Solutions to Exercise 2A

1a equality

1b implication

1c equivalence

1d for all

1e there exists

2a If a triangle has two equal angles, then it has two equal sides. True.

2b If the square of a number is odd, then the number is odd. True.

2c If I have four legs, then I am a horse. False.

2d If a number is even, then it ends with the digit 6. False.

2e Every rhombus is a square. False.

2f If $n \geq 0$, then $\sqrt{n} \in R$. True.

3a True

3b False

3c False

3d False

3e True

3f True

4a Not all cars are red. Alternatively, some cars are not red.

4b $a \leq b$

4c Hillary does not like both steak and pizza, i.e. she doesn't like steak or she doesn't like pizza.

4d Bill and Dave are both incorrect.

4e I live in Tasmania but not in Australia.

4f Nikhil doesn't study and he passes.

4g $x < -3$ or $x > 8$

4h $-5 \leq x < 0$

5a If my plants don't grow, then I haven't watered them.

5b If you live in Melbourne, then you live in Australia.

5c If a triangle doesn't have three equal angles, then it doesn't have three equal sides.

5d If I like motorists, then I don't like cycling.

5e If a number is not odd (i.e. it is even) then the previous number is not even (i.e. it is odd).

5f If $\frac{1}{a} \geq \frac{1}{b}$ then $a \leq b$ or at least one of a and b is negative.

6a If a number is divisible by both 3 and 5 then it is divisible by 15. Conversely, if a number is divisible by 15 then it is divisible by both 3 and 5.

6b If a triangle has two equal sides, then it has two equal angles. Conversely, if a triangle has two equal angles, then it has two equal sides.

6c If the only divisors of an integer n greater than 1 are 1 and n, then n is prime. Conversely, if n is prime (and implicitly, greater than 1) then its only divisors are 1 and n.

6d If a quadrilateral has a pair of opposite sides that are equal and parallel, then it is a parallelogram. Conversely, if a quadrilateral is a parallelogram, then it has a pair of opposite sides that are equal and parallel.

7a True.

7b False. $3x < x$ whenever $x < 0$, e.g. if $x = -1$.

7c True.

7d False. $x > x^2$ whenever $0 < x < 1$, e.g. if $x = \frac{1}{2}$.

7e False. When $x < 0$, $|-x| = -x \neq x$, e.g. if $x = -1$.

7f True.

8a False. For example, $2 > -3$ but $2^2 < (-3)^2$.

8b False. For example, $(-3)^2 > 2^2$ but $-3 < 2$.

8c True. We can divide this problem into cases based on the signs of a and b.

Case 1: If $a > b$ and a, b are both non-negative

Then:

$|a| > |b|$

$a^2 > b^2$

$a \times a^2 > b \times b^2$ (since all terms on LHS are positive and all terms on RHS are non-negative)

$a^3 > b^3$

Case 2: If $a > b$ and a, b are both non-positive

Then:

$b < a \leq 0$

Therefore:

$|b| > |a|$

$a = -|a|$

$a^3 = -|a|^3$

$b < 0$

$b = -|b|$

$b^3 = -|b|^3$

Since $|b| > |a|$

$-|b|^3 < -|a|^3$

Therefore $a^3 > b^3$.

Case 3: If $a > b$ and $a > 0$ but $b < 0$

Then $a^3 > 0$ and $b^3 < 0$

Therefore $a^3 > b^3$

So in all possible cases, $a^3 > b^3$.

8d True. If $a < b < 0$, then $\frac{1}{b} < \frac{1}{a} < 0$ so $\frac{1}{a} > \frac{1}{b}$.

8e False. If a and b have opposite signs then $|a + b| < |a| + |b|$.

For instance, $|2 + (-1)| = 1$ which is less than $|2| + |-1|$.

8f True. We can divide this problem into cases based on the signs of a and b.

Case 1: a and b both non-negative.

Then:

$|a| - |b| = a - b$

$\big||a| - |b|\big| = |a - b|$

So RHS equals LHS.

Case 2: a and b both negative.

Then:

$|a| - |b| = -a + b$

$\big||a| - |b|\big| = |-a + b|$

$= |a - b|$

So RHS equals LHS.

Case 3: a negative, b non-negative.

Then:

$|a - b| = \big|-|a| - |b|\big|$

$= |a| + |b|$

$= \big||a| + |b|\big|$

$\big||a| - |b|\big| \leq \big||a| + |b|\big|$

So LHS $\geq$ RHS.

Case 4: a non-negative, b negative.

Then:

$|a-b| = \big||a|+|b|\big|$

$\big||a|-|b|\big| \leq \big||a|+|b|\big|$

So LHS $\geq$ RHS.

So the result is true for all possible cases.

9a Rain only happens when there are clouds, but clouds can exist without rain, so this is an "implies" statement: $\Rightarrow$.

9b If $3a = 6$ then $a = 2$ so $5a$ must always equal 10.

Similarly, if $5a = 10$ then $a = 2$ so $3a$ must always equal 6.

Therefore the two sides are equivalent: $\Leftrightarrow$.

9c If $a > b$ then $-b$ is always greater than $-a$, and vice versa.

Therefore the two sides are equivalent: $\Leftrightarrow$.

9d If $x = 5$ then $x^2 = 25$.

But $x^2 = 25$ doesn't guarantee that $x = 5$, as $x = -5$ is also possible.

Therefore this is an "implies" statement: $\Rightarrow$.

9e If $x = 5$ then $x^3 = 125$, and if $x^3 = 125$ then x must equal 5, assuming x is real. Therefore the two sides are equivalent: $\Leftrightarrow$.

9f If a is an integer then a^2 is an integer.

But if (for example) $a^2 = 2$, then a is not an integer.

Therefore this is an "implies" statement: $\Rightarrow$.

10a False. If $\theta = \frac{5\pi}{6}$ then $\sin\theta = \frac{1}{2}$ but $\theta \neq \frac{\pi}{6}$.

10b False. If $\theta = \frac{\pi}{4}$ then $\tan\theta = \pm 1$ but $\sin\theta \neq -\frac{1}{\sqrt{2}}$.

10c True.

10d True.

11a If Jack does Extension 2 Mathematics then he is crazy.

11b Jack does Extension 2 Mathematics and he is not crazy.

11c If Jack does not do Extension 2 Mathematics then he is crazy.

11d If Jack is not crazy then he does Extension 2 Mathematics.

11e If Jack does not do Extension 2 Mathematics then he is not crazy.

11f Jack does Extension 2 Mathematics and he is not crazy.

12a For every integer, there exists an integer which is larger.

12b Any positive number added to its reciprocal gives at least two.

13a True, because the premise is false, or alternately because the conclusion is true. (Either is enough to prove the implication true.)

13b False, because the premise is true but the conclusion is not.

13c True, because the premise is false.

13d True, because the premise is true and the conclusion is true.

14a If a number is less than zero, then by definition it is a negative number, and vice versa.

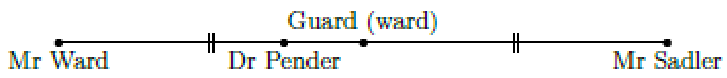
14b If $1 < 0$ then 1 is a negative number. This statement is true, because "$1 < 0$" is false.

15a Yes. "If I do not do my homework, then I will fail" is equivalent to "If I do not fail, then I have done my homework".

15b Cannot be determined. While studying hard implies passing, passing doesn't necessarily imply studying hard.

16 If either Anna or Bryan passed then Chris passed. Since the statement is false, Chris failed.

17



Mr Ward earns \$100 000, which is not divisible by 3.

So, the guard's nearest neighbour is not Mr Ward.

So, Mr Ward lives in Melbourne and so Dr Pender is the guard's nearest neighbour. Hence, the guard is Ward.

Pender is neither the fireman nor the guard.

Hence, the driver is Pender.

Solutions to Exercise 2B

1a Since both are even, let $a = 2m$ and $b = 2n$.

Then $a + b = 2m + 2n$

$= 2(m + n)$

Therefore $a + b$ is even.

1b Since both are odd, let $a = 2m + 1$ and $b = 2n + 1$.

Then $a + b = 2m + 1 + 2n + 1$

$= 2(m + n + 1)$

Therefore $a + b$ is even.

1c Since a is even and b is odd, let $a = 2m$ and $b = 2n + 1$.

Then $a + b = 2m + 2n + 1$

$= 2(m + n) + 1$

Therefore $a + b$ is odd.

2a Since both are even, let $a = 2m$ and $b = 2n$.

Then $ab = 2m \times 2n$

$= 2(2mn)$

Therefore ab is even.

2b Since both are odd, let $a = 2m + 1$ and $b = 2n + 1$.

Then $ab = 4mn + 2m + 2n + 1$

$= 2(2mn + m + n) + 1$

Therefore ab is odd.

2c Since a is even and b is odd, let $a = 2m$ and $b = 2n + 1$.

Then $ab = 2m(2n + 1)$

$= 2(2mn + m)$

Therefore ab is even.

3a If a is even, let $a = 2m$

Therefore $a^2 = 4m^2$

$= 2(2m^2)$

Therefore a^2 is even.

3b If a is odd, let $a = 2m + 1$

Therefore $a^2 = 4m^2 + 4m + 1$

$= 2(2m^2 + 2m) + 1$

Therefore a^2 is odd.

4a If b and $b + c$ are both divisible by a then let $b = ka$ and $b + c = la$.

$(b + c) - b = la - ka$

$c = (l - k)a$

Therefore c is divisible by a.

4a If b and $b - c$ are both divisible by a then let $b = ka$ and $b - c = la$.

$b - (b - c) = ka - la$

$c = (k - l)a$

Therefore c is divisible by a.

5 If b and c are both divisible by a then let $b = ka$ and $c = la$.

$bx + cy = kax + lay$

$= a(kx + ly)$

Therefore $bx + cy$ is divisible by a.

6 Note that $b = a + 1, c = a + 2, d = a + 3$.

6a Substituting these identities:

$a + d = a + a + 3$

$= 2a + 3$

$b + c = a + 1 + a + 2$

$= 2a + 3$

Therefore $a + d = b + c$.

6b Substituting again:

$ad = a(a + 3)$

$= a^2 + 3a$

$bc - 2 = (a + 1)(a + 2) - 2$

$= a^2 + 3a + 2 - 2$

$= a^2 + 3a$

Therefore $ad = bc - 2$.

6c Substituting again:

$a^2 + d^2 = a^2 + (a + 3)^2$

$= a^2 + a^2 + 6a + 9$

$= 2a^2 + 6a + 9$

$b^2 + c^2 + 4 = (a + 1)^2 + (a + 2)^2 + 4$

$= a^2 + 2a + 1 + a^2 + 4a + 4 + 4$

$= 2a^2 + 6a + 9$

Therefore $a^2 + d^2 = b^2 + c^2 + 4$.

7 Since $a - b$ is even, it can be expressed as $2m$ for some integer m.

$a^2 - b^2$

$= (a - b)(a + b)$

$= (a - b)\big((a - b) + 2b\big)$

$= 2m(2m + 2b)$

$= 4m(m + b)$

which must be a multiple of 4.

8 Let:

$2a + b = cn$ (1)

$3a + 2b = dn$ (2)

for integers c, d.

$4a + 2b = 2cn$ (3) (multiplying (1) by 2)

Subtracting (2) from (3):

$a = cn - dn$

$a = (c - d)n$ (4)

$2a = 2(c - d)n$ (5)

Subtracting (5) from (1):

$2a + b - 2a = cn - 2(c - d)n$

$b = (2d - c)n$

Hence a and b are both multiples of n.

9 Let:

$a^2 + a = 4c$ (1)

$a^2 - a = 4d$ (2)

Subtracting (2) from (1):

$2a = 4c - 4d$

$a = 2(c - d)$

Therefore a is even.

10 $a^3 - a$

$= a(a^2 - 1)$

$= (a-1)a(a+1)$

Since $(a-1)$ and a are consecutive, one of them must be a multiple of 2.

Since $(a-1)$, a, and $(a+1)$ are consecutive integers, one of them must be a multiple of 3.

Therefore $a^3 - a$ is a multiple of 2 and of 3, making it a multiple of 6.

11 If a is even, then there is an integer n such that $a = 2n$.

$a^3 + 2a^2$

$= (2n)^3 + 2(2n)^2$

$= 8n^3 + 8n^2$

$= 8(n^3 + n^2)$

So $a^3 + 2a^2$ must be a multiple of 8.

12 Part 1:

Suppose a number a is divisible by 6.

This means there is an integer b such that $a = 6b$.

We can rewrite this as $a = 2(3b)$ showing that a is divisible by 2.

We can also rewrite this as $a = 3(2b)$ showing that a is divisible by 3.

Therefore if a is divisible by 6, it is divisible by both 2 and 3.

Part 2:

Now suppose a number a is divisible by both 2 and 3.

This means there is an integer m such that $a = 2m$, and also that $2m$ is divisible by 3.

Since 3 is a prime number, either 3 divides 2 or 3 divides m.

Since 3 doesn't divide 2, it must divide m.

Therefore there is an integer n such that $m = 3n$.

Therefore $a = 2(3n)$

$a = 6n$

Therefore a is divisible by 6.

We have now proved that if a number is divisible by 6, it is divisible by 2 and 3, and we have also proved that if a number is divisible by 2 and 3, it is divisible by 6.

Therefore a number is divisible by 6 if and only if it is divisible by both 2 and 3.

13 Part 1:

Suppose an integer a is the sum of 7 consecutive numbers. That is:

$a = n + (n + 1) + (n + 2) + \cdots + (n + 6)$

for some integer n.

Therefore,

$a = 7n + 1 + 2 + \cdots + 6$

$= 7n + 21$

$= 7(n + 3)$

Therefore a is divisible by 7.

Part 2:

Suppose an integer a is divisible by 7. That is:

$a = 7m$ for some integer m.

Therefore,

$a = 7m + (-3) + (-2) + (-1) + 0 + (1) + (2) + (3)$

$= (m - 3) + (m - 2) + (m - 1) + m + (m + 1) + (m + 2) + (m + 3)$

So a is the sum of 7 consecutive numbers, beginning at $\frac{a}{7} - 3$.

We have now proved that if an integer is the sum of 7 consecutive integers, then it is a multiple of 7, and we have also proved that it is a multiple of 7 then it is the sum of 7 consecutive integers.

Therefore an integer is the sum of 7 consecutive integers if and only if it is divisible by 7.

14a If n is odd, there exists an integer m such that $n = 2m + 1$.

Consider the sum of n consecutive numbers: $a + (a + 1) + (a + 2) + \cdots + (a + n - 1)$ where a is the first number in the sequence.

This sum is equal to

$$an + 1 + 2 + \cdots + (n - 1)$$

$$= an + \frac{(n-1)n}{2}$$

$$= n\left(a + \frac{(n-1)}{2}\right)$$

$$= n\left(a + \frac{(2m+1-1)}{2}\right)$$

$$= n(a + m)$$

Therefore when n is odd, the sum is divisible by n.

14b By the same proof as above, the sum of n consecutive integers is equal to

$$n\left(a + \frac{(n-1)}{2}\right)$$

$$= n\left(a + \frac{n}{2} - \frac{1}{2}\right)$$

In this case, since n is even, $\frac{n}{2}$ is an integer, so $\left(a + \frac{n}{2} - \frac{1}{2}\right)$ cannot be an integer. There is a remainder of $\frac{n}{2}$.

Therefore the sum cannot be divisible by n when n is even.

15 Let the digits of a 4-digit number n be a, b, c, d.

That is,

$$n = 1000a + 100b + 10c + d$$

$$= (999a + 99b + 9c) + (a + b + c + d)$$

$$= 3(333a + 33b + 3c) + (a + b + c + d)$$

Part 1:

If n is divisible by 3, then $n = 3m$ for some integer m.

Therefore:

$3m = 3(333a + 33b + 3c) + (a + b + c + d)$

$(a + b + c + d) = 3(m - 333a - 33b - 3c)$

Therefore the sum of the digits $(a + b + c + d)$ is a multiple of 3.

Part 2:

If the sum of the digits is a multiple of 3, then $(a + b + c + d) = 3k$ for some integer k.

Therefore:

$n = 3(333a + 33b + 3c) + 3k$

$\quad = 3(333a + 33b + 3c + k)$

Therefore n is divisible by 3.

We have now proved the result in both directions, so a 4-digit number is divisible by 3 if and only if the sum of its digits is divisible by 3.

16a If n is divisible by 13, then $10x + y = 13m$ for some integer m.

Multiplying this by 4 gives:

$40x + 4y = 13(4m)$

$39x + x + 4y = 13(4m)$

$13(3x) + x + 4y = 13(4m)$

$x + 4y = 13(4m - 3x)$

Therefore $x + 4y$ is divisible by 13.

16b If $x + 4y$ is divisible by 13, then $x + 4y = 13k$ for some integer k.

Multiplying by 10:

$10x + 40y = 13(10k)$

$10x + y + 39y = 13(10k)$

$10x + y + 13(3y) = 13(10k)$

$10x + y = 13(10k - 3y)$

Therefore $10x + y$ is divisible by 13.

16c $\forall x, y \in \mathbf{Z}^+ : [\exists m \in \mathbf{Z} : 10x + y = 13m] \Leftrightarrow [\exists k \in \mathbf{Z} : x + 4y = 13k]$

16d From part a, if $10x + y$ is divisible by 13, then so is $x + 4y$.
So, if $8112 = 10 \times 811 + 2$ is divisible by 13,
then so is $811 + 4 \times 2 = 819$.
If $819 = 10 \times 81 + 9$ is divisible by 13,
then so is $81 + 4 \times 9 = 117$.
If $117 = 10 \times 11 + 7$ is divisible by 13,
then so is $11 + 4 \times 7 = 39$, which *is* divisible by 13.
We can reverse the above argument because we proved in part b
that the converse of part a is true.
So, because 39 is divisible by 13, so is 8112.

17a $(x - 1)(x^{n-1} + x^{n-2} + x^{n-3} + \cdots + x + 1)$

$= x^{n-1}(x - 1) + x^{n-2}(x - 1) + x^{n-3}(x - 1) + \cdots + x(x - 1) + 1(x - 1)$

$= x^n - x^{n-1} + x^{n-1} - x^{n-2} + x^{n-2} - \cdots - x^2 + x^2 - x + x - 1$

$= x^n + 0 + 0 + \cdots + 0 + 0 - 1$

$= x^n - 1$

17b i From part a:

$7^n - 1$

$= (7 - 1)(7^{n-1} + 7^{n-2} + 7^{n-3} + \cdots + 7 + 1)$

$= 6(7^{n-1} + 7^{n-2} + 7^{n-3} + \cdots + 7 + 1)$

Therefore $7^n - 1$ is a multiple of 6.

17b ii Note: we must also assume that a is positive here. Otherwise a counterexample is $a = -2, n = 2$.

From part a:

$a^n - 1 = (a - 1)(a^{n-1} + a^{n-2} + a^{n-3} + \cdots + a + 1)$

$(a^{n-1} + a^{n-2} + a^{n-3} + \cdots + a + 1) \geq a + 1 \geq 2$

If $a^n - 1$ is prime, then its divisors can only be itself and 1.

Since $(a^{n-1} + a^{n-2} + a^{n-3} + \cdots + a + 1) \neq 1$, its other divisor $(a - 1)$ must equal 1.

Therefore $a = 2$.

18a Any factor of n must be of the form $p^i q^j$ where $0 \leq i \leq a$ and $0 \leq j \leq b$.

This gives $a + 1$ possible values for i and $b + 1$ possible values for j.

Multiplying these together, there are $(a + 1)(b + 1)$ possible choices to make a factor of n.

18b $80\,000 = 2^7 \times 5^4$

Therefore there are $(7 + 1)(4 + 1) = 40$ different factors of 80 000.

19 If $a - c$ is a divisor of $ab + cd$, there is an integer n such that $ab + cd = n(a - c)$

$(a - c)(b - d)$

$= ab + cd - ad - bc$

$= (a - c)n - (ad + bc)$

$ad + bc$

$= (a - c)n + (a - c)(d - b)$

$= (a - c)(n + d - b)$

Therefore $a - c$ is a divisor of $ad + bc$.

20a $a^4 + 4b^4$

$= (a^4 + 4a^2b^2 + 4b^4) - 4a^2b^2$

$= (a^2 + 2b^2)^2 - (2ab)^2$

$= (a^2 - 2ab + 2b^2)(a^2 + 2ab + 2b^2)$

20b Put $a = 545$ and $b = 4^{136}$ into part a.

$545^4 + 4 \times (4^{136})^4$

$= (545^2 - 2 \times 545 \times 4^{136} + 2 \times 4^{272})(545^2 + 2 \times 545 \times 4^{136} + 2 \times 4^{272})$

The first factor cannot be 1, since the second factor is obviously less than $545^4 + 4^{545}$.

So $545^4 + 4^{545}$ is not prime.

21a Let n be an even number.

Then $n = 2k$, for some $k \in \mathbf{Z}$.

So $n^2 = 4k^2$, which is divisible by 4 since $k^2 \in \mathbf{Z}$.

21b Let m be an odd number.

Then $m = 2l + 1$, for some $l \in \mathbf{Z}$.

So, m^2

$= (2l + 1)^2$

$= 4l^2 + 4l + 1$

$= 4l(l + 1) + 1 \qquad (*)$

Now, $4l(l + 1)$ is divisible by 8 since either l or $l + 1$ is even.

So, from $*$, it follows that the remainder is 1 when m^2 is divided by 8.

21c Let $a = 2p + 1$ and $b = 2q + 1$, where $p, q \in \mathbf{Z}$.

Then $a^2 + b^2$

$= (4p(p + 1) + 1) + (4q(q + 1) + 1)$

$= 4p(p + 1) + 4q(q + 1) + 2$

So, the remainder is 2 when $a^2 + b^2$ is divided by 4.

Since $a^2 + b^2$ is even, it follows from part a that $a^2 + b^2$ is not a square.

(Even if $a^2 + b^2$ was odd, part b tells us that it could not be square because the remainder is 2 when it is divided by 8.)

22 Let p be a prime number greater than 30.

Then, when p is divided by $30 = 2 \times 3 \times 5$, we have $p = 30q + r$, for some quotient q and remainder r, where $0 \leq r < 30$.

If r is a composite, then it is divisible by 2, 3 or 5, since all the even composite numbers less than 30 are obviously divisible by 2. Also, 9, 15, 21, 27 are divisible by 3 and 25 is divisible by 5.

Hence, if r is composite, then $p = 30q + r$ is composite, since 30 is divisible by 2, 3 and 5, while r is divisible by 2, 3 or 5.

But we know that p is prime, so r cannot be composite. Hence, r is 1 or prime.

23a The factors of 6, excluding itself, are 1, 2, 3 and $1 + 2 + 3 = 6$.

The factors of 28, excluding itself, are 1, 2, 4, 7, 14 and $1 + 2 + 4 + 7 + 14 = 28$.

23b The sum of the factors that are powers of 2 is

$S_1 = 2^0 + 2^1 + \cdots + 2^{n-1}$ (geometric series)

$= 2^n - 1$

The sum of the factors that include $2^n - 1$ is:

$S_2 = (2^n - 1)(2^0 + 2^1 + \cdots + 2^{n-1})$ (*Note:* we must exclude the number itself)

$= (2^n - 1)(2^{n-1} - 1)$

$= 2^{n-1}(2^n - 1) - (2^n - 1)$

Hence, the sum of all the factors, excluding the number itself, is:

$S_1 + S_2 = 2^{n-1}(2^n - 1)$

Hence, $2^{n-1}(2^n - 1)$ is perfect.

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$1 = 2^0 \times 1$	$6 = 2^1 \times 3$
$2 = 2^1 \times 1$	$7 = 2^0 \times 7$
$3 = 2^0 \times 3$	$8 = 2^3 \times 1$
$4 = 2^2 \times 1$	$9 = 2^0 \times 9$
$5 = 2^0 \times 5$	$10 = 2^1 \times 5$

We will now use the pigeonhole principle, where the odd numbers 1, 3, 5, 7, 9 in the products above represent the pigeonholes.

Since we are choosing six numbers, there must be a pigeonhole with two (or more) numbers in it.

These two numbers have the same odd number in their product, so they differ in their power of 2.

It follows that the number with the lower power of 2 is a divisor (factor) of the number with the higher power of 2, and so our proof is complete.

Solutions to Exercise 2C

1 Suppose that $\log_7 13$ is rational, so $\log_7 13 = \frac{m}{n}$ where m, n are integers with no common factors other than 1, and n is positive.

Therefore $7^{\frac{m}{n}} = 13$

$7^m = 13^n$

Since $n \geq 1$ it follows that 13 is a factor of the RHS. But clearly 13 is not a factor of the LHS, so there is a contradiction.

Hence $\log_7 13$ is irrational.

2 Suppose that $\sqrt{5}$ is irrational, so $\sqrt{5} = \frac{m}{n}$ where m, n are integers with no common factors other than 1, and n is positive.

Therefore $5 = \frac{m^2}{n^2}$

$5n^2 = m^2$

Therefore m^2 is a multiple of 5, and since 5 is prime, m must be a multiple of 5.

Let $m = 5k$

Therefore $5n^2 = m^2 = 25k^2$

$n^2 = 5k^2$

Therefore n^2 is also a multiple of 5, and since 5 is prime, n must be a multiple of 5.

But this contradicts the assumption that m, n have no common factors.

Hence $\sqrt{5}$ is irrational.

3a "If a is even then a^2 is even".

3b Let $a = 2n$

Then $a^2 = 4n^2 = 2(2n)$

So a^2 is even.

By proving the contrapositive, we have proved that if a^2 is odd then a is odd.

4 Suppose m is even. Let $m = 2k$.

Therefore $m^2 + 4m + 7 = 4k^2 + 8k + 7$

$= 2(2k^2 + 4k + 3) + 1$

which must be odd.

Therefore, if m is even, $m^2 + 4m + 7$ is odd.

Since we have proved the contrapositive, if $m^2 + 4m + 7$ is even, then m is odd.

5 Suppose that a and b are both odd. Let $a = 2m + 1, b = 2n + 1$.

Therefore $ab = (2m + 1)(2n + 1)$

$= 4mn + 2m + 2n + 1$

$= 2(2mn + m + n) + 1$

which must be odd.

Hence, since we have proved the contrapositive, if ab is even then a and b cannot both be odd, i.e. at least one of them is even.

6a Proof by contradiction:

Suppose $\exists m, n \in \mathbf{Z}^+$ such that $\frac{m}{n} = \log_3 5$.

$\frac{m}{n} = \log_3 5$

$3^{\frac{m}{n}} = 5$

$3^m = 5^n$

But this contradicts the Fundamental Theorem of Arithmetic, since it would imply that 3^m has two different prime factorisations.

Therefore no such m, n exist.

Therefore $\log_3 5$ is irrational.

6b $\log_3 15$

$= \log_3(5 \times 3)$

$= \log_3 5 + \log_3 3$

$= \log_3 5 + 1$

Since $\log_3 5$ is irrational, $\log_3 5 + 1$ must also be irrational.

Therefore $\log_3 15$ is irrational.

7a Proof by contradiction:

Suppose $\exists m, n \in \mathbf{Z}^+$ such that $\frac{m}{n} = \sqrt{11}$ and the HCF of m and n is 1.

$$\frac{m^2}{n^2} = 11$$

$$m^2 = \frac{11}{n^2}$$

Therefore 11 divides m^2.

Since 11 is prime, if m is not divisible by 11 then m^2 would not be divisible by 11.

Therefore 11 must divide m, so we can write $m = 11k$.

Therefore $\dfrac{(11k)^2}{n^2} = 11$

$$\frac{11 \times 11k^2}{n^2} = 11$$

$$\frac{11k^2}{n^2} = 1$$

$$n^2 = 11k^2$$

Thus n^2 is divisible by 11.

Since 11 is prime, if n is not divisible by 11 then n^2 would not be divisible by 11.

Therefore n is divisible by 11.

That is, 11 is a common factor of m and n. But the HCF is 1, so there is a contradiction.

Hence $\sqrt{11}$ is irrational.

7b $\sqrt{44} = \sqrt{4 \times 11}$

$$= 2\sqrt{11}$$

Since $\sqrt{11}$ is irrational, $2\sqrt{11}$ is also irrational.

8 Proof by contradiction:

Suppose p_1 and p_2 are both greater than or equal to $\sqrt{n}$.

Since they cannot be equal to one another, at least one must be greater than $\sqrt{n}$.

Therefore $p_1 p_2 > \sqrt{n} \times \sqrt{n}$

Therefore $n > \sqrt{n} \times \sqrt{n}$

But $\sqrt{n} \times \sqrt{n} = n$ so this leads to a contradiction.

Therefore at least one of p_1 and p_2 is less than $\sqrt{n}$.

9a Proof by contradiction:

If n is odd, n^2 must be odd.

Therefore $n^2 + 2$ must also be odd, and cannot be divisible by 4.

This leads to a contradiction.

Therefore n must be even.

9b Since n is even, we know $n = 2m$ for some integer m.

Therefore $n^2 + 2 = (2m)^2 + 2$

$= 4m^2 + 2$

But this must have remainder 2 when divided by 4, leading to a contradiction.

Hence n cannot be either even or odd, leading to a contradiction.

Therefore no such number exists.

10a Every odd number a can be written in the form $2n + 1$.

If n is odd, then $n = 2m + 1$ for some m.

Therefore:

$a = 2(2m + 1) + 1$

$= 4m + 3$

$= 4m + 4 - 1$

$= 4(m + 1) - 1$

which is 1 less than a multiple of 4.

If n is even, then $n = 2m$ for some m.

Therefore:

$a = 2(2m) + 1$

$\quad = 4m + 1$

which is 1 more than a multiple of 4.

Therefore, a is either 1 more or 1 less than a multiple of 4.

10b Suppose the integers are $4n + 1$ and $4m + 1$.

Then their product is

$(4m + 1)(4n + 1)$

$= 16mn + 4m + 4n + 1$

$= 4(4mn + m + n) + 1$

$= 4k + 1$

where $k = 4mn + m + n$.

10c Proof by contradiction:

Suppose a composite number a of the form $4n - 1$ has no prime factors of the form $4n - 1$.

Since $4n - 1$ is odd, all prime factors must be odd.

From part a, any odd number must be of the form $4n + 1$ or $4n - 1$.

Since a has no prime factors of the form $4n - 1$, all its prime factors must be of the form $4n + 1$.

However, from part b, every time we multiply two numbers of the form $4n + 1$ we get another number of the form $4n + 1$.

Therefore, multiplying out all the prime factors of a results in a number of the form $4n + 1$.

Therefore a is of the form $4n + 1$.

A number cannot be both one greater and one less than a multiple of 4.

Therefore a cannot also be of the form $4n - 1$.

This creates a contradiction, proving that any composite number of the form $4n-1$ must have at least one prime factor of the form $4n-1$.

11 Proof by contradiction:

Suppose $\sqrt{4n-2}$ is rational.

Since $n \geq 1$, $\sqrt{4n-2} > 0$

Therefore $\sqrt{4n-2} = \frac{a}{b}$ for some $a, b \in \mathbf{Z}^+$ with HCF 1

$$4n-2 = \frac{a^2}{b^2}$$

LHS is an integer, therefore $\frac{a^2}{b^2}$ is an integer.

Therefore b^2 divides a^2.

But a and b have no common factors, so b must equal 1.

Therefore:

$$\sqrt{4n-2} = a$$

$$4n-2 = a^2$$

If a is odd, then a^2 must be odd, but $4n-2$ is even so this is impossible.

If a is even, then we can write $a = 2c$ for some $c \in \mathbf{Z}^+$.

Therefore:
$$4n-2 = 4c^2$$

$$c^2 = n - \frac{1}{2}$$

But the LHS must be an integer, and the RHS cannot be an integer, so this is also impossible.

Hence, assuming that $\sqrt{4n-2}$ is rational leads to a contradiction.

Therefore $\sqrt{4n-2}$ is irrational.

12 Proof by contradiction:

Suppose $\sqrt{3}+1$ is rational.

Therefore $\sqrt{3}$ is rational and can be written as $\frac{m}{n}$ for some $m, n \in \mathbf{Z}^+$ with HCF 1

$\sqrt{3} = \dfrac{m}{n}$

$3 = \dfrac{m^2}{n^2}$

$3n^2 = m^2$

Therefore 3 divides m^2 and hence m.

Therefore we can write $m = 3k$ for some $k \in \mathbf{Z}^+$.

Therefore:

$3n^2 = (3k)^2$

$3n^2 = 9k^2$

$n^2 = 3k^2$

Therefore 3 divides n^2 and hence n.

But this contradicts the assumption that m and n have no common factors.

Assuming $\sqrt{3} + 1$ is rational leads to a contradiction, therefore it is irrational.

13a Proof by contradiction:

Suppose $\sqrt{6}$ is rational.

Therefore $\sqrt{6}$ can be written as $\frac{m}{n}$ for some $m, n \in \mathbf{Z}^+$ with HCF 1

$\sqrt{6} = \dfrac{m}{n}$

$6 = \dfrac{m^2}{n^2}$

$6n^2 = m^2$

Therefore 2 and 3 both divide m^2.

Since each is prime, it follows that each divides m.

Therefore 6 divides m.

Therefore we can write $m = 6k$ for some $k \in \mathbf{Z}^+$.

Therefore:

$6n^2 = (6k)^2$

$6n^2 = 36k^2$

$n^2 = 6k^2$

Therefore 2 divides n^2 and hence 2 divides n.

But this contradicts the assumption that m and n have no common factors.

Assuming $\sqrt{6}$ is rational leads to a contradiction, therefore it is irrational.

13b Proof by contradiction:

Suppose $\sqrt{3}+\sqrt{2}$ is rational.

Then $\sqrt{3}+\sqrt{2} = \frac{m}{n}$ for some $m, n \in \mathbf{Z}^+$ with HCF 1

Therefore:

$$\left(\sqrt{3}+\sqrt{2}\right)^2 = \left(\frac{m}{n}\right)^2$$

$$3 + 2\sqrt{6} + 2 = \frac{m^2}{n^2}$$

$$2\sqrt{6} = \frac{m^2}{n^2} - 5$$

$$\sqrt{6} = \frac{m^2 - 5n^2}{2n^2}$$

which is rational, since all terms on the RHS are integers.

But we know from part a that $\sqrt{6}$ is irrational.

The assumption that $\sqrt{3}+\sqrt{2}$ is rational leads to a contradiction.

Therefore $\sqrt{3}+\sqrt{2}$ is irrational.

14 The contrapositive statement is: if n is not prime, then $2^n - 1$ is not prime.

If n is not prime, we can write $n = ab$ for some $a, b \in \mathbf{Z}^+$, with both greater than 1.

Then

$2^n - 1$

$= 2^{ab} - 1$

$= (2^a)^b - 1^b$

$= (2^a - 1)((2^a)^{b-1} + (2^a)^{b-2} + (2^a)^{b-3} + \cdots + (2^a)^1 + (2^a)^0)$

(using difference of powers)

Since $a > 1, a \geq 2$.

Therefore $2^a - 1$ is an integer greater than or equal to $2^2 - 1 = 3$.

Since $b > 1, b \geq 2$.

Therefore:

$((2^a)^{b-1} + (2^a)^{b-2} + (2^a)^{b-3} + \cdots + (2^a)^1 + (2^a)^0)$

$\geq (2^a)^1 + (2^a)^0$

$= 2^a + 1$

$\geq 2^2 + 1$

And $((2^a)^{b-1} + (2^a)^{b-2} + (2^a)^{b-3} + \cdots + (2^a)^1 + (2^a)^0)$ must be an integer since each term of the sum is an integer.

Therefore $2^n - 1$ is the product of two integers, each greater than 1, so it cannot be prime.

By proving the contrapositive, we have proved the original statement.

15 Proof by contradiction:

Suppose there are only finitely many prime numbers.

Therefore there must be a largest prime number, p.

Note that $p!$ is divisible by every number up to and including p, including all the prime numbers (since p is the largest prime number).

Hence, for every prime number q, there exists an integer m such that $\frac{p!}{q} = m$.

Therefore $\dfrac{p! + 1}{q} = m + \dfrac{1}{q}$

This cannot be an integer.

Therefore $p! + 1$ is not divisible by any prime number between 1 and p.

If $p! + 1$ can be factorised, then each of its prime factors must be larger than p.

If it cannot be factorised, then it is itself a prime number, larger than p.

Either way, p cannot be the largest prime number, so we have a contradiction.

Therefore there are infinitely many prime numbers.

16a If p is not prime, then:

$(\exists a, b \in \mathbf{Z}^+, p|ab \Rightarrow p \nmid a$ and $p \nmid b)$.

16b Let $a = p_1$, where p_1 is a prime factor of p, and let $b = \frac{p}{p_1}$.

Then $p|ab$, since:

$ab = p_1.\frac{p}{p_1}$

$\quad = p$

$p \nmid p_1 = a$, since $p_1 < p$ and $p \nmid \frac{p}{p_1} = b$, since $\frac{p}{p_1} < p$.

So, the theorem is proven.

17a

$$\frac{a}{b+\sqrt{c}} + \frac{d}{\sqrt{c}}$$

$$= \frac{a(b-\sqrt{c})}{b^2 - c} + \frac{d\sqrt{c}}{c}$$

$$= \frac{ac(b-\sqrt{c}) + d(b^2 - c)\sqrt{c}}{c(b^2 - c)}$$

$$= \frac{(abc) + (b^2d - ac - cd)\sqrt{c}}{c(b^2 - c)}$$

If this is rational, then:

$b^2d - ac - cd = 0$

$b^2d = c(a + d)$

17b Let $b = 1$ and suppose that $\frac{a}{1+\sqrt{c}} + \frac{d}{\sqrt{c}}$ is rational.

Then, from part a, $d = c(a + d)$ and so $a + d$ is a factor of d.

This is impossible since $a, d \in \mathbf{Z}^+$.

Hence, $\frac{a}{1+\sqrt{c}} + \frac{d}{\sqrt{c}}$ is not rational.

18 Assume that no digit occurs more than twice.

Then, because p^n is a 20-digit number, each of the digits 0, 1, …,9 occurs exactly twice.

So, the sum of the 20 digits is $2(0 + 1 + 2 + \cdots + 9) = 2(45) = 90$, which is divisible by 3.

From the divisibility test for 3, it follows that p^n is divisible by 3.

This is impossible since p is a prime number greater than 3.

Hence p^n is not divisible by 3.

We have a contradiction, so there must be a digit that occurs at least three times.

19 Assume that 2^k is the smallest power of 2 equal to $(36a + b)(36b + a)$.

It follows that $36a + b$ and $36b + a$ are both powers of 2, so a and b are both even, since $36a$ and $36b$ are even.

Let $36a + b = 2^m$, where $m \in \mathbf{Z}^+$ and $36b + a = 2^n$, where $n \in \mathbf{Z}^+$.

Since a, b are even, we can put $a = 2p$ and $b = 2q$, where $p, q \in \mathbf{Z}^+$.

Then $36(2p) + (2q) = 2^m$ and so, $36p + q = 2^{m-1}$.

Similarly, $36q + p = 2^{n-1}$, so

$(36p + q)(36q + p)$

$= 2^{m-1} \times 2^{n-1}$

$= 2^{m+n-2}$

$= 2^{k-2}$

This is a contradiction since 2^{k-2} is a smaller power of 2 than 2^k.

(Note that, since $p, q \in \mathbf{Z}^+$, they are possible values of a and b respectively.)

Solutions to Exercise 2D

1 Let $a = 1 + h, h > 0$.

Therefore $a^2 - 1 = (1 + h)^2 - 1$

$= 2h + h^2$

Since $2h$ and h^2 are both greater than 0, $a^2 - 1 > 0$ and therefore $a^2 > 1$.

Alternative proof:

$a^2 - 1 = (a + 1)(a - 1)$

Since $a > 1$, $a + 1 > 0$ and $a - 1 > 0$

Therefore $a^2 - 1$ is the product of two positive quantities and so must be positive. From there, the result follows as above.

2a LHS − RHS

$= a^2 + b^2 - 2ab$

$= (a - b)^2$

≥ 0

Therefore LHS ≥ RHS.

2b LHS − RHS

$= \frac{a^2}{b^2} + \frac{b^2}{a^2} - 2$

$= \left(\frac{a}{b} - \frac{b}{a}\right)^2$

≥ 0

Therefore LHS ≥ RHS.

2c $\quad$ LHS − RHS

$$= \frac{a^2 + b^2}{2} - \left(\frac{a+b}{2}\right)^2$$

$$= \frac{2a^2 + 2b^2 - a^2 - 2ab - b^2}{4}$$

$$= \frac{a^2 + b^2 - 2ab}{4}$$

$$= \frac{(a-b)^2}{4}$$

$$\geq 0$$

Therefore LHS ≥ RHS.

3 $\quad$ Since $a > 0$, let $a = b^2$

$$a + \frac{1}{a} - 2$$

$$= b^2 - 2 + \frac{1}{b^2}$$

$$= \left(b - \frac{1}{b}\right)^2$$

$$\geq 0$$

Therefore $a + \frac{1}{a} \geq 2$.

4a $\quad$ Since $a, b > 0$, let $a = p^2, b = q^2$ with $p, q > 0$.

$$\frac{1}{2}(a+b) - \sqrt{ab}$$

$$= \frac{p^2 + q^2}{2} - pq$$

$$= \frac{1}{2}(p^2 - 2pq + q^2)$$

$$= \frac{1}{2}(p-q)^2$$

$$\geq 0$$

Therefore $\frac{1}{2}(a+b) \geq \sqrt{ab}$

4b Let $a = p^2, b = q^2$ with $p, q > 0$

$\frac{1}{3}a + \frac{3}{4}b - \sqrt{ab}$

$= \frac{p^2}{3} + \frac{3q^2}{4} - \sqrt{p^2q^2}$

$= \frac{1}{3}p^2 + \frac{3}{4}q^2 - pq$

$= \frac{1}{12}(4p^2 - 12pq + 9q^2)$

$= \frac{1}{12}(2p - 3q)^2$

≥ 0

Therefore $\frac{1}{3}a + \frac{3}{4}b \geq \sqrt{ab}$.

5a Let $a = b + h, h > 0$ and $b > 0$

Then $(a^2 - b) - (b^2 - a)$

$= (b^2 + 2bh + h^2 - b) - (b^2 - b - h)$

$= 2bh + h^2 + h$

Since b and h are positive, and therefore $2bh$ and h^2 are also positive,

$(a^2 - b) > (b^2 - a)$

5b $a^3 - b^3 - (a^2b - ab^2)$

$= a^3 - a^2b + ab^2 - b^3$

$= (a - b)(a^2 + b^2)$

Since $a > b, a - b > 0$

Since $a \neq b$, a and b cannot both equal zero.

Therefore at least one of a^2, b^2 is greater than zero and the other is greater than or equal to zero.

Therefore $a^2 + b^2 > 0$

Therefore $(a-b)(a^2+b^2)>0$.

Therefore $a^3-b^3>(a^2b-ab^2)$.

6a Let $x=a^2, y=b^2$ where $a, b \geq 0$

$x+y-2\sqrt{xy}$

$=a^2+b^2-2ab$

$=(a-b)^2$

≥ 0

Therefore $x+y \geq 2\sqrt{xy}$.

6b From part a, we have $x+y \geq 2\sqrt{xy}$ and we know $2\sqrt{xy} \geq 0$.

Similarly $x+z \geq 2\sqrt{xz} \geq 0$

and $y+z \geq 2\sqrt{yz} \geq 0$.

Therefore

$(x+y)(x+z)(y+z) \geq \left(2\sqrt{xy}\right)\left(2\sqrt{xz}\right)\left(2\sqrt{yz}\right)$

$(x+y)(x+z)(y+z) \geq 8\sqrt{xy \times xz \times yz}$

$(x+y)(x+z)(y+z) \geq 8\sqrt{x^2y^2z^2}$

$(x+y)(x+z)(y+z) \geq 8xyz$

7a $p-q \neq 0$

Therefore:

$(p-q)^2>0$

$p^2-2pq+q^2>0$

$p^2+q^2>2pq$

7b $p^2+q^2>2pq$ (1)

$q^2+r^2>2qr$ (2) by the same proof as above

$r^2+p^2>2rp$ (3) by the same proof as above

Adding (1), (2) and (3) together:

$p^2 + q^2 + q^2 + r^2 + r^2 + p^2 > 2pq + 2qr + 2rp$

$2p^2 + 2q^2 + 2r^2 > 2pq + 2qr + 2rp$

$p^2 + q^2 + r^2 > pq + qr + rp$

7c $(p + q + r)^2 = 1$

$p^2 + q^2 + r^2 + 2pq + 2qr + 2rp = 1$ (1)

From part b, $p^2 + q^2 + r^2 > pq + qr + rp$

Substituting this into (1):

$2pq + 2qr + 2rp + pq + qr + rp < 1$

$3pq + 3qr + 3rp < 1$

$pq + qr + rp < \frac{1}{3}$

8a $(a^2 - b^2)^2 \geq 0$

$a^4 - 2a^2b^2 + b^4 \geq 0$

$a^4 + b^4 \geq 2a^2b^2$ (1)

Similarly:

$b^4 + c^4 \geq 2b^2c^2$ (2)

$c^4 + a^4 \geq 2c^2a^2$ (3)

Adding (1), (2) and (3) together:

$2a^4 + 2b^4 + 2c^4 \geq 2a^2b^2 + 2b^2c^2 + 2c^2a^2$

$a^4 + b^4 + c^4 \geq a^2b^2 + b^2c^2 + c^2a^2$

8b $(ab - bc)^2 \geq 0$

$a^2b^2 - 2ab^2c + b^2c^2 \geq 0$

$a^2b^2 + b^2c^2 \geq 2ab^2c$ (1)

Similarly:

$b^2c^2 + c^2a^2 \geq 2bc^2a$ (2)

$c^2a^2 + a^2b^2 \geq 2ca^2b \qquad (3)$

Adding (1), (2) and (3) together:

$2a^2b^2 + 2b^2c^2 + 2c^2a^2 \geq 2ab^2c + 2bc^2a + 2ca^2b$

$a^2b^2 + b^2c^2 + c^2a^2 \geq a^2bc + b^2ac + c^2ab$

8c $abcd = abc(a + b + c)$

$= a^2bc + b^2ac + c^2ab$

From parts a and b above,

$a^4 + b^4 + c^4 \geq a^2b^2 + b^2c^2 + c^2a^2 \geq a^2bc + b^2ac + c^2ab$

Therefore:

$a^4 + b^4 + c^4 \geq abcd$

9a $(a - b)^2 \geq 0$

$a^2 - 2ab + b^2 \geq 0$

$a^2 + b^2 \geq 2ab$

9b $a^2 + b^2 \geq 2ab \qquad (1)$

Similarly:

$b^2 + c^2 \geq 2bc \qquad (2)$

$c^2 + a^2 \geq 2ca \qquad (3)$

Adding (1), (2) and (3) together:

$2a^2 + 2b^2 + 2c^2 \geq 2ab + 2bc + 2ca$

$a^2 + b^2 + c^2 \geq ab + bc + ca$

9c From 9b, $a^2 + b^2 + c^2 \geq ab + bc + ca$

Therefore:

$a^2 + b^2 + c^2 - ab - bc - ca \geq 0$

$a + b + c > 0$ since all three are positive.

Therefore $(a+b+c)(a^2+b^2+c^2-ab-bc-ca) \geq 0$ as the product of two non-negative terms

Therefore $a^3+b^3+c^3-3abc \geq 0$

$a^3+b^3+c^3 \geq 3abc$

9d Let $a = \sqrt[3]{x}, b = \sqrt[3]{y}, c = \sqrt[3]{z}$

Substituting into the result from part c above:

$x+y+z \geq 3\sqrt[3]{x} \times \sqrt[3]{y} \times \sqrt[3]{z}$

$x+y+z \geq 3\sqrt[3]{xyz}$

$x+y+z \geq 3(xyz)^{\frac{1}{3}}$

10a Using the result from part a above, substitute $a = 1, b = \sqrt{x}$ to get the required result.

10b $1+x \geq 2\sqrt{x}$ (1) (from part a)

Similarly:

$1+y \geq 2\sqrt{y}$ (2)

$1+z \geq 2\sqrt{z}$ (3)

Multiplying (1), (2) and (3), and noting that all terms are positive:

$(1+x)(1+y)(1+z) \geq (2\sqrt{x})(2\sqrt{y})(2\sqrt{z})$

$(1+x)(1+y)(1+z) \geq 8\sqrt{xyz}$

Substituting $(1+x)(1+y)(1+z) = 8$:

$8 \geq 8\sqrt{xyz}$

$1 \geq \sqrt{xyz}$

11a

$$\left(\frac{a}{b}-\frac{b}{a}\right)^4=\left(\frac{a}{b}\right)^4-4\left(\frac{a}{b}\right)^3\left(\frac{b}{a}\right)+6\left(\frac{a}{b}\right)^2\left(\frac{b}{a}\right)^2-4\left(\frac{a}{b}\right)\left(\frac{b}{a}\right)^3+\left(\frac{b}{a}\right)^4$$

$$=\left(\frac{a}{b}\right)^4-4\left(\frac{a}{b}\right)^2+6-4\left(\frac{b}{a}\right)^2+\left(\frac{b}{a}\right)^4$$

$$=\frac{a^4}{b^4}-4\frac{a^2}{b^2}+6-4\frac{b^2}{a^2}+\frac{b^4}{a^4}$$

11b

$$\left(\frac{a}{b}-\frac{b}{a}\right)^4\geq 0$$

Using the equivalency from part a,

$$\frac{a^4}{b^4}-4\frac{a^2}{b^2}+6-4\frac{b^2}{a^2}+\frac{b^4}{a^4}\geq 0$$

$$\frac{a^4}{b^4}+\frac{b^4}{a^4}+6\geq 4\frac{a^2}{b^2}+4\frac{b^2}{a^2}$$

11c Since $x, y > 0$ we can write $x = a^2, y = b^2$.

Substituting into the result from part b then gives:

$$\frac{x^2}{y^2}+\frac{y^2}{x^2}+6\geq 4\left(\frac{x}{y}+\frac{y}{x}\right)$$

12a By the arithmetic mean – geometric mean inequality:

$$\sqrt{a^2b^2}\leq\frac{a^2+b^2}{2}$$

Therefore:

$$ab\leq\frac{a^2+b^2}{2}\qquad(1)$$

Likewise:

$$cd\leq\frac{c^2+d^2}{2}\qquad(2)$$

And applying the AM/GM inequality again:

$\sqrt{ab \times cd} \le \dfrac{ab + cd}{2}$ (3)

Substituting (1) and (2) into (3) gives:

$$\sqrt{ab \times cd} \le \frac{\frac{a^2 + b^2}{2} + \frac{c^2 + d^2}{2}}{2}$$

$$\sqrt{abcd} \le \frac{a^2 + b^2 + c^2 + d^2}{4}$$

12b Write $w = a^2, x = b^2, y = c^2, z = d^2$ and then substitute into the inequality proved in part a:

$$\sqrt{\sqrt{w}\sqrt{x}\sqrt{y}\sqrt{z}} \le \frac{w + x + y + z}{4}$$

$$\sqrt[4]{wxyz} \le \frac{w + x + y + z}{4}$$

13a See proof for question 9a.

13b See proof for question 9b to show that $a^2 + b^2 + c^2 \ge ab + bc + ca$, and then subtract $ab + bc + ca$ from both sides.

13c $(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$

$(a + b + c)^2 \ge ab + bc + ca + 2(ab + bc + ca)$ (substituting result from part b)

$(a + b + c)^2 \ge 3(ab + bc + ca)$

13d i By the triangle inequality,

$b \le a + c$

$b - c \le a$

and similarly:
$c - b \le a$

Therefore
$|b - c| \le |a|$

$(b - c)^2 \le a^2$

13d ii From part i:

$(b-c)^2 \le a^2$

$b^2 + c^2 - a^2 \le 2bc \quad (1)$

Similarly:

$a^2 + b^2 - c^2 \le 2ab \quad (2)$

$c^2 + a^2 - b^2 \le 2ca \quad (3)$

Adding (1), (2) and (3) together:

$a^2 + b^2 + c^2 \le 2ab + 2bc + 2ca \quad (4)$

$(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca \quad (5)$

Substituting (4) into (5):

$(a+b+c)^2 \le 2ab + 2bc + 2ca + 2ab + 2bc + 2ca$

$(a+b+c)^2 \le 4(ab + bc + ca)$

14a From question 9, we know $a^2 + b^2 \ge 2ab$.

Dividing by ab then gives

$\frac{a}{b} + \frac{b}{a} \ge 2$

14b $(a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$

$= 1 + \frac{a}{b} + \frac{a}{c} + \frac{b}{a} + 1 + \frac{b}{c} + \frac{c}{a} + \frac{c}{b} + 1$

$= 3 + \left(\frac{a}{b} + \frac{b}{a}\right) + \left(\frac{b}{c} + \frac{c}{b}\right) + \left(\frac{c}{a} + \frac{a}{c}\right)$

From question 14a, $\frac{a}{b} + \frac{b}{a} \ge 2$.

Similarly, $\frac{b}{c} + \frac{c}{b} \ge 2$ and $\frac{c}{a} + \frac{a}{c} \ge 2$.

Therefore:

$(a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \ge 3 + 2 + 2 + 2$

$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 9$

14c i Subtracting RHS from LHS gives:

$a^3+b^3-\left(\frac{a}{c}+\frac{b}{c}\right)abc$

$= a^3+b^3-(a+b)ab$

$= a^3-a^2b-ab^2+b^3$

$= (a^2-b^2)(a-b)$

$= (a+b)(a-b)^2$

$(a-b)^2 \geq 0$

Since a and b are positive, $(a+b) > 0$.

Therefore LHS − RHS is non-negative and so LHS ≥ RHS.

This proves that $a^3+b^3 \geq \left(\frac{a}{c}+\frac{b}{c}\right)abc$ (1)

Similarly:

$b^3+c^3 \geq \left(\frac{b}{a}+\frac{c}{a}\right)abc$ (2)

$c^3+a^3 \geq \left(\frac{c}{b}+\frac{a}{b}\right)abc$ (3)

14c ii Adding (1), (2) and (3) from above:

$2a^3+2b^3+2c^3 \geq \left(\frac{a}{b}+\frac{b}{a}+\frac{b}{c}+\frac{c}{b}+\frac{c}{a}+\frac{a}{c}\right)abc$

$2a^3+2b^3+2c^3 \geq (2+2+2)abc$ (using the result from part a)

$a^3+b^3+c^3 \geq 3abc$

14c iii Let:

$$x = \sqrt[3]{\frac{a}{b}}$$

$$y = \sqrt[3]{\frac{b}{c}}$$

$$z = \sqrt[3]{\frac{c}{a}}$$

From the result of part c ii:

$$x^3 + y^3 + z^3 \geq 3xyz$$

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq 3\sqrt[3]{\frac{abc}{bca}}$$

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq 3$$

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$$\frac{a}{b} + \frac{b}{a} + \frac{b}{c} + \frac{c}{b} + \frac{c}{a} + \frac{a}{c} \geq 2 + 2 + 2$$

$$\frac{a}{c} + \frac{b}{c} + \frac{b}{a} + \frac{c}{a} + \frac{c}{b} + \frac{a}{b} \geq 6 \quad \text{(rearranging and simplifying)}$$

$$\left(\frac{a+b}{c}\right) + \left(\frac{b+c}{a}\right) + \left(\frac{c+a}{b}\right) \geq 6$$

Multiplying by abc:

$$ab(a+b) + bc(b+c) + ca(c+a) \geq 6abc$$

16a $x^2 + y^2 \geq 2xy$ (as proved in question 9a)

$$x^2 + 2xy + y^2 \geq 4xy$$

$$(x+y)^2 \geq 4xy$$

$$\frac{x+y}{xy} \geq \frac{4}{x+y}$$

$$\frac{1}{x} + \frac{1}{y} \geq \frac{4}{x+y}$$

16b i

$$\frac{x}{y} + \frac{y}{x} \geq 2 \qquad \text{(see proof for question 14a)}$$

Dividing by xy:

$$\frac{1}{y^2} + \frac{1}{x^2} \geq \frac{2}{xy}$$

16b ii

$$\frac{1}{x} + \frac{1}{y} \geq \frac{4}{x+y} \qquad \text{(from part a)}$$

Squaring:

$$\frac{1}{x^2} + \frac{2}{xy} + \frac{1}{y^2} \geq \frac{16}{(x+y)^2}$$

From part b i,

$$\frac{2}{xy} \leq \frac{1}{x^2} + \frac{1}{y^2}$$

Therefore:

$$\frac{1}{x^2} + \frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{y^2} \geq \frac{16}{(x+y)^2}$$

$$\frac{2}{x^2} + \frac{2}{y^2} \geq \frac{16}{(x+y)^2}$$

$$\frac{1}{x^2} + \frac{1}{y^2} \geq \frac{8}{(x+y)^2}$$

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$$\begin{aligned} \text{LHS}^2 - \text{RHS}^2 &= (x-y)^2 - (|x| - |y|)^2 \\ &= x^2 - 2xy + y^2 - |x|^2 + 2|x||y| - |y|^2 \\ &= x^2 - 2xy + y^2 - x^2 + 2|xy| - y^2 \\ &= 2|xy| - 2xy \end{aligned}$$

Since $|xy| \geq xy$,

$\text{LHS}^2 - \text{RHS}^2 \geq 0$

$\text{LHS}^2 \geq \text{RHS}^2$

$(x-y)^2 \geq (|x| - |y|)^2$

Since both sides are non-negative, we can take the square root while preserving the inequality:

$|x - y| \geq \big||x| - |y|\big|$

18a $\left(\text{Re}(z)\right)^2 - |z|^2 = x^2 - (x^2 + y^2)$

$\left(\text{Re}(z)\right)^2 - |z|^2 = -y^2$

$\left(\text{Re}(z)\right)^2 - |z|^2 \leq 0$

Therefore:

$\left(\text{Re}(z)\right)^2 \leq |z|^2$

Since $|z|$ is positive, we can take square roots of both sides while preserving the inequality:

$\text{Re}(z) \leq |z|$

18b Squaring the LHS:

$|z + w|^2$

$= (z + w)\overline{(z + w)}$

$= z\bar{z} + z\bar{w} + \bar{z}w + w\bar{w}$

$= |z|^2 + |w|^2 + z\bar{w} + \bar{z}w$

Let $z = r(\cos\theta + i\sin\theta)$ and let $w = s(\cos\phi + i\sin\phi)$

with $r, s \geq 0$.

Then LHS^2

$$= |z|^2 + |w|^2 + rs(\cos\theta + i\sin\theta)(\cos\phi - i\sin\phi) \\ + rs(\cos\theta - i\sin\theta)(\cos\phi + i\sin\phi)$$

$= |z|^2 + |w|^2 + 2rs(\cos\theta\cos\phi + \sin\theta\sin\phi)$

$= |z|^2 + |w|^2 + 2rs\cos(\theta - \phi)$

$= |z|^2 + |w|^2 + 2|z||w|\cos(\theta - \phi)$

Squaring the RHS:

$(|z| + |w|)^2 = |z|^2 + |w|^2 + 2|z||w|$

Therefore $\text{RHS}^2 - \text{LHS}^2 = 2|z||w|(1 - \cos(\theta - \phi))$

If $z = 0$ or $w = 0$ or $\cos(\theta - \phi) = 1$:

$\text{RHS}^2 - \text{LHS}^2 = 0$

$\text{RHS}^2 = \text{LHS}^2$

RHS = LHS (since both are non-negative)

If none of these three conditions hold:

$2|z||w|(1 - \cos(\theta - \phi)) > 0$

Therefore:
$\text{RHS}^2 > \text{LHS}^2$

RHS > LHS (since both are non-negative)

Therefore $|z + w| \leq |z| + |w|$.

18c In the working for part b, we saw that equality holds only when one of z and w is zero, or when $\cos(\theta - \phi) = 1$. That is, $z = kw$ for some real $k > 0$.

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$$\frac{1+ab}{1+a} + \frac{1+bc}{1+b} + \frac{1+ca}{1+c}$$

$$= \frac{1+\frac{1}{c}}{1+a} + \frac{1+\frac{1}{a}}{1+b} + \frac{1+\frac{1}{b}}{1+c}, \text{since } abc = 1$$

$$= \frac{1+c}{c(1+a)} + \frac{1+a}{a(1+b)} + \frac{1+b}{b(1+c)}$$

$$\geq 3\sqrt[3]{\frac{(1+c)(1+a)(1+b)}{abc(1+a)(1+b)(1+c)}}, \text{using AM/GM with } n = 3$$

$$= 3\sqrt[3]{1}$$

$$= 3$$

20a i RHS – LHS

$$= a^2 - (a + b - c)(a + c - b)$$

$$= a^2 - (a^2 + \cancel{ac} - \cancel{ab} + \cancel{ab} + bc - b^2 - \cancel{ac} - c^2 + bc)$$

$$= a^2 - (a^2 - b^2 - c^2 + 2bc)$$

$= b^2 + c^2 - 2bc$

$= (b - c)^2$

≥ 0, since a square cannot be negative.

So, LHS $\leq$ RHS

Hence, $(a + b - c)(a + c - b) \leq a^2$.

20a ii From part i, we have:

$(a + b - c)(a + c - b) \leq a^2$, and

$(b + c - a)(b + a - c) \leq b^2$, and

$(c + a - b)(c + b - a) \leq c^2$

Multiplying, it follows that

$(a + b - c)^2(b + c - a)^2(a + c - b)^2 \leq a^2b^2c^2$

$(a + b - c)(b + c - a)(a + c - b) \leq abc$, since both sides are positive.

20b i By the cosine rule,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$1 - 2\sin^2\frac{A}{2} = \frac{b^2 + c^2 - a^2}{2bc}, \text{(using the double angle result } \cos 2\theta = 1 - 2\sin^2\theta)$$

$$2\sin^2\frac{A}{2} = 1 - \frac{b^2 + c^2 - a^2}{2bc}$$

$$\sin^2\frac{A}{2} = \frac{2bc - b^2 - c^2 + a^2}{4bc}$$

$$= \frac{a^2 - (b - c)^2}{4bc}$$

$$= \frac{(a + b - c) - (a + c - b)}{4bc}$$

20b ii Using part i, we have

$$\sin^2\frac{A}{2}\sin^2\frac{B}{2}\sin^2\frac{C}{2}$$

$$= \frac{(a+b-c)(a+c-b)}{4bc} \times \frac{(b+a-c)(b+c-a)}{4ac} \times \frac{(c+a-b)-(c+b-a)}{4ab}$$

$$= \frac{(a+b-c)^2(a+c-b)^2(b+c-a)^2}{64a^2b^2c^2}$$

$$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$$= \frac{(a+b-c)(a+c-b)(b+c-a)}{8abc}, \qquad \text{since LHS and RHS } > 0$$

So, $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \le \frac{abc}{8abc} = \frac{1}{8}$, using part a ii

Note: We can use the results in part a because $a, b, c > 0$, being the sides of a triangle and $a+b > c, a+c > b, b+c > a$ by the triangle inequality.

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$$\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b}$$

$$= \left(\frac{a}{b+c}+1\right) + \left(\frac{b}{a+c}+1\right) + \left(\frac{c}{a+b}+1\right) - 3$$

$$= \frac{a+b+c}{b+c} + \frac{a+b+c}{a+c} + \frac{a+b+c}{a+b} - 3$$

$$= (a+b+c)\left(\frac{1}{b+c} + \frac{1}{a+c} + \frac{1}{a+b}\right) - 3$$

$$= \frac{1}{2}\left((b+c)+(a+c)+(a+b)\right)\left(\frac{1}{b+c} + \frac{1}{a+c} + \frac{1}{a+b}\right) - 3$$

$$\ge \frac{1}{2} \times 3 \times \sqrt[3]{(b+c)(a+c)(a+b)} \times 3 \times \sqrt[3]{\frac{1}{b+c} + \frac{1}{a+c} + \frac{1}{a+b}} - 3$$

(by AM/GM with $n = 3$)

$$= \frac{9}{2}\sqrt[3]{1} - 3$$

$$= \frac{3}{2}$$

Solutions to Exercise 2E

1a A. When $n = 1$,

$$\begin{aligned}\text{LHS} &= 1\\ &= \frac{1}{2}n(n+1)\\ &= \text{RHS}\end{aligned}$$

so the result is true for $n = 1$.

B. Assume the statement is true for the positive integer $n = k$.

That is, assume that

$$\sum_{r=1}^{k} r = \frac{1}{2}k(k+1)$$

Now prove the statement for $n = k + 1$. That is, prove that

$$\sum_{r=1}^{k+1} r = \frac{1}{2}(k+1)(k+2)$$

By the induction hypothesis:

$$\sum_{r=1}^{k} r = \frac{1}{2}k(k+1)$$

Therefore

$$\begin{aligned}\sum_{r=1}^{k+1} r &= \left(\sum_{r=1}^{k} r\right) + k + 1\\ &= \frac{1}{2}k(k+1) + k + 1\\ &= \frac{1}{2}(k^2 + k + 2k + 2)\\ &= \frac{1}{2}(k+1)(k+2)\end{aligned}$$

as required.

C. It follows from parts A and B by mathematical induction that the result is true for all integers $n \geq 1$.

1b A. When $n = 1$,

$$\begin{aligned}\text{LHS} &= 1(2)\\ &= 2\\ &= \frac{1}{3}n(n+1)(n+2)\end{aligned}$$

so the result is true for $n = 1$.

B. Assume the statement is true for the positive integer $n = k$.

That is, assume that

$$\sum_{r=1}^{k} r(r+1) = \frac{1}{3}k(k+1)(k+2)$$

Now prove the statement for $n = k + 1$. That is, prove that

$$\sum_{r=1}^{k+1} r(r+1) = \frac{1}{3}(k+1)(k+2)(k+3)$$

By the induction hypothesis:

$$\begin{aligned}\sum_{r=1}^{k+1} r(r+1) &= \left(\sum_{r=1}^{k} r(r+1)\right) + (k+1)(k+2)\\ &= \frac{1}{3}k(k+1)(k+2) + (k+1)(k+2)\\ &= \frac{1}{3}\big(k(k+1)(k+2) + 3(k+1)(k+2)\big)\\ &= \frac{1}{3}(k+1)(k+2)(k+3)\end{aligned}$$

as required.

C. It follows from parts A and B by mathematical induction that the result is true for all integers $n \geq 1$.

1c A. When $n = 1$,

$$\text{LHS} = 1^2$$
$$= 1$$
$$= \frac{1}{6}n(n+1)(2n+1)$$

so the result is true for $n = 1$.

B. Assume the statement is true for the positive integer $n = k$.

That is, assume that

$$\sum_{r=1}^{k} r^2 = \frac{1}{6}k(k+1)(2k+1)$$

Now prove the statement for $n = k + 1$. That is, prove that

$$\sum_{r=1}^{k+1} r^2 = \frac{1}{6}(k+1)(k+2)(2(k+1)+1)$$
$$= \frac{1}{6}(k+1)(k+2)(2k+3)$$

By the induction hypothesis:

$$\sum_{r=1}^{k+1} r^2 = \left(\sum_{r=1}^{k} r^2\right) + (k+1)^2$$
$$= \frac{1}{6}k(k+1)(2k+1) + (k+1)^2$$
$$= \frac{1}{6}(k(k+1)(2k+1) + 6(k+1)^2)$$
$$= \frac{1}{6}(k+1)\big(2k^2 + k + 6(k+1)\big)$$
$$= \frac{1}{6}(k+1)(2k^2 + 7k + 6)$$
$$= \frac{1}{6}(k+1)(k+2)(2k+3)$$

as required.

C. It follows from parts A and B by mathematical induction that the result is true for all integers $n \geq 1$.

1d A. When $n = 1$,

$$\begin{aligned}\text{LHS} &= 1\\ &= \frac{1}{3}n(2n-1)(2n+1)\\ &= \text{RHS}\end{aligned}$$

so the result is true for $n = 1$.

B. Assume the statement is true for the positive integer $n = k$.

That is, assume that

$$\sum_{r=1}^{k}(2r-1)^2 = \frac{1}{3}k(2k-1)(2k+1)$$

Now prove the statement for $n = k + 1$. That is, prove that

$$\begin{aligned}\sum_{r=1}^{k+1}(2r-1)^2 &= \frac{1}{3}(k+1)(2(k+1)-1)(2(k+1)+1)\\ &= \frac{1}{3}(k+1)(2k+1)(2k+3)\\ &= \frac{1}{3}(4k^3+12k^2+11k+3)\end{aligned}$$

By the induction hypothesis:

$$\sum_{r=1}^{k}(2r-1)^2 = \frac{1}{3}k(2k-1)(2k+1)$$

Therefore

$$\begin{aligned}\sum_{r=1}^{k+1}(2r-1)^2 &= \left(\sum_{r=1}^{k}(2r-1)^2\right) + (2(k+1)-1)^2\\ &= \frac{1}{3}k(2k-1)(2k+1) + (2k+1)^2\\ &= \frac{1}{3}\big(k(2k-1)(2k+1)\big) + 4k^2 + 4k + 1\end{aligned}$$

$$= \frac{1}{3}(4k^3 - k + 12k^2 + 12k + 3)$$

$$= \frac{1}{3}(4k^3 + 12k^2 + 11k + 3)$$

as required.

C. It follows from parts A and B by mathematical induction that the result is true for all integers $n \geq 1$.

1e A. When $n = 1$,

$$\text{LHS} = \frac{1}{2}$$

$$= \frac{n}{n+1}$$

$$= \text{RHS}$$

so the result is true for $n = 1$.

B. Assume the statement is true for the positive integer $n = k$.

That is, assume that

$$\sum_{r=1}^{k} \frac{1}{r(r+1)} = \frac{k}{k+1}$$

Now prove the statement for $n = k + 1$. That is, prove that

$$\sum_{r=1}^{k+1} \frac{1}{r(r+1)} = \frac{k+1}{k+2}$$

By the induction hypothesis:

$$\sum_{r=1}^{k+1} \frac{1}{r(r+1)} = \left(\sum_{r=1}^{k} \frac{1}{r(r+1)}\right) + \frac{1}{(k+1)(k+2)}$$

$$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$$

$$= \frac{k(k+2)+1}{(k+1)(k+2)}$$

$$= \frac{k^2 + 2k + 1}{(k+1)(k+2)}$$

$$= \frac{(k+1)^2}{(k+1)(k+2)}$$

$$= \frac{k+1}{k+2}$$

as required.

C. It follows from parts A and B by mathematical induction that the result is true for all integers $n \geq 1$.

1f A. When $n = 1$,

$$\text{LHS} = \frac{1}{(2-1)(2+1)}$$

$$= \frac{1}{3}$$

$$= \frac{n}{2n+1}$$

$$= \text{RHS}$$

so the result is true for $n = 1$.

B. Assume the statement is true for the positive integer $n = k$.

That is, assume that

$$\sum_{r=1}^{k} \frac{1}{(2r-1)(2r+1)} = \frac{k}{2k+1}$$

Now prove the statement for $n = k + 1$. That is, prove that

$$\sum_{r=1}^{k+1} \frac{1}{(2r-1)(2r+1)} = \frac{k+1}{2(k+1)+1}$$

$$= \frac{k+1}{2k+3}$$

By the induction hypothesis:

$$\sum_{r=1}^{k+1} \frac{1}{(2r-1)(2r+1)} = \left(\sum_{r=1}^{k} \frac{1}{(2r-1)(2r+1)}\right) + \frac{1}{(2(k+1)-1)(2(k+1)+1)}$$

$$= \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)}$$

$$= \frac{k(2k+3)+1}{(2k+1)(2k+3)}$$

$$= \frac{2k^2+3k+1}{(2k+1)(2k+3)}$$

$$= \frac{(2k+1)(k+1)}{(2k+1)(2k+3)}$$

$$= \frac{k+1}{2k+3}$$

as required.

C. It follows from parts A and B by mathematical induction that the result is true for all integers $n \geq 1$.

2a A. When $n = 1$, $5^n + 3 = 8 = 2(4)$

so the result is true for $n = 1$.

B. Assume the statement is true for the positive integer $n = k$.

That is, assume that $5^k + 3 = 4a$ for some integer a.

Now prove the statement for $n = k + 1$. That is, prove that

$5^{k+1} + 3$

is a multiple of 4.

By the induction hypothesis:

$5^k + 3 = 4a$

$5^k = 4a - 3$

$5^{k+1} = 5(4a - 3)$

$\quad = 20a - 15$

$5^{k+1} + 3 = 20a - 12$

$= 4(5a - 3)$

which is a multiple of 4, as required.

C. It follows from parts A and B by mathematical induction that the result is true for all integers $n \geq 1$.

2b A. When $n = 1$, $2^{3n} + 6 = 8 + 6 = 14$

so the result is true for $n = 1$.

B. Assume the statement is true for the positive integer $n = k$.

That is, assume that $2^{3k} + 6 = 7a$ for some integer a.

Now prove the statement for $n = k + 1$. That is, prove that

$2^{3(k+1)} + 6$ is a multiple of 7.

By the induction hypothesis:

$2^{3k} + 6 = 7a$

$2^{3k} = 7a - 6$

$2^{3k+3} = 2^{3k} \times 2^3$

$= 2^{3k} \times 8$

$= 8(7a - 6)$

$= 56a - 48$

$2^{3(k+1)} + 6 = 56a - 48 + 6$

$= 56a - 42$

$= 7(8a - 6)$

which is a multiple of 7, as required.

C. It follows from parts A and B by mathematical induction that the result is true for all integers $n \geq 1$.

2c A. When $n = 1, 5^n + 2^{n+1} = 5 + 4 = 9$

which is a multiple of 3, so the result is true for $n = 1$.

B. Assume the statement is true for the positive integer $n = k$.

That is, assume that

$5^k + 2^{k+1} = 3a$ for some integer a

Now prove the statement for $n = k + 1$. That is, prove that

$5^{k+1} + 2^{k+2}$ is a multiple of 3.

By the induction hypothesis:

$5^k + 2^{k+1} = 3a$

$5(5^k + 2^{k+1}) = 15a$

$5^{k+1} + 2 \times 2^{k+1} + 3 \times 2^{k+1} = 15a$

$5^{k+1} + 2^{k+2} = 15a - 3\left(2^{k+1)}\right)$

$= 3(5a - 2^{k+1})$

which is a multiple of 3, as required.

C. It follows from parts A and B by mathematical induction that the result is true for all integers $n \geq 1$.

2d A. When $n = 1$,

$9^{n+2} - 4^n = 9^3 - 4$

$= 729 - 4$

$= 725$

so the result is true for $n = 1$.

B. Assume the statement is true for the positive integer $n = k$.

That is, assume that $9^{k+2} - 4^k = 5a$ for some integer a.

Now prove the statement for $n = k + 1$. That is, prove that

$9^{k+3} - 4^{k+1}$ is a multiple of 5.

By the induction hypothesis:

$9^{k+2} - 4^k = 5a$

Therefore

$9(9^{k+2} - 4^k) = 45a$

$9 \times 9^{k+2} - 9 \times 4^k = 45a$

$9^{k+3} - 4 \times 4^k - 5 \times 4^k = 45a$

$9^{k+3} - 4^{k+1} = 45a + 5 \times 4^k$

$= 5(9a + 4^k)$

which is a multiple of 5, as required.

C. It follows from parts A and B by mathematical induction that the result is true for all integers $n \geq 1$.

2e A. When $n = 1$, $6^n - 5n + 4 = 6 - 5 + 4 = 5$

so the result is true for $n = 1$.

B. Assume the statement is true for the positive integer $n = k$.

That is, assume that $6^k - 5k + 4 = 5a$ for some integer a.

Now prove the statement for $n = k + 1$. That is, prove that

$6^{k+1} - 5(k + 1) + 4$

is a multiple of 5.

By the induction hypothesis:

$6^k - 5k + 4 = 5a$

$6(6^k - 5k + 4) = 30a$

$6 \times 6^k - 30k + 24 = 30a$

$6^{k+1} - 5k - 5 - 25k + 29 = 30a$

$6^{k+1} - 5(k + 1) + 4 - 25k + 25 = 30a$

$6^{k+1} - 5(k + 1) + 4 = 30a + 25k - 25$

$6^{k+1} - 5(k + 1) + 4 = 5(6a + 5k - 5)$

which is a multiple of 5, as required.

C. It follows from parts A and B by mathematical induction that the result is true for all integers $n \geq 1$.

2f A. When $n = 1$,

$4^n + 6n - 1 = 4 + 6 - 1 = 9$

so the result is true for $n = 1$.

B. Assume the statement is true for the positive integer $n = k$.

That is, assume that $4^k + 6k - 1 = 9a$ for some integer a.

Now prove the statement for $n = k + 1$. That is, prove that

$4^{k+1} + 6(k + 1) - 1$ is a multiple of 9.

By the induction hypothesis:

$4^k + 6k - 1 = 9a$

$4(4^k + 6k - 1) = 36a$

$4 \times 4^k + 24k - 4 = 36a$

$4^{k+1} + 6k + 18k + 6 - 10 = 36a$

$4^{k+1} + 6(k + 1) + 18k - 1 - 9 = 36a$

$4^{k+1} + 6(k + 1) - 1 = 36a - 18k + 9$

$= 9(4a - 2k + 1)$

which is a multiple of 9, as required.

C. It follows from parts A and B by mathematical induction that the result is true for all integers $n \geq 1$.

3a A. When $n = 1$, LHS $= 1$

$$\text{RHS} = \frac{1}{6} \times 1 \times (1+1)(1+2)$$

$$= 1$$

So the result is true for $n = 1$.

B. Assume the statement is true for the positive integer $n = k$.

That is, assume that

$$1 + (1+2) + (1+2+3) + \cdots + (1+2+\cdots+k) = \tfrac{1}{6}k(k+1)(k+2)$$

Now prove the statement for $n = k+1$. That is, prove that

$$1 + (1+2) + (1+2+3) + \cdots + \big(1+2+\cdots+(k+1)\big)$$
$$= \frac{1}{6}(k+1)(k+2)(k+3)$$

From the inductive assumption:

$$\text{LHS} = \frac{1}{6}k(k+1)(k+2) + \big(1+2+\cdots+(k+1)\big)$$

$$= \frac{1}{6}k(k+1)(k+2) + \frac{(k+1)(k+2)}{2} \quad \text{(using the result from question 1a)}$$

$$= \frac{1}{6}(k+1)(k+2)(k+3)$$

$$= \text{RHS}$$

C. It follows from parts A and B by mathematical induction that the result is true for all integers $n \geq 1$.

3b A. When $n = 1$, LHS $= 1 =$ RHS

So the result is true for $n = 1$.

B. Assume the statement is true for the positive integer $n = k$.

That is, assume that

$$1^3 + 2^3 + \cdots + k^3 = (1+2+\cdots+k)^2$$

Now prove the statement for $n = k + 1$. That is, prove that

$$1^3 + 2^3 + \cdots + k^3 + (k+1)^3 = \big(1 + 2 + \cdots + k + (k+1)\big)^2$$

From the inductive assumption:

$\text{LHS} = (1 + 2 + \cdots + k)^2 + (k+1)^3$

$\text{LHS} - \text{RHS}$

$= (1 + 2 + \cdots + k)^2 + (k+1)^3 - \big(1 + 2 + \cdots + k + (k+1)\big)^2$

$= \left(\dfrac{k(k+1)}{2}\right)^2 + (k+1)^3 - \left(\dfrac{(k+1)(k+2)}{2}\right)^2$

(using the result from part a)

$= \dfrac{1}{4}\left((k^2+k)^2 + 4k^3 + 12k^2 + 12k + 4 - (k^2+3k+2)^2\right)$

$= \dfrac{1}{4}\left(k^4 + 2k^3 + k^2 + 4k^3 + 12k^2 + 12k + 4 - k^4 - 6k^3 - 13k^2 - 12k - 4\right)$

$= 0$

Therefore $\text{LHS} = \text{RHS}$.

C. It follows from parts A and B by mathematical induction that the result is true for all integers $n \geq 1$.

4a A. When $n = 0, n^2 + 2n = 0$

So the result is true for $n = 0$.

B. Assume the statement is true for the positive even integer $n = 2k$.

That is, assume that

$(2k)^2 + 2(2k) = 8l$ for some integer l.

Now prove the statement for $n = 2(k+1)$. That is, prove that

$(2k+2)^2 + 2(2k+2)$ is a multiple of 8.

$(2k+2)^2 + 2(2k+2)$

$= (2k)^2 + 2 \times 2 \times 2k + 2^2 + 2(2k) + 4$

$= (2k)^2 + 2(2k) + 8k + 8$

$= 8l + 8k + 8$ (from the inductive assumption)

$= 8(l + k + 1)$

Therefore $(2k + 2)^2 + 2(2k + 2)$ is a multiple of 8.

C. It follows from parts A and B by mathematical induction that the result is true for all even integers $n \geq 0$.

4b A. When $n = 0$, $n^3 + 2n = 0$

So the result is true for $n = 0$.

B. Assume the statement is true for the even integer $n = 2k$.

That is, assume that

$(2k)^3 + 2(2k) = 12l$ for some integer l.

Now prove the statement for $n = 2(k + 1)$. That is, prove that

$(2k + 2)^3 + 2(2k + 2)$ is a multiple of 12.

$(2k + 2)^3 + 2(2k + 2)$

$= (2k)^3 + 3 \times 2 \times (2k)^2 + 3 \times 2^2 \times 2k + 2^3 + 2(2k) + 4$

$= (2k)^3 + 2(2k) + 24k^2 + 24k + 12$

$= 12l + 24k^2 + 24k + 12$ (from the inductive assumption)

$= 12(l + 2k^2 + 2k + 1)$

Therefore $(2k + 2)^3 + 2(2k + 2)$ is a multiple of 12.

C. It follows from parts A and B by mathematical induction that the result is true for all even integers $n \geq 0$.

5a A. When $n = 1, 7^n + 2^n = 9$

So the result is true for $n = 1$.

B. Assume the statement is true for the odd integer $n = k$.

That is, assume that

$7^k + 2^k = 9l$ for some integer l.

Now prove the statement for $n = k + 2$. That is, prove that

$7^{k+2} + 2^{k+2}$ is a multiple of 9.

$7^{k+2} + 2^{k+2}$

$= 49 \times 7^k + 4 \times 2^k$

$= (45 + 4) \times 7^k + 4 \times 2^k$

$= 45 \times 7^k + 4(7^k + 2^k)$

$= 9(5 \times 7^k) + 4(9l)$ (from the inductive assumption)

$= 9(5 \times 7^k + 4l)$

Therefore $7^{k+2} + 2^{k+2}$ is a multiple of 9.

C. It follows from parts A and B by mathematical induction that the result is true for all odd integers $n \geq 1$.

5b A. When $n = 1, 7^n + 13^n + 19^n = 39$ which is a multiple of 13.

So the result is true for $n = 1$.

B. Assume the statement is true for the odd integer $n = k$.

That is, assume that

$7^k + 13^k + 19^k = 13l$ for some integer l.

Now prove the statement for $n = k + 2$. That is, prove that

$7^{k+2} + 13^{k+2} + 19^{k+2}$ is a multiple of 13.

$7^{k+2} + 13^{k+2} + 19^{k+2}$

$= 49 \times 7^k + 169 \times 13^k + 361 \times 19^k$

$= 49 \times 7^k + (49 + 120) \times 13^k + (49 + 312) \times 19^k$

$= 49 \times (7^k + 13^k + 19^k) + 120 \times 13^k + 312 \times 19^k$

$= 49 \times 13l + 120 \times 13^k + 13 \times 24 \times 19^k$ (from the inductive assumption)

$= 13(49l + 120 \times 13^{k-1} + 24 \times 19^k)$

Therefore $7^{k+2} + 13^{k+2} + 19^{k+2}$ is a multiple of 13, noting that $k - 1 \geq 2$ so 13^{k-1} is an integer.

C. It follows from parts A and B by mathematical induction that the result is true for all odd integers $n \geq 1$.

6a A. When $n = 1$, LHS $= 1$ and RHS $= 1$.

So the result is true for $n = 1$.

B. Assume the statement is true for integer $n = k \geq 1$.

That is, assume that

$k^2 \geq 3k - 2$

Therefore

$k^2 - 3k + 2 \geq 0$

Now prove the statement for $n = k + 1$. That is, prove that

$(k + 1)^2 \geq 3(k + 1) - 2$

LHS − RHS

$= k^2 + 2k + 1 - 3k - 3 + 2$

$= (k^2 - 3k + 2) + 2k - 2$

From the inductive assumption, $k^2 - 3k + 2 \geq 0$.

Since $k \geq 1$, $2k - 2 \geq 0$.

Therefore LHS − RHS ≥ 0

So $(k + 1)^2 \geq 3(k + 1) - 2$

C. It follows from parts A and B by mathematical induction that the result is true for all integers $n \geq 1$.

6b A. When $n = 4$, LHS $= 16$ and RHS $= 13$.

So the result is true for $n = 4$.

B. Assume the statement is true for integer $n = k \geq 4$.

That is, assume that

$2^k \geq 1 + 3k$

Therefore

$2^k - 1 - 3k \geq 0$

Now prove the statement for $n = k + 1$. That is, prove that

$2^{k+1} \geq 1 + 3(k + 1)$

LHS $-$ RHS

$= 2^{k+1} - 1 - 3(k + 1)$

$= 2^k + 2^k - 1 - 3k - 3$

$= (2^k - 1 - 3k) + 2^k - 3$

From the inductive assumption, $(2^k - 1 - 3k) \geq 0$.

Since $k \geq 4$, $2^k - 3 \geq 0$.

Therefore LHS $-$ RHS ≥ 0

So
$2^{k+1} \geq 1 + 3(k + 1)$

C. It follows from parts A and B by mathematical induction that the result is true for all integers $n \geq 4$.

7a A. When $n = 2$, $(1 + c)^n = 1 + 2c + c^2$

Since c is non-zero, $c^2 > 0$

Therefore:

$(1 + c)^n > 1 + 2c$

$(1 + c)^n > 1 + cn$

So the result is true for $n = 2$.

B. Assume the statement is true for integer $n = k \geq 2$.

That is, assume that

$(1+c)^k > 1 + ck$

Now prove the statement for $n = k + 1$. That is, prove that

$(1+c)^{k+1} > 1 + c(k+1)$

$(1+c)^{k+1}$

$= (1+c)(1+c)^k$

$> (1+c)(1+ck)$ (from the inductive assumption)

$= 1 + c(k+1) + kc^2$

$> 1 + c(k+1)$

Therefore:
$(1+c)^{k+1} > 1 + c(k+1)$

C. It follows from parts A and B by mathematical induction that the result is true for all integers $n \geq 2$.

7b Let $c = -\frac{1}{2n}$.

Since $n \geq 2$, $c \geq -\frac{1}{4}$

Therefore we can use the result from part a:

$(1+c)^n > 1 + cn$

$$\left(1 - \frac{1}{2n}\right)^n > 1 - \frac{n}{2n}$$

$$\left(1 - \frac{1}{2n}\right)^n > 1 - \frac{1}{2}$$

$$\left(1 - \frac{1}{2n}\right)^n > \frac{1}{2}$$

8a $x^2 > 2x + 1$

$x^2 - 2x - 1 > 0$

$x^2 - 2x + 1 > 2$

$(x - 1)^2 > 2$

$x - 1 > \sqrt{2}$ or $x - 1 < -\sqrt{2}$

$x > 1 + \sqrt{2}$ or $x < 1 - \sqrt{2}$

8b A. When $n = 5, 2^n = 32$ and $n^2 = 25$.

So the result is true for $n = 5$.

B. Assume the statement is true for integer $n = k \geq 5$.

That is, assume that

$2^k > k^2$

$2^k - k^2 > 0$

Now prove the statement for $n = k + 1$. That is, prove that

$2^{k+1} > (k + 1)^2$

Note that $k \geq 5 > 1 + \sqrt{2}$.

Therefore $k^2 > 2k + 1$ from part a.

$2^{k+1} - (k + 1)^2$

$= 2(2^k) - k^2 - 2k - 1$

$> 2(2^k) - k^2 - k^2$

$= 2(2^k - k^2)$

> 0 from the inductive assumption

Therefore
$2^{k+1} > (k + 1)^2$.

C. It follows from parts A and B by mathematical induction that the result is true for all integers $n \geq 5$.

9a A. When $n = 1, T_1 = 1 = \frac{1}{2}n(n+1)$.

So the result is true for $n = 1$.

B. Assume the statement is true for integer $n = k \geq 1$.

That is, assume that

$$T_k = \frac{1}{2}k(k+1)$$

Now prove the statement for $n = k + 1$. That is, prove that

$$T_{k+1} = \frac{1}{2}(k+1)(k+2)$$

From the inductive assumption, $T_k = \frac{1}{2}k(k+1)$.

T_{k+1}

$= T_k + k + 1$

$= \dfrac{1}{2}k(k+1) + k + 1$

$= \dfrac{1}{2}(k^2 + k + 2k + 2)$

$= \dfrac{1}{2}(k^2 + 3k + 2)$

$= \dfrac{1}{2}(k+1)(k+21)$

C. It follows from parts A and B by mathematical induction that the result is true for all integers $n \geq 1$.

9b A. When $= 1, T_1 = 1 = 2^1 - 1$.

So the result is true for $n = 1$.

B. Assume the statement is true for integer $n = k \geq 1$.

That is, assume that

$$T_k = 2^k - 1$$

Now prove the statement for $n = k + 1$. That is, prove that

$T_{k+1} = 2^{k+1} - 1$.

By definition,

T_{k+1}

$= 2T_k + 1$

$= 2(2^k - 1) + 1$ (from the inductive assumption)

$= 2(2^k) - 2 + 1$

$= 2^{k+1} - 1$

C. It follows from parts A and B by mathematical induction that the result is true for all integers $n \geq 1$.

9c A. When $n = 1$, $T_1 = 5 = 6 \times 2^{1-1} - 1$.

So the result is true for $n = 1$.

B. Assume the statement is true for integer $n = k \geq 1$.

That is, assume that

$T_k = 6 \times 2^{k-1} - 1$

Now prove the statement for $n = k + 1$. That is, prove that

$T_{k+1} = 6 \times 2^k - 1$

By definition,

T_{k+1}

$= 2T_k + 1$

$= 2(6 \times 2^{k-1} - 1) + 1$ (from the inductive assumption)

$= 6 \times 2 \times 2^{k-1} - 2 + 1$

$= 6 \times 2^k - 1$

C. It follows from parts A and B by mathematical induction that the result is true for all integers $n \geq 1$.

9d A. When $n = 1, T_1 = 1 = \frac{n}{2n-1}$.

So the result is true for $n = 1$.

B. Assume the statement is true for integer $n = k \geq 1$.

That is, assume that

$$T_k = \frac{k}{2k-1}$$

Now prove the statement for $n = k + 1$. That is, prove that

$$T_{k+1}$$

$$= \frac{k+1}{2(k+1)-1}$$

$$= \frac{k+1}{2k+1}$$

By definition,

$$T_{k+1}$$

$$= \frac{3T_k - 1}{4T_k - 1}$$

$$= \frac{3\frac{k}{2k-1} - 1}{4\frac{k}{2k-1} - 1}$$

$$= \frac{3k - (2k-1)}{4k - (2k-1)}$$

$$= \frac{k+1}{2k+1}$$

C. It follows from parts A and B by mathematical induction that the result is true for all integers $n \geq 1$.

10a

$$\frac{d}{dx}(x) = \lim_{h\to 0}\frac{(x+h)-x}{h}$$

$$= \lim_{h\to 0}\frac{h}{h}$$

$$= 1$$

10b A. When $n = 1$,

$$\frac{d}{dx}(x) = 1 = nx^{n-1}$$

So the result is true for $n = 1$.

B. Assume the statement is true for integer $n = k \geq 1$.

That is, assume that

$$\frac{d}{dx}(x^k) = kx^{k-1}$$

Now prove the statement for $n = k + 1$. That is, prove that

$$\frac{d}{dx}(x^{k+1}) = (k+1)x^k$$

We start with the identity

$$x^{k+1} = x \times x^k$$

Therefore, from the product rule,

$$\frac{d}{dx}(x^{k+1})$$

$$= x\frac{d}{dx}(x^k) + x^k\frac{d}{dx}(x)$$

$= xkx^{k-1} + x^k$ (from the inductive assumption and the result in part a)

$$= kx^k + x^k$$

$$= (k+1)x^k$$

C. It follows from parts A and B by mathematical induction that the result is true for all integers $n \geq 1$.

11 A. When $n = 3$, the polygon is a triangle with interior angles adding to $180°$.

$180° = (n - 2) \times 180°$

So the result is true for $n = 3$.

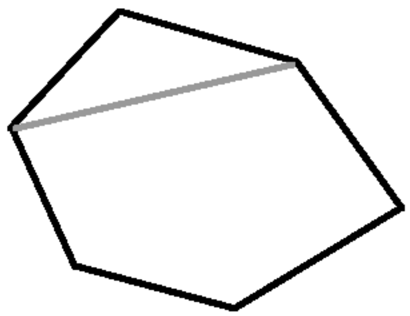
B. Assume the statement is true for integer $n = k \geq 3$.

That is, assume that the interior angle sum of a polygon with k sides is

$(k - 2) \times 180°$

Now prove the statement for $n = k + 1$. That is, prove that the interior angle sum of a polygon with $k + 1$ sides is $(k - 2 + 1) \times 180°$.

To show this, choose any two vertices of the polygon that are separated by just one other vertex, and draw a line between them:



This divides the $(k + 1)$-sided polygon into a triangle and a k-sided polygon.

The sum of internal angles of the $(k + 1)$-sided polygon equals the sum of internal angles of the triangle, and the sum of internal angles of the k-sided polygon.

From the inductive assumption, the internal angles of the k-sided polygon total $(k - 2) \times 180°$. Adding this to the $180°$ from the triangle gives

$(k - 2 + 1) \times 180°$ as required.

C. It follows from parts A and B by mathematical induction that the result is true for all integers $n \geq 3$. Since all polygons have at least three sides, this completes the proof.

12 A. When $n = 3$, we have a triangle with no diagonals.

$\frac{1}{2}3(3-3) = 0.$

So the result is true for $n = 3$.

B. Assume the statement is true for integer $n = k \geq 3$.

That is, assume that a k-sided polygon has $\frac{1}{2}k(k-3)$ diagonals.

Now prove the statement for $n = k+1$. That is, prove that a $(k+1)$-sided polygon has $\frac{1}{2}(k+1)(k-2)$ diagonals.

Consider a $(k+1)$-sided polygon P, with vertices labelled V_1 to V_{k+1} in clockwise order around the polygon.

Let P' be the k-sided polygon with vertices V_1 to V_k in clockwise order around the polygon, but not V_{k+1}.

By the inductive hypothesis, P' has $\frac{1}{2}k(k-3)$ diagonals. Note that each of these diagonals of P' is also a diagonal of P.

The other diagonals of P are:

One diagonal between V_k and V_1 (this wasn't counted in the diagonals of P' because it is an edge of P').

Diagonals between V_{k+1} and $V_2, V_3, \ldots, V_{k-1}$ (but not V_1 or V_k since these are edges of P). There are $k-2$ such diagonals.

Adding these together, P has a total of $\frac{1}{2}k(k-3) + 1 + k - 2$ diagonals.

$$\frac{1}{2}k(k-3) + 1 + k - 2$$

$$= \frac{1}{2}(k^2 - 3k + 2k - 2)$$

$$= \frac{1}{2}(k^2 - k - 2)$$

$$= \frac{1}{2}(k+1)(k+2)$$

as required.

C. It follows from parts A and B by mathematical induction that the result is true for all integers $n \geq 3$, and hence for all non-trivial polygons.

13 A. When $n = 0$, the plane forms a single region.

$$\frac{1}{2}(n^2 + n + 2) = 1$$

So the result is true for $n = 0$.

B. Assume the statement is true for integer $n = k \geq 0$.

That is, assume that k lines in the plane, with no two being parallel and no three concurrent, divide the plane into $\frac{1}{2}(k^2 + k + 2)$ regions.

Now prove the statement for $n = k + 1$. That is, prove that

$k + 1$ lines in the plane, with no two being parallel and no three concurrent, divide the plane into $\frac{1}{2}((k+1)^2 + (k+1) + 2) = \frac{1}{2}(k^2 + 3k + 4)$ regions.

From the inductive assumption, the first k lines divide the plane into $\frac{1}{2}(k^2 + k + 2)$ regions.

The $(k + 1)$th line crosses each of the other k lines, each at a separate point.

Therefore it passes through $k + 1$ of the regions formed by the first k lines, dividing each in two.

Therefore the total number of regions formed by $k + 1$ lines is

$\frac{1}{2}(k^2 + k + 2) + k + 1$

$$= \frac{1}{2}(k^2 + k + 2 + 2k + 2)$$
$$= \frac{1}{2}(k^2 + 3k + 4)$$

as required.

C. It follows from parts A and B by mathematical induction that the result is true for all integers $n \geq 0$.

14 A. When $n = 0, 2^n = 1$.

Every set with 0 members has 1 subset (the empty set) so the result is true for $n = 0$.

B. Assume the statement is true for integer $n = k \geq 0$.

That is, assume that every set with k members has 2^k subsets.

Now prove the statement for $n = k + 1$. That is, prove that every set with $k + 1$ members has 2^{k+1} subsets.

We have assumed that a set with k elements has 2^k subsets.

If we add a new element to the set, we get 2^k new subsets by adding the new element to each of the subsets of the original k-element set.

So, the new set has 2^k subsets that contain the new element as well as the original 2^k subsets that don't contain the new element.

So, a $(k + 1)$-element set has $2^k + 2^k = 2 \times 2^k = 2^{k+1}$ subsets.

C. It follows from parts A and B by mathematical induction that the result is true for all integers $n \geq 0$.

15a A. When $n = 1$:

$$\text{LHS} = \frac{1}{1^2} = 1$$

$$\text{RHS} = 2 - \frac{1}{1} = 1$$

So the result is true for $n = 1$.

B. Assume the statement is true for integer $n = k \geq 1$.

That is, assume that

$$\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{k^2} \leq 2 - \frac{1}{k}$$

Now prove the statement for $n = k + 1$. That is, prove that

$$\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{k^2} + \frac{1}{(k+1)^2} \leq 2 - \frac{1}{k+1}$$

From the inductive assumption:

$$\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{k^2} \leq 2 - \frac{1}{k}$$

Therefore:

$$\frac{1}{1^2}+\frac{1}{2^2}+\cdots+\frac{1}{k^2}+\frac{1}{(k+1)^2}$$

$$\leq 2-\frac{1}{k}+\frac{1}{(k+1)^2}$$

$$\leq 2-\frac{1}{k}+\frac{1}{k(k+1)}$$

$$=2+\frac{1-(k+1)}{k(k+1)}$$

$$=2-\frac{k}{k(k+1)}$$

$$=2-\frac{1}{k+1}$$

as required.

C. It follows from parts A and B by mathematical induction that the result is true for all integers $n \geq 1$.

15b A. When $n = 1$:

LHS $= \frac{1}{2} =$ RHS

So the result is true for $n = 1$.

B. Assume the statement is true for integer $n = k \geq 1$.

That is, assume that

$$\frac{1 \times 3 \times \ldots \times (2k-1)}{2 \times 4 \times \ldots \times 2k} \geq \frac{1}{2k}$$

Now prove the statement for $n = k + 1$. That is, prove that

$$\frac{1 \times 3 \times \ldots \times (2k-1) \times (2k+1)}{2 \times 4 \times \ldots \times 2k \times (2k+2)} \geq \frac{1}{2k+2}$$

From the inductive assumption:

$$\frac{1 \times 3 \times \ldots \times (2k-1)}{2 \times 4 \times \ldots \times 2k} \geq \frac{1}{2k}$$

Therefore:

$$\frac{1 \times 3 \times \ldots \times (2k-1)}{2 \times 4 \times \ldots \times 2k} \times \frac{2k+1}{2k+2} \geq \frac{1}{2k} \times \frac{2k+1}{2k+2}$$

$$\frac{1 \times 3 \times \ldots \times (2k-1) \times (2k+1)}{2 \times 4 \times \ldots \times 2k \times (2k+2)} \geq \frac{2k+1}{2k} \times \frac{1}{2k+2}$$

$$\frac{1 \times 3 \times \ldots \times (2k-1) \times (2k+1)}{2 \times 4 \times \ldots \times 2k \times (2k+2)} \geq \left(1 + \frac{1}{2k}\right) \times \frac{1}{2k+2}$$

$$\frac{1 \times 3 \times \ldots \times (2k-1) \times (2k+1)}{2 \times 4 \times \ldots \times 2k \times (2k+2)} \geq \frac{1}{2k+2}$$

as required.

C. It follows from parts A and B by mathematical induction that the result is true for all integers $n \geq 1$.

16 A. When $n = 1, n^3 - n = 0$ which is divisible by 24.

So the result is true for $n = 1$.

B. Assume the statement is true for odd integer $n = 2k - 1, k \geq 1$.

That is, assume that

$(2k-1)^3 - (2k-1) = 24l$ for some integer l.

That is,

$8k^3 - 12k^2 + 4k = 24l$

Now prove the statement for $n = 2k + 1$. That is, prove that

$(2k+1)^3 - (2k+1)$ is a multiple of 24.

$(2k+1)^3 - (2k+1)$

$= 8k^3 + 12k^2 + 4k$

$= (8k^3 - 12k^2 + 4k) + 24k^2$

$= 24l + 24k^2$ from the inductive assumption

$= 24(l + k^2)$

which is a multiple of 24.

C. It follows from parts A and B by mathematical induction that the result is true for all odd integers $n \geq 1$.

17a A. When $n = 1$,

$T_n = T_1 = a = a + (n-1)d$.

So the result is true for $n = 1$.

B. Assume the statement is true for integer $n = k \geq 1$.

That is, assume that

$T_k = a + (k-1)d$

Now prove the statement for $n = k + 1$. That is, prove that

$T_{k+1} = a + (k - 1 + 1)d$

$= a + kd$

From the inductive assumption,
$T_k = a + (k-1)d$

By definition,

T_{k+1}

$= T_k + d$
$= a + (k-1)d + d$ from the inductive assumption

$= a + kd$

as required.

C. It follows from parts A and B by mathematical induction that the result is true for all integers $n \geq 1$.

17b A. When $n = 1$,

$T_n = T_1 = a = ar^{1-1}$.

So the result is true for $n = 1$.

B. Assume the statement is true for integer $n = k \geq 1$.

That is, assume that

$T_k = ar^{k-1}$

Now prove the statement for $n = k + 1$. That is, prove that

$$T_{k+1} = ar^{k-1+1}$$
$$= ar^k$$

From the inductive assumption,
$T_k = ar^{k-1}$.

By definition,

T_{k+1}

$= rT_k$

$= r \times ar^{k-1}$
$= ar^k$

as required.

C. It follows from parts A and B by mathematical induction that the result is true for all integers $n \geq 1$.

17c A. When $n = 1$,

$S_n = S_1 = a$

$\frac{1}{2}n(2a + (n-1)d)$

$= \frac{1}{2}(2a + 0)$

$= a$

So the result is true for $n = 1$.

B. Assume the statement is true for integer $n = k \geq 1$.

That is, assume that

$S_k = \frac{1}{2}k(2a + (k-1)d)$

Now prove the statement for $n = k + 1$. That is, prove that

$$S_{k+1} = \frac{1}{2}(k+1)(2a + (k+1-1)d)$$

$$= \frac{1}{2}(k+1)(2a + kd)$$

From the inductive assumption,

$$S_k = \frac{1}{2}k(2a + (k-1)d)$$

S_{k+1}

$$= S_k + a + (k+1-1)d$$

$$= \frac{1}{2}k(2a + (k-1)d) + a + kd$$

$$= \frac{1}{2}(2ka + k(k-1)d + 2a + 2kd)$$

$$= \frac{1}{2}(2ka + 2a + k^2d + kd)$$

$$= \frac{1}{2}(k+1)(2a + kd)$$

as required.

C. It follows from parts A and B by mathematical induction that the result is true for all integers $n \geq 1$.

17d A. When $n = 1$,

$S_n = S_1 = a$

$$\frac{a(r^n - 1)}{r - 1}$$

$$= \frac{a(r-1)}{r-1}$$

$= a$ (assuming $r \neq 1$)

So the result is true for $n = 1$.

B. Assume the statement is true for integer $n = k \geq 1$.

That is, assume that

$$S_k = \frac{a(r^k - 1)}{r - 1}$$

Now prove the statement for $n = k + 1$. That is, prove that

$$S_{k+1} = \frac{a(r^{k+1} - 1)}{r - 1}$$

From the inductive assumption:

$$S_k = \frac{a(r^k - 1)}{r - 1}$$

By definition:

S_{k+1}

$= S_k + ar^k$

$= \dfrac{a(r^k - 1)}{r - 1} + ar^k$

$= \dfrac{a(r^k - 1) + ar^k(r - 1)}{r - 1}$

$= \dfrac{ar^k - a + ar^{k+1} - ar^k}{r - 1}$

$= \dfrac{ar^{k+1} - a}{r - 1}$

$= \dfrac{a(r^{k+1} - 1)}{r - 1}$

as required.

C. It follows from parts A and B by mathematical induction that the result is true for all integers $n \geq 1$.

18a $a^{n+1} + b^{n+1} - a^n b - b^n a$

$= (a^n - b^n)(a - b)$

$= (a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + ab^{n-2} + b^{n-1})(a - b)^2$

Given that $a, b > 0$,

$a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + ab^{n-2} + b^{n-1} > 0$

and $(a-b)^2 \geq 0$

Therefore:

$(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + ab^{n-2} + b^{n-1})(a-b)^2 \geq 0$

$a^{n+1} + b^{n+1} - a^n b - b^n a \geq 0$

$a^{n+1} + b^{n+1} \geq a^n b + b^n a$

18b A. When $n = 1$,

$$\left(\frac{a+b}{2}\right)^n = \frac{a^n + b^n}{2}$$

So the result is true for $n = 1$.

B. Assume the statement is true for integer $n = k \geq 1$.

That is, assume that

$$\left(\frac{a+b}{2}\right)^k \leq \frac{a^k + b^k}{2}$$

Now prove the statement for $n = k + 1$. That is, prove that

$$\left(\frac{a+b}{2}\right)^{k+1} \leq \frac{a^{k+1} + b^{k+1}}{2}$$

From the inductive assumption:

$$\left(\frac{a+b}{2}\right)^k \leq \frac{a^k + b^k}{2}$$

$$\left(\frac{a+b}{2}\right)^k \times \left(\frac{a+b}{2}\right) \leq \frac{a^k + b^k}{2} \times \left(\frac{a+b}{2}\right) \qquad \left(\text{since } \frac{a+b}{2} > 0\right)$$

$$\left(\frac{a+b}{2}\right)^{k+1} \leq \frac{1}{2} \times \frac{(a^k + b^k)(a+b)}{2}$$

$$\left(\frac{a+b}{2}\right)^{k+1} \leq \frac{1}{2} \times \frac{a^{k+1} + b^{k+1} + a^k b + ab^k}{2}$$

Using the result from part a:

$$\left(\frac{a+b}{2}\right)^{k+1} \le \frac{1}{2} \times \frac{a^{k+1}+b^{k+1}+a^{k+1}+b^{k+1}}{2}$$

$$\left(\frac{a+b}{2}\right)^{k+1} \le \frac{a^{k+1}+b^{k+1}}{2}$$

as required.

C. It follows from parts A and B by mathematical induction that the result is true for all integers $n \ge 1$.

19a $\sqrt{n+1}-\sqrt{n}$

$$= \frac{\left(\sqrt{n+1}-\sqrt{n}\right)\left(\sqrt{n+1}+\sqrt{n}\right)}{\sqrt{n+1}+\sqrt{n}}$$

$$= \frac{n+1-n}{\sqrt{n+1}+\sqrt{n}}$$

$$= \frac{1}{\sqrt{n+1}+\sqrt{n}}$$

$$> \frac{1}{\sqrt{n+1}+\sqrt{n+1}}$$

$$= \frac{1}{2\sqrt{n+1}}$$

Therefore

$$\sqrt{n+1}-\sqrt{n} > \frac{1}{2\sqrt{n+1}}$$

19b A. When $n = 7$,

$$1+\frac{1}{2}+\cdots+\frac{1}{n} = \frac{363}{140}$$

$$\left(\frac{363}{140}\right)^2 = \frac{131769}{19600}$$

which is less than 7

Therefore

$$1+\frac{1}{2}+\cdots+\frac{1}{n}<\sqrt{7}$$

So the result is true for $n=7$.

B. Assume the statement is true for integer $n=k\geq 7$.

That is, assume that

$$1+\frac{1}{2}+\cdots+\frac{1}{k}<\sqrt{k}$$

Now prove the statement for $n=k+1$. That is, prove that

$$1+\frac{1}{2}+\cdots+\frac{1}{k}+\frac{1}{k+1}<\sqrt{k+1}$$

From the inductive assumption,

$$1+\frac{1}{2}+\cdots+\frac{1}{k}+\frac{1}{k+1}<\sqrt{k}+\frac{1}{k+1} \qquad (1)$$

From the result in part a,

$$\sqrt{k+1}-\sqrt{k}>\frac{1}{2\sqrt{k+1}}$$

$$\sqrt{k+1}>\sqrt{k}+\frac{1}{2\sqrt{k+1}} \qquad (2)$$

Since $k\geq 7, \sqrt{k+1}>2$

Therefore

$$k+1>2\sqrt{k+1}$$

$$\frac{1}{k+1}<\frac{1}{2\sqrt{k+1}}$$

Combining with (2) gives:

$$\sqrt{k+1}>\sqrt{k}+\frac{1}{k+1}$$

Combining with (1) then gives

$$\sqrt{k+1}>1+\frac{1}{2}+\cdots+\frac{1}{k}+\frac{1}{k+1}$$

as required.

C. It follows from parts A and B by mathematical induction that the result is true for all integers $n \geq 7$.

20a $\sin(A+B) - \sin(A-B)$

$= \sin A \cos B + \cos A \sin B - \sin A \cos B + \cos A \sin B$

$= 2\cos A \sin B$

20b A. When $n = 1$,

$\text{LHS} = \dfrac{1}{2}$

$\text{RHS} = \dfrac{\sin\frac{\theta}{2}}{2\sin\frac{\theta}{2}} = \dfrac{1}{2}$

So the result is true for $n = 1$.

B. Assume the statement is true for integer $n = k \geq 1$.

That is, assume that

$$\frac{1}{2} + \cos\theta + \cos 2\theta + \cdots + \cos(k-1)\theta = \frac{\sin\left(k - \frac{1}{2}\right)\theta}{2\sin\frac{1}{2}\theta}$$

That is:

$$2\sin\frac{1}{2}\theta\left(\frac{1}{2} + \cos\theta + \cos 2\theta + \cdots + \cos(k-1)\theta\right) = \sin\left(k - \frac{1}{2}\right)\theta$$

Now prove the statement for $n = k + 1$. That is, prove that

$$2\sin\frac{1}{2}\theta\left(\frac{1}{2} + \cos\theta + \cos 2\theta + \cdots + \cos(k-1)\theta + \cos k\theta\right) = \sin\left(k + \frac{1}{2}\right)\theta$$

From the result in part a,

$$2\cos k\theta \sin\frac{1}{2}\theta = \sin\left(k + \frac{1}{2}\right)\theta - \sin\left(k - \frac{1}{2}\right)\theta$$

From the inductive assumption,

$$2\sin\frac{1}{2}\theta\left(\frac{1}{2} + \cos\theta + \cos 2\theta + \cdots + \cos(k-1)\theta\right) = \sin\left(k - \frac{1}{2}\right)\theta$$

Adding these two identities together:

$$2\sin\frac{1}{2}\theta\left(\frac{1}{2}+\cos\theta+\cos 2\theta+\cdots+\cos(k-1)\theta\right)+2\cos k\theta\sin\frac{1}{2}\theta$$
$$=\sin\left(k-\frac{1}{2}\right)\theta+\sin\left(k+\frac{1}{2}\right)\theta-\sin\left(k-\frac{1}{2}\right)\theta$$

$$2\sin\frac{1}{2}\theta\left(\frac{1}{2}+\cos\theta+\cos 2\theta+\cdots+\cos(k-1)\theta+\cos k\theta\right)=\sin\left(k+\frac{1}{2}\right)\theta$$

as required.

C. It follows from parts A and B by mathematical induction that the result is true for all integers $n \geq 1$.

21a $(x-y)^2 \geq 0$

$x^2-2xy+y^2 \geq 0$

$x^2+y^2 \geq 2xy$

$\frac{x}{y}+\frac{y}{x} \geq 2$ (dividing by xy)

21b A. When $n=1$,

$\text{LHS}=\frac{a_1}{a_1}=1$

$\text{RHS}=1^2=1$

So the result is true for $n=1$.

B. Assume the statement is true for integer $n=k \geq 1$.

That is, assume that

$$(a_1+a_2+\cdots+a_k)\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_k}\right) \geq k^2$$

Now prove the statement for $n=k+1$. That is, prove that

$$(a_1+a_2+\cdots+a_k+a_{k+1})\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_k}+\frac{1}{a_{k+1}}\right) \geq (k+1)^2$$

$$(a_1+a_2+\cdots+a_k+a_{k+1})\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_k}+\frac{1}{a_{k+1}}\right)$$

$$= (a_1 + a_2 + \cdots + a_k)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_k}\right) + a_{k+1}\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_k}\right) + (a_1 + a_2 + \cdots + a_k)\frac{1}{a_{k+1}} + \frac{a_{k+1}}{a_{k+1}}$$

$$\geq k^2 + a_{k+1}\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_k}\right) + (a_1 + a_2 + \cdots + a_k)\frac{1}{a_{k+1}} + 1$$

(from the inductive assumption)

$$= k^2 + \left(\frac{a_{k+1}}{a_1} + \frac{a_1}{a_{k+1}}\right) + \left(\frac{a_{k+1}}{a_2} + \frac{a_2}{a_{k+1}}\right) + \cdots + \left(\frac{a_{k+1}}{a_k} + \frac{a_k}{a_{k+1}}\right) + 1$$

$\geq k^2 + 2 + 2 + \cdots + 2 + 1$ (using the result from part a)

$= k^2 + 2k + 1$

$= (k + 1)^2$

Therefore

$$(a_1 + a_2 + \cdots + a_k + a_{k+1})\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_k} + \frac{1}{a_{k+1}}\right) \geq (k + 1)^2$$

as required.

C. It follows from parts A and B by mathematical induction that the result is true for all integers $n \geq 1$.

21c From part b:

$$(a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) \geq n^2$$

Let $n = 3, a_1 = \sin^2\theta, a_2 = \cos^2\theta, a_3 = \tan^2\theta$.

$$(a_1 + a_2 + a_3)\left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3}\right) \geq 3^2$$

$$(\sin^2\theta + \cos^2\theta + \tan^2\theta)\left(\frac{1}{\sin^2\theta} + \frac{1}{\cos^2\theta} + \frac{1}{\tan^2\theta}\right) \geq 3^2$$

$$(1 + \tan^2\theta)(\operatorname{cosec}^2\theta + \sec^2\theta + \cot^2\theta) \geq 9$$

$$\sec^2\theta\,(\operatorname{cosec}^2\theta + \sec^2\theta + \cot^2\theta) \geq 9$$

$$\operatorname{cosec}^2\theta + \sec^2\theta + \cot^2\theta \geq \frac{9}{\sec^2\theta}$$

$\operatorname{cosec}^2\theta + \sec^2\theta + \cot^2\theta \geq 9\cos^2\theta$ as required.

Note: The other parts of question 22 have similar solutions.

22b Confirm the result for $n = 1$ and $n = 2$.

$T_1 = 5^1 + 3^1 = 8$

$T_2 = 5^2 + 3^2 = 34$

So, the result is true for $n = 1$ and $n = 2$.

Assume the result is true for $n = k$ and $n = k + 1$, where $k \geq 3$.

That is, assume that $T_k = 5^k + 3^k$ and $T_{k+1} = 5^{k+1} + 3^{k+1}$. $\quad (*)$

Prove that the result is true for $n = k + 2$.

That is, prove that $T_{k+2} = 5^{k+2} + 3^{k+2}$.

LHS

$= T_{k+2}$

$= 8T_{k+1} - 15T_k$ (from the definition of the sequence)

$= 8(5^{k+1} + 3^{k+1}) - 15(5^k + 3^k)$

$= 8 \times 5 \times 5^k + 8 \times 3 \times 3^k - 15 \times 5^k - 15 \times 3^k$

$= 25 \times 5^k + 9 \times 3^k$

$= 5^{k+2} + 3^{k+2}$

$=$ RHS

So, the result is true for $n = k + 2$ if it is true for $n = k$ and $n = k + 1$.

But the result is true for $n = 1$ and $n = 2$.

So, by induction, the result is true for all positive integer values of n.

23a LHS

$$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{2k-1} - \frac{1}{2k}\right)$$

$$= \frac{1}{1 \times 2} + \frac{1}{3 \times 4} + \cdots + \frac{1}{2k(2k-1)}$$

$$> \frac{1}{(2k+1)(2k+2)} + \frac{1}{(2k+1)(2k+2)} + \cdots + \frac{1}{(2k+1)(2k+2)} \qquad (k \text{ terms})$$

(since $1 \times 2, 3 \times 4, \ldots, 2k(2k-1)$ are all less than $(2k+1)(2k+2)$)

$$= \frac{k}{(2k+1)(2k+2)}$$

$$= \text{RHS}$$

23b When $n = 2$,

$$\text{LHS} = 2\left(1 + \frac{1}{3}\right) = \frac{8}{3}$$

$$\text{RHS} = 3\left(\frac{1}{2} + \frac{1}{4}\right) = \frac{9}{4} < \text{LHS}$$

Hence, the result is true for $n = 2$.

Assume that the result is true for $n = k$, where $k \geq 2$.

That is, assume that

$$k\left(1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2k-1}\right)(k+1)\left(1 + \frac{1}{3} + \cdots \frac{1}{2k-1} + \frac{1}{2k+1}\right) \qquad (*)$$

Prove that the result is true for $n = k + 1$.

That is, prove that

$$(k+1)\left(1 + \frac{1}{3} + \cdots + \frac{1}{2k-1} + \frac{1}{2k+1}\right) > (k+2)\left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2k} + \frac{1}{2k+2}\right)$$

LHS

$$= (k+1)\left(1 + \frac{1}{3} + \cdots + \frac{1}{2k-1} + \frac{1}{2k+1}\right)$$

$$= k\left(1 + \frac{1}{3} + \cdots + \frac{1}{2k-1}\right) + 1\left(1 + \frac{1}{3} + \cdots + \frac{1}{2k-1}\right) + \frac{k+1}{2k+1}$$

$$> (k+1)\left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2k}\right) + \frac{k}{(2k+1)(2k+2)} + \left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2k}\right) + \frac{k+1}{2k+1} \text{ (using part a)}$$

(using $*$)

$$= (k+2)\left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2k}\right) + \frac{k + (k+1)(2k+2)}{(2k+1)(2k+2)}$$

$$= (k+2)\left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2k}\right) + \frac{2k^2 + 5k + 2}{(2k+1)(2k+2)}$$

$$= (k+2)\left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2k}\right) + \frac{\cancel{(2k+1)}(k+2)}{\cancel{(2k+1)}(2k+2)}$$

$$= (k+2)\left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2k} + \frac{1}{2k+2}\right)$$

$$= \text{RHS}$$

So, by induction, the result is true for all $n \geq 2$.

24 $S_2 = 1 \times 2 = 2$

Using the formula for S_n,

$S_2 = \frac{1}{24}(1)(2)(3)(8) = 2$

So, the result is true for $n = 2$.

Assume the result is true for $n = k$, where $k \geq 2$.

That is, assume that $S_k = \frac{1}{24}k(k-1)(k+1)(3k+2)$. $(*)$

Prove that the result is true for $n = k + 1$.

That is, prove that $S_{k+1} = \frac{1}{24}k(k+1)(k+2)(3k+5)$.

S_{k+1}

$= S_k + (k+1)(1 + 2 + \cdots + k)$

$= \frac{1}{24}k(k-1)(k+1)(3k+2) + (k+1) \times \frac{1}{2}k(k+1)$ (by $(*)$)

$= \frac{1}{24}k(k+1)\big((k-1)(3k+2) + 12(k+1)\big)$

$= \frac{1}{24}k(k+1)(3k^2 + 2k - 3k - 2 + 12k + 12)$

$= \frac{1}{24}k(k+1)(3k^2 + 11k + 10)$

$= \frac{1}{24}k(k+1)(k+2)(3k+5)$, as required.

So, by induction, the result is true for all integers $n \geq 2$.

25a There are $(n-1)$ players with whom Ben can swap tops.

Hence, there are $(n-2)$ players for whom there are D_{n-2} derangements.

So, the number of such derangements is $(n-1)D_{n-2}$.

25b Besides Ben, there are $(n-1)$ players for whom there are D_{n-1} derangements.

Then Ben must swap tops with one of the other $(n-1)$ players, who, at this stage, does NOT have his own top.

So, the number of such derangements is $D_{n-1}(n-1)$.

Now, either Ben does a direct swap with another player or he does not.

So, $D_n = (n-1)D_{n-1} + (n-1)D_{n-2}$, for $n > 2$.

25c D_n

$= (n-1)(D_{n-1} + D_{n-2})$

$= nD_{n-1} - D_{n-1} + (n-1)D_{n-2}$

So, $D_n - nD_{n-1} = -(D_{n-1} - (n-1)D_{n-2})$.

25d $D_1 = 0$ and $D_2 = 1$ (If there are 2 players, the only derangement is if they swap tops.)

When $n = 2$,

$D_n - nD_{n-1}$

$= D_2 - 2D_1$

$= 1 - 2(0)$

$= 1$

$= (-1)^2$

Hence, the result is true for $n = 2$.

Assume that the result is true for $n = k$, where $k \geq 2$.

That is, assume that $D_k - kD_{k-1} = (-1)^k$. $(*)$

Prove that the result is true for $n = k + 1$.

That is, prove that $D_{k+1} - (k+1)D_k = (-1)^{k+1}$.

LHS

$= D_{k+1} - (k+1)D_k$

$= -(D_k - kD_{k-1})$ (using part c)

$= (-1) \times (-1)^k$ (using $*$)

$= (-1)^{k+1}$

= RHS

Hence, by induction, the result is true for all integers $n > 1$.

25e When $n = 1$,

D_n

$= D_1$

$= 1! \times \sum_{r=0}^{1} \frac{(-1)^r}{r!}$

$= \frac{(-1)^0}{0!} + \frac{(-1)^1}{1!}$

$= 1 + (-1)$

$= 0$

Hence, the result is true for $n = 1$.

Assume that the result is true for $n = k$.

That is, assume that $D_k = k! \times \sum_{r=0}^{k} \frac{(-1)^r}{r!}$. $(*)$

Prove that the result is true for $n = k + 1$.

That is, prove that $D_{k+1} = (k+1)! \times \sum_{r=0}^{k+1} \frac{(-1)^r}{r!}$.

$D_{k+1} - (k+1)D_k = (-1)^{k+1}$ (from part d)

So, D_{k+1}

$= (k+1)D_k + (-1)^{k+1}$

$= (k+1) \times k! \times \sum_{r=0}^{k} \frac{(-1)^r}{r!} + (-1)^{k+1}$ $\big(\text{using } (*)\big)$

$= (k+1)! \times \sum_{r=0}^{k} \frac{(-1)^r}{r!} + (k+1)! \times \frac{(-1)^{k+1}}{(k+1)!}$

$= (k+1)! \times \left(\sum_{r=0}^{k} \frac{(-1)^r}{r!} + \frac{(-1)^{k+1}}{(k+1)!} \right)$

$= (k+1)! \times \sum_{r=0}^{k} \frac{(-1)^r}{r!}$, as required.

Hence, by induction, the result is true for all $n \in \mathbf{Z}^+$.

Solutions to Exercise 2F

1a Consider triangle AOB, as shown in the diagram.

Angle $AOB = \frac{360°}{12} = 30°$

Therefore

Area of $AOB = \frac{1}{2}|OA||OB| \sin 30°$

$= \dfrac{1}{4} \text{ cm}^2$

Since the inscribed dodecagon is made up of twelve such triangles, its area is

$12 \times \frac{1}{4} \text{cm}^2 = 3 \text{ cm}^2$.

1b i Let $t = \tan 15°$.

From the double-angle formula,

$\tan 30° = \dfrac{2t}{1 - t^2}$

$\dfrac{1}{\sqrt{3}} = \dfrac{2t}{1 - t^2}$

$1 - t^2 = 2t\sqrt{3}$

$t^2 + 2t\sqrt{3} - 1 = 0$

$t = \dfrac{-2\sqrt{3} \pm \sqrt{12 + 4}}{2}$

$= -\sqrt{3} \pm 2$

In the first quadrant, tan must be positive.

Therefore $t = 2 - \sqrt{3}$ so $\tan 15° = 2 - \sqrt{3}$.

1b ii Let M be the point where HG touches the circle.

Angle $MOH = 15°$ and $|OM| = 1$ cm

Therefore $|MH| = \left(2-\sqrt{3}\right)$ cm

Therefore

Area of $\Delta OGH = 2 \times \frac{1}{2} \times 1 \text{ cm} \times \left(2-\sqrt{3}\right) \text{ cm}$

$= \left(2-\sqrt{3}\right) \text{ cm}^2$

Therefore area of the circumscribed dodecagon is $12\left(2-\sqrt{3}\right) \text{ cm}^2$.

1c Since the circle lies entirely within the circumscribed dodecagon, and the inscribed dodecagon lies entirely within the circle, the circle's area must be between the two dodecagons.

Therefore $3 \text{ cm}^2 < \pi \text{ cm}^2 < 12\left(2-\sqrt{3}\right) \text{ cm}^2$

$3 < \pi < 12\left(2-\sqrt{3}\right) \doteqdot 3.22$

2a Let $x_0 = 0, x_1 = \frac{\pi}{6}, x_2 = \frac{\pi}{3}$

By Simpson's rule,

$$\text{Area } \doteqdot \frac{\pi}{6} \times \frac{1}{3}\left(0 + 4 \times \sin\frac{\pi}{6} + \sin\frac{\pi}{3}\right)$$

$$= \frac{\pi}{36}\left(4+\sqrt{3}\right)$$

2b We can find the exact area by integration:

$$\int_0^{\frac{\pi}{3}} \sin x \; dx$$

$$= \left[-\cos x\right]_0^{\frac{\pi}{3}}$$

$$= -\frac{1}{2} - (-1)$$

$$= \frac{1}{2}$$

Therefore

$$\frac{\pi}{36}\left(4+\sqrt{3}\right) \doteqdot \frac{1}{2}$$

$$\pi \doteqdot \frac{18}{4+\sqrt{3}}$$

$$= \frac{18\left(4-\sqrt{3}\right)}{\left(4+\sqrt{3}\right)\left(4-\sqrt{3}\right)}$$

$$= \frac{18\left(4-\sqrt{3}\right)}{16-3}$$

$$= \frac{18}{13}\left(4-\sqrt{3}\right) \text{ as required.}$$

Note that:

$$\frac{18}{13}\left(4-\sqrt{3}\right) = 3.140\,237\ldots \doteqdot 3.14$$

3 Area $ABDC$

$$= \frac{|AC|+|BD|}{2}|CD|$$

$$= \frac{\left(1+\frac{1}{2}\right)}{2}\times 1$$

$$= \frac{\left(\frac{3}{2}\right)}{2}$$

$$= \frac{3}{4} \text{ square units}$$

Let $Q = \left(\frac{3}{2}, 0\right)$ be the foot of the perpendicular drawn from P to the x-axis.

Construct the horizontal line that passes through P. Let E be the point where this line crosses MC and let F be the point where this line crosses the continuation of ND.

$$|EP| = |PF| = \frac{1}{2}$$

$$\angle MPE = \angle NPF$$

$$\angle MEP = \angle NFP = 90°$$

Therefore $\Delta MEP \equiv \Delta NFP$ (angle-side-angle)

Therefore $|ME| = |NF|$

Therefore $|MC| + |ND| = |EC| + |FD| = 2|PQ|$

Area $MNDC$

$= \dfrac{|MC| + |ND|}{2}|CD|$

$= \dfrac{2|PQ|}{2}|CD|$

$= |PQ||CD|$

$= \dfrac{2}{3} \times 1$

$= \dfrac{2}{3}$ square units

3b The exact area under the curve between $x = 1$ and $x = 2$ can be found by integration:

$\displaystyle\int_1^2 \frac{1}{x}\,dx$

$= [\ln x]_1^2$

$= \ln 2 - \ln 1$

$= \ln 2$

Since this area is completely covered by trapezium $ABDC$, and completely covers $MNDC$, it follows that:

area $MNDC < \ln 2 <$ area $ABDC$

That is, $\dfrac{2}{3} < \ln 2 < \dfrac{3}{4}$

4a The exact area is:

$$\int_{-1}^{0} e^x dx$$

$= [e^x]_{-1}^{0}$

$= (1 - e^{-1})$ square units

4b Area of $OABC$

$= \dfrac{|AO| + |BC|}{2}|OC|$

$= \dfrac{1 + e^{-1}}{2} \times 1$

$= \frac{1}{2}(1 + e^{-1})$ square units

4c Construct the horizontal line that passes through P. Let E be the point where this line crosses MO and let F be the point where this line crosses the continuation of NC.

$|EP| = |PF| = \dfrac{1}{2}$

$\angle MPE = \angle NPF$

$\angle MEP = \angle NFP = 90°$

Therefore $\Delta MEP \equiv \Delta NFP$ (angle-side-angle)

Therefore $|ME| = |NF|$

Therefore $|MO| + |NC| = |EO| + |FC| = 2|PQ|$

Area $OMNC$

$= \dfrac{|MO| + |NC|}{2}|CO|$

$= \dfrac{2|PQ|}{2}|CO|$

$= |PQ||CO|$

$= e^{-\frac{1}{2}} \times 1$

$= e^{-\frac{1}{2}}$ square units

4d The area under the curve is completely covered by $OABC$, and completely covers $MNCO$.

Therefore

area $MNCO < 1 - e^{-1} <$ area $OABC$

$e^{-\frac{1}{2}} < 1 - e^{-1} < \frac{1}{2}(1 + e^{-1})$

Looking at the right-hand inequality:

$1 - e^{-1} < \frac{1}{2}(1 + e^{-1})$

$2e - 2 < e + 1$ (multiplying both sides by $2e$)

$e < 3$

Looking at the left-hand inequality:

$e^{-\frac{1}{2}} < 1 - e^{-1}$

Let $k = e^{-\frac{1}{2}}$

Therefore $k < 1 - k^2$

$k^2 + k - 1 < 0$

$k^2 + k + \frac{1}{4} < \frac{5}{4}$

$\left(k + \frac{1}{2}\right)^2 < \left(\frac{\sqrt{5}}{2}\right)^2$

$k + \frac{1}{2} < \frac{\sqrt{5}}{2}$ (using the fact that both sides are positive)

$e^{-\frac{1}{2}} + \frac{1}{2} < \frac{\sqrt{5}}{2}$

$e^{-\frac{1}{2}} < \frac{\sqrt{5} - 1}{2}$

$e^{-1} < \left(\frac{\sqrt{5} - 1}{2}\right)^2$

$$e^{-1} < \frac{6-2\sqrt{5}}{4}$$

$$e^{-1} < \frac{3-\sqrt{5}}{2}$$

$$e > \frac{2}{3-\sqrt{5}} \quad \text{(taking reciprocal of both sides, noting that both are positive)}$$

$$e > \frac{2(3+\sqrt{5})}{(3+\sqrt{5})(3-\sqrt{5})}$$

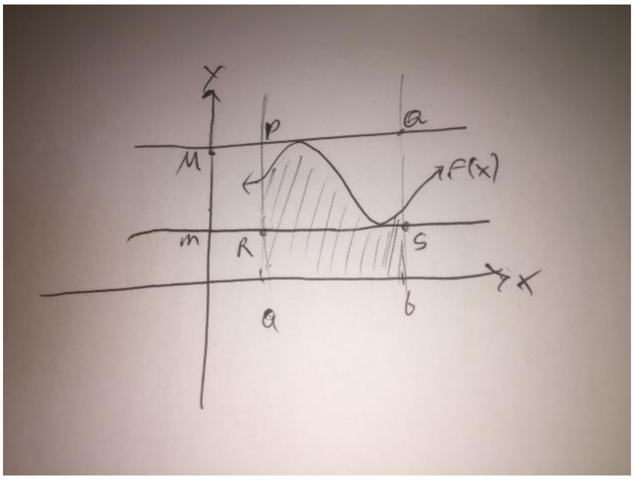
$$e > \frac{2(3+\sqrt{5})}{9-5}$$

$$e > \frac{3+\sqrt{5}}{2}$$

Therefore:

$$\frac{1}{2}(3+\sqrt{5}) < e < 3$$

5 Construct points $P = (a, M), Q = (b, M), R = (a, m), S = (b, m)$:



Since $m \le f(x) \le M$ in the interval $a \le x \le b$, it follows that

$$\int_a^b m\,dx \le \int_a^b f(x)\,dx \le \int_a^b M\,dx$$

$$\int_a^b m\,dx = [mx]_a^b = m(b-a)$$

$$\int_a^b M\,dx = [Mx]_a^b = M(b-a)$$

Therefore

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$$

6a Consider the area under the curve, between $x = 1$ and $x = n + 1$.

$$\int_1^n \frac{1}{x}\,dx = [\log x]_1^{n+1}$$

$$= \log(n+1) - \log 1$$

$$= \log(n+1)$$

The area under the curve must be less than or equal to the sum of upper rectangles over the same range.

Since the height of the upper rectangle beginning at $x = n$ is $\frac{1}{n}$, the sum of upper rectangles between $x = 1$ and $x = n + 1$ is

$$1 + \frac{1}{2} + \cdots + \frac{1}{n}$$

Therefore:

$$1 + \frac{1}{2} + \cdots + \frac{1}{n} \ge \log(n+1)$$

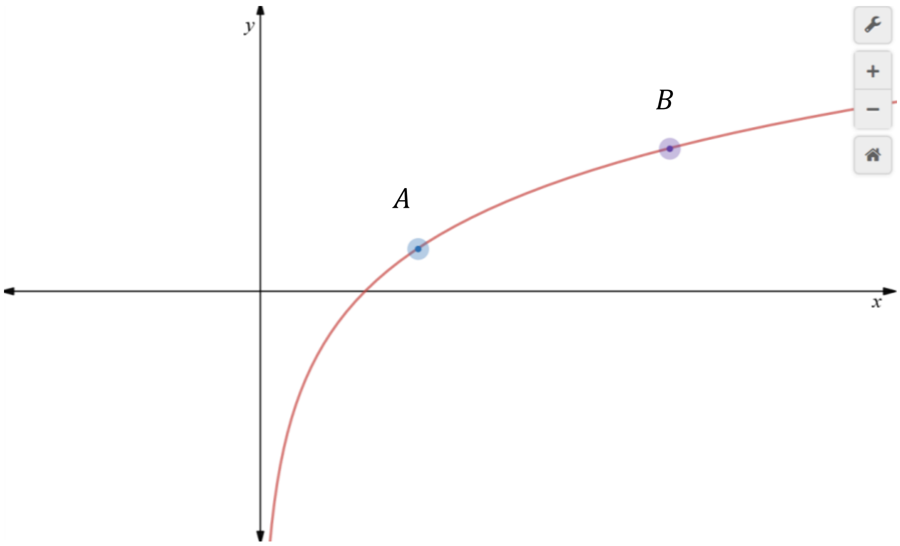
6b Since there is no upper limit to $\log(n+1)$, the infinite series becomes infinitely large.

7a $\frac{d}{dx}\ln x = \frac{1}{x}$

$$\frac{d^2}{dx^2}\ln x = -\frac{1}{x^2}$$

This is negative for all positive x and so the graph is concave down.

7b



7c $P = \left(a + \frac{2}{3}(b - a), \ln a + \frac{2}{3}(\ln b - \ln a)\right)$

$$= \left(\frac{a + 2b}{3}, \frac{\ln a + 2\ln b}{3}\right)$$

7d Since the graph of $\ln x$ is concave down, the value of the function at $x = \frac{a+2b}{3}$ must be greater than the y-value at P.

That is:

$$\ln\left(\frac{a + 2b}{3}\right) > \frac{\ln a + 2\ln b}{3}$$

8a

$$\frac{df}{dx} = x^n \frac{d}{dx}(e^{-x}) + e^{-x}\frac{d}{dx}(x^n)$$

$$= -x^n e^{-x} + nx^{n-1}e^{-x}$$

$$= x^{n-1}e^{-x}(n - x)$$

8b When $x = n, f(x) = n^n e^{-n}$

For $x > 0, x^{n-1}e^{-x} < 0.$

When $0 < x < n, n - x \geq 0$

Therefore $\frac{df}{dx} > 0$ here,

When $x > n, n - x < 0$

Therefore $\frac{df}{dx} < 0$ here.

So the gradient is zero or increasing for all points to the left of $x = n$, and decreasing for all points to the right, with zero gradient at $x = n$.

Hence $(n, n^n e^{-n})$ is a maximum turning point.

Alternately:

When $x = n, n - x = 0$ so $\frac{df}{dx} = 0.$

$$\frac{df}{dx} = x^{n-1}e^{-x}(n - x)$$

$$= nx^{n-1}e^{-x} - f(x)$$

$$\frac{d^2f}{dx^2} = nx^{n-2}e^{-x}(n - 1 - x) - x^{n-1}e^{-x}(n - x)$$

$$= x^{n-2}e^{-x}(n(n - 1 - x) - xn + x^2)$$

$$= x^{n-2}e^{-x}(x^2 - 2xn + n^2 - n)$$

At $x = n$,

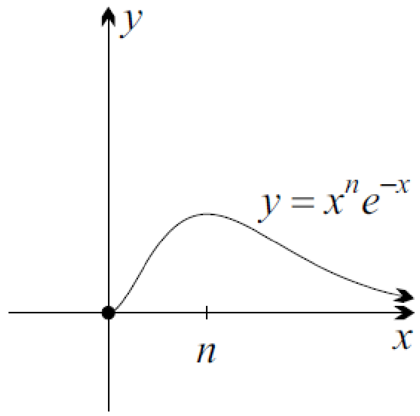
$$\frac{d^2f}{dx^2} = n^{n-2}e^{-n}(n^2 - 2n^2 + n^2 - n)$$

$$= -n^{n-1}e^{-n}$$

$$< 0$$

since $e^{-n} > 0$ and $-n^{n-1} < 0$.

So at this point the gradient is zero and the second derivative is negative, making it a maximum turning point.



8c For $x > n, f'(x) < 0$

Therefore $f(x) < f(n)$

$f(x) < n^n e^{-n}$

8d Let $x = n + 1$

From part c, it then follows that

$(n+1)^n e^{-(n+1)} < n^n e^{-n}$

$\dfrac{(n+1)^n}{n^n} < e^{-n} e^{n+1}$

$\left(\dfrac{n+1}{n}\right)^n < e^1$

$\left(1 + \dfrac{1}{n}\right)^n < e$

9a $f'(x) = 1 - \frac{2x}{1+x^2}$

Note that $(1 - x)^2 \geq 0$

with equality only when $x = 1$

$1 - 2x + x^2 \geq 0$

$1 + x^2 \geq 2x$

Therefore

$$\frac{2x}{1+x^2} \le 1$$

$$1 - \frac{2x}{1+x^2} \ge 0$$

So $f'(x) \ge 0$

with equality only when $x = 1$

9b $f(0) = 0 - \log_e(1 + 0^2)$

$= 0 - 0$

$= 0$

From part a, the gradient is positive for x between 0 and 1, hence the function is increasing in this domain and so $f(x) > 0$ for $0 < x < 1$.

At $x = 1, f(x) = 1 - \log_e 2$

$f(x) > 0$

For $x > 1$ the gradient is positive again, so the function is increasing in this range and so $f(x) \ge f(1) > 0$

So $f(x) > 0$ for all $x > 0$.

9c For positive x,

$x - \log_e(1 + x^2) > 0$

$x > \log_e(1 + x^2)$

$e^x > e^{\log_e(1+x^2)}$

$e^x > 1 + x^2$

10a $y = e^x\left(1-\frac{x}{10}\right)^{10}$

$$\frac{dy}{dx} = \left(1-\frac{x}{10}\right)^{10}\frac{d}{dx}(e^x) + e^x\frac{d}{dx}\left(1-\frac{x}{10}\right)^{10}$$

$$= e^x\left(1-\frac{x}{10}\right)^{10} + e^x \times 10 \times -\frac{1}{10} \times \left(1-\frac{x}{10}\right)^{9}$$

$$= e^x\left(1-\frac{x}{10}\right)^{9}\left(1-\frac{x}{10}-1\right)$$

$$= -\frac{x}{10}e^x\left(1-\frac{x}{10}\right)^{9}$$

Therefore $\frac{dy}{dx} = 0$ if $x = 0$ or $x = 10$.

Turning points are $(0,1)$ and $(10,0)$.

10b As $x \to \infty$, $e^x \to \infty$.

$$1-\frac{x}{10} \to -\infty$$

$$\left(1-\frac{x}{10}\right)^{10} \to \infty$$

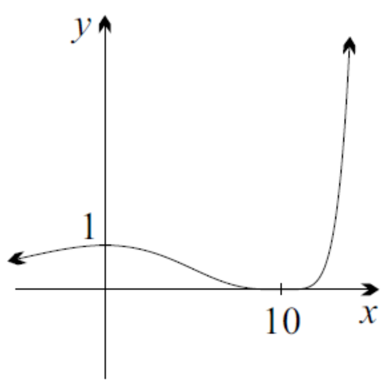
So $e^x\left(1-\frac{x}{10}\right)^{10} \to \infty$

As $x \to -\infty$, $e^x x^n$ tends to zero for any n and so

$$e^x\left(1-\frac{x}{10}\right)^{10} \to 0$$

since $\left(1-\frac{x}{10}\right)^{10}$ expands to a sum of powers of n.

10c



10d From the graph, $y \le 1$ for $x < 10$

Therefore, for $x < 10$,

$$e^x\left(1-\frac{x}{10}\right)^{10} \le 1$$

$$e^x \le \left(1-\frac{x}{10}\right)^{-10}$$

(noting that $1-\frac{x}{10} > 0$ for $x < 10$)

10e Letting $x = 1$,

$$e^1 \le \left(1-\frac{1}{10}\right)^{-10}$$

$$e \le \left(\frac{9}{10}\right)^{-10}$$

$$e \le \left(\frac{10}{9}\right)^{10}$$

Letting $x = -1$,

$$e^{-1} \le \left(1+\frac{1}{10}\right)^{-10}$$

$$e^{-1} \le \left(\frac{11}{10}\right)^{-10}$$

$$\left(\frac{11}{10}\right)^{10} \le e$$

11a By the cosine rule,

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$\cos A \ge -1$$

Therefore

$$a^2 \le b^2 + c^2 + 2bc$$

$$a^2 \le (b+c)^2$$

$$a \le b + c$$

Also:

$\cos A \leq 1$

Therefore

$a^2 \geq b^2 + c^2 - 2bc$

$a^2 \geq (b - c)^2$

$a \geq \sqrt{(b - c)^2}$

$a \geq |b - c|$

Note that $a = |b - c|$ if and only if $b = 0$ or $c = 0$ or $\cos A = 1$. This result will be used in part c.

11b Consider the triangle in the complex plane formed by z, w, and the origin O.

Let a be the distance between z and w, i.e. $|z - w|$.

Let b be the length of the line from O to z, i.e. $|z|$.

Let c be the length of the line from O to w, i.e. $|w|$.

From the result in part a,

$|b - c| \leq a \leq b + c$

$\big||z| - |w|\big| \leq |z - w| \leq |z| + |w|$

By replacing w in the above inequality with $-w$, we get:

$\big||z| - |-w|\big| \leq |z - (-w)| \leq |z| + |-w|$

$\big||z| - |w|\big| \leq |z + w| \leq |z| + |w|$

Combining these two results:

$\big||z| - |w|\big| \leq |z \pm w| \leq |z| + |w|$

11c Let $a = |z - (-w)|$.

Let $b = |z|$.

Let $c = |-w|$.

Let A be the angle between the line from the origin to z, and the line from the origin to w, in the complex plane.

From the result in part a:

$a = |b - c|$ if and only if $b = 0$ or $c = 0$ or $\cos A = 1$

$|z - (-w)| = \big||z| - |-w|\big|$ if and only if $|z| = 0$ or $|w| = 0$ or $\cos A = 1$

$|z + w| = \big||z| - |w|\big|$ if and only if $|z| = 0$ or $|w| = 0$ or $\cos A = 1$

That is, $|z| = 0$ or $|w| = 0$, or z and $-w$ have the same argument, or $z = kw$ for some negative k.

12a i $6^6 = 46656,\ \ 3 \times 5^6 = 46\,875$

12a ii $5 \times 6^6 = 233\,280,\ \ 2 \times 7^6 = 235\,298$

12b i $6^6 < \ 3 \times 5^6$

$3 > \left(\frac{6}{5}\right)^6$

$3^{\frac{1}{6}} > \frac{6}{5}$

$3^{-\frac{1}{6}} < \frac{5}{6}$

$1 - 3^{-\frac{1}{6}} > \frac{1}{6} \qquad (1)$

Gradient of AB equals

$$\frac{1 - 3^{-\frac{1}{6}}}{0 + \frac{1}{6}}$$

$= 6\left(1 - 3^{-\frac{1}{6}}\right)$

$> 6\left(\frac{1}{6}\right) \qquad$ substituting from (1)

So the gradient of AB is greater than 1.

$5 \times 6^6 < 2 \times 7^6$

$\frac{5}{2} < \left(\frac{7}{6}\right)^6$

$$\left(\frac{5}{2}\right)^{\frac{1}{6}} < \frac{7}{6}$$

Gradient of BC equals

$$\frac{\left(\frac{5}{2}\right)^{\frac{1}{6}} - 1}{\frac{1}{6}}$$

$$= 6\left(\left(\frac{5}{2}\right)^{\frac{1}{6}} - 1\right)$$

$$< 6\left(\frac{7}{6} - 1\right)$$

$$= 6\left(\frac{1}{6}\right)$$

$$= 1$$

So gradient of BC is less than 1.

12b ii Gradient of AB is greater than 1. Since the curve is concave up, gradient of $y = 3^x$ at $x = 0$ (i.e. B) is greater than the gradient of AB, so must be greater than 1.

Gradient of BC is less than 1. Since the curve is concave up, gradient of $y = \left(\frac{5}{2}\right)^x$ at $x = 0$ (i.e. B) is less than the gradient of BC, so must be less than 1.

Consider the function $f(x) = a^x$.

$$f(x) = e^{x \ln a}$$

$$\frac{df}{dx} = \ln a \times e^{x \ln a}$$

At $x = 0$,

$$\frac{df}{dx} = \ln a$$

Gradient of $y = 3^x$ at $x = 0$ is greater than 1. Therefore:

$$\ln 3 > 1$$

$$3 > e$$

Gradient of $y = \left(\frac{5}{2}\right)^x$ at $x = 0$ is less than 1. Therefore:

$\ln\frac{5}{2} < 1$

$\frac{5}{2} < e$

13a Let $u = t^2$, noting that $0 \le u < 1$

$1 + t^2 + t^4 + \cdots + t^{2N}$

$= 1 + u + u^2 + \cdots + u^N$

$= \frac{1 - u^{N+1}}{1 - u}$

$= \frac{1 - u^{N+1}}{1 - u}$

$< \frac{1}{1 - u}$ (since numerator and denominator are both positive)

$= \frac{1}{1 - t^2}$

Therefore

$1 + t^2 + t^4 + \cdots + t^{2N} < \frac{1}{1 - t^2}$

13b As shown above,

$1 + t^2 + t^4 + \cdots + t^{2N}$

$= \frac{1 - u^{N+1}}{1 - u}$

$= \frac{1 - t^{2N+2}}{1 - t^2}$

Therefore

$\frac{1}{1 - t^2} - (1 + t^2 + t^4 + \cdots + t^{2N})$

$= \frac{1}{1 - t^2} - \frac{1 - t^{2N+2}}{1 - t^2}$

$= \frac{t^{2N+2}}{1 - t^2}$

13c

$$\frac{1}{1-t^2}$$

$$=\frac{1}{(1+t)(1-t)}$$

$$=\frac{1}{2(1-t)}+\frac{1}{2(1+t)}$$

$$\int_0^x (1+t^2+t^4+\cdots+t^{2N})dt < \int_0^x \frac{1}{1-t^2}dt$$

$$\int_0^x (1+t^2+t^4+\cdots+t^{2N})dt < \frac{1}{2}\int_0^x \left(\frac{1}{(1-t)}+\frac{1}{(1+t)}\right)dt$$

$$\left[t+\frac{t^3}{3}+\frac{t^5}{5}+\cdots+\frac{t^{2N+1}}{2N+1}\right]_0^x < \frac{1}{2}[\log(1+t)-\log(1-t)]_0^x$$

$$x+\frac{x^3}{3}+\frac{x^5}{5}+\cdots+\frac{x^{2N+1}}{2N+1}-0 < \frac{1}{2}(\log(1+x)-\log(1-x)-0+0)$$

$$x+\frac{x^3}{3}+\frac{x^5}{5}+\cdots+\frac{x^{2N+1}}{2N+1} < \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$$

13d Over the interval, $t \le x$

Therefore $t^{2N+2} \le x^{2N+2}$

Furthermore,

$1-t^2>0$

Therefore

$$\frac{t^{2N+2}}{1-t^2} \le \frac{x^{2N+2}}{1-t^2}$$

Therefore, since the LHS term is less than or equal to the RHS term everywhere in the interval of integration and the upper limit of integration is greater than the lower limit:

$$\int_0^x \frac{t^{2N+2}}{1-t^2} \le \int_0^x \frac{x^{2N+2}}{1-t^2}$$

13e From part b:

$$\frac{1}{1-t^2} = 1 + t^2 + t^4 + \cdots + t^{2N} + \frac{t^{2N+2}}{1-t^2}$$

$$\int_0^x \frac{1}{1-t^2}\,dt = \int_0^x (1 + t^2 + t^4 + \cdots + t^{2N})\,dt + \int_0^x \frac{t^{2N+2}}{1-t^2}\,dt$$

Using the result from part d:

$$\int_0^x \frac{1}{1-t^2}\,dt \leq \int_0^x (1 + t^2 + t^4 + \cdots + t^{2N})\,dt + \int_0^x \frac{x^{2N+2}}{1-t^2}\,dt$$

$$\frac{1}{2}\log\left(\frac{1+x}{1-x}\right) \leq \left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots + \frac{x^{2N+1}}{2N+1}\right) + x^{2N+2}\int_0^x \frac{1}{1-t^2}\,dt$$

$$\frac{1}{2}\log\left(\frac{1+x}{1-x}\right) \leq \left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots + \frac{x^{2N+1}}{2N+1}\right) + \frac{x^{2N+2}}{2}\log\left(\frac{1+x}{1-x}\right)$$

And using the result from part c:

$$\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots + \frac{x^{2N+1}}{2N+1}\right) \leq \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$$
$$\leq \left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots + \frac{x^{2N+1}}{2N+1}\right) + \frac{x^{2N+2}}{2}\log\left(\frac{1+x}{1-x}\right)$$

Therefore

$$\left|\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots + \frac{x^{2N+1}}{2N+1}\right) - \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)\right| \leq \frac{x^{2N+2}}{2}\log\left(\frac{1+x}{1-x}\right)$$

As $N \to \infty, x^{2N+2} \to 0$ since $0 < x < 1$

Therefore, as $N \to \infty$,

$$\frac{x^{2N+2}}{2}\log\left(\frac{1+x}{1-x}\right) \to 0$$

Therefore

$$\lim_{N\to\infty}\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots + \frac{x^{2N+1}}{2N+1}\right) - \frac{1}{2}\log\left(\frac{1+x}{1-x}\right) = 0$$

$$\lim_{N\to\infty}\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots + \frac{x^{2N+1}}{2N+1}\right) = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$$

13f Substitute $x = \frac{1}{3}$

From part e:

$$\lim_{N\to\infty}\left(\frac{1}{3}+\frac{\left(\frac{1}{3}\right)^3}{3}+\frac{\left(\frac{1}{3}\right)^5}{5}+\cdots+\frac{\left(\frac{1}{3}\right)^{2N+1}}{2N+1}\right)$$

$$=\frac{1}{2}\log\left(\frac{1+\frac{1}{3}}{1-\frac{1}{3}}\right)$$

$$=\frac{1}{2}\log\left(\frac{3+1}{3-1}\right)$$

$$=\frac{1}{2}\log(2)$$

Therefore

$$\log(2) = 2\times\lim_{N\to\infty}\left(\frac{1}{3}+\frac{\left(\frac{1}{3}\right)^3}{3}+\frac{\left(\frac{1}{3}\right)^5}{5}+\cdots+\frac{\left(\frac{1}{3}\right)^{2N+1}}{2N+1}\right)$$

Summing a few terms of this series gives:

$\log(2) \doteqdot 0.693$

14a For $1 \le t \le x^{\frac{\alpha}{2}}$,

$$x^{-\frac{\alpha}{2}} \le \frac{1}{t} \le 1$$

Therefore:

$$\int_1^{x^{\frac{\alpha}{2}}} x^{-\frac{\alpha}{2}}\,dt \le \int_1^{x^{\frac{\alpha}{2}}} \frac{1}{t}\,dt \le \int_1^{x^{\frac{\alpha}{2}}} 1\,dt$$

$$x^{-\frac{\alpha}{2}}[t]_1^{x^{\frac{\alpha}{2}}} \le [\log t]_1^{x^{\frac{\alpha}{2}}} \le [t]_1^{x^{\frac{\alpha}{2}}}$$

$$x^{-\frac{\alpha}{2}}\left(x^{\frac{\alpha}{2}}-1\right) \le \log\left(x^{\frac{\alpha}{2}}\right)-\log(1) \le x^{\frac{\alpha}{2}}-1$$

$$1-x^{-\frac{\alpha}{2}} \le \frac{\alpha}{2}\log x \le x^{\frac{\alpha}{2}}-1$$

Since $x > 1, x^{-\frac{\alpha}{2}} < 1$

Therefore

$1 - x^{-\frac{\alpha}{2}} > 0$

and $x^{\frac{\alpha}{2}} - 1 < x^{\frac{\alpha}{2}}$

Therefore

$0 < \dfrac{\alpha}{2} \log x \leq x^{\frac{\alpha}{2}}$

14b Divide through by $\frac{\alpha}{2} x^{\alpha}$ to get:

$0 < \dfrac{\log x}{x^{\alpha}} < \dfrac{2}{\alpha} x^{-\frac{\alpha}{2}}$

As $x \to \infty, x^{-\frac{\alpha}{2}} \to 0$

Therefore $\frac{2}{\alpha} x^{-\frac{\alpha}{2}} \to 0$

Therefore

$\lim\limits_{x \to \infty} \left(\dfrac{\log x}{x^{\alpha}} \right) = 0$

15a When $n^k \leq x \leq n^{k+1}$,

$\dfrac{1}{x} \geq \dfrac{1}{n^{k+1}}$

Therefore

$\displaystyle\int_{n^k}^{n^{k+1}} \frac{1}{x}\, dx \geq \int_{n^k}^{n^{k+1}} \frac{1}{n^{k+1}}\, dx$

$\displaystyle\int_{n^k}^{n^{k+1}} \frac{1}{x}\, dx \geq \left[\frac{x}{n^{k+1}} \right]_{n^k}^{n^{k+1}}$

$\displaystyle\int_{n^k}^{n^{k+1}} \frac{1}{x}\, dx \geq \frac{n^{k+1} - n^k}{n^{k+1}}$

$\displaystyle\int_{n^k}^{n^{k+1}} \frac{1}{x}\, dx \geq 1 - \frac{1}{n}$

15b By dividing the interval of integration, we can see that

$$\int_1^{n^k} \frac{1}{x}\,dx = \int_1^{n} \frac{1}{x}\,dx + \int_n^{n^2} \frac{1}{x}\,dx + \cdots + \int_{n^{k-1}}^{n^k} \frac{1}{x}\,dx$$

Using the result from part a, each of the RHS integrals is $\geq 1 - \frac{1}{n}$

Therefore:

$$\int_1^{n^k} \frac{1}{x}\,dx \geq \left(1 - \frac{1}{n}\right) + \left(1 - \frac{1}{n}\right) + \cdots + \left(1 - \frac{1}{n}\right)$$

$$\int_1^{n^k} \frac{1}{x}\,dx \geq k\left(1 - \frac{1}{n}\right)$$

Since $n > 1,\ 1 - \frac{1}{n} > 0$

Therefore as $k \to \infty, k\left(1 - \frac{1}{n}\right) \to \infty$

Therefore

$\int_1^{n^k} \frac{1}{x}\,dx \to \infty$ as $k \to \infty$

16a When $n \leq t \leq n + x$,

$$\frac{1}{n+x} \leq \frac{1}{t} \leq \frac{1}{n}$$

Therefore

$$\int_n^{n+x} \frac{1}{n+x}\,dt \leq \int_n^{n+x} \frac{1}{t}\,dt \leq \int_n^{n+x} \frac{1}{n}\,dt$$

Therefore

$$\frac{x}{n+x} \leq \log(n+x) - \log(n) \leq \frac{x}{n}$$

$$\frac{x}{n+x} \leq \log\left(\frac{n+x}{n}\right) \leq \frac{x}{n}$$

$$\frac{x}{n+x} \leq \log\left(1 + \frac{x}{n}\right) \leq \frac{x}{n}$$

Multiply through by n:

$$\frac{x}{1 + \frac{x}{n}} \leq n\log\left(1 + \frac{x}{n}\right) \leq x$$

16b As $n \to \infty, 1 + \frac{x}{n} \to 1$

Therefore $\frac{x}{1+\frac{x}{n}} \to x$

Since $n \log\left(1 + \frac{x}{n}\right)$ is sandwiched between $\frac{x}{1+\frac{x}{n}}$ and x, and both of these tend to x as $n \to \infty$,

$$\lim_{n\to\infty} n \log\left(1 + \frac{x}{n}\right) = x$$

Therefore

$$\lim_{n\to\infty} e^{n \log\left(1+\frac{x}{n}\right)} = e^x$$

$$\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$$

16c The exact value of $e^{0.1}$ is 1.105 to three decimal places. By trial and error, n needs to be at least 9 to agree to three decimal places.

17a i $\int_2^{k+1} \ln(x-1)dx$

$= [uv]_2^{k+1} - \int_2^{k+1} vu^1$ (integration by parts)

Let $u = \ln(x-1)$, then $u^1 = \frac{1}{x-1}$

Let $v^1 = 1$, then $v = x$

Hence,

$[uv]_2^{k+1} - \int_2^{k+1} vu^1$

$= [x \ln(x-1)]_2^{k+1} - \int_2^{k+1} \frac{x}{x-1} dx$

$= (k+1)\ln k - \int_2^{k+1} \frac{(x-1)+1}{x-1} dx$

$=(k+1)\ln k - [x + \ln|x-1|]_2^{k+1}$

$=(k+1)\ln k - (k+1+\ln k - 2)$

$= k\ln k - k + 1$

17a ii Let $u = \ln x$, then $u^1 = \frac{1}{x}$

Let $v^1 = 1$, then $v = x$

$\int_2^{k+1} \ln x \, dx$

$= [x \ln x]_2^{k+1} - \int_2^{k+1} x . \frac{1}{x} dx$ (integration by parts)

$= [x \ln x - x]_2^{k+1}$

$= (k+1)\ln(k+1) - (k+1) - 2\ln 2 + 2$

$= (k+1)\ln(k+1) - \ln 4 - k + 1$

17b Area under $y = \ln(x-1)$ from $x = 2$ to $x = k+1$

$<$ sums of areas of $(k-1)$ rectangles

$<$ area under $y = \ln x$ from $x = 2$ to $x = k+1$

So,

$k \ln k - k + 1 < \ln 2 + \ln 3 + \cdots + \ln(k+1) < (k+1)\ln(k+1) - \ln 4 - k + 1$

$\ln k^k < \ln k! + (k-1) < \ln\left(\frac{(k+1)^{k+1}}{4}\right)$

$e^{\ln k^k} < e^{\ln k!} e^{k-1} < e^{\ln\left(\frac{1}{4}(k+1)^{k+1}\right)}$ (since e^x is an increasing function)

$k^k < k! \, e^{k-1} < \frac{1}{4}(k+1)^{k+1}$

18a $\int_a^b f(x) \, dx <$ area to the trapezium under chord $CD = (b-a).\frac{f(a)-f(b)}{2}$, and

$\int_a^b f(x) \, dx >$ area to the trapezium under chord $MN = (b-a).f\left(\frac{a+b}{2}\right)$

Proof that area of trapezium under MN is $(b-a).f\left(\frac{a+b}{2}\right)$:

Equation of MN is $y - f\left(\frac{a+b}{2}\right) = f'\left(\frac{a+b}{2}\right)\left(x - \frac{a+b}{2}\right)$

Let $x = a$ to get y-value of M:

$y = f\left(\frac{a+b}{2}\right) + f'\left(\frac{a+b}{2}\right)\left(\frac{a-b}{2}\right)$

Let $x = b$ to get y-value of N:

$y = f\left(\frac{a+b}{2}\right) + f'\left(\frac{a+b}{2}\right)\left(\frac{b-a}{2}\right)$

Hence, the area of the trapezium is:

$\frac{1}{2}.2f\left(\frac{a+b}{2}\right).(b-a)$

$= (b-a).f\left(\frac{a+b}{2}\right)$

18b Let $f(x) = \frac{1}{x^2}, a = n-1, b = n$ in part a.

Then $\frac{a+b}{2} = \frac{2n-1}{2}$, and so part a becomes:

$\frac{4}{(2n-1)^2} < \int_{n-1}^{n} x^{-2}\,dx < \frac{1}{2}\left(\frac{1}{(n-1)^2} + \frac{1}{n^2}\right)$

$\frac{4}{(2n-1)^2} < -\left[\frac{1}{x}\right]_{n-1}^{n} < \frac{1}{2}\left(\frac{1}{(n-1)^2} + \frac{1}{n^2}\right)$

$\frac{4}{(2n-1)^2} < \frac{1}{n-1} - \frac{1}{n} < \frac{1}{2}\left(\frac{1}{(n-1)^2} + \frac{1}{n^2}\right)$

18c Substitute $n = 2$, then $n = 3$, and so on, into part b.

$\frac{4}{3^2} < \frac{1}{1} - \frac{1}{2} < \frac{1}{2}\left(\frac{1}{1^2} + \frac{1}{2}\right)$, and

$\frac{4}{5^2} < \frac{1}{2} - \frac{1}{3} < \frac{1}{2}\left(\frac{1}{2^2} + \frac{1}{3^2}\right)$, and

$\frac{4}{7^2} < \frac{1}{3} - \frac{1}{4} < \frac{1}{2}\left(\frac{1}{3^2} + \frac{1}{4^2}\right)$, and so on.

Adding these double inequalities:

$\frac{4}{3^2} + \frac{4}{5^2} + \frac{4}{7^2} + \cdots < \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots$

$< \frac{1}{2}\left(\frac{1}{1^2} + \frac{1}{2^2}\right) + \frac{1}{2}\left(\frac{1}{2^2} + \frac{1}{3^2}\right) + \frac{1}{2}\left(\frac{1}{3^2} + \frac{1}{4^2}\right) + \cdots$

Hence, $4\left(\frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots\right) < 1 < \frac{1}{2} + \left(\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right)$

18d LHS

$= \frac{1}{2}\left(\frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots\right)$

$< \frac{1}{2}\left(\frac{1}{3^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{7^2} + \cdots\right)$ (since $\frac{1}{3^2} > \frac{1}{4^2}, \frac{1}{5^2} > \frac{1}{6^2}, \ldots$)

$=$ RHS

18e Multiplying both sides of part d by 4:

$$2\left(\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\cdots\right)<4\left(\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots\right)$$

Substituting into part c:

$$2\left(\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\cdots\right)<1<\frac{1}{2}+\left(\mathbf{\frac{1}{2^2}}+\mathbf{\frac{1}{3^2}}+\mathbf{\frac{1}{4^2}}+\cdots\right)$$

Hence,

$$\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\cdots<\frac{1}{2}\text{ and }\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}>\frac{1}{2}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^2}<\frac{1}{2}+\frac{1}{1^2}+\frac{1}{2^2}\text{ and }\sum_{n=1}^{\infty}\frac{1}{n^2}>\frac{1}{2}+\frac{1}{1^2}$$

Hence,

$$\frac{3}{2}<\sum_{n=1}^{\infty}\frac{1}{n^2}<\frac{7}{4}$$

19a $\int_1^n \ln x\,dx$

$= [uv]_1^n - \int_1^n vu^1$

Let $u = \ln x$, then $u^1 = \frac{1}{x}$

Let $v^1 = 1$, then $v = x$

Hence,

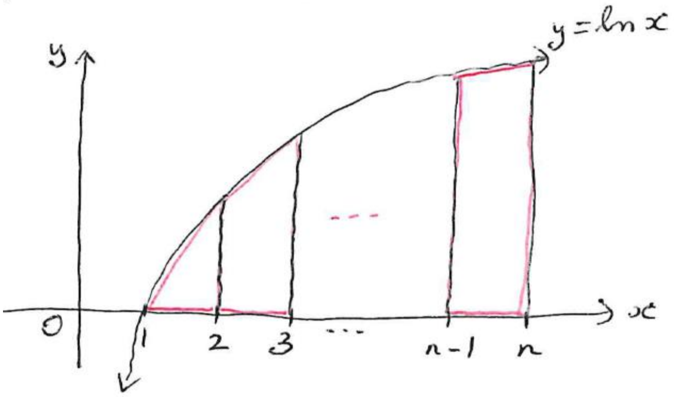
$[uv]_1^n - \int_1^n vu^1$

$= [x\ln x]_1^n - \int_1^n x.\frac{1}{x}dx$

$= [x\ln x - x]_1^n$

$= n\ln n - n + 1$

19b



$\int_1^n \ln x \, dx$

$> \frac{1}{2}\big((\ln 1 + \ln 2) + (\ln 2 + \ln 3) + \cdots + (\ln(n-1) + \ln n)\big)$

$= \frac{1}{2}\ln n + \ln\big(2 \times 3 \times \ldots \times (n-1)\big)$

$= \frac{1}{2}\ln n + \ln(n-1)!$

19c Combining part a and part b:

$n \ln n - n + 1 > \frac{1}{2}\ln n + \ln(n-1)!$

Adding $\frac{1}{2}\ln n$ to both sides:

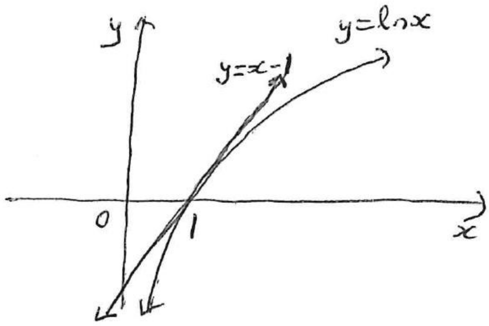
$\left(n + \frac{1}{2}\right)\ln n + 1 - n > \ln n!$

$e^{\ln n^{n+\frac{1}{2}}} . e^{1-n} > e^{\ln n!}$ (since e^x is a strictly increasing function)

Hence,

$n! < n^{n+\frac{1}{2}} . e^{1-n}$

20a



The tangent to $y = \ln x$ at $(1, 0)$ has equation:

$y - 0 = \frac{1}{1}(x - 1)$

So, $y = x - 1$

So, for $x > 0$, the tangent $y = x - 1$ is above the curve $y = \ln x$, except at $(1, 0)$, the point of tangency.

Hence, for $x > 0$, $\ln x \leq x - 1$.

20b We are given that $p_1 + p_2 + \cdots + p_n = 1$, where $p_1, p_2, \ldots, p_n > 0$.

We must prove that $\sum_{r-1}^{n} \ln(np_r) \leq 0$

LHS

$=\ln(np_1) + \ln(np_2) + \ldots \ln(np_n)$

$\leq (np_1 - 1) + (np_2 - 1) + \cdots + (np_n - 1)$ (using part a)

$= n(p_1 + p_2 + \cdots + p_n) - n$ (note that $np_i > 0$ for $i = 1, 2, \ldots, n$)

$= n(1) - n$

$= 0$

$=$ RHS

20c Let $x_1 + x_2 + \cdots + x_n = S$, and let $p_i = \frac{x_i}{S}$ for $i = 1, 2, \ldots, n$

Then $p_1 + p_2 + \cdots + p_n = \frac{S}{S} = 1$

Then, from part b,

$\sum_{r-1}^{n} \ln\left(n.\frac{x_r}{S}\right) \leq 0$

That is,

$$\ln\left(\frac{nx_1}{S}\right) + \ln\left(\frac{nx_2}{S}\right) + \cdots + \ln\left(\frac{nx_n}{S}\right) \le 0$$

$$\ln\left(\frac{n^n x_1 x_2 \ldots x_n}{S^n}\right) \le 0$$

$$\left(\frac{n}{S}\right)^n x_1 x_2 \ldots x_n \le 1$$

$$\left(\frac{S}{n}\right)^n \ge x_1 x_2 \ldots x_n$$

$$\frac{S}{n} \ge (x_1 x_2 \ldots x_n)^{\frac{1}{n}}$$

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \ge (x_1 x_2 \ldots x_n)^{\frac{1}{n}}$$

21a For $n = 1$,

LHS

$= e - S_1$

$= e - \left(1 + \frac{1}{1!}\right)$

$= e - 2$

RHS

$= e\int_0^1 xe^{-x}dx$

$= e\left([uv]_0^1 - \int_0^1 vu^1\right)$

Let $u = x$, then $u^1 = 1$

Let $v^1 = e^{-x}$, then $v = -e^{-x}$

Hence,

$e\left([uv]_0^1 - \int_0^1 vu^1\right)$

$= e\left([-xe^{-x}]_0^1 - \int_0^1 -e^{-x}\,dx\right)$

$= e[-xe^{-x} - e^{-x}]_0^1$

$= e(-e^{-1} - e^{-1} + 1)$

$= e - 2$

Hence, the result is true for $n = 1$

Assume that the result is true for some positive integer, $n = k$.

i.e., assume that $e - S_k = e\int_0^1 \frac{x^k}{k!}e^{-x}dx$. $(*)$

Prove that the result is true for $n = k + 1$.

i.e., prove that $e - S_{k+1} = e\int_0^1 \frac{x^{k+1}}{(k+1)!}e^{-x}dx$.

LHS

$= e - S_{k+1}$

$= e - \left(S_k + \frac{1}{(k+1)!}\right)$

$= e\int_0^1 \frac{x^k}{k!}e^{-x}dx - \frac{1}{(k+1)!}$ (using $(*)$)

$= e\left([uv]_0^1 - \int_0^1 vu'\right) - \frac{1}{(k+1)!}$

Let $u = e^{-x}$, then $u' = -e^{-x}$

Let $v' = \frac{x^k}{k!}$, then $v = \frac{x^{k+1}}{(k+1)!}$

Hence,

$e\left([uv]_0^1 - \int_0^1 vu'\right) - \frac{1}{(k+1)!}$

$= e\left(\left[\frac{x^{k+1}}{(k+1)!}.e^{-x}\right]_0^1 - \int_0^1 -\frac{x^{k+1}}{(k+1)!}.e^{-x}dx\right) - \frac{1}{(k+1)!}$

$= e\left(\left[\frac{1}{(k+1)!}.e^{-1}\right]_0^1 + \int_0^1 \frac{x^{k+1}}{(k+1)!}.e^{-x}\,dx\right) - \frac{1}{(k+1)!}$

$=$ RHS

So, by induction, the result is true for all $n \in \mathbb{Z}^+$.

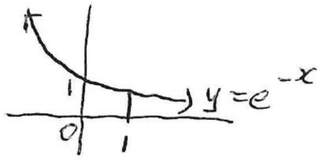
21b $e - S_n = e\int_0^1 \frac{x^n}{n!}e^{-x}dx > 0$, since $\frac{x^n}{n!}e^{-x} > 0$ for $0 < x < 1$.

Also,

$e - S_n$

$= e\int_0^1 \frac{x^n}{n!}e^{-x}dx$

$< 3\int_0^1 \frac{x^n}{n!}e^{-x}dx,$ since $0 < e^{-x} < 1$ for $0 < x < 1$.



$= 3\left[\frac{x^{n+1}}{(n+1)!}\right]_0^1$

$= \frac{3}{(n+1)!}$

21c $0 < e - S_n < \frac{3}{(n+1)!}$

Hence, $0 < (e - S_n)n! < \frac{3}{n+1}$

But $\frac{3}{n+1} \leq 1$ for $n = 2, 3, 4, \ldots$

So, $0 < (e - S_n)n! < 1$ for $n = 2, 3, 4, \ldots$

Hence, $(e - S_n)n!$ is not an integer for $n = 2, 3, 4, \ldots$

21d Suppose that $e = \frac{p}{q}$, where $p, q \in \mathbb{Z}^+$ and $q \neq 1$. (since $e \notin \mathbb{Z}^+$)

Then, from part c, we have $\left(\frac{p}{q} - S_n\right)n!$ is not an integer for $n = 2, 3, 4, \ldots$

So, by letting $n = q$, where $q > 1$, $\left(\frac{p}{q} - S_q\right)q!$ is not an integer.

But $\frac{pq!}{q} \in \mathbb{Z}^+$ and $S_q.q! = q!\left(1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{q!}\right) \in \mathbb{Z}^+$

So, $\left(\frac{p}{q} - S_q\right)q!$ is an integer which contradicts part c.

Hence, e cannot be written in the form $\frac{p}{q}$, where $p, q \in \mathbb{Z}^+$.

Solutions to Exercise 2G Review questions

1a If the opposite angles of a quadrilateral are supplementary, then it is cyclic. True.

1b If the sum of two numbers is even, then they are both odd. False.
(Counterexample: $1 + 1 = 2$.)

1c Every parallelogram is a rhombus. False.

2a Not all mathematicians are intelligent. (Alternately: there exists a mathematician who is not intelligent.)

2b Suzie doesn't like both of Physics and Chemistry, that is, she dislikes at least one of them.

2c I am on vacation and I am working.

3a If I don't have two wheels, then I am not a bicycle.

3b If a number's last digit is 6, then it is not odd. (i.e. it is even, assuming it's an integer.)

3c If a shape does not have four equal sides, then it is not a square.

4a If a number is even, it is divisible by 2; if a number is divisible by 2, it is even.

4b If a quadrilateral's diagonals bisect one another, it is a parallelogram; if a quadrilateral is a parallelogram, its diagonals bisect one another.

4c If a is divisible by b, $\exists c \in \mathbb{Z}$ such that $a = bc$; if $\exists c \in \mathbb{Z}$ such that $a = bc$, then a is divisible by b.

5a Let n be the first of the three numbers.

Then the other two are $n + 1$ and $n + 2$.

Therefore their sum is

$n + n + 1 + n + 2$

$= 3n + 3$

$= 3(n + 1)$

So their sum is divisible by 3.

5b Let $2n$ be the first of the three numbers.

Then the other two are $2n + 2$ and $2n + 4$.

Therefore their product is

$2n(2n + 2)(2n + 4)$

$= 2n \times 2(2 + 1) \times 2(2 + 2)$

$= 2 \times 2 \times 2n(n + 1)(n + 2)$

$= 8 \times n(n + 1)(n + 2)$

So their product is divisible by 8.

5c Let $2n$ be the first of the two numbers.

Then the other is $2n + 2$.

Their product is $4n^2 + 4n = 4n(n + 1)$

Case 1: If n is even, then $4n$ is a multiple of 8.

Therefore $4n(n + 1)$ is a multiple of 8.

Case 2: If n is odd, then $n + 1$ is even.

Therefore $4(n + 1)$ is a multiple of 8.

Therefore $4n(n + 1)$ is a multiple of 8.

So in either case, the product of the two numbers is divisible by 8.

6 If a square number is odd, then its square root is also odd.

Let $2n + 1$ be the square root.

Therefore the square number is

$(2n + 1)^2$

$= 4n^2 + 4n + 1$

$= 4(n^2 + n) + 1$

Since $4(n^2 + n)$ is divisible by 4, the remainder when $(2n + 1)^2$ is divisible by 4 is 1.

7 Suppose a number n is divisible by 15.

Therefore $n = 15a$ for some integer a.

Therefore $n = 3(5a)$ so it is divisible by 3.

Similarly, $n = 5(3a)$ so it is divisible by 5.

8 $n^3 - n$

$= n(n^2 - 1)$

$= (n - 1)(n)(n + 1)$

Since n is odd, $n - 1$ and $n + 1$ are consecutive even numbers, so their product must be divisible by 8. (See proof in question 5c.)

Since $n - 1, n,$ and $n + 1$ are three consecutive integers, one of them must be a multiple of 3, therefore their product must be divisible by 3.

Therefore $n^3 - n$ is divisible by both 3 and 8.

Since 3 and 8 have no common factors, it must therefore be divisible by $3 \times 8 = 24$.

9 If n is not divisible by 3, then it must be either one greater than or one less than a multiple of 3.

Therefore $n = 3m + a$, where m is an integer and a is either 1 or -1.

Therefore

$n^2 + 2$

$= (3m + a)^2 + 2$

$= 9m^2 + 6ma + a^2 + 2$

$= 9m^2 + 6ma + 1 + 2 \qquad (\text{since } a^2 = 1)$

$= 9m^2 + 6ma + 3$

$= 3(3m^2 + 2ma + 1)$

Therefore $n^2 + 2$ is divisible by 3.

10a $(x + 1)(x^{n-1} - x^{n-2} + x^{n-3} - x^{n-4} + \cdots + x^2 - x + 1)$

$= (x + 1)x^{n-1} - (x + 1)x^{n-2} + (x + 1)x^{n-3} - (x + 1)x^{n-4} + \cdots + (x + 1)x^2 - (x + 1)x + (x + 1)$

$= x^n + x^{n-1} - x^{n-1} - x^{n-2} + x^{n-2} + x^{n-3} - x^{n-3} - x^{n-4} + \cdots + x^3 + x^2 - x^2 - x + x + 1$

$= x^n + 0 + 0 + \cdots + 0 + 1$

$= x^n + 1$

10b i If n is odd,

$2^n + 1$

$= (2 + 1)(2^{n-1} - 2^{n-2} + \cdots - 2 + 1)$

$= 3(2^{n-1} - 2^{n-2} + \cdots - 2 + 1)$

Therefore $2^n + 1$ is divisible by 3.

10b ii If n is odd,

$2^{mn} + 1$

$= (2^m)^n + 1$

$= (2^m + 1)((2^m)^{n-1} - (2^m)^{n-2} + \cdots - (2^m) + 1)$

Therefore $2^{mn} + 1$ is divisible by $2^m + 1$.

11a Proof by contradiction:

Suppose $\sqrt{7} = \frac{m}{n}$ where m and n are integers with $n \geq 1$ and their HCF is 1. That is, they have been reduced to lowest terms.

Squaring and re-arranging gives

$7n^2 = m^2$

Thus m^2 is divisible by 7.

If m were not divisible by 7 then m^2 would not be divisible by 7.

Hence m is also divisible by 7. So let $m = 7p$ and write

$7n^2 = 49p^2$

or $n^2 = 7m^2$

Thus n^2 is divisible by 7.

Now if n were not divisible by 7 then n^2 would not be divisible by 7.

Hence n is also divisible by 7.

That is, 7 is a common factor of m and n.

But the HCF is 1, so there is a contradiction.

Hence $\sqrt{7}$ is irrational.

11b Proof by contradiction:

Suppose $\log_3 7 = \frac{m}{n}$ where m and n are integers with $n \geq 1$ and their HCF is 1. That is, they have been reduced to lowest terms.

By definition,

$3^{\frac{m}{n}} = 7$

Raising both sides to the power n,
$3^m = 7^n$

But this implies that 3 divides 7^n which is impossible as the only factors of this number are powers of 7.

This is a contradiction, hence $\log_3 7$ is irrational.

12 If a is not even, then a is odd and $a = 2m + 1$ for some integer m.

Therefore $a^2 = 4m^2 + 4m + 1$

$= 2(2m^2 + 2m) + 1$

Therefore a^2 is odd (and not even).

Proving this proves the contrapositive: if a^2 is even then a is odd.

13a $\left(\sqrt{x} - \sqrt{y}\right)^2 \geq 0$

$x - 2\sqrt{xy} + y \geq 0$

$x + y \geq 2\sqrt{xy}$

13b Substituting $y = \frac{1}{x}$ into the inequality derived above:

$x + \dfrac{1}{x} \geq 2\sqrt{\dfrac{x}{x}}$

$x + \dfrac{1}{x} \geq 2 \qquad (1)$

$(a + b)(1 + ab)$

$= a + a^2b + b + ab^2$

$= a + ab^2 + b + a^2b$

$= a(1 + b^2) + b(1 + a^2)$

$= ab\left(b + \dfrac{1}{b}\right) + ab\left(a + \dfrac{1}{a}\right)$

$\geq ab(2) + ab(2) \qquad$ using (1)

$= 4ab$

Therefore

$(a + b)(1 + ab) \geq 4ab$

14a A. When $n = 1$,

LHS $= 2 \times 2 = 4$

RHS $= 1 \times 2^2 = 4$

Therefore the result is true for $n = 1$.

B. Assume the statement is true for the positive integer $n = k$.

That is, assume that

$$\sum_{r=1}^{k} (r+1) \times 2^r = k \times 2^{k+1}$$

Now prove the statement for $n = k + 1$. That is, prove that

$$\sum_{r=1}^{k+1} (r+1) \times 2^r = (k+1) \times 2^{k+2}$$

From the inductive assumption:

$$\sum_{r=1}^{k+1} (r+1) \times 2^r$$

$$= \left(\sum_{r=1}^{k} (r+1) \times 2^r\right) + (k+1+1) \times 2^{k+1}$$

$$= k \times 2^{k+1} + (k+2) \times 2^{k+1}$$

$$= (2k+2) \times 2^{k+1}$$

$$= (k+1) \times 2^{k+2}$$

as required.

C. It follows from parts A and B by mathematical induction that the result is true for all integers $n \geq 1$.

14b A. When $n = 1$,

LHS $= 1(1 + 1) = 2$

RHS $= \frac{1}{12} \times 1 \times (1 + 1) \times (1 + 2) \times (3 + 1) = 1$

Therefore the result is true for $n = 1$.

B. Assume the statement is true for the positive integer $n = k$.

That is, assume that

$$\sum_{r=1}^{k} r^2(r + 1) = \frac{1}{12}k(k + 1)(k + 2)(3k + 1)$$

Now prove the statement for $n = k + 1$. That is, prove that

$$\sum_{r=1}^{k+1} r^2(r + 1) = \frac{1}{12}(k + 1)(k + 2)(k + 3)(3k + 4)$$

From the inductive assumption:

$$\sum_{r=1}^{k+1} r^2(r + 1)$$

$$= \left(\sum_{r=1}^{k} r^2(r + 1)\right) + (k + 1)^2(k + 2)$$

$$= \frac{1}{12}k(k + 1)(k + 2)(3k + 1) + (k + 1)^2(k + 2)$$

$$= \frac{1}{12}k(k + 1)(k + 2)(3k + 1) + (k + 1)^2(k + 2)$$

$$= \frac{1}{12}k(k + 1)(k + 2)(3k + 1) + (k^3 + 4k^2 + 5k + 2)$$

$$= \frac{1}{12}\left(k(k + 1)(k + 2)(3k + 1) + 12(k^3 + 4k^2 + 5k + 2)\right)$$

$$= \frac{1}{12}\left(k(k + 1)(k + 2)(3k + 1) + 12(k^3 + 4k^2 + 5k + 2)\right)$$

$$= \frac{1}{12}(3k^4 + 10k^3 + 9k^2 + 2k + 12k^3 + 48k^2 + 60k + 24)$$

$$= \frac{1}{12}(3k^4 + 22k^3 + 57k^2 + 62k + 24)$$

$= \dfrac{1}{12}(k+1)(k+2)(k+3)(3k+4)$

as required.

C. It follows from parts A and B by mathematical induction that the result is true for all integers $n \geq 1$.

15a A. When $n = 1$, $6^n + 4 = 10$

Therefore the result is true for $n = 1$.

B. Assume the statement is true for the positive integer $n = k$.

That is, assume that

$6^k + 4 = 5m$ for some integer m.

Now prove the statement for $n = k + 1$. That is, prove that

$6^{k+1} + 4$ is divisible by 5.

From the inductive assumption:

$6^k + 4 = 5m$

$6(6^k + 4) = 6^{k+1} + 24$

$6^{k+1} + 4$

$= 6(6^k + 4) - 20$

$= 6(5m) - 20$

$= 5(6m - 4)$

which is divisible by 5.

C. It follows from parts A and B by mathematical induction that the result is true for all integers $n \geq 1$.

15b A. When $n = 1$, $n^3 + 2n = 3$

Therefore the result is true for $n = 1$.

B. Assume the statement is true for the positive integer $n = k$.

That is, assume that

$k^3 + 2k = 3m$ for some integer m.

Now prove the statement for $n = k + 1$. That is, prove that

$(k + 1)^3 + 2(k + 1)$ is divisible by 3.

$(k + 1)^3 + 2(k + 1)$

$= k^3 + 3k^2 + 3k + 1 + 2k + 2$

$= k^3 + 3k^2 + 5k + 3$

$= (k^3 + 2k) + 3k^2 + 3k + 3$

$= 3(m + k^2 + k + 1)$ from the inductive assumption

which must be divisible by 3.

C. It follows from parts A and B by mathematical induction that the result is true for all integers $n \geq 1$.

16a A. When $n = 1$,

$$\text{LHS} = 1 - \frac{1}{2^2}$$

$$= \frac{3}{4}$$

$$\text{RHS} = \frac{1}{2}\left(1 + \frac{1}{2}\right)$$

$$= \frac{3}{4}$$

Therefore the result is true for $n = 1$.

B. Assume the statement is true for the positive integer $n = k$.

That is, assume that

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\ldots\left(1-\frac{1}{(k+1)^2}\right)=\frac{1}{2}\left(1+\frac{1}{k+1}\right)$$

Now prove the statement for $n = k + 1$. That is, prove that

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\ldots\left(1-\frac{1}{(k+1)^2}\right)\left(1-\frac{1}{(k+2)^2}\right)=\frac{1}{2}\left(1+\frac{1}{k+2}\right)$$

From the inductive assumption:

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\ldots\left(1-\frac{1}{(k+1)^2}\right)\left(1-\frac{1}{(k+2)^2}\right)$$

$$=\frac{1}{2}\left(1+\frac{1}{k+1}\right)\left(1-\frac{1}{(k+2)^2}\right)$$

$$=\frac{1}{2}\left(\frac{k+2}{k+1}\right)\left(\frac{(k+2)^2-1}{(k+2)^2}\right)$$

$$=\frac{1}{2}\left(\frac{1}{k+1}\right)\left(\frac{k^2+4k+3}{k+2}\right)$$

$$=\frac{1}{2}\left(\frac{1}{k+1}\right)\left(\frac{(k+3)(k+1)}{k+2}\right)$$

$$=\frac{1}{2}\left(\frac{k+3}{k+2}\right)$$

$$=\frac{1}{2}\left(1+\frac{1}{k+2}\right)$$

as required.

C. It follows from parts A and B by mathematical induction that the result is true for all integers $n \geq 1$.

16b As $n \to \infty$,

$$\frac{1}{2}\left(1+\frac{1}{n+1}\right)\to\frac{1}{2}(1+0)=\frac{1}{2}$$

Therefore the limit is $\frac{1}{2}$.

17a A. When $n = 2, n(n + 2) = 2(4) = 8$

Therefore the result is true for $n = 2$.

B. Assume the statement is true for the positive even integer $n = k$.

That is, assume that

$k(k + 2) = 4m$ for some integer m.

Now prove the statement for $n = k + 2$. That is, prove that

$(k + 2)(k + 4)$ is divisible by 4.

$(k + 2)(k + 4)$

$= k(k + 2) + 4(k + 2)$

$= 4m + 4(k + 2)$ (from the inductive assumption)

$= 4(m + k + 2)$

which is divisible by 4.

C. It follows from parts A and B by mathematical induction that the result is true for all even integers $n \geq 2$.

17b A. When $n = 1, 3^n + 7^n = 10$

Therefore the result is true for $n = 1$.

B. Assume the statement is true for the positive odd integer $n = k$.

That is, assume that

$3^n + 7^n = 10m$ for some integer m.

Now prove the statement for $n = k + 2$. That is, prove that

$3^{n+2} + 7^{n+2}$ is divisible by 10.

$3^{n+2} + 7^{n+2}$

$= 9 \times 3^n + 49 \times 7^n$

$= 9(3^n + 7^n) + 40 \times 7^n$

$= 9(10m) + 40 \times 7^n$ (by the inductive hypothesis)

$= 10(9m + 4 \times 7^n)$

which is divisible by 10.

C. It follows from parts A and B by mathematical induction that the result is true for all odd integers $n \geq 1$.

17c A. When $n = 1$, $4^n + 5^n + 6^n = 15$.

Therefore the result is true for $n = 1$.

B. Assume the statement is true for the positive odd integer $n = k$.

That is, assume that

$4^k + 5^k + 6^k = 15m$ for some integer m.

Now prove the statement for $n = k + 2$. That is, prove that

$4^{k+2} + 5^{k+2} + 6^{k+2}$ is divisible by 15.

$4^{k+2} + 5^{k+2} + 6^{k+2}$

$= 16 \times 4^k + 25 \times 5^k + 36 \times 6^k$

$= 16(4^k + 5^k + 6^k) + 9 \times 5^k + 20 \times 6^k$

$= 16(15m) + 45 \times 5^{k-1} + 60 \times 6^{k-1}$

$= 15(16m + 3 \times 5^{k-1} + 4 \times 6^{k-1})$

which is a multiple of 15.

(Note that 5^{k-1} and 6^{k-1} are integers because $k \geq 1$.)

C. It follows from parts A and B by mathematical induction that the result is true for all odd integers $n \geq 1$.

18 A. When $n = 1$, $2n^2 + 2n - 1 = 3$.

Therefore the result is true for $n = 1$.

B. Assume the statement is true for the positive integer $n = k$.

That is, assume that

$T_k = 2k^2 + 2k - 1$

Now prove the statement for $n = k + 1$. That is, prove that

$$\begin{aligned} T_{k+1} &= 2(k+1)^2 + 2(k+1) - 1 \\ &= 2k^2 + 6k + 3 \end{aligned}$$

By definition,

$$\begin{aligned} T_{k+1} &= T_k + 4(k+1) \\ &= 2k^2 + 2k - 1 + 4k + 4 \qquad \text{(from the inductive assumption)} \\ &= 2k^2 + 6k + 3 \end{aligned}$$

as required.

C. It follows from parts A and B by mathematical induction that the result is true for all integers $n \geq 1$.

19 A. When $n = 1$, $a_n = 1$.

Therefore the result is true for $n = 1$.

B. Assume the statement is true for the positive integer $n = k$.

That is, assume that

$a_k < 3$.

Now prove the statement for $n = k + 1$. That is, prove that

$a_{k+1} < 3$.

From the inductive assumption:

$a_k < 3$

$2a_k + 1 < 7$

$\sqrt{2a_k + 1} < \sqrt{7} < 3$

$a_{k+1} < 3$

as required.

C. It follows from parts A and B by mathematical induction that the result is true for all integers $n \geq 1$.

20 A. When $n = 7$,

$n! = 5040$

$3^n = 2187$

Therefore the result is true for $n = 7$.

B. Assume the statement is true for the positive integer $n = k, k \geq 7$.

That is, assume that

$k! > 3^k$

Now prove the statement for $n = k + 1$. That is, prove that

$(k + 1)! > 3^{k+1}$

From the inductive assumption:

$k! > 3^k$

$(k + 1)k! > (k + 1)3^k$

$(k + 1)! > (k + 1)3^k$

Further,

$(k + 1)3^k > 3 \times 3^k$ since $k + 1 > 3$

Therefore

$(k + 1)! > 3 \times 3^k$

$(k + 1)! > 3^{k+1}$

as required.

C. It follows from parts A and B by mathematical induction that the result is true for all integers $n \geq 7$.

21 A. When $n = 1$, the nth derivative of xe^{-x} is:

$$\frac{d}{dx}(xe^{-x}) = e^{-x} - xe^{-x}$$

$$= (-1)^1(x-1)e^{-x}$$

$$= (-1)^n(x-n)e^{-x}$$

Therefore the result is true for $n = 1$.

B. Assume the statement is true for the positive integer $n = k$.

That is, assume that the kth derivative of xe^{-x} is

$$(-1)^k(x-k)e^{-x}$$

Now prove the statement for $n = k + 1$. That is, prove that the $(k+1)$th derivative of xe^{-x} is

$$(-1)^{k+1}\big(x-(k+1)\big)e^{-x}$$

From the inductive assumption:

$$\frac{d^k}{dx^k}(xe^{-x}) = (-1)^k(x-k)e^{-x}$$

Therefore

$$\frac{d^{k+1}}{dx^{k+1}}(xe^{-x})$$

$$= \frac{d}{dx}(-1)^k(x-k)e^{-x}$$

$$= (-1)^k\left(\frac{d}{dx}(xe^{-x}) - \frac{d}{dx}(ke^{-x})\right)$$

$$= (-1)^k(e^{-x} - xe^{-x} + ke^{-x})$$

$$= (-1)^k\big((1+k)e^{-x} - xe^{-x}\big)$$

$$= -(-1)^k(xe^{-x} - (k+1)e^{-x})$$

$$= (-1)^{k+1}\big(x-(k+1)\big)e^{-x}$$

as required.

C. It follows from parts A and B by mathematical induction that the result is true for all integers $n \geq 1$.

22 We begin by showing that the sum of interior angles is equal to $180°(n-2)$ through induction.

A. When $n = 3$, the polygon is a triangle with interior angles adding to $180°$.

$180° = (n-2) \times 180°$

So the result is true for $n = 3$.

B. Assume the statement is true for integer $n = k \geq 3$.

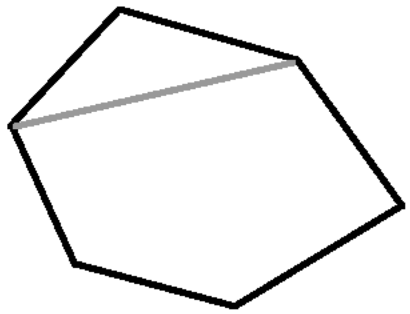
That is, assume that the interior angle sum of a polygon with k sides is

$(k-2) \times 180°$.

Now prove the statement for $n = k + 1$. That is, prove that the interior angle sum of a polygon with $k + 1$ sides is

$(k - 2 + 1) \times 180°$.

To show this, choose any two vertices of the polygon that are separated by just one other vertex, and draw a line between them:



This divides the $(k+1)$-sided polygon into a triangle and a k-sided polygon.

The sum of internal angles of the $(k+1)$-sided polygon equals the sum of internal angles of the triangle, and the sum of internal angles of the k-sided polygon.

From the inductive assumption, the internal angles of the k-sided polygon total $(k-2) \times 180°$. Adding this to the $180°$ from the triangle gives $(k - 2 + 1) \times 180°$ as required.

C. It follows from parts A and B by mathematical induction that the result is true for all integers $n \geq 3$. Since all polygons have at least three sides, this completes the proof that the sum of interior angles is equal to $180°(n-2)$.

At any vertex of the polygon, interior and exterior angles are complementary.

Therefore the sum of all exterior and all interior angles of the polygon equals $180° \times n$.

Therefore the sum of all exterior angles is

$180° \times n - 180°(n - 2)$

$= 180° \times 2$

$= 360°$

as required.

23a A. When $n = 1$, $2^{\mathrm{n}} = 2$

Therefore the result is true for $n = 1$.

B. Assume the statement is true for the positive integer $n = k$.

That is, assume that

$2^k > k$

Now prove the statement for $n = k + 1$. That is, prove that

$2^{k+1} > k + 1$

From the inductive assumption:

$2^k > k$

$2(2^k) > 2k$

$2^{k+1} > k + k$

$2^{k+1} > k + 1$

C. It follows from parts A and B by mathematical induction that the result is true for all integers $n \geq 1$.

23b For $n \geq 2$, and using the result proved above:

$1 < n < 2^n$

Taking nth roots:

$\sqrt[n]{1} < \sqrt[n]{n} < \sqrt[n]{2^n}$

$1 < \sqrt[n]{n} < 2$

23c Since $1 < \sqrt[n]{n} < 2$, it cannot be an integer and therefore it must be irrational. The *contrapositive* of the given fact has been used.

24a Using the result $\frac{x+y}{2} \geq \sqrt{xy}$ three times:

$$\frac{a+b}{2} \geq \sqrt{ab}$$

$$\frac{c+d}{2} \geq \sqrt{bc}$$

$$\frac{\left(\frac{a+b}{2}\right) + \left(\frac{c+d}{2}\right)}{2} \geq \sqrt{\left(\frac{a+b}{2}\right)\left(\frac{c+d}{2}\right)}$$

$$\frac{\left(\frac{a+b}{2}\right) + \left(\frac{c+d}{2}\right)}{2} \geq \sqrt{\sqrt{ab}\sqrt{cd}}$$

$$\frac{a+b+c+d}{4} \geq \sqrt[4]{abcd}$$

24b i

$$\frac{1}{4}\left(a+b+c+\frac{a+b+c}{3}\right)$$

$$= \frac{1}{4}(a+b+c)\left(1+\frac{1}{3}\right)$$

$$= (a+b+c)\left(\frac{1}{4}\right)\left(\frac{4}{3}\right)$$

$$= \frac{a+b+c}{3}$$

as required

24b ii

Let $d = \dfrac{a+b+c}{3}$

From part a,

$$\frac{a+b+c+d}{4} \geq \sqrt[4]{abcd}$$

$$\frac{1}{4}\left(a+b+c+\frac{a+b+c}{3}\right) \geq \sqrt[4]{abc\left(\frac{a+b+c}{3}\right)}$$

$$\frac{a+b+c}{3} \geq \sqrt[4]{abc}\sqrt[4]{\left(\frac{a+b+c}{3}\right)}$$

Dividing through by $\sqrt[4]{\left(\frac{a+b+c}{3}\right)}$:

$$\left(\frac{a+b+c}{3}\right)^{\frac{3}{4}} \geq (abc)^{\frac{1}{4}}$$

Raising both sides to the power $\frac{4}{3}$:

$$\left(\frac{a+b+c}{3}\right) \geq (abc)^{\frac{1}{3}}$$

$$\frac{a+b+c}{3} \geq \sqrt[3]{abc}$$

25 $abc = 100a + 10b + c$

$cba = 100c + 10b + a$

$abc - cba = (100a - 100c) - (a - c)$

Let $d = a - c$

Then $100a - 100c = d00$

$abc - cba = d00 - d$

Working through the subtraction, the digits of $abc - cba$ are $(d-1), 9, (10-d)$.

Since $a - c > 1$ we know that $d - 1$ is still greater than zero, hence $abc - cba$ is still a three-digit number.

When reversed, the digits of this difference are $(10-d), 9, (d-1)$.

Therefore, the difference added to its reverse is:

$100(d-1)+10(9)+1(10-d)+100(10-d)+10(9)+1(d-1)$

$=100(d-1+10-d)+10(9+9)+1(10-d+d-1)$

$=100(9)+10(18)+1(9)$

$=900+180+9$

$=1089$