Solutions to Exercise 11A

1a $\sin 2x - \cos x = 0$

$2\sin x\cos x - \cos x = 0$

$\cos x\,(2\sin x - 1) = 0$

Hence

$\cos x = 0$ or $\sin x = \frac{1}{2}$

1b For $\cos x = 0$

$x = \frac{\pi}{2}, \frac{3\pi}{2}$

For $\sin x = \frac{1}{2}$

$x = \frac{\pi}{6}, \frac{5\pi}{6}$

Thus

$x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{6}$

2a $\cos 2x - \cos x = 0$

$\cos^2 x - \sin^2 x - \cos x = 0$

$\cos^2 x - (1 - \cos^2 x) - \cos x = 0$

$2\cos^2 x - \cos x - 1 = 0$

$(2\cos x + 1)(\cos x - 1) = 0$

$\cos x = 1$ or $-\frac{1}{2}$

2b For $\cos x = 1$

$x = 0, 2\pi$

For $\cos x = -\frac{1}{2}$

$x = \frac{2\pi}{3}, \frac{4\pi}{3}$

Hence

$$x = 0, \frac{2\pi}{3}, \frac{4\pi}{3}, 2\pi$$

3a

$$\sin\left(x + \frac{\pi}{4}\right) = 2\cos\left(x - \frac{\pi}{4}\right)$$

$$\sin x \cos\frac{\pi}{4} + \sin\frac{\pi}{4}\cos x = 2\cos x \cos\frac{\pi}{4} + 2\sin x \sin\frac{\pi}{4}$$

$$\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x = \frac{2}{\sqrt{2}}\cos x + \frac{2}{\sqrt{2}}\sin x$$

$$0 = \frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x$$

$$\sin x + \cos x = 0$$

$$\sin x = -\cos x$$

$$\tan x = -1$$

3b $\tan x = -1$

$$x = \frac{3\pi}{4}, \frac{7\pi}{4}$$

4a

$$\sin\left(\theta + \frac{\pi}{6}\right) = 2\sin\left(\theta - \frac{\pi}{6}\right)$$

$$\sin\theta\cos\frac{\pi}{6} + \cos\theta\sin\frac{\pi}{6} = 2\sin\theta\cos\frac{\pi}{6} - 2\cos\theta\sin\frac{\pi}{6}$$

$$3\cos\theta\sin\frac{\pi}{6} = \sin\theta\cos\frac{\pi}{6}$$

$$3\tan\frac{\pi}{6} = \tan\theta$$

$$\frac{3}{\sqrt{3}} = \tan\theta$$

$$\tan\theta = \sqrt{3}$$

Hence $\theta = \frac{\pi}{3}, \frac{4\pi}{3}$

4b $\cos\left(\theta-\frac{\pi}{6}\right)=2\cos\left(\theta+\frac{\pi}{6}\right)$

$$\cos\theta\cos\frac{\pi}{6}+\sin\theta\sin\frac{\pi}{6}=2\cos\theta\cos\frac{\pi}{6}-2\sin\theta\sin\frac{\pi}{6}$$

$$3\sin\theta\sin\frac{\pi}{6}=\cos\theta\cos\frac{\pi}{6}$$

$$\tan\theta=\frac{1}{3}\cot\frac{\pi}{6}=\frac{1}{\sqrt{3}}$$

$$\theta=\frac{\pi}{6},\frac{7\pi}{6}$$

4c $\cos 4\theta\cos\theta+\sin 4\theta\sin\theta=\frac{1}{2}$

$$\cos(4\theta-\theta)=\frac{1}{2}$$

$$\cos 3\theta=\frac{1}{2}$$

Now since $0\le\theta\le 2\pi, 0\le 3\theta\le 6\pi$

$$3\theta=\frac{\pi}{3},\frac{5\pi}{3},\frac{7\pi}{3},\frac{11\pi}{3},\frac{13\pi}{3},\frac{17\pi}{3}$$

Hence

$$\theta=\frac{\pi}{9},\frac{5\pi}{9},\frac{7\pi}{9},\frac{11\pi}{9},\frac{13\pi}{9},\frac{17\pi}{9}$$

4d $\cos 3\theta=\cos 2\theta\cos\theta$

$$\cos(2\theta+\theta)=\cos 2\theta\cos\theta$$

$$\cos 2\theta\cos\theta-\sin 2\theta\sin\theta=\cos 2\theta\cos\theta$$

$$\sin 2\theta\sin\theta=0$$

Hence $\sin\theta=0$ or $\sin 2\theta=0$

For $\sin\theta=0$:

$$\theta=0,\pi,2\pi$$

For $\sin 2\theta=0$:

Since $0\le\theta\le 2\pi, 0\le 2\theta\le 4\pi$

$$2\theta=0,\pi,2\pi,3\pi,4\pi$$

Which gives $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$

So the solutions are $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$

5a $\sin 2x = \sin x$

$2 \sin x \cos x = \sin x$

$2 \sin x \cos x - \sin x = 0$

$\sin x\,(2 \cos x - 1) = 0$

$\sin x = 0$ or $\cos x = \frac{1}{2}$

For $\sin x = 0$:

$x = 0, \pi, 2\pi$

For $\cos x = \frac{1}{2}$:

$x = \frac{\pi}{3}, \frac{5\pi}{3}$

Hence

$x = 0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}, 2\pi$

5b $\sin 2x + \sqrt{3} \cos x = 0$

$2 \sin x \cos x + \sqrt{3} \cos x = 0$

$\cos x\left(2 \sin x + \sqrt{3}\right) = 0$

$\cos x = 0$ or $\sin x = -\frac{\sqrt{3}}{2}$

For $\cos x = 0$:

$x = \frac{\pi}{2}, \frac{3\pi}{2}$

For $\sin x = -\frac{\sqrt{3}}{2}$

$x = \frac{4\pi}{3}, \frac{5\pi}{3}$

Hence the solutions are

$x = \frac{\pi}{2}, \frac{4\pi}{3}, \frac{3\pi}{2}, \frac{5\pi}{3}$

5c $3\sin x + \cos 2x = 2$

$3\sin x + (1 - 2\sin^2 x) = 2$

$3\sin x + 1 - 2\sin^2 x = 2$

$2\sin^2 x - 3\sin x + 1 = 0$

Using the quadratic formula

$\sin x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4 \times 2 \times 1}}{2 \times 2}$

$= \frac{3 \pm \sqrt{1}}{4}$

$= 1 \text{ or } \frac{1}{2}$

For $\sin x = 1$:

$x = \frac{\pi}{2}$

For $\sin x = \frac{1}{2}$:

$x = \frac{\pi}{6}, \frac{5\pi}{6}$

Hence the solutions are

$x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$

5d $\cos 2x + 3\cos x + 2 = 0$

$2\cos^2 x - 1 + 3\cos x + 2 = 0$

$2\cos^2 x + 3\cos x + 1 = 0$

$(2\cos x + 1)(\cos x + 1) = 0$

Hence $\cos x = -1$ or $\cos x = -\frac{1}{2}$

For $\cos x = -1$:

$x = \pi$

For $\cos x = -\frac{1}{2}$:

$x = \frac{2\pi}{3}, \frac{4\pi}{3}$

Hence the solutions are

$x = \frac{2\pi}{3}, \pi, \frac{4\pi}{3}$

5e $\tan 2x + \tan x = 0$

$\frac{2\tan x}{1 - \tan^2 x} + \tan x = 0$

$2\tan x + \tan x - \tan^3 x = 0$

$\tan^3 x - 3\tan x = 0$

$\tan x\left(\tan x - \sqrt{3}\right)\left(\tan x + \sqrt{3}\right) = 0$

Hence $\tan x = 0$ or $\tan x = \pm\sqrt{3}$

For $\tan x = 0$:

$x = 0, \pi, 2\pi$

For $\tan x = \sqrt{3}$:

$x = \frac{\pi}{3}, \frac{4\pi}{3}$

For $\tan x = -\sqrt{3}$:

$x = \frac{2\pi}{3}, \frac{5\pi}{3}$

Hence the solutions are:

$x = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}, 2\pi$

5f $\sin 2x = \tan x$

$2\sin x\cos x = \frac{\sin x}{\cos x}$

$2\sin x\cos^2 x = \sin x$

$2\sin x\cos^2 x - \sin x = 0$

$\sin x\,(2\cos^2 x - 1) = 0$

Hence the solutions are $\sin x = 0$ or $\cos x = \pm\frac{1}{\sqrt{2}}$.

For $\sin x = 0$:

$x = 0, \pi, 2\pi$

For $\cos x = \frac{1}{\sqrt{2}}$:

$$x = \frac{\pi}{4}, \frac{7\pi}{4}$$

For $\cos x = -\frac{1}{\sqrt{2}}$:

$$x = \frac{3\pi}{4}, \frac{5\pi}{4}$$

Hence the solutions are

$$x = 0, \frac{\pi}{4}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{7\pi}{4}, 2\pi$$

6a $2\sin 2\theta + \cos\theta = 0$

$2(2\sin\theta\cos\theta) + \cos\theta = 0$

$\cos\theta\,(4\sin\theta + 1) = 0$

Hence $\cos\theta = 0$ or $\sin\theta = -\frac{1}{4}$

For $\cos\theta = 0$:

$\theta = 90°, 270°$

For $\sin\theta = -\frac{1}{4}$:

$\theta = 194°29', 345°\,31'$

Hence the solutions are

$\theta = 90°, 194°29', 270°, 345°\,31'$

6b $2\cos^2\theta + \cos 2\theta = 0$

$2\cos^2\theta + (2\cos^2\theta - 1) = 0$

$4\cos^2\theta = 1$

$\cos^2\theta = \frac{1}{4}$

$\cos\theta = \pm\frac{1}{2}$

$\theta = 60°, 120°, 240°, 300°$

6c $2\cos 2\theta + 4\cos\theta = 1$

$2(2\cos^2\theta - 1) + 4\cos\theta = 1$

$4\cos^2\theta + 4\cos\theta - 3 = 0$

Using the quadratic formula gives

$$\cos\theta = \frac{-4 \pm \sqrt{4^2 - 4\times 4\times -3}}{2\times 4}$$

$$= \frac{-4 \pm \sqrt{64}}{8}$$

$$= \frac{-4 \pm 8}{8}$$

$$= \frac{1}{2} \text{ or } -\frac{3}{2}$$

But $-1 \le \cos\theta \le 1$, hence

$\cos\theta = \frac{1}{2}$ gives the solutions

$\theta = 60°, 300°$

6d $8\sin^2\theta\cos^2\theta = 1$

$2(2\sin\theta\cos\theta)^2 = 1$

$\sin^2 2\theta = \frac{1}{2}$

$\sin 2\theta = \pm\frac{1}{\sqrt{2}}$

Since $0 \le \theta \le 360°$, hence $0 \le 2\theta \le 720°$

$2\theta = 45°, 135°, 225°, 315°, 405°, 495°, 585°, 675°$

$\theta = 22°30', 67°30', 112°, 30', 157°30', 202°30', 247°30', 292°30', 337°30'$

6e $3\cos 2\theta + \sin\theta = 1$

$3(1 - 2\sin^2\theta) + \sin\theta = 1$

$3 - 6\sin^2\theta + \sin\theta = 1$

$6\sin^2\theta - \sin\theta - 2 = 0$

Using the quadratic formula

$$\sin\theta = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \times 6 \times -2}}{2 \times 6}$$

$$= \frac{1 \pm \sqrt{49}}{12}$$

$$= \frac{1 \pm 7}{12}$$

$$= -\frac{1}{2}, \frac{3}{4}$$

For $\sin\theta = -\frac{1}{2}$:

$\theta = 210°, 330°$

For $\sin\theta = \frac{3}{4}$:

$\theta = 41°49', 138°11'$

Hence the solutions are

$\theta = 41°49', 138°11', 210°, 330°$

6f $\cos 2\theta = 3\cos^2\theta - 2\sin^2\theta$

$\cos^2\theta - \sin^2\theta = 3\cos^2\theta - 2\sin^2\theta$

$\sin^2\theta = 2\cos^2\theta$

$\tan^2\theta = 2$

$\tan\theta = \pm\sqrt{2}$

$\theta = 54°44', 125°16', 234°44', 305°16'$

6g $10\cos\theta + 13\cos\frac{1}{2}\theta = 5$

$$10\left(2\cos^2\frac{1}{2}\theta - 1\right) + 13\cos\frac{1}{2}\theta = 5$$

$$20\cos^2\frac{1}{2}\theta - 10 + 13\cos\frac{1}{2}\theta - 5 = 0$$

$$20\cos^2\frac{1}{2}\theta + 13\cos\frac{1}{2}\theta - 15 = 0$$

Now using the quadratic formula

$$\cos\frac{1}{2}\theta = \frac{-13 \pm \sqrt{13^2 - 4\times 20\times -15}}{2\times 20}$$

$$= \frac{-13 \pm 37}{2\times 20}$$

$$= -\frac{5}{4}, \frac{3}{5}$$

But $-1 \leq \cos\frac{1}{2}\theta \leq 1$

$$\cos\frac{1}{2}\theta = \frac{3}{5}$$

$$\frac{1}{2}\theta = 53°8'$$

$$\theta = 106°16'$$

6h $\tan\theta = 3\tan\frac{1}{2}\theta$

$$\frac{2\tan\frac{1}{2}\theta}{1-\tan^2\frac{1}{2}\theta} = 3\tan\frac{1}{2}\theta$$

$$2\tan\frac{1}{2}\theta = 3\tan\frac{1}{2}\theta - 3\tan^3\frac{1}{2}\theta$$

$$3\tan^3\frac{1}{2}\theta - \tan\frac{1}{2}\theta = 0$$

$$\tan\frac{1}{2}\theta\,(3\tan^2\theta - 1) = 0$$

Hence $\tan\frac{1}{2}\theta = 0$ or $\tan\theta = \pm\frac{1}{\sqrt{3}}$

Since $0° \leq \theta \leq 360°$, $0° \leq \frac{\theta}{2} \leq 180°$

For $\tan\frac{1}{2}\theta = 0$:

$\frac{1}{2}\theta = 0°, 180°$

$\theta = 0°, 360°$

For $\tan\frac{1}{2}\theta = \frac{1}{\sqrt{3}}$:

$\frac{1}{2}\theta = 30°$

$\theta = 60°$

For $\tan\frac{1}{2}\theta = -\frac{1}{\sqrt{3}}$:

$\frac{1}{2}\theta = 150°$

$\theta = 300°$

Hence the solutions are

$\theta = 0°, 60°, 300°, 360°$

6i $\cos^2 2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta$

$2\cos^2 2\theta = 1 - 1\cos 2\theta$

$2\cos^2 2\theta + \cos 2\theta - 1 = 0$

$(2\cos 2\theta - 1)(\cos 2\theta + 1) = 0$

Hence $\cos 2\theta = -1$ or $\frac{1}{2}$

Since $0° \leq \theta \leq 360°$, $0° \leq 2\theta \leq 720°$

For $\cos 2\theta = -1$:

$2\theta = 180°, 540°$

$\theta = 90°, 270°$

For $\cos 2\theta = \frac{1}{2}$:

$2\theta = 60°, 300°, 420°, 660°$

$\theta = 30°, 150°, 210°, 330°$

Hence

$\theta = 30°, 90°, 150°, 210°, 270°, 330°$

6j $\cos 2\theta + 3 = 3\sin 2\theta$

$\cos 2\theta + 3\cos^2\theta + 3\sin^2\theta = 3\sin 2\theta$

$\cos^2\theta - \sin^2\theta + 3\cos^2\theta + 3\sin^2\theta = 3(2\sin\theta\cos\theta)$

$4\cos^2\theta + 2\sin^2\theta = 6\sin\theta\cos\theta$

$4\cos^2\theta - 6\sin\theta\cos\theta + 2\sin^2\theta = 0$

$(2\cos\theta - \sin\theta)(2\cos\theta - 2\sin\theta) = 0$

$(2\cos\theta - \sin\theta)(\cos\theta - \sin\theta) = 0$

Dividing both sides by $\cos^2\theta$

$(2 - \tan\theta)(1 - \tan\theta) = 0$

Hence $\tan\theta = 1$ or 2

$\theta = 45°, 63°26', 225°, 243°26'$

7a

$\tan\left(\frac{\pi}{4} + \theta\right) = 3\tan(\frac{\pi}{4} - \theta)$

$\frac{\tan\frac{\pi}{4} + \tan\theta}{1 - \tan\frac{\pi}{4}\tan\theta} = \frac{3\tan\frac{\pi}{4} - 3\tan\theta}{1 + \tan\frac{\pi}{4}\tan\theta}$

$\frac{1 + \tan\theta}{1 - \tan\theta} = \frac{3(1 - \tan\theta)}{1 + \tan\theta}$

$(1 + \tan\theta)^2 = 3(1 - \tan\theta)^2$

$1 + 2\tan\theta + \tan^2\theta = 3 - 6\tan\theta + 3\tan^2\theta$

$2\tan^2\theta - 8\tan\theta + 2 = 0$

$\tan^2\theta - 4\tan\theta + 1 = 0$

7b Using the quadratic formula

$\tan\theta = \frac{4 \pm \sqrt{4^2 - 4 \times 1 \times 1}}{2}$

$$= \frac{4 \pm 2\sqrt{3}}{2}$$

$$= 2 \pm \sqrt{3}$$

$\theta = \tan^{-1}\left(2 \pm \sqrt{3}\right)$

$\theta = \frac{\pi}{12}, \frac{5\pi}{12}$

8a $2\cos x - 1 = 2(2\cos^2 x - 1)$

$2\cos x - 1 = 4\cos^2 x - 2$

$4\cos^2 x - 2\cos x - 3 = 0$

Using the quadratic formula

$$\cos x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \times 4 \times (-3)}}{2 \times 4}$$

$$= \frac{2 \pm 2\sqrt{5}}{2 \times 4}$$

$$= \frac{1}{4}(1 \pm \sqrt{5})$$

8b $x = \cos^{-1}\frac{1}{4}\left(1 \pm \sqrt{5}\right), 2\pi - \cos^{-1}\frac{1}{4}\left(1 \pm \sqrt{5}\right)$

$$= \frac{\pi}{5}, \frac{3\pi}{5}, \frac{7\pi}{5}, \frac{9\pi}{5}$$

9a $\sin(\alpha + \beta)\sin(\alpha - \beta)$

$= (\sin\alpha\cos\beta - \cos\alpha\sin\beta)(\sin\alpha\cos\beta + \cos\alpha\sin\beta)$
$= \sin^2\alpha\cos^2\beta - \cos^2\alpha\sin^2\beta$
$= \sin^2\alpha\,(1 - \sin^2\beta) - (1 - \sin^2\alpha)\sin^2\beta$
$= \sin^2\alpha - \sin^2\alpha\sin^2\beta - \sin^2\beta + \sin^2\alpha\sin^2\beta$
$= \sin^2\alpha - \sin^2\beta$

9b $\sin^2 3\theta - \sin^2\theta = \sin 2\theta$

$\sin(3\theta + \theta)\sin(3\theta - \theta) = \sin 2\theta$

$\sin 4\theta \sin 2\theta = \sin 2\theta$

$\sin 4\theta \sin 2\theta - \sin 2\theta = 0$

$\sin 2\theta (\cos 4\theta - 1) = 0$

Hence $\sin 2\theta = 0$ or $\cos 4\theta = 1$

Since $0 \le \theta \le \pi$, $0 \le 2\theta \le 2\pi$ and $0 \le 4\theta \le 2\pi$

For $\sin 2\theta$:

$2\theta = 0, \pi, 2\pi$

$\theta = 0, \frac{\pi}{2}, \pi$

For $\cos 4\theta$:

$4\theta = \frac{\pi}{2}, \frac{5\pi}{2}$

$\theta = \frac{\pi}{4}, \frac{3\pi}{4}$

Hence the solutions are

$\theta = 0, \frac{\pi}{8}, \frac{\pi}{2}, \frac{5\pi}{8}, \pi$

10a $\sin 3x$

$$\begin{aligned}
&= \sin x \cos 2x + \cos x \sin 2x \\
&= \sin x (1 - 2\sin^2 x) + \cos x (2 \sin x \cos x) \\
&= \sin x - 2\sin^3 x + 2 \sin x \cos^2 x \\
&= \sin x - 2\sin^3 x + 2 \sin x (1 - \sin^2 x) \\
&= \sin x - 2\sin^3 x + 2 \sin x - 2\sin^3 x \\
&= 3 \sin x - 4 \sin^3 x
\end{aligned}$$

10b $\sin 3x + \sin 2x = \sin x$

$3 \sin x - 4 \sin^3 x + 2 \sin x \cos x = \sin x$

$4 \sin^3 x - 2 \sin x - 2 \sin x \cos x = 0$

$2 \sin x (2 \sin^2 x - 1 - \cos x) = 0$

$2 \sin x (2(1 - \cos^2 x) - 1 - \cos x) = 0$

$2 \sin x (2 - 2\cos^2 x - 1 - \cos x) = 0$

$2 \sin x (1 - 2\cos^2 x - \cos x) = 0$

$\sin x\,(2\cos^2 x + \cos x - 1) = 0$

$\sin x\,(2\cos x - 1)(\cos x + 1) = 0$

Hence $\sin x = 0$, $\cos x = \frac{1}{2}$ or $\cos x = -1$

For $\sin x = 0$:

$x = 0, \pi, 2\pi$

For $\cos x = -1$:

$x = \pi$

For $\cos x = \frac{1}{2}$:

$x = \frac{\pi}{3}, \frac{5\pi}{3}$

Hence the solutions are

$x = 0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}, 2\pi$

11a

$\sin\left(\theta + \frac{\pi}{6}\right) = \cos\left(\theta - \frac{\pi}{4}\right)$

$\sin\theta\cos\frac{\pi}{6} + \cos\theta\sin\frac{\pi}{6} = \cos\theta\cos\frac{\pi}{4} + \sin\theta\sin\frac{\pi}{4}$

$\frac{\sqrt{3}}{2}\sin\theta + \frac{1}{2}\cos\theta = \frac{1}{\sqrt{2}}\cos\theta + \frac{1}{\sqrt{2}}\sin\theta$

Dividing both sides by $\cos\theta$

$\frac{\sqrt{3}}{2}\tan\theta + \frac{1}{2} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\tan\theta$

$\left(\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}\right)\tan\theta = \frac{1}{\sqrt{2}} - \frac{1}{2}$

$\left(\frac{\sqrt{6} - 2}{2\sqrt{2}}\right)\tan\theta = \frac{2 - \sqrt{2}}{2\sqrt{2}}$

$\tan\theta = \frac{2 - \sqrt{2}}{\sqrt{6} - 2}$

Hence

$$\begin{aligned}\tan\theta &= \frac{(2-\sqrt{2})(\sqrt{6}+2)}{(\sqrt{6}-2)(\sqrt{6}+2)}\\ &= \frac{2\sqrt{6}+4-\sqrt{12}-2\sqrt{2}}{6-4}\\ &= \frac{2\sqrt{6}+4-4\sqrt{3}-2\sqrt{2}}{2}\\ &= \sqrt{6}-\sqrt{3}-\sqrt{2}+2\end{aligned}$$

11b $\theta = \frac{7\pi}{24}, \frac{19\pi}{24}$

12a $\sec^2\alpha - 2\sec\alpha = 0$

$\sec\alpha\,(\sec\alpha - 2) = 0$

But $\sec\alpha \neq 0$, hence

$(\sec\alpha - 2) = 0$

$\sec\alpha = 2$

$\cos\alpha = \frac{1}{2}$

$\alpha = 60°, 300°$

12b $\sec^2\alpha - \tan\alpha - 3 = 0$

$1 + \tan^2\alpha - \tan\alpha - 3 = 0$

$\tan^2\alpha - \tan\alpha - 2 = 0$

$(\tan\alpha - 2)(\tan\alpha + 1) = 0$

$\tan\alpha = -1$ or 2

For $\tan\alpha = -1$:

$\alpha = 135°, 315°$

For $\tan\alpha = 2$:

$\alpha = 63°26', 243°26'$

Hence the solutions are $\alpha = 63°26', 135°, 243°26', 315°$

12c $\operatorname{cosec}^3 2\alpha = 4\operatorname{cosec} 2\alpha$

$$\frac{1}{\sin^3 2\alpha} = \frac{4}{\sin 2\alpha}$$

Note that $\sin 4\alpha \neq 0$ and $\sin 2\alpha \neq 0$

$\sin 2\alpha = 4\sin^3 2\alpha$

Now as $\sin 2\alpha \neq 0$

$4\sin^2 2\alpha = 1$

$$\sin 2\alpha = \pm\frac{1}{2}$$

Now, the domain is

$0 \leq \alpha \leq 360°$

which means

$0 \leq 2\alpha \leq 720°$

Hence

$2\alpha = 30°, 150°, 210°, 330°, 390°, 450°, 570°, 690°$

$\alpha = 15°, 75°, 105°, 165°, 195°, 255°, 285°, 345°$

12d

$$\sqrt{3}\operatorname{cosec}^2\frac{1}{2}\alpha + \cot\frac{1}{2}\alpha = \sqrt{3}$$

$$\sqrt{3}\left(\cot^2\frac{1}{2}\alpha + 1\right) + \cot\frac{1}{2}\alpha = \sqrt{3}$$

$$\sqrt{3}\cot^2\frac{1}{2}\alpha + \cot\frac{1}{2}\alpha = 0$$

$$\cot\frac{1}{2}\alpha\left(\sqrt{3}\cot\frac{1}{2}\alpha + 1\right) = 0$$

$\cot\frac{1}{2}\alpha = 0$ or $\cot\frac{1}{2}\alpha = -\frac{1}{\sqrt{3}}$

For $\cot\frac{1}{2}\alpha = 0, \frac{1}{2}\alpha = 90°$ and hence $\alpha = 180°$

For $\cot\frac{1}{2}\alpha = -\frac{1}{\sqrt{3}}, \frac{1}{2}\alpha = 120°$ and hence $2\alpha = 240°$

Thus the solutions are

$\alpha = 180°$ or $240°$

12e $\sqrt{3}\,\mathrm{cosec}^2\,\alpha = 4\cot\alpha$

$\sqrt{3}(\cot^2\alpha + 1) = 4\cot\alpha$

$\sqrt{3}\cot^2\alpha - 4\cot\alpha + \sqrt{3} = 0$

Using the quadratic formula

$$\cot\alpha = \frac{-(-4) \pm \sqrt{(-4)^2 - 4\times\sqrt{3}\times\sqrt{3}}}{2\sqrt{3}}$$

$$= \frac{4 \pm \sqrt{4}}{2\sqrt{3}}$$

$$= \frac{4 \pm 2}{2\sqrt{3}}$$

$$= \tfrac{1}{\sqrt{3}} \text{ or } \sqrt{3}$$

Hence

$\alpha = 30°, 60°, 210°, 240°$

12f $\cot\alpha + 3\tan\alpha = 5\,\mathrm{cosec}\,\alpha$

$$\frac{\cos\alpha}{\sin\alpha} + 3\frac{\sin\alpha}{\cos\alpha} = \frac{5}{\sin\alpha}$$

Multiplying both sides by $\sin\alpha\cos\alpha$

$\cos^2\alpha + 3\sin^2\alpha = 5\cos\alpha$

$\cos^2\alpha + 3(1 - \cos^2\alpha) = 5\cos\alpha$

$\cos^2\alpha + 3 - 3\cos^2\alpha = 5\cos\alpha$

$2\cos^2\alpha + 5\cos\alpha - 3 = 0$

Using the quadratic formula

$$\cos\alpha = \frac{-5 \pm \sqrt{5^2 - 4\times 2\times -3}}{2\times 2}$$

$$= \frac{-5 \pm \sqrt{49}}{4}$$

$$= \frac{-5 \pm 7}{4}$$

$$= -3 \text{ or } \tfrac{1}{2}$$

As $-1 \leq \cos\alpha \leq 1$

$$\cos\alpha = \frac{1}{2}$$

$$\alpha = 60°, 300°$$

13a $2\cos A\cos B = \cos(A+B) + \cos(A-B)$

Let $A = \frac{P+Q}{2}$ and let $B = \frac{P-Q}{2}$

$$2\cos\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right) = \cos\left(\frac{P+Q}{2} + \frac{P-Q}{2}\right) + \cos\left(\frac{P+Q}{2} - \frac{P-Q}{2}\right)$$

$$2\cos\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right) = \cos(P) + \cos(Q)$$

Hence

$$\cos P + \cos Q = 2\cos\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)$$

13b $\cos 4x + \cos x = 0$

$$2\cos\left(\frac{4x+x}{2}\right)\cos\left(\frac{4x-x}{2}\right) = 0$$

$$2\cos\left(\frac{5x}{2}\right)\cos\left(\frac{3x}{2}\right) = 0$$

Hence $\cos\left(\frac{5x}{2}\right) = 0$ or $\cos\left(\frac{3x}{2}\right) = 0$

For $\cos\left(\frac{5x}{2}\right) = 0$:

Note that since $0 \leq x \leq \pi$, $0 \leq \frac{5x}{2} \leq \frac{5\pi}{2}$

$$\frac{5x}{2} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$$

$$x = \frac{\pi}{5}, \frac{3\pi}{5}, \pi$$

For $\cos\left(\frac{3x}{2}\right) = 0$:

Note that since $0 \leq x \leq \pi$, $0 \leq \frac{3x}{2} \leq \frac{3\pi}{2}$

$$\frac{3x}{2} = \frac{\pi}{2}, \frac{3\pi}{2}$$

$x = \frac{\pi}{3}, \pi$

Hence the solutions are

$x = \frac{\pi}{5}, \frac{\pi}{3}, \frac{3\pi}{5}, \pi$

14a $\sin\theta + \cos\theta = \sin 2\theta$

Squaring both sides gives

$(\sin\theta + \cos\theta)^2 = \sin^2 2\theta$

$\sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta = \sin^2 2\theta$

$\sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta = \sin^2 2\theta$

$1 + 2\sin\theta\cos\theta = \sin^2 2\theta$

$\sin^2 2\theta - 2\sin\theta\cos\theta - 1 = 0$

$\sin^2 2\theta - \sin 2\theta - 1 = 0$

14b $\sin^2 2\theta - \sin 2\theta - 1 = 0$

Using the quadratic formula gives

$$\sin 2\theta = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \times 1 \times -1}}{2 \times 1}$$

$$= \frac{1 \pm \sqrt{5}}{2}$$

$$\theta = \frac{1}{2}\sin^{-1}\frac{1 \pm \sqrt{5}}{2} = 160°55', 289°5'$$

15a $\cos 3\theta$

$= \cos\theta\cos 2\theta - \sin\theta\sin 2\theta$
$= \cos\theta\,(2\cos^2\theta - 1) - \sin\theta\,(2\sin\theta\cos\theta)$
$= 2\cos^3\theta - \cos\theta - 2\sin^2\theta\cos\theta$
$= 2\cos^3\theta - \cos\theta - 2(1 - \cos^2\theta)\cos\theta$
$= 2\cos^3\theta - \cos\theta - 2\cos\theta + 2\cos^3\theta$
$= 4\cos^3\theta - 3\cos\theta$

15b $x^3 - 3x - 1 = 0$

Let $x = 2\cos\theta$

$(2\cos\theta)^3 - 3(2\cos\theta) - 1 = 0$

$8\cos^3\theta - 6\cos\theta - 1 = 0$

$4\cos^3\theta - 3\cos\theta - \dfrac{1}{2} = 0$

$\cos 3\theta - \frac{1}{2} = 0$ (from part a)

$\cos 3\theta = \dfrac{1}{2}$

$3\theta = 60°, 300°, 420°$

$\theta = 20°, 100°, 140°$

$$\begin{aligned} x &= 2\cos(20°), 2\cos(100°), 2\cos(140°) \\ &= 2\cos(20°), -2\sin(100° - 90°), -2\cos(180° - 140°) \\ &= 2\cos 20°, -2\sin 10°, -2\cos 40° \end{aligned}$$

15c $x^3 - 12x = 8\sqrt{3}$

Let $x = 4\cos\theta$

$(4\cos\theta)^3 - 12(4\cos\theta) = 8\sqrt{3}$

$64\cos^3\theta - 48\cos\theta = 8\sqrt{3}$

$4\cos^3\theta - 3\cos\theta = \dfrac{\sqrt{3}}{2}$

$\cos 3\theta = \frac{\sqrt{3}}{2}$ (from part a)

$\theta = 10°, 110°, 130°$

$x = 2\cos 10°, 2\cos 110°, 2\cos 130°$

$x \doteqdot -2.571, -1.368, 3.939$

16a $\tan 4x$

$= \dfrac{2\tan 2x}{1 - \tan^2 2x}$

$$= \frac{2\left(\frac{2t}{1-t^2}\right)}{1-\left(\frac{2t}{1-t^2}\right)^2}$$

$$= \frac{2\left(\frac{2t}{1-t^2}\right)}{1-\frac{4t^2}{(1-t^2)^2}}$$

$$= \frac{2(2t)(1-t^2)}{(1-t^2)^2-4t^2}$$

$$= \frac{4t(1-t^2)}{1-2t^2+t^4-4t^2}$$

$$= \frac{4t(1-t^2)}{1-6t^2+t^4}$$

16b $\tan 4x \tan x = 1$

$$\frac{4t(1-t^2)}{1-6t^2+t^4} \times t = 1$$

$$4t^2(1-t^2) = 1-6t^2+t^4$$

$$4t^2-4t^4 = 1-6t^2+t^4$$

$$5t^4-10t^2+1=0$$

16c

$$\frac{1}{2}(\cos(A-B)-\cos(A+B))$$
$$= \frac{1}{2}\big(\cos A\cos B+\sin A\sin B-(\cos A\cos B-\sin A\sin B)\big)$$
$$= \frac{1}{2}(2\sin A\sin B) = \sin A\sin B$$

$$\frac{1}{2}(\cos(A-B)+\cos(A+B))$$
$$= \frac{1}{2}\big(\cos A\cos B+\sin A\sin B+(\cos A\cos B-\sin A\sin B)\big)$$
$$= \frac{1}{2}(2\cos A\cos B)$$

$$= \cos A\cos B$$

16d $\sin A \sin B = \frac{1}{2}(\cos(A-B) - \cos(A+B))$ (1)

$\cos A \cos B = \frac{1}{2}(\cos(A-B) + \cos(A+B))$(2)

$\tan A \tan B = \frac{\cos(A-B)-\cos(A+B)}{\cos(A-B)+\cos(A+B)}$ (1) ÷ (2)

Hence $\tan 4x \tan x = \frac{\cos(3x)-\cos(5x)}{\cos(3x)+\cos(5x)}$

Thus for $\tan 4x \tan x = 1$,

$$\frac{\cos(3x) - \cos(5x)}{\cos(3x) + \cos(5x)} = 1$$

$$\cos(3x) - \cos(5x) = \cos(3x) + \cos(5x)$$

$$2\cos(5x) = 0$$

$$\cos(5x) = 0$$

$$5x = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$$

$$x = \frac{\pi}{10}, \frac{3\pi}{10}, \ldots$$

Hence $\frac{\pi}{10}$ and $\frac{3\pi}{10}$ both satisfy the equation.

16e $x = \tan\frac{\pi}{10}, -\tan\frac{\pi}{10}, \tan\frac{3\pi}{10}, -\tan\frac{3\pi}{10}$

Solutions to Exercise 11B

1a $R\sin\alpha = \sqrt{3}$ (1)

$R\cos\alpha = 1$ (2)

$\tan\alpha = \sqrt{3}$ (1) ÷ (2)

$\alpha = \dfrac{\pi}{3}$

$R^2\sin^2\alpha + R^2\cos^2\alpha = 3 + 1$ $(1)^2 + (2)^2$

$R^2(\sin^2\alpha + \cos^2\alpha) = 4$

$R^2 = 4$

$R = 2$

1b $R\sin\alpha = 3$ (1)

$R\cos\alpha = 3$ (2)

$\tan\alpha = 1$ (1) ÷ (2)

$\alpha = \dfrac{\pi}{4}$

$R^2\sin^2\alpha + R^2\cos^2\alpha = 3^2 + 3^2$ $(1)^2 + (2)^2$

$R^2(\sin^2\alpha + \cos^2\alpha) = 18$

$R^2 = 18$

$R = 3\sqrt{2}$

2a $R\sin\alpha = 5$ (1)

$R\cos\alpha = 12$ (2)

$\tan\alpha = \frac{5}{12}$ (1) ÷ (2)

$\alpha = \tan^{-1}\dfrac{5}{12} = 22°37'$

$R^2\sin^2\alpha + R^2\cos^2\alpha = 5^2 + 12^2$ $(1)^2 + (2)^2$

$R^2(\sin^2\alpha + \cos^2\alpha) = 25 + 144$

$R^2 = 169$

$R = 13$

2b $R\sin\alpha = 4$ (1)

$R\cos\alpha = 2$ (2)

$\tan\alpha = 2$ (1) ÷ (2)

$\alpha = \tan^{-1} 2 = 63°26'$

$R^2\sin^2\alpha + R^2\cos^2\alpha = 2^2 + 4^2$ $(1)^2 + (2)^2$

$R^2(\sin^2\alpha + \cos^2\alpha) = 4 + 16$

$R^2 = 20$

$R = 2\sqrt{5}$

3a $A\cos(x+\alpha) = A\cos x\cos\alpha - A\sin x\sin\alpha = \cos x - \sin x$

Equating coefficients gives

$A\cos\alpha = 1$ (1)

$A\sin\alpha = 1$ (2)

3b $A^2\sin^2\alpha + A^2\cos^2\alpha = 1^2 + 1^2$ $(1)^2 + (2)^2$

$A^2(\sin^2\alpha + \cos^2\alpha) = 2$

$A^2 = 2$

$A = \sqrt{2}$

3c $\tan\alpha = 1$ (1) ÷ (2)

$\alpha = \tan^{-1} 1 = \frac{\pi}{4}$

3d Note that since $-1 \le \cos\left(x + \frac{\pi}{4}\right) \le 1$, $-\sqrt{2} \le \sqrt{2}\cos\left(x + \frac{\pi}{4}\right) \le \sqrt{2}$ and hence $-\sqrt{2} \le \cos x - \sin x \le \sqrt{2}$. Hence the maximum value of the function is $\sqrt{2}$ and the minimum value is $-\sqrt{2}$.

The maximum value occurs when $\sqrt{2}\cos\left(x + \frac{\pi}{4}\right) = \sqrt{2}$ and hence

$\cos\left(x+\frac{\pi}{4}\right)=1$ (Note that $0\le x\le 2\pi$ so $\frac{\pi}{4}\le x+\frac{\pi}{4}\le\frac{9\pi}{4}$)

$$x+\frac{\pi}{4}=2\pi$$

$$x=\frac{7\pi}{4}$$

The minimum value occurs when $\sqrt{2}\cos\left(x+\frac{\pi}{4}\right)=-\sqrt{2}$ and hence

$\cos\left(x+\frac{\pi}{4}\right)=-1$ (Note that $0\le x\le 2\pi$ so $\frac{\pi}{4}\le x+\frac{\pi}{4}\le\frac{9\pi}{4}$)

$$x+\frac{\pi}{4}=\pi$$

$$x=\frac{3\pi}{4}$$

3e $\cos x-\sin x=-1$

$$\sqrt{2}\cos\left(x+\frac{\pi}{4}\right)=-1$$

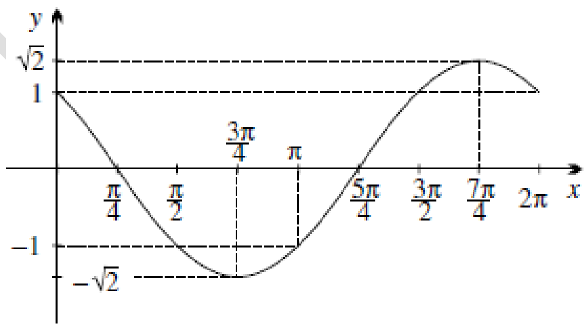
$$\cos\left(x+\frac{\pi}{4}\right)=-\frac{1}{\sqrt{2}}$$

$$x+\frac{\pi}{4}=\frac{3\pi}{4},\frac{5\pi}{4}$$

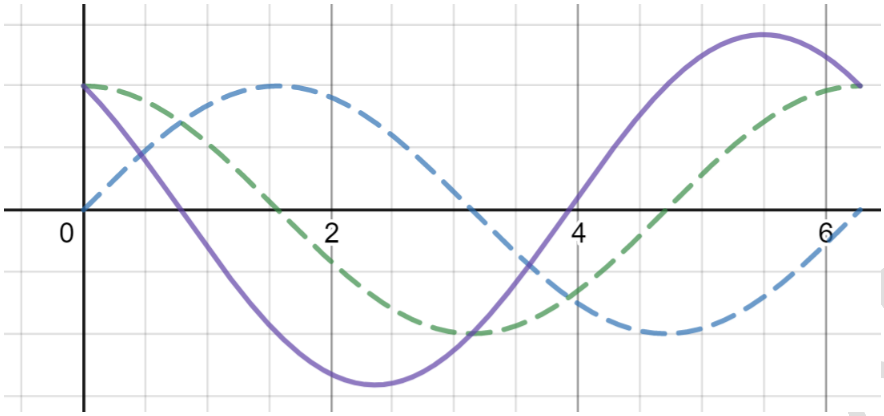
$$x=\frac{\pi}{2},\pi$$

3f The amplitude is equal to the value of A which is $\sqrt{2}$. The period is $\frac{2\pi}{1}=2\pi$.

Graph of $y=\cos x-\sin x$ is shown below.



4 Graph shows $y = \sin x$ (blue curve), $y = \cos x$ (green curve) and $y = \cos x - \sin x$ (purple curve).



This appears the same as the graph in the previous question.

5a $B\cos(x+\theta) = B\cos x\cos\theta - B\sin x\sin\theta \equiv \sqrt{3}\cos x - \sin x$

Equating coefficients gives

$B\cos\theta = \sqrt{3}$ (1)

$B\sin\theta = 1$ (2)

5b $B^2\sin^2\theta + B^2\cos^2\theta = 3 + 1$ $(1)^2 + (2)^2$

$B^2(\sin^2\theta + \cos^2\theta) = 4$

$B^2 = 4$

$B = 2$

5c $\tan\theta = \frac{1}{\sqrt{3}}$ $(2) \div (1)$

$\theta = \frac{\pi}{6}$

5d The greatest possible value is 2 and and the least value is -2 as $B = 2$ is the amplitude of the new periodic function.

Note that since $0 \le x \le 2\pi, \frac{\pi}{6} \le x + \frac{\pi}{6} \le \frac{13\pi}{6}$

For the point of maximum value

$2 = 2\cos\left(x + \frac{\pi}{6}\right)$

$\cos\left(x + \frac{\pi}{6}\right) = 1$

$x + \frac{\pi}{6} = 2\pi$

$x = \frac{11\pi}{6}$

For the point of minimum value

$-2 = 2\cos\left(x + \frac{\pi}{6}\right)$

$\cos\left(x + \frac{\pi}{6}\right) = -1$

$x + \frac{\pi}{6} = \pi$

$x = \frac{5\pi}{6}$

6a $A\sin(x - \alpha) = A\sin x\cos\alpha - A\cos x\sin\alpha = 4\sin x - 3\cos x$

Equating coefficients gives

$A\cos\alpha = 4 \quad (1)$

$A\sin\alpha = 3 \quad (2)$

6b $A^2\sin^2\alpha + A^2\cos^2\alpha = 3^2 + 4^2 \qquad (1)^2 + (2)^2$

$A^2(\sin^2\alpha + \cos^2\alpha) = 25$

$A^2 = 25$

$A = 5$

$\tan\alpha = \frac{3}{4} \qquad (1) \div (2)$

$\alpha = \tan^{-1}\frac{3}{4}$

6c $4\sin x - 3\cos x = 5$

$$5\sin\left(x - \tan^{-1}\frac{3}{4}\right) = 5$$

$$\sin\left(x - \tan^{-1}\frac{3}{4}\right) = 1$$

$$x - \tan^{-1}\frac{3}{4} = 90°$$

$$x = 90° + \tan^{-1}\frac{3}{4}$$

$$x \doteqdot 126°52'$$

7a $B\cos(x-\theta) = B\cos x\cos\theta + B\sin x\sin\theta \equiv 2\cos x + \sin x$

Equating coefficients gives

$B\cos\theta = 2 \quad (1)$

$B\sin\theta = 1 \quad (2)$

$B^2\sin^2\theta + B^2\cos^2\theta = 2^2 + 1^2 \qquad (1)^2 + (2)^2$

$B^2(\sin^2\theta + \cos^2\theta) = 4$

$B^2 = 4$

$B = \sqrt{5}$

$\tan\theta = \frac{1}{2} \qquad (2) \div (1)$

$$\theta = \tan^{-1}\frac{1}{2}$$

7b $2\cos x + \sin x = 1$

$$\sqrt{5}\cos\left(x - \tan^{-1}\frac{1}{2}\right) = 1$$

$$\cos\left(x - \tan^{-1}\frac{1}{2}\right) = \frac{1}{\sqrt{5}}$$

$$x = \cos^{-1}\frac{1}{\sqrt{5}} + \tan^{-1}\frac{1}{2}$$

$$x \doteqdot 323°8'$$

Testing 90° and 270° gives a second solution of 90°

8a $D\cos(x+\phi) = D\cos x\cos\phi - D\sin x\sin\phi \equiv \cos x - 3\sin x$

Equating coefficients gives

$D\cos\phi = 1 \quad (1)$

$D\sin\phi = 3 \quad (2)$

$D^2\sin^2\phi + D^2\cos^2\phi = 1 + 9 \qquad (1)^2 + (2)^2$

$D^2(\sin^2\phi + \cos^2\phi) = 10$

$D^2 = 10$

$D = \sqrt{10}$

$\tan\phi = 3 \qquad (2) \div (1)$

$\phi = \tan^{-1} 3$

8b $\cos x - 3\sin x = 3$

$\sqrt{10}\cos(x + \tan^{-1} 3) = 3$

$\cos(x + \tan^{-1} 3) = \dfrac{3}{\sqrt{10}}$

$x \doteqdot 306°52'$

Testing 90° and 270° gives a second solution of 270°

9a $C\sin(x+\alpha) = C\sin x\cos\alpha + C\cos x\sin\alpha = \sqrt{5}\sin x + 2\cos x$

Equating coefficients gives

$C\cos\alpha = \sqrt{5} \quad (1)$

$C\sin\alpha = 2 \quad (2)$

$C^2\sin^2\alpha + C^2\cos^2\alpha = 4 + 5 \qquad (1)^2 + (2)^2$

$A^2(\sin^2\alpha + \cos^2\alpha) = 9$

$C^2 = 9$

$C = 3$

$\tan\alpha = \frac{2}{\sqrt{5}} \qquad (1) \div (2)$

$\alpha = \tan^{-1}\dfrac{2}{\sqrt{5}}$

$\sqrt{5}\sin x + 2\cos x = 3\sin\left(x + \tan^{-1}\frac{2}{\sqrt{5}}\right)$

9b $\sqrt{5}\sin x + 2\cos x = -2$

$3\sin\left(x + \tan^{-1}\frac{2}{\sqrt{5}}\right) = -2$

$\sin\left(x + \tan^{-1}\frac{2}{\sqrt{5}}\right) = -\frac{2}{3}$

$x = 360° + \sin^{-1}\left(-\frac{2}{3}\right) - \tan^{-1}\frac{2}{\sqrt{5}} \doteqdot 276°23'$

Testing $0°, 180°$ and $360°$ gives a second solution of $x = 180°$

10a $3\sin x + 5\cos x = 4$

$\sqrt{34}\sin\left(x + \tan^{-1}\frac{5}{3}\right) = 4$

$\sin\left(x + \tan^{-1}\frac{5}{3}\right) = \frac{4}{\sqrt{34}}$

$x = 180° - \sin^{-1}\frac{4}{\sqrt{34}} - \tan^{-1}\frac{5}{3}, 360° + \sin^{-1}\frac{4}{\sqrt{34}} - \tan^{-1}\frac{5}{3}$

$x \doteqdot 77°39'$ or $344°17'$

10b $6\sin x - 5\cos x = 7$

$\sqrt{61}\sin\left(x - \tan^{-1}\frac{5}{6}\right) = 7$

$\sin\left(x - \tan^{-1}\frac{5}{6}\right) = \frac{7}{\sqrt{61}}$

$x = \sin^{-1}\frac{7}{\sqrt{61}} + \tan^{-1}\frac{6}{5}, 180° - \sin^{-1}\frac{7}{\sqrt{61}} + \tan^{-1}\frac{6}{5}$

$x \doteqdot 103°29'$ or $156°8'$

10c $7\cos x - 2\sin x = 5$

$\sqrt{7^2+2^2}\cos\left(x+\tan^{-1}\frac{2}{7}\right) = 5$

$\cos\left(x+\tan^{-1}\frac{2}{7}\right) = \frac{5}{\sqrt{53}}$

$x = \cos^{-1}\frac{5}{\sqrt{53}} - \tan^{-1}\frac{2}{7}, 360° - \cos^{-1}\frac{5}{\sqrt{53}} - \tan^{-1}\frac{2}{7}$

$x \doteqdot 30°41'$ or $297°26'$

10d $9\cos x + 7\sin x = 3$

$\sqrt{130}\sin\left(x+\tan^{-1}\frac{9}{7}\right) = 3$

$\sin\left(x+\tan^{-1}\frac{9}{7}\right) = \frac{3}{\sqrt{130}}$

$x = 180° - \sin^{-1}\frac{3}{\sqrt{130}} - \tan^{-1}\frac{9}{7}, 360° + \sin^{-1}\frac{3}{\sqrt{130}} - \tan^{-1}\frac{9}{7}$

$x \doteqdot 112°37'$ or $323°8'$

11a $A\sin\alpha = 1$ (1)

$A\cos\alpha = -\sqrt{3}$ (2)

$A^2\sin^2\alpha + A^2\cos^2\alpha = 1+3$ $(1)^2+(2)^2$

$A^2(\sin^2\alpha+\cos^2\alpha) = 4$

$A^2 = 4$

$A = 2$

$\tan\alpha = -\frac{1}{\sqrt{3}}$ $(1)\div(2)$

$\alpha = \frac{5\pi}{6}$

11b $A\sin\alpha = -5$ (1)

$A\cos\alpha = -5$ (2)

$A^2\sin^2\alpha + A^2\cos^2\alpha = 5^2+5^2$ $(1)^2+(2)^2$

$A^2(\sin^2\alpha + \cos^2\alpha) = 50$

$A^2 = 50$

$A = 5\sqrt{2}$

$\tan\alpha = 1$ $\quad$ (1) ÷ (2)

$\alpha = \dfrac{\pi}{4}$

12a $\quad A\sin\alpha = -4$ $\quad$ (1)

$A\cos\alpha = 5$ $\quad$ (2)

$A^2\sin^2\alpha + A^2\cos^2\alpha = 4^2 + 5^2$ $\quad$ $(1)^2 + (2)^2$

$A^2(\sin^2\alpha + \cos^2\alpha) = 41$

$A^2 = 41$

$A = \sqrt{41}$

$\tan\alpha = -\frac{4}{5}$ $\quad$ (1) ÷ (2)

$\alpha \doteqdot 321°21'$

12b $\quad A\sin\alpha = -11$ $\quad$ (1)

$A\cos\alpha = -2$ $\quad$ (2)

$A^2\sin^2\alpha + A^2\cos^2\alpha = 11^2 + 2^2$ $\quad$ $(1)^2 + (2)^2$

$A^2(\sin^2\alpha + \cos^2\alpha) = 125$

$A^2 = 125$

$A = 5\sqrt{5}$

$\tan\alpha = \frac{11}{2}$ $\quad$ (1) ÷ (2)

$\alpha \doteqdot 259°42'$

13a i $A\cos(x+\theta) = A\cos x\cos\theta - A\sin x\sin\theta \equiv \sqrt{3}\cos x + \sin x$

Equating coefficients gives

$A\cos\theta = \sqrt{3}$ (1)

$A\sin\theta = -1$ (2)

$A^2\sin^2\theta + A^2\cos^2\theta = \sqrt{3}^2 + (-1)^2$ $\quad (1)^2 + (2)^2$

$A^2(\sin^2\theta + \cos^2\theta) = 4$

$A^2 = 4$

$A = 2$

$\tan\theta = -\frac{1}{\sqrt{3}}$

$\theta = \frac{11\pi}{6}$

Hence $\sqrt{3}\cos x + \sin x = 2\cos\left(x + \frac{11\pi}{6}\right)$

13a ii $\sqrt{3}\cos x + \sin x = 1$

$2\cos\left(x + \frac{11\pi}{6}\right) = 1$

Since $0 \le x < 2\pi, \frac{11\pi}{6} \le x + \frac{11\pi}{6} < \frac{23\pi}{6}$

$x + \frac{11\pi}{6} = \frac{7\pi}{3}, \frac{11\pi}{3}$

$x = \frac{\pi}{2}, \frac{11\pi}{6}$

13b i $B\sin(x+\alpha)$

$= B\sin x\cos a + B\cos x\sin\alpha$

$\equiv \cos x - \sin x$

$= -\sin x + \cos x$

$B\cos\alpha = B\cos\alpha = -1$ $\quad$ (1)

$B\sin\alpha = 1$ $\quad$ (2)

$B^2\cos^2\alpha + B^2\sin^2\alpha = 1 + 1$ $\quad (1)^2 + (2)^2$

$B^2(\cos^2\alpha + \cos^2\alpha) = 1 + 1$

$B^2 = 2$

$B = \sqrt{2}$

$\tan\alpha = -1 \quad (2) \div 1$

$\alpha = \dfrac{3\pi}{4}$

Hence

$\sqrt{2}\sin\left(x + \dfrac{3\pi}{4}\right) = \cos x - \sin x$

13b ii $\cos x - \sin x = 1$

$\sqrt{2}\sin\left(x + \dfrac{3\pi}{4}\right) = 1$

$\sin\left(x + \dfrac{3\pi}{4}\right) = \dfrac{1}{\sqrt{2}}$

Now, note that since $0 \le x < 2\pi$, $0 \le x + \frac{3\pi}{4} < \frac{11\pi}{4}$

Hence

$x + \dfrac{3\pi}{4} = \dfrac{3\pi}{4}, \dfrac{9\pi}{4}$

$x = 0, \dfrac{3\pi}{2}$

13c i $C\sin(x + \beta)$

$= C\sin x\cos\beta + C\cos x\sin\beta$

$= \sin x - \sqrt{3}\cos x$

Equating coefficients gives

$C\cos\beta = 1 \quad (1)$

$C\sin\beta = -\sqrt{3} \quad (2)$

$C^2\sin^2\beta + C^2\cos^2\beta = 1^2 + \left(-\sqrt{3}\right)^2 \quad (1)^2 + (2)^2$

$C^2(\sin^2\beta + \cos^2\beta) = 4$

$C^2 = 4$

$C = 2$

$\tan\beta = -\sqrt{3}$ (2) ÷ (1)

$\beta = \dfrac{5\pi}{3}$

Hence $\sin x - \sqrt{3}\cos x = 2\cos\left(x + \frac{5\pi}{3}\right)$

13c ii $\sin x - \sqrt{3}\cos x = -1$

$2\cos\left(x + \dfrac{5\pi}{3}\right) = -1$

$\cos\left(x + \dfrac{5\pi}{3}\right) = -\dfrac{1}{2}$

Now, note that since $0 \le x < 2\pi$, $0 \le x + \frac{5\pi}{3} < \frac{11\pi}{3}$

$x + \dfrac{5\pi}{3} = \dfrac{11\pi}{3}, \dfrac{19\pi}{3}$

$x = \dfrac{\pi}{6}, \dfrac{3\pi}{2}$

13d i $D\cos(x - \phi) = D\cos x\cos\phi + D\sin x\sin\beta \equiv -\cos x - \sin x$

Equating coefficients gives

$D\cos\phi = -1$ (1)

$D\sin\phi = -1$ (2)

$D^2\sin^2\phi + D^2\cos^2\phi = 1^2 + (1)^2$ $(1)^2 + (2)^2$

$D^2(\sin^2\phi + \cos^2\phi) = 2$

$D^2 = 2$

$D = \sqrt{2}$

$\tan\phi = 1$ (2) ÷ (1)

$\phi = \dfrac{\pi}{4}$

Hence $-\cos x - \sin x = \sqrt{2}\cos\left(x - \frac{5\pi}{4}\right)$

13d ii $-\cos x - \sin x = 1$

$$\sqrt{2}\cos\left(x - \frac{5\pi}{4}\right) = 1$$

$$\cos\left(x - \frac{5\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

Now, note that since $0 \le x < 2\pi$, $-\frac{5\pi}{4} \le x - \frac{5\pi}{4} < \frac{3\pi}{4}$

$$x - \frac{5\pi}{4} = -\frac{\pi}{4}, \frac{\pi}{4}$$

$$x = \pi, \frac{3\pi}{2}$$

14a i $R\sin(x + \alpha)$

$= R\sin x\cos\alpha + R\cos x\sin\alpha$

$\equiv 2\cos x - \sin x$

$= -\sin x + 2\cos x$

$R\sin\alpha = 2$ (1)

$R\cos\alpha = -1$ (2)

$\tan\alpha = -2$ (1) ÷ (2)

$\alpha = -\tan^{-1}\frac{1}{2}$

$R^2\sin^2\alpha + R^2\cos^2\alpha = 1 + 4$ $(1)^2 + (2)^2$

$R^2(\sin^2\alpha + \cos^2\alpha) = 5$

$R^2 = 5$

$R = \sqrt{5}$

Hence $2\cos x - \sin x = \sqrt{5}\sin(x - \tan^{-1} 2)$

14a ii $2\cos x - \sin x = 1$

$$\sqrt{5}\sin(x - \tan^{-1} 2) = 1$$

$$\sin(x - \tan^{-1} 2) = \frac{1}{\sqrt{5}}$$

$x = -\sin^{-1}\frac{1}{\sqrt{5}} + \tan^{-1}2\,, 180° + \sin^{-1}\frac{1}{\sqrt{5}} + \tan^{-1}2$

$\quad = 36°52', 270°$

14b i $\quad S\cos(x-\beta)$

$= S\cos x\cos\beta + S\sin x\sin\beta$

$\equiv -3\sin x - 4\cos x$

$= -4\cos x - 3\sin x$

$S\sin\beta = -3 \qquad (1)$

$S\cos\beta = -4 \qquad (2)$

$\tan\beta = \frac{3}{4} \qquad (1)\div(2)$

$\beta = \tan^{-1}\frac{3}{4}$

$S^2\sin^2\beta + S^2\cos^2\beta = 9 + 16 \qquad (1)^2 + (2)^2$

$S^2(\sin^2\beta + \cos^2\beta) = 25$

$S^2 = 25$

$S = 5$

Hence $-3\sin x - 4\cos x = 5\cos\left(x - \tan^{-1}\frac{3}{4}\right)$

14b ii $\quad -3\sin x - 4\cos x = 2$

$5\cos\left(x - \tan^{-1}\frac{3}{4}\right) = 2$

$\cos\left(x - \tan^{-1}\frac{3}{4}\right) = \frac{2}{5}$

$x = \pi + \cos^{-1}\frac{2}{5} + \tan^{-1}\frac{3}{4}, \pi + \cos^{-1}\frac{2}{5} + \tan^{-1}\frac{3}{4}$

$\quad \doteqdot 2.63, 4.94$

15a $\quad 2\sec x - 2\tan x = 5$

Multiplying through $\cos x$

$2 - 2\sin x = 5\cos x$

$2\sin x + 5\cos x = 2$

$\sqrt{2^2 + 5^2}\sin\left(x + \tan^{-1}\frac{5}{2}\right) = 2$

$\sqrt{29}\sin\left(x + \tan^{-1}\frac{5}{2}\right) = 2$

$\sin\left(x + \tan^{-1}\frac{5}{2}\right) = \frac{2}{\sqrt{29}}$

$x = 360° + \sin^{-1}\frac{2}{\sqrt{29}} - \tan^{-1}\frac{5}{2}$

$\quad \doteqdot 313°36'$

15b $\quad 2\operatorname{cosec} x + 5\cot x = 3$

Multiplying through by $\sin x$

$2 + 5\cos x = 3\sin x$

$3\sin x - 5\cos x = 2$

$\sqrt{3^2 + 5^2}\sin\left(x - \tan^{-1}\frac{5}{3}\right) = 2$

$\sqrt{34}\sin\left(x - \tan^{-1}\frac{5}{3}\right) = 2$

$\sin\left(x - \tan^{-1}\frac{5}{3}\right) = \frac{2}{\sqrt{34}}$

$x = \sin^{-1}\frac{2}{\sqrt{34}} + \tan^{-1}\frac{5}{3}, 180° - \sin^{-1}\frac{2}{\sqrt{34}} + \tan^{-1}\frac{5}{3}$

$\quad \doteqdot 79°6', 218°59'$

16a $\quad \sin\theta + \cos\theta = \cos 2\theta$

$\sin\theta + \cos\theta = \cos^2\theta - \sin^2\theta$

$(\sin\theta + \cos\theta) = (\cos\theta - \sin\theta)(\cos\theta + \sin\theta)$

$0 = (\cos\theta - \sin\theta)(\cos\theta + \sin\theta) - (\sin\theta + \cos\theta)$

$0 = (\cos\theta - \sin\theta - 1)(\cos\theta + \sin\theta)$

Hence $\cos\theta - \sin\theta - 1 = 0$ or $\cos\theta + \sin\theta = 0$

Thus

$\cos\theta - \sin\theta - 1 = 0$

and

$\cos\theta - \sin\theta = 1$

or

$\cos\theta + \sin\theta = 0$

So

$\sin\theta = -\cos\theta$

$\tan\theta = -1$

16b For $\tan\theta = -1$ the solutions are $\theta = \frac{3\pi}{4}, \frac{7\pi}{4}$

For $\cos\theta - \sin\theta = 1$, $\sqrt{2}\cos\left(x + \frac{\pi}{4}\right) = 1$ which has solutions $x = 0, \frac{3\pi}{2}$.

Hence the solutions are $\theta = 0, \frac{3\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}$

17a $\sin x - \cos x = \sqrt{1.5}$

$$\sqrt{2}\cos\left(x + \frac{\pi}{4}\right) = \sqrt{1.5}$$

$$\cos\left(x + \frac{\pi}{4}\right) = \frac{\sqrt{3}}{2}$$

$$x = \frac{7\pi}{12}, \frac{11\pi}{12}$$

17b $\sqrt{3}\sin 2x - \cos 2x = 2$

$$\sqrt{4}\cos\left(x + \frac{\pi}{4}\right) = 2$$

$$\cos\left(2x + \frac{\pi}{4}\right) = \frac{1}{2}$$

$$2x + \frac{\pi}{4} = \frac{11\pi}{12}, \frac{35\pi}{12}$$

$$x = \frac{\pi}{3}, \frac{4\pi}{3}$$

17c

$$\sqrt{2}\cos\left(4x + \tan^{-1} - \frac{1}{1}\right) = 1$$

$$\cos\left(4x - \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

$$4x - \frac{\pi}{4} = -\frac{\pi}{4}, \frac{\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}, \frac{15\pi}{4}, \frac{17\pi}{4}, \frac{23\pi}{4}, \frac{25\pi}{4}, \frac{31\pi}{4}$$

$$x = 0, \frac{\pi}{8}, \frac{\pi}{2}, \frac{5\pi}{8}, \pi, \frac{9\pi}{8}, \frac{3\pi}{2}, \frac{13\pi}{8}, 2\pi$$

18a $A\cos(2x - \alpha)$

$= A\cos 2x\cos\alpha + A\sin 2x\sin\alpha$
$= (\sqrt{3}+1)\cos 2x + (\sqrt{3}-1)\sin 2x$

Equating coefficients gives

$A\cos\alpha = (\sqrt{3}+1)$ (1)

$A\sin\alpha = (\sqrt{3}-1)$ (2)

$A^2\sin^2\alpha + A^2\cos^2\alpha = (\sqrt{3}+1)^2 + (\sqrt{3}-1)^2$ $\quad (1)^2 + (2)^2$

$A^2(\sin^2\alpha + \cos^2\alpha) = 3 + 2\sqrt{3} + 1 + 3 - 2\sqrt{3} + 1$

$A^2 = 8$

$A = 2\sqrt{2}$

(1) ÷ (2) gives

$\tan\alpha = \frac{\sqrt{3}-1}{\sqrt{3}+1}$

$$= \frac{(\sqrt{3}-1)^2}{(\sqrt{3}+1)(\sqrt{3}-1)}$$

$$= \frac{3 - 2\sqrt{3} + 1}{2}$$

$$= 2 - 2\sqrt{3}$$

$\alpha = \tan^{-1}(2 - 2\sqrt{3}) = \frac{\pi}{12}$

$(\sqrt{3}+1)\cos 2x + (\sqrt{3}-1)\sin 2x = 2$

$2\sqrt{2}\cos(2x - \frac{\pi}{12}) = 2$

$\cos(2x - \frac{\pi}{12}) = \frac{1}{\sqrt{2}}$

$2x - \frac{\pi}{12} = -\frac{7\pi}{4}, -\frac{\pi}{4}, \frac{\pi}{4}, \frac{7\pi}{4}$

$x = -\frac{5\pi}{6}, -\frac{\pi}{12}, \frac{\pi}{6}, \frac{11\pi}{12}$

19a i $A\sin(x - \alpha) = A\sin x\cos\alpha - A\cos x\sin\alpha = \sin x - \cos x$

Equating coefficients gives

$A\cos\alpha = 1 \quad (1)$

$A\sin\alpha = 1 \quad (2)$

$A^2\sin^2\alpha + A^2\cos^2\alpha = 1^2 + 1^2 \qquad (1)^2 + (2)^2$

$A^2(\sin^2\alpha + \cos^2\alpha) = 2$

$A^2 = 2$

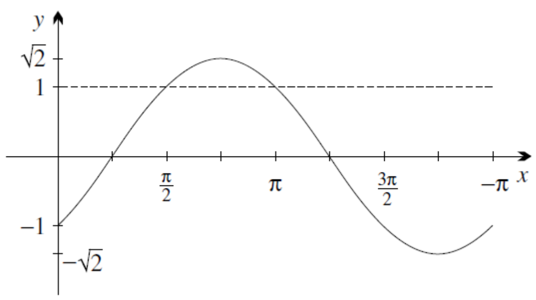
$A = \sqrt{2}$

$\tan\alpha = 1 \qquad (1) \div (2)$

$\alpha = \frac{\pi}{4}$

Hence $\sin x - \cos x = \sqrt{2}\sin\left(x - \frac{\pi}{4}\right)$

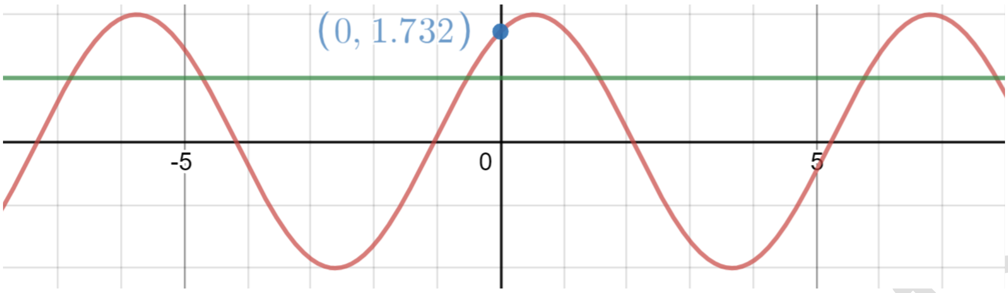
19a ii



19a iii $\frac{\pi}{2} < x < \pi$

19b i

$$\sin x + \sqrt{3}\cos x = 2\sin\left(x + \frac{\pi}{3}\right)$$



Noting that the points of intersection of the two graphs are when

$$2\sin\left(x + \frac{\pi}{3}\right) = 1$$

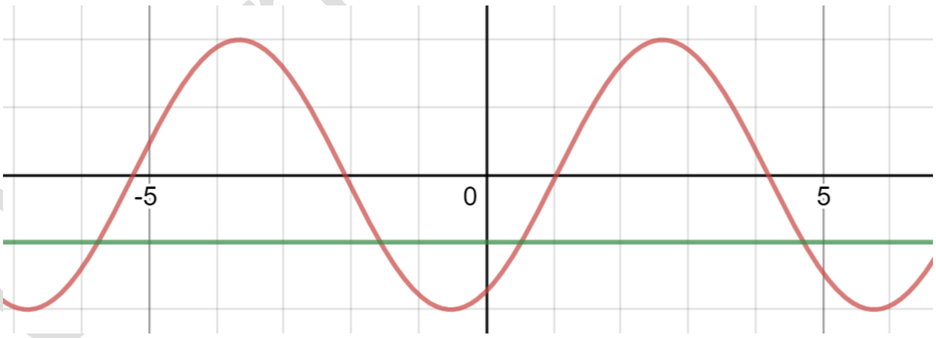
$$x = \frac{\pi}{2}, \frac{11\pi}{6}$$

We can read from the graph that $2\sin\left(x + \frac{\pi}{3}\right) \leq 1$ when $\frac{\pi}{2} \leq x \leq \pi$.

19b ii $\sin x - \sqrt{3}\cos x < -1$

$$2\sin\left(x - \tan^{-1}\sqrt{3}\right) < -1$$

$$\sin\left(x - \tan^{-1}\sqrt{3}\right) < -\frac{1}{2}$$



Noting that the solutions are $\frac{\pi}{6}, \frac{3\pi}{2}, 2\pi$, we see that the inequality holds when $0 \leq x < \frac{\pi}{6}$ or $\frac{3\pi}{2} < x \leq 2\pi$

19b iii $\left|\sqrt{3}\sin x + \cos x\right| < 1$

$$\left|2\sin\left(x+\frac{\pi}{6}\right)\right| < 1$$

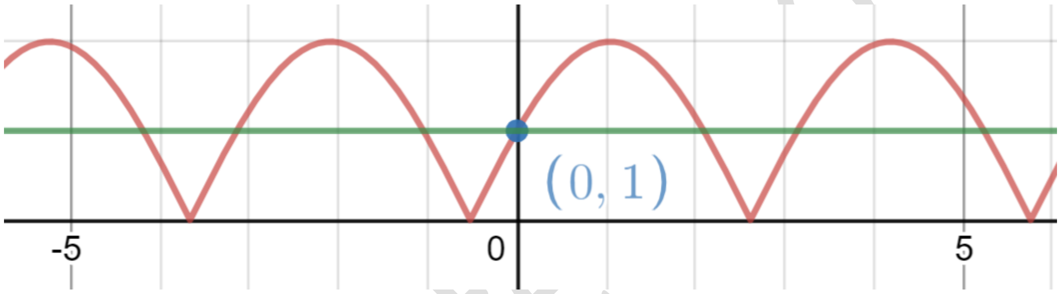
Solving the equation

$$\left|2\sin\left(x+\frac{\pi}{6}\right)\right| = 1$$

$$2\sin\left(x+\frac{\pi}{6}\right) = \pm 1$$

$$\sin\left(x+\frac{\pi}{6}\right) = \pm\frac{1}{2}$$

$$x = \frac{2\pi}{3}, \pi, \frac{5\pi}{3}, 2\pi$$



Hence, by observation of the graph, the inequality is satisfied when

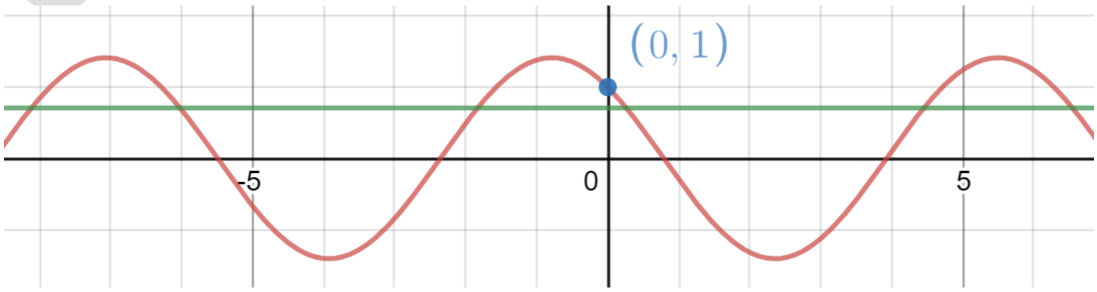
$\frac{2\pi}{3} < x < \pi$ or $\frac{5\pi}{3} < x < 2\pi$

19b iv

$$\cos x - \sin x \geq \frac{1}{2}\sqrt{2}$$

$$2\cos\left(x+\frac{\pi}{4}\right) \geq \frac{1}{2}\sqrt{2}$$

Solving for the intersection of the two graphs gives $x = \frac{\pi}{12}, \frac{17\pi}{12}, 2\pi$



Hence by observation of the graph, the inequality is satisfied when

$0 \leq x \leq \frac{\pi}{12}$ or $\frac{17\pi}{12} \leq x \leq 2\pi$

20a i $\cos\left(\theta - \frac{\pi}{2}\right) = \cos\theta\cos\frac{\pi}{2} + \sin\theta\sin\frac{\pi}{2}$

$= \cos\theta\,(0) + \sin\theta\,(1)$

$= \sin\theta$

20a ii $\sin\left(\theta + \frac{\pi}{2}\right) = \sin\theta\cos\frac{\pi}{2} + \cos\theta\sin\frac{\pi}{2}$

$= \sin\theta\,(0) + \cos\theta\,(1)$

$= \cos\theta$

20b $\sin x + \sqrt{3}\cos x$

$= 2\sin\left(x + \frac{\pi}{3}\right)$

$= 2\sin\left(x + \frac{\pi}{3} - 2\pi\right)$

$= 2\sin(x + \frac{5\pi}{3})$

$\sin x + \sqrt{3}\cos x$

$= 2\sin\left(x + \frac{\pi}{3}\right)$

$= 2\cos\left(x + \frac{\pi}{3} - \frac{\pi}{2}\right)$

$= 2\cos(x - \frac{\pi}{6})$

$\sin x + \sqrt{3}\cos x$

$= 2\sin\left(x + \frac{\pi}{3}\right)$

$= 2\cos\left(x - \frac{\pi}{6}\right)$

$= 2\cos\left(x - \frac{\pi}{6} + 2\pi\right)$

$= 2\cos(x + \frac{11\pi}{6})$

20c $\cos x - \sin x$

$= \sqrt{2}\cos\left(x + \frac{\pi}{4}\right)$

$\cos x - \sin x$

$= \sqrt{2}\cos\left(x + \frac{\pi}{4}\right)$

$= \sqrt{2}\cos\left(x + \frac{\pi}{4} - 2\pi\right)$

$= \sqrt{2}\cos\left(x - \frac{7\pi}{4}\right)$

$\cos x - \sin x$

$= \sqrt{2}\cos\left(x + \frac{\pi}{4}\right)$

$= \sqrt{2}\sin\left(x + \frac{\pi}{4} + \frac{\pi}{2}\right)$

$= \sqrt{2}\sin\left(x + \frac{3\pi}{4}\right)$

$\cos x - \sin x$

$= \sqrt{2}\cos\left(x + \frac{\pi}{4}\right)$

$= \sqrt{2}\sin\left(x + \frac{3\pi}{4}\right)$

$= \sqrt{2}\sin\left(x + \frac{3\pi}{4} - 2\pi\right)$

$= \sqrt{2}\sin\left(x - \frac{5\pi}{4}\right)$

21 $\sin(\theta + \pi)$

$= \sin\theta\cos\pi + \cos\theta\sin\pi$

$= \sin\theta\,(-1) + \cos\theta\,(0)$

$= -\sin\theta$

21b i $-\sqrt{3}\sin x + \cos x$

$= \sqrt{3}\sin(-x) + \cos(-x)$
$= 2\sin\left(-x + \frac{\pi}{6}\right)$
$= 2\sin\left(\pi - \left(-x + \frac{\pi}{6}\right)\right)$
$= 2\sin\left(x + \frac{5\pi}{6}\right)$

21b ii $-\sqrt{3}\sin x - \cos x$

$= -\left(\sqrt{3}\sin x + \cos x\right)$
$= -2\sin\left(x + \frac{\pi}{6}\right)$
$= 2\sin\left(-\left(x + \frac{\pi}{6}\right)\right)$
$= 2\sin\left(\pi + \left(x + \frac{\pi}{6}\right)\right)$
$= 2\sin\left(x + \frac{7\pi}{6}\right)$

21b iii $\sqrt{3}\sin x - \cos x$

$= -\left(-\sqrt{3}\sin x + \cos x\right)$
$= -\left(\sqrt{3}\sin(-x) + \cos(-x)\right)$
$= -2\sin\left(-x + \frac{\pi}{6}\right)$
$= -2\sin\left(-\left(x - \frac{\pi}{6}\right)\right)$
$= 2\sin\left(x - \frac{\pi}{6}\right)$

22a $\cos(x - \alpha) = \cos\beta$

$\cos x\cos\alpha + \sin x\sin\alpha = \cos\beta$

$\cot\alpha + \tan x = \dfrac{\cos\beta}{\sin\alpha\cos x}$

Squaring both sides gives

$$\cot^2\alpha + 2\cot\alpha\tan x + \tan^2 x = \frac{\cos^2\beta}{\sin^2\alpha\cos^2 x}$$

$$\cot^2\alpha + 2\cot\alpha\tan x + \tan^2 x = \frac{\cos^2\beta}{\sin^2\alpha}\sec^2 x$$

$$\cot^2\alpha + 2\cot\alpha\tan x + \tan^2 x = \frac{\cos^2\beta}{\sin^2\alpha}(1+\tan^2 x)$$

$$\left(1-\frac{\cos^2\beta}{\sin^2\alpha}\right)\tan^2 x + 2\cot\alpha\tan x + \cot^2\alpha - \frac{\cos^2\beta}{\sin^2\alpha}$$

Using the quadratic formula

$\tan x$

$$= \frac{-2\cot\alpha \pm \sqrt{(2\cot\alpha)^2 - 4\left(1-\frac{\cos^2\beta}{\sin^2\alpha}\right)\left(\cot^2\alpha - \frac{\cos^2\beta}{\sin^2\alpha}\right)}}{2\left(1-\frac{\cos^2\beta}{\sin^2\alpha}\right)}$$

$$= \frac{-2\cot\alpha \pm \sqrt{\left(\frac{2\cos\alpha}{\sin\alpha}\right)^2 - 4\left(\frac{\sin^2\alpha-\cos^2\beta}{\sin^2\alpha}\right)\left(\frac{\cos^2\alpha-\cos^2\beta}{\sin^2\alpha}\right)}}{2\left(1-\frac{\cos^2\beta}{\sin^2\alpha}\right)}$$

$$= \frac{-2\cot\alpha \pm 2\sqrt{\left(\frac{\cos\alpha}{\sin\alpha}\right)^2 - \left(\frac{\sin^2\alpha-\cos^2\beta}{\sin^2\alpha}\right)\left(\frac{\cos^2\alpha-\cos^2\beta}{\sin^2\alpha}\right)}}{2\left(1-\frac{\cos^2\beta}{\sin^2\alpha}\right)}$$

$$= \frac{-\cot\alpha \pm \sqrt{\left(\frac{\cos^2\alpha\sin^2\alpha - \sin^2\alpha\cos^2\alpha + \cos^2\alpha\cos^2\beta + \sin^2\alpha\cos^2\beta - \cos^4\beta}{\sin^4\alpha}\right)}}{\left(1-\frac{\cos^2\beta}{\sin^2\alpha}\right)}$$

$$= \frac{-\cot\alpha \pm \sqrt{\left(\frac{\cos^2\alpha\cos^2\beta + \sin^2\alpha\cos^2\beta - \cos^4\beta}{\sin^4\alpha}\right)}}{\left(1-\frac{\cos^2\beta}{\sin^2\alpha}\right)}$$

$$= \frac{-\cot\alpha \pm \sqrt{\left(\frac{(\sin^2\alpha + \cos^2\alpha)\cos^2\beta - \cos^4\beta}{\sin^4\alpha}\right)}}{\left(1-\frac{\cos^2\beta}{\sin^2\alpha}\right)}$$

$$= \frac{-\cot\alpha \pm \sqrt{\left(\frac{\cos^2\beta - \cos^4\beta}{\sin^4\alpha}\right)}}{\left(1 - \frac{\cos^2\beta}{\sin^2\alpha}\right)}$$

$$= \frac{-\frac{\cos\alpha}{\sin\alpha} \pm \frac{\cos\beta}{\sin^2\alpha}\sqrt{1-\cos^2\beta}}{\left(1 - \frac{\cos^2\beta}{\sin^2\alpha}\right)}$$

$$= \frac{-\frac{\cos\alpha}{\sin\alpha} \pm \frac{\cos\beta\sin\beta}{\sin^2\alpha}}{\left(1 - \frac{\cos^2\beta}{\sin^2\alpha}\right)}$$

$$= \frac{-\sin\alpha\cos\alpha \pm \cos\beta\sin\beta}{\sin^2\alpha - \cos^2\beta}$$

$$= \frac{\sin\alpha\cos\alpha \pm \cos\beta\sin\beta}{\cos^2\beta - \sin^2\alpha}$$

$$= \frac{\sin(\alpha \pm \beta)}{\cos(\alpha+\beta)} = \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}, \frac{\sin(\alpha-\beta)}{\cos(\alpha+\beta)}$$

$$= \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}, \frac{\sin(\alpha-\beta)}{\cos(\alpha-\beta)}$$

$$= \tan(\alpha+\beta), \tan(\alpha-\beta)$$

22b $A\cos(x-\theta) = A\cos x\cos\theta + A\sin x\sin\theta = 2\cos x + 11\sin x$

Equating coefficients gives

$A\cos\theta = 2 \quad (1)$

$A\sin\theta = 11 \quad (2)$

$A^2\sin^2\theta + A^2\cos^2\theta = 4 + 11^2$

$A^2(\sin^2\theta + \cos^2\theta) = 125$

$A^2 = 125$

$A = 5\sqrt{5}$

$\tan\theta = \frac{11}{2} \qquad (2) \div (1)$

$\theta = \tan^{-1}\frac{11}{2}$

Hence $2\cos x + 11\sin x = 5\sqrt{5}\cos\left(x - \tan^{-1}\frac{11}{2}\right)$

22c i $2\cos x + 11\sin x = 10$

$$5\sqrt{5}\cos\left(x - \tan^{-1}\frac{11}{2}\right) = 10$$

$$\cos\left(x - \tan^{-1}\frac{11}{2}\right) = \frac{2}{\sqrt{5}}$$

$$\cos\left(x - \tan^{-1}\frac{11}{2}\right) = \cos\left(\cos^{-1}\frac{2}{\sqrt{5}}\right)$$

Hence, from part a the solutions are

$$\begin{aligned} x &= \tan(\alpha + \beta) \\ &= \tan\left(\tan^{-1}\tfrac{11}{2} + \cos^{-1}\tfrac{2}{\sqrt{5}}\right) \\ &= \frac{\tan\left(\tan^{-1}\frac{11}{2}\right) + \tan\left(\cos^{-1}\frac{2}{\sqrt{5}}\right)}{1 - \tan\left(\tan^{-1}\frac{11}{2}\right)\tan\left(\cos^{-1}\frac{2}{\sqrt{5}}\right)} \\ &= \frac{\frac{11}{2} + \frac{1}{2}}{1 - \left(\frac{11}{2}\right)\left(\frac{1}{2}\right)} \\ &= \tfrac{4}{3} \end{aligned}$$

or

$$\begin{aligned} x &= \tan(\alpha - \beta) \\ &= \tan\left(\tan^{-1}\tfrac{11}{2} - \cos^{-1}\tfrac{2}{\sqrt{5}}\right) \\ &= \frac{\tan\left(\tan^{-1}\frac{11}{2}\right) - \tan\left(\cos^{-1}\frac{2}{\sqrt{5}}\right)}{1 + \tan\left(\tan^{-1}\frac{11}{2}\right)\tan\left(\cos^{-1}\frac{2}{\sqrt{5}}\right)} \\ &= \frac{\frac{11}{2} - \frac{1}{2}}{1 + \left(\frac{11}{2}\right)\left(\frac{1}{2}\right)} \\ &= -\tfrac{24}{7} \end{aligned}$$

22c ii $\tan x = \frac{4}{3}$, hence $x = \tan^{-1}\frac{4}{3}$

$\tan x = -\frac{24}{7}$, hence $x = \pi - \tan^{-1}\frac{24}{7}$

Thus, the roots are $\tan^{-1}\frac{4}{3}$ and $\pi - \tan^{-1}\frac{24}{7}$

22c iii

$$\tan\left(2\tan^{-1}\frac{4}{3}\right)$$

$$= \tan\left(\tan^{-1}\frac{4}{3} + \tan^{-1}\frac{4}{3}\right)$$

$$= \frac{\tan\tan^{-1}\frac{4}{3} + \tan\tan^{-1}\frac{4}{3}}{1 - \tan\tan^{-1}\frac{4}{3}\tan\tan^{-1}\frac{4}{3}}$$

$$= \frac{\frac{4}{3} + \frac{4}{3}}{1 - \left(\frac{4}{3}\right)\left(\frac{4}{3}\right)}$$

$$= -\frac{24}{7}$$

$$= \tan\left(\pi - \tan^{-1}\frac{24}{7}\right)$$

Thus, it follows that

$$2\tan^{-1}\frac{4}{3} = \pi - \tan^{-1}\frac{24}{7}$$

And thus, one root is twice the other.

Solutions to Exercise 11C

1a $\cos x - \sin x = 1$

$\frac{1-t^2}{1+t^2} - \frac{2t}{1+t^2} = 1$

$1 - t^2 - 2t = 1 + t^2$

$2t^2 + 2t = 0$

$t^2 + t = 0$

1b $t(t+1) = 0$

$t = -1$ or 0, hence

$\tan\frac{x}{2} = -1$ or 0

1c $t = -1$ or 0, hence

$\tan\frac{x}{2} = -1$ or 0

$x = 0, \frac{3\pi}{2}, 2\pi$

Now, testing points where $\tan\frac{1}{2}x$ is undefined which is where $x = \pi$ the solutions are

$x = 0, \frac{3\pi}{2}, 2\pi$

2a $\sqrt{3}\sin x + \cos x = 1$

$\sqrt{3} \times \frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2} = 1$

$2\sqrt{3}t + 1 - t^2 = 1 + t^2$

$2\sqrt{3}t - 2t^2 = 0$

$2\sqrt{3}t = 2t^2$

$t^2 = \sqrt{3}t$

2b $t^2 - \sqrt{3}t = 0$

$t(t - \sqrt{3}) = 0$

$t = 0$ or $\sqrt{3}$

$\tan\frac{x}{2} = 0$ or $\sqrt{3}$

Now, testing points where $\tan\frac{1}{2}x$ is undefined which is where $x = \pi$ the solutions are

$x = 0, \frac{2\pi}{3}, 2\pi$

3a $4\cos x + \sin x = 1$

Let $t = \tan\frac{1}{2}x$

$4\left(\frac{1 - t^2}{1 + t^2}\right) + \frac{2t}{1 + t^2} = 1$

$4(1 - t^2) + 2t = 1 + t^2$

$4 - 4t^2 + 2t = 1 + t^2$

$5t^2 - 2t - 3 = 0$

$(5t + 3)(t - 1) = 0$

3b $(5t + 3)(t - 1) = 0$

$t = 1$ or $-\frac{3}{5}$

So $\tan\frac{1}{2}x = 1$ or $-\frac{3}{5}$

Now, testing points where $\tan\frac{1}{2}x$ is undefined which is where $x = 180°$ the solutions are

$x = 90°$ or $x \doteqdot 298°4'$

4a $3\sin x - 2\cos x = 2$

$$3\left(\frac{2t}{1+t^2}\right) - 2\left(\frac{1-t^2}{1+t^2}\right) = 2$$

$$6t - 2(1-t^2) = 2(1+t^2)$$

$$6t - 2 + 2t^2 = 2 + 2t^2$$

$$6t - 4 = 0$$

$$3t - 2 = 0$$

4b

$$t = \frac{2}{3}$$

$$\tan\frac{1}{2}x = \frac{2}{3}$$

$$\frac{1}{2}x = \tan^{-1}\frac{2}{3}$$

$$x = 2\tan^{-1}\frac{2}{3}$$

$$\doteqdot 67°23'$$

Now, testing points where $\tan\frac{1}{2}x$ is undefined which is where $x = 180°$ the solutions are

$x = 180°$ or $x \doteqdot 67°23'$

5a $6\sin x - 4\cos x = 5$

$$6\left(\frac{2t}{1+t^2}\right) - 4\left(\frac{1-t^2}{1+t^2}\right) = 5$$

$$12t - 4 + 4t^2 = 5(1+t^2)$$

$$12t - 4 + 4t^2 = 5 + 5t^2$$

$$t^2 - 12t + 9 = 0$$

5b Using the quadratic formula

$$\tan\frac{1}{2}x$$
$$= t$$
$$= \frac{-(-12) \pm \sqrt{(-12)^2 - 4 \times 1 \times 9}}{2}$$
$$= \frac{12 \pm \sqrt{108}}{2}$$
$$= \frac{12 \pm 6\sqrt{3}}{2}$$
$$= 6 \pm 3\sqrt{3}$$

5c $\tan\frac{1}{2}x = 6 \pm 3\sqrt{3}$

$$\frac{1}{2}x = \tan^{-1}(6 \pm 3\sqrt{3})$$

Now, testing points where $\tan\frac{1}{2}x$ is undefined which is where $x = 180°$ the solutions are

$$x = 2\tan^{-1}\left(6 \pm 3\sqrt{3}\right)$$
$$= 77°35' \text{ or } 169°48'$$

6 Note for all following parts, as $0° \le x \le 360°$, $0° \le \frac{x}{2} \le 180°$

6a $5\sin x + 4\cos x = 5$

$$5\left(\frac{2t}{1+t^2}\right) + 4\left(\frac{1-t^2}{1+t^2}\right) = 5$$
$$10t + 4(1-t^2) = 5(1+t^2)$$
$$10t + 4 - 4t^2 = 5 + 5t^2$$
$$9t^2 - 10t + 1 = 0$$
$$(9t-1)(t-1) = 0$$
$$t = 1 \text{ or } 9$$
$$\tan\frac{1}{2}x = 1 \text{ or } \tan\frac{1}{2}x = 9$$

$\frac{1}{2}x = 45°$ or $\frac{1}{2}x = 6°20'25$

So $x = 90°$ or $x \doteqdot 12°41'$

6b $7\cos x - 6\sin x = 2$

$7\left(\frac{1-t^2}{1+t^2}\right) - 6\left(\frac{2t}{1+t^2}\right) = 2$

$7 - 7t^2 - 12t = 2 + 2t^2$

$9t^2 + 12t - 5 = 0$

$(3t-1)(3t+5) = 0$

Hence $t = \frac{1}{3}$ or $-\frac{5}{3}$

$\tan\frac{x}{2} = \frac{1}{3}$ or $-\frac{5}{3}$

$x \doteqdot 36°52', 241°56'$

6c

$3\left(\frac{2t}{1+t^2}\right) - 2\left(\frac{1-t^2}{1+t^2}\right) = 1$

$6t - 2 + 2t^2 = 1 + t^2$

$t^2 + 6t - 3 = 0$

Using the quadratic formula gives

$t = \frac{-6 \pm \sqrt{6^2 - 4 \times 1 \times -3}}{2}$

$= \frac{-6 \pm \sqrt{48}}{2}$

$= \frac{-6 \pm 4\sqrt{3}}{2}$

$= -3 \pm 2\sqrt{3}$

$\tan\frac{x}{2} = -3 \pm 2\sqrt{3}$

$x \doteqdot 49°48', 197°35'$

6d $5\cos x + 6\sin x = -5$

$$5\left(\frac{1-t^2}{1+t^2}\right) + 6\left(\frac{2t}{1+t^2}\right) = -5$$

$$5 - 5t^2 + 12t = -5 - 5t^2$$

$$12t = -10$$

$$t = -\frac{5}{6}$$

$$\tan\frac{x}{2} = -\frac{5}{6}$$

$x \doteqdot 100°23'$ or $x \doteqdot 280°23'$

However, after substitution we find that $x \doteqdot 100°23'$ is not a solution.

Since the terms in t^2 have cancelled out, we need to check $t = 180°$.

$$\begin{aligned} \text{LHS} &= 5\cos 180° + 6\sin 180° \\ &= 5 \times -1 + 6 \times 0 \\ &= -5 \\ &= \text{RHS} \end{aligned}$$

So the solutions are $x = 180°$ or $x \doteqdot 280°23'$.

7 $8\tan\theta - 4\sec\theta = 1$

$$\frac{8\sin\theta}{\cos\theta} - \frac{4}{\cos\theta} = 1$$

$$8\sin\theta - 4 = \cos\theta$$

$$8\left(\frac{2t}{1+t^2}\right) - 4 = \left(\frac{1-t^2}{1+t^2}\right)$$

$$16t - 4 - 4t^2 = 1 - t^2$$

$$3t^2 - 16t + 5 = 0$$

Using the quadratic formula

$$\begin{aligned} t &= \frac{-(-16) \pm \sqrt{(-16)^2 - 4 \times 3 \times 5}}{2 \times 3} \\ &= \frac{16 \pm 14}{6} \end{aligned}$$

$= 5, \frac{1}{3}$

$\tan\frac{1}{2}x = 5, \frac{1}{3}$

$x = 2\tan^{-1} 5\,, 2\tan^{-1}\frac{1}{3}$

8 $2\sin 2x + \cos 2x = 2$

Let $t = \tan x$

$2\left(\frac{2t}{1+t^2}\right) + \frac{1-t^2}{1+t^2} = 2$

$4t + 1 - t^2 = 2 + 2t^2$

$3t^2 - 4t + 1 = 0$

$(3t-1)(t-1) = 0$

$t = 1 \text{ or } \frac{1}{3}$

$\tan x = 1 \text{ or } \frac{1}{3}$

$x = 45°, 225° \text{ or } x \doteqdot 18.4°, 198.4°$

9a $a\cos x = 1 + \sin x$

$a\left(\frac{1-t^2}{1+t^2}\right) = 1 + \frac{2t}{1+t^2}$

$a - at^2 = 1 + t^2 + 2t$

$t^2 + at^2 + 2t + 1 - a = 0$

$(1+a)t^2 + 2t + (1-a) = 0$

Using the quadratic formula

$t = \frac{-2 \pm \sqrt{2^2 - 4(1+a)(1-a)}}{2(1+a)}$

$= \frac{-2 \pm \sqrt{2^2 - 4(1-a^2)}}{2(1+a)}$

$= \frac{-2 \pm \sqrt{4a^2}}{2(1+a)}$

$=\dfrac{-2\pm 2a}{2(1+a)}$

$=\dfrac{-1\pm a}{1+a}$

$=-1 \text{ or } \frac{-1+a}{1+a}$

Hence $t=\frac{a-1}{a+1}$ as $t=-1$ is not a solution for $0° < x < 90°$

9b $2\cos x-\sin x=1$

$2\cos x=1+\sin x$

$t=\dfrac{2-1}{2+1}=\dfrac{1}{3}$

Hence

$\tan\dfrac{1}{2}x=\dfrac{1}{3}$

$\dfrac{1}{2}x=18°26'$

$x=36°52'$

10 $6\cos\theta+17\sin\theta=18$

$6\left(\dfrac{1-t^2}{1+t^2}\right)+17\left(\dfrac{2t}{1+t^2}\right)=18$

$6(1-t^2)+34t=18+18t^2$

$24t^2-34t+12=0$

$12t^2-17t+6=0$

Hence, using the quadratic formula

$t=\dfrac{-(-17)\pm\sqrt{17^2-4\times 12\times 6}}{2\times 12}$

$=\dfrac{17\pm 1}{24}$

$=\frac{3}{4} \text{ or } \frac{2}{3}$

Hence the solutions are $\tan\frac{\theta_1}{2}=\frac{3}{4}$, $\tan\frac{\theta_2}{2}=\frac{2}{3}$

$\tan\left(\frac{\theta_1 - \theta_2}{2}\right)$

$= \frac{\tan\frac{\theta_1}{2} - \tan\frac{\theta_2}{2}}{1 + \tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2}}$

$= \frac{\frac{3}{4} - \frac{2}{3}}{1 + \left(\frac{3}{4}\right)\left(\frac{2}{3}\right)}$

$= \frac{1}{18}$

as required

11a $a\cos x + b\sin x = c$

$a\left(\frac{1-t^2}{1+t^2}\right) + b\left(\frac{2t}{1+t^2}\right) = c$

$a(1-t^2) + 2bt = c(1+t^2)$

$c(1+t^2) - a(1-t^2) - 2bt = 0$

$c + ct^2 - a + at^2 - 2bt = 0$

$(a+c)t^2 - 2bt - (a-c) = 0$

where $t = \tan\frac{x}{2}$

11b In order for the roots of this equation to be real, the discriminant must be greater than 0, hence

$\Delta = b^2 - 4ac \geq 0$

$(-2b)^2 - 4(a+c)\big(-(a-c)\big) \geq 0$

$4b^2 + 4(a+c)(a-c) \geq 0$

$4b^2 + 4(a^2 - c^2) \geq 0$

$4b^2 + 4(a^2 - c^2) \geq 0$

$4b^2 + 4a^2 - 4c^2 \geq 0$

$b^2 + a^2 - c^2 \geq 0$

$c^2 \leq a^2 + b^2$

11c The roots of the equation are given by the quadratic formula

$$t=\frac{-(-2b)\pm\sqrt{\Delta}}{2(a+c)}$$

$$=\frac{2b\pm\sqrt{4\mathrm{b}^2+4a^2-4c^2}}{2(a+c)}$$

$$=\frac{b\pm\sqrt{\mathrm{b}^2+a^2-c^2}}{(a+c)}$$

So let $\tan\frac{1}{2}\alpha=\frac{b+\sqrt{\mathrm{b}^2+a^2-c^2}}{(a+c)}$ and $\tan\frac{1}{2}\beta=\frac{b-\sqrt{\mathrm{b}^2+a^2-c^2}}{(a+c)}$

$$\tan\frac{1}{2}(\alpha+\beta)$$

$$=\frac{\tan\frac{1}{2}\alpha+\tan\frac{1}{2}\beta}{1-\tan\frac{1}{2}\alpha\tan\frac{1}{2}\beta}$$

$$=\frac{\frac{b+\sqrt{\mathrm{b}^2+a^2-c^2}}{(a+c)}+\frac{b-\sqrt{\mathrm{b}^2+a^2-c^2}}{(a+c)}}{1-\left(\frac{b+\sqrt{\mathrm{b}^2+a^2-c^2}}{(a+c)}\right)\left(\frac{b-\sqrt{\mathrm{b}^2+a^2-c^2}}{(a+c)}\right)}$$

$$=\frac{\frac{2b}{(a+c)}}{1-\left(\frac{b^2-(\mathrm{b}^2+a^2-c^2)}{(a+c)^2}\right)}$$

$$=\frac{\frac{2b}{(a+c)}}{1-\left(\frac{c^2-a^2}{(a+c)^2}\right)}$$

$$=\frac{\frac{2b}{(a+c)}}{1-\left(\frac{(c-a)(c+a)}{(a+c)^2}\right)}$$

$$=\frac{\frac{2b}{(a+c)}}{1-\left(\frac{c-a}{a+c}\right)}$$

$$=\frac{2b}{a+c-(c-a)}$$

$$=\frac{2b}{2a}$$

$$=\frac{b}{a}$$

12 $(2k-1)\left(\frac{1-t^2}{1+t^2}\right)+(k+2)\left(\frac{2t}{1+t^2}\right)=2k+1$

$(2k-1)(1-t^2)+2t(k+2)=(2k+1)(1+t^2)$

$(2k+1)(1+t^2)-(2k-1)(t^2-1)-2t(k+2)=0$

$2t^2-2t(k+2)+4k=0$

$t^2-t(k+2)+2k=0$

$$t=\frac{(k+2)\pm\sqrt{(k+2)^2-4\times1\times2k}}{2}$$

$$t=\frac{(k+2)\pm\sqrt{k^2+4k+4-4\times1\times2k}}{2}$$

$$t=\frac{(k+2)\pm\sqrt{k^2-4k+4}}{2}$$

$$t=\frac{(k+2)\pm\sqrt{(k-2)^2}}{2}$$

$$t=\frac{(k+2)\pm(k-2)}{2}$$

$t=k,2$

Noting that

$$\tan\theta=\frac{2t}{1-t^2}$$

it follows that

$$\tan\theta=\frac{4}{3},\frac{2k}{k^2-1}$$

13 $a\cos4\theta+b\sin4\theta=c$

$a(\cos^2\theta-\sin^2\theta)+2b\sin\theta\cos\theta=c$

Let $t=\tan\theta$

$$a\left(\left(\frac{1-t^2}{1+t^2}\right)^2-\left(\frac{2t}{1+t^2}\right)^2\right)+2b\left(\frac{2t}{1+t^2}\right)\left(\frac{1-t^2}{1+t^2}\right)=c$$

$a((1-t^2)^2-(2t)^2)+2b\times2t(1-t^2)=c(1+t^2)^2$

$a(1-2t^2+t^4-4t^2)+b(4t-4t^3)=c(1+2t^2+t^4)$

$(a-c)t^4-4bt^3-(2a+2c+4)t^2+4bt+(a-c)=0$

Hence it follows that the product of roots is

$$t_1 t_2 t_3 t_4 = \frac{a - c}{a - c} = 1$$

Since $t = \tan\theta$ and the solutions are $\theta_1, \theta_2, \theta_3, \theta_4$, it follows that

$$\tan\theta_1 \tan\theta_2 \tan\theta_3 \tan\theta_4 = 1$$

Solutions to Chapter review

1a $\sin 2x + \sin x = 0$

$2\sin x \cos x + \sin x = 0$

$\sin x\,(2\cos x + 1) = 0$

$\sin x = 0$ or $\cos x = -\frac{1}{2}$

For $\sin x = 0$, $x = 0, \pi, 2\pi$.

For $\cos x = -\frac{1}{2}$, $x = \frac{2\pi}{4}, \frac{4\pi}{3}$.

Together this gives $x = 0, \frac{2\pi}{4}, \pi, \frac{4\pi}{3}, 2\pi$

1b $\cos 2x + \cos x = 0$

$2\cos^2 x - 1 + \cos x = 0$

$(2\cos x - 1)(\cos x + 1) = 0$

$\cos x = \frac{1}{2}$ or -1

For $\cos x = \frac{1}{2}$, $x = \frac{\pi}{3}, \frac{5\pi}{3}$

For $\cos x = -1$, $x = \pi$

Together this gives $x = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$

1c $\cos 2x + 5\sin x + 2 = 0$

$1 - 2\sin^2 x + 5\sin x + 2 = 0$

$2\sin^2 x - 5\sin x - 3 = 0$

$(2\sin x + 1)(\sin x - 3) = 0$

$\sin x = -\frac{1}{2}$ or 3, but $-1 \le \sin x \le 1$ so the only solution is $\sin x = -\frac{1}{2}$

Hence, $x = \frac{7\pi}{6}, \frac{11\pi}{6}$

1d

$$2\sin\left(x-\frac{\pi}{6}\right)=\cos\left(x-\frac{\pi}{3}\right)$$

$$2\sin x\cos\frac{\pi}{6}-2\cos x\sin\frac{\pi}{6}=\cos x\cos\frac{\pi}{3}+\sin x\sin\frac{\pi}{3}$$

$$2\sin x\left(\frac{\sqrt{3}}{2}\right)-2\cos x\left(\frac{1}{2}\right)=\cos x\left(\frac{1}{2}\right)+\sin x\left(\frac{\sqrt{3}}{2}\right)$$

$$\left(\frac{\sqrt{3}}{2}\right)\sin x=3\left(\frac{1}{2}\right)\cos x$$

$$\tan x=\frac{3}{\sqrt{3}}$$

2a $R\sin(x-\alpha)=R\sin x\cos\alpha-R\cos x\sin\alpha=\sin x-\cos x$

Equating coefficients gives

$R\cos\alpha=1\quad(1)$

$R\sin\alpha=1\quad(2)$

$R^2\sin^2\alpha+R^2\cos^2\alpha=1^2+1^2\qquad(1)^2+(2)^2$

$R^2(\sin^2\alpha+\cos^2\alpha)=2$

$R^2=2$

$R=\sqrt{2}$

$\tan\alpha=1\qquad(2)\div(1)$

$\alpha=\frac{\pi}{4}$

Hence $\sin x-\cos x=\sqrt{2}\sin\left(x-\frac{\pi}{4}\right)$

2b $\sin x-\cos x=\sqrt{2}$

$$\sqrt{2}\sin\left(x-\frac{\pi}{4}\right)=\sqrt{2}$$

$$\sin\left(x-\frac{\pi}{4}\right)=1$$

$$x-\frac{\pi}{4}=\frac{\pi}{2}$$

$x = \frac{3\pi}{4}$

3a $A\cos(x-\theta) = A\cos x\cos\theta + A\sin x\sin\theta = \sqrt{3}\cos x + \sin x$

Equating coefficients gives

$A\cos\theta = \sqrt{3}$ (1)

$A\sin\theta = 1$ (2)

$A^2\sin^2\theta + A^2\cos^2\theta = 1^2 + \sqrt{3}^2$ $(1)^2 + (2)^2$

$A^2(\sin^2\theta + \cos^2\theta) = 4$

$A^2 = 4$

$A = 2$

$\tan\theta = \frac{1}{\sqrt{3}}$ $(2) \div (1)$

$\theta = \frac{\pi}{6}$

Hence $\sqrt{3}\cos x + \sin x = 2\cos\left(x - \frac{\pi}{6}\right)$

3b $\sqrt{3}\cos x + \sin x = -1$

$2\cos\left(x - \frac{\pi}{6}\right) = -1$

$\cos\left(x - \frac{\pi}{6}\right) = -\frac{1}{2}$

$x - \frac{\pi}{6} = \frac{2\pi}{3}, \frac{4\pi}{3}$

4a $R\sin(x+\alpha) = R\sin x\cos\alpha + R\cos x\sin\alpha = 2\sin x + \sqrt{5}\cos x$

Equating coefficients gives

$R\cos\alpha = 2$ (1)

$R\sin\alpha = \sqrt{5}$ (2)

$R^2\sin^2\alpha + R^2\cos^2\alpha = 2^2 + \sqrt{5}^2$ $(1)^2 + (2)^2$

$R^2(\sin^2\alpha + \cos^2\alpha) = 9$

$R^2 = 9$

$R = 3$

$\tan\alpha = \frac{\sqrt{5}}{2}$ (2) ÷ (1)

$\alpha = \tan^{-1}\frac{\sqrt{5}}{2}$

Hence $2\sin x + \sqrt{5}\cos x = 3\sin\left(x + \tan^{-1}\frac{\sqrt{5}}{2}\right)$

4b $2\sin x + \sqrt{5}\cos x = 3$

$3\sin\left(x + \tan^{-1}\frac{\sqrt{5}}{2}\right) = 3$

$\sin\left(x + \tan^{-1}\frac{\sqrt{5}}{2}\right) = 1$

$x + \tan^{-1}\frac{\sqrt{5}}{2} = 90°$

$x = 90° - \tan^{-1}\frac{\sqrt{5}}{2}$

$x \doteqdot 41.8°$

5a $A\cos(x + \theta) = A\cos x\cos\theta - A\sin x\sin\theta = 3\cos x - 2\sin x$

Equating coefficients gives

$A\cos\theta = 3$ (1)

$A\sin\theta = 2$ (2)

$A^2\sin^2\theta + A^2\cos^2\theta = 2^2 + 3^2$ $(1)^2 + (2)^2$

$A^2(\sin^2\theta + \cos^2\theta) = 13$

$A^2 = 13$

$A = \sqrt{13}$

$\tan\theta = \frac{2}{3}$ (2) ÷ (1)

$\theta = \tan^{-1}\frac{2}{3}$

Hence $3\cos x - 2\sin x = \sqrt{13}\cos\left(x + \tan^{-1}\frac{2}{3}\right)$

5b $3\cos x - 2\sin x = 1$

$\sqrt{13}\cos\left(x + \tan^{-1}\frac{2}{3}\right) = 1$

$\cos\left(x + \tan^{-1}\frac{2}{3}\right) = \frac{1}{\sqrt{13}}$

$x + \tan^{-1}\frac{2}{3} = \cos^{-1}\frac{1}{\sqrt{13}}, 2\pi - \cos^{-1}\frac{1}{\sqrt{13}}$

$x = \cos^{-1}\frac{1}{\sqrt{13}} - \tan^{-1}\frac{2}{3}, 2\pi - \cos^{-1}\frac{1}{\sqrt{13}} - \tan^{-1}\frac{2}{3}$

$x \doteqdot 40°12'$ or $252°25'$

6 $\sin x = \tan\frac{1}{2}x$

$\frac{2t}{1+t^2} = t$

$2t = t + t^3$

$t^3 - t = 0$

$t(t^2 - 1) = 0$

$t(t-1)(t+1) = 0$

$t = 0, \pm 1$

$\tan\frac{1}{2}x = 0, \pm 1$

$\frac{1}{2}x = 0, \frac{\pi}{4}, \frac{3\pi}{4}$ or π

$x = 0, \frac{\pi}{2}, \frac{3\pi}{2}$ or 2π

7a $7\sin x + \cos x = 5$

$7\left(\frac{2t}{1+t^2}\right) + \frac{1-t^2}{1+t^2} = 5$

$14t + 1 - t^2 = 5 + 5t^2$

$6t^2 - 14t + 4 = 0$

$3t^2 - 7t + 2 = 0 \quad \text{where } t = \tan\frac{x}{2}$

7b Using the quadratic formula

$$t = \frac{-(-7) \pm \sqrt{(-7)^2 - 4 \times 3 \times 2}}{2 \times 3}$$

$$= \frac{7 \pm \sqrt{25}}{6}$$

$$= \frac{7 \pm 5}{6}$$

$$= \frac{1}{3} \text{ or } 2$$

$$\tan\frac{x}{2} = \frac{1}{3} \text{ or } 2$$

$$\frac{x}{2} = \tan^{-1}\frac{1}{3} \text{ or } \tan^{-1} 2$$

$$x = 2\tan^{-1}\frac{1}{3} \text{ or } 2\tan^{-1} 2$$

8a $4\sin x - 2\cos x = 3$

Let $t = \tan\frac{x}{2}$

$$4\left(\frac{2t}{1+t^2}\right) - 2\left(\frac{1-t^2}{1+t^2}\right) = 3$$

$$4(2t) - 2(1 - t^2) = 3(1 + t^2)$$

$$8t - 2 + 2t^2 = 3 + 3t^2$$

$$t^2 - 8t + 5 = 0$$

Using the quadratic formula

$$t = \frac{-(-8) \pm \sqrt{(-8)^2 - 4 \times 1 \times 5}}{2 \times 1}$$

$$= \frac{8 \pm \sqrt{44}}{2}$$

$= \dfrac{8 \pm 2\sqrt{11}}{2}$

$= 4 \pm \sqrt{11}$

Since $0 \le x \le 2\pi$ then $0 \le \frac{x}{2} \le \pi$.

Recalling $t = \tan\frac{x}{2}$

$\tan\frac{x}{2} = 4 + \sqrt{11}$ or $4 - \sqrt{11}$

$\frac{x}{2} = \tan^{-1}\left(4 + \sqrt{11}\right)$ or $\tan^{-1}\left(4 - \sqrt{11}\right)$

$x = 2\tan^{-1}\left(4 + \sqrt{11}\right)$ or $2\tan^{-1}\left(4 - \sqrt{11}\right)$

$x \doteqdot 2.87$ or 1.20

9a $\cos 3x$

$= \cos x \cos 2x - \sin x \sin 2x$
$= \cos x\,(2\cos^2 x - 1) - \sin x\,(2\sin x \cos x)$
$= 2\cos^3 x - \cos x - 2\sin^2 x \cos x$
$= 2\cos^3 x - \cos x - 2(1 - \cos^2 x)\cos x$
$= 2\cos^3 x - \cos x - 2\cos x + 2\cos^3 x$
$= 4\cos^3 x - 3\cos x$

9b $\cos 3x + \sin 2x + \cos x = 0$

$4\cos^3 x - 3\cos x + \sin 2x + \cos x = 0$

$4\cos^3 x - 3\cos x + 2\sin x \cos x + \cos x = 0$

$4\cos^3 x - 2\cos x + 2\sin x \cos x = 0$

$2\cos x\,(2\cos^2 x - 1 + \sin x) = 0$

$2\cos x\,(2(1 - \sin^2 x) - 1 + \sin x) = 0$

$2\cos x\,(2 - 2\sin^2 x - 1 + \sin x) = 0$

$2\cos x\,(1 - 2\sin^2 x + \sin x) = 0$

$2\cos x\,(2\sin^2 x - \sin x - 1) = 0$

$2\cos x\,(2\sin x + 1)(\sin x - 1) = 0$

Hence the solutions occur when $\cos x = 0$, $\sin x = -\frac{1}{2}$ and $\sin x = 1$.

For $\cos x = 0, x = \frac{\pi}{2}, \frac{3\pi}{2}$

For $\sin x = -\frac{1}{2}, x = \frac{7\pi}{6}, \frac{11\pi}{6}$

For $\sin x = 1, x = \frac{\pi}{2}$

Hence the solutions are $x = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}$

