Solutions to Exercise 10A

1a Initially, $x = y = 0$, and, $\dot{x} = 30\sqrt{2}\cos 45°$, $\dot{x} = 30$; $\dot{y} = 30\sqrt{2}\sin 45°$, $\dot{y} = 30$

To begin, $\ddot{x} = 0$ (1)

Integrating, $\dot{x} = C_1$

When $t = 0$, $\dot{x} = 30$

$30 = C_1$

so $\dot{x} = 30$ (2)

To begin, $\ddot{y} = -10$ (3)

Integrating, $\dot{y} = -10t + C_2$

When $t = 0$, $\dot{y} = 30$

$30 = C_2$

so $\dot{y} = -10t + 30$ (4)

1b To begin, $\dot{x} = 30$

Integrating, $x = 30t + C_3$

When $t = 0$, $x = 0$

$0 = C_3$

so $x = 30t$ (5)

To begin, $\dot{y} = -10t + 30$

Integrating, $y = -5t^2 + 30t + C_4$

When $t = 0$, $y = 0$

$0 = C_4$

So $y = -5t^2 + 30t$ (6)

1c When $y = 0$,

From (6):

$0 = -5t^2 + 30t$

$5t^2 - 30t = 0$

$5t(t - 6) = 0$

$t = 0$ or $t = 6$

Particle returns to the ground at $t > 0$ seconds, so at $t = 6$ seconds.

1d When $t = 6$,

From (5):

$x = 30 \times 6$

$= 180$ m

The horizontal distance travelled by the particle is 180 m.

1e When $\dot{y} = 0$,

From (4):

$0 = -10t + 30$

$10t = 30$

$t = 3$ seconds

The particle reaches its greatest height after 3 seconds.

1f When $t = 3$,

From (6):

$y = -45 + 90$

$= 45$ m

The greatest height of the particle is 45 m.

2a Initially, $x = y = 0$, and, $\dot{x} = 40\cos 30°$, $\dot{y} = 40\sin 30°$

$\dot{x} = 20\sqrt{3}$, $\dot{y} = 20$

To begin, $\ddot{x} = 0$ (1)

Integrating, $\dot{x} = C_1$

When $t = 0$, $\dot{x} = 20\sqrt{3}$

$20\sqrt{3} = C_1$

so $\dot{x} = 20\sqrt{3}$ (2)

To begin, $\ddot{y} = -10$ (3)

Integrating, $\dot{y} = -10t + C_2$

When $t = 0$, $\dot{y} = 20$

$20 = C_2$

so $\dot{y} = -10t + 20$ (4)

2b To begin, $\dot{x} = 20\sqrt{3}$

Integrating, $x = 20t\sqrt{3} + C_3$

When $t = 0$, $x = 0$

$0 = C_3$

so $x = 20t\sqrt{3}$ (5)

To begin, $\dot{y} = -10t + 20$

Integrating, $y = -5t^2 + 20t + C_4$

When $t = 0$, $y = 0$

$0 = C_4$

so $y = -5t^2 + 20t$ (6)

2c i The particle returns to the ground when $y = 0$.

Substituting into (6):

$0 = -5t^2 + 20t$

$5t^2 - 20t = 0$

$5t(t - 4) = 0$

$t = 0$ or $t = 4$

Since $t > 0$, particle returns to the ground at $t = 4$ seconds.

2c ii From (5):

When $t = 4$,

$x = 20 \times 4 \times \sqrt{3}$

$\quad = 80\sqrt{3}$ m

The horizontal distance travelled by the particle is $80\sqrt{3}$ m.

2c iii Particle reaches greatest height when $\dot{y} = 0$.

Substituting into (4):

$0 = -10t + 20$

$10t = 20$

$t = 2$

Substituting into (6):

$y = -20 + 40$

$\quad = 20$ m

The greatest height reached above the ground is 20 m.

3a Initially, $x = y = 0$, and, $\dot{x} = 20 \cos 60°$, $\dot{x} = 10$; $\dot{y} = 20 \sin 60°$, $\dot{y} = 10\sqrt{3}$

To begin,	$\ddot{x} = 0$	(1)	To begin,	$\ddot{y} = -10$	(3)
Integrating,	$\dot{x} = C_1$		Integrating,	$\dot{y} = -10t + C_2$	

When $t = 0$, $\dot{x} = 10$

$10 = C_1$

so $\dot{x} = 10$ (2)

When $t = 0$, $\dot{y} = 10\sqrt{3}$

$10\sqrt{3} = C_2$

so $\dot{y} = -10t + 10\sqrt{3}$ (4)

3b To begin, $\dot{x} = 10$

Integrating, $x = 10t + C_3$

When $t = 0$, $x = 0$

$0 = C_3$

so $x = 10t$ (5)

To begin, $\dot{y} = -10t + 10\sqrt{3}$

Integrating, $y = -5t^2 + 10t\sqrt{3} + C_4$

When $t = 0$, $y = 0$

$0 = C_4$

so $y = -5t^2 + 10t\sqrt{3}$ (6)

3c i Since the particle starts at the origin and moves to a point with horizontal component x and vertical component y, we can consider a right-angled triangle whose hypotenuse equals the distance travelled.

Pythagoras' theorem can be used to calculate this hypotenuse.

$r^2 = x^2 + y^2$

$r = \sqrt{x^2 + y^2}$

From (5) and (6):

$$r = \sqrt{(10t)^2 + \left(-5t^2 + 10t\sqrt{3}\right)^2}$$

When $t = 1$,

$$r = \sqrt{10^2 + \left(-5 + 10\sqrt{3}\right)^2}$$

$$= \sqrt{100 + 25 - 100\sqrt{3} + 300}$$

$$= \sqrt{425 - 100\sqrt{3}}$$

$$\doteqdot 15.9 \text{ m}$$

So the distance of the particle from the origin after one second is about 15.9 m.

3c ii The particle's speed can be calculated by applying Pythagoras' theorem to the particle's velocity vector $\underset{\sim}{v} = \dot{x}\underset{\sim}{i} + \dot{y}\underset{\sim}{j}$.

$v^2 = \dot{x}^2 + \dot{y}^2$

$v = \sqrt{\dot{x}^2 + \dot{y}^2}$

From (2) and (4):

$v = \sqrt{10^2 + \left(-10t + 10\sqrt{3}\right)^2}$

When $t = 1$,

$v = \sqrt{10^2 + \left(-10 + 10\sqrt{3}\right)^2}$

$= \sqrt{100 + 100 - 200\sqrt{3} + 300}$

$= \sqrt{500 - 200\sqrt{3}}$

$\doteqdot 12.4 \text{ m/s}$

The speed of the particle after one second is about 12.4 m/s.

4a Initially, $x = y = 0$, and, $\dot{x} = 60\cos\left(\tan^{-1}\frac{4}{3}\right)$ $\dot{y} = 60\sin\left(\tan^{-1}\frac{4}{3}\right)$

$\dot{x} = 36$ $\dot{y} = 48$

To begin,	$\ddot{x} = 0$	(1)	To begin,	$\ddot{y} = -10$	(3)
Integrating,	$\dot{x} = C_1$		Integrating,	$\dot{y} = -10t + C_2$	
When $t = 0$,	$\dot{x} = 36$		When $t = 0$,	$\dot{y} = 48$	
	$36 = C_1$		$48 = C_2$		
so	$\dot{x} = 36$	(2)	so 3	$\dot{y} = -10t + 48$	(4)

4b

To begin,	$\dot{x} = 36$		To begin,	$\dot{y} = -10t + 48$	
Integrating,	$x = 36t + C_3$		Integrating,	$y = -5t^2 + 48t + C_4$	
When $t = 0$,	$x = 0$		When $t = 0$,	$y = 0$	
	$0 = C_3$		$0 = C_4$		
so	$x = 36t$	(5)	so	$y = -5t^2 + 48t$	(6)

4c $d^2 = x^2 + y^2$

$d = \sqrt{x^2 + y^2}$

From (5) and (6):

$d = \sqrt{(36t)^2 + (-5t^2 + 48t)^2}$

When $t = 3$,

$d = \sqrt{108^2 + (-45 + 144)^2}$

$= \sqrt{21\,465}$

$\doteqdot 146.5 \text{ m}$

The distance of the particle, from the point of projection, after 3 seconds is about 146.5 m.

4d $\underset{\sim}{v} = \dot{x}\underset{\sim}{i} + \dot{y}\underset{\sim}{j}$

From (2) and (4):

$\underset{\sim}{v} = 36\underset{\sim}{i} + (-10t + 48)\underset{\sim}{j}$

When $t = 3$,

$\underset{\sim}{v} = 36\underset{\sim}{i} + 18\underset{\sim}{j}$

$v^2 = 36^2 + 18^2$

$v = \sqrt{36^2 + 18^2}$

$= \sqrt{1296 + 324}$

$= \sqrt{1620}$

$= \sqrt{324 \times 5}$

$= 18\sqrt{5} \text{ m/s}$

$\tan\theta = \frac{18}{36} = \frac{1}{2}$

$\theta = \tan^{-1}\frac{1}{2}$

So, the velocity of the particle after three seconds is $18\sqrt{5}$ m/s at an angle of $\tan^{-1}\frac{1}{2}$ above the horizontal.

5a Initially, $x = y = 0$, and, $\dot{x} = 8$ $\dot{y} = 6$

To begin, $\ddot{x} = 0$ (1)
Integrating, $\dot{x} = C_1$
When $t = 0$, $\dot{x} = 8$
$8 = C_1$
so $\dot{x} = 8$ (2)

To begin, $\ddot{y} = -10$ (3)
Integrating, $\dot{y} = -10t + C_2$
When $t = 0$, $\dot{y} = 6$
$6 = C_2$
so $\dot{y} = -10t + 6$ (4)

Now $\underset{\sim}{v} = \dot{x}\underset{\sim}{i} + \dot{y}\underset{\sim}{j}$

so $\underset{\sim}{v} = 8\underset{\sim}{i} + (-10t + 6)\underset{\sim}{j}$

$\underset{\sim}{v} = 8\underset{\sim}{i} + (6 - 10t)\underset{\sim}{j}$

5b To begin, $\dot{x} = 8$
Integrating, $x = 8t + C_3$
When $t = 0$, $x = 0$
$0 = C_3$
so $x = 8t$ (5)

To begin, $\dot{y} = -10t + 6$
Integrating, $y = -5t^2 + 6t + C_4$
When $t = 0$, $y = 0$
$0 = C_4$
so $y = -5t^2 + 6t$ (6)

$\underset{\sim}{r} = x\underset{\sim}{i} + y\underset{\sim}{j}$

so $\underset{\sim}{r} = 8t\underset{\sim}{i} + (-5t^2 + 6t)\underset{\sim}{j}$

$\underset{\sim}{r} = (8t)\underset{\sim}{i} + (6t - 5t^2)\underset{\sim}{j}$

5c i $\underset{\sim}{v} = 8\underset{\sim}{i} + 6\underset{\sim}{j}$

$v^2 = 8^2 + 6^2$

$v = \sqrt{8^2 + 6^2}$

$= \sqrt{64 + 36}$

$= \sqrt{100}$

$= 10 \text{ m/s}$

The initial speed of the particle is 10 m/s.

5c ii $\underset{\sim}{r} = (8t)\underset{\sim}{i} + (6t - 5t^2)\underset{\sim}{j}$

When $t = 2$,

$\underset{\sim}{r} = 16\underset{\sim}{i} + (12 - 20)\underset{\sim}{j}$

$= 16\underset{\sim}{i} - 8\underset{\sim}{j}$

5c iii From (4):

When $\dot{y} = 0$,

$0 = -10t + 6$

$10t = 6$

$t = 0.6$ seconds

When $t = 0.6$,

$\underset{\sim}{r} = 4.8\underset{\sim}{i} + (3.6 - 1.8)\underset{\sim}{j}$

$= 4.8\underset{\sim}{i} + 1.8\underset{\sim}{j}$

6a To begin, $x = 40t$ (1)

Differentiate, $\dot{x} = 40$ (2)

When $t = 0$, $\dot{x} = 40$

To begin, $y = -5t^2 + 25t$ (3)

Differentiate, $\dot{y} = -10t + 25$ (4)

When $t = 0$, $\dot{y} = 25$

6b From (2) and (4):

$\underset{\sim}{v} = 40\underset{\sim}{i} + 25\underset{\sim}{j}$

$v^2 = 40^2 + 25^2$

$v = \sqrt{40^2 + 25^2}$

$= \sqrt{1600 + 625}$

$= \sqrt{2225}$

$= 5\sqrt{89}$ m/s

$\tan\theta = \dfrac{25}{40} = \dfrac{5}{8}$

$\theta = \tan^{-1}\dfrac{5}{8}$

$\theta \doteqdot 32°$

7a Initially, $x = y = 0$, and, $\dot{x} = V\cos\alpha$ $\dot{y} = V\sin\alpha$

To begin, $\ddot{x} = 0$ (1)

Integrating, $\dot{x} = C_1$

When $t = 0$, $\dot{x} = V\cos\alpha$

$V\cos\alpha = C_1$

so $\dot{x} = V\cos\alpha$ (2a)

Integrating, $x = Vt\cos\alpha + C_2$

When $t = 0$, $x = 0$

$0 = C_2$

so $x = Vt\cos\alpha$ (3a)

When $t = 2$, $8 = 2V\cos\alpha$

so $V\cos\alpha = 4$

To begin, $\ddot{y} = -10$ (4)

Integrating, $\dot{y} = -10t + C_3$

When $t = 0$, $\dot{y} = V\sin\alpha$

$V\sin\alpha = C_3$

so $\dot{y} = -10t + V\sin\alpha$ (5a)

Integrating, $y = -5t^2 + Vt\sin\alpha + C_4$

When $t = 0$, $y = 0$

$0 = C_4$

so $y = -5t^2 + Vt\sin\alpha$ (6a)

When $t = 2$, $-12 = -20 + 2V\sin\alpha$

so $V\sin\alpha = 4$

Hence, $\underset{\sim}{v} = 4\underset{\sim}{i} + 4\underset{\sim}{j}$

From (2a): $\dot{x} = 4$ (2b)

From (3a): $x = 4t$ (3b)

From (5a): $\dot{y} = -10t + 4$ (5b)

From (6a): $y = -5t^2 + 4t$ (6b)

7b $\underset{\sim}{r} = x\underset{\sim}{i} + y\underset{\sim}{j}$

From (3b) and (6b):

$\underset{\sim}{r} = 4t\underset{\sim}{i} + (-5t^2 + 4t)\underset{\sim}{j}$

When $t = 0.5$,

$\underset{\sim}{r} = 2\underset{\sim}{i} + (-1.25 + 2)\underset{\sim}{j}$

$= 2\underset{\sim}{i} + 0.75\underset{\sim}{j}$

8a Initially, $x = y = 0$, and, $\dot{x} = V\cos\theta$ $\dot{y} = V\sin\theta$

To begin,	$\ddot{x} = 0$	(1)	To begin,	$\ddot{y} = -10$	(4)
Integrating,	$\dot{x} = C_1$		Integrating,	$\dot{y} = -10t + C_3$	
When $t = 0$,	$\dot{x} = V\cos\theta$		When $t = 0$,	$\dot{y} = V\sin\theta$	
$V\cos\theta = C_1$			$V\sin\theta = C_3$		
so	$\dot{x} = V\cos\theta$	(2a)	so	$\dot{y} = -10t + V\sin\theta$	(5a)
Integrating,	$x = Vt\cos\theta + C_2$		Integrating,	$y = -5t^2 + Vt\sin\theta + C_4$	
When $t = 0$,	$x = 0$		When $t = 0$,	$y = 0$	
$0 = C_2$			$0 = C_4$		
so	$x = Vt\cos\theta$	(3a)	so	$y = -5t^2 + Vt\sin\theta$	(6a)
When $t = 2$,	$30 = 2V\cos\theta$		When $t = 2$,	$30 = -20 + 2V\sin\theta$	
so	$V\cos\theta = 15$		so	$V\sin\theta = 25$	
From (2a):	$\dot{x} = 15$	(2b)	From (5a):	$\dot{y} = -10t + 25$	(5b)
From (3a):	$x = 15t$	(3b)	From (6a):	$y = -5t^2 + 25t$	(6b)

8b $\underset{\sim}{V} = V\cos\theta\,\underset{\sim}{i} + V\sin\theta\,\underset{\sim}{j}$

So $\underset{\sim}{V} = 15\underset{\sim}{i} + 25\underset{\sim}{j}$

$V^2 = 15^2 + 25^2$

$V = \sqrt{15^2 + 25^2}$

$= \sqrt{225 + 625}$

$= \sqrt{850}$

$= 5\sqrt{34}$ m/s

$\tan\theta = \dfrac{25}{15} = \dfrac{5}{3}$

$\theta = \tan^{-1}\dfrac{5}{3}$

9a Initially, $x = y = 0$, and, $\dot{x} = 12\cos(-30°)$ $\dot{y} = 12\sin(-30°)$

$\dot{x} = 6\sqrt{3}$ $\dot{y} = -6$

To begin, $\ddot{x} = 0$ (1)

Integrating, $\dot{x} = C_1$

When $t = 0$, $\dot{x} = 6\sqrt{3}$

$6\sqrt{3} = C_1$

so $\dot{x} = 6\sqrt{3}$ (2)

To begin, $\ddot{y} = -10$ (3)

Integrating, $\dot{y} = -10t + C_2$

When $t = 0$, $\dot{y} = -6$

$-6 = C_2$

so $\dot{y} = -10t - 6$ (4)

9b To begin, $\dot{x} = 6\sqrt{3}$

Integrating, $x = 6t\sqrt{3} + C_3$

When $t = 0$, $x = 0$

$0 = C_3$

so $x = 6t\sqrt{3}$ (5)

To begin, $\dot{y} = -10t - 6$

Integrating, $y = -5t^2 - 6t + C_4$

When $t = 0$, $y = 0$

$0 = C_4$

so $y = -5t^2 - 6t$ (6)

9c The stone will hit the ground when $y = -11$.

Substituting into (6):

$-11 = -5t^2 - 6t$

$5t^2 + 6t - 11 = 0$

$(5t + 11)(t - 1) = 0$

$t = -\dfrac{11}{5}$ or $t = 1$

But $t \geq 0$, so the stone hits the ground when $t = 1$ second.

9d From (5):

$x = 6t\sqrt{3}$

When $t = 1$,

$x = 6\sqrt{3}$

$\quad \doteqdot 10.4 \text{ m}$

The stone hits the ground about 10.4 m from the base of the tower.

9e $\underset{\sim}{v} = \dot{x}\underset{\sim}{i} + \dot{y}\underset{\sim}{j}$

From (2) and (4):

$\underset{\sim}{v} = 6t\sqrt{3}\underset{\sim}{i} + (-10t - 6)\underset{\sim}{j}$

When $t = 1$,

$\underset{\sim}{v} = 6\sqrt{3}\underset{\sim}{i} - 16\underset{\sim}{j}$

$$\tan\theta = -\frac{16}{6\sqrt{3}}$$

$$= -\frac{8\sqrt{3}}{9}$$

$$\theta = \tan^{-1}\left(-\frac{8\sqrt{3}}{9}\right)$$

So the stone will hit the ground at an angle of $\tan^{-1}\frac{8\sqrt{3}}{9}$ below the horizontal.

10a Initially, $x = y = 0$, and, $\dot{x} = 6\cos 45°$, $\dot{x} = 3\sqrt{2}$ $\quad \dot{y} = 6\sin 45°$, $\dot{y} = 3\sqrt{2}$

To begin, $\ddot{x} = 0$ (1)

Integrating, $\dot{x} = C_1$

When $t = 0$, $\dot{x} = 3\sqrt{2}$

$3\sqrt{2} = C_1$

so $\dot{x} = 3\sqrt{2}$ (2)

To begin, $\dot{x} = 3\sqrt{2}$

Integrating, $x = 3t\sqrt{2} + C_2$

When $t = 0$, $x = 0$

$0 = C_2$

To begin, $\ddot{y} = -10$ (4)

Integrating, $\dot{y} = -10t + C_3$

When $t = 0$, $\dot{y} = 3\sqrt{2}$

$3\sqrt{2} = C_3$

so $\dot{y} = -10t + 3\sqrt{2}$ (5)

To begin, $\dot{y} = -10t + 3\sqrt{2}$

Integrating, $y = -5t^2 + 3t\sqrt{2} + C_4$

When $t = 0$, $y = 0$

$0 = C_4$

so $x = 3t\sqrt{2}$ (3)

so $y = -5t^2 + 3t\sqrt{2}$ (6)

The ball will hit the ground when $y = -2$.

Substituting into (6):

$-2 = -5t^2 + 3t\sqrt{2}$

$5t^2 - 3t\sqrt{2} - 2 = 0$

Using the quadratic formula:

$$t = \frac{3\sqrt{2} \pm \sqrt{\left(-3\sqrt{2}\right)^2 - 4 \times 5 \times (-2)}}{2 \times 5}$$

$$= \frac{3\sqrt{2} \pm \sqrt{18 + 40}}{10}$$

$$= \frac{3\sqrt{2} \pm \sqrt{58}}{10}$$

$t \doteqdot -0.34$ or $t \doteqdot 1.19$

But $t \geq 0$, so the stone hits the ground when $t \doteqdot 1.19$ seconds.

10b From (3):

$x = 3t\sqrt{2}$

When $t = \dfrac{3\sqrt{2} + \sqrt{58}}{10}$,

$$x = 3 \times \frac{3\sqrt{2} + \sqrt{58}}{10} \times \sqrt{2}$$

$\doteqdot 5.03$ m

The horizontal distance travelled by the ball is about 5.03 m.

11 Let V be the initial speed of the projectile

Initially, $x = y = 0$, and, $\dot{x} = V\cos 36°$ $\dot{y} = V\sin 36°$

To begin, $\ddot{x} = 0$ (1)

Integrating, $\dot{x} = C_1$

To begin, $\ddot{y} = -10$ (4)

Integrating, $\dot{y} = -10t + C_3$

When $t = 0$, $\dot{x} = V\cos 36°$

$V\cos 36° = C_1$

so $\dot{x} = V\cos 36°$ (2)

To begin, $\dot{x} = V\cos 36°$

Integrating, $x = Vt\cos 36° + C_2$

When $t = 0$, $x = 0$

$0 = C_2$

so $x = Vt\cos 36°$ (3)

When $t = 0$, $\dot{y} = V\sin 36°$

$V\sin 36° = C_3$

so $\dot{y} = -10t + V\sin 36°$ (5)

To begin, $\dot{y} = -10t + V\sin 36°$

Integrating, $y = -5t^2 + Vt\sin 36° + C_4$

When $t = 0$, $y = 0$

$0 = C_4$

so $y = -5t^2 + Vt\sin 36°$ (6)

From (3):

$$t = \frac{x}{V\cos 36°}$$

Substituting for t in (6):

$$y = -5\left(\frac{x}{V\cos 36°}\right)^2 + \left(\frac{Vx}{V\cos 36°}\right)\sin 36°$$

$$= \frac{-5x^2}{V^2\cos^2 36°} + x\tan 36°$$

If the projectile just clears the wall, it will reach a point where $x = 20$ and $y = 10$

$$10 = \frac{-5(20)^2}{V^2\cos^2 36°} + 20\tan 36°$$

$$10 = \frac{-2000}{V^2\cos^2 36°} + 20\tan 36°$$

$$10 - 20\tan 36° = \frac{-2000}{V^2\cos^2 36°}$$

$$V^2\cos^2 36° = \frac{-2000}{10 - 20\tan 36°}$$

$$V^2\cos^2 36° = \frac{2000}{20\tan 36° - 10}$$

$$V^2 = \frac{2000}{\cos^2 36°\,(20\tan 36° - 10)}$$

$$V = \sqrt{\frac{2000}{\cos^2 36°\,(20\tan 36° - 10)}}$$

$V \doteqdot 26\text{ m/s}$

The initial speed of the projectile is about 26 m/s.

12 Initially, $x = y = 0$, and, $\dot{x} = 20\cos\alpha$ $\dot{y} = 20\sin\alpha$

To begin,	$\ddot{x} = 0$	(1)	To begin,	$\ddot{y} = -10$	(4)
Integrating,	$\dot{x} = C_1$		Integrating,	$\dot{y} = -10t + C_3$	
When $t = 0$,	$\dot{x} = 20\cos\alpha$		When $t = 0$,	$\dot{y} = 20\sin\alpha$	
$20\cos\alpha = C_1$			$20\sin\alpha = C_3$		
so	$\dot{x} = 20\cos\alpha$	(2)	so	$\dot{y} = -10t + 20\sin\alpha$	(5)
To begin,	$\dot{x} = 20\cos\alpha$		To begin,	$\dot{y} = -10t + 20\sin\alpha$	
Integrating,	$x = 20t\cos\alpha + C_2$		Integrating,	$y = -5t^2 + 20t\sin\alpha + C_4$	
When $t = 0$,	$x = 0$		When $t = 0$,	$y = 0$	
$0 = C_2$			$0 = C_4$		
so	$x = 20t\cos\alpha$	(3)	so	$y = -5t^2 + 20t\sin\alpha$	(6)

As the ball reaches its highest point, $\dot{y} = 0$.

Substituting into (5):

$0 = -10t + 20\sin\alpha$

$10t = 20\sin\alpha$

$t = 2\sin\alpha$

Substituting for t in (6):

$y = -5(2\sin\alpha)^2 + 40\sin^2\alpha$

$\quad = -20\sin^2\alpha + 40\sin^2\alpha$

$\quad = 20\sin^2\alpha$

In order to clear the wall at its highest point, $y = 3$ when $\dot{y} = 0$.

$3 = 20\sin^2\alpha$

$\sin^2\alpha = \dfrac{3}{20}$

$$\sin\alpha = \sqrt{\frac{3}{20}}$$
$$= \sqrt{\frac{3}{20}} \times \frac{\sqrt{5}}{\sqrt{5}}$$
$$= \frac{\sqrt{15}}{\sqrt{100}}$$
$$= \frac{\sqrt{15}}{10}$$
$$\alpha = \sin^{-1}\frac{\sqrt{15}}{10}$$

13 For P_1

Initially, $x_1 = y_1 = 0$, and, $\dot{x}_1 = 20$ $\dot{y}_1 = 30$

To begin, $\ddot{x}_1 = 0$ (1a)
Integrating, $\dot{x}_1 = C_1$
When $t = 0$, $\dot{x}_1 = 20$
$20 = C_1$
so $\dot{x}_1 = 20$ (2a)
To begin, $\dot{x}_1 = 20$
Integrating, $x_1 = 20t + C_2$
When $t = 0$, $x_1 = 0$
$0 = C_2$
so $x_1 = 20t$ (3a)

To begin, $\ddot{y}_1 = -10$ (4a)
Integrating, $\dot{y}_1 = -10t + C_3$
When $t = 0$, $\dot{y}_1 = 30$
$30 = C_3$
so $\dot{y}_1 = -10t + 30$ (5a)
To begin, $\dot{y}_1 = -10t + 30$
Integrating, $y_1 = -5t^2 + 30t + C_4$
When $t = 0$, $y_1 = 0$
$0 = C_4$
so $y_1 = -5t^2 + 30t$ (6a)

For P_2

Initially, $x_2 = y_2 = 0$, and, $\dot{x}_2 = 60$ $\dot{y}_2 = 50$

To begin, $\ddot{x}_2 = 0$ (1b)
Integrating, $\dot{x}_2 = C_1$
When $t = 0$, $\dot{x}_2 = 60$
$60 = C_1$

To begin, $\ddot{y}_2 = -10$ (4b)
Integrating, $\dot{y}_2 = -10t + C_3$
When $t = 0$, $\dot{y}_2 = 50$
$50 = C_3$

so $\dot{x}_2 = 60$ (2b)

To begin, $\dot{x}_2 = 60$

Integrating, $x_2 = 60t + C_2$

When $t = 0$, $x_2 = 0$

$0 = C_2$

so $x_2 = 60t$ (3b)

so $\dot{y}_2 = -10t + 50$ (5b)

To begin, $\dot{y}_2 = -10t + 50$

Integrating, $y_2 = -5t^2 + 50t + C_4$

When $t = 0$, $y_2 = 0$

$0 = C_4$

so $y_2 = -5t^2 + 50t$ (6b)

Since P_2 is projected two seconds after P_1,

then $x_2 = 60(t-2)$

and $y_2 = -5(t-2)^2 + 50(t-2)$

If the particles collide, the same value of t must allow $x_1 = x_2$ and $y_1 = y_2$.

From (3a) and (3b):

$20t = 60(t-2)$

$20t = 60t - 120$

$40t = 120$

$t = 3$

From (6a) and (6b):

$-5t^2 + 30t = -5(t-2)^2 + 50(t-2)$

Check when $t = 3$,

$\text{LHS} = -45 + 90 = 45$

$\text{RHS} = -5 + 50 = 45$

Hence LHS = RHS.

P_1 and P_2 collide at 3 seconds after P_1 is projected (2 seconds after P_2 is projected).

14a Initially, $x = y = 0$, and, $\dot{x} = 20\sqrt{2}\cos\theta$ $\dot{y} = 20\sqrt{2}\sin\theta$

To begin, $\ddot{x} = 0$ (1)

Integrating, $\dot{x} = C_1$

When $t = 0$, $\dot{x} = 20\sqrt{2}\cos\theta$

$20\sqrt{2}\cos\theta = C_1$

so $\dot{x} = 20\sqrt{2}\cos\theta$ (2)

To begin, $\dot{x} = 20\sqrt{2}\cos\theta$

Integrating, $x = 20t\sqrt{2}\cos\theta + C_2$

When $t = 0$, $x = 0$

$0 = C_2$

so $x = 20t\sqrt{2}\cos\theta$ (3)

To begin, $\ddot{y} = -10$ (4)

Integrating, $\dot{y} = -10t + C_3$

When $t = 0$, $\dot{y} = 20\sqrt{2}\sin\theta$

$20\sqrt{2}\sin\theta = C_3$

so $\dot{y} = -10t + 20\sqrt{2}\sin\theta$ (5)

To begin, $\dot{y} = -10t + 20\sqrt{2}\sin\theta$

Integrating, $y = -5t^2 + 20t\sqrt{2}\sin\theta + C_4$

When $t = 0$, $y = 0$

$0 = C_4$

so $y = -5t^2 + 20t\sqrt{2}\sin\theta$ (6)

At time t, $x = 20$ and $y = 15$.

Substituting $x = 20$ into (3):

$20 = 20t\sqrt{2}\cos\theta$

so $\sqrt{2}\,t\cos\theta = 1$ (7)

Substituting $y = 15$ into (6):

$15 = -5t^2 + 20t\sqrt{2}\sin\theta$

so $4\sqrt{2}\,t\sin\theta - t^2 = 3$ (8)

14b From (7):

$$t = \frac{1}{\sqrt{2}\cos\theta}$$

Substituting for t in (8):

$$\frac{4\sqrt{2}}{\sqrt{2}\cos\theta}\sin\theta - \left(\frac{1}{\sqrt{2}\cos\theta}\right)^2 = 3$$

$$4\tan\theta - \frac{1}{2\cos^2\theta} = 3$$

$4\tan\theta - \dfrac{1}{2\cos^2\theta} - 3 = 0$

$\dfrac{1}{\cos^2\theta} - 8\tan\theta + 6 = 0$

$\sec^2\theta - 8\tan\theta + 6 = 0$

But $\sec^2\theta = \tan^2\theta + 1$

so $\tan^2\theta - 8\tan\theta + 7 = 0$

14c $\tan^2\theta - 8\tan\theta + 7 = 0$

$(\tan\theta - 1)(\tan\theta - 7) = 0$

$\tan\theta = 1$ or 7

$\theta = 45°$ or $\tan^{-1} 7$

$\theta = 45°$ or $\theta \doteqdot 81°52'$

15a Initially, $x = y = 0$, and, $\dot{x} = 50\cos\theta$ $\dot{y} = 50\sin\theta$

To begin, $\ddot{x} = 0$ (1)

Integrating, $\dot{x} = C_1$

When $t = 0$, $\dot{x} = 50\cos\theta$

$50\cos\theta = C_1$

so $\dot{x} = 50\cos\theta$ (2)

To begin, $\dot{x} = 50\cos\theta$

Integrating, $x = 50t\cos\theta + C_2$

When $t = 0$, $x = 0$

$0 = C_2$

so $x = 50t\cos\theta$ (3)

To begin, $\ddot{y} = -10$ (4)

Integrating, $\dot{y} = -10t + C_3$

When $t = 0$, $\dot{y} = 50\sin\theta$

$50\sin\theta = C_3$

so $\dot{y} = -10t + 50\sin\theta$ (5)

To begin, $\dot{y} = -10t + 50\sin\theta$

Integrating, $y = -5t^2 + 50t\sin\theta + C_4$

When $t = 0$, $y = 0$

$0 = C_4$

so $y = -5t^2 + 50t\sin\theta$ (6)

At point P, $x = 100$ and $y = 25$.

Substituting $x = 100$ into (3):

$100 = 50t\cos\theta$

$$t = \frac{2}{\cos\theta} \qquad (7)$$

Substituting $y = 25$ into (6):

$$25 = -5t^2 + 50t\sin\theta$$

$$t^2 - 10t\sin\theta + 5 = 0 \qquad (8)$$

Substituting (7) into (8):

$$\frac{4}{\cos^2\theta} - \frac{20\sin\theta}{\cos\theta} + 5 = 0$$

$$4\sec^2\theta - 20\tan\theta + 5 = 0$$

$$\sec^2\theta - 5\tan\theta + 1.25 = 0$$

But $\sec^2\theta = \tan^2\theta + 1$

$$\tan^2\theta - 5\tan\theta + 2.25 = 0$$

$$\left(\tan\theta - \frac{1}{2}\right)\left(\tan\theta - \frac{9}{2}\right) = 0$$

$$\tan\theta = \frac{1}{2} \text{ or } \frac{9}{2}$$

$$\theta = \tan^{-1}\frac{1}{2} \text{ or } \tan^{-1}\frac{9}{2}$$

15b i

For $\theta = \tan^{-1}\frac{1}{2}$,

consider a right-angled triangle with angle θ and opposite side 1 unit, adjacent side 2 units and hypotenuse $\sqrt{1^2 + 2^2} = \sqrt{5}$ units.

Hence, $\sin\theta = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$ and $\cos\theta = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$

Substituting for $\cos\theta$ in (7):

$$t = \frac{2}{\cos\theta}$$

$$= \frac{2}{\left(\frac{2\sqrt{5}}{5}\right)}$$

$$= \frac{5}{\sqrt{5}}$$

$$= \sqrt{5} \text{ seconds}$$

For $\theta = \tan^{-1}\frac{9}{2}$,

consider a right-angled triangle with angle θ and opposite side 9 units, adjacent side 2 units and hypotenuse $\sqrt{9^2 + 2^2} = \sqrt{85}$ units.

Hence, $\sin\theta = \frac{9}{\sqrt{85}} = \frac{9\sqrt{85}}{85}$ and $\cos\theta = \frac{2}{\sqrt{85}} = \frac{2\sqrt{85}}{85}$

Substituting for $\cos\theta$ in (7):

$$t = \frac{2}{\cos\theta}$$

$$= \frac{2}{\left(\frac{2\sqrt{85}}{85}\right)}$$

$$= \frac{85}{\sqrt{85}}$$

$$= \sqrt{85} \text{ seconds}$$

15b ii To begin, $\underset{\sim}{v} = \dot{x}\underset{\sim}{i} + \dot{y}\underset{\sim}{j}$

From (2) and (5):

$$\underset{\sim}{v} = 50\cos\theta\,\underset{\sim}{i} + (-10t + 50\sin\theta)\underset{\sim}{j}$$

When $\theta = \tan^{-1}\frac{1}{2}$, $\sin\theta = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$ and $\cos\theta = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$,

$$v^2 = (50\cos\theta)^2 + \left(-10\sqrt{5} + 50\sin\theta\right)^2$$

$$= \left(50 \times \frac{2\sqrt{5}}{5}\right)^2 + \left(-10\sqrt{5} + 50 \times \frac{\sqrt{5}}{5}\right)^2$$

$$= \left(20\sqrt{5}\right)^2 + \left(-10\sqrt{5} + 10\sqrt{5}\right)^2$$

$$= \left(20\sqrt{5}\right)^2$$

$v = 20\sqrt{5}$

$\doteqdot 44.7 \text{ m/s}$

$$\tan\theta = \frac{-10\sqrt{5} + 50\sin\theta}{50\cos\theta}$$

$$= \frac{-10\sqrt{5} + 10\sqrt{5}}{20\sqrt{5}}$$

$$= 0$$

So $\theta = 0°$

When $\theta = \tan^{-1}\frac{9}{2}$, $\sin\theta = \frac{9}{\sqrt{85}} = \frac{9\sqrt{85}}{85}$ and $\cos\theta = \frac{2}{\sqrt{85}} = \frac{2\sqrt{85}}{85}$

$$v^2 = (50\cos\theta)^2 + \left(-10\sqrt{85} + 50\sin\theta\right)^2$$

$$= \left(\frac{50 \times 2\sqrt{85}}{85}\right)^2 + \left(-10\sqrt{85} + \frac{50 \times 9\sqrt{85}}{85}\right)^2$$

$$= \left(\frac{20\sqrt{85}}{17}\right)^2 + \left(-10\sqrt{85} + \frac{90\sqrt{85}}{17}\right)^2$$

$$= \left(\frac{20 \times \sqrt{17} \times \sqrt{5}}{17}\right)^2 + \left(-10 \times \sqrt{17} \times \sqrt{5} + \frac{90 \times \sqrt{17} \times \sqrt{5}}{17}\right)^2$$

$$= \left(\frac{20\sqrt{5}}{\sqrt{17}}\right)^2 + \left(\frac{-80\sqrt{5}}{\sqrt{17}}\right)^2$$

$$= \frac{2000}{17} + \frac{32\,000}{17}$$

$$= \frac{34\,000}{17}$$

$$= 2000$$

$v = \sqrt{2000}$

$= 20\sqrt{5}$

$\doteqdot 44.7 \text{ m/s}$

$$\tan\theta = \frac{-10\sqrt{85} + 50\sin\theta}{50\cos\theta}$$

$$= \frac{-10\sqrt{85} + \left(\frac{90\sqrt{85}}{17}\right)}{\left(\frac{20\sqrt{85}}{17}\right)}$$

$$= \frac{\frac{-170\sqrt{85}}{17} + \frac{90\sqrt{85}}{17}}{\left(\frac{20\sqrt{85}}{17}\right)}$$

$$= \frac{\frac{-80\sqrt{85}}{17}}{\left(\frac{20\sqrt{85}}{17}\right)}$$

$$= -4$$

So $\theta \doteqdot 76.0°$.

For each possible angle of projection, the velocity of the particle is about 44.7 m/s at angles of 0° and 76.0° to the horizontal.

16a i Initially, $x = y = 0$, and, $\dot{x} = V\cos\theta$ $\dot{y} = V\sin\theta$

To begin,	$\ddot{x} = 0$	(1)	To begin,	$\ddot{y} = -g$	(4)
Integrating,	$\dot{x} = C_1$		Integrating,	$\dot{y} = -g + C_3$	
When $t = 0$,	$\dot{x} = V\cos\theta$		When $t = 0$,	$\dot{y} = V\sin\theta$	
$V\cos\theta = C_1$			$V\sin\theta = C_3$		
so	$\dot{x} = V\cos\theta$	(2)	so	$\dot{y} = -gt + V\sin\theta$	(5)
To begin,	$\dot{x} = V\cos\theta$		To begin,	$\dot{y} = -gt + V\sin\theta$	
Integrating,	$x = Vt\cos\theta + C_2$		Integrating,	$y = -\frac{g}{2}t^2 + Vt\sin\theta + C_4$	
When $t = 0$,	$x = 0$		When $t = 0$,	$y = 0$	
$0 = C_2$			$0 = C_4$		
so	$x = Vt\cos\theta$	(3)	so	$y = -\frac{g}{2}t^2 + Vt\sin\theta$	(6)

A particle reaches its greatest height when $\dot{y} = 0$.

Substituting into (5):

$0 = -gt + V\sin\theta$

$gt = V\sin\theta$

$t = \dfrac{V\sin\theta}{g}$

Substituting for t in (6):

$y = -\dfrac{g}{2}\left(\dfrac{V\sin\theta}{g}\right)^2 + \dfrac{V^2\sin^2\theta}{g}$

$= -\dfrac{V^2\sin^2\theta}{2g} + \dfrac{V^2\sin^2\theta}{g}$

$= \dfrac{V^2\sin^2\theta}{2g}$ metres

Hence the greatest height is $\dfrac{V^2\sin^2\theta}{2g}$ metres.

16a ii The particle returns to the ground when $y = 0$ and $t > 0$.

Substituting for y in (6):

$0 = -\dfrac{g}{2}t^2 + Vt\sin\theta$

$t\left(\dfrac{g}{2}t - V\sin\theta\right) = 0$

$t = 0$ or $t = \dfrac{2V\sin\theta}{g}$

But $t > 0$ so $t = \dfrac{2V\sin\theta}{g}$

Substituting for t in (3):

$x = V\left(\dfrac{2V\sin\theta}{g}\right)\cos\theta$

$x = \dfrac{2V^2\sin\theta\cos\theta}{g}$

But $\sin 2\theta = 2\sin\theta\cos\theta$

so $x = \dfrac{V^2\sin 2\theta}{g}$ metres

Hence the horizontal range is $\dfrac{V^2\sin 2\theta}{g}$ metres.

16b If the horizontal range is five times the maximum height,

$$\frac{V^2\sin^2\theta}{2g} \div \frac{V^2\sin 2\theta}{g} = \frac{1}{5}$$

$$\frac{gV^2\sin^2\theta}{2gV^2\sin 2\theta} = \frac{1}{5}$$

$$\frac{\sin^2\theta}{2\sin 2\theta} = \frac{1}{5}$$

$$5\sin^2\theta = 2\sin 2\theta$$

$$5\sin^2\theta = 2\times 2\sin\theta\cos\theta$$

$$5\sin^2\theta = 4\sin\theta\cos\theta$$

$$5\sin\theta = 4\cos\theta \text{ where } \sin\theta \neq 0$$

$$\frac{\sin\theta}{\cos\theta} = \frac{4}{5}$$

$$\tan\theta = \frac{4}{5}$$

$$\text{so } \theta = \tan^{-1}\frac{4}{5} = \arctan\frac{4}{5}$$

17 The usual equations of motion are:

$\dot{x} = V\cos\theta$ and $\dot{y} = -gt + V\sin\theta$

$x = Vt\cos\theta$ $\quad$ $y = -\frac{g}{2}t^2 + Vt\sin\theta$

The maximum height occurs when $\dot{y} = 0$.

That is, when $t = \dfrac{V\sin\theta}{g}$.

So when $t = \dfrac{V\sin\theta}{g}$, $y = 2$

Hence,

$$2 = -\frac{g}{2}\times\frac{V^2\sin^2\theta}{g^2} + V\sin\theta\times\frac{V\sin\theta}{g}$$

$$2 = \frac{V^2\sin^2\theta}{2g}$$

$\sin^2\theta = \dfrac{4g}{V^2}$

$\sin\theta = \dfrac{2\sqrt{g}}{V}$ (θ is acute)

The time of flight is $\dfrac{2V\sin\theta}{g}$.

When $t = \dfrac{2V\sin\theta}{g}$,

$x = V\cos\theta \times \dfrac{2V\sin\theta}{g}$

$x = \dfrac{2V^2\sin\theta\cos\theta}{g}$

If $\sin\theta = \dfrac{2\sqrt{g}}{V}$,

consider a right-angled triangle with angle θ and opposite side $2\sqrt{g}$ units, hypotenuse V units and adjacent side $\sqrt{V^2 - (2\sqrt{g})^2} = \sqrt{V^2 - 4g}$ units.

Hence, $\cos\theta = \dfrac{\sqrt{V^2 - 4g}}{V}$

Therefore,

$x = \dfrac{2V^2}{g} \times \dfrac{2\sqrt{g}}{V} \times \dfrac{\sqrt{V^2 - 4g}}{V}$

$x = \dfrac{4}{\sqrt{g}}\sqrt{V^2 - 4g}$

$x = \sqrt{\dfrac{16}{g}(V^2 - 4g)}$ as required.

18a $\dot{x} = V\cos\theta$ and $\dot{y} = -gt + V\sin\theta$

$x = Vt\cos\theta$ $\qquad$ $y = -\frac{g}{2}t^2 + Vt\sin\theta$

The greatest height occurs when $\dot{y} = 0$

That is, when $t = \dfrac{V\sin\theta}{g}$.

So when $t = \dfrac{V\sin\theta}{g}$, $y = h$

Hence,

$$h = -\frac{g}{2}\times\frac{V^2\sin^2\theta}{g^2} + V\sin\theta\times\frac{V\sin\theta}{g}$$

$$h = \frac{V^2\sin^2\theta}{2g}$$

$$V^2\sin^2\theta = 2gh \qquad (1)$$

18b

When $y = \dfrac{h}{2}$,

$$\frac{h}{2} = -\frac{gt^2}{2} + Vt\sin\theta$$

$$gt^2 - (2V\sin\theta)t + h = 0$$

Using the quadratic formula:

$$t = \frac{2V\sin\theta \pm \sqrt{4V^2\sin^2\theta - 4gh}}{2g}$$

$$t = \frac{V\sin\theta \pm \sqrt{V^2\sin^2\theta - gh}}{g}$$

From (1), $V^2\sin^2\theta = 2gh$ hence,

$$t = \frac{\sqrt{2gh} \pm \sqrt{2gh - gh}}{g}$$

$$t = \frac{\sqrt{2}\sqrt{gh} \pm \sqrt{gh}}{g}$$

$$t = \frac{\sqrt{gh}(\sqrt{2} \pm 1)}{g}$$

$$t = \frac{(\sqrt{2} \pm 1)\sqrt{h}}{\sqrt{g}}$$

Hence,

$t = \dfrac{(\sqrt{2}+1)\sqrt{h}}{\sqrt{g}}$ or $t = \dfrac{(\sqrt{2}-1)\sqrt{h}}{\sqrt{g}}$ as required.

18c At $y = h$, $\dot{y} = 0$, so the speed is equal to the horizontal speed, which is $V\cos\theta$.

At $y = \dfrac{h}{2}$, while the projectile is rising,

$$t = \frac{(\sqrt{2}-1)\sqrt{h}}{\sqrt{g}}$$

So $V\cos\theta$, and

$$\dot{y} = V\sin\theta - g \times \frac{(\sqrt{2}-1)\sqrt{h}}{\sqrt{g}}$$

$$\dot{y} = V\sin\theta - (\sqrt{2}-1)\sqrt{gh}$$

From (1) in part a:

$$V^2\sin^2\theta = 2gh$$

$$gh = \frac{V^2\sin^2\theta}{2}$$

$$\sqrt{gh} = \frac{V\sin\theta}{\sqrt{2}}$$

Hence

$$\dot{y} = V\sin\theta - (\sqrt{2}-1)\frac{V\sin\theta}{\sqrt{2}}$$

$$\dot{y} = \frac{\sqrt{2}V\sin\theta - \sqrt{2}V\sin\theta + V\sin\theta}{\sqrt{2}}$$

$$\dot{y} = \frac{V\sin\theta}{\sqrt{2}}$$

At $y = \frac{h}{2}$,

$$\text{speed} = \sqrt{(\dot{x})^2 + (\dot{y})^2}$$

$$= \sqrt{V^2 \cos^2\theta + \frac{V^2 \sin^2\theta}{2}}$$

$$= \frac{V}{\sqrt{2}}\sqrt{2\cos^2\theta + \sin^2\theta}$$

So the ratio of the speed is:

$$\frac{\frac{V}{\sqrt{2}}\sqrt{2\cos^2\theta + \sin^2\theta}}{V\cos\theta} = \frac{\sqrt{5}}{\sqrt{2}}$$

$$\frac{\sqrt{2\cos^2\theta + \sin^2\theta}}{\cos\theta} = \sqrt{5}$$

$$\sqrt{5}\cos\theta = \sqrt{2\cos^2\theta + \sin^2\theta}$$

$$5\cos^2\theta = 2\cos^2\theta + \sin^2\theta$$

$$5\cos^2\theta = 2\cos^2\theta + 1 - \cos^2\theta$$

$$4\cos^2\theta = 1$$

$$\cos^2\theta = \frac{1}{4}$$

$$\cos\theta = \frac{1}{2} \qquad (\text{as } \cos\theta > 0)$$

$$\theta = \frac{\pi}{3} \text{ or } 60°$$

Solutions to Exercise 10B

1a

$$y = -\frac{5}{324}x^2 + \frac{4}{3}x$$

When $x = 12,$

$$y = -\frac{5}{324} \times 12^2 + \frac{4}{3} \times 12$$

$$= -\frac{720}{324} + 16$$

$$= 13\frac{7}{9} \text{ m}$$

or $y \doteqdot 13.8$ m

The height of the particle is about 13.8 m.

1b When $y = 19,$

$$19 = -\frac{5}{324}x^2 + \frac{4}{3}x$$

$$5x^2 - 432x + 6156 = 0$$

Using the quadratic formula:

$$x = \frac{432 \pm \sqrt{432^2 - 4 \times 5 \times 6156}}{2 \times 5}$$

$$= \frac{432 \pm \sqrt{63\,504}}{10}$$

$$= \frac{432 \pm 252}{10}$$

$$= 43.2 - 25.2 \text{ or } 43.2 + 25.2$$

So $x = 18$ m or 68.4. m

The particle has travelled a horizontal distance of 18 m or 68.4 m.

1c i

$$\frac{dy}{dx} = -\frac{2\times 5}{324}x + \frac{4}{3}$$
$$= -\frac{5}{162}x + \frac{4}{3}$$

When $x = 0$,

$$\frac{dy}{dx} = -\frac{5}{162}\times 0 + \frac{4}{3}$$
$$= \frac{4}{3}$$

Let α = angle of projection

$$\tan\alpha = \frac{dy}{dx} = \frac{4}{3}$$

$$\alpha = \tan^{-1}\frac{4}{3}$$

So the angle of projection is $\tan^{-1}\frac{4}{3}$ above the horizontal.

1c ii When $x = 18$,

$$\frac{dy}{dx} = -\frac{5}{162}\times 18 + \frac{4}{3}$$
$$= \frac{7}{9}$$

Let α = angle of projection

$$\tan\alpha = \frac{dy}{dx} = \frac{7}{9}$$

$$\alpha = \tan^{-1}\frac{7}{9}$$

So the direction of motion is $\tan^{-1}\frac{7}{9}$ above the horizontal.

1c iii When $x = 54$,

$$\frac{dy}{dx} = -\frac{5}{162}\times 54 + \frac{4}{3}$$

$$= -\frac{1}{3}$$

Let $\alpha =$ angle of projection

$$\tan\alpha = \frac{dy}{dx} = -\frac{1}{3}$$

$$\alpha = \tan^{-1}\left(-\frac{1}{3}\right)$$

So the direction of motion is $\tan^{-1}\frac{1}{3}$ below the horizontal.

2a Initially, $t = 0 = x = y$.

$x = 48t$

so $t = \dfrac{x}{48}$ (1)

and

$y = -5t^2 + 20t$ (2)

Substituting (1) into (2):

$$y = -5\left(\frac{x}{48}\right)^2 + 20\left(\frac{x}{48}\right)$$

$$y = -\frac{5}{2304}x^2 + \frac{5}{12}x \qquad (3)$$

2b i When $y = 0$,

$$0 = -\frac{5}{2304}x^2 + \frac{5}{12}x$$

$$\frac{5}{2304}x^2 - \frac{5}{12}x = 0$$

$$5x^2 - 960x = 0$$

$$5x(x - 192) = 0$$

$$x = 0 \text{ or } 192$$

Therefore, horizontal range of particle is 192 m.

2b ii Using symmetry, greatest height is at midpoint of horizontal range.

$$x = \frac{192}{2} = 96$$

Thus, $x = 96$ m at greatest height.

Substituting into (3):

$$y = -\frac{5}{2304} \times 96^2 + \frac{5}{12} \times 96$$

$$= 20 \text{ m}$$

Alternatively, $\frac{dy}{dx} = 0$ for maximum/minimum height.

$$-\frac{5x}{1152} + \frac{5}{12} = 0$$

$$5x - \frac{5 \times 1152}{12} = 0$$

$$5x = 480$$

$$x = 96$$

$$\frac{d^2y}{dx^2} = -\frac{5}{1152} < 0, \text{so } x = 96 \text{ gives maximum height.}$$

Substituting $x = 96$ into (3):

$$y = -\frac{5}{2304} \times 96^2 + \frac{5}{12} \times 96$$

$$= 20 \text{ m}$$

The greatest height of the particle is 20 m.

2b iii Initially, $t = x = 0$

$$\frac{dy}{dx} = -\frac{5x}{1152} + \frac{5}{12}$$

When $x = 0$,

$$\frac{dy}{dx} = -\frac{5 \times 0}{1152} + \frac{5}{12}$$

$$= \frac{5}{12}$$

Let α = angle of projection

$\tan\alpha = \dfrac{dy}{dx} = \dfrac{5}{12}$

$\alpha \ = \tan^{-1}\dfrac{5}{12}$

$\alpha \doteqdot \ 22.6°$

The angle of projection is about 22.6°.

2b iv When $x = 120$,

$\dfrac{dy}{dx} = -\dfrac{5 \times 120}{1152} + \dfrac{5}{12}$

$\quad = -\dfrac{25}{48} + \dfrac{5}{12}$

$\quad = -\dfrac{5}{48}$

Let $\alpha \ =$ angle of projection

$\tan\alpha = \dfrac{dy}{dx} = -\dfrac{5}{48}$

$\alpha = \tan^{-1}\left(-\dfrac{5}{48}\right) \doteqdot -6°$

The direction of the particle will be 6° below the horizontal.

3a $|\underset{\sim}{v}| = \sqrt{3^2 + 1^2}$

$\quad = \sqrt{10}$

The initial speed is $\sqrt{10}$.

3b To begin, $\dot{x} = 3$

Integrating, $x = 3t + C_1$

When $t = 0$, $x = 0$

Thus, $C_1 = 0$

so $x = 3t$ (1)

To begin, $\dot{y} = 1 - 10t$

Integrating, $y = t - 5t^2 + C_2$

When $t = 0$, $y = 0$

Thus, $C_2 = 0$

so $y = t - 5t^2$ (2)

3c $x = 3t$

$t = \frac{x}{3}$ (3)

Substituting (3) into (2):

$$y = \frac{x}{3} - 5\left(\frac{x}{3}\right)^2$$

$$= \frac{x}{3} - \frac{5x^2}{9}$$

$$9y = 3x - 5x^2$$

3d i

$$\frac{dy}{dx} = \frac{1}{3} - \frac{10}{9}x$$

When $x = 0.15$,

$$\frac{dy}{dx} = \frac{1}{3} - \frac{10}{9} \times 0.15$$

$$= \frac{6}{18} - \frac{3}{18}$$

$$= \frac{3}{18}$$

$$= \frac{1}{6}$$

Let α = angle of projection

$$\tan\alpha = \frac{dy}{dx} = \frac{1}{6}$$

$\alpha = \tan^{-1}\left(\frac{1}{6}\right) \doteqdot 9.5°$

The direction of motion of the particle will be 9.5° above the horizontal.

3d ii

$\frac{dy}{dx} = \frac{1}{3} - \frac{10}{9}x$

When $t = 0.15, x = 3 \times 0.15 = 0.45$ (using $x = 3t$)

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{3} - \frac{10}{9} \times 0.45\\ &= \frac{6}{18} - \frac{9}{18}\\ &= -\frac{3}{18}\\ &= -\frac{1}{6}\end{aligned}$$

Let α = angle of projection

$\tan\alpha = \frac{dy}{dx} = -\frac{1}{6}$

$\alpha = \tan^{-1}\left(-\frac{1}{6}\right) \doteqdot -9.5°$

The direction of motion of the particle will be 9.5° below the horizontal.

4a To begin, $\dot{x} = 5$

Integrating, $x = 5t + C_1$

When $t = 0$, $x = 0$

Thus, $C_1 = 0$

so $x = 5t$ (1)

To begin, $\dot{y} = -10t$

Integrating, $y = -5t^2 + C_2$

When $t = 0$, $y = 0$

Thus, $C_2 = 0$

so $\quad y = -5t^2 \quad (2)$

From (1):

$t = \frac{x}{5}$

Substituting $t = \frac{x}{5}$ into (2):

$y = -5\left(\frac{x}{5}\right)^2$

$y = -\frac{1}{5}x^2$

4b Object lands at $y = -20$ m

$-20 = -\frac{1}{5}x^2$

$100 = x^2$

$x = 10 \text{ m} \quad (\text{as } x > 0)$

The object hits the ground 10 m from the base of the tower.

4c Object hits the ground when $x = 10$ m.

$t = \frac{10}{5}$

$\;\;= 2 \text{ s}$

$\frac{dy}{dx} = -\frac{2}{5}x$

When $x = 10$,

$\frac{dy}{dx} = -\frac{2}{5} \times 10$

$\quad\;\; = -4$

Let α = angle of projection

$\tan\alpha = \frac{dy}{dx} = -4$

$\alpha = \tan^{-1}(-4) \doteqdot -76°$

The direction of the particle will be 76° below the horizontal.

5a To begin, $\dot{x} = 24\cos 30° = 12\sqrt{3}$

Integrating, $x = 12\sqrt{3}t + C_1$

When $t = 0$, $x = 0$

Thus, $C_1 = 0$

so $x = 12\sqrt{3}t$ (1)

To begin, $\dot{y} = 24\sin 30° - 10t = 12 - 10t$

Integrating, $y = 12t - 5t^2 + C_2$

When $t = 0$, $y = 0$

Thus, $C_2 = 0$

so $y = 12t - 5t^2$ (2)

5b From (1):

$$t = \frac{x}{12\sqrt{3}}$$

Substituting $t = \frac{x}{12\sqrt{3}}$ into (2):

$$y = 12\left(\frac{x}{12\sqrt{3}}\right) - 5\left(\frac{x}{12\sqrt{3}}\right)^2$$

$$= \frac{1}{\sqrt{3}}x - \frac{5}{432}x^2$$

5c When the ball hits the ground, $x = D$ and $y = -6$.

$$-6 = \frac{1}{\sqrt{3}}D - \frac{5}{432}D^2$$

$$\frac{5}{432}D^2 - \frac{1}{\sqrt{3}}D - 6 = 0$$

$$5D^2 - \frac{432}{\sqrt{3}}D - 2592 = 0$$

$$5D^2 - \frac{432\sqrt{3}}{3}D - 2592 = 0$$

$$5D^2 - 144\sqrt{3}D - 2592 = 0$$

5d Using the quadratic formula:

$$D = \frac{144\sqrt{3} \pm \sqrt{\left(144\sqrt{3}\right)^2 - 4 \times 5 \times (-2592)}}{2 \times 5}$$

$$D = \frac{144\sqrt{3} \pm \sqrt{62\,208 + 51\,840}}{10}$$

$$D = \frac{144\sqrt{3} \pm \sqrt{114\,048}}{10}$$

$D \doteqdot -8.8$ or 58.7

But $D > 0$ so $D \doteqdot 58.7$ m.

6 To begin, $\dot{x} = V\cos 27°$

Integrating, $x = Vt\cos 27° + C_1$

When $t = 0$, $x = 0$

Thus, $C_1 = 0$

so $x = Vt\cos 27°$ (1)

To begin, $\dot{y} = -V\sin 27° - 10t$

Integrating, $y = -Vt\sin 27° - 5t^2 + C_2$

When $t = 0$, $y = 0$

Thus, $C_2 = 0$

so $y = -Vt\sin 27° - 5t^2$ (2)

From (1):

$$t = \frac{x}{V\cos 27°} \quad (3)$$

Substituting (3) into (2):

$$y = -V\left(\frac{x}{V\cos 27°}\right)\sin 27° - 5\left(\frac{x}{V\cos 27°}\right)^2$$

$$y = -\tan 27°\, x - \frac{5x^2}{V^2\cos^2 27°}$$

$$y = -x\tan 27° - \frac{5x^2}{V^2\cos^2 27°} \qquad \text{as required}$$

6b i Since $y = -60$ when $x = 35$,

$$-60 = -35\tan 27° - \frac{5(35)^2}{V^2\cos^2 27°}$$

$$60 - 35\tan 27° = \frac{5(35)^2}{V^2\cos^2 27°}$$

$$V^2\cos^2 27° = \frac{5(35)^2}{60 - 35\tan 27°}$$

$$V^2 = \frac{5(35)^2}{\cos^2 27°\,(60 - 35\tan 27°)}$$

$$V = \sqrt{\frac{5(35)^2}{\cos^2 27°\,(60 - 35\tan 27°)}} \qquad \text{as } V > 0$$

$V \doteqdot 13.5$ m/s

The initial speed of the stone is about 13.5 m/s.

6b ii

$$\frac{dy}{dx} = -\tan 27° - \frac{10x}{V^2\cos^2 27°}$$

When the stone lands in the ocean, $x = 35$ and $V \doteqdot 13.5$.

$$\frac{dy}{dx} \doteqdot -\tan 27° - \frac{10 \times 35}{13.5^2\cos^2 27°}$$

$$= -2.9285\ldots$$

Let α = angle of projection

$$\tan\alpha = \frac{dy}{dx} \doteqdot -2.93$$

$$\alpha = \tan^{-1}(-2.93) \doteqdot -71°$$

The direction of the stone will be 71° below the horizontal.

7a

$$y = x\tan\theta - \frac{gx^2\sec^2\theta}{2V^2}$$

$y = 0$ when particle lands

$$0 = x\tan\theta - \frac{gx^2}{2V^2\cos^2\theta}$$

$$0 = \frac{x\sin\theta}{\cos\theta} - \frac{gx^2}{2V^2\cos^2\theta}$$

$$0 = \frac{x\sin\theta\,(2V^2\cos\theta)}{2V^2\cos^2\theta} - \frac{gx^2}{2V^2\cos^2\theta}$$

$$0 = \frac{2V^2x\sin\theta\cos\theta}{2V^2\cos^2\theta} - \frac{gx^2}{2V^2\cos^2\theta}$$

$gx^2 - 2V^2x\sin\theta\cos\theta = 0$ (for non-zero range, $\cos\theta \neq 0$ and $V \neq 0$)

$x\,(gx - 2V^2\sin\theta\cos\theta) = 0$

$x = 0$ or $gx = 2V^2\sin\theta\cos\theta$

$x = 0$ is not a feasible solution so

$$x = \frac{2V^2\sin\theta\cos\theta}{g}$$

$$x = \frac{V^2\sin 2\theta}{g}$$ (as $\sin 2\theta = 2\sin\theta\cos\theta$)

So the horizontal range is $\dfrac{V^2\sin 2\theta}{g}$ as required.

.

7b $V = 30$ m/s and $x = 75$ m. Assume $g = 10$ m/s^2.

Substituting into $x = \dfrac{V^2\sin 2\theta}{g}$ gives:

$$75 = \frac{30^2 \times \sin 2\theta}{10}$$

$$\sin 2\theta = \frac{75 \times 10}{30^2}$$

$$\sin 2\theta = \frac{5}{6}$$

$$2\theta \doteqdot 56.4°, 123.6°$$

$$\theta \doteqdot 28.2°, 61.8°$$

8a

$$y = x\tan\theta - \frac{gx^2\sec^2\theta}{2V^2}$$

$$y = x\tan\theta - \frac{g}{2V^2\cos^2\theta}x^2$$

$$\frac{dy}{dx} = \tan\theta - \frac{g}{V^2\ \cos^2\theta}x$$

Vertex when $\frac{dy}{dx} = 0$

$$0 = \tan\theta - \frac{g}{V^2\cos^2\theta}x$$

$$\frac{g}{V^2\cos^2\theta}x = \tan\theta$$

$$\frac{g}{V^2\cos^2\theta}x = \frac{\sin\theta}{\cos\theta}$$

$$x = \frac{\sin\theta}{\cos\theta} \times \frac{V^2\cos^2\theta}{g}$$

$$x = \frac{V^2\sin\theta\cos\theta}{g} \qquad \text{where } \cos\theta \neq 0$$

$$x = \frac{V^2 \times 2\sin\theta\cos\theta}{2g}$$

$$x = \frac{V^2\sin 2\theta}{2g} \qquad (\text{since } \sin 2\theta = 2\cos\theta\sin\theta)$$

$$\frac{d^2y}{dx^2} = -\frac{g}{V^2\ \cos^2\theta} < 0.$$

Hence $x = \frac{V^2\sin 2\theta}{2g}$ gives the maximum turning point of the parabola.

When $x = \frac{V^2\sin 2\theta}{2g}$,

$$y = \left(\frac{V^2 \sin 2\theta}{2g}\right)\tan\theta - \frac{g}{2V^2\ \cos^2\theta}\left(\frac{V^2 \sin 2\theta}{2g}\right)^2$$

$$= \left(\frac{V^2 \sin 2\theta}{2g}\right)\tan\theta - \frac{g}{2V^2\ \cos^2\theta}\left(\frac{V^2 2\sin\theta\cos\theta}{2g}\right)^2$$

$$= \left(\frac{V^2 \sin\theta\cos\theta}{g}\right)\times\frac{\sin\theta}{\cos\theta} - \frac{g}{2V^2\cos^2\theta}\times\left(\frac{V^4 4\sin^2\theta\cos^2\theta}{4g^2}\right)$$

$$= \frac{V^2\sin^2\theta}{g} - \frac{V^2\sin^2\theta}{2g}$$

$$= \frac{2V^2\sin^2\theta}{2g} - \frac{V^2\sin^2\theta}{2g}$$

$$= \frac{V^2\sin^2\theta}{2g}$$

Thus the coordinates of the vertex are $\left(\frac{V^2\sin 2\theta}{2g}, \frac{V^2\sin^2\theta}{2g}\right)$.

8b $V = 20$ m/s and $y = 15$ m

At the greatest height,

$$y = \frac{V^2\sin^2\theta}{2g}$$

$$15 = \frac{(20)^2\sin^2\theta}{2\times 10}$$

$$\sin^2\theta = \frac{300}{400}$$

$$\sin^2\theta = \frac{3}{4}$$

$$\sin\theta = \pm\frac{\sqrt{3}}{2}$$

For angle of projection above the horizontal, $\sin\theta > 0$.

So $\theta = 60°$.

9a $x = Vt\cos\alpha \qquad (1)$

$y = -\frac{1}{2}gt^2 + Vt\sin\alpha \qquad (2)$

Rearranging (1):

$t = \dfrac{x}{V\cos\alpha}$

Substituting for t in (2):

$y = -\dfrac{1}{2}g\left(\dfrac{x}{V\cos\alpha}\right)^2 + V\left(\dfrac{x}{V\cos\alpha}\right)\sin\alpha$

$= -\dfrac{gx^2}{2V^2\cos^2\alpha} + \left(\dfrac{x}{\cos\alpha}\right)\sin\alpha$

$= -\dfrac{gx^2}{2V^2\cos^2\alpha} + x\tan\alpha$

$= -\dfrac{gx^2\sec^2\alpha}{2V^2} + x\tan\alpha$

$= -\dfrac{gx^2(1+\tan^2\alpha)}{2V^2} + x\tan\alpha$

$2yV^2 = -gx^2(1+\tan^2\alpha) + 2xV^2\tan\alpha$

$2yV^2 + gx^2(1+\tan^2\alpha) - 2xV^2\tan\alpha = 0$

$2yV^2 + gx^2 + gx^2\tan^2\alpha - 2xV^2\tan\alpha = 0$

$gx^2\tan^2\alpha - 2xV^2\tan\alpha + (2yV^2 + gx^2) = 0$

9b $V = 200$ m/s

$x = 3000$ m

$y = 500$ m

$gx^2\tan^2\alpha - 2xV^2\tan\alpha + (2yV^2 + gx^2) = 0$

$(10)(3000)^2\tan^2\alpha - 2(3000)(200)^2\tan\alpha + 2(500)(200)^2 + 10(3000)^2 = 0$

$90\,000\,000\tan^2\alpha - 240\,000\,000\tan\alpha + 130\,000\,000 = 0$

$9\tan^2\alpha - 24\tan\alpha + 13 = 0$

Using the quadratic formula:

$\tan\alpha = \dfrac{24 \pm \sqrt{24^2 - 4\times 9\times 13}}{2\times 9}$

$$\tan\alpha = \frac{24 \pm \sqrt{108}}{18}$$

$$\tan\alpha = \frac{24 \pm 6\sqrt{3}}{18}$$

$$\tan\alpha = \frac{4 \pm \sqrt{3}}{3}$$

$$\tan\alpha = \frac{4 + \sqrt{3}}{3} \text{ or } \tan\alpha = \frac{4 - \sqrt{3}}{3}$$

$\alpha \doteqdot 62°22'$ or $37°5'$

10a To begin, $\dot{x} = 34\cos\theta$

Integrating, $x = 34t\cos\theta + C_1$

When $t = 0$, $x = 0$

Thus, $C_1 = 0$

so $x = 34t\cos\theta$

and $t = \dfrac{x}{34\cos\theta}$ (1)

To begin, $\dot{y} = 34\sin\theta - 10t$

Integrating, $y = 34t\sin\theta - 5t^2 + C_2$

When $t = 0$, $y = 0$

Thus, $C_2 = 0$

so $y = 34t\sin\theta - 5t^2$ (2)

Substituting (1) into (2):

$$y = 34\left(\frac{x}{34\cos\theta}\right)\sin\theta - 5\left(\frac{x}{34\cos\theta}\right)^2$$

$$y = \left(\frac{x}{\cos\theta}\right)\sin\theta - \frac{5x^2}{1156\cos^2\theta}$$

$$y = x\tan\theta - \frac{5x^2\sec^2\theta}{1156}$$

10b $x = 30$ m

$y = 11$ m

$$11 = 30\tan\theta - \frac{5(30)^2\sec^2\theta}{1156}$$

$$11 = 30\tan\theta - \frac{4500(1+\tan^2\theta)}{1156}$$

$$12\,716 = 34\,680\tan\theta - 4500\,(1+\tan^2\theta)$$

$$12\,716 = 34\,680\tan\theta - 4500 - 4500\tan^2\theta$$

$$4500\;\tan^2\theta - 34\,680\tan\theta + 17\,216 = 0$$

$$1125\;\tan^2\theta - 8670\tan\theta + 4304 = 0$$

Using the quadratic formula:

$$\tan\theta = \frac{8670 \pm \sqrt{(-8670)^2 - 4\times 1125\times 4304}}{2\times 1125}$$

$$\tan\theta = \frac{8670 \pm \sqrt{55\,800\,900}}{2250}$$

$$\tan\theta = \frac{8670 \pm 7470}{2250}$$

$$\tan\theta = \frac{1200}{2250} \text{ or } \frac{16\,140}{2250}$$

$$\tan\theta = \frac{8}{15} \text{ or } \frac{538}{75}$$

$$\theta = \tan^{-1}\frac{8}{15} \text{ or } \tan^{-1}\frac{538}{75}$$

$$\theta = \arctan\frac{8}{15} \text{ or } \arctan\frac{538}{75}$$

11a Initially, $x = y = 0$, and, $\dot{x} = V\cos 45°$ $\dot{y} = V\sin 45°$

To begin, $\ddot{x} = 0$ To begin, $\ddot{y} = -g$

Integrating, $\dot{x} = C_1$ Integrating, $\dot{y} = -gt + C_2$

When $t = 0$, $\dot{x} = V\cos 45° = \frac{V}{\sqrt{2}}$ When $t = 0$, $\dot{y} = V\sin 45° = \frac{V}{\sqrt{2}}$

$\frac{V}{\sqrt{2}} = C_1$ $\frac{V}{\sqrt{2}} = C_2$

so $\dot{x} = \frac{V}{\sqrt{2}}$ so $\dot{y} = -gt + \frac{V}{\sqrt{2}}$

so $\dot{x} = \frac{V}{\sqrt{2}}$ (2) so $\dot{y} = -gt + \frac{V}{\sqrt{2}}$

Integrating, $x = \frac{Vt}{\sqrt{2}} + C_3$ Integrating, $y = \frac{-gt^2}{2} + \frac{Vt}{\sqrt{2}} + C_4$

When $t = 0$, $x = 0$ When $t = 0$, $y = 0$

$0 = C_3$ $0 = C_4$

so $x = \frac{Vt}{\sqrt{2}} = \frac{Vt\sqrt{2}}{2}$ (1) so $y = \frac{-gt^2}{2} + \frac{Vt\sqrt{2}}{2}$ (2)

11b From (1):

$$t = \frac{\sqrt{2}x}{V}$$

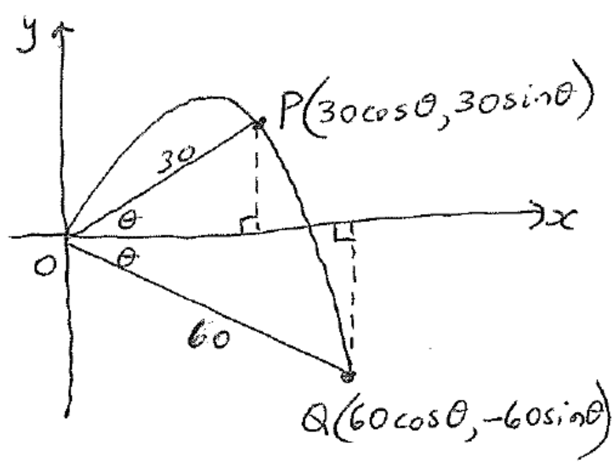
Substituting for t in (2):

$$y = \frac{-g}{2} \times \left(\frac{\sqrt{2}x}{V}\right)^2 + \frac{V\sqrt{2}}{2} \times \frac{\sqrt{2}x}{V}$$

$$y = \frac{-gx^2}{V^2} + x$$

$$y = x - \frac{gx^2}{V^2} \qquad (3)$$

11c



The coordinates of P and Q satisfy equation (3),

so $30\sin\theta = 30\cos\theta - \dfrac{g}{V^2} \times 900\cos^2\theta$

and $-60\sin\theta = 60\cos\theta - \dfrac{g}{V^2} \times 3600\cos^2\theta$

so $\sin\theta = \cos\theta - \dfrac{30g\cos^2\theta}{V^2}$

and $-\sin\theta = \cos\theta - \dfrac{60g\cos^2\theta}{V^2}$

Dividing through by $\cos\theta$,

$\tan\theta = 1 - \dfrac{30g\cos\theta}{V^2}$ and $-\tan\theta = 1 - \dfrac{60g\cos\theta}{V^2}$

So $\tan\theta = 1 - \dfrac{30g\cos\theta}{V^2}$ (4)

and $\tan\theta = \dfrac{60g\cos\theta}{V^2} - 1$

Hence, $1 - \dfrac{30g\cos\theta}{V^2} = \dfrac{60g\cos\theta}{V^2} - 1$

$V^2 - 30g\cos\theta = 60g\cos\theta - V^2$

$2V^2 = 90g\cos\theta$

$V^2 = 45g\cos\theta$

Substituting for V^2 into (4):

$\tan\theta = 1 - \dfrac{30g\cos\theta}{45g\cos\theta}$

$= 1 - \dfrac{2}{3}$

$= \dfrac{1}{3}$

$\theta = \tan^{-1}\dfrac{1}{3}$

12a $x = Vt\cos\alpha$ (1)

$y = Vt\sin\alpha - \frac{1}{2}gt^2$ (2)

Rearranging (1):

$$t = \frac{x}{V\cos\alpha}$$

Substituting for t in (2):

$$y = V\left(\frac{x}{V\cos\alpha}\right)\sin\alpha - \frac{1}{2}g\left(\frac{x}{V\cos\alpha}\right)^2$$

$$= \left(\frac{x}{\cos\alpha}\right)\sin\alpha - \frac{gx^2}{2V^2\cos^2\alpha}$$

$$= x\tan\alpha - \frac{gx^2}{2V^2\cos^2\alpha} \quad (3)$$

$y = 0$ when particle lands, so

$$0 = x\tan\alpha - \frac{gx^2}{2V^2\cos^2\alpha}$$

$$0 = \frac{x\sin\alpha}{\cos\theta} - \frac{gx^2}{2V^2\cos^2\alpha}$$

$$0 = \frac{x\sin\alpha\,(2V^2\cos\alpha)}{2V^2\cos^2\alpha} - \frac{gx^2}{2V^2\cos^2\alpha}$$

$$0 = \frac{2V^2x\sin\alpha\cos\alpha}{2V^2\cos^2\alpha} - \frac{gx^2}{2V^2\cos^2\alpha}$$

$gx^2 - 2V^2x\sin\alpha\cos\alpha = 0$ (for non-zero range, $\cos\theta \neq 0$ and $V \neq 0$)

$x\,(gx - 2V^2\sin\alpha\cos\alpha) = 0$

$x = 0$ or $gx = 2V^2\sin\alpha\cos\alpha$

$x = 0$ is not a feasible solution so

$$x = \frac{2V^2\sin\alpha\cos\alpha}{g}$$

$$x = \frac{V^2\sin 2\alpha}{g} \quad (\text{as } \sin 2\alpha = 2\sin\alpha\cos\alpha)$$

So the horizontal range is $\dfrac{V^2\sin 2\alpha}{g}$ as required.

12b From (3) above,

$$y = x\tan\alpha - \frac{gx^2}{2V^2\cos^2\alpha}$$

$$y = x\tan\alpha - \frac{gx^2}{2V^2\cos^2\alpha} \times \frac{x\tan\alpha}{x\tan\alpha}$$

$$y = x\tan\alpha\left(1 - \frac{gx^2}{2V^2\cos^2\alpha} \times \frac{1}{x\tan\alpha}\right)$$

$$y = x\tan\alpha\left(1 - \frac{gx^2}{2V^2\cos^2\alpha} \times \frac{\cos\alpha}{x\sin\alpha}\right)$$

$$y = x\tan\alpha\left(1 - \frac{gx}{2V^2\cos\alpha\sin\alpha}\right)$$

$$y = x\tan\alpha\left(1 - \frac{gx}{V^2\sin 2\alpha}\right)$$

$$y = x\tan\alpha\left(1 - \frac{x}{R}\right), \text{since } R = \frac{V^2\sin 2\alpha}{g}$$

$$y = x\left(1 - \frac{x}{R}\right)\tan\alpha$$

12c $\alpha = 45°$, so the path of the equation is:

$$y = x\left(1 - \frac{x}{R}\right)$$

$$\text{or } y = x - \frac{x^2}{R}$$

12c i Let $y = 4$, hence:

$$4 = x - \frac{x^2}{R}$$

$$x^2 - Rx + 4R = 0$$

Since $y = 4$ corresponds to $x = x_1$ and $x = x_2$,

x_1 and x_2 must be the roots of the equation $x^2 - Rx + 4R = 0$.

12c ii $(x_2 - x_1)^2 = (x_2 + x_1)^2 - 4x_2x_1$

Hence,

$6^2 = (x_2 + x_1)^2 - 4(x_2x_1)$

$36 = R^2 - 4(4R)$

$R^2 - 16R - 36 = 0$

$(R - 18)(R + 2) = 0$

$R = 18$, since $R > 0$

So the horizontal range, R, is 18 metres.

Solutions to Chapter review

1a Initially, $x = y = 0$, and, $\dot{x} = 60\cos 40°$ $\dot{y} = 60\sin 40°$

To begin,	$\ddot{x} = 0$ (1)	To begin,	$\ddot{y} = -10$	(3)
Integrating,	$\dot{x} = C_1$	Integrating,	$\dot{y} = -10t + C_2$	
When $t = 0$,	$\dot{x} = 60\cos 40°$	When $t = 0$,	$\dot{y} = 60\sin 40°$	
	$60\cos 40° = C_1$		$60\sin 40° = C_2$	
so	$\dot{x} = 60\cos 40°$ (2)	so	$\dot{y} = -10t + 60\sin 40°$	(4)

1b

To begin,	$\dot{x} = 60\cos 40°$	To begin,	$\dot{y} = -10t + 60\sin 40°$	
Integrating,	$x = 60t\cos 40° + C_3$	Integrating,	$y = -5t^2 + 60t\sin 40° + C_4$	
When $t = 0$,	$x = 0$	When $t = 0$,	$y = 0$	
	$0 = C_3$		$0 = C_4$	
so	$x = 60t\cos 40°$ (5)	so	$y = -5t^2 + 60t\sin 40°$	(6)

1c i The particle returns to the ground when $y = 0$,

Substituting into (6):

$0 = -5t^2 + 60t\sin 40°$

$0 = -5t(t - 12\sin 40°)$

$t = 0$ or $t = 12\sin 40° \doteqdot 7.7$ seconds

Particle returns to the ground at $t > 0$ seconds, so at $t \doteqdot 7.7$ seconds.

1c ii When $t = 12\sin 40°$, (5) becomes:

$x = 720\sin 40°\cos 40°$

$\doteqdot 354.5$ m

The horizontal distance travelled by the particle is about 354.5 m.

1c iii Particle reaches greatest height when $\dot{y} = 0$.

Substituting into (4):

$0 = -10t + 60\sin 40°$

$10t = 60\sin 40°$

$t = 6\sin 40°$

Substituting for t in (6):

$y = -5(6\sin 40°)^2 + 60(6\sin 40°)(\sin 40°)$

$= -180\sin^2 40 + 360\sin^2 40°$

$= 180\sin^2 40°$

$\doteqdot 74.4$ m

The greatest height reached by the particle above the ground is about 74.4 m.

2a Initially, $x = y = 0$, and, $\dot{x} = 30\cos 60°$ $\dot{y} = 30\sin 60°$

$\dot{x} = 15$ $\dot{y} = 15\sqrt{3}$

To begin,	$\ddot{x} = 0$	(1)	To begin,	$\ddot{y} = -10$	(4)
Integrating,	$\dot{x} = C_1$		Integrating,	$\dot{y} = -10t + C_3$	
When $t = 0$,	$\dot{x} = 15$		When $t = 0$,	$\dot{y} = 15\sqrt{3}$	
	$15 = C_1$			$15\sqrt{3} = C_3$	
so	$\dot{x} = 15$	(2)	so	$\dot{y} = -10t + 15\sqrt{3}$	(5)
Integrating,	$x = 15t + C_2$		Integrating,	$y = -5t^2 + 15t\sqrt{3} + C_4$	
When $t = 0$,	$x = 0$		When $t = 0$,	$y = 0$	
	$0 = C_2$			$0 = C_4$	
so	$x = 15t$	(3)	so	$y = -5t^2 + 15t\sqrt{3}$	(6)

Particle reaches greatest height when $\dot{y} = 0$.

Substituting into (5):

$0 = -10t + 15\sqrt{3}$

$10t = 15\sqrt{3}$

$t = \dfrac{3\sqrt{3}}{2}$

Substituting for t in (6):

$y = -5\left(\dfrac{3\sqrt{3}}{2}\right)^2 + 15 \times \dfrac{3\sqrt{3}}{2} \times \sqrt{3}$

$y = -5 \times \dfrac{27}{4} + 15 \times \dfrac{9}{2}$

$= -33.75 + 67.5$

$= 33.75$ m

The greatest height reached by the rock above the point of projection is 33.75 m.

2b The rock will hit the ground when $y = -40$.

Substituting into (6):

$-40 = -5t^2 + 15t\sqrt{3}$

$5t^2 - 15\sqrt{3}t - 40 = 0$

Using the quadratic formula:

$$t = \frac{15\sqrt{3} \pm \sqrt{\left(15\sqrt{3}\right)^2 - 4 \times 5 \times (-40)}}{2 \times 5}$$

$t = \dfrac{15\sqrt{3} \pm \sqrt{1475}}{10}$

$t = \dfrac{15\sqrt{3} \pm 5\sqrt{59}}{10}$

$t = \dfrac{3\sqrt{3} - \sqrt{59}}{2}$ or $t = \dfrac{3\sqrt{3} + \sqrt{59}}{2}$

$t \doteqdot -1.24$ or $t \doteqdot 6.44$

Since $t > 0$, the stone hits the ground when $t \doteqdot 6.44$ seconds.

When $t = \dfrac{3\sqrt{3} + \sqrt{59}}{2}$,

$x = 15t$ becomes

$$x = \frac{15(3\sqrt{3} + \sqrt{59})}{2}$$

$$\doteqdot 96.58 \text{ m}$$

The rock lands about 96.58 m from the base of the cliff.

2c The particle's speed can be calculated by applying Pythagoras' theorem to the particle's velocity vector.

That is
$$\underset{\sim}{v} = \dot{x}\underset{\sim}{i} + \dot{y}\underset{\sim}{j}$$
$$v^2 = \dot{x}^2 + \dot{y}^2$$
$$v = \sqrt{\dot{x}^2 + \dot{y}^2}$$

From (2) and (5):

$$v = \sqrt{15^2 + \left(-10t + 15\sqrt{3}\right)^2}$$

When $t = \dfrac{3\sqrt{3} + \sqrt{59}}{2}$,

$$v = \sqrt{15^2 + \left(-10\left(\frac{3\sqrt{3} + \sqrt{59}}{2}\right) + 15\sqrt{3}\right)^2}$$

$$= \sqrt{15^2 + \left(-15\sqrt{3} - 5\sqrt{59} + 15\sqrt{3}\right)^2}$$

$$= \sqrt{225 + 1475}$$

$$= \sqrt{1700}$$

$$\doteqdot 41.23 \text{ m/s}$$

The rock hits the ground at a speed of about 41.23 m/s.

3a Initially, $x = y = 0$

And $\dot{x} = 25\cos\left(\tan^{-1}\frac{4}{3}\right)$ $\quad \dot{y} = 25\sin\left(\tan^{-1}\frac{4}{3}\right)$

$\dot{x} = 15$ $\quad \dot{y} = 20$

To begin, $\ddot{x} = 0$ (1) To begin, $\ddot{y} = -10$ (3)

Integrating, $\dot{x} = C_1$ Integrating, $\dot{y} = -10t + C_2$

When $t = 0$, $\dot{x} = 15$

$15 = C_1$

so $\dot{x} = 15$ (2)

When $t = 0$, $\dot{y} = 20$

$20 = C_2$

so $\dot{y} = -10t + 20$ (4)

3b To begin, $\dot{x} = 15$

Integrating, $x = 15t + C_3$

When $t = 0$, $x = 0$

$0 = C_3$

so $x = 15t$ (5)

To begin, $\dot{y} = -10t + 20$

Integrating, $y = -5t^2 + 20t + C_4$

When $t = 0$, $y = 0$

$0 = C_4$

so $y = -5t^2 + 20t$ (6)

3c The particle's distance can be calculated by applying Pythagoras' theorem.

$r^2 = x^2 + y^2$

$r = \sqrt{x^2 + y^2}$

From (5) and (6):

$r = \sqrt{(15t)^2 + (-5t^2 + 20t)^2}$

When $t = 1$,

$r = \sqrt{15^2 + (-5 + 20)^2}$

$= \sqrt{225 + 225}$

$= \sqrt{450}$

$= 15\sqrt{2}$

The distance of the particle from point of projection after one second is $15\sqrt{2}$ m.

3d $\underset{\sim}{v} = \dot{x}\underset{\sim}{i} + \dot{y}\underset{\sim}{j}$

From (2) and (4):

$\underset{\sim}{v} = 15\underset{\sim}{i} + (-10t + 20)\underset{\sim}{j}$

When $t = 1$,

$\underset{\sim}{v} = 15\underset{\sim}{i} + 10\underset{\sim}{j}$

$v^2 = 15^2 + 10^2$

$v = \sqrt{15^2 + 10^2}$

$= \sqrt{325}$

$= 5\sqrt{13}$ m/s

$\tan\theta = \dfrac{10}{15} = \dfrac{2}{3}$

$\theta = \tan^{-1}\dfrac{2}{3}$

3e Particle reaches greatest height when $\dot{y} = 0$.

Substituting into (4):

$0 = -10t + 20$

$10t = 20$

$t = 2$

Substituting $t = 2$ into (6):

$y = -20 + 40$

$= 20$ m

So the greatest height, H, is 20 m.

Particle reaches the ground when $y = 0$.

Substituting into (6):

$0 = -5t^2 + 20t$

$5t^2 - 20t = 0$

$5t(t - 4) = 0$

$t = 0$ or $t = 4$

The particle reaches its horizontal range, R, at $t = 4$.

From (5):

$x = 15 \times 4 = 60$, and hence $R = 60$

$\dfrac{R}{H} = \dfrac{60}{20} = 3$

So $R = 3H$.

4a Initially, $x = y = 0$, and, $\dot{x} = 10\cos 45°$ $\dot{y} = 10\sin 45°$

$\dot{x} = 5\sqrt{2}$ $\dot{y} = 5\sqrt{2}$

To begin,	$\ddot{x} = 0$	(1)	To begin,	$\ddot{y} = -10$	(4)
Integrating,	$\dot{x} = C_1$		Integrating,	$\dot{y} = -10t + C_3$	
When $t = 0$,	$\dot{x} = 5\sqrt{2}$		When $t = 0$,	$\dot{y} = 5\sqrt{2}$	
	$5\sqrt{2} = C_1$			$5\sqrt{2} = C_3$	
so	$\dot{x} = 5\sqrt{2}$	(2)	so	$\dot{y} = -10t + 5\sqrt{2}$	(5)
Integrating,	$x = 5t\sqrt{2} + C_2$		Integrating,	$y = -5t^2 + 5t\sqrt{2} + C_4$	
When $t = 0$,	$x = 0$		When $t = 0$,	$y = 0$	
	$0 = C_2$			$0 = C_4$	
so	$x = 5t\sqrt{2}$	(3)	so	$y = -5t^2 + 5t\sqrt{2}$	(6)

From (3):

$$t = \frac{x}{5\sqrt{2}}$$

Substituting for t in (6):

$$y = -5\left(\frac{x}{5\sqrt{2}}\right)^2 + \frac{5x\sqrt{2}}{5\sqrt{2}}$$

$$y = \frac{-5x^2}{50} + x$$

$$y = x - \frac{1}{10}x^2 \qquad (7)$$

4b Particle reaches the ground when $y = 0$.

Substituting into (7):

$$0 = x - \frac{x^2}{10}$$

$$x^2 - 10x = 0$$

$$x(x - 10) = 0$$

$x = 0$ or $x = 10$

For horizontal range, $x > 0$, so $x = 10$ m.

By symmetry, the greatest height is reached at half the horizontal range

so, $x = 5$

From (7):

$$y = 5 - \frac{25}{10}$$

$$= 2.5 \text{ m}$$

Hence the horizontal range is 10 m and the greatest height is 2.5 m.

4c i From (7):

$$y = x - \frac{x^2}{10}$$

When $x = 8$,

$$y = 8 - \frac{64}{10}$$

$$= 1.6 \text{ m}$$

The stone hits the wall at a height of 1.6 m.

4c ii From (7):

$$\frac{dy}{dx} = 1 - \frac{x}{5}$$

When $x = 8$,

$$\frac{dy}{dx} = -\frac{3}{5}$$

$\tan\theta = -\frac{3}{5}$

$\theta = \tan^{-1}\left(-\frac{3}{5}\right)$

The direction of the object when it hits the wall is $\tan^{-1}\frac{3}{5}$ below the horizontal.

4d i From (7):

$y = x - \frac{x^2}{10}$

When $y = 2.1$,

$2.1 = x - \frac{x^2}{10}$

$\frac{x^2}{10} - x + 2.1 = 0$

$x^2 - 10x + 21 = 0$

$(x-3)(x-7) = 0$

So $x = 3$ or $x = 7$

It must be the case that $x = 3$ m since the object collided with the ceiling when travelling up.

4d ii From (7):

$\frac{dy}{dx} = 1 - \frac{x}{5}$

When $x = 3$,

$\frac{dy}{dx} = \frac{2}{5}$

$\tan\theta = \frac{2}{5}$

$\theta = \tan^{-1}\frac{2}{5}$

The angle at which the stone hits the ceiling is $\tan^{-1}\frac{2}{5}$ above the horizontal.

5a Initially, $x = y = 0$, and, $\dot{x} = 24$ $\dot{y} = 18$

To begin,	$\ddot{x} = 0$	(1)	To begin,	$\ddot{y} = -10$	(3)
Integrating,	$\dot{x} = C_1$		Integrating,	$\dot{y} = -10t + C_2$	
When $t = 0$,	$\dot{x} = 24$		When $t = 0$,	$\dot{y} = 18$	
	$24 = C_1$			$18 = C_2$	
so	$\dot{x} = 24$	(2)	so	$\dot{y} = -10t + 18$	(4)

To begin, $\underset{\sim}{v} = \dot{x}\underset{\sim}{i} + \dot{y}\underset{\sim}{j}$

From (2) and (5): $\underset{\sim}{v} = 24\underset{\sim}{i} + (-10t + 18)\underset{\sim}{j} = 24\underset{\sim}{i} + (18 - 10t)\underset{\sim}{j}$

5b

To begin,	$\dot{x} = 24$		To begin,	$\dot{y} = -10t + 18$	
Integrating,	$x = 24t + C_3$		Integrating,	$y = -5t^2 + 18t + C_4$	
When $t = 0$,	$x = 0$		When $t = 0$,	$y = 0$	
	$0 = C_3$			$0 = C_4$	
so	$x = 24t$	(5)	so	$y = -5t^2 + 18t$	(6)

To begin, $\underset{\sim}{r} = x\underset{\sim}{i} + y\underset{\sim}{j}$

From (5) and (6): $\underset{\sim}{r} = (24t)\underset{\sim}{i} + (-5t^2 + 18t)\underset{\sim}{j}$

$\underset{\sim}{r} = (24t)\underset{\sim}{i} + (18t - 5t^2)\underset{\sim}{j}$

5c i When $t = 0$,

$\underset{\sim}{v} = 24\underset{\sim}{i} + 18\underset{\sim}{j}$

$v^2 = 24^2 + 18^2$

$v = \sqrt{24^2 + 18^2}$

$= \sqrt{900}$

$= 30\text{ m/s}$

The initial speed of the particle is 30 m/s.

5c ii When $t = 4$,

$\underline{r} = 96\underline{i} + (72 - 80)\underline{j}$

$\underline{r} = 96\underline{i} - 8\underline{j}$

5c iii Particle reaches greatest height when $\dot{y} = 0$.

From (4):

$0 = -10t + 18$

$10t = 18$

$t = 1.8$

When $t = 1.8$,

$\underline{r} = 43.2\underline{i} + (32.4 - 16.2)\underline{j}$

$\underline{r} = 43.2\underline{i} + 16.2\underline{j}$

6a Initially, $x = y = 0$, and, $\dot{x} = V\cos\alpha$ $\dot{y} = V\sin\alpha$

To begin, $\ddot{x} = 0$ (1)

Integrating, $\dot{x} = C_1$

When $t = 0$, $\dot{x} = V\cos\alpha$

$V\cos\alpha = C_1$

so $\dot{x} = V\cos\alpha$ (2a)

Integrating, $x = Vt\cos\alpha + C_2$

When $t = 0$, $x = 0$

$0 = C_2$

so $x = Vt\cos\alpha$ (3a)

When $t = 2$, $24\sqrt{5} = 2V\cos\alpha$

so $V\cos\alpha = 12\sqrt{5}$

From (2a): $\dot{x} = 12\sqrt{5}$ (2b)

From (3a): $x = 12t\sqrt{5}$ (3b)

To begin, $\ddot{y} = -10$ (4)

Integrating, $\dot{y} = -10t + C_3$

When $t = 0$, $\dot{y} = V\sin\alpha$

$V\sin\alpha = C_3$

so $\dot{y} = -10t + V\sin\alpha$ (5a)

Integrating, $y = -5t^2 + Vt\sin\alpha + C_4$

When $t = 0$, $y = 0$

$0 = C_4$

so $y = -5t^2 + Vt\sin\alpha$ (6a)

When $t = 2$, $28 = -20 + 2V\sin\alpha$

so $V\sin\alpha = 24$

From (5a): $\dot{y} = -10t + 24$ (5b)

From (6a): $y = -5t^2 + 24t$ (6b)

To begin, $\underset{\sim}{v} = \dot{x}\underset{\sim}{i} + \dot{y}\underset{\sim}{j}$

From (2b) and (5b):

$\underset{\sim}{v} = 12\sqrt{5}\underset{\sim}{i} + (-10t + 24)\underset{\sim}{j}$

When $t = 0$,

$\underset{\sim}{v} = 12\sqrt{5}\underset{\sim}{i} + 24\underset{\sim}{j}$

6b To begin, $\underset{\sim}{r} = x\underset{\sim}{i} + y\underset{\sim}{j}$

From (3b) and (6b):

$\underset{\sim}{r} = 12t\sqrt{5}\underset{\sim}{i} + (-5t^2 + 24t)\underset{\sim}{j}$

When $t = 3$,

$\underset{\sim}{r} = 36\sqrt{5}\underset{\sim}{i} + (-45 + 72)\underset{\sim}{j}$

$\underset{\sim}{r} = 36\sqrt{5}\underset{\sim}{i} + 27\underset{\sim}{j}$

6c The ball will be rising if $\dot{y} > 0$ or falling if $\dot{y} < 0$.

From (5b):

$\dot{y} = -10t + 24$

When $t = 3$,

$\dot{y} = -30 + 24$

$\dot{y} = -6$

Since $\dot{y} = -6 < 0$, the ball must be falling.

7a Initially, $x = y = 0$, and, $\dot{x} = V\cos\theta$ $\dot{y} = V\sin\alpha$

To begin,	$\ddot{x} = 0$	(1)	To begin,	$\ddot{y} = -10$	(4)
Integrating,	$\dot{x} = C_1$		Integrating,	$\dot{y} = -10t + C_3$	
When $t = 0$,	$\dot{x} = V\cos\theta$		When $t = 0$,	$\dot{y} = V\sin\theta$	
	$V\cos\theta = C_1$			$V\sin\theta = C_3$	
so	$\dot{x} = V\cos\theta$	(2a)	so	$\dot{y} = -10t + V\sin\theta$	(5a)

Integrating,	$x = Vt\cos\theta + C_2$		Integrating,	$y = -5t^2 + Vt\sin\theta + C_4$	
When $t = 0$,	$x = 0$		When $t = 0$,	$y = 0$	
	$0 = C_2$			$0 = C_4$	
so	$x = Vt\cos\theta$	(3a)	so	$y = -5t^2 + Vt\sin\theta$	(6a)
When $t = 3$,	$108 = 3V\cos\theta$		When $t = 3$,	$0 = -45 + 3V\sin\theta$	
so	$V\cos\theta = 36$		so	$V\sin\theta = 15$	
From (2a):	$\dot{x} = 36$	(2b)	From (5a):	$\dot{y} = -10t + 15$	(5b)
From (3a):	$x = 36t$	(3b)	From (6a):	$y = -5t^2 + 15t$	(6b)

To begin, $\underset{\sim}{v} = \dot{x}\underset{\sim}{i} + \dot{y}\underset{\sim}{j}$

From (2b) and (5b):

$\underset{\sim}{v} = 36\underset{\sim}{i} + (-10t + 15)\underset{\sim}{j}$

When $t = 0$,

$\underset{\sim}{v} = 36\underset{\sim}{i} + 15\underset{\sim}{j}$

$V^2 = 36^2 + 15^2$

$V = \sqrt{36^2 + 15^2}$

$\quad = \sqrt{1521}$

$\quad = 39 \text{ m/s}$

$\tan\theta = \frac{15}{36} = \frac{5}{12}$

$\theta = \tan^{-1}\frac{5}{12} = \text{artan}\,\frac{5}{12}$

7b Particle reaches greatest height when $\dot{y} = 0$.

Substituting into (5b):

$0 = -10t + 15$

$10t = 15$

$t = 1.5$

Substituting for t in (6b):

$y = -11.25 + 22.5$

$= 11.25 \text{ m}$

The greatest height reached by the particle is 11.25 m.

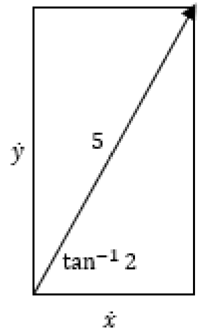
8a $V = 5 \text{ m/s}$ and $\tan\alpha = 2$ or $\alpha = \tan^{-1} 2$

For a right-angled triangle with angle α, opposite side is 2 units, adjacent side is 1 unit and hypotenuse is $\sqrt{2^2 + 1^2} = \sqrt{5}$. Hence $\cos\alpha = \frac{1}{\sqrt{5}}$ and $\sin\alpha = \frac{2}{\sqrt{5}}$.

$\dot{x} = 5\cos\alpha$

$= 5 \times \frac{1}{\sqrt{5}}$

$= \sqrt{5} \text{ m/s}$



$\dot{y} = 5\sin\alpha$

$= 5 \times \frac{2}{\sqrt{5}}$

$= 2\sqrt{5} \text{ m/s}$

So $\underset{\sim}{v} = \sqrt{5}\underset{\sim}{i} + 2\sqrt{5}\underset{\sim}{j}$ m/s

Hence, the initial values are $\dot{x} = \sqrt{5}$ and $\dot{y} = 2\sqrt{5}$.

8b Initially, $x = y = 0$, and, $\dot{x} = \sqrt{5}$ $\quad \dot{y} = 2\sqrt{5}$

To begin,	$\ddot{x} = 0$	(1)	To begin,	$\ddot{y} = -10$	(4)
Integrating,	$\dot{x} = C_1$		Integrating,	$\dot{y} = -10t + C_3$	
When $t = 0$,	$\dot{x} = \sqrt{5}$		When $t = 0$,	$\dot{y} = 2\sqrt{5}$	
	$\sqrt{5} = C_1$			$2\sqrt{5} = C_3$	
so	$\dot{x} = \sqrt{5}$	(2)	so	$\dot{y} = -10t + 2\sqrt{5}$	(5)
Integrating,	$x = \sqrt{5}t + C_2$		Integrating,	$y = -5t^2 + 2t\sqrt{5} + C_4$	
When $t = 0$,	$x = 0$		When $t = 0$,	$y = 0$	
	$0 = C_2$			$0 = C_4$	
so	$x = t\sqrt{5}$	(3)	so	$y = -5t^2 + 2t\sqrt{5}$	(6)

8c From (6):

$y = -5t^2 + 2t\sqrt{5}$

When $t = 1$,

$y = -5 + 2\sqrt{5}$

$\quad \doteqdot -0.53$

At one second, the apple would be below the height at which it was thrown. However, since the apple is caught at the same height that it was thrown, it must have been caught before one second. Therefore, it spent less than one second in the air.

8d The apple reaches greatest height when $\dot{y} = 0$.

Substituting into (5):

$0 = -10t + 2\sqrt{5}$

$10t = 2\sqrt{5}$

$t = \dfrac{\sqrt{5}}{5}$

Substituting for t in (6):

$y = -5\left(\dfrac{\sqrt{5}}{5}\right)^2 + 2\sqrt{5} \times \dfrac{\sqrt{5}}{5}$

$\quad = -1 + 2$

$\quad = 1$

The greatest height of the apple above the point of release is 1 metre.

8e The apple lands when $y = 0$.

Substituting into (6):

$0 = -5t^2 + 2t\sqrt{5}$

$5t^2 - 2\sqrt{5}t = 0$

$t(5t - 2\sqrt{5}) = 0$

$t = 0 \text{ or } t = \dfrac{2\sqrt{5}}{5}\text{ s}$

For the flight of the apple,

$$t = \frac{2\sqrt{5}}{5} = 2\frac{\sqrt{5}}{5}\text{ s}$$

Substituting for t in (3):

$$x = \sqrt{5} \times \frac{2\sqrt{5}}{5}$$

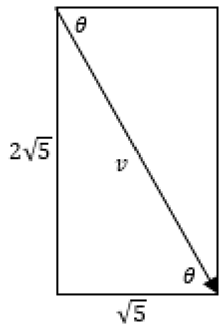
$$x = 2\text{ m}$$

Hence the horizontal distance travelled by the apple is 2 m.

8f Adam catches the apple at $t = \frac{2\sqrt{5}}{5}$ seconds.

From (2): $\dot{x} = \sqrt{5}$

Substituting for t in (5):



$$\dot{y} = -10 \times \frac{2\sqrt{5}}{5} + 2\sqrt{5}$$

$$\dot{y} = -4\sqrt{5} + 2\sqrt{5}$$

$$\dot{y} = -2\sqrt{5}$$

$$v^2 = \left(\sqrt{5}\right)^2 + \left(-2\sqrt{5}\right)^2$$

$$v = \sqrt{5 + 20}$$

$$v = 5\text{ m/s}$$

So, the final speed is equal to the initial speed.

$$\tan\theta = \frac{-2\sqrt{5}}{\sqrt{5}}$$

$$\theta = \tan^{-1}\left(\frac{-2\sqrt{5}}{\sqrt{5}}\right)$$

$$\theta = \tan^{-1}(-2)$$

$$\theta = -\tan^{-1} 2$$

The final and initial angles of inclination are equal in magnitude, though it is an angle of inclination initially, and an angle of depression finally.

8g From (3):

$$t = \frac{x}{\sqrt{5}}$$

Substituting for t in (6):

$$y = -5\left(\frac{x}{\sqrt{5}}\right)^2 + 2 \times \frac{x}{\sqrt{5}} \times \sqrt{5}$$
$$= -x^2 + 2x$$
$$= 2x - x^2$$

9 Let V be the initial speed of the projectile

Initially, $x = y = 0$, and, $\dot{x} = V\cos 26°$ $\dot{y} = V\sin 26°$

To begin, $\ddot{x} = 0$ (1)
Integrating, $\dot{x} = C_1$
When $t = 0$, $\dot{x} = V\cos 26°$
$V\cos 26° = C_1$
so $\dot{x} = V\cos 26°$ (2)
To begin, $\dot{x} = V\cos 26°$
Integrating, $x = Vt\cos 26° + C_2$
When $t = 0$, $x = 0$
$0 = C_2$
so $x = Vt\cos 26°$ (3)

To begin, $\ddot{y} = -10$ (4)
Integrating, $\dot{y} = -10t + C_3$
When $t = 0$, $\dot{y} = V\sin 26°$
$V\sin 26° = C_3$
so $\dot{y} = -10t + V\sin 26°$ (5)
To begin, $\dot{y} = -10t + V\sin 26°$
Integrating, $y = -5t^2 + Vt\sin 26° + C_4$
When $t = 0$, $y = 0$
$0 = C_4$
so $y = -5t^2 + Vt\sin 26°$ (6)

From (3):

$$t = \frac{x}{V\cos 26°}$$

Substituting for t in (6):

$$y = -5\left(\frac{x}{V\cos 26°}\right)^2 + \left(\frac{Vx}{V\cos 26°}\right)\sin 26°$$
$$= \frac{-5x^2}{V^2\cos^2 26°} + x\tan 26°$$

If the ball just clears the tree, it will reach a point where $x = 30$ and $y = 12$.

$$12 = \frac{-5(30)^2}{V^2\cos^2 26°} + 30\tan 26°$$

$$\frac{-4500}{V^2\cos^2 26°} = 12 - 30\tan 26°$$

$$-\frac{4500}{V^2} = \cos^2 26° \times (12 - 30\tan 26°)$$

$$V^2 = -\frac{4500}{\cos^2 26° \times (12 - 30\tan 26°)}$$

$$V = \sqrt{-\frac{4500}{\cos^2 26° \times (12 - 30\tan 26°)}}$$

$V \doteqdot 46$ m/s

The initial speed of the ball is about 46 m/s.

10a Initially, $x = y = 0$, and, $\dot{x} = V\cos\theta$ $\dot{y} = V\sin\theta$

Assuming $x = Vt\cos\theta$ (1)

$y = -\frac{1}{2}gt^2 + Vt\sin\theta$ (2)

From (1):

$$t = \frac{x}{V\cos\theta}$$

Substituting for t in (2):

$$y = -\frac{1}{2}g\left(\frac{x}{V\cos\theta}\right)^2 + \frac{Vx\sin\theta}{V\cos\theta}$$

$$= -\frac{gx^2}{2V^2\cos^2\theta} + x\tan\theta$$

$$= -\frac{gx^2\sec^2\theta}{2V^2} + x\tan\theta$$

$$= x\tan\theta - \frac{gx^2\sec^2\theta}{2V^2} \qquad (3)$$

10b If initially $V = 25$, then the particle passes through the point $x = 30$ and $y = 6$.

Substituting $x = 30$, $y = 6$ and $g = 10$ in (3):

$$6 = 30\tan\theta - \frac{10 \times 30^2 \sec^2\theta}{2 \times 25^2}$$

$$6 = 30\tan\theta - \frac{9000\sec^2\theta}{1250}$$

$$6 = 30\tan\theta - \frac{36\sec^2\theta}{5}$$

$$30 = 150\tan\theta - 36\sec^2\theta$$

$$36\sec^2\theta - 150\tan\theta + 30 = 0$$

$$6\sec^2\theta - 25\tan\theta + 5 = 0$$

But $\sec^2\theta = 1 + \tan^2\theta$

$$6\,(1 + \tan^2\theta) - 25\tan\theta + 5 = 0$$

$$6 + 6\tan^2\theta - 25\tan\theta + 5 = 0$$

$$6\tan^2\theta - 25\tan\theta + 11 = 0$$

$$(2\tan\theta - 1)(3\tan\theta - 11) = 0$$

$$\tan\theta = \frac{1}{2} \text{ or } \tan\theta = \frac{11}{3}$$

So $\theta = \tan^{-1}\frac{1}{2}$ or $\theta = \tan^{-1}\frac{11}{3}$

or $\theta = \arctan\frac{1}{2}$ or $\theta = \arctan\frac{11}{3}$

11a Initially, $x = y = 0$, and, $\dot{x} = 15\cos 60°$ $\dot{y} = 15\sin 60°$

$\dot{x} = \frac{15}{2}$ $\dot{y} = \frac{15\sqrt{3}}{2}$

To begin, $\ddot{x} = 0$ (1)

Integrating, $\dot{x} = C_1$

When $t = 0$, $\dot{x} = \frac{15}{2}$

$\frac{15}{2} = C_1$

so $\dot{x} = \frac{15}{2}$ (2)

Integrating, $x = \frac{15}{2}t + C_2$

When $t = 0$, $x = 0$

$0 = C_2$

so $x = \frac{15}{2}t$ (3)

To begin, $\ddot{y} = -10$ (4)

Integrating, $\dot{y} = -10t + C_3$

When $t = 0$, $\dot{y} = \frac{15\sqrt{3}}{2}$

$\frac{15\sqrt{3}}{2} = C_3$

so $\dot{y} = -10t + \frac{15\sqrt{3}}{2}$ (5)

Integrating, $y = -5t^2 + \frac{15\sqrt{3}}{2}t + C_4$

When $t = 0$, $y = 0$

$0 = C_4$

so $y = -5t^2 + \frac{15\sqrt{3}}{2}t$ (6)

11b From (3):

$$t = \frac{2x}{15}$$

Substituting for t in (6):

$$y = -5\left(\frac{2x}{15}\right)^2 + \frac{15\sqrt{3}}{2} \times \frac{2x}{15}$$

$$= -\frac{20x^2}{225} + \sqrt{3}x$$

$$= -\frac{4}{45}x^2 + \sqrt{3}x \qquad (7)$$

11c i The ball will land when the equation of its path intersects with the equation of the slope.

$$\theta = 30°$$

$$\tan\theta = \frac{\sqrt{3}}{3}$$

So the gradient is $m = \frac{\sqrt{3}}{3}$.

Hence $y = \frac{\sqrt{3}}{3}x$ is the equation of the slope.

Equating with (7):

$$\frac{\sqrt{3}}{3}x = -\frac{4}{45}x^2 + \sqrt{3}x$$

$$\frac{4}{45}x^2 = \sqrt{3}x - \frac{\sqrt{3}}{3}x$$

$$\frac{4}{45}x = \sqrt{3} - \frac{\sqrt{3}}{3} \qquad \text{assuming } x \neq 0$$

$$\frac{4}{45}x = \frac{3\sqrt{3} - \sqrt{3}}{3}$$

$$x = \frac{45}{4} \times \frac{2\sqrt{3}}{3}$$

$$x = \frac{15\sqrt{3}}{2}$$

When $x = \frac{15\sqrt{3}}{2}$,

$$y = \frac{15\sqrt{3}}{2} \times \frac{\sqrt{3}}{3}$$

$$y = \frac{15}{2}$$

Using Pythagoras' theorem to calculate the distance up the slope:

$$r^2 = \left(\frac{15\sqrt{3}}{2}\right)^2 + \left(\frac{15}{2}\right)^2$$

$$r = \sqrt{\frac{675}{4} + \frac{225}{4}}$$

$$= \sqrt{225}$$

$$= 15 \text{ m}$$

The ball landed 15 m up the hill.

11c ii From (3):

$$t = \frac{2x}{15}$$

When $x = \frac{15\sqrt{3}}{2}$,

$$t = \frac{2}{15} \times \frac{15\sqrt{3}}{2}$$

$$= \sqrt{3} \text{ seconds}$$

The time of the flight is $\sqrt{3}$ seconds.