Solutions to Exercise 6A Foundation questions

1a i $v = 6$

$$v = \frac{dx}{dt}$$

$$\frac{1}{v} = \frac{dt}{dx}$$

$$\frac{1}{6} = \frac{dt}{dx}$$

$$t = \frac{1}{6}x + C$$

Substituting $x = 1$ when $t = 0$,

$$0 = \frac{1}{6}(1) + C$$

$$C = -\frac{1}{6}$$

$$t = \frac{1}{6}(x - 1)$$

1a ii $v = 6$

$$v = \frac{dx}{dt}$$

$$x = 6t + C$$

Substituting $x = 1$ when $t = 0$,

$$1 = C$$

$$\therefore x = 6t + 1$$

1b i $v = -6x^{-2}$

$v = \dfrac{dx}{dt}$

$-6x^2 = \dfrac{dt}{dx}$

$t = -\dfrac{1}{18}x^3 + C$

Substituting $x = 1$ when $t = 0$,

$0 = -\dfrac{1}{18}(1) + C$

$\therefore C = \dfrac{1}{18}$

$\therefore t = \dfrac{1}{18}(1 - x^3)$

1b ii $v = -6x^{-2}$

$t = \dfrac{1}{18}(1 - x^3)$

$18t = 1 - x^3$

$x^3 = 1 - 18t$

$\therefore x = (1 - 18t)^{\frac{1}{3}}$

1c i $v = -6x^3$

$-6x^3 = \dfrac{dx}{dt}$

$t = \dfrac{1}{12}x^{-2} + C$

Substituting $x = 1$ when $t = 0$

$0 = \dfrac{1}{12} + C$

$\therefore C = -\dfrac{1}{12}$

$\therefore t = \dfrac{1}{12}x^{-2} - \dfrac{1}{12} = \dfrac{1}{12}(x^{-2} - 1)$

1c ii $v = -6x^3$

$t = \frac{1}{12}(x^{-2} - 1)$

$x^{-2} = 12t + 1$

$x = (12t + 1)^{-\frac{1}{2}}$

1d i $v = e^{-2x}$

$e^{-2x} = \frac{dx}{dt}$

$t = \frac{1}{2}e^{2x} + C$

Substituting $x = 1$ when $t = 0$

$0 = \frac{1}{2}e^2 + C$

$\therefore C = -\frac{1}{2}e^2$

$\therefore t = \frac{1}{2}(e^{2x} - e^2)$

1d ii $v = e^{-2x}$

$t = \frac{1}{2}(e^{2x} - e^2)$

$2t + e^2 = e^{2x}$

$\therefore x = \frac{1}{2}\ln(2t + e^2)$

1e i $v = 1 + x^2$

$1 + x^2 = \frac{dx}{dt}$

$\frac{1}{1 + x^2} = \frac{dt}{dx}$

$t = \tan^{-1} x + C$

Substituting $x = 1$ when $t = 0$

$0 = \tan^{-1} 1 + C$

$\therefore C = -\frac{\pi}{4}$

$\therefore t = \tan^{-1} x - \frac{\pi}{4}$

1e ii $v = 1 + x^2$

$t = \tan^{-1} x - \frac{\pi}{4}$

$t + \frac{\pi}{4} = \tan^{-1} x$

$x = \tan\left(t + \frac{\pi}{4}\right)$

1f i $v = \cos^2 x$

$\cos^2 x = \frac{dx}{dt}$

$\frac{1}{\cos^2 x} = \frac{dt}{dx}$

$t = \tan x + C$

Substituting $x = 1$ when $t = 0$

$0 = \tan 1 + C$

$\therefore C = -\tan 1$

$\therefore t = \tan x - \tan 1$

1f ii $v = \cos^2 x$

$t = \tan x - \tan 1$

$t + \tan 1 = \tan x$

$x = \tan^{-1}(t + \tan 1)$

2a $v = 6$

$\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$= \frac{d}{dx}\left(\frac{1}{2}6^2\right)$

$= \frac{d}{dx}(18) = 0$

2b $v = -6x^{-2}$

$\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$= \frac{d}{dx}\left(\frac{1}{2}(-6x^{-2})^2\right)$

$= \frac{d}{dx}(18x^{-4})$

$= -72x^{-5}$

2c $v = -6x^3$

$\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$= \frac{d}{dx}\left(\frac{1}{2}(-6x^3)^2\right)$

$= \frac{d}{dx}(18x^6)$

$= 108x^5$

2d $v = e^{-2x}$

$$\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$= \frac{d}{dx}\left(\frac{1}{2}(e^{-2x})^2\right)$$

$$= \frac{d}{dx}\left(\frac{1}{2} \times e^{-4x}\right)$$

$$= -2e^{-4x}$$

2e $v = 1 + x^2$

$$\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$= \frac{d}{dx}\left(\frac{1}{2}(1 + x^2)^2\right)$$

$$= \frac{d}{dx}\left(\frac{1}{2}(1 + x^4 + 2x^2)\right)$$

$$= \frac{d}{dx}\left(\frac{1}{2} + \frac{1}{2}x^4 + x^2\right)$$

$$= 2x + 2x^3$$

$$= 2x(1 + x^2)$$

2f $v = \cos^2 x$

$$\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$= \frac{d}{dx}\left(\frac{1}{2}(\cos^2 x)^2\right)$$

$$= \frac{d}{dx}\left(\frac{1}{2}(\cos^4 x)\right)$$

$$= -2\cos^3 x \sin x$$

3a $\ddot{x} = 6x^2$

From above

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 6x^2$$

$$\because \frac{d}{dx}(v^2) = 12x^2$$

$$\therefore v^2 = 4x^3 + C$$

for some constant C.

Now apply the conditions $v(0) = 0$ and $x(0) = 0$

$$0 = 0 + C$$

$$\because C = 0$$

$$\therefore v^2 = 4x^3$$

3b

$$\ddot{x} = \frac{1}{e^x}$$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{1}{e^x}$$

$$\because \frac{d}{dx}(v^2) = \frac{2}{e^x}$$

$$\therefore v^2 = -\frac{2}{e^x} + C$$

for some constant C.

Now apply the conditions $v(0) = 0$ and $x(0) = 0$

$$0 = -2 + C$$

$$\because C = 2$$

$$\therefore v^2 = -\frac{2}{e^x} + 2$$

$$= 2(1 - e^{-x})$$

3c

$$\ddot{x} = \frac{1}{2x+1}$$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{1}{2x+1}$$

$$\because \frac{d}{dx}(v^2) = \frac{2}{2x+1}$$

$$\therefore v^2 = \ln|2x+1| + C$$

for some constant C.

Now apply the conditions $v(0) = 0$ and $x(0) = 0$

$$0 = \ln|1| + C$$

$$\because C = 0$$

$$\therefore v^2 = \ln|2x+1|$$

3d

$$\ddot{x} = \frac{1}{4+x^2}$$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{1}{4+x^2}$$

$$\because \frac{d}{dx}(v^2) = \frac{2}{4+x^2}$$

$$\therefore v^2 = \tan^{-1}\frac{x}{2} + C$$

for some constant C.

Now apply the conditions $v(0) = 0$ and $x(0) = 0$

$$0 = \tan^{-1} 0 + C$$

$$\because C = 0$$

$$\therefore v^2 = \tan^{-1}\frac{x}{2}$$

4a

$\ddot{x} = \frac{2}{v^2}$ and when $t = 0, v = 0$

$\frac{dv}{dt} = \frac{2}{v^2}$

$\frac{dt}{dv} = \frac{v^2}{2}$

$t = \frac{1}{6}v^3 + C$

for some constant C.

Now apply the conditions $v(0) = 0$ and $t(0) = 0$

$0 = 0 + C$

$\because C = 0$

$\therefore t = \frac{1}{6}v^3$

4b $\ddot{x} = v^2$ and when $t = 0, v = \frac{1}{2}$

$\frac{dv}{dt} = v^2$

$\frac{dt}{dv} = \frac{1}{v^2}$

$t = -\frac{1}{v} + C$

for some constant C.

Now apply the conditions $v(0) = \frac{1}{2}$ and $t(0) = 0$

$0 = -2 + C$

$\because C = 2$

$\therefore t = 2 - \frac{1}{v}$

4c $\ddot{x} = 2 + v$ and when $t = 0, v = 1$

$$\frac{dv}{dt} = 2 + v$$

$$\frac{dt}{dv} = \frac{1}{2 + v}$$

$$t = \ln|2 + v| + C$$

for some constant C.

Now apply the conditions $v(0) = 1$ and $t(0) = 0$

$$0 = \ln|2 + 1| + C$$

$$\because C = -\ln|3|$$

$$\therefore t = \ln|2 + v| - \ln|3| = \ln\left|\frac{2 + v}{3}\right|$$

5a

$$\ddot{x} = \frac{v^2}{4} \text{ and when } x = 0, v = 1$$

$$v\frac{dv}{dx} = \frac{v^2}{4}$$

$$\frac{dx}{dv} = \frac{4}{v}$$

$$x = 4\ln|v| + C$$

for some constant C.

Now apply the conditions $v(0) = 1$ and $x(0) = 0$

$$0 = 0 + C$$

$$\because C = 0$$

$$\therefore x = 4\ln|v|$$

5b

$\ddot{x} = \dfrac{3}{v}$ and when $x = 0, v = 6$

$v\dfrac{dv}{dx} = \dfrac{3}{v}$

$\dfrac{dx}{dv} = \dfrac{v^2}{3}$

$x = \dfrac{v^3}{9} + C$

for some constant C.

Now apply the conditions $v(0) = 6$ and $x(0) = 0$

$0 = 24 + C$

$\because C = -24$

$\therefore x = \dfrac{v^3}{9} - 24$

5c $\ddot{x} = 2 + v$ and when $x = 0, v = 0$

$v\dfrac{dv}{dx} = 2 + v$

$\dfrac{dx}{dv} = \dfrac{v}{2 + v}$

$x = 2 + v - 2\ln|2 + v| + C$

for some constant C.

Now apply the conditions $v(0) = 6$ and $x(0) = 0$

$0 = 2 - 2\ln|2| + C$

$\because C = 2\ln|2| - 2$

$\therefore x = v + 2\ln|2| - 2\ln|2 + v|$

$\quad = v + 2\ln\left|\dfrac{2}{2 + v}\right|$

6a From Newton's second law, after dividing through by the mass,

$$\frac{dv}{dt} = 3t - 2.$$

Integrating, $v = \frac{3}{2}t^2 - 2t + C$

At $= 0$, $v = 0$, so $C = 0$ and

$$v = \frac{3}{2}t^2 - 2t$$

$$\frac{dx}{dt} = \frac{3}{2}t^2 - 2t$$

Integrating, $\frac{dx}{dt} = \frac{3}{2}t^2 - 2t$

$$x = \frac{1}{2}t^3 - t^2 + D$$

At $t = 0$, $x = 0$, so $D = 0$ and

$$x = \frac{1}{2}t^3 - t^2$$

Hence at $t = 4$, $x = 16$

6b From Newton's second law, after dividing through by the mass,

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{4}{3}x + \frac{2}{3}$$

$$\frac{d}{dx}(v^2) = \frac{8}{3}x + \frac{4}{3}$$

Integrating, $v^2 = \frac{4}{3}x^2 + \frac{4}{3}x + C$

At $= 0$, $v = 0$, so $C = 0$

$$v^2 = \frac{4}{3}x^2 + \frac{4}{3}x$$

Hence at $x = 3$, $v = 4$

6c

From Newton's second law, after dividing through by the mass,

$$\frac{dv}{dt} = \frac{1}{0.25v + 0.5}$$

$$\frac{dt}{dv} = 0.25v + 0.5$$

Integrating, $t = \frac{1}{8}v^2 + \frac{1}{2}v + C$

At $= 0$, $v = 0$, so $C = 0$

$$t = \frac{1}{8}v^2 + \frac{1}{2}v$$

Hence at $v = 4$, $t = 4$

6d From Newton's second law, after dividing through by the mass,

$$v\frac{dv}{dx} = \frac{1}{0.5v + 1}$$

$$\frac{dv}{dx} = \frac{1}{0.5v^2 + v}$$

$$\frac{dx}{dv} = 0.5v^2 + v$$

Integrating, $x = \frac{1}{6}v^3 + \frac{1}{2}v^2 + C$

At $= 0$, $x = 0$, so $C = 0$

$$x = \frac{1}{6}v^3 + \frac{1}{2}v^2$$

Hence at $v = 3$, $x = 9$

Solutions to Exercise 6A Development questions

7 $F_1 = 12\hat{\imath} + 23\hat{\jmath}$

$F_2 = 9\hat{\imath} - 7\hat{\jmath}$

$F_3 = -5\hat{\imath} + 14\hat{\jmath}$

F_r

$= F_1 + F_2 + F_3$

$= 16\hat{\imath} + 30\hat{\jmath}$

$F = ma; \quad m = 2$

a_r

$= \dfrac{16\hat{\imath} + 30\hat{\jmath}}{2}$

$= 8\hat{\imath} + 15\hat{\jmath}$

$|a_r|$

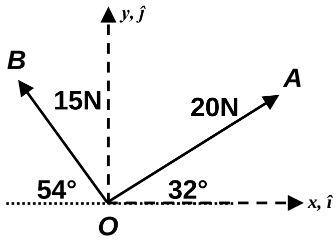
$= \sqrt{8^2 + 15^2}$

$= 17 \text{ m/s}^2$

$\widehat{a_r} = \tan^{-1}\left(\frac{15}{8}\right)$

The acceleration is 17 m/s^2 at an angle of $\tan^{-1}\left(\frac{15}{8}\right)$ above the horizontal.

8a Superimposing a Cartesian Coordinate system on the forces diagram given in the question, we can see that $\overrightarrow{OA}$ is at 32° to the horizontal i.e. $\overrightarrow{OA}$ is 32° from the positive x-axis ($\hat{\imath}$, using vector notation). The force is also pointing up in a NE direction, which corresponds with the positive y-axis ($\hat{\jmath}$, using vector notation). Using trigonometry, we then breakdown $\overrightarrow{OA}$ into the corresponding Fx and Fy components using vector notation.



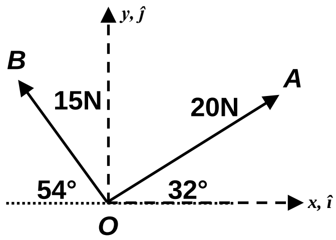
$\overrightarrow{OA}$

$= 20\text{N}$

$= 20\cos 32°\,\hat{\imath} + 20\sin 32°\,\hat{\jmath}$

$= 16.96\hat{\imath} + 10.6\hat{\jmath}$

Similarly, we can see that $\overrightarrow{OB}$ is at 54° to the horizontal from the negative x-axis ($\hat{\imath}$, using vector notation), or $\overrightarrow{OB}$ is at $(180° - 54°)$ from the positive x-axis. The force is also pointing up in a NW direction, which again corresponds with the positive y-axis ($\hat{\jmath}$, using vector notation). Using trigonometry, we then breakdown $\overrightarrow{OB}$ into the corresponding Fx and Fy components using vector notation.



$\overrightarrow{OB}$

$= 15\text{N at } (180 - 54)°$

$= 15\cos(180 - 54)°\,\hat{\imath} + 15\sin(180 - 54)°\,\hat{\jmath}$

$= -15\cos 54°\,\hat{\imath} + 15\sin 54°\,\hat{\jmath}$

$= -8.82\hat{\imath} + 12.14\hat{\jmath}$

8b F_r

$= \overrightarrow{OA} + \overrightarrow{OB}$

$= 16.96\hat{\imath} + 10.6\hat{\jmath} + (-8.82\hat{\imath} + 12.14\hat{\jmath})$

$= 8.14\hat{\imath} + 22.74\hat{\jmath}$

$|F_r|$

$= \sqrt{8.14^2 + 22.74^2}$

$= 24.15 \ldots$

$|F_r| \doteqdot 24 \text{ N}$

8c Direction of $F_r = \widehat{F_r}$

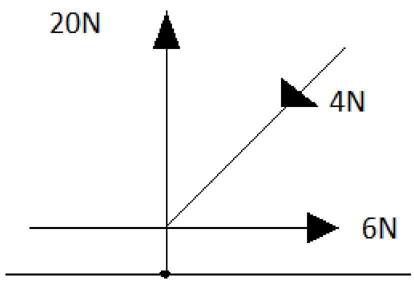
$\widehat{F_r}$

$= \tan^{-1}\left(\frac{22.74}{8.14}\right)$

$= 70.3 \ldots°$

$\widehat{F_r} \doteqdot 70°$ above the horizontal

9a $F_1 = 20\,\hat{\jmath}$; $F_2 = 6\,\hat{\imath}$; $F_3 = -2\sqrt{2}\hat{\imath} - 2\sqrt{2}\hat{\jmath}$



F_r

$= F_1 + F_2 + F_3$

$= 20\,\hat{\jmath} + 6\,\hat{\imath} + (-2\sqrt{2}\hat{\imath} - 2\sqrt{2}\hat{\jmath})$

$= (6 - 2\sqrt{2}\,)\hat{\imath} + (20 - 2\sqrt{2})\hat{\jmath}$

9b $|F_r|$

$= \sqrt{\left(6-2\sqrt{2}\right)^2 + \left(20-2\sqrt{2}\right)^2}$

$= 17.462\ldots$

$\doteqdot 17.5 \text{ N}$

$\widehat{F_r} = \tan^{-1}\left(\dfrac{20-2\sqrt{2}}{6-2\sqrt{2}}\right)$

$\widehat{F_r} \doteqdot 79.5°$ above the horizontal

10 Initial velocity $= u$

Final velocity $= v$

Time $= t$

Acceleration $= \ddot{x} = a$

Displacement $= x = s$

$\ddot{x} = \dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right)$

$a = \dfrac{d}{ds}\left(\dfrac{1}{2}v^2\right)$

Integrating for s

$as = \left[\dfrac{v^2}{2}\right]_u^v$

$as = \dfrac{v^2-u^2}{2}$

Hence $a = \dfrac{\frac{1}{2}(v^2-u^2)}{s-0}$

Rearranging gives,

$v^2 = u^2 + 2as$

11a Using the result from question 10 and given $u = 20\ m/s$ and $a = -10\ m/s^2$ [Note: $a = g = \ddot{x}$], take the distance as variable x [Note: $s = x$],

v^2

$= u^2 + 2ax$

$= (20)^2 + 2(-10)x$

$\therefore v^2 = 400 - 20x$

At the greatest height, x_{max}, the velocity of the ball will be zero for a moment after which it will start descending back to the ground. So, $v = 0$

Rearranging $v^2 = 400 - 20x$

$$x_{max} = \frac{400 - 0}{20}$$

$x_{max} = 20$ metres

11b When the ball is rising, from Newton's first and second laws of motion, the ball tries to maintain its inertial state of going up with a velocity of 20 m/s while being acted upon by a gravitational force pulling it towards the ground with an acceleration of -10 m/s^2.

Hence, when the ball is rising, it follows the positive equation:

$v = \sqrt{400 - 20x}$

11c At 11b we determined,

$v = \sqrt{400 - 20x}$

$\frac{dx}{dt} = \sqrt{400 - 20x}$

Rearranging the formula,

$dt = \frac{dx}{\sqrt{400 - 20x}}$

Integrating both sides,

LHS:

$\int dt = t$

RHS:

$\int \frac{dx}{\sqrt{400-20x}} = \frac{-1}{10}\sqrt{400 - 20x} + C$

Hence:

$t = \frac{-1}{10}\sqrt{400 - 20x} + C$

At $t = 0, x = 0$

$0 = \frac{-1}{10}\sqrt{400 - 0} + C$

$-C = -\left(\frac{1}{10}\right)(20)$

$C = 2$

$\therefore t = 2 - \frac{\sqrt{400 - 20x}}{10}$

Rearranging the solution for t to determine x,

$\frac{\sqrt{400 - 20x}}{10} = 2 - t$

$\sqrt{400 - 20x} = 20 - 10t$

$400 - 20x = (20 - 10t)^2$

$-20x = (400 - 400t + 100t^2) - 400$

$$x = \frac{-400t + 100t^2}{-20}$$

$\therefore x = 20t - 5t^2$

To find the time t taken to reach maximum height,

$x = x_{max} = 20$ m [solved at 11a]

$20 = 20t - 5t^2$

$t^2 - 4t + 4 = 0$

$(t - 2)(t - 2) = 0$

$\therefore t = 2$ seconds

12a $u = 1 \text{ km/s} = 1000 \text{ m/s};\ v = 10 \text{ m/s when } x = 100 \text{ m}$

$\ddot{x} = -kv^2$

$v\dfrac{dv}{dx} = -kv^2$

$v\dfrac{dv}{v^2} = -kdx$

$\dfrac{dv}{v} = -kdx$

Integrating both sides,

$\ln v = -kx + C$

Using the given information of $u = 1000$ m/s when $x = 0$ m

$\ln 1000 = -k(0) + C$

$\therefore C = \ln 1000$

$C = 3\ln 10$

Now substituting $v = 10$ m/s when $x = 100$ m

$\ln v = -kx + \ln 1000$

$\ln 10 = -k(100) + 3\ln 10$

$\ln 10 - 3\ln 10 = -k(100)$

$-2\ln 10 = -k(100)$

$k = \dfrac{2\ln 10}{100}$

$k = \dfrac{\ln 10}{50}$

$\therefore \ln v = -\dfrac{\ln 10}{50}x + 3\ln 10$

$\ln v - 3\ln 10 = -\dfrac{\ln 10}{50}x$

$\dfrac{-50\,(\ln v - 3\ln 10)}{\ln 10} = x$

$\dfrac{-50\ln v + 150\ln 10}{\ln 10} = x$

$x = 150 - \dfrac{50\ln v}{\ln 10}$

At $v = 1$ m/s

$$x = 150 - \frac{50(\ln 1)}{\ln 10}$$

$$x = 150 - \frac{50(0)}{\ln 10}$$

$\mathrm{x} = 150$ metres

12b Given $u = 1$ km/s $= 1000$ m/s; $t = 1$s; $v = 10$ m/s

$\ddot{x} = -kv^2$

$\frac{dv}{dt} = -kv^2$

$\frac{dv}{-v^2} = kdt$

$-v^{-2}dv = kdt$

Integrating both sides,

$kt = \frac{1}{v} + C$

At $t = 0, u = 1000$ m/s

$0 = \frac{1}{1000} + C$

$\therefore C = -\frac{1}{1000}$

At $t = 1s, v = 10$ m/s

$k(1) = \left(\frac{1}{10} - \frac{1}{1000}\right)$

$\therefore k = \frac{99}{1000}$

$\frac{99}{1000}t = \frac{1}{v} - \frac{1}{1000}$

$\therefore t = \frac{1000}{99}\left(\frac{1}{v} - \frac{1}{1000}\right)$

For $v = 1$ m/s,

$t = \frac{1000}{99}\left(\frac{1}{1} - \frac{1}{1000}\right)$

$t = \frac{999}{99}$

$t = 10\frac{1}{11}$s

13 Given $\ddot{x} = e^{-x}$m/s²; and initial conditions $v = u = 2$ m/s when $t = 0$ and $x = 0$

$\ddot{x} = e^{-x}$

$v\dfrac{dv}{dx} = e^{-x}$

$vdv = e^{-x}dx$

Integrating both sides,

$\dfrac{v^2}{2} = -e^{-x} + C$

Using initial conditions,

$\dfrac{2^2}{2} = -e^{0} + C$

$2 = -1 + C$

$C = 3$

$\dfrac{v^2}{2} = 3 - e^{-x}$

$\therefore v^2 = 6 - 2e^{-x}$

The particle has an acceleration of e^{-x} m/s², which is always greater than zero, and the particle has an initial velocity of 2 m/s. So, from the beginning onwards, the velocity of the particle is always positive and 2 m/s is a minimum, i.e. the velocity can never fall below 2 m/s.

After an infinite amount of time, the particle will reach a limiting velocity of $\sqrt{6}$ m/s.

14a Given $v = 6 - 2x$ m/s; and x is at the origin

$$\frac{dv}{dt} = v\frac{dv}{dx}$$

$$\ddot{x} = v\frac{dv}{dx}$$

Substituting $v = 6 - 2x$ m/s,

$$\ddot{x} = \frac{dv}{dx}(6 - 2x)$$

Differentiating v with respect to x,

$$\ddot{x} = (6 - 2x)(-2)$$

$$\ddot{x} = 4x - 12$$

At the origin $x = 0$

$$\therefore \ddot{x} = -12 \text{ m/s}^2$$

14b Given $v = 6 - 2x$ m/s,

$$\frac{dx}{dt} = 6 - 2x$$

$$dt = \frac{dx}{2(3 - x)}$$

Integrating both sides,

$$\int dt = \frac{1}{2}\int \frac{dx}{3 - x}$$

$$t = \frac{1}{2}\left[\left(\frac{1}{-1}\right) ln|3 - x| + C\right]$$

$$t = -\frac{1}{2} ln|3 - x| + C$$

At $t = 0, x = 0,$ so

$$0 = -\frac{1}{2} ln|3 - 0| + C$$

$$C = \frac{1}{2} ln\, 3$$

$$\therefore t = -\frac{1}{2} ln|3 - x| + \frac{1}{2} ln\, 3$$

t

$$= -\frac{1}{2} ln\left(\frac{|3 - x|}{3}\right)$$

$$= -\frac{1}{2} ln\left|1 - \frac{x}{3}\right|$$

$$-2t = ln\left|1 - \frac{x}{3}\right|$$

Taking exponents of both sides

$$e^{-2t} = 1 - \frac{x}{3}$$

$\therefore x = 3(1 - e^{-2t})$ metres

14c As $t \to \infty$, the particle reaches (tends towards) the point $x = 3$ metres

15a $x_i = 0\ m$; $u = 0\ m/s$; $a = 2(1+v)\ m/s^2$; $v = 20\ m/s$

$a = \frac{dv}{dt}$

$\frac{dv}{dt} = 2(1+v)$

Rearranging and integrating both sides,

$2dt = \frac{dv}{(1+v)}$

$2t = \frac{1}{1} ln|1+v| + C$

Using initial conditions,

$0t = ln|1+0| + C$

$C = 0$

$\therefore t = \frac{1}{2} ln|1+v|$

When $v = 20\ m/s$,

$t = \frac{1}{2} ln|1+20|$

$t = \frac{1}{2} ln\ 21$

$t \doteqdot 1.52$ seconds

15b

a

$= v\dfrac{dv}{dx}$

$= 2(1+v)$

$2dx = \dfrac{v}{1+v}dv$

$2dx = \left(1 - \dfrac{1}{1+v}\right)dv$

Integrating both sides

$2x + C = v - \frac{1}{1}\ln(1+v)$

Using initial conditions $x_i = 0\ m,\ \ u = 0\ m/s$,

$2(0) + C = 0 - \ln(1+0)$

$C = 0$

$2x = v - \ln(1+v)$

Rearranging for x,

$x = \dfrac{1}{2}v - \dfrac{1}{2}\ln(1+v)$

When $v = 20$,

$x = \dfrac{1}{2}20 - \dfrac{1}{2}\ln(1+20)$

$x = 10 - \dfrac{1}{2}\ln(21)$

$\therefore x \doteqdot 8.48$ m

16a Mass $= m$; $v_i = u = 0\ m/s$; $x_i = 0\ m$; $v = u + \frac{x}{k}\ m/s$

a

$= v\dfrac{dv}{dx}$

$= \dfrac{dv}{dx}\left(u + \dfrac{x}{k}\right)$

$= \left(\dfrac{1}{k}\right)\left(u + \dfrac{x}{k}\right)$

$= \left(\dfrac{1}{k}\right)(v)$

$= \dfrac{v}{k}$

F

$= ma$

$= \dfrac{mv}{k}$

Hence, $F \propto v$ and the constant of proportionality is $\frac{m}{k}$

16bi $v_A = 3u$ at point A

$v_A = u + \dfrac{x_A}{k}$

$u + \dfrac{x_A}{k} = 3u$

$x_A = k(3u - u)$

$x_A = 2ku$

16bii

$$v = \frac{dx}{dt}$$

$$\frac{dx}{dt} = u + \frac{x}{k}$$

Rearranging for t and integrating,

$$dt = \frac{dx}{u + \frac{x}{k}}$$

$$t + C = \frac{1}{\frac{1}{k}} \ln \left| u + \frac{x}{k} \right|$$

Using initial conditions, $x_i = 0\ m, t = 0s$

$$0 + C = k \ln \left| u + \frac{0}{k} \right|$$

$$C = k \ln u$$

t

$$= k \left[\ln \left| u + \frac{x}{k} \right| - \ln u\right]$$

$$= k \ln \left| \frac{u + \frac{x}{k}}{u} \right|$$

$$= k \ln \left| \left(u + \frac{x}{k}\right) \left(\frac{1}{u}\right) \right|$$

$$= k \ln \left| \left(\frac{u}{u} + \frac{x}{uk}\right) \right|$$

$$\therefore t = k \ln \left| 1 + \frac{x}{uk} \right|$$

At point A, $x_A = 2u\text{k}$

t_A

$$= k \ln \left| 1 + \frac{2uk}{uk} \right|$$

$$= k \ln|1 + 2|$$

$$= k \ln 3$$

17a $m = 0.5$ kg; $F = \left(x - \frac{1}{2}\right)$ N; $x_i = +5\ m$; $u = 0\ m/s$

$$F = ma$$

$$F = m\frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$x - \frac{1}{2} = m\frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$\left(x - \frac{1}{2}\right)dx = \left(\frac{1}{2}\right)\left(\frac{1}{2}v^2\right)$$

$$\frac{v^2}{4} = \int \left(x - \frac{1}{2}\right)dx$$

$$\frac{v^2}{4} = \frac{x^2 - x}{2} + C$$

Using initial conditions $x_i = +5\ m$, $u = 0\ m/s$,

$$\frac{0^2}{4} = \frac{5^2 - 5}{2} + C$$

$$C = -10$$

$$v^2 = 2(x^2 - x - 20)$$

$$v^2 = 2(x - 5)(x + 4)$$

As $v^2 \geq 0$, i.e. v^2 cannot be negative, so the particle cannot be at a position where $x < 5$. Hence the particle can never be at the origin.

17b Given $x_i = +5\ m$; $v = 2\sqrt{5}\ m/s$; $v^2 = 20\ (m/s)^2$

$v^2 = 2(x^2 - x - 20)$

$20 = 2(x^2 - x - 20)$

$x^2 - x - 20 = 10$

$x^2 - x - 30 = 0$

$(x + 5)(x - 6) = 0$

$x = -5$ or 6

Given that $x \geq 5$, $x = -5$ cannot be the chosen solution, furthermore the particle cannot pass through the origin. $\therefore x = 6$ m is the correct solution, this is the position where the velocity is $2\sqrt{5}$ m/s. Thus, the particle moves forwards with increasing velocity from this $x = 6$ m position.

18a Given $m = 2$ kg; $F = 6x^2$ N; $x_i = 1\ m$; $u_i = -\sqrt{2}$ m/s

$F = ma$

$6x^2 = m\frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$6x^2 = 2(\frac{1}{2})\frac{d}{dx}(v^2)$

$6x^2 = \frac{d}{dx}(v^2)$

Rearranging and integrating,

$v^2 = 6\int x^2\,dx$

$v^2 = 2x^3 + C$

Using initial conditions $x_i = 1$ m, $u_i = -\sqrt{2}$ m/s,

$\left(-\sqrt{2}\right)^2 = 2(1)^3 + C$

$2 = 2 + C$

$C = 0$

$\therefore v^2 = 2x^3$

18b From part a) we have:

$$\frac{dx}{dt} = \sqrt{2x^3}$$

Rearranging,

$$\frac{dx}{\sqrt{2x^3}} = dt$$

$$dt = \frac{1}{\sqrt{2}} x^{-\frac{3}{2}}\, dx$$

Integrating both sides,

$$t + C$$

$$= \frac{\frac{1}{\sqrt{2}} x^{-\frac{3}{2}+1}}{-\frac{3}{2}+1}$$

$$= \frac{\frac{1}{\sqrt{2}} x^{-\frac{1}{2}}}{-\frac{1}{2}}$$

$$= (-2)\frac{1}{\sqrt{2}}\, x^{-\frac{1}{2}}$$

$$= -\sqrt{2}\, x^{-\frac{1}{2}}$$

$$= -\sqrt{\frac{2}{x}}$$

Using initial conditions $t = 0s,\ x_i = 1\text{ m},\ u_i = -\sqrt{2}\text{ m/s}$,

$$0 + C = -\sqrt{\frac{2}{1}}$$

$$C = -\sqrt{2}$$

$$\therefore t - \sqrt{2} = -\sqrt{\frac{2}{x}}$$

Rearranging for x,

$$x = \frac{2}{\left(t - \sqrt{2}\right)^2}$$

The particle starts at position $x = 1$ and moves away from the origin and then it disappears at time $t = \sqrt{2}$ seconds.

19a Given $\ddot{x} = 3(1 - x^2)\ m/s^2$; $u = 4\ m/s$; $x_i = 0\ m$

$\ddot{x} = 3(1 - x^2)$

$\frac{d}{dx}\left(\frac{v^2}{2}\right) = 3(1 - x^2)$

Rearranging and integrating,

$\frac{v^2}{2}$

$= 3\int (1 - x^2)\,dx$

$= 3(x - \frac{x^3}{3}) + C$

$= (3x - x^3) + C$

Using initial conditions $u = 4\ m/s$, $x_i = 0\ m$,

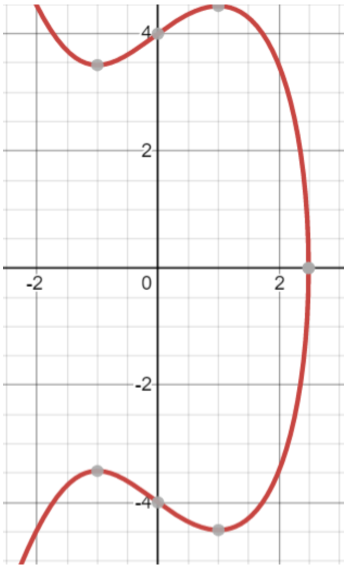
$\frac{4^2}{2} = 3(0) - (0)^3 + C$

$C = 8$

$\frac{v^2}{2} = 3x - x^3 + 8$

$\therefore v^2 = 6x - 2x^3 + 16$

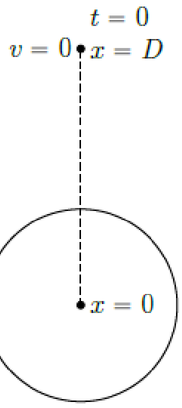
19b $v^2 = 6x - 2x^3 + 16$



From the sketch, we can see that the particle does change direction.

Solutions to Exercise 6A Enrichment questions

20



$$\ddot{x} = \frac{d\left(\frac{1}{2}v^2\right)}{dx} = -kx^{-2}$$

$$\int \frac{d\left(\frac{1}{2}v^2\right)}{dx}\,dx = \int kx^{-2}\,dx$$

$$\frac{1}{2}v^2$$

$$= \frac{-kx^{-1}}{-1} + C$$

$$= kx^{-1} + C$$

When $t = 0, x = D$ and $v = 0$

So, $C = -kD^{-1}$

Hence,

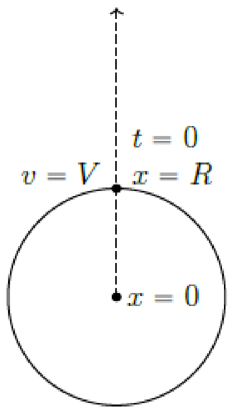
$$\frac{1}{2}v^2 = \frac{k}{x} - \frac{k}{D}$$

$$v^2 = 2k\left(\frac{1}{x} - \frac{1}{D}\right)$$

So, the speed is

$$\sqrt{2k\left(\frac{1}{x} - \frac{1}{D}\right)} = \sqrt{\frac{2k(D-x)}{Dx}}$$

21a



$\ddot{x} = \dfrac{-k}{x^2}$

When $t = 0$, $x = R$ and $\ddot{x} = -g$. So

$-g = \dfrac{-k}{R^2}$

Hence, $k = gR^2$

21b From question 20,

$\frac{1}{2}v^2$

$= \frac{k}{x} + C$

$= \frac{gR^2}{x} + C$

When $x = R, v = V$.Hence,

$\text{C} = \frac{1}{2}v^2 - gR$

$\frac{1}{2}v^2 = \frac{gR^2}{x} + \frac{1}{2}V^2 - gR$

And

$v^2 = \frac{2gR^2}{x} + V^2 - 2gR$

At v= 0, $x = H$ (the greatest height)

Hence,

$0 = \frac{2gR^2}{H} + V^2 - 2gR$

$\frac{2gR^2}{H} = 2gR - V^2$

$H = \frac{2gR^2}{2gR - V^2}$

21c H does not exist if:

$V^2 \geq 2gR$

i.e., $V^2 \geq 2 \times 0.0098 \times 6400\ \text{km/s}$

So, $V \geq 11.2\ \ \text{km/s}$

22a $v = 8 - 3e^{-2t}$

$\ddot{x} = 6e^{-2t} = 2(8 - v)$

$v\dfrac{dv}{dx} = 2(8 - v)$

$\dfrac{dv}{dx} = \dfrac{2(8 - v)}{v}$

$\dfrac{dx}{dv} = \dfrac{v}{2(8 - v)}$

$2dx = \dfrac{v}{8 - v}dv$

$2\int dx = \int \dfrac{v}{8 - v}dv$

22b Let the required distance be D m

$2\int_0^D dx$

$= \int_0^7 \dfrac{v}{8 - v}dv$

$= \int_0^7 \dfrac{-(8 - v) + 8}{8 - v}dv$

$= \int_0^7 \dfrac{8}{8 - v} - 1\, dv$

Hence:

$2D$

$= [-8\ln|8 - v| - 1]_0^7$

$= 8\ln 8 - 7$

≈ 4.82 m

Solutions to Exercise 6B Foundation questions

1a The amplitude is 12.

The period is $\frac{2\pi}{\frac{\pi}{2}} = 4$

1b

$v = -6\pi \sin\frac{\pi}{2}t$

$\ddot{x} = -3\pi^2 \cos\frac{\pi}{2}t$

Thus, $\ddot{x} = -\left(\frac{\pi}{2}\right)^2 x$

$= -\frac{\pi^2}{4}x$

1c At $t = 0$, $x = 12\cos 0 = 12$ cm

At $t = 0$, $v = -6\pi \sin 0 = 0$ cm/s

1d Solving $x = 0$ gives

$0 = 12\cos\frac{\pi}{2}t$

$\cos\frac{\pi}{2}t = 0$

$\frac{\pi}{2}t = \frac{\pi}{2}, \frac{3\pi}{2}$

$t =$ after 1 second

1e As the next time the particles hits the origin in at $t = 3$, it is 2 seconds between each visit.

2a The amplitude is 2.

The period is $\frac{2\pi}{4\pi} = \frac{1}{2}$ s

2b Max and min are 2 and -2. The points of intersection with the x-axis are $\frac{\frac{1}{2}}{2} = \frac{1}{4}$.

$\therefore \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$

2c $v = 8\pi \cos 4\pi t$

$\ddot{x} = -32\pi^2 \sin 4\pi t$

2d $x = 2 \sin 4\pi t$

$\ddot{x} = -32\pi^2 \sin 4\pi t$

$= -16\pi^2 (2 \sin 4\pi t)$

$= -16\pi^2 x$

2e When $v = 0$,

$0 = 8\pi \cos 4\pi t$

$\cos 4\pi t = 0$

$4\pi t = \frac{\pi}{2}, \frac{3\pi}{2}$

$t = \frac{1}{8}, \frac{3}{8}$

At $t = \frac{1}{8}$, $\ddot{x} = -32\pi^2$

At $t = \frac{3}{8}$, $\ddot{x} = 32\pi^2$

2f Greatest speed is when acceleration is 0.

When $\ddot{x} = 0$,

$0 = -32\pi^2 \sin 4\pi t$

$\sin 4\pi t = 0$

$4\pi t = 0, \pi$

$t = \dfrac{1}{4}$

When $t = \frac{1}{4}$,

$v = 8\pi \cos \pi$

$= 8\pi \text{ m/s}$

3a When $t = 0, x = 0$

3b $v = 4\cos \pi t$

$\ddot{x} = -4\pi \sin \pi t$

$= -\pi^2 \times \dfrac{4}{\pi} \sin \pi t$

$= -\pi^2 x$

3c The amplitude is $\frac{4}{\pi}$. The period is $\frac{2\pi}{\pi} = 2$ s

3d The maximum distance is the amplitude. $\therefore \frac{4}{\pi}$

Maximum speed $= \frac{4}{\pi} \times \pi = 4$ seconds

3e Max and Min of x at $\frac{4}{\pi}$ and $-\frac{4}{\pi}$.

When $x = 0$,

$0 = \frac{4}{\pi}\sin \pi t$

$\sin \pi t = 0$

$t = 1, 2$

$v = 4\cos \pi t$

Max and Min of v at 4 and -4.

$\ddot{x} = -4\pi \sin \pi t$

Max and min are -4π and 4π

3f When $v = -4$ m/s,

$-4 = 4\cos \pi t$

$\cos \pi t = -1$

$t = 1$

When $v = 4$ m/s,

$4 = 4\cos \pi t$

$\cos \pi t = 1$

$t = 2$

3g When $\ddot{x} = -4\pi$ m/s,

$-4\pi = -4\pi \sin \pi t$

$\sin \pi t = 1$

$t = \frac{1}{2}$

When $\ddot{x} = 4\pi$ m/s,

$4\pi = -4\pi \sin \pi t$

$\sin \pi t = -1$

$t = \frac{3}{2}$

4a

$$n = \frac{2\pi}{\pi} = 2$$

$$a = \frac{4}{2} = 2$$

$$x = 2\sin 2t$$

4b $a = 6$

$$n = \frac{4}{6} = \frac{2}{3}$$

$$x = 6\sin\frac{2}{3}t$$

5a $x = b\sin nt + c\cos nt$

$$v = bn\cos nt - cn\sin nt$$

$$\ddot{x} = -bn^2\sin nt - cn^2\cos nt$$

$$= -n^2x$$

5b i When $t = 0, x = 3$

$3 = b\sin 0 + c\cos 0$

Thus, $c = 3$

When $t = 0, v = 0$

$0 = bn\cos 0 - cn\sin 0$

Thus, $b = 0$

So, $x = 3\cos nt$

5b ii

$$n = \frac{2\pi}{1} = 2\pi$$

When $t = 0, x = 5$

$5 = \ b \sin 0 + c \cos 0$

Thus, $c = 5$

When $t = 0, v = 0$

$0 = \ bn \cos 0 - cn \sin 0$

Thus, $b = 0$

So, $x = 5 \cos 2\pi t$

$t = \frac{1}{4}$ as it takes 1 s to complete cycle of 2π so it takes $\frac{1}{4}$ s to return to origin.

6a $x = a\sin \pi t$

$v = a\pi \cos \pi t$

$\ddot{x} = -a\pi^2 \sin \pi t = -\pi^2 x$

6b

$$a = \frac{4\pi}{\pi} = 4$$

6c When $v = 2\pi$,

As this is slower than the starting velocity, this value is negative

$-2\pi = 4\pi \cos \pi t$

$$-\frac{1}{2} = \cos \pi t$$

$$t = \frac{1}{3}, \frac{2}{3}$$

7a $v = 6\sin 3t$

$\ddot{x} = 18\cos 3t$

A particle is stationary when the velocity is 0.

When $x = 10$,

$10 = 12 - 2\cos 3t$

$$t = \frac{2\pi}{3}$$

When $t = \frac{2\pi}{3}$,

$v = 6\sin 2\pi = 0$

7b $a = 2$

$$t = \frac{2\pi}{3}$$

Centre $x = 12$

7c Moving between $10 \leq x \leq 14$

$$t = \frac{\frac{2\pi}{3}}{2} = \frac{\pi}{3}$$

7d When $x = 10$,

$10 = 12 - 2\cos 3t$

$1 = \cos 3t$

$t = \frac{2\pi}{3}, \frac{4\pi}{3}$

When $t = \frac{2\pi}{3}, \frac{4\pi}{3}$,

$v = 0, \ddot{x} = 18 \text{ cm/s}^2$

7e When $x = 12$,

$12 = 12 - 2\cos 3t$

$0 = \cos 3t$

$t = \frac{\pi}{6}, \frac{\pi}{2}$

When $t = \frac{\pi}{6}, \frac{\pi}{2}$

$v = 6 \text{ cm/s}$

$\ddot{x} = 0 \text{ cm/s}^2$

8a Amplitude $= 6$, period $= \frac{2\pi}{2}$, initial phase: $\frac{\pi}{2}$

8b

$\dot{x} = 12\cos\left(2t + \frac{\pi}{2}\right)$

$\ddot{x} = -24\sin\left(2t + \frac{\pi}{2}\right)$

$\quad = -2^2 x$

8c When $x = 0$,

$0 = 6\sin\left(2t + \frac{\pi}{2}\right)$

$\sin\left(2t + \frac{\pi}{2}\right) = 0$

$2t + \frac{\pi}{2} = \pi$ and $2t + \frac{\pi}{2} = 2\pi$

$t = \frac{\pi}{4}, \frac{3\pi}{4}$

When $t = \frac{\pi}{4}$, $v = -12$

When $t = \frac{3\pi}{4}$, $v = 12$

8d Velocity is maximum when acceleration is zero.

When $\ddot{x} = 0$,

$0 = -24\sin\left(2t + \frac{\pi}{2}\right)$

$2t + \frac{\pi}{2} = 2\pi$ and $2t + \frac{\pi}{2} = 4\pi$

$t = \frac{3\pi}{4}, \frac{7\pi}{4}$

When $t = \frac{3\pi}{4}, \frac{7\pi}{4}$, $x = 0$

8e As the period is π, $t = \pi, 2\pi$

$\dot{x} = 12\cos\left(2(\pi) + \frac{\pi}{2}\right) = 0$

$\ddot{x} = -24\sin\left(2(\pi) + \frac{\pi}{2}\right) = -24$

$\dot{x} = 12\cos\left(2(2\pi) + \frac{\pi}{2}\right) = 0$

$\ddot{x} = -24\sin\left(2(2\pi) + \frac{\pi}{2}\right) = -24$

Solutions to Exercise 6B Development questions

9a Given amplitude, $a = 120$ m; period, $T = 24$ s; $x_0 = 0$ m; $v_0 > 0$ m/s

$$T = \frac{2\pi}{n}$$

$$24 = \frac{2\pi}{n}$$

$$n = \frac{\pi}{12}$$

$$x = 120\ \sin\frac{\pi}{12}t$$

$$\dot{x} = v$$

$$v$$

$$= \left(\frac{\pi}{12}\right)120\ \cos\frac{\pi}{12}t$$

$$= 10\pi\ \cos\frac{\pi}{12}t$$

The maximum speed occurs as it passes through the centre of motion where:

$$|\dot{x}|$$

$$= (na),$$

$$= \left(\frac{\pi}{12}\right)(120)$$

$$= 10\pi \text{ m/s}$$

$$\therefore |\dot{x}| \doteqdot 31.4 \text{ m/s}$$

9bi When $x = 30$ m to the right of the origin,

$$x = 120\ \sin\frac{\pi}{12}t$$

$$30 = 120\ \sin\frac{\pi}{12}t$$

$$\frac{1}{4} = \sin\frac{\pi}{12}t$$

$$t = \frac{12}{\pi}\sin^{-1}\frac{1}{4}$$

$t \doteqdot 0.9652$ seconds

9bii When $x = 30$ m to the left of the origin,

$$x = 120\ \sin\frac{\pi}{12}t$$

$$-30 = 120\ \sin\frac{\pi}{12}t$$

$$-\frac{1}{4} = \sin\frac{\pi}{12}t$$

t

$$= \frac{12}{\pi}\sin^{-1}\left(-\frac{1}{4}\right)$$

$= \frac{T}{2} + \frac{12}{\pi}\sin^{-1}\frac{1}{4}$ (add half a period since the particle is now move left)

$$= 12 + \frac{12}{\pi}\sin^{-1}\frac{1}{4}$$

t $\doteqdot$ 12.97 seconds

9c Half maximum speed $= \frac{10\pi}{2} = 5\pi$

Hence $v = 5\pi$ or -5π

When $v = 5\pi$,

$$v = 10\pi\cos\frac{\pi}{12}t$$

$$5\pi = 10\pi\cos\frac{\pi}{12}t$$

$$\frac{1}{2} = \cos\frac{\pi}{12}t$$

t

$$= \frac{12}{\pi}\cos^{-1}\frac{1}{2}$$

$$= \frac{12}{\pi}\times\frac{\pi}{3}$$

$\mathrm{t} = 4$ seconds

When $v = -5\pi$,

$$-5\pi = 10\pi\cos\frac{\pi}{12}t$$

$$-\frac{1}{2} = \cos\frac{\pi}{12}t$$

t

$$= \frac{12}{\pi}\cos^{-1}\left(-\frac{1}{2}\right)$$

$$= \frac{T}{2} - \frac{12}{\pi}\cos^{-1}\frac{1}{2}$$

$$= \frac{24}{2} - \frac{12}{\pi}\times\frac{\pi}{3}$$

$$= 12 - 4$$

$\mathrm{t} = 8$ seconds

10a Given amplitude, $a = 4$ cm; period, $T = \frac{\pi}{2}$ s; $x_0 = 4$ cm; $v_0 > 0$ m/s

$T = \frac{2\pi}{n}$

$\frac{\pi}{2} = \frac{2\pi}{n}$

$n = 4$

The particle is at the positive extreme initially and is moving towards O.

$x = A\cos nt$

$x = 4\cos 4t$

$\dot{x} = v$

v

$= (-4)4 \sin 4t$

$= -16 \sin 4t$

10bi For $x = +2$ cm

$x = 4\cos 4t$

$2 = 4\cos 4t$

$\frac{1}{2} = \cos 4t$

$t = \frac{1}{4}\cos^{-1}\frac{1}{2}$

$t = \frac{1}{4} \times \frac{\pi}{3}$

$t = \frac{\pi}{12}$ s

10bii For $x = -2$ cm

$x = 4\cos 4t$

$-2 = 4\cos 4t$

$-\frac{1}{2} = \cos 4t$

t

$= \frac{1}{4}\cos^{-1}\left(-\frac{1}{2}\right)$

$= \frac{T}{2} - \frac{1}{4}\cos^{-1}\frac{1}{2}$

$= \frac{\pi}{4} - \frac{\pi}{12}$

$t = \frac{\pi}{6}$ s

10c $|v_{max}| = 16$ cm/s

For half maximum speed:

$v = -8$ cm/s and $v = 8$ cm/s

When $v = -8$ cm/s ,

$v = -16 \sin 4t$

$-8 = -16 \sin 4t$

$\frac{1}{2} = \sin 4t$

t

$= \frac{1}{4} \sin^{-1} \frac{1}{2}$

$= \frac{1}{4} \times \frac{\pi}{6}$

$t = \frac{\pi}{24}$ seconds

When $v = 8$ cm/s ,

$v = -16 \sin 4t$

$8 = -16 \sin 4t$

$-\frac{1}{2} = \sin 4t$

t

$= \frac{1}{4} \sin^{-1} \left(-\frac{1}{2}\right)$

$= \frac{T}{2} - \frac{1}{4} \sin^{-1} \frac{1}{2}$

$= \frac{\pi}{4} - \frac{\pi}{24}$

$= \frac{5\pi}{24}$ seconds

Therefore, the first two times when the speed is half the maximum speed are: π/24 s and 5π/24 s

11 Given $x = \sin^2 t$,

Using $\cos 2\theta = 1 - 2\sin^2\theta$

$$\sin^2\theta = \frac{1 - \cos 2\theta}{2}$$

$$\therefore \sin^2 t = \frac{1 - \cos 2t}{2}$$

$$x = \frac{1}{2} - \frac{1}{2}\cos 2t$$

Using the form $x = x_0 - a\cos nt$:

The centre of the motion is the line $x_0 = \frac{1}{2}$ cm,

The amplitude of the motion, $a = \frac{1}{2}$ cm

The range of the SHM:

$$\left(\frac{1}{2} - \frac{1}{2}\right) \le x \le \left(\frac{1}{2} + \frac{1}{2}\right)$$

$$0 \le x \le 1$$

The Period, T:

$$T = \frac{2\pi}{n}$$

$$T = \frac{2\pi}{2}$$

$$T = \pi \text{ s}$$

12a Given $x = 3 - 2\cos^2 2t$

Using $\cos 2\theta = 2\cos^2\theta - 1$

$$2\cos^2\theta = 1 + \cos 2\theta$$

$$\therefore 2\cos^2 2t = 1 + \cos 4t$$

$$x = 3 - (1 + \cos 4t)$$

$$x = 2 - \cos 4t$$

12b Using the form $x = x_0 - a\cos nt$:

The centre of the motion is the line $x_0 = 2$ cm,

The amplitude of the motion, $a = 1$ cm,

The range of the SHM:

$(2-1) \le x \le (2+1)$

$1 \le x \le 3$

The Period, T:

$$T = \frac{2\pi}{n}$$

$$T = \frac{2\pi}{4}$$

$$T = \frac{1}{2}\pi \text{ s}$$

12c

$$v = \frac{dx}{dt} = 4\sin 4t$$

Maximum speed $= 4\text{cm/s}$

The maximum speed occurs at the centre and as the particle starts at one of the extremes, it will occur after:

$$\frac{T}{4}$$

$$= \frac{\pi}{2} \times \frac{1}{4}$$

$$= \frac{\pi}{8}\text{s}$$

13a Given $x = 2 + 3\cos t + 3\sqrt{3}\sin t$

$\dot{x} = v$

$v = -3\sin t + 3\sqrt{3}\cos t$

$\ddot{x}$

$= -3\cos t - 3\sqrt{3}\sin t$

$= -\left(3\cos t + 3\sqrt{3}\sin t\right)$

$= -(x - 2)$

Which is a function of the form:

$\ddot{x} = -n^2(x - c)$, where $n = 1$ and $c = 2$.

This confirms that this is SHM.

13b The centre of the motion is the line $x = 2$ cm

13c The Period, T, of the motion is:

T

$= \frac{2\pi}{n}$

$= \frac{2\pi}{1}$

$= 2\pi$ s

13d Express $3\cos t + 3\sqrt{3}\sin t$ in the form of

$A\cos t\,(x-\theta)$, $A > 0,\ 0 < \theta < \frac{\pi}{2}$

x

$= 6\left(\dfrac{1}{2}\cos t + \dfrac{\sqrt{3}}{2}\sin t\right)$

$= 6\left(\cos\dfrac{\pi}{3}\cos t + \sin\dfrac{\pi}{3}\sin t\right)$ (since we are aiming for $0 < \theta < \dfrac{\pi}{2}$)

$= 6\left(\cos t\cos\dfrac{\pi}{3} + \sin t\sin\dfrac{\pi}{3}\right)$

Using $\cos(A-B) = \cos A\cos B + \sin A\sin B$,

$x = 6\cos\left(t - \dfrac{\pi}{3}\right)$

Which is of the form $A\cos(x-\theta)$where $A = 6, x = t$ and $\theta = \frac{\pi}{3}$

13e $x = 2 + 6\cos\left(t - \frac{\pi}{3}\right)$

The amplitude is 6 cm

The initial phase $\varphi = -\frac{\pi}{3}$

13f The interval of particle = range of oscillation

Range: $(2-6) \le x \le (2+6)$

Range: $-4 \le x \le 8$

14a Given $x = b\sin nt + c\cos nt$, where $n > 0$

Period, $T = 4\pi\ s$; $x_0 = 6$; $v_0 = 3$

$x = b\sin nt + c\cos nt$

$T = \dfrac{2\pi}{n}$

$4\pi = \dfrac{2\pi}{n}$

$n = \dfrac{1}{2}$

$x = b\sin\dfrac{t}{2} + c\cos\dfrac{t}{2}$

Using initial conditions, $x_0 = 6, t_0 = 0$ s,

$6 = b\sin\dfrac{1}{2}(0) + c\cos\dfrac{1}{2}(0)$

$6 = 0 + c(1)$

Hence $c = 6$

$\dot{x} = \dfrac{dx}{dt}$

$\dfrac{dx}{dt} = v$

v

$= \left(\dfrac{1}{2}\right)b\cos\dfrac{t}{2} - \left(\dfrac{1}{2}\right)6\sin\dfrac{t}{2}$

$= \dfrac{b}{2}\cos\dfrac{t}{2} - 3\sin\dfrac{t}{2}$

$= bn\cos nt - cn\sin nt$

Using initial conditions $v_0 = 3, t_0 = 0$ s,

$3 = \dfrac{b}{2}\cos\left(\dfrac{1}{2}\right)(0) - 3\sin\left(\dfrac{1}{2}\right)(0)$

$3 = \dfrac{b}{2}$

Hence $b = 6$

Thus:

$x = 6\left(\sin\frac{t}{2} + \cos\frac{t}{2}\right)$

The particle is at the origin for the first two times when:

$0 = 6\left(\sin\frac{t}{2} + \cos\frac{t}{2}\right)$

$0 = \sin\frac{t}{2} + \cos\frac{t}{2}$

$\sin\frac{t}{2} = -\cos\frac{t}{2}$

$\tan\frac{t}{2} = -1$

$\frac{t}{2} = \frac{3\pi}{4}, \frac{7\pi}{4}$

$\therefore t = \frac{3\pi}{2}$ s and $\frac{7\pi}{2}$ s

14b Given Period, $T = 6\ s$; $x_0 = -2$; $v_0 = \dot{x} = 3$

$$T = \frac{2\pi}{n}$$

$$6 = \frac{2\pi}{n}$$

$$n = \frac{\pi}{3}$$

$$x = b\sin\frac{\pi}{3}t + c\cos\frac{\pi}{3}t$$

$$\dot{\mathrm{x}} = v$$

$$v = \left(\frac{\pi}{3}\right)b\cos\frac{\pi}{3}t - \left(\frac{\pi}{3}\right)c\sin\frac{\pi}{3}t$$

Using initial conditions,

$t_0 = 0$ s; $x_0 = -2$,

$$-2 = b\sin\frac{\pi}{3}0 + c\cos\frac{\pi}{3}0$$

$$c = -2$$

Similarly:

At $t_0 = 0$ s; $v_0 = 3$,

$$3 = \left(\frac{\pi}{3}\right)b\cos\frac{\pi}{3}(0) - \left(\frac{\pi}{3}\right)c\sin\frac{\pi}{3}(0)$$

$$3 = \left(\frac{\pi}{3}\right)b$$

$$b = \frac{9}{\pi}$$

Hence:

$$x = \frac{9}{\pi}\sin\frac{\pi}{3}t - 2\cos\frac{\pi}{3}t$$

The particle is at the centre the first two times when $x = 0$:

$$0 = \frac{9}{\pi}\sin\frac{\pi}{3}t - 2\cos\frac{\pi}{3}t$$

$$\frac{9}{\pi}\sin\frac{\pi}{3}t = 2\cos\frac{\pi}{3}t$$

$\tan\frac{\pi}{3}t = \frac{2\pi}{9}$

$t = \frac{3}{\pi}\tan^{-1}\frac{2\pi}{9}$

$t \doteqdot 0.582 \text{ and } \left(\frac{T}{2} + 0.582\right)$

$t \doteqdot 0.582 \text{ and } 3.582 \text{ s}$

15a Given $x = a\sin(nt + \alpha)$; $x_0 = 0$ m; $v_0 = 5$ m/s; Period $= 6$ s

$\dot{x} = v$

v

$= (n)a\cos(nt + \alpha)$

$= an\cos(nt + \alpha)$

$T = \frac{2\pi}{n}$

$6 = \frac{2\pi}{n}$

$n = \frac{\pi}{3}$

Using initial conditions, $x_0 = 0$; $t_0 = 0$ s; $v_0 = 5$ m/s

$0 = a\sin\left(\frac{\pi}{3}(0) + \alpha\right) = a\sin\alpha$

Since $0 \le a < 2\pi$,

$\alpha = 0$

Similarly:

$v = an\cos(nt + \alpha)$

$5 = a\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{3}(0) + (0)\right)$

$5 = \frac{a\pi}{3}$

$a = \frac{15}{\pi}$ m

15b Given $x = a\sin(nt + \alpha)$; $t_0 = 0$ s; $x_0 = -5$ m; $v_0 = 0$ m/s; Period $= 3\pi$ s

$v = an\cos(nt + \alpha)$, previously determined at 15a

$$T = \frac{2\pi}{n}$$

$$3\pi = \frac{2\pi}{n}$$

$$n = \frac{2}{3}$$

Using initial conditions:

$$x_0 = a\sin\left(\frac{2}{3}(0) + \alpha\right)$$

$$-5 = a\sin\alpha$$

and,

$$v_0 = \frac{2}{3}a\cos\left(\frac{2}{3}(0) + \alpha\right)$$

$$0 = \frac{2}{3}a\cos\left(\frac{2}{3}(0) + \alpha\right)$$

$$\frac{2a}{3}\cos\alpha = 0$$

The second condition requires:

$$\alpha = \frac{\pi}{2} \text{ or } \frac{3\pi}{2}$$

Using this with $x_0 = -5 = a\sin\alpha$, we then have noting $a > 0$:

$$a = 5 \text{ and } \alpha = \frac{3\pi}{2}$$

15c $x = a\sin(nt + \alpha)$; $t_0 = 0$ s; $x_0 = 1$ m; $v_0 = -1$ m/s; Period $= 2\pi$ s

$v = an\cos(nt + \alpha)$, previously determined at 15a

$$T = \frac{2\pi}{n}$$

$$2\pi = \frac{2\pi}{n}$$

$$n = 1$$

Using initial conditions:

$$x_0 = a\sin((0) + \alpha)$$

$$1 = a\sin\alpha$$

and,

$$v_0 = a\cos((0) + \alpha)$$

$$-1 = a\cos\alpha$$

Dividing the two requirements we have:

$$\frac{x_0}{v_0} = \frac{1}{-1} = \frac{a\sin\alpha}{a\cos\alpha}$$

$$\tan\alpha = -1$$

$$\alpha = \frac{3\pi}{4}$$

Using this we then have:

$$1 = a\sin\frac{3\pi}{4}$$

$$a = \sqrt{2}\text{ m}$$

16a $x = a\cos(2t + \alpha),\ \ a > 0, -\pi < \alpha < \pi;\ \ x_0 = 0;\ \ v_0 = 6$; assume $t_0 = 0$ s

$$\frac{dx}{dt} = (-2)a\sin(2t + \alpha)$$

$v = -2a\sin(2t + \alpha)$

Using initial conditions:

$x_0 = a\cos(2(0) + \alpha)$

$0 = a\cos\alpha$

Since $-\pi < a \leq \pi$,

$$\alpha = \frac{\pi}{2} \text{ or } -\frac{\pi}{2}$$

The second initial condition gives:

$v_0 = -2a\sin(2(0) + \alpha)$

$6 = -2a\sin\alpha$

$-3 = a\sin\alpha$

Given that $a > 0$,

$\therefore \alpha = -\frac{\pi}{2}$

$\therefore a = 3$

16b $x = a\cos(2t + \alpha)$, $a > 0, -\pi < \alpha < \pi$; $x_0 = 1$; $v_0 = -2\sqrt{3}$; assume $t_0 = 0$ s

$v = -2a\sin(2t + \alpha)$, previously determined at 16a

Using initial conditions:

$x_0 = a\cos(2(0) + \alpha)$

$1 = a\cos\alpha$

and,

$v_0 = -2a\sin(2(0) + \alpha)$

$-2\sqrt{3} = -2a\sin\alpha$

$\sqrt{3} = a\sin\alpha$

Dividing the initial conditions we have:

$$\frac{\sqrt{3}}{1} = \frac{a\sin\alpha}{a\cos\alpha}$$

$\sqrt{3} = \tan\alpha$

Hence:

$$\alpha = -\frac{5\pi}{6} \text{ or } \frac{\pi}{3}$$

Given that $v_0 < 0$,

$$\therefore \alpha = \frac{\pi}{3}$$

Hence:

$$1 = a\cos\frac{\pi}{3}$$

$$1 = a\left(\frac{1}{2}\right)$$

$\therefore a = 2$

17 Given $x = a\cos\left(\frac{\pi}{8}t + \alpha\right)$; $a > 0$; $0 < \alpha < 2\pi$;

$x = 0$ when t $= 2$ s; $v = -4$ cm/s when $t = 4$ s

$$\frac{dx}{dt} = \left(-\frac{\pi}{8}\right)a\sin\left(\frac{\pi}{8}t + \alpha\right)$$

$$v = -a\frac{\pi}{8}\sin\left(\frac{\pi}{8}t + \alpha\right)$$

Using the first conditions gives:

$$x_{t=2} = a\cos\left(\frac{\pi}{8}(2) + \alpha\right)$$

$$0 = a\cos\left(\frac{\pi}{4} + \alpha\right)$$

$$\cos\left(\frac{\pi}{4} + \alpha\right) = 0$$

Thus:

$$\alpha = \frac{\pi}{4} \text{ or } \frac{5\pi}{4}$$

The second conditions gives:

$$v_{t=4} = -a\frac{\pi}{8}\sin\left(\frac{\pi}{8}(4) + \alpha\right)$$

$$-4 = -a\frac{\pi}{8}\sin\left(\frac{\pi}{2} + \alpha\right)$$

$$\frac{32}{\pi} = a\sin\left(\frac{\pi}{2} + \alpha\right)$$

Using now the result, $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$. We have that:

$$\sin\left(\frac{\pi}{2} + x\right) = \cos x$$

And so:

$$\frac{32}{\pi} = a\cos\alpha$$

Given $a > 0$ and $0 < \alpha < 2\pi$, and that:

$$\cos\frac{\pi}{4} > 0 \text{ and } \cos\frac{5\pi}{4} < 0$$

We conclude that:

$$\alpha = \frac{\pi}{4}$$

Hence:

$$\frac{32}{\pi} = a\sin\left(\frac{\pi}{2} + \frac{\pi}{4}\right)$$

$$\frac{32}{\pi} = a\sin\frac{3\pi}{4}$$

$$a = \frac{32\sqrt{2}}{\pi}$$

$$a \doteqdot 14.405\ cm$$

18 Given $x = a\sin(nt + \alpha)$, $a > 0, 0 < \alpha < 2\pi$; Period, $T = 8\pi$ s

When $t = 1$ s, $x_1 = 3$ m, and $v_1 = -1$ m/s

$$v = \frac{dx}{dt} = an\cos(nt + \alpha)$$

$$T = \frac{2\pi}{n}$$

$$8\pi = \frac{2\pi}{n}$$

$$n = \frac{1}{4}$$

Hence:

$$x = a\sin\left(\frac{1}{4}t + \alpha\right)$$

Using conditions for x and v,

$$x_1 = a\sin\left(\frac{1}{4}(1) + \alpha\right)$$

$$3 = a\sin\left(\frac{1}{4} + \alpha\right)$$

$$v_1 = \left(\frac{1}{4}\right)a\cos\left(\frac{1}{4}(1) + \alpha\right)$$

$$-1 = \left(\frac{1}{4}\right)a\cos\left(\frac{1}{4} + \alpha\right)$$

Diving these results we have:

$$\frac{x_1}{v_1} = \frac{3}{-1} = \frac{a\sin(\frac{1}{4} + \alpha)}{\left(\frac{1}{4}\right)a\cos(\frac{1}{4} + \alpha)}$$

$$-3 = 4\frac{a\sin(\frac{1}{4} + \alpha)}{a\cos(\frac{1}{4} + \alpha)}$$

$$-\frac{3}{4} = \tan\left(\frac{1}{4} + \alpha\right)$$

$$\frac{1}{4} + \alpha = \tan^{-1}\left(-\frac{3}{4}\right)$$

Now the first condition on the displacement requires $\sin > 0$ while the second condition on the velocity requires $\cos < 0$, hence, $\left(\frac{1}{4} + \alpha\right)$ is in the 2nd quadrant. Thus:

$$\frac{1}{4} + \alpha = \pi - \tan^{-1}\left(\frac{3}{4}\right)$$

$$\alpha = -\frac{1}{4} + \pi - \tan^{-1}\left(\frac{3}{4}\right)$$

$$\alpha \doteqdot 2.248$$

Hence:

3

$$= a\sin\left(\frac{1}{4} - \frac{1}{4} + \pi - \tan^{-1}\left(\frac{3}{4}\right)\right)$$

$$= a\sin\left(\pi - \tan^{-1}\left(\frac{3}{4}\right)\right)$$

$$= a\sin\left(\tan^{-1}\left(\frac{3}{4}\right)\right)$$

$$= a\frac{3}{5}$$

Therefore:

$$a = 5 \text{ m}$$

19a Given, Period, $T = \frac{\pi}{2}$ s; $x_0 = 3$ m; $v_0 = 16$ m/s; assume $t_0 = 0$ s

$$T = \frac{2\pi}{n}$$

$$\frac{\pi}{2} = \frac{2\pi}{n}$$

$$n = 4$$

$$x$$

$$= b \sin nt + c \cos nt$$

$$= b \sin 4t + c \cos 4t$$

$$\frac{dx}{dt} = v$$

$$v = 4b \cos 4t - 4c \sin 4t$$

Using initial conditions:

$$3 = b\, sin\, 4(0) + c\, cos\, 4(0)$$

$$3 = c\ (1)$$

$$c = 3$$

And

$$16 = 4b \cos 4(0) - 4c \sin 4(0)$$

$$16 = 4b\ (1)$$

$$b = 4$$

$$\therefore x = 4 \sin 4t + 3 \cos 4t$$

19b Given $x = 4\sin 4t + 3\cos 4t$; $a > 0, 0 < \varepsilon < 2\pi$

x

$= 3\cos 4t + 4\sin 4t$

$= 5\left(\frac{3}{5}\cos 4t + \frac{4}{5}\sin 4t\right)$

$x = 5(\cos\varepsilon\cos 4t + \sin\varepsilon\sin 4t)$

$x = 5(\cos 4t\cos\varepsilon + \sin 4t\sin\varepsilon)$

Using $\cos(A - B) = \cos A\cos B + \sin A\sin B$, gives:

$x = 5\cos(4t - \varepsilon)$

Where:

$\varepsilon = cos^{-1}\frac{3}{5} = \sin^{-1}\frac{4}{5}$

Thus:

$\varepsilon = \tan^{-1}\frac{4}{3}$

$\varepsilon \doteqdot 37^\circ$

This is of the form:

$x = a\cos(nt - \varepsilon)$

With:

$a = 5$ m

$\varepsilon = \tan^{-1}\frac{4}{3}$

19c Amplitude of SHM is 5 m as previously determined at 19b

$x = a\cos(nt - \varepsilon)$

$\dot{x} = -(n)a\ \sin(nt - \varepsilon)$

Substituting $a = 5, n = 4,\ \varepsilon = \tan^{-1}\frac{4}{3}$, (previously determined at 19b)

$\dot{x} = -(4)5\ \sin(4t - \varepsilon)$

$v = -20\ \sin(4t - \varepsilon)$

So the maximum speed is 20 m/s

19d When $x = 0$,

$x = 4\sin 4t + 3\cos 4t$

$0 = 4\sin 4t + 3\cos 4t$

$4\sin 4t = -3\cos 4t$

$\dfrac{\sin 4t}{\cos 4t} = -\dfrac{3}{4}$

$\tan 4t = -\dfrac{3}{4}$

Thus:

t

$= \dfrac{T}{2} - \dfrac{1}{4}\tan^{-1}\dfrac{3}{4}$ (half period becaus $v_0 > 0$)

$= \dfrac{\frac{\pi}{2}}{2} - \dfrac{1}{4}\tan^{-1}\dfrac{3}{4}$

$= \dfrac{\pi}{4} - \dfrac{1}{4}\tan^{-1}\dfrac{3}{4}$

Also, when $x = 0$

$0 = 5\cos(4t - \varepsilon)$

$\cos(4t - \varepsilon) = 0$

$4t - \varepsilon = \dfrac{\pi}{2}$

$4t - \tan^{-1}\dfrac{4}{3} = \dfrac{\pi}{2}$

$4t = \dfrac{\pi}{2} + \tan^{-1}\dfrac{4}{3}$

$t = \dfrac{\pi}{8} + \dfrac{1}{4}\tan^{-1}\dfrac{4}{3}$

Since $\tan^{-1}\dfrac{4}{3} = \dfrac{\pi}{2} - \tan^{-1}\dfrac{3}{4}$, $\left(\text{using}\cot x = \tan\left(\dfrac{\pi}{2} - x\right)\right)$

$t = \dfrac{\pi}{8} + \dfrac{1}{4}\tan^{-1}\dfrac{4}{3}$ becomes

t

$= \frac{\pi}{8} + \frac{1}{4}\left(\frac{\pi}{2} - \tan^{-1}\frac{3}{4}\right)$

$= \frac{\pi}{8} + \frac{\pi}{8} - \frac{1}{4}\tan^{-1}\frac{3}{4}$

$= \frac{\pi}{4} - \frac{1}{4}\tan^{-1}\frac{3}{4}$

Hence the two answers obtained are the same.

20 $T = 24$ hours; Range $= 19°\text{C} - 9°\text{C} = 10°\text{C}$; $x_0 = 9$. The centre of motion is given as:

c

$= \frac{19 + 9}{2}$

$= \frac{28}{2}$

$= 14°\text{C}$

Amplitude, A is given by:

A

$= 19 - 9$

$= 5°\text{C}$

And n, is given by:

$n = \frac{2\pi}{T}$

$n = \frac{2\pi}{24}$

$n = \frac{\pi}{12}$ hours

t will be in hours after 4: 00 am. All this gives:

x

$= a\cos(nt + \alpha) + c$

$= -5\cos\left(\frac{\pi}{12}t\right) + 14$

$= 14 - 5\cos\left(\frac{\pi}{12}t\right)$

20a When $x = 14°\text{C}$

$14 = 14 - 5\cos\left(\frac{\pi}{12}t\right)$

$\cos\left(\frac{\pi}{12}t\right) = 0$

$t = \frac{12}{\pi}\cos^{-1} 0$

$t = \frac{\pi}{2} \text{ or } \frac{3\pi}{2}$

$\therefore t = 6 \text{ or } 18$

That is, 6 or 18 hours after 4:00 am, the temperature will be 14°.

In the given time interval, it will be 10:00 am.

20b When $x = 11°\text{C}$

$11 = 14 - 5\cos\left(\frac{\pi}{12}t\right)$

$\cos\left(\frac{\pi}{12}t\right) = \frac{3}{5}$

$t = \frac{12}{\pi}\cos^{-1}\frac{3}{5}$

$t = 3.542$ ...hours after 4:00 am

$t = 3$ hours 32.52 ... minutes after 4:00 am

The temperature will be 11°C at 7:33am

20c When $x = 17°\text{C}$

$17 = 14 - 5\cos\left(\frac{\pi}{12}t\right)$

$\cos\left(\frac{\pi}{12}t\right) = -\frac{3}{5}$

$t = \frac{12}{\pi}\cos^{-1}\left(-\frac{3}{5}\right)$

$t = 8.457$ ...hours after 4:00 am

$t = 8$ hours 27.479 ... minutes after 4:00 am

The temperature will be 17°C at 12:27am

21 Range $= 16 - 10 = 6$ m

$$T = \frac{2\pi}{n}$$

$$\frac{T}{2}$$

$$= (16{:}00 - 9{:}00)$$

$$= 7 \text{ hours}$$

$$T = 14 \text{ hours}$$

So,

$$14 = \frac{2\pi}{n}$$

$$n = \frac{\pi}{7}$$

Centre of motion,

$$c = \frac{16 + 10}{2} = 13$$

Amplitude, A,

$$A$$

$$= 16 - c$$

$$= 16 - 13$$

$$\therefore A = 3 \text{ m}$$

Hence:

$$x$$

$$= a\cos(nt + \alpha) + c$$

$$= -3\cos\left(\frac{\pi}{7}t\right) + 13$$

$$x = 13 - 3\cos\left(\frac{\pi}{7}t\right)$$

When $x = 12$ m:

$$12 = 13 - 3\cos\left(\frac{\pi}{7}t\right)$$

$$\cos\left(\frac{\pi}{7}t\right) = \frac{1}{3}$$

$$t = \frac{7}{\pi}\cos^{-1}\frac{1}{3}$$

$t = 2.742$ …hours

Hence for t ≑ 2 hours 45 minutes after 9:00 am the tide will be high enough for ships to pass. The water will then remain above 12 m as it continues to increase until 4:00 pm before receding back down. However, the water will remain above 12 m for a whole period minus 2 hours 45 minutes. This is, the water will remain above 12 m for, 14 hours − 2 hours 45 minutes = 11 hours 15 minutes after 9:00 am.

Hence, the allowed timings are after 11:45 am and before 8:15 pm.

22 Consider a SHM with time period T, amplitude A, frequency n, and phase angle α. Say, the equation for displacement of the particle is: $x = A\sin(nt + \alpha)$

So, the velocity of the particle is given by: $v = An\cos(nt + \alpha)$

In SHM, a particle travels a total distance of $4A$ in time period T. So, the average velocity is:

$$v_{avg} = \frac{4A}{T}$$

Also, since

$$T = \frac{2\pi}{n},$$

$$v_{avg} = \frac{2An}{\pi}$$

The maximum velocity of a particle in SHM is: $v_{max} = An$

The ratio of average speed to maximum speed is then:

$$\frac{v_{avg}}{v_{max}} = \frac{\frac{2An}{\pi}}{An} = \frac{2}{\pi}$$

Thus, the ratio is $2:\pi$

Solutions to Exercise 6B Enrichment questions

23a $x = 4\sin\left(3t + \frac{\pi}{6}\right) + 2\sin 3t$

$\dot{x} = 12\cos\left(3t + \frac{\pi}{6}\right) + 6\cos 3t$

$\ddot{x} = -36\sin\left(3t + \frac{\pi}{6}\right) - 18\sin 3t = -9x$

So, $\ddot{x}$ is of the form $\ddot{x} = -n^2x$, hence the motion is SHM.

23b x

$= 4\sin 3t\cos\frac{\pi}{6} + 4\cos 3t\sin\frac{\pi}{6} + 2\sin 3t$

$= 2\sqrt{3}\sin 3t + 2\cos 3t + 2\sin 3t$

$= 2\left(\sqrt{3} + 1\right)\sin 3t + 2\cos 3t$

$= A\sin(3t + \alpha)$

Where:

A^2

$= 4\left(\sqrt{3} + 1\right)^2 + 4$

$= 4\left(4 + 2\sqrt{3}\right) + 4$

$= 20 + 8\sqrt{3}$

So, the amplitude A is $\sqrt{4\left(5 + 2\sqrt{3}\right)} = 2\sqrt{5 + 2\sqrt{3}}$.

24 Let $x = a\cos(nt + \alpha)$

When $t = 0, x = 1$ so, $a\cos\alpha = 1$ (1)

When $t = 1, x = 5$ so, $a\cos(n + \alpha) = 5$ (2)

When $t = 2, x = 5$ so, $a\cos(2n + \alpha) = 5$ (3)

We must show that $n = \cos^{-1}\frac{3}{5}$.

Expanding (2):

$a\cos n\cos\alpha - a\sin n\sin\alpha = 5$

Using (1):

$\cos n - a\sin n\sin\alpha = 5$ (4)

Expanding (3):

$a\cos 2n\cos\alpha - a\sin 2n\sin\alpha = 5$

Using (1):

$\cos 2n - a\sin 2n\sin\alpha = 5$

$2\cos^2 n - 1 - 2a\sin n\cos n\sin\alpha = 5$

$\cos^2 n - a\sin n\cos n\sin\alpha = 3$

$\cos n\,(\cos n - a\sin n\sin\alpha) = 3$

Using (4):

$\cos n\,(\cos n + 5 - \cos n) = 3$

Hence,

$\cos n = \frac{3}{5}$

$n = \cos^{-1}\frac{3}{5}$, as required.

Solutions to Exercise 6C Foundation questions

1a $v = -6\sin 2t$

$\ddot{x} = -12\cos 2t = -4x$

$36\sin^2 2t + 36\cos^2 2t = 36$

$v^2 = 36 - 36\cos^2 2t = 36 - 4x^2$

$v^2 = 4(9 - x^2)$

1b When $x = 2$,

$v = \pm\sqrt{4(9 - 2^2)}$

$= \pm 2\sqrt{5}\ \text{m/s}$

$\ddot{x} = -4(2)$

$= -8\ \text{m/s}^2$

2a

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -n^2(x - c)$

$v^2 = D - n^2(x - c)^2$

At each extreme of motion, the velocity is 0.

$0 = D - n^2a^2$

$D = n^2a^2$

$v^2 = -n^2((x - c)^2 - a^2)$

$c = 0$, $a = 5$ and $n = 3$

Thus, $v^2 = 9(25 - x^2)$

2b $v = \pm\sqrt{9(25 - 3^2)}$

$= \pm 12\ \text{m/s}$

$\ddot{x} = -9(3)$

$= -27\ \text{m/s}^2$

2c When $t = 0, x = 0$

$v = \pm\sqrt{9(25)}$

$= \pm 15 \text{ m/s}$

3a

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -n^2(x - c)$

$v^2 = D - n^2(x - c)^2$

$c = 0$, and $n = \left|\sqrt{16}\right| = 4$

$v^2 = D - 4^2x^2$

$v = 24$ when $x = 0$

$576 = D - 16(0)$

Thus,

$v^2 = 576 - 16x^2$

$= 16(36 - x^2)$

3b Period $= \frac{2\pi}{4} = \frac{\pi}{2}$ and Amplitude $= 6$

3c When $x = 2$,

$|v| = \sqrt{16(36 - 2^2)}$

$= 16\sqrt{2} \text{ cm/s}$

$\ddot{x} = -16(2)$

$= -32 \text{ cm/s}^2$

4a

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -n^2(x-c)$$

$$v^2 = D - n^2(x-c)^2$$

At each extreme of motion, the velocity is 0.

$$0 = D - n^2a^2$$

$$D = n^2a^2$$

$$v^2 = -n^2((x-c)^2 - a^2)$$

$c = 0$, $a = 6$ and $n = 2$

Thus, $v^2 = 4(36 - x^2)$

Therefore, the period is $\frac{2\pi}{2} = \pi$ and maximum speed is 6 cm/s.

4b i $x = 6\cos 2t$

4b ii $x = -6\cos 2t$

4b iii $x = 6\sin 2t$

4b iv $x = -6\sin 2t$

5a

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -n^2(x - c)$$

$$v^2 = D - n^2(x - c)^2$$

$$n = \left|\sqrt{256}\right| = 16$$

$$c = 0$$

$$v^2 = D - 16^2(x)^2$$

$v = 0$ when $x = 2$

$$0 = D - 16^2(4)$$

Thus, $v^2 = 1024 - 16x^2 = 16(64 - x^2)$

When $x = 0$,

$$|v| = \sqrt{16(64)}$$

$$= 32 \text{ m/s}$$

5b

$$\ddot{x} = -\frac{1}{4}x$$

$$v^2 = D - n^2(x - c)^2$$

$$n = \left|\sqrt{\frac{1}{4}}\right| = \frac{1}{2}$$

$$c = 0$$

$$v^2 = D - \frac{1}{2}^2 (x)^2$$

When $x = 0, v = 4.$

$$16 = \frac{1}{4}a$$

$$a = 64$$

$$v^2 = \frac{1}{2}(64 - x^2)$$

When $v = 0,$

$$0 = \frac{1}{2}(64 - x^2)$$

$$0 = 64 - x^2$$

$$x^2 = 64$$

$$x = 8 \text{ cm}$$

6a When $x = 0, v = 4$

$$n = \frac{2\pi}{\frac{\pi}{2}} = 4$$

$$16 = 4^2(a^2)$$

$$a = 1 \text{ m}$$

6b

$$n = \frac{2\pi}{3}$$

$$v^2 = \frac{2\pi^2}{3}\ (2^2 - x^2)$$

When $x = 0$,

$$|v| = \sqrt{\frac{4\pi}{9}(4)}$$

$$= \frac{4\pi}{3}\ \text{m/s}$$

Solutions to Exercise 6C Development questions

7a Given the range of the SHM is $A \leq x \leq B$, distance $AB = 20$ cm, Period $T = 8$ s

To determine amplitude, a

$$a = \frac{AB}{2}$$

$$a = \frac{20}{2}$$

$$a = 10 \text{ cm}$$

$$T = \frac{2\pi}{n}$$

$$8 = \frac{2\pi}{n}$$

$$n = \frac{\pi}{4}$$

The maximum speed of particle in SHM occurs at the when $|v| = na$

$$|v| = na$$

$$na = \frac{\pi}{4} \times 10$$

$$v = \frac{5\pi}{2} \text{ cm/s}$$

Assuming the particle started from O (though, it won't affect the maximum speed or acceleration):

x

$= a\sin\,(nt + \beta) + c$

$= a\sin nt$

$= 10\sin\frac{\pi}{4}t$

Then:

$\dot{x} = v$

v

$= \left(\frac{\pi}{4}\right) 10 \cos\frac{\pi}{4}t$

$= \frac{5\pi}{2}\cos\frac{\pi}{4}t$

$\ddot{x} = \frac{dv}{dx}$

$\ddot{x}$

$= -\left(\frac{\pi}{4}\right)\frac{5\pi}{2}\sin\frac{\pi}{4}t$

$= -\frac{5}{8}\pi^2 \sin\frac{\pi}{4}t$

The maximum acceleration occurs at the extremes of motion,

$\therefore \ddot{x} = \frac{5}{8}\pi^2 \text{cm/s}^2$

7b $x = 6$ cm,

Using the relations: $v^2 = n^2(a^2 - x^2)$ and $\ddot{x} = n^2(c - x)$

$v^2 = \left(\frac{\pi}{4}\right)^2 (10^2 - 6^2)$

$|v|$

$= \sqrt{\frac{\pi^2}{16}(100 - 36)}$

$= \sqrt{\frac{\pi^2}{16}(64)}$

$= \sqrt{4\pi^2}$

$v = \pm 2\pi$ cm/s

$\ddot{x}$

$= \pm\left(\frac{\pi}{4}\right)^2 (0 - 6)$

$= \pm\left(\frac{\pi^2}{16}\right)(-6)$

$= \pm\frac{3\pi^2}{8}\text{cm/s}^2$

8 SHM $a = 5$ m; when $x = 2$, $\ddot{x} = \pm 4$ m/s^2

Using $\ddot{x} = n^2(c - x)$

$c = 0$

$\ddot{x} = n^2(-x)$

$-4 = n^2(-2)$

$n^2 = 2$

$n = \sqrt{2}$

Using $v^2 = n^2(a^2 - x^2)$

v^2

$= \sqrt{2}^2(5^2 - x^2)$

$= 2(25 - x^2)$

At $x = 0$,

v_0

$= \sqrt{2(25 - 0^2)}$

$= 5\sqrt{2}$ m/s

At $x = 4$,

v_4

$= \sqrt{n^2(a^2 - x^2)}$

$= \sqrt{2(25 - 16)}$

$= \sqrt{2(9)}$

$= 3\sqrt{2}$ m/s

9 $T = \pi; \quad v_{max} = 8 \text{ m/s}$

$T = \dfrac{2\pi}{n}$

$\pi = \dfrac{2\pi}{n}$

$n = 2$

$v_{max} = an$

$a = \dfrac{v_{max}}{n}$

$a = \dfrac{8}{2}$

$a = 4 \text{ m}$

Using $v^2 = n^2(a^2 - x^2)$

When $x = 3$,

${v_3}^2$

$= 2^2(4^2 - 3^2)$

$= 4(16 - 9)$

$v_3 = 2\sqrt{7} \text{ m/s}$

10 Given $\ddot{x} \propto x_0$; when $x = 4$ cm, $v = 20$ cm/s, $\ddot{x} = -6\frac{2}{3}$ cm/s^2

Using the relation:

$\ddot{x} = n^2(c - x)$

With:

$c = 0$

$-6\dfrac{2}{3} = n^2(-4)$

$n^2 = \dfrac{5}{3}$

v^2

$= -n^2((x - c)^2 - a^2)$

$= -n^2(x^2 - a^2)$

$= n^2(a^2 - x^2)$

Using the conditions:

$400 = \dfrac{5}{3}(a^2 - 16)$

$a^2 = \left(400 \times \dfrac{3}{5}\right) + 16$

$a = \sqrt{\left(400 \times \dfrac{3}{5}\right) + 16}$

$a = 16$ cm

Substituting a into $v^2 = -n^2(x^2 - a^2)$

$v^2 = -\dfrac{5}{3}(x^2 - 16^2)$

11 SHM with centre O, passes through O with initial speed $|v_{max}| = 10\sqrt{3}$ cm/s

$x = a \sin nt$

$\dot{x} = v$

$v = na \cos nt$

$v_{max} = na$

$10\sqrt{3} = na$

$n = \dfrac{10\sqrt{3}}{a}$

Let x_m be the position of x halfway between its mean position and an endpoint

$x_m = \dfrac{a}{2}$

v^2

$= n^2(a^2 - x^2)$

$= \dfrac{300}{a^2}\left(a^2 - \dfrac{a^2}{4}\right)$

$= 300 - \dfrac{300}{4}$

$= 225$

$v = \pm 15$ cm/s

$\therefore$ Speed at x_m is 15 cm/s

12 SHM with centre O, $v_0 = V = v_{max} = na$

$x = a\sin nt$ and $v = na\cos nt$

v^2

$= n^2(a^2 - x^2)$

$= n^2a^2 - n^2x^2$

At the origin:

$v^2 = V^2$

$V^2 = n^2a^2$

The particle comes to rest when $v = 0$, using the result above:

0

$= n^2a^2 - n^2x^2$

$= V^2 - n^2x^2$

$V^2 = n^2x^2$

$x = \pm\dfrac{V}{n}$

Hence proved.

13a SHM, amplitude $= a$ and $\ddot{x} = -n^2 x,\ n > 0$

$\ddot{x} = \dfrac{d^2x}{dt^2}$

$\dfrac{d^2x}{dt^2} = \dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right)$

$\dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right) = -n^2 x$

Integrating both sides w.r.t. x

$\dfrac{1}{2}v^2 = -n^2 \displaystyle\int x\ dx$

v^2

$= -2n^2 \displaystyle\int x\ \ dx$

$= -2n^2 \dfrac{x^2}{2} + C$

$= -n^2x^2 + C$

As the particle is in SHM,

At $x = a,\ \ v = 0,$

$0 = -n^2a^2 + C$

$C = n^2a^2$

Therefore,

v^2

$= n^2a^2 - n^2x^2$

$= n^2(a^2 - x^2)$

13bi $x = 0$

$v_0^2 = n^2(a^2 - 0^2)$

$|v| = (na),$

$v_0 = \pm na$

Speed at the origin $= |v_0| = na$

13bii $x = \frac{a}{2}$, v_m = speed midway between the particle's mean position and endpoint

$v^2 = n^2(a^2 - x^2)$

v_m^2

$= n^2\left(a^2 - \frac{a^2}{4}\right)$

$= n^2a^2\left(1 - \frac{1}{4}\right)$

$= n^2a^2\left(\frac{3}{4}\right)$

v_m

$= \pm\sqrt{\frac{3n^2a^2}{4}}$

$= \pm\frac{\sqrt{3}}{2}na$

$|v_m| = \frac{1}{2}\sqrt{3}\,na$

$\therefore$ Speed $= |v_m| = \frac{1}{2}\sqrt{3}na$

Acceleration is:

$\ddot{x}$

$= -n^2x$

$= -n^2\left(\frac{a}{2}\right)$

$\ddot{x} = -\frac{1}{2}an^2$

14a Given $v^2 = -9x^2 + 18x + 27$ prove SHM

v^2

$= -9x^2 + 18x + 27$

$= -9(x^2 - 2x - 3)$

$= -9\left((x^2 - 2x + 1) - 4\right)$

$= -3^2((x-1)^2 - 2^2)$

$= 3^2(2^2 - (x-1)^2)$

This function is of the form $v^2 = n^2(a^2 - (x-c)^2)$

Where $n = 3, a = 2,$ and $c = 1$, hence this satisfies SHM.

$\therefore$ Centre, c of the motion is $x = 1$

T

$= \dfrac{2\pi}{n}$

$= \dfrac{2\pi}{3}$

$\therefore$ Period, $T = \frac{2\pi}{3}$

$\therefore$ Amplitude, $a = 2$

If v^2 is SHM then $\ddot{x} = -n^2(x-c)$

$\ddot{x}$

$= \dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right)$

$= \dfrac{1}{2}\dfrac{d}{dx}(-9x^2 + 18x + 27)$

$\ddot{x} = -(2)\dfrac{9x}{2} + \dfrac{18}{2}$

$\ddot{x} = -9x + 9$

$\ddot{x} = -9(x-1)$

Thus:

$\ddot{x} = -n^2(x-c)$

Hence SHM proved.

14bi Given $v^2 = 80 + 64x - 16x^2$ prove SHM

$v^2 = 80 + 64x - 16x^2$

$v^2 = 16(5 + 4x - x^2)$

$v^2 = 16\big(9 - (x^2 - 4x + 4)\big)$

$v^2 = 4^2(3^2 - (x-2)^2)$

This function is of the form $v^2 = n^2(a^2 - (x-c)^2)$

Where $n = 4, a = 3,$ and $c = 2,$ hence this satisfies SHM.

$\therefore$ Centre, c of the motion is $x = 2$

T

$= \dfrac{2\pi}{n}$

$= \dfrac{2\pi}{4}$

$= \dfrac{\pi}{2}$

$\therefore$ Period, $T = \frac{\pi}{2}$

$\therefore$ Amplitude, $a = 3$

If v^2 is SHM then $\ddot{x} = -n^2(x-c)$

$\ddot{x}$

$= \dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right)$

$= \dfrac{1}{2}\dfrac{d}{dx}(80 + 64x - 16x^2)$

$= -(2)\dfrac{16x}{2} + \dfrac{64}{2}$

$= -16x + 32$

$= -16(x-2)$

Thus:

$\ddot{x} = -n^2(x-c)$

Hence SHM proved.

14bii Given $v^2 = -9x^2 + 108x - 180$ prove SHM

v^2

$= -9x^2 + 108x - 180$

$= 9(-20 + 12x - x^2)$

$= 9\left(16 - (x^2 - 12x + 36)\right)$

$= 3^2(4^2 - (x-6)^2)$

This function is of the form $v^2 = n^2(a^2 - (x-c)^2)$

Where $n = 3, a = 4,$ and $c = 6,$ hence this satisfies SHM.

$\therefore$ Centre, c of the motion is $x = 6$

T

$= \dfrac{2\pi}{n}$

$= \dfrac{2\pi}{3}$

$\therefore$ Period, $T = \frac{2\pi}{3}$

$\therefore$ Amplitude, $a = 4$

If v^2 is SHM then $\ddot{x} = -n^2(x-c)$

$\ddot{x}$

$= \dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right)$

$= \dfrac{1}{2}\dfrac{d}{dx}(-9x^2 + 108x - 180)$

$= -(2)\dfrac{9x}{2} + \dfrac{108}{2}$

$= -9x + 54$

$= -9(x-6)$

Thus:

$\ddot{x} = -n^2(x-c)$

Hence SHM proved.

14biii Given $v^2 = -2x^2 - 8x - 6$ prove SHM

v^2

$= -2x^2 - 8x - 6$

$= 2(-x^2 - 4x - 3)$

$= 2\left(1 - (x^2 + 4x + 4)\right)$

$= \left(\sqrt{2}\right)^2(1^2 - (x+2)^2)$

This function is of the form $v^2 = n^2(a^2 - (x-c)^2)$

Where $n = \sqrt{2}, a = 1,$ and $c = -2,$ hence this satisfies SHM.

$\therefore$ Centre, c of the motion is $x = -2$

T

$= \dfrac{2\pi}{n}$

$= \dfrac{2\pi}{\sqrt{2}}$

$\therefore$ Period, $T = \sqrt{2}\pi$

$\therefore$ Amplitude, $a = 1$

If v^2 is SHM then $\ddot{x} = -n^2(x-c)$

$\ddot{x}$

$= \dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right)$

$= \dfrac{1}{2}\dfrac{d}{dx}(-2x^2 - 8x - 6\ )$

$= -(2)\dfrac{2x}{2} - \dfrac{8}{2}$

$= -2x - 4$

$= -2(x+2)$

Thus:

$\ddot{x} = -n^2(x-c)$

Hence SHM proved.

14biv Given $v^2 = 8 - 10x - 3x^2$ prove SHM

v^2

$= 8 - 10x - 3x^2$

$= 3\left(\frac{8}{3} - \frac{10}{3}x - x^2\right)$

$= 3\left(\frac{8}{3} + \frac{25}{9} - \left(x^2 + \frac{10}{3}x + \frac{25}{9}\right)\right)$

$= 3\left(\frac{49}{9} - \left(x + \frac{5}{3}\right)^2\right)$

$= \left(\sqrt{3}\right)^2\left(\left(\frac{7}{3}\right)^2 - \left(x + \frac{5}{3}\right)^2\right)$

This function is of the form $v^2 = n^2(a^2 - (x - c)^2)$

Where $n = \sqrt{3}, a = \frac{7}{3}$, and $c = -\frac{5}{3}$, hence this satisfies SHM.

$\therefore$ Centre, c of the motion is $x = -\frac{5}{3}$

T

$= \frac{2\pi}{n}$

$= \frac{2\pi}{\sqrt{3}}$

$= \frac{2\sqrt{3}}{3}\pi$

$\therefore$ Period, $T = \frac{2\pi}{\sqrt{3}}$ or $\frac{2\sqrt{3}}{3}\pi$

$\therefore$ Amplitude, $a = \frac{7}{3}$ or $2\frac{1}{3}$

If v^2 is SHM then $\ddot{x} = -n^2(x - c)$

$\ddot{x}$

$= \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$= \frac{1}{2}\frac{d}{dx}(8 - 10x - 3x^2)$

$= -(2)\dfrac{3x}{2} - \dfrac{10}{2}$

$= -3x - 5$

$= -3(x + \dfrac{5}{3})$

Thus:

$\ddot{x} = -n^2(x - c)$

Hence SHM proved.

15ai Given $x = \sin^2 5t$

Using $\cos 2y = 1 - 2\sin^2 y$

$\sin^2 y = \dfrac{1}{2} - \dfrac{1}{2}\cos 2y$

Hence:

$x = \dfrac{1}{2} - \dfrac{1}{2}\cos 10t$

$v = \dfrac{dx}{dt}$

$\dot{x} = 5\sin 10t$

$\dfrac{d^2x}{dt^2} = \ddot{x}$

$\ddot{x}$

$= 50\cos 10t$

$= 50\cos 10t + 50 - 50$

$= -100\left(\left(\dfrac{1}{2} - \dfrac{1}{2}\cos 10t\right) - \dfrac{1}{2}\right)$

$= -10^2\left(x - \dfrac{1}{2}\right)$

$\ddot{x} = -n^2(x - c)$

Hence SHM proved.

15aii Given $x = \sin^2 5t$

And $\sin 2y = 2\sin y \,.\cos y$:

$\dot{x}$

$= \dfrac{dx}{dt}$

$= 2\sin 5t \,.\cos 5t \,.\, 5$

$= 5\sin 10t$

$\ddot{x}$

$= \dfrac{d^2x}{dt^2}$

$= 50\cos 10t$

$= 50(2\sin^2 5t - 1)$

$= -10^2\left(\sin^2 5t - \frac{1}{2}\right)$

$= -10^2\left(x - \dfrac{1}{2}\right)$

Thus:

$\ddot{x} = -n^2(x - c)$

Hence SHM proved.

15b Previously determined $\ddot{x} = -10^2\left(x - \frac{1}{2}\right)$

$\therefore$ Centre, c of the motion is $x = \frac{1}{2}$ metres

Hence:

$Range = 0 \leq x \leq 1$

Period is:

T

$= \dfrac{2\pi}{n}$

$= \dfrac{2\pi}{10}$

$= \dfrac{\pi}{5}$

$\therefore$ Period, $T = \frac{\pi}{5}$ minutes

Given, $x = \sin^2 5t$ at $t = 0$, the particle is at the origin initially, it then moves away, and the next time $x = 0$, is when $5t = \pi$ or $t = \frac{\pi}{5}$

16a Given $\ddot{x} = -9(x - 7)$ and $a = 7$cm

Centre c of the motion is given from this equation to be:

$c = 7$

So, the particle oscillates in the range:

$(7 - a) \leq x \leq (7 + a)$

$Range = 0 \leq x \leq 14$

That is, the origin is a point of extremity for this motion.

The velocity of a particle in SHM at the extremes is zero.

Hence, for this motion, the velocity at the origin is zero.

16b Given $\ddot{x} = -9(x-7)$

$\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -9(x-7)$

Integrating both sides w.r.t x

$\frac{v^2}{2} = -9\int (x-7)dx$

v^2

$= 2[-9\int (x-7)dx]$

$= -18\left(\left(\frac{x^2}{2}\right) - 7x\right) + C$

$= -9(x^2 - 14x) + C$

At $x = 0, v = 0$ [determined in 16a]

$0^2 = -9\left(0^2 - 14(0)\right) + C$

$C = 0$

Therefore

v^2

$= -9(x^2 - 14x)$

$= -9\left((x^2 - 14x + 49) - 49\right)$

$= -9((x-7)^2 - 7^2)$

$= 9(49 - (x-7)^2)$

$v = v_{max}$ when $(x - c) = 0$. Using above:

${v_{max}}^2 = -9((7-7)^2 - 7^2)$

$v_{max} = \pm 21$ cm/s

Maximum speed

$|v_{max}| = 21$ cm/s

16c Given $\ddot{x} = -9(x - 7)$

At the origin $x = 0$

$\ddot{x}$

$= -9(0 - 7)$

$= 63\ \text{cm/s}^2$

Observe that the acceleration at origin of the particle is towards the positive direction and at its maximum value.

Thus, even if the particle has zero velocity at the origin, the restoring acceleration of the particle is maximum and makes it move away from the origin.

17a Given $x = 4\cos 3t - 6\sin 3t$. Let

$\cos\theta = \dfrac{4}{\sqrt{4^2 + 6^2}}$ and $\sin\theta = \dfrac{6}{\sqrt{4^2 + 6^2}}$

This gives:

x

$= 4\cos 3t - 6\sin 3t$

$= \sqrt{52}(\cos\theta\cos 3t - \sin\theta\sin 3t)$

$= \sqrt{52}\cos(3t - \theta)$

$\dfrac{dx}{dt} = \dot{x}$

$\dot{x} = -3\sqrt{52}\sin(3t - \theta)$

$\dfrac{d^2x}{dt^2}$

$= \ddot{x}$

$= -9\sqrt{52}\cos(3t - \theta)$

$= -9x$

Hence, the acceleration is proportional to the displacement but in the opposite direction, and so the motion is simple harmonic.

17b Taking amplitude a, Period T

As previously determined,

$\ddot{x}$

$= -9x$

$= -3^2x$

Hence:

$n = 3$

T

$= \dfrac{2\pi}{n}$

$= \dfrac{2\pi}{3}$

$x = \sqrt{52}\cos(3t - \theta)$

$x = a\cos nt$ in the general form and so:

$a = \sqrt{52} = 2\sqrt{13}$

The maximum speed, v_{max}

$|v_{max}|$

$= na$

$= (3)(2\sqrt{13})$

$= 6\sqrt{13}$

The maximum acceleration is $|\ddot{x}_{max}| = n^2a$ and occurs at the extremes of motion. The minimum acceleration, $\ddot{x}_{min}$, is zero and occurs as the object passes through the centre of motion, its mean position. Therefore, the acceleration when the particle is halfway between its mean position and one of the extreme positions $(x = \frac{a}{2})$, $|\ddot{x}_m|$ is:

$|\ddot{x}_m|$

$= \left(\dfrac{n^2a}{2}\right)$

$$= 3^2\left(\frac{2\sqrt{13}}{2}\right)$$

$$= 9\sqrt{13}$$

18a Given $x = 3 + \sin 4t + \sqrt{3}\cos 4t$. Let:

$$\cos\alpha = \frac{1}{\sqrt{1^2 + \sqrt{3}^2}} = \frac{1}{2} \text{ and } \sin\alpha = \frac{\sqrt{3}}{2}$$

Then:

x

$= 3 + 2(\sin 4t\cos\alpha + \sin\alpha\cos 4t)$

$= 3 + 2(\sin(4t + \alpha))$

$\dot{x} = 8\cos(4t + \alpha)$

and

$\ddot{x}$

$= -32\sin(4t + \alpha)$

$= -16(2\sin(4t + \alpha))$

$= -16\big((3 + 2\sin(4t + \alpha)) - 3\big)$

$= -16(x - 3)$

Hence proved. From the standard form we see that. The centre $\boldsymbol{c}$ of the motion is $x = 3$. Also we see from the standard from that we must have $n = 4$ and so the period is:

T

$$= \frac{2\pi}{n}$$

$$= \frac{2\pi}{4}$$

$$= \frac{\pi}{2}$$

18b From the solution in part a:

$x = 3 + \sin 4t + \sqrt{3}\cos 4t$ and with:

$\cos\alpha = \dfrac{1}{\sqrt{1^2 + \sqrt{3}^2}} = \dfrac{1}{2}$ and $\sin\alpha = \dfrac{\sqrt{3}}{2}$

$x = 3 + 2\sin(4t + \alpha)$

From the definition above we have:

$\tan\alpha = \sqrt{3}$

$\alpha = \tan^{-1}\sqrt{3}$

$\alpha = \dfrac{\pi}{3}$

$\therefore x = 3 + 2\sin\left(4t + \frac{\pi}{3}\right)$

18c When the particle is at the centre in SHM its speed is the maximum.

$|v_{max}|$

$= na$

$= 4 \times 2$

$= 8$

From part a) the centre of motion is $c = 3$ and when $x = c$ we have:

$3 = 3 + 2\sin\left(4t + \dfrac{\pi}{3}\right)$

$0 = \sin\left(4t + \dfrac{\pi}{3}\right)$

For first three times:

$4t + \dfrac{\pi}{3} = \pi, 2\pi, 3\pi$

$t = \dfrac{\pi}{6}, \dfrac{5\pi}{12}, \dfrac{2\pi}{3}$

19a Given $x = 10 + 8\sin 2t + 6\cos 2t$, let:

$\cos\alpha = \dfrac{8}{10}$ and $\sin\alpha = \dfrac{6}{10}$

Then:

$\alpha = \tan^{-1}\dfrac{3}{4}$

and,

x

$= 10 + 10(\cos\alpha\sin 2t + \sin\alpha\cos 2t)$

$= 10 + 10\sin(2t + \alpha)$

$\dot{x} = 20\cos(2t + \alpha)$

So,

$\ddot{x}$

$= -40\sin(2t + \alpha)$

$= -4\big((10 + 10\sin(2t + \alpha)) - 10\big)$

$= -4(x - 10)$

This is of the form $\ddot{x} = -n^2(x - c)$ where $n = 2$ and $c = 10$. Hence, the motion is SHM. From this this equation the centre of motion is $x = 10$, and $n = 2$. The period is then:

T

$= \dfrac{2\pi}{n}$

$= \dfrac{2\pi}{2}$

$= \pi$

Since $x = 10 + 10\sin(2t + \alpha)$ is of the form:

$x = x_0 + a\sin(nt + \alpha)$

Amplitude, $a = 10$

19b The particle first reaches origin when $x = 0$

$$0 = 10 + 10\sin(2t + \alpha)$$

$$\sin(2t + \alpha) = -1$$

$$(2t + \alpha) = \sin^{-1}(-1)$$

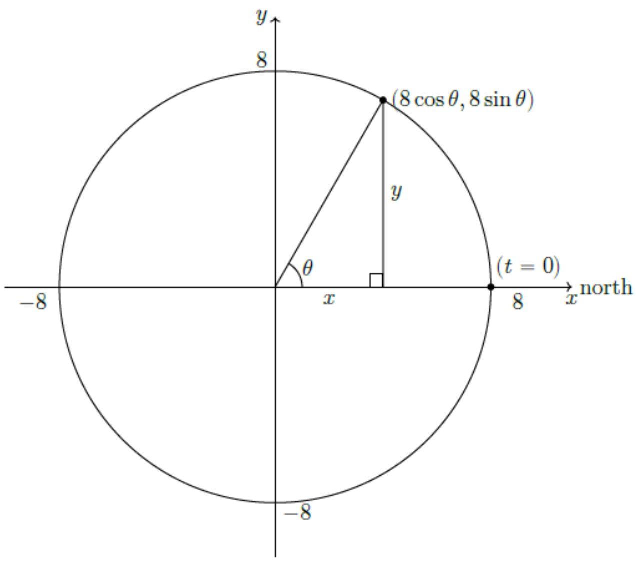
$$2t + \alpha = \frac{3\pi}{2}$$

$$t = \frac{3\pi}{4} - \frac{1}{2}\tan^{-1}\frac{3}{4}$$

$$t = 2.034$$

Solutions to Exercise 6C Enrichment questions

20a The positive x-axis points north.



$$\frac{x}{8} = \cos\theta$$

$x = 8\cos\theta$

Similarly, $y = 8\sin\theta$

$$\frac{d\theta}{dt} = 1 \text{ rev/min}$$

$$\frac{d\theta}{dt} = 2\pi \text{ rad/min}$$

So, $\theta = 2\pi t + C$

When, $t = 0, \theta = 0$

So, $C = 0$ and $\theta = 2\pi t$

Hence, $x = 8\cos 2\pi t$ and $y = 8\sin 2\pi t$.

20b $\quad \dot{x} = -16\pi \sin 2\pi t \quad$ and $\quad \dot{y} = 16\pi \cos 2\pi t$

So, $\ddot{x} = -32\pi^2 \cos 2\pi t \quad$ and $\quad \ddot{y} = -32\pi^2 \sin 2\pi t$

$\ddot{x} = -4\pi^2 x \qquad \ddot{y} = -4\pi^2 y$

20ci

$$\frac{y}{x} = \tan\theta = \frac{1}{\sqrt{3}}$$

Hence:

$$\theta = \frac{\pi}{6} \text{ or, } \theta = \frac{7\pi}{6}$$

20cii

$$\frac{\dot{x}}{\dot{y}} = -\tan\theta = -\sqrt{3}$$

$$\theta = \frac{\pi}{3} \text{ or, } \theta = \frac{4\pi}{3}$$

20ciii $\quad -16\pi \sin\theta = 16\pi\cos\theta$

$\tan\theta = -1$

Hence:

$$\theta = \frac{3\pi}{4} \text{ or, } \theta = \frac{7\pi}{4}$$

21a

$$\int \frac{d\left(\frac{1}{2}v^2\right)}{dx}\,dx = \int -n^2 x\,dx$$

$\frac{1}{2}v^2 = -\frac{1}{2}n^2x^2 + C$

When, $V = 0, x = \pm a$

$C = \frac{1}{2}n^2a^2$

$V^2 = n^2(a^2 - x^2)$

21b When $x = x_1, |v| = v_1$

So, $v_1{}^2 = n^2(a^2 - x_1{}^2)$ (1)

When $x = x_2, |v| = v_2$

So, $v_2{}^2 = n^2(a^2 - x_2{}^2)$ (2)

From (1),

$$a^2 = \frac{v_1{}^2}{n^2} + x_1{}^2$$

Substitute into (2):

$$v_2{}^2 = n^2\left(\frac{v_1{}^2}{n^2} + x_1{}^2 - x_2{}^2\right)$$

$$v_2{}^2 - v_1{}^2 = n^2(x_1{}^2 - x_2{}^2)$$

$$n = \sqrt{\frac{v_2{}^2 - v_1{}^2}{x_1{}^2 - x_2{}^2}}$$

$$T = \frac{2\pi}{n} = 2\pi\sqrt{\frac{x_1{}^2 - x_2{}^2}{v_2{}^2 - v_1{}^2}}$$

(2) ÷ (1):

$$\frac{V_2{}^2}{V_1{}^2} = \frac{a^2 - x_2{}^2}{a^2 - x_1{}^2}$$

$$a^2V_2{}^2 - V_2{}^2x_1{}^2 = a^2V_1{}^2 - V_1{}^2x_2{}^2$$

$$a^2\left(V_1{}^2 - V_2{}^2\right) = V_1{}^2x_2{}^2 - V_2{}^2x_1{}^2$$

$$a = \sqrt{\frac{V_1{}^2x_2{}^2 - V_2{}^2x_1{}^2}{V_1{}^2 - V_2{}^2}}$$

21c

$$a = \sqrt{\frac{1024 - 324}{64 - 36}} = 5 \text{ cm}$$

$$T = 2\pi\sqrt{\frac{16 - 9}{64 - 36}} = 2\pi\sqrt{\frac{1}{4}} = \pi \text{ seconds}$$

$|v|_{max}$

$= na$

$= 2 \times 5$

$= 10 \text{ cm/s}$

22 Max speed $= na = V$

v^2

$= n^2(a^2 - x^2)$

$= V^2 - n^2x^2$

So when $x = \frac{1}{2}a$

v^2

$= V^2 - \dfrac{1}{4}n^2a^2$

$= V^2 - \dfrac{1}{4}V^2$

$= \dfrac{3}{4}V^2$

So,

$v = \dfrac{\pm\sqrt{3}}{2}V$

Also, when $v = \frac{1}{2}V$,

$\dfrac{1}{4}V^2 = V^2 - n^2x^2$

$n^2x^2 = \dfrac{3}{4}V^2$

$\dfrac{V^2}{a^2}x^2 = \dfrac{3}{4}V^2$

$x^2 = \dfrac{3}{4}a^2$

$x = \pm\dfrac{\sqrt{3}}{2}a$

Hence:

v^2

$= V^2 - n^2x^2$

$= V^2 - \dfrac{V^2}{a^2}x^2$

So,

$$|v| = V\sqrt{1 - \frac{x^2}{a^2}}$$

And,

$$n^2x^2 = V^2 - v^2$$

$$\frac{V^2}{a^2}x^2 = V^2 - v^2$$

$$x^2 = \frac{a^2}{V^2}(V^2 - v^2)$$

So,

$$|x| = a\sqrt{1 - \frac{v^2}{V^2}}$$

23a $x = \sin t - 2\sin(t - \alpha)$

$\dot{x} = \cos t - 2\cos(t - \alpha)$

$\ddot{x} = -\sin t - 2\sin(t - \alpha) = -x$

Thus, the acceleration has the form $\ddot{x} = -nx^2$, so the motion is SHM with

$$T = \frac{2\pi}{n} = 2\pi$$

23b x

$= \sin t - 2\sin t\cos\alpha + 2\cos t\sin\alpha$

$= (1 - 2\cos\alpha)\sin t + (2\sin\alpha)\cos t$

$= A\sin(t + \theta)$

Where:

A^2

$= (1 - 2\cos\alpha)^2 + (2\sin\alpha)^2$

$= 1 - 4\cos\alpha + 4(\cos^2\alpha + \sin^2\alpha)$

$= 5 - 4\cos\alpha$

So, the amplitude (i.e., the maximum value of x) is $A = \sqrt{5 - 4\cos\alpha}$

The maximum possible value of A occurs when $\cos\alpha = -1$.

So, the maximum value is $\sqrt{9} = 3$ and it occurs when $\alpha = \pi$.

When $\alpha = \pi$,

x

$= \sin t - 2\sin(t - \pi)$

$= \sin t - 2(-\sin t)$

$= 3\sin t$

The minimum possible value of A occurs when $\cos\alpha = 1$.

So, the min value is $\sqrt{1} = 1$ and it occurs when $\alpha = 0$.

When $\alpha = 0$,

x

$= \sin t - 2\sin t$

$= -\sin t$

23c The balls are level when $x_1 = x_2$.

Sin t

$= 2\sin(t-\alpha)$

$= 2\sin t\cos\alpha + 2\cos t\sin\alpha$

tan t

$= 2\tan t\cos\alpha - 2\sin\alpha$

$= \dfrac{2\sin\alpha}{2\cos\alpha - 1}$

$= \dfrac{4T}{1+T^2} / \left(\dfrac{2-2T^2}{1+T^2} - 1\right)$, where $T = tan\dfrac{\alpha}{2}$

$= \dfrac{4T}{2-2T^{2-}(1+T^2)}$

$= \dfrac{4T}{1-3T^2}$

The balls will be level twice in the interval $0 \le t < 2\pi$,

once in the interval $0 \le t < \frac{\pi}{2}$ and once in the interval $\pi \le t < \frac{3\pi}{2}$.

23d When $t = 0, x = A$

$2\sin\alpha = \sqrt{5 - 4\cos\alpha}$

$4\sin^2\alpha = 5 - 4\cos\alpha$

$4(1 - \cos^2\alpha) = 5 - 4\cos\alpha$

$4\cos^2\alpha - 4\cos\alpha + 1 = 0$

$(2\cos\alpha - 1)^2 = 0$

Hence:

$\cos\alpha = \dfrac{1}{2}$

$\alpha = \dfrac{\pi}{3}$ or, $\dfrac{5\pi}{3}$

When $\alpha = \dfrac{\pi}{3}$

x

$= \sin t - 2\sin\left(t - \dfrac{\pi}{3}\right)$

$= \sin t - 2\left(\dfrac{1}{2}\sin t - \dfrac{\sqrt{3}}{2}\cos t\right)$

$= \sqrt{3\cos t}$

When $\alpha = \dfrac{5\pi}{3}$

x

$= \sin t - 2\left(\dfrac{1}{2}\sin t + \dfrac{\sqrt{3}}{2}\cos t\right)$

$= -\sqrt{3\cos t}$

Solutions to Exercise 6D Foundation questions

1a Let direction of motion be positive, mass $= M$ kg.

So braking force $= -\dfrac{2}{3}M$ N, hence

braking acceleration $= \dfrac{-\frac{2}{3}M}{M} = -\dfrac{2}{3}$ m/s^2

and resistive force $= -\dfrac{1}{180}Mv^2$ N, hence

resistive acceleration $= \dfrac{-\frac{1}{180}Mv^2}{M} = -\dfrac{1}{180}v^2$ m/s^2

Therefore

$$\ddot{x} = -\frac{2}{3} - \frac{1}{180}v^2$$

$$\frac{dv}{dt} = -\frac{2}{3} - \frac{1}{180}v^2$$

$$= \frac{-120 - v^2}{180}$$

Now $\dfrac{dv}{dt} = v\dfrac{dv}{dx}$, so

$$\frac{dv}{dx} = \frac{-120 - v^2}{180v}$$

$$\frac{dx}{dv} = \frac{180v}{-120 - v^2}$$

$$= \frac{-90 \times 2v}{120 + v^2}$$

$x = -90\ln(120 + v^2) + C$ (since $120 + v^2 > 0$)

When $x = 0, v = 288$ km/h $= 80$ m/s.

$0 = -90\ln(120 + 80^2) + C$

$C = 90\ln(120 + 80^2)$

Therefore

$x = -90\ln(120 + v^2) + 90\ln(120 + 80^2)$

$$= 90\ln\left(\frac{120 + 80^2}{120 + v^2}\right)$$

1b When $v = 0$,

$$x = 90\ln\left(\frac{120 + 80^2}{120 + 0^2}\right)$$

$$= 360 \text{ metres}$$

Therefore the car travels 360 metres to bring it to rest.

2a Let direction of motion be positive, mass $= 10\,000$ kg.

So propelling force $= 10\,000$ N, hence

propelling acceleration $= \dfrac{10\,000}{10\,000} = 1 \text{ m/s}^2$

and resistive force $= -100v^2$ N, hence

resistive acceleration $= \dfrac{-100v^2}{10\,000} = -\dfrac{1}{100}v^2 \text{ m/s}^2$

Therefore

$$\ddot{x} = 1 - \frac{1}{100}v^2$$

Maximum speed is when $\ddot{x} = 0$.

$$1 - \frac{1}{100}v^2 = 0$$

$$v^2 = 100$$

$$|v| = 10 \text{ m/s}$$

Hence the maximum speed is 10 m/s or 36 km/h.

2b From part a:

$$\frac{dv}{dt} = 1 - \frac{1}{100}v^2$$

$$= \frac{100 - v^2}{100}$$

Now $\frac{dv}{dt} = v\frac{dv}{dx}$, so

$$\frac{dv}{dx} = \frac{100 - v^2}{100v}$$

$$\frac{dx}{dv} = \frac{100v}{100 - v^2}$$

$$= \frac{-50 \times -2v}{100 - v^2}$$

$x = -50\ln(100 - v^2) + C$ (since $|v| < 10$ m/s so $100 - v^2 > 0$)

When $x = 0, v = 0$ m/s.

$0 = -50\ln 100 + C$

$C = 50\ln 100$

Therefore

$x = -50\ln(100 - v^2) + 50\ln 100$

$$= 50\ln\left(\frac{100}{100 - v^2}\right)$$

2c When $x = 50$ m,

$$50 = 50\ln\left(\frac{100}{100 - v^2}\right)$$

$$\ln\left(\frac{100}{100 - v^2}\right) = 1$$

$$\frac{100}{100 - v^2} = e^1$$

$$100 = e^1(100 - v^2)$$

$$\frac{100}{e} = 100 - v^2$$

$$v^2 = 100 - \frac{100}{e}$$

$$|v| = \sqrt{100 - 100e^{-1}}\ \text{ m/s}$$

Percentage of maximum speed

$$= \frac{\sqrt{100 - 100e^{-1}}}{10} \times 100\%$$

$$= 79.506\ldots\%$$

$$\doteqdot 80\%$$

After 50 m, the monorail is travelling at a speed that is about 80% of its maximum speed.

3a Let direction of motion be positive, mass $= m$ kg.

So propelling force $= \dfrac{P}{v}$ N, hence

propelling acceleration $= \dfrac{\frac{P}{v}}{m} = \dfrac{P}{mv}$ m/s^2

and resistive force $= -kv^2$ N, hence

resistive acceleration $= \dfrac{-kv^2}{m}$ m/s^2

Therefore

$$\ddot{x} = \frac{P}{mv} - \frac{k}{m}v^2$$

$$\ddot{x} = \frac{P - kv^3}{mv}$$

Maximum speed of u m/s occurs when $\ddot{x} = 0$.

$$0 = \frac{P - ku^3}{mu}$$

$$P - ku^3 = 0$$

$$k = \frac{P}{u^3}$$

3b From part a:

$$\ddot{x} = \frac{dv}{dt} = \frac{P - kv^3}{mv}$$

$$\frac{dv}{dt} = \frac{P - \frac{P}{u^3}v^3}{mv}$$

$$= \frac{Pu^3 - Pv^3}{mu^3 v}$$

Now $\frac{dv}{dt} = v\frac{dv}{dx}$, so

$$\frac{dv}{dx} = \frac{Pu^3 - Pv^3}{mu^3 v^2}$$

$$= \frac{P}{m}\left(\frac{u^3 - v^3}{u^3 v^2}\right)$$

$$= \frac{P}{m}\left(\frac{u^3}{u^3 v^2} - \frac{v^3}{u^3 v^2}\right)$$

$$= \frac{P}{m}\left(\frac{1}{v^2} - \frac{1}{u^3}\right)$$

3c From part b:

$$\frac{dv}{dx}=\frac{Pu^3-Pv^3}{mu^3v^2}$$

$$\frac{dx}{dv}=\frac{mu^3v^2}{Pu^3-Pv^3}$$

$$=\frac{-\frac{1}{3P}mu^3\times -3Pv^2}{Pu^3-Pv^3}$$

$$x=-\frac{1}{3P}mu^3\ln(Pu^3-Pv^3)+C \qquad (\text{since } u>v \text{ so } Pu^3-Pv^3>0)$$

Distance travelled when speed changes from $\frac{1}{3}u$ to $\frac{2}{3}u$

$$=\left(-\frac{1}{3P}mu^3\ln\left(Pu^3-P\left(\frac{2}{3}u\right)^3\right)+C\right)-\left(-\frac{1}{3P}mu^3\ln\left(Pu^3-P\left(\frac{1}{3}u\right)^3\right)+C\right)$$

$$=-\frac{1}{3P}mu^3\ln\left(Pu^3-\frac{8}{27}Pu^3\right)+\frac{1}{3P}mu^3\ln\left(Pu^3-\frac{1}{27}Pu^3\right)$$

$$=\frac{1}{3P}mu^3\ln\left(\frac{Pu^3-\frac{1}{27}Pu^3}{Pu^3-\frac{8}{27}Pu^3}\right)$$

$$=\frac{1}{3P}mu^3\ln\left(\frac{1-\frac{1}{27}}{1-\frac{8}{27}}\right)$$

$$=\frac{1}{3P}mu^3\ln\left(\frac{\frac{26}{27}}{\frac{19}{27}}\right)$$

$$=\frac{1}{3P}mu^3\ln\frac{26}{19}$$

$$=\frac{mu^3\ln\frac{26}{19}}{3P}\quad\text{metres}$$

3d

Maximum braking force $= -B$ N, hence

maximum braking acceleration $= -\dfrac{B}{m}$ m/s^2

and resistive force $= -kv^2 = -\dfrac{Pv^2}{u^3}$ N, hence

resistive acceleration $= -\dfrac{Pv^2}{mu^3}$ m/s^2

Therefore

$$\ddot{x} = -\frac{B}{m} - \frac{Pv^2}{mu^3}$$

$$\ddot{x} = \frac{-Bu^3 - Pv^2}{mu^3}$$

$$\frac{dv}{dt} = \frac{-Bu^3 - Pv^2}{mu^3}$$

Now $\dfrac{dv}{dt} = v\dfrac{dv}{dx}$, so

$$\frac{dv}{dx} = \frac{-Bu^3 - Pv^2}{mu^3 v}$$

$$\frac{dx}{dv} = \frac{mu^3 v}{-Bu^3 - Pv^2}$$

$$= \frac{-mu^3 v}{Bu^3 + Pv^2}$$

$$= \frac{-\frac{1}{2P}mu^3 \times 2Pv}{Bu^3 + Pv^2}$$

$$x = -\frac{1}{2P}mu^3 \ln(Bu^3 + Pv^2) + C \qquad (\text{since } Bu^3 + Pv^2 > 0)$$

Minimum distance travelled when speed changes from u to 0

$$= \left(-\frac{1}{2P}mu^3 \ln(Bu^3 + 0) + C\right) - \left(-\frac{1}{2P}mu^3 \ln(Bu^3 + Pu^2) + C\right)$$

$$= -\frac{1}{2P}mu^3 \ln(Bu^3) + \frac{1}{2P}mu^3 \ln(Bu^3 + Pu^2)$$

$$= \frac{1}{2P}mu^3 \ln\left(\frac{Bu^3 + Pu^2}{Bu^3}\right)$$

$$= \frac{1}{2P} mu^3 \ln\left(1 + \frac{Pu^2}{Bu^3}\right)$$

$$= \frac{mu^3}{2P} \ln\left(1 + \frac{P}{Bu}\right) \text{ metres}$$

Solutions to Exercise 6D Development questions

4a The negative sign of the forces indicates that the forces are resistive. The force in the spring is resistive to the displacement of the door and acts towards the origin. The resistive force in the dashpot is in the opposite direction to the velocity.

4b The forces acting on the door are:

$m\ddot{y} = -3m\dot{y} - 2my$

$\ddot{y} + 3\dot{y} + 2y = 0$

4c Let $y = f(t)$ and $y = g(t)$ be solutions of $\ddot{y} + 3\dot{y} + 2y = 0$. Then we have,

$f''(t) + 3f'(t) + 2f(t) = 0 \qquad (1)$

$g''(t) + 3g'(t) + 2g(t) = 0 \qquad (2)$

Now let,

$y = Af(t) + Bg(t) \qquad$ where A, B are constants

Then we have,

$\dot{y} = Af'(t) + Bg'(t)$

$\ddot{y} = Af''(t) + Bg''(t)$

Substituting the above values of y and its derivatives into $\ddot{y} + 3\dot{y} + 2y$:

$\ddot{y} + 3\dot{y} + 2y$

$= Af''(t) + Bg''(t) + 3\big(Af'(t) + Bg'(t)\big) + 2\big(Af(t) + Bg(t)\big)$

$= A\big(f''(t) + 3f'(t) + 2f(t)\big) + B\big(g''(t) + 3g'(t) + 2g(t)\big)$

$= A(0) + B(0) \qquad$ from (1) and (2)

$= 0$

Hence proved.

4d $y = e^{kt}$ is a solution of the differential equation. Differentiating we have,

$\dot{y} = ke^{kt}$ and $\ddot{y} = k^2e^{kt}$

Substituting the above values of y and its derivatives in the equation of motion:

$\ddot{y} + 3\dot{y} + 2y = 0$

$k^2e^{kt} + 3ke^{kt} + 2e^{kt} = 0$

$(k^2 + 3k + 2)e^{kt} = 0$

$(k + 1)(k + 2)e^{kt} = 0$

Since $e^{kt} > 0$ this gives,

$k = -1$ or -2

Hence, possible values of the constant k are $k = -1$ and -2.

4e At $t = 0$, $y = 0$ and $\frac{dy}{dt} = 1$. Now we have,

$y = Ae^{-2t} + Be^{-t}$

and so,

$\frac{dy}{dt} = -2Ae^{-2t} - Be^{-t}$

At $t = 0$, using $y = 0$ we have,

$0 = A(e^0) + B(e^0)$

Or

$A = -B$

Now, at $t = 0$ we also have $\frac{dy}{dt} = 1$, which becomes,

$1 = -2A - B$

Subbing in $A = -B$ we have,

$1 = -2A + A$

Hence, $A = -1$ & $B = 1$

Therefore, $y = e^{-t} - e^{-2t}$ is a solution.

5a The driving force of the engine is F Newtons and the resistive force of the water is proportional to v^2, and so the water force on the submarine is $-kv^2$ where k is a constant (- since it is a resistive force). Hence, the net force on the submarine is given by,

$$m\ddot{x} = (F - kv^2)$$

Now, $\ddot{x} = \dfrac{dv}{dt}$

Hence,

$$\frac{dv}{dt} = \frac{1}{m}(F - kv^2)$$

5b We have from the chain rule and results from the notes,

$$\frac{dv}{dt} = \frac{vdv}{dx} = vv' \qquad (1)$$

$$\int \frac{vv'}{a + bv^2}\,dx = \frac{1}{2b}\log(a + bv^2) + C \qquad (2)$$

Using (1) and (2) with the result of part a):

$$vv' = \frac{1}{m}(F - kv^2)$$

Integrate both sides w.r.t x between the limits v_1 and v_2

$$\int_{v_1}^{v_2} \frac{vv'}{F - kv^2}\,dx = \frac{1}{m}\int dx$$

$$\left[-\frac{1}{2k}\ln(F - kv^2)\right]_{v_1}^{v_2} = \frac{x}{m}$$

$$x = \frac{m}{2k}\left[-\ln(F - kv_2^2) - (-\ln(F - kv_1^2))\right]$$

$$x = \frac{m}{2k}\ln\frac{F - kv_1^2}{F - kv_2^2}$$

6a From the question we have that $m = 1, f_r = v + v^3$ and $v_0 = Q$.

Since, the only force acting is the resisting one we have

$$m\ddot{x} = -(v + v^3)$$

or

$$\frac{dv}{dt} = -(v + v^3)$$

Integrating both sides we have,

$$\int_0^t dt = -\int_Q^v \frac{dv}{v + v^3}$$

Applying partial fractions, the RHS becomes,

$$\frac{1}{v + v^3} = \frac{1}{v} - \frac{1}{2}\left(\frac{2v}{1 + v^2}\right)$$

Hence, we have,

$$\int_0^t dt = -\int_Q^v \left(\frac{1}{v} - \frac{1}{2}\left(\frac{2v}{1 + v^2}\right)\right) dv$$

t

$$= -\left[\ln v - \frac{1}{2}\ln(1 + v^2)\right]_Q^v$$

$$= -\left[\ln\frac{v}{Q} - \frac{1}{2}\ln\frac{1 + v^2}{1 + Q^2}\right]$$

$$= -\left[\frac{1}{2}\ln\frac{v^2}{Q^2} - \frac{1}{2}\ln\frac{1 + v^2}{1 + Q^2}\right]$$

$$= \frac{1}{2}\ln\left(\frac{Q^2}{v^2} \times \frac{1 + v^2}{1 + Q^2}\right)$$

6b Using the result of part a) we have,

$$t = \frac{1}{2}\ln\left(\frac{Q^2}{v^2} \times \frac{1+v^2}{1+Q^2}\right)$$

Hence, taking the exp of both sides and rearranging we have,

$$\frac{1+v^2}{v^2} = \frac{1+Q^2}{Q^2}e^{2t}$$

$$\frac{v^2}{1+v^2} = \frac{Q^2}{1+Q^2}e^{-2t}$$

$$1 - \frac{1}{1+v^2} = \frac{Q^2}{1+Q^2}e^{-2t}$$

$$1 + v^2 = \frac{1}{1 - \frac{Q^2}{1+Q^2}e^{-2t}}$$

$$v^2 = \frac{\frac{Q^2}{1+Q^2}e^{-2t}}{1 - \frac{Q^2}{1+Q^2}e^{-2t}}$$

$$v^2 = \frac{Q^2}{e^{2t}(1+Q^2) - Q^2}$$

6c From part b) we have,

$$v^2 = \frac{Q^2}{e^{2t}(1+Q^2) - Q^2}$$

Now as $t \to \infty$, $e^{2t} \to \infty$, and because Q is constant the denominator must tend towards infinity. Thus, since in the limit v^2 is a constant divided by a value tending towards infinity, we conclude that as $t \to \infty$, $v^2 \to 0$ and so also that $v \to 0$.

Now $Q > 0$ initially, which means that the speed decreases from some positive value to zero as time tends towards infinity.

Hence, v is always positive.

6d From the chain rule we have that,

$$\frac{dv}{dt} = -(v + v^3) = v\frac{dv}{dx}$$

And so, noting that that at $t = 0, x = 0$ and $v = Q$ we have,

$$v\frac{dv}{dx} = -v(1 + v^2)$$

$$\int_0^x dx = -\int_Q^v \frac{dv}{1 + v^2}$$

Substituting $v = \tan a$, we get

$$dv = \sec^2 a\, da$$

and using the result $(1 + \tan^2 a) = \sec^2 a$ the integral becomes,

$$x = -\int_{\tan(Q)}^{\tan(v)} da$$

or

$$x$$

$$= -[\tan^{-1} v]_Q^v$$

$$= -[\tan^{-1} v - \tan^{-1} Q]$$

$$= \tan^{-1} Q - \tan^{-1} v$$

Now, taking the limit as $x \to \infty$ and using the result from part c)

$$\lim_{t\to\infty} x = \lim_{t\to\infty}(\tan^{-1} Q - \tan^{-1} v)$$

$$\lim_{t\to\infty} x = \tan^{-1} Q - \lim_{t\to\infty}\left(\tan^{-1}\sqrt{\frac{Q^2}{e^{2t}(1+Q^2)-Q^2}}\right)$$

$$\lim_{t\to\infty} x = \tan^{-1} Q - \lim_{t\to\infty} \tan^{-1} 0$$

$$\lim_{t\to\infty} x = \tan^{-1} Q$$

6e $x = \tan^{-1} Q - \tan^{-1} v$

Let $\tan^{-1} Q = q$ and $\tan^{-1} v = r$

So, $Q = \tan q$ and $v = \tan r$

Then using the angle difference formula for tan we have

$$\tan(q - r) = \frac{\tan q - \tan r}{1 + \tan q \tan r}$$

$$\tan(q - r) = \frac{Q - v}{1 + Qv}$$

$$(q - r) = \tan^{-1} \frac{Q - v}{1 + Qv}$$

Hence,

x

$= \tan^{-1} Q - \tan^{-1} v$

$= (q - r)$

$= \tan^{-1} \dfrac{Q - v}{1 + Qv}$

7a

$$\frac{1}{(2 - v)(3 + v)} = \frac{A}{2 - v} + \frac{B}{3 + v}$$

$(3A + Av) + (2B - Bv) = 1$

Collecting like terms we have

$3A + 2B = 1$

and

$(A - B)v = 0$

Hence,

$A = B$

and

$5A = 1$

Hence, $A = B = \frac{1}{5}$

7b i From the question we have,

$$\frac{dv}{dt} = \frac{10^4}{m}(6 - v - kv^2); \quad m = 4.5 \times 10^6 \text{ kg}; \quad v_{max} = 2\frac{\text{m}}{\text{s}}; \quad F = 10^4(6 - v) \text{ N}$$

When the speed reaches its maximum value, the acceleration is zero and so,

$$0 = \frac{10^4}{m}(6 - v_{max} - kv_{max}^2)$$

$$6 - 2 - 4k = 0$$

$$k = 1$$

7b ii We have from the question and part i)

$$\frac{dv}{dt} = \frac{10^4}{m}(6 - v - v^2)$$

Now we also know that the body starts at rest and so,

$$\frac{10^4}{m}\int_0^t dt = \int_0^v \frac{dv}{6 - v - v^2} = \frac{1}{5}\int \left(\frac{1}{2 - v} + \frac{1}{3 + v}\right) dv$$

From part a) we have,

$$\int_0^v \frac{dv}{6 - v - v^2} = \frac{1}{5}\int_0^v \left(\frac{1}{2 - v} + \frac{1}{3 + v}\right) dv$$

And so

$$\frac{10^4}{m}t = \left[\frac{1}{5}\ln\frac{3 + v}{2 - v}\right]_0^v$$

$$\frac{10^4}{m}t = \frac{1}{5}\ln\frac{3 + v}{2 - v} - \frac{1}{5}\ln\frac{3}{2}$$

At $v = 1.5$ m/s we have,

$$t = \frac{4.5 \times 10^6}{5 \times 10^4}\left(\ln\left(\frac{3 + 1.5}{2 - 1.5}\right) - \ln\frac{3}{2}\right)$$

$$t = 161.25 \text{ seconds}$$

$$t = 2 \text{ minutes and } 41 \text{ seconds}$$

8a For the first 3 seconds, only friction is acting and so the equation of motion is:

$M\ddot{x} = -2M$

Hence,

$\ddot{x} = -2 \text{ m/s}^2$

From the question we also have that $v_0 = 72\text{m/s}$. Now, since acceleration is constant integrating w.r.t time gives $v = v_0 + at$, where v_0 is the velocity at $t = 0$ and a is the constant acceleration. Integrating again w.r.t time (and taking $x = 0$ at $t = 0$) then gives, $x = v_0 t + \frac{1}{2}at^2$. Subbing in now $t = 3s$ we have,

v_3

$= v_0 + at$

$= 72 - 3 \times 2$

$= 66 \text{ m/s}$

and

x_3

$= 72 \times 3 - \dfrac{1}{2} \times 2 \times 3^2$

$= 207\text{m}$

8b After the first 3 seconds, the equation of motion is:

$\ddot{x} = v\dfrac{dv}{dx} = -2 - \dfrac{v^2}{10^4}$

Rearranging and integrating both sides we have

$\displaystyle\int dx = -5000 \int \frac{2v}{20000 + v^2} dv$

$x = -5000[\ln(20000 + v^2)] + C$

At $t = 3$, $x = 207$ and $v = 66$. Subbing these in we find

$207 = -5000 \ln(20000 + 66^2) + C$

$C = 207 + 5000 \ln(20000 + 66^2)$

Therefore

$x = 207 + 5000 \ln\left(\dfrac{20000 + 66^2}{20000 + v^2}\right)$

8c From above we have

$$x = 207 + 5000 \ln \left(\frac{20000 + 66^2}{20000 + v^2} \right)$$

This equation is valid till the limit of $v = 36$, or until

x

$$= 207 + 5000 \ \ln \left(\frac{20000 + 66^2}{20000 + 36^2} \right)$$

$= 207 + 671.3$

$= 878 .3$ m

So to the nearest meter the thrust will shut off 878 m after touchdown.

8d Since only the break act, the equation of motion after the speed has reduced to 36 m/s is:

$$\ddot{x} = \ -2 = v \frac{dv}{dx}$$

Rearranging and integrating from when the thrust shuts off to taxi speed we have,

x

$$= \ 878.3 + \int_{36}^{7} \left(-\frac{v}{2} \right) dv$$

$$= 878.3 + \int_{7}^{36} \frac{v}{2} dv$$

$= \ 878.3 + 311.75$

$= 1190.05$ m

Hence, from touchdown it takes 1190 m to the nearest meter before the jet is at taxi speed.

9a $F = mP$; $f_r = mkv$; $v_0 = V_i$

Equation of the motion of the box:

$m\ddot{x} = (F - f_r) = mP - mkv$

$a = P - kv$

When net force is zero, $a = 0$, and so:

$P - kV_0 = 0$

or,

$V_0 = \dfrac{P}{k}$

9b We have from part a),

$$\frac{dv}{dt} = (P - kv)$$

separating variables and integrating we have,

$$\int dt = \int \frac{dv}{P - kv}$$

$$t$$

$$= -\frac{1}{k}\ln(P - kv) + C$$

$$= -\frac{1}{k}\ln\left(1 - \frac{v}{V_0}\right) + C$$

At $t = 0$, $v = V_i$ and so,

$$C$$

$$= \frac{1}{k}\ln(P - kV_i)$$

$$= \frac{1}{k}\ln\left(1 - \frac{V_i}{V_0}\right)$$

Therefore,

$$t$$

$$= -\frac{1}{k}\ln\frac{P - kv}{P - kV_i}$$

$$= -\frac{1}{k}\ln\frac{1 - \frac{k}{P}v}{1 - \frac{k}{P}V_i}$$

$$= -\frac{1}{k}\ln\frac{1 - \frac{v}{V_0}}{1 - \frac{V_i}{V_0}}$$

Rearranging and taking the exponential of both sides we have,

$$\left(1 - \frac{V_i}{V_0}\right)e^{-kt} = 1 - \frac{v}{V_0}$$

Hence,

$$v = V_0\left(1 - \left(1 - \frac{V_i}{V_0}\right)e^{-kt}\right)$$

Taking the limit as $t \to \infty$ we have,

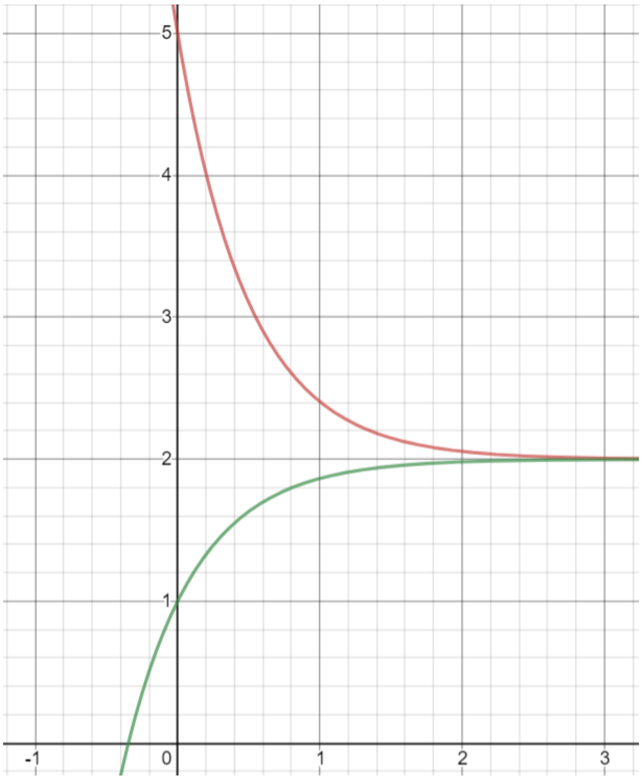
$$\lim_{t\to\infty} v = \lim_{t\to\infty} V_0\left(1 - \left(1 - \frac{V_i}{V_0}\right)e^{-kt}\right) = V_0(1-0)$$

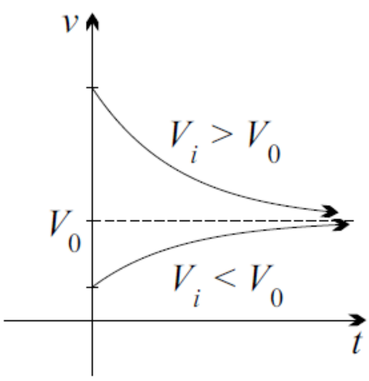
Therefore

$$\lim_{t\to\infty} v = V_0$$

9c $V_i > V_0$ = Red Curve; $V_i = 5$; $V_0 = 2$

$V_i < V_0$ = Green Curve $V_i = 1$; $V_0 = 2$





9d $v_1 = \frac{1}{3}V_0$; $v_2 = \frac{2}{3}V_0$

From above part b) we have,

$$t = -\frac{1}{k}\ln\frac{1-\frac{v}{V_0}}{1-\frac{V_i}{V_0}}$$

Hence,

$t_2 - t_1$

$$= -\frac{1}{k}\ln\frac{1-\frac{2V_0}{3V_0}}{1-\frac{V_i}{V_0}} + \frac{1}{k}\ln\frac{1-\frac{V_0}{3V_0}}{1-\frac{V_i}{V_0}}$$

$$= -\frac{1}{k}\ln\frac{1-\frac{2V_0}{3V_0}}{1-\frac{V_i}{V_0}} + \frac{1}{k}\ln\frac{1-\frac{V_0}{3V_0}}{1-\frac{V_i}{V_0}}$$

$$= \frac{1}{k}\ln 2$$

10a From the question,

$$\frac{d^2x}{dt^2} = \frac{5-2x}{x^3}; \qquad x_0 = 1; \qquad v_0 = 0$$

Now,

$$\frac{d^2x}{dt^2} = \frac{5-2}{1} = 3\ (> 0\ )\text{ at } x = 1$$

At $x = 1$ the acceleration is positive. So, the particle will start to move in positive x direction.

10b

$$\frac{d^2x}{dt^2} = v\frac{dv}{dx} = \frac{5-2x}{x^3}$$

Integrating gives,

$$\frac{v^2}{2} = \int \frac{5-2x}{x^3}dx$$

or

v^2

$$= 2\int \frac{5-2x}{x^3}dx$$

$$= 10\int \frac{dx}{x^3} - 4\int \frac{dx}{x^2}$$

$$= -5x^{-2} + 4x^{-1} + C$$

$$= \frac{4}{x} - \frac{5}{x^2} + C$$

At $t = 0$, $x = 1$ and $v = 0$. Subbing these in we find

$0 = 4 - 5 + C$

$C = 1$

Therefore

v^2

$$= \frac{4}{x} - \frac{5}{x^2} + 1$$

$$= \frac{x^2 + 4x - 5}{x^2}$$

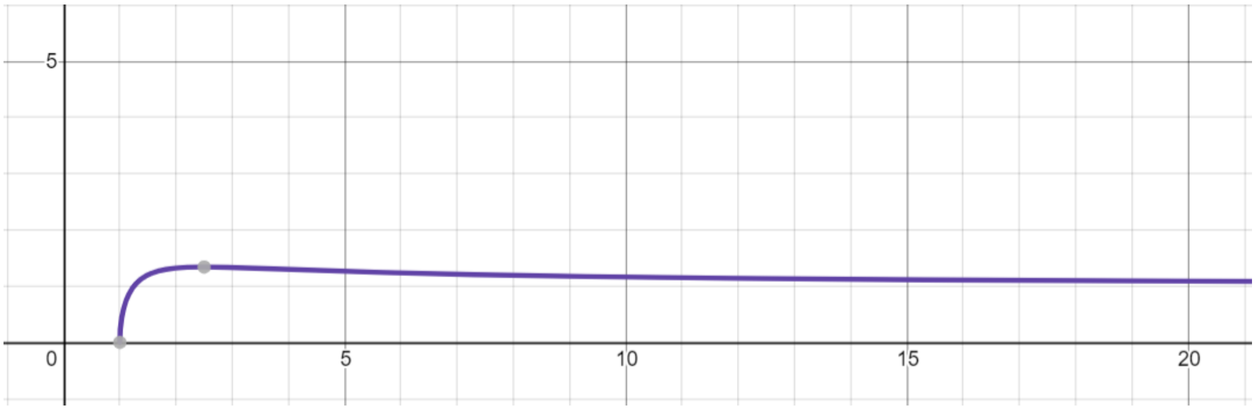
Thus,

$$v = \pm\frac{\sqrt{x^2 + 4x - 5}}{x}$$

Since the particle starts at rest at $x = 1$ and accelerates to the right we conclude that $v \geq 0$ for $x \geq 1$, hence:

$$v = \frac{\sqrt{x^2 + 4x - 5}}{x} \quad \text{for } x \geq 1$$

10c Graph of $v = \dfrac{\sqrt{x^2 + 4x - 5}}{x}$. Consider $x \geq \dfrac{5}{2}$



As can be seen from the above graph, the velocity approaches 1 from above.

11a i $x_0 = 0$; $\quad v_0 = V_0\ (> 0)$

$$v\frac{dv}{dx} = -kv^{\frac{3}{2}}$$

Integrating w.r.t x gives,

$$\int_0^x dx = -\frac{1}{k}\int_{V_0}^{v}\frac{dv}{\sqrt{v}}$$

or

x

$$= -\left[\frac{1}{k} \times 2\sqrt{v}\right]_{V_0}^{v}$$

$$= \left[\frac{2}{k}\sqrt{v}\right]_{v}^{V_0}$$

$$= \frac{2}{k}\left(\sqrt{V_0} - \sqrt{v}\right)$$

$$= \frac{2}{k}\left(1 - \sqrt{\frac{v}{V_0}}\right)\sqrt{V_0}$$

Rearranging we have,

$$\sqrt{\frac{v}{V_0}} = 1 - \frac{kx}{2\sqrt{V_0}}$$

11a ii From above we have,

$$\sqrt{\frac{v}{V_0}} = 1 - \frac{kx}{2\sqrt{V_0}}$$

Now since the LHS must be positive we conclude that the maximum value of x must be when the RHS is 0, or when $x = \frac{2\sqrt{V_0}}{k}$. Also, since initially the object is moving to the right ($V_0 > 0$), and because there is no other force acting on it except friction, the minimum value of x be when $x = 0$. Hence, x can only take on positive values, further these values lie between.

$$x_{max} = \frac{2\sqrt{V_0}}{k} \quad \text{and} \quad x_{min} = 0$$

Thus,

$$0 \le x \le \frac{2\sqrt{V_0}}{k}$$

And when the particle stops moving, $v = 0$ giving

$$x = \frac{2\sqrt{V_0}}{k}$$

11b i From part a)

$$\sqrt{\frac{v}{V_0}} = 1 - \frac{kx}{2\sqrt{V_0}}$$

Rearranging

$$v = \frac{dx}{dt} = V_0\left(1 - \frac{kx}{2\sqrt{V_0}}\right)^2$$

Let $\frac{k}{2\sqrt{V_0}} = C$, then

$$\frac{dx}{dt} = V_0(1 - Cx)^2$$

Integrating

$$\int dt = V_0 \int (1 - Cx)^{-2}\, dx$$

$$t = \frac{V_0}{C}(1 - Cx)^{-1} + D$$

At $t = 0, x = 0$, we get:

$$D = -\frac{V_0}{C}$$

Hence,

$$\frac{C}{V_0}t = (1 - Cx)^{-1} - 1 = \frac{1}{1 - Cx} - 1$$

Or

$$\frac{V_0 + Ct}{V_0} = \frac{1}{1 - Cx}$$

Rearranging for x now gives,

$$1 - Cx = \frac{V_0}{V_0 + Ct}$$

And so,

x

$$= \frac{1}{C}\left(1 - \frac{V_0}{V_0 + Ct}\right)$$

$$= \frac{2\sqrt{V_0}}{k}\left(1 - \frac{V_0}{V_0 + \frac{kt}{2\sqrt{V_0}}}\right)$$

$$x = \frac{2\sqrt{V_0}}{k}\left(1 - \frac{2}{2 + \frac{kt}{\sqrt{V_0^3}}}\right)$$

11b ii $\lim_{t\to\infty} x$

$$= \lim_{t\to\infty} \frac{2\sqrt{V_0}}{k}\left(1 - \frac{2}{2 + \frac{kt}{\sqrt{V_0^3}}}\right)$$

$$= \frac{2\sqrt{V_0}}{k}(1 - 0)$$

$$= \frac{2\sqrt{V_0}}{k}$$

And

$\lim_{t\to\infty} v$

$$= \lim_{t\to\infty} V_0\left(1 - \frac{kx}{2\sqrt{V_0}}\right)^2$$

$$= V_0\left(1 - \frac{k}{2\sqrt{V_0}}\frac{2\sqrt{V_0}}{k}\right)^2$$

$$= 0$$

Which matches the answer to a)(ii)

Solutions to Exercise 6D Enrichment questions

12a

Let

$u = \sqrt{v}$

$v = u^2$

$\frac{dv}{du} = 2u$

Also:

$v^{\frac{3}{2}}$

$= v\sqrt{v}$

$= u^2 u$

$= u^3$

Now:

$\frac{dv}{dt} = 1 - v^{\frac{3}{2}}$

$\frac{dv}{du}.\frac{du}{dt} = 1 - u^3$

$2u.\frac{du}{dt} = 1 - u^3$

$\frac{dt}{du} = \frac{2u}{1 - u^3}$

12b Let

$$\frac{2u}{(1-u)(1+u+u^2)} = \frac{A}{1-u} + \frac{Bu+C}{1+u+u^2}$$

Then, $2u = A(1+u+u^2) + (Bu+C)(1-u)$

Put $u = 1$:

$2 = 3A$

$A = \frac{3}{2}$

Put $u = 0$:

$0 = \frac{2}{3} + C$

$C = -\frac{2}{3}$

Put $u = -1$

$-2 = \frac{2}{3} + \left(-B - \frac{2}{3}\right)(2)$

$-2 = \frac{2}{3} - 2B - \frac{4}{3}$

$2B = \frac{4}{3}$

$B = \frac{2}{3}$

Hence,

$\frac{dt}{du}$

$= \frac{2}{3}.\frac{1}{1-u} + \frac{2}{3}.\frac{u-1}{1+u+u^2}$

$= \frac{2}{3}.\left(\frac{1}{1-u} + \frac{\frac{1}{2}(1+2u) - \frac{3}{2}}{1+u+u^2}\right)$

$= \frac{2}{3}.\left(\frac{1}{1-u} + \frac{1}{2}.\frac{1+2u}{1+u+u^2} - \frac{3}{2}.\frac{1}{\left(\frac{1}{2}+u\right)^2 + \frac{3}{4}}\right)$

12c t

$$= \int \frac{dt}{du}\, du$$

$$= \frac{2}{3}(-\ln|1-u|) + \frac{1}{2}\ln(1+u+u^2) - \frac{3}{2}.\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2\left(u+\frac{1}{2}\right)}{\sqrt{3}}\right) + C$$

$$= \frac{2}{3}\left(\frac{1}{2}\ln\frac{1+u+u^2}{(1-u)^2}\right) - \sqrt{3}\tan^{-1}\left(\frac{2u+1}{\sqrt{3}}\right) + C$$

When $t = 0, v = u^2 = 0$, so $u = 0$

So,

C

$$= -\frac{2}{3}\left(\frac{1}{2}\ln 1 - \sqrt{3}\tan^{-1}\frac{1}{\sqrt{3}}\right)$$

$$= \frac{2}{3}.\sqrt{3}.\frac{\pi}{6}$$

$$= \frac{\pi\sqrt{3}}{9}$$

So, replacing u with $\sqrt{v}$,

$$t = \frac{2}{3}\left(\frac{1}{2}\ln\frac{1+\sqrt{v}+v}{\left(1-\sqrt{v}\right)^2}\right) - \sqrt{3}\left(\tan^{-1}\left(\frac{2\sqrt{v}+1}{\sqrt{3}}\right) - \frac{\pi}{6}\right)$$

Now,

$$\tan\left(\tan^{-1}\left(\frac{2\sqrt{v}+1}{\sqrt{3}}\right) - \tan^{-1}\frac{1}{\sqrt{3}}\right)$$

$$= \frac{\frac{2\sqrt{v}+1}{\sqrt{3}} - \frac{1}{\sqrt{3}}}{1 + \frac{2\sqrt{v}+1}{\sqrt{3}}.\frac{1}{\sqrt{3}}}.\frac{3}{3}$$

$$= \frac{\sqrt{3}\left(2\sqrt{v}+1\right) - \sqrt{3}}{3 + 2\sqrt{v} + 1}$$

$$= \frac{2\sqrt{3v}}{2\left(2+\sqrt{v}\right)}$$

$$= \frac{\sqrt{3v}}{2+\sqrt{v}}$$

Hence,

$$\tan^{-1}\left(\frac{2\sqrt{v}+1}{\sqrt{3}}\right) - \frac{\pi}{6} = \tan^{-1}\frac{\sqrt{3v}}{2+\sqrt{v}}$$

and the result is proven.

Solutions to Exercise 6E Foundation questions

1a Resistive acceleration (acting downwards)

$$= \frac{\text{resistive force (acting downwards)}}{\text{mass}}$$

$$= -\frac{0.2v^2}{5}$$

$g =$ acceleration due to gravity (acting downwards)

$$= -10$$

Hence

$$a = -10 - \frac{0.2v^2}{5}$$

$$\frac{dv}{dt} = \frac{-50 - 0.2v^2}{5}$$

$$= \frac{-250 - v^2}{25}$$

1b

$$\frac{dv}{dt} = \frac{-250 - v^2}{25}$$

$$\frac{dt}{dv} = \frac{25}{-250 - v^2}$$

$$= \frac{-25}{250 + v^2}$$

$$= -25 \times \frac{1}{\left(5\sqrt{10}\right)^2 + v^2}$$

$$t = -25 \times \frac{1}{5\sqrt{10}} \tan^{-1} \frac{v}{5\sqrt{10}} + C$$

$$= -\frac{5}{\sqrt{10}} \tan^{-1} \frac{v}{5\sqrt{10}} + C$$

When $t = 0, v = 40$.

$$0 = -\frac{5}{\sqrt{10}} \tan^{-1} \frac{40}{5\sqrt{10}} + C$$

$$C = \frac{5}{\sqrt{10}} \tan^{-1} \frac{8}{\sqrt{10}}$$

Therefore

$$t = -\frac{5}{\sqrt{10}} \tan^{-1} \frac{v}{5\sqrt{10}} + \frac{5}{\sqrt{10}} \tan^{-1} \frac{8}{\sqrt{10}}$$

Maximum height when $v = 0$:

$$t = -\frac{5}{\sqrt{10}} \tan^{-1} \frac{0}{5\sqrt{10}} + \frac{5}{\sqrt{10}} \tan^{-1} \frac{8}{\sqrt{10}}$$

$$= 0 + \frac{5}{\sqrt{10}} \tan^{-1} \frac{8}{\sqrt{10}}$$

$$= 1.888 \ldots$$

$$\doteqdot 1.9 \text{ seconds}$$

Maximum height is reached after about 1.9 seconds.

1c

$$\frac{dv}{dt} = v\frac{dv}{dy}$$

$$\frac{-250 - v^2}{25} = v\frac{dv}{dy}$$

$$\frac{dv}{dy} = \frac{-250 - v^2}{25v}$$

$$\frac{dy}{dv} = \frac{25v}{-250 - v^2}$$

$$= \frac{-25v}{250 + v^2}$$

$$= \frac{-\frac{25}{2} \times 2v}{250 + v^2}$$

$$y = -\frac{25}{2}\ln(250 + v^2) + C \qquad (\text{since } 250 + v^2 > 0)$$

When $y = 0, v = 40$.

$$0 = -\frac{25}{2}\ln(250 + 40^2) + C$$

$$C = \frac{25}{2}\ln 1850$$

Therefore

$$y = -\frac{25}{2}\ln(250 + v^2) + \frac{25}{2}\ln 1850$$

Maximum height is reached when $v = 0$.

$$y = -\frac{25}{2}\ln 250 + \frac{25}{2}\ln 1850$$

$$= \frac{25}{2}\ln\frac{1850}{250}$$

$$= 25.0185$$

$$\doteqdot 25 \text{ metres}$$

Maximum height, correct to the nearest metre, is 25 metres.

2a Resistive acceleration (acting downwards)

$$= \frac{\text{resistive force (acting downwards)}}{\text{mass}}$$

$$= -\frac{0.2v}{0.5}$$

g = acceleration due to gravity (acting downwards)

$$= -10$$

Hence

$$\ddot{x} = -10 - \frac{0.2v}{0.5}$$

$$= \frac{-5 - 0.2v}{0.5}$$

$$= \frac{-50 - 2v}{5}$$

2b

$$\ddot{x} = \frac{dv}{dt}, \qquad \text{so}$$

$$\frac{dv}{dt} = \frac{-50 - 2v}{5}$$

$$\frac{dt}{dv} = \frac{5}{-50 - 2v}$$

$$= \frac{-\frac{5}{2}}{25 + v}$$

$$t = -\frac{5}{2}\ln|25 + v| + C$$

When $t = 0, v = 40$.

$$0 = -\frac{5}{2}\ln|25 + 40| + C$$

$$C = \frac{5}{2}\ln 65$$

Therefore

$$t = -\frac{5}{2}\ln|25 + v| + \frac{5}{2}\ln 65$$

Maximum height is reached when $v = 0$.

$$t = -\frac{5}{2}\ln 25 + \frac{5}{2}\ln 65$$

$$= \frac{5}{2}(\ln 65 - \ln 25)$$

$$= \frac{5}{2}\ln\frac{65}{25}$$

$$= \frac{5}{2}\ln\frac{13}{5}$$

Hence the object takes $\frac{5}{2}\ln\frac{13}{5}$ seconds to reach its maximum height.

2c

$$\frac{dv}{dt} = v\frac{dv}{dy}$$

$$\frac{-50-2v}{5} = v\frac{dv}{dy}$$

$$\frac{dv}{dy} = \frac{-50-2v}{5v}$$

$$\frac{dy}{dv} = \frac{5v}{-50-2v}$$

$$= \frac{-5v}{50+2v}$$

$$= -\frac{5}{2} + \frac{125}{50+2v} \qquad \text{(by long division)}$$

$$y = -\frac{5}{2}v + \frac{125}{2}\ln|50+2v| + C$$

When $y = 0, v = 40$.

$$0 = -\frac{5}{2}(40) + \frac{125}{2}\ln|50+80| + C$$

$$C = 100 - \frac{125}{2}\ln 130$$

Therefore

$$y = -\frac{5}{2}v + \frac{125}{2}\ln|50+2v| + 100 - \frac{125}{2}\ln 130$$

Maximum height reached when $v = 0$.

$$y = 0 + \frac{125}{2}\ln 50 + 100 - \frac{125}{2}\ln 130$$

$$= 100 + \frac{125}{2}\ln\frac{50}{130}$$

$$= 100 + \frac{125}{2}\ln\frac{5}{13}$$

Maximum height reached is $\left(100 + \frac{125}{2}\ln\frac{5}{13}\right)$ metres.

3a Resistive acceleration (acting upwards)

$$= \frac{\text{resistive force (acting upwards)}}{\text{mass}}$$

$$= \frac{0.1v^2}{100}$$

$g =$ acceleration due to gravity (acting downwards)

$$= -9.8$$

Hence

$$\ddot{x} = -9.8 + \frac{0.1v^2}{100}$$

$$= -9.8 + \frac{v^2}{1000}$$

$$= \frac{-9800 + v^2}{1000}$$

Terminal velocity occurs when $\ddot{x} = 0$.

$$0 = \frac{-9800 + v^2}{1000}$$

$$0 = -9800 + v^2$$

$$9800 = v^2$$

$$|v| = \sqrt{9800}$$

$$= 70\sqrt{2}$$

$$\doteqdot 99$$

The terminal speed is about 99 m/s.

3b

$$\frac{dv}{dt} = v\frac{dv}{dy}$$

$$v\frac{dv}{dy} = \frac{-9800 - v^2}{1000}$$

$$\frac{dv}{dy} = \frac{-9800 - v^2}{1000v}$$

$$\frac{dy}{dv} = \frac{1000v}{-9800 - v^2}$$

$$= \frac{-500 \times 2v}{9800 + v^2}$$

$y = -500\ln(9800 + v^2) + C$ (since $9800 + v^2 > 0$)

When $y = 0$, $v = 0.8 \times 70\sqrt{2} = 56\sqrt{2}$

$$0 = -500\ln\left(9800 + \left(56\sqrt{2}\right)^2\right) + C$$

$C = 500\ln(16\,072)$

Therefore

$y = -500\ln(9800 + v^2) + 500\ln(16\,072)$

When object is dropped, $v = 0$.

$$y = -500\ln(9800) + 500\ln(16\,072)$$

$$= 500\ln\left(\frac{16\,072}{9800}\right)$$

$$= 247.348\ldots$$

$$\doteqdot 247 \text{ metres}$$

The point at which the object was dropped was about 247 metres above the ground.

Solutions to Exercise 6E Development questions

4ai $g = 10$; $u = 20$; $f_r = \frac{1}{40}v^2$

The equation of motion of the object is:

$$\ddot{x} = v\frac{dv}{dx} = -\left(g + \frac{1}{40}v^2\right)$$

$$dx = -\frac{40v}{400 + v^2}dv$$

$$\int dx = -20\int \frac{2v}{400 + v^2}dv$$

$$x = -20\ln(400 + v^2) + C$$

At $x = 0, v = 20$.

$$C = 20\ln(800)$$

Therefore

$$x = 20\ln\left(\frac{800}{400 + v^2}\right)$$

At maximum height, $v = 0$

x

$$= 20\ln\left(\frac{800}{400}\right)$$

$$= 20\ln 2 \text{ metres}$$

4a ii

$$\frac{dv}{dt} = \ddot{x} = -\left(g + \frac{1}{40}v^2\right)$$

$$\int dt = -\int \frac{40dv}{400 + v^2}$$

$$t = -40\int \frac{dv}{400 + v^2}$$

Using $v = 20\tan a$,

$dv = 20\sec^2 a\, da$ and $1 + \tan^2 a = \sec^2 a$

t

$$= -\frac{40}{20}\tan^{-1}\frac{v}{20} + C$$

$$= -2\tan^{-1}\frac{v}{20} + C$$

At $t = 0$, $v = 20$ and so.

$$0 = C - 2\tan^{-1} 1 = C - \frac{2\pi}{4}$$

$$C = \frac{\pi}{2}$$

Therefore

$$t = \frac{\pi}{2} - 2\tan^{-1}\frac{v}{20}$$

At $v = 0$, $(x = x_{max})$

t

$$= \frac{\pi}{2} - 2\tan^{-1} 0$$

$$= \frac{\pi}{2} \text{ seconds}$$

4b i Equation of motion of the object as it falls is:

$$\ddot{x} = g - \frac{1}{40}v^2$$

4b ii

$$\ddot{x} = g - \frac{1}{40}v^2 = v\frac{dv}{dx}$$

$$dx = \frac{40v}{400 - v^2}dv$$

$$\int dx = -20\int \frac{-2v}{400 - v^2}dx$$

$$x = -20\ln(400 - v^2) + C$$

Now, since the object is falling down, the new stationary point conditions will be $x = 0, v = 0$ (peak of fall) as the direction of the motion is changed. Hence,

$$0 = -20\ln 400 + C$$

$$C = 20\ln 400$$

Therefore

$$x = -20\ln(400 - v^2) + 20\ln 400$$

$$x = 20\ln\frac{400}{400 - v^2}$$

When the object returns to the starting point, which is now $x = 20\ln 2$,

$$20\ \ln 2 = 20\ln\frac{400}{400 - v^2}$$

$$800 - 2v^2 = 400$$

$$v^2 = 200$$

$$\text{Speed} = |v| = 10\sqrt{2}\ \text{m/s}$$

5a $v_i = V_0$; $f_r = mkv$;

$$m\ddot{x} = (mg - mkv)$$

$$\ddot{x} = (g - kv)$$

$$\frac{dv}{dt} = (g - kv)$$

$$\int dt = \int \frac{dv}{g - kv}$$

$$t = -\frac{1}{k}\ln(g - kv) + C$$

At $t = 0,\ v = V_0$.

$$C = \frac{1}{k}\ln(g - kV_0)$$

Therefore

$$t = -\frac{1}{k}\ln(g - kv) + \frac{1}{k}\ln(g - kV_0)$$

$$t = \frac{1}{k}\ln\frac{g - kV_0}{g - kv}$$

5b

$t = \frac{1}{k}\ln\frac{g - kV_0}{g - kv}$ from part a), rearranging this we get,

$$e^{kt} = \frac{g - kV_0}{g - kv}$$

$$kv = g - (g - kV_0)e^{-kt}$$

$$v = \frac{g}{k}(1 - e^{-kt}) + V_0e^{-kt}$$

The result, now as $t \to \infty$,

$$\lim_{t\to\infty} v$$

$$= \lim_{t\to\infty}\left(\frac{g}{k}(1 - e^{-kt}) + V_0e^{-kt}\right)$$

$$= \frac{g}{k}(1 - 0) + V_0(0)$$

$$= \frac{g}{k}$$

5c From above,

$$v = \frac{g}{k} - \frac{g}{k}e^{-kt} + V_0 e^{-kt} = \frac{dx}{dt}$$

$$\int dx = \int \left(\frac{g}{k} + \left(V_0 - \frac{g}{k}\right)e^{-kt}\right)dt$$

$$x = \frac{g}{k}t + \left(V_0 - \frac{g}{k}\right)\left(-\frac{1}{k}e^{-kt}\right) + C$$

At $t = 0, x = 0,$ and so

$$C = \left(V_0 - \frac{g}{k}\right)\left(\frac{1}{k}\right)$$

Hence,

$$x = \frac{g}{k}t + \frac{kV_0 - g}{k^2}(1 - e^{-kt})$$

5d From part c, the object will fall as:

$$x = \frac{g}{k}t + \frac{kV_0 - g}{k^2}(1 - e^{-kt})$$

The terminal velocity of the object is:

$$\frac{g}{k} = \frac{10}{k} = 20$$

Hence,

$$k = \frac{1}{2}$$

Now, for the first object

$$x_1$$

$$= \frac{g}{k}t + \frac{(kV_0 - g)}{k^2}(1 - e^{-kt})$$

$$= 20t + \frac{\left(\frac{20}{2} - 10\right)}{k^2}(1 - e^{-kt})$$

$$= 20t + 0$$

The other object is released from rest and so $V_0 = 0$, which gives:

$$x_2$$

$$= \frac{g}{k}t + \frac{g}{k^2}\left(1 - e^{-\frac{1}{2}t}\right)$$

$$= 20t + 40\left(1 - e^{-\frac{1}{2}t}\right)$$

Thus, the distance between the objects after t seconds will be:

$$d$$

$$= |x_1 - x_2|$$

$$= \left|20t - \left(20t + 40\left(1 - e^{-\frac{1}{2}t}\right)\right|$$

$$= 40\left(1 - e^{-\frac{1}{2}t}\right)$$

The limiting distance is when the stone thrown with terminal velocity hits the ground. As we don't know the height, we assume the time of descent is very large and $t \to \infty$. Hence, the limiting distance between objects is,

$$\lim_{t\to\infty} d = \lim_{t\to\infty} 40(1 - e^{-kt}) = 40(1 - 0) = 40 \text{ m}$$

6a $m = 10$; $k = \frac{1}{10}$; $f_r = kv^2$; $v_i = u$; $g = 10$

At the highest point (point A), $v = 0$. The equation of motion is,

$$\ddot{x} = -g - \frac{k}{m}v^2 = \frac{dv}{dt}$$

Integrating

$$\int_0^t dt = -\int_u^v \frac{dv}{g + \frac{k}{m}v^2}$$

$$t = \frac{1}{10}\int_v^u \frac{dv}{1 + (\frac{1}{10\sqrt{10}}\ v)^2}$$

Let $\frac{1}{10\sqrt{10}}v = \tan a$, then $\frac{1}{10\sqrt{10}}\ dv = \sec^2 a\, da$ and using this with

$1 + \tan^2 a = \sec^2 a$, the integral evaluates to

t

$$= \sqrt{10}\left[\tan^{-1}\frac{v}{10\sqrt{10}}\right]_v^u$$

$$= \sqrt{10}\left(\tan^{-1}\frac{u}{10\sqrt{10}} - \tan^{-1}\frac{v}{10\sqrt{10}}\right)$$

At point A, $v = 0$ and so we have:

t

$$= \sqrt{10}\left(\tan^{-1}\frac{u}{10\sqrt{10}} - \tan^{-1}\frac{0}{10\sqrt{10}}\right)$$

$$= \sqrt{10}\tan^{-1}\frac{u}{10\sqrt{10}}$$

6b

$$v\frac{dv}{dx} = -g - \frac{k}{m}v^2$$

$$\int dx = -\frac{1}{2}\int \frac{2v\,dv}{g + \frac{k}{m}v^2}$$

x

$$= -\frac{m}{2k}\ln\left(g + \frac{k}{m}v^2\right) + C$$

$$= -50\ln\left(10 + \frac{1}{100}v^2\right) + C$$

At $t = 0, x = 0$ and $v = u$.

$$C = 50\ln\left(\frac{1000 + u^2}{100}\right)$$

Therefore

$$x = 50\ln\left(\frac{1000 + u^2}{1000 + v^2}\right)$$

At maximum height, $v = 0$.

$$x = 50\ln\left(\frac{1000 + u^2}{1000}\right) \text{ metres}$$

6c $v_f = \omega$

The object will fall a distance of OA. Taking down to be the positive direction the equation of motion becomes,

$$v\frac{dv}{dx} = \left(g - \frac{k}{m}v^2\right)$$

This is the same equation of motion as in part b), with the sign of g reversed. Hence since g is just a constant the derivation of the equation in part b) still applies if we switch g with $-g$. Furth, because we are considering the peak of the motion to be the new starting point $u = 0$. Putting this all together we have,

x

$$= 50\ln\left(\frac{-1000}{-1000 + v^2}\right)$$

$$= -50\ln\left(\frac{1000 - v^2}{1000}\right)$$

Rearranging and taking the exponential of each side we have,

$$e^{-\frac{x}{50}} = \frac{1000 - v^2}{1000}$$

Solving now for v^2 gives

$$v^2 = 1000 - (1000)e^{-\frac{x}{50}}$$

Now in part b) we showed that the height of OA was $x = 50\ln\frac{1000+u^2}{1000}$. Hence, the particle would have to also fall this distance to reach the ground. Subbing this value of x we find that the velocity when the object hits the ground is.

ω^2

$$= 1000 - 1000e^{-\ln\frac{1000+u^2}{1000}}$$

$$= 1000 - 1000 \times \frac{1000}{1000 + u^2}$$

$$= 1000 \times \frac{1000 + u^2 - 1000}{(1000 + u^2)}$$

$$= \frac{1000u^2}{1000 + u^2}$$

7a i The equation of motion is:

$m\ddot{x} = mg - mkv$

$\ddot{x} = (g - kv)$

Terminal velocity is when $\ddot{x} = 0$. Hence,

$g - kV_T = 0$

$V_T = \dfrac{g}{k}$

7a ii

$\dfrac{dv}{dt} = (g - kv)$

Integrating we have

$\int dt = \int \dfrac{1}{g - kv}\, dv$

$t = -\dfrac{1}{k}\ln(g - kv) + C$

At $t = 0,\ v = 0$.

$C = \dfrac{1}{k}\ln g$

Therefore

t

$= -\dfrac{1}{k}\ln(g - kv) + \dfrac{1}{k}\ln g$

$= -\dfrac{1}{k}\ln\left(\dfrac{g - kv}{g}\right)$

Solving for v gives

v

$= \dfrac{1}{k}(g - ge^{-kt})$

$= V_T(1 - e^{-kt})$

7b i The equation of motion for the second particle:

$$\ddot{x} = -(g + kv) = \frac{dv}{dt}$$

At maximum height $v = 0$, as such integrating to find max height time gives

$$\int_0^{t_{max}} dt = -\int_U^0 \frac{dv}{g + kv}$$

$$t_{max}$$

$$= \int_0^U \frac{dv}{g + kv}$$

$$= \left[\frac{1}{k}\ln(g + kv)\right]_0^U$$

$$= \frac{1}{k}\ln\left(\frac{g + kU}{g}\right)$$

7b ii The speed of the first particle is given by:

$$v_1 = V_T(1 - e^{-kt})$$

The time taken by second particle to reach maximum height is given from above:

$$T_2 = \frac{1}{k}\ln\left(\frac{g + kU}{g}\right)$$

Substituting the value of time in the equation for velocity:

$$v_1$$

$$= V_T\left(1 - e^{-k \times \frac{1}{k}\ln\frac{g+kU}{g}}\right)$$

$$= V_T\left(1 - \frac{g}{g + kU}\right)$$

$$= \frac{kUV_T}{g + kU}$$

$$= \frac{kUV_T}{k\left(\frac{g}{k} + U\right)}$$

$$= \frac{V_T U}{V_T + U} \quad \text{as } V_T = \frac{g}{K}$$

8a The terminal velocity for a particle falling under gravity is:

$$\ddot{x} = (g - kv)$$

At $\ddot{x} = 0$,

$$g - kV_T = 0$$

$$V_T = \frac{g}{k}$$

8b The initial velocity of projection for P_2 is:

$$2V_T = \frac{2g}{k}$$

The equation of motion for P_2:

$$\ddot{x} = -(g + kv) = \frac{dv}{dt}$$

Integrating gives,

$$\int_0^t dt = -\int_{v_0}^{v} \frac{dv}{g + kv}$$

$$t$$

$$= \int_{v}^{v_0} \frac{dv}{g + kv}$$

$$= \left[\frac{1}{k}\ln(g + kv)\right]_v^{v_0}$$

$$= \frac{1}{k}\left\{\ln\left(g + \frac{2kg}{k}\right) - \ln(g + kv)\right\}$$

$$= \frac{1}{k}\ln\left(\frac{3g}{g + kv}\right)$$

8c Velocity of P_1 is 30% of its terminal velocity:

$$v_1 = 0.3\frac{g}{k}$$

The equation of motion is:

$$\frac{dv}{dt} = (g - kv)$$

Integrating we find that,

$$t = -\frac{1}{k}\ln(g - kv) + C$$

Since P_1 is dropped, at $t = 0, v = 0$. Hence,

$$C = \frac{1}{k}\ln g$$

Hence,

$$t = \frac{1}{k}\ln\left(\frac{g}{g - kv}\right)$$

Subbing in v_1 we have

$$t$$

$$= \frac{1}{k}\ln\left(\frac{1}{1 - 0.3}\right)$$

$$= \frac{1}{k}\ln\frac{10}{7}$$

Substituting the value of time in the equation from part b gives:

$$t = \frac{1}{k}\ln\left(\frac{3g}{g + kv_2}\right) = \frac{1}{k}\ln\frac{10}{7}$$

Rearranging and solving we have

$$\frac{3\frac{g}{k}}{\frac{g}{k} + v_2} = \frac{10}{7}$$

$$\frac{21g}{k} = \frac{10g}{k} + 10v_2$$

$$v_2 = \frac{11g}{10k} \text{ m/s}$$

9a i From the question

$$f(x) = x - \frac{g^2}{x} - 2g\ln\frac{x}{g} \qquad \text{where } x \geq g$$

Hence,

$f(g)$

$= g - \frac{g^2}{g} - 2g\ln\left(\frac{g}{g}\right)$

$= g - g - 2g\ln 1$

$= 0$

9a ii

$f'(x)$

$= \frac{d}{dx}f(x)$

$= 1 + \frac{g^2}{x^2} - 2g\left(\frac{1}{x}\right)$

If $\frac{g}{x} = k$, then factorising gives

$1 + k^2 - 2k = (1-k)^2$

Hence,

$f'(x) = \left(1 - \frac{g}{x}\right)^2$

9a iii From above,

$f'(x) = \left(1 - \frac{g}{x}\right)^2$

Now $f'(x) > 0$ for all values of x since it is squared. Hence, $f(x)$ is an increasing function and as $f(g) = 0$, $f(x) > 0$ for $x > g$.

9b i

$$\frac{d^2y}{dt^2} = -(g + kv) = \frac{dv}{dt}$$

Integrating gives,

$$\int dt = -\int \frac{dv}{g + kv}$$

$$t = -\frac{1}{k}\ln(g + kv) + C$$

At $t = 0$, $v = V_0$.

$$C = \frac{1}{k}\ln(g + kV_0)$$

Therefore

$$t = \frac{1}{k}\ln\left(\frac{g + kV_0}{g + kv}\right)$$

Rearranging for v gives,

$$g + kv = (g + kV_0)e^{-kt}$$

Hence,

$$v = \frac{(g + kV_0)e^{-kt} - g}{k} = \frac{dy}{dt}$$

Integrating again now gives,

$$\int dy = \frac{1}{k}\int (g + kV_0)e^{-kt}\,dt - \frac{g}{k}\int dt$$

$$y = -\frac{1}{k^2}(g + kV_0)e^{-kt} - \frac{g}{k}t + D$$

At $t = 0, y = 0$ and so

$$D = \frac{1}{k^2}(g + kV_0)$$

Therefore

$$y = \frac{1}{k^2}(g + kV_0)(1 - e^{-kt}) - \frac{g}{k}t$$

$$k^2y = (g + kV_0)(1 - e^{-kt}) - gkt$$

9b ii At maximum height, $v = 0$. From above we have:

$$t = \frac{1}{k}\ln\left(\frac{g + kV_0}{g + kv}\right)$$

And so, T is given by,

T

$$= \frac{1}{k}\ln\left(\frac{g + kV_0}{g + 0}\right)$$

$$= \frac{1}{k}\ln\left(\frac{g + kV_0}{g}\right)$$

9b iii The question for part b) stated that we assume the equation of motion is true for all $t \geq 0$. Hence, the equation derived in part b)(i) still applies. Subbing in $t = 2T$ we have:

k^2y

$$= (g + kV_0)(1 - e^{-kt}) - gkt$$

$$= (g + kV_0)\left(1 - \left(\frac{g}{g + kV_0}\right)^2\right) - gk\left(\frac{2}{k}\ln\left(\frac{g + kV_0}{g}\right)\right)$$

$$= g + kV_0 - \frac{g^2}{g + kV_0} - 2g\ln\left(\frac{g + kV_0}{g}\right)$$

9biv Let $(g + kV_0) = X$

Then the equation above can be rewritten as,

$$f(X) = X - \frac{g^2}{X} - 2g\ln\frac{X}{g}$$

This is the form from part a, where we proved that $f(X)$ is an increasing function and that $f(X) > 0$ for $X > g$. Now, $X = g + kV_0 > g$ since $k, V_0 > 0$. Thus, using the result of part a) $f(X) = ky^2 > 0$. Hence, $y > 0$. Now it took at time T to reach the peak from $y = 0$, but after another time interval of T we have that $y > 0$. Thus, the downwards journey must take longer than the upward journey.

10a i Object of mass 1 kg dropped under gravity:

The forces acting on the object are gravity and air resistance.

Gravity pulls the object downwards while the air resistance resists the motion of the object. So, the equation of motion is (taking down to be positive):

$$m\ddot{x} = mg - kv$$

$$\ddot{x} = g - \frac{k}{m}v$$

$$\ddot{x} = 10 - \frac{1}{10}v$$

10a ii

$$\ddot{x} = v\frac{dv}{dx} = 10 - \frac{1}{10}v$$

$$\frac{dv}{dx} = \frac{100 - v}{10v}$$

Integrating using partial fractions gives

$$\int dx = 10\int \frac{v}{100 - v}dv$$

$$x$$

$$= 10\int \left(\frac{100}{100 - v} - 1\right)dv$$

$$= -1000\ln(100 - v) - 10v + C$$

At $t = 0$, $v = 0$, $x = 0$. Which gives

$$C = 1000\ln 100$$

Therefore

$$x = -1000\ln\frac{100 - v}{100} - 10v$$

Now using $x = 40$ and $v = V$ we have

$$40 = -1000\ln\left(1 - \frac{V}{100}\right) - 10V$$

Or,

$$V + 100\ln\left(1 - \frac{V}{100}\right) + 4 = 0$$

10b i After the parachute opens, the new equation of motion will be:

$\ddot{x}$

$= (g - kv^2)$

$= 10 - \dfrac{1}{10}v^2$

$= \dfrac{100 - v^2}{10}$

10b ii

$$\ddot{x} = (g - kv^2) = \frac{100 - v^2}{10} = v\frac{dv}{dx}$$

Integrating gives,

$$\int dx = 5\int \frac{2v}{100 - v^2}\,dx$$

$x = -5\ln(100 - v^2) + C$

At the start of this second journey, $x = 0$ and $v = V$. Hence,

$C = 5\ln(100 - V^2)$

Therefore

$$x = -5\ln\left(\frac{100 - v^2}{100 - V^2}\right)$$

Solving this equation for v after taking the exponential of both sides gives,

$v^2 = 100 - (100 - V^2)e^{-\frac{x}{5}}$

Terminal velocity occurs when $\ddot{x} = 0$, Hence, at terminal velocity,

$\dfrac{100 - V_T^2}{10} = 0$

$V_T^2 = 100$

$V_T = 10\ \text{ms}^{-1}$

10b iii

$$\ddot{x} = (g - kv^2) = \frac{100 - v^2}{10} = \frac{dv}{dt}$$

Integrating using partial fractions gives:

$$\int dt = 10 \int \frac{dv}{100 - v^2}$$

t

$$= 10 \int \frac{1}{(10 + v)(10 - v)} dv$$

$$= \frac{1}{2} \int \left(\frac{1}{10 + v} + \frac{1}{10 - v} \right) dv$$

$$= \frac{1}{2} (\ln(10 + v) - \ln(10 - v)) + C$$

$$= \frac{1}{2} \ln \left(\frac{10 + v}{10 - v} \right) + C$$

At the start of this new journey, $t = 0$, $v = V$. Hence,

$$C = -\frac{1}{2} \ln \left(\frac{10 + V}{10 - V} \right)$$

Therefore

$$t = \frac{1}{2} \ln \frac{(10 + v)(10 - V)}{(10 - v)(10 + V)}$$

10b iv $V = 25.7$ m/s. The terminal velocity after the parachute opens is $V_T = 10$ m/s and so 105% of $V_t = 10.5$ m/s.

Using equation from part b iii with $v = 10.5$ and $V = 25.7$ we find that,

t

$$= \frac{1}{2} \ln \frac{(10 + 10.5)(10 - 25.7)}{(10 - 10.5)(10 + 25.7)}$$

$\doteqdot 1.446$ s

11a $V = 30$, $\theta = 45°$, $k = \frac{1}{3}$, $g = 10$

The equations of motion are,

$$\frac{d^2x}{dt^2} = -k\frac{dx}{dt} \text{ and } \frac{d^2y}{dt^2} = -g - k\frac{dy}{dt}$$

Looking at the horizontal component we have:

$$\frac{dv_x}{dt} = -kv_x$$

Which integrating gives:

$$\int dt = -\frac{1}{k}\int \frac{dv_x}{v_x}$$

$$t = -\frac{1}{k}\ln v_x + C$$

At $t = 0$, $v_x = v_{x0} = V\cos 45° = 15\sqrt{2}$ m/s. Hence:

$$C = \frac{1}{k}\ln v_{x0}$$

Therefore,

$$t = \frac{1}{k}\ln\frac{v_{x0}}{v_x}$$

Rearranging for v_x we have,

$$v_x = v_{x0}e^{-kt} = \frac{dx}{dt}$$

Integrating gives:

$$\int dx = v_{x0}\int e^{-kt}\,dt$$

$$x = -\frac{1}{k}v_{x0}e^{-kt} + D$$

At $t = 0$, $x = 0$. Hence:

$$D = \frac{1}{k}v_{x0}$$

Therefore

$$x = \frac{1}{k}v_{x0}(1 - e^{-kt})$$

Which subbing in the values for v_{x0} and k becomes:

$$x = 45\sqrt{2}\left(1 - e^{-\frac{t}{3}}\right)$$

11b

$$\frac{dv_y}{dt} = -(g + kv_y)$$

Integrating gives:

$$\int dt = -\int \frac{dv_y}{g + kv_y}$$

$$t = -\frac{1}{k}\ln(g + kv_y) + C$$

At $t = 0,\ \ v_y = v_{y0} = V\sin 45° = 15\sqrt{2}$ m/s. Hence:

$$C = \frac{1}{k}\ln(g + kv_{y0})$$

Therefore

$$t = -\frac{1}{k}\ln\left(\frac{g + kv_y}{g + kv_{y0}}\right)$$

Rearranging for v_y gives:

$$v = \frac{(g + kv_{y0})e^{-kt} - g}{k} = \frac{dy}{dt}$$

Integrating again gives:

$$\int dy = \frac{1}{k}\int (g + kv_{y0})e^{-kt}\,dt - \frac{g}{k}\int dt$$

$$y = -\frac{1}{k^2}(g + kv_{y0})e^{-kt} - \frac{g}{k}t + D$$

At $t = 0,\ \ y = 0$. Hence:

$$D = \frac{1}{k^2}(g + kv_{y0})$$

Therefore

$$y = \frac{1}{k^2}(g + kv_{y0})(1 - e^{-kt}) - \frac{g}{k}t$$

Subbing in the values for v_{y0}, g and k we get:

$$y = 45(2 + \sqrt{2})(1 - e^{-t/3}) - 30t$$

11c We have from the previous two questions that,

$y = 45(2+\sqrt{2})\left(1 - e^{-\frac{t}{3}}\right) - 30t$ and

$x = 45\sqrt{2}\left(1 - e^{-\frac{t}{3}}\right)$

Solving the second equation for t gives:

$t = -3\ln\left(1 - \dfrac{x}{45\sqrt{2}}\right)$

Substituting this equation into the equation for y:

y

$= 45(2+\sqrt{2})\left(1 - e^{\ln\left(1-\frac{x}{45\sqrt{2}}\right)}\right) + 90\ln\left(1 - \dfrac{x}{45\sqrt{2}}\right)$

$= 45(2+\sqrt{2})\left(1 - \left(1 - \dfrac{x}{45\sqrt{2}}\right)\right) + 90\ln\left(\dfrac{45\sqrt{2} - x}{45\sqrt{2}}\right)$

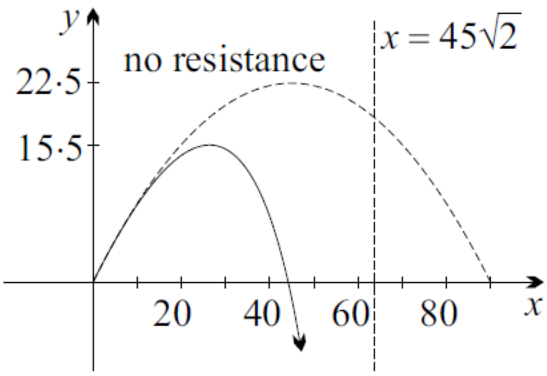
$= 45(2+\sqrt{2})\dfrac{x}{45\sqrt{2}} - 90\ln\left(\dfrac{45\sqrt{2}}{45\sqrt{2} - x}\right)$

$= 45\sqrt{2}(\sqrt{2}+1)\dfrac{x}{45\sqrt{2}} - 90\ln\left(\dfrac{45\sqrt{2}}{45\sqrt{2} - x}\right)$

$= (1+\sqrt{2})x - 90\ln\left(\dfrac{45\sqrt{2}}{45\sqrt{2} - x}\right)$

11d Graph of

$$y = (1 + \sqrt{2})x - 90 \ln\left(\frac{45\sqrt{2}}{45\sqrt{2} - x}\right)$$



The range from the plot is approximately 44 m.

11e When there is no air resistance:

$$\frac{d^2x}{dt^2} = 0 \quad \text{and} \quad \frac{d^2y}{dt^2} = -g$$

$$x = V\cos 45°\ t \quad \text{and} \quad y = V\sin 45°\ t - \tfrac{1}{2}gt^2$$

$$x = 15\sqrt{2}t \quad \text{and} \quad y = 15\sqrt{2}\,t - \tfrac{1}{2}gt^2$$

$$t = \frac{x}{15\sqrt{2}}$$

Substituting for t in equation for y:

$$y = 15\sqrt{2} \times \frac{x}{15\sqrt{2}} - \frac{1}{2} \times 10 \times \left(\frac{x}{15\sqrt{2}}\right)^2$$

$$y = x - \frac{5x^2}{450}$$

$$y = x - \frac{x^2}{90}$$

$$y = x\left(1 - \frac{x}{90}\right)$$

Solutions to Exercise 6E Enrichment questions

12 $\ddot{y} = -(g + kv)$

When $\ddot{y} = 0$, $v = V_T$

Hence,

$0 = -g - kV_T$

$V_T = -\dfrac{g}{k}$

From 9b,

$t = \dfrac{1}{k}m\left(\dfrac{g + kV_0}{g + kV}\right)(1)$

From which,

$V = \dfrac{g + kV_0}{k}e^{-kt} - \dfrac{g}{k}(2)$

Then, after integrating with respect to t, the height is given by:

$y_h = \dfrac{g + kV_0}{k^2}(1 - e^{-kt}) - \dfrac{g}{k}t(3)$(See 9bi)

From 9bii, the time T to reach the maximum height is:

$T = \dfrac{1}{k}\ln\left(\dfrac{g + kV_0}{g}\right)(4)$

If, however, we consider the downwards motion separately, as in Worked Example 18, we have $y = 0$ as the starting point, downwards as the positive direction, and so y is the distance fallen, which we will call y_f.

Note that t= T is the initial value of T.

From part c, replacing t with $t - T$,

y_f

$= \dfrac{g}{k}\left((t - T) + \dfrac{1}{k}e^{-k(t-T)} - \dfrac{1}{k}\right)$

$= \dfrac{g}{k}\left((t - T) + \dfrac{g}{k^2}e^{kt}.e^{-kt} - \dfrac{g}{k^2}\right)$

From (4),

$T = \dfrac{1}{k}\ln\left(\dfrac{g + kV_0}{g}\right)$

$$e^{kT} = \frac{g + kV_0}{g}$$

Hence,

$$y_f = \frac{g}{k}t - \frac{g}{k^2}\ln\left(\frac{g + kV_0}{g}\right) + \frac{g + kV_0}{k^2}e^{-kt} - \frac{g}{k^2}\,(5)$$

If the two approaches are consistent, adding y_h and y_f will give the maximum height H.

First find H. When $t = T$, $y_h = H$

Substituting into (3):

H

$$= \frac{g + kV_0}{k^2}(1 - e^{-kT}) - \frac{g}{k}T, \left(\text{where } T = \frac{1}{k}\ln\left(\frac{g + kV_0}{g}\right)\right)$$

$$= \frac{g + kV_0}{k^2}\left(1 - \frac{g}{g + kV_0}\right) - \frac{g}{k^2}\ln\left(\frac{g + kV_0}{g}\right)$$

So,

$$H = \frac{V_0}{k} - \frac{g}{k^2}\ln\left(\frac{g + kV_0}{g}\right)\,(6)$$

Finally, adding (3) and (5), for $t \geq T$, gives,

$y_h + y_f$

$$= \frac{g + kV_0}{k^2}(1 - e^{-kt}) - \frac{g}{k}t + \frac{g}{k}t - \frac{g}{k^2}\ln\left(\frac{g + kV_0}{g}\right) + \frac{g + kV_0}{k^2}e^{-kt} - \frac{g}{k^2}$$

$$= \frac{V_0}{k} - \frac{g}{k^2}\ln\left(\frac{g + kV_0}{g}\right)$$

$= H$

Hence, the two approaches are consistent.

13ai

$$\frac{du}{dt} \approx \frac{u(t+\delta t) - u(t)}{\delta t}$$

Hence,

$u(t+\delta t)$

$\approx \delta t . \dot{u} + u(t)$

$= u(t) + \delta t\left(-k(u^2+v^2)^{\frac{1}{2}}.u(t)\right)$

$= u(t) + \delta t\left(1 - k.\delta t(u^2+v^2)^{\frac{1}{2}}\right)$

13aii

$$\frac{dx}{dt} \approx \frac{x(t+\delta t)-x(t)}{\delta t}$$

Hence,

$\mathrm{x}(t+\delta t)$

$\approx \delta t.\dot{x} + x(t)$

$=\delta t.u(t) + x(t)$

Also,

$$\frac{dy}{dt} \approx \frac{y(t+\delta t)-y(t)}{\delta t}$$

Hence,

$y(t+8t)$

$\approx \delta t.\dot{y} + y(t)$

$= \delta t.v(t) + y(t)$

Finally,

$$\frac{dv}{dt} \approx \frac{v(t+\delta t)-v(t)}{\delta t}$$

Hence,

$\mathrm{v}(t+\delta t)$

$\approx \delta t.\dot{v} + v(t)$

$= v(t) - \delta t\left(g + k(u^2+v^2)^{\frac{1}{2}}.v(t)\right)$

$= v(t)\left(1 - k.\delta t(u^2+v^2)^{\frac{1}{2}}\right) - g.\delta t$

13bi $u(0.1)$

$$\approx u(0)\left(1 - \frac{1}{90}(0.1)(u(0)^2 + v(0)^2)^{\frac{1}{2}}\right)$$

$$= 15\sqrt{2}\left(1 - \frac{1}{900}(450 + 450)^{\frac{1}{2}}\right)$$

$$= 15\sqrt{2}\left(1 - \frac{1}{30}\right)$$

$$\approx 20.51$$

The calculations for 13bii and 13c are similar.

13d $R \approx 54$

Solutions to Exercise 6F Foundation questions

1a $x = 20t$

$y = -5t^2 + 20\sqrt{3}t$

$\dot{x} = 20$

$\dot{y} = -10t + 20\sqrt{3}$

1b

Using $v_o \sin\theta\, t - \frac{1}{2}gt^2 = 0$

$t\left(v_o \sin\theta - \frac{1}{2}gt\right) = 0$

$t = 0$ or $t = \frac{2v_o \sin\theta}{g} = \frac{2 \times 40 \sin 60°}{10} = 4\sqrt{3}$ seconds

$R = xt$

$= v_0 \cos\theta\, t$

$= \frac{2v_0 \cos\theta\, v_o \sin\theta}{g}$

$= \frac{v_0^2 \sin 2\theta}{g}$

$= \frac{40^2 \sin 2(60°)}{10}$

$= 80\sqrt{3}$ metres

1c $v_y = v_0 \sin\theta - gt = 0$

$t = \frac{v_0 \sin\theta}{g}$

$= \frac{40 \sin 60°}{10}$

$= 2\sqrt{3}$

Maximum height:

$y(2\sqrt{3}) = 40 \sin 60° \,(2\sqrt{3}) - \frac{1}{2}(10)(2\sqrt{3})^2$

$= 60$ metres

1d

$$R = xt$$

$$= v_0 \cos\theta\, t$$

$$= \frac{2v_0 \cos\theta\, v_o \sin\theta}{g}$$

$$= \frac{v_0^2 \sin 2\theta}{g}$$

$$= \frac{40^2 \sin 2(30°)}{10}$$

$$= 80\sqrt{3} \text{ metres}$$

It is false.

2a

$$x = 39 \cos\left(\tan^{-1}\frac{12}{5}\right) t$$

$$= 15t$$

$$y = -5t^2 + 39 \sin\left(\tan^{-1}\frac{12}{5}\right) t$$

$$= -5t^2 + 36t$$

$$\dot{x} = 15$$

$$\dot{y} = -10t + 36$$

2b

$$t = \frac{x}{v_0 \cos\theta}$$

$$= \frac{30}{39 \cos\left(\tan^{-1}\frac{12}{5}\right)}$$

$$= 2$$

$$y = -5(2)^2 + 36(2)$$

$$= 52$$

2c $\dot{x} = 15$

$\dot{y} = -10t + 36$

At $t = 2$,

$v = \sqrt{15^2 + 16^2}$

$\doteqdot 21.9$ m/s

2d $v_y = v_0 \sin\theta - gt = 0$

$t = \dfrac{v_0 \sin\theta}{g}$

$= \dfrac{39 \sin\left(\tan^{-1}\frac{12}{5}\right)}{10}$

$= 3.6$

Therefore, after.

3a $x = 20\cos(30°)\, t$

$= 10\sqrt{3}t$

$y = -5t^2 + 20\sin(30°)\, t$

$= -5t^2 + 10t$

3b $-75 = -5t^2 + 10t$

$-5t^2 + 10t + 75 = 0$

$t^2 - 2t - 15 = 0$

$(t+3)(t-5) = 0$

$t = -3, 5$

When $t = 5$,

$x = 20\cos(30°)\,(5)$

$= 50\sqrt{3}$

3c

$$v_y = v_0 \sin\theta - gt = 0$$

$$t = \frac{v_0 \sin\theta}{g}$$

$$= \frac{20 \sin 30^\circ}{10}$$

$$= 1$$

Maximum height:

$$y(1) = 20 \sin 30^\circ\,(1) - \frac{1}{2}(10)(1)^2$$

$$= 5 \text{ metres}$$

Thus, $75 + 5 = 80$ metres.

3d $\dot{x} = 10\sqrt{3}$

$$\dot{y} = -10t + 10$$

At $t = 5$,

$$v = \sqrt{\left(10\sqrt{3}\right)^2 + (-40)^2}$$

$$\doteqdot 44 \text{ m/s}$$

The direction is given by:

$$\tan\theta = \frac{40}{10\sqrt{3}}$$

$$\theta \doteqdot 67^\circ$$

3e $x = 10\sqrt{3}t$

$t = \dfrac{x}{10\sqrt{3}}$

$y = -5t^2 + 10t$

$= -5\left(\dfrac{x}{10\sqrt{3}}\right)^2 + 10\left(\dfrac{x}{10\sqrt{3}}\right)$

$= -\dfrac{1}{60}x^2 + \dfrac{1}{\sqrt{3}}x$

4a

$$\frac{363.6 \times 1000}{1 \times 60 \times 60} = 101 \text{ m/s}$$

4b $x = 101t$

$y = -5t^2$

4c $-600 = -5t^2$

$t^2 = 120$

$t = 2\sqrt{30}$

4d $\dot{x} = 101$

$\dot{y} = -10t$

At $t = 2\sqrt{30}$,

$v = \sqrt{101^2 + \left(-20\sqrt{30}\right)^2}$

$= 149 \text{ m/s}$

$\tan\theta = \dfrac{20\sqrt{30}}{101}$

$\theta \doteqdot 47° \, 19'$

4e At $t = 2\sqrt{30}$,

$$x = \frac{101 \times 2\sqrt{30}}{1000}$$

$$\doteqdot 1.106 \text{ km}$$

5a $x = 50\cos(45°)\,t$

$$= 25\sqrt{2}t$$

$$y = -5t^2 + 50\sin(45)\,t$$

$$= -5t^2 + 25\sqrt{2}t$$

When $x = 75$,

$$t = \frac{75}{25\sqrt{2}}$$

$$= \frac{3}{2}\sqrt{2}$$

At $t = \frac{3}{2}\sqrt{2}$,

$$y = -5\left(\frac{3}{2}\sqrt{2}\right)^2 + 25\sqrt{2}\left(\frac{3}{2}\sqrt{2}\right)$$

$$= 52.5$$

5b $\dot{x} = 25\sqrt{2}$

$$\dot{y} = -10t + 25\sqrt{2}$$

At $t = \frac{3}{2}\sqrt{2}$,

$$v = \sqrt{\left(25\sqrt{2}\right)^2 + \left(-10\left(\frac{3}{2}\sqrt{2}\right) + 25\sqrt{2}\right)^2}$$

$$= 5\sqrt{58} \text{ m/s}$$

$$\tan\theta = \frac{10\sqrt{2}}{25\sqrt{2}} = \frac{2}{5}$$

$$\theta = \tan^{-1}\frac{2}{5}$$

6a The components of x and y of the displacement are given by

$x = v_0 \cos\theta\, t$

$y = v_0 \sin\theta\, t - \frac{1}{2}gt^2$

As $v_0 = V$, $g = 10$ and the angle is equal to θ:

$x = V\cos\theta\, t$

$y = V\sin\theta\, t - 5t^2$

6b When $t = 2, x = 24\sqrt{5}$.

$24\sqrt{5} = V\cos\theta \times 2$

$V\cos\theta = 12\sqrt{5}$

When $= 2, y = 28$.

$28 = V\sin\theta\,(2) - 5(2)^2$

$2V\sin\theta = 48$

$V\sin\theta = 24$

6c $V^2\sin^2\theta = 576$

$V^2\cos^2\theta = 720$

$V^2\cos^2\theta + V^2\sin^2\theta = 1296$

$V^2(\cos^2\theta + \sin^2\theta) = 1296$

$V = \left|\sqrt{1296}\right| = 36$

$36\sin\theta = 24$

$\sin\theta = \frac{24}{36}$

$\theta = 41^\circ\, 49'$

6d $v_y = v_0 \sin\theta - gt = 0$

$$t = \frac{v_0 \sin\theta}{g}$$

$$= \frac{36 \sin 41° \, 49'}{10}$$

$$= 2.4$$

Total flight time $= 2.4(2) = 4.8$

$x(4.8) = 36 \cos 41° \, 49' \, (4.8)$

$= 129$ metres

Solutions to Exercise 6F Development questions

7a $\quad x = Vt\cos\alpha \quad$ and $\quad y = Vt\sin\alpha - \frac{1}{2}gt^2$

$$t = \frac{x}{V\cos\alpha}$$

Substituting for t in equation for y:

y

$$= V\left(\frac{x}{V\cos\alpha}\right)\sin\alpha - \frac{g}{2}\left(\frac{x}{V\cos\alpha}\right)^2$$

$$= \frac{x\sin\alpha}{\cos\alpha} - \frac{gx^2}{2V^2\cos^2\alpha}$$

$$= \frac{x}{\cos^2\alpha}\left(\sin\alpha\cos\alpha - \frac{gx}{2V^2}\right)$$

7b $\quad$ For horizontal range, $y = 0$. Using the equation above:

$$0 = \frac{x}{\cos^2\alpha}\left(\sin\alpha\cos\alpha - \frac{gx}{2V^2}\right)$$

We solve for $\frac{x}{\cos^2\alpha} \neq 0$ and so:

$$V^2(2\sin\alpha\cos\alpha) = gx$$

Using the double angle identity this becomes:

$$x = R = \frac{V^2\sin 2\alpha}{g}$$

7c i $\quad V = 30; \quad g = 10; \quad R = 45$. Subbing these values into above equation:

$$45 = \frac{30^2\sin 2\alpha}{10}$$

Hence,

$$\sin 2\alpha = \frac{1}{2}$$

And

$$\alpha = \frac{1}{2}\sin^{-1}\frac{1}{2}$$

Therefore, $\alpha = 15°$ or $75°$

7c ii For $\alpha = 15°$,

$$y_1 = \frac{x}{\cos^2 15°}\left(\sin 15° \cos 15° - \frac{10x}{1800}\right)$$

For $\alpha = 75°$,

$$y_2 = \frac{x}{\cos^2 75°}\left(\sin 75° \cos 75° - \frac{10x}{1800}\right)$$

Evaluating y_1 and y_2 at $x = 40$:

$y_1 \doteqdot 1.1908$ m and $y_2 \doteqdot 16.5868$ m

So, the ball will reach 45 m if thrown at 75° but not for 15°.

8a $V = 20, \quad \theta = \tan^{-1}\frac{4}{3}$

From the triangle formed by θ we also have that $\sin\theta = \frac{4}{5}$ and $\cos\theta = \frac{3}{5}$. Now:

x

$= V\cos\theta\, t$

$= 20 \times \frac{3}{5} \times t$

$= 12t$

And

y

$= V\sin\theta\ t - \frac{1}{2}gt^2$

$= 20 \times \frac{4}{5} \times t - 5t^2$

$= 16t - 5t^2$

8b

R

$= \dfrac{V^2 \sin 2\theta}{g}$

$= \dfrac{V^2\, 2\sin\theta\cos\theta}{g}$

$= \dfrac{400 \times 2 \times \frac{4}{5} \times \frac{3}{5}}{10}$

$= 38.4\text{ m}$

The ball reaches maximum height when the vertical velocity component is zero. From part a) we have,

$y = 16t - 5t^2$

Hence:

$\dfrac{dy}{dt} = 16 - gt$

At maximum height, $\frac{dy}{dt} = 0$, which gives:

$t = \dfrac{16}{g} = 1.6\text{ s}$

Thus:

y

$= 16 \times 1.6 - 5 \times 1.6^2$

$= 12.8\text{ m}$

$= H$

8c i

$\alpha = \tan^{-1}\dfrac{1}{5}$

$\sin\alpha = \dfrac{1}{\sqrt{26}}$ and $\cos\alpha = \dfrac{5}{\sqrt{26}}$

Trajectory of the incline is:

$y = x\tan\alpha = \dfrac{x}{5}$

The horizontal position when it reaches maximum height (at $t = 1.6$ s) is:

x

$= 12t$

$= 12 \times 1.6$

$= 19.2$ m

The height of the incline platform at 19.2 m is:

$H_p = \dfrac{19.2}{5} = 3.84$ m

So, the difference in height between the ball and the incline is:

$H - H_p$

$= 12.8 - 3.84$

$= 8.96$ m

Hence the ball is about 9 m above the road when it reaches its greatest height.

8c ii The time of flight is given when the ball hits the incline, this is when:

$$y = 16t - 5t^2 = y_{road} = \frac{x}{5}$$

The horizontal position of the ball is $x = 12t$, hence we want to solve:

$$16t - 5t^2 = \frac{12t}{5}$$

Or

$$\frac{t}{25}\left(\frac{68}{25} - t\right) = 0$$

Solving for t $(t \neq 0)$ we find that the time of flight must be,

$$T = \frac{68}{25} = 2.72 \text{ s}$$

Now, the horizontal distance the ball has moved is given by:

$$x = 12T = 32.64 \text{ m}$$

But we have to calculate the distance travelled by the ball up the inclination, hence that distance will be:

$$\frac{32.64}{\cos(\tan^{-1} 0.2)} = 33.3 \text{ m}$$

9a Ball thrown at angle θ from 1 m above the ground with a velocity of $u = 20$ m/s.

$$\frac{d^2x}{dt^2} = 0 \quad \text{and} \quad \frac{d^2y}{dt^2} = -g$$

Integrating:

$$\frac{dx}{dt} = C_1 \quad \text{and} \quad \frac{dy}{dt} = -gt + C_2$$

At $t = 0$, $v_x = V\cos\theta$ and $v_y = V\sin\theta$.

So $C_1 = V\cos\theta$ and $C_2 = V\sin\theta$.

Therefore

$$\frac{dx}{dt} = V\cos\theta \quad \text{and} \quad \frac{dy}{dt} = V\sin\theta - gt$$

Integrating:

$$x = Vt\cos\theta + C_3 \quad \text{and } y = Vt\sin\theta - \frac{gt^2}{2} + C_4$$

At $t = 0$, $x = 0$ and $y = 1$.

So $C_3 = 0$ and $C_4 = 1$.

Therefore:

$$x = Vt\cos\theta \quad \text{and } y = Vt\sin\theta - \frac{gt^2}{2} + 1$$

Since $V = 20$ m/s and $g = 10$ m/s^2,

$$x = 20t\cos\theta \quad \text{and } y = 20t\sin\theta - 5t^2 + 1$$

Hence:

$$x = 20t\cos\theta \quad \text{and } y = -5t^2 + 20t\sin\theta + 1$$

9b $x = 16$, $y = 16$, $t = T$, subbing into the equations we just derived gives:

$$16 = 20T\cos\theta \quad \text{and} \quad 16 - 1 = 20T\sin\theta - 5T^2$$

Hence:

$$4 = 5T\cos\theta \quad \text{and} \quad 3 = 4T\sin\theta - T^2$$

9c From part b):

$4 = 5T\cos\theta \qquad (1)$

$3 = 4T\sin\theta - T^2 \qquad (2)$

Using (1):

$$T = \frac{4}{5\cos\theta}$$

Substituting for T in (2):

$$3 = 4 \times \left(\frac{4}{5\cos\theta}\right)\sin\theta - \left(\frac{16}{25\cos^2\theta}\right)$$

$$3 = \frac{16}{5}\tan\theta - \frac{16}{25}\sec^2\theta$$

$$75 = 80\tan\theta - 16(\tan^2\theta + 1)$$

$$16\tan^2\theta - 80\tan\theta + 91 = 0$$

9d Let $\tan\theta = X$. Then from the part c) we have:

$$16X^2 - 80X + 91 = 0$$

Using the quadratic formula gives,

$$X = \frac{80 \pm \sqrt{80^2 - 4 \times 16 \times 91}}{32}$$

Hence:
$X = 3.25$ or $X = 1.75$

And so:

$\tan\theta = 1.75 \quad \text{or} \quad 3.25$

Which gives:

$\theta = 60.255\ldots^\circ \quad \text{or} \quad 72.897\ldots^\circ$

So, to the nearest minute, the two possible values of θ are $60°15'$ and $72°54'$.

10a $h = 2.4, \quad V = 144 \text{ km/h} = 40 \text{ m/s}, \quad \alpha = 7^\circ$

The equations of motion for the ball are:

$$\frac{d^2x}{dt^2} = 0 \quad \text{and} \quad \frac{d^2y}{dt^2} = -g$$

Integrating gives:

$$\frac{dx}{dt} = C_1 \quad \text{and} \quad \frac{dy}{dt} = -gt + C_2$$

At $t = 0$, $\dfrac{dx}{dt} = V\cos 7^\circ$ and $\dfrac{dy}{dt} = -V\sin 7^\circ$.

So $C_1 = V\cos 7^\circ$ and $C_2 = -V\sin 7^\circ$.

Therefore

$$\frac{dx}{dt} = V\cos 7^\circ \quad \text{and} \quad \frac{dy}{dt} = -gt - V\sin 7^\circ$$

Integrating again gives:

$$x = Vt\cos 7^\circ + C_3 \quad \text{and } y = -\frac{gt^2}{2} - Vt\sin 7^\circ + C_4$$

At $t = 0$, $x = 0$, $y = 2.4$.

So $C_3 = 0$ and $C_4 = 2.4$.

Therefore

$$x = Vt\cos 7^\circ \text{ and } y = -\frac{gt^2}{2} - Vt\sin 7^\circ + 2.4$$

Since $V = 40$ m/s and $g = 10$ m/s^2,

$x = 40t\cos 7^\circ$ and $y = 2.4 - 40t\sin 7^\circ - 5t^2$

10b When the ball hits the pitch, $y = 0$. Hence from part a):

$5t^2 + 40t \sin 7° - 2.4 = 0$

Using the quadratic formula:

t

$= \dfrac{-40 \sin 7° \pm \sqrt{(40 \sin 7°)^2 - 4 \times 5 \times (-2.4)}}{10}$

$= \dfrac{1}{10}(3.596 \ldots)$ (ignoring the negative value)

$\doteqdot 0.36$ s

10c From part a)

$x = 40t\cos 7°$ (1)

$y = 2.4 - 40t\sin 7° - 5t^2$ (2)

Using (1):

$t = \dfrac{x}{40\cos 7°}$

Substituting for t in (2):

$y = 2.4 - x\tan 7° - \dfrac{x^2}{320\cos^2 7°}$

The ball hits the pitch after 0.36 s and at $y = 0$. Hence, the horizontal distance is:

x

$= Vt\cos 7°$

$= 40 \times 0.36 \times \cos 7°$

$\doteqdot 14.29$ m

The angle at which the ball hits the pitch is the slope of the tangent to the trajectory at the point of contact. Using the equation derived above:

$\dfrac{dy}{dx}\Big|_{x=14.29}$

$= -\tan 7° - \dfrac{2x}{320\cos^2 7°}$

$= -\left(\tan 7° + \dfrac{2 \times 14.29}{320\cos^2 7°}\right)$

$= -0.213\ldots$

Thus, the angle δ that the ball hits the pitch is given by,

$\delta \doteqdot \tan^{-1} -0.213 \doteqdot -12.05°$

The negative sign indicates that it is an angle of declination. The angle at which the ball hits the pitch is approximately 12°.

10d The ball was pitched at 14.29m from the bowler. The length of the wicket is 19m.

So, the length for the batsman is: $(19 - 14.29) = 4.72$ m < 5.

Hence, it is not a short-pitched delivery.

11a As the particles are projected at the same time, each particle will have the same flight time the moment they collide. Also, when they collide, their x- and y-coordinates will be the same.

At time $= t$ both the particles will collide and have $y_1 = y_2$. Hence:

$$y_1 = -\frac{1}{2}gt^2 + tV_1 \sin\theta_1$$

$$y_2 = -\frac{1}{2}gt^2 + tV_2 \sin\theta_2$$

And,

$$-\frac{1}{2}gt^2 + tV_1 \sin\theta_1 = -\frac{1}{2}gt^2 + tV_2 \sin\theta_2$$

Therefore:

$$V_1 \sin\theta_1 = V_2 \sin\theta_2$$

11b i $AB = 200$; $V_1 = 30$; $\theta_1 = \sin^{-1}\frac{4}{5}$; $\theta_2 = \sin^{-1}\frac{3}{5}$; $g = 10$

Using the condition obtained in part a) for collision of the particles:

$$V_1 \sin\theta_1 = V_2 \sin\theta_2$$

$$30 \times \frac{4}{5} = V_2 \times \frac{3}{5}$$

Hence,

$$V_2 = 40 \text{ m/s}$$

The x-coordinate at which the particles collide is the same for both the projectiles, this is:

$$V_1 t \cos\theta_1 = 200 - V_2 t \cos\theta_2$$

Now,

$$\cos\theta_1 = \frac{3}{5} \text{ and } \cos\theta_2 = \frac{4}{5}$$

This gives with $V_1 = 30$ and $V_2 = 40$:

$$18t = 200 - 32t$$

$$50t = 200$$

$$t = 4 \text{ s}$$

11b ii The height of both the particles is same at time of collision:

$$y_1 = y_2 = V_1 t_c \sin\theta_1 - 5t_c^2$$

$$y_1 = y_2 = 30 \times 4 \times \frac{4}{5} - 5 \times 16$$

$$y_1 = y_2 = 16 \text{ m}$$

11b iii The coordinates of collision of the particles are:

$$x = V_1 \cos\theta_1\, t_c = 30 \times \tfrac{3}{5} \times 4 = 72 \text{ m} \quad \text{and} \quad y = 16 \text{ m}$$

Now, by rearranging the equations for x to get equations for t and subbing this into the equations for y like in example 21, the trajectories of the particles are:

$$y_1 = x_1 \tan\theta_1 - \frac{5x_1^2}{324} \qquad \text{and} \qquad y_2 = (200 - x_2)\tan\theta_2 - \frac{5(200 - x_2)^2}{1024}$$

Differentiating y_1 and y_2 with respect to x_1 and x_2, gives the line slopes as:

$$m_1 = \tan\theta_1 - \frac{5x_1}{162} \quad \text{and} \quad m_2 = -\tan\theta_2 + \frac{5(200 - x_2)}{512}$$

At the point of collision, $x_1 = (200 - x_2) = 72$ and so:

$$m_1 = \frac{4}{3} - \frac{20}{9} \qquad \text{and} \qquad m_2 = -\frac{3}{4} + \frac{5}{4}$$

$$m_1 = -\frac{8}{9} \text{ and } m_2 = \frac{1}{2}$$

And the angle between these two lines is given by:

$\tan\delta$

$$= \frac{m_1 - m_2}{1 + m_1 \times m_2}$$

$$= \frac{-\frac{8}{9} - \left(\frac{1}{2}\right)}{1 + \left(\frac{8}{9} \times \frac{1}{2}\right)}$$

$$= -\frac{25}{26}$$

Hence $\delta = 68°$, and so the obtuse angle is: $180° - 68° = 112°$

12a $V = 30$; $\alpha = 60^\circ$; $\ddot{x} = -\frac{1}{6}\dot{x}$; $\ddot{y} = -10 - \frac{1}{6}\dot{y}$

$$\ddot{x} = \frac{dv_x}{dt} = -\frac{1}{6}v_x$$

Integrating gives

$$\int dt = -6\int \frac{dv_x}{v_x}$$

$$t = -6\ln v_x + C$$

At $t = 0$; $v_x = 30\cos 60° = 15$. Hence

$$C = 6\ln 15$$

Therefore

$$t = 6\ln\frac{15}{v_x}$$

Which rearranging becomes:

$$v_x = 15e^{-\frac{t}{6}} = \dot{x}$$

Now, for the vertical direction

$$\ddot{y} = \frac{dv_y}{dt} = -10 - \frac{1}{6}v_y$$

Integrating gives:

$$\int dt = -\int \frac{dv_y}{10 + \frac{v_y}{6}}$$

$$t = -6\ln\left(10 + \frac{v_y}{6}\right) + D$$

At $t = 0$, $v_y = 30\sin 60° = 15\sqrt{3}$. Hence

$$D = 6\ln\left(10 + \frac{15\sqrt{3}}{6}\right)$$

Therefore

$$t = 6\ln\left(\frac{10 + \frac{5\sqrt{3}}{2}}{10 + \frac{v_y}{6}}\right)$$

Rearranging we have

$$10 + \frac{v_y}{6} = 5\left(2 + \frac{\sqrt{3}}{2}\right)e^{-\frac{t}{6}}$$

$$v_y = 15\left(\left(4 + \sqrt{3}\right)e^{-\frac{t}{6}} - 4\right) = \dot{y}$$

Hence

$$\dot{x} = 15e^{-\frac{t}{6}}$$

$$\dot{y} = 15\left(\left(4 + \sqrt{3}\right)e^{-\frac{1}{6}t} - 4\right)$$

12b From part a)

$$v_x = 15e^{-\frac{t}{6}} = \frac{dx}{dt}$$

Integrating gives

$$\int dx = 15\int e^{-\frac{t}{6}}\,dt$$

$$x = -90e^{-\frac{t}{6}} + C$$

At $t = 0$, $x = 0$, and so

$$C = 90$$

Therefore

x

$$= -90e^{-\frac{t}{6}} + 90$$

$$= 90\left(1 - e^{-\frac{1}{6}t}\right)$$

12c At greatest height, $\dot{y} = 0$. Hence, using the result from part a) and solving for t we have:

$$15\left(\left(4 + \sqrt{3}\right)e^{-\frac{t}{6}} - 4\right) = 0$$

$$e^{-\frac{t}{6}} = \frac{4}{4 + \sqrt{3}}$$

$$t = 6\ln\left(\frac{4 + \sqrt{3}}{4}\right)$$

12d Using equations from part b and part c:

$x = 90\left(1 - e^{-\frac{t}{6}}\right)$, where $t = 6\ln\left(\frac{4+\sqrt{3}}{4}\right)$

Hence,

x

$= 90\left(1 - e^{-\frac{6}{6}\ln\left(\frac{4+\sqrt{3}}{4}\right)}\right)$

$= 90\left(1 - \frac{4}{4+\sqrt{3}}\right)$

$= 27.195\ldots$

$\doteqdot 27$ m

13a $V = 20;\quad g = 10;\quad R = 20$

The equations of the motion of the ball are:

$\ddot{x} = 0$ and $\ddot{y} = -g$

$\dot{x} = V\cos\alpha$ and $\dot{y} = V\sin\alpha - gt$

$x = V\cos\alpha\ t$ and $y = Vt\sin\alpha - \frac{1}{2}gt^2$

Rearranging the equation for x we have,

$t = \frac{x}{V\cos\alpha}$

Subbing this into the equation for y, we get the trajectory of the ball:

$y = x\tan\alpha - \frac{gx^2}{2V^2\cos^2\alpha}$

The wall is at 20 m from the point of hitting the ball. So, when the ball hits the wall, $x = 20$ m. Hence:

h

$= 20\tan\alpha - \frac{10\times 20^2}{2\times 20^2\times\cos^2\alpha}$

$= 20\tan\alpha - 5\sec^2\alpha$

13b

$$\frac{d}{d\alpha}(\sec\alpha)$$

$$= \frac{d}{d\alpha}\left(\frac{1}{\cos\alpha}\right)$$

$$= \frac{d}{d\alpha}(\cos^{-1}\alpha)$$

$$= -\cos^{-2}\alpha \times -\sin\alpha$$

$$= \frac{\sin\alpha}{\cos^2\alpha}$$

$$= \frac{1}{\cos\alpha} \times \frac{\sin\alpha}{\cos\alpha}$$

$$= \sec\alpha\tan\alpha$$

13c From part a, $h = 20\tan\alpha - 5\sec^2\alpha$. Differentiating and using part b), we have:

$h'(\alpha) = 20\sec^2\alpha - 5 \times 2\sec\alpha \times \sec\alpha\tan\alpha$

For maximum value of h, $h'(\alpha) = 0$. Hence:

$20\sec^2\alpha - 5 \times 2\sec\alpha \times \sec\alpha\tan\alpha = 0$

$20\sec^2\alpha - 10\sec^2\alpha\tan\alpha = 0$

$10\sec^2\alpha\,(2 - \tan\alpha) = 0$

$\sec^2\alpha \neq 0$ so $2 - \tan\alpha = 0$

Therefore, $\tan\alpha = 2$ at the maximum height

13d Taking $\tan\alpha = 2$, we have that:

$\sin\alpha = \frac{2}{\sqrt{2^2+1^2}} = \frac{2}{\sqrt{5}}$ and $\cos\alpha = \frac{1}{\sqrt{2^2+1^2}} = \frac{1}{\sqrt{5}}$

Thus, $\sec\alpha = \sqrt{5}$ and using $h = 20\tan\alpha - 5\sec^2\alpha$, we have at maximum height,

h_{max}

$= 20(2) - 5\left(\sqrt{5}\right)^2$

$= 15$ m

13e From part a, the trajectory of the ball is:

$$y = x\tan\alpha - \frac{x^2\sec^2\alpha}{80}$$

The coordinates of impact point are: $(20, 15)$, and so the gradient of impact is:

$$\frac{dy}{dx}\Big|_{20,15}$$

$$= \tan\alpha - \frac{x\sec^2\alpha}{40}$$

$$= 2 - \frac{20\times 5}{40}$$

$$= -0.5$$

Hence, if θ is the angle of impact we have,

$$\tan\theta = -0.5$$

Or

$$\theta$$

$$= -26.565\ldots°$$

$$= 63.434\ldots° \text{ below the horizontal}$$

$$\doteqdot 63°26' \text{ below the horizontal}$$

The time of flight of the ball can be found from the horizontal velocity. Multiplying velocity by the time we get the distance to the wall, this is:

$$20 = 20\cos\alpha\, T$$

Rearranging for time of flight T and using the results form part d) we have,

$$T$$

$$= \frac{1}{\cos\alpha}$$

$$= \sec\alpha$$

$$= \sqrt{5}$$

Now calculating the speed of the ball when it hits the wall we have the horizontal and vertical components of speed as:

$$v_x$$

$$= 20\cos\alpha$$

$= 20 \times \frac{1}{\sqrt{5}}$

$= 4\sqrt{5}$

And:

v_y

$= 20 \sin \alpha - 10T$

$= 20 \times \frac{2}{\sqrt{5}} - 10 \times \sqrt{5}$

$= 8\sqrt{5} - 10\sqrt{5}$

$= -2\sqrt{5}$

Thus, the speed with which the ball hits the wall is then:

V

$= \sqrt{\left(4\sqrt{5}\right)^2 + \left(2\sqrt{5}\right)^2}$

$= \sqrt{80 + 20}$

$= 10 \text{ m/s}$

14a $y_0 = 40;\quad V = 20;\quad g = 10;$

The equations of motion for the stone are:

$\ddot{x} = 0$ and $\ddot{y} = -g$

At $t = 0$, $\dot{x} = V\cos\theta$ and $\dot{y} = V\sin\theta$

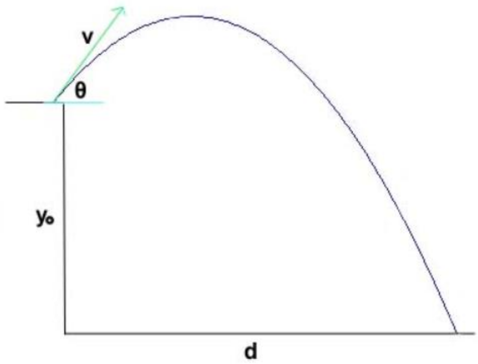
Integrating the equations of motion then and using the initial conditions we have,

$\dot{x} = V\cos\theta$ and $\dot{y} = V\sin\theta - gt$

At $t = 0$, $x = 0$ and $y = 0$ (assuming cliff as level zero). Using these conditions and integrating again we have:

$x(t) = Vt\cos\theta$ and $y(t) = Vt\sin\theta - \frac{1}{2}gt^2$

14b



Using part a) we have:

$x(T) = 20T\cos\theta$

$\cos\theta = \dfrac{x(T)}{20T}$

Using trigonometry rules, we then have that:

$$\sin\theta = \sqrt{1 - \left(\frac{x(T)}{20T}\right)^2}$$

Now, the cliff is 40 m high and we have taken the top of the cliff to be 0, hence $y(T) = -40$ m, and we have using the equation in part a)

$-40 = 20T\sin\theta - 5T^2$

Rearranging we have:

$$\frac{5T^2 - 40}{20T} = \sin\theta = \sqrt{1 - \left(\frac{x(T)}{20T}\right)^2}$$

$(5T^2 - 40)^2 = (20T)^2 - \left(x(T)\right)^2$

$\left(x(T)\right)^2 = 400T^2 - (5T^2 - 40)^2$

14c From above $\left(x(T)\right)^2 = 400T^2 - (5T^2 - 40)^2$ and for maximum $\left(x(T)\right)^2$, we need

$\frac{d}{dT}\left(x(T)\right)^2 = 0$

Hence:

0

$= 800T - 20T(5T^2 - 40)$

$= -100T^3 + 1600T$

$= -100T(T^2 - 16)$

Thus,

$T = 0$ or $T^2 - 16 = 0$

$T \neq 0$ as the stone is being thrown at that time and so $T = 4$ s maximises the expression. To find the value of θ we sub this value of T, back into the equation for X(T) and then this into the equation for $\cos\theta$. This gives:

$x(T)$

$= \sqrt{400(16) - (5(16) - 40)^2}$

$= 40\sqrt{3}$

And:

$\cos\theta$

$= \frac{x(T)}{20T}$

$= \frac{40\sqrt{3}}{20 \times 4}$

$= \frac{\sqrt{3}}{2}$

Thus, the angle of θ that maximises the horizontal distance is $\theta = 30^\circ$.

15a The equation of motion with respect to time are:

$\ddot{x} = 0$ and $\ddot{y} = -g$

$\dot{x} = V\cos\theta$ and $\dot{y} = V\sin\theta - gt$

$x = Vt\cos\theta$ and $y = Vt\sin\theta - \frac{1}{2}gt^2$

At maximum height, $\dot{y} = 0.$, and so we have:

$V\sin\theta - gt_{max} = 0$

$t_{max} = \dfrac{V\sin\theta}{g}$

Substituting for t in the equation for y:

y_{max}

$= V\left(\dfrac{V\sin\theta}{g}\right)\sin\theta - \dfrac{1}{2}g\left(\dfrac{V\sin\theta}{g}\right)^2$

$= \dfrac{V^2\sin^2\theta}{g} - \dfrac{V^2\sin^2\theta}{2g}$

$= \dfrac{V^2\sin^2\theta}{2g}$

15b i As y_{max} is the same we have:

$\dfrac{V^2\sin^2\theta}{2g} = \dfrac{9V^2\sin^2\frac{1}{2}\theta}{8g}$

$4\sin^2\theta = 9\sin^2\dfrac{\theta}{2}$

Using $\cos 2x = 1 - 2\sin^2 x$ we get,

$4\sin^2\theta = 9 \times \dfrac{1}{2}(1 - \cos\theta)$

$8(1 - \cos^2\theta) = 9 - 9\cos\theta$

$8 - 8\cos^2\theta = 9 - 9\cos\theta$

$8\cos^2\theta - 9\cos\theta + 1 = 0$

$(8\cos\theta - 1)(\cos\theta - 1) = 0$

$\cos\theta = \dfrac{1}{8}$ or $\cos\theta = 1$

Thus, since the particles are projected up, $\cos\theta \neq 1$ and so $\theta = \cos^{-1}\frac{1}{8}$

15b ii Now $\cos\theta = \frac{1}{8}$, so using $\sin^2\theta + \cos^2\theta = 1$,

$$\sin\theta = \frac{\sqrt{63}}{8} \quad \text{and} \quad \sin\frac{\theta}{2} = \sqrt{\frac{1}{2} - \frac{1}{16}} = \frac{\sqrt{7}}{4}$$

Therefore:

$$\frac{3}{2}\sin\frac{\theta}{2} = \frac{\sqrt{63}}{8}$$

As such we see that

$$V\sin\theta = \frac{3}{2}V\sin\frac{\theta}{2}$$

So, yes. Since the particles have the same initial vertical components of their velocity and are acted upon by the same vertical force, they will reach the max height at the same time

16a $v_i = V$, Angle of projectile $= \alpha$

The equations of motion are:

$\ddot{x} = 0$ and $\ddot{y} = -g$

$\dot{x} = V\cos\alpha$ and $\dot{y} = V\sin\alpha - gt$

$x = Vt\cos\alpha$ and $y = Vt\sin\alpha - \frac{1}{2}gt^2$

The range of the projectile is when the projectile hits the ground for the first time. That is, when $y = 0$. This is when:

$$Vt\sin\alpha = \frac{1}{2}gt^2$$

$$t = \frac{2V\sin\alpha}{g}$$

Substituting for t in the equation for x gives:

$$x = V\left(\frac{2V\sin\alpha}{g}\right)\cos\alpha$$

$$x = \frac{V^2\sin 2\alpha}{g} \qquad (\text{using } \sin 2\theta = 2\sin\theta\cos\theta)$$

16b i

$$R = \frac{V^2 \sin 2\alpha}{g}$$

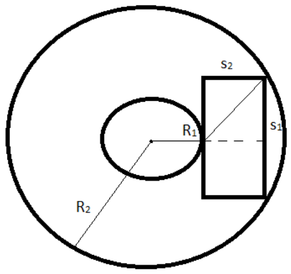
α varies between 15° and 45° and so the ranges will vary as:

$$\frac{V^2 \sin(2 \times 15°)}{g} \le R_s \le \frac{V^2 \sin(2 \times 45°)}{g}$$

$$\frac{V^2 \sin\ 30°}{g} \le R_s \le \frac{V^2 \sin 90°}{g}$$

$$\frac{V^2}{2g} \le R_s \le \frac{V^2}{g}$$

16b ii Suppose one of the sides s_1 of the rectangular garden is tangent to the inner circle and the other side is s_2.



For the minimum value of R_1 and R_2, the optimal case is when s_1 is the larger side and is tangent to the inner circle at its midpoint.

From the diagram we can derive the formula for the sagitta of a segment, by using Pythagoras theorem, and noting that R_2 must touch the circle and top corner of the rectangle. Hence we find:

$$R_2^2 \geq \left((R_1 + s_2)^2 + \left(\frac{s_1}{2}\right)^2\right)$$

Subbing in the values from the question and results from part i) and taking equality in the equation (this gives smallest possible value of R_2) we have:

$$\left(\frac{V^2}{g}\right)^2 = \left(\frac{V^2}{2g} + 3\right)^2 + 3^2$$

$$\frac{V^4}{g^2} = \frac{V^4}{4g^2} + \frac{3V^2}{g} + 18$$

$$\frac{3}{4}\frac{V^4}{g^2} - \frac{3V^2}{g} - 18 = 0$$

$$\frac{1}{4}\frac{V^4}{g^2} - \frac{V^2}{g} - 6 = 0$$

Using the quadratic formula we get:

$$\frac{V^2}{g} = \frac{1 \pm \sqrt{1 + 4\left(\frac{1}{4}\right)6}}{\frac{1}{2}} = \frac{1 \pm \sqrt{7}}{\frac{1}{2}}$$

Rearranging gives:

$$\frac{V^2}{2g} = 1 \pm \sqrt{7}$$

But all the quantities in the LHS are positive. Thus:

$$\frac{V^2}{2g} = 1 + \sqrt{7}$$

This, is the condition to have the smallest value of R_2 and still contain the rectangle. R_2 can be larger than this however, and so the more general expression becomes:

$$\frac{V^2}{2g} \geq 1 + \sqrt{7}$$

17a The equations of motion are:

$\ddot{x} = -k\dot{x} = -kv_x$ and

$\ddot{y} = -g - k\dot{y} = -g - kv_y$

Integrating once the equation for x we get,

$$\int dt = -\int \frac{dv_x}{kv_x}$$

$$t = -\frac{1}{k}\ln(v_x) + C$$

At $t = 0$, $v_x = V\cos\alpha$. Hence:

$$C = \frac{1}{k}\ln(V\cos\alpha)$$

Therefore

$$t = \frac{1}{k}\ln\left(\frac{V\cos\alpha}{v_x}\right)$$

$$kt = \ln\left(\frac{V\cos\alpha}{v_x}\right)$$

$$e^{kt} = \frac{V\cos\alpha}{v_x}$$

$$v_x = \frac{V\cos\alpha}{e^{kt}}$$

$$v_x = V\cos\alpha\ e^{-kt}$$

Hence

$$\dot{x} = V\cos\alpha\ e^{-kt}$$

Integrating the equation for y now we have:

$$\int dt = -\int \frac{dv_y}{g + kv_y}$$

$$t = -\frac{1}{k}\ln\left(g + kv_y\right) + D$$

At $t = 0$, $v_y = V\sin\alpha$. Hence,

$$D = \frac{1}{k}\ln(g + kV\sin\alpha)$$

Therefore

$$t = \frac{1}{k}\ln\left(\frac{g + kV\sin\alpha}{g + kv_y}\right)$$

$$kt = \ln\left(\frac{g + kV\sin\alpha}{g + kv_y}\right)$$

$$e^{kt} = \frac{g + kV\sin\alpha}{g + kv_y}$$

$$g + kv_y = (g + kV\sin\alpha)e^{-kt}$$

$$kv_y = (g + kV\sin\alpha)e^{-kt} - g$$

$$v_y = \frac{1}{k}(g + kV\sin\alpha)e^{-kt} - \frac{g}{k}$$

$$v_y = \left(\frac{g}{k} + V\sin\alpha\right)e^{-kt} - \frac{g}{k}$$

Hence

$$\dot{y} = \left(\frac{g}{k} + V\sin\alpha\right)e^{-kt} - \frac{g}{k}$$

17b From above we have:

$$v_x = V\cos\alpha\ e^{-kt} = \dot{x} = \frac{dx}{dt}$$

Integrating gives:

$$\int dx = \int V\cos\alpha\ e^{-kt}dt$$

$$x = -\frac{V\cos\alpha}{k}e^{-kt} + C$$

At $t = 0,\ x = 0$. Hence:

$$C = \frac{V\cos\alpha}{k}$$

Therefore

x

$$= -\frac{V\cos\alpha}{k}e^{-kt} + \frac{V\cos\alpha}{k}$$

$$= \frac{V\cos\alpha}{k}(1 - e^{-kt})$$

Similarly for the y component we have:

$$v_y = \left(\frac{g}{k} + V\sin\alpha\right)e^{-kt} - \frac{g}{k} = \dot{y} = \frac{dy}{dt}$$

Integrating gives:

$$\int dy = \int \left(\left(\frac{g}{k} + V\sin\alpha\right)e^{-kt} - \frac{g}{k}\right)dt$$

$$y = -\left(\frac{g}{k^2} + \frac{V\sin\alpha}{k}\right)e^{-kt} - \frac{g}{k}t + D$$

At $t = 0,\ y = 0$. Hence:

$$D = \frac{g}{k^2} + \frac{V\sin\alpha}{k}$$

Therefore

y

$$= -\left(\frac{g}{k^2} + \frac{V\sin\alpha}{k}\right)e^{-kt} - \frac{g}{k}t + \frac{g}{k^2} + \frac{V\sin\alpha}{k}$$

$$= \left(\frac{g}{k^2} + \frac{V\sin\alpha}{k}\right) - \left(\frac{g}{k^2} + \frac{V\sin\alpha}{k}\right)e^{-kt} - \frac{g}{k}t$$

$$y = \left(\frac{g}{k^2} + \frac{V\sin\alpha}{k}\right)(1 - e^{-kt}) - \frac{g}{k}t$$

17c At maximum height, $\dot{y} = 0$. Using the result from part a) gives:

$$\left(\frac{g}{k} + V\sin\alpha\right)e^{-kt} - \frac{g}{k} = 0$$

$$\frac{g}{k} = \left(\frac{g}{k} + V\sin\alpha\right)e^{-kt}$$

$$g = (g + kV\sin\alpha)e^{-kt}$$

$$e^{-kt} = \frac{g}{g + kV\sin\alpha}$$

Substituting this into the equation for x we get:

x

$$= \frac{V\cos\alpha}{k}(1 - e^{-kt})$$

$$= \frac{V\cos\alpha}{k}\left(1 - \frac{g}{g + kV\sin\alpha}\right)$$

$$= \frac{V\cos\alpha}{k}\left(\frac{g + kV\sin\alpha - g}{g + kV\sin\alpha}\right)$$

$$= \frac{V\cos\alpha}{k(g + kV\sin\alpha)}(g + kV\sin\alpha - g)$$

$$= \frac{V\cos\alpha \times kV\sin\alpha}{k(g + kV\sin\alpha)}$$

$$= \frac{V^2\cos\alpha\sin\alpha}{(g + kV\sin\alpha)}$$

$$= \frac{V^2(2\sin\alpha\cos\alpha)}{2(g + kV\sin\alpha)}$$

$$= \frac{V^2\sin 2\alpha}{2(g + kV\sin\alpha)}$$

Solutions to Exercise 6F Enrichment questions

18a The path has equation:

$$y = -\frac{g}{2V^2}(1 + \tan^2\alpha)x^2 + x\tan\alpha \; (*)$$

Now,

$$k = \frac{V^2}{2g}$$

and so,

$$\frac{1}{k} = \frac{2g}{V^2}$$

$$\frac{1}{4k} = \frac{g}{2V^2}$$

Hence, $(*)$ becomes,

$$y = -\frac{1}{4k}(1 + \tan^2\alpha)x^2 + x\tan\alpha$$

$$4ky = -x^2 - x^2\tan^2\alpha + 4kx\tan\alpha$$

$$x^2\tan^2\alpha - 4kx\tan\alpha + (4ky + x^2) = 0$$

18b $X^2\tan^2\alpha - 4kX\tan\alpha + (4kY + X^2) = 0(**)$

If there are two real and distinct solutions for $\tan\alpha$, then $\Delta > 0$, and so,

$$(-4kX)^2 - 4.X^2(4kY + X^2) > 0$$

$$16k^2X^2 - 16X^2Y - 4X^4 > 0$$

$$4k^2 - 4kY - X^2 > 0 \quad (X \neq 0)$$

$$X^2 < 4k^2 - 4kY$$

18c The product of the roots of the equation $(**)$ is:

$\tan\alpha_1 \tan\alpha_2$

$= \dfrac{4kY + X^2}{X^2}$

$= 1 + \dfrac{4kY}{X^2}$

> 1, since $k, X.Y > 0$ (A)

Suppose that α_1 and α_2 are both less than $45°$. Then $\tan\alpha_1$ and $\tan\alpha_2$ are both less than one. So, $\tan\alpha_1 \tan\alpha_2 < 1$, which contradicts (A).

Hence, α_1 and α_2 cannot both be less than $45°$.

19a P has coordinates $(d\cos\beta, d\sin\beta)$.

19b $d\cos\beta = Vt\cos\alpha$ (1)

$d\sin\beta = Vt\cos\beta - \dfrac{1}{2}gt^2(2)$

From (1):

$t = \dfrac{d\cos\beta}{V\cos\alpha}$

Substituting into (2):

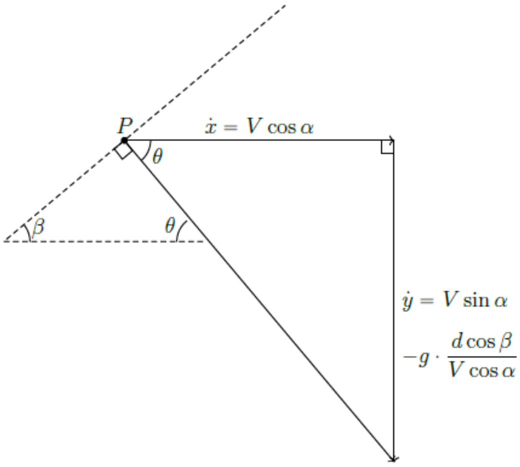
$d\sin\beta = V\sin\alpha . \dfrac{d\cos\beta}{V\cos\alpha} - \dfrac{g}{2} . \dfrac{d^2\cos^2\beta}{V^2\cos^2\alpha}$

$\sin\beta = \tan\alpha\cos\beta - \dfrac{gd\cos^2\beta}{2V^2\cos^2\alpha} \quad (d \neq 0)$

Hence,

$d = \dfrac{2V^2\cos^2\alpha}{g\cos^2\beta}(\tan\alpha\cos\beta - \sin\beta)$

13c



At P,

$$t = \frac{d\cos\beta}{V\cos\alpha}$$

$$\theta + \beta = \frac{\pi}{2}$$

$$\theta = \frac{\pi}{2} - \beta$$

$$\tan\theta = \frac{|\dot{y}|}{|\dot{x}|} \text{(note that } \dot{y} < 0)$$

So,

$$\tan\left(\frac{\pi}{2} - \beta\right) = \frac{\frac{gd\cos\beta}{V\cos\alpha} - V\sin\alpha}{V\cos\alpha}$$

Hence,

$$\cot\beta = \frac{gd\cos\beta}{V^2\cos^2\alpha} - \tan\alpha$$

13d From part b,

$\frac{gd\cos\beta}{V^2\cos^2\alpha}$

$= \frac{2}{\cos\beta}(\tan\alpha\cos\beta - \sin\beta)$

$= 2\tan\alpha - 2\tan\beta$

Substituting into part c,

$\cot\beta = (2\tan\alpha - 2\tan\beta) - \tan\alpha$

$\tan\alpha = \cot\beta + 2\tan\beta$

20a $y = 2\sin(x-\theta)\cos x$

20ai y'

$= 2\sin(x-\theta).(-\sin x) + 2\cos(x-\theta).\cos x$

$= 2(\cos(x-\theta)\cos x - \sin(x-\theta)\sin x)$

$= 2\cos(2x-\theta)$

20aii Integrating I,

$\int \frac{dy}{dx}dx = \int 2\cos(2x-\theta)\,dx$

$y = 2\sin(2x-\theta) + C$

When $x = 0, y = 2\sin(-\theta) = -2\sin\theta$

$-2\sin\theta = \sin(-\theta) + C$

$C = -\sin\theta$

Hence,

$2\sin(x-\theta)\cos x = \sin(2x-\theta) - \sin\theta$

20bi The Cartesian equation of the parabola is:

$$y = \frac{-g\sec^2\alpha}{2V^2}x^2 + x\tan\alpha$$

So,

$$Y = \frac{-g\sec^2\alpha}{2V^2}X^2 + X\tan\alpha\ (*)$$

Now, $X = R\cos\beta$ and $Y = R\sin\beta$

So,

$$\frac{Y}{X} = \tan\beta$$

Substituting into $(*)$

$$\frac{Y}{X} = \frac{-g\sec^2\alpha}{2V^2}\mathrm{X} + \tan\alpha$$

$$\tan\beta = \frac{-g}{2V^2\cos^2\alpha}\mathrm{X} + \tan\alpha$$

$$X = \frac{2V^2\cos^2\alpha}{g}(\tan\alpha - \tan\beta)$$

20bii R

$$= \frac{X}{\cos\beta}$$

$$= \frac{2V^2\cos^2\alpha(\tan\alpha - \tan\beta)}{g\cos\beta}$$

$$= \frac{2V^2\cos^2\alpha\left(\dfrac{\sin\alpha}{\cos\alpha} - \dfrac{\sin\beta}{\cos\beta}\right)}{g\cos\beta}\cdot\frac{\cos\beta}{\cos\beta}$$

$$= \frac{2V^2\cos a(\sin\alpha\cos\beta - \cos\alpha\sin\beta)}{g\cos^2\beta}$$

$$= \frac{2V^2\cos a\sin(\alpha - \beta)}{g\cos^2\beta}$$

20biii Using part aii with $x = \alpha$ and $\theta = \beta$,

$$R = \frac{V^2(\sin(2\alpha - \beta) - \sin\beta)}{g\cos^2\beta}$$

If α varies such that $\alpha > \beta$ and β is constant,

$$R \leq \frac{V^2(1 - \sin\beta)}{g(1 - \sin^2\beta)} (\text{since } -1 \leq \sin(2\alpha - \beta) \leq 1)$$

So

$$R \leq \frac{V^2}{g(1 + \sin\beta)}$$

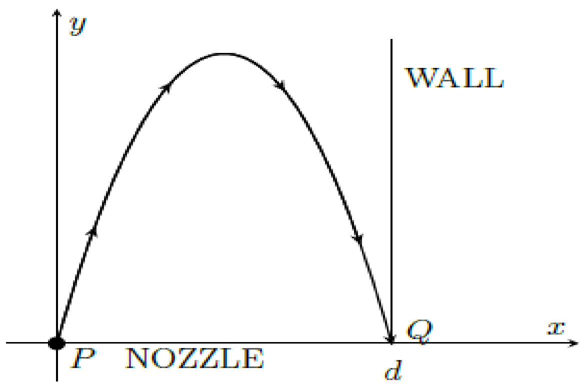
20biv Given $\beta = 14^\circ$ the maximum value of R occurs when,

$\sin(2\alpha - 14^\circ) = 1$

$2\alpha - 14^\circ = 90^\circ$

$\alpha = 52^\circ$

21a



The horizontal range is $d = \frac{V^2 \sin 2\alpha}{g}$.

The maximum range is $d = \frac{V^2}{g}$, since $|\sin 2\alpha| \leq 1$.

So $V = \sqrt{gd}$ if the water can just reach the bottom of the wall.

Thus, if $V > \sqrt{gd}$, the water can hit the wall above ground level.

21b

$$x = Vt\cos\alpha \text{ and } y = Vt\sin\alpha - \frac{gt^2}{2}$$

When $x = d$,

$$t = \frac{d}{V\cos\alpha},$$

Hence:

y

$$= d\tan\alpha - \frac{g}{2}.\frac{d^2}{V^2\cos^2\alpha}$$

$$= d\tan\alpha - \frac{g}{2}.\frac{d^2}{4gd\cos^2\alpha}, \qquad (\text{since } V = 2\sqrt{gd})$$

$$= d\tan\alpha - \frac{d}{8\cos^2\alpha}$$

$$= d\left(\tan\alpha - \frac{1}{8}(1+\tan^2\alpha)\right)$$

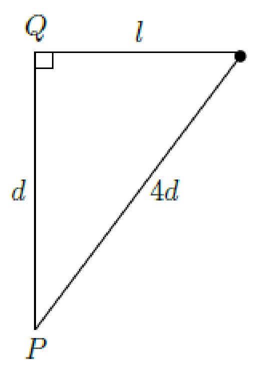
$$= \frac{d}{8}(-\tan^2\alpha + 8\tan\alpha - 1)$$

$$= \frac{d}{8}(15 - (\tan\alpha - 4)^2)$$

Hence, the greatest height that can be reached up the wall is

$$y_{max} = \frac{15d}{8}$$

when $\alpha = \tan^{-1} 4$.



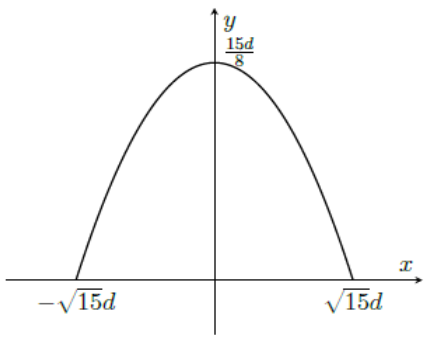
The maximum horizontal range given $V = 2\sqrt{gd}$ is,

$$\frac{V^2}{g} = \frac{4gd}{g} = 4d$$

Let ℓ be the greatest horizontal distance from Q that water can reach along the wall. Then:

$$l^2 = 16d^2 - 15d^2$$

$$l = \sqrt{15}d$$



Assuming that the portion of the wall that can be sprayed is a parabolic segment, the parabola has equation:

$y = A(\sqrt{15}d - x)(\sqrt{15}d + x)$, that is,

$y = A(15d^2 - x^2)$, for some constant A.

When $x = 0, y = \dfrac{15d}{8}$

So, $\dfrac{15d}{8} = A\,15d^2$ and so, $A = \dfrac{1}{8d}$. So, the required area is:

$$2\int_0^{\sqrt{15}d} \frac{1}{8d}(15d^2 - x^2)dx$$

$$= \frac{1}{4d}\left[15d^2x - \frac{1}{3}x^3\right]_0^{\sqrt{15}d}$$

$$= \frac{1}{4d}\left(15\sqrt{15}d^3 - \frac{1}{3}15\sqrt{15}d^3\right)$$

$$= \frac{1}{4d}.10\sqrt{15}d^3$$

$$= \frac{5\sqrt{15}}{2}d^2$$

Proving that the segment is parabolic is considered to be beyond the scope of the question.

Solutions to Exercise 6G Foundation questions

1a The components of y of the displacement is given by

$$y = v_0 \sin\theta\, t - \frac{1}{2}gt^2$$

As $v_0 t = H$, $g = 10$ and the angle is equal to 90°:

$$y = H - 5t^2$$

1b When $t = 0, y = 2.$

$$2 = H - 5(0)^2$$

$$H = 2$$

When $t = \frac{1}{5}, y = 0.$

$$0 = H - 5\left(\frac{1}{5}\right)^2$$

$$H = \frac{1}{5}$$

$$H = 2 + 2 - \frac{1}{5}$$

$$= 3.8$$

2a $v_y = v_0 \sin\theta - gt = 0$

$$t = \frac{v_0 \sin\theta}{g}$$

$$= \frac{16 \sin 90°}{10}$$

$$= 1\frac{3}{5} \text{ s}$$

Maximum height:

$$y\left(1\frac{3}{5}\right) = 16 \sin 90° \left(1\frac{3}{5}\right) - \frac{1}{2}(10)\left(1\frac{3}{5}\right)^2$$

$$= 12\frac{4}{5} \text{ metres}$$

2b The equation for $y_1 = -5t^2 + vt$

As y_2 is fired one second later, $t_2 = t - 1$.

Thus, $y_2 = -5(t-1)^2 + v(t-1)$

2c At $t = 1\frac{3}{5}, y_2 = 12\frac{4}{5}$.

$12.8 = -5(0.6)^2 + v(0.6)$

$v = 24\dfrac{1}{3}$ m/s

3a $B - Mg = -Md$

$B - (M-m)g = (M-m)a$

3b $Ma - ma = B - Mg + mg$

$-ma - mg = B - Mg - Ma$

$-ma - mg = -Md + Mg - Mg - Ma \quad (B = -Md + Mg)$

$ma + mg = Md + Ma$

$m(a+g) = M(a+d)$

$m = M\dfrac{a+d}{a+g}$

4a $v^2 = u^2 + 2as$

Thus,

$v^2 = V_0^2 + \dfrac{2gR^2}{r} - 2gR$

4b $V_0 = \sqrt{gR}$

$\doteqdot 1680$ m/s

5a $T = mg\cos\theta$

$\dfrac{d^2\theta}{dt^2} = -\dfrac{g\sin\theta}{L}$

5b

$a = \dfrac{dv}{dt} = \dfrac{dx}{dt} \times \dfrac{dv}{dx}$ by the chain rule

$= v\frac{dv}{dx}$ since $d = \frac{dx}{dt}$

Now, $v\frac{dv}{dx} = \frac{d}{dv}\left(\frac{1}{2}v^2\right) \times \frac{dv}{dx}$

By the chain rule, $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

Thus substituting ϖ as v, $\frac{d^2\theta}{dt^2} = \frac{d}{d\theta}(\frac{1}{2}\,\varpi^2)$

5c

$\dfrac{d}{d\theta}\left(\dfrac{1}{2}\varpi^2\right) = a$

$\dfrac{1}{2}\varpi^2 = \dfrac{T - T_0}{mL}$

$\varpi^2 = \dfrac{2}{mL}(T - T_0)$

6 Max $= 0.4$ and Min $= -1$. Therefore, $A = 0.7$. Shift down 0.3

Cycle repeats every 12 hours and 25 mins.

Period $= \frac{2\pi}{\frac{149}{12}}$

$x = -0.7\cos\left(\dfrac{2\pi}{\frac{149}{12}}t\right) - 0.3$

Using this equation to graph the harmonic motion,

The first time that $y = 0$ is when $t = 3.9794$ and returns to that position when

$t = 8.4372$ hours past 1:10 a.m.

Thus, the times are 5:09 a.m. to 9:36 a.m.

Solutions to Exercise 6G Development questions

7a

$$\tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)$$

$$=\frac{\tan\frac{\pi}{4}+\tan\frac{\alpha}{2}}{1-\tan\frac{\pi}{4}\tan\frac{\alpha}{2}} \qquad \text{using } \tan(a+b)=\frac{\tan a+\tan b}{1-\tan a\tan b}$$

$$=\frac{1+\tan\frac{\alpha}{2}}{1-\tan\frac{\alpha}{2}}$$

$$=\frac{1+\frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}}{1-\frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}}$$

$$=\frac{\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}}$$

$$=\frac{\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}}\times\frac{\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}}$$

$$=\frac{\left(\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}\right)^2}{\cos^2\frac{\alpha}{2}-\sin^2\frac{\alpha}{2}}$$

$$=\frac{\cos^2\frac{\alpha}{2}+2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}+\sin^2\frac{\alpha}{2}}{\cos\alpha} \qquad \text{using } \cos 2x=\cos^2 x-\sin^2 x$$

$$=\frac{1+\sin\alpha}{\cos\alpha} \qquad \text{using } \cos^2 x+\sin^2 x=1 \text{ and } \sin 2x=2\sin x\cos x$$

$$=\frac{\sin\alpha+1}{\cos\alpha}$$

7b

$$y = -\frac{gx^2}{2V^2}\sec^2\beta + x\tan\beta$$

Using $\sec^2 x = 1 + \tan^2 x$:

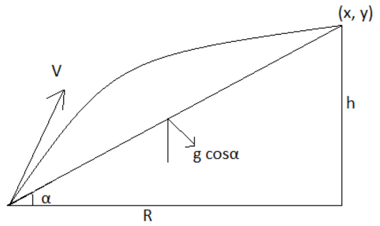
$$y = -\frac{gx^2}{2V^2}(1 + \tan^2\beta) + x\tan\beta$$

$$y = -\frac{gx^2}{2V^2} - \frac{gx^2}{2V^2}\tan^2\beta + x\tan\beta$$

$$\frac{gx^2}{2V^2}\tan^2\beta - x\tan\beta + \left(y + \frac{gx^2}{2V^2}\right) = 0$$

7c i This case is the same as the case of projectile motion up an inclined plane with an angle of elevation α.

The equations of motion from part b) are:



$$\frac{gx^2}{2V^2}\tan^2\beta - x\tan\beta + \left(y + \frac{gx^2}{2V^2}\right) = 0$$

$$gx^2\tan^2\beta - 2V^2x\tan\beta + (2V^2y + gx^2) = 0$$

As stated in the question, since there is only one solution for $\tan\beta$,

$$\text{sum of the roots} = 2\tan\beta = -\frac{b}{a} = \frac{2V^2x}{gx^2}$$

$$\tan\beta = \frac{V^2}{gx}$$

Substituting in the equation:

$$gx^2\left(\frac{V^2}{gx}\right)^2 - 2V^2x\left(\frac{V^2}{gx}\right) + (2V^2y + gx^2) = 0$$

$$\frac{V^4}{g} - \frac{2V^4}{g} + 2V^2y + gx^2 = 0$$

$$-\frac{V^4}{g} + 2V^2y + gx^2 = 0$$

$$V^4 - 2gyV^2 - g^2x^2 = 0 \qquad \text{(multiplying by} - g\text{)}$$

7c ii $y = h$ and $x = h\cot\alpha$

From part c i, we get,

$V^4 - 2gyV^2 - g^2x^2 = 0$

V^2

$= \dfrac{2gy \pm \sqrt{(2gy)^2 - 4(1)(-g^2x^2)}}{2}$ (using quadratic formula)

$= \dfrac{2gy \pm \sqrt{4g^2h^2 + 4g^2x^2}}{2}$

$= gh \pm g\sqrt{h^2 + x^2}$

$= gh \pm g\sqrt{h^2 + (h\cot\alpha)^2}$

$= gh \pm g\sqrt{h^2(1 + \cot^2\alpha)}$

$= gh \pm gh\sqrt{(\operatorname{cosec}^2\alpha)}$

$= gh(1 \pm \operatorname{cosec}\alpha)$

$= gh(1 + \operatorname{cosec}\alpha)$ (as$V^2 > 0$)

7c iii

$$\frac{gx^2}{2V^2}\tan^2\beta - x\tan\beta + \left(y + \frac{gx^2}{2V^2}\right) = 0$$

From part a, c i and c ii, we have:

$$V^2 = gh(1+\operatorname{cosec}\alpha); \qquad \tan\beta = \frac{V^2}{gx}; \qquad x = h\cot\alpha; \qquad \tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right) = \frac{1+\sin\alpha}{\cos\alpha}$$

$\tan\beta$

$$= \frac{V^2}{gx}$$

$$= \frac{V^2}{gh\cot\alpha}$$

$$= \frac{gh(1+\operatorname{cosec}\alpha)}{gh\cot\alpha}$$

$$= \frac{\sin\alpha}{\cos\alpha}\left(1+\frac{1}{\sin\alpha}\right)$$

$$= \frac{\sin\alpha + 1}{\cos\alpha}$$

$$= \tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)$$

Hence,

$$\beta = \frac{\pi}{4}+\frac{\alpha}{2}$$

8a

$$\frac{gx^2}{2V^2}\tan^2\theta - x\tan\theta + \left(y + \frac{gx^2}{2V^2}\right) = 0$$

Let the solutions for this quadratic in $\tan\theta$ be t_1 and t_2 and the coefficients of the quadratic be a, b and c. Then using the sum and products of roots:

$t_1 + t_2$

$= -\dfrac{b}{a}$

$= \dfrac{x}{\frac{gx^2}{2V^2}}$

$= \dfrac{2V^2}{gx}$

$t_1.t_2$

$= \dfrac{c}{a}$

$= \dfrac{\left(y + \frac{gx^2}{2V^2}\right)}{\frac{gx^2}{2V^2}}$

$= \dfrac{2V^2y}{gx^2} + 1$

Now, since the equation is quadratic in $\tan\theta$, $t_1 = \tan\theta_1$ and $t_2 = \tan\theta_2$, and:

$\tan(\theta_1 + \theta_2)$

$= \dfrac{\tan\theta_1 + \tan\theta_2}{1 - \tan\theta_1\tan\theta_2}$

$= \dfrac{t_1 + t_2}{1 - t_1.t_2}$

$= \dfrac{\frac{2V^2}{gx}}{1 - \left(\frac{2V^2y}{gx^2} + 1\right)}$

$= \dfrac{2V^2}{gx} \times -\dfrac{gx^2}{2V^2y}$

$= -\dfrac{x}{y}$

8b $\theta_1 = \theta_2$ for some point $P\ (a, b)$

At the point P, the height is b and the range is a.

So, the angle of elevation for point P from the point of firing is:

$$\tan\alpha = \frac{b}{a}$$

From question 7c i and 7c ii, we have:

$$V^2 = gy(1 + \operatorname{cosec}\alpha) = gb(1 + \operatorname{cosec}\alpha)$$

From part a, using $\tan\theta_1 + \tan\theta_2 = \dfrac{2V^2}{gx}$:

$$2\tan\theta = \frac{2V^2}{gx}$$

$$\text{So } \tan\theta = \frac{V^2}{gx} = \frac{V^2}{ga}$$

$\tan\theta$

$$= \frac{gb(1 + \operatorname{cosec}\alpha)}{ga}$$

$$= \tan\alpha\left(\frac{\sin\alpha + 1}{\sin\alpha}\right)$$

$$= \frac{\sin\alpha + 1}{\cos\alpha}$$

$$= \tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right) \qquad \text{(from question 7a)}$$

Hence,

$\theta = \dfrac{\pi}{4} + \dfrac{\alpha}{2}$, where $\tan\alpha = \dfrac{b}{a}$ and the equation has only one solution.

8c

$$\frac{gx^2}{2V^2}\tan^2\theta - x\tan\theta + \left(y + \frac{gx^2}{2V^2}\right) = 0$$

The discriminant of the above equation is zero (for the equation of the parabola) as only one solution for θ.

$$x^2 - 4\left(\frac{gx^2}{2V^2}\right)\left(y + \frac{gx^2}{2V^2}\right) = 0$$

$$x^2 - \frac{2gx^2y}{V^2} - \frac{g^2x^4}{V^4} = 0$$

$$x^2\left(1 - \frac{2gy}{V^2} - \frac{g^2x^2}{V^4}\right) = 0$$

$$\frac{g^2x^2}{V^4} = -\frac{2gy}{V^2} + 1 \qquad \text{(as } x \neq 0\text{)}$$

$$x^2 = \frac{V^4}{g^2}\left(-\frac{2gy}{V^2} + 1\right)$$

$$x^2 = -\frac{2V^2y}{g} + \frac{V^4}{g^2}$$

$$x^2 = -\frac{2V^2}{g}\left(y - \frac{V^2}{2g}\right)$$

$$\text{or} \quad y = \frac{V^2}{2g} - \frac{gx^2}{2V^2}$$

For the vertex of the parabola, $\frac{dy}{dx} = 0$.

$$-\frac{gx}{V^2} = 0$$

$$x = 0$$

$$\text{so } y = \frac{V^2}{2g}$$

Therefore, coordinates of vertex are $\left(0, \frac{V^2}{2g}\right)$.

The focal length of the parabola is:

$$\frac{\frac{2V^2}{g}}{4} = \frac{V^2}{2g}$$

9 The equations for the positions of jth ball bearing at time t are:

$x = Vt\cos\alpha_j \quad (1)$

and $y = Vt\sin\alpha_j - \frac{1}{2}gt^2 \quad (2)$

$x^2 = \left(Vt\cos\alpha_j\right)^2 \quad (3)$

and $y^2 = \left(Vt\sin\alpha_j\right)^2 - Vgt^3\sin\alpha_j + \frac{g^2t^4}{4} \quad (4)$

From (2): $\sin\alpha_j = \frac{y + \frac{1}{2}gt^2}{Vt}$

Substituting into (4):

y^2

$= \left(Vt\sin\alpha_j\right)^2 - gt^2\left(y + \frac{gt^2}{2}\right) + \frac{g^2t^4}{4}$

$= \left(Vt\sin\alpha_j\right)^2 - gt^2y - \frac{g^2t^4}{2} + \frac{g^2t^4}{4}$

$= \left(Vt\sin\alpha_j\right)^2 - gt^2y - \frac{g^2t^4}{4}$

Therefore

$x^2 + y^2$

$= \left(Vt\cos\alpha_j\right)^2 + \left(Vt\sin\alpha_j\right)^2 - gt^2y - \frac{g^2t^4}{4}$

$= (Vt)^2\left(\sin^2\alpha_j + \cos^2\alpha_j\right) - gt^2y - \frac{g^2t^4}{4}$

$x^2 + y^2 + gt^2y + \frac{g^2t^4}{4} = (Vt)^2\left(\sin^2\alpha_j + \cos^2\alpha_j\right)$

$x^2 + \left(y + \frac{1}{2}gt^2\right)^2 = (Vt)^2$

Hence, the particles are on a circle with radius Vt and centre $(0, -\frac{1}{2}gt^2)$ at any given time t.

10a

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = g \qquad 0 \le x \le l$$

$$\frac{1}{2}v^2 = \int_0^l g\,dx$$

$$\frac{v^2}{2} = [gx]_0^l$$

$$v^2 = 2g(l-0)$$

$$v^2 = 2gl$$

10b i

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = g - gk(x-l) \quad x > l$$

$$\frac{1}{2}v^2 = \int \big(g - gk(x-l)\big)dx$$

$$v^2$$

$$= 2g\left(\int dx - k\int (x-l)\,dx\right)$$

$$= 2gx - gk(x-l)^2 + C$$

At $x = l$, $v^2 = 2gl$.

$$2gl = 2gl - 0 + C$$

$$C = 0$$

Therefore

$v^2 = 2gx - gk(x-l)^2$ for $x > l$

10b ii $v^2 = 2gx - gk(x-l)^2$

When the jumper is at the maximum distance, $v^2 = 0$.

$2gx_{max} - gk(x_{max} - l)^2 = 0$

$2x_{max} - k(x_{max}^2 - 2lx_{max} + l^2) = 0$

$2x_{max} - kx_{max}^2 + 2klx_{max} - kl^2 = 0$

$kx_{max}^2 - 2(1+kl)x_{max} + kl^2 = 0$

x_{max}

$= \dfrac{2(1+kl) \pm \sqrt{\left(2(1+kl)\right)^2 - 4k(kl^2)}}{2k}$ using the quadratic formula

$= \dfrac{2(1+kl) \pm \sqrt{4k^2l^2 + 8kl + 4 - 4k^2l^2}}{2k}$

$= \dfrac{1}{k}\left((1+kl) \pm \sqrt{1+2kl}\right)$

$= \dfrac{1}{k}\left(1 + kl + \sqrt{1+2kl}\right)$ since $k > 0, l > 0$

10b iii

$x_{max} = \dfrac{1}{k}\left(1 + kl + \sqrt{1+2kl}\right)$

$kl = 4$

$k = \dfrac{4}{l}$

Therefore

x_{max}

$= \dfrac{l}{4}\left(1 + 4 + \sqrt{1+2(4)}\right)$

$= 2l$

11a

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -\frac{10R^2}{(R+x)^2}$$

The negative sign indicates that the acceleration due to gravity acts in the downwards direction. So, when the object is moving upwards, it will feel resistance and the speed will reduce to zero (deceleration). When an object is falling down, the speed will increase (acceleration).

11b

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -\frac{10R^2}{(R+x)^2}$$

$$\frac{v^2}{2}$$

$$= -10R^2 \int \frac{dx}{(R+x)^2}$$

$$= 10R^2\left(\frac{1}{R+x}\right) + C$$

At $x = 0, v = V_0$.

$$C = \frac{V_0^2}{2} - 10R$$

Therefore

$$\frac{v^2}{2} = 10R^2\left(\frac{1}{R+x}\right) + \frac{V_0^2}{2} - 10R$$

For maximum height, $v = 0$ and $x = h$.

$$0 = 10R^2\left(\frac{1}{R+h}\right) + \frac{V_0^2}{2} - 10R$$

$$\frac{1}{R+h}$$

$$= \frac{10R - \frac{V_0^2}{2}}{10R^2}$$

$$= \frac{20R - V_0^2}{20R^2}$$

$$R + h = \frac{20R^2}{20R - V_0^2}$$

$$h$$

$$= \frac{20R^2}{20R - V_0^2} - R$$

$$= \frac{20R^2 - R(20R - V_0^2)}{20R - V_0^2}$$

$$= \frac{RV_0^2}{20R - V_0^2}$$

11c

$$h = \frac{RV_0^2}{20R - V_0^2}$$

Given $R = 6.4 \times 10^6$ m, $\quad V_0 = 500$ m/s:

h

$$= \frac{6.4 \times 10^6 \times 500^2}{20 \times 6.4 \times 10^6 - 500^2}$$

$= 12\,524.461\ldots$

$\doteqdot 12\,524$ m

12a

$$\frac{dB}{dt} = -k(B - W) \quad \text{and} \quad \frac{dW}{dt} = \frac{3}{4}k(B - W)$$

Differentiating $\frac{3}{4}B + W$:

$$\frac{d}{dt}\left(\frac{3}{4}B + W\right)$$

$$= \frac{3}{4}\frac{dB}{dt} + \frac{dW}{dt}$$

$$= \frac{3}{4}\big(-k(B - W)\big) + \frac{3}{4}k(B - W)$$

$$= \frac{3}{4}k(-B + W + B - W)$$

$= 0$

As $\frac{d}{dt}\left(\frac{3}{4}B + W\right) = 0$,

$\frac{3}{4}B + W$ is a constant.

Therefore

$\frac{3}{4}B + W = C$, where C is a constant

12b Substituting $B = 120°\text{C}$, $W = 22°\text{C}$:

$$C = \frac{3}{4}B + W = 112°\text{C}$$

$$\frac{3}{4}B + W = 112$$

$$W = 112 - \frac{3}{4}B$$

Now,

$$\frac{dB}{dt}$$

$$= -k(B - W)$$

$$= -k\left(B - \left(112 - \frac{3}{4}B\right)\right)$$

$$= -k\left(B - 112 + \frac{3}{4}B\right)$$

$$= -k\left(\frac{7}{4}B - 112\right)$$

12c

$$\frac{dB}{dt} = -k\left(\frac{7}{4}B - 112\right)$$

$$\frac{dt}{dB} = \frac{1}{-k\left(\frac{7}{4}B - 112\right)}$$

$$dt = \frac{dB}{-k\left(\frac{7}{4}B - 112\right)}$$

$$\int dt$$

$$= -\frac{1}{k}\int \frac{dB}{\frac{7}{4}B - 112}$$

$$= -\frac{4}{7k}\int \frac{\frac{7}{4}}{\frac{7}{4}B - 112}\,dB$$

$$t = -\frac{4}{7k}\ln\left|\frac{7}{4}B - 112\right| + C$$

At $t = 0,\ B = 120$

$$C = \frac{4}{7k}\ln 98$$

Therefore

$$t = -\frac{4}{7k}\ln\left|\frac{7}{4}B - 112\right| + \frac{4}{7k}\ln 98$$

$$-\frac{7kt}{4} = \ln\frac{\frac{7}{4}B - 112}{98}$$

$$e^{-\frac{7kt}{4}} = \frac{\frac{7}{4}B - 112}{98}$$

$$\frac{7}{4}B - 112 = 98e^{-\frac{7kt}{4}}$$

$$B = \frac{4}{7}\left(112 + 98e^{-\frac{7kt}{4}}\right)$$

$$B = 4\left(16 + 14e^{-\frac{7kt}{4}}\right)$$

12d When $t = 10$, $B = 92°\text{C}$.

$B = 4\left(16 + 14e^{-\frac{7kt}{4}}\right)$

$92 = 4\left(16 + 14e^{-\frac{7}{4}k(10)}\right)$

$23 = 16 + 14e^{-\frac{35k}{2}}$

$14e^{-\frac{35k}{2}} = 7$

$e^{-\frac{35k}{2}} = \frac{1}{2}$

$e^{\frac{35k}{2}} = 2$

$\frac{35k}{2} = \ln 2$

$k = \frac{2}{35}\ln 2$

When $t = 20$,

B

$= 4\left(16 + 14e^{-\frac{7}{4}\times\frac{2}{35}\ln 2\times 20}\right)$

$= 4\left(16 + 14e^{-2\ln 2}\right)$

$= 78°\text{C}$

12e The eventual temperature of water bath occurs when the heat energy exchange results in equilibrium condition. $B = W = T$

$\frac{dB}{dt} = 0$

$-k\left(\frac{7}{4}T - 112\right) = 0$

$\frac{7}{4}T - 112 = 0$

$T = 112 \times \frac{4}{7}$

$T = 64°\text{C}$

13a

$$P = \frac{21\ 000}{7 + 3e^{-\frac{t}{3}}}$$

Let $u = 7 + 3e^{-\frac{t}{3}}$

and $\dfrac{du}{dt} = -e^{-\frac{t}{3}}$

Therefore

$$P = 21\ 000\left(\frac{1}{u}\right) = 21\ 000u^{-1}$$

$$\frac{dP}{dt}$$

$$= \frac{dP}{du} \times \frac{du}{dt}$$

$$= -21\ 000u^{-2} \times \left(-e^{-\frac{t}{3}}\right)$$

$$= -\frac{21\ 000}{\left(7 + 3e^{-\frac{t}{3}}\right)^2} \times \left(-e^{-\frac{t}{3}}\right)$$

$$= \frac{21\ 000}{7 + 3e^{-\frac{t}{3}}} \times \left(-\frac{1}{7 + 3e^{-\frac{t}{3}}}\right) \times \left(-e^{-\frac{t}{3}}\right)$$

$$= P\left(\frac{e^{-\frac{t}{3}}}{7 + 3e^{-\frac{t}{3}}}\right)$$

$$= \frac{P}{3}\left(\frac{\left(7 + 3e^{-\frac{t}{3}}\right) - 7}{7 + 3e^{-\frac{t}{3}}}\right)$$

$$= \frac{P}{3}\left(1 - \frac{7}{7 + 3e^{-\frac{t}{3}}}\right)$$

$$= \frac{P}{3}\left(1 - \frac{21\ 000}{3000\left(7 + 3e^{-\frac{t}{3}}\right)}\right)$$

$$= \frac{P}{3}\left(1 - \frac{P}{3000}\right)$$

13b At the end of 2008, $t = 0$.

When $t = 0$,

P

$= \dfrac{21000}{7+3}$

$= 2100$

13c Eventually, population growth stops when:

$\dfrac{dP}{dt} = 0$

$\dfrac{P}{3}\left(1 - \dfrac{P}{3000}\right) = 0$

$P = 3000$

13d Rate of growth at the end of 2008:

r

$= \dfrac{dP}{dt}$

$= \dfrac{P_{2008}}{3}\left(1 - \dfrac{P_{2008}}{3000}\right)$

$= \dfrac{2100}{3}\left(1 - \dfrac{2100}{3000}\right)$

$= 700\left(1 - \dfrac{7}{10}\right)$

$= 210$

Annual % rate of growth

$= \dfrac{r}{P_{2008}} \times 100$

$= \dfrac{210}{2100} \times 100\%$

$= 10\%$

14a y

$= bx - ax^2$

$= -a\left(x^2 - \frac{b}{a}x\right)$

$= -a\left(x^2 - \frac{b}{a}x + \frac{b^2}{4a} - \frac{b^2}{4a}\right)$

$y - \frac{b^2}{4}$

$= -a\left(x^2 - \frac{b}{a}x + \frac{b^2}{4a}\right)$

$= -a\left(x^2 - \frac{b}{a}x + \frac{b^2}{4a^2}\right)$

$= -a\left(x - \frac{b}{2a}\right)^2$

$-\frac{1}{a}\left(y - \frac{b^2}{4}\right) = \left(x - \frac{b}{2a}\right)^2$

or $\left(x - \frac{b}{2a}\right)^2 = -\frac{1}{a}\left(y - \frac{b^2}{4}\right)$

Comparing with the standard equation for a parabola: $(x-h)^2 = 4A(y-k)$,

$4A = -\frac{1}{a}$

$A = -\frac{1}{4a}$

The focal length is positive so the focal length for this parabola is $\frac{1}{4a}$.

14b

y

$= x\tan\alpha - x^2\dfrac{g\sec^2\alpha}{2V^2}$

$= -\dfrac{g\sec^2\alpha}{2V^2}\left(x^2 - x\tan\alpha \times \dfrac{2V^2}{g\sec^2\alpha}\right)$

$= -\dfrac{g\sec^2\alpha}{2V^2}\left(x^2 - x \times \dfrac{\sin\alpha}{\cos\alpha} \times \dfrac{2V^2\cos^2\alpha}{g}\right)$

$= -\dfrac{g\sec^2\alpha}{2V^2}\left(x^2 - \dfrac{2V^2\sin\alpha\cos\alpha}{g}x\right)$

$= -\dfrac{g\sec^2\alpha}{2V^2}\left(x^2 - \dfrac{V^2\sin 2\alpha}{g}x + \left(\dfrac{V^2\sin 2\alpha}{2g}\right)^2 - \left(\dfrac{V^2\sin 2\alpha}{2g}\right)^2\right)$

$= -\dfrac{g\sec^2\alpha}{2V^2}\left(x^2 - \dfrac{V^2\sin 2\alpha}{g}x + \left(\dfrac{V^2\sin 2\alpha}{2g}\right)^2\right) - \dfrac{g\sec^2\alpha}{2V^2} \times -\left(\dfrac{V^2\sin 2\alpha}{2g}\right)^2$

$= -\dfrac{g\sec^2\alpha}{2V^2}\left(x^2 - \dfrac{V^2\sin 2\alpha}{g}x + \left(\dfrac{V^2\sin 2\alpha}{2g}\right)^2\right) + \dfrac{g\sec^2\alpha}{2V^2} \times \left(\dfrac{V^2\sin\alpha\cos\alpha}{g}\right)^2$

$= -\dfrac{g\sec^2\alpha}{2V^2}\left(x - \dfrac{V^2\sin 2\alpha}{2g}\right)^2 + \dfrac{V^2\sin^2\alpha\cos^2\alpha}{2g\cos^2\alpha}$

$= -\dfrac{g\sec^2\alpha}{2V^2}\left(x - \dfrac{V^2\sin 2\alpha}{2g}\right)^2 + \dfrac{V^2\sin^2\alpha}{2g}$

$y - \dfrac{V^2\sin^2\alpha}{2g} = -\dfrac{g\sec^2\alpha}{2V^2}\left(x - \dfrac{V^2\sin 2\alpha}{2g}\right)^2$

$-\dfrac{2V^2}{g\sec^2\alpha}\left(y - \dfrac{V^2\sin^2\alpha}{2g}\right) = \left(x - \dfrac{V^2\sin 2\alpha}{2g}\right)^2$

or $\left(x - \dfrac{V^2\sin 2\alpha}{2g}\right)^2 = -\dfrac{2V^2}{g\sec^2\alpha}\left(y - \dfrac{V^2\sin^2\alpha}{2g}\right)$

Standard equation for a parabola is $(x-h)^2 = 4A(y-k)$

where the coordinates of the vertex are (h, k),

the equation of its directrix is $y = k - A$ and

the coordinates of the focus are $(h, k + A)$.

14b i The coordinates of the vertex are (h, k) or

$$\left(\frac{V^2 \sin 2\alpha}{2g}, \frac{V^2 \sin^2 \alpha}{2g}\right)$$

14b ii

$$4A = -\frac{2V^2}{g \sec^2 \alpha}$$

A

$$= -\frac{V^2}{2g \sec^2 \alpha}$$

$$= -\frac{V^2 \cos^2 \alpha}{2g}$$

The focal length of the trajectory is $\dfrac{V^2 \cos^2 \alpha}{2g}$.

14b iii The equation of the directrix is $y = k - A$ or

y

$$= \frac{V^2 \sin^2 \alpha}{2g} - \left(-\frac{V^2 \cos^2 \alpha}{2g}\right)$$

$$= \frac{V^2}{2g}(\sin^2 \alpha + \cos^2 \alpha)$$

$$= \frac{V^2}{2g}$$

The coordinates of focus are $(h, k + A)$ or

$$\left(\frac{V^2 \sin 2\alpha}{2g}, \frac{V^2 \sin^2 \alpha}{2g} - \frac{V^2 \cos^2 \alpha}{2g}\right)$$

$$= \left(\frac{V^2 \sin 2\alpha}{2g}, \frac{V^2 (\sin^2 \alpha - \cos^2 \alpha)}{2g}\right)$$

$$= \left(\frac{V^2 \sin 2\alpha}{2g}, -\frac{V^2 (\cos^2 \alpha - \sin^2 \alpha)}{2g}\right)$$

$$= \left(\frac{V^2 \sin 2\alpha}{2g}, -\frac{V^2 \cos 2\alpha}{2g}\right)$$

15a

$$\frac{dy}{dt} = -k\sqrt{y}$$

Integrating gives:

$$\int dt = -\frac{1}{k}\int \frac{dy}{\sqrt{y}}$$

$$t = -\frac{2}{k}\sqrt{y} + C$$

At time $t = 0, y = y_0$.

$$0 = -\frac{2}{k}\sqrt{y_0} + C$$

$$C = \frac{2}{k}\sqrt{y_0}$$

Therefore

$$t = -\frac{2}{k}\sqrt{y} + \frac{2}{k}\sqrt{y_0}$$

At $t = T, y = 0$

$$T = \frac{2}{k}\sqrt{y_0}$$

$$k = \frac{2}{T}\sqrt{y_0}$$

Hence

t

$$= -\frac{2}{k}\sqrt{y} + \frac{2}{k}\sqrt{y_0}$$

$$= -\frac{2}{k}\left(\sqrt{y} - \sqrt{y_0}\right)$$

$$= -\frac{T\sqrt{y}}{\sqrt{y_0}} + \frac{T\sqrt{y_0}}{\sqrt{y_0}} \qquad \left(\text{substituting } k = \frac{2}{T}\sqrt{y_0}\right)$$

$$\frac{t}{T} = -\frac{\sqrt{y}}{\sqrt{y_0}} + 1$$

$$\frac{\sqrt{y}}{\sqrt{y_0}} = 1 - \frac{t}{T}$$

15b At $t = 15,\ y = \frac{y_0}{2}$

$$\sqrt{\frac{\frac{y_0}{2}}{y_0}} = 1 - \frac{15}{T}$$

$$\sqrt{\frac{y_0}{2y_0}} = 1 - \frac{15}{T}$$

$$\sqrt{\frac{1}{2}} = 1 - \frac{15}{T}$$

$$\frac{15}{T} = \frac{\sqrt{2} - 1}{\sqrt{2}}$$

$$\frac{T}{15} = \frac{\sqrt{2}}{\sqrt{2} - 1}$$

T

$$= \frac{15\sqrt{2}}{\sqrt{2} - 1}$$

$$= \frac{15\sqrt{2}}{\sqrt{2} - 1} \times \frac{\sqrt{2} + 1}{\sqrt{2} + 1}$$

$$= 15\left(2 + \sqrt{2}\right)$$

$\doteqdot 51.2$ seconds

16a

$$y = \frac{3}{2}x^{\frac{2}{3}}$$

$$\frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 + mgy = E$$

At $t = 0$, bead is released from rest from point $\left(1, \frac{3}{2}\right)$.

$\frac{dx}{dt} = \dot{x} = 0$ and $\frac{dy}{dt} = \dot{y} = 0$ at $t = 0$

$$\frac{1}{2}m \times 0 + \frac{1}{2}m \times 0 + mg \times \frac{3}{2} = E$$

$$E = \frac{3}{2}mg$$

16b

$$y=\frac{3}{2}x^{\frac{2}{3}}$$

$$\frac{dy}{dt}=\frac{dy}{dx}\times\frac{dx}{dt}$$

$$\dot{y}=x^{-\frac{1}{3}}\times\dot{x}$$

$$\dot{x}=x^{\frac{1}{3}}\dot{y}$$

$$\dot{x}^2$$

$$=x^{\frac{2}{3}}\dot{y}^2$$

$$=\frac{2}{3}y\dot{y}^2$$

Using the equation for E:

$$\frac{1}{2}m\dot{x}^2+\frac{1}{2}m\dot{y}^2+mgy=E$$

$$\frac{1}{2}m\left(\frac{2}{3}y\dot{y}^2\right)+\frac{1}{2}m\dot{y}^2+mgy=\frac{3}{2}mg$$

$$\frac{2}{3}y\dot{y}^2+\dot{y}^2+2gy=3g$$

$$\dot{y}^2\left(\frac{2}{3}y+1\right)+2gy=3g$$

$$\dot{y}^2\left(\frac{2}{3}y+1\right)=g(3-2y)$$

$$\dot{y}^2(2y+3)=3g(3-2y)$$

$$\dot{y}^2=\frac{3g(3-2y)}{3+2y}$$

16c At the origin, $x = 0$ and $y = 0$.

$\dot{x}$

$= x^{\frac{1}{3}}\dot{y}$

$= 0 \times \dot{y}$

$= 0$

$\dot{y}$

$= \pm\sqrt{\dfrac{3g(3-2y)}{3+2y}}$

$= \pm\sqrt{\dfrac{3g(3-0)}{3+0}}$

$= \pm\sqrt{3g}$

Since y is decreasing to that point on the curve,

$\dot{y} = -\sqrt{3g}$

16d

$$\int_0^{\alpha} \sqrt{\frac{1+u}{1-u}}\,du = \sin^{-1}\alpha + 1 - \sqrt{1-\alpha^2} \qquad \text{for } 0 \le \alpha \le 1$$

As our motion is also restricted in the same interval,

$$\dot{y} = \frac{dy}{dt} = -\sqrt{\frac{3g(3-2y)}{3+2y}}$$

$$\frac{dt}{dy} = -\frac{1}{\sqrt{3g}}\sqrt{\frac{3+2y}{3-2y}}$$

$$dt = -\frac{1}{\sqrt{3g}}\sqrt{\frac{3+2y}{3-2y}}\,dy$$

$$\int_0^t dt = -\frac{1}{\sqrt{3g}}\int_{\frac{3}{2}}^{0}\sqrt{\frac{1+\frac{2y}{3}}{1-\frac{2y}{3}}}\,dy$$

$$t = \frac{1}{\sqrt{3g}}\int_0^{\frac{3}{2}}\sqrt{\frac{1+\frac{2y}{3}}{1-\frac{2y}{3}}}\,dy$$

Let $u = \frac{2y}{3}$

$$\frac{du}{dy} = \frac{2}{3}$$

$$dy = \frac{3}{2}du$$

When $y = 0, u = 0$

When $y = \frac{3}{2}, u = 1$

Therefore

t

$$= \frac{1}{\sqrt{3g}}\int_0^1 \frac{3}{2}\sqrt{\frac{1+u}{1-u}}\,du$$

$$= \frac{3}{2\sqrt{3g}} \int_0^1 \sqrt{\frac{1+u}{1-u}}\, du$$

$$= \frac{3}{2\sqrt{3g}} \left(\sin^{-1} 1 + 1 - \sqrt{1 - 1^2}\right) \quad \text{(using the given relationship with } \alpha = 1\text{)}$$

$$= \frac{3}{2\sqrt{3g}} \left(\frac{\pi}{2} + 1 - \sqrt{0}\right)$$

$$= \frac{3}{2\sqrt{3g}} \left(\frac{\pi}{2} + 1\right)$$

$$= \frac{\sqrt{3}}{2\sqrt{g}} \left(\frac{\pi}{2} + 1\right)$$

16e

$$\int_0^\alpha \sqrt{\frac{1+u}{1-u}}\, du$$

$$= \int_0^\alpha \frac{\sqrt{1+u}}{\sqrt{1-u}} \times \frac{\sqrt{1+u}}{\sqrt{1+u}} du$$

$$= \int_0^\alpha \frac{1+u}{\sqrt{1-u^2}}\, du$$

$$= \int_0^\alpha \frac{1}{\sqrt{1-u^2}}\, du - \frac{1}{2}\int_0^\alpha \frac{-2u}{\sqrt{1-u^2}}\, du$$

$$= \left[\sin^{-1} u - \sqrt{1-u^2}\right]_0^\alpha$$

$$= \left(\sin^{-1} \alpha - \sqrt{1-\alpha^2}\right) - \left(\sin^{-1} 0 - \sqrt{1-0^2}\right)$$

$$= \sin^{-1} \alpha - \sqrt{1-\alpha^2} + 1$$

$$= \sin^{-1} \alpha + 1 - \sqrt{1-\alpha^2}$$

Looking at the denominator we see that interestingly at $\alpha = 1$, the integral becomes improper.

Solutions to Exercise 6H Chapter review

1a i $v = 2x - 1$

$$\frac{dx}{dt} = 2x - 1$$

$$\int dt = \int \frac{1}{2x - 1}\, dx$$

$$t = \frac{1}{2}\int \frac{2}{2x - 1}\, dx$$

$$t = \frac{1}{2}\ln|2x - 1| + C$$

At $t = 0,\ x = 1$.

$$0 = \frac{1}{2}\ln|1| + C$$

$$C = 0$$

Therefore

$$t = \frac{1}{2}\ln|2x - 1|$$

1a ii

$$t = \frac{1}{2}\ln|2x - 1|$$

$$2t = \ln|2x - 1|$$

$$e^{2t} = 2x - 1$$

$$x = \frac{1}{2}(e^{2t} + 1)$$

1b i $v = -6x^2$

$$\frac{dx}{dt} = -6x^2$$

$$\int dt = -\int \frac{dx}{6x^2}$$

$$t = \frac{1}{6x} + C$$

At $t = 0, x = 1.$

$$C = -\frac{1}{6}$$

Therefore

$$t = \frac{1}{6x} - \frac{1}{6}$$

$$t = \frac{1}{6}(x^{-1} - 1)$$

1b ii

$$t = \frac{1}{6}(x^{-1} - 1)$$

$$6t = x^{-1} - 1$$

$$x^{-1} = 1 + 6t$$

$$x = \frac{1}{1 + 6t}$$

2a $v = 2x - 1$

$$\ddot{x} = \frac{dv}{dt} = v\frac{dv}{dx}$$

$\ddot{x}$

$$= (2x - 1)(2)$$

$$= 2(2x - 1)$$

2b $v = -6x^2$

$\ddot{x} = \frac{dv}{dt} = v\frac{dv}{dx}$

$\ddot{x}$

$= -6x^2(-12x)$

$= 72x^3$

3a $\ddot{x} = 6$

$\ddot{x} = \frac{dv}{dt} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 6$

$\frac{1}{2}v^2 = \int 6\,dx$

$v^2 = \int 12\,dx$

$v^2 = 12x + C$

At $x = 0, v = 0.$

$C = 0$

Therefore

$v^2 = 12x$

3b $\ddot{x} = \sin 6x$

$$\ddot{x} = \frac{dv}{dt} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \sin 6x$$

$$v^2 = 2\int \sin 6x\, dx$$

$$v^2 = -\frac{1}{3}\cos 6x + C$$

At $v = 0,\ \ x = 0$, hence

$$C = \frac{1}{3}$$

Therefore

$$v^2 = -\frac{1}{3}\cos 6x + \frac{1}{3}$$

$$v^2 = \frac{1}{3}(1 - \cos 6x)$$

4 $\ddot{x} = 4v$

$$\frac{dv}{dt} = 4v$$

$$\int dt = \int \frac{dv}{4v}$$

$$t = \frac{1}{4}\ln v + C$$

At $t = 0,\ \ v = 2$, hence

$$C = -\frac{1}{4}\ln 2$$

Therefore

$$t = \frac{1}{4}\ln v - \frac{1}{4}\ln 2$$

$$t = \frac{1}{4}\ln\frac{v}{2}$$

5

$\ddot{x} = \frac{1}{3v}$

$\frac{dv}{dt} = v\frac{dv}{dx} = \frac{1}{3v}$

$\frac{dv}{dx} = \frac{1}{3v^2}$

$\int dx = \int 3v^2 dv$

$x = v^3 + C$

At $x = 0, \quad v = 1$, hence

$C = -1$

Therefore

$x = v^3 - 1$

6a $F = 18t, \quad m = 3$

$$F = m\ddot{x} = m\frac{d^2x}{dt^2} = 18t$$

$$\frac{d^2x}{dt^2} = 6t$$

$$\frac{dx}{dt} = 3t^2 + C$$

Initially particle is at rest at origin, so $\frac{dx}{dt} = 0$ when $t = 0$, hence:

$C = 0$

Therefore

$$\frac{dx}{dt} = 3t^2$$

$$\int dx = 3\int t^2 dt$$

$$x = t^3 + D$$

Using initial conditions, $t = 0$ and $x = 0$:

$D = 0$

Therefore

$x = t^3$

At $t = 2$,

$x = 2^3 = 8$

6b $F = 4x - 3, \quad m = 2$

$F = m\ddot{x} = m\dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right) = 4x - 3$

$\dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right) = \dfrac{1}{2}(4x - 3)$

$v^2 = \int (4x - 3)\, dx$

$v^2 = 2x^2 - 3x + C$

At $t = 0, x = 0, v = 0,$ hence

$C = 0$

Therefore

$v^2 = 2x^2 - 3x$

At $x = 4$,

v^2

$= 2(16) - 3(4)$

$= 20$

$v = 2\sqrt{5}$ (taking positive value of v)

6c

$$F = \frac{1}{1+v}, \quad m = 0.5$$

$$F = m\ddot{x} = m\frac{dv}{dt} = \frac{1}{1+v}$$

$$\frac{dv}{dt} = \frac{2}{1+v}$$

$$\int dt = \frac{1}{2}\int (1+v)\, dv$$

$$t = \frac{1}{2}\left(v + \frac{v^2}{2}\right) + C$$

At $t = 0, \quad v = 0,$ hence

$C = 0$

Therefore

$$t = \frac{v}{2} + \frac{v^2}{4}$$

At $v = 6,$

t

$$= \frac{6}{2} + \frac{36}{4}$$

$= 12$

6d $F = 2 + v^2, \quad m = 2$

$F = m\ddot{x} = mv\dfrac{dv}{dx} = (2 + v^2)$

$\dfrac{dv}{dx} = \dfrac{2 + v^2}{2v}$

$\int dx = \int \dfrac{2v}{2 + v^2}\, dv$

$x = \ln(2 + v^2) + C \qquad$ (since $2 + v^2$ is positive)

At $x = 0, \quad v = 0$, hence

$C = -\ln 2$

Therefore

x

$= \ln(2 + v^2) - \ln 2$

$= \ln\left(\dfrac{2 + v^2}{2}\right)$

At $v = 2$,

x

$= \ln\left(\dfrac{2 + 4}{2}\right)$

$= \ln 3$

7 $m = 2$, $F_1 = 40$ N in NE direction

$F_1 = 20\sqrt{2}\hat{\imath} + 20\sqrt{2}\hat{\jmath}$

$F_2 = 20\hat{\imath} + 40\hat{\jmath}$

The resultant force is:

F_r

$= F_1 + F_2$

$= (20\sqrt{2} + 20)\hat{\imath} + (20\sqrt{2} + 40)\hat{\jmath}$

$= 20\left((\sqrt{2} + 1)\hat{\imath} + (\sqrt{2} + 2)\hat{\jmath}\right)$

Now $F_r = m\ddot{x}_r$

$20\left((\sqrt{2} + 1)\hat{\imath} + (\sqrt{2} + 2)\hat{\jmath}\right) = 2\ddot{x}_r$

$\ddot{x}_r = 10\left((\sqrt{2} + 1)\hat{\imath} + (\sqrt{2} + 2)\hat{\jmath}\right)$

$|\ddot{x}_r|$

$= 10\sqrt{(\sqrt{2} + 1)^2 + (\sqrt{2} + 2)^2}$

$= 10\sqrt{3 + 2\sqrt{2} + 6 + 4\sqrt{2}}$

$= 10\sqrt{9 + 6\sqrt{2}}$

$= 41.815 \ldots$

Magnitude of resultant acceleration is about 41.8 m/s^2

The direction of resultant acceleration is same as that of the resultant force.

Direction of $\ddot{x}_r$:

$\tan\alpha = \dfrac{\sqrt{2} + 2}{\sqrt{2} + 1}$

$\alpha = 54.735 \ldots°$

Angle north of east is about 54.7° so direction of acceleration is 035.3°T.

8a The plane undergoes a constant deceleration until it stops completely. So, the forces acting on the plane are just the deceleration. Hence,

$\ddot{x} = -k$

$v_0 = 100, \quad d = 2000$

$\ddot{x} = \dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right) = -k$

$\dfrac{1}{2}v^2 = -k\displaystyle\int dx$

$v^2 = -2kx + C$

At $x = 0, \quad v = 100$, hence

$C = 10\ 000$

Therefore

$v^2 = 10\ 000 - 2kx$

At $x = 2000, \quad v = 0.$

$0 = 10\ 000 - 4000k$

$k = \dfrac{5}{2}$

Therefore

$v^2 = 10\ 000 - 5x$

8b i $v^2 = 10\ 000 - 5x$

At $x = 1$ km $= 1000$ m,

$v^2 = 10\ 000 - 5000$

$v^2 = 5000$

$v = 50\sqrt{2}$ m/s (taking positive v as plane continues to move in forward direction)

8b ii $v^2 = 10\,000 - 5x$

At $v = 50$,

$2500 = 10\,000 - 5x$

$x = 1500$ m

8c $v^2 = 10\,000 - 5x$

Now, because of Newtons first law of motion while the brakes are being applied, the plane continues to move in the forward direction resisting change in inertia. So, since the velocity is positive initially we must have $v = \sqrt{10\,000 - 5x}$

8d $v = \sqrt{10\,000 - 5x}$

$$\frac{dx}{dt} = \sqrt{10\,000 - 5x}$$

$$\int dt = \int \frac{dx}{\sqrt{10\,000 - 5x}}$$

$$t = -\frac{2}{5}\sqrt{10\,000 - 5x} + C$$

When $t = 0, x = 0,$ hence

$$0 = -\frac{2}{5}\sqrt{10\,000} + C$$

$$C = \frac{2}{5} \times 100$$

$$C = 40$$

Therefore

$$t = -\frac{2}{5}\sqrt{10\,000 - 5x} + 40$$

$$\sqrt{10\,000 - 5x} = \frac{5}{2}(40 - t)$$

$$10\,000 - 5x = \frac{25}{4}(40 - t)^2$$

$$5x = 10\,000 - \frac{25}{4}(1600 - 80t + t^2)$$

$$x = 2000 - \frac{5}{4}(1600 - 80t + t^2)$$

$$x = 2000 - 2000 + 100t - \frac{5}{4}t^2$$

$$x = 100t - \frac{5}{4}t^2$$

The plane stops when $x = 2000$.

t

$$= -\frac{2}{5}\sqrt{10\,000 - 5x} + 40$$

$$= -\frac{2}{5}\sqrt{10\,000 - 10\,000} + 40$$

$$= 40 \text{ s}$$

9a $\quad v_0 = \frac{1}{2}\text{ km/s} = 500\text{ m/s}, \quad v_1 = 250\text{ m/s}, \quad x_1 = 50\text{ m}$

$$\ddot{x} = v\frac{dv}{dx} = -kv$$

$$\frac{dv}{dx} = -k$$

$$\int dv = -k\int dx$$

$$v = -kx + C$$

At $x = 0, \;\; v_0 = 500$, hence

$$C = 500$$

Therefore

$$v = 500 - kx$$

At $x = 50, \;\; v = 250$, hence

$$k = 5$$

Therefore

$$v = \frac{dx}{dt} = 500 - 5x$$

$$\int dt = \int \frac{dx}{500 - 5x}$$

$$\int dt = -\frac{1}{5}\int \frac{-5}{500 - 5x}\,dx$$

$$t = -\frac{1}{5}\ln|500 - 5x| + D$$

At $t = 0, x = 0$, hence

$$D = \frac{1}{5}\ln 500$$

Therefore

$$t = \frac{1}{5}\ln\left(\frac{500}{500 - 5x}\right) \qquad \text{for } x < 100$$

$$5t = \ln\left(\frac{100}{100 - x}\right)$$

$$\frac{100}{100 - x} = e^{5t}$$

$100 - x = 100e^{-5t}$

x

$= 100 - 100e^{-5t}$

$= 100(1 - e^{-5t})$

9b

$t = \frac{1}{5}\ln\left(\frac{100}{100 - x}\right)$

At $x = 50$,

$t = \frac{1}{5}\ln\left(\frac{100}{100 - 50}\right)$

$t = \frac{1}{5}\ln 2$

$v = 500 - 5x$

Eventually, the torpedo will stop after a distance of 100 m as the velocity reduces to zero.

10a

$\ddot{x} = -\frac{1}{2}e^{-x}\ \mathrm{m/s^2}$

$\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = -\frac{1}{2}e^{-x}$

v^2

$= -\int e^{-x}\,dx$

$= e^{-x} + C$

At $x = 0$, $v = 1$, hence

$C = 0$

Therefore

$v^2 = e^{-x}$

10b v is always positive as v is initially positive (1 m/s) and is never zero ($e^{-x} \neq 0$).

$v^2 = e^{-x}$

$v = e^{-\frac{x}{2}}$

$\frac{dx}{dt} = e^{-\frac{x}{2}}$

$\int dt = \int e^{\frac{x}{2}}\,dx$

$t = 2e^{\frac{x}{2}} + C$

At $t = 0$, $x = 0$, hence

$C = -2$

Therefore

t

$= 2e^{\frac{x}{2}} - 2$

$= 2\left(e^{\frac{x}{2}} - 1\right)$

$e^{\frac{x}{2}} = \frac{t}{2} + 1$

$\frac{x}{2} = \ln\left(\frac{t}{2} + 1\right)$

$x = 2\ln\left(\frac{1}{2}(t+2)\right)$

10c

$x = 2\ln\left(\frac{1}{2}(t+2)\right), v = e^{-\frac{x}{2}}$

As $t \to \infty$, $x \to \infty$ and $v \to 0$

11a $\quad \ddot{x} = a + bv^2 \qquad a, b > 0$

$$\ddot{x} = v\frac{dv}{dx} = a + bv^2$$

$$\frac{dv}{dx} = \frac{a + bv^2}{v}$$

$$\int dx = \int \frac{v}{a + bv^2}\, dv$$

x

$$= \frac{1}{2b}\int \frac{2bv}{a + bv^2}\, dv$$

$$= \frac{1}{2b}\ln(a + bv^2) + C \qquad \text{(since } a, b > 0 \text{ so } a + bv^2 \text{ is positive)}$$

At $t = 0, \quad x = 0, \quad v = 0$, hence

$$C = -\frac{1}{2b}\ln a$$

Therefore

$$x = \frac{1}{2b}\ln\left(\frac{a + bv^2}{a}\right)$$

$$2bx = \ln\left(\frac{a + bv^2}{a}\right)$$

e^{2bx}

$$= \frac{a + bv^2}{a}$$

$$= 1 + \frac{b}{a}v^2$$

$$\frac{b}{a}v^2 = e^{2bx} - 1$$

$$v^2 = \frac{a}{b}(e^{2bx} - 1)$$

11b

$$\ddot{x} = a + bv^2 = \frac{dv}{dt}$$

$$\int dt = \int \frac{dv}{a + bv^2} \quad \text{(using a table of integrals)}$$

$$t = \frac{1}{\sqrt{ab}} \tan^{-1}\left(\frac{\sqrt{b}\,v}{\sqrt{a}}\right) + C$$

At $t = 0$, $v = 0$, hence

$$C = 0$$

Therefore

$$t = \frac{1}{\sqrt{ab}} \tan^{-1}\left(\frac{\sqrt{b}\,v}{\sqrt{a}}\right)$$

$$\sqrt{ab}\,t = \tan^{-1}\left(\frac{\sqrt{b}\,v}{\sqrt{a}}\right)$$

$$\frac{\sqrt{b}\,v}{\sqrt{a}} = \tan\sqrt{ab}\,t$$

$$v = \frac{\sqrt{a}}{\sqrt{b}} \tan\sqrt{ab}\,t$$

12a

$$\ddot{x} = -16x = v\frac{dv}{dx}$$

$$\int v\,dv = -16\int x\,dx$$

$$\frac{1}{2}v^2 = -8x^2 + C$$

At $t = 0$, $x = 1$, $v = 4$, hence

$$C = 16$$

Therefore

$$\frac{1}{2}v^2 = -8x^2 + 16$$

$$v^2 = 32 - 16x^2 = 16(2 - x^2)$$

12b At $x = x_{max}$, $v = 0$.

$0 = 16(2 - (x_{max})^2)$

$(x_{max})^2 = 2$

$x_{max} = \sqrt{2}$

12c $\ddot{x} = -16x$

Amplitude $a = x_{max} = \sqrt{2}$

Let $x = \sqrt{2}\cos(nt + \alpha)$

$\dot{x} = -\sqrt{2}n\sin(nt + \alpha)$ and

$\ddot{x} = -\sqrt{2}n^2\cos(nt + \alpha)$

Substituting for $\ddot{x}$ and x in $\ddot{x} = -16x$:

$-\sqrt{2}n^2\cos(nt + \alpha) = -16\sqrt{2}\cos(nt + \alpha)$

$n^2 = 16$

$n = 4$

At $t = 0$, $x = 1$, so $x = \sqrt{2}\cos(nt + \alpha)$ becomes:

$1 = \sqrt{2}\cos(0 + \alpha)$

$\cos\alpha = \dfrac{1}{\sqrt{2}}$

$\alpha = \dfrac{\pi}{4}$

Therefore

$x = \sqrt{2}\cos\left(4t + \dfrac{\pi}{4}\right)$

13a $x = 5 + \sqrt{3}\sin 3t - \cos 3t$

$\dot{x} = 3\sqrt{3}\cos 3t + 3\sin 3t$

$\ddot{x}$

$= -9\sqrt{3}\sin 3t + 9\cos 3t$

$= -9\left(\sqrt{3}\sin 3t - \cos 3t\right)$

$= -9\left(\left(5 + \sqrt{3}\sin 3t - \cos 3t\right) - 5\right)$

$= -3^2(x - 5)$

which is of the form $\ddot{x} = -n^2(x - c)$

As the equation satisfies the conditions of restoring force of SHM, the given particle is under SHM.

13b $\sqrt{3}\sin 3t - \cos 3t$

$= 2\left(\frac{\sqrt{3}}{2}\sin 3t - \frac{1}{2}\cos 3t\right)$

$= 2\left(\cos\frac{\pi}{6}\sin 3t - \sin\frac{\pi}{6}\cos 3t\right)$

$= 2\sin\left(3t - \frac{\pi}{6}\right)$

Therefore

$R = 2, \quad \alpha = \frac{\pi}{6}$

Hence

$x = 5 + 2\sin\left(3t - \frac{\pi}{6}\right)$

13c The amplitude is: $a = R = 2$

The centre of motion is: $x = 5$

13d

$x = 5 + 2\sin(3t - \frac{\pi}{6})$

When $v = v_{max}$, $x = 5$

$5 = 5 + 2\sin\left(3t - \frac{\pi}{6}\right)$

$\sin(3t - \frac{\pi}{6}) = 0$

$3t = \frac{\pi}{6}$

$t = \frac{\pi}{18}$ s

14 $v_{max} = 2$, $\ddot{x}_{max} = 6$

For SHM: $v_{max} = an$ and $\ddot{x}_{max} = an^2$

$an = 2$ and $an^2 = 6$

$\frac{an^2}{an} = \frac{6}{2}$

$n = 3$

$a = \frac{2}{3}$

So, the amplitude is: $a = \frac{2}{3}$ and the frequency is: $n = 3$

Period of motion is:

$T = \frac{2\pi}{n} = \frac{2\pi}{3}$ s

15a $a = 5$ cm, $m = 0.5$ kg, cycles $= 60/\text{s}$, $n = 2\pi \times 60 = 120\pi$

v_{max}

$= an$

$= 5 \times 120\pi$

$= 1884.95 \ldots$ cm/s

$\doteqdot 18.8$ m/s

15b $\ddot{x}_{max} = an^2$

$a = 5 \text{ cm} = 0.05 \text{ m}$

F_{max}

$= m\ddot{x}_{max}$

$= 0.5 \times an^2$

$= 0.5 \times 0.05 \times (120\pi)^2$

$= 3553.057 \ldots$

$= 3553 \text{ N}$

16a At $x = 0.01$ m, $|v| = 0.09$ m/s and $x = -0.02$ m, $|v| = 0.06$ m/s. For SHM

$v^2 = -n^2((x-c)^2 - a^2)$

Here, $c = 0$. This gives,

$(0.09)^2 = -n^2((0.01)^2 - a^2)$ and $(0.06)^2 = -n^2((-0.02)^2 - a^2)$

$$\left(\frac{0.09}{0.06}\right)^2 = \frac{(0.01)^2 - a^2}{(-0.02)^2 - a^2}$$

$$\frac{9}{4} = \frac{0.0001 - a^2}{0.0004 - a^2}$$

$0.0004 - 4a^2 = 0.0036 - 9a^2$

$5a^2 = 0.0032$

a

$$= \sqrt{\frac{32}{50\,000}}$$

$$= \frac{4\sqrt{2}}{100\sqrt{5}}$$

$$= \frac{\sqrt{2}}{25\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}}$$

$$= \frac{\sqrt{10}}{125}$$

Hence amplitude is $\frac{\sqrt{10}}{125}$ metres.

16b

$$a = \frac{\sqrt{10}}{125},$$

$$(0.09)^2 = -n^2((0.01)^2 - a^2)$$

$$n^2$$

$$= \frac{(0.09)^2}{\frac{10}{15\,625} - 0.0001}$$

$$= \frac{0.0081}{0.00054}$$

$$= 15$$

$$n = \sqrt{15}$$

$$T = \frac{2\pi}{n} = \frac{2\pi}{\sqrt{15}}$$

Hence period is $\frac{2\pi}{\sqrt{15}}$ seconds.

16c Let $x = a\cos nt$

$x = \frac{\sqrt{10}}{125}\cos\frac{2\pi t}{\sqrt{15}}$

When $x = 0.01$,

$0.01 = \frac{\sqrt{10}}{125}\cos\frac{2\pi t_1}{\sqrt{15}}$

t_1

$= \frac{\sqrt{15}}{2\pi}\cos^{-1}\frac{1.25}{\sqrt{10}}$

$= 0.717\ 75\ \ldots\ \text{s}$

When $x = -0.02$,

$-0.02 = \frac{\sqrt{10}}{125}\cos\frac{2\pi t_2}{\sqrt{15}}$

t_2

$= \frac{\sqrt{15}}{2\pi}\cos^{-1}\left(\frac{-2.5}{\sqrt{10}}\right)$

$= 1.530\ 24\ \ldots\ \text{s}$

$t_2 - t_1$

$= 1.530\ 24\ \ldots - 0.717\ 75\ \ldots$

$\doteqdot 0.81\ \text{s}$

So, it takes about 0.81 seconds for the particle to go from A to B.

17 The equation of motion for the aircraft is:

$$M\ddot{x} = T - kv^2$$

$$M\,v\frac{dv}{dx} = T - kv^2$$

$$\frac{dv}{dx} = \frac{T - kv^2}{Mv}$$

$$\int dx = \frac{M}{2k}\int \frac{2kv}{T - kv^2}\,dv$$

$$x = -\frac{M}{2k}\ln(T - kv^2) + C \qquad \text{(since } T - kv^2 \text{ is positive as aircraft is moving)}$$

At $x = 0$, $v = 0$, hence

$$C = \frac{M}{2k}\ln T$$

Therefore

$$x = \frac{M}{2k}\ln\left(\frac{T}{T - kv^2}\right)$$

So, when the speed of the aircraft is V, the distance travelled is:

$$x = \frac{M}{2k}\ln\left(\frac{T}{T - kV^2}\right)$$

18a $m = 2$ kg, $v_0 = 20$ m/s

The equation of motion of the object when it is travelling upwards is:

$$m\ddot{y} = mv\frac{dv}{dy} = -mg - \frac{v^2}{10}$$

$$2v\frac{dv}{dy} = -20 - \frac{v^2}{10}$$

$$\frac{dv}{dy} = -\frac{200 + v^2}{20v}$$

$$\int dy = -10\int \frac{2v\,dv}{200 + v^2}$$

$y = -10\ln(200 + v^2) + C$ (since $200 + v^2$ is positive)

At $y = 0$, $v = 20$, hence

$C = 10\ln 600$

Therefore

$$y = 10\ln\frac{600}{200 + v^2}$$

At maximum height, $v = 0$. Hence:

$$h = 10\ln\frac{600}{200 + 0}$$

$h = 10\ln 3$

Hence maximum height is $10\ln 3$ metres.

18b For half its maximum height, $y = 5\ln 3$.

$$y = 10\ln\frac{600}{200 + v^2}$$

$$5\ln 3 = 10\ln\frac{600}{200 + v^2}$$

$$10\ln 3^{\frac{1}{2}} = 10\ln\frac{600}{200 + v^2}$$

$$\sqrt{3} = \frac{600}{200 + v^2}$$

$$200\sqrt{3} + \sqrt{3}v^2 = 600$$

$$\sqrt{3}v^2 = 600 - 200\sqrt{3}$$

$$3v^2 = 600\sqrt{3} - 600$$

$$v^2 = 200\sqrt{3} - 200$$

$$|v| = 12.100\ldots$$

Hence speed of object is about 12.1 m/s.

19a $m = 0.03$ kg, $H = 6$ m, $f_r = 0.6v$ N, $g = 10$ m/s^2

Taking downwards as positive, equation of motion for the balloon is:

$$m\ddot{y} = mg - f_r$$

$$0.03\ddot{y} = 0.3 - 0.6v$$

$$\ddot{y} = 10 - 20v$$

19b Terminal velocity is reached when the resultant forces/acceleration is zero.

$$\ddot{y} = 0 = 10 - 20v_t$$

$$20v_t = 10$$

$$v_t = 0.5 \text{ m/s}$$

Terminal speed is 0.5 m/s.

19c

$$\ddot{y} = \frac{dv}{dt} = 10 - 20v$$

$$\int dt$$

$$= \int \frac{1}{10 - 20v}\, dv$$

$$= -\frac{1}{20}\int \frac{-20}{10 - 20v}\, dv$$

$t = -\frac{1}{20}\ln(10 - 20v) + C$ (since terminal velocity is 0.5 so $10 - 2v$ is positive)

At $t = 0$, $v = 0$, hence

$$0 = -\frac{1}{20}\ln 10 + C$$

$$C = \frac{1}{20}\ln 10$$

$$t = \frac{1}{20}\ln\left(\frac{1}{1 - 2v}\right)$$

$$e^{20t} = \frac{1}{1 - 2v}$$

$$e^{-20t} = 1 - 2v$$

$$v = \frac{1}{2}(1 - e^{-20t})$$

19d For half its terminal speed, $v = 0.25$.

t

$$= \frac{1}{20}\ln\left(\frac{1}{1 - 2v}\right)$$

$$= \frac{1}{20}\ln\left(\frac{1}{1 - 0.5}\right)$$

$$= \frac{1}{20}\ln 2$$

$= 0.034\,657\ldots$

Time to reach half its terminal speed is about 0.35 seconds.

20a Taking downwards as positive, the equation of motion of the particle is:

$m\ddot{y} = mg - mkv$

$\ddot{y} = \dfrac{dv}{dt} = g - kv$

At $\dfrac{dv}{dt} = 0, v = V$

$g - kV = 0$

$k = \dfrac{g}{V}$

Therefore

$\dfrac{dv}{dt}$

$= g - \dfrac{g}{V}v$

$= \dfrac{g}{V}(V - v)$

20b

$$\frac{dv}{dt} = g - kv$$

$$\int dt = \int \frac{1}{g - kv}\, dv$$

$t = -\frac{1}{k}\ln(g - kv) + C$ (since particle is falling so $g - kv$ is positive)

At $t = 0,\ v = 0$, hence

$$C = \frac{1}{k}\ln g$$

Therefore

t

$$= \frac{1}{k}\ln\left(\frac{g}{g - kv}\right)$$

$= \frac{V}{g}\ln\left(\frac{g}{g - \frac{gv}{V}}\right)$ (since $k = \frac{g}{V}$)

$$= \frac{V}{g}\ln\left(\frac{1}{1 - \frac{v}{V}}\right)$$

$$= \frac{V}{g}\ln\left(\frac{V}{V - v}\right)$$

At $v = \frac{1}{2}V$,

$$t = \frac{V}{g}\ln\left(\frac{V}{V - \frac{1}{2}V}\right)$$

$$t = \frac{V}{g}\ln 2$$

20c

$$\ddot{y} = v\frac{dv}{dy} = \frac{g}{V}(V - v)$$

$$\frac{dv}{dy} = \frac{g}{Vv}(V - v)$$

$$\int dy$$

$$= \frac{V}{g}\int \frac{v}{V - v}\, dv$$

$$= -\frac{V}{g}\int \frac{V - v - V}{V - v} dv$$

$$= -\frac{V}{g}\int \left(\frac{V - v}{V - v} + \frac{-V}{V - v}\right) dv$$

$$= -\frac{V}{g}\int \left(1 + \frac{-V}{V - v}\right) dv$$

$$y = -\frac{V}{g}(v + V\ln(V - v)) + C$$

At $t = 0, v = 0$ and $y = 0$, hence

$$y = -\frac{V}{g}(V\ln V) + C$$

$$C = \frac{V^2}{g}\ln V$$

Therefore

$$y$$

$$= -\frac{V}{g}(v + V\ln(V - v)) + \frac{V^2}{g}\ln V$$

$$= -\frac{V}{g}v - \frac{V^2}{g}\ln(V - v) + \frac{V^2}{g}\ln V$$

$$= \frac{V^2}{g}\ln\left(\frac{V}{V - v}\right) - \frac{V}{g}v$$

At $v = \frac{1}{2}V$,

y

$= \dfrac{V^2}{g}\ln\left(\dfrac{V}{V-\frac{1}{2}V}\right) - \dfrac{V}{g}\left(\dfrac{1}{2}V\right)$

$= \dfrac{V^2}{g}\ln 2 - \dfrac{V^2}{2g}$

$= \dfrac{V^2}{g}\left(\ln 2 - \dfrac{1}{2}\right)$

21a Particle P_2 is dropped from point B.

Taking upwards as positive, the equation of motion is:

$m\ddot{y} = mv\dfrac{dv}{dy} = -mg + mkv^2$

$v\dfrac{dv}{dy} = -g + kv^2$

When terminal speed, V, is reached, the forces on the particle are balanced and the resultant acceleration is zero.

$0 = -g + kV^2$

$V = \sqrt{\dfrac{g}{k}}$

21b The equation of motion for particle P_1 is:

$$\ddot{y} = v\frac{dv}{dy} = -(g + kv^2)$$

$$\frac{dv}{dy} = -\frac{g + kv^2}{v}$$

$$\int dy = -\int \frac{v}{g + kv^2}\, dv$$

$$y = -\frac{1}{2k}\ln(g + kv^2) + C$$

At $t = 0$, $y = 0$ and $v = U$, hence

$$C = \frac{1}{2k}\ln(g + kU^2)$$

Therefore

y

$$= -\frac{1}{2k}\ln(g + kv^2) + \frac{1}{2k}\ln(g + kU^2)$$

$$= \frac{1}{2k}\ln\left(\frac{g + kU^2}{g + kv^2}\right)$$

The height and velocity of P_1 is y_1 and v_1. Hence,

$$y_1 = \frac{1}{2k}\ln\left(\frac{g + kU^2}{g + kv_1^2}\right)$$

21c H = height P_1 would reach if it did not collide with P_2, $= y_{1max}$

At y_{1max}, $v_1 = 0$. Hence,

$$y_{1max} = H = \frac{1}{2k}\ln\left(\frac{g + kU^2}{g + k(0)}\right)$$

$$H = \frac{1}{2k}\ln\left(1 + \frac{kU^2}{g}\right)$$

Since $V = \sqrt{\frac{g}{k}}$

$$V^2 = \frac{g}{k}$$

Therefore

$$H = \frac{1}{2k}\ln\left(1 + \frac{U^2}{V^2}\right)$$

21d Given,

$$y_2 = \frac{1}{2k}\ln\left|\frac{g}{g - kv_2^2}\right|$$

At the point of collision:

$$v_2 = \frac{1}{2}V = \frac{1}{2}\sqrt{\frac{g}{k}}$$

y_2

$$= \frac{1}{2k}\ln\left|\frac{g}{g - k\left(\frac{1}{4}\frac{g}{k}\right)}\right|$$

$$= \frac{1}{2k}\ln\left|\frac{g}{g - \frac{g}{4}}\right|$$

$$= \frac{1}{2k}\ln\left(\frac{1}{\frac{3}{4}}\right)$$

$$= \frac{1}{2k}\ln\frac{4}{3}$$

21e $H = y_1 + y_2$

$$\frac{1}{2k}\ln\left(1+\frac{U^2}{V^2}\right) = \frac{1}{2k}\ln\left(\frac{g+kU^2}{g+kv_1^2}\right) + \frac{1}{2k}\ln\frac{4}{3}$$

$$\frac{1}{2k}\ln\frac{3}{4}\left(1+\frac{U^2}{V^2}\right) = \frac{1}{2k}\ln\left(\frac{g+kU^2}{g+kv_1^2}\right)$$

$$\frac{3}{4}\left(1+\frac{U^2}{V^2}\right) = \frac{g+kU^2}{g+kv_1^2}$$

$$\frac{3}{4}\left(1+\frac{U^2}{V^2}\right) = \frac{1+\frac{k}{g}U^2}{1+\frac{k}{g}v_1^2}$$

$$\frac{3}{4}\left(1+\frac{U^2}{V^2}\right) = \frac{1+\frac{U^2}{V^2}}{1+\frac{v_1^2}{V^2}} \qquad \left(\text{since } \frac{g}{k} = V^2\right)$$

$$1+\frac{v_1^2}{V^2} = \frac{4}{3}$$

$$\frac{v_1}{V} = \sqrt{\frac{4}{3}-1}$$

$$v_1 = \frac{V}{\sqrt{3}}$$

22a $v = 48\underset{\sim}{i} + 36\underset{\sim}{j}$

$v_{x0} = 48$ and $v_{y0} = 36$

The equations of motion for the particle are:

$\ddot{x} = 0$ and $\ddot{y} = -g$

Integrating both with respect to time, we get:

$\dot{x} = C_1$ and $\dot{y} = C_2 - gt$

Given the initial velocity vector is: $v_{x0} = 48$ and $v_{y0} = 36$

$\dot{x} = 48$ and $\dot{y} = 36 - gt = 36 - 10t$

Integrating again with respect to time, we get:

$x = 48t + C_3$ and $y = 36t - 5t^2 + C_4$

At $t = 0$, $x = 0, y = 0,$ hence

$C_3 = 0$ and $C_4 = 0$

Therefore

$x = 48t$ and $y = 36t - 5t^2$

22b The maximum height of the projectile is reached when $\dot{y} = 0$.

$0 = 36 - 10t$

$t = \dfrac{36}{10} = 3.6$ s

Maximum height, H, found when $t = 3.6$ is substituted into $y = 36t - 5t^2$.

H

$= 36(3.6) - 5(3.6)^2$

$= 64.8$ m

22c To find horizontal range, consider when $y = 0$.

$0 = 36t - 5t^2$

$0 = t(36 - 5t)$

$t = 0, \text{or } t = \frac{36}{5} = 7.2 \text{ s}$

Horizontal range, R, found when $t = 7.2$ is substituted into $x = 48t$

R

$= 48 \times 7.2$

$= 345.6 \text{ m}$

22d When $t = 1.6$ s,

$\dot{x} = 48$

$\dot{y} = 36 - 5t = 36 - 5(1.6) = 28$

Velocity as component after 1.6 s is $48\underset{\sim}{i} + 28\underset{\sim}{j}$.

23a $x = 12t \qquad (1)$

$y = 9t - 5t^2 \qquad (2)$

From (1): $t = \frac{x}{12}$

Substituting for t in (2):

$y = 9\left(\frac{x}{12}\right) - 5\left(\frac{x}{12}\right)^2$

$y = \frac{3}{4}x - \frac{5}{144}x^2$

23b For horizontal range, R, consider when $y = 0$.

$9t - 5t^2 = 0$

$t(9 - 5t) = 0$

$t = 0 \text{ or } t = \frac{9}{5} = 1.8 \text{ s}$

Horizontal range, R, found when $t = 1.8$ is substituted into $x = 12t$.

R

$= 12(1.8)$

$= 21.6 \text{ m}$

For maximum height, H,

$\frac{dy}{dt} = 0$

$9 - 10t = 0$

$t = \frac{9}{10} = 0.9 \text{ s}$

Maximum height, H, found when $t = 0.9$ is substituted into $y = 9t - 5t^2$.

H

$= 9(0.9) - 5(0.9)^2$

$= 4.05 \text{ m}$

23c

$y = \frac{3}{4}x - \frac{5}{144}x^2$

$\frac{dy}{dx} = \frac{3}{4} - \frac{5}{72}x$

At $x = 0$,

$\frac{dy}{dx} = \frac{3}{4}$

Hence the gradient at $x = 0$ is $\frac{3}{4}$.

So, the angle of projection is $\tan^{-1}\frac{3}{4}$.

23d $x = 12t$ and $y = 9t - 5t^2$

$\dot{x} = 12$ and $\dot{y} = 9 - 10t$

At $t = 0$,

$v_x = 12$ and $v_y = 9$

Initial velocity is $12\hat{\imath} + 9\hat{\jmath}$.

Magnitude of initial velocity is $\sqrt{12^2 + 9^2}$ or 15 m/s.

Angle of projection is $\tan^{-1}\dfrac{9}{12}$ or $\tan^{-1}\dfrac{3}{4}$.

23e At $y = 4$,

$4 = 9t - 5t^2$

$5t^2 - 9t + 4 = 0$

$(5t - 4)(t - 1) = 0$

$t = 0.8$ or $t = 1$

Particle is 4 m high when $t = 0.8$ s and $t = 1.0$ s.

So, the horizontal displacements are:

$x_{0.8} = 12(0.8) = 9.6$ m

$x_{1.0} = 12(1) = 12$ m

24a $\alpha = \tan^{-1} 2, V = 5$

so $\tan\alpha = 2$, and

$\cos\alpha = \dfrac{1}{\sqrt{2^2 + 1^2}} = \dfrac{1}{\sqrt{5}}$ and $\sin\alpha = \dfrac{2}{\sqrt{2^2 + 1^2}} = \dfrac{2}{\sqrt{5}}$

Therefore

$\dot{x}_0 = V\cos\alpha$ and $\dot{y}_0 = V\sin\alpha$

$\dot{x}_0 = 5\left(\dfrac{1}{\sqrt{5}}\right)$ and $\dot{y}_0 = 5\left(\dfrac{2}{\sqrt{5}}\right)$

$\dot{x}_0 = \sqrt{5}$ and $\dot{y}_0 = 2\sqrt{5}$

24b $\ddot{x} = 0$

Integrating with respect to time:

$\dot{x} = C_1$

At $t = 0$, $\dot{x}_0 = \sqrt{5}$, hence

$C_1 = \sqrt{5}$

Therefore

$\dot{x} = \sqrt{5}$

Integrating with respect to time:

$x = \sqrt{5}t + C_2$

At $t = 0$, $x = 0$, hence

$C_2 = 0$

Therefore

$x = \sqrt{5}t$ or $x = t\sqrt{5}$

$\ddot{y} = -10$

Integrating with respect to time:

$\dot{y} = -10t + C_3$

At $t = 0$, $\dot{y}_0 = 2\sqrt{5}$, hence $C_3 = 2\sqrt{5}$

Therefore

$\dot{y} = 2\sqrt{5} - 10t$

Integrating with respect to time:

$y = 2\sqrt{5}t - 5t^2 + C_4$

At $t = 0$, $y = 0$, hence $C_4 = 0$

Therefore

$y = 2\sqrt{5}t - 5t^2$

24c For greatest height: $\dot{y} = 0$

$0 = 2\sqrt{5} - 10t$

$t = \dfrac{1}{\sqrt{5}}$ seconds

Greatest height, H, found when $t = \dfrac{1}{\sqrt{5}}$ is substituted into $y = 2\sqrt{5}t - 5t^2$.

H

$= 2\sqrt{5}\left(\dfrac{1}{\sqrt{5}}\right) - 5\left(\dfrac{1}{\sqrt{5}}\right)^2$

$= 2 - 1$

$= 1$ metre

24d For time of flight for the apple, consider when $y = 0$.

$2\sqrt{5}t - 5t^2 = 0$

$t\left(2\sqrt{5} - 5t\right) = 0$

$t = 0$ or $t = \dfrac{2\sqrt{5}}{5}$

So apple was in the air for $\dfrac{2\sqrt{5}}{5}$ or $\dfrac{2}{5}\sqrt{5}$ seconds.

Horizontal distance, R, is found by substituting for t in $x = t\sqrt{5}$.

R

$= \dfrac{2}{5}\sqrt{5} \times \sqrt{5}$

$= 2$ metres

24e $\dot{x} = \sqrt{5}$ and $\dot{y} = 2\sqrt{5} - 10t$

Adam caught the apple when $t = \dfrac{2\sqrt{5}}{5}$ s, so

$\dot{x} = \sqrt{5}$ and $\dot{y} = 2\sqrt{5} - \dfrac{20\sqrt{5}}{5} = -2\sqrt{5}$

v_f

$= \sqrt{(\sqrt{5})^2 + (-2\sqrt{5})^2}$

$= \sqrt{25}$

$= 5$ m/s

Hence the final speed is the same as the initial speed.

The angle at which the projectile was caught is:

$\tan^{-1}\left(-\dfrac{2\sqrt{5}}{\sqrt{5}}\right)$

$= \tan^{-1}(-2)$

$= \tan^{-1} 2$ below the horizontal

Hence, the final direction is the opposite of the initial direction.

24f $x = \sqrt{5}t$ (1)

$y = 2\sqrt{5}t - 5t^2$ (2)

From (1): $t = \dfrac{x}{\sqrt{5}}$

Substituting for t in (2):

$y = 2\sqrt{5}\left(\dfrac{x}{\sqrt{5}}\right) - 5\left(\dfrac{x}{\sqrt{5}}\right)^2$

$y = 2x - x^2$

25a $\dot{x}_0 = 200$ and $\dot{y}_0 = 0$

25b $\ddot{x} = 0$ and $\ddot{y} = -g$

Integrating with respect to time and, as there is assumed to be no air resistance, the constants will be the initial velocities. (Assume upwards is positive)

$\dot{x} = 200$ and $\dot{y} = -gt = -10t$

Integrating with respect to time and with origin at the window, the constants will be zero.

$x = 200t$ and $y = -5t^2$

For Cartesian equation, consider:

$x = 200t$ (1)

$y = -5t^2$ (2)

From (1): $t = \dfrac{x}{200}$

Substituting for t in (2):

$y = -5\left(\dfrac{x}{200}\right)^2$

$y = -\dfrac{x^2}{8000}$

or $y = -\dfrac{1}{8000}x^2$

25c For the horizontal range, $y = -45$.

$-45 = -5t^2$

$t = 3$ s

Horizontal range, R, is found by substituting $t = 3$ into $x = 200t$.

R

$= 200(3)$

$= 600$ m

Horizontal distance that the bullet travels is 600 metres.

25d The bullet hits the ground when $t = 3$.

When $t = 3$,

$\dot{x} = 200$

$\dot{y} = -10(3) = -30$

The angle at which the bullet hits the ground is:

$\tan^{-1}\left(-\frac{30}{200}\right)$

$= \tan^{-1}\left(-\frac{3}{20}\right)$

$= 8.530\ 76\ \ldots°$

$= 8°32'$ below the horizontal.

26a Let $\alpha = 45°$, $\theta = 30°$, $V = 30$

$\ddot{x} = 0$ and $\ddot{y} = -10$

Integrating with respect to time:

$\dot{x} = C_1$ and $\dot{y} = -10t + C_2$

At $t = 0$, $v_x = V\cos\alpha$ and $v_y = V\sin\alpha$, hence

$C_1 = 30\cos 45°$ and $C_2 = 30\sin 45°$

$C_1 = 15\sqrt{2}$ and $C_2 = 15\sqrt{2}$

Therefore

$\dot{x} = 15\sqrt{2}$ and $\dot{y} = -10t + 15\sqrt{2}$

Integrating with respect to time:

$x = 15\sqrt{2}t + C_3$ and $y = -5t^2 + 15\sqrt{2}t + C_4$

At $t = 0$, $x = 0$ and $y = 0$, hence

$C_3 = 0$ and $C_4 = 0$

Therefore

$x = 15\sqrt{2}t$ and $y = 15\sqrt{2}t - 5t^2$

26b $x = 15\sqrt{2}t$ (1)

$y = 15\sqrt{2}t - 5t^2$ (2)

From (1): $t = \dfrac{x}{15\sqrt{2}}$

Substituting for t in (2):

y

$= 15\sqrt{2}\left(\dfrac{x}{15\sqrt{2}}\right) - 5\left(\dfrac{x}{15\sqrt{2}}\right)^2$

$= x - \dfrac{1}{90}x^2$

26c The equation of line of inclined hill is:

y

$= \tan 30°\, x$

$= \frac{1}{\sqrt{3}} x$

P is the point of intersection of inclined hill and the parabolic trajectory:

$\frac{1}{\sqrt{3}} x = x - \frac{x^2}{90}$

$\frac{x^2}{90} - x + \frac{1}{\sqrt{3}} x = 0$

$x\left(\frac{x}{90} - \left(1 - \frac{1}{\sqrt{3}}\right)\right) = 0$

$x_p = 0 \text{ or } x_p = \frac{90(\sqrt{3} - 1)}{\sqrt{3}} = 30(3 - \sqrt{3})$

$y_p = \frac{1}{\sqrt{3}} x_p$ so

$y_p = 0$ or $y_p = 30(\sqrt{3} - 1)$

So, the distance OP is:

OP

$= \sqrt{\left(30(3 - \sqrt{3}) - 0\right)^2 + \left(30(\sqrt{3} - 1) - 0\right)^2}$

$= 30\sqrt{(3 - \sqrt{3})^2 + (\sqrt{3} - 1)^2}$

$= 30\sqrt{9 - 6\sqrt{3} + 3 + 3 - 2\sqrt{3} + 1}$

$= 30\sqrt{16 - 8\sqrt{3}}$

$= 60\sqrt{4 - 2\sqrt{3}}$

27a $x = Vt\cos\alpha \qquad (1)$

$y = Vt\sin\alpha - \frac{1}{2}gt^2 \qquad (2)$

From (1): $t = \frac{x}{V\cos\alpha}$

Substituting for t in (2):

y

$= V\left(\frac{x}{V\cos\alpha}\right)\sin\alpha - \frac{1}{2}g\left(\frac{x}{V\cos\alpha}\right)^2$

$= x\tan\alpha - \frac{\frac{1}{2}gx^2}{V^2}\sec^2\alpha$

$= x\tan\alpha - \frac{1}{2}g\frac{x^2}{V^2}\tan^2\alpha - \frac{1}{2}g\frac{x^2}{V^2} \qquad (\text{using } 1 + \tan^2\alpha = \sec^2\alpha)$

$2V^2y = 2V^2x\tan\alpha - gx^2\tan^2\alpha - gx^2$

$gx^2\tan^2\alpha - 2xV^2\tan\alpha + (2yV^2 + gx^2) = 0$

27b $V = 200, \quad g = 10, \quad x = 3000, \quad y = 500$

$gx^2\tan^2\alpha - 2xV^2\tan\alpha + (2yV^2 + gx^2) = 0$

$9 \times 10^7\tan^2\alpha - 24 \times 10^7\tan\alpha + (4 \times 10^7 + 9 \times 10^7) = 0$

$9\tan^2\alpha - 24\tan\alpha + 13 = 0$

$\tan\alpha$

$= \frac{24 \pm \sqrt{24^2 - 4(9)(13)}}{2(9)}$

$= \frac{24 \pm \sqrt{108}}{18}$

$= \frac{4 \pm \sqrt{3}}{3}$

$\alpha = 62.373\,65\ldots° \text{ or } 37.088\,66\ldots°$

$\alpha = 62°22' \text{ or } 37°5'$

28a $V = 14, \quad R = 10, \quad H_w = 8$

$\ddot{x} = 0$

Integrating with respect to time:

$\dot{x} = C_1$

At $t = 0$, $v_x = V\cos\theta$, hence $C_1 = V\cos\theta$.

Therefore

$\dot{x} = V\cos\theta$

Integrating with respect to time:

$x = Vt\cos\theta + C_2$

At $t = 0$, $x = 0$, hence $C_2 = 0$.

Therefore

x

$= Vt\cos\theta$

$= 14t\cos\theta$

$\ddot{y} = -9.8$

Integrating with respect to time:

$\dot{y} = -9.8t + C_3$

At $t = 0$, $v_y = V\sin\theta$, hence $C_3 = V\sin\theta$.

Therefore

$\dot{y} = -9.8t + V\sin\theta$

or $\dot{y} = V\sin\theta - 9.8t$

Integrating with respect to time:

$y = Vt\sin\theta - 4.9t^2 + C_4$

At $t = 0$, $y = 0$, hence $C_4 = 0$.

Therefore

y

$= Vt\sin\theta - 4.9t^2$

$= 14t\sin\theta - 4.9t^2$

28b $x = 14t\cos\theta \qquad (1)$

$y = 14t\sin\theta - 4.9t^2 \qquad (2)$

From (1): $t = \dfrac{x}{14\cos\theta}$

Substituting for t in (2):

y

$= 14\left(\dfrac{x}{14\cos\theta}\right)\sin\theta - 4.9\left(\dfrac{x}{14\cos\theta}\right)^2$

$= x\tan\theta - \dfrac{4.9}{196}x^2\sec^2\theta$

$= x\tan\theta - \dfrac{1}{40}x^2(\tan^2\theta + 1)$

$= mx - \left(\dfrac{m^2+1}{40}\right)x^2 \qquad$ where $m = \tan\theta$

28c At $x = 10, \quad y = h$

h

$= 10m - \frac{100}{40}m^2 - \frac{100}{40}$

$= 10m - 2.5m^2 - 2.5$

$2.5m^2 - 10m + (2.5 + h) = 0$

m

$= \frac{10 \pm \sqrt{100 - 4(2.5)(2.5 + h)}}{5}$

$= \frac{10 \pm \sqrt{75 - 10h}}{5}$

$= \frac{10 \pm 5\sqrt{3 - \frac{10}{25}h}}{5}$

$= 2 \pm \sqrt{3 - 0.4h}$

Rearranging and squaring both sides:

$(m - 2)^2 = 3 - 0.4h$

h

$= \frac{3 - (m - 2)^2}{0.4}$

$= 7.5 - \frac{(m - 2)^2}{0.4}$

For maximum h, $\quad \frac{dh}{dm} = 0.$

$-5(m - 2) = 0$

$m = 2$

When $m = 2, h = 7.5$ m

28d $y_l = 3.9, \quad y_h = 5.9, \quad x = 10$

$y = mx - \left(\frac{m^2+1}{40}\right)x^2$ and,

$m = = \frac{10 \pm \sqrt{75 - 10h}}{5}$

For $y_l = 3.9$, this becomes

m_l

$= \frac{10 \pm \sqrt{75 - 39}}{5}$

$= \frac{10 \pm \sqrt{36}}{5}$

$m_l = \frac{4}{5}$ or $\frac{16}{5}$

or

$m_l = 0.8$ or 3.2

When $y = y_h = 5.9$,

$m_h = \frac{10 \pm \sqrt{75 - 59}}{5}$

$m_h = \frac{10 \pm \sqrt{16}}{5}$

$m_h = \frac{6}{5}$ or $\frac{14}{5}$

or

$m_h = 1.2$ or 2.8

Hence, the paintball will pass through the hole for $0.8 \le m \le 1.2$ or

$2.8 \le m \le 3.2$.

29a $\ddot{x} = -k\dot{x}$

$\ddot{x} = \dfrac{dv}{dt} = -kv$

$\displaystyle\int dt = -\int \frac{1}{kv}\, dv$

$t = -\dfrac{1}{k}\ln kv + C$

At $t = 0$, $v = u\cos\alpha$, hence

$C = \dfrac{1}{k}\ln(ku\cos\alpha)$

Therefore

t

$= -\dfrac{1}{k}\ln kv + \dfrac{1}{k}\ln(ku\cos\alpha)$

$= \dfrac{1}{k}\ln\dfrac{u\cos\alpha}{v}$

$kt = \ln\dfrac{u\cos\alpha}{v}$

$e^{kt} = \dfrac{u\cos\alpha}{v}$

$e^{-kt} = \dfrac{v}{u\cos\alpha}$

$v = u\cos\alpha\, e^{-kt}$

Therefore

$\dot{x} = ue^{-kt}\cos\alpha$

29b

$$\dot{y} = \frac{1}{k}\left((ku\sin\alpha + g)e^{-kt} - g\right)$$

Differentiating with respect to time:

$\ddot{y}$

$= -k\left(\frac{1}{k}(ku\sin\alpha + g)e^{-kt}\right)$

$= -k\left(\frac{1}{k}\left((ku\sin\alpha + g)e^{-kt} - g + g\right)\right)$

$= -k\left(\frac{1}{k}\left((ku\sin\alpha + g)e^{-kt} - g\right) + \frac{g}{k}\right)$

$= -k\left(\frac{1}{k}\left((ku\sin\alpha + g)e^{-kt} - g\right)\right) - g$

$= -k(\dot{y}) - g$

Therefore

$\ddot{y} = -k\dot{y} - g$

29c When the particle reaches its maximum height, the vertical velocity is zero.

$\dot{y} = 0$

$\frac{1}{k}\left((ku\sin\alpha + g)e^{-kt} - g\right) = 0$

$(ku\sin\alpha + g)e^{-kt} = g$

$e^{-kt} = \frac{g}{ku\sin\alpha + g}$

$-kt = \ln\left(\frac{g}{ku\sin\alpha + g}\right)$

t

$= -\frac{1}{k}\ln\left(\frac{g}{ku\sin\alpha + g}\right)$

$= \frac{1}{k}\ln\left(\frac{ku\sin\alpha + g}{g}\right)$

$= \frac{1}{k}\ln\left(\frac{ku\sin\alpha}{g} + \frac{g}{g}\right)$

$= \frac{1}{k}\ln\left(\frac{ku}{g}\sin\alpha + 1\right)$

or $t = \frac{1}{k}\ln\left(1 + \frac{ku}{g}\sin\alpha\right)$

29d The limiting value of horizontal displacement is when the time is very large.

$\dot{x} = ue^{-kt}\cos\alpha$

$\dfrac{dx}{dt} = ue^{-kt}\cos\alpha$

$\displaystyle\int dx = u\cos\alpha \int e^{-kt}dt$

$x = -\dfrac{u}{k}\cos\alpha\, e^{-kt} + C$

When $t = 0, x = 0$, hence

$0 = -\dfrac{u}{k}\cos\alpha + C$

$C = \dfrac{u}{k}\cos\alpha$

Therefore

$x = -\dfrac{u}{k}\cos\alpha\, e^{-kt} + \dfrac{u}{k}\cos\alpha$

$\lim\limits_{t\to\infty} x = \lim\limits_{t\to\infty}\left(-\dfrac{u}{k}\cos\alpha\, e^{-kt} + \dfrac{u}{k}\cos\alpha\right)$

As $t \to \infty, e^{-kt} \to 0$, so

$\lim\limits_{t\to\infty} x$

$= 0 + \dfrac{u}{k}\cos\alpha$

$= \dfrac{u}{k}\cos\alpha$

Hence, the limiting value of horizontal displacement is $\dfrac{u}{k}\cos\alpha$.