Solutions to Exercise 8A

1a Magnitude of $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = 100 \text{ km} - 40 \text{ km} = 60 \text{ km}$

$\overrightarrow{BC}$ is towards west direction which means the magnitude to be considered negative as per the quadrant rule.

Direction of $\overrightarrow{AC}$ is 90°T.

1b Magnitude of $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$

In this case, using Pythagoras theorem of triangles,

$$AC^2 = AB^2 + BC^2$$

$$= 6^2 + 4^2$$

$$= 36 + 16$$

$$= 52$$

$$AC = \sqrt{52} = 7.211 \ldots \doteqdot 7$$

The magnitude of $\overrightarrow{AC}$ is 7 km.

Direction of $\overrightarrow{AC}$ is

$$\tan\theta = \frac{AB}{BC} = \frac{4}{6}$$

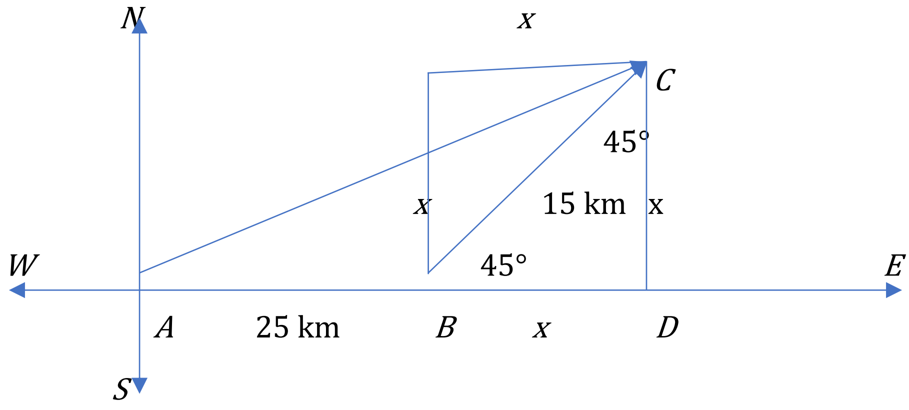
$$\theta = \tan^{-1}\frac{4}{6} = 33.690 \ldots \doteqdot 34°$$

The tangent is in fourth quadrant where it is negative,

therefore, bearing from A to C is $180° - 34° = 146°$.

Direction of $\overrightarrow{AC}$ is 146°T.

1c



According to Pythagoras' Theorem

$BC^2 = BD^2 + DC^2$

$15^2 = x^2 + x^2$

$2x^2 = 15^2$

$x = \dfrac{15\sqrt{2}}{2} = 10.606 \ldots \doteqdot 10.6$

Magnitude of $\overrightarrow{AC} = \overrightarrow{AD} + \overrightarrow{CD}$

In this case, using Pythagoras' Theorem,

$$\begin{aligned} AC^2 &= AD^2 + CD^2 \\ &= (25 + 10.6)^2 + 10.6^2 \\ &= 1267.36 + 112.36 \\ &= 1379.72 \end{aligned}$$

$AC = \sqrt{1379.72} = 37.144 \ldots \doteqdot 37$

The magnitude of $\overrightarrow{AC}$ is 37 km.

Direction of $\overrightarrow{AC}$ is

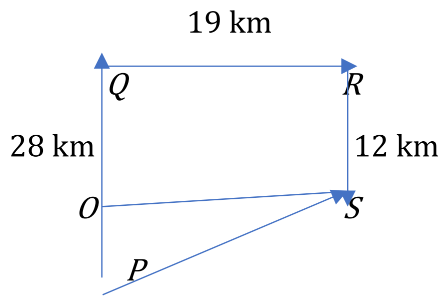
$\tan\theta = \dfrac{AD}{DC} = \dfrac{35.6}{10.6}$

$\theta = \tan^{-1}\dfrac{35.6}{10.6} = 73.418 \ldots \doteqdot 73°$

The tangent is in first quadrant where it is positive.

Direction of $\overrightarrow{AC}$ is 073°T.

2a



2b According to the picture, QR is parallel to OS which means

$QR = OS = 19$ km

Similarly, OQ is parallel to RS, therefore

$OQ = RS = 12$ km

Hence, $OP = PQ - OQ = 28 - 12 = 16$ km

Using Pythagoras theorem

$$\begin{aligned} PS^2 &= OP^2 + OS^2 \\ &= 16^2 + 19^2 \\ &= 256 + 361 \\ &= 617 \end{aligned}$$

$PS = \sqrt{617} = 24.839\ldots$

Magnitude of $\overrightarrow{PS} \doteqdot 24.8$ km

Direction of $\overrightarrow{PS}$ is

$$\tan\theta = \frac{OS}{OP} = \frac{19}{16}$$

$$\theta = \tan^{-1}\frac{19}{16} = 49.899\ldots \doteqdot 50\,°$$

The tangent is in first quadrant where it is positive.

Direction of $\overrightarrow{AC}$ is 050°T.

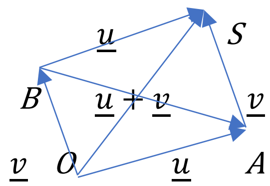
3 The opposite sides WX and ZY are parallel and equal hence $WXYZ$ is a parallelogram.

4 The opposite sides BA and CD are parallel and equal, and $\angle BAD = 90°$. Hence, $ABCD$ is a parallelogram with an angle being $90°$, therefore, it is a rectangle.

5a All the sides of the quadrilateral are equal therefore it is a rhombus.

5b The opposite sides of a rhombus are parallel, so $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ have opposite directions.

6



As per the above diagram, $OBSA$ is a parallelogram where $OB \parallel AS$ and $BS \parallel OA$

Hence, $BS = OA = \underset{\sim}{u}$ and $OB = AS = \underset{\sim}{v}$

$\underset{\sim}{u} + \underset{\sim}{v} = \overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AS} = \overrightarrow{OS}$

And $\underset{\sim}{v} + \underset{\sim}{u} = \overrightarrow{OB} + \overrightarrow{OA} = \overrightarrow{OB} + \overrightarrow{BS} = \overrightarrow{OS}$

So, $\underset{\sim}{u} + \underset{\sim}{v} = \underset{\sim}{v} + \underset{\sim}{u}$ as required.

7 $\overrightarrow{UV} + \overrightarrow{VW} + \overrightarrow{WU}$

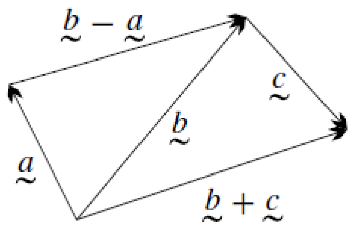
$= \overrightarrow{UW} + \overrightarrow{WU}$

$= \overrightarrow{UU}$

$= \underset{\sim}{0}$

It gives a zero vector.

8



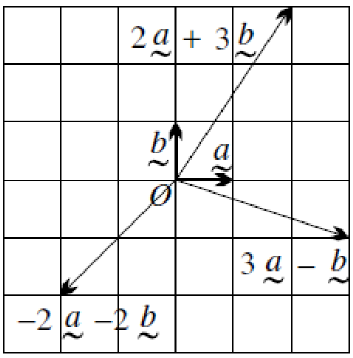
9a $\overrightarrow{AC} + \overrightarrow{AD} = \overrightarrow{AD}$

9b $\overrightarrow{BC} + \overrightarrow{CA} = \overrightarrow{BA}$

9c $\overrightarrow{AD} - \overrightarrow{AB} = \overrightarrow{AD} + \overrightarrow{BA} = \overrightarrow{BA} + \overrightarrow{AD} = \overrightarrow{BD}$

9d $\overrightarrow{AC} - \overrightarrow{BC} = \overrightarrow{AC} + \overrightarrow{CB} = \overrightarrow{AB}$

10



11a $\underset{\sim}{f}$

11b $\underset{\sim}{d}$

11c $\underset{\sim}{h}$

11d $\underset{\sim}{b}$

11e $\underset{\sim}{c}$

11f $\underset{\sim}{a}$

11g $\underset{\sim}{e}$

11h $\underset{\sim}{g}$

12a $\overrightarrow{AB} = \overrightarrow{AC} + \overrightarrow{CB}$

$= \overrightarrow{AC} - \overrightarrow{BC}$

$= \underset{\sim}{u} - \underset{\sim}{v}$

12b $\overrightarrow{AM} = \overrightarrow{AC} + \overrightarrow{CM}$

$= \overrightarrow{AC} - \overrightarrow{MC}$

$= \overrightarrow{AC} - \frac{1}{2}\overrightarrow{BC}$

$= \underset{\sim}{u} - \frac{1}{2}\underset{\sim}{v}$

13a $\overrightarrow{AM}$ and $\overrightarrow{MB}$ have the same length and direction.

Hence, $\overrightarrow{AM} = \underset{\sim}{u} = \overrightarrow{MB}$

Similarly, $\overrightarrow{PN}$ and $\overrightarrow{NQ}$ have the same length and direction.

Hence, $\overrightarrow{PN} = \underset{\sim}{v} = \overrightarrow{NQ}$

13b $\overrightarrow{AN} = \overrightarrow{AP} + \overrightarrow{PN} = \underset{\sim}{a} + \underset{\sim}{v}$

and $\overrightarrow{AN} = \overrightarrow{AM} + \overrightarrow{MN} = \underset{\sim}{u} + \underset{\sim}{p}$

Hence,

$\underset{\sim}{a} + \underset{\sim}{v} = \underset{\sim}{u} + \underset{\sim}{p}$

$\underset{\sim}{a} = \underset{\sim}{p} + \underset{\sim}{u} - \underset{\sim}{v}$

Similarly, $\overrightarrow{MQ} = \overrightarrow{MB} + \overrightarrow{BQ} = \underset{\sim}{u} + \underset{\sim}{b}$

and $\overrightarrow{MQ} = \overrightarrow{MN} + \overrightarrow{NQ} = \underset{\sim}{p} + \underset{\sim}{v}$

Hence,

$\underset{\sim}{u} + \underset{\sim}{b} = \underset{\sim}{p} + \underset{\sim}{v}$

$\underset{\sim}{b} = \underset{\sim}{p} - \underset{\sim}{u} + \underset{\sim}{v}$

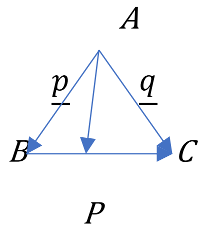
13c LHS $= \underset{\sim}{a} + \underset{\sim}{b}$

$= \left(\underset{\sim}{p} + \underset{\sim}{u} - \underset{\sim}{v}\right) + \left(\underset{\sim}{p} + \underset{\sim}{v} - \underset{\sim}{u}\right)$

$= 2\underset{\sim}{p}$

$=$ RHS

14a



$\overrightarrow{BC} = \overrightarrow{BA} + \overrightarrow{AC}$

$= -\overrightarrow{AB} + \overrightarrow{AC}$

$= \overrightarrow{AC} - \overrightarrow{AB}$

$= \underset{\sim}{q} - \underset{\sim}{p}$

Given, $BP : PC = 1 : 2$

Hence, $\overrightarrow{BP} = \frac{1}{3}\overrightarrow{BC} = \frac{1}{3}(\underset{\sim}{q} - \underset{\sim}{p})$

14b $\overrightarrow{AP} = \underset{\sim}{r}$, hence

$\overrightarrow{AP} = \overrightarrow{AC} + \overrightarrow{CP}$

$$\overrightarrow{AP} = \overrightarrow{AC} - \overrightarrow{PC}$$

$$\underset{\sim}{r} = \underset{\sim}{q} - \frac{2}{3}\left(\underset{\sim}{q} - \underset{\sim}{p}\right)$$

$$\underset{\sim}{r} = \underset{\sim}{q} - \frac{2}{3}\underset{\sim}{q} + \frac{2}{3}\underset{\sim}{p}$$

$$\underset{\sim}{r} = \frac{1}{3}\underset{\sim}{q} + \frac{2}{3}\underset{\sim}{p}$$

Hence, proved.

15a $\overrightarrow{AC} = \overrightarrow{AD} + \overrightarrow{DC}$

$= \overrightarrow{AD} - \overrightarrow{CD}$

$= \underset{\sim}{w} - \underset{\sim}{u}$

$\overrightarrow{BC} = \overrightarrow{BA} + \overrightarrow{AC} = \underset{\sim}{v} + \underset{\sim}{w} - \underset{\sim}{u}$

As M is the midpoint of $\overrightarrow{BC}$,

$\overrightarrow{BM} = \frac{1}{2}\overrightarrow{BC} = \frac{1}{2}(\underset{\sim}{v} + \underset{\sim}{w} - \underset{\sim}{u})$

$\overrightarrow{MB}$ is in opposite direction of $\overrightarrow{BM}$.

Hence, $\overrightarrow{MB} = -\frac{1}{2}\left(\underset{\sim}{v} + \underset{\sim}{w} - \underset{\sim}{u}\right) = \frac{1}{2}(\underset{\sim}{u} - \underset{\sim}{v} - \underset{\sim}{w})$

15b $\overrightarrow{MA} = \overrightarrow{MB} + \overrightarrow{BA}$

$$= \frac{1}{2}\left(\underset{\sim}{u} - \underset{\sim}{v} - \underset{\sim}{w}\right) + \underset{\sim}{v}$$

$$= \frac{1}{2}\left(\underset{\sim}{u} + \underset{\sim}{v} - \underset{\sim}{w}\right)$$

16a $\overrightarrow{WX} = \overrightarrow{WR} + \overrightarrow{RX}$

$= \overrightarrow{RX} - \overrightarrow{RW}$

$= \underset{\sim}{x} - \underset{\sim}{w}$

$\overrightarrow{WP} = \frac{1}{2}\overrightarrow{WX} = \frac{1}{2}(\underset{\sim}{x} - \underset{\sim}{w})$

16b $\overrightarrow{RP} = \overrightarrow{RX} + \overrightarrow{XP}$

$= \overrightarrow{RX} - \overrightarrow{PX}$

$= \underline{x} - \frac{1}{2}(\underline{x} - \underline{w})$

$= \underline{x} - \frac{1}{2}\underline{x} + \frac{1}{2}\underline{w}$

$= \frac{1}{2}\underline{x} + \frac{1}{2}\underline{w}$

$= \frac{1}{2}(\underline{w} + \underline{x})$

16c $\overrightarrow{YZ} = \overrightarrow{YR} + \overrightarrow{RZ}$

$= \overrightarrow{RZ} - \overrightarrow{RY}$

$= \underline{z} - \underline{y}$

$\overrightarrow{RQ} = \overrightarrow{RZ} + \overrightarrow{ZQ}$

$= \overrightarrow{RZ} - \overrightarrow{QZ}$

$= \overrightarrow{RZ} - \frac{1}{2}\overrightarrow{YZ}$

$= \underline{z} - \frac{1}{2}(\underline{z} - \underline{y})$

$= \underline{z} - \frac{1}{2}\underline{z} + \frac{1}{2}\underline{y}$

$= \frac{1}{2}\underline{z} + \frac{1}{2}\underline{y}$

$= \frac{1}{2}(\underline{y} + \underline{z})$

16d $\underline{w} + \underline{x} + \underline{y} + \underline{z}$

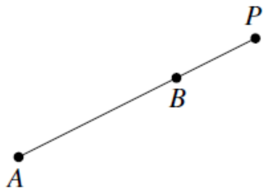
$= \overrightarrow{RW} + \overrightarrow{RX} + \overrightarrow{RY} + \overrightarrow{RZ}$

$= \overrightarrow{RR} + \overrightarrow{RR}$

$= \overrightarrow{RR}$

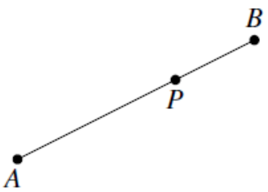
$= \underline{0}$

17a i



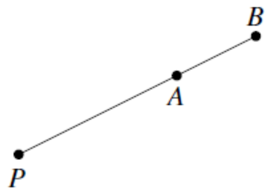
The magnitude of $\overrightarrow{AP}$ is more than $\overrightarrow{AB}$ as given $\overrightarrow{AP} = k\overrightarrow{AB}$ and $k > 1$.

17a ii



The magnitude of $\overrightarrow{AP}$ is less than $\overrightarrow{AB}$ but more than zero vector as given $\overrightarrow{AP} = k\overrightarrow{AB}$ and $0 < k < 1$.

17a iii



The magnitude of $\overrightarrow{AP}$ is less than $\overrightarrow{AB}$ as given $\overrightarrow{AP} = k\overrightarrow{AB}$ and $k < 0$.

17b i

$$\overrightarrow{AP} = \frac{3}{2}\overrightarrow{PB}$$

$$\overrightarrow{PB} = \frac{2}{3}\overrightarrow{AP}$$

And, $\overrightarrow{AB} = \overrightarrow{AP} + \overrightarrow{PB}$

$$= \overrightarrow{AP} + \frac{2}{3}\overrightarrow{AP}$$

$$= \frac{5}{3}\overrightarrow{AP}$$

$$\overrightarrow{AP} = \frac{3}{5}\overrightarrow{AB}$$

Given, $\overrightarrow{AP} = k\overrightarrow{AB}$

Hence, $k = \frac{3}{5}$.

17b ii $\overrightarrow{AP} = -\frac{3}{2}\overrightarrow{PB}$

$\overrightarrow{PB} = -\frac{2}{3}\overrightarrow{AP}$

And, $\overrightarrow{AB} = \overrightarrow{AP} + \overrightarrow{PB}$

$$= \overrightarrow{AP} + \left(-\frac{2}{3}\overrightarrow{AP}\right)$$

$$= \overrightarrow{AP} - \frac{2}{3}\overrightarrow{AP}$$

$$= \frac{1}{3}\overrightarrow{AP}$$

$$\overrightarrow{AP} = 3\overrightarrow{AB}$$

Given, $\overrightarrow{AP} = k\overrightarrow{AB}$

Hence, $k = 3$.

17b iii $\overrightarrow{AP} = -\frac{2}{3}\overrightarrow{PB}$

$\overrightarrow{PB} = -\frac{3}{2}\overrightarrow{AP}$

And, $\overrightarrow{AB} = \overrightarrow{AP} + \overrightarrow{PB}$

$$= \overrightarrow{AP} + \left(-\frac{3}{2}\overrightarrow{AP}\right)$$

$$= \overrightarrow{AP} - \frac{3}{2}\overrightarrow{AP}$$

$$= -\frac{1}{2}\overrightarrow{AP}$$

$$\overrightarrow{AP} = -2\overrightarrow{AB}$$

Given, $\overrightarrow{AP} = k\overrightarrow{AB}$

Hence, $k = -2$.

17c $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = \overrightarrow{OB} - \overrightarrow{OA} = \underset{\sim}{b} - \underset{\sim}{a}$

And, $\overrightarrow{AP} = \overrightarrow{AO} + \overrightarrow{OP} = \overrightarrow{OP} - \overrightarrow{OA} = \underset{\sim}{p} - \underset{\sim}{a}$

Given, $\overrightarrow{AP} = k\overrightarrow{AB}$

$\underset{\sim}{p} - \underset{\sim}{a} = k(\underset{\sim}{b} - \underset{\sim}{a})$

$\underset{\sim}{p} - \underset{\sim}{a} = k\underset{\sim}{b} - k\underset{\sim}{a}$

$\underset{\sim}{p} = \underset{\sim}{a} - k\underset{\sim}{a} + k\underset{\sim}{b} = (1-k)\underset{\sim}{a} + k\underset{\sim}{b}$

18 The triangles are similar by the SAS similarity test — the angles between $\underset{\sim}{a}$ and $\underset{\sim}{b}$, and between $\lambda\underset{\sim}{a}$ and $\lambda\underset{\sim}{b}$ are equal, and the matching sides are in ratio1:λ. It now follows that the head of the vector $\lambda\underset{\sim}{b}$ is the head of the vector $\lambda(\underset{\sim}{a} + \underset{\sim}{b})$.

19a Two zero vectors each have zero length and no direction, and so are equal.

19b Rome for administration (in the distant past), Greenwich UK for longitude, Jerusalem and Mecca for religious ceremonies, the North and South Poles for maps. The obelisk in Macquarie Place, Sydney, remains the origin for road distances in NSW. It is inscribed on the front,

'This Obelisk was erected in Macquarie Place A.D. 1818, to Record that all the public roads leading to the interior of the colony are measured from it. L. Macquarie Esq Governor'.

20a $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$ and we are given $\overrightarrow{AB} = \underset{\sim}{u}$ and $\overrightarrow{AC} = \underset{\sim}{v}$

$\overrightarrow{BC} = \underset{\sim}{v} - \underset{\sim}{u}$ and $\overrightarrow{BC} = \frac{1}{2}\overrightarrow{BX}$, so $\overrightarrow{BX} = \frac{1}{2}(\underset{\sim}{v} - \underset{\sim}{u})$

$\overrightarrow{AX} = \overrightarrow{AB} + \overrightarrow{BX}$ and so $\overrightarrow{AX} = \underset{\sim}{u} + \frac{1}{2}(\underset{\sim}{v} - \underset{\sim}{u}) = \frac{1}{2}(\underset{\sim}{u} + \underset{\sim}{v})$

$\overrightarrow{AP} = \frac{2}{3}\overrightarrow{AX}$ and so $\overrightarrow{AP} = \frac{2}{3} \times \frac{1}{2}(\underset{\sim}{u} + \underset{\sim}{v}) = \frac{1}{3}(\underset{\sim}{u} + \underset{\sim}{v})$

$$\overrightarrow{PY} = \overrightarrow{PA} + \overrightarrow{AY}$$
$$= -\frac{1}{3}(\underset{\sim}{u} + \underset{\sim}{v}) + \frac{1}{2}\underset{\sim}{v}$$
$$= -\frac{1}{3}\underset{\sim}{u} + \frac{1}{6}\underset{\sim}{v}$$

So $\overrightarrow{PY} = \frac{1}{6}(\underset{\sim}{v} - 2\underset{\sim}{u})$.

20b To prove collinearity we need to show that $\overrightarrow{BP} = k\overrightarrow{PY}$.

$$\overrightarrow{PY} = \frac{1}{6}(\underset{\sim}{v} - 2\underset{\sim}{u})$$

$$\overrightarrow{BP} = \overrightarrow{BA} + \overrightarrow{AP}$$
$$= -\underset{\sim}{u} + \frac{1}{3}(\underset{\sim}{u} + \underset{\sim}{v})$$
$$= -\frac{2}{3}\underset{\sim}{u} + \frac{1}{3}\underset{\sim}{v}$$
$$= \frac{1}{3}(\underset{\sim}{v} - 2\underset{\sim}{u})$$

So $\overrightarrow{BP} = 2\overrightarrow{PY}$ and hence the points B, P and Y are collinear.

20c The three medians of a triangle are concurrent, and their point of intersection trisects each median. [A *median* of a triangle is the line joining a vertex to the midpoint of the opposite side.]

21a $\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$

$$\overrightarrow{OP} = \overrightarrow{OA} + \frac{1}{4}\overrightarrow{AB}$$
$$= \overrightarrow{OA} + \frac{1}{4}\left(\overrightarrow{AO} + \overrightarrow{OB}\right)$$
$$= \overrightarrow{OA} + \frac{1}{4}\left(\overrightarrow{OB} - \overrightarrow{OA}\right)$$
$$= \frac{1}{4}\left(3\overrightarrow{OA} + \overrightarrow{OB}\right)$$

We are given $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OB} = \underset{\sim}{b}$ and so $\overrightarrow{OP} = \frac{1}{4}(3\underset{\sim}{a} + \underset{\sim}{b})$.

21b $\overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OQ}$

$$\begin{aligned}\overrightarrow{PQ} &= \overrightarrow{PO} + \overrightarrow{OD} + \overrightarrow{DQ} \\ &= -\overrightarrow{OP} + \overrightarrow{OD} + \frac{3}{4}\overrightarrow{DC} \\ &= -\overrightarrow{OP} + \overrightarrow{OD} + \frac{3}{4}\left(\overrightarrow{DO} + \overrightarrow{OC}\right) \\ &= -\overrightarrow{OP} + \frac{1}{4}\overrightarrow{OD} + \frac{3}{4}\overrightarrow{OC}\end{aligned}$$

We are given $\overrightarrow{OP} = \frac{1}{4}(3\underset{\sim}{a} + \underset{\sim}{b})$, $\overrightarrow{OC} = \underset{\sim}{c}$ and $\overrightarrow{OD} = \underset{\sim}{d}$ and so

$$\overrightarrow{PQ} = -\frac{1}{4}(3\underset{\sim}{a} + \underset{\sim}{b}) + \frac{1}{4}\underset{\sim}{d} + \frac{3}{4}\underset{\sim}{c}.$$

$$\overrightarrow{PQ} = -\frac{3}{4}\underset{\sim}{a} - \frac{1}{4}\underset{\sim}{b} + \frac{1}{4}\underset{\sim}{d} + \frac{3}{4}\underset{\sim}{c}$$

So $\overrightarrow{PQ} = \frac{1}{4}(3\underset{\sim}{c} + \underset{\sim}{d} - 3\underset{\sim}{a} - \underset{\sim}{b})$.

21c $\overrightarrow{OM} = \overrightarrow{OP} + \overrightarrow{PM}$

$\overrightarrow{OM} = \overrightarrow{OP} + \frac{1}{2}\overrightarrow{PQ}$ and we are given $\overrightarrow{OP} = \frac{1}{4}(3\underset{\sim}{a} + \underset{\sim}{b})$ and $\overrightarrow{PQ} = \frac{1}{4}(3\underset{\sim}{c} + \underset{\sim}{d} - 3\underset{\sim}{a} - \underset{\sim}{b})$

$$\begin{aligned}\overrightarrow{OM} &= \frac{1}{4}(3\underset{\sim}{a} + \underset{\sim}{b}) + \frac{1}{8}(3\underset{\sim}{c} + \underset{\sim}{d} - 3\underset{\sim}{a} - \underset{\sim}{b}) \\ &= \frac{3}{8}\underset{\sim}{a} - \frac{1}{8}\underset{\sim}{b} + \frac{3}{8}\underset{\sim}{c} + \frac{1}{8}\underset{\sim}{d}\end{aligned}$$

So $\overrightarrow{OM} = \frac{1}{8}(3\underset{\sim}{a} - \underset{\sim}{b} + 3\underset{\sim}{c} + \underset{\sim}{d})$.

Solutions to Exercise 8B

1a $\quad \underset{\sim}{a} = 8\underset{\sim}{i} + 6\underset{\sim}{j}$

$x = 8$ and $y = 6$

Hence, $\left|\underset{\sim}{a}\right| = \sqrt{x^2 + y^2}$

$= \sqrt{8^2 + 6^2}$

$= \sqrt{64 + 36}$

$= \sqrt{100}$

$= 10$

1b $\quad 2\underset{\sim}{a} = 2 \times \left(8\underset{\sim}{i} + 6\underset{\sim}{j}\right) = 16\underset{\sim}{i} + 12\underset{\sim}{j}$

1c $\quad 2\underset{\sim}{a} = 16\underset{\sim}{i} + 12\underset{\sim}{j}$

$x = 16$ and $y = 12$

Hence, $\left|2\underset{\sim}{a}\right| = \sqrt{x^2 + y^2}$

$= \sqrt{16^2 + 12^2}$

$= \sqrt{256 + 144}$

$= \sqrt{400}$

$= 20$

1d $\quad -5\underset{\sim}{a} = -5 \times \left(8\underset{\sim}{i} + 6\underset{\sim}{j}\right) = -40\underset{\sim}{i} - 30\underset{\sim}{j}$

1e $\quad -5\underset{\sim}{a} = -40\underset{\sim}{i} - 30\underset{\sim}{j}$

$x = -40$ and $y = -30$

Hence, $\left|-5\underset{\sim}{a}\right| = \sqrt{x^2 + y^2}$

$= \sqrt{(-40)^2 + (-30)^2}$

$= \sqrt{1600 + 900}$

$= \sqrt{2500}$

$= 50$

2a $\underset{\sim}{a} = 2\underset{\sim}{i} + 3\underset{\sim}{j}$ and $\underset{\sim}{b} = \underset{\sim}{i} - 4\underset{\sim}{j}$

$\underset{\sim}{a} + \underset{\sim}{b} = \left(2\underset{\sim}{i} + 3\underset{\sim}{j}\right) + \left(\underset{\sim}{i} - 4\underset{\sim}{j}\right)$

$= (2 + 1)\underset{\sim}{i} + (3 - 4)\underset{\sim}{j}$

$= 3\underset{\sim}{i} - \underset{\sim}{j}$

2b $\underset{\sim}{a} + \underset{\sim}{b} = 3\underset{\sim}{i} - \underset{\sim}{j}$

$x = 3$ and $y = -1$

Hence, $\left|\underset{\sim}{a} + \underset{\sim}{b}\right| = \sqrt{x^2 + y^2}$

$= \sqrt{3^2 + (-1)^2}$

$= \sqrt{9 + 1}$

$= \sqrt{10}$

2c $\underset{\sim}{a} = 2\underset{\sim}{i} + 3\underset{\sim}{j}$ and $\underset{\sim}{b} = \underset{\sim}{i} - 4\underset{\sim}{j}$

$\underset{\sim}{a} - \underset{\sim}{b} = \left(2\underset{\sim}{i} + 3\underset{\sim}{j}\right) - \left(\underset{\sim}{i} - 4\underset{\sim}{j}\right)$

$= (2 - 1)\underset{\sim}{i} + (3 - (-4))\underset{\sim}{j}$

$= (2 - 1)\underset{\sim}{i} + (3 + 4)\underset{\sim}{j}$

$= \underset{\sim}{i} + 7\underset{\sim}{j}$

2d $\underset{\sim}{a} - \underset{\sim}{b} = \underset{\sim}{i} + 7\underset{\sim}{j}$

$x = 1$ and $y = 7$

Hence, $\left|\underset{\sim}{a} - \underset{\sim}{b}\right| = \sqrt{x^2 + y^2}$

$= \sqrt{1^2 + 7^2}$

$= \sqrt{1 + 49}$

$= \sqrt{50}$

$= 5\sqrt{2}$

2e $\quad \underset{\sim}{a} = 2\underset{\sim}{i} + 3\underset{\sim}{j}$ and $\underset{\sim}{b} = \underset{\sim}{i} - 4\underset{\sim}{j}$

$-3\underset{\sim}{a} - 2\underset{\sim}{b} = -3\left(2\underset{\sim}{i} + 3\underset{\sim}{j}\right) - 2\left(\underset{\sim}{i} - 4\underset{\sim}{j}\right)$

$= -6i - 9j - 2i + 8\underset{\sim}{j}$

$= (-6 - 2)\underset{\sim}{i} + (-9 + 8)\underset{\sim}{j}$

$= -8\underset{\sim}{i} - \underset{\sim}{j}$

2f $\quad -3\underset{\sim}{a} - 2\underset{\sim}{b} = -8\underset{\sim}{i} - \underset{\sim}{j}$

$x = -8$ and $y = -1$

Hence, $\left|-3\underset{\sim}{a} - 2\underset{\sim}{b}\right| = \sqrt{x^2 + y^2}$

$= \sqrt{(-8)^2 + (-1)^2}$

$= \sqrt{64 + 1}$

$= \sqrt{65}$

3a $\quad \underset{\sim}{a} = \begin{bmatrix} -17 \\ 3 \end{bmatrix} \quad \underset{\sim}{b} = \begin{bmatrix} 5 \\ -11 \end{bmatrix} \quad \underset{\sim}{c} = \begin{bmatrix} -7 \\ -13 \end{bmatrix}$

$\underset{\sim}{a} + \underset{\sim}{b} - \underset{\sim}{c} = \begin{bmatrix} -17 + 5 - (-7) \\ 3 + (-11) - (-13) \end{bmatrix}$

$= \begin{bmatrix} -17 + 5 + 7 \\ 3 - 11 + 13 \end{bmatrix}$

$= \begin{bmatrix} -5 \\ 5 \end{bmatrix}$

3b $\underset{\sim}{a} + \underset{\sim}{b} - \underset{\sim}{c} = \begin{bmatrix} -5 \\ 5 \end{bmatrix}$

$x = -5$ and $y = 5$

$$\begin{aligned}|\underset{\sim}{a} + \underset{\sim}{b} - \underset{\sim}{c}| &= \sqrt{x^2 + y^2} \\ &= \sqrt{(-5)^2 + 5^2} \\ &= \sqrt{25 + 25} \\ &= \sqrt{50} \\ &= 5\sqrt{2}\end{aligned}$$

3c $\underset{\sim}{a} = \begin{bmatrix} -17 \\ 3 \end{bmatrix}$ $\underset{\sim}{b} = \begin{bmatrix} 5 \\ -11 \end{bmatrix}$ $\underset{\sim}{c} = \begin{bmatrix} -7 \\ -13 \end{bmatrix}$

$-3\underset{\sim}{a} = \begin{bmatrix} 51 \\ -9 \end{bmatrix}$

$-5\underset{\sim}{b} = \begin{bmatrix} -25 \\ 55 \end{bmatrix}$

$2\underset{\sim}{c} = \begin{bmatrix} -14 \\ -26 \end{bmatrix}$

$$\begin{aligned}-3\underset{\sim}{a} - 5\underset{\sim}{b} + 2\underset{\sim}{c} &= \begin{bmatrix} 51 - 25 - 14 \\ -9 + 55 - 26 \end{bmatrix} \\ &= \begin{bmatrix} 12 \\ 20 \end{bmatrix}\end{aligned}$$

3d $-3\underset{\sim}{a} - 5\underset{\sim}{b} + 2\underset{\sim}{c} = \begin{bmatrix} 12 \\ 20 \end{bmatrix}$

$x = 12$ and $y = 20$

$$\begin{aligned}|-3\underset{\sim}{a} - 5\underset{\sim}{b} + 2\underset{\sim}{c}| &= \sqrt{x^2 + y^2} \\ &= \sqrt{12^2 + 20^2} \\ &= \sqrt{144 + 400} \\ &= \sqrt{544} \\ &= 4\sqrt{34}\end{aligned}$$

4a $\underset{\sim}{u} = 2\underset{\sim}{i} + \underset{\sim}{j}$ and $\underset{\sim}{v} = -\underset{\sim}{i} + 2\underset{\sim}{j}$

$\underset{\sim}{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \underset{\sim}{v} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$$\begin{aligned}\underset{\sim}{u} + \underset{\sim}{v} &= \begin{bmatrix} 2 + (-1) \\ 1 + 2 \end{bmatrix} \\ &= \begin{bmatrix} 2 - 1 \\ 1 + 2 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 3 \end{bmatrix}\end{aligned}$$

4b $\underset{\sim}{a} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \quad \underset{\sim}{b} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

$\underset{\sim}{a} = 3\underset{\sim}{i} + 2\underset{\sim}{j}$ and $\underset{\sim}{b} = 4\underset{\sim}{i} + \underset{\sim}{j}$

$$\begin{aligned}\underset{\sim}{a} - \underset{\sim}{b} &= \left(3\underset{\sim}{i} + 2\underset{\sim}{j}\right) - \left(4\underset{\sim}{i} + \underset{\sim}{j}\right) \\ &= (3 - 4)\underset{\sim}{i} + (2 - 1)\underset{\sim}{j} \\ &= -\underset{\sim}{i} + \underset{\sim}{j}\end{aligned}$$

5a $\underset{\sim}{u} = \underset{\sim}{i} + 2\underset{\sim}{j}$

$x = 1$ and $y = 2$

$$\begin{aligned}\text{Hence, } \left|\underset{\sim}{u}\right| &= \sqrt{x^2 + y^2} \\ &= \sqrt{1^2 + 2^2} \\ &= \sqrt{1 + 4} \\ &= \sqrt{5}\end{aligned}$$

$$\begin{aligned}\hat{\underset{\sim}{u}} &= \frac{\underset{\sim}{u}}{\left|\underset{\sim}{u}\right|} \\ &= \frac{\underset{\sim}{i} + 2\underset{\sim}{j}}{\sqrt{5}} \\ &= \frac{1}{\sqrt{5}}\underset{\sim}{i} + \frac{2}{\sqrt{5}}\underset{\sim}{j}\end{aligned}$$

5b $\underset{\sim}{v} = -4\underset{\sim}{i} + 3\underset{\sim}{j}$

$x = -4$ and $y = 3$

Hence, $|\underset{\sim}{v}| = \sqrt{x^2 + y^2}$

$= \sqrt{(-4)^2 + 3^2}$

$= \sqrt{16 + 9}$

$= \sqrt{25}$

$= 5$

$\hat{\underset{\sim}{v}} = \frac{\underset{\sim}{v}}{|\underset{\sim}{v}|}$

$= \frac{-4\underset{\sim}{i} + 3\underset{\sim}{j}}{5}$

$= -\frac{4}{5}\underset{\sim}{i} + \frac{3}{5}\underset{\sim}{j}$

5c $\underset{\sim}{w} = \underset{\sim}{u} + \underset{\sim}{v}$

$\underset{\sim}{u} + \underset{\sim}{v} = \left(\underset{\sim}{i} + 2\underset{\sim}{j}\right) + \left(-4\underset{\sim}{i} + 3\underset{\sim}{j}\right)$

$= (1 + (-4))\underset{\sim}{i} + (2 + 3)\underset{\sim}{j}$

$= (1 - 4)\underset{\sim}{i} + (2 + 3)\underset{\sim}{j}$

$= -3\underset{\sim}{i} + 5\underset{\sim}{j}$

$\underset{\sim}{w} = -3\underset{\sim}{i} + 5\underset{\sim}{j}$

$x = -3$ and $y = 5$

Hence, $|\underset{\sim}{w}| = \sqrt{x^2 + y^2}$

$= \sqrt{(-3)^2 + 5^2}$

$= \sqrt{9 + 25}$

$= \sqrt{34}$

$\hat{\underset{\sim}{w}} = \frac{\underset{\sim}{w}}{|\underset{\sim}{w}|}$

$$= \frac{-3\underset{\sim}{i} + 5\underset{\sim}{j}}{\sqrt{34}}$$

$$= -\frac{3}{\sqrt{34}}\underset{\sim}{i} + \frac{5}{\sqrt{34}}\underset{\sim}{j}$$

6a $\underset{\sim}{a} = \begin{bmatrix} 5 \\ -12 \end{bmatrix}$

$2\underset{\sim}{a} = \begin{bmatrix} 10 \\ -24 \end{bmatrix}$

$$\left|2\underset{\sim}{a}\right| = \sqrt{x^2 + y^2}$$

$$= \sqrt{10^2 + (-24)^2}$$

$$= \sqrt{100 + 576}$$

$$= \sqrt{676}$$

$$= 26$$

$$\left|\underset{\sim}{a}\right| = \sqrt{x^2 + y^2}$$

$$= \sqrt{5^2 + (-12)^2}$$

$$= \sqrt{25 + 144}$$

$$= \sqrt{169}$$

$$= 13$$

LHS: $\left|2\underset{\sim}{a}\right| = 26$

RHS: $2\left|\underset{\sim}{a}\right| = 2 \times 13 = 26$

Hence, LHS $=$ RHS.

6b $\underset{\sim}{b} = \begin{bmatrix} -15 \\ -8 \end{bmatrix}$

$-\underset{\sim}{b} = \begin{bmatrix} 15 \\ 8 \end{bmatrix}$

$$\left|-\underset{\sim}{b}\right| = \sqrt{x^2 + y^2}$$

$$= \sqrt{15^2 + 8^2}$$

$$= \sqrt{225 + 64}$$

$= \sqrt{289}$

$= 17$

$|\underset{\sim}{b}| = \sqrt{x^2 + y^2}$

$= \sqrt{(-15)^2 + (-8)^2}$

$= \sqrt{225 + 64}$

$= \sqrt{289}$

$= 17$

LHS: $|-\underset{\sim}{b}| = 17$

RHS: $|\underset{\sim}{b}| = 17$

Hence, LHS $=$ RHS.

6c $\quad \underset{\sim}{a} = \begin{bmatrix} 5 \\ -12 \end{bmatrix} \quad \underset{\sim}{b} = \begin{bmatrix} -15 \\ -8 \end{bmatrix}$

$\underset{\sim}{a} + \underset{\sim}{b} = \begin{bmatrix} 5 + (-15) \\ (-12) + (-8) \end{bmatrix}$

$= \begin{bmatrix} 5 - 15 \\ -12 - 8 \end{bmatrix}$

$= \begin{bmatrix} -10 \\ -20 \end{bmatrix}$

$|\underset{\sim}{a} + \underset{\sim}{b}| = \sqrt{x^2 + y^2}$

$= \sqrt{(-10)^2 + (-20)^2}$

$= \sqrt{100 + 400}$

$= \sqrt{500}$

$= 5\sqrt{10}$

$|\underset{\sim}{a}| = \sqrt{x^2 + y^2}$

$= \sqrt{5^2 + (-12)^2}$

$= \sqrt{25 + 144}$

$= \sqrt{169}$

$= 13$

$$|\underset{\sim}{b}| = \sqrt{x^2 + y^2}$$
$$= \sqrt{(-15)^2 + (-8)^2}$$
$$= \sqrt{225 + 64}$$
$$= \sqrt{289}$$
$$= 17$$

LHS: $|\underset{\sim}{a} + \underset{\sim}{b}| = 5\sqrt{10} = 15.811 \ldots$

RHS: $|\underset{\sim}{a}| + |\underset{\sim}{b}| = 13 + 17 = 30$

Hence, LHS $<$ RHS.

6d LHS: $|\underset{\sim}{a} + \underset{\sim}{b}| = 5\sqrt{10} = 15.811 \ldots$

RHS: $|\underset{\sim}{a}| - |\underset{\sim}{b}| = 13 - 17 = -4$

Hence, LHS $>$ RHS.

7a $\overrightarrow{OP} = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$ $\overrightarrow{OQ} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\overrightarrow{OR} = \begin{bmatrix} 8 \\ -3 \end{bmatrix}$

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$$
$$= \begin{bmatrix} 3 - (-1) \\ 2 - 6 \end{bmatrix}$$
$$= \begin{bmatrix} 4 \\ -4 \end{bmatrix}$$

$$\text{Gradient} = \frac{y}{x} = \frac{-4}{4} = -1$$

$$\overrightarrow{QR} = \overrightarrow{OR} - \overrightarrow{OQ}$$
$$= \begin{bmatrix} 8 - 3 \\ (-3) - 2 \end{bmatrix}$$
$$= \begin{bmatrix} 5 \\ -5 \end{bmatrix}$$

$$\text{Gradient} = \frac{y}{x} = \frac{-5}{5} = -1$$

P, Q and R are collinear as the gradient is -1.

7b $\overrightarrow{OA} = 3\underset{\sim}{i} + 8\underset{\sim}{j}$ $\overrightarrow{OB} = -\underset{\sim}{i} + 3\underset{\sim}{j}$ and $\overrightarrow{OC} = -4\underset{\sim}{i} - \underset{\sim}{j}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= \left(-\underset{\sim}{i} + 3\underset{\sim}{j}\right) - \left(3\underset{\sim}{i} + 8\underset{\sim}{j}\right)$

$= \left((-1) - 3\right)\underset{\sim}{i} + (3 - 8)\underset{\sim}{j}$

$= -4\underset{\sim}{i} - 5\underset{\sim}{j}$

$\text{Gradient} = \dfrac{y}{x} = \dfrac{-5}{-4} = \dfrac{5}{4}$

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$

$= \left(-4\underset{\sim}{i} - \underset{\sim}{j}\right) - \left(-\underset{\sim}{i} + 3\underset{\sim}{j}\right)$

$= \left((-4) - (-1)\right)\underset{\sim}{i} + \left((-1) - 3\right)\underset{\sim}{j}$

$= -3\underset{\sim}{i} - 4\underset{\sim}{j}$

$\text{Gradient} = \dfrac{y}{x} = \dfrac{-3}{-4} = \dfrac{3}{4}$

A, B and C are not collinear as the gradient is not same.

8a $\underset{\sim}{a} = \overrightarrow{OA} = 4\underset{\sim}{i} - 7\underset{\sim}{j}$

Consider O to be the origin and $\overrightarrow{OA}$ represented as the position vector.

8b $\underset{\sim}{b} = \overrightarrow{OB} = 6\underset{\sim}{i} + 3\underset{\sim}{j}$

Consider O to be the origin and $\overrightarrow{OB}$ represented as the position vector.

8c $M = \overrightarrow{OM}$ is the mid-point of $\overrightarrow{AB}$.

$\overrightarrow{AB} = \underset{\sim}{a} + \underset{\sim}{b}$

$= \left(4\underset{\sim}{i} - 7\underset{\sim}{j}\right) + \left(6\underset{\sim}{i} + 3\underset{\sim}{j}\right)$

$= (4 + 6)\underset{\sim}{i} + (-7 + 3)\underset{\sim}{j}$

$= 10\underset{\sim}{i} - 4\underset{\sim}{j}$

$M = \overrightarrow{OM}$

$= \frac{1}{2}\overrightarrow{AB}$

$= \frac{1}{2}(10\underset{\sim}{i} - 4\underset{\sim}{j})$

$= 5\underset{\sim}{i} - 2\underset{\sim}{j}$

9a $\overrightarrow{OP} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$

9b $2\overrightarrow{OP} = \begin{bmatrix} 8 \\ -2 \end{bmatrix}$

$\overrightarrow{OQ} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$

$2\overrightarrow{OP} - \overrightarrow{OQ} = \begin{bmatrix} 8-(-3) \\ -2-5 \end{bmatrix} = \begin{bmatrix} 11 \\ -7 \end{bmatrix}$

9c $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$

$= \begin{bmatrix} -3-4 \\ 5-(-1) \end{bmatrix}$

$= \begin{bmatrix} -7 \\ 6 \end{bmatrix}$

9d $\overrightarrow{QP} = \overrightarrow{OP} - \overrightarrow{OQ}$

$= \begin{bmatrix} 4-(-3) \\ (-1)-5 \end{bmatrix}$

$= \begin{bmatrix} 7 \\ -6 \end{bmatrix}$

10a $\underset{\sim}{a} = \overrightarrow{OA} = \underset{\sim}{i} - \underset{\sim}{j}$

Consider O to be the origin and $\overrightarrow{OA}$ represented as the position vector.

$\underset{\sim}{b} = \overrightarrow{OB} = 7\underset{\sim}{i} + 3\underset{\sim}{j}$

Consider O to be the origin and $\overrightarrow{OB}$ represented as the position vector.

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$
$$= \left(7\underset{\sim}{i} + 3\underset{\sim}{j}\right) - \left(\underset{\sim}{i} - \underset{\sim}{j}\right)$$
$$= (7-1)\underset{\sim}{i} + (3-(-1))\underset{\sim}{j}$$
$$= 6\underset{\sim}{i} + 4\underset{\sim}{j}$$

10b $|AB| = \sqrt{x^2 + y^2}$
$$= \sqrt{6^2 + 4^2}$$
$$= \sqrt{36 + 16}$$
$$= \sqrt{52}$$
$$= 2\sqrt{13}$$

10c

$$\widehat{AB} = \frac{\overrightarrow{AB}}{|AB|}$$
$$= \frac{6\underset{\sim}{i} + 4\underset{\sim}{j}}{2\sqrt{13}}$$
$$= \frac{6}{2\sqrt{13}}\underset{\sim}{i} + \frac{4}{2\sqrt{13}}\underset{\sim}{j}$$
$$= \frac{3}{\sqrt{13}}\underset{\sim}{i} + \frac{2}{\sqrt{13}}\underset{\sim}{j}$$

11a $\underset{\sim}{a} = 2\underset{\sim}{i} + 2\underset{\sim}{j}$

$$\left|\underset{\sim}{a}\right| = \sqrt{x^2 + y^2}$$
$$= \sqrt{2^2 + 2^2}$$
$$= \sqrt{4 + 4}$$
$$= \sqrt{8}$$
$$= 2\sqrt{2}$$

$\tan\theta = \frac{y}{x} = \frac{2}{2} = 1$

x and y are both positive in 1st quadrant hence,

$\theta = \frac{\pi}{4}$

11b $\underset{\sim}{b} = \begin{bmatrix} 1 \\ -\sqrt{3} \end{bmatrix}$

$|\underset{\sim}{b}| = \sqrt{x^2 + y^2}$

$= \sqrt{1^2 + (-\sqrt{3})^2}$

$= \sqrt{1 + 3}$

$= \sqrt{4}$

$= 2$

$\tan\theta = \frac{y}{x} = \frac{-\sqrt{3}}{1} = -\sqrt{3}$

x is positive and y is negative hence θ in 4th quadrant.

$\theta = -\frac{\pi}{3}$

11c $\underset{\sim}{c} = -3\sqrt{3}\underset{\sim}{i} + 3\underset{\sim}{j}$

$|\underset{\sim}{c}| = \sqrt{x^2 + y^2}$

$= \sqrt{(-3\sqrt{3})^2 + 3^2}$

$= \sqrt{27 + 9}$

$= \sqrt{36}$

$= 6$

$\tan\theta = \frac{y}{x} = \frac{3}{-3\sqrt{3}} = -\frac{1}{\sqrt{3}}$

x is negative and y is positive hence θ in 2nd quadrant.

$\theta = \frac{5\pi}{6}$

11d $\underset{\sim}{d} = \begin{bmatrix} -\sqrt{6} \\ -\sqrt{6} \end{bmatrix}$

$|\underset{\sim}{d}| = \sqrt{x^2 + y^2}$

$= \sqrt{(-\sqrt{6})^2 + (-\sqrt{6})^2}$

$= \sqrt{6 + 6}$

$= \sqrt{12}$

$= 2\sqrt{3}$

$\tan\theta = \frac{y}{x} = \frac{-\sqrt{6}}{-\sqrt{6}} = 1$

x is negative and y is negative hence θ in 3rd quadrant.

$\theta = -\frac{3\pi}{4}$

12a Let $\underset{\sim}{a}$ be the vector.

$|\underset{\sim}{a}| = 4$

$\sqrt{x^2 + y^2} = 4$

$x^2 + y^2 = 16 \qquad (1)$

$\theta = -\frac{\pi}{4}$

$\tan\theta = -1$

θ in 4th quadrant so x is positive and y is negative.

$\frac{y}{x} = -1$

$y = -x$

Substituting the value of y in equation (1):

$x^2 + y^2 = 16$

$x^2 + (-x)^2 = 16$

$x^2 + x^2 = 16$

$2x^2 = 16$

$x^2 = 8$

$x = 2\sqrt{2}$ (as x is positive)

$y = -x = -2\sqrt{2}$

$\underset{\sim}{a} = 2\sqrt{2}\underset{\sim}{i} - 2\sqrt{2}\underset{\sim}{j}$

12b Let $\underset{\sim}{a}$ be the vector.

$|\underset{\sim}{a}| = 2\sqrt{6}$

$\sqrt{x^2 + y^2} = 2\sqrt{6}$

$x^2 + y^2 = 24$ (1)

$\theta = \frac{2\pi}{3}$

$\tan\theta = \tan\frac{2\pi}{3} = -\sqrt{3}$

θ in 2nd quadrant so x is negative and y is positive.

$\frac{y}{x} = -\sqrt{3}$

$y = -\sqrt{3}x$

Substituting the value of y in equation (1):

$x^2 + y^2 = 24$

$x^2 + (-\sqrt{3}x)^2 = 24$

$x^2 + 3x^2 = 24$

$4x^2 = 24$

$x^2 = 6$

$x = -\sqrt{6}$ (as x is negative)

$y = -\sqrt{3}x = -\sqrt{6} \times -\sqrt{3} = 3\sqrt{2}$

$\underset{\sim}{a} = -\sqrt{6}\underset{\sim}{i} + 3\sqrt{2}\underset{\sim}{j}$

12c Let $\underset{\sim}{a}$ be the vector.

$|\underset{\sim}{a}| = 2$

$\sqrt{x^2 + y^2} = 2$

$x^2 + y^2 = 4 \qquad (1)$

$\theta = -\dfrac{5\pi}{6}$

$\tan\theta = \tan\left(-\dfrac{5\pi}{6}\right) = \dfrac{1}{\sqrt{3}}$

θ in 3rd quadrant so x is negative and y is negative.

$\dfrac{y}{x} = \dfrac{1}{\sqrt{3}}$

$y = \dfrac{1}{\sqrt{3}}x$

Substituting the value of y in equation (1):

$x^2 + y^2 = 4$

$x^2 + \left(\dfrac{1}{\sqrt{3}}x\right)^2 = 4$

$x^2 + \dfrac{1}{3}x^2 = 4$

$\dfrac{4}{3}x^2 = 4$

$x^2 = 3$

$x = -\sqrt{3} \qquad$ (as x is negative)

$y = \dfrac{1}{\sqrt{3}}x = \dfrac{1}{\sqrt{3}} \times -\sqrt{3} = -1$

$\underset{\sim}{a} = -\sqrt{3}\underset{\sim}{i} - \underset{\sim}{j}$

12d Let $\underset{\sim}{a}$ be the vector.

$|\underset{\sim}{a}| = 2\sqrt{2}$

$\sqrt{x^2 + y^2} = 2\sqrt{2}$

$x^2 + y^2 = 8 \qquad (1)$

$\theta = \frac{5\pi}{12}$

$\tan\theta$

$= \tan\frac{5\pi}{12}$

$= \tan\left(\frac{2\pi}{3} - \frac{\pi}{4}\right)$

$= \frac{\tan\frac{2\pi}{3} - \tan\frac{\pi}{4}}{1 + \tan\frac{2\pi}{3}\tan\frac{\pi}{4}}$

$= \frac{-\sqrt{3} - 1}{1 + (-\sqrt{3})(1)}$

$= \frac{(-\sqrt{3} - 1)(1 + \sqrt{3})}{(1 - \sqrt{3})(1 + \sqrt{3})}$

$= \frac{-\sqrt{3} - 3 - 1 - \sqrt{3}}{-2}$

$= \frac{-4 - 2\sqrt{3}}{-2}$

$= 2 + \sqrt{3}$

θ in 1st quadrant so x is positive and y is positive.

$\frac{y}{x} = 2 + \sqrt{3}$

$y = (2 + \sqrt{3})\,x$

Substituting the value of y in equation (1):

$x^2 + y^2 = 8$

$x^2 + ((2 + \sqrt{3})x)^2 = 8$

$x = \sqrt{3} - 1$

$y = (2 + \sqrt{3})x = (2 + \sqrt{3}) \times (\sqrt{3} - 1) = \sqrt{3} + 1$

$\underset{\sim}{a} = -(\sqrt{3} - 1)\underset{\sim}{i} + (\sqrt{3} + 1)\,\underset{\sim}{j}$

13 $\underset{\sim}{a} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$ $\underset{\sim}{b} = \begin{bmatrix} -3 \\ -5 \end{bmatrix}$ $\underset{\sim}{c} = \begin{bmatrix} 24 \\ 8 \end{bmatrix}$

$\underset{\sim}{c} = \lambda_1 \underset{\sim}{a} + \lambda_2 \underset{\sim}{b}$

$\begin{bmatrix} 24 \\ 8 \end{bmatrix} = \lambda_1 \begin{bmatrix} 2 \\ -2 \end{bmatrix} + \lambda_2 \begin{bmatrix} -3 \\ -5 \end{bmatrix}$

$\begin{bmatrix} 24 \\ 8 \end{bmatrix} = \begin{bmatrix} 2\lambda_1 \\ -2\lambda_1 \end{bmatrix} + \begin{bmatrix} -3\lambda_2 \\ -5\lambda_2 \end{bmatrix}$

$\begin{bmatrix} 24 \\ 8 \end{bmatrix} = \begin{bmatrix} 2\lambda_1 - 3\lambda_2 \\ -2\lambda_1 - 5\lambda_2 \end{bmatrix}$

Hence:

$2\lambda_1 - 3\lambda_2 = 24 \quad (1)$

$-2\lambda_1 - 5\lambda_2 = 8 \quad (2)$

Adding the equations (1) and (2) we get:

$-8\lambda_2 = 32$

$\lambda_2 = -4$

Substituting the value of λ_2 in equation (1) we get:

$2\lambda_1 - 3\lambda_2 = 24$

$2\lambda_1 - 3 \times -4 = 24$

$2\lambda_1 + 12 = 24$

$2\lambda_1 = 12$

$\lambda_1 = 6$

14a Let O be the origin.

$\overrightarrow{OA} = \begin{bmatrix} 2\sqrt{3} \\ 3 \end{bmatrix}$

$\overrightarrow{OB} = \begin{bmatrix} 3\sqrt{3} \\ 4 \end{bmatrix}$

$\overrightarrow{OC} = \begin{bmatrix} 2\sqrt{3} \\ 5 \end{bmatrix}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{bmatrix} 3\sqrt{3} - 2\sqrt{3} \\ 4 - 3 \end{bmatrix} = \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix}$

$\overrightarrow{CB} = \overrightarrow{OB} - \overrightarrow{OC} = \begin{bmatrix} 3\sqrt{3} - 2\sqrt{3} \\ 4 - 5 \end{bmatrix} = \begin{bmatrix} \sqrt{3} \\ -1 \end{bmatrix}$

14b $|\overrightarrow{AB}| = \sqrt{(\sqrt{3})^2 + 1^2}$

$= \sqrt{3+1}$

$= \sqrt{4}$

$= 2$

$|\overrightarrow{CB}| = \sqrt{(\sqrt{3})^2 + (-1)^2}$

$= \sqrt{3+1}$

$= \sqrt{4}$

$= 2$

14c $\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \begin{bmatrix} 2\sqrt{3} - 2\sqrt{3} \\ 5-3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

$|\overrightarrow{AC}| = \sqrt{0^2 + 2^2}$

$= \sqrt{0+4}$

$= \sqrt{4}$

$= 2$

$|\overrightarrow{AB}| = |\overrightarrow{CB}| = |\overrightarrow{AC}|$

Hence, ABC is an equilateral triangle as each side has a length of 2.

15 Let O be the origin.

$\overrightarrow{OA} = \begin{bmatrix} -7 \\ -5 \end{bmatrix}$

$\overrightarrow{OB} = \begin{bmatrix} 8 \\ -2 \end{bmatrix}$

$\overrightarrow{OC} = \begin{bmatrix} 10 \\ 9 \end{bmatrix}$

$\overrightarrow{OD} = \begin{bmatrix} -5 \\ 6 \end{bmatrix}$

$\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = \begin{bmatrix} -5-(-7) \\ 6-(-5) \end{bmatrix} = \begin{bmatrix} 2 \\ 11 \end{bmatrix}$

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \begin{bmatrix} 10-8 \\ 9-(-2) \end{bmatrix} = \begin{bmatrix} 2 \\ 11 \end{bmatrix}$

Hence, $\overrightarrow{AD} = \overrightarrow{BC}$.

Therefore, $ABCD$ is a parallelogram as opposite sides are equal and parallel.

16a Let O be the origin.

$\overrightarrow{OP} = \begin{bmatrix} -3 \\ -4 \end{bmatrix}$

$\overrightarrow{OQ} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$

$\overrightarrow{OR} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$

$\overrightarrow{OS} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \begin{bmatrix} 2-(-3) \\ -2-(-4) \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$

$\overrightarrow{SR} = \overrightarrow{OR} - \overrightarrow{OS} = \begin{bmatrix} 4-(-1) \\ 3-1 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$

Hence, $\overrightarrow{PQ} = \overrightarrow{SR}$.

16b $|\overrightarrow{PQ}| = \sqrt{5^2 + 2^2}$

$= \sqrt{25 + 4}$

$= \sqrt{29}$

$\overrightarrow{QR} = \overrightarrow{OR} - \overrightarrow{OQ} = \begin{bmatrix} 4-2 \\ 3-(-2) \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$

$|\overrightarrow{QR}| = \sqrt{2^2 + 5^2}$

$= \sqrt{4 + 25}$

$= \sqrt{29}$

Hence, $|\overrightarrow{PQ}| = |\overrightarrow{QR}|$.

16c $PQRS$ is a rhombus as opposite sides are parallel: $\overrightarrow{PQ} = \overrightarrow{SR}$ and adjacent sides are equal $|\overrightarrow{PQ}| = |\overrightarrow{QR}|$.

17a $\overrightarrow{OA} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$

$\overrightarrow{OB} = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$

$\overrightarrow{OC} = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \begin{bmatrix} 2-5 \\ 7-(-3) \end{bmatrix} = \begin{bmatrix} -3 \\ 10 \end{bmatrix}$

$\frac{1}{2}\overrightarrow{AC} = \begin{bmatrix} -\frac{3}{2} \\ 5 \end{bmatrix}$

$\overrightarrow{OA} + \frac{1}{2}\overrightarrow{AC}$

$= \begin{bmatrix} 5 \\ -3 \end{bmatrix} + \begin{bmatrix} -\frac{3}{2} \\ 5 \end{bmatrix}$

$= \begin{bmatrix} 5+(-\frac{3}{2}) \\ -3+5 \end{bmatrix}$

$= \begin{bmatrix} \frac{7}{2} \\ 2 \end{bmatrix}$

$\overrightarrow{OB} = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$

$\frac{1}{2}\overrightarrow{OB} = \begin{bmatrix} \frac{7}{2} \\ 2 \end{bmatrix}$

Hence, $\overrightarrow{OA} + \frac{1}{2}\overrightarrow{AC} = \frac{1}{2}\overrightarrow{OB}$.

17b $OABC$ is a parallelogram as diagonals bisect each other.

18 Let O be the origin hence,

$\overrightarrow{OW} = \begin{bmatrix} -6 \\ 4 \end{bmatrix}$

$\overrightarrow{OX} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$

$\overrightarrow{OY} = \begin{bmatrix} 4 \\ 9 \end{bmatrix}$

$\overrightarrow{OZ} = \begin{bmatrix} a \\ b \end{bmatrix}$

$\overrightarrow{WZ} = \overrightarrow{OZ} - \overrightarrow{OW} = \begin{bmatrix} a-(-6) \\ b-4 \end{bmatrix} = \begin{bmatrix} a+6 \\ b-4 \end{bmatrix}$

$\overrightarrow{XY} = \overrightarrow{OY} - \overrightarrow{OX} = \begin{bmatrix} 4-6 \\ 9-2 \end{bmatrix} = \begin{bmatrix} -2 \\ 7 \end{bmatrix}$

As $WXYZ$ is a parallelogram, opposite sides are equal hence,

$\overrightarrow{WZ} = \overrightarrow{XY}$

$\begin{bmatrix} a+6 \\ b-4 \end{bmatrix} = \begin{bmatrix} -2 \\ 7 \end{bmatrix}$

$a + 6 = -2$

$a = -8$

$b - 4 = 7$

$b = 11$

19 The three possible position vectors representing the point D are $\underset{\sim}{a} + \underset{\sim}{b} - \underset{\sim}{c}$, $\underset{\sim}{b} + \underset{\sim}{c} - \underset{\sim}{a}$ and $\underset{\sim}{c} + \underset{\sim}{a} - \underset{\sim}{b}$.

Solutions to Exercise 8C

1a $\quad \underset{\sim}{a} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ $\underset{\sim}{b} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

$\underset{\sim}{a} \cdot \underset{\sim}{b} = 3 \times 2 + 1 \times 4 = 6 + 4 = 10$

1b $\quad \underset{\sim}{a} = \begin{bmatrix} -8 \\ -5 \end{bmatrix}$ $\underset{\sim}{b} = \begin{bmatrix} 6 \\ -14 \end{bmatrix}$

$\underset{\sim}{a} \cdot \underset{\sim}{b} = (-8) \times 6 + (-5) \times (-14) = -48 + 70 = 22$

1c $\quad \underset{\sim}{a} = \begin{bmatrix} 6u \\ -2v \end{bmatrix}$ $\underset{\sim}{b} = \begin{bmatrix} 3v \\ 9u \end{bmatrix}$

$\underset{\sim}{a} \cdot \underset{\sim}{b} = 6u \times 3v + (-2v) \times 9u = 18uv - 18uv = 0$

1d $\quad \underset{\sim}{a} = \begin{bmatrix} x-1 \\ x-2 \end{bmatrix}$ $\underset{\sim}{b} = \begin{bmatrix} x-1 \\ x+2 \end{bmatrix}$

$\underset{\sim}{a} \cdot \underset{\sim}{b} = (x-1) \times (x-1) + (x-2) \times (x+2)$

$= (x-1)^2 + x^2 - 4$

$= x^2 - 2x + 1 + x^2 - 4$

$= 2x^2 - 2x - 3$

2a $\quad |\underset{\sim}{a}| = 6 \qquad |\underset{\sim}{b}| = 5 \qquad \theta = 60°$

$\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}| \times |\underset{\sim}{b}| \cos\theta$

$= 6 \times 5 \cos 60°$

$= 6 \times 5 \times \frac{1}{2}$

$= 15$

2b $\quad |\underset{\sim}{a}| = 4 \qquad |\underset{\sim}{b}| = 3 \qquad \theta = 45°$

$\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}| \times |\underset{\sim}{b}| \cos\theta$

$= 4 \times 3 \cos 45°$

$= 4 \times 3 \times \frac{1}{\sqrt{2}}$

$= 12 \times \frac{1}{\sqrt{2}}$

$= 6\sqrt{2}$

3a $|\underset{\sim}{u}| = 4$ $|\underset{\sim}{v}| = 5$ $\underset{\sim}{u} \cdot \underset{\sim}{v} = -10$

$\underset{\sim}{u} \cdot \underset{\sim}{v} = |\underset{\sim}{u}| \times |\underset{\sim}{v}| \cos\theta$

$-10 = 4 \times 5 \cos\theta$

$20 \cos\theta = -10$

$\cos\theta = -\frac{1}{2}$

$\theta = 120°$

3b $|\underset{\sim}{u}| = 3$ $|\underset{\sim}{v}| = 5$ $\underset{\sim}{u} \cdot \underset{\sim}{v} = 12$

$\underset{\sim}{u} \cdot \underset{\sim}{v} = |\underset{\sim}{u}| \times |\underset{\sim}{v}| \cos\theta$

$12 = 3 \times 5 \cos\theta$

$15 \cos\theta = 12$

$\cos\theta = \frac{4}{5} = 0.8$

$\theta \doteqdot 37°$

4a $4\underset{\sim}{i} \cdot 2\underset{\sim}{j} = 8 \times \underset{\sim}{i} \cdot \underset{\sim}{j} = 8 \times 0 = 0$

4b $-5\underset{\sim}{i} \cdot 3\underset{\sim}{j} = -15 \times \underset{\sim}{i} \cdot \underset{\sim}{j} = -15 \times 0 = 0$

4c $4\underset{\sim}{i} \cdot 2\underset{\sim}{i} = 8 \times \underset{\sim}{i} \cdot \underset{\sim}{i} = 8 \times 1 = 8$

4d $-5\underset{\sim}{j} \cdot 3\underset{\sim}{j} = -15 \times \underset{\sim}{j} \cdot \underset{\sim}{j} = -15 \times 1 = -15$

5a $(4\underset{\sim}{i} + 2\underset{\sim}{j}) \cdot (4\underset{\sim}{i} + 2\underset{\sim}{j})$

$= 4\underset{\sim}{i} \cdot 4\underset{\sim}{i} + 4\underset{\sim}{i} \cdot 2\underset{\sim}{j} + 2\underset{\sim}{j} \cdot 4\underset{\sim}{i} + 2\underset{\sim}{j} \cdot 2\underset{\sim}{j}$

$= 16\underset{\sim}{i} \cdot \underset{\sim}{i} + 8\underset{\sim}{i} \cdot \underset{\sim}{j} + 8\underset{\sim}{j} \cdot \underset{\sim}{i} + 4\underset{\sim}{j} \cdot \underset{\sim}{j}$

$= 16 \times 1 + 8 \times 0 + 8 \times 0 + 4 \times 1$

$= 16 + 4$

$= 20$

5b $(-5\underset{\sim}{i} + 3\underset{\sim}{j}) \cdot (-5\underset{\sim}{i} + 3\underset{\sim}{j})$

$= (-5)\underset{\sim}{i} \cdot (-5)\underset{\sim}{i} + (-5)\underset{\sim}{i} \cdot 3\underset{\sim}{j} + 3\underset{\sim}{j} \cdot (-5)\underset{\sim}{i} + 3\underset{\sim}{j} \cdot 3\underset{\sim}{j}$

$= 25\underset{\sim}{i} \cdot \underset{\sim}{i} - 15\underset{\sim}{i} \cdot \underset{\sim}{j} - 15\underset{\sim}{j} \cdot \underset{\sim}{i} + 9\underset{\sim}{j} \cdot \underset{\sim}{j}$

$= 25 \times 1 - 15 \times 0 - 15 \times 0 + 9 \times 1$

$= 25 + 9$

$= 34$

5c $(4\underset{\sim}{i} + 2\underset{\sim}{j}) \cdot (-5\underset{\sim}{i} + 3\underset{\sim}{j})$

$= 4\underset{\sim}{i} \cdot (-5)\underset{\sim}{i} + 4\underset{\sim}{i} \cdot 3\underset{\sim}{j} + 2\underset{\sim}{j} \cdot (-5)\underset{\sim}{i} + 2\underset{\sim}{j} \cdot 3\underset{\sim}{j}$

$= -20\underset{\sim}{i} \cdot \underset{\sim}{i} + 12\underset{\sim}{i} \cdot \underset{\sim}{j} - 10\underset{\sim}{j} \cdot \underset{\sim}{i} + 6\underset{\sim}{j} \cdot \underset{\sim}{j}$

$= -20 \times 1 + 12 \times 0 - 10 \times 0 + 6 \times 1$

$= -20 + 6$

$= -14$

6a $\underset{\sim}{u} = \begin{bmatrix} -4 \\ 5 \end{bmatrix}$ $\underset{\sim}{v} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$

$\underset{\sim}{u} \cdot \underset{\sim}{v} = (-4) \times 7 + 5 \times 6$

$= -28 + 30$

$= 2$

$\underset{\sim}{u}$ and $\underset{\sim}{v}$ are not perpendicular as $\underset{\sim}{u} \cdot \underset{\sim}{v} \neq 0$.

6b $\underset{\sim}{u} = \begin{bmatrix} -4 \\ -6 \end{bmatrix}$ $\underset{\sim}{v} = \begin{bmatrix} 18 \\ -12 \end{bmatrix}$

$\underset{\sim}{u} \cdot \underset{\sim}{v} = (-4) \times 18 + (-6) \times (-12)$

$= -72 + 72$

$= 0$

$\underset{\sim}{u}$ and $\underset{\sim}{v}$ are perpendicular as $\underset{\sim}{u} \cdot \underset{\sim}{v} = 0$.

6c $\underset{\sim}{u} = \begin{bmatrix} -1 \\ a^{-2} \end{bmatrix}$ $\underset{\sim}{v} = \begin{bmatrix} a^{-1} \\ a \end{bmatrix}$

$\underset{\sim}{u} \cdot \underset{\sim}{v} = (-1) \times a^{-1} + a^{-2} \times a$

$= -a^{-1} + a^{-1}$

$= 0$

$\underset{\sim}{u}$ and $\underset{\sim}{v}$ are perpendicular as $\underset{\sim}{u} \cdot \underset{\sim}{v} = 0$.

7a Consider O to be the origin and $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$ represent position vectors.

Hence,

$\overrightarrow{OA} = 2\underset{\sim}{i} + 5\underset{\sim}{j}$ $\overrightarrow{OB} = 5\underset{\sim}{i} + 14\underset{\sim}{j}$ and $\overrightarrow{OC} = -2\underset{\sim}{i} + 13\underset{\sim}{j}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= \left(5\underset{\sim}{i} + 14\underset{\sim}{j}\right) - \left(2\underset{\sim}{i} + 5\underset{\sim}{j}\right)$

$= (5 - 2)\underset{\sim}{i} + (14 - 5)\underset{\sim}{j}$

$= 3\underset{\sim}{i} + 9\underset{\sim}{j}$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$

$= \left(-2\underset{\sim}{i} + 13\underset{\sim}{j}\right) - \left(2\underset{\sim}{i} + 5\underset{\sim}{j}\right)$

$= ((-2) - 2)\underset{\sim}{i} + (13 - 5)\underset{\sim}{j}$

$= -4\underset{\sim}{i} + 8\underset{\sim}{j}$

7b $\overrightarrow{AB} \cdot \overrightarrow{AC} = 3 \times (-4) + 9 \times 8$

$= -12 + 72$

$= 60$

7c $|\overrightarrow{AB}| = \sqrt{x^2 + y^2}$

$= \sqrt{3^2 + 9^2}$

$= \sqrt{9 + 81}$

$= \sqrt{90}$

$= 3\sqrt{10}$

$|\overrightarrow{AC}| = \sqrt{x^2 + y^2}$

$= \sqrt{(-4)^2 + 8^2}$

$= \sqrt{16 + 64}$

$= \sqrt{80}$

$= 4\sqrt{5}$

$\overrightarrow{AB} \cdot \overrightarrow{AC} = |\overrightarrow{AB}| \times |\overrightarrow{AC}| \cos\theta$

$= 3\sqrt{10} \times 4\sqrt{5} \cos 45°$

$= 3\sqrt{10} \times 4\sqrt{5} \times \frac{1}{\sqrt{2}}$

$= 60\sqrt{2} \times \frac{1}{\sqrt{2}}$

$= 60$

Hence, it is confirmed with part b above.

8a Consider O to be the origin and $\overrightarrow{OP}$, $\overrightarrow{OQ}$ and $\overrightarrow{OR}$ represent position vectors.

Hence,

$\overrightarrow{OP} = \sqrt{3}\underset{\sim}{i} + 8\underset{\sim}{j}$ $\overrightarrow{OQ} = 3\sqrt{3}\underset{\sim}{i} + 14\underset{\sim}{j}$ and $\overrightarrow{OR} = 5\sqrt{3}\underset{\sim}{i} + 12\underset{\sim}{j}$

$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$

$= \left(3\sqrt{3}\underset{\sim}{i} + 14\underset{\sim}{j}\right) - \left(\sqrt{3}\underset{\sim}{i} + 8\underset{\sim}{j}\right)$

$= (3\sqrt{3} - \sqrt{3})\underset{\sim}{i} + (14 - 8)\underset{\sim}{j}$

$= 2\sqrt{3}\underset{\sim}{i} + 6\underset{\sim}{j}$

$$\overrightarrow{PR} = \overrightarrow{OR} - \overrightarrow{OP}$$
$$= \left(5\sqrt{3}\underset{\sim}{i} + 12\underset{\sim}{j}\right) - \left(\sqrt{3}\underset{\sim}{i} + 8\underset{\sim}{j}\right)$$
$$= (5\sqrt{3} - \sqrt{3})\underset{\sim}{i} + (12 - 8)\underset{\sim}{j}$$
$$= 4\sqrt{3}\underset{\sim}{i} + 4\underset{\sim}{j}$$

8b $$\overrightarrow{PQ} \cdot \overrightarrow{PR} = 2\sqrt{3} \times 4\sqrt{3} + 6 \times 4$$
$$= 24 + 24$$
$$= 48$$

8c $$\left|\overrightarrow{PQ}\right| = \sqrt{x^2 + y^2}$$
$$= \sqrt{(2\sqrt{3})^2 + 6^2}$$
$$= \sqrt{12 + 36}$$
$$= \sqrt{48}$$
$$= 4\sqrt{3}$$
$$\left|\overrightarrow{PR}\right| = \sqrt{x^2 + y^2}$$
$$= \sqrt{(4\sqrt{3})^2 + 4^2}$$
$$= \sqrt{48 + 16}$$
$$= \sqrt{64}$$
$$= 8$$

$$\overrightarrow{PQ} \cdot \overrightarrow{PR} = \left|\overrightarrow{PQ}\right| \times \left|\overrightarrow{PR}\right| \cos\theta$$
$$= 4\sqrt{3} \times 8 \cos 30°$$
$$= 4\sqrt{3} \times 8 \times \frac{\sqrt{3}}{2}$$
$$= 32\sqrt{3} \times \frac{\sqrt{3}}{2}$$
$$= 48$$

Hence, it is confirmed with part b above.

9a $\underset{\sim}{a} = 4\underset{\sim}{i} + 3\underset{\sim}{j} \quad \underset{\sim}{b} = 5\underset{\sim}{j}$

$\underset{\sim}{a} \cdot \underset{\sim}{b} = (4\underset{\sim}{i} + 3\underset{\sim}{j}) \cdot (5\underset{\sim}{j})$

$= 4\underset{\sim}{i} \cdot 0 + 4\underset{\sim}{i} \cdot 5\underset{\sim}{j} + 3\underset{\sim}{j} \cdot 0 + 3\underset{\sim}{j} \cdot 5\underset{\sim}{j}$

$= 0 + 20\underset{\sim}{i} \cdot \underset{\sim}{j} + 0 + 15\underset{\sim}{j} \cdot \underset{\sim}{j}$

$= 0 + 20 \times 0 + 0 + 15 \times 1$

$= 15$

$|\underset{\sim}{a}| = \sqrt{x^2 + y^2}$

$= \sqrt{4^2 + 3^2}$

$= \sqrt{16 + 9}$

$= \sqrt{25}$

$= 5$

$|\underset{\sim}{b}| = \sqrt{x^2 + y^2}$

$= \sqrt{5^2}$

$= \sqrt{25}$

$= 5$

$\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}| \times |b| \cos\theta$

$15 = 5 \times 5 \cos\theta$

$25 \cos\theta = 15$

$\cos\theta = \frac{15}{25} = \frac{3}{5}$

9b $\underset{\sim}{a} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \ \underset{\sim}{v} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

$\underset{\sim}{a} \cdot \underset{\sim}{b} = 2 \times 3 + 2 \times (-1)$

$= 6 - 2$

$= 4$

$$|\underset{\sim}{a}| = \sqrt{x^2 + y^2}$$

$$= \sqrt{2^2 + 2^2}$$

$$= \sqrt{4 + 4}$$

$$= \sqrt{8}$$

$$= 2\sqrt{2}$$

$$|\underset{\sim}{b}| = \sqrt{x^2 + y^2}$$

$$= \sqrt{3^2 + (-1)^2}$$

$$= \sqrt{9 + 1}$$

$$= \sqrt{10}$$

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}| \times |b| \cos\theta$$

$$4 = 2\sqrt{2} \times \sqrt{10} \cos\theta$$

$$4\sqrt{5}\cos\theta = 4$$

$$\cos\theta = \frac{4}{4\sqrt{5}} = \frac{1}{\sqrt{5}}$$

9c $\underset{\sim}{a} = \begin{bmatrix} -6 \\ 4 \end{bmatrix}$ $\underset{\sim}{v} = \begin{bmatrix} -8 \\ -2 \end{bmatrix}$

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = (-6) \times (-8) + 4 \times (-2)$$

$$= 48 - 8$$

$$= 40$$

$$|\underset{\sim}{a}| = \sqrt{x^2 + y^2}$$

$$= \sqrt{(-6)^2 + 4^2}$$

$$= \sqrt{36 + 16}$$

$$= \sqrt{52}$$

$$= 2\sqrt{13}$$

$$|\underset{\sim}{b}| = \sqrt{x^2 + y^2}$$

$$= \sqrt{(-8)^2 + (-2)^2}$$

$$= \sqrt{64 + 4}$$

$= \sqrt{68}$

$= 2\sqrt{17}$

$\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}| \times |b| \cos\theta$

$40 = 2\sqrt{13} \times 2\sqrt{17} \cos\theta$

$4\sqrt{221} \cos\theta = 40$

$\cos\theta = \dfrac{40}{4\sqrt{221}} = \dfrac{10}{\sqrt{221}}$

10 $\underset{\sim}{u} = \lambda^2 \underset{\sim}{i} + 2\underset{\sim}{j} \quad \underset{\sim}{v} = 3\underset{\sim}{i} - (2 + 2\lambda)\underset{\sim}{j}$

$\underset{\sim}{u}$ and $\underset{\sim}{v}$ are perpendicular.

Hence,

$\underset{\sim}{u} \cdot \underset{\sim}{v} = 0$

$\left(\lambda^2 \underset{\sim}{i} + 2\underset{\sim}{j}\right) \cdot \left(3\underset{\sim}{i} - (2 + 2\lambda)\underset{\sim}{j}\right) = 0$

$\lambda^2 \underset{\sim}{i} \cdot 3\underset{\sim}{i} - \lambda^2 \underset{\sim}{i} \cdot (2 + 2\lambda)\underset{\sim}{j} + 2\underset{\sim}{j} \cdot 3\underset{\sim}{i} - 2\underset{\sim}{j} \cdot (2 + 2\lambda)\underset{\sim}{j} = 0$

$3\lambda^2 \underset{\sim}{i} \cdot \underset{\sim}{i} - (2 + 2\lambda)\lambda^2 \underset{\sim}{i} \cdot \underset{\sim}{j} + 6\underset{\sim}{j} \cdot \underset{\sim}{i} - (2 + 2\lambda)2\underset{\sim}{j} \cdot \underset{\sim}{j} = 0$

$3\lambda^2 \times 1 + (2 + 2\lambda)\lambda^2 \times 0 + 6 \times 0 - (2 + 2\lambda)2 \times 1 = 0$

$3\lambda^2 - (2 + 2\lambda)2 = 0$

$3\lambda^2 - 4 - 4\lambda = 0$

$3\lambda^2 - 4\lambda - 4 = 0$

$(3\lambda + 2)(\lambda - 2) = 0$

$\lambda = -\dfrac{2}{3}$ or $\lambda = 2$

11a i $|\underset{\sim}{a}| = 6 \qquad |\underset{\sim}{b}| = 2 \qquad \theta = \frac{\pi}{3}$

$\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}| \times |\underset{\sim}{b}| \cos\theta$

$= 6 \times 2 \cos\dfrac{\pi}{3}$

$= 12 \times \dfrac{1}{2}$

$= 6$

11a ii $|\underset{\sim}{a}| = 6$ $\quad |\underset{\sim}{b}| = 2$ $\quad \theta = \frac{\pi}{3}$

$$2\underset{\sim}{a} \cdot (-5)\underset{\sim}{b} = -10\underset{\sim}{a} \cdot \underset{\sim}{b}$$

$$= -10 \times 6$$

$$= -60$$

11a iii $|\underset{\sim}{a}| = 6$ $\quad |\underset{\sim}{b}| = 2$ $\quad \theta = \frac{\pi}{3}$

$$4\underset{\sim}{a} \cdot 0\underset{\sim}{b} = 0\underset{\sim}{a} \cdot \underset{\sim}{b}$$

$$= 0 \times 6$$

$$= 0$$

11a iv $|\underset{\sim}{a}| = 6$ $\quad |\underset{\sim}{b}| = 2$ $\quad \theta = \frac{\pi}{3}$

$$\underset{\sim}{a} \cdot (\underset{\sim}{a} + \underset{\sim}{b}) = \underset{\sim}{a} \cdot \underset{\sim}{a} + \underset{\sim}{a} \cdot \underset{\sim}{b}$$

$$= |\underset{\sim}{a}|^2 + |\underset{\sim}{a}| \times |\underset{\sim}{b}| \cos\theta$$

$$= 6^2 + 6 \times 2 \cos\frac{\pi}{3}$$

$$= 36 + 6 \times 2 \times \frac{1}{2}$$

$$= 36 + 6$$

$$= 42$$

11a v $|\underset{\sim}{a}| = 6$ $\quad |\underset{\sim}{b}| = 2$ $\quad \theta = \frac{\pi}{3}$

$$\underset{\sim}{b} \cdot (\underset{\sim}{a} + \underset{\sim}{b}) = \underset{\sim}{b} \cdot \underset{\sim}{a} + \underset{\sim}{b} \cdot \underset{\sim}{b}$$

$$= |\underset{\sim}{b}| \times |\underset{\sim}{a}| \cos\theta + |\underset{\sim}{b}|^2$$

$$= 2 \times 6 \cos\frac{\pi}{3} + 2^2$$

$$= 2 \times 6 \times \frac{1}{2} + 4$$

$= 4 + 6$

$= 10$

11a vi $|\underset{\sim}{a}| = 6$ $\quad |\underset{\sim}{b}| = 2$ $\quad \theta = \frac{\pi}{3}$

$(\underset{\sim}{a} - \underset{\sim}{b}) \cdot (\underset{\sim}{a} + \underset{\sim}{b})$

$= \underset{\sim}{a} \cdot \underset{\sim}{a} + \underset{\sim}{a} \cdot \underset{\sim}{b} - \underset{\sim}{b} \cdot \underset{\sim}{a} - \underset{\sim}{b} \cdot \underset{\sim}{b}$

$= \underset{\sim}{a} \cdot \underset{\sim}{a} - \underset{\sim}{b} \cdot \underset{\sim}{b}$

$= |\underset{\sim}{a}|^2 - |\underset{\sim}{b}|^2$

$= 6^2 - 2^2$

$= 36 - 4$

$= 32$

11b i $|\underset{\sim}{a}| = 6$ $\quad |\underset{\sim}{b}| = 2$ $\quad \theta = \frac{2\pi}{3}$

$\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}| \times |\underset{\sim}{b}| \cos \theta$

$= 6 \times 2 \cos \frac{2\pi}{3}$

$= 12 \times -\frac{1}{2}$

$= -6$

11b ii $|\underset{\sim}{a}| = 6$ $\quad |\underset{\sim}{b}| = 2$ $\quad \theta = \frac{2\pi}{3}$

$2\underset{\sim}{a} \cdot (-5)\underset{\sim}{b} = -10\underset{\sim}{a} \cdot \underset{\sim}{b}$

$= -10 \times -6$

$= 60$

11b iii $|\underset{\sim}{a}| = 6$ $\quad |\underset{\sim}{b}| = 2$ $\quad \theta = \frac{2\pi}{3}$

$4\underset{\sim}{a} \cdot 0\underset{\sim}{b} = 0\underset{\sim}{a} \cdot \underset{\sim}{b}$

$= 0 \times -6$

$= 0$

11b iv $|\underset{\sim}{a}| = 6 \qquad |\underset{\sim}{b}| = 2 \qquad \theta = \frac{2\pi}{3}$

$\underset{\sim}{a} \cdot (\underset{\sim}{a} + \underset{\sim}{b})$

$= \underset{\sim}{a} \cdot \underset{\sim}{a} + \underset{\sim}{a} \cdot \underset{\sim}{b}$

$= |\underset{\sim}{a}|^2 + |\underset{\sim}{a}| \times |\underset{\sim}{b}| \cos\theta$

$= 6^2 + 6 \times 2 \cos\frac{2\pi}{3}$

$= 36 + 6 \times 2 \times -\frac{1}{2}$

$= 36 - 6$

$= 30$

11b v $|\underset{\sim}{a}| = 6 \qquad |\underset{\sim}{b}| = 2 \qquad \theta = \frac{2\pi}{3}$

$\underset{\sim}{b} \cdot (\underset{\sim}{a} + \underset{\sim}{b})$

$= \underset{\sim}{b} \cdot \underset{\sim}{a} + \underset{\sim}{b} \cdot \underset{\sim}{b}$

$= |\underset{\sim}{b}|^2 + |\underset{\sim}{b}| \times |\underset{\sim}{a}| \cos\theta$

$= 2^2 + 2 \times 6 \cos\frac{2\pi}{3}$

$= 4 + 2 \times 6 \times -\frac{1}{2}$

$= 4 - 6$

$= -2$

11b vi $|\underset{\sim}{a}| = 6 \qquad |\underset{\sim}{b}| = 2 \qquad \theta = \frac{2\pi}{3}$

$(\underset{\sim}{a} - \underset{\sim}{b}) \cdot (\underset{\sim}{a} + \underset{\sim}{b})$

$= \underset{\sim}{a} \cdot \underset{\sim}{a} + \underset{\sim}{a} \cdot \underset{\sim}{b} - \underset{\sim}{b} \cdot \underset{\sim}{a} - \underset{\sim}{b} \cdot \underset{\sim}{b}$

$= \underset{\sim}{a} \cdot \underset{\sim}{a} - \underset{\sim}{b} \cdot \underset{\sim}{b}$

$= |\underset{\sim}{a}|^2 - |\underset{\sim}{b}|^2$

$= 6^2 - 2^2$

$= 36 - 4$

$= 32$

11c i $|\underset{\sim}{a}| = 6$ $\quad |\underset{\sim}{b}| = 2$ $\quad \theta = \frac{\pi}{2}$

$\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}| \times |\underset{\sim}{b}| \cos\theta$

$= 6 \times 2\cos\frac{\pi}{2}$

$= 12 \times 0$

$= 0$

11c ii $|\underset{\sim}{a}| = 6$ $\quad |\underset{\sim}{b}| = 2$ $\quad \theta = \frac{\pi}{2}$

$2\underset{\sim}{a} \cdot (-5)\underset{\sim}{b} = -10\underset{\sim}{a} \cdot \underset{\sim}{b}$

$= -10 \times 0$

$= 0$

11c iii $|\underset{\sim}{a}| = 6$ $\quad |\underset{\sim}{b}| = 2$ $\quad \theta = \frac{\pi}{2}$

$4\underset{\sim}{a} \cdot 0\underset{\sim}{b} = 0\underset{\sim}{a} \cdot \underset{\sim}{b}$

$= 0 \times 0$

$= 0$

11c iv $|\underset{\sim}{a}| = 6$ $\quad |\underset{\sim}{b}| = 2$ $\quad \theta = \frac{\pi}{2}$

$\underset{\sim}{a} \cdot (\underset{\sim}{a} + \underset{\sim}{b})$

$= \underset{\sim}{a} \cdot \underset{\sim}{a} + \underset{\sim}{a} \cdot \underset{\sim}{b}$

$= |\underset{\sim}{a}|^2 + |\underset{\sim}{a}| \times |\underset{\sim}{b}| \cos\theta$

$= 6^2 + 6 \times 2\cos\frac{\pi}{2}$

$= 36 + 6 \times 2 \times 0$

$= 36 + 0$

$= 36$

11c v $|\underset{\sim}{a}| = 6$ $|\underset{\sim}{b}| = 2$ $\theta = \frac{\pi}{2}$

$\underset{\sim}{b} \cdot (\underset{\sim}{a} + \underset{\sim}{b})$

$= \underset{\sim}{b} \cdot \underset{\sim}{a} + \underset{\sim}{b} \cdot \underset{\sim}{b}$

$= |\underset{\sim}{b}| \times |\underset{\sim}{a}| \cos\theta + |\underset{\sim}{b}|^2$

$= 2 \times 6 \cos\frac{\pi}{2} + 2^2$

$= 2 \times 6 \times 0 + 4$

$= 0 + 4$

$= 4$

11c vi $|\underset{\sim}{a}| = 6$ $|\underset{\sim}{b}| = 2$ $\theta = \frac{\pi}{3}$

$(\underset{\sim}{a} - \underset{\sim}{b}) \cdot (\underset{\sim}{a} + \underset{\sim}{b})$

$= \underset{\sim}{a} \cdot \underset{\sim}{a} + \underset{\sim}{a} \cdot \underset{\sim}{b} - \underset{\sim}{b} \cdot \underset{\sim}{a} - \underset{\sim}{b} \cdot \underset{\sim}{b}$

$= \underset{\sim}{a} \cdot \underset{\sim}{a} - \underset{\sim}{b} \cdot \underset{\sim}{b}$

$= |\underset{\sim}{a}|^2 - |\underset{\sim}{b}|^2$

$= 6^2 - 2^2$

$= 36 - 4$

$= 32$

12a Consider O to be the origin and $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ and $\overrightarrow{OD}$ represent position vectors.

Hence,

$\overrightarrow{OA} = -3\underset{\sim}{i} - 6\underset{\sim}{j}$ $\overrightarrow{OB} = \underset{\sim}{i} - 4\underset{\sim}{j}$ $\overrightarrow{OC} = -2\underset{\sim}{i} + 2\underset{\sim}{j}$ and $\overrightarrow{OD} = -6\underset{\sim}{i}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= (\underset{\sim}{i} - 4\underset{\sim}{j}) - (-3\underset{\sim}{i} - 6\underset{\sim}{j})$

$= (1 - (-3))\underset{\sim}{i} + ((-4) - (-6))\underset{\sim}{j}$

$= (1 + 3)\underset{\sim}{i} + (-4 + 6)\underset{\sim}{j}$

$= 4\underset{\sim}{i} + 2\underset{\sim}{j}$

$\overrightarrow{DC} = \overrightarrow{OC} - \overrightarrow{OD}$

$= \left(-2\underset{\sim}{i} + 2\underset{\sim}{j}\right) - \left(-6\underset{\sim}{i}\right)$

$= ((-2) - (-6))\underset{\sim}{i} + 2\underset{\sim}{j}$

$= (-2 + 6)\underset{\sim}{i} + 2\underset{\sim}{j}$

$= 4\underset{\sim}{i} + 2\underset{\sim}{j}$

Hence, $\overrightarrow{AB} = \overrightarrow{DC}$.

12b $\overrightarrow{AB} = 4\underset{\sim}{i} + 2\underset{\sim}{j}$

$\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA}$

$= \left(-6\underset{\sim}{i}\right) - \left(-3\underset{\sim}{i} - 6\underset{\sim}{j}\right)$

$= ((-6) - (-3))\underset{\sim}{i} - (-6)\underset{\sim}{j}$

$= (-6 + 3)\underset{\sim}{i} + 6\underset{\sim}{j}$

$= -3\underset{\sim}{i} + 6\underset{\sim}{j}$

By using the method, $\underset{\sim}{u} \cdot \underset{\sim}{v} = x_1 \times x_2 + y_1 \times y_2$

$\overrightarrow{AB} \cdot \overrightarrow{AD} = 4 \times (-3) + 2 \times 6$

$\overrightarrow{AB} \cdot \overrightarrow{AD} = -12 + 12 = 0$

12c From part a, $\overrightarrow{AB} = \overrightarrow{DC}$

From part b, $\overrightarrow{AB} \cdot \overrightarrow{AD} = 0$

This means two sides of the quadrilateral are equal and parallel and two sides are perpendicular to each other hence, a right-angled parallelogram or rectangle.

13a Consider O to be the origin and $\overrightarrow{OP}$, $\overrightarrow{OQ}$, $\overrightarrow{OR}$ and $\overrightarrow{OS}$ represent position vectors.

Hence,

$\overrightarrow{OP} = -8\underset{\sim}{i} + 3\underset{\sim}{j}$, $\overrightarrow{OQ} = 3\underset{\sim}{i} + 7\underset{\sim}{j}$, $\overrightarrow{OR} = 7\underset{\sim}{i} + 18\underset{\sim}{j}$ and $\overrightarrow{OS} = -4\underset{\sim}{i} + 14\underset{\sim}{j}$

$$\begin{aligned}\overrightarrow{PQ} &= \overrightarrow{OQ} - \overrightarrow{OP}\\ &= \left(3\underset{\sim}{i} + 7\underset{\sim}{j}\right) - \left(-8\underset{\sim}{i} + 3\underset{\sim}{j}\right)\\ &= (3 - (-8))\underset{\sim}{i} + (7 - 3)\underset{\sim}{j}\\ &= (3 + 8)\underset{\sim}{i} + (7 - 3)\underset{\sim}{j}\\ &= 11\underset{\sim}{i} + 4\underset{\sim}{j}\end{aligned}$$

$$\begin{aligned}\overrightarrow{PR} &= \overrightarrow{OR} - \overrightarrow{OP}\\ &= \left(7\underset{\sim}{i} + 18\underset{\sim}{j}\right) - \left(-8\underset{\sim}{i} + 3\underset{\sim}{j}\right)\\ &= (7 - (-8))\underset{\sim}{i} + (18 - 3)\underset{\sim}{j}\\ &= (7 + 8)\underset{\sim}{i} + (18 - 3)\underset{\sim}{j}\\ &= 15\underset{\sim}{i} + 15\underset{\sim}{j}\end{aligned}$$

$$\begin{aligned}\frac{1}{2}\overrightarrow{PR} &= \frac{1}{2}\left(15\underset{\sim}{i} + 15\underset{\sim}{j}\right)\\ &= \frac{15}{2}\underset{\sim}{i} + \frac{15}{2}\underset{\sim}{j}\end{aligned}$$

$$\begin{aligned}\overrightarrow{QS} &= \overrightarrow{OS} - \overrightarrow{OQ}\\ &= \left(-4\underset{\sim}{i} + 14\underset{\sim}{j}\right) - \left(3\underset{\sim}{i} + 7\underset{\sim}{j}\right)\\ &= (-4 - 3)\underset{\sim}{i} + (14 - 7)\underset{\sim}{j}\\ &= -7\underset{\sim}{i} + 7\underset{\sim}{j}\end{aligned}$$

$$\begin{aligned}\frac{1}{2}\overrightarrow{QS} &= \frac{1}{2}\left(-7\underset{\sim}{i} + 7\underset{\sim}{j}\right)\\ &= -\frac{7}{2}\underset{\sim}{i} + \frac{7}{2}\underset{\sim}{j}\end{aligned}$$

$\text{LHS} = \frac{1}{2}\overrightarrow{PR}$

$= \frac{15}{2}\underset{\sim}{i} + \frac{15}{2}\underset{\sim}{j}$

$\text{RHS} = \overrightarrow{PQ} + \frac{1}{2}\overrightarrow{QS}$

$= \left(11\underset{\sim}{i} + 4\underset{\sim}{j}\right) + \left(-\frac{7}{2}\underset{\sim}{i} + \frac{7}{2}\underset{\sim}{j}\right)$

$= \left(11 - \frac{7}{2}\right)\underset{\sim}{i} + (4 + \frac{7}{2})\underset{\sim}{j}$

$= \frac{15}{2}\underset{\sim}{i} + \frac{15}{2}\underset{\sim}{j}$

Hence, $\text{LHS} = \text{RHS}$.

13b $\overrightarrow{PR} = 15\underset{\sim}{i} + 15\underset{\sim}{j}$

$\overrightarrow{QS} = -7\underset{\sim}{i} + 7\underset{\sim}{j}$

By using the method, $\underset{\sim}{u} \cdot \underset{\sim}{v} = x_1 \times x_2 + y_1 \times y_2$

$\overrightarrow{PR} \cdot \overrightarrow{QS} = 15 \times (-7) + 15 \times 7$

$= -105 + 105$

$= 0$

13c From part a, $\frac{1}{2}\overrightarrow{PR} = \overrightarrow{PQ} + \frac{1}{2}\overrightarrow{QS}$

From part b, $\overrightarrow{PR} \cdot \overrightarrow{QS} = 0$

This means the diagonals of the quadrilateral bisect each other at 90° hence, a rhombus.

14a $\overrightarrow{OA} = \begin{bmatrix} -3 \\ 3 \end{bmatrix}, \overrightarrow{OP} = \begin{bmatrix} 2 \\ 9 \end{bmatrix}, \overrightarrow{OQ} = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$

$\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA}$

$= \begin{bmatrix} 2 - (-3) \\ 9 - 3 \end{bmatrix}$

$= \begin{bmatrix} 5 \\ 6 \end{bmatrix}$

$$\overrightarrow{AQ} = \overrightarrow{OQ} - \overrightarrow{OA}$$

$$= \begin{bmatrix} 10 - (-3) \\ 0 - 3 \end{bmatrix}$$

$$= \begin{bmatrix} 13 \\ -3 \end{bmatrix}$$

14b $\left|\overrightarrow{AP}\right| = \sqrt{x^2 + y^2}$

$$= \sqrt{5^2 + 6^2}$$

$$= \sqrt{25 + 36}$$

$$= \sqrt{61}$$

$$\left|\overrightarrow{AQ}\right| = \sqrt{x^2 + y^2}$$

$$= \sqrt{13^2 + (-3)^2}$$

$$= \sqrt{169 + 9}$$

$$= \sqrt{178}$$

By using the method, $\underline{u} \cdot \underline{v} = x_1 \times x_2 + y_1 \times y_2$

$$\overrightarrow{AP} \cdot \overrightarrow{AQ} = 5 \times 13 + 6 \times (-3)$$

$$= 65 - 18$$

$$= 47$$

$$\left|\overrightarrow{AP}\right| \times \left|\overrightarrow{AQ}\right| \cos\theta = \overrightarrow{AP} \cdot \overrightarrow{AQ}$$

$$\sqrt{61} \times \sqrt{178} \cos\theta = 47$$

$$\cos\theta = \frac{47}{\sqrt{61} \times \sqrt{178}} = 0.451 \ldots$$

$$\theta = 63.189 \ldots°$$

Hence $\angle PAQ \doteqdot 63°$

15 Consider O to be the origin and $\overrightarrow{OP}$, $\overrightarrow{OQ}$, $\overrightarrow{OR}$ and $\overrightarrow{OS}$ represent position vectors.

Hence,

$\overrightarrow{OP} = \underline{i} + 2\underline{j}$, $\overrightarrow{OQ} = 8\underline{i} + 3\underline{j}$, $\overrightarrow{OR} = 6\underline{i} + 13\underline{j}$ and $\overrightarrow{OS} = 4\underline{i} + 9\underline{j}$

PR and QS are the diagonals of the quadrilateral.

$$\overrightarrow{PR} = \overrightarrow{OR} - \overrightarrow{OP}$$

$$= \left(6\underset{\sim}{i} + 13\underset{\sim}{j}\right) - \left(\underset{\sim}{i} + 2\underset{\sim}{j}\right)$$

$$= (6-1)\underset{\sim}{i} + (13-2)\underset{\sim}{j}$$

$$= 5\underset{\sim}{i} + 11\underset{\sim}{j}$$

$$\overrightarrow{QS} = \overrightarrow{OS} - \overrightarrow{OQ}$$

$$= \left(4\underset{\sim}{i} + 9\underset{\sim}{j}\right) - \left(8\underset{\sim}{i} + 3\underset{\sim}{j}\right)$$

$$= (4-8)\underset{\sim}{i} + (9-3)\underset{\sim}{j}$$

$$= -4\underset{\sim}{i} + 6\underset{\sim}{j}$$

$$\left|\overrightarrow{PR}\right| = \sqrt{x^2 + y^2}$$

$$= \sqrt{5^2 + 11^2}$$

$$= \sqrt{25 + 121}$$

$$= \sqrt{146}$$

$$\left|\overrightarrow{QS}\right| = \sqrt{x^2 + y^2}$$

$$= \sqrt{(-4)^2 + 6^2}$$

$$= \sqrt{16 + 36}$$

$$= \sqrt{52}$$

$$= 2\sqrt{13}$$

By using the method, $\underset{\sim}{u} \cdot \underset{\sim}{v} = x_1 \times x_2 + y_1 \times y_2$

$$\overrightarrow{PR} \cdot \overrightarrow{QS} = 5 \times (-4) + 11 \times 6$$

$$= -20 + 66$$

$$= 46$$

$$\left|\overrightarrow{PR}\right| \times \left|\overrightarrow{QS}\right| \cos\theta = \overrightarrow{PR} \cdot \overrightarrow{QS}$$

$$\sqrt{146} \times 2\sqrt{13}\cos\theta = 46$$

$$\cos\theta = \frac{46}{\sqrt{146} \times 2\sqrt{13}} = 0.527\ldots$$

$$\theta = 58.134\ldots°$$

$\doteqdot 58°8'$

16 Consider O to be the centre of the circle, $\overrightarrow{OP} = r\cos\theta\, \underset{\sim}{i} + r\sin\theta\, \underset{\sim}{j}$

$\overrightarrow{OA} = -r\underset{\sim}{i}$ $\overrightarrow{OB} = r\underset{\sim}{i}$

Hence,

$$\overrightarrow{PA} = \overrightarrow{OA} - \overrightarrow{OP}$$
$$= -r\underset{\sim}{i} - \left(r\cos\theta\, \underset{\sim}{i} + r\sin\theta\, \underset{\sim}{j}\right)$$
$$= (-r - r\cos\theta)\underset{\sim}{i} - r\sin\theta\, \underset{\sim}{j}$$

$$\overrightarrow{PB} = \overrightarrow{OB} - \overrightarrow{OP}$$
$$= r\underset{\sim}{i} - \left(r\cos\theta\, \underset{\sim}{i} + r\sin\theta\, \underset{\sim}{j}\right)$$
$$= (r - r\cos\theta)\underset{\sim}{i} - r\sin\theta\, \underset{\sim}{j}$$

$$\left|\overrightarrow{PA}\right| = \sqrt{(-r - r\cos\theta)^2 + (-r\sin\theta)^2}$$
$$= \sqrt{r^2 + 2r^2\cos\theta + r^2\cos^2\theta + r^2\sin^2\theta}$$
$$= \sqrt{r^2 + r^2\cos^2\theta + r^2\sin^2\theta + 2r^2\cos\theta}$$
$$= \sqrt{r^2 + r^2(\cos^2\theta + \sin^2\theta) + 2r^2\cos\theta}$$
$$= \sqrt{r^2 + r^2 \times 1 + 2r^2\cos\theta}$$
$$= \sqrt{2r^2 + 2r^2\cos\theta}$$
$$= \sqrt{2r^2(1 + \cos\theta)}$$
$$= \sqrt{2r^2 \times 2\cos^2\frac{\theta}{2}} \qquad \text{(using } \cos 2\theta = 2\cos^2\theta - 1\text{)}$$
$$= \sqrt{4r^2\cos^2\frac{\theta}{2}}$$
$$= 2r\cos\frac{\theta}{2}$$

$$\left|\overrightarrow{PB}\right| = \sqrt{(r - r\cos\theta)^2 + (-r\sin\theta)^2}$$
$$= \sqrt{r^2 - 2r^2\cos\theta + r^2\cos^2\theta + r^2\sin^2\theta}$$
$$= \sqrt{r^2 + r^2\cos^2\theta + r^2\sin^2\theta - 2r^2\cos\theta}$$

$= \sqrt{r^2 + r^2(\cos^2\theta + \sin^2\theta) - 2r^2\cos\theta}$

$= \sqrt{r^2 + r^2 \times 1 - 2r^2\cos\theta}$

$= \sqrt{2r^2 - 2r^2\cos\theta}$

$= \sqrt{2r^2(1 - \cos\theta)}$

$= \sqrt{2r^2 \times 2\sin^2\frac{\theta}{2}}$ (using $\cos 2\theta = 1 - 2\sin^2\theta$)

$= \sqrt{4r^2\sin^2\frac{\theta}{2}}$

$= 2r\sin\dfrac{\theta}{2}$

By using the method, $\underset{\sim}{u} \cdot \underset{\sim}{v} = x_1 \times x_2 + y_1 \times y_2$

$\overrightarrow{PA} \cdot \overrightarrow{PB}$

$= (-r - r\cos\theta) \times (r - r\cos\theta) + (-r\sin\theta) \times (-r\sin\theta)$

$= -(r^2 - r^2\cos^2\theta) + r^2\sin^2\theta$

$= -r^2 + r^2\cos^2\theta + r^2\sin^2\theta$

$= -r^2 + r^2(\cos^2\theta + \sin^2\theta)$

$= -r^2 + r^2 \times 1$

$= 0$

The dot product is 0, hence angle between $\overrightarrow{PA}$ and $\overrightarrow{PB}$. That is, $\angle APB$ is $90°$.

17a Consider O to be the origin hence, $\overrightarrow{OA} = 2\underset{\sim}{i} + \underset{\sim}{j}$

$\overrightarrow{OB} = 10\underset{\sim}{i} + 4\underset{\sim}{j}$

$\overrightarrow{OC} = 5\underset{\sim}{i} + 13\underset{\sim}{j}$

$\overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB}$

$= \left(2\underset{\sim}{i} + \underset{\sim}{j}\right) - \left(10\underset{\sim}{i} + 4\underset{\sim}{j}\right)$

$= (2 - 10)\underset{\sim}{i} + (1 - 4)\underset{\sim}{j}$

$= -8\underset{\sim}{i} - 3\underset{\sim}{j}$

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$

$= \left(5\underset{\sim}{i} + 13\underset{\sim}{j}\right) - \left(10\underset{\sim}{i} + 4\underset{\sim}{j}\right)$

$= (5 - 10)\underset{\sim}{i} + (13 - 4)\underset{\sim}{j}$

$= -5\underset{\sim}{i} + 9\underset{\sim}{j}$

$\left|\overrightarrow{BA}\right| = \sqrt{(-8)^2 + (-3)^2}$

$= \sqrt{64 + 9}$

$= \sqrt{73}$

$\left|\overrightarrow{BC}\right| = \sqrt{(-5)^2 + 9^2}$

$= \sqrt{25 + 81}$

$= \sqrt{106}$

By using the method, $\underset{\sim}{u} \cdot \underset{\sim}{v} = x_1 \times x_2 + y_1 \times y_2$

$\overrightarrow{BA} \cdot \overrightarrow{BC} = (-8) \times (-5) + (-3) \times 9$

$= 40 - 27$

$= 13$

$\left|\overrightarrow{BA}\right| \times \left|\overrightarrow{BC}\right| \cos \angle ABC = \overrightarrow{BA} \cdot \overrightarrow{BC}$

$\sqrt{73} \times \sqrt{106} \cos \angle ABC = 13$

$\cos \angle ABC = \dfrac{13}{\sqrt{73} \times \sqrt{106}}$

$= \dfrac{13}{\sqrt{7738}}$

17b $\sin\theta = \sqrt{1 - \cos^2\theta}$

$\sin \angle ABC = \sqrt{1 - \left(\dfrac{13}{\sqrt{7738}}\right)^2}$

$= \sqrt{1 - \dfrac{169}{7738}}$

$= \sqrt{\dfrac{7569}{7738}}$

$$= \frac{87}{\sqrt{7738}}$$

17c Area of triangle ABC

$$A = \frac{1}{2}\left|\overrightarrow{BA}\right| \times \left|\overrightarrow{BC}\right| \sin\angle ABC$$

$$= \frac{1}{2}\sqrt{73} \times \sqrt{106} \times \frac{87}{\sqrt{7738}}$$

$$= \frac{1}{2} \times \sqrt{7738} \times \frac{87}{\sqrt{7738}}$$

$$= \frac{87}{2}$$

$$= 43.5$$

Area of triangle ABC is 43.5 square units.

18a $\overrightarrow{PA} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ $\quad \overrightarrow{PB} = \begin{bmatrix} a \\ b \end{bmatrix}$ $\quad \left|\overrightarrow{PB}\right| = 4\sqrt{5}$

$$\left|\overrightarrow{PA}\right| = \sqrt{3^2 + 1^2}$$

$$= \sqrt{9+1}$$

$$= \sqrt{10}$$

Area of triangle APB is 10 square units.

$$\frac{1}{2}\left|\overrightarrow{PA}\right| \times \left|\overrightarrow{PB}\right| \sin\theta = A$$

$$\frac{1}{2}\sqrt{10} \times 4\sqrt{5}\sin\theta = 10$$

$$\frac{1}{2} \times 20\sqrt{2}\sin\theta = 10$$

$$\sin\theta = \frac{1}{\sqrt{2}}$$

$$\cos\theta = \pm\sqrt{1-\sin^2\theta}$$

$$= \pm\sqrt{1-\left(\frac{1}{\sqrt{2}}\right)^2}$$

$$= \pm\sqrt{1-\frac{1}{2}}$$

$$= \pm\sqrt{\frac{1}{2}}$$

$$= \pm\frac{1}{\sqrt{2}}$$

$$\overrightarrow{PA} \cdot \overrightarrow{PB} = |\overrightarrow{PA}| \times |\overrightarrow{PB}| \cos\theta$$

$$= \sqrt{10} \times 4\sqrt{5} \times \pm\frac{1}{\sqrt{2}}$$

$$= \pm 20$$

By using the method, $\underset{\sim}{u} \cdot \underset{\sim}{v} = x_1 \times x_2 + y_1 \times y_2$

$$\overrightarrow{PA} \cdot \overrightarrow{PB} = 3 \times a + 1 \times b$$

$$\pm 20 = 3a + b$$

Hence, $3a + b = 20$ or $3a + b = -20$.

18b All the possibilities can be:

$$\overrightarrow{PB} = \begin{bmatrix} 8 \\ -4 \end{bmatrix}, \begin{bmatrix} 4 \\ 8 \end{bmatrix}, \begin{bmatrix} -8 \\ 4 \end{bmatrix}, \begin{bmatrix} -4 \\ -8 \end{bmatrix}$$

19a $\overrightarrow{OA} = \underset{\sim}{u} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$, $\overrightarrow{OB} = \underset{\sim}{v} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$ and $\angle AOB = \theta$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$\overrightarrow{AB} = \underset{\sim}{u} - \underset{\sim}{v}$$

$$|\underset{\sim}{u}|^2 = {x_1}^2 + {y_1}^2$$

$$|\underset{\sim}{v}|^2 = {x_2}^2 + {y_2}^2$$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = x_1 \times x_2 + y_1 \times y_2$$

Using the cosine rule,

$$|AB|^2 = |\underset{\sim}{u}|^2 + |\underset{\sim}{v}|^2 - 2\underset{\sim}{u} \cdot \underset{\sim}{v}$$

$$|AB|^2 = |\underset{\sim}{u}|^2 + |\underset{\sim}{v}|^2 - 2|\underset{\sim}{u}| \times |\underset{\sim}{v}| \cos\theta$$

19b $|AB|^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$

19c $(x_1 - x_2)^2 + (y_1 - y_2)^2 = |\underset{\sim}{u}|^2 + |\underset{\sim}{v}|^2 - 2|\underset{\sim}{u}| \times |\underset{\sim}{v}| \cos\theta$

$x_1{}^2 - 2(x_1 \times x_2) + x_2{}^2 + y_1{}^2 - 2(y_1 \times y_2) + y_2{}^2 = |\underset{\sim}{u}|^2 + |\underset{\sim}{v}|^2 - 2|\underset{\sim}{u}| \times |\underset{\sim}{v}| \cos\theta$

$x_1{}^2 + y_1{}^2 + x_2{}^2 + y_2{}^2 - 2(x_1 \times x_2) - 2(y_1 \times y_2) = |\underset{\sim}{u}|^2 + |\underset{\sim}{v}|^2 - 2|\underset{\sim}{u}| \times |\underset{\sim}{v}| \cos\theta$

$|\underset{\sim}{u}|^2 + |\underset{\sim}{v}|^2 - 2(x_1 \times x_2 + y_1 \times y_2) = |\underset{\sim}{u}|^2 + |\underset{\sim}{v}|^2 - 2|\underset{\sim}{u}| \times |\underset{\sim}{v}| \cos\theta$

$-2(x_1 \times x_2 + y_1 \times y_2) = -2|\underset{\sim}{u}| \times |\underset{\sim}{v}| \cos\theta$

$x_1 \times x_2 + y_1 \times y_2 = |\underset{\sim}{u}| \times |\underset{\sim}{v}| \cos\theta$

$x_1x_2 + y_1y_2 = |\underset{\sim}{u}||\underset{\sim}{v}| \cos\theta$

Hence, proved.

20a $\overrightarrow{PR} = \overrightarrow{PO} + \overrightarrow{OQ} + \overrightarrow{QR}$ and we are given $\overrightarrow{OP} = \underset{\sim}{a} + \underset{\sim}{b}$ and $\overrightarrow{OQ} = 3\underset{\sim}{a} - 2\underset{\sim}{b}$

$\overrightarrow{OP} = \overrightarrow{RQ} = \underset{\sim}{a} + \underset{\sim}{b}$

$\overrightarrow{PR} = -(\underset{\sim}{a} + \underset{\sim}{b}) + 3\underset{\sim}{a} - 2\underset{\sim}{b} - (\underset{\sim}{a} + \underset{\sim}{b})$

So $\overrightarrow{PR} = \underset{\sim}{a} - 4\underset{\sim}{b}$.

20b If $OPQR$ is a square then $\overrightarrow{OP} \cdot \overrightarrow{OR} = 0$.

$$\begin{aligned}\overrightarrow{OR} &= \overrightarrow{OQ} + \overrightarrow{QR} \\ &= 3\underset{\sim}{a} - 2\underset{\sim}{b} - (\underset{\sim}{a} + \underset{\sim}{b}) \\ &= 2\underset{\sim}{a} - 3\underset{\sim}{b}\end{aligned}$$

So $(\underset{\sim}{a} + \underset{\sim}{b}) \cdot (2\underset{\sim}{a} - 3\underset{\sim}{b}) = 0$.

$$2(\underset{\sim}{a} \cdot \underset{\sim}{a}) - 3(\underset{\sim}{a} \cdot \underset{\sim}{b}) + 2(\underset{\sim}{a} \cdot \underset{\sim}{b}) - 3(\underset{\sim}{b} \cdot \underset{\sim}{b}) = 0$$
$$2|\underset{\sim}{a}|^2 - (\underset{\sim}{a} \cdot \underset{\sim}{b}) - 3|\underset{\sim}{b}|^2 = 0$$

So $\underset{\sim}{a} \cdot \underset{\sim}{b} = 2|\underset{\sim}{a}|^2 - 3|\underset{\sim}{b}|^2$ (1)

If $OPQR$ is a square then $\overrightarrow{PR} \cdot \overrightarrow{OQ} = 0$ and we are given $\overrightarrow{PR} = \underset{\sim}{a} - 4\underset{\sim}{b}$ and $\overrightarrow{OQ} = 3\underset{\sim}{a} - 2\underset{\sim}{b}$.

So $\left(\underset{\sim}{a}-4\underset{\sim}{b}\right)\cdot\left(3\underset{\sim}{a}-2\underset{\sim}{b}\right)=0$.

$$\begin{aligned}3\left(\underset{\sim}{a}\cdot\underset{\sim}{a}\right)-2\left(\underset{\sim}{a}\cdot\underset{\sim}{b}\right)-12\left(\underset{\sim}{a}\cdot\underset{\sim}{b}\right)+8\left(\underset{\sim}{b}\cdot\underset{\sim}{b}\right)&=0\\3\left|\underset{\sim}{a}\right|^2-14\left(\underset{\sim}{a}\cdot\underset{\sim}{b}\right)+8\left|\underset{\sim}{b}\right|^2&=0\end{aligned}$$

So $3\left|\underset{\sim}{a}\right|^2-14\left(\underset{\sim}{a}\cdot\underset{\sim}{b}\right)+8\left|\underset{\sim}{b}\right|^2=0 \quad (2)$

Substituting (1) into (2) we obtain:

$$\begin{aligned}3\left|\underset{\sim}{a}\right|^2-28\left|\underset{\sim}{a}\right|^2+42\left|\underset{\sim}{b}\right|^2+8\left|\underset{\sim}{b}\right|^2&=0\\-25\left|\underset{\sim}{a}\right|^2&=-50\left|\underset{\sim}{b}\right|^2\\\left|\underset{\sim}{a}\right|^2&=2\left|\underset{\sim}{b}\right|^2\end{aligned}$$

So $\left|\underset{\sim}{a}\right|^2=2\left|\underset{\sim}{b}\right|^2$.

21a i A condition for the diagonals AC and BD of quadrilateral $ABCD$ to be perpendicular is $\left(\underset{\sim}{c}-\underset{\sim}{a}\right)\cdot\left(\underset{\sim}{d}-\underset{\sim}{b}\right)=0$.

21a ii A condition for the diagonals AC and BD of quadrilateral $ABCD$ to be the same length is $\left|\underset{\sim}{c}-\underset{\sim}{a}\right|=\left|\underset{\sim}{d}-\underset{\sim}{b}\right|$ (other answers are possible).

21b Given $\underset{\sim}{a}=\begin{bmatrix}1\\5\end{bmatrix}$, $\underset{\sim}{b}=\begin{bmatrix}5\\8\end{bmatrix}$, $\underset{\sim}{c}=\begin{bmatrix}8\\4\end{bmatrix}$ and $\underset{\sim}{d}=\begin{bmatrix}m\\n\end{bmatrix}$.

$$\begin{aligned}\underset{\sim}{c}-\underset{\sim}{a}&=\begin{bmatrix}8\\4\end{bmatrix}-\begin{bmatrix}1\\5\end{bmatrix}\\&=\begin{bmatrix}7\\-1\end{bmatrix}\end{aligned}$$

$$\begin{aligned}\underset{\sim}{d}-\underset{\sim}{b}&=\begin{bmatrix}m\\n\end{bmatrix}-\begin{bmatrix}5\\8\end{bmatrix}\\&=\begin{bmatrix}m-5\\n-8\end{bmatrix}\end{aligned}$$

$$\begin{bmatrix}7\\-1\end{bmatrix}\cdot\begin{bmatrix}m-5\\n-8\end{bmatrix}=0\Rightarrow 7\left(m-5\right)+\left(8-n\right)=0$$

So $7\left(m-5\right)=n-8 \quad (1)$.

$$\left|\underset{\sim}{c}-\underset{\sim}{a}\right|=\left|\underset{\sim}{d}-\underset{\sim}{b}\right|\Rightarrow 5\sqrt{2}=\sqrt{\left(m-5\right)^2+\left(n-8\right)^2}$$

So $50=(m-5)^2+(n-8)^2 \quad (2)$.

Substituting (1) into (2) we obtain:

$$\begin{aligned}50&=(m-5)^2+49(m-5)^2\\(m-5)^2&=1\\m-5&=\pm1\\m&=4,6\end{aligned}$$

Substituting into (1) and solving for n we obtain $n=1,15$.

So $m=4$ and $n=1$ or $m=6$ and $n=15$.

22a Given $\underset{\sim}{a}+\underset{\sim}{b}+\underset{\sim}{c}=\underset{\sim}{0}$.

$(\underset{\sim}{a}+\underset{\sim}{b}+\underset{\sim}{c})\cdot(\underset{\sim}{a}+\underset{\sim}{b}+\underset{\sim}{c})=0$

Expanding the LHS we obtain:

$$\begin{aligned}(\underset{\sim}{a}+\underset{\sim}{b}+\underset{\sim}{c})\cdot(\underset{\sim}{a}+\underset{\sim}{b}+\underset{\sim}{c})&=\underset{\sim}{a}\cdot\underset{\sim}{a}+\underset{\sim}{a}\cdot\underset{\sim}{b}+\underset{\sim}{a}\cdot\underset{\sim}{c}+\underset{\sim}{b}\cdot\underset{\sim}{a}+\underset{\sim}{b}\cdot\underset{\sim}{b}+\underset{\sim}{b}\cdot\underset{\sim}{c}+\underset{\sim}{c}\cdot\underset{\sim}{a}+\underset{\sim}{c}\cdot\underset{\sim}{b}+\underset{\sim}{c}\cdot\underset{\sim}{c}\\&=|\underset{\sim}{a}|^2+|\underset{\sim}{b}|^2+|\underset{\sim}{c}|^2+2(\underset{\sim}{a}\cdot\underset{\sim}{b}+\underset{\sim}{a}\cdot\underset{\sim}{c}+\underset{\sim}{b}\cdot\underset{\sim}{c})\end{aligned}$$

$$|\underset{\sim}{a}|^2+|\underset{\sim}{b}|^2+|\underset{\sim}{c}|^2+2(\underset{\sim}{a}\cdot\underset{\sim}{b}+\underset{\sim}{a}\cdot\underset{\sim}{c}+\underset{\sim}{b}\cdot\underset{\sim}{c})=0$$

So $|\underset{\sim}{a}|^2+|\underset{\sim}{b}|^2+|\underset{\sim}{c}|^2=-2(\underset{\sim}{a}\cdot\underset{\sim}{b}+\underset{\sim}{b}\cdot\underset{\sim}{c}+\underset{\sim}{c}\cdot\underset{\sim}{a})$.

A dot product is negative when the angle is obtuse.

22b In Box 13, we are given the cosine rule in vector form which can be solved for $\underset{\sim}{a}\cdot\underset{\sim}{b}$ to obtain $2\underset{\sim}{a}\cdot\underset{\sim}{b}=|\underset{\sim}{a}|^2+|\underset{\sim}{b}|^2-|\underset{\sim}{a}-\underset{\sim}{b}|^2$.

In triangle ABC, $\underset{\sim}{a}+\underset{\sim}{b}+\underset{\sim}{c}=\underset{\sim}{0}$.

To adapt the result in Box 13 here, we firstly replace $\underset{\sim}{a}-\underset{\sim}{b}$ with $-\underset{\sim}{b}$ and replace $\underset{\sim}{b}$ with $-\underset{\sim}{c}$

Thus using Box 13 we obtain:

$$\begin{aligned}2\underset{\sim}{a}\cdot(-\underset{\sim}{c})&=|\underset{\sim}{a}|^2+|-\underset{\sim}{c}|^2-|-\underset{\sim}{b}|^2\\-2\underset{\sim}{a}\cdot\underset{\sim}{c}&=|\underset{\sim}{a}|^2+|\underset{\sim}{c}|^2-|\underset{\sim}{b}|^2\end{aligned}$$

Similarly, $-2\underset{\sim}{a}\cdot\underset{\sim}{b}=|\underset{\sim}{a}|^2+|\underset{\sim}{b}|^2-|\underset{\sim}{c}|^2$ and $-2\underset{\sim}{b}\cdot\underset{\sim}{c}=|\underset{\sim}{b}|^2+|\underset{\sim}{c}|^2-|\underset{\sim}{a}|^2$

Adding the three versions of this cyclic formula gives the desired result.

So $|\underset{\sim}{a}|^2+|\underset{\sim}{b}|^2+|\underset{\sim}{c}|^2=-2(\underset{\sim}{a}\cdot\underset{\sim}{b}+\underset{\sim}{b}\cdot\underset{\sim}{c}+\underset{\sim}{c}\cdot\underset{\sim}{a})$.

22c i For an equilateral triangle of side length 1:

$$\text{LHS}=|\underset{\sim}{a}|^2+|\underset{\sim}{b}|^2+|\underset{\sim}{c}|^2=1+1+1=3$$

$$\begin{aligned}\text{RHS}&=-2(\underset{\sim}{a}\cdot\underset{\sim}{b}+\underset{\sim}{b}\cdot\underset{\sim}{c}+\underset{\sim}{c}\cdot\underset{\sim}{a})\\&=-2(1\times1\cos120^\circ+1\times1\cos120^\circ+1\times1\cos120^\circ)\\&=-2(3\times\cos120^\circ)\\&=-2\times3\times-\frac{1}{2}\\&=3\end{aligned}$$

So $\text{LHS}=\text{RHS}$.

They are both 3. When calculating the RHS, be careful to take the exterior angles as the angles between the vectors.

22c ii For a right-angled isosceles triangle whose equal sides have length 1:

$$\text{LHS}=|\underset{\sim}{a}|^2+|\underset{\sim}{b}|^2+|\underset{\sim}{c}|^2=\left(\sqrt{2}\right)^2+1+1=4$$

$$\begin{aligned}\text{RHS}&=-2(\underset{\sim}{a}\cdot\underset{\sim}{b}+\underset{\sim}{b}\cdot\underset{\sim}{c}+\underset{\sim}{c}\cdot\underset{\sim}{a})\\&=-2\left(\sqrt{2}\times1\cos135^\circ+1\times1\cos90^\circ+1\times\sqrt{2}\cos135^\circ\right)\\&=-2\left(2\sqrt{2}\times\cos135^\circ\right)\\&=-2\times2\sqrt{2}\times-\frac{1}{\sqrt{2}}\\&=4\end{aligned}$$

So $\text{LHS}=\text{RHS}$.

They are both 4.

22c iii For a right-angled triangle with hypotenuse of length 2 and one side of length 1:

$$\text{LHS} = |\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2 + |\underset{\sim}{c}|^2 = 2^2 + \left(\sqrt{3}\right)^2 + 1 = 8$$

$$\begin{aligned}\text{RHS} &= -2\left(\underset{\sim}{a}\cdot\underset{\sim}{b} + \underset{\sim}{b}\cdot\underset{\sim}{c} + \underset{\sim}{c}\cdot\underset{\sim}{a}\right)\\ &= -2\left(2\times\sqrt{3}\cos 150^\circ + \sqrt{3}\times 1\cos 90^\circ + 1\times 2\cos 120^\circ\right)\\ &= -2(-3-1)\\ &= 8\end{aligned}$$

So $\text{LHS} = \text{RHS}$.

They are both 8.

23a Given $\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c} + \underset{\sim}{d} = \underset{\sim}{0}$.

$$\left(\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c} + \underset{\sim}{d}\right)\cdot\left(\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c} + \underset{\sim}{d}\right) = 0$$

Expanding the LHS we obtain:

$$\left(\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c} + \underset{\sim}{d}\right)\cdot\left(\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c} + \underset{\sim}{d}\right) =$$

$$\begin{aligned}&= \underset{\sim}{a}\cdot\underset{\sim}{a} + \underset{\sim}{a}\cdot\underset{\sim}{b} + \underset{\sim}{a}\cdot\underset{\sim}{c} + \underset{\sim}{a}\cdot\underset{\sim}{d} + \underset{\sim}{b}\cdot\underset{\sim}{a} + \underset{\sim}{b}\cdot\underset{\sim}{b} + \underset{\sim}{b}\cdot\underset{\sim}{c} + \underset{\sim}{b}\cdot\underset{\sim}{d}\\ &\quad + \underset{\sim}{c}\cdot\underset{\sim}{a} + \underset{\sim}{c}\cdot\underset{\sim}{b} + \underset{\sim}{c}\cdot\underset{\sim}{c} + \underset{\sim}{c}\cdot\underset{\sim}{d} + \underset{\sim}{d}\cdot\underset{\sim}{a} + \underset{\sim}{d}\cdot\underset{\sim}{b} + \underset{\sim}{d}\cdot\underset{\sim}{c} + \underset{\sim}{d}\cdot\underset{\sim}{d}\\ &= |\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2 + |\underset{\sim}{c}|^2 + |\underset{\sim}{d}|^2 + 2\left(\underset{\sim}{a}\cdot\underset{\sim}{b} + \underset{\sim}{a}\cdot\underset{\sim}{c} + \underset{\sim}{a}\cdot\underset{\sim}{d} + \underset{\sim}{b}\cdot\underset{\sim}{c} + \underset{\sim}{b}\cdot\underset{\sim}{d} + \underset{\sim}{c}\cdot\underset{\sim}{d}\right)\end{aligned}$$

$$|\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2 + |\underset{\sim}{c}|^2 + |\underset{\sim}{d}|^2 + 2\left(\underset{\sim}{a}\cdot\underset{\sim}{b} + \underset{\sim}{a}\cdot\underset{\sim}{c} + \underset{\sim}{a}\cdot\underset{\sim}{d} + \underset{\sim}{b}\cdot\underset{\sim}{c} + \underset{\sim}{b}\cdot\underset{\sim}{d} + \underset{\sim}{c}\cdot\underset{\sim}{d}\right) = 0$$

So $\underset{\sim}{a}\cdot\underset{\sim}{b} + \underset{\sim}{a}\cdot\underset{\sim}{c} + \underset{\sim}{a}\cdot\underset{\sim}{d} + \underset{\sim}{b}\cdot\underset{\sim}{c} + \underset{\sim}{b}\cdot\underset{\sim}{d} + \underset{\sim}{c}\cdot\underset{\sim}{d} = -\frac{1}{2}\left(|\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2 + |\underset{\sim}{c}|^2 + |\underset{\sim}{d}|^2\right)$.

23b i For a rectangle with sides k and l:

$|\underset{\sim}{a}| = |\underset{\sim}{c}| = k$ and $|\underset{\sim}{b}| = |\underset{\sim}{d}| = l$

$$\begin{aligned}\text{RHS} &= -\frac{1}{2}\left(|\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2 + |\underset{\sim}{c}|^2 + |\underset{\sim}{d}|^2\right)\\ &= -\frac{1}{2}\left(k^2 + l^2 + k^2 + l^2\right)\\ &= -k^2 - l^2\end{aligned}$$

$$\begin{aligned}\text{LHS} &= \underline{a}\cdot\underline{b}+\underline{a}\cdot\underline{c}+\underline{a}\cdot\underline{d}+\underline{b}\cdot\underline{c}+\underline{b}\cdot\underline{d}+\underline{c}\cdot\underline{d}\\ &= k\times l\times\cos 90^\circ+k\times k\times\cos 180^\circ+k\times l\times\cos 90^\circ\\ &\quad +k\times l\times\cos 90^\circ+l\times l\times\cos 180^\circ+k\times l\times\cos 90^\circ\\ &= k\times k\times\cos 180^\circ+l\times l\times\cos 180^\circ\\ &= -k^2-l^2\end{aligned}$$

So $\text{LHS}=\text{RHS}$.

They are both $-k^2-l^2$.

23b ii For a parallelogram with sides k and l and angle θ at A:

$|\underline{a}|=|\underline{c}|=k$ and $|\underline{b}|=|\underline{d}|=l$

$$\begin{aligned}\text{RHS} &= -\frac{1}{2}\left(|\underline{a}|^2+|\underline{b}|^2+|\underline{c}|^2+|\underline{d}|^2\right)\\ &= -\frac{1}{2}\left(k^2+l^2+k^2+l^2\right)\\ &= -k^2-l^2\end{aligned}$$

Using the result $\cos(180^\circ-\theta)=-\cos\theta$ and evaluating the LHS we obtain:

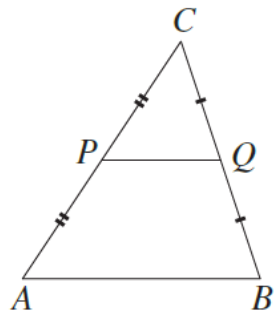
$$\begin{aligned}\text{LHS} &= \underline{a}\cdot\underline{b}+\underline{a}\cdot\underline{c}+\underline{a}\cdot\underline{d}+\underline{b}\cdot\underline{c}+\underline{b}\cdot\underline{d}+\underline{c}\cdot\underline{d}\\ &= k\times l\times\cos\theta+k\times k\times\cos 180^\circ+k\times l\times\cos(180^\circ-\theta)\\ &\quad +k\times l\times\cos(180^\circ-\theta)+l\times l\times\cos 180^\circ+k\times l\times\cos\theta\\ &= k\times l\times\cos\theta+k\times k\times\cos 180^\circ-k\times l\times\cos\theta\\ &\quad -k\times l\times\cos\theta+l\times l\times\cos 180^\circ+k\times l\times\cos\theta\\ &= -k^2-l^2\end{aligned}$$

So $\text{LHS}=\text{RHS}$.

They are both $-k^2-l^2$.

Solutions to Exercise 8D

1a



$\overrightarrow{AC} = \underset{\sim}{a}$ and $\overrightarrow{CB} = \underset{\sim}{b}$

P is the midpoint of AC and Q is the midpoint of BC

Hence, $AP = PC$ and $QC = QB$

$\overrightarrow{AB} = \overrightarrow{AC} + \overrightarrow{CB}$

$= \underset{\sim}{a} + \underset{\sim}{b}$

1b $\overrightarrow{PC} = \frac{1}{2}\left(\overrightarrow{AC}\right) = \frac{1}{2}\underset{\sim}{a}$

$\overrightarrow{CQ} = \frac{1}{2}\left(\overrightarrow{CB}\right) = \frac{1}{2}\underset{\sim}{b}$

$\overrightarrow{PQ} = \overrightarrow{PC} + \overrightarrow{CQ}$

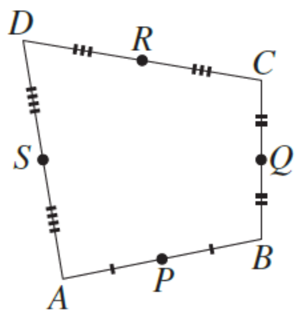
$= \frac{1}{2}\underset{\sim}{a} + \frac{1}{2}\underset{\sim}{b}$

$= \frac{1}{2}(\underset{\sim}{a} + \underset{\sim}{b})$

1c $\overrightarrow{PQ} = \frac{1}{2}\left(\underset{\sim}{a} + \underset{\sim}{b}\right) = \frac{1}{2}\overrightarrow{AB}$

Hence, $PQ \parallel AB$ and $PQ = \frac{1}{2}AB$.

2a



$\overrightarrow{AB} = \underline{a}$, $\overrightarrow{BC} = \underline{b}$, $\overrightarrow{AD} = \underline{d}$ and $\overrightarrow{DC} = \underline{c}$

$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC} = \underline{a} + \underline{b}$

$\overrightarrow{AD} + \overrightarrow{DC} = \overrightarrow{AC} = \underline{d} + \underline{c}$

As, $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC} = \overrightarrow{AD} + \overrightarrow{DC}$

Hence, $\overrightarrow{AC} = \underline{a} + \underline{b} = \underline{d} + \underline{c}$.

2b P is the midpoint of AB and Q is the midpoint of BC.

Hence, $AP = PB$ and $BQ = QC$.

$\overrightarrow{PB} = \frac{1}{2}\left(\overrightarrow{AB}\right) = \frac{1}{2}\underline{a}$

$\overrightarrow{BQ} = \frac{1}{2}\left(\overrightarrow{BC}\right) = \frac{1}{2}\underline{b}$

$$\begin{aligned}\overrightarrow{PQ} &= \overrightarrow{PB} + \overrightarrow{BQ}\\ &= \frac{1}{2}\underline{a} + \frac{1}{2}\underline{b}\\ &= \frac{1}{2}(\underline{a} + \underline{b})\end{aligned}$$

2c R is the midpoint of CD and S is the midpoint of DA.

Hence, $AS = SD$ and $DR = RC$

$\overrightarrow{DR} = \frac{1}{2}\left(\overrightarrow{DC}\right) = \frac{1}{2}\underline{c}$

$\overrightarrow{SD} = \frac{1}{2}\left(\overrightarrow{AD}\right) = \frac{1}{2}\underline{d}$

$\overrightarrow{SR} = \overrightarrow{SD} + \overrightarrow{DR}$

$$= \frac{1}{2}\underline{d} + \frac{1}{2}\underline{c}$$

$$= \frac{1}{2}(\underline{d} + \underline{c})$$

2d From part a, $\overrightarrow{AC} = \underline{a} + \underline{b} = \underline{d} + \underline{c}$

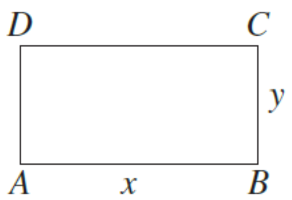
$\overrightarrow{PQ} = \frac{1}{2}(\underline{a} + \underline{b}) = \frac{1}{2}\overrightarrow{AC}$

$\overrightarrow{SR} = \frac{1}{2}(\underline{d} + \underline{c}) = \frac{1}{2}\overrightarrow{AC}$

Hence, $\overrightarrow{PQ} = \overrightarrow{SR}$.

2e Since $\overrightarrow{PQ} = \overrightarrow{SR}$, the line joining midpoints of the adjacent sides of the quadrilateral are parallel and equal to each other, hence, $ABCD$ is a parallelogram.

3a



$\overrightarrow{AB} = \underline{a}$, $\overrightarrow{BC} = \underline{b}$ and $\overrightarrow{CD} = \underline{c}$

$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$

$= \underline{a} + \underline{b}$

$\overrightarrow{BD} = \overrightarrow{BC} + \overrightarrow{CD}$

$= \underline{b} + \underline{c}$

3b $\underline{a} \cdot \underline{b} = |\underline{a}||\underline{b}| \cos\theta$

θ is the angle between $\underline{a}$ and $\underline{b}$.

$ABCD$ is a rectangle hence, $\theta = 90°$.

$|\underline{a}| = x$ and $|\underline{b}| = y$

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = x \times y \times \cos 90°$$
$$= x \times y \times 0$$
$$= 0$$

3c $\underset{\sim}{a} \cdot \underset{\sim}{a} = |\underset{\sim}{a}|^2 = x^2$

3d $|\underset{\sim}{a} + \underset{\sim}{b}|^2 = |\overrightarrow{AC}|^2 = x^2 + y^2$

$|\overrightarrow{AC}| = \sqrt{x^2 + y^2}$

$(\underset{\sim}{a} + \underset{\sim}{b}) \cdot (\underset{\sim}{a} + \underset{\sim}{b})$

$= |\underset{\sim}{a} + \underset{\sim}{b}||\underset{\sim}{a} + \underset{\sim}{b}| \cos\theta$

$= \sqrt{x^2 + y^2} \times \sqrt{x^2 + y^2} \times \cos 0°$

$= (x^2 + y^2) \times 1$

$= x^2 + y^2$

$|\underset{\sim}{b} + \underset{\sim}{c}|^2 = |\overrightarrow{BD}|^2 = x^2 + y^2$

As $ABCD$ is a rectangle, $|\overrightarrow{AB}| = |\overrightarrow{CD}|$.

$|\overrightarrow{BD}| = \sqrt{x^2 + y^2}$

$(\underset{\sim}{b} + \underset{\sim}{c}) \cdot (\underset{\sim}{b} + \underset{\sim}{c})$

$= |\underset{\sim}{b} + \underset{\sim}{c}||\underset{\sim}{b} + \underset{\sim}{c}| \cos\theta$

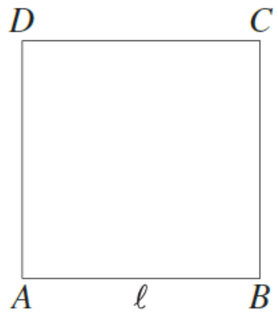
$= \sqrt{x^2 + y^2} \times \sqrt{x^2 + y^2} \times \cos 0°$

$= (x^2 + y^2) \times 1$

$= x^2 + y^2$

3e The diagonals of a rectangle have equal length.

4a



$\overrightarrow{AB} = \underset{\sim}{a}$ and $\overrightarrow{BC} = \underset{\sim}{b}$

$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$

$= \underset{\sim}{a} + \underset{\sim}{b}$

$\overrightarrow{BD} = \overrightarrow{AD} - \overrightarrow{AB}$

$= \underset{\sim}{b} - \underset{\sim}{a}$

4b $\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}| \cos\theta$

θ is the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

$ABCD$ is a square hence, $\theta = 90°$.

$|\underset{\sim}{a}| = l$ and $|\underset{\sim}{b}| = l$

$\underset{\sim}{a} \cdot \underset{\sim}{b} = l \times l \times \cos 90°$

$= x \times y \times 0$

$= 0$

4c $\underset{\sim}{a} \cdot \underset{\sim}{a} = |\underset{\sim}{a}|^2 = l^2$

4d $|\underset{\sim}{a} + \underset{\sim}{b}|^2 = |\overrightarrow{AC}|^2 = l^2 + l^2 = 2l^2$

$|\overrightarrow{AC}| = \sqrt{2l^2} = \sqrt{2}l$

$|\underset{\sim}{b} - \underset{\sim}{a}|^2 = |\overrightarrow{BD}|^2 = l^2 + l^2$

$|\overrightarrow{BD}| = \sqrt{2l^2} = \sqrt{2}l$

$(\underset{\sim}{a} + \underset{\sim}{b}) \cdot (\underset{\sim}{b} - \underset{\sim}{a})$

$= |\underset{\sim}{a} + \underset{\sim}{b}||\underset{\sim}{b} - \underset{\sim}{a}| \cos\theta$

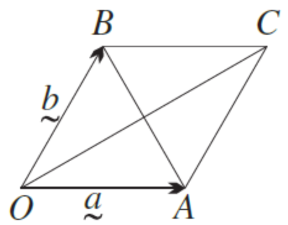
$= \sqrt{2}l \times \sqrt{2}l \times \cos 90°$

$= 2l^2 \times 0$

$= 0$

4e The diagonals of the square meet at right angles.

5a



$\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OB} = \underset{\sim}{b}$

$|\underset{\sim}{a}| = |\underset{\sim}{b}|$ because the sides of rhombus are equal.

5b $\underset{\sim}{a} \cdot \underset{\sim}{a} = |\underset{\sim}{a}|^2$

$\underset{\sim}{b} \cdot \underset{\sim}{b} = |\underset{\sim}{b}|^2 = |\underset{\sim}{a}|^2$ as $|\underset{\sim}{a}| = |\underset{\sim}{b}|$

Hence, $\underset{\sim}{a} \cdot \underset{\sim}{a} = \underset{\sim}{b} \cdot \underset{\sim}{b}$.

5c $\overrightarrow{OC} = \overrightarrow{OB} + \overrightarrow{BC}$

$= \overrightarrow{OB} + \overrightarrow{OA}$

$= \underset{\sim}{b} + \underset{\sim}{a}$

$= \underset{\sim}{a} + \underset{\sim}{b}$

$\overrightarrow{BA} = \overrightarrow{BO} + \overrightarrow{OA}$

$= \overrightarrow{OA} - \overrightarrow{OB}$

$= \underset{\sim}{a} - \underset{\sim}{b}$

5d $\overrightarrow{OC} \cdot \overrightarrow{BA}$

$= (\underset{\sim}{a} + \underset{\sim}{b}) \cdot (\underset{\sim}{a} - \underset{\sim}{b})$

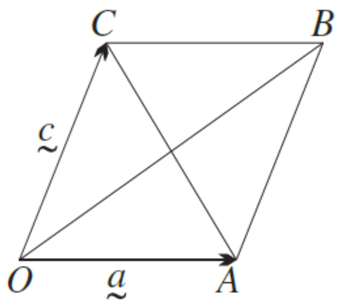
$= |\underset{\sim}{a}|^2 - |\underset{\sim}{b}|^2$

$= |\underset{\sim}{a}|^2 - |\underset{\sim}{a}|^2$, as $|\underset{\sim}{a}| = |\underset{\sim}{b}|$

$= 0$

5e Diagonals of a rhombus are perpendicular.

6a



The opposite sides of a parallelogram are equal, hence $\overrightarrow{CB} = \underset{\sim}{a}$.

6b $\overrightarrow{OB} = \overrightarrow{OC} + \overrightarrow{CB}$

$= \underset{\sim}{c} + \underset{\sim}{a}$

6c $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$

$= \overrightarrow{OC} - \overrightarrow{CB}$

$= \underset{\sim}{c} - \underset{\sim}{a}$

6d $|\underset{\sim}{c} + \underset{\sim}{a}| = |\underset{\sim}{c} - \underset{\sim}{a}|$ because $|\overrightarrow{OB}| = |\overrightarrow{AC}|$ (diagonals of this parallelogram are equal).

6e Since $|\underset{\sim}{c}+\underset{\sim}{a}| = |\underset{\sim}{c}-\underset{\sim}{a}|$,

$|\underset{\sim}{c}+\underset{\sim}{a}|^2 = |\underset{\sim}{c}-\underset{\sim}{a}|^2$

$|\underset{\sim}{c}+\underset{\sim}{a}|^2 - |\underset{\sim}{c}-\underset{\sim}{a}|^2 = 0$

$(\underset{\sim}{c}+\underset{\sim}{a})\cdot(\underset{\sim}{c}+\underset{\sim}{a}) - (\underset{\sim}{c}-\underset{\sim}{a})\cdot(\underset{\sim}{c}-\underset{\sim}{a}) = 0$

$\underset{\sim}{c}\cdot\underset{\sim}{c} + 2(\underset{\sim}{a}\cdot\underset{\sim}{c}) + \underset{\sim}{a}\cdot\underset{\sim}{a} - (\underset{\sim}{c}\cdot\underset{\sim}{c} - 2(\underset{\sim}{a}\cdot\underset{\sim}{c}) + \underset{\sim}{a}\cdot\underset{\sim}{a}) = 0$

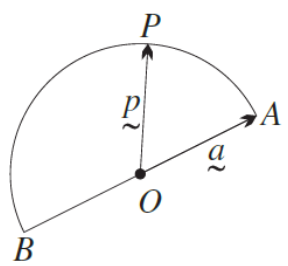
$|\underset{\sim}{c}|^2 + 2(\underset{\sim}{a}\cdot\underset{\sim}{c}) + |\underset{\sim}{a}|^2 - |\underset{\sim}{c}|^2 + 2(\underset{\sim}{a}\cdot\underset{\sim}{c}) - |\underset{\sim}{a}|^2 = 0$

$4(\underset{\sim}{a}\cdot\underset{\sim}{c}) = 0$

$\underset{\sim}{a}\cdot\underset{\sim}{c} = 0$

6f It is a rectangle as $\underset{\sim}{a}\cdot\underset{\sim}{c} = 0$

7a



$\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OP} = \underset{\sim}{p}$

Since $\overrightarrow{AB}$ is the diameter and $\overrightarrow{OB}$ is the radius in the opposite direction to $\overrightarrow{OA}$,

$\overrightarrow{OB} = -\underset{\sim}{a}$

7b $\overrightarrow{AP} = \overrightarrow{AO} + \overrightarrow{OP}$

$= \overrightarrow{OP} - \overrightarrow{OA}$

$= \underset{\sim}{p} - \underset{\sim}{a}$

$\overrightarrow{BP} = \overrightarrow{BP} + \overrightarrow{OP}$

$= \overrightarrow{OP} - \overrightarrow{OB}$

$= \underset{\sim}{p} - (-\underset{\sim}{a})$

$= \underset{\sim}{p} + \underset{\sim}{a}$

7c $\quad \overrightarrow{AP} \cdot \overrightarrow{BP} = (\underset{\sim}{p} - \underset{\sim}{a}) \cdot (\underset{\sim}{p} + \underset{\sim}{a})$

$= \underset{\sim}{p} \cdot \underset{\sim}{p} + \underset{\sim}{p} \cdot \underset{\sim}{a} - \underset{\sim}{a} \cdot \underset{\sim}{p} - \underset{\sim}{a} \cdot \underset{\sim}{a}$

$= \underset{\sim}{p} \cdot \underset{\sim}{p} - \underset{\sim}{a} \cdot \underset{\sim}{a}$

$= \left|\underset{\sim}{p}\right|^2 - \left|\underset{\sim}{a}\right|^2$

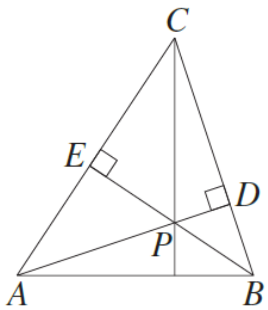
$= \left|\underset{\sim}{p}\right|^2 - \left|\underset{\sim}{p}\right|^2 \qquad$ (since $\overrightarrow{OA}$ and $\overrightarrow{OP}$ are radii and hence $\left|\underset{\sim}{a}\right| = \left|\underset{\sim}{p}\right|$)

$= 0$

Therefore, $\overrightarrow{AP}$ and $\overrightarrow{BP}$ are perpendicular. Hence, $\angle APB = 90°$.

7d $\quad$ An angle in a semi-circle is a right angle.

8a



$\overrightarrow{OA} = \underset{\sim}{a}$, $\overrightarrow{OB} = \underset{\sim}{b}$, $\overrightarrow{OC} = \underset{\sim}{c}$ and $\overrightarrow{OP} = \underset{\sim}{p}$

$\left(\underset{\sim}{p} - \underset{\sim}{a}\right) \cdot (\underset{\sim}{c} - \underset{\sim}{b})$

$= (\overrightarrow{OP} - \overrightarrow{OA}) \cdot (\overrightarrow{OC} - \overrightarrow{OB})$

$= \overrightarrow{AP} \cdot \overrightarrow{BC}$

Since P lies on the altitude from A to BC, $\overrightarrow{AP}$ is perpendicular to $\overrightarrow{BC}$.

Hence $\overrightarrow{AP} \cdot \overrightarrow{BC} = 0$.

Therefore $\left(\underline{p}-\underline{a}\right)\cdot(\underline{c}-\underline{b})=0$.

8b $\left(\underline{p}-\underline{b}\right)\cdot(\underline{a}-\underline{c})$

$=(\overrightarrow{OP}-\overrightarrow{OB})\cdot\left(\overrightarrow{OA}-\overrightarrow{OC}\right)$

$=\overrightarrow{BP}\cdot\overrightarrow{CA}$

Since P lies on the altitude from B to CA, $\overrightarrow{BP}$ is perpendicular to $\overrightarrow{CA}$.

Hence $\overrightarrow{BP}\cdot\overrightarrow{CA}=0$.

Therefore $\left(\underline{p}-b\right)\cdot(\underline{a}-\underline{c})=0$.

8c $\left(\underline{p}-\underline{a}\right)\cdot(\underline{c}-\underline{b})=0$ and $\left(\underline{p}-\underline{b}\right)\cdot(\underline{a}-\underline{c})=0$

$\left(\underline{p}-\underline{a}\right)\cdot(\underline{c}-\underline{b})+\left(\underline{p}-\underline{b}\right)\cdot(\underline{a}-\underline{c})=0$

$\underline{p}\cdot\underline{c}-\underline{p}\cdot\underline{b}-\underline{a}\cdot\underline{c}+\underline{a}\cdot\underline{b}+\underline{p}\cdot\underline{a}-\underline{p}\cdot\underline{c}-\underline{b}\cdot\underline{a}+\underline{b}\cdot\underline{c}=0$

$-\underline{p}\cdot\underline{b}-\underline{a}\cdot\underline{c}+\underline{p}\cdot\underline{a}+\underline{b}\cdot\underline{c}=0$

$\underline{p}\cdot\underline{a}-\underline{p}\cdot\underline{b}-\underline{a}\cdot\underline{c}+\underline{b}\cdot\underline{c}=0$

$\underline{p}(\underline{a}-\underline{b})-\underline{c}(\underline{a}-\underline{b})=0$

$\left(\underline{p}-\underline{c}\right)\cdot(\underline{a}-\underline{b})=0$

8d From part c,

$\left(\underline{p}-\underline{c}\right)\cdot(\underline{a}-\underline{b})=0$

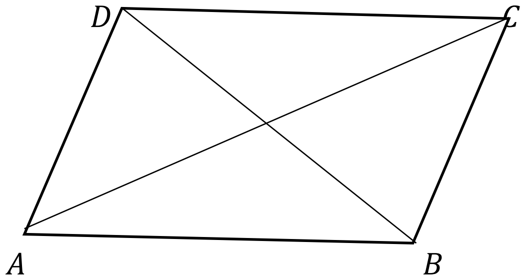
$\left(\overrightarrow{OP}-\overrightarrow{OC}\right)\cdot\left(\overrightarrow{OA}-\overrightarrow{OB}\right)=0$

$\overrightarrow{CP}\cdot\overrightarrow{BA}=0$

Hence $\overrightarrow{CP}$ is perpendicular to $\overrightarrow{BA}$ and P lies on the altitude from C to BA.

Hence, the three altitudes are concurrent.

9



Let $ABCD$ be a parallelogram.

Let $\underset{\sim}{a}$, $\underset{\sim}{b}$, $\underset{\sim}{c}$ and $\underset{\sim}{d}$ be the respective position vectors of A, B, C and D relative to an origin O.

As $ABCD$ is a parallelogram, then opposite sides of the parallelogram are equal.

$\overrightarrow{AB} = \overrightarrow{DC}$

$\overrightarrow{OB} - \overrightarrow{OA} = \overrightarrow{OC} - \overrightarrow{OD}$

$\underset{\sim}{b} - \underset{\sim}{a} = \underset{\sim}{c} - \underset{\sim}{d}$

$\underset{\sim}{b} + \underset{\sim}{d} = \underset{\sim}{a} + \underset{\sim}{c}$

We can also write,

$\frac{1}{2}(\underset{\sim}{b} + \underset{\sim}{d}) = \frac{1}{2}(\underset{\sim}{a} + \underset{\sim}{c})$

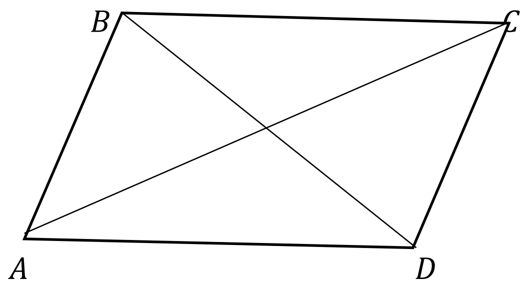
$\frac{1}{2}(\overrightarrow{OB} + \overrightarrow{OD}) = \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OC})$

$\frac{1}{2}\overrightarrow{BD} = \frac{1}{2}\overrightarrow{AC}$

$\overrightarrow{BD}$ and $\overrightarrow{AC}$ are the diagonals of the parallelogram.

Hence, this shows that the diagonals of the parallelogram bisect each other.

10



Let $ABCD$ be a parallelogram.

Let $\underset{\sim}{a}$, $\underset{\sim}{b}$, $\underset{\sim}{c}$ and $\underset{\sim}{d}$ be the respective position vectors of A, B, C and D relative to an origin O.

AC and BD are the two diagonals that intersect at P.

Hence,

$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$ and $\overrightarrow{BD} = \overrightarrow{BC} + \overrightarrow{CD} = \overrightarrow{BC} - \overrightarrow{AB}$

Now,

$|\overrightarrow{AC}|^2 + |\overrightarrow{BD}|^2 = |\overrightarrow{AB} + \overrightarrow{BC}|^2 + |\overrightarrow{BC} - \overrightarrow{AB}|^2$

We know that, $\underset{\sim}{a} \cdot \underset{\sim}{a} = |\underset{\sim}{a}|^2$

$|\overrightarrow{AC}|^2 + |\overrightarrow{BD}|^2 = |\overrightarrow{AB} + \overrightarrow{BC}| \cdot |\overrightarrow{AB} + \overrightarrow{BC}| + |\overrightarrow{BC} - \overrightarrow{AB}| \cdot |\overrightarrow{BC} - \overrightarrow{AB}|$

$|\overrightarrow{AC}|^2 + |\overrightarrow{BD}|^2 = |\overrightarrow{AB}|^2 + 2\overrightarrow{AB} \cdot \overrightarrow{BC} + |\overrightarrow{BC}|^2 + |\overrightarrow{BC}|^2 - 2\overrightarrow{AB} \cdot \overrightarrow{BC} + |\overrightarrow{AB}|^2$

$|\overrightarrow{AC}|^2 + |\overrightarrow{BD}|^2 = |\overrightarrow{AB}|^2 + |\overrightarrow{BC}|^2 + |\overrightarrow{BC}|^2 + |\overrightarrow{AB}|^2$

$ABCD$ is a parallelogram, so $|\overrightarrow{BC}| = |\overrightarrow{AD}|$ and $|\overrightarrow{AB}| = |\overrightarrow{CD}|$

Therefore, we can write

$|\overrightarrow{AC}|^2 + |\overrightarrow{BD}|^2 = |\overrightarrow{AB}|^2 + |\overrightarrow{AD}|^2 + |\overrightarrow{BC}|^2 + |\overrightarrow{CD}|^2$

Hence, proved.

11 Let $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OC} = \underset{\sim}{c}$.

Then $\overrightarrow{OM} = \frac{1}{2}\underset{\sim}{a}$, $\overrightarrow{OP} = k_1(\underset{\sim}{a} + \underset{\sim}{c})$ and $\overrightarrow{MP} = k_2\left(\underset{\sim}{c} - \frac{1}{2}\underset{\sim}{a}\right)$, where $|k_1|, |k_2| < 1$.

Since $\overrightarrow{OP} = \overrightarrow{OM} + \overrightarrow{MP}$,

$k_1(\underset{\sim}{a} + \underset{\sim}{c}) = \frac{1}{2}\underset{\sim}{a} + k_2\left(\underset{\sim}{c} - \frac{1}{2}\underset{\sim}{a}\right)$

from which we get $\left(k_1 + \frac{1}{2}k_2 - \frac{1}{2}\right)\underset{\sim}{a} = (k_2 - k_1)\underset{\sim}{c}$.

But $\underset{\sim}{a}$ and $\underset{\sim}{c}$ are not scalar multiples of each other, since they have

different directions. (They are linearly independent.)

So $k_1 + \frac{1}{2}k_2 - \frac{1}{2} = 0$ and $k_2 - k_1 = 0$

from which we get $k_1 = k_2 = \frac{1}{3}$, so $\overrightarrow{OP} = \frac{1}{3}\overrightarrow{OB}$.

12a $\overrightarrow{AP} = \overrightarrow{AO} + \overrightarrow{OP}$ and we are given $\overrightarrow{OP} = \frac{1}{2}\overrightarrow{OB}$

$$\begin{aligned}\overrightarrow{AP} &= \overrightarrow{AO} + \frac{1}{2}\overrightarrow{OB} \\ &= -\underset{\sim}{a} + \frac{1}{2}\underset{\sim}{b}\end{aligned}$$

Given that $\overrightarrow{AC} = \lambda_1\overrightarrow{AP}$, we obtain $\overrightarrow{AC} = \lambda_1\left(-\underset{\sim}{a} + \frac{1}{2}\underset{\sim}{b}\right)$.

So $\overrightarrow{AC} = \lambda_1\left(\frac{1}{2}\underset{\sim}{b} - \underset{\sim}{a}\right)$.

$\overrightarrow{BQ} = \overrightarrow{BO} + \overrightarrow{OQ}$ and we are given $\overrightarrow{OQ} = \frac{1}{2}\overrightarrow{OA}$

$$\begin{aligned}\overrightarrow{BQ} &= \overrightarrow{BO} + \frac{1}{2}\overrightarrow{OA} \\ &= -\underset{\sim}{b} + \frac{1}{2}\underset{\sim}{a}\end{aligned}$$

Given that $\overrightarrow{BC} = \lambda_2\overrightarrow{BQ}$, we obtain $\overrightarrow{BC} = \lambda_2\left(-\underset{\sim}{b} + \frac{1}{2}\underset{\sim}{a}\right)$.

So $\overrightarrow{BC} = \lambda_2\left(\frac{1}{2}\underset{\sim}{a} - \underset{\sim}{b}\right)$.

12b $\overrightarrow{BC}$ can also be obtained as follows:

$$\begin{aligned}\overrightarrow{BC} &= \overrightarrow{BA} + \overrightarrow{AC}\\ &= \overrightarrow{BO} + \overrightarrow{OA} + \overrightarrow{AC}\\ &= -\underset{\sim}{b} + \underset{\sim}{a} + \lambda_1\left(\frac{1}{2}\underset{\sim}{b} - \underset{\sim}{a}\right)\end{aligned}$$

So $\lambda_2\left(\frac{1}{2}\underset{\sim}{a} - \underset{\sim}{b}\right) = -\underset{\sim}{b} + \underset{\sim}{a} + \lambda_1\left(\frac{1}{2}\underset{\sim}{b} - \underset{\sim}{a}\right)$.

$$-\lambda_2\underset{\sim}{b} + \frac{\lambda_2}{2}\underset{\sim}{a} = (1-\lambda_1)\underset{\sim}{a} + \left(\frac{\lambda_1}{2} - 1\right)\underset{\sim}{b}$$

Since $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are non-zero vectors that are not parallel, we have:

$$\frac{\lambda_2}{2} = 1 - \lambda_1 \quad (1)$$

$$-\lambda_2 = \frac{\lambda_1}{2} - 1 \quad (2)$$

Multiply (1) by 2 and add to (2):

$$\begin{aligned}0 &= 2 - 2\lambda_1 + \frac{\lambda_1}{2} - 1\\ 1 &= \frac{3\lambda_1}{2}\\ \lambda_1 &= \frac{2}{3}\end{aligned}$$

Substituting $\lambda_1 = \frac{2}{3}$ into (1) we obtain $\lambda_2 = \frac{2}{3}$.

We have shown that $AC : CP = BC : CQ = 2 : 1$.

By symmetry, the point of intersection of the medians AP and OR must also divide AP in the ratio $2:1$ and therefore must be C.

Hence the three medians are concurrent at C.

Solutions to Exercise 8E

1a $\underset{\sim}{a} = \underset{\sim}{i} + \underset{\sim}{j}\,,\, \underset{\sim}{b} = \underset{\sim}{i}$

$$Proj_{\underset{\sim}{b}}\underset{\sim}{a} = \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{\underset{\sim}{b} \cdot \underset{\sim}{b}} \times \underset{\sim}{b}$$

$$= \frac{1 \times 1 + 1 \times 0}{1 \times 1 + 0 \times 0} \times \underset{\sim}{i}$$

$$= \frac{1}{1} \times \underset{\sim}{i}$$

$$= \underset{\sim}{i}$$

1b $\underset{\sim}{a} = \underset{\sim}{i} + 2\underset{\sim}{j}\,,\, \underset{\sim}{b} = \underset{\sim}{j}$

$$Proj_{\underset{\sim}{b}}\underset{\sim}{a} = \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{\underset{\sim}{b} \cdot \underset{\sim}{b}} \times \underset{\sim}{b}$$

$$= \frac{1 \times 0 + 2 \times 1}{0 \times 0 + 1 \times 1} \times \underset{\sim}{j}$$

$$= \frac{2}{1} \times \underset{\sim}{j}$$

$$= 2\underset{\sim}{j}$$

1c $\underset{\sim}{a} = -3\underset{\sim}{i} + 2\underset{\sim}{j}\,,\, \underset{\sim}{b} = \underset{\sim}{i}$

$$Proj_{\underset{\sim}{b}}\underset{\sim}{a} = \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{\underset{\sim}{b} \cdot \underset{\sim}{b}} \times \underset{\sim}{b}$$

$$= \frac{-3 \times 1 + 2 \times 0}{1 \times 1 + 0 \times 0} \times \underset{\sim}{i}$$

$$= \frac{-3}{1} \times \underset{\sim}{i}$$

$$= -3\underset{\sim}{i}$$

2a $\underset{\sim}{a} = 2\underset{\sim}{i} + 3\underset{\sim}{j}\,,\, \underset{\sim}{b} = \underset{\sim}{i}$

Length of $Proj_{\underset{\sim}{b}}\underset{\sim}{a}$

$$= \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{b}|}$$

$$= \frac{2 \times 1 + 3 \times 0}{\sqrt{1^2 + 0^2}}$$

$$= \frac{2}{1}$$

$$= 2$$

2b $\underset{\sim}{a} = -2\underset{\sim}{i} - 4\underset{\sim}{j}\,, \underset{\sim}{b} = \underset{\sim}{j}$

Length of $Proj_{\underset{\sim}{b}}\underset{\sim}{a}$

$$= \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{b}|}$$

$$= \frac{-2 \times 0 - 4 \times 1}{\sqrt{0^2 + 1^2}}$$

$$= \frac{-4}{1}$$

$$= -4$$

Hence, length is 4.

2c $\underset{\sim}{a} = -6\sqrt{2}\underset{\sim}{i} + 8\sqrt{2}\underset{\sim}{j}\,, \underset{\sim}{b} = \underset{\sim}{i}$

Length of $Proj_{\underset{\sim}{b}}\underset{\sim}{a}$

$$= \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{b}|}$$

$$= \frac{-6\sqrt{2} \times 1 + 8\sqrt{2} \times 0}{\sqrt{1^2 + 0^2}}$$

$$= \frac{-6\sqrt{2}}{1}$$

$$= -6\sqrt{2}$$

Hence, length is $6\sqrt{2}$.

3a $\underset{\sim}{a} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\underset{\sim}{b} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$

$$Proj_{\underset{\sim}{b}}\underset{\sim}{a} = \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{\underset{\sim}{b} \cdot \underset{\sim}{b}} \times \underset{\sim}{b}$$

$$= \frac{2 \times 4 + 1 \times 0}{4 \times 4 + 0 \times 0} \times \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$= \frac{8}{16} \times \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$= \frac{1}{2} \times \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

3b $\underset{\sim}{a} = 3\underset{\sim}{i} + 3\underset{\sim}{j}$, $\underset{\sim}{b} = 2\underset{\sim}{j}$

$$Proj_{\underset{\sim}{b}}\underset{\sim}{a} = \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{\underset{\sim}{b} \cdot \underset{\sim}{b}} \times \underset{\sim}{b}$$

$$= \frac{3 \times 0 + 3 \times 2}{0 \times 0 + 2 \times 2} \times 2\underset{\sim}{j}$$

$$= \frac{6}{4} \times 2\underset{\sim}{j}$$

$$= 3\underset{\sim}{j}$$

3c $\underset{\sim}{a} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$, $\underset{\sim}{b} = \begin{bmatrix} -6 \\ 0 \end{bmatrix}$

$$Proj_{\underset{\sim}{b}}\underset{\sim}{a} = \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{\underset{\sim}{b} \cdot \underset{\sim}{b}} \times \underset{\sim}{b}$$

$$= \frac{5 \times (-6) + (-3) \times 0}{(-6) \times (-6) + 0 \times 0} \times \begin{bmatrix} -6 \\ 0 \end{bmatrix}$$

$$= \frac{-30}{36} \times \begin{bmatrix} -6 \\ 0 \end{bmatrix}$$

$$= \frac{-5}{6} \times \begin{bmatrix} -6 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

4a $\quad |\overrightarrow{OA}| = 6, \quad \angle AOB = 30°$

$$Proj_{\underline{\overrightarrow{OB}}}\underline{\overrightarrow{OA}} = |\overrightarrow{OA}| \cos 30°$$

$$= 6 \times \cos 30°$$

$$= 6 \times \frac{\sqrt{3}}{2}$$

$$= 3\sqrt{3}$$

4b $\quad |\overrightarrow{OA}| = 6\sqrt{6}, \quad \angle AOB = 45°$

$$Proj_{\underline{\overrightarrow{OB}}}\underline{\overrightarrow{OA}} = |\overrightarrow{OA}| \cos 45°$$

$$= 6\sqrt{6} \times \cos 45°$$

$$= 6\sqrt{6} \times \frac{1}{\sqrt{2}}$$

$$= 6\sqrt{3}$$

5 $\quad \underline{a} = \begin{bmatrix} 10 \\ -2 \end{bmatrix}, \ \underline{b} = \begin{bmatrix} 1 \\ -7 \end{bmatrix}$

Length of $Proj_{\underline{b}}\underline{a}$

$$= \frac{\underline{a} \cdot \underline{b}}{|\underline{b}|}$$

$$= \frac{10 \times 1 + (-2) \times (-7)}{\sqrt{1^2 + (-7)^2}}$$

$$= \frac{24}{\sqrt{50}}$$

$$= \frac{24}{5\sqrt{2}}$$

$$= \frac{12\sqrt{2}}{5}$$

6a $\underset{\sim}{a} = \begin{bmatrix}1\\2\end{bmatrix}$ $\underset{\sim}{b} = \begin{bmatrix}2\\2\end{bmatrix}$

$$Proj_{\underset{\sim}{b}}\underset{\sim}{a} = \frac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{\underset{\sim}{b}\cdot\underset{\sim}{b}} \times \underset{\sim}{b}$$

$$= \frac{1\times 2 + 2\times 2}{2\times 2 + 2\times 2} \times \begin{bmatrix}2\\2\end{bmatrix}$$

$$= \frac{6}{8} \times \begin{bmatrix}2\\2\end{bmatrix}$$

$$= \frac{3}{4} \times \begin{bmatrix}2\\2\end{bmatrix}$$

$$= \begin{bmatrix}\frac{3}{2}\\\frac{3}{2}\end{bmatrix}$$

6b $\underset{\sim}{a} = \underset{\sim}{i} + \underset{\sim}{j}$, $\underset{\sim}{b} = 3\underset{\sim}{i} - \underset{\sim}{j}$

$$Proj_{\underset{\sim}{b}}\underset{\sim}{a} = \frac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{\underset{\sim}{b}\cdot\underset{\sim}{b}} \times \underset{\sim}{b}$$

$$= \frac{1\times 3 + 1\times(-1)}{3\times 3 + (-1)\times(-1)} \times \left(3\underset{\sim}{i} - \underset{\sim}{j}\right)$$

$$= \frac{2}{10} \times (3\underset{\sim}{i} - \underset{\sim}{j})$$

$$= \frac{1}{5} \times (3\underset{\sim}{i} - \underset{\sim}{j})$$

$$= \frac{3}{5}\underset{\sim}{i} - \frac{1}{5}\underset{\sim}{j}$$

6c $\underset{\sim}{a} = \begin{bmatrix}-5\\5\end{bmatrix}$ $\underset{\sim}{b} = \begin{bmatrix}-6\\8\end{bmatrix}$

$$Proj_{\underset{\sim}{b}}\underset{\sim}{a} = \frac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{\underset{\sim}{b}\cdot\underset{\sim}{b}} \times \underset{\sim}{b}$$

$$= \frac{(-5)\times(-6) + 5\times 8}{(-6)\times(-6) + 8\times 8} \times \begin{bmatrix}-6\\8\end{bmatrix}$$

$$= \frac{70}{100} \times \begin{bmatrix}-6\\8\end{bmatrix}$$

$$= \frac{7}{10} \times \begin{bmatrix} -6 \\ 8 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{21}{5} \\ \frac{28}{5} \end{bmatrix}$$

7a $\underset{\sim}{a} = \underset{\sim}{i} + \underset{\sim}{j}, \underset{\sim}{b} = 3\underset{\sim}{i} + \underset{\sim}{j}$

$$Proj_{\underset{\sim}{b}}\underset{\sim}{a} = \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{\underset{\sim}{b} \cdot \underset{\sim}{b}} \times \underset{\sim}{b}$$

$$= \frac{1 \times 3 + 1 \times 1}{3 \times 3 + 1 \times 1} \times \left(3\underset{\sim}{i} + \underset{\sim}{j}\right)$$

$$= \frac{4}{10} \times (3\underset{\sim}{i} + \underset{\sim}{j})$$

$$= \frac{2}{5} \times (3\underset{\sim}{i} + \underset{\sim}{j})$$

$$= \frac{6}{5}\underset{\sim}{i} + \frac{2}{5}\underset{\sim}{j}$$

7b $\underset{\sim}{a} = 4\underset{\sim}{i} - 3\underset{\sim}{j}, \underset{\sim}{b} = 6\underset{\sim}{i} + 2\underset{\sim}{j}$

$$Proj_{\underset{\sim}{b}}\underset{\sim}{a} = \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{\underset{\sim}{b} \cdot \underset{\sim}{b}} \times \underset{\sim}{b}$$

$$= \frac{4 \times 6 + (-3) \times 2}{6 \times 6 + 2 \times 2} \times \left(6\underset{\sim}{i} + 2\underset{\sim}{j}\right)$$

$$= \frac{18}{40} \times (6\underset{\sim}{i} + 2\underset{\sim}{j})$$

$$= \frac{9}{20} \times (6\underset{\sim}{i} + 2\underset{\sim}{j})$$

$$= \frac{27}{10}\underset{\sim}{i} + \frac{9}{10}\underset{\sim}{j}$$

8a $\underset{\sim}{a} = -2\underset{\sim}{i}\,,\ \underset{\sim}{b} = -3\underset{\sim}{i} - 2\underset{\sim}{j}$

Length of $Proj_{\underset{\sim}{b}}\underset{\sim}{a}$

$$= \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{b}|}$$

$$= \frac{(-2) \times (-3) + 0 \times (-2)}{\sqrt{(-3)^2 + (-2)^2}}$$

$$= \frac{6}{\sqrt{13}}$$

Hence, the magnitude of $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b}$ is $\frac{6}{\sqrt{13}}$.

8b $\underset{\sim}{a} = 6\underset{\sim}{i} - 4\underset{\sim}{j}\,,\ \underset{\sim}{b} = -3\underset{\sim}{i} + 6\underset{\sim}{j}$

Length of $Proj_{\underset{\sim}{b}}\underset{\sim}{a}$

$$= \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{b}|}$$

$$= \frac{6 \times (-3) + (-4) \times 6}{\sqrt{(-3)^2 + 6^2}}$$

$$= \frac{-18 - 24}{\sqrt{(-3)^2 + 6^2}}$$

$$= \frac{-42}{\sqrt{45}}$$

$$= \frac{-42}{3\sqrt{5}}$$

$$= \frac{-14}{\sqrt{5}}$$

Hence, the magnitude of $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b}$ is $\frac{14}{\sqrt{5}}$.

9 Let O be the origin and $\overrightarrow{OA} = -3\underset{\sim}{i} - 7\underset{\sim}{j}$ and $\overrightarrow{OB} = \underset{\sim}{i} + 5\underset{\sim}{j}$ be the position vectors.

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$= \left(\underset{\sim}{i} + 5\underset{\sim}{j}\right) - \left(-3\underset{\sim}{i} - 7\underset{\sim}{j}\right)$$

$$= \underset{\sim}{i} + 5\underset{\sim}{j} + 3\underset{\sim}{i} + 7\underset{\sim}{j}$$

$$= 4\underset{\sim}{i} + 12\underset{\sim}{j}$$

Let $\underset{\sim}{b} = -6\underset{\sim}{i} + 4\underset{\sim}{j}$

$$Proj_{\underset{\sim}{b}}\overrightarrow{AB} = \frac{\overrightarrow{AB} \cdot \underset{\sim}{b}}{\underset{\sim}{b} \cdot \underset{\sim}{b}} \times \underset{\sim}{b}$$

$$= \frac{4 \times (-6) + 12 \times 4}{(-6) \times (-6) + 4 \times 4} \times \left(-6\underset{\sim}{i} + 4\underset{\sim}{j}\right)$$

$$= \frac{24}{52} \times (-6\underset{\sim}{i} + 4\underset{\sim}{j})$$

$$= \frac{6}{13} \times (-6\underset{\sim}{i} + 4\underset{\sim}{j})$$

$$= -\frac{36}{13}\underset{\sim}{i} + \frac{24}{13}\underset{\sim}{j}$$

10 Let O be the origin and let $\overrightarrow{OA} = \underset{\sim}{i} + 3\underset{\sim}{j}$, $\overrightarrow{OB} = 6\underset{\sim}{i} + 18\underset{\sim}{j}$, $\overrightarrow{OC} = 9\underset{\sim}{i} + 4\underset{\sim}{j}$ and $\overrightarrow{OD} = 19\underset{\sim}{i} + 24\underset{\sim}{j}$ be the position vectors.

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$= \left(6\underset{\sim}{i} + 18\underset{\sim}{j}\right) - \left(\underset{\sim}{i} + 3\underset{\sim}{j}\right)$$

$$= 6\underset{\sim}{i} + 18\underset{\sim}{j} - \underset{\sim}{i} - 3\underset{\sim}{j}$$

$$= 5\underset{\sim}{i} + 15\underset{\sim}{j}$$

$$\overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC}$$

$$= \left(19\underset{\sim}{i} + 24\underset{\sim}{j}\right) - \left(9\underset{\sim}{i} + 4\underset{\sim}{j}\right)$$

$$= 19\underset{\sim}{i} + 24\underset{\sim}{j} - 9\underset{\sim}{i} - 4\underset{\sim}{j}$$

$$= 10\underset{\sim}{i} + 20\underset{\sim}{j}$$

Length of $Proj_{\overrightarrow{CD}}\overrightarrow{AB}$

$$= \frac{\overrightarrow{AB} \cdot \overrightarrow{CD}}{|\overrightarrow{CD}|}$$

$$= \frac{5 \times 10 + 15 \times 20}{\sqrt{10^2 + 20^2}}$$

$$= \frac{50 + 300}{\sqrt{100 + 400}}$$

$$= \frac{350}{\sqrt{500}}$$

$$= \frac{350}{10\sqrt{5}}$$

$$= \frac{35}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}}$$

$$= 7\sqrt{5}$$

11 Let $\underset{\sim}{a} = \lambda \underset{\sim}{i} + 4\underset{\sim}{j}$ and $\underset{\sim}{b} = 12\underset{\sim}{i} - 5\underset{\sim}{j}$

Length of $Proj_{\underset{\sim}{b}}\underset{\sim}{a}$

$$= \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{b}|}$$

$$= \frac{\lambda \times 12 + 4 \times (-5)}{\sqrt{12^2 + (-5)^2}}$$

$$= \frac{12\lambda - 20}{\sqrt{144 + 25}}$$

$$= \frac{12\lambda - 20}{\sqrt{169}}$$

$$= \frac{12\lambda - 20}{13}$$

Now $\left|\frac{12\lambda - 20}{13}\right| = \frac{140}{13}$

$$\frac{12\lambda - 20}{13} = \frac{140}{13} \text{ or } -\left(\frac{12\lambda - 20}{13}\right) = \frac{140}{13}$$

$$12\lambda - 20 = 140 \text{ or } -(12\lambda - 20) = 140$$

$12\lambda - 20 = 140$ or $-12\lambda + 20 = 140$

$12\lambda = 160$ or $-12\lambda = 120$

$\lambda = \frac{40}{3}$ or $\lambda = -10$

12 For any scalar λ and $\underset{\sim}{x} = \underset{\sim}{i}$,

$\lambda\underset{\sim}{u} = \lambda\left(\underset{\sim}{i} + \underset{\sim}{j}\right) = \lambda\underset{\sim}{i} + \lambda\underset{\sim}{j}$

$$Proj_{\underset{\sim}{x}}\lambda\underset{\sim}{u} = \frac{\underset{\sim}{u}\cdot\underset{\sim}{x}}{\underset{\sim}{x}\cdot\underset{\sim}{x}} \times \underset{\sim}{x}$$

$$= \frac{\lambda\times 1 + \lambda\times 0}{1\times 1 + 0\times 0} \times \underset{\sim}{i}$$

$$= \frac{\lambda}{1} \times \underset{\sim}{i}$$

$$= \lambda\underset{\sim}{i}$$

$$= \lambda(Proj_{\underset{\sim}{x}}\underset{\sim}{u})$$

Hence, proved.

Let $\underset{\sim}{u} = a\underset{\sim}{i} + b\underset{\sim}{j}$ and $\underset{\sim}{v} = c\underset{\sim}{i} + d\underset{\sim}{j}$.

$$Proj_{\underset{\sim}{x}}\underset{\sim}{u} = \frac{\underset{\sim}{u}\cdot\underset{\sim}{x}}{\underset{\sim}{x}\cdot\underset{\sim}{x}} \times \underset{\sim}{x}$$

$$= \frac{a\times 1 + b\times 0}{1\times 1 + 0\times 0} \times \underset{\sim}{i}$$

$$= \frac{a}{1} \times \underset{\sim}{i}$$

$$= a\underset{\sim}{i}$$

$$Proj_{\underset{\sim}{x}}\underset{\sim}{v} = \frac{\underset{\sim}{v}\cdot\underset{\sim}{x}}{\underset{\sim}{x}\cdot\underset{\sim}{x}} \times \underset{\sim}{x}$$

$$= \frac{c\times 1 + d\times 0}{1\times 1 + 0\times 0} \times \underset{\sim}{i}$$

$$= \frac{c}{1} \times \underset{\sim}{i}$$

$$= c\underset{\sim}{i}$$

$\underset{\sim}{u} + \underset{\sim}{v}$

$= a\underset{\sim}{i} + b\underset{\sim}{j} + c\underset{\sim}{i} + d\underset{\sim}{j}$

$= (a+c)\underset{\sim}{i} + (b+d)\underset{\sim}{j}$

$$Proj_{\underset{\sim}{x}}(\underset{\sim}{u} + \underset{\sim}{v}) = \frac{(\underset{\sim}{u} + \underset{\sim}{v}) \cdot \underset{\sim}{x}}{\underset{\sim}{x} \cdot \underset{\sim}{x}} \times \underset{\sim}{x}$$

$$= \frac{(a+c) \times 1 + (b+d) \times 0}{1 \times 1 + 0 \times 0} \times \underset{\sim}{i}$$

$$= \frac{(a+c)}{1} \times \underset{\sim}{i}$$

$$= (a+c)\underset{\sim}{i}$$

$$= a\underset{\sim}{i} + c\underset{\sim}{i}$$

$$= Proj_{\underset{\sim}{x}}\underset{\sim}{u} + Proj_{\underset{\sim}{x}}\underset{\sim}{v}$$

Hence, proved.

13a The line l has equation $x + 3y + 10 = 0$.

$3y = -x - 10 \Rightarrow y = -\frac{1}{3}x - \frac{10}{3}$

Hence the gradient of l is $-\frac{1}{3}$.

13b A vector $\underset{\sim}{v}$ that is parallel to l is $\underset{\sim}{v} = -3\underset{\sim}{i} + \underset{\sim}{j}$. Note that this one such vector.

13c When $x = 2, y = -4$.

Substituting $x = 2, y = -4$ into $x + 3y + 10 = 0$ we obtain:

$\text{LHS} = 2 - 12 + 10 = 0$

So $\text{LHS} = \text{RHS}$ and hence l passes through point A.

13d $\overrightarrow{AP} = \overrightarrow{AO} + \overrightarrow{OP}$

$$\begin{aligned}\overrightarrow{AP} &= \overrightarrow{OP} - \overrightarrow{OA}\\ &= \left(25\underset{\sim}{i} - 5\underset{\sim}{j}\right) - \left(2\underset{\sim}{i} - 4\underset{\sim}{j}\right)\\ &= 23\underset{\sim}{i} - \underset{\sim}{j}\end{aligned}$$

13e The length of the projection of $\overrightarrow{AP}$ onto $\underset{\sim}{v}$ is given by $\left|\overrightarrow{AP} \cdot \hat{\underset{\sim}{v}}\right|$.

$$\begin{aligned}\left|\overrightarrow{AP} \cdot \hat{\underset{\sim}{v}}\right| &= \left|\left(23\underset{\sim}{i} - \underset{\sim}{j}\right) \cdot \frac{1}{\sqrt{10}}\left(-3\underset{\sim}{i} + \underset{\sim}{j}\right)\right|\\ &= \left|\frac{1}{\sqrt{10}}(-69-1)\right|\\ &= \left|-7\sqrt{10}\right|\\ &= 7\sqrt{10}\end{aligned}$$

13f Let the perpendicular distance from P to l be d.

$$\begin{aligned}d^2 + \left(7\sqrt{10}\right)^2 &= AP^2\\ d^2 &= 530 - 490\\ &= 40\end{aligned}$$

Hence $d = 2\sqrt{10}$.

14 Let P be the point $(0,0)$.

The line l has equation $ax + by + c = 0$.

$$by = -ax - c \Rightarrow y = -\frac{a}{b}x - \frac{c}{b}$$

Hence the gradient of l is $-\frac{a}{b}$.

A vector $\underset{\sim}{v}$ that is parallel to l is $\underset{\sim}{v} = -b\underset{\sim}{i} + a\underset{\sim}{j}$. Note that this one such vector.

Let A be the point (x_1, y_1) on l.

$\overrightarrow{AP} = \overrightarrow{AO} + \overrightarrow{OP}$

$$\begin{aligned}\overrightarrow{AP} &= \overrightarrow{OP} - \overrightarrow{OA}\\ &= \left(0\underset{\sim}{i} + 0\underset{\sim}{j}\right) - \left(x_1\underset{\sim}{i} + y_1\underset{\sim}{j}\right)\\ &= -x_1\underset{\sim}{i} - y_1\underset{\sim}{j}\end{aligned}$$

The length of the projection of $\overrightarrow{AP}$ onto $\underset{\sim}{v}$ is given by $\left|\overrightarrow{AP}\cdot\hat{\underset{\sim}{v}}\right|$.

$$\begin{aligned}\left|\overrightarrow{AP}\cdot\hat{\underset{\sim}{v}}\right| &= \left|\left(-x_1\underset{\sim}{i} - y_1\underset{\sim}{j}\right)\cdot\frac{1}{\sqrt{a^2+b^2}}\left(-b\underset{\sim}{i} + a\underset{\sim}{j}\right)\right|\\ &= \left|\frac{1}{\sqrt{a^2+b^2}}\left(bx_1 - ay_1\right)\right|\end{aligned}$$

Let the perpendicular distance from P to l be d.

$$d^2 + \left(\frac{1}{\sqrt{a^2+b^2}}\left(bx_1 - ay_1\right)\right)^2 = AP^2$$

$$\begin{aligned}d^2 &= \left(x_1^{\,2} + y_1^{\,2}\right) - \frac{1}{a^2+b^2}\left(bx_1 - ay_1\right)^2\\ &= \frac{\left(a^2+b^2\right)\left(x_1^{\,2}+y_1^{\,2}\right) - \left(bx_1 - ay_1\right)^2}{a^2+b^2}\\ &= \frac{a^2x_1^{\,2} + a^2y_1^{\,2} + b^2x_1^{\,2} + b^2y_1^{\,2} - \left(b^2x_1^{\,2} - 2abx_1y_1 + a^2y_1^{\,2}\right)}{a^2+b^2}\\ &= \frac{a^2x_1^{\,2} + 2abx_1y_1 + b^2y_1^{\,2}}{a^2+b^2}\\ &= \frac{\left(ax_1 + by_1\right)^2}{a^2+b^2}\end{aligned}$$

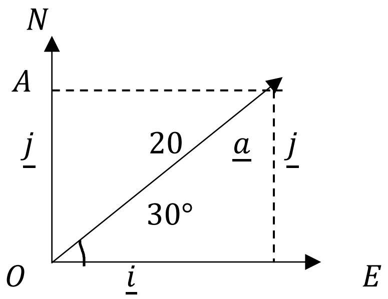
Substituting $x = x_1$ and $y = y_1$ into $ax + by + c = 0$ and rearranging we obtain $-c = ax_1 + by_1$.

Substituting $-c = ax_1 + by_1$ into $d^2 = \dfrac{\left(ax_1+by_1\right)^2}{a^2+b^2}$ we obtain $d^2 = \dfrac{(-c)^2}{a^2+b^2}$.

Hence $d = \dfrac{|c|}{\sqrt{a^2+b^2}}$.

Solutions to Exercise 8F

1



Initial speed is 20 m/s.

Using trigonometry,

$\cos 30° = \dfrac{\underline{i}}{\underline{a}}$

$\dfrac{\sqrt{3}}{2} = \dfrac{\underline{i}}{20}$

$2\underline{i} = 20\sqrt{3}$

$\underline{i} = 10\sqrt{3}$

$\tan 30° = \dfrac{\underline{j}}{\underline{i}}$

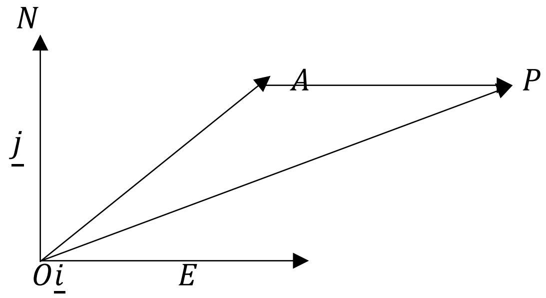
$\dfrac{1}{\sqrt{3}} = \dfrac{\underline{j}}{10\sqrt{3}}$

$\sqrt{3}\underline{j} = 10\sqrt{3}$

$\underline{j} = 10$

Initial horizontal component of velocity is $10\sqrt{3}$ m/s and initial vertical component of velocity is 10 m/s.

2



Let $\overrightarrow{OA} = 4\underset{\sim}{i} + 5\underset{\sim}{j}$ be the position vector and $\overrightarrow{AP} = 3\underset{\sim}{i} - 2\underset{\sim}{j}$ be the velocity vector.

Let's say, in t seconds, it moves from $\overrightarrow{AP}$ hence,

$\overrightarrow{OP} = \overrightarrow{OA} + t \times \overrightarrow{AP}$

When $t = 7$ s,

$\overrightarrow{OP} = 4\underset{\sim}{i} + 5\underset{\sim}{j} + 7 \times (3\underset{\sim}{i} - 2\underset{\sim}{j})$

$\overrightarrow{OP} = 4\underset{\sim}{i} + 5\underset{\sim}{j} + 21\underset{\sim}{i} - 14\underset{\sim}{j}$

$\overrightarrow{OP} = 25\underset{\sim}{i} - 9\underset{\sim}{j}$

Hence, position vector after 7 seconds is $25\underset{\sim}{i} - 9\underset{\sim}{j}$.

3 Let $\underset{\sim}{u} = \left(2\underset{\sim}{i} - 3\underset{\sim}{j}\right)$ N, $\underset{\sim}{v} = \left(4\underset{\sim}{i} + \underset{\sim}{j}\right)$ N and $\underset{\sim}{w} = \left(-3\underset{\sim}{i} + 3\underset{\sim}{j}\right)$ N

Resultant vector $\underset{\sim}{a} = \underset{\sim}{u} + \underset{\sim}{v} + \underset{\sim}{w}$

$\underset{\sim}{a} = (2\underset{\sim}{i} - 3\underset{\sim}{j}) + (4\underset{\sim}{i} + \underset{\sim}{j}) + (-3\underset{\sim}{i} + 3\underset{\sim}{j})$

$= 2\underset{\sim}{i} - 3\underset{\sim}{j} + 4\underset{\sim}{i} + \underset{\sim}{j} - 3\underset{\sim}{i} + 3\underset{\sim}{j}$

$= 3\underset{\sim}{i} + \underset{\sim}{j}$

The magnitude of resultant vector is

$|\underset{\sim}{a}| = \sqrt{3^2 + 1^2}$

$= \sqrt{9 + 1}$

$= \sqrt{10}$ N

4 The resultant force is

$\underset{\sim}{F} = 30\underset{\sim}{i} + 16\underset{\sim}{j}$

The magnitude of resultant force is

$|\underset{\sim}{F}| = \sqrt{30^2 + 16^2}$

$= \sqrt{900 + 256}$

$= 34 \text{ N}$

The direction of resultant force, if θ is the angle between $\underset{\sim}{i}$ and $\underset{\sim}{F}$

and using simple trigonometry:

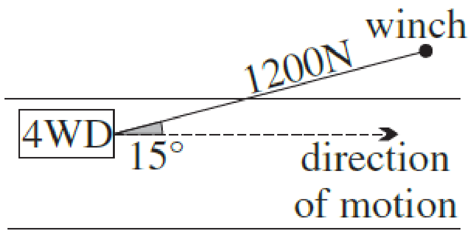
$\tan\theta = \dfrac{16}{30}$

$\theta = 28.072\ldots°$

$\doteqdot 28°$

The resultant force is 34 N at about 28° to the 30 N force.

5a



Magnitude of the component of the force in the direction of motion

$= 1200\cos 15°$

$= 1159.110\ldots$

$\doteqdot 1159 \text{ N}$

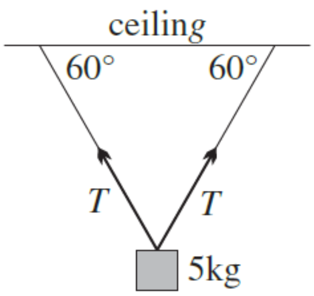
5b Magnitude of the component of the force in the perpendicular direction of motion

$= 1200 \sin 15°$

$= 310.582 \ldots$

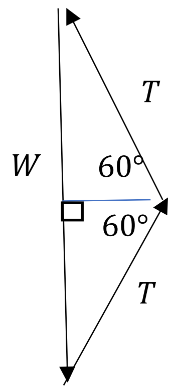
$\doteqdot 311 \text{ N}$

6



The weight force, W, has magnitude $|W| = 5 \times 9.8 = 49 \text{ N}$ (using $F = mg$).

For no movement, the resultant force due to tension and weight is zero as shown in the vector diagram below.



Using trigonometry,

$$\sin 60° = \frac{\frac{1}{2}|W|}{|T|}$$

$$|T| = \frac{\frac{1}{2}|W|}{\sin 60°}$$

$$= \frac{24.5}{\sin 60°}$$

$$= 28.290 \ldots$$

$$\doteqdot 28.3 \text{ N}$$

Alternatively, the magnitude of the vertical components of the tension forces should equal the magnitude of the vertical weight force.

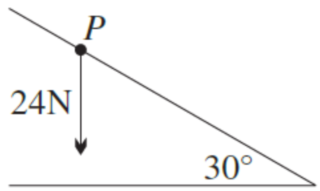
$$2|T| \sin 60° = 49$$

$$|T| \sin 60° = 24.5$$

$$|\underset{\sim}{T}| = \frac{24.5}{\sin 60°}$$

$$\doteqdot 28.3 \text{ N}$$

7a



Component of the weight down the plane

$$= 24 \sin 30°$$

$$= 24 \times \frac{1}{2}$$

$$= 12 \text{ N}$$

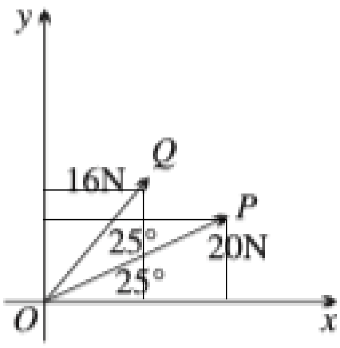
7b Component of the weight perpendicular to the plane

$$= 24 \cos 30°$$

$$= 24 \times \frac{\sqrt{3}}{2}$$

$$= 12\sqrt{3} \text{ N}$$

8a



Let $\underline{i}$ and $\underline{j}$ be the unit vectors of horizontal and vertical plane respectively.

For $\overrightarrow{OP}$, horizontal position vector is $(20\cos 25°)\underline{i}$ while vertical position vector is $(20\sin 25°)\underline{j}$.

Hence, $\overrightarrow{OP} = (20\cos 25°)\underline{i} + (20\sin 25°)\underline{j}$.

The angle between $\overrightarrow{OQ}$ and plane will be $25° + 25° = 50°$.

For $\overrightarrow{OQ}$, horizontal position vector is $(16\cos 50°)\underline{i}$ while vertical position vector is $(16\sin 50°)\underline{j}$.

Hence, $\overrightarrow{OQ} = (16\cos 50°)\underline{i} + (16\sin 50°)\underline{j}$.

8b Resultant vector of the two forces:

$$\begin{aligned}\underline{R} &= \overrightarrow{OP} + \overrightarrow{OQ}\\ &= (20\cos 25°)\underline{i} + (20\sin 25°)\underline{j} + (16\cos 50°)\underline{i} + (16\sin 50°)\underline{j}\\ &= (20\cos 25° + 16\cos 50°)\underline{i} + (20\sin 25° + 16\sin 50°)\underline{j}\\ &= (28.410\ldots)\underline{i} + (20.709\ldots)\underline{j}\\ &\doteqdot 28\underline{i} + 21\underline{j}\end{aligned}$$

Magnitude of resultant force:

$$\begin{aligned}|\underline{R}| &= \sqrt{28^2 + 21^2}\\ &= \sqrt{784 + 441}\\ &= 35\text{ N}\end{aligned}$$

Let θ be the angle between $\underline{i}$ and $\underline{R}$.

For the direction of the resultant force, $\underset{\sim}{R}$:

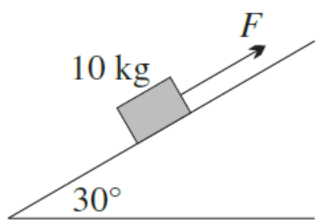
$$\tan\theta = \frac{20\sin 25° + 16\sin 50°}{20\cos 25° + 16\cos 50°} \qquad \text{(using exact values from above)}$$

$\theta = 36.088\ldots°$

$\quad \doteqdot 36°$

The resultant force is 35 N in a direction of about 36° above the horizontal.

9



The weight force, W, has magnitude $|W| = 10 \times 9.8 = 98$ N (using $F = mg$).

Magnitude of the component of the force F acting parallel to the plane

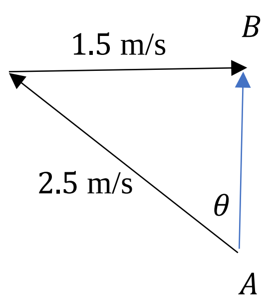
$= |W|\sin 30°$

$= 98\sin 30°$

$= 98 \times \dfrac{1}{2}$

$= 49$ N

10



Let θ be the angle between the line AB and the direction Sam should row from A.

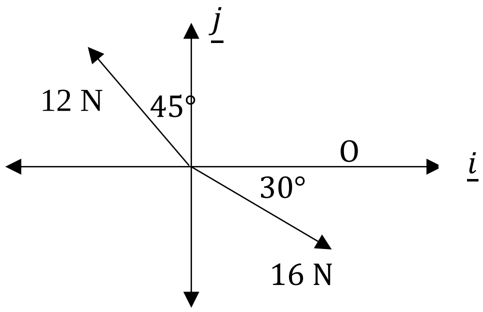
$\sin\theta = \dfrac{1.5}{2.5}$

$\theta = 36.869\ldots°$

$\quad \doteqdot 37°$

Sam should row in the direction of 37° from the line AB.

11



Using components, the force vector for Brutus is:

$\underset{\sim}{F_B} = (-12\sin 45°)i + (12\cos 45°)j$

The force vector for Nitro is:

$\underset{\sim}{F_N} = (16\cos 30°)i + (-16\sin 30°)j$

Resultant vector of the two forces:

$\underset{\sim}{R} = \underset{\sim}{F_B} + \underset{\sim}{F_N}$

$= (-12\sin 45°)i + (12\cos 45°)j + (16\cos 30°)i + (-16\sin 30°)j$

$= (-12\sin 45° + 16\cos 30°)\underset{\sim}{i} + (12\cos 45° - 16\sin 30°)\underset{\sim}{j}$

$= \left(-12\times\dfrac{1}{\sqrt{2}} + 16\times\dfrac{\sqrt{3}}{2}\right)\underset{\sim}{i} + \left(12\times\dfrac{1}{\sqrt{2}} - 16\times\dfrac{1}{2}\right)\underset{\sim}{j}$

$= (-6\sqrt{2} + 8\sqrt{3})\underset{\sim}{i} + (6\sqrt{2} - 8)\underset{\sim}{j}$

$= (8\sqrt{3} - 6\sqrt{2})\underset{\sim}{i} + (6\sqrt{2} - 8)\underset{\sim}{j}$

Magnitude of resultant force:

$|\underset{\sim}{R}| = \sqrt{(8\sqrt{3} - 6\sqrt{2})^2 + (6\sqrt{2} - 8)^2}$

$\quad = 5.393\ldots$

$\doteqdot 5.4$ N

Let θ be the angle between $\underset{\sim}{i}$ and $\underset{\sim}{R}$.

For the direction of the resultant force, $\underset{\sim}{R}$:

$\tan\theta = \dfrac{6\sqrt{2}-8}{8\sqrt{3}-6\sqrt{2}}$ (using exact values from above)

$\theta = 5.162\ldots°$

$\doteqdot 5.2°$

The resultant force is about 5.4 N in a direction of about 5.2° north of east.

12 Let the vectors be $\underset{\sim}{a} = 9\underset{\sim}{i} - 2\underset{\sim}{j}$, $\underset{\sim}{b} = -3\underset{\sim}{i} + 10\underset{\sim}{j}$ and $\underset{\sim}{c} = 18\underset{\sim}{i} - \underset{\sim}{j}$.

and the resultant force be $\underset{\sim}{F} = \underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c}$

$\underset{\sim}{F} = \left(9\underset{\sim}{i} - 2\underset{\sim}{j}\right) + \left(-3\underset{\sim}{i} + 10\underset{\sim}{j}\right) + (18\underset{\sim}{i} - \underset{\sim}{j})$

$= 9\underset{\sim}{i} - 3\underset{\sim}{i} + 18\underset{\sim}{i} - 2\underset{\sim}{j} + 10\underset{\sim}{j} - \underset{\sim}{j}$

$= 24\underset{\sim}{i} + 7\underset{\sim}{j}$

The magnitude of the resultant force is:

$\left|\underset{\sim}{F}\right| = \sqrt{24^2 + 7^2}$

$= \sqrt{576 + 49}$

$= 25$ N

The magnitude of acceleration of the object is:

$|a| = \dfrac{\left|\underset{\sim}{F}\right|}{m}$

$= \dfrac{25}{5}$

$= 5\ \text{m/s}^2$

The direction of the acceleration of the object is in the direction of the resultant force.

$\tan\theta = \dfrac{7}{24}$

$\theta = \tan^{-1}\dfrac{7}{24}$ above the horizontal

13a At 12 noon, the position vector is $\overrightarrow{OA} = 40\underset{\sim}{i} + 16\underset{\sim}{j}$

After 5 minutes, the position vector is $\overrightarrow{OB} = 33\underset{\sim}{i} + 40\underset{\sim}{j}$

Let the displacement vector (after 5 minutes) be $\overrightarrow{AB}$ which is:

$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{OB} - \overrightarrow{OA} \\ &= (33\underset{\sim}{i} + 40\underset{\sim}{j}) - (40\underset{\sim}{i} + 16\underset{\sim}{j}) \\ &= 33\underset{\sim}{i} + 40\underset{\sim}{j} - 40\underset{\sim}{i} - 16\underset{\sim}{j} \\ &= -7\underset{\sim}{i} + 24\underset{\sim}{j}\end{aligned}$$

At 12.15 pm (after $3 \times 5 = 15$ minutes), the position vector will be:

$$\overrightarrow{OA} + 3\overrightarrow{AB}$$

$$= 40\underset{\sim}{i} + 16\underset{\sim}{j} + 3\left(-7\underset{\sim}{i} + 24\underset{\sim}{j}\right)$$

$$= 40\underset{\sim}{i} + 16\underset{\sim}{j} - 21\underset{\sim}{i} + 72\underset{\sim}{j}$$

$$= 19\underset{\sim}{i} + 88\underset{\sim}{j}$$

13b After 1 hour (that is, after $12 \times 5 = 60$ minutes), the displacement vector of the plane will be:

$$\begin{aligned}12\overrightarrow{AB} &= 12\left(-7\underset{\sim}{i} + 24\underset{\sim}{j}\right) \\ &= -84\underset{\sim}{i} + 288\underset{\sim}{j}\end{aligned}$$

Hence, the velocity vector of the plane will be $\left(-84\underset{\sim}{i} + 288\underset{\sim}{j}\right)$ km/h.

14a The weight of the object is $5g = 5 \times 9.8 = 49\ N$

14b Let $\underset{\sim}{F}$ represent the resultant force.

$$\underset{\sim}{F} = \left(50\sin 40° - 75\sin 20°\right)\underset{\sim}{i} + \left(50\cos 40° + 75\cos 20° - 5g\right)\underset{\sim}{j}$$

$$\begin{aligned}\left|\underset{\sim}{F}\right| &= \sqrt{\left(50\sin 40° - 75\sin 20°\right)^2 + \left(50\cos 40° + 75\cos 20° - 5g\right)^2} \\ &= 60.13\ldots\end{aligned}$$

The magnitude of the three forces acting on the object is 60 N (correct to the nearest N).

14c The angle that $\underset{\sim}{F}$ makes with the upward vertical direction is $(90° - \theta)$ where $\theta = \tan^{-1}\left(\dfrac{50\cos 40° + 75\cos 20° - 5g}{50\sin 40° - 75\sin 20°}\right)$.

$\theta = 83.80...°$

So the angle that $\underset{\sim}{F}$ makes with the upward vertical direction is $6°$ (correct to the nearest degree).

15a $\underset{\sim}{v} = \left(3 - 2\sqrt{2}\right)\underset{\sim}{i} + \left(5 - 2\sqrt{2}\right)\underset{\sim}{j}$

15b The speed of the boat is given by $|\underset{\sim}{v}|$.

$$\begin{aligned} |\underset{\sim}{v}| &= \sqrt{\left(3 - 2\sqrt{2}\right)^2 + \left(5 - 2\sqrt{2}\right)^2} \\ &= 2.17... \end{aligned}$$

So the speed of the boat is 2.2 m/s (correct to two significant figures).

The bearing on which the boat is travelling is given by $(90° - \theta)$ where $\theta = \tan^{-1}\left(\dfrac{5 - 2\sqrt{2}}{3 - 2\sqrt{2}}\right)$.

$\theta = 85.48...°$

So the bearing on which the boat is travelling is $4.5°\text{T}$ (correct to the nearest tenth of a degree).

16a In the horizontal direction, we have $T_1 \cos 60° = T_2 \cos 30°$.

$$\frac{T_1}{2} = \frac{\sqrt{3}T_2}{2} \Rightarrow T_1 = \sqrt{3}T_2$$

16b In the vertical direction, we have $T_1 \sin 60° + T_2 \sin 30° = 9.8m$.

$$\frac{\sqrt{3}T_1}{2} + \frac{T_2}{2} = 9.8m \Rightarrow \sqrt{3}T_1 + T_2 = 19.6m$$

16c Given $T_1 = \sqrt{3}T_2$ and $\sqrt{3}T_1 + T_2 = 19.6m$.

Substituting $T_1 = \sqrt{3}T_2$ into $\sqrt{3}T_1 + T_2 = 19.6m$ and simplifying we obtain:
$4T_2 = 19.6m$

So $T_2 = 4.9m$.

We are given $T_2 = 98$.

Solving for m we obtain $m = 20$.

So the mass of the flowerpot is 20 kg.

17a The tension in the string is T newtons.

$$T - 3g = \frac{3g}{2} \Rightarrow T = \frac{9g}{2}$$

So $T = 44.1$.

17b $mg - T = \frac{mg}{2} \Rightarrow T = \frac{mg}{2}$

From part a, $T = \frac{9g}{2}$.

So $m = 9$.

18 When the two forces act at $90°$ to each other, the resultant force is $2\sqrt{7}$ N.

So $p^2 + q^2 = \left(2\sqrt{7}\right)^2$.

$p^2 + q^2 = 28 \quad (1)$

When the two forces act at $30°$ to each other, the resultant force is $2\sqrt{13}$ N.

So $p^2+q^2-2pq\cos150°=\left(2\sqrt{13}\right)^2$.

$p^2+q^2+\sqrt{3}pq=52 \quad (2)$

Substituting (1) into (2) we obtain:

$$\begin{aligned}28+\sqrt{3}pq&=52\\ \sqrt{3}pq&=24\\ q&=\frac{24}{\sqrt{3}p}\end{aligned}$$

Substituting $q=\dfrac{24}{\sqrt{3}p}$ into $p^2+q^2=28$ we obtain:

$$\begin{aligned}p^2+\frac{576}{3p^2}&=28\\ p^4+192&=28p^2\\ p^4-28p^2+192&=0\\ \left(p^2-16\right)\left(p^2-12\right)&=0\\ p&=2\sqrt{3},4\ \left(p>0\right)\end{aligned}$$

When $p=2\sqrt{3},4$, $q=4,2\sqrt{3}$

So $p=2\sqrt{3}$ and $q=4$.

19a Resolving parallel to the plane we obtain $3a=T-3g\sin\theta$.

19b Resolving parallel to the plane we obtain $2a=2g\sin2\theta-T$.

19c When the system is in equilibrium, $a=0$.

$T-3g\sin\theta=0\Rightarrow T=3g\sin\theta$

Substituting $T=3g\sin\theta$ into $2g\sin2\theta-T=0$ we obtain:

$2g\sin2\theta-3g\sin\theta=0$

$$\begin{aligned}4g\sin\theta\cos\theta-3g\sin\theta&=0\\ g\sin\theta\left(4\cos\theta-3\right)&=0\end{aligned}$$

$g\sin\theta \neq 0$ and so $4\cos\theta - 3 = 0$.

So the system is in equilibrium when $\cos\theta = \dfrac{3}{4}$.

20a Resolving parallel to the plane for the object of mass m_1 we obtain:

$$T - m_1 g\sin 30° = m_1 a$$
$$T - \frac{m_1 g}{2} = m_1 a \quad (1)$$

Resolving parallel to the plane for the object of mass m_2 we obtain:

$$m_2 g\sin 60° - T = m_2 a$$
$$\frac{\sqrt{3}m_1 g}{2} - T = m_2 a \quad (2)$$

$(1)+(2)$ gives:

$$\frac{\sqrt{3}m_2 g - m_1 g}{2} = (m_1 + m_2)a$$
$$a = \frac{g\left(\sqrt{3}m_2 - m_1\right)}{2(m_1 + m_2)}$$

When $a > 0$, the object of mass m_1 will accelerate towards the pulley.

So we require $\sqrt{3}m_2 - m_1 > 0 \Rightarrow m_1 < \sqrt{3}m_2$.

20b Given $u = 0$, $a = \dfrac{g\left(\sqrt{3}m_2 - m_1\right)}{2(m_1 + m_2)}$ and $s = d$, we can use the kinematics formula $v^2 = u^2 + 2as$.

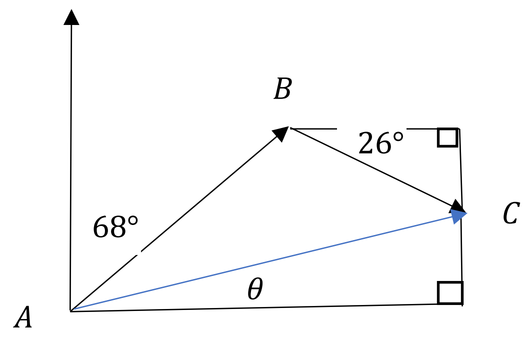
Substituting into $v = \sqrt{u^2 + 2as}$ we obtain:

$$v = \sqrt{\frac{2dg\left(\sqrt{3}m_2 - m_1\right)}{2(m_1 + m_2)}}$$
$$= \sqrt{\frac{dg\left(\sqrt{3}m_2 - m_1\right)}{m_1 + m_2}}$$

Solutions to Chapter review

1 $|\overrightarrow{AB}| = 133$ km and angle is 068°T or N68°E.

$|\overrightarrow{BC}| = 98$ km and angle is 116°T or 26° south of east.



$\overrightarrow{AB} = 133 \sin 68°\, \underset{\sim}{i} + 133 \cos 68°\, \underset{\sim}{j}$

$\overrightarrow{BC} = 98 \cos 26°\, \underset{\sim}{i} - 98 \sin 26°\, \underset{\sim}{j}$

Hence,

$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$

$= 133 \sin 68°\, \underset{\sim}{i} + 133 \cos 68°\, \underset{\sim}{j} + 98 \cos 26°\, \underset{\sim}{i} - 98 \sin 26°\, \underset{\sim}{j}$

$= (133 \sin 68° + 98 \cos 26°)\underset{\sim}{i} + (133 \cos 68° - 98 \sin 26°)\underset{\sim}{j}$

$|\overrightarrow{AC}| = \sqrt{(133 \sin 68° + 98 \cos 26°)^2 + (133 \cos 68° - 98 \sin 26°)^2}$

$|\overrightarrow{AC}| = 211.508\ldots$

$\doteqdot 211.5$ km

The magnitude of $\overrightarrow{AC}$ is about 211.5 km.

Let θ be the angle north of east for $\overrightarrow{AC}$.

$\tan\theta = \dfrac{133 \cos 68° - 98 \sin 26°}{133 \sin 68° + 98 \cos 26°}$

$\theta = 1.859\ldots°$

$\doteqdot 2°$

So θ is about 2° north of east.

Hence, the direction of $\overrightarrow{AC}$ is 088°T.

2a $\overrightarrow{AE} + \overrightarrow{ED} = \overrightarrow{AD}$

2b $\overrightarrow{AD} + \overrightarrow{DC} = \overrightarrow{AC}$

2c $\overrightarrow{DA} + \overrightarrow{AC} = \overrightarrow{DC}$

2d $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} + \overrightarrow{EA}$

$= \overrightarrow{AC} + \overrightarrow{CE} + \overrightarrow{EA}$

$= \overrightarrow{AE} + \overrightarrow{EA}$

$= \overrightarrow{AA}$

$= 0$

2e $\overrightarrow{AD} - \overrightarrow{AC}$

$= \overrightarrow{AD} + \overrightarrow{CA}$

$= \overrightarrow{CA} + \overrightarrow{AD}$

$= \overrightarrow{CD}$

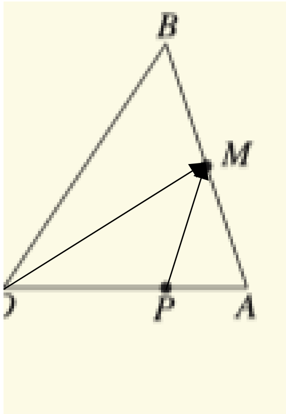
2f $\overrightarrow{EB} - \overrightarrow{ED}$

$= \overrightarrow{EB} + \overrightarrow{DE}$

$= \overrightarrow{DE} + \overrightarrow{EB}$

$= \overrightarrow{DB}$

3a



$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$= \underset{\sim}{b} - \underset{\sim}{a}$$

3b

$$\overrightarrow{AM} = \frac{1}{2}\overrightarrow{AB}$$

$$= \frac{1}{2}(\underset{\sim}{b} - \underset{\sim}{a})$$

$$\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM}$$

$$= \underset{\sim}{a} + \frac{1}{2}(\underset{\sim}{b} - \underset{\sim}{a})$$

$$= \underset{\sim}{a} + \frac{1}{2}\underset{\sim}{b} - \frac{1}{2}\underset{\sim}{a}$$

$$= \frac{1}{2}\underset{\sim}{a} + \frac{1}{2}\underset{\sim}{b}$$

$$= \frac{1}{2}(\underset{\sim}{a} + \underset{\sim}{b})$$

3c $\overrightarrow{PM} = \overrightarrow{OM} - \overrightarrow{OP}$

From part b, we know that

$$\overrightarrow{OM} = \frac{1}{2}\underset{\sim}{a} + \frac{1}{2}\underset{\sim}{b}$$

To find $\overrightarrow{OP}$, we can use:

$$\frac{\overrightarrow{OP}}{\overrightarrow{OA}} = \frac{2}{3}$$

$$\overrightarrow{OP} = \frac{2}{3}\overrightarrow{OA}$$

$$= \frac{2}{3}\underset{\sim}{a}$$

Hence

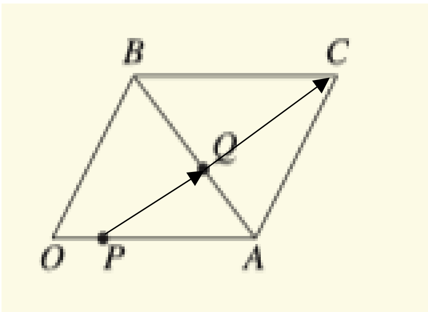
$$\overrightarrow{PM} = \overrightarrow{OM} - \overrightarrow{OP}$$

$$= \frac{1}{2}\underset{\sim}{a} + \frac{1}{2}\underset{\sim}{b} - \frac{2}{3}\underset{\sim}{a}$$

$$= \frac{1}{2}\underset{\sim}{b} - \frac{1}{6}\underset{\sim}{a}$$

$$= \frac{1}{6}(3\underset{\sim}{b} - \underset{\sim}{a})$$

4a



$$\frac{\overrightarrow{PA}}{\overrightarrow{OA}} = \frac{3}{4}$$

$$\overrightarrow{PA} = \frac{3}{4}\overrightarrow{OA}$$

$$= \frac{3}{4}\underset{\sim}{a}$$

4b

$$\frac{\overrightarrow{AQ}}{\overrightarrow{AB}} = \frac{3}{7}$$

$$\overrightarrow{AQ} = \frac{3}{7}\overrightarrow{AB}$$

$$\overrightarrow{AQ} = \frac{3}{7}\left(\overrightarrow{OB} - \overrightarrow{OA}\right)$$

$$= \frac{3}{7}(\underset{\sim}{b} - \underset{\sim}{a})$$

4c $\overrightarrow{PQ} = \overrightarrow{PA} + \overrightarrow{AQ}$

$$= \frac{3}{4}\underset{\sim}{a} + \frac{3}{7}(\underset{\sim}{b} - \underset{\sim}{a})$$

$$= \frac{3}{4}\underset{\sim}{a} + \frac{3}{7}\underset{\sim}{b} - \frac{3}{7}\underset{\sim}{a}$$

$$= \frac{9}{28}\underset{\sim}{a} + \frac{3}{7}\underset{\sim}{b}$$

4d $\overrightarrow{QC} = \overrightarrow{QA} + \overrightarrow{AC} = \overrightarrow{AC} - \overrightarrow{AQ}$

$OACB$ is a parallelogram, hence, $\overrightarrow{AC} = \overrightarrow{OB} = \underset{\sim}{b}$

$$\overrightarrow{QC} = \underset{\sim}{b} - \frac{3}{7}(\underset{\sim}{b} - \underset{\sim}{a})$$

$$= \underset{\sim}{b} + \frac{3}{7}\underset{\sim}{b} - \frac{3}{7}\underset{\sim}{a}$$

$$= \frac{3}{7}\underset{\sim}{a} + \frac{4}{7}\underset{\sim}{b}$$

4e $\overrightarrow{PC} = \overrightarrow{PA} + \overrightarrow{AC}$

$$= \frac{3}{4}\underset{\sim}{a} + \underset{\sim}{b}$$

and

$\overrightarrow{PQ} + \overrightarrow{QC}$

$$= \frac{9}{28}\underset{\sim}{a} + \frac{3}{7}\underset{\sim}{b} + \frac{3}{7}\underset{\sim}{a} + \frac{4}{7}\underset{\sim}{b}$$

$$= \frac{3}{4}\underset{\sim}{a} + \underset{\sim}{b}$$

$$= \overrightarrow{PC}$$

Hence, P, Q and C are collinear.

5a Let O be the origin hence, $\overrightarrow{OA} = -4\underset{\sim}{i} + 2\underset{\sim}{j}$ and $\overrightarrow{OB} = 2\underset{\sim}{i} + 10\underset{\sim}{j}$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$= (2\underset{\sim}{i} + 10\underset{\sim}{j}) - (-4\underset{\sim}{i} + 2\underset{\sim}{j})$$

$$= 2\underset{\sim}{i} + 10\underset{\sim}{j} + 4\underset{\sim}{i} - 2\underset{\sim}{j}$$

$$= 6\underset{\sim}{i} + 8\underset{\sim}{j}$$

5b $$\left|\overrightarrow{AB}\right| = \sqrt{6^2 + 8^2}$$

$$= \sqrt{100}$$

$$= 10$$

5c

$$\widehat{AB} = \frac{\overrightarrow{AB}}{\left|\overrightarrow{AB}\right|}$$

$$= \frac{6\underset{\sim}{i} + 8\underset{\sim}{j}}{10}$$

$$= \frac{3}{5}\underset{\sim}{i} + \frac{4}{5}\underset{\sim}{j}$$

6a $$\underset{\sim}{v} = \begin{bmatrix} 2a \\ a \end{bmatrix}$$

$$\left|\underset{\sim}{v}\right| = \sqrt{(2a)^2 + (a)^2}$$

$$= \sqrt{5a^2}$$

$$= \sqrt{5}a$$

6b

$$\hat{\underset{\sim}{v}} = \frac{\underset{\sim}{v}}{|\underset{\sim}{v}|}$$

$$= \frac{1}{\sqrt{5}a}\begin{bmatrix} 2a \\ a \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}$$

6c $\quad \underset{\sim}{v} + \underset{\sim}{v}$

$$= \begin{bmatrix} 2a \\ a \end{bmatrix} + \begin{bmatrix} 2a \\ a \end{bmatrix}$$

$$= \begin{bmatrix} 4a \\ 2a \end{bmatrix}$$

6d $\quad \underset{\sim}{v} \cdot \underset{\sim}{v} = |\underset{\sim}{v}|^2$

$$= \left(\sqrt{5}a\right)^2$$

$$= 5a^2$$

7a $\quad \underset{\sim}{a} = \begin{bmatrix} x-1 \\ 1-x \end{bmatrix}$ and $\underset{\sim}{b} = \begin{bmatrix} x+1 \\ 1+x \end{bmatrix}$

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = x_1x_2 + y_1y_2$$

$$= (x-1)(x+1) + (1-x)(1+x)$$

$$= x^2 - 1 + 1 - x^2$$

$$= 0$$

As $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$, vectors are perpendicular.

7b $\underset{\sim}{a} = \begin{bmatrix} 5x \\ 5x-1 \end{bmatrix}$ and $\underset{\sim}{b} = \begin{bmatrix} 1-2x \\ 2x \end{bmatrix}$

$\underset{\sim}{a} \cdot \underset{\sim}{b} = x_1x_2 + y_1y_2$

$= 5x(1-2x) + (5x-1)2x$

$= 5x - 10x^2 + 10x^2 - 2x$

$= 3x$

As $\underset{\sim}{a} \cdot \underset{\sim}{b} \neq 0$, vectors are not perpendicular (unless $x = 0$).

8 $\underset{\sim}{a} = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$ and $\underset{\sim}{b} = \begin{bmatrix} 10 \\ 2 \end{bmatrix}$

$\underset{\sim}{a} \cdot \underset{\sim}{b} = x_1x_2 + y_1y_2$

$= -5 \times 10 + 3 \times 2$

$= -50 + 6$

$= -44$

Also, where θ is the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$,

$\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}| \cos\theta$

$|\underset{\sim}{a}| = \sqrt{(-5)^2 + 3^2}$

$= \sqrt{25 + 9}$

$= \sqrt{34}$

$|\underset{\sim}{b}| = \sqrt{10^2 + 2^2}$

$= \sqrt{100 + 4}$

$= \sqrt{104}$

So $\underset{\sim}{a} \cdot \underset{\sim}{b} = \sqrt{34} \times \sqrt{104} \cos\theta$

Therefore

$\sqrt{34} \times \sqrt{104} \cos\theta = -44$

$\cos\theta = \dfrac{-44}{\sqrt{3536}}$

$\theta = 137.726\ldots°$

$\doteqdot 137°44'$

9a Let O be the origin hence, $\overrightarrow{OP} = -4\underline{i} - 5\underline{j}$, $\overrightarrow{OQ} = 10\underline{i} + 5\underline{j}$, $\overrightarrow{OR} = 5\underline{i} + 12\underline{j}$ and $\overrightarrow{OS} = -9\underline{i} + 2\underline{j}$.

$$\begin{aligned}\overrightarrow{PQ} &= \overrightarrow{OQ} - \overrightarrow{OP} \\ &= (10\underline{i} + 5\underline{j}) - (-4\underline{i} - 5\underline{j}) \\ &= 10\underline{i} + 5\underline{j} + 4\underline{i} + 5\underline{j} \\ &= 14\underline{i} + 10\underline{j}\end{aligned}$$

$$\begin{aligned}\overrightarrow{SR} &= \overrightarrow{OR} - \overrightarrow{OS} \\ &= (5\underline{i} + 12\underline{j}) - (-9\underline{i} + 2\underline{j}) \\ &= 5\underline{i} + 12\underline{j} + 9\underline{i} - 2\underline{j} \\ &= 14\underline{i} + 10\underline{j}\end{aligned}$$

Hence, $\overrightarrow{PQ} = \overrightarrow{SR}$.

9b

$$\begin{aligned}\overrightarrow{PS} &= \overrightarrow{OS} - \overrightarrow{OP} \\ &= (-9\underline{i} + 2\underline{j}) - (-4\underline{i} - 5\underline{j}) \\ &= -9\underline{i} + 2\underline{j} + 4\underline{i} + 5\underline{j} \\ &= -5\underline{i} + 7\underline{j}\end{aligned}$$

$$\begin{aligned}\overrightarrow{PQ} \cdot \overrightarrow{PS} &= (14\underline{i} + 10\underline{j}) \cdot (-5\underline{i} + 7\underline{j}) \\ &= 14 \times (-5) + 10 \times 7 \\ &= -70 + 70 \\ &= 0\end{aligned}$$

9c $\overrightarrow{PQ} = \overrightarrow{SR}$ and $\overrightarrow{PQ} \cdot \overrightarrow{PS} = 0$

This means opposite sides are equal and adjacent sides are 90° with each other.

Hence, $PQRS$ is a rectangle.

10a $\underset{\sim}{a} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$ and $\underset{\sim}{b} = \begin{bmatrix} -3 \\ -3 \end{bmatrix}$

$$Proj_{\underset{\sim}{b}}\underset{\sim}{a} = \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{\underset{\sim}{b}.\underset{\sim}{b}} \times \underset{\sim}{b}$$

$$= \frac{5 \times (-3) + (-2) \times (-3)}{(-3) \times (-3) + (-3) \times (-3)} \times \begin{bmatrix} -3 \\ -3 \end{bmatrix}$$

$$= -\frac{1}{2} \times \begin{bmatrix} -3 \\ -3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{2} \\ \frac{3}{2} \end{bmatrix}$$

10b $\underset{\sim}{a} = 4\underset{\sim}{i} - \underset{\sim}{j}$ and $\underset{\sim}{b} = 6\underset{\sim}{i} + 2\underset{\sim}{j}$

$$Proj_{\underset{\sim}{b}}\underset{\sim}{a} = \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{\underset{\sim}{b}.\underset{\sim}{b}} \times \underset{\sim}{b}$$

$$= \frac{4 \times 6 + (-1) \times 2}{6 \times 6 + 2 \times 2} \times (6\underset{\sim}{i} + 2\underset{\sim}{j})$$

$$= \frac{11}{20} \times \left(6\underset{\sim}{i} + 2\underset{\sim}{j}\right)$$

$$= \frac{33}{10}\underset{\sim}{i} + \frac{11}{10}\underset{\sim}{j}$$

11 $\underset{\sim}{a} = \begin{bmatrix} -8 \\ 9 \end{bmatrix}$ and $\underset{\sim}{b} = \begin{bmatrix} 3 \\ 12 \end{bmatrix}$

Length of $Proj_{\underset{\sim}{b}}\underset{\sim}{a}$

$$= \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{b}|}$$

$$= \frac{(-8) \times 3 + 9 \times 12}{\sqrt{3^2 + 12^2}}$$

$$= \frac{84}{\sqrt{153}}$$

12 Let O be the origin hence $\overrightarrow{OA} = -3\underset{\sim}{i} + \underset{\sim}{j}$, $\overrightarrow{OB} = 4\underset{\sim}{i} + 8\underset{\sim}{j}$ and $\overrightarrow{OC} = 2\underset{\sim}{i} - 5\underset{\sim}{j}$.

$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{AO} + \overrightarrow{OB}\\ &= \overrightarrow{OB} - \overrightarrow{OA}\\ &= (4\underset{\sim}{i} + 8\underset{\sim}{j}) - (-3\underset{\sim}{i} + \underset{\sim}{j})\\ &= 4\underset{\sim}{i} + 8\underset{\sim}{j} + 3\underset{\sim}{i} - \underset{\sim}{j}\\ &= 7\underset{\sim}{i} + 7\underset{\sim}{j}\end{aligned}$$

$$\begin{aligned}\overrightarrow{BC} &= \overrightarrow{BO} + \overrightarrow{OC}\\ &= \overrightarrow{OC} - \overrightarrow{OB}\\ &= (2\underset{\sim}{i} - 5\underset{\sim}{j}) - (4\underset{\sim}{i} + 8\underset{\sim}{j})\\ &= 2\underset{\sim}{i} - 5\underset{\sim}{j} - 4\underset{\sim}{i} - 8\underset{\sim}{j}\\ &= -2\underset{\sim}{i} - 13\underset{\sim}{j}\end{aligned}$$

$$\begin{aligned}\overrightarrow{AB} \cdot \overrightarrow{BC} &= (7\underset{\sim}{i} + 7\underset{\sim}{j}) \cdot (-2\underset{\sim}{i} - 13\underset{\sim}{j})\\ &= 7 \times (-2) + 7 \times (-13)\\ &= -105\end{aligned}$$

Also, $\overrightarrow{AB} \cdot \overrightarrow{BC} = |\overrightarrow{AB}||\overrightarrow{BC}| \cos\theta$

Let θ be $\angle ABC$.

$$\begin{aligned}|\overrightarrow{AB}| &= \sqrt{7^2 + 7^2}\\ &= \sqrt{49 + 49}\\ &= 7\sqrt{2}\end{aligned}$$

$$\begin{aligned}|\overrightarrow{BC}| &= \sqrt{(-2)^2 + (-13)^2}\\ &= \sqrt{4 + 169}\\ &= \sqrt{173}\end{aligned}$$

Therefore

$$|\overrightarrow{AB}||\overrightarrow{BC}| \cos\theta = \overrightarrow{AB} \cdot \overrightarrow{BC}$$

$$7\sqrt{2} \times \sqrt{173} \cos\theta = -105$$

$$\cos\theta = \frac{-105}{7\sqrt{2}\times\sqrt{173}}$$

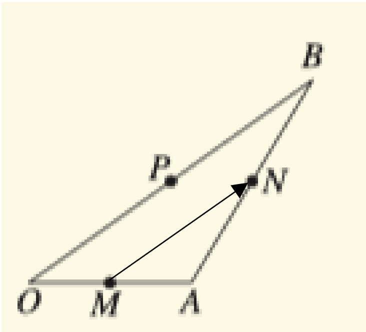
However $\angle ABC$ is an acute angle.

$$\cos\angle ABC = \frac{105}{7\sqrt{2}\times\sqrt{173}}$$

$$\angle ABC = 36.253\ldots°$$

$$\doteqdot 36°$$

13a



$$\overrightarrow{MA} = \frac{1}{2}\overrightarrow{OA}$$

$$= \frac{1}{2}\underset{\sim}{a}$$

13b $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$$= \underset{\sim}{b} - \underset{\sim}{a}$$

$$\overrightarrow{AN} = \frac{1}{2}\overrightarrow{AB}$$

$$= \frac{1}{2}(\underset{\sim}{b} - \underset{\sim}{a})$$

13c $\overrightarrow{MN} = \overrightarrow{MA} + \overrightarrow{AN}$

$$= \frac{1}{2}\underset{\sim}{a} + \frac{1}{2}(\underset{\sim}{b} - \underset{\sim}{a})$$

$$= \frac{1}{2}\underset{\sim}{a} + \frac{1}{2}\underset{\sim}{b} - \frac{1}{2}\underset{\sim}{a}$$

$= \frac{1}{2}\underset{\sim}{b}$

13d $\overrightarrow{MN} = \frac{1}{2}\underset{\sim}{b} = \overrightarrow{PB}$

This means a pair of opposite sides are equal and parallel. Hence, $MNBP$ is a parallelogram.

14a $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$\underset{\sim}{p} = \underset{\sim}{b} - \underset{\sim}{a}$

$\overrightarrow{MB} = \frac{1}{2}\overrightarrow{AB}$

$= \frac{1}{2}(\underset{\sim}{b} - \underset{\sim}{a})$

$\overrightarrow{OM} = \overrightarrow{OB} + \overrightarrow{BM}$

$= \overrightarrow{OB} - \overrightarrow{MB}$

$\underset{\sim}{m} = \underset{\sim}{b} - \frac{1}{2}(\underset{\sim}{b} - \underset{\sim}{a})$

$= \underset{\sim}{b} - \frac{1}{2}\underset{\sim}{b} + \frac{1}{2}\underset{\sim}{a}$

$= \frac{1}{2}\underset{\sim}{b} + \frac{1}{2}\underset{\sim}{a}$

$= \frac{1}{2}(\underset{\sim}{b} + \underset{\sim}{a})$

$= \frac{1}{2}(\underset{\sim}{a} + \underset{\sim}{b})$

14b $\left|\underset{\sim}{p}\right|^2 + 4\left|\underset{\sim}{m}\right|^2$

$= (\underset{\sim}{b} - \underset{\sim}{a}) \cdot (\underset{\sim}{b} - \underset{\sim}{a}) + 4\left(\frac{1}{2}(\underset{\sim}{b} + \underset{\sim}{a}) \cdot \frac{1}{2}(\underset{\sim}{b} + \underset{\sim}{a})\right)$

$= \underset{\sim}{b} \cdot \underset{\sim}{b} - \underset{\sim}{b} \cdot \underset{\sim}{a} - \underset{\sim}{a} \cdot \underset{\sim}{b} + \underset{\sim}{a} \cdot \underset{\sim}{a} + 4\left(\frac{1}{4}\underset{\sim}{b} \cdot \underset{\sim}{b} + \frac{1}{4}\underset{\sim}{b} \cdot \underset{\sim}{a} + \frac{1}{4}\underset{\sim}{a} \cdot \underset{\sim}{b} + \frac{1}{4}\underset{\sim}{a} \cdot \underset{\sim}{a}\right)$

$= \underset{\sim}{b} \cdot \underset{\sim}{b} - \underset{\sim}{b} \cdot \underset{\sim}{a} - \underset{\sim}{a} \cdot \underset{\sim}{b} + \underset{\sim}{a} \cdot \underset{\sim}{a} + \underset{\sim}{b} \cdot \underset{\sim}{b} + \underset{\sim}{b} \cdot \underset{\sim}{a} + \underset{\sim}{a} \cdot \underset{\sim}{b} + \underset{\sim}{a} \cdot \underset{\sim}{a}$

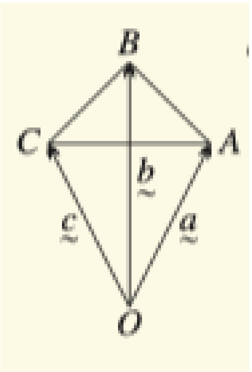
$= \underset{\sim}{b} \cdot \underset{\sim}{b} + \underset{\sim}{a} \cdot \underset{\sim}{a} + \underset{\sim}{b} \cdot \underset{\sim}{b} + \underset{\sim}{a} \cdot \underset{\sim}{a}$

$= 2(\underset{\sim}{b} \cdot \underset{\sim}{b} + \underset{\sim}{a} \cdot \underset{\sim}{a})$

$= 2\left(\left|\underset{\sim}{b}\right|^2 + \left|\underset{\sim}{a}\right|^2\right)$

$= 2\left(\left|\underset{\sim}{a}\right|^2 + \left|\underset{\sim}{b}\right|^2\right)$

15a



Adjacent sides of a kite (rhombus) are equal in length.

Therefore, $\left|\underset{\sim}{a}\right|^2 = \left|\underset{\sim}{c}\right|^2$

Hence, $\underset{\sim}{a} \cdot \underset{\sim}{a} = \underset{\sim}{c} \cdot \underset{\sim}{c}$.

15b Adjacent sides of a kite (rhombus) are equal.

Therefore, $\left|\overrightarrow{AB}\right|^2 = \left|\overrightarrow{CB}\right|^2$.

So $\overrightarrow{AB} \cdot \overrightarrow{AB} = \overrightarrow{CB} \cdot \overrightarrow{CB}$

Now $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$

$= \overrightarrow{OB} - \overrightarrow{OA}$

$= \underset{\sim}{b} - \underset{\sim}{a}$

and $\overrightarrow{CB} = \overrightarrow{CO} + \overrightarrow{OB}$

$= \overrightarrow{OB} - \overrightarrow{OC}$

$= \underline{b} - \underline{c}$

Hence, $(\underline{b} - \underline{a}) \cdot (\underline{b} - \underline{a}) = (\underline{b} - \underline{c}) \cdot (\underline{b} - \underline{c})$.

15c $\overrightarrow{CA} = \overrightarrow{OA} - \overrightarrow{OC}$

$= \underline{a} - \underline{c}$

$\overrightarrow{OB} = \overrightarrow{OC} + \overrightarrow{CB}$

$= \underline{c} + \underline{b} - \underline{c}$

$= \underline{b}$

$\overrightarrow{CA} \cdot \overrightarrow{OB} = (\underline{a} - \underline{c}) \cdot \underline{b}$

$= \underline{a} \cdot \underline{b} - \underline{c} \cdot \underline{b}$

From part b,

$(\underline{b} - \underline{a}) \cdot (\underline{b} - \underline{a}) = (\underline{b} - \underline{c}) \cdot (\underline{b} - \underline{c})$.

$\underline{b} \cdot \underline{b} - \underline{b} \cdot \underline{a} - \underline{a} \cdot \underline{b} + \underline{a} \cdot \underline{a} = \underline{b} \cdot \underline{b} - \underline{b} \cdot \underline{c} - \underline{c} \cdot \underline{b} + \underline{c} \cdot \underline{c}$

$-\underline{b} \cdot \underline{a} - \underline{a} \cdot \underline{b} + \underline{a} \cdot \underline{a} = -\underline{b} \cdot \underline{c} - \underline{c} \cdot \underline{b} + \underline{c} \cdot \underline{c}$

$-\underline{b} \cdot \underline{a} - \underline{a} \cdot \underline{b} = -\underline{b} \cdot \underline{c} - \underline{c} \cdot \underline{b}$ (since $\underline{a} \cdot \underline{a} = \underline{c} \cdot \underline{c}$ from part a)

$\underline{b} \cdot \underline{a} + \underline{a} \cdot \underline{b} = \underline{b} \cdot \underline{c} + \underline{c} \cdot \underline{b}$

$2\underline{a} \cdot \underline{b} = 2\underline{c} \cdot \underline{b}$

$\underline{a} \cdot \underline{b} = \underline{c} \cdot \underline{b}$

Therefore

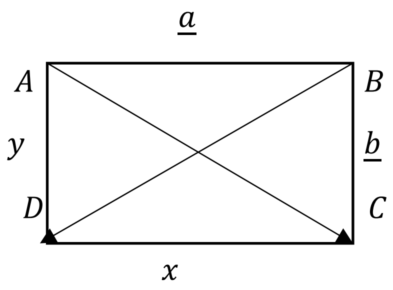
$\overrightarrow{CA} \cdot \overrightarrow{OB} = \underline{a} \cdot \underline{b} - \underline{c} \cdot \underline{b}$

$= \underline{a} \cdot \underline{b} - \underline{a} \cdot \underline{b}$

$= 0$

Hence, diagonals of a kite are perpendicular.

16a



$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$

$= \underline{a} + \underline{b}$

$\overrightarrow{BD} = \overrightarrow{BC} + \overrightarrow{CD}$

$= \overrightarrow{BC} - \overrightarrow{DC}$

$= \overrightarrow{BC} - \overrightarrow{AB}$

$= \underline{b} - \underline{a}$

16b $(\underline{a} + \underline{b}) \cdot (\underline{a} + \underline{b})$

$= \underline{a} \cdot \underline{a} + \underline{a} \cdot \underline{b} + \underline{b} \cdot \underline{a} + \underline{b} \cdot \underline{b}$

$= \underline{a} \cdot \underline{a} + 2\underline{a} \cdot \underline{b} + \underline{b} \cdot \underline{b}$

$= |\underline{a}|^2 + 2\underline{a} \cdot \underline{b} + |\underline{b}|^2$

$= x^2 + 2\underline{a} \cdot \underline{b} + y^2$

$= x^2 + y^2 + 2\underline{a} \cdot \underline{b}$

$(\underline{b} - \underline{a}) \cdot (\underline{b} - \underline{a})$

$= \underline{b} \cdot \underline{b} - \underline{b} \cdot \underline{a} - \underline{a} \cdot \underline{b} + \underline{a} \cdot \underline{a}$

$= \underline{b} \cdot \underline{b} - 2\underline{a} \cdot \underline{b} + \underline{a} \cdot \underline{a}$

$= |\underline{b}|^2 - 2\underline{a} \cdot \underline{b} + |\underline{a}|^2$

$= y^2 - 2\underline{a} \cdot \underline{b} + x^2$

$= x^2 + y^2 - 2\underline{a} \cdot \underline{b}$

16c If the parallelogram is a rectangle, then $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$.

$(\underset{\sim}{a} + \underset{\sim}{b}) \cdot (\underset{\sim}{a} + \underset{\sim}{b}) = x^2 + y^2 + 2\underset{\sim}{a} \cdot \underset{\sim}{b} = x^2 + y^2$

$(\underset{\sim}{b} - \underset{\sim}{a}) \cdot (\underset{\sim}{b} - \underset{\sim}{a}) = x^2 + y^2 - 2\underset{\sim}{a} \cdot \underset{\sim}{b} = x^2 + y^2$

Hence

$(\underset{\sim}{a} + \underset{\sim}{b}) \cdot (\underset{\sim}{a} + \underset{\sim}{b}) = (\underset{\sim}{b} - \underset{\sim}{a}) \cdot (\underset{\sim}{b} - \underset{\sim}{a})$

$\overrightarrow{AC} \cdot \overrightarrow{AC} = \overrightarrow{BD} \cdot \overrightarrow{BD}$

$|\overrightarrow{AC}|^2 = |\overrightarrow{BD}|^2$

So the diagonals are equal.

Conversely, if the diagonals are equal:

$|\overrightarrow{AC}|^2 = |\overrightarrow{BD}|^2$

$\overrightarrow{AC} \cdot \overrightarrow{AC} = \overrightarrow{BD} \cdot \overrightarrow{BD}$

$(\underset{\sim}{a} + \underset{\sim}{b}) \cdot (\underset{\sim}{a} + \underset{\sim}{b}) = (\underset{\sim}{b} - \underset{\sim}{a}) \cdot (\underset{\sim}{b} - \underset{\sim}{a})$

$x^2 + y^2 + 2\underset{\sim}{a} \cdot \underset{\sim}{b} = x^2 + y^2 - 2\underset{\sim}{a} \cdot \underset{\sim}{b}$

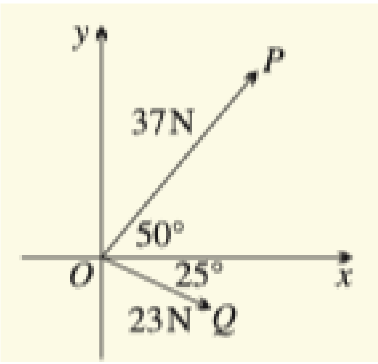
$2\underset{\sim}{a} \cdot \underset{\sim}{b} = -2\underset{\sim}{a} \cdot \underset{\sim}{b}$

$4\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$

$\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$

So the parallelogram is a rectangle.

17a



$\overrightarrow{OP} = (37\cos 50°)\underset{\sim}{i} + (37\sin 50°)\underset{\sim}{j}$

$\overrightarrow{OQ} = (23\cos 25°)\underset{\sim}{i} - (23\sin 25°)\underset{\sim}{j}$

17b Let the resultant force be $\underset{\sim}{F} = \overrightarrow{OP} + \overrightarrow{OQ}$.

$\underset{\sim}{F} = (37\cos 50°)i + (37\sin 50°)j + (23\cos 25°)i - (23\sin 25°)j$

$\quad = (37\cos 50° + 23\cos 25°)i + (37\sin 50° - 23\sin 25°)j$

The magnitude of the resultant force is:

$|\underset{\sim}{F}| = \sqrt{(37\cos 50° + 23\cos 25°)^2 + (37\sin 50° - 23\sin 25°)^2}$

$\quad = 48.358\ldots$

$\quad \doteqdot 48.4\text{ N}$

Let θ be the angle of the resultant force above the horizontal.

$$\tan\theta = \frac{37\sin 50° - 23\sin 25°}{37\cos 50° + 23\cos 25°}$$

$\theta = 22.650\ldots°$

$\quad \doteqdot 22.7°$

The direction of the resultant force is 22.7° above the horizontal.

18 $\underset{\sim}{v} = \left(8 - 2\sqrt{2}\right)\underset{\sim}{i} + \left(-2 - 2\sqrt{2}\right)\underset{\sim}{j}$

The speed of the boat is given by $|\underset{\sim}{v}|$.

$|\underset{\sim}{v}| = \sqrt{\left(8 - 2\sqrt{2}\right)^2 + \left(-2 - 2\sqrt{2}\right)^2}$

$\quad = 7.075\ldots$

$\quad \doteqdot 7.08$

So the speed of the boat is 7.08 km/h (correct to two decimal places).

The bearing on which the boat is travelling is given by $180° - \theta$ where

$$\theta = \tan^{-1}\left(\frac{8 - 2\sqrt{2}}{\left|-2 - 2\sqrt{2}\right|}\right).$$

$\theta = 46.96\ldots°$

So the bearing on which the boat is travelling is $133°\text{T}$ (correct to the nearest degree).

19 Resolving forces vertically we obtain:

$T\sin 45° + T\sin 45 = 2g$

Solving for T we obtain $T = \dfrac{2g}{2\sin 45°} = 13.85\ldots$.

So the tension in each of the chains is 14 N correct to the nearest newton.

20a The tension in the string is T newtons.

Considering the 3 kg object we have:

$3a = 3g - T \quad (1)$

Considering the 1 kg object we have:

$a = T - g \quad (2)$

Adding (1) and (2) we obtain:

$4a = 2g \Rightarrow a = \dfrac{g}{2}$

So $a = 4.9$ (m/s^2).

20b Substituting $a = \dfrac{g}{2}$ into (2) for example we obtain:

$T = \dfrac{g}{2} + g = \dfrac{3g}{2}$

So $T = 14.7$ (N).