2}\right)\right)$$

$$= \frac{1}{8}\left(\omega^2 + 1 + 1 + \frac{1}{\omega^2} + \omega^3 + \omega + \frac{1}{\omega} + \frac{1}{\omega^3}\right)\left(\omega^2 + \frac{1}{\omega^2}\right)$$

$$= \frac{1}{8}\left(\omega^4 + 1 + 2\omega^2 + \frac{2}{\omega^2} + 1 + \frac{1}{\omega^4} + \omega^5 + \omega + \omega^3 + \frac{1}{\omega} + \omega + \frac{1}{\omega^3} + \frac{1}{\omega} + \frac{1}{\omega^5}\right)$$

$$= \frac{1}{8}\left(\left(\frac{1}{\omega^3} + \frac{1}{\omega^2} + \frac{1}{\omega} + 1 + \omega + \omega^2 + \omega^3 + \omega^4 + \omega^5\right)\right.$$
$$\left. + \left(\frac{1}{\omega^5} + \frac{1}{\omega^4} + \frac{1}{\omega^3} + \frac{1}{\omega^2} + \frac{1}{\omega} + 1 + \omega + \omega^2 + \omega^3\right) - \left(\omega^3 + \frac{1}{\omega^3}\right)\right)$$

Now, using the result from part b, we have that

$$1 + \omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 + \omega^7 + \omega^8 = 0$$

Dividing this equation by ω^3 gives

$$\frac{1}{\omega^3} + \frac{1}{\omega^2} + \frac{1}{\omega} + 1 + \omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 = 0$$

And dividing it by ω^5 gives

$$\frac{1}{\omega^5} + \frac{1}{\omega^4} + \frac{1}{\omega^3} + \frac{1}{\omega^2} + \frac{1}{\omega} + 1 + \omega + \omega^2 + \omega^3 = 0$$

Finally,

$$\omega^3 + \frac{1}{\omega^3} = 2\cos\frac{6\pi}{9} = 2\cos\frac{\pi}{3} = -1$$

Subbing all of this into above gives,

$$\cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{4\pi}{9} = \frac{1}{8}\big(0 + 0 - (-1)\big) = \frac{1}{8}$$

11a $\rho^7 = \left(\cos\frac{2\pi}{7} + i\cos\frac{2\pi}{7}\right)^7$

$$= \left(\text{cis}\frac{2\pi}{7}\right)^7$$

$$= \text{cis}(2\pi)$$

$$= 1$$

Hence $\rho^7 - 1 = 0$, factorising gives

$(\rho - 1)(1 + \rho + \rho^2 + \cdots + \rho^6) = 0$

Since $\rho \neq 1$,

$(1 + \rho + \rho^2 + \cdots + \rho^6) = 0$

11b Since the equation has real coefficients, and α is complex, the complex conjugate must also be a root. Hence

$$\beta = \overline{\alpha}$$

$$= \overline{\rho + \rho^2 + \rho^4}$$

$$= \overline{\rho} + \overline{\rho^2} + \overline{\rho^4}$$

$$= \overline{\text{cis}\left(\frac{2\pi}{7}\right)} + \overline{\text{cis}\left(\frac{2\pi}{7}\right)^2} + \overline{\text{cis}\left(\frac{2\pi}{7}\right)^4}$$

$$= \overline{\text{cis}\left(\frac{2\pi}{7}\right)} + \overline{\text{cis}\left(\frac{4\pi}{7}\right)} + \overline{\text{cis}\left(\frac{8\pi}{7}\right)}$$

$$= \text{cis}\left(-\frac{2\pi}{7}\right) + \text{cis}\left(-\frac{4\pi}{7}\right) + \text{cis}\left(-\frac{8\pi}{7}\right)$$

$$= \text{cis}\left(2\pi - \frac{2\pi}{7}\right) + \text{cis}\left(2\pi - \frac{4\pi}{7}\right) + \text{cis}\left(2\pi - \frac{8\pi}{7}\right)$$

$$= \text{cis}\left(\frac{12\pi}{7}\right) + \text{cis}\left(\frac{10\pi}{7}\right) + \text{cis}\left(\frac{6\pi}{7}\right)$$

$$= \text{cis}\left(\frac{2\pi}{7}\right)^6 + \text{cis}\left(\frac{2\pi}{7}\right)^5 + \text{cis}\left(\frac{2\pi}{7}\right)^3$$

$$= \rho^6 + \rho^5 + \rho^3$$

So $\beta = \rho^3 + \rho^5 + \rho^6$

11c Using the sum of roots we have,

$\alpha + \beta = -a$

Hence

$a = -(\alpha + \beta)$

$= -(\rho + \rho^2 + \rho^4 + \rho^3 + \rho^5 + \rho^6)$

$= -(-1)$ (from part b)

$= 1$

Using the product of roots,

$\alpha\beta = b$

$b = \alpha\beta$

$= (\rho + \rho^2 + \rho^4)(\rho^3 + \rho^5 + \rho^6)$

$= \rho^4 + \rho^6 + \rho^7 + \rho^5 + \rho^7 + \rho^8 + \rho^7 + \rho^9 + \rho^{10}$

$= \rho^4 + \rho^6 + 1 + \rho^5 + 1 + \rho + 1 + \rho^2 + \rho^3$ (since $\rho^7 = 1$)

$= 3 + \rho + \rho^2 + \rho^4 + \rho^3 + \rho^5 + \rho^6$

$= 3 + (-1)$ (from part b)

$= 2$

11d From part c we know that at the root α the polynomial has the form $\alpha^2 + \alpha + 2 = 0$. Solving for α gives

$$\alpha = \frac{-1 \pm \sqrt{1-8}}{2} = \frac{-1 \pm \sqrt{-7}}{2} = \frac{-1 \pm \sqrt{7}i}{2}$$

Also we have

$$\alpha$$

$$= \rho + \rho^2 + \rho^4$$

$$= \text{cis}\left(\frac{2\pi}{7}\right) + \text{cis}\left(\frac{4\pi}{7}\right) + \text{cis}\left(\frac{8\pi}{7}\right)$$

$$= \cos\left(\frac{2\pi}{7}\right) + i\sin\left(\frac{2\pi}{7}\right) + \cos\left(\frac{4\pi}{7}\right) + i\sin\left(\frac{4\pi}{7}\right) + \cos\left(\frac{8\pi}{7}\right) + i\sin\left(\frac{8\pi}{7}\right)$$

Equating real and imaginary parts in the expressions for α, we have

$$\pm\frac{\sqrt{7}}{2} = \sin\left(\frac{2\pi}{7}\right) + \sin\left(\frac{4\pi}{7}\right) + \sin\left(\frac{8\pi}{7}\right)$$

$$\pm\frac{\sqrt{7}}{2} = \sin\left(\frac{2\pi}{7}\right) + \sin\left(\pi - \frac{4\pi}{7}\right) + \sin\left(\pi - \frac{8\pi}{7}\right)$$

$$\pm\frac{\sqrt{7}}{2} = \sin\left(\frac{2\pi}{7}\right) + \sin\left(\frac{3\pi}{7}\right) + \sin\left(-\frac{\pi}{7}\right)$$

$$\pm\frac{\sqrt{7}}{2} = \sin\left(\frac{2\pi}{7}\right) + \sin\left(\frac{3\pi}{7}\right) - \sin\left(\frac{\pi}{7}\right)$$

Now because sin is an increasing function in the first quadrant, we have $\sin\left(\frac{2\pi}{7}\right) > \sin\left(\frac{\pi}{7}\right)$. Hence, $\sin\left(\frac{2\pi}{7}\right) + \sin\left(\frac{3\pi}{7}\right) - \sin\left(\frac{\pi}{7}\right) > 0$ and as such we find

$$\frac{\sqrt{7}}{2} = \sin\left(\frac{2\pi}{7}\right) + \sin\left(\frac{3\pi}{7}\right) - \sin\left(\frac{\pi}{7}\right)$$

Solutions to Exercise 3C Enrichment questions

12ai $\text{cis } 4\theta$

$= cis\ \theta^4$ (By de Moivre)
$= \cos^4\theta + 4i\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta - 4i\cos\theta\sin^3\theta + \sin^4\theta$
Equating real and imaginary parts:

$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$

$\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$

12aii So for $\cos 4\theta \neq 0$ we have

$\tan\theta$

$$= \frac{4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta}{\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta} \times \frac{\frac{1}{\cos^4\theta}}{\frac{1}{\cos^4\theta}} \qquad (\cos\theta \neq 0)$$

$$= \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$$

12aiii Let $\theta = \tan^{-1}\frac{1}{3}$, then $\tan\theta = \frac{1}{3}$ and taking tan of the RHS of the equation gives,

$\tan\left(4\tan^{-1}\frac{1}{3}\right)$

$= \tan 4\theta$

$$= \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$$

$$= \frac{4.\frac{1}{3} - 4.\frac{1}{27}}{1 - 6.\frac{1}{9} + \frac{1}{81}} \times \frac{81}{81}$$

$$= \frac{108 - 12}{81 - 54 + 1}$$

$$= \frac{24}{7}$$

Taking the inverse then gives $4\tan^{-1}\frac{1}{3} = \tan^{-1}\frac{24}{7}$.

Note: $\theta < \frac{\pi}{4}\left(\tan\theta = \frac{1}{3}\right)$ and $\tan 4\theta > 0$, so all angles are in the first quadrant.

12b Let z be a fourth root, then $iz, -z, -iz$ are also fourth roots.

Now let $z = r\text{cis}\ \theta$, where θ is acute. Then,

$$r^4\text{cis}\ 4\theta = 7 + 24i = 25\left(\frac{7}{25} + \frac{24}{25}\ i\right)$$

Hence, $r^4 = 25$, i.e., $r = \sqrt{5}$, and $\tan 4\theta = \frac{24}{7}$, i.e. $4\theta = \tan^{-1}\frac{24}{7}$

It follows from part a that $\theta = \tan^{-1}\frac{1}{3}$.

Hence, $z = \sqrt{5}\,(cis\ \theta)$, where θ is acute and $\tan\theta = \frac{1}{3}$.

Thus, using Pythagoras the diagonal of the right angle formed by θ is $\sqrt{10}$, and we can calculate $\sin\theta$ and $\cos\theta$, giving

z

$$= \sqrt{5}\left(\frac{3}{\sqrt{10}} + i\ \frac{1}{\sqrt{10}}\right)$$

$$= \frac{3}{\sqrt{2}} + \frac{i}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}}(3 + i)$$

The other roots are then:

$$iz = \frac{1}{\sqrt{2}}(-1 + 3i)$$

$$-z = -\frac{1}{\sqrt{2}}(3 + i)$$

$$-iz = \frac{1}{\sqrt{2}}(1 - 3i)$$

13a The LHS is a 7th degree polynomial, so there should be 7 roots.

Clearly $z = 0$ is not a solution of $(z+1)^8 - z^8 = 0$.

Diving by z then gives gives,

$$\left(1+\frac{1}{z}\right)^8 - 1 = 0$$

Or $w^8 - 1 = 0$, where $w = 1 + \frac{1}{z}$, excluding $w = 1$ since z is undefined there.

Let $w = \text{cis}\,\theta$ and $1 = \text{cis}\,2k\pi$, for k an integer, then

$$\text{cis}\,8\theta = \text{cis}\,2k\pi$$

So,

$$\theta = \frac{k\pi}{4}, k = \pm1, \pm2, \pm3, 4 \ \ (k \neq 0, \text{since } \ w \neq 1)$$

Thus,

$$1 + \frac{1}{z} = \text{cis}\,\frac{k\pi}{4}$$

Or,

$$z = \frac{1}{\text{cis}\,\frac{k\pi}{4} - 1} \times \frac{\overline{cis}\,\frac{k\pi}{8}}{\overline{cis}\,\frac{k\pi}{8}} \quad \text{(Noting the half angle in the given roots)}$$

$$z = \frac{\cos\frac{k\pi}{8} - i\sin\frac{k\pi}{8}}{\text{cis}\frac{k\pi}{8} - \overline{cis}\,\frac{k\pi}{8}}$$

$$z = \frac{\cos\frac{k\pi}{8} - i\sin\frac{k\pi}{8}}{2\text{i}\sin\frac{k\pi}{8}}$$

$$z = -\frac{1}{2}\left(i\cot\frac{k\pi}{8} + 1\right) \ k = \pm1, \pm2, \pm3, 4$$

Thus, we have

$$z = -\frac{1}{2} \text{ (for } k = 4)$$

or

$$z = -\frac{1}{2}\left(1 \pm i\cot\frac{k\pi}{8}\right) k = 1, 2, 3 \text{ (since cot is odd)}$$

13b By the factor theorem for polynomials, and the roots of part a,

$$(z+1)^8 - z^8$$

$$= 8\left(z+\frac{1}{2}\right)\left(z+\frac{1}{2}\left(1+i\cot\frac{\pi}{8}\right)\right)\left(z+\frac{1}{2}\left(1-i\cot\frac{\pi}{8}\right)\right)$$

$$\times\left(z+\frac{1}{2}\left(1+i\cot\frac{\pi}{4}\right)\right)\left(z+\frac{1}{2}\left(1-i\cot\frac{\pi}{4}\right)\right)$$

$$\times\left(z+\frac{1}{2}\left(1+i\cot\frac{3\pi}{8}\right)\right)\left(z+\frac{1}{2}\left(1-i\cot\frac{3\pi}{8}\right)\right)$$

$$= 4(2z+1)\left(z^2+z+\frac{1}{4}\left(1+\cot^2\frac{\pi}{8}\right)\right)\left(z^2+z+\frac{1}{4}\left(1+\cot^2\frac{\pi}{4}\right)\right)$$

$$\times\left(z^2+z+\frac{1}{4}\left(1+\cot^2\frac{3\pi}{8}\right)\right) \qquad (*)$$

$$= 4(2z+1)\left(z^2+z+\frac{1}{4}\csc^2\frac{\pi}{8}\right)\left(z^2+z+\frac{1}{2}\right)\left(z^2+z+\frac{1}{4}\left(1+\csc^2\frac{3\pi}{8}\right)\right)$$

$$= \frac{1}{8}(2z+1)\left(4z^2+4z+\csc^2\frac{\pi}{8}\right)(2z^2+2z+1)\left(4z^2+4z++\csc^2\frac{3\pi}{8}\right)$$

13c

Sub $\left(z+\frac{1}{2}\right)=\frac{\cos 2\theta}{2}$ into $(*)$ above to get:

RHS $(*)$

$$= 4\cos 2\theta\left(\frac{\cos^2 2\theta}{4}+\frac{1}{4}\cot^2\frac{\pi}{8}\right)\left(\frac{\cos^2 2\theta}{4}+\frac{1}{4}\cot^2\frac{\pi}{4}\right)\left(\frac{\cos^2 2\theta}{4}+\frac{1}{4}\cot^2\frac{3\pi}{8}\right)$$

$$= \frac{1}{16}\cos 2\theta\left(\cos^2 2\theta+\cot^2\frac{\pi}{8}\right)(\cos^2 2\theta+1)\left(\cos^2 2\theta+\cot^2\frac{\pi}{8}\right)$$

Then we have.

LHS $(*)$

$$= \left(\frac{\cos 2\theta}{2}+\frac{1}{2}\right)^8-\left(\frac{\cos 2\theta}{2}-\frac{1}{2}\right)^8$$

$$= (\cos^2\theta)^8-(\sin^2\theta)^8 \qquad \text{(double angle formulae)}$$

$$= \cos^{16}\theta-\sin^{16}\theta$$

14 $w^3 = 1, w \neq 1$, so $w = \text{cis}\ \frac{2\pi}{3}$ or $\overline{\text{cis}}\ \frac{2\pi}{3}$

Note: the conjugate roots since the polynomial equation has real roots.

For $w = \text{cis}\ \frac{2\pi}{3}$, suppose $w^k = 1$, then $\text{cis}\ \frac{2k\pi}{3} = 1$. (by De Moivre)

So $\text{cis}\ \frac{2k\pi}{3}$ is a multiple of 2π.

Hence, k is a multiple of 3.

Likewise, for $w = \overline{\text{cis}}\ \frac{2\pi}{3}$.

14a If $w^3 = 1$ it follows that $(w^3)^k = 1$.

Hence, $(w^k)^3 - 1 = 0$ or $(w^k - 1)(w^{2k} + w^k + 1) = 0$.

Either $w^k - 1 = 0$, in which case k is a multiple of 3. And,

$w^{2k} + w^k + 1$

$= (w^k)^2 + (w^k) + 1$

$= 1 + 1 + 1$

$= 3$

Or

$w^{2k} + w^k + 1 = 0$

in which case k is not a multiple of 3.

Thus,

$w^{2k} + w^k + 1 = 3$ if k is a multiple of 3 and $= 0$ otherwise.

14b

$$(1 + w)^n = \sum_{r=0}^{n} {}^nC_r w^r \text{ and } (1 + w^2)^n = \sum_{r=0}^{n} {}^nC_r w^{2r}$$

14c RHS

$$=\frac{1}{3}(2^n+(1+w)^n+(1+w^2)^n)$$

$$=\frac{1}{3}((1+1)^n+(1+w)^n+(1+w^2)^n)$$

$$=\frac{1}{3}\sum_{r=0}^{n}\left({}^nC_r+{}^nC_rw^r+{}^nC_rw^{2r}\right)$$

$$=\frac{1}{3}\sum_{r=0}^{n}{}^nC_r(1+w^r+w^{2r})$$

$$=\frac{1}{3}({}^nC_0.3+{}^nC_3.3+{}^nC_6.3+\cdots+{}^nC_{3l}.3) \qquad \text{(All other terms zero by part a.)}$$
$$={}^nC_0+{}^nC_3+{}^nC_6+\cdots+{}^nC_{3l}$$
$$=\text{LHS}$$

14d Since n is a multiple of 6, $3l=n$ with l even. From part c we have,

$$ {}^nC_0+{}^nC_3+{}^nC_6+\cdots+{}^nC_n=\frac{1}{3}(2^n+(1+w)^n+(1+w^2)^n) \qquad (*)$$

Let $k=1$ in $1+w^k+w^{2k}$, to give,

$1+w+w^2=0$ (By part a.)

Hence, $1+w=-w^2$ and $1+w^2=-w$

Thus, RHS of $(*)$ becomes

$$=\frac{1}{3}(2^n+(-w^2)^n+(-w)^n)$$

$$=\frac{1}{3}(2^n+(w^n)^2+w^n) \qquad \text{(Since n is even.)}$$

$$=\frac{1}{3}(2^n+(w^{2n}+w^n+1)-1)$$

$$=\frac{1}{3}(2^n+3-1) \qquad \text{(By part a, since n is a multiple of 3.)}$$

$$=\frac{1}{3}(2^n+2)$$

15a $(z+1)^{2n} + (z-1)^{2n} = 0$

So, $z \neq 1, -1$. Hence, by rearranging,

$$\left(\frac{(z+1)}{(z-1)}\right)^{2n} = -1 = \text{cis}\,(2k+1)\pi, \qquad \text{for integer } k$$

Thus, by de Moivre,

$$\frac{z+1}{z-1}$$

$$= \text{cis}\,\frac{(2k+1)\pi}{2n}$$

$$= \text{cis}\,2\alpha, \text{where } \alpha = \frac{(2k+1)\pi}{4n}$$

Also, for principal values $-\pi < \frac{(2k+1)\pi}{2n} \leq \pi$. Hence,

$$-2n < 2k + 1 \leq 2n$$

$$-2n - 1 < 2k \leq 2n - 1$$

Hence,

$$-n - \frac{1}{2} < k \leq n - \frac{1}{2} \text{ or } -n \leq k \leq n - 1$$

Now,

$$z + 1 = (z-1)\text{cis}\,2\alpha$$

or

$$z(\text{cis}\,2\alpha - 1) = \text{cis}\,2\alpha + 1$$

Thus, we have

$$z$$

$$= \frac{\text{cis}\,2\alpha + 1}{\text{cis}\,2\alpha - 1}$$

$$= \frac{\text{cis}\,2\alpha + 1}{\text{cis}\,2\alpha - 1} \times \frac{\overline{cis}\,\alpha}{\overline{cis}\,\alpha} (\text{Using the half angle result.})$$

$$= \frac{\text{cis}\,\alpha + \overline{cis}\,\alpha}{\text{cis}\,\alpha - \overline{cis}\,\alpha} (\text{See also Exercise 3A, Q18 and Exercise 3C, Q13.})$$

$$= \frac{2\cos\alpha}{2i\sin\alpha}$$

$$= -i\cot\alpha$$

$= -i\cot\dfrac{(2k+1)\pi}{4n}, -n \le k \le n-1$

Writing these in conjugate pairs (a polynomial equation with real coefficients):

$$z = -i\cot\frac{\pi}{4n}, i\cot\frac{\pi}{4n}, -i\cot\frac{3\pi}{4n}, i\cot\frac{3\pi}{4n}, \ldots, -i\cot\frac{(2n-1)\pi}{4n}, i\cot\frac{(2n-1)\pi}{4n}$$

$(k=0)$ $(k=-1)$ $(k=1)$ $(k=-2)$ $(k=n-1)$ $(k=-n)$

15b $OP_1{}^2 + OP_2{}^2 + \cdots + OP_{2n}{}^2$

$= |z_1|^2 + |z_2|^2 + \cdots + |z_{2n}|^2$

$= |z_1{}^2| + |z_2{}^2| + \cdots + |z_{2n}{}^2|$

$= -z_1{}^2 - z_2{}^2 + \cdots - z_{2n}{}^2$ (Since each root is imaginary.)

$= -(z_1{}^2 + z_2{}^2 + \cdots + z_{2n}{}^2)$

Which is the opposite of the sum of squares of roots.

Now, (sum of square roots) = (sum of square roots)^2 – 2(sum of roots in pairs)

Also, $(z+1)^{2n} + (z-1)^{2n} = 0$

Hence, the leading terms are:

$z^{2n} + {}^{2n}C_1\, z^{2n-1} + {}^{2n}C_2\, z^{2n-2} + \cdots + 1 + z^{2n} - {}^{2n}C_1\, z^{2n-1} + {}^{2n}C_2\, z^{2n-2} + \cdots + 1 = 0$

Cancelling opposite terms and dividing by 2, gives

$z^{2n} + {}^{2n}C_2\, z^{2n-2} + \cdots + 1 = 0$

Hence, looking at the above polynomial we see that the sum of the roots $= 0$ and the sum of the roots in pairs $= {}^{2n}C_2$.

Thus,

$OP_1{}^2 + OP_2{}^2 + \cdots + OP_{2n}{}^2$

$= 0^2 - 2.{}^{2n}C_2$

$= 2.\dfrac{(2n)!}{(2n-2)!\,.2!}$

$= 2n(2n-1)$

Solutions to Exercise 3D Foundation questions

1a $\left(e^{i\theta}\right)^3 = e^{i\theta\times 3} = e^{3i\theta}$

1b $\left(e^{-i\theta}\right)^6 = e^{-i\theta\times 6} = e^{-6i\theta}$

1c $\left(e^{2i\theta}\right)^4 = e^{2i\theta\times 4} = e^{8i\theta}$

1d $\left(e^{-5i\theta}\right)^{-2} = e^{-5i\theta\times -2} = e^{10i\theta}$

2a $e^{i\theta}\times e^{-2i\theta} = e^{i\theta-2i\theta} = e^{-i\theta}$

2b

$$\frac{e^{6i\theta}}{e^{3i\theta}} = e^{6i\theta-3i\theta} = e^{3i\theta}$$

2c $\left(e^{4i\theta}\right)^{-2}\times\left(e^{-2i\theta}\right)^{-5}$

$= e^{-8i\theta}\times e^{10i\theta}$

$= e^{-8i\theta+10i\theta} = e^{2i\theta}$

2d

$$\frac{\left(e^{2i\theta}\right)^3\times\left(e^{-3i\theta}\right)^{-4}}{(e^{-i\theta})^2}$$

$$= \frac{e^{6i\theta}\times e^{12i\theta}}{e^{-2i\theta}}$$

$$= \frac{e^{6i\theta+12i\theta}}{e^{-2i\theta}}$$

$$= \frac{e^{18i\theta}}{e^{-2i\theta}}$$

$= e^{18i\theta-(-2i\theta)}$

$= e^{20i\theta}$

3a $2i = 2e^{\frac{\pi}{2}i}$ (note that $i = e^{\frac{\pi}{2}i}$)

3b $1 + i$

$= \sqrt{1^2 + 1^2}\, e^{i\times\tan^{-1}\frac{1}{1}}$

$= \sqrt{2}\, e^{i\frac{\pi}{4}}$

3c -6

$= \sqrt{0^2 + (-6)^2}\, e^{i\left(\pi-\tan^{-1}\frac{0}{6}\right)}$

$= 6e^{i\pi}$

3d $-1 + \sqrt{3}i$

$= \sqrt{(-1)^2 + \left(\sqrt{3}\right)^2}\, e^{i\left(\pi-\tan^{-1}\frac{\sqrt{3}}{1}\right)}$

$= 2e^{\frac{2i\pi}{3}}$

3e $-3 - 3i$

$= \sqrt{(-3)^2 + (-3)^2}e^{i\left(-\pi+\tan^{-1}\frac{3}{3}\right)}$

$= \sqrt{18}e^{-\frac{3i\pi}{4}}$

$= 3\sqrt{2}e^{-\frac{3i\pi}{4}}$

3f $2\sqrt{3} - 2i$

$= \sqrt{\left(2\sqrt{3}\right)^2 + (-2)^2}e^{i\tan^{-1}(-\frac{2}{2\sqrt{3}})}$

$= \sqrt{16}e^{-\frac{i\pi}{6}}$

$= 4e^{-\frac{i\pi}{6}}$

4a $5e^{i\pi}$

$= 5(\cos\pi + i\sin\pi)$

$= 5(-1 + 0i)$

$= -5$

4b

$e^{\frac{i\pi}{3}}$

$= \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$

$= \frac{1}{2} + \frac{\sqrt{3}}{2}i$

4c

$4e^{-\frac{i\pi}{2}}$

$= 4\left(\cos\frac{\pi}{2} - i\sin\frac{\pi}{2}\right)$

$= 4(0 - i)$

$= -4i$

4d

$2e^{\frac{5i\pi}{6}}$

$= 2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$

$= 2\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$

$= -\sqrt{3} + i$

4e $2\sqrt{2}e^{-\frac{i\pi}{4}}$

$= 2\sqrt{2}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)$

$= 2\sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)$

$= 2 - 2i$

4f $4\sqrt{3}e^{-\frac{2i\pi}{3}}$

$= 4\sqrt{3}\left(\cos\frac{2\pi}{3} - i\sin\frac{2\pi}{3}\right)$

$= 4\sqrt{3}\left(-\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)$

$= -2\sqrt{3} - 6i$

Solutions to Exercise 3D Development questions

5a $zw = \left(1+\sqrt{3}i\right)(1-i)$

$$= \left(2e^{i\frac{\pi}{3}}\right)\left(\sqrt{2}e^{-i\frac{\pi}{4}}\right)$$

$$= 2\sqrt{2}e^{i\frac{\pi}{12}}$$

5b

$$\frac{w}{z} = \frac{(1-i)}{\left(1+\sqrt{3}i\right)}$$

$$= \frac{\left(\sqrt{2}e^{-i\frac{\pi}{4}}\right)}{\left(2e^{i\frac{\pi}{3}}\right)}$$

$$= \frac{1}{\sqrt{2}}e^{-i\frac{7\pi}{12}}$$

5c $z^3w = \left(1+\sqrt{3}i\right)^3(1-i)$

$$= \left(2e^{i\frac{\pi}{3}}\right)^3\left(\sqrt{2}e^{-i\frac{\pi}{4}}\right)$$

$$= 8e^{i\pi}\left(\sqrt{2}e^{-i\frac{\pi}{4}}\right)$$

$$= 8\sqrt{2}e^{i\frac{3\pi}{4}}$$

5d

$$\frac{z^2}{w} = \frac{\left(1+\sqrt{3}i\right)^2}{(1-i)}$$

$$= \frac{\left(2e^{i\frac{\pi}{3}}\right)^2}{\left(\sqrt{2}e^{-i\frac{\pi}{4}}\right)}$$

$$= \frac{4e^{\frac{i2\pi}{3}}}{\left(\sqrt{2}e^{-i\frac{\pi}{4}}\right)}$$

$$= 2\sqrt{2}e^{\frac{i11\pi}{12}}$$

6a $\left(\sqrt{3}+i\right)^6$

$= \left(2e^{i\frac{\pi}{6}}\right)^6$

$= (2)^6 e^{i\pi}$

$= 2^6(-1)$

$= -64$

6b $(-1+i)^5$

$= \left(\sqrt{2}e^{i\frac{3\pi}{4}}\right)^5$

$= \left(\sqrt{2}\right)^5 e^{i\frac{15\pi}{4}}$

$= \left(\sqrt{2}\right)^4 \sqrt{2}e^{-i\frac{\pi}{4}}$

$= \left(\sqrt{2}\right)^4 (1-i)$

$= 4-4i$

6c

$\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)^{-8}$

$= \left(e^{-i\frac{\pi}{3}}\right)^{-8}$

$= e^{i\frac{8\pi}{3}}$

$= e^{i\frac{2\pi}{3}}$

$= -\frac{1}{2}+\frac{\sqrt{3}}{2}i$

6d $\left(-3-3\sqrt{3}i\right)^4$

$= \left(6e^{-i\frac{2\pi}{3}}\right)^4$

$= 1296\, e^{-i\frac{8\pi}{3}}$

$= 1296\, e^{-i\frac{2\pi}{3}}$

$= 1296\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)$

$= -648-648\sqrt{3}i$

7a $z^{10}-w^{10}=2i$

$\left(\frac{1+i}{\sqrt{2}}\right)^{10}-\left(\frac{1-i}{\sqrt{2}}\right)^{10}$

$= \left(e^{i\frac{\pi}{4}}\right)^{10}-\left(e^{-i\frac{\pi}{4}}\right)^{10}$

$= e^{i\frac{10\pi}{4}}-e^{-i\frac{10\pi}{4}}$

$= e^{i\frac{2\pi}{4}}-e^{-i\frac{2\pi}{4}}$

$= e^{i\frac{\pi}{2}}-e^{-i\frac{\pi}{2}}$

$= i-(-i)$

$= 2i$

7b $1+z+z^2+z^3+z^4$

$= 1+\left(e^{i\frac{\pi}{4}}\right)+\left(e^{i\frac{\pi}{4}}\right)^2+\left(e^{i\frac{\pi}{4}}\right)^3+\left(e^{i\frac{\pi}{4}}\right)^4$

$= 1+e^{i\frac{\pi}{4}}+e^{i\frac{\pi}{2}}+e^{i\frac{3\pi}{4}}+e^{i\pi}$

$= 1+\frac{1}{\sqrt{2}}(1+i)+i+\frac{1}{\sqrt{2}}(-1+i)-1$

$= \frac{2}{\sqrt{2}}i+i$

$= \left(\sqrt{2}+1\right)i$

8a $\left(1+\sqrt{3}i\right)^5(1-i)^4+\left(1-\sqrt{3}i\right)^5(1+i)^4$

$= (2)^5 e^{i\frac{5\pi}{3}}\left(\sqrt{2}\right)^4 e^{-i\frac{4\pi}{4}} + (2)^5 e^{-i\frac{5\pi}{3}}\left(\sqrt{2}\right)^4 e^{i\frac{4\pi}{4}}$

$= 32e^{i\left(\frac{5\pi}{3}-2\pi\right)}(-4) + 32e^{-i\left(\frac{5\pi}{3}-2\pi\right)}(-4)$

$= 32e^{-i\frac{\pi}{3}}(-4) + 32e^{i\frac{\pi}{3}}(-4)$

$= -128e^{-i\frac{\pi}{3}} - 128e^{i\frac{\pi}{3}}$

$= -128\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i+\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\right)$

$= -128$

8b

$\frac{\left(1+\sqrt{3}i\right)^5}{(1-i)^4}+\frac{\left(1-\sqrt{3}i\right)^5}{(1+i)^4}$

$= \frac{32e^{-i\frac{\pi}{3}}}{(-4)}+\frac{32e^{i\frac{\pi}{3}}}{(-4)}$ (Using the exponetional forms found in part a)

$= 8e^{-i\frac{\pi}{3}} - 8e^{i\frac{\pi}{3}}$

$= -8\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i+\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\right)$

$= -8$

9a $1 + z^4$

$= 1 + \left(e^{i\theta}\right)^4$

$= 1 + e^{4i\theta}$

$= 1 + \cos 4\theta + i \sin 4\theta$

$= 1 + (\cos^2 2\theta - \sin^2 2\theta) + i(2 \sin 2\theta \cos 2\theta)$

$= 1 - \sin^2 2\theta + \cos^2 2\theta + 2i \sin 2\theta \cos 2\theta$

$= \cos^2 2\theta + \cos^2 2\theta + 2i \sin \theta \cos \theta$

$= 2\cos^2 2\theta + 2i \sin 2\theta \cos 2\theta$

$= 2\cos 2\theta\,(\cos 2\theta + i \sin 2\theta)$

$= 2\cos 2\theta \operatorname{cis} 2\theta$

9b

$\dfrac{1 + z^4}{1 + z^{-4}}$

$= \dfrac{z^4(1 + z^4)}{z^4 + 1}$

$= z^4$

$= (\operatorname{cis} \theta)^4$

$= \operatorname{cis} 4\theta$

10a $(1 - i)z^2$

$= \left(\sqrt{2}e^{-\frac{i\pi}{4}}\right)\left(re^{i\theta}\right)^2$

$= \sqrt{2}r^2e^{-\frac{i\pi}{4}}e^{i2\theta}$

$= \sqrt{2}r^2e^{i\left(2\theta - \frac{\pi}{4}\right)}$

$= \sqrt{2}r^2e^{i\frac{1}{4}(8\theta - \pi)}$

10b

$$\frac{1+\sqrt{3}i}{z}$$

$$=\frac{2e^{\frac{i\pi}{3}}}{re^{i\theta}}$$

$$=\frac{2e^{\frac{i\pi}{3}}e^{-i\theta}}{r}$$

$$=\frac{2e^{i\left(\frac{\pi}{3}-\theta\right)}}{r}$$

$$=\frac{2e^{\frac{1}{3}i(\pi-3\theta)}}{r}$$

11a $(1+i)^n$

$$=\left(\sqrt{2}e^{i\frac{\pi}{4}}\right)^n$$

$$=\left(\sqrt{2}\right)^n e^{i\frac{n\pi}{4}}$$

This is real when the imaginary part of the exponent is a multiple of $2\lambda\pi$ or $2\lambda\pi \pm \pi$, that is, when $\frac{n\pi}{4} = 2\pi\lambda \pm \pi$ or $2\lambda\pi$ where λ is an integer.

So $n = 8\lambda \pm 4$ or 8λ. Hence $n = 0, 4, 8$ ….

Therefore $(1+i)^n$ is real when n is divisible by 4.

11b $(1-i)^n$

$= \left(\sqrt{2}e^{-i\frac{\pi}{4}}\right)^n$

$= \left(\sqrt{2}\right)^n e^{-i\frac{n\pi}{4}}$

This is purely imaginary when the imaginary part of the exponent is of the form $2\lambda\pi \pm \frac{\pi}{2}$ where λ is an integer, that is, when $-\frac{n\pi}{4} = 2\pi\lambda \pm \frac{\pi}{2}$ where λ is an integer. So,

$n\pi = -4\left(2\lambda\pi \pm \frac{\pi}{2}\right)$

Absorbing the minus sign into λ then gives,

$n = 8\lambda \pm 2$ where λ is an integer

Hence the positive values of n are $n = 2, 6, 10 \ldots$

11c $\left(\sqrt{3}-i\right)^n$

$= \left(2e^{-i\frac{\pi}{6}}\right)^n$

$= 2^n e^{-\frac{in\pi}{6}}$

This is real when the imaginary part of the exponent is a multiple of $2\lambda\pi$ or $2\lambda\pi \pm \pi$ where λ is an integer, that is, when $-\frac{n\pi}{6} = 2\pi\lambda \pm \pi$ or $2\lambda\pi$ where λ is an integer. Hence,

$n = -6(2\lambda \pm 1)$ or $n = -12\lambda$

Absorbing the minus signs into λ we that the positive values of n are,

$n = 0, 6, 12, 18, \ldots$

That is, n is divisible by 6.

11d $\left(1+\sqrt{3}i\right)^n$

$= \left(2e^{\frac{i\pi}{3}}\right)^n$

$= 2^n e^{\frac{in\pi}{3}}$

This is purely imaginary when the imaginary part of the exponent is of the form $2\lambda\pi \pm \frac{\pi}{2}$ where λ is an integer, that is, when $\frac{n\pi}{3} = 2\pi\lambda \pm \frac{\pi}{2}$ where λ is an integer. So,

$n = 3\left(2\lambda \pm \frac{1}{2}\right)$

Hence, the positive values of n are,

$n = \frac{3}{2}, \frac{9}{2}, \frac{15}{2}, \ldots$

12a i $e^{ni\theta} + e^{-ni\theta}$

$= \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$

$= 2\cos n\theta$

12a ii $e^{ni\theta} - e^{-ni\theta}$

$= \cos n\theta + i\sin n\theta - (\cos n\theta - i\sin n\theta)$

$= 2i\sin n\theta$

12b i Using part a ii with $n = 3$

$e^{3i\theta} - e^{-3i\theta} = 2i\sin 3\theta$

12b ii Using part a i with $n = 1$

$\left(e^{i\theta} + e^{-i\theta}\right)^2$

$= (2\cos\theta)^2$

$= 4\cos^2\theta$

12biii Using part a ii with $n = 1$

$\left(e^{i\theta} - e^{-i\theta}\right)^3$

$= (2i \sin\theta)^3$

$= -8i \sin^3\theta$

12b iv $e^{2i\theta} + e^{i\theta} + 2 + e^{-i\theta} + e^{-2i\theta}$

$= \left(e^{2i\theta} + e^{-2i\theta}\right) + \left(e^{i\theta} + e^{-i\theta}\right) + 2$

$= 2\cos 2\theta + 2\cos\theta + 2$ (Using part a i)

$= 2(2\cos^2\theta - 1) + 2\cos\theta + 2$

$= 4\cos^2\theta + 2\cos\theta$

$= 2\cos\theta\,(2\cos\theta + 1)$

12v $e^{3i\theta} - e^{i\theta} + e^{-i\theta} - e^{-3i\theta}$

$= \left(e^{3i\theta} - e^{-3i\theta}\right) - \left(e^{i\theta} - e^{-i\theta}\right)$

$= 2i\sin 3\theta - 2i\sin\theta$ (Using part a ii)

13a Using question 12a i with $n = 1$,

$e^{i\theta} + e^{-i\theta} = 2\cos\theta$

$\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$

13b $\cos(-\theta)$

$= \frac{1}{2}\left(e^{-i\theta} + e^{-(-i\theta)}\right)$

$= \frac{1}{2}\left(e^{-i\theta} + e^{i\theta}\right)$

$= \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$

$= \cos\theta$

Hence $\cos\theta$ is an even function.

13c Using 12a ii as we did in part a, we have

$$\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$$

Hence

$\sin(-\theta)$

$$= \frac{1}{2i}\left(e^{-i\theta} - e^{-(-i\theta)}\right)$$

$$= \frac{1}{2i}\left(e^{-i\theta} - e^{i\theta}\right)$$

$$= -\frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$$

$$= -\sin\theta$$

Hence $\sin\theta$ is an odd function.

13d

$$\tan(-\theta) = \frac{\sin(-\theta)}{\cos(-\theta)} = \frac{-\sin\theta}{\cos\theta} = -\tan\theta$$

Hence $\tan\theta$ is an odd function.

$$\cot(-\theta) = \frac{1}{\tan(-\theta)} = \frac{1}{-\tan\theta} = -\cot\theta$$

Hence $\cot\theta$ is an odd function.

$$\sec(-\theta) = \frac{1}{\cos(-\theta)} = \frac{1}{\cos\theta} = \sec\theta$$

Hence $\sec\theta$ is an even function.

$$\operatorname{cosec}(-\theta) = \frac{1}{\sin(-\theta)} = \frac{1}{-\sin\theta} = -\operatorname{cosec}\theta$$

Hence $\operatorname{cosec}\theta$ is an odd function.

14a

$$\left(z + 2e^{\frac{i\pi}{2}}\right)\left(z - 2e^{\frac{i\pi}{2}}\right)$$

$$= z^2 - \left(2e^{\frac{i\pi}{2}}\right)^2$$

$$= z^2 - 4e^{i\pi}$$

$$= z^2 + 4$$

14b

$$\left(z-e^{\frac{i\pi}{3}}\right)\left(z-e^{-\frac{i\pi}{3}}\right)$$

$$=z^2-ze^{-\frac{i\pi}{3}}-ze^{\frac{i\pi}{3}}+\left(e^{\frac{i\pi}{3}}\right)\left(e^{-\frac{i\pi}{3}}\right)$$

$$=z^2-z\left(e^{-\frac{i\pi}{3}}+e^{\frac{i\pi}{3}}\right)+e^0$$

$$=z^2-z\left(2\cos\frac{\pi}{3}\right)+1$$

$$=z^2-2z\cos\frac{\pi}{3}+1$$

$$=z^2-z+1$$

14c

$$(z+2)\left(z-2e^{\frac{i\pi}{3}}\right)\left(z-2e^{-\frac{i\pi}{3}}\right)$$

$$=(z+2)(z^2-2z+4)\text{ (Using the result of part b)}$$

$$=z^3+8$$

14d

$$\left(z-\sqrt{2}e^{\frac{i\pi}{4}}\right)\left(z-\sqrt{2}e^{-\frac{i\pi}{4}}\right)\left(z-\sqrt{2}e^{\frac{3i\pi}{4}}\right)\left(z-\sqrt{2}e^{-\frac{3i\pi}{4}}\right)$$

$$=\left(z^2-z\sqrt{2}\left(e^{\frac{i\pi}{4}}+e^{-\frac{i\pi}{4}}\right)+\left(-\sqrt{2}e^{\frac{i\pi}{4}}\right)\left(-\sqrt{2}e^{-\frac{i\pi}{4}}\right)\right)$$

$$\left(z^2-z\sqrt{2}\left(e^{\frac{3i\pi}{4}}+e^{-\frac{3i\pi}{4}}\right)+\left(-\sqrt{2}e^{\frac{3i\pi}{4}}\right)\left(-\sqrt{2}e^{-\frac{3i\pi}{4}}\right)\right)$$

$$=\left(z^2-2\sqrt{2}z\cos\frac{\pi}{4}+2\right)\left(z^2-2\sqrt{2}z\cos\frac{3\pi}{4}+2\right)$$

$$=\left(z^2-z2\sqrt{2}\left(\frac{1}{\sqrt{2}}\right)+2\right)\left(z^2-z2\sqrt{2}\left(-\frac{1}{\sqrt{2}}\right)+2\right)$$

$$=(z^2-2z+2)(z^2+2z+2)$$

$$=z^4+2z^3+2z^2-2z^3-4z^2-4z+2z^2+4z+4$$

$$=z^4+4$$

15a $re^{i\theta} = se^{i\phi}$

$|re^{i\theta}| = |se^{i\phi}|$

$|r||e^{i\theta}| = |s||e^{i\phi}|$

$|r|(1) = |s|(1)$

$|r| = |s|$

And since $r > 0$ and $s > 0$,

$r = s$

15b $re^{i\theta} = se^{i\phi}$

Since $r = s$,

$e^{i\theta} = e^{i\phi}$

$\frac{e^{i\theta}}{e^{i\phi}} = 1$

$e^{i(\theta-\phi)} = 1$ (1)

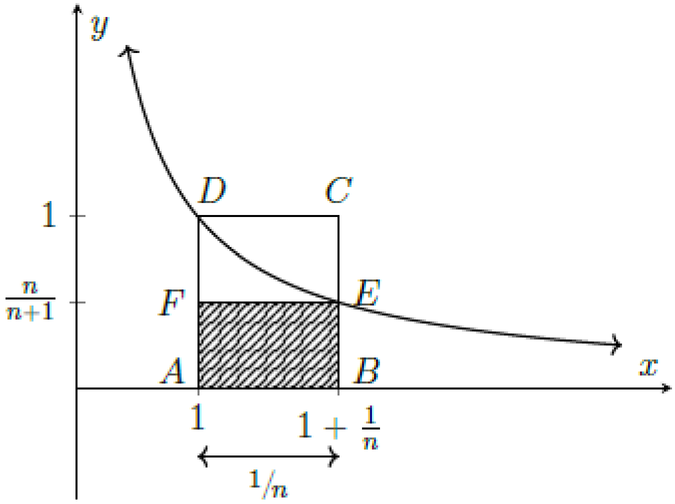
Since ϕ and θ are principle values we have by definition that, $-\pi < \phi, \theta \leq \pi$. Thus, $\phi - \theta < \pi - (-\pi) < 2\pi$ and $\phi - \theta > -\pi - \pi > -2\pi$.

Hence $-2\pi < \phi - \theta < 2\pi$ and within this range to satisfy (1) we have, $\theta - \phi = 0$ and so $\theta = \phi$.

15c If two complex numbers are equal, then they represent the same point in the Argand diagram. Hence the moduli are equal, and because the principal argument is unique between $-\pi$ and π it must also be equal.

Solutions to Exercise 3D Enrichment questions

16a



First not from the diagram that, Area ABEF $\leq$ area under curve $\leq$ area ABCD. Thus,

$$\frac{n}{1+n} \times \frac{1}{n} \leq \int_1^{1+\frac{1}{n}} \frac{1}{x} dx \leq 1 \times \frac{1}{n}$$

$$\frac{1}{1+n} \leq [\log x]_1^{1+\frac{1}{n}} \leq \frac{1}{n}$$

$$\frac{1}{1+n} \leq \log\left(1+\frac{1}{n}\right) - \log 1 \leq \frac{1}{n}$$

But $\log 1 = 0$, hence,

$$\frac{1}{1+n} \leq \log\left(1+\frac{1}{n}\right) \leq \frac{1}{n}$$

16b Since n is a positive integer we can multiply through by it to give,

$$\frac{n}{1+n} \leq \log\left(1+\frac{1}{n}\right)^n \leq 1 \text{(By the log laws.)}$$

Taking exponentials of each part:

$$e^{\frac{n}{1+n}} \leq \left(1+\frac{1}{n}\right)^n \leq e(*)$$

Then take the limit as $n \to \infty$,

$$\lim_{n\to\infty} e^{\frac{n}{n+1}} \leq \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \leq e$$

$$e \leq \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \leq e$$

Hence, $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$

16c In $(*)$ of part b, replace n with $\frac{n}{x}$ then,

$$e^{\frac{n}{x+n}} \leq \left(1+\frac{x}{n}\right)^{\frac{n}{x}} \leq e$$

Raise to the power of x:

$$\left(e^{\frac{n}{x+n}}\right)^x \leq \left(1+\frac{x}{n}\right)^n \leq e^x$$

Once again, take the limit as $n \to \infty$.

$$e^x \leq \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n \leq e^x$$

That is, $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$

17a $e^t \geq 1$ for $0 \leq t \leq x$, and so,

$$\int_0^x e^t \, dt \geq \int_0^x 1 \, dt$$

$$[e^t]_0^x \geq [1]_0^x$$

$$e^x - 1 \geq x$$

Hence, $e^x \geq 1 + x$

17b From part a, $e^t \geq 1 + t$ for $0 \leq t \leq x$, as such

$$\int_0^x e^t \, dt \geq \int_0^x 1 + t \, dt$$

$$e^x - 1 \geq x + \frac{x^2}{2}$$

Hence,

$$e^x \geq 1 + x + \frac{x^2}{2}$$

17c Induction is useful here, with step A done above.

B: Assume that the result is true for $n = k$, that is, assume that:

$$e^x \geq \sum_{n=0}^{k} \frac{x^n}{n!} (*)$$

Now prove the result true for $n = k + 1$, that is:

$$e^x \geq \sum_{n=0}^{k+1} \frac{x^n}{n!}$$

Let $x = t$ in $(*)$ and integrate from 0 to x to get:

$$\int_0^x e^t dt \geq \int_0^x \sum_{n=0}^{k} \frac{t^n}{n!} dt$$

$$[e^t]_0^x \geq \left[\sum_{n=0}^{k} \frac{t^{n+1}}{(n+1)!}\right]_0^x$$

$$e^x - 1 \geq \sum_{n=0}^{k} \frac{x^{n+1}}{(n+1)!} - 0$$

$$e^x \geq 1 + \sum_{n=1}^{k+1} \frac{x^n}{n!} \text{(Replacing } n+1 \text{ with } n\text{)}$$

$$e^x \geq \sum_{n=0}^{k+1} \frac{x^n}{n!} \text{(Since } 1 = \frac{(\mathrm{x})^0}{0!}, \mathrm{x} \neq 0$$

C: From parts A and B, by mathematical induction, the result is true for all m; viz

$e^x \geq \sum_{n=0}^{m} \frac{x^n}{n!}$, and taking the limit as $m \to \infty$, gives

$$e^x \geq 1 + \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

17d

$$E(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Let $h(x) = e^{-x}.E(x)$, then we have,

$h'(x) = -e^{-x}.E(x) + e^{-x}.E'(x)$

Now let, $E_m(x) = \sum_{n=0}^{m} \frac{x^n}{n!}$ $\quad (*)$ (see special note at end of question)

Then,

$E'_m(x)$

$$= \sum_{n=0}^{m} \frac{nx^{n-1}}{n!}$$

$$= 0 + 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{m-1}}{(m-1)!}$$

$$= \sum_{n=0}^{m-1} \frac{x^n}{n!}$$

So

$E'(x)$

$$= \lim_{m\to\infty} \sum_{n=0}^{m-1} \frac{x^n}{n!}$$

$$= \sum_{n=0}^{m} \frac{x^n}{n!}$$

$= E(x)$

Hence, $h'(x) = -e^{-x}.E(x) + e^{-x}.E'(x) = 0$

This is true for all x.

17e Thus, $h(x)$ must be constant.

But $h(0) = e^{-0}.E(0)$

$= 1 \times 1$

$= 1$

Thus, $e^{-x}.E(x) = 1$

Hence, $E(x) = e^x$, this is,

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Q17 <u>Special Note:</u>

Some readers may find it strange that the infinite series for $E'(x)$ is determined using the finite series $E_m(x)$ and $E'_m(x)$.

This is done because, in general, it is not valid to simply differentiate the terms of a series expansion.

That is, if $E(x) = \sum_{n=0}^{\infty} a^n$, then it does not necessarily follow that,

$$f'(x) = \sum_{n=0}^{\infty} \frac{da^n}{dx}$$

For example, $f(x) = \sum_{n=0}^{\infty} 2^{-n} \cos(4^n x)$ is a Fourier series that converges, but

$\frac{d}{dx}\text{RHS} = -\sum_{n=0}^{\infty} 2^n \sin(4^n x)$ does not converge, so cannot be equal to $f'(x)$.

The first series converges because the coefficients 2^{-n} form a GP with

$|\text{ratio}| < 1$.

The second series does <u>not</u> converge because the coefficients 2^n form a GP with

$|\text{ratio}| > 1$.

In Q17d, by differentiating the finite sum and taking the limit, the above problem is avoided.

18a $\quad c(x) = a_0 + a_2x^2 + a_4x^4 + \cdots$

18ai $\quad$ At $x = 0$, $\cos 0 = 1$ and $c(0) = a_0$.

Hence, $a_0 = 1$

18aii $\quad c'(x) = 2a_2x + 4a_4x^3 + 6a_6x^5 + \cdots$

$c''(x) = 2a_2 + 12a_4x^2 + 30a_6x^4 + \cdots$

and

$-c(x) = -a_0 - a_2x^2 - a_4x^4 - \cdots$

So, if $c''(x) = -c(x)$ then, equating coefficients of like powers of x:

x^0: $\quad 2a_2 = -a_0 = -1$

$$a_2 = -\frac{1}{2} = \frac{-1}{2!}$$

x^2: $\quad 12a_4 = -a_2 = \frac{1}{2}$

$$a_4 = \frac{1}{24} = \frac{-1}{4!}$$

18aiii Like Q17, this can be done by induction.

Here is just an outline of the proof:

Assuming $a_{2k} = \dfrac{(-1)^k}{(2k)!}$

Then differentiating $c(x)$ twice and comparing with $-c(x)$, gives

$(2k+2)(2k+1)2a_{2k+2} = -2a_{2k}$

Hence,

$2a_{2k+2}$

$= \dfrac{(-1)^{k+1}}{(2k+2)(2k+1)(2k)!}$

$= \dfrac{(-1)^{k+1}}{(2k+2)!}$

So, by the induction step, we find,

$c(x)$

$= \dfrac{1}{0!} - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \cdots$

$= \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

18b $s(x) = a_1 x + a_3 x^3 + a_5 x^5 + \cdots$

18bi At $x = 0$, $\sin 0 = 0$ and $s(0) = 0$, no information.

Differentiating at $x = 0$,

$\dfrac{d}{dx} \sin x \,|_{x=0} = \cos x \,|_{x=0} = 1$

At $x = 0$,

$\dfrac{d}{dx} s(x) \,|_{x=0} = a_1 + 3a_3 x^2 + 5a_5 x^4 + \cdots \,|_{x=0} = a_1$

Hence, $a_1 = 1$

18bii Differentiating again:

$s''(x) = 6a_3x + 20a_5x^3 + 42a_7x^5 + \cdots$

$-s(x) = -a_1x - a_3x^3 - a_5x^5 - \cdots$

So, if $s''(x) = -s(x)$ then, equating coefficients of like powers of x:

x^1: $6a_3 = -a_1 = -1$

$$a_3 = -\frac{1}{6} = \frac{-1}{3!}$$

x^3: $20a_5 = -a_3 = \dfrac{+1}{3!}$

$$a_5 = \frac{1}{5 \times 4 \times 3!} = \frac{+1}{5!}$$

18biii Once again, induction may be used. Here is just an outline of the proof:

Assuming $a_{2k+1} = \dfrac{(-1)^k}{(2k+1)!}$

Then comparing the terms in $s''(x)$ with $-s(x)$, we have

$(2k+3)(2k+2)2a_{2k+3}$

$= -a_{2k+1}$

$= \dfrac{(-1)^k}{(2k+1)!}$

$2a_{2k+3}$

$= \dfrac{(-1)^{k+1}}{(2k+3)(2k+2)(2k+1)!}$

$= \dfrac{(-1)^{k+1}}{(2k+3)!}$

So,

$S(x)$

$= x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots$

$= \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

18ci

$$c(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

Assuming we can differentiate term by term (which, like Q17, can be shown by taking the limit of a partial sum):

$c'(x)$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 2n x^{(2n-1)}}{(2n)!}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n 2n x^{(2n-1)}}{(2n)!} \text{ (Since when } n = 0\text{, the first term is 0)}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n x^{(2n-1)}}{(2n-1)!} \qquad \text{(Cancelling } 2n\text{)}$$

$$= (-1)\sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{(2n-1)}}{(2n-1)!}$$

$$= (-1)\sum_{n=0}^{\infty} \frac{(-1)^n x^{(2n+1)}}{(2n+1)!} \qquad \text{(Replacing } n \text{ with } n+1\text{)}$$

$$= -s(x)$$

Also,

$s'(x)$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}(2n+1)}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$= c(x)$$

18cii $h(x) = (c(x) - \cos x)^2 + (s(x) - \sin x)^2$

So,

$h'(x)$

$= 2(c(x) - \cos x)(c'(x) + \sin x) + 2(s(x) - \sin x)(s'(x) - \cos x)$

$= 2(c(x) - \cos x)(-s(x) + \sin x) + 2(s(x) - \sin x)(c(x) - \cos x)$

$= -2(c(x) - \cos x)(s(x) - \sin x) + 2(s(x) - \sin x)(c(x) - \cos x)$

$= 0$

Hence, $h(x)$ is constant. And subbing in $x = 0$ gives,

$h(0)$

$= (c(0) - \cos 0)^2 + (s(0) - \sin 0)^2$

$= 0^2 + 0^2$

$= 0$

Hence, $h(x) = 0$ for all real x.

18ciii Since $h(x) = 0$,

$(c(x) - \cos x)^2 + (s(x) - \sin x)^2 = 0$

The only time the sum of two square reals is zero is if each is zero. Hence,

$c(x) - \cos x = 0$ and $s(x) - \sin x = 0$

Thus, $c(x) = \cos x$ and $s(x) = \sin x$, for all real x.

19a, b $e^{i\theta}$

$= E(i\theta)$

$= \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!}$ (By question 17)

$= 1 + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^6}{6!} + \frac{(i\theta)^8}{8!} + \cdots + \frac{i\theta}{1!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^7}{7!} + \cdots$

$= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \frac{\theta^8}{8!} + \cdots + i\left(\frac{\theta}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \frac{\theta^7}{7!} + \cdots\right)$

$= \cos\theta + i\sin\theta$ (By question 18)

Solutions to Exercise 3E Foundation questions

1a $2i = 2e^{i\frac{\pi}{2}}$ (note that $i = e^{i\frac{\pi}{2}}$)

1b $2e^{i\frac{\pi}{2}} = 2e^{i\left(\frac{\pi}{2}+2k\pi\right)}$

1c $z = re^{i\theta}$

$z^2 = r^2e^{2i\theta}$

$2e^{i\left(\frac{\pi}{2}+2k\pi\right)} = 2i = z^2 = r^2e^{2i\theta}$

Hence $r^2e^{2i\theta} = 2e^{i\left(\frac{\pi}{2}+2k\pi\right)}$, thus $r^2 = 2$ and $2\theta = \frac{\pi}{2} + 2k\pi = \frac{\pi+4k\pi}{2}$ and so $r = \sqrt{2}, \theta = \frac{(4k+1)\pi}{4}$

1d $z = \sqrt{2}e^{\frac{(4k+1)i\pi}{4}} = \sqrt{2}e^{i\frac{\pi}{4}}, \sqrt{2}e^{\frac{-3i\pi}{4}}$

1e $z = \sqrt{2}e^{i\frac{\pi}{4}} = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) = 1 + i$

$z = \sqrt{2}e^{-i\frac{3\pi}{4}} = \sqrt{2}\left(\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right) = \sqrt{2}\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) - 1 - i$

2a $-1 = e^{i\pi}$

2b $e^{i(\pi+2k\pi)}$

2c $z^4 = \left(re^{i\theta}\right)^4 = r^4e^{4i\theta}$

Hence $r^4e^{4i\theta} = e^{i(\pi+2k\pi)}$

$r = 1$ and

$4\theta = (\pi + 2k\pi)$

$$\theta = \frac{\pi + 2k\pi}{4}$$

2d

$$z = e^{-i\frac{3\pi}{4}}, e^{-i\frac{\pi}{4}},\ \ e^{i\frac{\pi}{4}}, e^{i\frac{3\pi}{4}}$$

2e $z = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i, \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i, \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i, -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$

3a $-i = e^{-\frac{i\pi}{2}}$

3b $-i = e^{-i\left(\frac{\pi}{2}+2k\pi\right)}$

3c $z = re^{i\theta}$

$$\left(re^{i\theta}\right)^3 = e^{-i\left(\frac{\pi}{2}+2k\pi\right)}$$

$$r^3e^{3i\theta} = e^{-i\left(\frac{\pi}{2}+2k\pi\right)}$$

$$r = 1$$

$$3\theta = -\left(\frac{\pi}{2} + 2k\pi\right)$$

$$\theta = -\frac{1}{3}\left(\frac{\pi}{2} + 2k\pi\right)$$

$$\theta = -\frac{1}{3}\left(\frac{\pi + 4k\pi}{2}\right)$$

$$\theta = -\frac{(4k+1)\pi}{6}$$

3d $z = e^{\frac{i\pi}{2}}, e^{-\frac{i\pi}{6}}, e^{-\frac{5i\pi}{6}}$

4a $e^{ni\theta} + e^{-ni\theta}$

$$= \cos n\theta + i\sin n\theta + \cos(-n\theta) + i\sin(-n\theta)$$

$$= \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$$

$$= 2\cos n\theta$$

4b $\left(e^{i\theta}+e^{-i\theta}\right)^3$

$= e^{3i\theta}+3e^{2i\theta}e^{-i\theta}+3e^{i\theta}e^{-2i\theta}+e^{-3i\theta}$

$= e^{3i\theta}+3e^{i\theta}+3e^{-i\theta}+e^{-3i\theta}$

$= e^{3i\theta}+3\left(e^{i\theta}+e^{-i\theta}\right)+e^{-3i\theta}$

$= \left(e^{3i\theta}+e^{-3i\theta}\right)+3\left(e^{i\theta}+e^{-i\theta}\right)$

4c $\cos^3\theta$

$= \left(\frac{1}{2}\times 2\cos\theta\right)^3$

$= \left(\frac{1}{2}\times\left(e^{i\theta}+e^{-i\theta}\right)\right)^3$

$= \frac{1}{2^3}\left(e^{i\theta}+e^{-i\theta}\right)^3$

$= \frac{1}{8}\left(\left(e^{3i\theta}+e^{-3i\theta}\right)+3\left(e^{i\theta}+e^{-i\theta}\right)\right)$

$= \frac{1}{8}(2\cos 3\theta+6\cos\theta)$

$= \frac{1}{4}\cos 3\theta+\frac{3}{4}\cos\theta$

5

$e^{in\theta} - e^{-in\theta}$

$= \cos n\theta + i\sin n\theta - (\cos n\theta - i\sin n\theta)$

$= 2i\sin n\theta$

Hence

$\sin n\theta = \dfrac{1}{2i}\left(e^{in\theta} - e^{-in\theta}\right)$

$\sin^3\theta$

$= \left(\dfrac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)\right)^3$

$= \dfrac{1}{(2i)^3}\left(e^{i\theta} - e^{-i\theta}\right)^3$

$= \dfrac{1}{(2i)^3}\left(e^{3i\theta} - 3e^{2i\theta}e^{-i\theta} + 3e^{i\theta}e^{-2i\theta} - e^{-3i\theta}\right)$

$= \dfrac{1}{(2i)^3}\left(e^{3i\theta} - e^{-3i\theta} - 3e^{i\theta} + 3e^{-i\theta}\right)$

$= \dfrac{1}{(2i)^3}\left(e^{3i\theta} - e^{-3i\theta} - 3\left(e^{i\theta} - 3e^{-i\theta}\right)\right)$

$= \dfrac{1}{(2i)^3}(2i\sin 3\theta - 6i\sin\theta)$

$= -\dfrac{1}{8}(2\sin 3\theta - 6\sin\theta)$

$= \dfrac{3}{4}\sin\theta - \dfrac{1}{4}\sin 3\theta$

Solutions to Exercise 3E Development questions

6a Consider the equation

$z^4 + 16 = 0$

Let $z = re^{i\theta}$ be a root of the equation, then

$\left(re^{i\theta}\right)^4 + 16 = 0$

$r^4 e^{4i\theta} = -16$

Taking the modulus of both sides we see that $r^4 = 16$, and so $e^{4i\theta} = -1$.

Hence $r = 2$ (r is always positive) and $4\theta = \pi + 2n\pi$ where n is an integer. Thus,

$\theta = \dfrac{(2n+1)\pi}{4}$, where n is an integer

This gives the roots of the equation as,

$z = 2e^{\pm\frac{i\pi}{4}}, 2e^{\pm\frac{3i\pi}{4}}, \ldots$

Hence writing $z^4 + 16$ as a product of factors gives

$z^4 + 16 = \left(z - 2e^{\frac{i\pi}{4}}\right)\left(z - 2e^{-\frac{i\pi}{4}}\right)\left(z - 2e^{\frac{3i\pi}{4}}\right)\left(z - 2e^{-\frac{3i\pi}{4}}\right)$

6b $z^4 + 16$

$= \left(z - 2e^{\frac{i\pi}{4}}\right)\left(z - 2e^{-\frac{i\pi}{4}}\right)\left(z - 2e^{\frac{3i\pi}{4}}\right)\left(z - 2e^{-\frac{3i\pi}{4}}\right)$

$= \left(z^2 - 2ze^{\frac{i\pi}{4}} - 2ze^{-\frac{i\pi}{4}} + 4e^0\right)\left(z^2 - 2ze^{\frac{3i\pi}{4}} - 2ze^{-\frac{3i\pi}{4}} + 4e^0\right)$

$= \left(z^2 - 2z\left(e^{\frac{i\pi}{4}} + e^{-\frac{i\pi}{4}}\right) + 4\right)\left(z^2 - 2z\left(e^{\frac{3i\pi}{4}} + e^{-\frac{3i\pi}{4}}\right) + 4\right)$

$= \left(z^2 - 2z\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4} + \cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right) + 4\right)$

$\left(z^2 - 2z\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4} + \cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4}\right) + 4\right)$

$= \left(z^2 - 2z\left(2\cos\frac{\pi}{4}\right) + 4\right)\left(z^2 - 2z\left(2\cos\frac{3\pi}{4}\right) + 4\right)$

$= \left(z^2 - 2z\left(\frac{2}{\sqrt{2}}\right) + 4\right) + \left(z^2 - 2z\left(-\frac{2}{\sqrt{2}}\right) + 4\right)$

$= (z^2 - 2\sqrt{2}z + 4)(z^2 + 2\sqrt{2}z + 4)$

6c $z^4 + 16$

$= (z^4 + 8z^2 + 16) - 8z^2$

$= (z^2 + 4)^2 - \left(2\sqrt{2}z\right)^2$

$= \left(z^2 + 4 - 2\sqrt{2}z\right)\left(z^2 + 4 + 2\sqrt{2}z\right)$

$= (z^2 - 2\sqrt{2}z + 4)(z^2 + 2\sqrt{2}z + 4)$

7a Consider the equation $z^5 + 1 = 0$. Let $z = r\,\mathrm{e}^{\mathrm{i}\theta}$ be a root of the equation, then

$r^5\mathrm{e}^{\mathrm{i}5\theta} + 1 = 0$

$r^5\mathrm{e}^{\mathrm{i}5\theta} = -1$

Taking the modulus, we see that $r = 1$ and so $e^{i5\theta} = -1$. Thus, $5\theta = \pi + 2n\pi$ where n is an integer, and so

$\theta = \dfrac{(2n+1)\pi}{5}$, where n is an integer.

Hence, the roots are,

$z = e^{\pm\frac{i\pi}{5}}, e^{\pm\frac{3i\pi}{5}}, e^{\pm\frac{5i\pi}{5}}, \ldots$

$z = e^{\pm\frac{i\pi}{5}}, e^{\pm\frac{3i\pi}{5}}, -1, \ldots$

Thus, writing $z^5 + 1$ as a product of factors gives

$$z^5 + 1 = \left(z - (-1)\right)\left(z - e^{\frac{i\pi}{5}}\right)\left(z - e^{-\frac{i\pi}{5}}\right)\left(z - e^{\frac{3i\pi}{5}}\right)\left(z - e^{-\frac{3i\pi}{5}}\right)$$

$$= (z+1)\left(z - e^{\frac{i\pi}{5}}\right)\left(z - e^{-\frac{i\pi}{5}}\right)\left(z - e^{\frac{3i\pi}{5}}\right)\left(z - e^{-\frac{3i\pi}{5}}\right)$$

7b $z^5 + 1$

$$= (z+1)\left(z - e^{\frac{i\pi}{5}}\right)\left(z - e^{-\frac{i\pi}{5}}\right)\left(z - e^{\frac{3i\pi}{5}}\right)\left(z - e^{-\frac{3i\pi}{5}}\right)$$

$$= (z+1)\left(z^2 - \left(e^{\frac{i\pi}{5}} + e^{-\frac{i\pi}{5}}\right)z + \left(e^{\frac{i\pi}{5}}e^{-\frac{i\pi}{5}}\right)\right)\left(z^2 - \left(e^{3\frac{i\pi}{5}} + e^{-3\frac{i\pi}{5}}\right)z + \left(e^{3\frac{i\pi}{5}}e^{-3\frac{i\pi}{5}}\right)\right)$$

$$= (z+1)\left(z^2 - \left(e^{\frac{i\pi}{5}} + e^{-\frac{i\pi}{5}}\right)z + e^0\right)\left(z^2 - \left(e^{3\frac{i\pi}{5}} + e^{-3\frac{i\pi}{5}}\right)z + e^0\right)$$

$$= (z+1)\left(z^2 - \left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5} + \cos\frac{\pi}{5} - i\sin\frac{\pi}{5}\right)z + 1\right)\left(z^2 - \left(\cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5} + \cos\frac{3\pi}{5} - i\sin\frac{3\pi}{5}\right)z + 1\right)$$

$$= (z+1)(z^2 - 2z\cos\frac{\pi}{5} + 1)(z^2 + 2z\cos\left(\pi - \frac{3\pi}{5}\right) + 1)$$

$$= (z+1)(z - 2\cos\frac{\pi}{5} + 1)(z + 2\cos\frac{2\pi}{5} + 1)$$

7c The sum of the roots of the equation $z^5 + 1$ is

$$\text{cis}\left(\frac{3\pi}{5}\right) + \text{cis}\left(-\frac{3\pi}{5}\right) + \text{cis}\left(\frac{\pi}{5}\right) + \text{cis}\left(-\frac{\pi}{5}\right) - 1 = 0$$

$$2\cos\left(\frac{3\pi}{5}\right) + 2\cos\left(\frac{\pi}{5}\right) - 1 = 0$$

$$-2\cos\left(\pi - \frac{3\pi}{5}\right) + 2\cos\left(\frac{\pi}{5}\right) - 1 = 0$$

Hence,

$$2\cos\left(\frac{2\pi}{5}\right) - 2\cos\left(\frac{\pi}{5}\right) + 1 = 0 \ (1)$$

Using the double angle identity, we have,

$$\cos\frac{2\pi}{5} = 2\left(\cos\frac{\pi}{5}\right)^2 - 1$$

Subbing into (1) gives

$$4\left(\cos\frac{\pi}{5}\right)^2 - 2 - 2\cos\frac{\pi}{5} + 1 = 0$$

$$4\left(\cos\frac{\pi}{5}\right)^2 - 2\cos\frac{\pi}{5} - 1 = 0$$

Solving we have

$$\cos\frac{\pi}{5} = \frac{2 \pm \sqrt{(4+16)}}{8}$$

Since $\cos\frac{\pi}{5} > 0$ we only take the positive root and so,

$$\cos\frac{\pi}{5} = \frac{1+\sqrt{5}}{4}$$

Back subbing this into the identity then gives,

$$\cos\frac{2\pi}{5} = 2 \times \frac{\left(1+\sqrt{5}\right)^2}{16} - 1$$

$$= \frac{6 + 2\sqrt{5} - 8}{8}$$

$$= \frac{-1+\sqrt{5}}{4}$$

$$= \frac{\sqrt{5}-1}{4}$$

8a Noting that $\text{cis}(\theta) = e^{i\theta}$

it follows that

$\cos\theta + i\sin\theta = e^{i\theta}$ (1)

And hence that

$\cos(-\theta) + i\sin(-\theta) = e^{-i\theta}$

$\cos\theta - i\sin\theta = e^{-i\theta}$ (2)

(1) + (2):

$2\cos\theta = e^{i\theta} + e^{-i\theta}$

$$\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$$

(1) − (2):

$2i\sin\theta = e^{i\theta} - e^{-i\theta}$

$$\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$$

8b i $\cos 2\theta = \frac{1}{2}\left(e^{2i\theta} + e^{-2i\theta}\right)$

$$= \frac{1}{4}\left(e^{2i\theta} + e^{0} + e^{0} + e^{-2i\theta}\right) + \frac{1}{4}\left(e^{2i\theta} - e^{0} - e^{0} + e^{-2i\theta}\right)$$

$$= \left(\frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)\right)^2 - \left(\frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)\right)^2$$

$$= \cos^2\theta - \sin^2\theta$$

8b ii $\sin 2\theta$

$$= \frac{1}{2i}\left(e^{2i\theta} - e^{-2i\theta}\right)$$

$$= \frac{1}{2i}\left(e^{i\theta} + e^{-i\theta}\right)\left(e^{i\theta} - e^{-i\theta}\right)$$

$$= 2\left(\frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)\frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)\right)$$

$$= 2\cos\theta\sin\theta$$

8b iii $\cos(\alpha + \beta)$

$$= \frac{1}{2}\left(e^{i(\alpha+\beta)} + e^{-i(\alpha+\beta)}\right)$$

$$= \frac{1}{4}\left(e^{i(\alpha+\beta)} + e^{i(\alpha-\beta)} + e^{i(\beta-\alpha)} + e^{-i(\alpha+\beta)}\right)$$

$$+ \frac{1}{4}\left(e^{i(\alpha+\beta)} - e^{i(\alpha-\beta)} - e^{i(\beta-\alpha)} + e^{-i(\alpha+\beta)}\right)$$

$$= \frac{1}{2}\left(e^{i(\alpha)} + e^{-i(\alpha)}\right)\frac{1}{2}\left(e^{i(\beta)} + e^{-i(\beta)}\right) - \frac{1}{2i}\left(e^{i(\alpha)} - e^{-i(\alpha)}\right)\frac{1}{2i}\left(e^{i(\beta)} - e^{-i(\beta)}\right)$$

$$= \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

8b iv $\sin(\alpha + \beta)$

$$= \frac{1}{2i}\left(e^{i(\alpha+\beta)} - e^{-i(\alpha+\beta)}\right)$$

$$= \frac{1}{4i}\left(e^{i(\alpha+\beta)} + e^{i(\alpha-\beta)} - e^{i(\beta-\alpha)} - e^{-i(\alpha+\beta)}\right)$$

$$+ \frac{1}{4i}\left(e^{i(\alpha+\beta)} - e^{i(\alpha-\beta)} + e^{i(\beta-\alpha)} - e^{-i(\alpha+\beta)}\right)$$

$$= \frac{1}{2i}\left(e^{i(\alpha)} - e^{-i(\alpha)}\right)\frac{1}{2}\left(e^{i(\beta)} + e^{-i(\beta)}\right) + \frac{1}{2}\left(e^{i(\alpha)} + e^{-i(\alpha)}\right)\frac{1}{2i}\left(e^{i(\beta)} - e^{-i(\beta)}\right)$$

$$= \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

9a $\cos^6\theta$

$$=\left(\frac{1}{2}\left(e^{i\theta}+e^{-i\theta}\right)\right)^6$$

$$=\frac{1}{2^6}\left(e^{6i\theta}+6e^{4i\theta}+15e^{2i\theta}+20+15e^{-2i\theta}+6e^{-4i\theta}+e^{-6i\theta}\right)$$

$$=\frac{1}{2^6}\left(\left(e^{6i\theta}+e^{-6i\theta}\right)+6\left(e^{4i\theta}+e^{-4i\theta}\right)+15\left(e^{2i\theta}+e^{-2i\theta}\right)+20\right)$$

$$=\frac{1}{2^6}(2\cos 6\theta+12\cos 4\theta+30\cos 2\theta+20)$$

$$=\frac{1}{32}(\cos 6\theta+6\cos 4\theta+15\cos 2\theta+10)$$

9b

$$\int_0^{\frac{\pi}{4}}\cos^6\theta\,d\theta$$

$$=\int_0^{\frac{\pi}{4}}\frac{1}{2^5}(\cos 6\theta+6\cos 4\theta+15\cos 2\theta+10)\,d\theta$$

$$=\frac{1}{2^5}\int_0^{\frac{\pi}{4}}(\cos 6\theta+6\cos 4\theta+15\cos 2\theta+10)\,d\theta$$

$$=\frac{1}{2^5}\left[\frac{1}{6}\sin 6\theta+\frac{6}{4}\sin 4\theta+\frac{15}{2}\sin 2\theta+10\theta\right]_0^{\frac{\pi}{4}}$$

$$=\frac{1}{2^5}\left(\frac{1}{6}\sin\frac{3\pi}{2}+\frac{6}{4}\sin\pi+\frac{15}{2}\sin\frac{\pi}{2}+\frac{10\pi}{4}-0\right)$$

$$=\frac{1}{2^5}\left(-\frac{1}{6}+0+\frac{15}{2}+\frac{10\pi}{4}\right)$$

$$=\frac{1}{2^5}\left(\frac{44}{6}+\frac{10\pi}{4}\right)$$

$$=\frac{15\pi+44}{192}$$

10a $\sin^3\theta$

$$= \left(\frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)\right)^3$$

$$= \frac{1}{8i^3}\left(e^{3i\theta} - 3e^{i\theta} + 3e^{-i\theta} - e^{-3i\theta}\right)$$

$$= -\frac{1}{8i}\left(\left(e^{3i\theta} - e^{-3i\theta}\right) - 3\left(e^{i\theta} - e^{-i\theta}\right)\right)$$

$$= -\frac{1}{8i}(2i\sin 3\theta - 6i\sin\theta)$$

$$= -\frac{1}{4}(\sin 3\theta - 3\sin\theta)$$

$\sin^5\theta$

$$= \left(\frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)\right)^5$$

$$= \frac{1}{32i^5}\left(e^{5i\theta} - 5e^{3i\theta} + 10e^{i\theta} - 10e^{-i\theta} + 5e^{-3i\theta} - e^{-5i\theta}\right)$$

$$= \frac{1}{32i}\left(\left(e^{5i\theta} - e^{-5i\theta}\right) - 5\left(e^{3i\theta} - e^{-3i\theta}\right) + 10\left(e^{i\theta} - e^{-i\theta}\right)\right)$$

$$= \frac{1}{32i}(2i\sin 5\theta - 10i\sin 3\theta + 20i\sin\theta)$$

$$= \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$$

10b $\sin^3\theta\cos^2\theta$

$$= \sin^3\theta\,(1 - \sin^2\theta)$$

$$= \sin^3\theta - \sin^5\theta$$

$$= -\frac{1}{4}(\sin 3\theta - 3\sin\theta) - \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$$

$$= \frac{1}{16}(2\sin\theta + \sin 3\theta - \sin 5\theta)$$

10c

$$\int_0^{\frac{\pi}{3}} \sin^3\theta \cos^2\theta \, d\theta$$

$$= \int_0^{\frac{\pi}{3}} \frac{1}{16}(2\sin\theta + \sin 3\theta - \sin 5\theta)\, d\theta$$

$$= \frac{1}{16}\left[-2\cos\theta - \frac{1}{3}\cos 3\theta + \frac{1}{5}\cos 5\theta\right]_0^{\frac{\pi}{3}}$$

$$= \frac{1}{16}\left(-2\cos\frac{\pi}{3} - \frac{1}{3}\cos\pi + \frac{1}{5}\cos\frac{5\pi}{3} - \left(-2\cos 0 - \frac{1}{3}\cos 0 + \frac{1}{5}\cos 0\right)\right)$$

$$= \frac{1}{16}\left(-1 + \frac{1}{3} + \frac{1}{10} - \left(-2 - \frac{1}{3} + \frac{1}{5}\right)\right)$$

$$= \frac{47}{480}$$

11a $e^{ni\theta} + e^{-ni\theta}$

$= \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$

$= 2\cos n\theta$

11b Since

$5z^4 - 11z^3 + 16z^2 - 11z + 5 = 0$

$z^2(5z^2 - 11z + 16 - 11z^{-1} + 5z^{-2}) = 0$

$z^2(5(z^2 + z^{-2}) - 11(z + z^{-1}) + 16) = 0$

Since the roots have modulus 1, we have $z \neq 0$, and so,

$5(2\cos 2\theta) - 11(2\cos\theta) + 16 = 0$

$5\cos 2\theta - 11\cos\theta + 8 = 0$

11c Using the result from part b,

$5\cos 2\theta - 11\cos\theta + 8 = 0$

$5(2\cos^2\theta - 1) - 11\cos\theta + 8 = 0$

$10\cos^2\theta - 5 - 11\cos\theta + 8 = 0$

$10\cos^2\theta - 11\cos\theta + 3 = 0$

$(5\cos\theta - 3)(2\cos\theta - 1) = 0$

Hence, we have

$\cos\theta = \dfrac{3}{5}$ or $\dfrac{1}{2}$

When $\cos\theta = \dfrac{3}{5}$, $\quad \sin\theta = \pm\dfrac{\sqrt{5^2 - 3^2}}{5} = \pm\dfrac{4}{5}$

When $\cos\theta = \dfrac{1}{2}$, $\quad \sin\theta = \pm\dfrac{\sqrt{2^2 - 1}}{2} = \pm\dfrac{\sqrt{3}}{2}$

Hence the roots are $z = \dfrac{3}{5} \pm \dfrac{4}{5}i, \dfrac{1}{2} \pm \dfrac{\sqrt{3}}{2}i$

12 $1 - i$

$= \sqrt{2}e^{-i\frac{\pi}{4}}$

$= e^{\ln\sqrt{2}}e^{-i\frac{\pi}{4}}$

$= e^{\ln\sqrt{2} - \frac{i\pi}{4}}$

Comparing this with e^{a+ib} gives $a = \ln\sqrt{2} = \frac{1}{2}\ln 2$ and $b = -\frac{\pi}{4}$.

13a $\cos(A + B) + \cos(A - B)$

$= \cos A\cos B - \sin A\sin B + \cos A\cos B + \sin A\sin B$

$= 2\cos A\cos B$

13b Let $A = \frac{\alpha+\beta}{2}$ and $B = \frac{\alpha-\beta}{2}$, thus

$2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$

$= \cos\left(\left(\frac{\alpha+\beta}{2}\right)+\left(\frac{\alpha-\beta}{2}\right)\right)+\cos\left(\left(\frac{\alpha+\beta}{2}\right)-\left(\frac{\alpha-\beta}{2}\right)\right)$

$= \cos\alpha + \cos\beta$

13c $\sin(A+B)+\sin(A-B)$

$= \sin A\cos B + \cos A\sin B + \sin A\cos B - \cos A\sin B$

$= 2\sin A\cos B$

Let $A = \frac{\alpha+\beta}{2}$ and $B = \frac{\alpha-\beta}{2}$, thus

$2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$

$= \sin\left(\left(\frac{\alpha+\beta}{2}\right)+\left(\frac{\alpha-\beta}{2}\right)\right)+\sin\left(\left(\frac{\alpha+\beta}{2}\right)-\left(\frac{\alpha-\beta}{2}\right)\right)$

$= \sin\alpha + \sin\beta$

as required

13d $e^{i\alpha}+e^{i\beta}$

$= \cos\alpha + i\sin\alpha + \cos\beta + i\sin\beta$

$= (\cos\alpha+\cos\beta)+i(\sin\alpha+\sin\beta)$

$= 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)+i\left(2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)\right)$ (using part b and c)

$= 2\cos\left(\frac{\alpha-\beta}{2}\right)\left[\cos\left(\frac{\alpha+\beta}{2}\right)+i\sin\left(\frac{\alpha+\beta}{2}\right)\right]$

$= 2\cos\left(\frac{\alpha-\beta}{2}\right)e^{\frac{i}{2}(\alpha+\beta)}$

14ai $\cos\alpha + \cos\beta$

$$= \frac{1}{2}\left(e^{i\alpha} + e^{-i\alpha}\right) + \frac{1}{2}\left(e^{i\beta} + e^{-i\beta}\right)$$

$$= \frac{1}{2}\left(e^{i\alpha} + e^{i\beta} + e^{-i\alpha} + e^{-i\beta}\right)$$

$$= \frac{1}{2}\left(e^{\frac{i}{2}(\alpha+\beta)} + e^{-\frac{i}{2}(\alpha+\beta)}\right)\left(e^{\frac{i}{2}(\alpha-\beta)} + e^{-\frac{i}{2}(\alpha-\beta)}\right)$$

$$= 2\left(\frac{1}{2}\left(e^{\frac{i}{2}(\alpha+\beta)} + e^{-\frac{i}{2}(\alpha+\beta)}\right)\right)\left(\frac{1}{2}\left(e^{\frac{i}{2}(\alpha-\beta)} + e^{-\frac{i}{2}(\alpha-\beta)}\right)\right)$$

$$= 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

14aii $\sin\alpha + \sin\beta$

$$= \frac{1}{2i}\left(e^{i\alpha} - e^{-i\alpha}\right) + \frac{1}{2i}\left(e^{i\beta} - e^{-i\beta}\right)$$

$$= \frac{1}{2i}\left(e^{i\alpha} + e^{i\beta} - e^{-i\alpha} - e^{-i\beta}\right)$$

$$= \frac{1}{2i}\left(e^{\frac{i}{2}(\alpha+\beta)} - e^{-\frac{i}{2}(\alpha+\beta)}\right)\left(e^{\frac{i}{2}(\alpha-\beta)} + e^{-\frac{i}{2}(\alpha-\beta)}\right)$$

$$= 2\left(\frac{1}{2i}\left(e^{\frac{i}{2}(\alpha+\beta)} - e^{-\frac{i}{2}(\alpha+\beta)}\right)\right)\left(\frac{1}{2}\left(e^{\frac{i}{2}(\alpha-\beta)} + e^{-\frac{i}{2}(\alpha-\beta)}\right)\right)$$

$$= 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

14b $\tan\theta$

$$= \frac{\sin\theta}{\cos\theta}$$

$$= \frac{\frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)}{\frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)}$$

$$= \frac{\left(e^{i\theta} - e^{-i\theta}\right)}{i\left(e^{i\theta} + e^{-i\theta}\right)}$$

14c $\tan 2\theta$

$$= \frac{\left(e^{2i\theta} - e^{-2i\theta}\right)}{i(e^{2i\theta} + e^{-2i\theta})}$$

$$= \frac{\frac{2}{i}\left(e^{2i\theta} - e^{-2i\theta}\right)}{2(e^{2i\theta} + e^{-2i\theta})}$$

$$= \frac{\frac{2}{i}\left(e^{2i\theta} - e^{-2i\theta}\right)}{e^{2i\theta} + e^{-2i\theta} + e^{2i\theta} + e^{-2i\theta}}$$

$$= \frac{\frac{2}{i}\left(e^{2i\theta} - e^{-2i\theta}\right)}{e^{2i\theta} + 2 + e^{-2i\theta} + e^{2i\theta} - 2 + e^{-2i\theta}}$$

$$= \frac{\frac{2}{i}\left(e^{i\theta} - e^{-i\theta}\right)\left(e^{i\theta} + e^{-i\theta}\right)}{(e^{i\theta} + e^{-i\theta})^2 + (e^{i\theta} - e^{-i\theta})^2}$$

$$= \frac{2\left(\frac{\left(e^{i\theta} - e^{-i\theta}\right)}{i(e^{i\theta} + e^{-i\theta})}\right)}{1 + \frac{(e^{i\theta} - e^{-i\theta})^2}{(e^{i\theta} + e^{-i\theta})^2}}$$

$$= \frac{2\left(\frac{\left(e^{i\theta} - e^{-i\theta}\right)}{i(e^{i\theta} + e^{-i\theta})}\right)}{1 - \left(\frac{\left(e^{i\theta} - e^{-i\theta}\right)}{i(e^{i\theta} + e^{-i\theta})}\right)^2}$$

$$= \frac{2\tan\theta}{1 - \tan^2\theta}$$

15a $z + z^2 + z^3 + \cdots + z^n$

This is a geometric series with $a = z$ and $r = z$. Hence, the sum of the geometric series is

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$= \frac{z(z^n - 1)}{z - 1}$$

$$= \frac{z^{n+1} - z}{z - 1}$$

15b Using part a we have

$$z + z^2 + \cdots + z^n = \frac{z^{n+1} - z}{z - 1}$$

Putting $z = e^{i\theta}$ gives

$$e^{i\theta} + \left(e^{i\theta}\right)^2 + \cdots + \left(e^{i\theta}\right)^n = \frac{\left(e^{i(n+1)\theta} - e^{i\theta}\right)}{e^{i\theta} - 1}$$

$$e^{i\theta} + e^{2i\theta} + \cdots + e^{ni\theta} = \frac{\left(e^{i(n+1)\theta} - e^{i\theta}\right)}{e^{i\theta} - 1}$$

$$e^{i\theta} + e^{2i\theta} + \cdots + e^{ni\theta} = \frac{e^{i\theta}\left(e^{in\theta} - 1\right)}{e^{i\theta} - 1}$$

$$e^{i\theta} + e^{2i\theta} + \cdots + e^{ni\theta} = \frac{\left(e^{in\theta} - 1\right)}{1 - e^{-i\theta}}$$

$$e^{i\theta} + e^{2i\theta} + \cdots + e^{ni\theta} = \frac{e^{\frac{1}{2}in\theta}\left(e^{\frac{1}{2}in\theta} - e^{-\frac{1}{2}in\theta}\right)}{e^{-\frac{1}{2}i\theta}(e^{\frac{1}{2}i\theta} - e^{-\frac{1}{2}i\theta})}$$

$$e^{i\theta} + e^{2i\theta} + \cdots + e^{ni\theta} = \frac{e^{\frac{1}{2}i(n+1)\theta}\left(e^{\frac{1}{2}in\theta} - e^{-\frac{1}{2}in\theta}\right)}{(e^{\frac{1}{2}i\theta} - e^{-\frac{1}{2}i\theta})}$$

$$e^{i\theta} + e^{2i\theta} + \cdots + e^{ni\theta} = \frac{e^{\frac{1}{2}i(n+1)\theta}\left(2i\sin\frac{1}{2}n\theta\right)}{2i\sin\frac{1}{2}\theta}$$

$$e^{i\theta} + e^{2i\theta} + \cdots + e^{ni\theta} = \frac{e^{\frac{1}{2}i(n+1)\theta}\left(\sin\frac{1}{2}n\theta\right)}{\sin\frac{1}{2}\theta}$$

Expanding both sides, we get

$$\cos\theta + i\sin\theta + \cos 2\theta + i\sin 2\theta + \cdots + \cos n\theta + i\sin n\theta$$

$$= \frac{\left(\cos\frac{1}{2}(n+1)\theta + i\sin\frac{1}{2}(n+1)\theta\right)\left(\sin\frac{1}{2}n\theta\right)}{\sin\frac{1}{2}\theta}$$

Equating the imaginary component in the above equation gives

$$\sin\theta + \sin 2\theta + \cdots + \sin n\theta = \frac{\sin\frac{1}{2}(n+1)\theta\left(\sin\frac{1}{2}n\theta\right)}{\sin\frac{1}{2}\theta}$$

15c Let $\theta = \frac{\pi}{n}$, then using the result from part b we have

$$\sin\frac{\pi}{n} + \sin\frac{2\pi}{n} + \sin\frac{3\pi}{n} + \cdots + \sin\frac{(n-1)\pi}{n} + \sin\frac{n\pi}{n} = \frac{\sin\frac{1}{2}n\left(\frac{\pi}{n}\right)\sin\frac{1}{2}(n+1)\frac{\pi}{n}}{\sin\frac{1}{2}\left(\frac{\pi}{n}\right)}$$

$$\sin\frac{\pi}{n} + \sin\frac{2\pi}{n} + \sin\frac{3\pi}{n} + \cdots + \sin\frac{(n-1)\pi}{n} + \sin\pi = \frac{\sin\frac{1}{2}\pi\sin\frac{1}{2}\left(1+\frac{1}{n}\right)\pi}{\sin\frac{1}{2}\left(\frac{\pi}{n}\right)}$$

$$\sin\frac{\pi}{n} + \sin\frac{2\pi}{n} + \sin\frac{3\pi}{n} + \cdots + \sin\frac{(n-1)\pi}{n} + 0 = \frac{\sin\frac{1}{2}\pi\sin\frac{1}{2}\left(1+\frac{1}{n}\right)\pi}{\sin\frac{1}{2}\left(\frac{\pi}{n}\right)}$$

$$\sin\frac{\pi}{n} + \sin\frac{2\pi}{n} + \sin\frac{3\pi}{n} + \cdots + \sin\frac{(n-1)\pi}{n} = \frac{(1)\sin\left(\frac{\pi}{2}+\frac{\pi}{2n}\right)}{\sin\frac{1}{2}\left(\frac{\pi}{n}\right)}$$

$$\sin\frac{\pi}{n} + \sin\frac{2\pi}{n} + \sin\frac{3\pi}{n} + \cdots + \sin\frac{(n-1)\pi}{n} = \frac{\cos\left(\frac{\pi}{2n}\right)}{\sin\frac{1}{2}\left(\frac{\pi}{n}\right)}$$

$$\sin\frac{\pi}{n} + \sin\frac{2\pi}{n} + \sin\frac{3\pi}{n} + \cdots + \sin\frac{(n-1)\pi}{n} = \cot\frac{\pi}{2n}$$

Solutions to Exercise 3E Enrichment questions

16 LHS

$$= \frac{e^{(\alpha+\beta)i} - e^{-(\alpha+\beta)i}}{i(e^{(\alpha+\beta)i} + e^{-(\alpha+\beta)i})}$$

RHS

$$= \left(\frac{\dfrac{e^{i\alpha} - e^{-i\alpha}}{i(e^{i\alpha} + e^{-i\alpha})} + \dfrac{e^{i\beta} - e^{-i\beta}}{i(e^{i\beta} + e^{-i\beta})}}{1 + \dfrac{e^{i\alpha} - e^{-i\alpha}}{e^{i\alpha} + e^{-i\alpha}} \cdot \dfrac{e^{i\beta} - e^{-i\beta}}{e^{i\beta} + e^{-i\beta}}} \right)$$

$$= \frac{(e^{i\alpha} - e^{-i\alpha})(e^{i\beta} + e^{-i\beta}) + (e^{i\beta} - e^{-i\beta})(e^{i\alpha} + e^{-i\alpha})}{i[(e^{i\alpha} + e^{-i\alpha})(e^{i\beta} + e^{-i\beta}) + (e^{i\alpha} - e^{-i\alpha})(e^{i\beta} - e^{-i\beta})]}$$

$$= \frac{e^{i(\alpha+\beta)} + e^{i(\alpha-\beta)} - e^{i(\beta-\alpha)} - e^{-i(\alpha+\beta)} + e^{i(\alpha+\beta)} + e^{i(\beta-\alpha)} - e^{i(\alpha-\beta)} - e^{-i(\alpha+\beta)}}{i[e^{i(\alpha+\beta)} + e^{i(\alpha-\beta)} + e^{i(\beta-\alpha)} + e^{-i(\alpha+\beta)} + e^{i(\alpha+\beta)} - e^{i(\alpha-\beta)} - e^{i(\beta-\alpha)} + e^{-i(\alpha+\beta)}]}$$

$$= \frac{2(e^{i(\alpha+\beta)} - e^{-i(\alpha+\beta)})}{2i(e^{i(\alpha+\beta)} + e^{-i(\alpha+\beta)})}$$

$$= \frac{e^{i(\alpha+\beta)} - e^{-i(\alpha+\beta)}}{i(e^{i(\alpha+\beta)} - e^{-i(\alpha+\beta)})}$$

$= \text{LHS}$

17a $z^{2n+1} = 1$

So, $e^{i(2n+1)\theta} = e^{i2k\pi}$, where $z = e^{i\theta}$ and k is an integer

Thus $(2n+1)\theta = 2k\pi$, which gives,

$\theta = \dfrac{2k\pi}{(2n+1)}$, and for principal values $-n \le k \le n$

<u>Note:</u> The answers in the textbook are equivalent when converted to principal values. The proof is left as an exercise.

17b $z^{2n+1} - 1 = 0$

So, by the factor theorem and part a:

$z^{2n+1} - 1$

$= \prod_{k=-n}^{n} \left(z - e^{i\theta}\right),\ \theta = \dfrac{2k\pi}{2n+1}$

$= (z-1)\prod_{k=1}^{n} \left(z - e^{i\theta}\right)\left(z - e^{-i\theta}\right)$

$= (z-1)\prod_{k=1}^{n} \left(z^2 - \left(e^{i\theta} + e^{-i\theta}\right)z + 1\right)$

$= (z-1)\prod_{k=1}^{n} \left(z^2 - 2\left(\cos\dfrac{2k\pi}{2n+1}\right)z + 1\right)$

But $z^{2n+1} - 1 = (z-1)(z^{2n} + z^{2n-1} + \cdots + z + 1)$ (By G.P. theory.)

Hence,

$z^{2n} + z^{2n-1} + \cdots + z + 1 = \prod_{k=1}^{n} \left(z^2 - 2\left(\cos\dfrac{2k\pi}{2n+1}\right)z + 1\right)$

17c When $z = 1$,

LHS

$= 1^{2n} + 1^{2n-1} + \cdots + 1^2 + 1^1 + 1$

$= 2n + 1$

RHS

$$= \prod_{k=1}^{n} \left(2 - 2\left(\cos \frac{2k\pi}{2n+1}\right) z\right)$$

$$= \prod_{k=1}^{n} \left(2.\,2.\sin^2 \frac{k\pi}{2n+1}\right) \text{(double angle)}$$

Hence,

$2n + 1$

$$= \prod_{k=1}^{n} \left(2 \sin \frac{k\pi}{2n+1}\right)^2$$

$$= \left(\prod_{k=1}^{n} 2 \sin \frac{k\pi}{2n+1}\right)^2$$

Thus

$$\prod_{k=1}^{n} 2 \sin \frac{k\pi}{2n+1} = \sqrt{2n+1}$$

Viz:

$$2^n \sin \frac{\pi}{2n+1} \, \sin \frac{2\pi}{2n+1} \, \sin \frac{3\pi}{2n+1} \ldots \sin \frac{n\pi}{2n+1} = \sqrt{2n+1}$$

Solutions to Exercise 3F Chapter review

1a $(\cos\theta + i\sin\theta)^3(\cos 2\theta + i\sin 2\theta)^2$

$= (\text{cis }\theta)^3(\text{cis }2\theta)^2$

$= \text{cis }3\theta\ \text{cis }4\theta$

$= \text{cis }7\theta$

1b

$$\frac{(\cos\theta + i\sin\theta)^4}{(\cos\theta - i\sin\theta)^2}$$

$$= \frac{(\text{cis }\theta)^4}{(\text{cis}(-\theta))^2}$$

$$= \frac{\text{cis }4\theta}{\text{cis}(-2\theta)}$$

$= \text{cis }6\theta$

2

$$\frac{\left(e^{-\frac{i\pi}{7}}\right)^3}{\left(e^{i\frac{\pi}{7}}\right)^4}$$

$$= \frac{e^{-i\frac{3\pi}{7}}}{e^{i\frac{4\pi}{7}}}$$

$$= e^{-i\frac{7\pi}{7}}$$

$= e^{-i\pi}$

$= -1$

3a $1 - i$

$$= \sqrt{1+1}\text{cis}\left(\tan^{-1} -\frac{1}{1}\right)$$

$$= \sqrt{2}\ \text{cis}\left(-\frac{\pi}{4}\right)$$

3b $(1-i)^{13}$

$= \left(\sqrt{2}\operatorname{cis}\left(-\frac{\pi}{4}\right)\right)^{13}$

$= \left(\sqrt{2}\right)^{13}\operatorname{cis}\left(-\frac{\pi}{4}\times 13\right)$

$= 2^6\sqrt{2}\operatorname{cis}\left(-\frac{13\pi}{4}\right)$

$= 2^6\sqrt{2}\operatorname{cis}\left(-\frac{13\pi}{4}+4\pi\right)$

$= 2^6\sqrt{2}\operatorname{cis}\left(\frac{3\pi}{4}\right)$

$= 2^6\sqrt{2}\left(-\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)$

$= -64+64i$

4a $\left(\sqrt{3}+i\right)^{12}+\left(\sqrt{3}-i\right)^{12}$

$= \left(2e^{\frac{\pi}{6}}\right)^{12}+\left(2e^{-\frac{\pi}{6}}\right)^{12}$

$= 2^{12}e^{2\pi}+2^{12}e^{-2\pi}$

$= 2^{12}+2^{12}$

$= 2\times 2^{12}$

$= 2^{13}$

4b i $\left(\sqrt{3}+i\right)^n+\left(\sqrt{3}-i\right)^n$

$=\left(2e^{\frac{\pi}{6}}\right)^n+\left(2e^{-\frac{\pi}{6}}\right)^n$

$=2^{12}e^{\frac{n\pi}{6}}+2^{12}e^{-\frac{n\pi}{6}}$

$=2^{12}e^{\frac{n\pi}{6}}+2^{12}e^{-\frac{n\pi}{6}}$

$=2^{12}(e^{\frac{n\pi}{6}}+e^{-\frac{n\pi}{6}})$

$=2^{12}(\cos\frac{n\pi}{6}+i\sin\frac{n\pi}{6}+\cos\frac{n\pi}{6}-i\sin\frac{n\pi}{6})$

$=2^{13}\cos\frac{n\pi}{6}$

which is real

4b ii $\left(\sqrt{3}+i\right)^n+\left(\sqrt{3}-i\right)^n$

is rational when $2^{13}\cos\frac{n\pi}{6}$ is rational and hence when $\cos\frac{n\pi}{6}$ is rational. This is when n is even or a multiple of 3.

5a $\cos 6\theta+i\sin 6\theta$

$=\text{cis } 6\theta$

$=(\text{cis }\theta)^6$

$=(\cos\theta+i\sin\theta)^6$

$=\cos^6\theta+6i\cos^5\theta\sin\theta+15i^2\cos^4\theta\sin^2\theta+20i^3\cos^3\theta\sin^3\theta$
$\quad+15i^4\cos^2\theta\sin^4\theta+6i^5\cos\theta\sin^5\theta+i^6\sin^6\theta$

$=\cos^6\theta+6i\cos^5\theta\sin\theta-15\cos^4\theta\sin^2\theta-20i\cos^3\theta\sin^3\theta$
$\quad+15\cos^2\theta\sin^4\theta+6i\cos\theta\sin^5\theta-\sin^6\theta$

$=(\cos^6\theta-15\cos^4\theta\sin^2\theta+15\cos^2\theta\sin^4\theta-\sin^6\theta)$
$\quad+i(6\cos^5\theta\sin\theta-20\cos^3\theta\sin^3\theta+6\cos\theta\sin^5\theta)$

Equating the real components of the above equation gives

$\cos 6\theta=\cos^6\theta-15\cos^4\theta\sin^2\theta+15\cos^2\theta\sin^4\theta-\sin^6\theta$

Equating the imaginary components of the above equation gives

$\sin 6\theta=6\cos^5\theta\sin\theta-20\cos^3\theta\sin^3\theta+6\cos\theta\sin^5\theta$

5b $\tan 6\theta$

$$= \frac{\sin 6\theta}{\cos 6\theta}$$

$$= \frac{6\cos^5\theta\sin\theta - 20\cos^3\theta\sin^3\theta + 6\cos\theta\sin^5\theta}{\cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta}$$

$$= \frac{6\cos^5\theta\sin\theta - 20\cos^3\theta\sin^3\theta + 6\cos\theta\sin^5\theta}{\cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta} \div \frac{\cos^6\theta}{\cos^6\theta}$$

$$= \frac{6\tan\theta - 20\tan^3\theta + 6\tan^5\theta}{1 - 15\tan^2\theta + 15\tan^4\theta - \tan^6\theta}$$

$$= \frac{6t - 20t^3 + 6t^5}{1 - 15t^2 + 15t^4 - t^6}$$

$$= \frac{2t(3 - 10t^2 + 3t^4)}{1 - 15t^2 + 15t^4 - t^6}$$

6a

$$\left(z + \frac{1}{z}\right)^4$$

$$= z^4 + 4z^3\left(\frac{1}{z}\right) + 6z^2\left(\frac{1}{z^2}\right) + 4z\left(\frac{1}{z^3}\right) + \left(\frac{1}{z^4}\right)$$

$$= z^4 + 4z^2 + 6 + 4z^{-2} + z^{-4}$$

$$\left(z - \frac{1}{z}\right)^4$$

$$= z^4 - 4z^3\left(\frac{1}{z}\right) + 6z^2\left(\frac{1}{z^2}\right) - 4z\left(\frac{1}{z^3}\right) + \left(\frac{1}{z^4}\right)$$

$$= z^4 - 4z^2 + 6 - 4z^{-2} + z^{-4}$$

6b Adding the above two results from part a,

$$\left(z+\frac{1}{z}\right)^4+\left(z-\frac{1}{z}\right)^4=2(z^4+6+z^{-4})$$

$$\left(z+\frac{1}{z}\right)^4+\left(z-\frac{1}{z}\right)^4=2(z^4+z^{-4}+6)$$

$$(2\cos\theta)^4+(2i\sin\theta)^4=2(2\cos 4\theta+6)$$

$$16\cos^4\theta+16\sin^4\theta=4(\cos 4\theta+3)$$

$$\cos^4\theta+\sin^4\theta=\frac{1}{4}(\cos 4\theta+3)$$

7a If ω is a cube root of -1 it follows that

$\omega^3=-1$

Now

$(-\omega^2)^3$

$=-\omega^6$

$=-(\omega^3)^2$

$=-(-1)^2$

$=-1$

Hence $-\omega^2$ is a cube root of -1.

7b $(6\omega+1)(6\omega^2-1)$

$=36\omega^3+6(\omega^2-\omega)-1$

$=36(-1)-6(\omega-\omega^2)-1$

$=-36-6(\omega-\omega^2-1)-6-1$

Since $\omega-\omega^2-1$ is the sum of the cube roots of -1, and because there is no coefficient of ω^2 in $\omega^3-1=0$, it follows that $\omega-\omega^2-1=0$. Hence,

$(6\omega+1)(6\omega^2-1)$

$=-36-6(\omega-\omega^2-1)-6-1$

$=-36-6(0)-6-1$

$=-43$

8 $z^3 - 8i = 0$

$z^3 = 8i$

Let $z = re^{i\theta}$, then $z^3 = r^3e^{3i\theta}$. Taking the modulus of both sides we have $r^3 = 8$, and so we must have $r = 2$ and $e^{i3\theta} = i$. Hence, we must have $3\theta = \frac{\pi}{2} + 2n\pi$, where n is an integer, and so $\theta = \frac{(1+4n)\pi}{6}$. Hence, taking $n = -1, 0, 1$ we see that the roots are,

$z = 2e^{-\frac{i\pi}{2}}, 2e^{\frac{i\pi}{6}}, 2e^{\frac{i5\pi}{6}}$

9a $2 + 2i$

$= \sqrt{2^2 + 2^2}\ \text{cis}\left(\frac{\pi}{4}\right)$

$= 2\sqrt{2}\ \text{cis}\left(\frac{\pi}{4}\right)$

Let $z = r\ \text{cis}(\theta)$ be a cube root of $2\sqrt{2}\ \text{cis}\left(\frac{\pi}{4}\right)$. It follows that

$z^3 = 2\sqrt{2}\ \text{cis}\left(\frac{\pi}{4}\right)$

$r^3\ \text{cis}(3\theta) = 2\sqrt{2}\ \text{cis}\left(\frac{\pi}{4}\right)$

Taking the modulus, we see that $r^3 = 2\sqrt{2}$ and hence $r = \sqrt{2}$. Then we must also have that $3\theta = 2n\pi + \frac{\pi}{4} = \frac{(8n+1)\pi}{4}$, where n is an integer. Thus,

$\theta = \dfrac{(8n+1)\pi}{12}$ where n is an integer

and so

$z = \sqrt{2}\ \text{cis}\left(\dfrac{(8n+1)\pi}{12}\right)$

Taking $n = -1, 0, 1$ we see that the 3 roots are,

$z = \sqrt{2}\ \text{cis}\left(\dfrac{k\pi}{12}\right)$ for $k = -7, 1, 9$

9b Let $z = r \operatorname{cis} \theta$ be a sixth root of i, it follows that

$$z^6 = r^6 \operatorname{cis} 6\theta = i = \operatorname{cis}\frac{\pi}{2}$$

Hence comparing modulus, we see that $r = 1$, and by comparing argument we see that, $6\theta = 2n\pi + \frac{\pi}{2}$ where n is an integer. Thus,

$$\theta = \frac{(4n+1)\pi}{12}$$

and so, the sixth roots have the form

$$z = \operatorname{cis}\left(\frac{(4n+1)\pi}{12}\right) \text{ where } n \text{ is an integer}$$

Taking $n = -3,\ -2,\ -1, 0, 1, 2$, we have the six roots as,

$$z = \operatorname{cis}\left(\frac{k\pi}{12}\right) \text{ for } k = -11, -7, -3, 1, 5, 9$$

10a z

$$= 4\sqrt{3}e^{\frac{i\pi}{3}} - 4e^{\frac{5i\pi}{6}}$$

$$= 4\sqrt{3}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) - 4\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$$

$$= 4\sqrt{3}\left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right) - 4\left(-\frac{\sqrt{3}}{2} + \frac{i}{2}\right)$$

$$= 4\sqrt{3} + 4i$$

$$= 8\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)$$

$$= 8e^{\frac{i\pi}{6}}$$

10b Using part a we have

$$\frac{z}{8} + i\left(\frac{z}{8}\right)^2 + \left(\frac{z}{8}\right)^3$$

$$= \frac{8e^{\frac{i\pi}{6}}}{8} + i\left(\frac{8e^{\frac{i\pi}{6}}}{8}\right)^2 + \left(\frac{8e^{\frac{i\pi}{6}}}{8}\right)^3$$

$$= e^{\frac{i\pi}{6}} + i\left(e^{\frac{i\pi}{6}}\right)^2 + \left(e^{\frac{i\pi}{6}}\right)^3$$

$$= e^{\frac{i\pi}{6}} + ie^{\frac{i\pi}{3}} + e^{\frac{i\pi}{2}}$$

$$= \cos\frac{\pi}{6} + i\sin\frac{\pi}{6} + i\cos\frac{\pi}{3} + i^2\sin\frac{\pi}{3} + \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$$

$$= \frac{\sqrt{3}}{2} + \frac{i}{2} + \frac{i}{2} - \frac{\sqrt{3}}{2}i + 0 + i$$

$$= 2i$$

10c Let $\lambda = re^{i\theta}$ be a cube root of z. It follows that

$$\lambda^3 = r^3e^{3i\theta} = 8e^{\frac{i\pi}{6}}$$

Comparing modulus, we see that $r^3 = 8$ and so $r = 2$. Then comparing argument, we see that $3\theta = 2n\pi + \frac{\pi}{6} = \frac{(12n+1)\pi}{6}$, where n is an integer. Hence,

$$\theta = \frac{(12n+1)\pi}{18}$$

Thus, the roots are of the form

$$\lambda = 2e^{\frac{(12n+1)\pi}{18}} \quad \text{where } n \text{ is an integer}$$

Taking $n = -1, 0, 1$, we see that the three cube roots of z are,

$$\lambda = 2e^{-\frac{11i\pi}{18}}, 2e^{\frac{i\pi}{18}}, 2e^{\frac{13i\pi}{18}}$$

11a $(z - z^{-1})^7$

$= z^7 - 7(z^6)(z^{-1}) + 21(z^5)(z^{-2}) - 35(z^4)(z^{-3}) + 35(z^3)(z^{-4}) - 21(z^2)(z^{-5})$
$+7(z)(z^{-6}) - z^{-7}$

$= z^7 - 7z^5 + 21z^3 - 35z + 35z^{-1} - 21z^{-3} + 7z^{-5} - z^{-7}$

$= (z^7 - z^{-7}) - 7(z^5 - z^{-5}) + 21(z^3 - z^{-3}) - 35(z - z^{-1})$

11b $z = \cos\theta + i\sin\theta = \operatorname{cis}\theta$

$z - z^{-1} = \operatorname{cis}\theta - \operatorname{cis}(-\theta)$

$= \cos\theta + i\sin\theta - (\cos\theta - i\sin\theta)$

$= 2i\sin\theta$

$z^n = \cos n\theta + i\sin n\theta = \operatorname{cis} n\theta$

$z^n - z^{-n} = \operatorname{cis} n\theta - \operatorname{cis}(-n\theta)$

$= \cos n\theta + i\sin n\theta - (\cos n\theta - i\sin n\theta)$

$= 2i\sin n\theta$

11c $\sin^7\theta$

$= i^8\sin^7\theta$

$= \left(\dfrac{z - z^{-1}}{2i}\right)^7$

$= \dfrac{i}{128}(z - z^{-1})^7$

$= \dfrac{i}{128}\big((z^7 - z^{-7}) - 7(z^5 - z^{-5}) + 21(z^3 - z^{-3}) - 35(z - z^{-1})\big)$

$= \dfrac{i}{128}\big(2i\sin 7\theta - 7(2i\sin 5\theta) + 21(2i\sin 3\theta) - 35(2i\sin\theta)\big)$

$= -\dfrac{1}{128}\big(2\sin 7\theta - 7(2\sin 5\theta) + 21(2\sin 3\theta) - 35(2\sin\theta)\big)$

$= \dfrac{1}{64}(35\sin\theta - 21\sin 3\theta + 7\sin 5\theta - \sin 7\theta)$

11d Using part c

$$\int (35\sin\theta - 64\sin^7\theta)\,d\theta$$

$$= \int \left(35\sin\theta - 64\left(\frac{1}{64}(35\sin\theta - 21\sin 3\theta + 7\sin 5\theta - \sin 7\theta)\right)\right)d\theta$$

$$= \int \big(35\sin\theta - (35\sin\theta - 21\sin 3\theta + 7\sin 5\theta - \sin 7\theta)\big)\ d\theta$$

$$= \int (21\sin 3\theta - 7\sin 5\theta + \sin 7\theta)\,d\theta$$

$$= -7\cos 3\theta + \frac{7}{5}\cos 5\theta - \frac{1}{7}\cos 7\theta + C$$

12a $\cos 5\theta + i\sin 5\theta$

$= \text{cis}\ 5\theta$

$= (\text{cis}\ \theta)^5$

$= (\cos\theta + i\sin\theta)^5$

$= \cos^5\theta + 5i\cos^4\theta\sin\theta + 10i^2\cos^3\theta\sin^2\theta + 10i^3\cos^2\theta\sin^3\theta$
$\qquad + 5i^4\cos\theta\sin^4\theta + i^5\sin^5\theta$

$= \cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - 10i\cos^2\theta\sin^3\theta$
$\qquad + 5\cos\theta\sin^4\theta + i\sin^5\theta$

Equating real components in the above equation gives

$\cos 5\theta$

$= \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$

$= \cos^5\theta - 10\cos^3\theta\,(1-\cos^2\theta) + 5\cos\theta\,(1-\cos^2\theta)^2$

$= \cos^5\theta - 10\cos^3\theta\,(1-\cos^2\theta) + 5\cos\theta\,(1 - 2\cos^2\theta + \cos^4\theta)$

$= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

12b $16x^4 - 20x^2 + 5 = 0$

Let $x = \cos\theta$, then we have

$16\cos^4\theta - 20\cos^2\theta + 5 = 0$

$\cos\theta\,(16\cos^4\theta - 20\cos^2\theta + 5) = 0 \times \cos\theta$

$16\cos^5\theta - 20\cos^3\theta + 5\cos\theta = 0$

$\cos 5\theta = 0$ (using part a)

Thus, we must have

$5\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}$

$\theta = \frac{\pi}{10}, \frac{3\pi}{10}, \frac{\pi}{2}, \frac{7\pi}{10}, \frac{9\pi}{10}$

We omit $\frac{\pi}{2}$ as that solution was introduced when we multiplied the equation by $\cos\theta$.

So the solutions are $\theta = \frac{\pi}{10}, \frac{3\pi}{10}, \frac{7\pi}{10}, \frac{9\pi}{10}$

Hence the solutions are $x = \cos\frac{\pi}{10}, \cos\frac{3\pi}{10}, \cos\frac{7\pi}{10}, \cos\frac{9\pi}{10}$

12c The product of the roots is

$\cos\frac{\pi}{10}\cos\frac{3\pi}{10}\cos\frac{7\pi}{10}\cos\frac{9\pi}{10} = \frac{5}{16}$

$\cos\frac{\pi}{10}\cos\frac{3\pi}{10}\left(-\cos\left(\pi - \frac{7\pi}{10}\right)\right)\left(-\cos\left(\pi - \frac{9\pi}{10}\right)\right) = \frac{5}{16}$

$\cos\frac{\pi}{10}\cos\frac{3\pi}{10}\cos\frac{3\pi}{10}\cos\frac{\pi}{10} = \frac{5}{16}$

$\left(\cos\frac{\pi}{10}\cos\frac{3\pi}{10}\right)^2 = \frac{5}{16}$

$\cos\frac{\pi}{10}\cos\frac{3\pi}{10} = \pm\frac{\sqrt{5}}{4}$

But $\cos\frac{\pi}{10}, \cos\frac{3\pi}{10} > 0$ and so

$\cos\frac{\pi}{10}\cos\frac{3\pi}{10} = \frac{\sqrt{5}}{4}$

12d Let $u = 2x^2 - 1$

$4u^2 - 2u - 1$

$= 4(2x^2 - 1)^2 - 2(2x^2 - 1) - 1$

$= 4(4x^4 - 4x^2 + 1) - (4x^2 - 2) - 1$

$= 16x^4 - 20x^2 + 5$

$= 0$

12e $x = \cos\frac{\pi}{10}$ is a solution to $16x^4 - 20x^2 + 5 = 0$, and using the double angle identity we have,

$\cos\dfrac{\pi}{5}$

$= 2\cos^2\dfrac{\pi}{10} - 1$

$= 2x^2 - 1$

From part d, $u = 2x^2 - 1$ is a solution to the equation $4u^2 - 2u - 1 = 0$. So, letting $u = \cos\frac{\pi}{5}$, the solutions to the equation become,

$\cos\dfrac{\pi}{5}$

$= \dfrac{-(-2) \pm \sqrt{(-2)^2 - 4(4)(-1)}}{2(4)}$

$= \dfrac{2 \pm \sqrt{4 + 16}}{8}$

$= \dfrac{2 \pm 2\sqrt{5}}{8}$

$= \dfrac{1 \pm \sqrt{5}}{4}$

But since $\cos\dfrac{\pi}{5} > 0$, we have $\cos\dfrac{\pi}{5} = \dfrac{1 + \sqrt{5}}{4}$

13a Begin by noting that

$e^{i\theta} = \cos\theta + i\sin\theta \qquad (1)$

and that

$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta)$

Hence

$e^{-i\theta} = \cos\theta - i\sin\theta \qquad (2)$

$(1) + (2)$:

$e^{i\theta} + e^{-i\theta}$

$= \cos\theta + i\sin\theta + \cos\theta - i\sin\theta$

$= 2\cos\theta$

Hence

$\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$

$(1) - (2)$:

$e^{i\theta} - e^{-i\theta}$

$= \cos\theta + i\sin\theta - (\cos\theta - i\sin\theta)$

$= 2i\sin\theta$

Hence

$\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$

13b i $2\cos^2\theta$

$$= 2\left(\frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)\right)^2$$

$$= 2\left(\frac{1}{4}\left(e^{2i\theta} + 2e^0 + e^{-2i\theta}\right)\right)$$

$$= \frac{1}{2}\left(e^{2i\theta} + 2 + e^{-2i\theta}\right)$$

$$= \frac{1}{2}(\cos 2\theta + i\sin 2\theta + 2 + \cos 2\theta - i\sin 2\theta)$$

$$= \frac{1}{2}(2 + 2\cos 2\theta)$$

$$= 1 + \cos 2\theta$$

13b ii $2\sin^2\theta$

$$= 2\left(\frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)\right)^2$$

$$= 2\left(\frac{1}{-4}\left(e^{2i\theta} - 2e^0 + e^{-2i\theta}\right)\right)$$

$$= -\frac{1}{2}\left(e^{2i\theta} - 2 + e^{-2i\theta}\right)$$

$$= -\frac{1}{2}(\cos 2\theta + i\sin 2\theta - 2 + \cos 2\theta - i\sin 2\theta)$$

$$= -\frac{1}{2}(-2 + 2\cos 2\theta)$$

$$= 1 - \cos 2\theta$$

13b iii $\cos(\alpha - \beta)$

$$= \frac{1}{2}\left(e^{i(\alpha-\beta)} + e^{-i(\alpha-\beta)}\right)$$

$$= \frac{1}{4}\left(2e^{i(\alpha-\beta)} + 2e^{i(-\alpha+\beta)}\right)$$

$$= \frac{1}{4}\left(e^{i(\alpha+\beta)} + e^{i(\alpha-\beta)} + e^{i(-\alpha+\beta)} + e^{i(-\alpha-\beta)} - e^{i(\alpha+\beta)} + e^{i(\alpha-\beta)} + e^{i(-\alpha+\beta)} - e^{i(-\alpha-\beta)}\right)$$

$$= \frac{1}{4}\left(e^{i(\alpha+\beta)} + e^{i(\alpha-\beta)} + e^{i(-\alpha+\beta)} + e^{i(-\alpha-\beta)}\right) - \frac{1}{4}\left(e^{i(\alpha+\beta)} - e^{i(\alpha-\beta)} - e^{i(-\alpha+\beta)} + e^{i(-\alpha-\beta)}\right)$$

$$= \frac{1}{2}\left(e^{i\alpha} + e^{-i\alpha}\right) \times \frac{1}{2}\left(e^{i\beta} + e^{-i\beta}\right) + \frac{1}{2i}\left(e^{i\alpha} - e^{-i\alpha}\right) \times \frac{1}{2i}\left(e^{i\beta} - e^{-i\beta}\right)$$

$$= \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

13b iv $\sin(\alpha - \beta)$

$$= \frac{1}{2i}\left(e^{i(\alpha-\beta)} - e^{-i(\alpha-\beta)}\right)$$

$$= \frac{1}{4i}\left(2e^{i(\alpha-\beta)} - 2e^{-i(\alpha-\beta)}\right)$$

$$= \frac{1}{4i}\left(e^{i(\alpha+\beta)} + e^{i(\alpha-\beta)} - e^{i(-\alpha+\beta)} - e^{i(-\alpha-\beta)} - e^{i(\alpha+\beta)} + e^{i(\alpha-\beta)} - e^{i(-\alpha+\beta)} + e^{i(-\alpha-\beta)}\right)$$

$$= \frac{1}{4i}\left(\left(e^{i\alpha} - e^{-i\alpha}\right)\left(e^{i\beta} + e^{-i\beta}\right) - \left(e^{i\alpha} + e^{-i\alpha}\right)\left(e^{i\beta} - e^{-i\beta}\right)\right)$$

$$= \left(\frac{1}{2i}\left(e^{i\alpha} - e^{-i\alpha}\right)\right)\left(\frac{1}{2}\left(e^{i\beta} + e^{-i\beta}\right)\right) - \left(\frac{1}{2i}\left(e^{i\alpha} + e^{-i\alpha}\right)\right)\left(\frac{1}{2}\left(e^{i\beta} - e^{-i\beta}\right)\right)$$

$$= \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

14a Let $z = r\text{cis}\,\theta$ be a seventh root of -1, then

$z^7 = r^7(\text{cis}\,\theta)^7 = r^7\text{cis}\,7\theta = -1$

It follows by comparing modulus, then that $r = 1$ and so $\text{cis}\,7\theta = -1$

Hence $\cos 7\theta = -1$ and so $7\theta = \pm\pi, \pm 3\pi, \pm 5\pi, 7\pi$

Thus $\theta = \pm\frac{\pi}{7}, \pm\frac{3\pi}{7}, \pm\frac{5\pi}{7}, \pi$ and so the roots are

$$z = \text{cis}\left(\pm\frac{\pi}{7}\right), \text{cis}\left(\pm\frac{3\pi}{7}\right), \text{cis}\left(\pm\frac{5\pi}{7}\right), \text{cis}(\pi)$$

This is,

$$z = \text{cis}\left(\pm\frac{\pi}{7}\right), \text{cis}\left(\pm\frac{3\pi}{7}\right), \text{cis}\left(\pm\frac{5\pi}{7}\right), -1$$

14b i The roots of the equation $z^7 = -1$ are the same as the roots of the equation $z^7 + 1 = 0$. Since there is no coefficient of z^6, it follows that the sum of the roots of the equation is equal to zero. Hence

$$\text{cis}\left(\frac{\pi}{7}\right) + \text{cis}\left(-\frac{\pi}{7}\right) + \text{cis}\left(\frac{3\pi}{7}\right) + \text{cis}\left(-\frac{3\pi}{7}\right) + \text{cis}\left(\frac{5\pi}{7}\right) + \text{cis}\left(-\frac{5\pi}{7}\right) + (-1) = 0$$

$$\left(\cos\frac{\pi}{7} + i\sin\frac{\pi}{7}\right) + \left(\cos\frac{\pi}{7} - i\sin\frac{\pi}{7}\right) + \left(\cos\frac{3\pi}{7} + i\sin\frac{3\pi}{7}\right) + \left(\cos\frac{3\pi}{7} - i\sin\frac{3\pi}{7}\right)$$
$$+ \left(\cos\frac{5\pi}{7} + i\sin\frac{5\pi}{7}\right) + \left(\cos\frac{5\pi}{7} - i\sin\frac{5\pi}{7}\right) + (-1) = 0$$

$$2\cos\frac{\pi}{7} + 2\cos\frac{3\pi}{7} + 2\cos\frac{5\pi}{7} - 1 = 0$$

$$2\cos\frac{\pi}{7} + 2\cos\frac{3\pi}{7} + 2\cos\frac{5\pi}{7} = 1$$

$$\cos\frac{\pi}{7} + \cos\frac{3\pi}{7} + \cos\frac{5\pi}{7} = \frac{1}{2}$$

14b ii Writing $z^7 + 1$ as a product of factors (using part a) gives

$$z^7 + 1$$

$$= (z - (-1))\left(z - \operatorname{cis}\left(\frac{\pi}{7}\right)\right)\left(z - \operatorname{cis}\left(-\frac{\pi}{7}\right)\right)\left(z - \operatorname{cis}\left(\frac{3\pi}{7}\right)\right)$$

$$\left(z - \operatorname{cis}\left(-\frac{3\pi}{7}\right)\right)\left(z - \operatorname{cis}\left(\frac{5\pi}{7}\right)\right)\left(z - \operatorname{cis}\left(-\frac{5\pi}{7}\right)\right)$$

$$= (z + 1)\left(z^2 - z\left(\operatorname{cis}\left(\frac{\pi}{7}\right) + \operatorname{cis}\left(-\frac{\pi}{7}\right)\right) + \operatorname{cis}\left(\frac{\pi}{7}\right)\operatorname{cis}\left(-\frac{\pi}{7}\right)\right)$$

$$\left(z^2 - z\left(\operatorname{cis}\left(\frac{3\pi}{7}\right) + \operatorname{cis}\left(-\frac{3\pi}{7}\right)\right) + \operatorname{cis}\left(\frac{3\pi}{7}\right)\operatorname{cis}\left(-\frac{3\pi}{7}\right)\right)$$

$$\left(z^2 - z\left(\operatorname{cis}\left(\frac{5\pi}{7}\right) + \operatorname{cis}\left(-\frac{5\pi}{7}\right)\right) + \operatorname{cis}\left(\frac{5\pi}{7}\right)\operatorname{cis}\left(-\frac{5\pi}{7}\right)\right)$$

$$= (z + 1)\left(z^2 - 2z\cos\frac{\pi}{7} + \operatorname{cis}(0)\right)$$

$$\left(z^2 - 2z\cos\frac{3\pi}{7} + \operatorname{cis}(0)\right)\left(z^2 - 2z\cos\frac{5\pi}{7} + \operatorname{cis}(0)\right)$$

$$= (z + 1)\left(z^2 - 2z\cos\frac{\pi}{7} + 1\right)\left(z^2 - 2z\cos\frac{3\pi}{7} + 1\right)\left(z^2 - 2z\cos\frac{5\pi}{7} + 1\right)$$

14b iii $(z + 1)(z^6 - 7^5 + z^4 - z^3 + z^2 - z + 1)$

$$= z^7 - 7^6 + z^5 - z^4 + z^3 - z^2 + z + z^6 - 7^5 + z^4 - z^3 + z^2 - z + 1$$

$$= z^7 + 1$$

Hence

$$(z + 1)(z^6 - 7^5 + z^4 - z^3 + z^2 - z + 1)$$

$$= z^7 + 1$$

$$= (z + 1)\left(z^2 - 2z\cos\frac{\pi}{7} + 1\right)\left(z^2 - 2z\cos\frac{3\pi}{7} + 1\right)\left(z^2 - 2z\cos\frac{5\pi}{7} + 1\right)$$

Thus

$$z^6 - z^5 + z^4 - z^3 + z^2 - z + 1$$

$$= \left(z^2 - 2z\cos\frac{\pi}{7} + 1\right)\left(z^2 - 2z\cos\frac{3\pi}{7} + 1\right)\left(z^2 - 2z\cos\frac{5\pi}{7} + 1\right)$$

14c Dividing both sides of the identity in part b iii by z^3 gives

$z^3 - z^2 + z - 1 + z^{-1} - z^{-2} + z^{-3}$

$= \dfrac{\left(z^2 - 2z\cos\frac{\pi}{7} + 1\right)}{z}\dfrac{\left(z^2 - 2z\cos\frac{3\pi}{7} + 1\right)}{z}\dfrac{\left(z^2 - 2z\cos\frac{5\pi}{7} + 1\right)}{z}$

$(z^3 + z^{-3}) - (z^2 + z^{-2}) + (z + z^{-1}) - 1$

$= (z - 2\cos\frac{\pi}{7} + z^{-1})(z - 2\cos\frac{3\pi}{7} + z^{-1})(z - 2\cos\frac{5\pi}{7} + z^{-1})$

$= \left((z + z^{-1}) - 2\cos\frac{\pi}{7}\right)\left((z + z^{-1}) - 2\cos\frac{3\pi}{7}\right)\left((z + z^{-1}) - 2\cos\frac{5\pi}{7}\right)$

We have already seen that $z^n + z^{-n} = 2\cos n\theta$ and so we have

$2\cos 3\theta - 2\cos 2\theta + 2\cos\theta - 1$

$= \left(2\cos\theta - 2\cos\frac{\pi}{7}\right)\left(2\cos\theta - 2\cos\frac{3\pi}{7}\right)\left(2\cos\theta - 2\cos\frac{5\pi}{7}\right)$

$= 8\left(\cos\theta - \cos\frac{\pi}{7}\right)\left(\cos\theta - \cos\frac{3\pi}{7}\right)\left(\cos\theta - \cos\frac{5\pi}{7}\right)$

15a Let $z = re^{i\theta}$ be a fifth root of unity, so that

$z^5 = r^5e^{5i\theta} = 1 = e^{i2\pi n}$, where n is an integer

Comparing modulus, we see that $r = 1$ and comparing argument we see that we must have $5\theta = 2\pi n$ where n is an integer. Hence,

$\theta = \dfrac{2\pi n}{5}$

Taking $n = -2, -1, 0, 1, 2$ we find that the 5 roots of unity are,

$z = e^0, e^{\pm\frac{i2\pi}{5}}, e^{\pm\frac{i4\pi}{5}}$

15b i $u + v$

$= \alpha + \alpha^4 + \alpha^2 + \alpha^3$

$= \alpha + \alpha^2 + \alpha^3 + \alpha^4$

Now we can factorise the equation $z^5 - 1$ as

$z^5 - 1$

$= (z - 1)(z^4 + z^3 + z^2 + z + 1)$

Then since α is a root of the equation $z^5 - 1$, it follows that

$(\alpha - 1)(\alpha^4 + \alpha^3 + \alpha^2 + \alpha + 1) = 0$

and since $\alpha = e^{\frac{i2\pi}{5}} \neq 1$ it must be the case that,

$\alpha^4 + \alpha^3 + \alpha^2 + \alpha + 1 = 0$

or

$u + v = \alpha^4 + \alpha^3 + \alpha^2 + \alpha = -1$

Now squaring $u - v$ gives

$(u - v)^2$

$= u^2 - 2uv + v^2$

$= (\alpha + \alpha^4)^2 - 2(\alpha + \alpha^4)(\alpha^2 + \alpha^3) + (\alpha^2 + \alpha^3)^2$

$= (\alpha^2 + 2\alpha^5 + \alpha^8) - 2(\alpha^3 + \alpha^4 + \alpha^6 + \alpha^7) + (\alpha^4 + 2\alpha^5 + \alpha^6)$

$= (\alpha^2 + 2 + \alpha^3) - 2(\alpha^3 + \alpha^4 + \alpha + \alpha^2) + (\alpha^4 + 2 + \alpha)$ (Noting that $\alpha^5 = 1$)

$= 4 - (\alpha^3 + \alpha^4 + \alpha + \alpha^2)$

$= 4 - (-1)$ (from the working above $\alpha^4 + \alpha^3 + \alpha^2 + \alpha = -1$)

$= 5$

Hence, taking the square root we must have,

$u - v = \pm\sqrt{5}$

Now,

$\alpha^4 = \text{cis}\left(\frac{8\pi}{5}\right) = \text{cis}\left(2\pi - \frac{8\pi}{5}\right) = \text{cis}\left(-\frac{2\pi}{5}\right) = \alpha^{-1}$

and

$$\alpha^3 = \text{cis}\left(\frac{6\pi}{5}\right) = \text{cis}\left(2\pi - \frac{6\pi}{5}\right) = \text{cis}\left(-\frac{4\pi}{5}\right) = \alpha^{-2}$$

Hence, using the fact that $z^n + z^{-n} = 2\cos n\theta$ we have that,

$$u = \alpha + \alpha^4 = \alpha + \alpha^{-1} = 2\cos\frac{2\pi}{5}$$

and

$$v = \alpha^2 + \alpha^3 = \alpha^2 + \alpha^{-2} = 2\cos\frac{4\pi}{5}$$

Thus, we see that $u = 2\cos\frac{2\pi}{5} > 0$ and $v = 2\cos\frac{4\pi}{5} < 0$. So we see that both $u = 2\cos\frac{2\pi}{5} > 0$ and $-v = -2\cos\frac{4\pi}{5} > 0$, and as such conclude that $u - v > 0$. Thus, we can omit the negative sign and have,

$$u - v = \sqrt{5}$$

15b ii Using part i we have,

$$2u = u + v + u - v = -1 + \sqrt{5}$$

Thus,

$$u = \frac{-1 + \sqrt{5}}{2}$$

Now, we also have from the working in part i that, $u = 2\cos\frac{2\pi}{5}$. Thus, we see that

$$u = 2\cos\frac{2\pi}{5} = \frac{-1 + \sqrt{5}}{2}$$

or that

$$\cos\frac{2\pi}{5} = \frac{-1 + \sqrt{5}}{4}$$

16a $z^n + z^{-n}$

$= (\text{cis}\,\theta)^n + (\text{cis}\,\theta)^{-n}$

$= \text{cis}\,n\theta + \text{cis}(-n\theta)$

$= \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$

$= 2\cos n\theta$

16b $\sin(A+B)-\sin(A-B)$

$= \sin A\cos B+\cos A\sin B-(\sin A\cos B-\cos A\sin B)$

$= 2\cos A\sin B$

16c $(z^{2n}+z^{2n-2}+z^{2n-4}+\cdots+z^{-2n})\sin\theta$

$= \big((z^{2n}+z^{-2n})+(z^{2n-2}+z^{-2n+2})+\cdots+(z^2+z^{-2})+z^0\big)\sin\theta$

$= (2\cos 2n\theta+2\cos(2n-2)\theta+\cdots+\cos 0)\sin\theta$

$= 2\cos 2n\theta\sin\theta+2\cos(2n-2)\theta\sin\theta+\cdots+\cos 0\sin\theta$

$= (\sin(2n\theta+\theta)-\sin(2n\theta-\theta))+(\sin(2(n-1)\theta+\theta)-\sin(2(n-1)\theta-\theta))$

$+\cdots+\sin\theta$

$= (\sin(2n\theta+\theta)-\sin(2n\theta-\theta))+(\sin(2n\theta-\theta)-\sin(2n\theta-2\theta))$

$+\cdots(\sin 2\theta-\sin\theta)+\sin\theta$

$= \sin(2n\theta+\theta)$

$= \sin(2n+1)\theta$

16d Using the result in part c with $n=3$

$(z^6+z^4+z^2+z^0+z^{-2}+z^{-4}+z^{-6})\sin\theta=\sin 7\theta$

$z^6+z^4+z^2+z^0+z^{-2}+z^{-4}+z^{-6}=\dfrac{\sin 7\theta}{\sin\theta}$

$(z^6+z^{-6})+(z^4+z^{-4})+(z^2+z^{-2})+1=\dfrac{\sin 7\theta}{\sin\theta}$

$2\cos 6\theta+2\cos 4\theta+2\cos 2\theta+1=\dfrac{\sin 7\theta}{\sin\theta}$

$2(4\cos^3 2\theta-3\cos 2\theta)+2(2\cos^2 2\theta-1)+2\cos 2\theta+1=\dfrac{\sin 7\theta}{\sin\theta}$

$8\cos^3 2\theta+4\cos^2 2\theta-4\cos 2\theta-1=\dfrac{\sin 7\theta}{\sin\theta}$

17 $\sin\alpha - \sin\beta$

$= \frac{1}{2i}\left(e^{i\alpha} - e^{-i\alpha}\right) - \frac{1}{2i}\left(e^{i\beta} - e^{-i\beta}\right)$

$= \frac{1}{2i}\left(e^{i\alpha} - e^{i\beta} + e^{-i\beta} - e^{-i\alpha}\right)$

$= \frac{1}{2i}\left(e^{i\left(\frac{\alpha+\beta}{2}\right)} + e^{-i\left(\frac{\alpha+\beta}{2}\right)}\right)\left(e^{i\left(\frac{\alpha-\beta}{2}\right)} - e^{-i\left(\frac{\alpha-\beta}{2}\right)}\right)$

$= 2\left(\left[\frac{1}{2}\left(e^{i\left(\frac{\alpha+\beta}{2}\right)} + e^{-i\left(\frac{\alpha+\beta}{2}\right)}\right)\right]\left[\frac{1}{2i}\left(e^{i\left(\frac{\alpha-\beta}{2}\right)} - e^{-i\left(\frac{\alpha-\beta}{2}\right)}\right)\right]\right)$

$= 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$

18a $(1 + 2\omega + 3\omega^2 + 4\omega^3 + \cdots + n\omega^{n-1})(\omega - 1)$

$= \omega - 1 + 2\omega^2 - 2\omega + 3\omega^3 - 3\omega^2 + 4\omega^4 - 4\omega^3 + \cdots + n\omega^n - n\omega^{n-1}$

$= -1 - \omega - \omega^2 - \omega^3 - \cdots - \omega^{n-1} + n\omega^n$

$= -(1 + \omega + \omega^2 + \cdots + \omega^{n-1}) + n\omega^n$

Since ω is an nth root of unity, $\omega^n = 1$ and $1 + \omega + \omega^2 + \cdots + \omega^{n-1} = 0$. This follows from the definition and the fact that we can factorise.

$\omega^n - 1 = (\omega - 1)(1 + +\omega^2 + \cdots + \omega^{n-1}) = 0$

and since $\omega \neq 0$, we must have that $1 + +\omega^2 + \cdots + \omega^{n-1} = 0$. Using these results, we find that,

$(1 + 2\omega + 3\omega^2 + 4\omega^3 + \cdots + n\omega^{n-1})(\omega - 1)$

$= -0 + n(1)$

$= n$

18b

$$\frac{1}{z^2-1}=\frac{z^{-1}}{z-z^{-1}}$$

Let $z=\operatorname{cis}\theta$ then we have using above

$$\frac{1}{(\operatorname{cis}\theta)^2-1}=\frac{(\operatorname{cis}\theta)^{-1}}{\operatorname{cis}\theta-(\operatorname{cis}\theta)^{-1}}$$

$$\frac{1}{\operatorname{cis}2\theta-1}=\frac{\operatorname{cis}(-\theta)}{\operatorname{cis}\theta-\operatorname{cis}(-\theta)}$$

$$\frac{1}{\cos 2\theta+i\sin 2\theta-1}=\frac{\cos\theta-i\sin\theta}{\cos\theta+i\sin\theta-(\cos\theta-i\sin\theta)}$$

$$\frac{1}{\cos 2\theta+i\sin 2\theta-1}=\frac{\cos\theta-i\sin\theta}{2i\sin\theta}$$

18c Considering the above equation, let $\theta=\frac{\pi}{n}$. It follows that

$$\frac{1}{\cos\frac{2\pi}{n}+i\sin\frac{2\pi}{n}-1}=\frac{\cos\frac{\pi}{n}-i\sin\frac{\pi}{n}}{2i\sin\frac{\pi}{n}}$$

Using the definition of ω we have

$$\frac{1}{\omega-1}$$

$$=\frac{\cos\frac{\pi}{n}}{2i\sin\frac{\pi}{n}}-\frac{i\sin\frac{\pi}{n}}{2i\sin\frac{\pi}{n}}$$

$$=-i\frac{\cos\frac{\pi}{n}}{2\sin\frac{\pi}{n}}-\frac{\sin\frac{\pi}{n}}{2\sin\frac{\pi}{n}}$$

$$=-\frac{1}{2}-i\frac{\cos\frac{\pi}{n}}{2\sin\frac{\pi}{n}}$$

Hence the real part of $\frac{1}{\omega-1}$ is $-\frac{1}{2}$.

18d Now, to begin let $n = 5$, then we have that $\omega = \text{cis}(\frac{2\pi}{5})$, and we see that

$$\omega^5 = \text{cis}\left(\frac{2\pi}{5}\right)^5 = \text{cis}(2\pi) = 1$$

Hence, ω is a fifth root of unity. Then we can apply the result of part a and write.

$$(1 + 2\omega + 3\omega^2 + 4\omega^3 + 5\omega^5)(\omega - 1) = 5$$

Since, $\omega \neq 1$ we can divide the above expression by $(\omega - 1)$ and get,

$$1 + 2\omega + 3\omega^2 + 4\omega^3 + 5\omega^5 = \frac{5}{\omega - 1}$$

Subbing in ω we have,

$$\frac{5}{\omega - 1} = 1 + 2\text{cis}\left(\frac{2\pi}{5}\right) + 3\text{cis}\left(\frac{2\pi}{5}\right)^2 + 4\text{cis}\left(\frac{2\pi}{5}\right)^3 + 5\text{cis}\left(\frac{2\pi}{5}\right)^4$$

$$\frac{5}{\omega - 1} = 1 + 2\text{cis}\left(\frac{2\pi}{5}\right) + 3\text{cis}\left(\frac{4\pi}{5}\right) + 4\text{cis}\left(\frac{6\pi}{5}\right) + 5\text{cis}\left(\frac{8\pi}{5}\right)$$

Now, taking the real part of both sides of the equation, recalling that $\text{Re}(\text{cis}\theta) = \cos\theta$, we have for the RHS

$$\text{Re}\left(1 + 2\text{cis}\left(\frac{2\pi}{5}\right) + 3\text{cis}\left(\frac{4\pi}{5}\right) + 4\text{cis}\left(\frac{6\pi}{5}\right) + 5\text{cis}\left(\frac{8\pi}{5}\right)\right)$$

$$= \text{Re}(1) + \text{Re}\left(2\text{cis}\left(\frac{2\pi}{5}\right)\right) + \text{Re}\left(3\text{cis}\left(\frac{4\pi}{5}\right)\right) + \text{Re}\left(4\text{cis}\left(\frac{6\pi}{5}\right)\right) + \text{Re}\left(5\text{cis}\left(\frac{8\pi}{5}\right)\right)$$

$$= 1 + 2\text{Re}\left(\text{cis}\left(\frac{2\pi}{5}\right)\right) + 3\text{Re}\left(\text{cis}\left(\frac{4\pi}{5}\right)\right) + 4\text{Re}\left(\text{cis}\left(\frac{6\pi}{5}\right)\right) + 5\text{Re}\left(\text{cis}\left(\frac{8\pi}{5}\right)\right)$$

$$= 1 + 2\cos\frac{2\pi}{5} + 3\cos\frac{4\pi}{5} + 4\cos\frac{6\pi}{5} + 5\cos\frac{8\pi}{5}$$

Now, for the LHS using the result of part c, we have

$$\text{Re}\left(\frac{5}{\omega - 1}\right)$$

$$= 5\text{Re}\left(\frac{1}{\omega - 1}\right)$$

$$= 5\left(-\frac{1}{2}\right)$$

$$= -\frac{5}{2}$$

Hence, equating the real parts of the RHS and LHS of the equation we see that,

$$1+2\cos\frac{2\pi}{5}+3\cos\frac{4\pi}{5}+4\cos\frac{6\pi}{5}+5\cos\frac{8\pi}{5}=-\frac{5}{2}$$

18e Now using the result from part d, we have that

$$1+2\cos\frac{2\pi}{5}+3\cos\frac{4\pi}{5}+4\cos\frac{6\pi}{5}+5\cos\frac{8\pi}{5}$$

$$=1+2\cos\frac{2\pi}{5}-3\cos\left(\pi-\frac{4\pi}{5}\right)-4\cos\left(\pi-\frac{6\pi}{5}\right)+5\cos\left(\frac{8\pi}{5}-2\pi\right)$$

$$=1+2\cos\frac{2\pi}{5}-3\cos\frac{\pi}{5}-4\cos\left(\frac{\pi}{5}\right)+5\cos\left(\frac{2\pi}{5}\right)$$

$$=1+7\cos\frac{2\pi}{5}-7\cos\frac{\pi}{5}$$

$$=\ 1+7\left(2\cos^2\frac{\pi}{5}-1\right)-7\cos\frac{\pi}{5}\ \text{(Using the double angle identity)}$$

$$=\ 14\cos^2\frac{\pi}{5}-7\cos\frac{\pi}{5}-6$$

$$=\ -\frac{5}{2}\ \text{(from part d)}$$

Thus, we have

$$14\cos^2\frac{\pi}{5}-7\cos\frac{\pi}{5}-6=\ -\frac{5}{2},\text{or}$$

$$4\cos^2\frac{\pi}{5}-2\cos\frac{\pi}{5}-1=\ 0$$

Solving this quadratic equation gives

$$\cos\frac{\pi}{5}$$

$$=\frac{2\pm\sqrt{4-4(4)(-1)}}{8}$$

$$=\frac{2\pm\sqrt{20}}{8}$$

$$=\frac{1\pm\sqrt{5}}{4}$$

and since $\cos\frac{\pi}{5}>0$, we can conclude that $\cos\frac{\pi}{5}=\frac{1+\sqrt{5}}{4}$