Solutions to Exercise 1A

1a Each term is 150 more than the previous term. The sequence is as follows:
850, 1000, 1150, 1300, 1450, 1600, 1750, 1900, 2050, 2200, 2350, 2500, 2650, 2800

1b Looking at the sequence from part (a), $T_9 = 2050$.
Alex's stamp collection first exceeded 2000 stamps after 9 months.

2a Each term is 10 more than the previous term. The sequence is as follows:
6, 16, 26, 36, 46, 56, 66

2b Each term is double the previous term. The sequence is as follows:
3, 6, 12, 24, 48, 96, 192

2c Each term is 4 less than the previous term. The sequence is as follows:
38, 34, 30, 26, 22, 18, 14

2d Each term is half the previous term. The sequence is as follows:
$24, 12, 6, 3, 1\frac{1}{2}, \frac{3}{4}, \frac{3}{8}$

2e Each term is the previous term multiplied by -1. The sequence is as follows:
$-1, 1, -1, 1, -1, 1, -1$

2f Each term is squared. The sequence is as follows:
1, 4, 9, 16, 25, 36, 49

2g Each term is of the form $T_n = \frac{n}{n+1}$. The sequence is as follows:
$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{7}{8}$

2h Each term is the previous term divided by -2. The sequence is as follows:
$16, -8, 4, -2, 1, -\frac{1}{2}, \frac{1}{4}$

3a $T_n = 5n - 2$
$T_1 = 5 - 2 = 3$
$T_2 = 10 - 2 = 8$
$T_3 = 15 - 2 = 13$
$T_4 = 20 - 2 = 18$

3b $T_n = 5^n$
$T_1 = 5^1 = 5$
$T_2 = 5^2 = 25$
$T_3 = 5^3 = 125$
$T_4 = 5^4 = 625$

3c $T_n = 6 - 2n$
$T_1 = 6 - 2 = 4$
$T_2 = 6 - 4 = 2$
$T_3 = 6 - 6 = 0$
$T_4 = 6 - 8 = -2$

3d $T_n = 7 \times 10^n$
$T_1 = 7 \times 10 = 70$
$T_2 = 7 \times 100 = 700$
$T_3 = 7 \times 1\,000 = 7000$
$T_4 = 7 \times 10\,000 = 70\,000$

3e $T_n = n^3$
$T_1 = 1^3 = 1$
$T_2 = 2^3 = 8$
$T_3 = 3^3 = 27$
$T_4 = 4^3 = 64$

3f $T_n = n(n + 1)$
$T_1 = 1(2) = 2$
$T_2 = 2(3) = 6$
$T_3 = 3(4) = 12$
$T_4 = 4(5) = 20$

3g $T_n = (-1)^n$
$T_1 = (-1)^1 = -1$
$T_2 = (-1)^2 = 1$
$T_3 = (-1)^3 = -1$
$T_4 = (-1)^4 = 1$

3h $T_n = (-3)^n$
$T_1 = (-3)^1 = -3$
$T_2 = (-3)^2 = 9$
$T_3 = (-3)^3 = -27$
$T_4 = (-3)^4 = 81$

4a Start with 11 as the first term, and add 50 to find the next term:
$11, 61, 111, 161$

4b Start with 15 as the first term, then subtract 3 to find the next term:
$15, 12, 9, 6$

4c Start with 5 as the first term, then double it to find the next term:
$5, 10, 20, 40$

4d Start with -100 as the first term, then divide by 5 to find the next term:
$-100, -20, -4, -\frac{4}{5}$

5 Each term is 5 more than the previous term. The sequence is as follows:
$7, 12, 17, 22, 27, 32, 37, 42, 47, 52, 57, 62$

5a Counting the number of terms:
$\mathbf{7}, \mathbf{12}, \mathbf{17}, \mathbf{22}, \mathbf{27}, 32, 37, 42, 47, 52, 57, 62$
5 terms are less than 30.

5b Counting the number of terms:
$7, 12, 17, \mathbf{22}, \mathbf{27}, \mathbf{32}, \mathbf{37}, 42, 47, 52, 57, 62$
4 terms lie between 20 and 40.

5c $7, 12, 17, 22, 27, 32, 37, 42, 47, \mathbf{52}, 57, 62$
The tenth term is 52.

5d 37 is the 7th term.

5e This sequence is in the form: $T_n = 5n + 2$
Put $T_n = 87$
Then $87 = 5n + 2$
$85 = 5n$
$n = 17$
Hence 87 is the 17th term.

5f This sequence is in the form: $T_n = 5n + 2$
Put $T_n = 201$
Then $201 = 5n + 2$
$199 = 5n$
$n = 39\frac{4}{5}$
Hence 201 is not a term of this sequence.

6 Each term is double the previous term:
$\frac{3}{4}, 1\frac{1}{2}, 3, 6, 12, 24, 48, 96, 192, 384, 768, 1536$

6a Counting the number of terms:
$\mathbf{\frac{3}{4}}, \mathbf{1\frac{1}{2}}, \mathbf{3}, \mathbf{6}, \mathbf{12}, \mathbf{24}, \mathbf{48}, \mathbf{96}, \mathbf{192}, \mathbf{384}, 768, 1536$
10 terms are less than 400

6b Counting the number of terms:
$\frac{3}{4}, 1\frac{1}{2}, 3, 6, 12, \mathbf{24}, \mathbf{48}, \mathbf{96}, 192, 384, 768, 1536$
3 terms are between 20 and 100.

6c 384 is the 10th term.

6d 192 is the 9th term.

6e Looking at the sequence:
Yes, $T_8 = 96$

6f Looking at the sequence:
No, 100 is not a term in the sequence.

7a $T_n = 12 + n$
$T_1 = 12 + 1 = 13$
$T_2 = 12 + 2 = 14$
$T_3 = 12 + 3 = 15$
$T_4 = 12 + 4 = 16$
$T_5 = 12 + 5 = 17$
The first term is 12, and every term after that is 1 more than the previous one.

7b $T_n = 4 + 5n$
$T_1 = 4 + 5 = 9$
$T_2 = 4 + 10 = 14$
$T_3 = 4 + 15 = 19$
$T_4 = 4 + 20 = 24$
$T_5 = 4 + 25 = 29$
The first term is 9, and every term after that is 5 more than the previous one.

7c $T_n = 15 - 5n$
$T_1 = 15 - 5 = 10$
$T_2 = 15 - 10 = 5$
$T_3 = 15 - 15 = 0$
$T_4 = 15 - 20 = -5$
$T_5 = 15 - 25 = -10$
The first term is 10, and every term after that is 5 less than the previous one.

7d $T_n = 3 \times 2^n$
$T_1 = 3 \times 2 = 6$
$T_2 = 3 \times 4 = 12$
$T_3 = 3 \times 8 = 24$
$T_4 = 3 \times 16 = 48$

$T_5 = 3 \times 32 = 96$
The first term is 6, and every term after that is double the previous one.

7e $T_n = 7 \times (-1)^n$
$T_1 = 7 \times (-1)^1 = -7$
$T_2 = 7 \times (-1)^2 = 7$
$T_3 = 7 \times (-1)^3 = -7$
$T_4 = 7 \times (-1)^4 = 7$
$T_5 = 7 \times (-1)^5 = -7$
The first term is -7, and every term after that is the previous one multiplied by -1.

7f $T_n = 80 \times \left(\frac{1}{2}\right)^n$
$T_1 = 80 \times \left(\frac{1}{2}\right)^1 = 40$
$T_2 = 80 \times \left(\frac{1}{2}\right)^2 = 20$
$T_3 = 80 \times \left(\frac{1}{2}\right)^3 = 10$
$T_4 = 80 \times \left(\frac{1}{2}\right)^4 = 5$
$T_n = 80 \times \left(\frac{1}{2}\right)^5 = 2\frac{1}{2}$
The first term is 40, and every term after that is half the previous term.

8a $T_n = 3n + 1$
Put $T_n = 40$
Then $40 = 3n + 1$
$39 = 3n$
$n = 13$
Hence 40 is the 13th term.

8b $T_n = 3n + 1$
Put $T_n = 30$
Then $30 = 3n + 1$
$29 = 3n$
$n = 9.666666\ ....$
Hence this is not a term in the sequence as 9.666666 .... is not an integer

8c $T_n = 3n + 1$

Put $T_n = 100$

Then $100 = 3n + 1$

$99 = 3n$

$n = 33$

Hence this is a term in the sequence as 33 is an integer.

$T_n = 3n + 1$

Put $T_n = 200$

Then $200 = 3n + 1$

$119 = 3n$

$n = 39.666666\ ....$

Hence this is not a term in the sequence as 39.666666 .... is not an integer.

$T_n = 3n + 1$

Put $T_n = 1000$

Then $1000 = 3n + 1$

$999 = 3n$

$n = 333$

Hence this is a term in the sequence as 333 is an integer.

9a $T_n = 10n - 6$

Put $T_n = 44$

Then $44 = 10n - 6$

$50 = 10n$

$n = 5$

Put $T_n = 200$

Then $200 = 10n - 6$

$206 = 10n$

$n = 20\frac{6}{10}$

Put $T_n = 306$

Then $306 = 10n - 6$

$312 = 10n$

$n = 31\frac{2}{10}$

Hence 200 and 306 are not terms in this sequence. 44 is the 5th term in the sequence.

9b $T_n = 2n^2$

Put $T_n = 40$

Then $40 = 2n^2$

$20 = n^2$

$n = 4.47214\ ...$

Put $T_n = 72$
Then $72 = 2n^2$
$36 = n^2$
$n = 6$
Put $T_n = 200$
Then $200 = 2n^2$
$100 = n^2$
$n = 10$
Hence 40 is not a term in this sequence. 72 is the 6th term in this sequence, and 200 is the 10th term in this sequence.

9c $T_n = 2^n$
Put $T_n = 8$
Then $8 = 2^n$
$n = \log_2 8$
$n = 3$
Put $T_n = 96$
Then $96 = 2^n$
$n = \log_2 96$
$n = 6.58496 \ldots$
Put $T_n = 128$
Then $128 = 2^n$
$n = \log_2 128$
$n = 7$
Hence 96 is not a term in this sequence. 8 is the 3rd term in the sequence and 128 is the 7th term in the sequence.

10a $T_n = 10n + 4$
Put $T_n < 100$
Then $10n + 4 < 100$
$10n < 96$
$n < 9.6$
Hence there are 9 terms less than 100.

10b $T_n = 10n + 4$
Put $T_n > 56$
Then $10n + 4 > 56$
$10n > 52$
$n > 5.2$
Hence $T_6 = 64$ is the first term greater then 56.

11a $T_n = 2n - 5$
Put $T_n < 100$
Then $2n - 5 < 100$
$2n < 105$
$n < 52.5$
Hence there are 52 terms less than 100.

11b $T_n = 7n - 44$
Put $T_n > 100$
Then $7n - 44 > 100$
$7n > 144$
$n > 20.57$
Hence the first term greater than 100 is $T_{21} = 103$.

12a $T_1 = 5$
$T_2 = T_1 + 12 = 5 + 12 = 17$
$T_3 = T_2 + 12 = 17 + 12 = 29$
$T_4 = T_3 + 12 = 19 + 12 = 41$

12b $T_1 = 12$
$T_2 = T_1 - 10 = 12 - 10 = 2$
$T_3 = T_2 - 10 = 2 - 10 = -8$
$T_4 = T_3 - 10 = -8 - 10 = -18$

12c $T_1 = 20$
$T_2 = \frac{1}{2}T_1 = \frac{1}{2} \times 20 = 10$
$T_3 = \frac{1}{2}T_2 = \frac{1}{2} \times 10 = 5$
$T_4 = \frac{1}{2}T_3 = \frac{1}{2} \times 5 = 2\frac{1}{2}$

12d $T_1 = 1$
$T_2 = -T_1 = -1 \times 1 = -1$
$T_3 = -T_2 = -1 \times -1 = 1$
$T_4 = -T_3 = -1 \times 1 = -1$

13a This is an AP as all terms have the same common difference, so
$d = T_2 - T_1 = 21 - 16 = 5$
$T_n = T_{n-1} + d = T_{n-1} + 5$

13b This is a GP as all terms have the same common ratio, so
$r = \frac{T_2}{T_1} = \frac{14}{7} = 2$
$T_n = rT_{n-1} = 2T_{n-1}$

13c This is an AP as all terms have the same common difference, so
$d = T_2 - T_1 = 2 - 9 = -7$
$T_n = T_{n-1} + d = T_{n-1} - 7$

13d This is a GP as all terms have the same common ratio, so
$r = \frac{T_2}{T_1} = \frac{4}{-4} = -1$
$T_n = rT_{n-1} = -T_{n-1}$

14a $T_1 = \sin 90° = 1$
$T_2 = \sin 180° = 0$
$T_3 = \sin 270° = -1$
$T_4 = \sin 360° = 0$

Terms are zero where n is even.

14b $T_1 = \cos 90° = 0$
$T_2 = \cos 180° = -1$
$T_3 = \cos 270° = 0$
$T_4 = \cos 360° = 1$

Terms are zero where n is odd.

14c $T_1 = \cos 180° = -1$
$T_2 = \cos 360° = 1$
$T_3 = \cos 540° = -1$
$T_4 = \cos 720° = 1$

No terms are zero.

14d $T_1 = \sin 180° = 0$
$T_2 = \sin 360° = 0$
$T_3 = \sin 540° = 0$
$T_4 = \sin 720° = 0$

All terms are zero.

15a $T_n = n^2 - 3n$
Put $T_n = 28$
Then $n^2 - 3n = 28$
$n^2 - 3n - 28 = 0$
$(n - 7)(n + 4) = 0$
$n = -4 \text{ or } 7$

But $n \geq 1$ so $T_7 = 28$

Put $T_n = 70$
Then $n^2 - 3n = 70$
$n^2 - 3n - 70 = 0$
$(n - 10)(n + 7) = 0$
$n = 10 \text{ or } -7$

But $n \geq 1$ so $T_{10} = 70$

15b $T_n = n^2 - 3n$
Put $T_n < 18$
Then $n^2 - 3n < 18$
$n^2 - 3n - 18 < 0$
$(n - 6)(n + 3) < 0$
$-3 < n < 6$
Now n is an integer greater than or equal to 1.
$1 \leq n \leq 5$
So there are 5 terms less than 18.

16a $T_n = \frac{3}{32} \times 2^n$
Put $T_n = 1\frac{1}{2}$
Then $\frac{3}{32} \times 2^n = 1\frac{1}{2}$
$2^n = 16$
$n = 4$

So $T_4 = 1\frac{1}{2}$

Put $\quad T_n = 96$

Then $\quad \frac{3}{32} \times 2^n = 96$

$2^n = 1024$

$n = 10$

So $T_{10} = 96$

16b $\quad T_n = \frac{3}{32} \times 2^n$

Put $\quad T_n > 10$

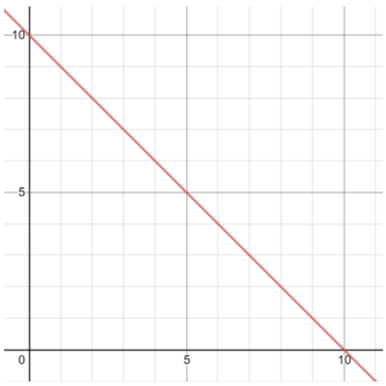
Then $\quad \frac{3}{32} \times 2^n > 10$

$2^n > 106\frac{2}{6}$

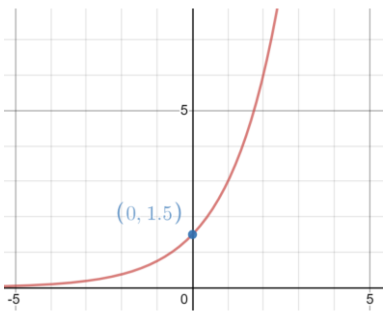
By trial and error the lowest integer solution is $n = 7$.

So $T_7 = 12$ is the first term greater than 10

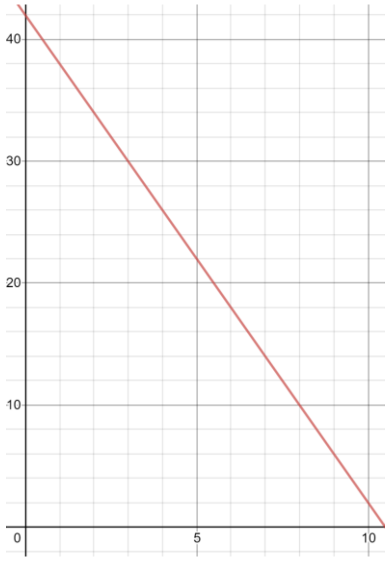
17a $\quad y = 10x - 4$



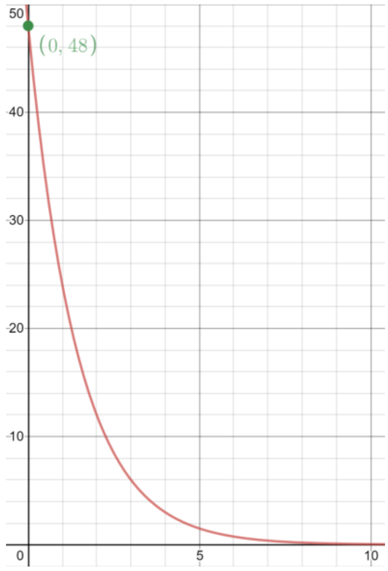
17b $\quad y = 2^{x-1} \times 3$



17c $y = 42 - 4x$

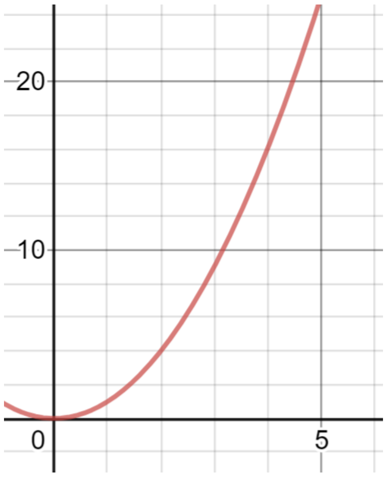


17d $y = 48 \times 2^{-x}$



17e Here $T_n = (-1)^n$, but there is no curve and no real-valued function.

17f $y = x^2$



17g $y = \frac{x}{x-1}$

17h Here $T_n = (-2)^{5-n}$, but there is no curve and no real-valued function.

18a $T_1 + T_2 + T_3 + T_4$

$= \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \frac{1}{5}$

$= 1 - \frac{1}{5}$

$= \frac{4}{5}$

$T_1 + T_2 + \cdots + T_n$

$= \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{n} - \frac{1}{n+1}$

$= 1 - \frac{1}{n+1}$

$$= \frac{n+1-1}{n+1}$$

$$= \frac{n}{n+1}$$

18b $T_n = \frac{1}{n} - \frac{1}{n+1}$

$$= \frac{n+1-n}{n(n+1)}$$

$$= \frac{1}{n(n+1)}$$

Put $T_n = \frac{1}{30}$

Then $\frac{1}{n(n+1)} = \frac{1}{30}$

$30 = n^2 + n$

$n^2 + n - 30 = 0$

$(n+6)(n-5) = 0$

$n = -6$ or $n = 5$

But $n \geq 1$ so $T_5 = \frac{1}{30}$

19a $T_n = \frac{n-1}{n}$

Put $T_n = 0.9$

Then $\frac{n-1}{n} = 0.9$

$n - 1 = 0.9n$

$0.1n = 1$

$n = 10$

So $T_{10} = 0.9$

Put $T_n = 0.99$

Then $\frac{n-1}{n} = 0.99$

$n - 1 = 0.99n$

$0.01n = 1$

$n = 100$

So $T_{100} = 0.99$

19b $T_{n+1} : T_n$

$$= \frac{n+1-1}{n+1} : \frac{n-1}{n}$$

$$= \frac{n}{n+1} : \frac{n-1}{n}$$

So:

$$\frac{T_n}{T_{n+1}} = \frac{\frac{n-1}{n}}{\frac{n}{n+1}}$$

$$= \frac{n-1}{n} \times \frac{n+1}{n}$$

$$= \frac{n^2-1}{n^2}$$

$$\frac{T_n}{T_{n+1}} + \frac{1}{n^2} = \frac{n^2-1}{n^2} + \frac{1}{n^2}$$

$$= \frac{n^2-1+1}{n^2}$$

$$= \frac{n^2}{n^2}$$

$$= 1$$

19c $T_2 \times T_3 \times \ldots \times T_n = \frac{1}{2} \times \frac{2}{3} \times \ldots \times \frac{n-2}{n-1} \times \frac{n-1}{n} = \frac{1}{n}$

19d $T_{n+1} - T_{n-1}$

$$= \frac{n+1-1}{n+1} - \frac{n-1-1}{n-1}$$

$$= \frac{n}{n+1} - \frac{n-2}{n-1}$$

$$= \frac{n(n-1)}{(n+1)(n-1)} - \frac{(n-2)(n+1)}{(n+1)(n-1)}$$

$$= \frac{n(n-1)-(n-2)(n+1)}{(n+1)(n-1)}$$

$$= \frac{n^2-n-(n^2-n-2)}{(n+1)(n-1)}$$

$$= \frac{2}{n^2-1}$$

20a $F_1 = 1$
$F_2 = 1$
$F_3 = F_1 + F_2 = 1 + 1 = 2$
$F_4 = F_3 + F_2 = 2 + 1 = 3$
$F_5 = F_4 + F_3 = 3 + 2 = 5$
$F_6 = 8$
$F_7 = 13$
$F_8 = 21$
$F_9 = 34$
$F_{10} = 55$
$F_{11} = 89$
$F_{12} = 144$

20b $L_1 = 1$
$L_2 = 3$
$L_3 = L_1 + L_2 = 1 + 3 = 4$
$L_4 = L_3 + L_2 = 3 + 4 = 7$
$L_5 = L_4 + L_3 = 4 + 7 = 11$
$L_6 = 18$
$L_7 = 29$
$L_8 = 47$
$L_9 = 76$
$L_{10} = 123$
$L_{11} = 199$
$L_{12} = 322$

20c The sum of two odd integers is even, and the sum of an even and an odd integer is odd.

20d The first is $2, 4, 6, 10, 16, \ldots$, which is $2F_{n+1}$.

The second is $0, 2, 2, 4, 6, \ldots$, which is $2F_{n-1}$.

21 Investigation question – answers will vary.

22a The 20th number is 10, and -20 is the 41st number on the list.

22b Start by writing down the successive diagonals $1, 2, \frac{1}{2}, 3, \frac{2}{2}, \frac{1}{3}, 4, \frac{3}{2}, \frac{2}{3}, 14, \ldots$

Then remove every fraction that can be cancelled because it has previously been listed.

22c The number x is not on the list because it differs from the nth number on the list at the nth decimal place.

Solutions to Exercise 1B

1a Each term is 5 more than the previous term. The sequence is as follows:
$3, 8, 13, 18, 23, 28$

1b Each term is 10 less than the previous term. The sequence is as follows:
$35, 25, 15, 5, -5, -15$

1c Each term is $1\frac{1}{2}$ more than the previous term. The sequence is as follows:
$4\frac{1}{2}, 6, 7\frac{1}{2}, 9, 10\frac{1}{2}, 12$

2a Start at 3 and add 2. The sequence is as follows:
$3, 5, 7, 9$

2b Start at 7 and subtract 4. The sequence is as follows:
$7, 3, -1, -5$

2c Start at 30 and subtract 11. The sequence is as follows:
$30, 19, 8, -3$

2d Start at -9 and add 4. The sequence is as follows:
$-9, -5, -1, 3$

2e Start at $3\frac{1}{2}$ and subtract 2. The sequence is as follows:
$3\frac{1}{2}, 1\frac{1}{2}, -\frac{1}{2}, -2\frac{1}{2}$

2f Start at 0.9 and add 0.7. The sequence is as follows:
$0.9, 1.6, 2.3, 3.0$

3a $T_2 - T_1 = 7 - 3 = 4$
$T_3 - T_2 = 11 - 7 = 4$
Hence this sequence is an AP with $a = 3$ and $d = 4$.

3b $T_2 - T_1 = 7 - 11 = -4$
$T_3 - T_2 = 3 - 7 = -4$
Hence this sequence is an AP with $a = 11$ and $d = -4$.

3c $T_2 - T_1 = 34 - 23 = 11$
$T_3 - T_2 = 45 - 34 = 11$
Hence this sequence is an AP with $a = 23$ and $d = 11$.

3d $T_2 - T_1 = (-7) - (-12) = 5$
$T_3 - T_2 = (-7) - (-2) = 5$
Hence this sequence is an AP with $a = -12$ and $d = 5$.

3e $T_2 - T_1 = 20 - (-40) = 60$
$T_3 - T_2 = (-10) - 20 = -30$
Hence this sequence is not an AP, as the differences are not all the same.

3f $T_2 - T_1 = 11 - 1 = 10$
$T_3 - T_2 = 111 - 11 = 100$
Hence this sequence is not an AP, as the differences are not all the same.

3g $T_2 - T_1 = (-2) - 8 = -10$
$T_3 - T_2 = (-12) - (-2) = -10$
Hence this sequence is an AP with $a = 8$ and $d = -10$.

3h $T_2 - T_1 = 0 - (-17) = 17$
$T_3 - T_2 = 17 - 0 = 17$
Hence this sequence is an AP with $a = -17$ and $d = 17$.

3i $T_2 - T_1 = 7\frac{1}{2} - 10 = -2\frac{1}{2}$
$T_3 - T_2 = 5 - 7\frac{1}{2} = -2\frac{1}{2}$
Hence this sequence is an AP with $a = 10$ and $d = -2\frac{1}{2}$.

4a $a = 7$ and $d = 6$
$T_n = 7 + 6(n - 1)$

$= 7 + 6n - 6$
$= 1 + 6n$
$T_{11} = 1 + 6 \times 11 = 67$

4b $a = 15$ and $d = -7$
$T_n = 15 - 7(n - 1)$
$= 15 - 7n + 7$
$= 22 - 7n$
$T_{11} = 22 - 7 \times 11 = -55$

4c $a = 10\frac{1}{2}$ and $d = 4$
$T_n = 10\frac{1}{2} + 4(n - 1)$
$= 10\frac{1}{2} + 4n - 4$
$= 6\frac{1}{2} + 4n$
$T_{11} = 6\frac{1}{2} + 4 \times 11 = 50\frac{1}{2}$

5a $a = 1$ and $d = 4$
$T_n = 1 + 4(n - 1)$
$= 1 + 4n - 4$
$= -3 + 4n$

5b $a = 100$ and $d = -7$
$T_n = 100 - 7(n - 1)$
$= 100 - 7n + 7$
$= 107 - 7n$

5c $a = -13$ and $d = 6$
$T_n = -13 + 6(n - 1)$
$= -13 + 6n - 6$
$= -19 + 6n$

6a $T_2 - T_1 = 16 - 6 = 10$
$T_3 - T_2 = 26 - 16 = 10$
Hence this sequence is an AP with $a = 6$ and $d = 10$.

6b $T_9 = 6 + 10(9 - 1)$
$T_9 = 86$
$T_{21} = 6 + 10(21 - 1) = 206$
$T_{100} = 6 + 10(100 - 1) = 996$

6c $T_n = 6 + 10(n - 1)$
$= 6 + 10n - 10$
$= 10n - 4$

7a $T_2 - T_1 = 11 - 8 = 3$
$T_3 - T_2 = 14 - 11 = 3$
Hence this sequence is an AP with $a = 8$ and $d = 3$.
$T_n = 8 + 3(n - 1)$
$= 8 + 3n - 3$
$= 5 + 3n$

7b $T_2 - T_1 = 15 - 21 = -6$
$T_3 - T_2 = 9 - 15 = -6$
Hence this sequence is an AP with $a = 21$ and $d = -6$.
$T_n = 21 - 6(n - 1)$
$= 21 - 6n + 6$
$= 27 - 6n$

7c $T_2 - T_1 = 4 - 8 = -4$
$T_3 - T_2 = 2 - 4 = -2$
Hence this sequence is not an AP, as the differences are not all the same.

7d $T_2 - T_1 = 1 - (-3) = 4$
$T_3 - T_2 = 5 - 1 = 4$
Hence this sequence is an AP with $a = -3$ and $d = 4$.
$T_n = -3 + 4(n - 1)$
$= -3 + 4n - 4$
$= 4n - 7$

7e $T_2 - T_1 = 3 - 1\frac{3}{4} = 1\frac{1}{4}$

$T_3 - T_2 = 4\frac{1}{4} - 3 = 1\frac{1}{4}$

Hence this sequence is an AP with $a = 1\frac{3}{4}$ and $d = 1\frac{1}{4}$.

$$\begin{aligned} T_n &= 1\tfrac{3}{4} + 1\tfrac{1}{4}(n-1) \\ &= 1\tfrac{3}{4} + 1\tfrac{1}{4}n - 1\tfrac{1}{4} \\ &= 1\tfrac{1}{4}n + \tfrac{1}{2} \\ &= \tfrac{1}{4}(2+5n) \end{aligned}$$

7f $T_2 - T_1 = -5 - 12 = -17$

$T_3 - T_2 = -22 - (-5) = -17$

Hence this sequence is an AP with $a = 12$ and $d = -17$.

$$\begin{aligned} T_n &= 12 - 17(n-1) \\ &= 12 - 17n + 17 \\ &= 29 - 17n \end{aligned}$$

7g $T_2 - T_1 = 2\sqrt{2} - \sqrt{2} = \sqrt{2}$

$T_3 - T_2 = 3\sqrt{2} - 2\sqrt{2} = \sqrt{2}$

Hence this sequence is an AP with $a = \sqrt{2}$ and $d = \sqrt{2}$.

$$\begin{aligned} T_n &= \sqrt{2} + \sqrt{2}(n-1) \\ &= \sqrt{2} + n\sqrt{2} - \sqrt{2} \\ &= n\sqrt{2} \end{aligned}$$

7h $T_2 - T_1 = 4 - 1 = 3$

$T_3 - T_2 = 9 - 4 = 5$

$T_4 - T_3 = 16 - 9 = 7$

Hence this sequence is not an AP, as the differences are not all the same.

7i $T_2 - T_1 = 1 — 2\frac{1}{2} = 3\frac{1}{2}$

$T_3 - T_2 = 4\frac{1}{2} - 1 = 3\frac{1}{2}$

Hence this sequence is an AP with $a = -2\frac{1}{2}$ and $d = 3\frac{1}{2}$.

$$\begin{aligned} T_n &= -2\tfrac{1}{2} + \tfrac{7}{2}(n-1) \\ &= -2\tfrac{1}{2} + \tfrac{7}{2}n - 3\tfrac{1}{2} \\ &= \tfrac{7}{2}n - 6 \end{aligned}$$

8a $T_2 - T_1 = 160 - 165 = -5$
$T_3 - T_2 = 160 - 165 = -5$
Hence this sequence is an AP with $a = 165$ and $d = -5$.
$$\begin{aligned} T_n &= 165 - 5(n-1) \\ &= 165 - 5n + 5 \\ &= 170 - 5n \end{aligned}$$

8b Put $T_n = 40$
Then: $40 = 165 - 5(n-1)$
$40 = 165 - 5n + 5$
$40 = 170 - 5n$
$130 = 5n$
$n = 26$
There are 26 terms in the sequence.

8c Put $T_n < 0$
Then: $0 > 165 - 5(n-1)$
$0 > 165 - 5n + 5$
$0 > 170 - 5n$
$5n > 170$
$n > 34$
The first negative term is $T_{35} = -5$

9a $T_2 - T_1 = 17 - 20 = -3$
$T_3 - T_2 = 14 - 17 = -3$
Hence this sequence is an AP with $a = 20$ and $d = -3$.
Put $T_n < 0$
Then: $0 > 20 - 3(n-1)$
$0 > 20 - 3n + 3$
$0 > 23 - 3n$
$3n > 23$
$n > 7.66\ldots$
Hence the first negative term is $T_8 = -1$

9b $T_2 - T_1 = 79 - 82 = -3$
$T_3 - T_2 = 76 - 79 = -3$
Hence this sequence is an AP with $a = 82$ and $d = -3$.
Put $T_n < 0$
Then: $0 > 82 - 3(n-1)$

$0 > 82 - 3n + 3$
$0 > 85 - 3n$
$3n > 85$
$n > 28.33\ldots$
Hence the first negative term is $T_{29} = -2$

9c $T_2 - T_1 = 24 - 24\frac{1}{2} = -\frac{1}{2}$
$T_3 - T_2 = 23\frac{1}{2} - 24 = -\frac{1}{2}$
Hence this sequence is an AP with $a = 24\frac{1}{2}$ and $d = -\frac{1}{2}$.
Put $T_n < 0$
Then: $0 > 24\frac{1}{2} - \frac{1}{2}(n-1)$
$0 > 24\frac{1}{2} - \frac{1}{2}n + \frac{1}{2}$
$0 > 25 - \frac{1}{2}n$
$\frac{1}{2}n > 25$
$n > 50$
Hence the first negative term is $T_{51} = -\frac{1}{2}$

10a $T_2 - T_1 = 12 - 10 = 2$
$T_3 - T_2 = 14 - 12 = 2$
Hence this sequence is an AP with $a = 10$ and $d = 2$.
Put $T_n = 30$
Then: $30 = 10 + 2(n-1)$
$30 = 10 + 2n - 2$
$30 = 8 + 2n$
$22 = 2n$
$n = 11$
There are 11 terms in the sequence.

10b $T_2 - T_1 = 4 - 1 = 3$
$T_3 - T_2 = 7 - 4 = 3$
Hence this sequence is an AP with $a = 1$ and $d = 3$.
Put $T_n = 100$
Then: $100 = 1 + 3(n-1)$
$100 = 1 + 3n - 3$
$100 = 3n - 2$
$102 = 3n$

$n = 34$

There are 34 terms in the sequence.

10c $T_2 - T_1 = 100 - 105 = -5$

$T_3 - T_2 = 95 - 100 = -5$

Hence this sequence is an AP with $a = 105$ and $d = -5$.

Put $T_n = 30$

Then: $30 = 105 - 5(n-1)$

$30 = 105 - 5n + 5$

$30 = 110 - 5n$

$-80 = -5$

$n = 16$

There are 16 terms in the sequence.

10d $T_2 - T_1 = 92 - 100 = -8$

$T_3 - T_2 = 84 - 92 = -8$

Hence this sequence is an AP with $a = 100$ and $d = -8$.

Put $T_n = 4$

Then: $4 = 100 - 8(n-1)$

$4 = 100 - 8n + 8$

$4 = 108 - 8n$

$-104 = -8n$

$n = 13$

There are 13 terms in the sequence.

10e $T_2 - T_1 = \left(-10\frac{1}{2}\right) - (-12) = 1\frac{1}{2}$

$T_3 - T_2 = (-9) - \left(-10\frac{1}{2}\right) = 1\frac{1}{2}$

Hence this sequence is an AP with $a = -12$ and $d = 1\frac{1}{2}$.

Put $T_n = 0$

Then: $0 = -12 + \frac{3}{2}(n-1)$

$0 = -12 + \frac{3}{2}n - \frac{3}{2}$

$0 = \frac{3}{2}n - \frac{27}{2}$

$\frac{3}{2}n = \frac{27}{2}$

$n = 9$

There are 9 terms in the sequence.

10f $T_2 - T_1 = 5 - 2 = 3$

$T_3 - T_2 = 8 - 5 = 3$

Hence this sequence is an AP with $a = 2$ and $d = 3$.

Put $T_n = 2000$

Then: $2000 = 2 + 3(n - 1)$

$2000 = 2 + 3n - 3$

$2000 = 3n - 1$

$2001 = 3n$

$n = 667$

There are 667 terms in the sequence.

11a $T_n = 7 + 4n$

$T_1 = 7 + 4 = 11$

$T_2 = 7 + 8 = 15$

$T_3 = 7 + 12 = 19$

$T_4 = 7 + 16 = 23$

Hence this sequence is an AP with $a = 11$ and $d = 4$.

11b $T_{25} + T_{50} = (7 + 4 \times 25) + (7 + 4 \times 50)$

$= 107 + 207$

$= 314$

$T_{50} - T_{25} = (7 + 4 \times 50) - (7 + 4 \times 25)$

$= 207 - 107$

$= 100$

11c $5T_1 + 4T_2 = 5(7 + 4 \times 1) + 4(7 + 4 \times 2)$

$= 5(11) + 4(15)$

$= 115$

$T_{27} = 7 + 4 \times 27 = 115$

Hence, $5T_1 + 4T_2 = T_{27}$.

11d Put $T_n = 815$

Then $815 = 7 + 4n$

$808 = 4n$

$n = 202$

Hence 815 is the 202nd term in this sequence.

11e Put $T_n = 1000$

Then $1000 = 7 + 4n$

$993 = 4n$

$n = 248\frac{1}{4}$

Hence the last term less than 1000 is $T_{248} = 999$, and the first term greater than 1000 is $T_{249} = 1003$.

11f Put $T_n > 200$

Then $200 < 7 + 4n$

$193 < 4n$

$n > 48\frac{1}{4}$

Put $T_n < 300$

Then $300 > 7 + 4n$

$293 > 4n$

$n < 73\frac{1}{4}$

Hence, the terms $T_{49} = 203, \ldots, T_{73} = 299$ are between 200 and 300. There are 25 terms between 200 and 300.

12a i $T_2 - T_1 = 16 - 8 = 8$

$T_3 - T - 2 = 24 - 16 = 8$

This is an AP with $d = 8$ and $a = 8$

$$\begin{aligned} T_n &= a + (n-1)d \\ &= 8 + 8(n-1) \\ &= 8 + 8n - 8 \\ &= 8n \end{aligned}$$

12a ii Put $T_n > 500$

Then: $8n > 500$

$n > 62.5$

Hence the first term greater than 500 is $T_{63} = 504$

Put $T_n < 850$

Then: $8n < 850$

$n < 106.25$

Hence the last term less than 850 is $T_{106} = 848$

12a iii $106 - 63 + 1 = 44$ gives 44 multiples of 8 between 500 and 850.

12b Considering the AP with $a = 11, d = 11$

$T_n = 11 + 11(n - 1)$
$T_n = 11n$

Put $T_n > 1000$
$11n > 1000$
$n > 90.9$

Hence the first term above 1000 is $T_{91} = 1001$

Put $T_n < 2000$
$11n < 2000$
$n < 181.81$

Hence the first term below 2000 is $T_{181} = 1991$

$181 - 91 + 1 = 91$ gives 91 multiples of 11 between 1000 and 2000

12c Considering the AP with $a = 7, d = 7$

$T_n = 7 + 7(n - 1)$
$T_n = 7n$

Put $T_n > 800$
$7n > 800$
$n > 114.28$

Hence the first term above 800 is $T_{115} = 805$

Put $T_n < 2000$
$7n < 2000$
$n < 285.71$

Hence the first term below 2000 is $T_{285} = 1995$

$285 - 115 + 1 = 171$ gives 171 multiples of 7 between 800 and 2000

13a $T_4 = 16$ and $a = 7$
Put $16 = 7 + (4 - 1)d$
$16 = 7 + 3d$
$9 = 3d$
$d = 3$
The first four terms in the sequence are 7, 10, 13, 16.

13b $T_{11} = 108$ and $a = 28$

Put $108 = 28 + (11 - 1)d$

$108 = 28 + 10d$

$80 = 10d$

$d = 8$

$T_{20} = 28 + 8(20 - 1)$

$= 180$

13c $T_{20} = -6$ and $a = 32$

Put $-6 = 32 + (20 - 1)d$

$-6 = 32 + 19d$

$-38 = 19d$

$d = -2$

$T_{100} = 32 - 2(100 - 1)$

$= -166$

14a $T_1 = 500$

$T_2 = T_1 + 300 = 500 + 300 = \$800$

$T_3 = T_2 + 300 = 800 + 300 = \$1100$

$T_4 = T_3 + 300 = 1100 + 300 = \$1400$

14b $T_n = a + (n - 1)d$

$= 500 + (n - 1)(300)$

$= 500 + 300n - 300$

$= 200 + 300n$

$T_{15} = 200 + 300(15) = \$4700$

14c $T_n = a + (n - 1)d$

$= 500 + (n - 1)(300)$

$= 500 + 300n - 300$

$= 200 + 300n$

14d Put $T_n < 10\,000$

$200 + 300n < 10\,000$

$300n < 9800$

$n < 32.666 \ldots$

So the maximum number of windows whose total cost is less than \$10 000 is 32.

15a The track is 160 km before building resumes. It is 20 km longer one month later, and each term is 20 km more than the previous term. The sequence is as follows: $180, 200, 220, \dots$

15b $T_n = 180 + 20(n-1)$
$T_{12} = 180 + 20(12-1)$
$= 400$
After 12 months, there is 400 km of track.

15c $T_n = 180 + 20(n-1)$
$= 180 + 20n - 20$
$= 160 + 20n$

15d Put $T_n = 540$
Then $540 = 160 + 20n$
$380 = 20n$
$n = 19$
It took 19 months to complete the track.

16a $A_1 = 2000 \times 1.06 = \$2120$
$A_2 = 2120 \times 1.06 = \$2240$
$A_3 = 2240 \times 1.06 = \$2360$
$A_4 = 2360 \times 1.06 = \$2480$

16b Put $A_1 = 2120$
$a + (1-1)d = 2120$

$a + 0 = 2120$

$a = 2120$

Put $A_2 = 2240$

$a + (2-1)d = 2240$

$2120 + d = 2240$

$d = 120$

Hence $A_n = 2120 + 120(n-1) = 2000 + 120n$

Thus $A_{12} = 2000 + 120(12) = 3340$

16c Put $T_n > 6000$

$2000 + 120n > 6000$

$120n > 4000$

$n > 33.33$

Hence, it will take 34 years before the total amount exceeds \$6000.

17a $f(1) = 9, f(2) = 6, f(3) = 3$

$T_2 - T_1 = 6 - 9 = -3$

$T_3 - T_2 = 3 - 6 = -3$

This is an AP with $a = 9$, $d = -3$. Hence:

$$\begin{aligned} T_n &= a + (n-1)d \\ &= 9 - 3(n-1) \\ &= 9 - 3n + 3 \\ &= 12 - 3n \end{aligned}$$

17b i $T_2 - T_1 = -1 - (-3) = 2$

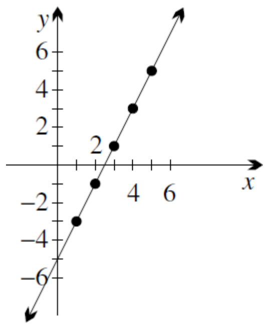
$T_3 - T_2 = 1 - (-1) = 2$

This is an AP with $a = -3$, $d = 2$. Hence:

$$\begin{aligned} T_n &= a + (n-1)d \\ &= -3 + 2(n-1) \\ &= -3 + 2n - 2 \\ &= 2n - 5 \end{aligned}$$

The linear function that generates this is $f(x) = 2x - 5$.

17b ii



18a This is an AP with $a = 5x - 9$ and $d = (5x - 5) - (5x - 9) = 4$. Hence:

$$\begin{aligned} T_n &= a + (n-1)d \\ &= 5x - 9 + (n-1) \times 4 \\ &= 5x - 9 + 4n - 4 \\ &= 5x + 4n - 13 \end{aligned}$$

Put $T_{11} = 36$

$36 = 5x + 4(11) - 13$

$36 = 5x + 31$

$5 = 5x$

$x = 1$

18b This is an AP with $a = 16$ and $d = (16 + 6x) - 16 = 6x$. Hence:

$$\begin{aligned} T_n &= a + (n-1)d \\ &= 16 + (n-1) \times 6x \end{aligned}$$

Put $T_{11} = 36$

$36 = 16 + (11 - 1) \times 6x$

$36 = 16 + 60x$

$20 = 60x$

$x = \dfrac{1}{3}$

19a This is an AP with $a = \log_3 2$ and

$d = \log_3 4 - \log_3 2$

$d = \log_3(2 \times 2) - \log_3 2$

$d = \log_3 2 + \log_3 2 - \log_3 2$

$d = \log_3 2$

Hence:

$$\begin{aligned} T_n &= a + (n-1)d \\ &= \log_3 2 + (n-1)\log_3 2 \\ &= \log_3 2 + n\log_3 2 - \log_3 2 \\ &= n\log_3 2 \end{aligned}$$

19b This is an AP with $a = \log_a 54$ and
$d = \log_a 18 - \log_a 54$
$d = \log_a 18 - \log_a (3 \times 18)$
$d = \log_a 18 - \log_a 3 - \log_a 18$
$d = -\log_a 3$
Hence:
$T_n = a + (n-1)d$
$= \log_a 54 + (n-1)(-\log_a 3)$
$= \log_a (2 \times 3^3) + (n-1)(-\log_a 3)$
$= \log_a 2 + 3\log_a 3 - n\log_a 3 + \log_a 3$
$= \log_a 2 + (4-n)\log_a 3$

19c This is an AP with $a = x - 3y$ and
$d = (2x + y) - (x - 3y)$
$d = x + 4y$
Hence:
$T_n = a + (n-1)d$
$= x - 3y + (n-1)(x + 4y)$
$= x - 3y + nx + 4ny - x - 4y$
$= nx + (4n - 7)y$

19d This is an AP with $a = 5 - 6\sqrt{5}$ and
$d = 1 + \sqrt{5} - (5 - 6\sqrt{5})$
$d = -4 + 7\sqrt{5}$
Hence:
$T_n = a + (n-1)d$
$= 5 - 6\sqrt{5} + (n-1)(-4 + 7\sqrt{5})$
$= 5 - 6\sqrt{5} - 4n + 7\sqrt{5}n + 4 - 7\sqrt{5}$
$= 9 - 4n + (7n - 13)\sqrt{5}$

19e This is an AP with $a = 1.36$ and
$d = -0.52 - 1.36$
$d = -1.88$
Hence:
$T_n = a + (n-1)d$
$= 1.36 + (n-1)(-1.88)$
$= 3.24 - 1.88n$

19f This is an AP with $a = \log_a 3x^2 = \log_a 3 + 2\log_a x$ and
$d = \log_a 3x - \log_a 3x^2$
$d = \log_a 3 + \log_a x - \log_a 3 - 2\log_a x$
$d = -\log_a x$
Hence:
$T_n = \log_a 3 + 2\log_a x + (n-1)(-\log_a x)$
$T_n = \log_a 3 + (3-n)\log_a x$

20 This is an AP with $a = 100, d = -3$. So $T_n = 100 + (n-1)(-3) = 103 - 3n$.

Put $T_n^2 < 400$

$(103 - 3n)^2 < 400$

$10\,609 - 618n + 9n^2 < 400$

$9n^2 - 618n + 10\,209 < 0$

By the quadratic formula, the solutions to the equation $9n^2 - 618n + 10209 = 0$ are:

$$n = \frac{618 \pm \sqrt{618^2 - 4 \times 9 \times 10209}}{2 \times 9} = 27.66, 41$$

Thus, as we want the region of the parabola below the axis,

$27.66 < n < 41$

So for integer solutions

$28 \le n \le 40$

Hence, the 13 terms $T_{28} = 19, \dots, T_{40} = -17$ have squares less than 400.

21a $f(x) = mx + b$

When $x = 1, f(1) = m + b$; so $T_1 = m + b$

When $x = 2, f(2) = 2m + b$; so $T_2 = 2m + b$

First term: $a = m + b$

Difference: $d = 2m + b - (m + b) = m$

21b $T_1 = a$ and $T_2 = a + d$

For a linear function, $f(x) = mx + b$

When $x = 1, f(1) = m + b$; so $T_1 = a = m + b$ or $b = a - m$

When $x = 2, f(2) = 2m + b$; so $T_2 = a + d = 2m + b$

$a + d = 2m + b$ becomes:

$m + b + d = 2m + b$

$b + d - b = 2m - m$

$m = d$

So the gradient is d and the y-intercept is $a - m$ or $a - d$.

22a Take an arbitrary λ and μ. For T_n we have $T_n = \lambda a_1 + (n-1)\lambda d_1$ and for U_n we have $U_n = \mu a_2 + (n-1)\mu d_1$. Hence the sum of the sequences will be:

$T_n + U_n$

$= \lambda a_1 + (n-1)\lambda d_1 + \mu a_2 + (n-1)\mu d_1$

$= (\lambda a_1 + \mu a_2) + (n-1)(\lambda d_1 + \mu d_2)$

This is the equation of an AP with $a = \lambda a_1 + \mu a_2$, $d = \lambda d_1 + \mu d_2$

22b $A(1,0)$ has $T_n = 1 + (n-1)(0) = 1$ and thus is $1, 1, 1, \ldots$

$A(0,1)$ has $T_n = 0 + (n-1)(1) = n - 1$ and thus is $0, 1, 2 \ldots$

$A(a, d)$ is:

$$\begin{aligned} T_n &= a + (n-1)d \\ &= a + 0d + (0 + (n-1)d) \\ &= a(1 + 0d) + d\big(0 + (n-1)(1)\big) \\ &= aA(1,0) + dA(0,1) \end{aligned}$$

Hence, T_n is of the form $\lambda A(1,0) + \mu A(0,1)$ where $\lambda = a$ and $\mu = d$.

Solutions to Exercise 1C

1a Each term is 2 times the previous term. The next three terms are:
$8, 16, 32$

1b Each term is $\frac{1}{3}$ of the previous term. The next three terms are:
$3, 1, \frac{1}{3}$

1c Each term is 2 times the previous term. The next three terms are:
$-56, -112, -224$

1d Each term is $\frac{1}{5}$ of the previous term. The next three terms are:
$-20, -4, -\frac{4}{5}$

1e Each term is -2 times the previous term. The next three terms are:
$-24, 48, -96$

1f Each term is -2 times the previous term. The next three terms are:
$200, -400, 800$

1g Each term is -1 times the previous term. The next three terms are:
$-5, 5, -5$

1h Each term is $-\frac{1}{10}$ of the previous term. The next three terms are:
$1, -\frac{1}{10}, \frac{1}{100}$

1i Each term is 10 times the previous term. The next three terms are:
$40, 400, 4000$

2a Start with 12. Each term is 2 times the previous term. The sequence is:
$12, 24, 48, 96$

2b Start with 5. Each term is -2 times the previous term. The sequence is:
$5, -10, 20, -40$

2c Start with 18. Each term is $\frac{1}{3}$ of the previous term. The sequence is:
$18, 6, 2, \frac{2}{3}$

2d Start with 18. Each term is $-\frac{1}{3}$ of the previous term. The sequence is:
$18, -6, 2, -\frac{2}{3}$

2e Start with 6. Each term is $-\frac{1}{2}$ of the previous term. The sequence is:
$6, -3, \frac{3}{2}, -\frac{3}{4}$

2f Start with -7. Each term is -1 times the previous term. The sequence is:
$-7, 7, -7, 7$

3a $\frac{T_3}{T_2} = \frac{16}{8} = 2$
$\frac{T_2}{T_1} = \frac{8}{4} = 2$
This is a GP with $a = 4$ and $r = 2$.

3b $\frac{T_3}{T_2} = \frac{4}{8} = \frac{1}{2}$
$\frac{T_2}{T_1} = \frac{8}{16} = \frac{1}{2}$
This is a GP with $a = 16$ and $r = \frac{1}{2}$.

3c $\frac{T_3}{T_2} = \frac{6}{4} = \frac{3}{2}$
$\frac{T_2}{T_1} = \frac{4}{2} = 2$
This is not a GP, as the ratios are not all the same.

3d $\frac{T_3}{T_2} = \frac{-10}{-100} = \frac{1}{10}$

$\frac{T_2}{T_1} = \frac{-100}{-1000} = \frac{1}{10}$

This is a GP with $a = -1000$ and $r = \frac{1}{10}$.

3e $\frac{T_3}{T_2} = \frac{-20}{40} = -\frac{1}{2}$

$\frac{T_2}{T_1} = \frac{40}{-80} = -\frac{1}{2}$

This is a GP with $a = -80$ and $r = -\frac{1}{2}$.

3f $\frac{T_3}{T_2} = \frac{29}{29} = 1$

$\frac{T_2}{T_1} = \frac{29}{29} = 1$

This is a GP with $a = 29$ and $r = 1$.

3g $\frac{T_3}{T_2} = \frac{9}{4}$

$\frac{T_2}{T_1} = \frac{4}{1} = 4$

This is not a GP, as the ratios are not all the same.

3h $\frac{T_3}{T_2} = \frac{-14}{14} = -1$

$\frac{T_2}{T_1} = \frac{14}{-14} = -1$

This is a GP with $a = -14$ and $r = -1$.

3i $\frac{T_3}{T_2} = \frac{\frac{1}{6}}{1} = \frac{1}{6}$

$\frac{T_2}{T_1} = \frac{1}{6}$

This is a GP with $a = 6$ and $r = \frac{1}{6}$.

4a $T_n = ar^{n-1}$

$$\begin{aligned} T_4 &= 5 \times 2^3 \\ &= 5 \times 8 \\ &= 40 \end{aligned}$$

4b $T_n = ar^{n-1}$

$T_4 = 300 \times \left(\frac{1}{10}\right)^3$

$= 300 \times \frac{1}{1000}$

$= \frac{3}{10}$

4c $T_n = ar^{n-1}$

$T_4 = -7 \times 2^3$

$= -7 \times 8$

$= -56$

4d $T_n = ar^{n-1}$

$T_4 = -64 \times \left(\frac{1}{2}\right)^3$

$= -64 \times \frac{1}{8}$

$= -8$

4e $T_n = ar^{n-1}$

$T_4 = 11 \times (-2)^3$

$= 11 \times -8$

$= -88$

4f $T_n = ar^{n-1}$

$T_4 = -15 \times (-2)^3$

$T_4 = -15 \times -8$

$T_4 = 120$

5a $T_n = ar^{n-1}$ with $a = 1, r = 3$

$T_n = 1 \times 3^{n-1} = 3^{n-1}$

5b $T_n = ar^{n-1}$ with $a = 5, r = 7$

$T_n = 5 \times 7^{n-1}$

5c $T_n = ar^{n-1}$ with $a = 8, r = \frac{1}{3}$

$T_n = 8 \times (-\frac{1}{3})^{n-1}$

6a $\frac{T_3}{T_2} = \frac{28}{14} = 2$

$\frac{T_2}{T_1} = \frac{14}{7} = 2$

This is a GP with $a = 7$ and $r = 2$.

6b $T_n = ar^{n-1}$

$T_6 = 7 \times 2^5$

$= 7 \times 32$

$= 224$

$T_{50} = 7 \times 2^{49}$

6c $T_n = ar^{n-1}$

$a = 7$ and $r = 2$

$T_n = 7 \times 2^{n-1}$

7a $\frac{T_3}{T_2} = \frac{90}{-30} = -3$

$\frac{T_2}{T_1} = \frac{-30}{10} = -3$

This is a GP with $a = 10$ and $r = -3$.

7b $T_n = ar^{n-1}$

$T_6 = 10 \times (-3)^5$

$= 10 \times -243$

$= -2430$

$T_{25} = 10 \times (-3)^{24}$

$= 10 \times 3^{24}$

7c $T_n = ar^{n-1}$

$a = 10$ and $r = -3$

$T_n = 10 \times (-3)^{n-1}$

8a $\frac{T_3}{T_2} = \frac{40}{20} = 2$

$\frac{T_2}{T_1} = \frac{20}{10} = 2$

This is a GP with $a = 10$ and $r = 2$.

$T_n = ar^{n-1}$

$\quad = 10 \times 2^{n-1}$

$T_6 = 10 \times 2^5$

$\quad = 10 \times 32$

$\quad = 320$

8b $\frac{T_3}{T_2} = \frac{20}{60} = \frac{1}{3}$

$\frac{T_2}{T_1} = \frac{60}{180} = \frac{1}{3}$

This is a GP with $a = 180$ and $r = \frac{1}{3}$.

$T_n = ar^{n-1}$

$\quad = 180 \times \left(\frac{1}{3}\right)^{n-1}$

$T_6 = 180 \times \left(\frac{1}{3}\right)^5$

$\quad = 180 \times \frac{1}{243}$

$\quad = \frac{20}{27}$

8c $\frac{T_3}{T_2} = \frac{100}{81}$

$\frac{T_2}{T_1} = \frac{81}{64}$

This is not a GP, as the ratios are not the same.

8d $\frac{T_3}{T_2} = \frac{65}{50} = \frac{13}{10}$

$\frac{T_2}{T_1} = \frac{50}{35} = \frac{10}{7}$

This is not a GP, as the ratios are not the same.

8e $\frac{T_3}{T_2} = \frac{12}{3} = 4$

$\frac{T_2}{T_1} = \frac{3}{\frac{3}{4}} = 4$

This is a GP with $a = \frac{3}{4}$ and $r = 4$.

$T_n = ar^{n-1}$

$\quad = \frac{3}{4} \times 4^{n-1}$

$$\begin{aligned} T_6 &= \tfrac{3}{4} \times 4^5 \\ &= \tfrac{3}{4} \times 1024 \\ &= 768 \end{aligned}$$

8f $\frac{T_3}{T_2} = \frac{-12}{-24} = \frac{1}{2}$

$\frac{T_2}{T_1} = \frac{-24}{-48} = \frac{1}{2}$

This is a GP with $a = -48$ and $r = \frac{1}{2}$.

$$\begin{aligned} T_n &= ar^{n-1} \\ &= -48 \times \left(\tfrac{1}{2}\right)^{n-1} \\ T_6 &= -48 \times \left(\tfrac{1}{2}\right)^5 \\ &= -48 \times \tfrac{1}{32} \\ &= -\tfrac{3}{2} \end{aligned}$$

9a $\frac{T_2}{T_1} = \frac{-1}{1} = -1$

This is a GP with $a = 1$ and $r = -1$

$$\begin{aligned} T_n &= ar^{n-1} \\ &= 1 \times (-1)^{n-1} \\ T_6 &= 1 \times (-1)^5 \\ &= -1 \end{aligned}$$

9b $\frac{T_2}{T_1} = \frac{4}{-2} = -2$

This is a GP with $a = -2$ and $r = -2$

$$\begin{aligned} T_n &= ar^{n-1} \\ &= -2 \times (-2)^{n-1} \\ &= (-2)^n \\ T_6 &= (-2)^6 \\ &= 64 \end{aligned}$$

9c $\frac{T_2}{T_1} = \frac{24}{-8} = -3$

This is a GP with $a = -8$ and $r = -3$

$T_n = ar^{n-1}$

$= -8 \times (-3)^{n-1}$

$T_6 = -8 \times (-3)^5$

$= 1944$

9d $\frac{T_2}{T_1} = \frac{-30}{60} = -\frac{1}{2}$

This is a GP with $a = 60$ and $r = -\frac{1}{2}$

$T_n = ar^{n-1}$

$= 60 \times \left(-\frac{1}{2}\right)^{n-1}$

$T_6 = 60 \times \left(-\frac{1}{2}\right)^5$

$= -\frac{15}{8}$

9e $\frac{T_2}{T_1} = \frac{512}{-1024} = -\frac{1}{2}$

This is a GP with $a = -1024$ and $r = -\frac{1}{2}$

$T_n = ar^{n-1}$

$= -1024 \times \left(-\frac{1}{2}\right)^{n-1}$

$T_6 = -1024 \times \left(-\frac{1}{2}\right)^5$

$= 32$

9f $\frac{T_2}{T_1} = \frac{-\frac{3}{8}}{\frac{1}{16}} = -6$

This is a GP with $a = \frac{1}{16}$ and $r = -6$

$T_n = ar^{n-1}$

$= \frac{1}{16} \times (-6)^{n-1}$

$T_6 = \frac{1}{16} \times (-6)^5$

$= -486$

10a $a = 1$ and $r = 2$
$T_n = ar^{n-1}$
$T_n = 2^{n-1}$
$64 = 2^{n-1}$
$2^6 = 2^{n-1}$
$n - 1 = 6$
$n = 7$
Hence there are 7 terms in this finite sequence.

10b $a = -1$ and $r = 3$
$T_n = ar^{n-1}$
$T_n = -3^{n-1}$
$-81 = -3^{n-1}$
$81 = 3^{n-1}$
$3^4 = 3^{n-1}$
$n - 1 = 4$
$n = 5$
Hence there are 5 terms in this finite sequence.

10c $a = 8$ and $r = 5$
$T_n = ar^{n-1}$
$T_n = 8 \times 5^{n-1}$
$125\,000 = 8 \times 5^{n-1}$
$15\,625 = 5^{n-1}$
$5^6 = 5^{n-1}$
$n - 1 = 6$
$n = 7$
Hence there are 7 terms in this finite sequence.

10d $a = 7$ and $r = 2$
$T_n = ar^{n-1}$
$T_n = 7 \times 2^{n-1}$
$224 = 7 \times 2^{n-1}$
$32 = 2^{n-1}$
$2^5 = 2^{n-1}$
$n - 1 = 5$
$n = 6$
Hence there are 6 terms in this finite sequence.

10e $a = 2$ and $r = 7$

$T_n = ar^{n-1}$

$T_n = 2 \times 7^{n-1}$

$4802 = 2 \times 7^{n-1}$

$2401 = 7^{n-1}$

$7^4 = 7^{n-1}$

$n - 1 = 4$

$n = 5$

Hence there are 5 terms in this finite sequence.

10f $a = \frac{1}{25}$ and $r = 5$

$T_n = ar^{n-1}$

$T_n = \frac{1}{25} \times 5^{n-1}$

$T_n = 5^{-2} \times 5^{n-1}$

$T_n = 5^{n-3}$

$625 = 5^{n-3}$

$5^4 = 5^{n-3}$

$n - 3 = 4$

$n = 7$

Hence there are 7 terms in this finite sequence.

11a $T_n = 25r^{n-1}$

Put $T_4 = 200$

$25r^{4-1} = 200$

$r^3 = 8$

$r = 2$

$T_1 = 25$

$T_2 = 50$

$T_3 = 100$

$T_4 = 200$

$T_5 = 400$

11b i $T_6 = 96, a = 3$

$ar^5 = 96$

$3r^5 = 96$

$r^5 = 32$

$r = 2$

11b ii $T_7 = 0.001, a = 1000$

$ar^6 = 0.001$

$1000r^6 = 0.001$

$r^6 = 0.000\,001$

$r = \pm 0.1$

11b iii $T_6 = 32, a = 32$

$ar^5 = -243$

$32r^5 = -243$

$r^5 = -\dfrac{243}{32}$

$r = -\dfrac{3}{2}$

11b iv $T_7 = 40, a = 5$

$5r^6 = 40$

$r^6 = 8$

$r = \pm\sqrt{2}$

12a $T_1 = 50$

$T_2 = 100$

$T_3 = 200$

$T_4 = 400$

$T_5 = 800$

$T_6 = 1600$

$a = 50, r = \frac{T_2}{T_1} = \frac{100}{50} = 2$

12b Put $T_n = 6400$

$25 \times 2^n = 6400$

$2^n = 256$

$n = 8$

Hence $T_8 = 6400$

12c $T_{50} \times T_{25} = 25(2)^{50} \times 25(2)^{25} = 25^2 \times 2^{75} = 5^4 \times 2^{75}$

$T_{50} \div T_{25} = 25(2)^{50} \div 25(2)^{25} = 2^{25}$

12d $T_9 \times T_{11} = 25(2)^9 \times 25(2)^{11} = 25 \times 25 \times (2)^{20} = 25 \times T_{20}$

12e There are 6 terms, they are:
$T_6 = 1600$
$T_7 = 3200$
$T_8 = 6400$
$T_9 = 12\,800$
$T_{10} = 25\,600$
$T_{11} = 51\,200$

12f $T_{12} = 25 \times 2^{12} = 102\,400$ whereas
$T_{11} = 25 \times 2^{11} = 51\,200$. Hence T_{11} is the last term less than 100 000 and
$T_{12} = 102\,400$ is the first term greater than 100 000.

13 Start with 0.1. Each term is 2 times the previous term. The sequence is:
0.1, 0.2, 0.4, …
Hence, this is a GP with $a = 0.1$ and $r = 2$.
T_{101} is equivalent to the thickness from 100 successive folds.
$T_n = ar^{n-1}$
$T_{101} = 0.1 \times 2^{100}$
$T_{101} = \frac{2^{100}}{10}$ mm $\doteqdot 1.27 \times 10^{23}$ km $\doteqdot 1.34 \times 10^{10}$ light years
This is close to the present estimate of the distance to the Big Bang.

14a $A_1 = P \times 1.07, A_2 = P \times (1.07)^2, A_3 = P \times (1.07)^3$

14b This is a GP with $a = P \times 1.07$ and $r = 1.07$. Hence:

$A_n = (P \times 1.07) \times (1.07)^{n-1} = P \times (1.07)^n$

14c By trial and error it will take 11 full years to double, 35 years to increase tenfold.

15a $W_1 = 20\,000 \times \frac{80}{100} = 20\,000 \times 0.8$
$W_2 = 20\,000 \times \frac{80}{100} \times \frac{80}{100} = 20\,000 \times (0.8)^2$
$W_3 = 20\,000 \times \frac{80}{100} \times \frac{80}{100} \times \frac{80}{100} = 20\,000 \times (0.8)^3$

Hence, by observation:

$W_n = 20000 \times (0.8)^n$

15b The first term below \$2000 is $W_{11} = 1717.99$, hence it takes 11 years for the value to fall below \$2000.

16a This is a GP with $r = \frac{T_2}{T_1} = \frac{2\sqrt{3}}{\sqrt{6}} = \frac{2}{\sqrt{2}} = \frac{2\sqrt{2}}{2} = \sqrt{2}$ and $a = \sqrt{6}$.

Hence $T_n = ar^{n-1} = \sqrt{6}\left(\sqrt{2}\right)^{n-1} = \frac{\sqrt{6}}{\sqrt{2}}\left(\sqrt{2}\right)^n = \sqrt{3}\left(\sqrt{2}\right)^n$

16b This is a GP with $r = \frac{T_2}{T_1} = \frac{a^2x^3}{ax} = ax^2$ and $a = ax$.

Hence $T_n = ar^{n-1} = ax(ax^2)^{n-1} = axa^{n-1}x^{2n-2} = a^nx^{2n-1}$

16c This is a GP with $r = \frac{T_2}{T_1} = -\frac{1}{-\frac{x}{y}} = \frac{y}{x}$ and $a = -\frac{x}{y}$.

Hence $T_n = ar^{n-1} = \left(-\frac{x}{y}\right)\left(\frac{y}{x}\right)^{n-1} = -\left(\frac{y}{x}\right)^{-1}\left(\frac{y}{x}\right)^{n-1} = -\left(\frac{y}{x}\right)^{n-2} = -x^{2-n}y^{n-2}$

17a This is a GP with $r = \frac{T_2}{T_1} = \frac{2x^2}{2x} = x$ and $a = 2x$.

Hence $T_n = ar^{n-1} = 2x(x)^{n-1} = 2x^n$

Put $T_6 = 2$

$2x^6 = 2$

$x^6 = 1$

$x = \pm 1$

17b This is a GP with $r = \frac{T_2}{T_1} = \frac{x^2}{x^4} = x^{-2}$ and $a = x^4$.

Hence $T_n = ar^{n-1} = x^4(x^{-2})^{n-1} = x^4x^{2-2n} = x^{6-2n}$

Put $T_6 = 3^6$

$x^{6-12} = 3^6$

$x^{-6} = 3^6$

$x^6 = 3^{-6}$

$x^6 = \left(\frac{1}{3}\right)^6$

$x = \pm\frac{1}{3}$

17c This is a GP with $r = \frac{T_2}{T_1} = \frac{2^{-12}x}{2^{-16}x} = 2^{-12+16} = 2^4$ and $a = 2^{-16}x$.

Hence $T_n = ar^{n-1} = 2^{-16}x(2^4)^{n-1} = 2^{-16}x2^{4n-4} = 2^{4n-20}x$

Put $T_6 = 96$

$2^{4(6)-20}x = 96$

$2^4x = 96$

$16x = 96$

$x = 6$

18a $\frac{T_2}{T_1} = \frac{2^2}{2^5} = 2^{-3}$

$\frac{T_3}{T_2} = \frac{2^{-1}}{2^2} = 2^{-3}$

$\frac{T_4}{T_3} = \frac{2^{-4}}{2^{-1}} = 2^{-3}$

Hence this is a GP with $a = T_1 = 2^5$ and $r = 2^{-3}$.

$T_n = ar^{n-1}$

$= 2^5 \times (2^{-3})^{n-1}$

$= 2^5 \times 2^{3-3n}$

$= 2^{8-3n}$

18b $T_2 - T_1$

$= \log_2 24 - \log_2 96$

$= \log_2(3 \times 2^3) - \log_2(3 \times 2^5)$

$= \log_2 3 + 3\log_2 2 - (\log_2 3 + 5\log_2 2)$

$= -2\log_2 2$

$= -2$

$T_3 - T_2$

$= \log_2 6 - \log_2 24$

$= \log_2(3 \times 2) - \log_2(3 \times 2^3)$

$= \log_2 3 + \log_2 2 - (\log_2 3 + 3\log_2 2)$

$= -2\log_2 2$

$= -2$

Hence this is an AP with common ratio $r = -2$ and $a = T_1 = \log_2 96$

$T_n = a + (n-1)d$

$= \log_2 96 + (n-1)(-2)$

$= \log_2(3 \times 2^5) + (n-1)(-2)$

$= \log_2 3 + 5\log_2 2 - 2(n-1)$

$= \log_2 3 + 5 - 2n + 2$
$= 7 - 2n + \log_2 3$

19a $T_1 = f(1) = \frac{4}{25} \times 5^1 = \frac{4}{5}$

$T_2 = f(2) = \dfrac{4}{25} \times 5^2 = 4$

$T_3 = f(3) = \dfrac{4}{25} \times 5^3 = 20$

$T_4 = f(4) = \dfrac{4}{25} \times 5^4 = 100$

$T_5 = f(4) = \dfrac{4}{25} \times 5^5 = 500$

$T_n = f(n) = \dfrac{4}{25} \times 5^n$

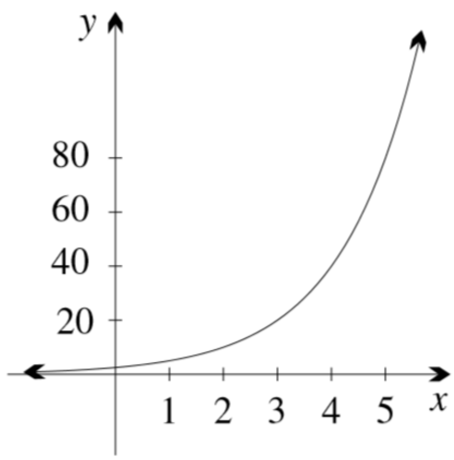
19b i This is a GP with $a = 5, r = \frac{10}{5} = 2$, hence:

$T_n = 5 \times 2^{n-1}$

$= \dfrac{5}{2} \times 2^n$

$f(x) = \frac{5}{2} \times 2^x$

19b ii

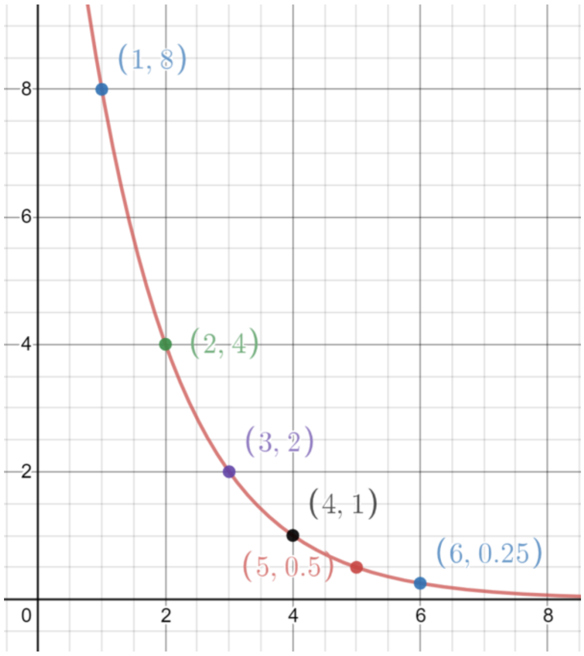


20a $\frac{T_n}{T_{n-1}} = \frac{kb^n}{kb^{n-1}} = b$

Hence all terms have the same common ratio so this is a GP with $a = kb^1 = kb$ and $r = b$.

20b Suppose that T_n is a GP with first term a and ratio r. This means that $T_n = ar^{n-1}$. The exponential function $f(x) = ar^{x-1}$ is such that $T_n = f(n)$.

20c



21a $a = f(1) = cb, r = \frac{f(2)}{f(1)} = \frac{cb^2}{cb} = b$

First term is cb and common ratio is b.

21b We know for a sequence with first term a and ratio r that

$$T_n = a \times r^{n-1} = a \times \frac{r^n}{r} = \frac{a}{r} \times r^n$$

Hence a generating function for this sequence would be $f(x) = \frac{a}{r} \times r^x$.

22a

$$\frac{V_n}{V_{n-1}}$$

$$= \frac{T_n U_n}{T_{n-1} U_{n-1}}$$

$$= \frac{ar^{n-1} AR^{n-1}}{ar^{n-1-1} AR^{n-1-1}}$$

$$= \frac{ar^{n-1} AR^{n-1}}{ar^{n-2} AR^{n-2}}$$

$$= rR$$

Hence all term have the same common ratio of rR.

The first term is $V_1 = T_1 U_1 = ar^{1-1} AR^{1-1} = aA$.

22b Suppose that W_n is a GP, then we have that:

$$W_n W_{n+2} = {W_{n+1}}^2$$

$$(ar^{n-1} + AR^{n-1})(ar^{n+1} + AR^{n+1}) = (ar^n + AR^n)^2$$

$$a^2 r^{2n} + aAr^{n-1}R^{n+1} + aAr^{n+1}R^{n-1} + A^2 R^{2n} = a^2 r^{2n} + 2aAr^n R^n + A^2 R^{2n}$$

$$aAr^{n-1}R^{n+1} + aAr^{n+1}R^{n-1} = 2aAr^n R^n$$

$$aAr^{n-1}R^{n-1}(R^2 + r^2) = 2aAr^n R^n$$

$$(R^2 + r^2) = \frac{2aAr^n R^n}{aAr^{n-1}R^{n-1}}$$

$$R^2 + r^2 = 2rR$$

$$R^2 - 2Rr + r^2 = 0$$

$$(R - r)^2 = 0$$

$$r = R$$

Hence, if $r = R$ then:

$$W_n = ar^{n-1} + AR^{n-1}$$

$$= ar^{n-1} + Ar^{n-1}$$

$$= (a + A)r^{n-1} \text{ where } a + A \neq 0$$

Solutions to Exercise 1D

1a $T_2 - T_1 = T_3 - T_2$
$m - 5 = 17 - m$
$2m = 22$
$m = 11$

1b $T_2 - T_1 = T_3 - T_2$
$m - 32 = 14 - m$
$2m = 46$
$m = 23$

1c $T_2 - T_1 = T_3 - T_2$
$m - (-12) = (-50) - m$
$2m = -62$
$m = -31$

1d $T_2 - T_1 = T_3 - T_2$
$m - (-23) = 7 - m$
$2m = -16$
$m = -8$

1e $T_2 - T_1 = T_3 - T_2$
$22 - m = 32 - 22$
$22 - m = 10$
$m = 12$

1f $T_2 - T_1 = T_3 - T_2$
$-5 - (-20) = m - (-5)$
$15 = m + 5$
$m = 10$

2a $\frac{T_2}{T_1} = \frac{T_3}{T_2}$
$\frac{g}{2} = \frac{18}{g}$
$g^2 = 36$
$g = 6 \text{ or } -6$

2b $\frac{T_2}{T_1} = \frac{T_3}{T_2}$

$\frac{g}{48} = \frac{3}{g}$

$g^2 = 144$

$g = 12 \text{ or } -12$

2c $\frac{T_2}{T_1} = \frac{T_3}{T_2}$

$\frac{g}{-10} = \frac{-90}{g}$

$g^2 = 900$

$g = 30 \text{ or } -30$

2d $\frac{T_2}{T_1} = \frac{T_3}{T_2}$

$\frac{g}{-98} = \frac{-2}{g}$

$g^2 = 196$

$g = 14 \text{ or } -14$

2e $\frac{T_2}{T_1} = \frac{T_3}{T_2}$

$\frac{20}{g} = \frac{80}{20}$

$\frac{20}{g} = 4$

$4g = 20$

$g = 5$

2f $\frac{T_2}{T_1} = \frac{T_3}{T_2}$

$\frac{4}{-1} = \frac{g}{4}$

$g = -16$

3a i $T_2 - T_1 = T_3 - T_2$

$x - 4 = 16 - x$

$2x = 20$

$x = 10$

3a ii $\frac{T_2}{T_1} = \frac{T_3}{T_2}$

$\frac{x}{4} = \frac{16}{x}$

$x^2 = 64$

$x = 8 \text{ or } -8$

3b i $T_2 - T_1 = T_3 - T_2$

$x - 1 = 49 - x$

$2x = 50$

$x = 25$

3b ii $\frac{T_2}{T_1} = \frac{T_3}{T_2}$

$\frac{x}{1} = \frac{49}{x}$

$x^2 = 49$

$x = 7 \text{ or } -7$

3c i $T_2 - T_1 = T_3 - T_2$

$x - 16 = 25 - x$

$2x = 41$

$x = 20\frac{1}{2}$

3c ii $\frac{T_2}{T_1} = \frac{T_3}{T_2}$

$\frac{x}{16} = \frac{25}{x}$

$x^2 = 400$

$x = 20 \text{ or } -20$

3d i $T_2 - T_1 = T_3 - T_2$

$x - (-5) = -20 - x$

$2x = -25$

$x = -12\frac{1}{2}$

3d ii $\frac{T_2}{T_1} = \frac{T_3}{T_2}$

$\frac{x}{-5} = \frac{-20}{x}$

$x^2 = 100$

$x = 10 \text{ or} -10$

3e i $T_2 - T_1 = T_3 - T_2$

$10 - x = 50 - 10$

$10 - x = 40$

$x = -30$

3e ii $\frac{T_2}{T_1} = \frac{T_3}{T_2}$

$\frac{10}{x} = \frac{50}{10}$

$\frac{10}{x} = 5$

$5x = 10$

$x = 2$

3f i $T_2 - T_1 = T_3 - T_2$

$12 - x = 24 - 12$

$12 - x = 12$

$x = 0$

3f ii $\frac{T_2}{T_1} = \frac{T_3}{T_2}$

$\frac{12}{x} = \frac{24}{12}$

$\frac{12}{x} = 2$

$2x = 12$

$x = 6$

3g i $T_2 - T_1 = T_3 - T_2$

$-1 - x = 1 - (-1)$

$-1 - x = 2$

$x = -3$

3g ii $\frac{T_2}{T_1} = \frac{T_3}{T_2}$

$\frac{-1}{x} = \frac{1}{-1}$

$-\frac{1}{x} = -1$

$x = 1$

3h i $T_2 - T_1 = T_3 - T_2$

$6 - x = -12 - 6$

$6 - x = -18$

$x = 24$

3h ii $\frac{T_2}{T_1} = \frac{T_3}{T_2}$

$\frac{6}{x} = \frac{-12}{6}$

$\frac{6}{x} = -2$

$-2x = 6$

$x = -3$

3i i $T_2 - T_1 = T_3 - T_2$

$30 - 20 = x - 30$

$10 = x - 30$

$x = 40$

3i ii $\frac{T_2}{T_1} = \frac{T_3}{T_2}$

$\frac{30}{20} = \frac{x}{30}$

$x = \frac{900}{20}$

$x = 45$

3j i $T_2 - T_1 = T_3 - T_2$

$24 - (-36) = x - 24$

$60 = x - 24$

$x = 84$

3j ii $\frac{T_2}{T_1} = \frac{T_3}{T_2}$

$-\frac{24}{36} = \frac{x}{24}$

$x = -\frac{576}{36}$

$x = -16$

3k i $T_2 - T_1 = T_3 - T_2$

$-3 - \left(-\frac{1}{4}\right) = x - (-3)$

$-2\frac{3}{4} = x + 3$

$x = -5\frac{3}{4}$

3k ii $\frac{T_2}{T_1} = \frac{T_3}{T_2}$

$\frac{-3}{-\frac{1}{4}} = -\frac{x}{3}$

$-\frac{x}{3} = 12$

$x = -36$

3l i $T_2 - T_1 = T_3 - T_2$

$-7 - 7 = x - (-7)$

$-14 = x + 7$

$x = -21$

3l ii $\frac{T_2}{T_1} = \frac{T_3}{T_2}$

$\frac{-7}{7} = \frac{x}{-7}$

$-\frac{x}{7} = -1$

$x = 7$

4a $T_n = 7 + (n - 1)d$

Put $T_6 = 42$

$7 + (6 - 1)d = 42$

$7 + 5d = 42$

$5d = 42 - 7$

$5d = 35$

$d = 7$

$T_n = 7 + 7(n - 1)$
$T_n = 7n$

$T_1 = 7$
$T_2 = 14$
$T_3 = 21$
$T_4 = 28$
$T_5 = 35$
$T_6 = 42$

4b $T_n = 27r^{n-1}$

Put $T_4 = 8$

$8 = 27r^{4-1}$
$8 = 27r^3$
$r^3 = \frac{8}{27}$
$r = \frac{2}{3}$

$T_1 = 27$
$T_2 = 18$
$T_3 = 12$
$T_4 = 8$

4c $T_n = 48 + (n - 1)d$

Put $T_5 = 3$

$3 = 48 + (5 - 1)d$
$-45 = 4d$
$d = -11\frac{1}{4}$
$T_n = 48 - 11\frac{1}{4}(n - 1)$

$T_1 = 48$

$T_2 = 36\frac{3}{4}$

$T_3 = 25\frac{1}{2}$

$T_4 = 14\frac{1}{4}$

$T_5 = 3$

4d $T_n = 48r^{n-1}$

Put $T_5 = 3$

$3 = 48r^{5-1}$
$\frac{1}{16} = r^4$
$r = \pm\frac{1}{2}$

When $r = \frac{1}{2}$

$T_n = 3(2)^{n-1}$

$T_1 = 48$
$T_2 = 24$
$T_3 = 12$
$T_4 = 6$
$T_5 = 3$

When $r = -\frac{1}{2}$

$T_n = 3(-2)^{n-1}$

$T_1 = 48$
$T_2 = -24$
$T_3 = 12$
$T_4 = -6$
$T_5 = 3$

5a $T_n = a + (n-1)d$
$T_{10} = 18$ gives $18 = a + 9d$ (1)
$T_{20} = 48$ gives $48 = a + 19d$ (2)
Subtract (1) from (2):
$30 = 10d$
$d = 3$
Substitute $d = 3$ into (1):
$18 = a + 9 \times 3$
$a = -9$

5b $T_n = a + (n-1)d$

$T_5 = 24$ gives $24 = a + 4d$ (1)

$T_9 = -12$ gives $-12 = a + 8d$ (2)

Subtract (1) from (2):

$-36 = 4d$

$d = -9$

Substitute $d = -9$ into (1):

$24 = a + 4 \times -9$

$a = 60$

5c $T_n = a + (n-1)d$

$T_4 = 6$ gives $6 = a + 3d$ (1)

$T_{12} = 34$ gives $34 = a + 11d$ (2)

Subtract (1) from (2):

$28 = 8d$

$d = 3\frac{1}{2}$

Substitute $d = 3\frac{1}{2}$ into (1):

$6 = a + 3 \times 3\frac{1}{2}$

$a = -4\frac{1}{2}$

6a $\frac{T_6}{T_3} = \frac{ar^{6-1}}{ar^{3-1}} = \frac{ar^5}{ar^2} = r^3$

$$\frac{T_6}{T_3} = \frac{128}{16} = 8$$

$r^3 = 8$

$r = 2$

$T_3 = ar^{3-1} = ar^2 = a(2)^2 = 4a$ and $T_3 = 16$

$4a = 16$

$a = 4$

6b $\frac{T_6}{T_2} = \frac{ar^{6-1}}{ar^{2-1}} = \frac{ar^5}{ar} = r^4$

$$\frac{T_6}{T_2} = \frac{27}{\frac{1}{3}} = 81$$

$r^4 = 81$

$r = 3$ or -3

When $r = 3$,

$T_6 = ar^{6-1} = ar^5 = a(3)^5 = 243a$ and $T_6 = 27$

$243a = 27$

$a = \frac{1}{9}$

When $r = -3$,

$T_6 = ar^{6-1} = ar^5 = a(-3)^5 = -243a$ and $T_6 = 27$

$-243a = 27$

$a = -\frac{1}{9}$

6c $\frac{T_9}{T_5} = \frac{ar^{9-1}}{ar^{5-1}} = \frac{ar^8}{ar^4} = r^4$

$\frac{T_9}{T_5} = \frac{24}{6} = 4$

$r^4 = 4$

$r = \sqrt{2}$ or $-\sqrt{2}$

When $r = \sqrt{2}$,

$T_5 = ar^{5-1} = ar^4 = a\left(\sqrt{2}\right)^4 = 4a$ and $T_5 = 6$

$4a = 6$

$a = \frac{3}{2}$

When $r = -\sqrt{2}$,

$T_5 = ar^{5-1} = ar^4 = a\left(-\sqrt{2}\right)^4 = 4a$ and $T_5 = 6$

$4a = 6$

$a = \frac{3}{2}$

7a $T_n = a + (n-1)d$
$T_3 = 7$ gives $7 = a + 2d$ (1)
$T_7 = 31$ gives $31 = a + 6d$ (2)
Subtract (1) from (2):

$24 = 4d$
$d = 6$
Substitute $d = 6$ into (1):
$7 = a + 2 \times 6$
$a = -5$
$T_8 = -5 + 7 \times 6 = 37$

7b $T_n = a + (n-1)d$
$d = -7$
$T_{10} = 3$ gives $3 = a + 9 \times -7$
$3 = a - 63$
$a = 66$
$T_2 = 66 - 7 = 59$

7c $T_n = ar^{n-1}$
$r = 2$
$T_6 = 6$ gives $6 = a \times 2^5$
$a = \frac{6}{32} = \frac{3}{16}$
$T_2 = \frac{3}{16} \times 2^1 = \frac{3}{8}$

8a $3^n > 1\,000\,000$

$\ln 3^n > \ln 1\,000\,000$
$n \ln 3 > \ln 1\,000\,000$
$n > \dfrac{\ln 1\,000\,000}{\ln 3}$
$n > 12.575 \ldots$

The smallest integer solution is $n = 13$.

8b $5^n < 1\,000\,000$

$\ln 5^n < \ln 1\,000\,000$
$n \ln 5 < \ln 1\,000\,000$
$n < \dfrac{\ln 1\,000\,000}{\ln 5}$
$n < 8.584 \ldots$

The largest integer solution is $n = 8$.

8c $7^n > 1\,000\,000\,000$

$\ln 7^n > \ln 1\,000\,000\,000$
$n \ln 7 > \ln 1\,000\,000\,000$
$n > \dfrac{\ln 1\,000\,000\,000}{\ln 7}$
$n > 10.64 \ldots$

The smallest integer solution is $n = 11$.

8d $12^n < 1\,000\,000\,000$

$\ln 12^n < \ln 1\,000\,000\,000$
$n \ln 12 < \ln 1\,000\,000\,000$
$n < \dfrac{\ln 1\,000\,000\,000}{\ln 12}$
$n < 8.339 \ldots$

The largest integer solution is $n = 8$.

9a $\frac{T_2}{T_1} = \frac{4}{2} = 2$ and $\frac{T_3}{T_2} = \frac{8}{4} = 2$
Hence the sequence is a GP with $r = 2$ and $a = 2$.
$T_n = ar^{n-1}$
$T_n = 2 \times 2^{n-1}$
$= 2^1 \times 2^{n-1}$
$= 2^n$

9b $T_n < 1\,000\,000$
$2^n < 1\,000\,000$
$\log_{10} 2^n < \log_{10} 1\,000\,000$
$n \log_{10} 2 < \log_{10} 1\,000\,000$
$n < \frac{\log_{10} 1\,000\,000}{\log_{10} 2}$
$n < 19.93 \ldots$
Hence there are 19 terms less than 1 000 000.

9c $T_n < 1\,000\,000\,000$
$2^n < 1\,000\,000\,000$
$\log_{10} 2^n < \log_{10} 1\,000\,000\,000$
$n \log_{10} 2 < \log_{10} 1\,000\,000\,000$
$n < \frac{\log_{10} 1\,000\,000\,000}{\log_{10} 2}$

$n < 29.8973\ldots$
Hence there are 29 terms less than 1 000 000 000.

9d $T_n < 10^{20}$
$2^n < 10^{20}$
$\log_{10} 2^n < \log_{10} 10^{20}$
$n \log_{10} 2 < 20$
$n < \frac{20}{\log_{10} 2}$
$n < 66.43\ldots$
Hence there are 66 terms less than 10^{20}.

9e Using the answers to parts b and c, there are 10 terms between 1 000 000 and 1 000 000 000.

9f Using the answers to parts c and d, there are 37 terms between 1 000 000 000 and 10^{20}.

10a $\frac{T_2}{T_1} = \frac{14}{98} = \frac{1}{7}$

This is a GP with $a = 98, r = \frac{1}{7}$ so $T_n = 98\left(\frac{1}{7}\right)^{n-1}$.

$T_n > 10^{-6}$

$98\left(\frac{1}{7}\right)^{n-1} > 10^{-6}$

$\left(\frac{1}{7}\right)^{n-1} > \frac{10^{-6}}{98}$

$\ln\left(\frac{1}{7}\right)^{n-1} > \ln\frac{10^{-6}}{98}$

$(n-1)\ln\frac{1}{7} > \ln\frac{10^{-6}}{98}$

$n - 1 < \frac{\ln\frac{10^{-6}}{98}}{\ln\frac{1}{7}}$ (Note that $\ln\frac{1}{7} < 0$, hence we must reverse the sign)

$n - 1 < 9.46\ldots$

$n < 10.46\ldots$

Hence there are 10 terms greater than 10^{-6}.

10b $\frac{T_2}{T_1} = \frac{5}{25} = \frac{1}{5}$

This is a GP with $a = 25, r = \frac{1}{5}$ so $T_n = 25\left(\frac{1}{5}\right)^{n-1}$.

$T_n > 10^{-6}$

$25\left(\frac{1}{5}\right)^{n-1} > 10^{-6}$

$\left(\frac{1}{5}\right)^{n-1} > \frac{10^{-6}}{25}$

$\ln\left(\frac{1}{5}\right)^{n-1} > \ln\frac{10^{-6}}{25}$

$(n-1)\ln\frac{1}{5} > \ln\frac{10^{-6}}{25}$

$n - 1 < \frac{\ln\frac{10^{-6}}{25}}{\ln\frac{1}{5}}$

$n - 1 < 10.58 \ldots$

$n < 11.58 \ldots$

Hence there are 11 terms greater than 10^{-6}.

10c $\frac{T_2}{T_1} = \frac{0.9}{1} = 0.9$

This is a GP with $a = 1, r = 0.9$ so $T_n = (0.9)^{n-1}$.

$T_n > 10^{-6}$

$(0.9)^{n-1} > 10^{-6}$

$(n-1)\ln 0.9 > \ln 10^{-6}$

$n - 1 < \frac{\ln 10^{-6}}{\ln 0.9}$

$n - 1 < 131.13 \ldots$

$n < 132.13 \ldots$

Hence there are 132 terms greater than 10^{-6}.

11a This is a GP with $T_n = 0.97^n$, where T_n is the intensity of the light, and n represents the number of sheets of glass.
For 50 sheets of glass:
$T_{50} = 0.97^{50} = 0.22$ or 22%
Hence the light's intensity is reduced by $1 - 22\% = 78\%$ after passing through 50 sheets of glass.

11b $T_n < 0.01$
$0.97^n < 0.01$
$\ln(0.97^n) < \ln(0.01)$
$n\ln(0.97) < \ln(0.01)$
$n > \frac{\ln(0.01)}{\ln(0.97)}$
$n > 151.19 \ldots$
$T_{152} = 0.97^{152} = 0.975 \ldots \%$
Hence a minimum of 152 sheets of glass are required to reduce the light's intensity to below 1%.

12a $T_6 + T_8 = 44$

$a + (6-1)d + a + (8-1)d = 44$

$2a + 12d = 44$ (1)

$T_{10} + T_{13} = 35$
$a + (10-1)d + a + (13-1)d = 35$
$a + 9d + a + 12d = 35$

$2a + 21d = 35$ (2)
$9d = -9$ (2)−(1)
$d = -1$ (3)
$2a + 12(-1) = 44$ (3) in (1)
$2a = 56$
$a = 28$

So $a = 28$ and $d = -1$.

12b $T_2 + T_3 = 4$

$ar^{2-1} + ar^{3-1} = 4$
$ar^1 + ar^2 = 4$

$ar(1+r) = 4$ (1)

$T_4 + T_5 = 36$
$ar^3 + ar^4 = 36$

$ar^3(1 + r) = 36$ (2)
$r^2 = 9$ (2) $\div$ (1)

$r = \pm 3$

When $r = -3$, (3)

$a(-3)(1 - 3) = 4$ (3) in (1)

$6a = 4$
$a = \frac{2}{3}$

When $r = 3$ (4)

$a(3)(1 + 3) = 4$ (4) in (1)

$12a = 4$
$a = \frac{1}{3}$

So $a = \frac{2}{3}$ and $r = -3$, or $a = \frac{1}{3}$ and $r = 3$

12c $T_4 + T_6 + T_8 = -6$
As this is an AP, $T_8 = T_6 + 2d$ and $T_4 = T_6 - 2d$, hence

$T_6 + 2d + T_6 + T_6 - 2d = -6$
$3T_6 = -6$
$T_6 = -2$

13a $T_2 - T_1 = T_3 - T_2$
$17 - (x - 1) = (x + 15) - 17$
$18 - x = x - 2$
$2x = 20$
$x = 10$
The numbers are: $9, 17, 25$.

13b $T_2 - T_1 = T_3 - T_2$
$(x - 4) - (2x + 2) = 5x - (x - 4)$
$-x - 6 = 4x + 4$
$5x = -10$

$x = -2$
The numbers are: $-2, -6, -10$.

13c $T_2 - T_1 = T_3 - T_2$
$5 - (x - 3) = (2x + 7) - 5$
$8 - x = 2x + 2$
$3x = 6$
$x = 2$
The numbers are: $-1, 5, 11$.

13d $T_2 - T_1 = T_3 - T_2$
$x - (3x - 2) = (x + 10) - x$
$-2x + 2 = 10$
$-2x = 8$
$x = -4$
The numbers are: $-14, -4, 6$.

14a $\frac{T_3}{T_2} = \frac{T_2}{T_1}$
$$\frac{x}{x+1} = \frac{x+1}{x}$$
$x^2 = (x + 1)^2$
$x^2 = x^2 + 2x + 1$
$2x = -1$
$$x = -\frac{1}{2}$$
The numbers are: $-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}$.

14b $\frac{T_3}{T_2} = \frac{T_2}{T_1}$
$$\frac{5-x}{2} = \frac{2}{2-x}$$
$(5 - x)(2 - x) = 4$
$10 - 7x + x^2 = 4$
$x^2 - 7x + 6 = 0$
$(x - 1)(x - 6) = 0$

$x = 1$ or $x = 6$
When $x = 1$, the numbers are: $1, 2, 4$.
When $x = 6$, the numbers are: $-4, 2, -1$.

15a i $T_2 - T_1 = T_3 - T_2$
$24 - x = 96 - 24$
$24 - x = 72$
$x = -48$
The numbers are: $-48, 24, 96$.

15a ii $\frac{T_2}{T_1} = \frac{T_3}{T_2}$
$\frac{24}{x} = \frac{96}{24}$
$\frac{24}{x} = 4$
$4x = 24$
$x = 6$
The numbers are: $6, 24, 96$.

15b i $T_2 - T_1 = T_3 - T_2$
$x - 0.2 = 0.000\,02 - x$
$2x = 0.200\,02$
$x = 0.100\,01$
The numbers are: $0.2, 0.100\,01, 0.000\,02$.

15b ii $\frac{T_2}{T_1} = \frac{T_3}{T_2}$
$\frac{x}{0.2} = \frac{0.000\,02}{x}$
$x^2 = 0.000\,004$
$x = 0.002$ or -0.002
The numbers are: $0.2, 0.002, 0.000\,02$ or $0.2, -0.002, 0.000\,02$.

15c i $T_2 - T_1 = T_3 - T_2$
$0.2 - x = 0.002 - 0.2$
$-x = -0.398$
$x = 0.398$
The numbers are: $0.398, 0.2, 0.002$.

15c ii $\frac{T_2}{T_1} = \frac{T_3}{T_2}$

$\frac{0.2}{x} = \frac{0.002}{0.2}$

$\frac{0.2}{x} = 0.01$

$0.01x = 0.2$

$x = 20$

The numbers are: $20, 0.2, 0.002$.

15d i $T_2 - T_1 = T_3 - T_2$

$(x+1) - (x-4) = (x+11) - (x+1)$

$5 = 10$ FALSE

Hence, these numbers cannot form an AP.

15d ii $\frac{T_2}{T_1} = \frac{T_3}{T_2}$

$\frac{x+1}{x-4} = \frac{x+11}{x+1}$

$(x+1)(x+1) = (x+11)(x-4)$

$x^2 + 2x + 1 = x^2 + 7x - 44$

$2x + 1 = 7x - 44$

$5x = 45$

$x = 9$

The numbers are: $5, 10, 20$.

15e i $T_2 - T_1 = T_3 - T_2$

$(x+2) - (x-2) = (5x-2) - (x+2)$

$4 = 4x - 4$

$4x = 8$

$x = 2$

The numbers are: $0, 4, 8$.

15e ii $\frac{T_2}{T_1} = \frac{T_3}{T_2}$

$\frac{x+2}{x-2} = \frac{5x-2}{x+2}$

$(x+2)(x+2) = (5x-2)(x-2)$

$x^2 + 4x + 4 = 5x^2 - 12x + 4$

$0 = 4x^2 - 16x$

$0 = x^2 - 4x$

$0 = x(x-4)$

$x = 0$ or $x = 4$
The numbers are: $-2, 2, -2$ or $2, 6, 18$.

15f i $T_2 - T_1 = T_3 - T_2$
$x - (\sqrt{5}+1) = (\sqrt{5}-1) - x$
$2x = 2\sqrt{5}$
$x = \sqrt{5}$
The numbers are: $\sqrt{5}+1, \sqrt{5}, \sqrt{5}-1$.

15f ii $\frac{T_2}{T_1} = \frac{T_3}{T_2}$
$\frac{x}{\sqrt{5}+1} = \frac{\sqrt{5}-1}{x}$
$x^2 = (\sqrt{5}+1)(\sqrt{5}-1)$
$x^2 = 5 - 1 = 4$
$x = -2$ or 2
The numbers are: $\sqrt{5}+1, -2, \sqrt{5}-1$ or $\sqrt{5}+1, 2, \sqrt{5}-1$.

15g i $T_2 - T_1 = T_3 - T_2$
$x - \sqrt{2} = \sqrt{8} - x$
$x - \sqrt{2} = 2\sqrt{2} - x$
$2x = 3\sqrt{2}$
$x = \frac{3}{2}\sqrt{2}$
The numbers are: $\sqrt{2}, \frac{3}{2}\sqrt{2}, \sqrt{8}$.

15g ii $\frac{T_2}{T_1} = \frac{T_3}{T_2}$
$\frac{x}{\sqrt{2}} = \frac{\sqrt{8}}{x}$
$x^2 = \sqrt{2 \times 8}$
$x^2 = \sqrt{16}$
$x^2 = 4$
$x = -2$ or 2
The numbers are: $\sqrt{2}, -2, \sqrt{8}$ or $\sqrt{2}, 2, \sqrt{8}$.

15h i $T_2 - T_1 = T_3 - T_2$
$x - 2^4 = 2^6 - x$
$2x = 2^4 + 2^6$
$2x = 80$
$x = 40$
The numbers are: $2^4, 40, 2^6$.

15h ii $\frac{T_2}{T_1} = \frac{T_3}{T_2}$

$\frac{x}{2^4} = \frac{2^6}{x}$

$x^2 = 2^{10}$

$x = 2^5$ or -2^5

The numbers are: $2^4, 2^5, 2^6$ or $2^4, -2^5, 2^6$.

15i i $T_2 - T_1 = T_3 - T_2$

$x - 7 = -7 - x$

$x = 0$

The numbers are: $7, 0, -7$.

15i ii $\frac{T_2}{T_1} = \frac{T_3}{T_2}$

$\frac{x}{7} = \frac{-7}{x}$

$x^2 = -49$ This is a false statement.

These numbers cannot form a GP.

16a For a GP:

$\frac{T_3}{T_2} = \frac{T_2}{T_1}$

$\frac{1}{b} = \frac{b}{a}$

$a = b^2$ (1)

For an AP:

$T_3 - T_2 = T_2 - T_1$

$10 - a = a - b$

$2a - b - 10 = 0$ (2)

$2b^2 - b - 10 = 0$ (1) in (2)

$(2b - 5)(b + 2) = 0$

Hence $b = \frac{5}{2} = 2\frac{1}{2}$ and $a = \left(\frac{5}{2}\right)^2 = 6\frac{1}{4}$ or $b = -2$ and $a = (-2)^2 = 4$.

16b For a GP:

$\frac{T_3}{T_2} = \frac{T_2}{T_1}$

$$\frac{a+b}{1}=\frac{1}{a}$$

$a^2+ab=1$ (1)

For an AP:

$T_3-T_2=T_2-T_1$

$$a-b-\frac{1}{2}=\frac{1}{2}-b$$

$a=1$ (2)

$1+b=1$ (2) in (1)

$b=0$

Hence $a=1, b=0$

17a For an AP, the terms must be of the form

$T_1=x-d$

$T_2=x$

$T_2=x+d$

as given.

For a GP:

$$\frac{T_2}{T_1}=\frac{T_3}{T_2}$$

$$\frac{x}{x-d}=\frac{x+d}{x}$$

$x^2=(x-d)(x+d)$

$x^2=x^2-d^2$

$0=-d^2$

$d^2=0$

$d=0$

Hence $T_1=T_2=T_3=x$ so all terms are the same.

17b In an AP:

$T_1=a$

$T_4=a+3d$

$T_7 = a + 6d$

$T_7 - T_4 = a + 6d - (a + 3d) = 3d$

$T_4 - T_1 = a + 3d - (a) = 3d$

So $T_7 - T_4 = T_4 - T_1$and thus T_1, T_4 and T_7 form an AP as they have the same common difference.

17c In an GP:

$T_1 = a$

$T_4 = ar^3$

$T_7 = ar^6$

$$\frac{T_7}{T_4} = \frac{ar^6}{ar^3} = \frac{r^6}{r^3} = r^3$$

$$\frac{T_4}{T_1} = \frac{ar^3}{a} = \frac{r^3}{1} = r^3$$

So $\frac{T_7}{T_4} = \frac{T_4}{T_1}$ and thus T_1, T_4 and T_7 form a GP as they have the same common ratio.

18a For an AP, the terms must be of the form:

$T_1 = a$

$T_2 = a + d$

$T_4 = a + 3d$

To form a GP, terms must have the same common ratio so

$$\frac{a + 3d}{a + d} = \frac{a + d}{a}$$

$a(a + 3d) = (a + d)^2$

$a^2 + 3ad = a^2 + 2ad + d^2$

$ad - d^2 = 0$

$d(a - d) = 0$

Hence $d = 0$ or $a = d$.

If $d = 0$, then the AP sequence is constant.

If $a = d$ then $T_n = a + (n-1)a = na = nT_1$ and so the terms are positive integer multiples of the first term.

18b For an AP, the terms must be of the form:

$T_1 = a$

$T_2 = a + d$

$T_5 = a + 4d$

To form a GP, terms must have the same common ratio so

$$\frac{a+4d}{a+d} = \frac{a+d}{a}$$

$a(a+4d) = (a+d)^2$

$a^2 + 4ad = a^2 + 2ad + d^2$

$2ad - d^2 = 0$

$d(2a - d) = 0$

Hence $d = 0$ or $d = 2a$.

If $d = 0$, then the AP sequence is constant.

If $d = 2a$ then $T_n = a + (n-1) \times 2a = (2n-1)a = (2n-1)T_1$ and so the terms are odd positive integer multiples of the first term.

18c For an AP, the terms must be of the form:

$T_1 = a$

$T_2 = ar$

$T_4 = ar^3$

To form an AP, terms must have the same common difference so

$ar^3 - ar = ar - a$

$r^3 - r = r - 1$

$r^3 - 2r + 1 = 0$

$(r-1)(r^2 - r - 1) = 0$

Hence $r = 1$ or $r^2 - r - 1 = 0$

Using the quadratic formula:

$$r = \frac{1 \pm \sqrt{1^2 - 4(1)(-1)}}{2}$$

$$= \frac{1 \pm \sqrt{5}}{2}$$

So $r = 1,\ \frac{1}{2} + \frac{1}{2}\sqrt{5}$ or $\frac{1}{2} - \frac{1}{2}\sqrt{5}$

18d For the GP:

$T_1 = a,\ T_2 = ar, T_3 = ar^2$

$S_1 = a,\ S_2 = a + ar, S_3 = a + ar + ar^2$

However, each term is one more than the sum of all the previous terms.

$T_2 = ar$ and $T_2 = 1 + S_1 = 1 + a$ so $ar = 1 + a$

or $r = \frac{1+a}{a}$ (1)

$T_3 = ar^2$ and $T_3 = 1 + S_2 = 1 + a + ar$ so

$ar^2 = 1 + a + ar$ (2)

Substituting (1) into (2):

$$a \times \left(\frac{1+a}{a}\right)^2 = 1 + a + a \times \frac{1+a}{a}$$

$$\frac{(1+a)^2}{a} = 1 + a + 1 + a$$

$(1 + a)^2 = a(2 + 2a)$

$1 + 2a + a^2 = 2a + 2a^2$

$1 = a^2$

$a = \pm 1$

Substituting in (1):

When $a = 1, r = \frac{1+a}{a} = \frac{1+1}{1} = 2$

When $a = -1, r = \frac{1+a}{a} = \frac{1-1}{-1} = 0$ (Not possible values for the GP.)

So the GP is 1, 2, 4, 8, …

19a The square of any real number cannot be negative so $(a - b)^2 \geq 0$.

19b $(a-b)^2 \geq 0$

$a^2 - 2ab + b^2 \geq 0$

$a^2 - 2ab + b^2 + 4ab \geq 0 + 4ab$

$a^2 + 2ab + b^2 \geq 4ab$

$(a+b)^2 \geq 4ab$

$a + b \geq 2\sqrt{ab}$

$\dfrac{1}{2}(a+b) \geq 2ab$

$m \geq g$

Hence $g \leq m$

19c Set $a = 1$ for both sets of sequences so $m = \frac{1}{2}(1+b)$ or $b = 2m - 1$

If $m = 1$ then $g = \frac{1}{2}(1+1) = 1$ and $b = 2 \times 1 - 1 = 1$.

Hence we have that 1, 1, 1 is trivially an AP and a GP.

If $m = 5$ then $g = \frac{1}{2}(1+5) = 3$ and $b = 2 \times 5 - 1 = 9$.

Hence we have that 1, 5, 9 is an AP whilst 1, 3, 9 is a GP.

20a Put $T_{13} = \frac{1}{2}T_1$

$ar^{12} = \dfrac{1}{2}a$

$r^{12} = \dfrac{1}{2}$

$r = \left(\frac{1}{2}\right)^{\frac{1}{12}}$ (taking $r > 0$ as pipes do not have negative lengths)

20b $T_8 = ar^7 = T_1\left(\left(\frac{1}{2}\right)^{\frac{1}{12}}\right)^7 = T_1\left(\frac{1}{2}\right)^{\frac{7}{12}} \doteqdot 0.667T_1 \doteqdot \frac{2}{3}T_1$

20c $T_5 = ar^4 = T_1\left(\frac{1}{2}\right)^{\frac{4}{12}} \doteqdot 0.7937T_1 \doteqdot \frac{4}{5}T_1$

20d Put $T_n = \frac{3}{4}T_1$

$$ar^{n-1} = \frac{3}{4}a$$

$$\left(\left(\frac{1}{2}\right)^{\frac{1}{12}}\right)^{n-1} = \frac{3}{4}$$

$$2^{\frac{1-n}{12}} = \frac{3}{4}$$

By trial and error, the closest integer solution is $n = 6$ so the sixth pipe is about three-quarters of the length of the first pipe.

Put $T_n = \frac{5}{6}T_1$

$$ar^{n-1} = \frac{5}{6}a$$

$$\left(\left(\frac{1}{2}\right)^{\frac{1}{12}}\right)^{n-1} = \frac{5}{6}$$

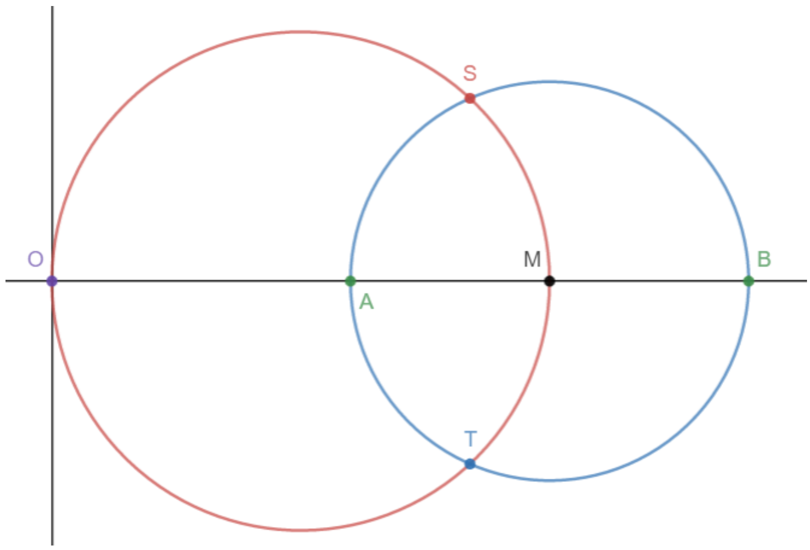
$$2^{\frac{1-n}{12}} = \frac{5}{6}$$

By trial and error, the closest integer solution is $n = 4$ so the fourth pipe is about five-sixths of the length of the first pipe.

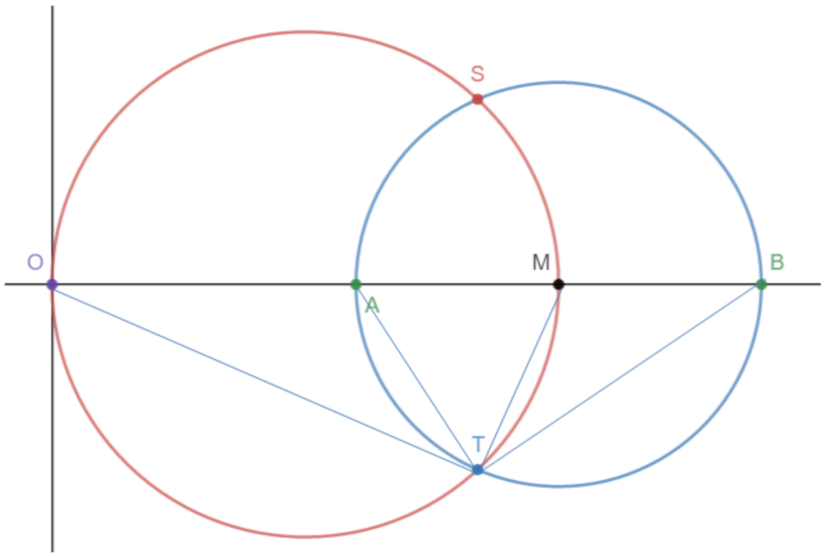
20e $T_3 = ar^2 = T_1\left(\frac{1}{2}\right)^{\frac{2}{12}} \doteqdot 0.8908T_1 \doteqdot \frac{8}{9}T_1$

$$T_2 = ar^1 = T_1\left(\frac{1}{2}\right)^{\frac{1}{12}} \doteqdot 0.9439T_1 \doteqdot \frac{17}{18}T_1$$

21a-d



21e $\angle OTM = 90°$ because it is an angle in a semi-circle, so OT is a tangent.



As OT is a tangent, it follows that $\angle OTA = \angle OBT$ as angle of a triangle inscribed in a circle is equal to the angle formed by the opposing chord and tangent. $\angle BOT = \angle AOT$ as they are the same angle. Hence OAT and OTB are equiangular and thus similar. As OAT and OTB are similar triangles, by the equal ratio of sides in similar triangles $\frac{OT}{OB} = \frac{OA}{OT}$. Thus $OT^2 = OA \times OB$.

Solutions to Exercise 1E

1a $S_4 = 3 + 5 + 7 + 9 = 24$

1b $S_4 = 2 + 6 + 18 + 54 = 80$

1c $S_4 = 2 + 1 + \frac{1}{2} + \frac{1}{4} = 3\frac{3}{4}$

1d $S_4 = 32 - 16 + 8 - 4 = 20$

2a $1 - 2 + 3 - 4 + 5 - 6$: by alternating positive and negative numbers

The sums are:

n	4	5	6
T_n	-4	5	-6
S_n	-2	3	-3

2b $81 + 27 + 9 + 3 + 1, \frac{1}{3}$: dividing by 3

The sums are:

n	4	5	6
T_n	3	1	$\frac{1}{3}$
S_n	120	121	$121\frac{1}{3}$

2c $30 + 20 + 10 + 0 - 10 - 20$: subtract 10

The sums are:

n	4	5	6
T_n	0	-10	-20
S_n	60	50	30

2d $0.1 + 0.01 + 0.001 + 0.0001 + 0.00001 + 0.0000001$: dividing by 3

The sums are:

n	4	5	6
T_n	0.0001	0.00001	0.000001
S_n	0.1111	0.111 11	0.111 111

3a

n	1	2	3	4	5	6	7
T_n	2	5	8	11	14	17	20
S_n	2	7	15	26	40	57	77

3b

n	1	2	3	4	5	6	7
T_n	40	38	36	34	32	30	28
S_n	40	78	114	148	180	210	238

3c

n	1	2	3	4	5	6	7
T_n	2	-4	6	-8	10	-12	14
S_n	2	-2	4	-4	6	-6	8

3d

n	1	2	3	4	5	6	7
T_n	7	−7	7	−7	7	−7	7
S_n	7	0	7	0	7	0	7

4a $\sum_{n=1}^{6} 2n = 2(1) + 2(2) + 2(3) + 2(4) + 2(5) + 2(6) = 42$

4b $\sum_{n=1}^{6}(3n + 2) = 5 + 8 + 11 + 14 + 17 + 20 = 75$

4c $\sum_{k=3}^{7}(18 - 3n) = 9 + 6 + 3 + 0 + (-3) = 15$

4d $\sum_{n=5}^{8} n^2 = 5^2 + 6^2 + 7^2 + 8^2 = 174$

4e $\sum_{n=1}^{4} n^3 = 1^3 + 2^3 + 3^3 + 4^3 = 100$

4f $\sum_{n=0}^{5} 2^n = 2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 = 63$

4g $\sum_{n=2}^{4} 3^n = 3^2 + 3^3 + 3^4 = 117$

4h $\sum_{l=1}^{31}(-1)^l = (-1)^1 + (-1)^2 + \cdots + (-1)^{31} = (-1) + 1 + (-1) + \cdots + (-1) = -1$

4i $\sum_{l=1}^{40}(-1)^{l-1} = (-1)^0 + (-1)^1 + \cdots + (-1)^{39} = (1) + (-1) + \cdots + (-1) = 0$

4j $\sum_{n=5}^{105} 4 = 4 + 4 \ldots 4 = 101 \times 4 = 404$

4k $\sum_{n=0}^{4}(-1)^n(n + 5) = 5 - 6 + 7 - 8 + 9 = 7$

4l $\sum_{n=0}^{4}(-1)^{n+1}(n + 5) = -5 + 6 - 7 + 8 - 9 = -7$

5a By looking at the diagram, it forms the shape of a square. The area of a square, with side length n, is n^2. Consequently, applying this logic, as the formation of the first n odd positive integers forms a square, the sum of them, which is equivalent to the area of the square, is n^2.

5b The way to calculate the sum is using the equation below:

Sum $=$ total elements in main diagonal $+$ half of remaining elements

By looking at the picture,

Total elements in matrix: n^2

Total elements in main diagonal: n

$$\begin{aligned}\text{Sum} &= n + \frac{n^2 - n}{2}\\ &= \frac{2n + n^2 - n}{2}\\ &= \frac{n^2 + n}{2}\\ &= \frac{1}{2}n(n+1)\end{aligned}$$

5c $T_1 = 1$
$T_2 = T_1 + 2 = 3$
$T_3 = T_2 + 3 = 6$
$T_4 = T_3 + 4 = 10$
$T_5 = T_4 + 5 = 15$
$T_6 = T_5 + 6 = 21$
$T_7 = T_6 + 7 = 28$
$T_8 = T_7 + 8 = 36$
$T_9 = T_8 + 9 = 45$
$T_{10} = T_9 + 10 = 55$
$T_{11} = T_{10} + 11 = 66$
$T_{12} = T_{11} + 12 = 78$
$T_{13} = T_{12} + 13 = 91$
$T_{14} = T_{13} + 14 = 105$
$T_{15} = T_{14} + 15 = 120$

6a

T_n	1	3	5	7	9	11	13
S_n	1	4	9	16	25	36	49

6b

T_n	2	4	8	16	32	64	128
S_n	2	6	14	30	62	126	254

6c

T_n	-3	-5	-7	-9	-11	-13	-15
S_n	-3	-8	-15	-24	-35	-48	-63

6d

T_n	8	-8	8	-8	8	-8	8
S_n	8	0	8	0	8	0	8

7a

n	1	2	3	4	5	6	7	8
T_n	1	1	1	2	3	5	8	13
S_n	1	2	3	5	8	13	21	34

7b

n	1	2	3	4	5	6	7	8
T_n	3	1	3	4	7	11	18	29
S_n	3	4	7	11	18	29	47	76

8a $S_1 = 3 - 1 = 2$
$S_2 = 3^2 - 1 = 8$
$S_3 = 3^3 - 1 = 26$
$S_4 = 3^4 - 1 = 80$
$S_5 = 3^5 - 1 = 242$

8b $T_1 = S_1 = 2$
$T_2 = S_2 - S_1 = 8 - 2 = 6$
$T_3 = S_3 - S_2 = 26 - 8 = 18$
$T_4 = S_4 - S_3 = 80 - 26 = 54$
$T_5 = S_5 - S_4 = 242 - 80 = 162$

8c $S_{n-1} = 3^{n-1} - 1$

$$\begin{aligned} T_n &= S_n - S_{n-1} \\ &= (3^n - 1) - (3^{n-1} - 1) \\ &= 3^n - 3^{n-1} \\ &= 3^{n-1}(3 - 1) \\ &= 2 \times 3^{n-1} \end{aligned}$$

9a $S_1 = 10$
$S_2 = 30$
$S_3 = 70$
$S_4 = 150$
$S_5 = 310$

$T_1 = S_1 = 10$
$T_2 = S_2 - S_1 = 30 - 10 = 20$
$T_3 = S_3 - S_2 = 70 - 30 = 40$
$T_4 = S_4 - S_3 = 150 - 70 = 80$
$T_5 = S_5 - S_4 = 310 - 150 = 160$

$S_{n-1} = 10(2^{n-1} - 1)$

$$\begin{aligned} T_n &= S_n - S_{n-1} \\ &= 10(2^n - 1) - 10(2^{n-1} - 1) \\ &= 10(2^n - 1 - 2^{n-1} + 1) \\ &= 10(2^n - 2^{n-1}) \\ &= 10 \times 2^{n-1}(2 - 1) \\ &= \frac{10}{2} \times 2^n(1) \\ &= 5 \times 2^n \end{aligned}$$

9b $S_1 = 16$
$S_2 = 96$
$S_3 = 496$
$S_4 = 2496$
$S_5 = 12496$

$T_1 = S_1 = 16$
$T_2 = S_2 - S_1 = 96 - 16 = 80$
$T_3 = S_3 - S_2 = 496 - 96 = 400$
$T_4 = S_4 - S_3 = 2496 - 496 = 2000$
$T_5 = S_5 - S_4 = 12496 - 2496 = 1000$

$S_{n-1} = 4(5^{n-1} - 1)$

$$\begin{aligned} T_n &= S_n - S_{n-1} \\ &= 4(5^n - 1) - 4(5^{n-1} - 1) \\ &= 4(5^n - 1 - 5^{n-1} + 1) \\ &= 4(5^n - 5^{n-1}) \\ &= 4 \times 5^{n-1}(5 - 1) \\ &= 4 \times 5^{n-1}(4) \\ &= 16 \times 5^{n-1} \end{aligned}$$

9c $S_1 = \frac{3}{4}$
$S_2 = \frac{15}{4}$
$S_3 = \frac{63}{4}$
$S_4 = \frac{255}{4}$
$S_5 = \frac{1023}{4}$

$T_1 = S_1 = \frac{3}{4}$
$T_2 = S_2 - S_1 = \frac{15}{4} - \frac{3}{4} = 3$
$T_3 = S_3 - S_2 = \frac{63}{4} - \frac{15}{4} = 12$
$T_4 = S_4 - S_3 = \frac{255}{4} - \frac{63}{4} = 48$
$T_5 = S_5 - S_4 = \frac{1023}{4} - \frac{255}{4} = 192$

$S_{n-1} = \frac{1}{4}(4^{n-1} - 1)$

$T_n = S_n - S_{n-1}$

$$\begin{aligned}
&= \frac{1}{4}(4^{n-1}-1) - \frac{1}{4}(4^{n-1}-1)\\
&= \frac{1}{4}(4^{n}-1) - \frac{1}{4}(4^{n-1}-1)\\
&= \frac{1}{4}(4^{n}-4^{n-1})\\
&= \frac{1}{4}4^{n-1}(4-1)\\
&= 4^{n-2}(3)\\
&= 3 \times 4^{n-2}
\end{aligned}$$

10a
$$\begin{aligned}
T_n &= S_n - S_{n-1}\\
&= 3n(n+1) - 3(n-1)(n-1+1)\\
&= 3n(n+1) - 3n(n-1)\\
&= 3n\big(n+1-(n-1)\big)\\
&= 3n(2)\\
&= 6n
\end{aligned}$$

$T_1 = 6, T_2 = 12, T_3 = 18$

10b
$$\begin{aligned}
T_n &= S_n - S_{n-1}\\
&= 5n - n^2 - (5(n-1) - (n-1)^2)\\
&= 5n - n^2 - \big(5(n-1) - (n^2 - 2n + 1)\big)\\
&= 5n - n^2 - (5n - 5 - n^2 + 2n - 1)\\
&= 5n - n^2 - (-n^2 + 7n - 6)\\
&= 6 - 2n
\end{aligned}$$

$T_1 = 4, T_2 = 2, T_3 = 0$

10c
$$\begin{aligned}
T_n &= S_n - S_{n-1}\\
&= 4n - 4(n-1)\\
&= 4
\end{aligned}$$
$T_1 = 4, T_2 = 4, T_3 = 4$

10d
$$\begin{aligned}
T_n &= S_n - S_{n-1}\\
&= n^3 - (n-1)^3\\
&= \big(n-(n-1)\big)(n^2 + n(n-1) + (n-1)^2)\\
&= (n^2 + n^2 - n + n^2 - 2n + 1)\\
&= 3n^2 - 3n + 1
\end{aligned}$$
$T_1 = 1, T_2 = 7, T_3 = 19$

10e $T_n = S_n - S_{n-1}$
$= 1 - 3^{-n} - \left(1 - 3^{-(n-1)}\right)$
$= 1 - 3^{-n} - (1 - 3^{1-n})$
$= 3^{1-n} - 3^{-n}$
$= 3^{-n}(3 - 1)$
$= 3^{-n}(2)$
$= 2 \times 3^{-n}$
$T_1 = \frac{2}{3}, T_2 = \frac{2}{9}, T_3 = \frac{2}{27}$

10f $T_n = S_n - S_{n-1}$
$= \left(\frac{1}{7}\right)^n - 1 - \left(\left(\frac{1}{7}\right)^{n-1} - 1\right)$
$= \left(\frac{1}{7}\right)^n - \left(\frac{1}{7}\right)^{n-1}$
$= \left(\frac{1}{7}\right)^{n-1}\left(\frac{1}{7} - 1\right)$
$= \left(\frac{1}{7}\right)^{n-1}\left(-\frac{6}{7}\right)$
$= -6\left(\frac{1}{7}\right)^{n-1}\left(\frac{1}{7}\right)$
$= -6\left(\frac{1}{7}\right)^n$
$T_1 = -\frac{6}{7}, T_2 = -\frac{6}{49}, T_3 = -\frac{6}{343}$

11a $\sum_{n=1}^{40} n^3$

11b $\sum_{n=1}^{40} \frac{1}{n}$

11c $\sum_{n=1}^{20} (n + 2)$

11d $\sum_{n=1}^{12} 2^n$

11e $\sum_{n=1}^{10} (-1)^n n$

11f $\sum_{n=1}^{10}(-1)^{n+1}n$ or $\sum_{n=1}^{10}(-1)^{n-1}n$

12a $T_1 = S_1 = 1 + 4 + 3 = 8$

$$\begin{aligned} T_n &= S_n - S_{n-1} \\ &= n^2 + 4n + 3 - ((n-1)^2 + 4(n-1) + 3) \\ &= n^2 + 4n + 3 - (n^2 - 2n + 1 + 4n - 4 + 3) \\ &= 2n + 3 \quad \text{for } n \geq 2 \end{aligned}$$

12b $T_1 = S_1 = 7(3^1 - 4) = -7$

$$\begin{aligned} T_n &= S_n - S_{n-1} \\ &= 7(3^n - 4) - 7(3^{n-1} - 4) \\ &= 7(3^n - 3^{n-1}) \\ &= 7 \times 3^{n-1}(3 - 1) \\ &= 7 \times 3^{n-1}(2) \\ &= 14 \times 3^{n-1} \quad \text{for } n \geq 2 \end{aligned}$$

12c $T_1 = S_1 = \frac{1}{1}$

$$\begin{aligned} T_n &= S_n - S_{n-1} \\ &= \frac{1}{n} - \frac{1}{n-1} \\ &= \frac{n-1}{n(n-1)} - \frac{n}{n(n-1)} \\ &= \frac{-1}{n(n-1)} \quad \text{for } n \geq 2 \end{aligned}$$

12d $T_1 = S_1 = 1 + 1 + 1 = 3$

$$\begin{aligned} T_n &= S_n - S_{n-1} \\ &= n^3 + n^2 + n - ((n-1)^3 + (n-1)^2 + (n-1)) \\ &= n^3 + n^2 + n - (n-1)((n-1)^2 + (n-1) + 1) \\ &= n^3 + n^2 + n - (n-1)(n^2 - 2n + 1 + n - 1 + 1) \\ &= n^3 + n^2 + n - (n-1)(n^2 - n + 1) \\ &= n^3 + n^2 + n - (n^3 - n^2 + n - n^2 + n - 1) \\ &= 3n^2 - n + 1 \quad \text{for } n \geq 1 \end{aligned}$$

The formula holds for $n = 1$ when $S_0 = 0$.

13a $T_1 = S_1 = 2^1 = 2$

$$T_n = S_n - S_{n-1}$$
$$= 2^n - 2^{n-1}$$
$$= 2^{n-1}(2-1)$$
$$= 2^{n-1} \quad \text{for } n \geq 2$$

13b

T_n	2	2	4	8	16	32	64
S_n	2	4	8	16	32	64	128

13c The derivative of e^x is the original function e^x. Remove the initial term 2 from the sequence in part b, and the successive differences are the original sequence.

T_n	2	4	8	16	32	64
S_n	4	8	16	32	64	128
$S_n - T_n$	2	4	8	16	32	64

14a $n^3 - (n-1)^3 = n^3 - (n^3 - 3n^2 + 3n + 1) = 3n^2 - 3n + 1$

14b $T_1 = S_1 = 1^3 = 1$

$$T_n = S_n - S_{n-1}$$
$$= n^3 - (n-1)^3$$
$$= 3n^2 - 3n + 1 \quad \text{for } n \geq 2$$

14c $U_1 = T_1 = 1,$

$$U_n = T_{n+1} - T_n$$
$$= (3(n+1)^2 - 3(n+1) + 1) - (3n^2 - 3n + 1)$$
$$= (3n^2 + 6n + 3 - 3n - 3 + 1) - (3n^2 - 3n + 1)$$
$$= 6n \quad \text{for } n \geq 2$$

14d

T_n	1	7	19	37	61	91
S_n	1	6	12	18	24	30

14e The derivative of x^3 is the quadratic $3x^2$, and its derivative is the linear function $6x$. Taking successive differences once gives a quadratic, and taking them twice gives a linear function.

15a

$$\sum_{r=1}^{10}\left(\frac{1}{r}-\frac{1}{r+1}\right)=\frac{1}{1}-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\cdots+\frac{1}{10}-\frac{1}{11}=\frac{1}{1}-\frac{1}{11}=\frac{11}{11}-\frac{1}{11}=\frac{10}{11}$$

15b

$$\frac{1}{\sqrt{k+1}+\sqrt{k}}$$

$$=\frac{1\times\left(\sqrt{k+1}-\sqrt{k}\right)}{\left(\sqrt{k+1}+\sqrt{k}\right)\left(\sqrt{k+1}-\sqrt{k}\right)}$$

$$=\frac{\sqrt{k+1}-\sqrt{k}}{(k+1)-k}$$

$$=\frac{\sqrt{k+1}-\sqrt{k}}{1}$$

$$=\sqrt{k+1}-\sqrt{k}$$

$$\sum_{k=1}^{15}\frac{1}{\sqrt{k+1}+\sqrt{k}}$$

$$=\sum_{k=1}^{15}\left(\sqrt{k+1}-\sqrt{k}\right)$$

$$=\sqrt{2}-\sqrt{1}+\sqrt{3}-\sqrt{2}+\sqrt{4}-\sqrt{3}+\sqrt{5}-\sqrt{4}\ldots+\sqrt{16}-\sqrt{15}$$

$$=\sqrt{16}-\sqrt{1}$$

$$=4-1$$

$$=3$$

15c

$$\sum_{r=1}^{4}\left(\sum_{s=1}^{4}\left(\sum_{t=1}^{4} rst\right)\right)$$

$$= \sum_{r=1}^{4}\left(\sum_{s=1}^{4}(rs + 2rs + 3rs + 4rs)\right)$$

$$= \sum_{r=1}^{4}\left(\sum_{s=1}^{4} rs(1 + 2 + 3 + 4)\right)$$

$$= \sum_{r=1}^{4}\left(\sum_{s=1}^{4} 10rs\right)$$

$$= \sum_{r=1}^{4}\big(10r(1) + 10r(2) + 10r(3) + 10r(4)\big)$$

$$= \sum_{r=1}^{4} 10r(1 + 2 + 3 + 4)$$

$$= \sum_{r=1}^{4} 100r$$

$$= 100(1) + 100(2) + 100(3) + 100(4)$$

$$= 1000$$

Solutions to Exercise 1F

1 $S_7 = 2 + 5 + 8 + 11 + 14 + 17 + 20$

$S_7 = 20 + 17 + 14 + 11 + 8 + 5 + 2$

$2S_7 = 7 \times 22 = 154$

$S_7 = \frac{154}{2} = 77$

2a $n = 100$

$S_n = \frac{1}{2}n(a + l)$

$a = 1, l = 100$

$S_{100} = \frac{1}{2} \times 100(100 + 1) = 5050$

2b $n = 50$

$S_n = \frac{1}{2}n(a + l)$

$a = 1, l = 99$

$S_{50} = \frac{1}{2} \times 50(99 + 1) = 2500$

2c $n = 50$

$S_n = \frac{1}{2}n(a + l)$

$a = 2, l = 100$

$S_{50} = \frac{1}{2} \times 50(100 + 2) = 2550$

2d $n = 100$

$S_n = \frac{1}{2}n(a + l)$

$a = 3, l = 300$

$S_{100} = \frac{1}{2} \times 100(300 + 3) = 15\,150$

2e $n = 50$

$S_n = \frac{1}{2} n(a + l)$

$a = 101, l = 199$

$S_{50} = \frac{1}{2} \times 50(101 + 199) = 7500$

2f $n = 9000$

$S_n = \frac{1}{2} n(a + l)$

$a = 1001, l = 10000$

$S_{100} = \frac{1}{2} \times 100(100 + 1) = 49\,504\,500$

3a $S_6 = \frac{1}{2} \times 6(10 + 5 \times 10) = 3(60) = 180$

3b $S_6 = \frac{1}{2} \times 6(16 + 5 \times 2) = 3(26) = 78$

3c $S_6 = \frac{1}{2} \times 6(-6 + 5 \times -9) = 3(51) = -153$

3d $S_6 = \frac{1}{2} \times 6(-14 + 5 \times -12) = 3(-74) = -222$

4a $a = 2, d = 3$

$S_{12} = \frac{1}{2} \times 12(4 + 11 \times 3) = 6(37) = 222$

4b $a = 40, d = -7$

$$S_{21} = \frac{1}{2} \times 21(80 + 20 \times -7) = 10.5(-60) = -630$$

4c $a = -6, d = 4$

$$S_{200} = \frac{1}{2} \times 200(-12 + 199 \times 4) = 100(784) = 78\,400$$

4d $a = 33, d = -3$

$$S_{23} = \frac{1}{2} \times 23(66 + 22 \times -3) = 11.5(0) = 0$$

4e $a = -10, d = 2.5$

$$S_{13} = \frac{1}{2} \times 13(-20 + 12 \times 2.5) = 6.5(-10) = 65$$

4f $a = 10.5, d = -0.5$

$$S_{40} = \frac{1}{2} \times 40(21 + 39 \times -0.5) = 20(1.5) = 30$$

5a $150 = 50 + (n-1)1$
$100 = n - 1$
$n = 101$
$S_{101} = 50.5(50 + 150) = 50.5(200) = 10\,100$

5b $92 = 8 + (n-1)7$

$$\frac{84}{7} = n - 1$$

$n = 13$

$S_{13} = 6.5(8 + 92) = 6.5(100) = 650$

5c $60 = -10 + (n-1)7$

$\frac{70}{7} = n - 1$

$n = 11$

$S_{11} = 5.5(-10 + 60) = 5.5(50) = 275$

5d $301 = 4 + (n-1)3$

$\frac{297}{3} = n - 1$

$n = 100$

$S_{100} = 50(4 + 301) = 50(305) = 15\,250$

5e $51.5 = 6.5 + (n-1)4.5$

$\frac{45}{4.5} = n - 1$

$n = 11$

$S_{11} = 5.5(6.5 + 51.5) = 5.5(58) = 319$

5f $13\frac{2}{3} = -1\frac{1}{3} + (n-1)1\frac{2}{3}$

$\frac{15}{\frac{5}{3}} = n - 1$

$n = 10$

$S_{10} = 5\left(-1\frac{1}{3} + 13\frac{2}{3}\right) = 5\left(12\frac{1}{3}\right) = 61\frac{2}{3}$

6a $1000 = 2 + (n-1)2$

$\frac{998}{2} = n - 1$

$n = 500$

$S_{500} = 250(2 + 1000) = 250(1002) = 250\,500$

6b $3000 = 1000 + (n-1)1$

$2000 = n - 1$

$n = 2001$

$S_{2001} = 1000.5(1000 + 3000) = 1000.5(4000) = 4\,002\,000$

6c $S_{40} = \frac{1}{2} \times 40(2 + 39 \times 4) = 20(158) = 3160$

6d $S_{12} = \frac{1}{2} \times 12(20 + 11 \times 20) = 6(240) = 1440$

7a This is an AP with $a = 5$ and $d = 10 - 5 = 5$. Hence:

$$\begin{aligned} S_n &= \frac{n}{2}(2a + (n-1)d) \\ &= \frac{n}{2}\big(2(5) + 5(n-1)\big) \\ &= \frac{n}{2}(10 + 5n - 5) \\ &= \frac{n}{2}(5 + 5n) \end{aligned}$$

7b This is an AP with $a = 10$ and $d = 13 - 10 = 3$. Hence:

$$\begin{aligned} S_n &= \frac{n}{2}(2a + (n-1)d) \\ &= \frac{n}{2}\big(2(10) + 3(n-1)\big) \\ &= \frac{n}{2}(20 + 3n - 3) \\ &= \frac{n}{2}(17 + 3n) \end{aligned}$$

7c This is an AP with $a = 3$ and $d = 7 - 3 = 4$. Hence:

$$\begin{aligned} S_n &= \frac{n}{2}(2a + (n-1)d) \\ &= \frac{n}{2}\big(2(3) + 4(n-1)\big) \\ &= \frac{n}{2}(6 + 4n - 4) \\ &= \frac{n}{2}(2 + 4n) \\ &= n(1 + 2n) \end{aligned}$$

7d This is an AP with $a = -9$ and $d = -4 - (-9) = 5$. Hence:

$$\begin{aligned} S_n &= \frac{n}{2}(2a + (n-1)d) \\ &= \frac{n}{2}\big(2(-9) + 5(n-1)\big) \\ &= \frac{n}{2}(-18 + 5n - 5) \\ &= \frac{n}{2}(5n - 23) \end{aligned}$$

7e This is an AP with $a = 5$ and $d = 4\frac{1}{2} - 5 = -\frac{1}{2}$. Hence:

$$\begin{aligned} S_n &= \frac{n}{2}(2a + (n-1)d) \\ &= \frac{n}{2}\left(2(5) + \left(-\frac{1}{2}\right)(n-1)\right) \\ &= \frac{n}{2}\left(10 - \frac{n}{2} + \frac{1}{2}\right) \\ &= \frac{n}{4}(21 - n) \end{aligned}$$

7f This is an AP with $a = (1 - \sqrt{2})$ and $d = 1 - (1 - \sqrt{2}) = \sqrt{2}$. Hence:

$$\begin{aligned} S_n &= \frac{n}{2}(2a + (n-1)d) \\ &= \frac{n}{2}\big(2(1 - \sqrt{2}) + \sqrt{2}(n-1)\big) \\ &= \frac{n}{2}(2 - 2\sqrt{2} + n\sqrt{2} - \sqrt{2}) \\ &= \frac{n}{2}(2 + n\sqrt{2} - 3\sqrt{2}) \end{aligned}$$

8a n positive integers are: $1 + 2 + 3 + 4 \ldots.$

$a = 1, d = 1$

$$\begin{aligned} S_n &= \frac{n}{2}(2a + (n-1)d) \\ &= \frac{n}{2}\big(2 + (n-1)\big) \\ &= \frac{1}{2}n(n+1) \end{aligned}$$

8b Odd positive integers are: $1 + 3 + 5 + 7 + \cdots$

$a = 1, d = 2$

$$S_n = \frac{n}{2}(2a + (n-1)d)$$
$$= \frac{n}{2}(2 + (n-1)2)$$
$$= \frac{1}{2}n(2 + 2n - 2)$$
$$= \frac{1}{2}n(2n)$$
$$= n^2$$

8c Positive integers divisible by 3 are: $3 + 6 + 9 + 12 + 15 + \cdots$

$a = 3, d = 3$

$$S_n = \frac{n}{2}(2a + (n-1)d)$$
$$= \frac{n}{2}(6 + (n-1)3)$$
$$= \frac{1}{2}n(3n + 3)$$
$$= \frac{3}{2}n(n + 1)$$

8d Odd positive multiples of 100 are: $100 + 300 + 500 + 700 + \cdots$

$a = 100, d = 200$

$$S_n = \frac{n}{2}(2a + (n-1)d)$$
$$= \frac{n}{2}(200 + (n-1)200)$$
$$= \frac{1}{2}n(200 + 200n - 200)$$
$$= \frac{1}{2}n(200n)$$
$$= 100n^2$$

9a $15 \times 0 + 15 \times 2 + 15 \times 4 + 15 \times 6 + 15 \times 8 + 15 \times 10 = 450$

The mean number of legs is $\frac{450}{90} = 5$. No creature has this number of legs.

9b $1200 \times \left(\frac{6+17}{2}\right) + 100 \times 30 + 60 = 16\,860$ years

9c His earnings are a GP with $a = 28\,000$, $r = 1600$ and $n = 10$. Hence:

$$S_{10} = \frac{a(r^n - 1)}{r - 1}$$

$$= \frac{28\,000(1600^{10} - 1)}{1600 - 1}$$

$$= \$352\,000$$

10a When $k = 1$: $a = 598$

When $k = 200$: $l = 200$

$200 = 598 + (n-1)(-2)$

$-\frac{398}{-2} = n - 1$

$n = 200$

$S_{200} = 100(598 + 200) = 100(798) = 79\,800$

10b When $k = 1$: $a = 90$

When $k = 61$: $l = -90$

$-90 = 90 + (n-1)(-3)$

$\frac{0}{3} = n - 1$

$n = 1$

$S_1 = 0.5(90 - 90) = 0.5(0) = 0$

10c When $k = 1$: $a = -47$

When $k = 40$: $l = 70$

$70 = -47 + (n - 1)3$

$\frac{117}{3} = n - 1$

$n = 40$

$S_{40} = 20(-47 + 70) = 20(23) = 460$

10d When $k = 10$: $a = 53$

When $k = 30$: $l = 153$

$153 = 53 + (n - 1)5$

$\frac{100}{5} = n - 1$

$n = 21$

$S_{21} = 10.5(53 + 153) = 10.5(206) = 2163$

11a For the AP:

$a = \log_a 2$

$d = \log_a 4 - \log_a 2$

$\quad = \log_a 2^2 - \log_a 2$

$\quad = 2\log_a 2 - \log_a 2$

$\quad = \log_a 2$

For the last term:

$T_n = \log_a 1024$

$a + (n - 1)d = \log_a 1024$

$\log_a 2 + (n - 1)\log_a 2 = \log_a 1024$

$n\log_a 2 = \log_a 1024$

$n\log_a 2 = \log_a 2^{10}$

$n\log_a 2 = 10\log_a 2$

$n = 10$

$$S_n = \frac{1}{2}n(a + l)$$
$$S_{10} = \frac{1}{2} \times 10 \times (\log_a 2 + \log_a 1024)$$
$$= 5(\log_a 2 + 10\log_a 2)$$
$$= 5 \times 11\log_a 2$$
$$= 55\log_a 2$$

11b For the AP:

$$a = \log_5 243$$
$$d = \log_5 81 - \log_5 243$$
$$= \log_5 3^4 - \log_5 3^5$$
$$= 4\log_5 3 - 5\log_5 3$$
$$= -\log_5 3$$

For the last term:

$$T_n = \log_5 \frac{1}{243}$$
$$a + (n-1)d = \log_5 \frac{1}{243}$$
$$\log_5 243 + (n-1)(-\log_5 3) = \log_5 3^{-5}$$
$$\log_5 3^5 + (n-1)(-\log_5 3) = \log_5 3^{-5}$$
$$5\log_5 3 + (n-1)(-\log_5 3) = -5\log_5 3$$
$$(n-1)(-\log_5 3) = -10\log_5 3$$
$$n - 1 = 10$$
$$n = 11$$
$$S_n = \frac{1}{2}n(a + l)$$
$$S_{11} = \frac{1}{2} \times 11 \times \left(\log_5 243 + \log_5 \frac{1}{243}\right)$$
$$= \frac{11}{2}(\log_5 243 - \log_5 243)$$
$$= 0$$

11c For the AP:

$a = \log_b 36$

$d = \log_b 18 - \log_b 36$

$= \log_b \frac{18}{36}$

$= \log_b \frac{1}{2}$

$= -\log_b 2$

For the last term:

$T_n = \log_b \frac{9}{8}$

$a + (n-1)d = \log_b \frac{9}{8}$

$\log_b 36 + (n-1)(-\log_b 2) = \log_b \frac{9}{8}$

$(n-1)(-\log_b 2) = \log_b \frac{9}{8} - \log_b 36$

$(n-1)(-\log_b 2) = \log_b \left(\frac{9}{8} \div 36\right)$

$(n-1)(-\log_b 2) = \log_b \frac{1}{32}$

$(n-1)(-\log_b 2) = \log_b 2^{-5}$

$(n-1)(-\log_b 2) = -5\log_b 2$

$n - 1 = 5$

$n = 6$

$S_n = \frac{1}{2}n(a + l)$

$S_6 = \frac{1}{2} \times 6 \times \left(\log_b 36 + \log_b \frac{9}{8}\right)$

$= 3\log_b \left(36 \times \frac{9}{8}\right)$

$= 3\log_b \left(\frac{81}{2}\right)$

$= 3\left(\log_b 81 - \log_b 2\right)$

$= 3\,(\log_b 3^4 - \log_b 2)$

$= 3\,(4\log_b 3 - \log_b 2)$

11d For the AP:

$a = \log_x \dfrac{27}{8}$

$d = \log_x \dfrac{9}{4} - \log_x \dfrac{27}{8}$

$d = \log_x \left(\dfrac{9}{4} \div \dfrac{27}{8}\right)$

$d = \log_x \left(\dfrac{9}{4} \times \dfrac{8}{27}\right)$

$d = \log_x \dfrac{2}{3}$

$S_n = \dfrac{1}{2} n(2a + (n-1)d)$

$S_{10} = \dfrac{1}{2} \times 10 \times \left(2 \log_x \dfrac{27}{8} + (10-1) \log_x \dfrac{2}{3}\right)$

$= 5\left(2 \log_x \dfrac{27}{8} + 9 \log_x \dfrac{2}{3}\right)$

$= 5\left(\log_x \left(\dfrac{27}{8}\right)^2 + \log_x \left(\dfrac{2}{3}\right)^9\right)$

$= 5\left(\log_x \dfrac{3^6}{2^6} + \log_x \dfrac{2^9}{3^9}\right)$

$= 5 \log_x \left(\dfrac{3^6 \times 2^9}{2^6 \times 3^9}\right)$

$= 5 \log_x \left(\dfrac{2^3}{3^3}\right)$

$= 5\,(\log_x 2^3 - \log_x 3^3)$

$= 5\,(3\log_x 2 - 3\log_x 3)$

$= 15\,(\log_x 2 - \log_x 3)$

12a $S_n = \frac{n}{2}(a + l)$

$-5 = \frac{10}{2}(-23 + l)$

$-1 = -23 + l$

$l = 22$

12b $S_n = \frac{n}{2}(a + l)$

$28 = \frac{40}{2}\left(a + 8\frac{1}{2}\right)$

$\frac{28}{20} = a + 8\frac{1}{2}$

$a = \frac{28}{20} - 8\frac{1}{2}$

$= -7.1$

12c $S_n = \frac{n}{2}(2a + (n - 1)d)$

$348 = \frac{8}{2}(2(5) + (8 - 1)d)$

$87 = 10 + (8 - 1)d$

$87 = 10 + 7d$

$7d = 77$

$d = 11$

12d $S_n = \frac{n}{2}(2a + (n - 1)d)$

$-15 = \frac{15}{2}\left(2a + (n - 1)\frac{2}{7}\right)$

$-2 = \left(2a + (15 - 1)\frac{2}{7}\right)$

$-2 = \left(2a + (14)\frac{2}{7}\right)$

$-2 = 2a + 4$

$2a + 4 = -2$

$2a = -6$

$a = -3$

13a $a = 60, d = -8, n = n$

$$S_n = \frac{n}{2}(2a + (n-1)d)$$

$$= \frac{n}{2}\big(120 + (n-1)(-8)\big)$$

$$= \frac{n}{2}(120 - 8n + 8)$$

$$= \frac{n}{2}(128 - 8n)$$

$$= 4n(16 - n)$$

13b i $0 = 4n(16 - n)$

Either $4n = 0$ and therefore $n = 0$

or $16 - n = 0$ and $n = 16$

Therefore, 16 terms.

13b ii To make it negative, it would be more than 16 terms.

13c $220 = 4n(16 - n)$

$220 = 64n - 4n^2$

$0 = -4n^2 + 64n - 220$

$0 = -4(n^2 - 16n + 55)$

$0 = -4(n - 11)(n - 5)$

Therefore, either 5 terms or 11 terms.

13d $-144 = 4n(16 - n)$

$-144 = 64n - 4n^2$

$0 = -4n^2 + 64n + 144$

$0 = -4(n^2 - 16n - 36)$

$0 = -4(n - 18)(n + 2)$

Therefore, $n = 18$ or $n = -2$, but n must be a positive integer

13e $156 < 4n(16 - n)$

$0 < -4n^2 + 64n - 156$

$0 < -4(n^2 - 16n + 39)$

$0 < -4(n - 13)(n - 3)$

$0 > (n - 13)(n - 3)$

$3 < n < 13$

Therefore, $n = 4, 5, 6, \dots, 12$.

13f $4n(16 - n) > 256$

$-4n^2 + 64n - 256 > 0$

$-4(n^2 + 16n - 64) > 0$

$(n - 8)^2 < 0$
$S_n > 256$ gives $(n - 8)^2 < 0$, which has no solutions.

Therefore, the sum cannot exceed 256.

14a $a = 42, d = 40 - 42 = -2$. Hence:

$$S_n = \frac{1}{2}n(2a + (n - 1)d)$$
$$= \frac{1}{2}n\big(2 \times 42 + (n - 1)(-2)\big)$$
$$= \frac{1}{2}n(84 - 2n + 2)$$
$$= \frac{1}{2}n(86 - 2n)$$
$$= n(43 - n)$$

Put $S_n = 0$
$n(43 - n) = 0$

$n = 0$ or 43

Hence 43 terms.

14b $a = 60$, $d = 57 - 60 = -3$. Hence:

$$\begin{aligned} S_n &= \frac{1}{2}n(2a + (n-1)d) \\ &= \frac{1}{2}n\big(2 \times 60 + (n-1)(-3)\big) \\ &= \frac{1}{2}n(120 - 3n + 3) \\ &= \frac{1}{2}n(123 - 3n) \\ &= \frac{3}{2}n(41 - n) \end{aligned}$$

Put $S_n = 0$

$$\frac{3}{2}n(41 - n) = 0$$

$$n(41 - n) = 0$$

$$n = 0 \text{ or } 41$$

Hence 41 terms.

14c $a = 45$, $d = 51 - 45 = 6$. Hence:

$$\begin{aligned} S_n &= \frac{1}{2}n(2a + (n-1)d) \\ &= \frac{1}{2}n(2 \times 45 + (n-1) \times 6) \\ &= \frac{1}{2}n(90 + 6n - 6) \\ &= \frac{1}{2}n(84 + 6n) \\ &= n(42 + 3n) \\ &= 3n(n + 14) \end{aligned}$$

Put $S_n = 153$

$$3n(n + 14) = 153$$

$$3n^2 + 42n = 153$$

$$3n^2 + 42n - 153 = 0$$

$$n^2 + 14n - 51 = 0$$

$$(n + 17)(n - 3) = 0$$

$$n = -17 \text{ or } 3$$

Hence 3 terms.

14d $a = 2\frac{1}{2}, d = 3 - 2\frac{1}{2} = \frac{1}{2}$. Hence:

$$\begin{aligned} S_n &= \frac{1}{2}n(2a + (n-1)d) \\ &= \frac{1}{2}n\left(2 \times 2\frac{1}{2} + (n-1) \times \frac{1}{2}\right) \\ &= \frac{1}{2}n\left(5 + \frac{1}{2}n - \frac{1}{2}\right) \\ &= \frac{1}{2}n\left(4\frac{1}{2} + \frac{1}{2}n\right) \\ &= \frac{1}{4}n(9+n) \end{aligned}$$

Put $S_n = 22\frac{1}{2}$

$$\frac{1}{4}n(9+n) = 22\frac{1}{2}$$
$$n(9+n) = 90$$
$$n^2 + 9n - 90 = 0$$
$$(n+15)(n-6) = 0$$
$$n = 6 \text{ or } -15$$

Hence 6 terms.

15a Put $S_n = 0$

$$\frac{n}{2}(a + l) = 0$$

$$\frac{n}{2}(a + 32) = 0$$

$$a = -32$$

$$T_n = a + (n-1)d = -32 + (n-1)(4) = 4n - 36$$

Put $T_n = l = 32$

$$4n - 36 = 32$$

$$4n = 68$$

$$n = 17$$

15b Put $S_n = 55$

$$\frac{n}{2}(a + l) = 55$$

$$\frac{n}{2}(a - 10) = 55 \qquad (1)$$

$T_n = a + (n-1)(-3)$

Put $T_n = l = -10$

$a + (n-1)(-3) = -10$

$a - 3n + 3 = -10$

$a - 3n = -13$

$a = 3n - 13 \quad (2)$

Substituting this into (1) gives

$\frac{n}{2}(3n - 13 - 10) = 55$

$3n^2 - 23n = 110$

$3n^2 - 26n - 110 = 0$

$n = \frac{23 \pm \sqrt{23^2 - 4(3)(-110)}}{2(3)}$

$= \frac{23 \pm 43}{2(3)}$

$= 11, -\frac{10}{3}$

So $n = 11$ as n must be a positive integer.

Substituting this into (2) gives $a = 20$.

16a The number of logs in a row is an AP with $a = 10$ and $d = 1$

$S_n = \frac{1}{2}n(2a + (n-1)d)$

$= \frac{1}{2}n(20 + (n-1) \times 1)$

$= \frac{1}{2}n(n + 19)$

Put $S_n = 390$

$390 = \frac{1}{2}n(n + 19)$

$780 = n^2 + 19n$

$n^2 + 19n - 780 = 0$

$(n-20)(n+39)=0$

$n=20$ is the only positive solution, hence there are 20 rows, the bottom row will have $T_{20}=10+(20-1)\times 1=29$ logs.

16b Distance per second is an AP with $a=5$ and $d=10$.

Hence the total distance travelled after n seconds is:

$$S_n=\frac{1}{2}n(2a+(n-1)d)$$

$$=\frac{1}{2}n(2\times 5+(n-1)\times 10)$$

$$=5n^2$$

For a total distance of 245 m, set $S_n=245$, hence:

$245=5n^2$

$n^2=49$

$n=\pm 7$

As time is positive, it will be 7 seconds until the stone hits the ground.

16c The distance with each trip and back forms an AP with $a=20\times 2=40$ and $l=30\times 2=60$.

Now, $S_n=550$ and hence:

$550=\frac{1}{2}n(a+l)$

$1100=n(40+60)$

$n=11$

So there are 11 trips.

Now considering one way trips:

$T_{11}=30$

$30=20+(11-1)d$

$10=10d$

$d=1$

So the deposits are 1 km apart.

17a $T_4 + T_1 = 16$

$a + 3d + a = 16$

$2a + 3d = 16 \quad (1)$

$T_3 + T_8 = 4$

$a + 2d + a + 7d = 4$

$2a + 9d = 4 \quad (2)$

$6d = -12 \quad (2) - (1)$

$d = -2 \quad (3)$

$2a - 6 = 16 \quad (3) \text{ in } (1)$

$a = 11$

$S_n = \frac{1}{2}n(2a + (n-1)d)$

$S_{10} = 5(2 \times 11 + 9 \times -2)$

$= 5(22 - 18)$

$= 20$

17b $S_{10} = 0$

$\frac{1}{2} \times 10 \times (2a + (10-1)d) = 0$

$5(2a + 9d) = 0$

$2a + 9d = 0 \quad (1)$

$T_{10} = -9$

$a + (10-1)d = -9$

$a + 9d = -9 \quad (2)$

$a = 9 \quad (1) - (2)$

Substituting $a = 9$ into (2) gives:

$9 + 9d = -9$

$9d = -18$

$d = -2$

$T_1 = a = 9$

$T_2 = a + (2-1)d = 9 - 2 = 7$

17c $S_{16} = 96$

$\frac{1}{2}(16)(2a + (16-1)d) = 96$

$8(2a + 15d) = 96$

$2a + 15d = 12 \quad (1)$

$T_2 + T_4 = 45$

$a + d + a + 3d = 45$

$2a + 4d = 45 \quad (2)$

$11d = -33 \quad (1) - (2)$

$d = -3$

Substituting $d = -3$ into (1) gives:

$2a + 15(-3) = 12$

$2a = 57$

$a = 28\frac{1}{2}$

$T_4 = a + 3d = 28\frac{1}{2} + 3(-3) = 19\frac{1}{2}$

$S_n = \frac{1}{2}n(2a + (n-1)d)$

$S_4 = 2\left(2 \times 28\frac{1}{2} + 3 \times -3\right)$

$= 2(57 - 9)$

$= 96$

18a This is an AP with $a = 1$, $l = 24$ and $n = 24$. Hence:

$1 + 2 + \cdots + 24 = S_{24} = \frac{24}{2}(1 + 24) = 300$

18b This is an AP with $a = \frac{1}{n}$, $l = \frac{n}{n}$ and $n = n$. Hence:

$$\frac{1}{n} + \frac{2}{n} + \cdots + \frac{n}{n} = S_n = \frac{n}{2}\left(\frac{1}{n} + \frac{n}{n}\right) = \frac{n}{2}\left(\frac{1+n}{n}\right) = \frac{n+1}{2}$$

18c The sequence is $\left(\frac{1}{1}\right) + \left(\frac{1}{2} + \frac{2}{2}\right) + \left(\frac{1}{3} + \frac{2}{3} + \frac{3}{3}\right) + \cdots$

Now the number of terms in each set of brackets is 1, 2, 3, …

Hence, using part a, we see that $\left(\frac{1}{1}\right) + \left(\frac{1}{2} + \frac{2}{2}\right) + \left(\frac{1}{3} + \frac{2}{3} + \frac{3}{3}\right) + \cdots + \left(\frac{1}{24} + \cdots + \frac{24}{24}\right)$ will have 300 terms. Rearranging this sequence, and using part b we get:

$$\begin{aligned}
&\frac{1+1}{2} + \frac{2+1}{2} + \frac{3+1}{2} + \cdots + \frac{24+1}{2} \\
&= \frac{1}{2}(1 + 1 + 2 + 1 + 3 + 1 + \cdots + 24 + 1) \\
&= \frac{1}{2}\big((1 + 1 + \cdots + 1) + (1 + 2 + 3 + \cdots + 24)\big) \\
&= \frac{1}{2}(24 + 300) \\
&= 162
\end{aligned}$$

19a This is an AP with $a = 1$, $l = n$, $n = n$ and hence:

$$\begin{aligned}
S_n &= \frac{n}{2}(a + l) \\
&= \frac{n}{2}(1 + n) \\
&= \frac{1}{2}n(n+1)
\end{aligned}$$

19b i $S_n = \frac{1}{2}n(n+1)$ is divisible by 5 if n is divisible by 5 (in which case n ends in 0 or 5) or $n + 1$ is divisible by 5 (in which case n ends in 4 or 9).

19b ii S_n is even if n is divisible by 4 or $n + 1$ is divisible by 4 (in which case the remainder is 3 when $n + 1$ is divided by 4).

19c i 29 is prime, so S_n is divisible by 29 if n is divisible by 29 or $n + 1$ is divisible by 29. So the smallest value of n is $n = 28$. $S_{28} = \frac{1}{2} \times 28 \times 29 = 14 \times 29$

19c ii We want prime factors of 7 and 5 in order to obtain the smallest value of n. So we require consecutive integers n and $n+1$ of which one is divisible by 7 and the other by 5. By trial and error, the smallest value of n that fulfils this requirement is $n = 14$.

$$S_{14} = \frac{1}{2} \times 14 \times 15 = 7 \times 15 = 7 \times 5 \times 3$$

19c iii We want prime factors of 2 and 13 in order to obtain the smallest value of n. So we require consecutive integers n and $n+1$ of which one is divisible by 2 and the other by 13. By trial and error, the smallest value of n that fulfils this requirement is $n = 12$.

$$S_{12} = \frac{1}{2} \times 12 \times 13 = 6 \times 13 = 3 \times 2 \times 13$$

19c iv We want prime factors of 2 and 19 in order to obtain the smallest value of n. So we require consecutive integers n and $n+1$ of which one is divisible by 2 and the other by 19. By trial and error, the smallest value of n that fulfils this requirement is $n = 19$.

$$S_{19} = \frac{1}{2} \times 19 \times 20 = 19 \times 10 = 19 \times 2 \times 10$$

19c v We want two distinct prime numbers (eg 2, 3, 5, 7, 11,…) in order to obtain the smallest value of n. So we require consecutive integers n and $n+1$ of which one is divisible by a prime number and the other by a different prime number. By trial and error, the smallest value of n that fulfils this requirement is $n = 3$ where S_n is divisible by the two distinct primes 2 and 3.

$$S_3 = \frac{1}{2} \times 3 \times 4 = 3 \times 2$$

19c vi We want three distinct prime numbers in order to obtain the smallest value of n. So we require consecutive integers n and $n+1$ of which one is divisible by one or more distinct prime numbers and the other by one or more different prime numbers. By trial and error, the smallest value of n that fulfils this requirement is $n = 11$ where S_n is divisible by the three distinct primes 2, 3 and 11.

$$S_{11} = \frac{1}{2} \times 11 \times 12 = 11 \times 6 = 11 \times 2 \times 3$$

19c vii We want four distinct prime numbers in order to obtain the smallest value of n. So we require consecutive integers n and $n + 1$ of which one is divisible by one or more distinct prime numbers and the other by one or more different prime numbers. By trial and error, the smallest value of n that fulfils this requirement is $n = 20$ where S_n is divisible by the three distinct primes 2, 3, 5 and 7.

$$S_{20} = \frac{1}{2} \times 20 \times 21 = 10 \times 21 = 2 \times 5 \times 3 \times 7$$

Solutions to Exercise 1G

1 $3S_6 = (2 \times 3) + (6 \times 3) + (18 \times 3) + (54 \times 3) + (162 \times 3) + (486 \times 3)$

$= 6 + 18 + 54 + 162 + 486 + 1458$

$S_6 = 2 + 6 + 18 + 54 + 162 + 486$

$3S_6 - S_6 = (6 - 2) + (18 - 6) + (54 - 18) + (162 - 54) + (486 - 163) + (1458 - 486)$

$2S_6 = 1456$

$S_6 = 728$

2 If one speaker was going to St Ives, the rest are going the other way:

Number going other way $= 7^0 + 7^1 + 7^2 + 7^3 + 7^4$

This is a GP with $a = 1$ and $r = 7$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_5 = \frac{1 \times (7^5 - 1)}{7 - 1}$$

$= 2801$

3a GP with $a = 1$ and $r = 3$

$$S_7 = \frac{a(r^7 - 1)}{r - 1}$$

$$S_7 = \frac{1 \times (3^7 - 1)}{3 - 1}$$

$= 1093$

3b GP with $a = 1$ and $r = -3$

$$S_7 = \frac{a(1 - r^7)}{1 - r}$$

$$S_7 = \frac{1 \times (1 - (-3)^7)}{1 - (-3)}$$

$= 547$

4a GP with $a = 1$ and $r = 2$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_{10} = \frac{1 \times (2^{10} - 1)}{2 - 1}$$

$$= 1023$$

$$S_n = \frac{1(2^n - 1)}{2 - 1}$$

$$= 2^n - 1$$

4b GP with $a = 2$ and $r = 3$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_5 = \frac{2 \times (3^5 - 1)}{3 - 1}$$

$$= 242$$

$$S_n = \frac{2(3^n - 1)}{3 - 1}$$

$$= 3^n - 1$$

4c GP with $a = -1$ and $r = 10$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_5 = \frac{-1 \times (10^5 - 1)}{10 - 1}$$

$$= -11\ 111$$

$$S_n = \frac{-1(10^n - 1)}{10 - 1}$$

$$= -\tfrac{1}{9}(10^n - 1)$$

4d GP with $a = -1$ and $r = 5$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_5 = \frac{-1 \times (5^5 - 1)}{5 - 1}$$

$$= -781$$

$$S_n = \frac{-1(5^n - 1)}{5 - 1}$$

$$= -\tfrac{1}{4}(5^n - 1)$$

4e GP with $a = 1$ and $r = -2$

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

$$S_{10} = \frac{1 \times (1 - (-2)^{10})}{1 - (-2)}$$

$$= -341$$

$$S_n = \frac{1(1 - (-2^n))}{1 - (-2)}$$

$$= \tfrac{1}{3}(1 - (-2)^n)$$

4f GP with $a = 2$ and $r = -3$

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

$$S_5 = \frac{2 \times (1 - (-3)^5)}{1 - (-3)}$$

$$= 122$$

$$S_n = \frac{2(1 - (-3^n))}{1 - (-3)}$$

$$= \tfrac{1}{2}(1 - (-3)^n)$$

4g GP with $a = -1$ and $r = -10$

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$S_5 = \frac{-1 \times (1-(-10)^5)}{1-(-10)}$$

$$= -9091$$

$$S_n = \frac{-1(1-(-10)^n))}{1-(-10)}$$

$$= -\tfrac{1}{11}(1-(-10)^n)$$

4h GP with $a = -1$ and $r = -5$

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$S_5 = \frac{-1 \times (1-(-5)^5)}{1-(-5)}$$

$$= -521$$

$$S_n = \frac{-1(1-(-5)^n))}{1-(-5)}$$

$$= -\tfrac{1}{6}(1-(-5)^n)$$

5a GP with $a = 8$ and $r = \frac{1}{2}$

$$S_n = \frac{a(r^n-1)}{r-1}$$

$$S_{10} = \frac{8 \times \left(\left(\frac{1}{2}\right)^{10} - 1\right)}{\frac{1}{2} - 1}$$

$$= \frac{8 \times \left(\frac{1}{1024} - 1\right)}{-\frac{1}{2}}$$

$$= -16\left(\frac{1}{1024} - 1\right)$$

$$= 16\left(\frac{1023}{1024}\right)$$

$$= \frac{1023}{64}$$

$$S_n = \frac{8\left(\left(\frac{1}{2}\right)^n - 1\right)}{\frac{1}{2} - 1}$$

$$= -16\left(\left(\frac{1}{2}\right)^n - 1\right)$$

$$= 16\left(1 - \left(\frac{1}{2}\right)^n\right)$$

5b GP with $a = 9$ and $r = \frac{1}{3}$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_6 = \frac{9 \times \left(\left(\frac{1}{3}\right)^6 - 1\right)}{\frac{1}{3} - 1}$$

$$= \frac{9 \times \left(\frac{1}{729} - 1\right)}{-\frac{2}{3}}$$

$$= -\frac{27}{2}\left(\frac{1}{729} - 1\right)$$

$$= -\frac{27}{2}\left(\frac{-728}{729}\right)$$

$$= \frac{364}{27}$$

$$S_n = \frac{9\left(\left(\frac{1}{3}\right)^n - 1\right)}{\frac{1}{3} - 1}$$

$$= -\frac{27}{2}\left(\left(\frac{1}{3}\right)^n - 1\right)$$

$$= \frac{27}{2}\left(1 - \left(\frac{1}{3}\right)^n\right)$$

5c GP with $a = 45$ and $r = \frac{1}{3}$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_5 = \frac{45 \times \left(\left(\frac{1}{3}\right)^5 - 1\right)}{\frac{1}{3} - 1}$$

$$= \frac{45 \times \left(\frac{1}{243} - 1\right)}{-\frac{2}{3}}$$

$$= -\frac{135}{2}\left(\frac{1}{243} - 1\right)$$

$$= -\frac{135}{2}\left(\frac{-242}{243}\right)$$

$$= \frac{605}{9}$$

$$S_n = \frac{45\left(\left(\frac{1}{3}\right)^n - 1\right)}{\frac{1}{3} - 1}$$

$$= -\frac{135}{2}\left(\left(\frac{1}{3}\right)^n - 1\right)$$

$$= \frac{135}{2}\left(1 - \left(\frac{1}{3}\right)^n\right)$$

5d GP with $a = \frac{2}{3}$ and $r = \frac{3}{2}$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_5 = \frac{\frac{2}{3} \times \left(\left(\frac{3}{2}\right)^5 - 1\right)}{\frac{3}{2} - 1}$$

$$= \frac{\frac{2}{3} \times \left(\frac{243}{32} - 1\right)}{\frac{1}{2}}$$

$$= \frac{4}{3}\left(\frac{1}{1024} - 1\right)$$

$$= \frac{4}{3}\left(\frac{211}{32}\right)$$

$$= \frac{211}{24}$$

$$S_n = \frac{\frac{2}{3} \times \left(\left(\frac{3}{2}\right)^n - 1\right)}{\frac{3}{2} - 1}$$

$$= \frac{4}{3}\left(\left(\frac{3}{2}\right)^n - 1\right)$$

5e GP with $a = 8$ and $r = -\frac{1}{2}$

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

$$S_{10} = \frac{8 \times \left(1 - \left(-\frac{1}{2}\right)^{10}\right)}{1 - \left(-\frac{1}{2}\right)}$$

$$= \frac{8 \times \left(1 - \frac{1}{1024}\right)}{\frac{3}{2}}$$

$$= \frac{16}{3}\left(\frac{1023}{1024}\right)$$

$$= \frac{341}{64}$$

$$S_n = \frac{8 \times \left(1 - \left(-\frac{1}{2}\right)^n\right)}{1 - \left(-\frac{1}{2}\right)}$$

$$= \frac{16}{3}\left(1 - \left(-\frac{1}{2}\right)^n\right)$$

5f GP with $a = 9$ and $r = -\frac{1}{3}$

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$S_6 = \frac{9 \times \left(1-\left(-\frac{1}{3}\right)^6\right)}{1-\left(-\frac{1}{3}\right)}$$

$$= \frac{9 \times \left(1-\frac{1}{729}\right)}{\frac{4}{3}}$$

$$= \frac{27}{4}\left(\frac{728}{729}\right)$$

$$= \frac{182}{27}$$

$$S_n = \frac{9 \times \left(1-\left(-\frac{1}{3}\right)^n\right)}{1-\left(-\frac{1}{3}\right)}$$

$$= \frac{27}{4}\left(1-\left(-\frac{1}{3}\right)^n\right)$$

5g GP with $a = -45$ and $r = -\frac{1}{3}$

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$S_5 = \frac{-45 \times \left(1-\left(-\frac{1}{3}\right)^5\right)}{1-\left(-\frac{1}{3}\right)}$$

$$= \frac{-45 \times \left(1-\frac{1}{243}\right)}{\frac{4}{3}}$$

$$= -\frac{135}{4}\left(\frac{242}{243}\right)$$

$$= -\frac{305}{9}$$

$$S_n = \frac{-45 \times \left(1 - \left(-\frac{1}{3}\right)^n\right)}{1 - \left(-\frac{1}{3}\right)}$$

$$= \frac{-45 \times \left(1 - (-\frac{1}{3})^n\right)}{\frac{4}{3}}$$

$$= -\frac{135}{4}\left(1 - \left(-\frac{1}{3}\right)^n\right)$$

5h GP with $a = \frac{2}{3}$ and $r = -\frac{3}{2}$

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

$$S_5 = \frac{\frac{2}{3} \times \left(1 - \left(-\frac{3}{2}\right)^5\right)}{1 - \left(-\frac{3}{2}\right)}$$

$$= \frac{\frac{2}{3} \times \left(1 + \frac{243}{32}\right)}{\frac{5}{2}}$$

$$= \frac{4}{15}\left(\frac{275}{32}\right)$$

$$= \frac{55}{24}$$

$$S_n = \frac{\frac{2}{3} \times \left(1 - \left(-\frac{3}{2}\right)^n\right)}{1 - \left(-\frac{3}{2}\right)}$$

$$= \frac{\frac{2}{3} \times \left(1 - \left(-\frac{3}{2}\right)^n\right)}{\frac{5}{2}}$$

$$= \frac{4}{15}\left(1 - \left(-\frac{3}{2}\right)^n\right)$$

6a GP with $a = 1$ and $r = 1.2$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$
$$= \frac{1(1.2^n - 1)}{1.2 - 1}$$
$$= \frac{1.2^n - 1}{0.2}$$
$$= 5(1.2^n - 1)$$
$$S_{10} = 5(1.2^{10} - 1) \doteqdot 25.96$$

6b GP with $a = 1$ and $r = 0.95$

$$S_n = \frac{a(1 - r^n)}{1 - r}$$
$$= \frac{1(1 - 0.95^n)}{1 - 0.95}$$
$$= \frac{1 - 0.95^n}{0.05}$$
$$= 20(1 - 0.95^n)$$
$$S_{10} = 20(1 - 0.95^n) \doteqdot 8.025$$

6c GP with $a = 1$ and $r = 1.01$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$
$$= \frac{1(1.01^n - 1)}{1.01 - 1}$$
$$= \frac{1.01^n - 1}{0.01}$$
$$= 100(1.01^n - 1)$$
$$S_{10} = 100(1.01^{10} - 1) \doteqdot 10.46$$

6d GP with $a = 1$ and $r = 0.99$

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$= \frac{1(1-0.99^n)}{1-0.99}$$

$$= \frac{1-0.99^n}{0.01}$$

$$= 100(1-0.99^n)$$

$$S_{10} = 100(1-0.99^{10}) \doteqdot 9.562$$

7a i Number of grains in last square $= 2^{63}$

7a ii GP with $a = 1$ and $r = 2$

$$S_n = \frac{a(r^n-1)}{r-1}$$

$$S_{64} = \frac{1 \times (2^{64}-1)}{2-1}$$

$$S_{64} = 2^{64} - 1$$

Number of grains in whole chessboard $= 2^{64} - 1$

7b $1\text{ L} = 1^{-12}\text{ km}^3 = 3 \times 10^4$ grains

Volume of wheat

$= \frac{2^{64}-1}{3\times10^4}$ litres

$= 6.148 \times 10^{14}$ litres

$= 1^{-12} \times 6.148 \times 10^{14}\text{ km}^3$

$= 615\text{ km}^3$

8a GP with $a = 1$ and $r = \sqrt{2}$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$= \frac{1 \times \left((\sqrt{2})^n - 1\right)}{\sqrt{2} - 1}$$

$$= \frac{\left((\sqrt{2})^n - 1\right)}{\sqrt{2} - 1} \times \frac{\sqrt{2} + 1}{\sqrt{2} + 1}$$

$$= \frac{\left(\left(\sqrt{2}\right)^n - 1\right)(\sqrt{2} + 1)}{2 - 1}$$

$$= \left(\left(\sqrt{2}\right)^n - 1\right)(\sqrt{2} + 1)$$

$$S_{10} = \left(\left(\sqrt{2}\right)^{10} - 1\right)(\sqrt{2} + 1)$$

$$= (32 - 1)(\sqrt{2} + 1)$$

$$= 31(\sqrt{2} + 1)$$

8b GP with $a = 2$ and $r = -\sqrt{5}$

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

$$= \frac{2 \times \left(1 - (-\sqrt{5})^n\right)}{1 - (-\sqrt{5})}$$

$$= \frac{2\left(1 - (-\sqrt{5})^n\right)}{1 + \sqrt{5}} \times \frac{1 - \sqrt{5}}{1 - \sqrt{5}}$$

$$= \frac{2\left(1 - \left(-\sqrt{5}\right)^n\right)(1 - \sqrt{5})}{1 - 5}$$

$$= -\frac{1}{2}\left(1 - \left(-\sqrt{5}\right)^n\right)(1 - \sqrt{5})$$

$$= \frac{1}{2}\left(1 - \left(-\sqrt{5}\right)^n\right)(\sqrt{5} - 1)$$

$$S_{10} = \frac{1}{2}\left(1 - \left(-\sqrt{5}\right)^{10}\right)(\sqrt{5} - 1)$$

$$= -1562(\sqrt{5} - 1)$$

9a $T_1 = 3 \times 2^1 = 6$

$T_2 = 3 \times 2^2 = 12$

$T_3 = 3 \times 2^3 = 24$

GP with $a = 6$ and $r = 2$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_7 = \frac{6 \times (2^7 - 1)}{2 - 1}$$

$= 6 \times 127$

$= 762$

9b $T_1 = 3^2 = 9$

$T_2 = 3^3 = 27$

$T_3 = 3^4 = 81$

GP with $a = 9$ and $r = 3$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_6 = \frac{9 \times (3^6 - 1)}{3 - 1}$$

$= \frac{9}{2} \times 728$

$= 3276$

9c $T_1 = 3 \times 2^2 = 12$

$T_2 = 3 \times 2^1 = 6$

$T_3 = 3 \times 2^0 = 3$

GP with $a = 12$ and $r = \frac{1}{2}$

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

$$S_8 = \frac{12 \times \left(1 - \left(\frac{1}{2}\right)^8\right)}{1 - \frac{1}{2}}$$

$$= 2 \times 12 \times \left(1 - \left(\frac{1}{2}\right)^8\right)$$

$$= 24\left(\frac{255}{256}\right)$$

$$= \frac{6120}{256}$$

$$= \frac{765}{32}$$

10a $T_1 = a = \frac{1}{8}$

$T_5 = ar^4 = 162$

$\frac{1}{8}r^4 = 162$

$r^4 = 1296$

$r = \pm 6$

When $r = 6$, first five terms are: $\frac{1}{8}, \frac{3}{4}, \frac{9}{2}, 27, 162$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_6 = \frac{\frac{1}{8}(6^5 - 1)}{6 - 1}$$

$$= \frac{1}{40}(7776 - 1)$$

$$= 194\tfrac{3}{8}$$

When $r = -6$, first five terms are: $\frac{1}{8}, -\frac{3}{4}, \frac{9}{2}, -27, 162$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_6 = \frac{\frac{1}{8}((-6)^5 - 1)}{(-6) - 1}$$

$$= -\frac{1}{56}(-7776 - 1)$$

$$= 138\tfrac{7}{8}$$

10b $T_1 = a = -\frac{3}{4}$

$T_4 = ar^3 = 6$

$$-\frac{3}{4}r^3 = 6$$

$$r^3 = -8$$

$$r = -2$$

$$S_n = \frac{-\frac{3}{4}((-2)^n - 1)}{-2 - 1}$$

$$S_6 = \frac{-\frac{3}{4}((-2)^6 - 1)}{-2 - 1}$$

$$= \frac{-\frac{3}{4}((-2)^6 - 1)}{-2 - 1}$$

$$= \frac{63}{4}$$

$$= 15\frac{3}{4}$$

10c $T_2 = ar = 0.08,\ T_3 = ar^2 = 0.4$

$$0.08r = 0.4$$

$$r = 5$$

$$ar = 0.08$$

$$5a = 0.08$$

$$a = 0.016$$

$$S_n = \frac{0.016(5^n - 1)}{5 - 1}$$

$$S_8 = \frac{0.016(5^8 - 1)}{4}$$

$$= 1562.496$$

10d $r = 2$

$S_8 = 1785$

$$1785 = \frac{a(2^8 - 1)}{2 - 1}$$

$$1785 = 255a$$

$$a = 7$$

10e $r = -\frac{1}{2}$, $S_8 = 425$

$$425 = \frac{a\left(1 - \left(-\frac{1}{2}\right)^8\right)}{1 - \left(-\frac{1}{2}\right)}$$

$$425 = \frac{85}{128}a$$

$$a = 640$$

11a i Amount $= 6 \times \left(\frac{1}{2}\right)^9 \doteqdot 0.011\,72$ tonnes

11a ii GP with $a = 6$ and $r = \frac{1}{2}$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_9 = \frac{6 \times \left(\left(\frac{1}{2}\right)^9 - 1\right)}{\frac{1}{2} - 1}$$

$$= \frac{6\left(\left(\frac{1}{2}\right)^9 - 1\right)}{-\frac{1}{2}}$$

$$= -12\left(\frac{1}{512} - 1\right)$$

$$= \frac{6132}{512}$$

$$\doteqdot 11.98 \text{ tonnes}$$

11b Amount $= 20 \times \left(\frac{1}{2}\right)^{12} = 4.9 \times 10^{-3}\text{g}$

11c i GP with $a = P$ and $r = 1.1$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_{10} = \frac{P \times (1.1^{10} - 1)}{1.1 - 1}$$

$$= \frac{P \times (1.1^{10} - 1)}{0.1}$$

$$= 10P(1.1^{10} - 1)$$

11c ii $S_{10} = 10P(1.1^{10} - 1)$

$900 = 10P(1.1^{10} - 1)$

$90 = P(1.1^{10} - 1)$

$P = \$56.47$

12a For the Abletown Show: GP with $a = 5000$, $r = 1.05$, hence:

$$S_6 = \frac{5000((1.05)^6 - 1)}{1.05 - 1}$$

$$= \frac{5000((1.05)^6 - 1)}{0.05}$$

$$= 100\,000((1.05)^6 - 1)$$

$$= 34\,010$$

For the Bush Creek Show: GP with $a = 5000$, $r = 0.95$, hence:

$$S_6 = \frac{5000((0.95)^6 - 1)}{0.95 - 1}$$

$$= \frac{5000((0.95)^6 - 1)}{-0.05}$$

$$= -100\,000((0.95)^6 - 1)$$

$$= 26\,491$$

12b For the Abletown Show, $T_n = 5000 \times (1.05)^{n-1}$

For the Bush Creek Show, $T_n = 5000 \times (0.95)^{n-1}$

Put $5000 \times (1.05)^{n-1} > 10 \times 5000 \times (0.95)^{n-1}$

$(1.05)^{n-1} > 10 \times (0.95)^{n-1}$

$\dfrac{(1.05)^{n-1}}{(0.95)^{n-1}} > 10$

$\left(\dfrac{1.05}{0.95}\right)^{n-1} > 10$

$\ln\left(\dfrac{1.05}{0.95}\right)^{n-1} > \ln 10$

$(n-1)\ln\left(\dfrac{1.05}{0.95}\right) > \ln 10$

$n - 1 > \dfrac{\ln 10}{\ln \dfrac{1.05}{0.95}}$

$n > \dfrac{\ln 10}{\ln \dfrac{1.05}{0.95}} + 1$

$n > 24.0066 \ldots$

The number attending the Abletown Show first exceeds ten times the number attending the Bush Creek show in the 25th year.

12c For the Abletown Show: GP with $a = 5000, r = 1.05$, hence:

$S_{25} = \dfrac{5000((1.05)^{25} - 1)}{1.05 - 1}$

$= \dfrac{5000((1.05)^{25} - 1)}{0.05}$

$= 100\,000((1.05)^{25} - 1)$

$= 238\,635$

For the Bush Creek Show: GP with $a = 5000, r = 0.95$, hence:

$$S_{25} = \frac{5000((0.95)^{25} - 1)}{0.95 - 1}$$

$$= \frac{5000((0.95)^{25} - 1)}{-0.05}$$

$$= -100\,000((0.95)^{25} - 1)$$

$$= 72\,261$$

Thus, the ratio is $\frac{238\,635}{72\,261} \doteqdot 3.30$.

13a This is a GP with $a = 7, r = \frac{14}{2} = 2$, hence:

$$S_n = \frac{7(2^n - 1)}{2 - 1} = 7(2^n - 1)$$

13b Put $S_n = 1785$

$7(2^n - 1) = 1785$

$2^n - 1 = 255$

$2^n = 256$

$2^n = 2^8$

$n = 8$

13c $T_n = ar^{n-1} = 7 \times 2^{n-1}$

Put $T_n < 70\,000$

$7 \times 2^{n-1} < 70\,000$

$2^{n-1} < 10\,000$

$(n - 1)\ln 2 < \ln 10\,000$

$$n - 1 < \frac{\ln 10\,000}{\ln 2}$$

$n < 14.28 \ldots$

Hence there are 14 terms less than 70 000.

13d Put $S_n > 70\,000$

$7(2^n - 1) > 70\,000$

$2^n - 1 > 10\,000$

$2^n > 10\,001$

By trial and error the lowest integer solution is 14, hence, the first S_n greater than 70 000 is $S_{14} = 114\,681$.

Alternatively:

$n \ln 2 > \ln 10\,001$

$n > \dfrac{\ln 10\,001}{\ln 2}$

$n > 13.28 \ldots$

13e Need to prove that: $S_n = T_{n+1} - 7$

$S_n = 7(2^n - 1)$

$T_{n+1} - 7 = 7 \times 2^{n+1-1} - 7 = 7 \times 2^n - 7 = 7(2^n - 1) = S_n$

as required

14a For the GP of powers of 3, $a = 3$ and $r = 3$.

$T_1 = 3$ is the first term greater than 2.

For the last term less than 10^{20}:

$T_n < 10^{20}$

$3 \times 3^{n-1} < 10^{20}$

$3^n < 10^{20}$

$n < \log_3 10^{20}$

$n < 41.91 \ldots$

Hence there are 41 powers of 3.

14b $S_n = \frac{a(r^n-1)}{r-1}$

$= \frac{3(3^n - 1)}{3 - 1}$

$= \frac{3}{2}(3^n - 1)$

Hence the smallest value of n for which $S_n > 10^{20}$ is:

$\frac{3}{2}(3^n - 1) > 10^{20}$

$3^n - 1 > \frac{2}{3} \times 10^{20}$

$3^n > \frac{2}{3} \times 10^{20} + 1$

$n > \log_3\left(\frac{2}{3} \times 10^{20} + 1\right)$

$n > 41.5 \ldots$

Hence the smallest value for which $S_n > 10^{20}$ is 42.

15a This is a GP with $a = 5, r = 2$.

$S_n = \frac{5(2^n - 1)}{2 - 1}$

$315 = \frac{5(2^n - 1)}{2 - 1}$

$2^n - 1 = 63$

$2^n = 64$

$2^n = 2^6$

$n = 6$

15b This is a GP with $a = 5, r = -2$.

$S_n = \frac{5((-2)^n - 1)}{(-2) - 1}$

$-425 = \frac{5((-2)^n - 1)}{-3}$

$(-2)^n - 1 = 255$

$(-2)^n = 256$

$(-2)^n = (-2)^8$

$n = 8$

15c This is a GP with $a = 18, r = \frac{1}{3}$.

$$S_n = \frac{18\left(1-\left(\frac{1}{3}\right)^n\right)}{1-\frac{1}{3}}$$

$$26\frac{8}{9} = \frac{18\left(1-\left(\frac{1}{3}\right)^n\right)}{1-\frac{1}{3}}$$

$$1-\left(\frac{1}{3}\right)^n = \frac{242}{243}$$

$$\left(\frac{1}{3}\right)^n = \frac{1}{243}$$

$$\left(\frac{1}{3}\right)^n = \left(\frac{1}{3}\right)^5$$

$n = 5$

15d This is a GP with $a = 48, r = -\frac{1}{2}$.

$$S_n = \frac{48\left(1-\left(-\frac{1}{2}\right)^n\right)}{1-\left(-\frac{1}{2}\right)}$$

$$32\frac{1}{4} = \frac{48\left(1-\left(-\frac{1}{2}\right)^n\right)}{\frac{3}{2}}$$

$$1-\left(-\frac{1}{2}\right)^n = \frac{129}{128}$$

$$\left(-\frac{1}{2}\right)^n = -\frac{1}{128}$$

$n = 7$

16a The terms in the numerator form an AP with $a = 2$ and $d = 2$.

The sum of these terms is given by:

$$\begin{aligned}S_n &= \frac{1}{2}n(2a + (n-1)d)\\ &= \frac{1}{2}n(4 + (n-1) \times 2)\\ &= \frac{1}{2}n(2n+2)\\ &= n^2 + n\end{aligned}$$

The terms in the denominator form an AP with $a = 1$ and $d = 2$.

The sum of these terms is given by:

$$\begin{aligned}S_n &= \frac{1}{2}n(2a + (n-1)d)\\ &= \frac{1}{2}n(2 + (n-1) \times 2)\\ &= \frac{1}{2}n(2n)\\ &= n^2\end{aligned}$$

Thus the nth term of the sequence is:

$$T_n = \frac{n^2 + n}{n} = \frac{n+1}{n}$$

16b The terms in the numerator form an GP with $a = 1$ and $r = 2$.

The sum of these terms is given by:

$$\begin{aligned}S_n &= \frac{a(r^n - 1)}{r - 1}\\ &= \frac{1(2^n - 1)}{2 - 1}\\ &= 2^n - 1\end{aligned}$$

The terms in the denominator form an GP with $a = 1$ and $d = 4$.

The sum of these terms is given by:

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$= \frac{1(4^n - 1)}{3 - 1}$$

$$= \frac{4^n - 1}{3}$$

Thus the nth term of the sequence is:

$$T_n = \frac{2^n - 1}{\frac{4^n - 1}{3}}$$

$$= \frac{3(2^n - 1)}{(2^2)^n - 1}$$

$$= \frac{3(2^n - 1)}{(2^n)^2 - 1}$$

$$= \frac{3(2^n - 1)}{(2^n - 1)(2^n + 1)}$$

$$= \frac{3}{2^n + 1}$$

17a $S_{2n}: S_n$

$$\frac{a(r^{2n} - 1)}{r - 1} : \frac{a(r^n - 1)}{r - 1}$$

$$(r^{2n} - 1) : (r^n - 1)$$

$$(r^n - 1)(r^n + 1) : (r^n - 1)$$

$$(r^n + 1) : 1$$

$$S_{12} : S_6 = 65 : 1$$

$$(r^6 + 1) : 1 = 65 : 1$$

Hence:

$$r^6 + 1 = 65$$

$$r^6 = 64$$

$$r = \pm 2$$

17b $\sum n: S_n$

$\frac{a((r^2)^n - 1)}{r^2 - 1} : \frac{a(r^n - 1)}{r - 1}$

$\frac{r^{2n} - 1}{r^2 - 1} : \frac{r^n - 1}{r - 1}$

$\frac{(r^n - 1)(r^n + 1)}{(r - 1)(r + 1)} : \frac{r^n - 1}{r - 1}$

$\frac{r^n + 1}{r + 1} : 1$

$(r^n + 1):(r + 1)$

17c $R_n = T_{n+1} + T_{n+2} + \cdots + T_{2n} = S_{2n} - S_n$

$R_n: S_n$

$(S_{2n} - S_n): S_n$

$\left(\frac{a(r^{2n} - 1)}{r - 1} - \frac{a(r^n - 1)}{r - 1}\right) : \frac{a(r^n - 1)}{r - 1}$

$\left((r^{2n} - 1) - (r^n - 1)\right):(r^n - 1)$

$(r^{2n} - r^n):(r^n - 1)$

$r^n(r^n - 1):(r^n - 1)$

$r^n: 1$

$R_8: S_8 = 1: 81$

$r^8: 1 = 1: 81$

$r^8: 1 = \frac{1}{81}: 1$

$r^8 = \frac{1}{81}$

$r = \pm\frac{1}{\sqrt{3}}$

$r = 3^{-\frac{1}{2}}$ or $-3^{-\frac{1}{2}}$

18 $S_{10} = T_1 + T_2 + \cdots + T_{10} = 2$

Hence:

$\frac{a(r^{10}-1)}{r-1} = 2$ (1)

$S_{30} = T_1 + T_2 + \cdots + T_{10} + T_{11} + \cdots + T_{30} = 2 + 12 = 14$

Hence:

$\frac{a(r^{30}-1)}{r-1} = 14$ (2)

$\frac{r^{30}-1}{r^{10}-1} = 7$ (2) ÷ (1)

$\dfrac{(r^{10})^3 - 1}{r^{10} - 1} = 7$

$\dfrac{(r^{10} - 1)((r^{10})^2 + r^{10} + 1)}{r^{10} - 1} = 7$

$(r^{10})^2 + r^{10} + 1 = 7$

$(r^{10})^2 + r^{10} - 6 = 0$

$(r^{10} + 3)(r^{10} - 2) = 0$

$r^{10} = 2$ (note that $r^{10} > 0$ and hence $r^{10} = -3$ is not a solution)

Substituting this into (1) gives:

$\dfrac{a(2-1)}{r-1} = 2$

$\dfrac{a}{r-1} = 2$

Thus we conclude that:

$T_{31} + T_{32} + \cdots + T_{60}$

$= S_{60} - S_{30}$

$= \dfrac{a(r^{60} - 1)}{r - 1} - 14$

$= \dfrac{a}{r-1} \times (r^{60} - 1) - 14$

$= 2 \times ((r^{10})^6 - 1) - 14$ (as $\frac{a}{r-1} = 2$)

$= 2(2^6 - 1) - 14$ (as $r^{10} = 2$)

$= 112$

19 Since $l = T_n = ar^{n-1}$ we have that $lr = T_n r = ar^n$ hence:

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$= \frac{ar^n - a}{r - 1}$$

$$= \frac{lr - a}{r - 1}$$

$$= \frac{a - rl}{1 - r} \qquad \text{as required}$$

19a i This is a GP with $a = 1, r = 2, l = 1\,048\,576$. Hence:

$$S_n = \frac{lr - a}{r - 1} = \frac{1\,048\,576 \times 2 - 1}{2 - 1} = 2\,097\,151$$

19a ii This is a GP with $a = 1, r = \frac{1}{3}, l = \frac{1}{2187}$. Hence:

$$S_n = \frac{a - rl}{1 - r} = \frac{1 - \frac{1}{2187} \times \frac{1}{3}}{1 - \frac{1}{3}} = \frac{6560}{4374}$$

19b Put $S_n = 85$

$$\frac{lr - a}{r - 1} = 85$$

$$\frac{64r - 1}{r - 1} = 85$$

$$64r - 1 = 85r - 85$$

$$21r = 84$$

$$r = 4$$

Put $S_n = 85$

$$\frac{a(r^n - 1)}{r - 1} = 85$$

$$\frac{1(4^n - 1)}{4 - 1} = 85$$

$$\frac{1(4^n - 1)}{3} = 85$$

$$4^n - 1 = 255$$

$$4^n = 256$$

$$n = 4$$

19c Put $S_n = -910$

$$\frac{lr - a}{r - 1} = -910$$

$$\frac{l(-3) - 5}{(-3) - 1} = -910$$

$$l(-3) - 5 = -910 \times -4$$

$$-3l - 5 = 3640$$

$$-3l = 3645$$

$$l = -1215$$

Put $S_n = -910$

$$\frac{5((-3)^n - 1)}{-3 - 1} = -910$$

$$5(-3)^n - 5 = 3640$$

$$5(-3)^n = 3645$$

$$(-3)^n = 729$$

$$(-3)^n = (-3)^6$$

$$n = 6$$

20a For the first GP, $T_n = 2 \times 3^n = (2 \times 3) \times 3^{n-1}$ so $a = 2 \times 3 = 6$ and $r = 3$.

For the second GP, $T_n = 3 \times 2^n = (3 \times 2) \times 2^{n-1}$ so $a = 3 \times 2 = 6$ and $r = 2$.

Thus, the sum is given by:

$$S_n = \frac{6(3^n - 1)}{3 - 1} + \frac{6(2^n - 1)}{2 - 1}$$

$$= \frac{6(3^n - 1)}{2} + \frac{6(2^n - 1)}{1}$$
$$= 3 \times 3^n - 3 + 6 \times 2^n - 6$$
$$= 3 \times 3^n + 6 \times 2^n - 9$$

20b The term consists of an AP with $a = 5, d = 2$ and a GP with $a = 2, r = 2$. Hence the sum is:

$$S_n = \frac{1}{2}n(2 \times 5 + (n - 1) \times 2) + \frac{2(2^n - 1)}{2 - 1}$$
$$= \frac{1}{2}n(2n + 8) + 2(2^n - 1)$$
$$= n^2 + 4n + 2 \times 2^n - 2$$
$$= 2 \times 2^n + n^2 + 4n - 2$$

20c Put $T_1 = 10$

$a + d + b \times 2 = 10$

$a + d + 2b = 10$ (1)

Put $T_2 = 19$

$a + 2d + b \times 2^2 = 19$

$a + 2d + 4b = 19$ (2)

Put $T_3 = 34$

$a + 3d + b \times 2^3 = 34$

$a + 3d + 8b = 34$ (3)

Hence:

$d + 2b = 9$ (2) − (1) = (4)

$d + 4b = 15$ (3) − (2) = (5)

Subtracting (4) from (5) gives:

$2b = 6$

$b = 3$

Substituting $b = 3$ into (4) gives $d = 3$.

Substituting $b = 3, d = 3$ into (1) gives $a = 1$.

Hence, $a = 1, b = 3$ and $d = 3$.

Hence T_n is formed by the sum of the terms of an AP with $a = 1 + 3 = 4$, $d = 3$ and a GP with $a = 3 \times 2 = 6$, $r = 2$. Thus the total sum is:

$$S_n = \frac{1}{2}n(2 \times 4 + (n-1) \times 3) + \frac{6(2^n - 1)}{2 - 1}$$

$$= \frac{1}{2}n(3n + 5) + 6(2^n - 1)$$

$$= \frac{3}{2}n^2 + \frac{5}{2}n + 6 \times 2^n - 6$$

$$= \frac{3}{2}n^2 + \frac{5}{2}n - 6 + 6 \times 2^n$$

Solutions to Exercise 1H

1a

n	1	2	3	4	5	6
T_n	18	6	2	$\frac{2}{3}$	$\frac{2}{9}$	$\frac{2}{27}$
S_n	18	24	26	$26\frac{2}{3}$	$26\frac{8}{9}$	$26\frac{26}{27}$

1b $S_\infty = \frac{a}{1-r} = \frac{18}{1-\frac{1}{3}} = \frac{18}{\left(\frac{2}{3}\right)} = \frac{3}{2}(18) = 27$

1c $S_\infty - S_6 = 27 - 26\frac{26}{27} = \frac{1}{27}$

2a

n	1	2	3	4	5	6
T_n	24	-12	6	-3	$1\frac{1}{2}$	$-\frac{3}{4}$
S_n	24	12	18	15	$16\frac{1}{2}$	$15\frac{3}{4}$

2b $S_\infty = \frac{a}{1-r} = \frac{24}{1-(-\frac{1}{2})} = \frac{24}{\frac{3}{2}} = \frac{2}{3}(24) = 16$

2c $S_\infty - S_6 = 16 - 15\frac{3}{4} = \frac{1}{4}$

3a $a = 8, r = \frac{1}{2}$, hence:

$$S_\infty = \frac{a}{1-r}$$

$$= \frac{8}{1-\frac{1}{2}}$$

$$= \frac{8}{\frac{1}{2}}$$

$$= 16$$

3b $a = -4, r = \frac{1}{2}$, hence:

$$S_\infty = \frac{a}{1-r}$$

$$= \frac{-4}{1-\frac{1}{2}}$$

$$= \frac{-4}{\frac{1}{2}}$$

$$= -8$$

3c $a = 1, r = -\frac{1}{3}$, hence:

$$S_\infty = \frac{a}{1-r}$$

$$= \frac{1}{1-\left(-\frac{1}{3}\right)}$$

$$= \frac{1}{\frac{4}{3}}$$

$$= \frac{3}{4}$$

3d $a = 36, r = -\frac{1}{3}$, hence:

$$S_\infty = \frac{a}{1-r}$$

$$= \frac{36}{1-\left(-\frac{1}{3}\right)}$$

$$= \frac{36}{\frac{4}{3}}$$

$$= 27$$

3e $a = 60, r = -\frac{1}{2}$, hence:

$$S_\infty = \frac{a}{1-\left(-\frac{1}{2}\right)}$$

$$= \frac{60}{\frac{3}{2}}$$

$$= 60 \times \frac{2}{3}$$

$$= 40$$

3f $a = 60,\ r = \frac{-12}{60} = -\frac{1}{5}$, hence:

$$S_\infty = \frac{a}{1-r}$$

$$= \frac{60}{1-\left(-\frac{1}{5}\right)}$$

$$= \frac{60}{\frac{6}{5}}$$

$$= 60 \times \frac{5}{6}$$

$$= 50$$

4a $r = \frac{\left(-\frac{1}{2}\right)}{1} = -\frac{1}{2}$, hence there is a limiting sum as $|r| < 1$. Now $a = 1$ so:

$$S_\infty = \frac{a}{1-r}$$

$$= \frac{1}{1-\left(-\frac{1}{2}\right)}$$

$$= \frac{1}{\frac{3}{2}}$$

$$= \frac{2}{3}$$

4b $r = -\frac{6}{4} = -1.5$, hence there is no limiting sum as $|r| > 1$.

4c $r = \frac{4}{12} = \frac{1}{3}$, hence there is a limiting sum as $|r| < 1$. Now $a = 12$ so:

$$S_\infty = \frac{a}{1-r}$$

$$= \frac{12}{1-\frac{1}{3}}$$

$$= \frac{12}{\frac{2}{3}}$$

$$= 12 \times \frac{3}{2}$$

$$= 18$$

4d $r = \frac{100}{1000} = \frac{1}{10}$, hence there is a limiting sum as $|r| < 1$. Now $a = 1000$ so:

$$S_\infty = \frac{1000}{1 - \frac{1}{10}}$$

$$= \frac{1000}{\frac{9}{10}}$$

$$= \frac{10\ 000}{9}$$

$$= 1111\frac{1}{9}$$

4e $r = \frac{\frac{2}{5}}{-2} = -\frac{1}{5}$, hence there is a limiting sum as $|r| < 1$. Now $a = -2$ so:

$$S_\infty = \frac{-2}{1 - \left(-\frac{1}{5}\right)}$$

$$= \frac{-2}{\frac{6}{5}}$$

$$= \frac{-10}{6}$$

$$= -\frac{5}{3}$$

4f $r = \frac{-1}{1} = -1$, hence there is no limiting sum as $|r| > 1$.

5a The ball must travel 8 metres downwards to the ground, then it bounces back up to half the height which is $8 \times \frac{1}{2} = 4$ m. This means a total of $8 + 4 = 12$ m is travelled down-and-up.

Successive down-and-up distances are formed by taking the previous down-and-up distance and then halving the distance to go down and halving the distance to go back up. This means that each successive down and up sequence is half that of the previous. Hence it forms a GP with $r = \frac{1}{2}$.

5b The distance travelled in each up-and-down sequence is given by a GP with $a = 12$ and $r = \frac{1}{2}$. Thus the total distance travelled is given by:

$$\begin{aligned} S_\infty &= \frac{a}{1-r} \\ &= \frac{12}{1-\frac{1}{2}} \\ &= \frac{12}{\frac{1}{2}} \\ &= 24 \end{aligned}$$

Ball 'eventually' travelled 24 metres.

6a

n	1	2	3	4	5	6
T_n	10	10	10	10	10	10
S_n	10	20	30	40	50	60

$S_n \to \infty$ as $n \to \infty$

6b

n	1	2	3	4	5	6
T_n	10	-10	10	-10	10	-10
S_n	10	0	10	0	10	0

S_n oscillates between 0 and 10 as $n \to \infty$

6c

n	1	2	3	4	5	6
T_n	10	20	40	80	160	320
S_n	10	30	70	150	310	630

$S_n \to \infty$ as $n \to \infty$

6d

n	1	2	3	4	5	6
T_n	10	-20	40	-80	160	-320
S_n	10	-10	30	-50	110	-210

S_n oscillates between larger and larger positive and negative numbers as $n \to \infty$

7a For the series $a = 80$ and $r = \frac{40}{80} = \frac{1}{2}$

Thus $S_4 = \frac{a(1-r^n)}{1-r} = \frac{80\left(1-\left(\frac{1}{2}\right)^4\right)}{1-\frac{1}{2}} = 150$

and $S_\infty = \frac{a}{1-r} = \frac{80}{1-\frac{1}{2}} = \frac{80}{\frac{1}{2}} = 160$

So $S_\infty - S_4 = 160 - 150 = 10$

7b For the series $a = 100$ and $r = \frac{10}{100} = \frac{1}{10}$.

Thus $S_4 = \frac{a(1-r^n)}{1-r} = \frac{100\left(1-\left(\frac{1}{10}\right)^4\right)}{1-\frac{1}{10}} = 111\frac{1}{10}$

and $S_\infty = \frac{a}{1-r} = \frac{100}{1-\frac{1}{10}} = \frac{100}{\left(\frac{9}{10}\right)} = \frac{1000}{9} = 111\frac{1}{9}$

So $S_\infty - S_4 = 111\frac{1}{9} - 111\frac{1}{10} = \frac{1}{90}$

7c For the series $a = 100$ and $r = \frac{-80}{100} = -\frac{4}{5}$.

Thus $S_4 = \frac{a(1-r^n)}{1-r} = \frac{100\left(1-\left(-\frac{4}{5}\right)^4\right)}{1-\left(-\frac{4}{5}\right)} = \frac{164}{5}$

and $S_\infty = \frac{a}{1-r} = \frac{100}{1-(-\frac{4}{5})} = \frac{100}{\left(\frac{9}{5}\right)} = \frac{500}{9}$

So $S_\infty - S_4 = \frac{500}{9} - \frac{164}{5} = 22\frac{34}{45}$

8a The numbers installing reflective house numbers in each subsequent month is 20% of that in the previous month. This is equivalent to multiplying the number in the previous month by $20\% = 0.2$. Hence, this gives us a GP as all terms have a common ratio of 0.2.

8b This is a GP with $a = 10\,000 \times 0.2 = 2000$ and $r = 0.2$

$$S_\infty = \frac{a}{1-r}$$
$$= \frac{2000}{1-0.2}$$
$$= 2500$$

8c $S_n = \frac{a(r^n-1)}{r-1}$

$$S_4 = \frac{2000(0.2^4-1)}{0.3-1}$$
$$= 2496$$

$S_\infty - S_4 = 2500 - 2496 = 4$

9a This is a GP with $a = 1000$ and $r = 0.9$.

$$S_\infty = \frac{a}{1-r}$$
$$= \frac{1000}{1-0.9}$$
$$= \frac{1000}{0.1}$$
$$= 10\,000$$

Thus the advertisements will eventually bring in 10 000 sales.

9b The first 10 advertisements $S_{10} = \frac{a(1-r^n)}{1-r} = \frac{1000(1-0.9^{10})}{1-0.9} = 6513.2\ldots$

Thus $S_\infty - S_{10} = 10\,000 - 6513.2\ldots \doteqdot 3487$

10a This is a GP with $a = 1$ and $r = 1.01$.

As $|r| = 1.01 > 1$, no limiting sum exists.

10b This is a GP with $a = 1$ and $r = 1.01^{-1}$.

As $|r| = 1.01^{-1} < 1$, there is a limiting sum which is:

$$\begin{aligned} S_\infty &= \frac{a}{1-r} \\ &= \frac{1}{1-1.01^{-1}} \\ &= \frac{1}{1-\frac{1}{1.01}} \\ &= \frac{1}{\frac{1.01}{1.01}-\frac{1}{1.01}} \\ &= \frac{1}{\left(\frac{0.01}{1.01}\right)} \\ &= \frac{1.01}{0.01} \\ &= 101 \end{aligned}$$

10c This is a GP with $a = 16\sqrt{5}$ and $r = \frac{4\sqrt{5}}{16\sqrt{5}} = \frac{1}{4}$.

$$\begin{aligned} S_\infty &= \frac{a}{1-r} \\ &= \frac{16\sqrt{5}}{1-\frac{1}{4}} \\ &= \frac{16\sqrt{5}}{\frac{3}{4}} \\ &= \frac{4}{3}\left(16\sqrt{5}\right) \\ &= \frac{64\sqrt{5}}{3} \end{aligned}$$

10d This is a GP with $a = 7$ and $r = \frac{\sqrt{7}}{7}$.

$$\begin{aligned}S_\infty &= \frac{a}{1-r}\\ &= \frac{7}{1-\frac{\sqrt{7}}{7}}\\ &= \frac{7}{\frac{7-\sqrt{7}}{7}}\\ &= \frac{49}{7-\sqrt{7}}\\ &= \frac{49}{7-\sqrt{7}} \times \frac{7+\sqrt{7}}{7+\sqrt{7}}\\ &= \frac{49(7+\sqrt{7})}{49-7}\\ &= \frac{49(7+\sqrt{7})}{42}\\ &= \frac{7(7+\sqrt{7})}{6}\end{aligned}$$

10e This is a GP with $a = 4$ and $r = -\frac{2\sqrt{2}}{4} = -\frac{\sqrt{2}}{2}$.

$$\begin{aligned}S_\infty &= \frac{a}{1-r}\\ &= \frac{4}{1-\left(-\frac{\sqrt{2}}{2}\right)}\\ &= \frac{4}{1+\frac{\sqrt{2}}{2}}\\ &= \frac{8}{2+\sqrt{2}}\\ &= \frac{8}{2+\sqrt{2}} \times \frac{2-\sqrt{2}}{2-\sqrt{2}}\\ &= \frac{8(2-\sqrt{2})}{4-2}\\ &= \frac{8(2-\sqrt{2})}{2}\\ &= 4(2-\sqrt{2})\end{aligned}$$

10f This is a GP with $a = 5$ and $r = -\frac{2\sqrt{5}}{5} = -\frac{2}{\sqrt{5}}$.

$$\begin{aligned}
S_\infty &= \frac{a}{1-r} \\
&= \frac{5}{1-\left(-\frac{2}{\sqrt{5}}\right)} \\
&= \frac{5}{1+\frac{2}{\sqrt{5}}} \\
&= \frac{5\sqrt{5}}{\sqrt{5}+2} \\
&= \frac{5\sqrt{5}}{\sqrt{5}+2} \times \frac{\sqrt{5}-2}{\sqrt{5}-2} \\
&= \frac{5\sqrt{5}(\sqrt{5}-2)}{5-4} \\
&= 5\sqrt{5}(\sqrt{5}-2) \\
&= 25-10\sqrt{5} \\
&= 5(5-2\sqrt{5})
\end{aligned}$$

10g This is a GP with $a = 9$ and $r = \frac{3\sqrt{10}}{9} = \frac{\sqrt{10}}{3}$. But $r = \frac{\sqrt{10}}{3} > 1$ and hence there is no limiting sum.

10h This is a GP with $a = 1$ and $r = 1 - \sqrt{3}$.

$$\begin{aligned}
S_\infty &= \frac{a}{1-r} \\
&= \frac{1}{1-(1-\sqrt{3})} \\
&= \frac{1}{\sqrt{3}} \\
&= \frac{\sqrt{3}}{3}
\end{aligned}$$

11a

$$\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n=\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\cdots$$

This is a GP with $a=\frac{1}{3}$ and $r=\frac{1}{3}$, hence the sum is given by:

$$\begin{aligned}S_\infty&=\frac{a}{1-r}\\&=\frac{\frac{1}{3}}{1-\frac{1}{3}}\\&=\frac{\frac{1}{3}}{\frac{2}{3}}\\&=2\end{aligned}$$

11b

$$\sum_{n=1}^{\infty}7\times\left(\frac{1}{2}\right)^n=\frac{7}{2}+\frac{7}{2^2}+\frac{7}{2^3}+\cdots$$

This is a GP with $a=\frac{7}{2}$ and $r=\frac{1}{2}$, hence the sum is given by:

$$\begin{aligned}S_\infty&=\frac{a}{1-r}\\&=\frac{\frac{7}{2}}{1-\frac{1}{2}}\\&=\frac{\frac{7}{2}}{\frac{1}{2}}\\&=7\end{aligned}$$

11c

$$\sum_{n=1}^{\infty}40\times\left(-\frac{3}{5}\right)^n=-40\left(\frac{3}{5}\right)+40\left(\frac{3}{5}\right)^2-40\left(\frac{3}{5}\right)^3+\cdots$$

This is a GP with $a = -\frac{120}{5} = -24$ and $r = -\frac{3}{5}$, hence the sum is given by:

$$S_\infty = \frac{a}{1-r}$$

$$= \frac{-24}{1-\left(-\frac{3}{5}\right)}$$

$$= \frac{-24}{\frac{8}{5}}$$

$$= -15$$

12a The left-hand side forms a GP with $a = 5$ and $r = x$.

$$S_\infty = \frac{a}{1-r}$$

$$= \frac{5}{1-x}$$

Hence solving the equation gives:

$$S_\infty = 10$$

$$\frac{5}{1-x} = 10$$

$$5 = 10 - 10x$$

$$10x = 5$$

$$x = \frac{1}{2}$$

12b The left-hand side forms a GP with $a = 5$ and $r = -x$.

$$S_\infty = \frac{a}{1-r}$$

$$= \frac{5}{1-(-x)}$$

$$= \frac{5}{1+x}$$

Hence solving the equation gives:

$$S_\infty = 15$$

$$\frac{5}{1+x} = 15$$

$$5 = 15 + 15x$$

$$15x = -10$$

$$x = -\frac{2}{3}$$

12c The left-hand side forms a GP with $a = x$ and $r = \frac{1}{3}$.

$$S_\infty = \frac{a}{1-r}$$

$$= \frac{x}{1-\frac{1}{3}}$$

$$= \frac{x}{\frac{2}{3}}$$

$$= \frac{3x}{2}$$

Hence solving the equation gives:

$$S_\infty = 2$$

$$\frac{3}{2}x = 2$$

$$x = \frac{4}{3}$$

12d The left-hand side forms a GP with $a = x$ and $r = -\frac{1}{3}$.

$$S_\infty = \frac{a}{1-r}$$

$$= \frac{x}{1-\left(-\frac{1}{3}\right)}$$

$$= \frac{x}{\frac{4}{3}}$$

$$= \frac{3x}{4}$$

Hence solving the equation gives:

$S_\infty = 2$

$\frac{3}{4}x = 2$

$x = \frac{8}{3}$

13a The first sequence is a GP with common ratio $\frac{ar^2}{ar} = \frac{ar}{a} = r$, $S_\infty = \frac{a}{1-r}$.

The second sequence is a GP with common ratio $\frac{ar^2}{-ar} = \frac{-ar}{a} = -r$,
$S_\infty = \frac{a}{1-(-r)} = \frac{a}{1+r}$.

The third sequence is a GP with common ratio $\frac{ar^4}{ar^2} = \frac{ar^2}{a} = r^2$, $S_\infty = \frac{a}{1-r^2}$.

The fourth sequence is a GP with common ratio $\frac{ar^5}{ar^3} = \frac{ar^3}{ar} = r^2$, $S_\infty = \frac{ar}{1-r^2}$.

Hence the ratio of limiting sums is given by:

$\frac{a}{1-r} : \frac{a}{1+r} : \frac{a}{1-r^2} : \frac{ar}{1-r^2}$

$\frac{1}{1-r} : \frac{1}{1+r} : \frac{1}{1-r^2} : \frac{r}{1-r^2}$

$\frac{1-r^2}{1-r} : \frac{1-r^2}{1+r} : \frac{1-r^2}{1-r^2} : \frac{r(1-r^2)}{1-r^2}$

$\frac{(1+r)(1-r)}{1-r} : \frac{(1+r)(1-r)}{1+r} : 1 : r$

$1+r : 1-r : 1 : r$

13b i This is a GP with $a = 48, r = \frac{1}{2}$ so $S_\infty = \frac{a}{1-r} = \frac{48}{1-\frac{1}{2}} = 96$

13b ii This is a GP with $a = 48, r = -\frac{1}{2}$ so $S_\infty = \frac{a}{1-r} = \frac{48}{1-\left(-\frac{1}{2}\right)} = \frac{48}{\frac{3}{2}} = 32$

13b iii This is a GP with $a = 48, r = \frac{1}{4} = \left(\frac{1}{2}\right)^2$ so $S_\infty = \frac{a}{1-r} = \frac{48}{1-\frac{1}{4}} = 64$

13b iv This is a GP with $a = 24$, $r = \frac{1}{4} = \left(\frac{1}{2}\right)^2$ so $S_\infty = \frac{a}{1-r} = \frac{24}{1-\frac{1}{4}} = 32$

The ratio of the sequences is $96: 32: 64: 32 = 3: 1: 2: 1$

If we apply the above formula we obtain $1 + \frac{1}{2}: 1 - \frac{1}{2}: 1: \frac{1}{2} = \frac{3}{2}: \frac{1}{2}: 1: \frac{1}{2} = 3: 1: 2: 1$ which is the same as directly calculating the ratio. So we have verified the formula proven above.

14a This is a GP with $a = 7$ and $r = x$.

For a limiting sum, $|r| < 1$ so $|x| < 1$ or $-1 < x < 1$

So $S_\infty = \frac{a}{1-r} = \frac{7}{1-x}$.

14b This is a GP with $a = 2x$ and $r = \frac{6x^2}{2x} = 3x$.

For a limiting sum, $|r| < 1$ so $|3x| < 1$ or $-\frac{1}{3} < x < \frac{1}{3}$

So $S_\infty = \frac{a}{1-r} = \frac{2x}{1-3x}$.

14c This is a GP with $a = 1$ and $r = x - 1$.

For a limiting sum, $|r| < 1$ so $|x - 1| < 1$ or $-1 < x - 1 < 1$ or $0 < x < 2$

So $S_\infty = \frac{1}{1-(x-1)} = \frac{1}{2-x}$.

14d This is a GP with $a = 1$ and $r = 1 + x$.

For a limiting sum, $|r| < 1$ so $|1 + x| < 1$ or $-1 < 1 + x < 1$ or $-2 < x < 0$

So $S_\infty = \frac{a}{1-r} = \frac{1}{1-(1+x)} = -\frac{1}{x}$.

15a This is a GP with $a = 1$ and $r = x^2 - 1$

To have a limiting sum, $|r| < 1$, so:

$|x^2 - 1| < 1$

$-1 < x^2 - 1 < 1$

$0 < x^2 < 2$

Hence:

$-\sqrt{2} < x < \sqrt{2}$ and $x \neq 0$ (since $x^2 \neq 0$)

$$S_\infty = \frac{a}{1-r} = \frac{1}{1-(x^2-1)} = \frac{1}{2-x^2}$$

15b This is a GP with $a = 1$ and $r = \frac{1}{1+x^2}$.

To have a limiting sum, $|r| < 1$, so:

$$\left|\frac{1}{1+x^2}\right| < 1$$

$1 < |1 + x^2|$

$x^2 + 1 > 1$ or $x^2 + 1 < -1$

Hence:

$x^2 > 0$ or $x^2 < -2$ (not possible)

Thus there is a limiting sum for $x \neq 0$.

$$S_\infty = \frac{a}{1-r} = \frac{1}{1-\frac{1}{1+x^2}} = \frac{1+x^2}{1+x^2-1} = \frac{1+x^2}{x^2}$$

16a We know that a GP has a limiting sum if $|r| < 1$; that is:

$-1 < r < 1$

$-1 < -r < 1$

$1 - 1 < 1 - r < 1 + 1$

$0 < 1 - r < 2$ as required.

16b Suppose that we have $a = 8$, $S_\infty = 2$

$$\frac{8}{1-r} = 2$$

$$\frac{1}{1-r} = \frac{1}{4}$$

$1 - r = 4$

This does not lie in the bound $0 < 1 - r < 2$ and thus we can conclude that there is no limiting sum.

16c Since $0 < 1 - r < 2$, then $\frac{1}{1-r} > \frac{1}{2}$

$$S_\infty = \frac{a}{1-r}$$

$$S_\infty = a \times \frac{1}{1-r}$$

$$S_\infty > a \times \frac{1}{2}$$

$$S_\infty > \frac{1}{2}a$$

16d Note that if $a < 0$

$\frac{1}{1-r} > \frac{1}{2}$ becomes

$$\frac{a}{1-r} < \frac{a}{2}$$

So $S_\infty < \frac{1}{2}a$

16d i Since $a > 0, S_\infty > \frac{1}{2}a$

$$S_\infty > \frac{1}{2} \times 6$$

$$S_\infty > 3$$

16d ii Since $a < 0, S_\infty < \frac{1}{2}a$

$$S_\infty < \frac{1}{2} \times (-8)$$

$$S_\infty < -4$$

16d iii Since $a > 0, S_\infty > \frac{1}{2}a$

16d iv Since $a < 0, S_\infty < \frac{1}{2}a$

17a This is a GP with $a = v$ and $r = v$

$$w = S_\infty = \frac{a}{1-r} = \frac{v}{1-v}$$

17b $w = \frac{v}{1-v}$

$$w - wv = v$$

$$w = v + wv$$

$$w = v(1+w)$$

$$v = \frac{w}{1+w}$$

17c This is a GP with $a = w$ and $r = -\frac{w^2}{w} = -w$

$$S_\infty = \frac{a}{1-r}$$

$$= \frac{w}{1-(-w)}$$

$$= \frac{w}{1+w}$$

$$= v \qquad \text{(from part b)}$$

17d If $v = \frac{1}{3}$ then $w = \frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2}$.

Hence the limiting sum is:

$$S_\infty = \frac{w}{1+w}$$

$$= \frac{\frac{1}{2}}{1+\frac{1}{2}}$$

$$= \frac{\frac{1}{2}}{\frac{3}{2}}$$

$$= \frac{1}{3}$$

$$= v$$

18a $S_\infty = 5T_1$

$$\frac{a}{1-r} = 5a$$

$$\frac{1}{1-r} = 5$$

$$1 - r = \frac{1}{5}$$

$$r = \frac{4}{5}$$

18b $T_2 = ar = 6$

$$r = \frac{6}{a}$$

$$S_\infty = 27$$

$$\frac{a}{1-r} = 27$$

$$\frac{a}{1-\frac{6}{a}} = 27$$

$$\frac{a^2}{a-6} = 27$$

$$a^2 = 27a - 162$$

$$a^2 - 27a + 162 = 0$$

$$(a-9)(a-18) = 0$$

$$a = 9 \text{ or } a = 18$$

When $a = 9$ and $r = \frac{6}{9} = \frac{2}{3}$, the first three terms are 9, 6 and 4.

When $a = 18$ and $r = \frac{6}{18} = \frac{1}{3}$, the first three terms are 18, 6 and 2.

18c $S_\infty - S_1 = 5T_1$

$S_\infty - T_1 = 5T_1$

$S_\infty = 6T_1$

$\frac{a}{1-r} = 6a$

$\frac{1}{1-r} = 6$

$\frac{1}{6} = 1 - r$

$r = 1 - \frac{1}{6}$

$r = \frac{5}{6}$

Hence the ratio of the sum of the terms is $r = \frac{5}{6}$.

18d The sum of all terms from the third term on is equal to the sum of all terms with the sum of the first two terms subtracted from it. That is:

$$\begin{aligned} S &= S_\infty - S_2 \\ &= \frac{a}{1-r} - \frac{a(1-r^2)}{1-r} \\ &= \frac{a}{1-r} - \frac{a-ar^2}{1-r} \\ &= \frac{a-a+ar^2}{1-r} \\ &= \frac{ar^2}{1-r} \end{aligned}$$

18d i $S = T_1$

$\frac{ar^2}{1-r} = a$

$\frac{r^2}{1-r} = 1$

$r^2 = 1 - r$

$r^2 + r - 1 = 0$

Using the quadratic formula:

$$r = \frac{-1 \pm \sqrt{1^2 - 4(1)(-1)}}{2(1)}$$
$$= \frac{-1 \pm \sqrt{1+4}}{2}$$
$$= \frac{-1 \pm \sqrt{5}}{2}$$

But $r = -\frac{1}{2} - \sqrt{5} < -1$, so it is not a possible solution, hence the solution is

$$r = -\frac{1}{2} + \frac{1}{2}\sqrt{5}$$

18d ii $S = T_2$

$$\frac{ar^2}{1-r} = ar$$

$$\frac{r^2}{1-r} = r$$

$$r^2 = r - r^2$$

$$2r^2 - r = 0$$

$$r(2r-1) = 0$$

$r = 0$ or $\frac{1}{2}$ but $r \neq 0$

Hence, $r = \frac{1}{2}$.

18d iii $S = T_1 + T_2$

$$\frac{ar^2}{1-r} = a + ar$$

$$\frac{r^2}{1-r} = 1 + r$$

$$r^2 = (1+r)(1-r)$$

$$r^2 = 1 - r^2$$

$$2r^2 = 1$$

$$r^2 = \frac{1}{2}$$

$r = \pm\dfrac{1}{\sqrt{2}}$

$r = \dfrac{1}{\sqrt{2}} \times \dfrac{\sqrt{2}}{\sqrt{2}}$ or $r = -\dfrac{1}{\sqrt{2}} \times \dfrac{\sqrt{2}}{\sqrt{2}}$

$r = \dfrac{\sqrt{2}}{2}$ or $r = -\dfrac{\sqrt{2}}{2}$

19 Suppose we consider the sequence $4 + \frac{4}{3} + \frac{4}{9}$... which is the extension of the sequence to the left of the term.

Starting at the first term after 4, the sequence has $a = \frac{4}{3}$, $r = \frac{1}{3}$ and hence:

$$S_\infty = \frac{\frac{4}{3}}{1 - \frac{1}{3}}$$

$$= \frac{\frac{4}{3}}{-\frac{2}{3}}$$

$$= -2$$

This is the same as the limiting sum found in the calculation in the question. Hence the 'meaning' of this sum can be given as the sum of all terms in the sequence 'prior' to the first term.

Solutions to Exercise 1I

1a $0.\dot{3} = 0.333\ldots = 0.3 + 0.03 + 0.003 + \cdots$

This is a GP with $a = 0.3$ and $r = 0.1$. Hence:

$$0.\dot{3} = S_\infty = \frac{a}{1-r} = \frac{0.3}{1-0.1} = \frac{0.3}{0.9} = \frac{3}{9} = \frac{1}{3}$$

1b $0.\dot{1} = 0.111\ldots = 0.1 + 0.01 + 0.001 + \cdots$

This is a GP with $a = 0.1$ and $r = 0.1$. Hence:

$$0.\dot{1} = S_\infty = \frac{a}{1-r} = \frac{0.1}{1-0.1} = \frac{0.1}{0.9} = \frac{1}{9}$$

1c $0.\dot{7} = 0.777\ldots = 0.7 + 0.07 + 0.007 + \cdots$

This is a GP with $a = 0.7$ and $r = 0.1$. Hence:

$$0.\dot{7} = S_\infty = \frac{a}{1-r} = \frac{0.7}{1-0.1} = \frac{0.7}{0.9} = \frac{7}{9}$$

1d $0.\dot{6} = 0.666\ldots = 0.6 + 0.06 + 0.006 + \cdots$

This is a GP with $a = 0.6$ and $r = 0.1$. Hence:

$$0.\dot{6} = S_\infty = \frac{a}{1-r} = \frac{0.6}{1-0.1} = \frac{0.6}{0.9} = \frac{6}{9} = \frac{2}{3}$$

2a $0.\dot{2}\dot{7} = 0.2727\ldots = 0.27 + 0.0027 + 0.000027 + \cdots$

This is a GP with $a = 0.27$ and $r = 0.01$. Hence:

$$0.\dot{2}\dot{7} = S_\infty = \frac{a}{1-r} = \frac{0.27}{1-0.01} = \frac{0.27}{0.99} = \frac{27}{99} = \frac{3}{11}$$

2b $0.\dot{8}\dot{1} = 0.8181\ldots = 0.81 + 0.0081 + 0.000081 + \cdots$

This is a GP with $a = 0.81$ and $r = 0.01$. Hence:

$$0.\dot{8}\dot{1} = S_\infty = \frac{a}{1-r} = \frac{0.81}{1-0.01} = \frac{0.81}{0.99} = \frac{81}{99} = \frac{9}{11}$$

2c $0.\dot{0}\dot{9} = 0.0909\ldots = 0.09 + 0.0009 + 0.000009 + \cdots$

This is a GP with $a = 0.09$ and $r = 0.01$. Hence:

$$0.\dot{0}\dot{9} = S_\infty = \frac{a}{1-r} = \frac{0.09}{1-0.01} = \frac{0.09}{0.99} = \frac{9}{99} = \frac{1}{11}$$

2d $0.\dot{1}\dot{2} = 0.1212\ldots = 0.12 + 0.0012 + 0.000012 + \cdots$

This is a GP with $a = 0.12$ and $r = 0.01$. Hence:

$$0.\dot{1}\dot{2} = S_\infty = \frac{a}{1-r} = \frac{0.12}{1-0.01} = \frac{0.12}{0.99} = \frac{12}{99} = \frac{4}{33}$$

2e $0.\dot{7}\dot{8} = 0.7878\ldots = 0.78 + 0.0078 + 0.000078 + \cdots$

This is a GP with $a = 0.78$ and $r = 0.01$. Hence:

$$0.\dot{7}\dot{8} = S_\infty = \frac{a}{1-r} = \frac{0.78}{1-0.01} = \frac{0.78}{0.99} = \frac{78}{99} = \frac{26}{33}$$

2f $0.\dot{0}2\dot{7} = 0.027027\ldots = 0.027 + 0.000027 + 0.000000027 + \cdots$

This is a GP with $a = 0.027$ and $r = 0.001$. Hence:

$$0.\dot{0}2\dot{7} = S_\infty = \frac{a}{1-r} = \frac{0.027}{1-0.001} = \frac{0.027}{0.999} = \frac{27}{999} = \frac{1}{37}$$

2g $0.\dot{1}3\dot{5} = 0.135135\ldots = 0.135 + 0.000135 + 0.000000135 + \cdots$

This is a GP with $a = 0.135$ and $r = 0.001$. Hence:

$$0.\dot{1}3\dot{5} = S_\infty = \frac{a}{1-r} = \frac{0.135}{1-0.001} = \frac{0.135}{0.999} = \frac{135}{999} = \frac{5}{37}$$

2h $0.\dot{1}8\dot{5} = 0.185185\ldots = 0.185 + 0.000185 + 0.000000185 + \cdots$

This is a GP with $a = 0.185$ and $r = 0.001$. Hence:

$$0.\dot{1}8\dot{5} = S_\infty = \frac{a}{1-r} = \frac{0.185}{1-0.001} = \frac{0.185}{0.999} = \frac{185}{999} = \frac{5}{27}$$

3a $12.\dot{4} = 12.444\ldots = 12 + 0.4 + 0.04 + 0.004 + \cdots$

All terms after 12 form a GP with $a = 0.4$ and $r = 0.1$. Hence:

$$12.\dot{4} = 12 + S_\infty = 12 + \frac{a}{1-r} = 12 + \frac{0.4}{1-0.1} = 12 + \frac{0.4}{0.9} = 12\frac{4}{9}$$

3b $7.\dot{8}\dot{1} = 7.8181\ldots = 0.81 + 0.0081 + 0.000081 + \cdots$

All terms after 7 form a GP with $a = 0.81$ and $r = 0.01$. Hence

$$7.\dot{8}\dot{1} = 7 + S_\infty = 7 + \frac{a}{1-r} = 7 + \frac{0.81}{1-0.01} = 7 + \frac{0.81}{0.99} = 7\frac{9}{11}$$

3c $8.4\dot{6} = 8.466\ldots = 8.4 + 0.06 + 0.006 + \cdots$

All terms after 8.4 form a GP with $a = 0.06$ and $r = 0.1$. Hence:

$$8.4\dot{6} = 8.4 + S_\infty = 8.4 + \frac{a}{1-r} = 8.4 + \frac{0.06}{1-0.1} = 8.4 + \frac{0.06}{0.9} = 8.4 + \frac{6}{90} = 8\frac{7}{15}$$

3d $0.2\dot{3}\dot{6} = 0.23636\ldots = 0.2 + 0.036 + 0.00036 + \cdots$

All terms after 0.2 form a GP with $a = 0.036$ and $r = 0.01$. Hence:

$$\begin{aligned} 0.2\dot{3}\dot{6} &= 0.2 + S_\infty \\ &= 0.2 + \frac{a}{1-r} \\ &= 0.2 + \frac{0.036}{1-0.01} \\ &= 0.2 + \frac{0.036}{0.99} \\ &= 0.2 + \frac{2}{55} \\ &= \frac{11}{55} + \frac{2}{55} \\ &= \frac{13}{55} \end{aligned}$$

4a $0.\dot{9} = 0.99999\ldots = 0.9 + 0.09 + 0.009 + 0.0009 + \cdots$

This is a GP with $a = 0.9$ and $r = \frac{0.09}{0.9} = 0.1$, so the sum will be:

$$S_\infty = \frac{a}{1-r}$$

$$= \frac{0.9}{1 - 0.1}$$

$$= \frac{0.9}{0.9}$$

$$= 1$$

Thus $0.\dot{9} = 1$

4b $2.7\dot{9} = 2.799999 \ldots = 2.7 + 0.09 + 0.009 + 0.0009 + \cdots$

All terms after 2.7 form a GP with $a = 0.09$ and $r = \frac{0.009}{0.09} = 0.1$, so the sum will be

$$S_\infty = \frac{a}{1 - r} = \frac{0.09}{1 - 0.1} = \frac{0.09}{0.9} = 0.1$$

Hence $2.7\dot{9} = 2.7 + S_\infty = 2.7 + 0.1 = 2.8$

5a $0.\dot{9}5\dot{7} = 0.957\ 957\ 957 \ldots = 0.957 + 0.000\ 957 + 0.000\ 000\ 957 + \cdots$

This is a GP with $a = 0.957$ and $r = 0.001$. Hence:

$$0.\dot{9}5\dot{7} = S_\infty = \frac{a}{1 - r} = \frac{0.957}{1 - 0.001} = \frac{0.957}{0.999} = \frac{957}{999} = \frac{29}{303}$$

5b $0.\dot{2}47\dot{5} = 0.247\ 524\ 75 \ldots = 0.2475 + 0.000\ 0247\ 5 + \cdots$

This is a GP with $a = 0.2475$ and $r = 0.0001$. Hence:

$$0.\dot{2}47\dot{5} = S_\infty = \frac{a}{1 - r} = \frac{0.2475}{1 - 0.0001} = \frac{0.2475}{0.9999} = \frac{2475}{9999} = \frac{25}{101}$$

5c $0.\dot{2}30\ 76\dot{9} = 0.230\ 769\ 230\ 769 \ldots = 0.2307\ 69 + 0.000\ 002\ 307\ 69 + \cdots$

This is a GP with $a = 0.230\ 769$ and $r = 0.000\ 01$. Hence:

$$0.\dot{2}30\ 76\dot{9} = S_\infty = \frac{a}{1 - r} = \frac{0.230\ 769}{1 - 0.000\ 01} = \frac{0.230\ 769}{0.999\ 99} = \frac{230\ 769}{99\ 999} = \frac{3}{13}$$

5d $0.\dot{4}28\ 57\dot{1} = 0.428\ 571\ ... = 0.428\ 571 + 0.000\ 004\ 285\ 71 + \cdots$

This is a GP with $a = 0.230\ 769$ and $r = 0.000\ 01$. Hence:

$$0.\dot{4}2857\dot{1} = S_\infty = \frac{a}{1-r} = \frac{0.428\ 571}{1 - 0.000\ 01} = \frac{0.428\ 571}{0.999\ 99} = \frac{428\ 571}{999\ 990} = \frac{3}{7}$$

5e $0.25\dot{5}\dot{7} = 0.255\ 757\ 575\ 7\ ... = 0.25 + (0.0057 + 0.000\ 057 + \cdots)$

The bracketed terms form a GP with $a = 0.0057$ and $r = 0.01$. Hence:

$$\begin{aligned} 0.25\dot{5}\dot{7} &= 0.25 + S_\infty \\ &= 0.25 + \frac{a}{1-r} \\ &= \frac{1}{4} + \frac{0.0057}{1 - 0.01} \\ &= \frac{1}{4} + \frac{0.0057}{0.99} \\ &= \frac{1}{4} + \frac{57}{9900} \\ &= \frac{211}{825} \end{aligned}$$

5f $1.1\dot{0}3\dot{7} = 1.103\ 703\ 703\ 7\ ... = 1.1 + (0.0037 + 0.000\ 003\ 7 + \cdots)$

The bracketed terms form a GP with $a = 0.0037$ and $r = 0.001$. Hence:

$$\begin{aligned} 1.1\dot{0}3\dot{7} &= 1.1 + S_\infty \\ &= 1.1 + \frac{a}{1-r} \\ &= 1.1 + \frac{0.0037}{1 - 0.001} \\ &= 1.1 + \frac{0.0037}{0.999} \\ &= \frac{10989}{9990} + \frac{37}{9990} \\ &= 1\frac{14}{135} \end{aligned}$$

5g $0.0\dot{0}0\ 27\dot{1} = 0.000\ 271\ 002\ 710\ 027\ 1\ ...$
$= 0.000\ 271 + 0.000\ 000\ 002\ 71 + 0.000\ 000\ 000\ 000\ 027\ 1 + \cdots$

This is a GP with $a = 0.000\ 271$ and $r = 0.0001$. Hence:

$$
\begin{aligned}
0.0\dot{0}0\ 27\dot{1} &= S_\infty \\
&= \frac{a}{1-r} \\
&= \frac{0.000\ 271}{1-0.0001} \\
&= \frac{0.000\ 271}{0.9999} \\
&= \frac{1}{3690}
\end{aligned}
$$

5h $$
\begin{aligned}
7.7\dot{7}1\ 428\ \dot{5} &= 7.771\ 428\ 571\ 428\ 571\ 428\ 5\ \ldots \\
&= 7.7 + (0.071\ 428\ 5 + 0.000\ 007\ 142\ 85 + \cdots)
\end{aligned}
$$

The bracketed terms form a GP with $a = 0.071\ 428\ 5$ and $r = 0.000\ 01$. Hence:

$$
\begin{aligned}
0.0\dot{0}0\ 27\dot{1} &= 7.7 + S_\infty \\
&= 7.7 + \frac{a}{1-r} \\
&= 7.7 + \frac{0.071\ 428\ 5}{1-0.000\ 01} \\
&= 7.7 + \frac{0.071\ 428\ 5}{0.999\ 99} \\
&= 7\frac{27}{35}
\end{aligned}
$$

6 $\sqrt{2} = 1.414\ 213\ 562\ \ldots$ which has no obvious repeating pattern.

If $\sqrt{2}$ were a recurring decimal, then we could use the methods of this section to write it as a fraction.

7a Notice that $\frac{1}{9} = 0.\dot{1}$, $\frac{1}{99} = 0.\dot{0}\dot{1}$, $\frac{1}{999} = 0.\dot{0}\dot{0}\dot{1}$, and so on. If the denominator of a fraction can be made a string of nines, then the fraction will be a multiple of one of these recurring decimals.

7b Periods: $1, 6, 1, 2, 6, 3, 3, 5, 4, 5 = 0.\dot{9}, 5.\dot{9}, 0.\dot{9}, 1.\dot{9}, 5.\dot{9}, 2.\dot{9}, 2.\dot{9}, 4.\dot{9}, 3.\dot{9}, 4.\dot{9}$

8a $$
\begin{aligned}
0.46\dot{9} &= 0.469\ 999\ 999\ 9\ \ldots \\
&= 0.46 + (0.009 + 0.0009 + 0.000\ 09 + \cdots)
\end{aligned}
$$

The bracketed terms form a GP with $a = 0.009$ and $r = 0.1$. Hence:

$$\begin{aligned} 0.46\dot{9} &= 0.46 + S_\infty \\ &= 0.46 + \frac{a}{1-r} \\ &= 0.46 + \frac{0.009}{1-0.01} \\ &= 0.46 + \frac{0.009}{0.9} \\ &= 0.46 + \frac{1}{100} \\ &= 0.46 + 0.01 \\ &= 0.47 \end{aligned}$$

8b The infinite string of 9s can be removed and the last digit that is not a 9 is increased by 1.

8c The last digit of any decimal can be reduced by 1 and then all following terms replaced with an infinite string of 9s.

8d The fourth sentence should be changed to, ‘Imagine that each real number T_n in the sequence is written as an infinite decimal string of digits $0.dddddd\ldots$, where each d represents a digit. Add an infinite string of zeroes to every terminating decimal, and if there is an infinite string of 9s, rewrite the decimal as a terminating decimal.’

9-10 Answers are contained in the question.

Chapter 1 Review

1 The first 12 terms of the sequence are:
$50, 41, 32, 23, 14, 5, -4, -13, -22, -31, -40, -49$

1a The positive terms are $50, 41, 32, 23, 14, 5$

Counting, there are a total of 6.

1b The terms between 0 and 40 are $32, 23, 14, 5$

Counting, there are a total of 4.

1c $T_{10} = -31$

1d $T_8 = -13$

1e No, extending the sequence gives:
$50, 41, 32, 23, 14, 5, -4, -13, -22, -31, -40, -49, -58, -67, -76, -85, -95, -104 \ldots$

which does not contain -100.

1f $T_{11} = -40$

2a $T_1 = 58 - 6(1) = 52$

$T_{20} = 58 - 6(20) = -62$

$T_{100} = 58 - 6(100) = -542$

$T_{1\,000\,000} = 58 - 6(1\,000\,000) = -5\,999\,942$

2b Solving $T_n = 20$

$58 - 6n = 20$

$6n = 38$

$n = 6.33 \ldots$

As n is not an integer, 20 is not a term in the sequence.

Solving $T_n = 10$

$58 - 6n = 10$

$6n = 48$

$n = 8$

Thus $T_8 = 10$ is a term.

Solving $T_n = -56$

$58 - 6n = -56$

$6n = 114$

$n = 19$

Thus $T_{19} = -56$ is a term.

Solving $T_n = -100$

$58 - 6n = -100$

$6n = 158$

$n = 26.33 \ldots$

As n is not an integer, -100 is not a term in the sequence.

2c $T_n < -200$

$58 - 6n < -200$

$6n > 258$

$n > 43$

Hence the first term less than -200 is $T_{44} = -206$.

2d $T_n > -600$

$58 - 6n > -600$

$6n < 658$

$n < 109.666$

Hence the last term greater than -600 is $T_{109} = -596$

3a $S_1 = T_1 = 4$

$S_2 = T_1 + S_1$

Hence:

$T_2 = S_2 - S_1 = 11 - 4 = 7$

Similarly:

$T_3 = S_3 - S_2 = 18 - 11 = 7$

$T_4 = S_4 - S_3 = 25 - 18 = 7$

$T_5 = S_5 - S_4 = 32 - 25 = 7$

$T_6 = S_6 - S_5 = 39 - 32 = 7$

Hence giving the sequence:

$4, 7, 7, 7, 7, 7, \ldots$

3b $S_1 = T_1 = 0$

$S_2 = T_1 + S_1$

Hence:

$T_2 = S_2 - S_1 = 1 - 0 = 1$

Similarly:

$T_3 = S_3 - S_2 = 3 - 1 = 2$

$T_4 = S_4 - S_3 = 6 - 3 = 3$

$T_5 = S_5 - S_4 = 10 - 6 = 4$

$T_6 = S_6 - S_5 = 15 - 10 = 5$

$T_7 = S_7 - S_6 = 21 - 15 = 6$

Hence giving the sequence:

$0, 1, 2, 3, 4, 5, 6, \ldots$

3c $T_1 = S_1 = 1^2 + 5 = 6$

$$\begin{aligned} T_n &= S_n - S_{n-1} \\ &= n^2 + 5 - ((n-1)^2 + 5) \\ &= n^2 + 5 - (n^2 - 2n + 1 + 5) \\ &= 2n - 1 \text{ for } n > 1 \end{aligned}$$

3d $T_1 = S_1 = 3^1 = 3$

$$\begin{aligned} T_n &= S_n - S_{n-1} \\ &= 3^n - 3^{n-1} \\ &= 3^{n-1}(3 - 1) \\ &= 2 \times 3^{n-1} \text{ for } n > 1 \end{aligned}$$

4a

$$\sum_{n=3}^{6}(n^2 - 1) = 8 + 15 + 24 + 35 = 82$$

4b

$$\sum_{n=-2}^{2}(5n - 3) = -13 + (-8) + (-3) + 2 + 8 = -15$$

4c

$$\sum_{n=0}^{6}(-1)^n = 1 + (-1) + 1 + (-1) + 1 + (-1) + 1 = 1$$

4d

$$\sum_{n=1}^{6}\left(\frac{1}{2}\right)^n = \frac{1}{2} + \frac{1}{4} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} = \frac{63}{64}$$

5a The first eight terms are:

$T_1 = 5 \times (-1)^1 = -5$
$T_2 = 5 \times (-1)^2 = 5$
$T_3 = 5 \times (-1)^3 = -5$
$T_4 = 5 \times (-1)^4 = 5$
$T_5 = 5 \times (-1)^5 = -5$
$T_6 = 5 \times (-1)^6 = 5$
$T_7 = 5 \times (-1)^7 = -5$
$T_8 = 5 \times (-1)^8 = 5$

Thus the sequence is:

$-5, 5, -5, 5, -5, 5, -5, 5$

5b The sum of first seven terms is:

$$S_7 = \frac{a(1-r^n)}{1-r} = \frac{-5(1-(-1)^7)}{1-(-1)} = -\frac{5(2)}{2} = -5$$

$$S_8 = \frac{a(1-r^n)}{1-r} = \frac{-5(1-(-1)^8)}{1-(-1)} = -\frac{5(0)}{2} = 0$$

5c For this particular sequence, one simply adds -5 if the previous term is 0 and adds 5 if the previous term was -5. This means that $T_n = -5$ if n is odd and $T_n = 0$ of n is even.

5d Noting that $T_n = -5$ if n is odd and $T_n = 0$ if n is even.

$T_{20} = 0$

$T_{75} = -5$

$T_{111} = -5$

6a $T_2 - T_1 = 83 - 76 = 7$
$T_3 - T_2 = 90 - 83 = 7$

Hence there is a common difference between the terms so it is an AP with common difference of 7.

6b $T_2 - T_1 = 100 - (-21) = -121$
$T_3 - T_2 = (-21) - (-142) = -121$

Hence there is a common difference between the terms so it is an AP with common difference of -121.

6c $T_2 - T_1 = 9 - 4 = 5$
$T_3 - T_2 = 4 - 1 = 3$

$$\frac{T_3}{T_2} = \frac{9}{4} = 2.25$$

$$\frac{T_2}{T_1} = \frac{4}{1} = 4$$

There is no common ratio nor common difference. Hence it is neither an AP nor a GP.

6d

$$\frac{T_3}{T_2} = \frac{54}{18} = 3$$

$$\frac{T_2}{T_1} = \frac{18}{6} = 3$$

Hence as there is a common ratio, this is a GP with $r = 3$.

6e $T_2 - T_1 = 10 - 6 = 4$
$T_3 - T_2 = 15 - 10 = 5$

$$\frac{T_3}{T_2} = \frac{15}{10} = \frac{3}{2}$$

$$\frac{T_2}{T_1} = \frac{10}{6} = \frac{5}{3}$$

There is no common ratio nor common difference. Hence it is neither an AP nor a GP.

6f

$$\frac{T_3}{T_2} = \frac{12}{-24} = -\frac{1}{2}$$

$$\frac{T_2}{T_1} = \frac{-24}{48} = -\frac{1}{2}$$

Hence as there is a common ratio, this is a GP with $r = -\frac{1}{2}$.

7a $a = 23, d = 35 - 23 = 12$

7b Since $T_n = a + (n-1)d = 23 + 12(n-1)$

$T_{20} = 23 + 20(20 - 1) = 251$

$T_{600} = 23 + 20(600 - 1) = 7211$

7c $T_n = a + (n-1)d = 23 + 12(n-1) = 23 + 12n - 12 = 11 + 12n$

7d If 143 is a term, then:

$143 = 11 + 12n$

$12n = 132$

$n = 11$

As n is a positive integer, 143 is a term.

If 173 is a term, then:

$173 = 11 + 12n$

$12n = 162$

$n = 13.5$

As n is not a positive integer, 173 is not a term.

7e In order for the term to be greater than 1000, we must have:

$T_n > 1000$

$11 + 12n > 1000$

$12n > 989$

$n > 82.4$

So the smallest integer that satisfies this inequality is $n = 83$.

Hence the first term greater than 1000 is $T_{83} = 1007$.

In order for the term to be less than 2000 we must have:

$11 + 12n < 2000$

$12n < 1989$

$n < 165.75$

The largest integer that satisfies this inequality is $n = 165$.

Hence the last term less than 2000 is $T_{165} = 1991$.

7f $165 - 83 + 1 = 83$ (count both T_{83} and T_{165})

8a The amount charged forms an AP with $a = 20$, $d = 16$

8b $T_n = a + (n-1)d = 20 + 16(n-1) = 20 + 16n - 16 = 4 + 16n$

8c Note that as at most \$200 can be spent:

$T_n \leq 200$

$4 + 16n \leq 200$

$16n \leq 196$

$n \leq 12.25$

Hence the largest number of cases that can be bought is 12. Furthermore, as $T_{12} = 196$, the twelve cases will cost a total of \$196, hence, there will be \$4 left in change.

8d $T_n = 292$

$4 + 16n = 292$

$16n = 288$

$n = 18$

Hence the neighbour purchased 18 cases.

9a $a = 50, r = \frac{100}{50} = 2$

9b $T_n = ar^{n-1} = 50 \times 2^{n-1}$ (or 25×2^n)

9c $T_8 = 50(2)^{8-1} = 6400, T_{12} = 50(2)^{12-1} = 102\ 400$

9d If 1600 is a term, then:

$1600 = 50(2)^{n-1},$

$32 = 2^{n-1}$

$n - 1 = 5$

$n = 6$

As n is a positive integer, 1600 is a term.

If 4800 is a term, then:

$4800 = 50(2)^{n-1}$

$96 = 2^{n-1}$

As 96 is not a power of 2, n cannot be a positive integer. This means that 4800 is not a term.

9e $T_4 \times T_5 = 50 \times 2^{4-1} \times 50 \times 2^{5-1} = 320\ 000$

9f $ar^{n-1} < 10\,000\,000$

$50 \times 2^{n-1} < 10\,000\,000$

$2^{n-1} < 200\,000$

$n - 1 < \log_2 200\,000$

$n < \log_2 200\,000 + 1$

$n < 18.6$

Hence, rounding down, we can conclude that there are 18 terms.

10a The number of visitors on each subsequent day, is given by multiplying the number on the previous day by $\frac{1}{3}$, hence, by definition we are describing a GP with $a = 486, r = \frac{1}{3}$.

10b 486, 162, 54, 18, 6, 2 (we do not go further as fractions here are nonsensical)

10c 4 days (there are 4 terms greater than 10 in the above sequence)

10d

$$S_6 = \frac{a(1 - r^n)}{1 - r}$$

$$= \frac{486\left(1 - \left(\frac{1}{3}\right)^6\right)}{1 - \frac{1}{3}}$$

$$= \frac{486\left(1 - \left(\frac{1}{3}\right)^6\right)}{\frac{2}{3}}$$

$$= 728$$

Total number of visitors was 728.

10e

$$S_\infty = \frac{a}{1-r}$$
$$= \frac{486}{1-\frac{1}{3}}$$
$$= 729$$

The 'eventual' number of visitors is 729.

11a Since terms 1 and 2 and terms 2 and 3 must have the same difference we have:

$x - 15 = 135 - x$

$2x = 150$

$x = 75$

11b Since terms 1 and 2 and terms 2 and 3 must have the same ratio we have:

$$\frac{x}{15} = \frac{135}{x}$$

$x^2 = 135 \times 15$

$x^2 = 2025$

$x = \pm 45$

12a For this AP, $a = 51, d = 11$ so:

$$S_{41} = \frac{1}{2}n(2a + (n-1)d)$$
$$= \frac{1}{2} \times 41 \times (2 \times 51 + (41-1) \times 11)$$
$$= 11\ 111$$

12b For this AP, $a = 100$, $d = -25$ so:

$$S_{41} = \frac{1}{2}n(2a + (n-1)d)$$

$$= \frac{1}{2} \times 41 \times (2 \times 100 + (41-1) \times -25)$$

$$= -16\,400$$

12c For this AP, $a = -35$, $d = 3$ so:

$$S_{41} = \frac{1}{2}n(2a + (n-1)d)$$

$$= \frac{1}{2} \times 41 \times (2 \times -35 + (41-1) \times 3)$$

$$= 1025$$

13a $a = 23$ and $d = 27 - 23 = 4$

Thus, we find the number of terms by solving the equation:

$199 = 23 + (n-1) \times 4$

$176 = 4(n-1)$

$n - 1 = 44$

$n = 45$

Hence:

$$S_{45} = \frac{1}{2}n(a + l) = \frac{1}{2} \times 45 \times (23 + 4) = 4995$$

13b $a = 200$ and $d = 197 - 200 = 3$

Thus, we find the number of terms by solving the equation:

$-100 = 200 + (n-1) \times -3$

$-300 = -3(n-1)$

$n - 1 = 100$

$n = 101$

Hence:

$$S_{101} = \frac{1}{2}n(a+l) = \frac{1}{2} \times 101 \times (200 - 100) = 5050$$

13c $a = 12$ and $d = 12\frac{1}{2} - 12 = \frac{1}{2}$

Thus, we find the number of terms by solving the equation:

$$50 = 12 + (n-1) \times \frac{1}{2}$$

$$38 = \frac{1}{2}(n-1)$$

$$n - 1 = 76$$

$$n = 77$$

Hence:

$$S_{77} = \frac{1}{2}n(a+l) = \frac{1}{2} \times 77 \times (12 + 50) = 2387$$

14a For this GP, $a = 3$ and $r = 2$, hence:

$$S_6 = \frac{a(r^n - 1)}{r - 1} = \frac{3(2^6 - 1)}{2 - 1} = 189$$

14b For this GP, $a = 6$ and $r = -2$, hence:

$$S_6 = \frac{a(r^n - 1)}{r - 1} = \frac{6((-2)^6 - 1)}{-2 - 1} = -1092$$

14c For this GP, $a = -80$ and $r = \frac{1}{2}$, hence:

$$S_6 = \frac{a(r^n - 1)}{r - 1} = \frac{-80\left(\left(\frac{1}{2}\right)^6 - 1\right)}{\frac{1}{2} - 1} = -157\frac{1}{2}$$

15a This is a GP with $a = 240$ and $r = \frac{48}{240} = \frac{1}{5}$.

$$\begin{aligned}S_\infty &= \frac{a}{1-r}\\ &= \frac{240}{1-\frac{1}{5}}\\ &= \frac{240}{\frac{4}{5}}\\ &= \frac{5}{4} \times 240\\ &= 300\end{aligned}$$

15b $r = \frac{9}{-6} = -\frac{3}{2} < -1$, so there is no limiting sum.

15c This is a GP with $a = -405$ and $r = \frac{-135}{405} = -\frac{1}{3}$.

$$\begin{aligned}S_\infty &= \frac{a}{1-r}\\ &= \frac{-135}{1-\left(-\frac{1}{3}\right)}\\ &= \frac{-130}{\frac{4}{3}}\\ &= \frac{3}{4} \times (-130)\\ &= -303\frac{3}{4}\end{aligned}$$

16a This is a GP with common ratio $r = \frac{(2+x)^2}{2+x} = 2 + x$.

In order to have a limiting sum we must have $|r| < 1$ and hence $|2 + x| < 1$. This implies that $-1 < 2 + x < 1$ and so there will be a limiting sum if $-3 < x < -1$.

16b

$$S_\infty = \frac{a}{1-r} = \frac{2+x}{1-(2+x)} = \frac{2+x}{-x-1} = -\frac{2+x}{1+x}$$

17a $0.\dot{3}\dot{9} = 0.393\,939\ldots = 0.39 + 0.0039 + 0.000\,039 + \cdots$

This is a GP with $a = 0.39$ and $r = 0.01$. Hence:

$$0.\dot{3}\dot{9} = S_\infty = \frac{a}{1-r} = \frac{0.39}{1-0.01} = \frac{0.39}{0.99} = \frac{39}{99} = \frac{13}{33}$$

17b $0.\dot{4}6\dot{8} = 0.468\,468\ldots = 0.468 + 0.000\,468 + 0.000\,000\,468 + \cdots$

This is a GP with $a = 0.468$ and $r = 0.001$. Hence:

$$0.\dot{0}2\dot{7} = S_\infty = \frac{a}{1-r} = \frac{0.468}{1-0.001} = \frac{0.468}{0.999} = \frac{468}{999} = \frac{52}{111}$$

17c $12.30\dot{4}\dot{5} = 12.304\,545\,454\,5\ldots$

$$= 12.30 + 0.0045 + 0.000045 + 0.00000045 + \cdots$$

All terms after 12.30 form a GP with $a = 0.0045$ and $r = 0.01$. Hence:

$$\begin{aligned} 12.30\dot{4}\dot{5} &= 12.30 + S_\infty \\ &= 12.30 + \frac{a}{1-r} \\ &= 12.30 + \frac{0.045}{1-0.01} \\ &= 12.30 + \frac{0.045}{0.999} \\ &= 12.30 + \frac{445}{999} \\ &= 12\frac{335}{1100} \\ &= 12\frac{67}{220} \end{aligned}$$

18a $T_2 = 21$

$a + (2-1)d = 21$

$a + d = 21 \qquad (1)$

$T_9 = 56$

$a + (9-1)d = 56$

$a + 8d = 56 \qquad (2)$

$7d = 35 \qquad (2) - (1)$

$d = 5$ (3)

$a + 5 = 21$ (3) in (1)

$a = 16$

$T_{100} = a + (n-1)d = 16 + (100-1) \times 5 = 511$

18b $T_3 = 10$

$a + (3-1)d = 10$

$a + 2d = 10$ (1)

$T_{12} = -89$

$a + (12-1)d = -89$

$a + 11d = -89$ (2)

$9d = -99$ (2) − (1)

$d = -11$ (3)

$a + 2(-11) = 10$ (3) in (1)

$a = 32$

Hence:

$$S_{20} = \frac{1}{2}n(2a + (n-1)d) = \frac{1}{2} \times 20 \times \big(2 \times 32 + (20-1)(-11)\big) = -1450$$

18c $T_3 = 3$

$ar^{3-1} = 3$

$ar^2 = 3$ (1)

$T_8 = -96$

$ar^{8-1} = -96$

$ar^7 = -96$ (2)

$r^5 = -32$ (2) ÷ (1)

$r = -2$ (3)

$a(-2)^2 = 3$ (3) in (1)

$a = \frac{3}{4}$

Hence:

$T_6 = \frac{3}{4}(-2)^{6-1} = -24$

18d We are given that $T_1 = a = 1$ and that $S_{10} = -215$

Hence:

$S_{10} = -215$

$\frac{1}{2} \times 10 \times (2 \times 1 + (10-1) \times d) = -215$

$5(2 + 9d) = -215$

$2 + 9d = -43$

$9d = -45$

$d = -5$

18e The AP has $a = 4\frac{1}{2}$ and $d = -1$.

If $S_n = 8$ where n is the number of terms:

$\frac{1}{2}n\left(2 \times 4\frac{1}{2} + (n-1) \times -1\right) = 8$

$n(9 - n + 1) = 16$

$n(10 - n) = 16$

$10n - n^2 = 16$

$n^2 - 10n + 16 = 0$

$(n-2)(n-8) = 0$

Thus there are either 2 or 8 terms.

18f

$$S_\infty = \frac{a}{1-r}$$

$$45 = \frac{60}{1-r}$$

$$1 - r = \frac{60}{45}$$

$$1 - r = \frac{4}{3}$$

$$r = 1 - \frac{4}{3}$$

$$r = -\frac{1}{3}$$

18g $S_{10} = 682$

$$\frac{a(r^n - 1)}{r - 1} = 682$$

$$\frac{a((-2)^{10} - 1)}{-2 - 1} = 682$$

$$\frac{a(1024 - 1)}{-3} = 682$$

$$a = 682 \times \frac{-3}{1023} = -2$$

$$T_n = ar^{n-1}$$

$$T_4 = -2 \times (-2)^3 = 16$$