Solutions to Exercise 14A

1a This is an arithmetic sequence with $a = 102$, $d = 2$ and $n = 500$. Hence the sum is $S_n = \frac{n}{2}(2a + (n-1)d) = \frac{500}{2}(204 + (500-1) \times 2) = 300\,500$

1b This is an arithmetic sequence with $a = 15$, $l = -10$ and $n = 50$ (including the 1st and last term). Hence the sum is $S_n = \frac{n}{2}(a + l) = \frac{50}{2}\big((15) + (-10)\big) = 125$

1c i The common difference is $d = T_2 - T_1 = 97 - 100 = -3$

1c ii The nth term is
$T_n = a + (n-1)d = 100 + (n-1)(-3) = 100 - 3n + 3 = 103 - 3n$

1c iii $S_n = \frac{n}{2}(2a + (n-1)d) = \frac{n}{2}\big(2(100) + (n-1)(-3)\big) = \frac{n}{2}(203 - 3n)$

In order for S_n to be positive, it must be the case that $\frac{n}{2}(203 - 3n) \leq 0$, since $\frac{n}{2} > 0$, we require that $203 \leq 3n$ and thus $67\frac{2}{3} \leq n$. Thus, there are at least 68 terms for which S_n is negative.

2a i $\frac{T_3}{T_2} = \frac{4500}{3000} = 1.5$

$$\frac{T_2}{T_1} = \frac{3000}{2000} = 1.5$$

Hence all terms have the same common ratio of 1.5. Thus, by definition this is a GP.

2a ii $S_n = \frac{2000(r^n - 1)}{r - 1} = \frac{2000(1.5^n - 1)}{1.5 - 1}$, hence $S_5 \frac{2000(1.5^5 - 1)}{0.5} = 26375$

2a iii $|r| = \left|\frac{3}{2}\right| = \frac{3}{2} > 1$, hence there is no limiting sum

2b i

$$\frac{T_3}{T_2} = \frac{2}{6} = \frac{1}{3}$$

$$\frac{T_2}{T_1} = \frac{6}{18} = \frac{1}{3}$$

Hence all terms have the same common ratio of $\frac{1}{3}$. Thus, by definition this is a GP.

2b ii This GP has a limiting sum as $|r| = \left|\frac{1}{3}\right| = \frac{1}{3} < 1$. The limiting sum is

$$S_\infty = \frac{a}{1-r} = \frac{18}{1-\frac{1}{3}} = \frac{18}{\left(\frac{2}{3}\right)} = \frac{3}{2}(18) = 27$$

2b iii

$$\begin{aligned} S_n &= \frac{a(r^n - 1)}{r-1} \\ &= \frac{18\left(\left(\frac{1}{3}\right)^n - 1\right)}{\frac{1}{3} - 1} \\ &= \frac{18\left(\left(\frac{1}{3}\right)^n - 1\right)}{-\frac{2}{3}} \\ &= -\frac{3}{2}\left(18\left(\left(\frac{1}{3}\right)^n - 1\right)\right) \\ &= 27\left(1 - \left(\frac{1}{3}\right)^n\right) \end{aligned}$$

Hence

$$S_{10} = 27\left(1 - \left(\frac{1}{3}\right)^{10}\right) = 26.9995\ldots \doteqdot 27 = S_\infty \text{ (to 3 decimal places)}$$

3a The secretaries salary is an AP with $a = 60\,000$ and $d = 4000$, hence

$$T_n = 60\,000 + (n-1) \times 4000$$

and

$S_n = \frac{1}{2}(n)(2 \times 60\,000 + (n-1) \times 4000)$

After 10 years the annual salary will be $T_{10} = 60\,000 + (10-1) \times 4000 =$ \$96 000 and the total earnings will be $S_{10} = \frac{1}{2}(10)(2 \times 60\,000 + (10-1) \times 4000) =$ \$780 000.

3b To find the year we solve for $T_n = 84\,000$

$60\,000 + (n-1) \times 4000 = 84\,000$

$(n-1) \times 4000 = 24\,000$

$n - 1 = 6$

$n = 7$

Hence, his salary will be \$84 000 in year 7.

4a By definition, her salary is a GP. A 5% increase implies that each year, her salary will be equal to the previous year's salary multiplied by 1.05. That is, the salaries between each year have a common ratio of $r = 1.05$.

4b This is a GP with $a = 80\,000$ and $r = 1.05$. Hence

$T_n = 80\,000(1.05)^{n-1}$

and

$$S_n = \frac{80\,000((1.05)^n - 1)}{1.05 - 1}$$

After 10 years her annual salary will be $T_{10} = 80\,000(1.05)^{10-1} =$ \$124 106 and her total earnings will be $S_{10} = \frac{80\,000((1.05)^{10}-1)}{1.05-1} =$ \$1 006 231.

5a i All the terms are the same.

5a ii The terms are decreasing.

5b If $r = 0$, then $T_2 \div T_1 = 0$, so $T_2 = 0$. Hence $T_3 \div T_2 = T_3 \div 0$ is undefined.

5c i The terms alternate in sign as you are multiplying by a negative number.

5c ii All the terms are the same as you are multiplying by 1.

5c iii The terms will take the form $a, -a, a, -a, \ldots$ where a is the first term of the sequence.

5c iv The terms are decreasing in absolute value and will hence tend towards 0.

6a Lawrence's wage increases by a fixed amount of \$5000 per annum and hence there will be a common difference of \$5000 between each year's salary. Hence his salary is an AP with $a = \$50\,000$ and $d = \$5000$.

Thus $T_n = a + (n-1)d = 50\,000 + (n-1) \times 5000$. This means that:

$T_1 = \$50\,000$
$T_2 = \$55\,000$
$T_3 = \$60\,000$

6b By definition Julian's salary is a GP as a 15% increase per annum means that each year his salary will be the previous year's salary multiplied by 1.15. That is, the salaries between each year have a common ratio of $r = 1.15$. Thus we have a GP with $a = 40\,000$ and $r = 1.15$. Hence $T_n = 40\,000(1.15)^{n-1}$. Applying this formula gives

$T_1 = \$40\,000$
$T_2 = \$46\,000$
$T_3 = \$52\,900$

6c Julian's wage is greater when

$50\,000 + (n-1) \times 5000 < 40\,000(1.15)^{n-1}$

Trial and error finds that the lowest value of n to satisfy this inequality is $n = 6$. For Julian

$T_6 \doteqdot \$80\,454.29$

For Lawrence

$T_6 = \$75\,000$

Hence the difference is \$5454.29 which is \$5454 to the nearest dollar.

7a i This is an AP with $a = 50\ 000$ and $d = 3000$. Hence

$$T = a + (n-1)d$$
$$= 50\ 000 + (n-1) \times 3000$$
$$= 50\ 000 + 3000n - 3000$$
$$= 47\ 000 + 3000n$$

7a ii To have at least twice the original salary we must have

$T_n > 100\ 000$

$47\ 000 + 3000n > 100\ 000$

$3000n > 53\ 000$

$n > 17.66 \ldots$

Hence, the salary will be at least twice the original salary after the 18th year.

7b This describes an AP with $a = 50\ 000$ and $r = 1.04$.

Hence $T_n = 50\ 000(1.04)^{n-1}$. Hence the salary after the 10th year will be

$T_{10} = 50\ 000(1.04)^{10-1} = 50\ 000(1.04)^9 = \$71\ 166$

8a To the first trough and return: $6 + 6 = 12$m

To the second trough and return: $6 + 5 + 5 + 6 = 22$m

To the third trough and return: $6 + 5 + 5 + 5 + 5 + 6 = 32$m

8b Observing that we have an AP with $a = 12$ and $d = 22 - 12 = 10$, we conclude that:

$T_n = a + (n-1)d = 12 + (n-1) \times 10 = 12 + 10n - 10 = 10n + 2$

8c i To find the number of troughs, we solve the equation $T_n = 62$

$10n + 2 = 62$

$10n = 60$

$n = 6$

Hence there are 6 feed troughs.

8c ii The total distance travelled to feed n troughs will be

$S_n = \frac{1}{2}n(2a + (n-1)d) = \frac{1}{2}n(2 \times 12 + (n-1) \times 10)$.

Thus in order to fill 6 troughs, the total distance to be travelled is

$S_6 = \frac{1}{2}(6)(24 + (6-1) \times 10) = 222$ metres

9a Note that as $120 \div 7 = 17.14$ there will be 17 Sundays between the initial advertisement and Christmas. Thus, there are a total of 18 advertisements.

9b The first advertisement is published for 120 days and each subsequent advertisement will be published for 7 days less than the previous. Hence the number of days each advertisement is published for is given by an AP with $a = 120$ and $d = -7$. Hence the total number of days that all advertisements are published for will be given by the sum

$$S_n = \frac{1}{2}(n)(2a + (n-1)d) = \frac{1}{2}(n)\big(2 \times 120 + (n-1)(-7)\big)$$

Hence

$S_{18} = \frac{1}{2}(18)\big(240 + (18-1)(-7)\big) = 1089$ days

9c The last advertisement is $7 \times 17 = 119$ days after the first advertisement. Christmas is 120 days after the first advertisement, so Christmas must be on the next day which is a Monday.

10a Let x be the number of infections on 1st August. As the number of infections forms a AP, all terms must have a common difference and so

$x - 10\,000 = 160\,000 - x$

Hence

$2x = 170\,000$

And thus

$x = 85\,000$

So there are 85 000 infections on 1st August.

10b Let x be the number of infections on 1st August. As the number of infections forms a GP, all terms must have a common ratio and so

$$\frac{x}{10\ 000} = \frac{160\ 000}{x}$$

Hence

$x^2 = 1\ 600\ 000\ 000$

And thus

$x = 40\ 000$

So there are 40 000 infections on 1st August.

11a If the salary increases by a fixed amount then the salary forms an AP with

$T_n = 60\ 000 + (n-1)D$. Hence if $T_{10} = 117\ 600$ then

$60\ 000 + (10-1)D = 117\ 600$

$9D = 57600$

$D = 6400$

11b If the salary increases by a fixed amount then the salary forms an AP with

$T_n = 60\ 000 + (n-1)D$ and

$S_n = \frac{1}{2}(n)(2a + (n-1)D) = \frac{1}{2}(n)(2 \times 60\ 000 + (n-1)D)$.

Hence if $S_{10} = 942\ 000$

$$942\ 000 = \frac{1}{2}(10)(2 \times 60\ 000 + (10-1)D)$$

$188\ 400 = 120\ 000 + 9D$

$9D = 68\ 400$

$D = 7600$

11c If $D = 4400$ then the salary forms an AP with

$T_n = 60\ 000 + (n-1) \times 4400 = 4400n + 55600$

The salary exceeds \$120 000 when

$T_n > 120\ 000$

$4400n + 55600 > 120\ 000$

$4400n > 64400$

$n > 14.6$

Hence the salary first exceeds \$120 000 after the 15th year.

11d If $D = 4000$ then the salary forms an AP with $a = 60\ 000$ and $d = 4000$. Hence

$T_n = 60\ 000 + (n-1) \times 4000 = 4000n + 56\ 000$

and

$S_n = \frac{1}{2}(n)(2a + (n-1)d) = \frac{1}{2}(n)(2 \times 60\ 000 + (n-1) \times 4000)$

So

$S_{13} = \$1\ 092\ 000$

and

$S_{14} = \$1\ 204\ 000$

So the total earnings first exceed \$1 200 000 during the 14th year.

12 This is a GP with $a = F$ and $T_5 = \frac{1}{2}F$. Since $T_n = ar^{n-1}$ it follows that:

$\frac{1}{2}F = Fr^{5-1}$

$\frac{1}{2} = r^4$

$r = \frac{1}{2^{\frac{1}{4}}}$

Over time, the limiting sum will be $S_\infty = \frac{a}{1-r^n} = \frac{F}{1-\frac{1}{2^{\frac{1}{4}}}} \doteqdot 6.29F$ (2 decimal places)

13a i The common ratio of this sequence is $r = -\tan^2 x$. The sequence converges when $|r| < 1$. That is, when $|\tan^2 x| < 1$.

This is when $-1 < \tan x < 1$ which is when $0 < |x| < \frac{\pi}{4}$ in the given domain.

13a ii When the series does converge, the limit is

$$S_\infty = \frac{a}{1-r} = \frac{1}{1-(-\tan^2 x)} = \frac{1}{1+\tan^2 x} = \frac{1}{\sec^2 x} = \cos^2 x$$

13a iii When $\sin x = 0$, the series is not a GP because the ratio cannot be zero. But the series is then $1 + 0 + 0 + \ldots$, which trivially converges to 1. When $\sin x = 0$, then $\cos x = 0$ or -1, so $\sec^2 x = 0$, which means that the given formula for S_∞ is still correct

13b i

$$\frac{T_2}{T_1} = \frac{\cos^2 x}{1} = \cos^2 x$$

$$\frac{T_3}{T_2} = \frac{\cos^4 x}{\cos^2 x} = \cos^2 x$$

Thus as all terms have a common ratio of $\cos^2 x$.

13b ii The angles do not converge when

$|r| \geq 1$

$|\cos^2 x| \geq 1$

$\cos^2 x = 1$

$\cos x = \pm 1$

$x = 0, \pi, 2\pi$

13b iii $S_\infty = \frac{a}{1-r} = \frac{1}{1-\cos^2 x} = \frac{1}{\sin^2 x} = \operatorname{cosec}^2 x$

13b iv When $\cos x = 0$, the series is not a GP because the ratio cannot be zero. But the series is then $1 + 0 + 0 + \cdots$, which trivially converges to 1. When $\cos x = 0$, then $\sin x = 1$ or -1, so $\operatorname{cosec}^2 x = 1$, which means that the given formula for S_∞ is still correct.

13c i

$$\frac{T_2}{T_1} = \frac{\sin^2 x}{1} = \sin^2 x$$

$$\frac{T_3}{T_2} = \frac{\sin^4 x}{\sin^2 x} = \sin^2 x$$

Thus, as all terms have a common ratio of $\sin^2 x$.

13c ii The angles do not converge when

$|r| \geq 1$

$|\sin^2 x| \geq 1$

$\sin^2 x = 1$

$\sin x = \pm 1$

$$x = \frac{\pi}{2}, \frac{3\pi}{2}$$

13c iii $S_\infty = \frac{a}{1-r} = \frac{1}{1-\sin^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$

13c iv When sin x = 0, the series is not a GP because the ratio cannot be zero. But the series is then $1 + 0 + 0 + \ldots$, which trivially converges to 1. When $\sin x = 0$, then $\cos x = 1$ or -1, so $\sec^2 x = 1$, which means that the given formula for S_∞ is still correct.

14a Let $x_{\text{bee}} = 2Vt$ be the position of the bee and $x_{\text{dozer}} = 36 - Vt$ be the position of bulldozer B at a given time t. The bee reaches the bulldozer when they have the same x-position. This is when

$x_{\text{bee}} = x_{\text{dozer}}$

$2Vt = 36 - Vt$

$36 = 3Vt$

$$t = \frac{12}{V}$$

Hence, $x_{\text{dozer}} = 36 - Vt = 36 - V\left(\frac{12}{V}\right) = 36 - 12 = 24$. Thus the bee reaches the bulldozer B when it is at $x = 24$.

14b Firstly note that $x_A = Vt$, hence, when the bee hits bulldozer B, $x_A = V\left(\frac{12}{V}\right) = 12$. For the next part of the question, we shall consider $t = 0$ to be when the bee hits bulldozer A.

Hence the new equation for bulldozer A will be $x_A = 12 + Vt$ and $x_{bee} = 24 - 2Vt$. Thus, bulldozer A hits the bee when

$x_A = x_{bee}$

$12 + Vt = 24 - 2Vt$

$3Vt = 12$

$t = \dfrac{4}{V}$

$x_A = 12 + V\left(\dfrac{4}{V}\right) = 12 + 4 = 16$

Hence, they hit each other at $x = 16$.

14c i The bee will keep being able to fly between the two bulldozers until the bulldozers hit one another. As both bulldozers are travelling at the same speed, they will hit one another at the midpoint between them, $x = 18$.

14c ii The total time that the bee flies for is the time taken for the bulldozers to intercept. This time is given by solving the equation

$x_a = Vt$

$18 = Vt$

$t = \dfrac{18}{V}$

Thus, the total distance travelled by the bee is given by

distance = speed × time $= 2V\left(\frac{18}{V}\right) = 36$ metres.

This is the original distance between the two bulldozers.

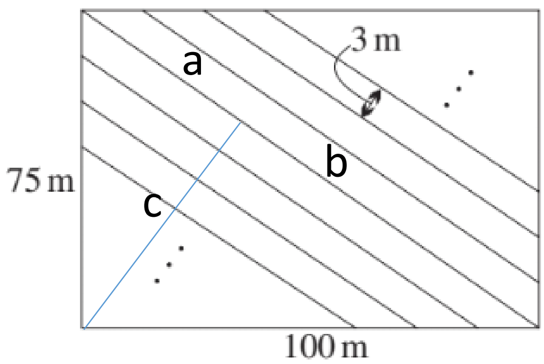
15a Using Pythagoras' Theorem:

Length of diagonal

$= \sqrt{75^2 + 100^2}$

$= 125 \text{ m}$

15b



Using Pythagoras' Theorem:

$a^2 + c^2 = 75^2$

$b^2 + c^2 = 100^2$

Subtracting these two equations gives:

$b^2 - a^2 = 100^2 - 75^2$

$(b - a)(b + a) = 4735$

$(b - a) \times 125 = 4735$

$b - a = 35$

Noting that $a + b = 125$,

we have that $b = 80$ and $a = 45$.

Substituting this back into $a^2 + c^2 = 75^2$gives $c = 60$.

Now using similar triangles we have that the length of the row on either side of the diagonal will satisfy the equation

$$\frac{l}{125} = \frac{60 - 3}{60}$$

$$l = 125 \times \frac{57}{60}$$

$$= 118.75 \text{ m}$$

15c Using similar triangles we have that the length of the row on either side of the diagonal will satisfy the equation

$$\frac{l}{125} = \frac{60 - 6}{60}$$

$$l = 125 \times \frac{53}{60}$$

$$= 112.5 \text{ m}$$

15d $125 - 118.75 = 118.75 - 112.5 = 6.25$

Hence the lengths form an arithmetic sequence with $a = 125$ and $d = 6.25$.

15e There will be $n = \frac{60}{3} = 20$ rows on one 'side' of the paddock.

For one side of the paddock the total length of vines will be

$$S_n = \frac{n}{2}(a + l)$$

$$= \frac{20}{2}(125 + 0)$$

$$= 1250$$

Thus the total length of all rows of vines will be $2 \times 1250 - 125 = 2375$ m.

Solutions to Exercise 14B

1a 5

1b 14

1c 3

1d 15

1e 4

1f 8

1g 14

1h 11

2a $2^n > 7000$

$\log_2 2^n > \log_2 7000$

$n > \dfrac{\ln 7000}{\ln 2}$

$n > \dfrac{\ln 7000}{\ln 2}$

$n > 12.77$

The smallest integer solution is $n = 13$.

2b $3^n > 20\ 000$

$\log_3 3^n > \log_3 20\ 000$

$n > \dfrac{\ln 20000}{\ln 3}$

$n > 9.01$

The smallest integer solution is $n = 10$.

2c $\left(\frac{1}{2}\right)^n < 0.004$

$\log_{\frac{1}{2}}\left(\frac{1}{2}\right)^n > \log_{\frac{1}{2}} 0.004$

$n > \dfrac{\ln 0.004}{\ln \frac{1}{2}}$

$n > 7.97$

Thus the smallest integer solution is $n = 8$.

2d $\left(\frac{1}{3}\right)^n < 0.0002$

$\log_{\frac{1}{3}}\left(\frac{1}{3}\right)^n > \log_{\frac{1}{3}} 0.0002$

$n > \dfrac{\ln 0.0002}{\ln \frac{1}{3}}$

$n > 7.75$

Thus the smallest integer solution is $n = 8$.

3a

$\dfrac{T_2}{T_1} = \dfrac{11}{10} = 1.1$

$\dfrac{T_3}{T_2} = \dfrac{12.1}{11} = 1.1$

Hence all terms have a common ratio of 1.1 and thus this sequence forms a GP with $a = 10$ and $r = 1.1$.

3b $a = 10$ and $r = 1.1$

3c $T_n = ar^{n-1} = 10(1.1)^{n-1}$, hence $T_{15} = 10(1.1)^{15-1} = 10(1.1)^{14} \doteqdot 37.97$

3d 19 ($T_{18} \doteqdot 55.6$ and $T_{19} \doteqdot 61.16$)

3e $10(1.1)^{n-1} < 60$

$1.1^{n-1} < 6$

$n - 1 < \frac{\ln 6}{\ln 1.1}$

$n < \frac{\ln 6}{\ln 1.1} + 1$

$n < 19.80$

Hence there are 19 terms which satisfy this inequality so there are 19 terms less than 60.

4a By definition her salary is a GP as a 5% increase means that each year her salary will be the previous multiplied by 1.05 . That the salary between each year has a common ratio of $r = 1.05$.

4b This is a GP with $a = 40\ 000$ and $r = 1.05$.

Thus $T_n = ar^{n-1} = 40\ 000(1.05)^{n-1}$ and so her annual salary after 10 years is

$T_{10} = 40\ 000(1.05)^{10-1} = 40\ 000(1.05)^9 = \$62\ 053$

Furthermore $S_n = \frac{a(r^n-1)}{r-1} = \frac{40\ 000(1.05^n-1)}{1.05-1}$ so her total earnings after 10 years will be

$S_{10} = \frac{40\ 000(1.05^{10} - 1)}{1.05 - 1} = \$503\ 116$

4c Her salary first exceeds \$70 000 when $T_n > 70\ 000$ this is when

$40\ 000(1.05)^{n-1} > 70\ 000$

$(1.05)^{n-1} > 1.75$

$n - 1 > \frac{\ln 1.75}{\ln 1.05}$

$n > \frac{\ln 1.75}{\ln 1.05} + 1$

$n > 12.45$

$T_{13} = 40\ 000(1.05)^{13-1} = 40\ 000(1.05)^{12} = \$71\ 834$

Hence her salary first exceeds \$70 000 after the 13th year.

5 The salary is given by a GP with $a = 50\ 000$ and $r = 1.04$ so the salary after the nth year will be $T_n = 50\ 000(1.04)^{n-1}$. To have twice the original salary we must have

$T_n > 100\ 000$

$50\ 000(1.04)^{n-1} > 100\ 000$

$(1.04)^{n-1} > 2$

$n - 1 > \dfrac{\ln 2}{\ln 1.04}$

$n > \dfrac{\ln 2}{\ln 1.04} + 1$

$n > 18.67$

$T_{19} = 50\ 000(1.04)^{19-1} = 50\ 000(1.04)^{18} = \$101\ 291$

Thus it will be twice the original salary in the 19th year.

6a SC 50: $100\% - 50\% = 50\%$

SC 75: $100\% - 75\% = 25\%$

SC 90: $100\% - 90\% = 10\%$

6b The first layer of SC 50 stops 50% of UV rays. The second layer then removes a further 50% of the 50% that have passed through. This means the second layer stops $50\% \times 50\% = 25\%$ of the total UV. Together the first and second layer block $50\% + 25\% = 75\%$ of the UV which is the same as that of SC 75.

6c By the same logic as above the nth SC 50 shade sail will block $(50\%)^n = (0.5)^n$ of the total sunlight. This means the around of sunlight blocked by the nth sail is a GP with $a = 0.5$ and $r = 0.5$ so the total around blocked by n sails is

$S_n = \dfrac{0.5(0.5^n - 1)}{0.5 - 1} = 1 - 0.5^n$

Hence to cut out 90% of rays we require

$S_n > 0.9$

$1 - 0.5^n > 0.9$

$0.5^n < 0.1$

$n > \dfrac{\ln 0.1}{\ln 0.5}$

$n > 3.32$

This means that at least 4 SC 50 shade sails are required.

6d By the same logic as above the nth SC 50 shade sail will block $(50\%)^n = (0.5)^n$ of the total sunlight. This means the around of sunlight blocked by the nth sail is a GP with $a = 0.5$ and $r = 0.5$ so the total around blocked by n sails is

$S_n = \dfrac{0.5(0.5^n - 1)}{0.5 - 1} = 1 - 0.5^n$

Hence to cut out 99% of rays we require

$S_n > 0.99$

$1 - 0.5^n > 0.99$

$0.5^n < 0.01$

$n > \dfrac{\ln 0.01}{\ln 0.5}$

$n > 6.64$

This means that at least 7 SC 50 shade sails are required.

7a $r = \frac{T_2}{T_1} = \frac{2}{3}$

This is a GP with $a = 3$ and $r = \frac{2}{3}$ so $T_n = 3 \times \left(\frac{2}{3}\right)^{n-1}$

7b The ball will have travelled the height of the roof on the '0th' bounce. Hence the height of the roof is. $T_0 = 3 \times \left(\frac{2}{3}\right)^{-1} = \frac{3}{\left(\frac{2}{3}\right)} = \frac{3}{2}(3) = 4.5$ metres

7c i If $T_n < 0.01$ then

$3 \times \left(\dfrac{2}{3}\right)^{n-1} < 0.01$

$3 \times \left(\dfrac{2}{3}\right)^{n-1} < \dfrac{1}{100}$

$$\left(\frac{2}{3}\right)^{n-1} < \frac{1}{300}$$

$$\frac{1}{\left(\frac{2}{3}\right)^{n-1}} > 300$$

$$\left(\frac{3}{2}\right)^{n-1} > 300$$

as required

7c ii Solving the inequality above gives

$$\left(\frac{3}{2}\right)^{n-1} > 300$$

$$n - 1 > \frac{\ln 300}{\ln \frac{3}{2}}$$

$$n > \frac{\ln 300}{\ln \frac{3}{2}} + 1$$

$$n > 15.07$$

Hence it will have bounced 16 times.

8a There are $20\,000 \times 0.10 = 2\,000$ graphics calculators sold per month.

8b The number of graphics calculators sold forms an AP with $a = 2\,000$ and $d = 150$.

$$\begin{aligned}\text{Thus } T_n &= a + (n-1)d \\ &= 2000 + (n-1)(150) \\ &= 1850 + 150n\end{aligned}$$

This means that 6 months from now there will be $T_6 = 2750$ graphics calculators sold.

8c All calculators will be graphics calculators when $T_n = 20\,000$, hence, this is when $1850 + 150n = 20\,000$. This is when $150n = 18150$ and thus when $n = 121$. This will be after 10 years and 1 month.

9a Annual sales form a GP with $a = 200\,000$ and $r = 1.2$.

Hence $T_n = 200\,000(1.2)^{n-1}$. The annual sales exceed \$1 000 000 when

$$200\,000(1.2)^{n-1} > 1\,000\,000$$

$$(1.2)^{n-1} > 5$$

$$n - 1 > \frac{\ln 5}{\ln 1.2}$$

$$n > \frac{\ln 5}{\ln 1.2} + 1$$

$$n > 9.83$$

Thus annual sales exceed \$1 000 000 in the 10th year.

9b Annual sales form a GP with $a = 200\,000$ and $r = 1.2$.

Hence $S_n = \frac{200\,000((1.2)^n - 1)}{1.2 - 1}$. The total sales exceed \$2 000 000 when

$$\frac{200\,000((1.2)^n - 1)}{1.2 - 1} > 2\,000\,000$$

$$\frac{200\,000((1.2)^n - 1)}{0.2} > 2\,000\,000$$

$$200\,000((1.2)^n - 1) > 400\,000$$

$$(1.2)^n - 1 > 2$$

$$(1.2)^n > 3$$

$$n > \frac{\ln 3}{\ln 1.2}$$

$$n > 6.03$$

Thus total sales exceed \$2 000 000 in the 7th year.

10a Increasing by 100% means doubling, increasing by 200% means trebling, increasing by 300% means multiplying by 4, and so on.

10b Solve $(1.25)^n > 4$.

$n > \dfrac{\ln 4}{\ln 1.25}$

$n > 6.21$

The smallest integer solution is $n = 7$.

11a This is a GP with $a = 3$ and $r = \frac{2}{3}$, thus

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$= \frac{3\left(\left(\frac{2}{3}\right)^n - 1\right)}{\frac{2}{3} - 1}$$

$$= \frac{3\left(\left(\frac{2}{3}\right)^n - 1\right)}{-\frac{1}{3}}$$

$$= 9\left(1 - \left(\frac{2}{3}\right)^n\right)$$

11b $|r| = \left|\frac{2}{3}\right| < 1$, hence there is a limiting sum as the common ratio is less than 1.

$$S = \frac{a}{1 - r} = \frac{3}{1 - \frac{2}{3}} = \frac{3}{\frac{1}{3}} = 9$$

11c $S - S_n < 0.01$

$$9 - 9\left(1 - \left(\frac{2}{3}\right)^n\right) < 0.01$$

$$9 - 9 + 9\left(\frac{2}{3}\right)^n < 0.01$$

$$9\left(\frac{2}{3}\right)^n < 0.01$$

$$\left(\frac{2}{3}\right)^n < \frac{0.01}{9}$$

$\ln\left(\frac{2}{3}\right)^n < \ln\frac{0.01}{9}$

$n\ln\left(\frac{2}{3}\right) < \ln\frac{0.01}{9}$

$n > \frac{\ln\frac{0.01}{9}}{\ln\left(\frac{2}{3}\right)}$ (note we switch the inequality sign as $\ln\left(\frac{2}{3}\right) < 0$)

$n > 16.78$

Hence the smallest value of n for which $S - S_n < 0.01$ is $n = 17$.

12a $A = \frac{1}{2}bh = \frac{1}{2} \times \cos\theta \times \sin\theta$

12b For the second triangle, the base satisfies $\cos\theta = \frac{b}{\sin\theta}$ and the height satisfies $\sin\theta = \frac{a}{\sin\theta}$. Thus the area of the second triangle is $A = \frac{1}{2}bh = \frac{1}{2}(\cos\theta\sin\theta)(\sin\theta\sin\theta)$. Hence, the ratio of areas is $\frac{\frac{1}{2}(\cos\theta\sin\theta)(\sin\theta\sin\theta).}{\frac{1}{2}\times\cos\theta\times\sin\theta} = \sin^2\theta$.

12b The areas form a GP with $a = \frac{1}{2} \times \cos\theta \times \sin\theta$ and $r = \sin^2\theta$. The limiting sum is thus $S_\infty = \frac{a}{1-r} = \frac{\frac{1}{2}\times\cos\theta\times\sin\theta}{1-\sin^2\theta} = \frac{\frac{1}{2}\times\cos\theta\times\sin\theta}{\cos^2\theta} = \frac{\frac{1}{2}\sin\theta}{\cos\theta} = \frac{1}{2}\tan\theta$.

13a By observation $\theta_n = \frac{1}{\sqrt{n}}$

13b

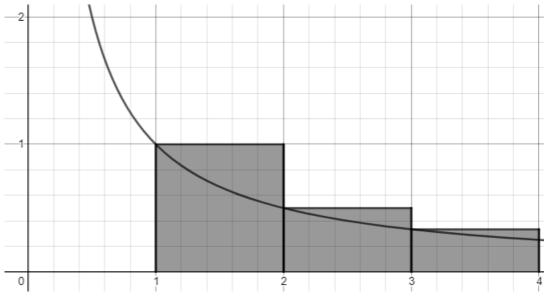
$$\sum_{n=1}^{k}\theta_n \geq \sum_{n=1}^{k}\frac{1}{2}\tan\theta_n$$

$$= \sum_{n=1}^{k}\frac{1}{2}\tan\frac{1}{\sqrt{n}}$$

$$\geq \sum_{n=1}^{k}\frac{1}{2}\frac{1}{n}$$

$$\geq \frac{1}{2}\sum_{n=1}^{k}\frac{1}{n}$$

13c



By observation of the above diagram

$$\sum_{n=1}^{k}\frac{1}{n} \geq \int_{1}^{k}\frac{1}{n}\,dn$$

13d Evaluating the integral gives $\int_1^k \frac{1}{n}\,dn = \ln k$, since $\ln k$ is unbounded, it follows by comparison that $\sum_{n=1}^{k}\frac{1}{n}$ and in turn $\sum_{n=1}^{k}\theta_n$ must also be unbounded. Thus we conclude that the spiral keeps turning without bound.

Solutions to Exercise 14C

1a i $\quad I = PRn = 5000(0.06)(3) = \$900$

1a ii $\quad A = P + I = 5000 + 900 = \$5900$

1b i $\quad I = PRn = 12000(0.0615)(7) = \$5166$

1b ii $\quad A = P + I = 12000 + 5166 = \$17166$

2a i $\quad A = P(1+R)^n = 5000(1.06)^{(3)} = \$5955.08$

2a ii $\quad I = A - P = 5955.08 - 5000 = \$955.08$

2b i $\quad A = P(1+R)^n = 12\,000(1.0615)^{(7)} = \$18\,223.06$

2b ii $\quad I = A - P = 18\,223.06 - 12\,000 = \$6223.06$

3a i $\quad A = P(1-R)^n = 5000(1-0.06)^{(3)} = 5000(0.94)^{(3)} = \$4152.92$

3a ii $\quad D = P - A = 5000 - 4152.92 = \$847.08$

3b i $\quad A = P(1-R)^n = 12\,000(1-0.0615)^{(7)} = 12\,000(0.9385)^{(7)} = \$7695.22$

3b ii $\quad D = P - A = 12\,000 - 7695.22 = \$4304.78$

4a $\quad A = P(1+R)^n = P\left(1 + \frac{12}{100\times 12}\right)^{2\times 12} = 400(1.01)^{12} = \$507.89$

4b $A = P(1+R)^n = P\left(1+\frac{7.28}{100\times52}\right)^{1\times52} = 10000\left(1+\frac{7.28}{100\times52}\right)^{52} = \$10\,754.61$

5a $A_n = P + I = P + PRn = P(1+Rn) = 10\,000(1+0.065\times n)$

5b $A_{15} = \$19\,750$, $A_{16} = \$20400$, hence the investment exceeds \$20 000 at the end of 16 years but not at the end of 15 years.

6a $A_n = P(1-R)^n = 229\,000(1-0.15)^n = 229\,000(0.85)^n$.

Hence the net worth of the fleet in 5 years will be $A_5 = \$101\,608.52$.

6b The loss in value will be $\$229\,000 - \$101\,608.52 = \$127\,391.48$

7a The final amount for Juno is

$A = P(1+R)^n = 20\,000(1+0.066)^1 = \$21\,320$

The final amount for Howard is

$A = P + PRn = 20\,000 + 20\,000(0.0675)(1) = \$21350$

So Howard has the better investment

7b The final amount for Juno is

$$A = P(1+R)^n = 20\,000\left(1+\frac{0.066}{12}\right)^{12} = \$21\,360.67$$

The final amount for Howard is

$A = P + PRn = 20\,000 + 20\,000(0.0675)(1) = \$21350$

So Juno has the better investment by \$10.67

8a $A = P + PRn = 5000 + 5000(0.07)(3) = \$6050$

8b Using the simple interest formula

$I = PRn$

$13\,824 = P(0.06)(9)$

Hence

$$P = \frac{13\,824}{0.06 \times 9} = \$25600$$

8c The amount of interest earned is

$I = A - P = 31\,222.50 - 23\,000 = \$8222.50$

Now using the simple interest formula

$I = PRn$

$8222.50 = 23\,000(0.0325)(n)$

Hence

$n = 11$

8d The total interest earned is

$I = A - P = 22\,610 - 17\,000 = \$5610$

Now using the simple interest formula

$I = PRn$

$5610 = 17\,000R(6)$

$$R = \frac{5610}{17\,000 \times 6} = 0.055$$

So the interest rate is 5.5%.

9a $A = P(1 + R)^n$

$32\,364 = P(1 + 0.15)^{10}$

$P = \frac{32\,364}{(1.15)^{10}} = \$8000$ (to the nearest dollar)

9b $A = P(1 + R)^n$

$40\,559.20 = P(1 + 0.07)^{18}$

$P = \frac{40\,559.20}{(1+0.07)^{18}} = \$12\,000$ (to the nearest dollar)

9c $A = P(1+R)^n$

$22\,884.96 = P\left(1+\frac{0.045}{12}\right)^{3\times 12}$

$P = \frac{22\,884.96}{\left(1+\frac{0.045}{12}\right)^{3\times 12}} = \$20\,000$ (to the nearest dollar)

10 $A = P(1-R)^n$

$14\,235 = P(1-0.107)^3$

$P = \frac{14\,235}{(1-0.107)^3} = \$19\,990$

11a $A = P(1+R)^n = 6000\left(1+\frac{0.0825}{12}\right)^{3\times 12} = \$7678.41$

11b $I = A - P = 7678.41 - 6000 = \$1678.41$

11c For simple interest

$I = PRn$

$1678.41 = 6000R(3)$

$R = 0.093245$

Hence a simple interest rate of 9.32% per annum is required to yield the same amount.

12a $A = P(1+R)^n = 10\,000\left(1+\frac{0.04}{12}\right)^{5\times 12} = \$12\,209.97$

12b The interest earned over the 5 years is

$I = A - P = 12\,209.97 - 10\,000 = \$2209.97$

For simple interest

$I = PRn$

$2209.97 = 10\,000R(5)$

$R = 0.044\,199\,4$

Hence a simple interest rate of 4.4% per annum is required to yield the same amount.

12c It will fully exceed \$15 000 when

$A > 15000$

$10\,000\left(1+\frac{0.04}{12}\right)^n > 15000$

$\left(1+\frac{0.04}{12}\right)^n > 1.5$

$n > \frac{\ln 1.5}{\ln\left(1+\frac{0.04}{12}\right)}$

$n > 121.84\ldots$

Hence the smallest integer solution is $n = 122$ months.

13 For depreciation

$A = P(1-R)^n$

$P(1-0.175)^6 = 350\,000$

Hence

$P = \frac{350\,000}{(1-0.175)^6} = \$1\,110\,054.631$

To the nearest dollar, the original value of the asset was \$1 110 000.

14 $A_n = Pr^n$

$A_6 = 45\,108.91,\ P = 30\,000$

$30\,000r^6 = 45\,108.91$

$r^6 = 1.5036$

$r = 1.5036^{\frac{1}{6}} = 1.07\ldots$

Hence the interest rate is 7% per annum.

15a In order for the investment to increase by a factor of 10, it must be the case that

$A_n \geq 60\,000$

$6000(1.12)^n \geq 60\,000$

$(1.12)^n \geq 10$

$n \geq \dfrac{\ln 10}{\ln 1.12}$

$n \geq 20.32$

Hence the smallest number of years required for the investment to double is 21 years.

15b $A_n = P(1+R)^n = 100\,000\left(1 + \frac{0.0825}{12}\right)^n = 100\,000(1.006875)^n$ where n is in months

In order for the investment to double, it must be the case that

$A_n \geq 200\,000$

$100\,000(1.006875)^n \geq 200\,000$

$(1.006875)^n \geq 2$

$n \geq \dfrac{\ln 2}{\ln 1.006875}$

$n \geq 101.17$

Hence the smallest number of years required for the investment to double is 102 months. This is 8 years and 6 months.

15c $A_n = P(1-0.15)^n = P \times 0.85^n$

In order for the car to be less than 10% of its initial cost we require $A_n < 0.1P$

$P \times 0.85^n < 0.1P$

$0.85^n < 0.1$

$n > \log_{0.85} 0.1$

$n > \dfrac{\ln 0.1}{\ln 0.85}$

$n > 14.17$

Thus, the value will first be less than 10% of the cost after 15 years.

16 $1.015^n \doteqdot 1.1956$

$\ln 1.015^n \doteqdot \ln 1.1956$

$n \ln 1.015 \doteqdot \ln 1.1956$

$n \doteqdot \dfrac{\ln 1.1956}{\ln 1.015} \doteqdot 11.999$

Hence there are 12 quarters (3 years) in the period of investment.

17 Thirwin $A = 10\,000(1 + 0.072)^1 = \$10\,720$

Neri $A = 10\,000(1 + 0.072) \times 1 = \$10\,720$

Sid $A = 10\,000\left(1 + \frac{0.07}{12}\right)^{12} = \$10\,722.90$

Nee $A = 10\,000(1 + 0.081) - 50 - 50 = \$10\,710$

Thus Sid is furthest ahead after one year.

18a $I = PRn = I = 15\,000 \times 0.07 \times 5 = \$5250$

18b $A = P + I = 15\,000 + 5250 = \$20250$

18c $A = P(1 + R)^n$

$20250 = 15000(1 + R)^5$

$(1 + R)^5 = 1.35$

$(1 + R) = 1.35^{\frac{1}{5}}$

$R = 1.35^{\frac{1}{5}} - 1$

$R = 0.0619 = 6.19\%$ per annum

19a $A = P(1 - R)^n$

$A = 54\ 391.22(1 - 0.09)^3 = \$40\ 988$

19b $A = P(1 + R)^n$

$54\ 391.22 = P(1 + 0.09)^3$

$P = \dfrac{54\ 391.22}{(1 + 0.09)^3} = \$42\ 000$

20a i $A = 1000\left(1 + \frac{0.12}{1}\right)^1 = \$1120$

20a ii $A = 1000\left(1 + \frac{0.12}{4}\right)^4 = \$1125.51$

20a iii $A = 1000\left(1 + \frac{0.12}{12}\right)^{12} = \$1126.83$

20a iv $A = 1000\left(1 + \frac{0.12}{365}\right)^{365} = \$1127.47$

20b If the compounding were continuous we would have

$$A = \lim_{n\to\infty} 1000\left(1 + \frac{0.12}{n}\right)^n = 1000e^{0.12} = 1127.50$$

20c For 10 years compounding annually $A = 1000\left(1 + \frac{0.12}{1}\right)^{10} = \$1120$

For 10 years compounding continuously

$$A = \lim_{n\to\infty} 1000\left(1 + \frac{0.12}{n}\right)^{10n}$$

$$= 1000\left(\lim_{n\to\infty}\left(1 + \frac{0.12}{n}\right)^n\right)^{10}$$

$$= 1000(e^{0.12})^{10}$$

$$= \$3320.12$$

21a $A_n = P(1+R)^n$

21b By the binomial theorem

$$\begin{aligned} A_n &= P(1+R)^n \\ &= P\sum_{k=0}^{n}\binom{n}{k}R^k1^{n-k} \\ &= P\left(\binom{n}{0}R^0 1^{n-0} + \binom{n}{1}R^1 1^{n-1} + \sum_{k=2}^{n}\binom{n}{k}R^k1^{n-k}\right) \\ &= P\left(1 + nR + \sum_{k=2}^{n}\binom{n}{k}R^k1^{n-k}\right) \\ &= P + PRn + P\sum_{k=2}^{n}\binom{n}{k}R^k1^{n-k} \end{aligned}$$

21c P is the principal, PRn is the simple interest and $\sum_{k=2}^{n}\binom{n}{k}R^k1^{n-k}$ is the result of compound interest over and above simple interest.

Solutions to Exercise 14D

1a i On the 31st December 2023, the first instalment will have been compounded 4 times, hence the value of the first instalment is $500 \times 1.1^4 = \$732.05$.

1a ii On the 31st December 2023, the second instalment will have been compounded 3 times, hence the value of the second instalment is $500 \times 1.1^3 = \$665.50$.

1a iii On the 31st December 2023, the third instalment will have been compounded 2 times, hence the value of the third instalment is $500 \times 1.1^2 = \$605$.

1a iv On the 31st December 2023, the fourth instalment will have been compounded once, hence the value of the fourth instalment is $500 \times 1.1 = \$550$.

1a v The total value will be $732.05 + 665.50 + 605 + 550 = \$2552.55$

1b i \$550, \$605, \$665.50, \$732.05. These terms form a GP with common ratio 1.1 as

$$\frac{732.05}{665.50} = \frac{665.50}{605} = \frac{605}{550} = 1.1$$

1b ii The first term is 550, the common ratio is 1.1 and there are 4 terms.

1b iii $S_n = \frac{a(r^n-1)}{r-1} = \frac{550(1.1^4-1)}{1.1-1} = \$2552.55$ which matches the answer in part **a v**.

2a i On the 31st March 2020, the first instalment will have been compounded 5 times, hence the value of the first instalment is $1200 \times 1.05^5 = \$1531.54$.

2a ii On the 31st March 2020, the second instalment will have been compounded 4 times, hence the value of the second instalment is $1200 \times 1.05^4 = \$1458.61$.

2a iii On the 31st March 2020, the third instalment will have been compounded 3 times, hence the value of the third instalment is $1200 \times 1.05^3 = \$1389.15$.

On the 31st March 2020, the fourth instalment will have been compounded 2 times, hence the value of the fourth instalment is $1200 \times 1.05^2 = \$1323$.

On the 31st March 2020, the fifth instalment will have been compounded 1 time, hence the value of the fifth instalment is $1200 \times 1.05^1 = \$1260$.

2a iv The total value of the superannuation is

$1531.54 + 1458.61 + 1389.15 + 1389.15 + 1323 + 1260 = \$6962.30$

2b i $1260, $1323, $1389.15, $1458.61, $1531.54

These terms form a GP with common ratio 1.05 as

$$\frac{1531.54}{1458.61} = \frac{1458.61}{1389.15} = \frac{1389.15}{1323} = \frac{1323}{1260} = 1.05$$

2b ii The first term is $1260, the common ratio is 1.05 and there ae 5 terms

2b iii $S_n = \frac{a(r^n-1)}{r-1} = \frac{1260(1.05^4-1)}{1.05-1} = \$6962.30$ which matches the answer in part **a iv.**

3a i On the target date, the first instalment will have been compounded 15 times, hence the value of the first instalment is 1500×1.07^{15}.

3a ii On the target date, the second instalment will have been compounded 14 times, hence the value of the second instalment is 1500×1.07^{14}.

3a iii On the target date, the last instalment will have been compounded 1 time, hence the value of the last instalment is 1500×1.07.

3a iv The series for A_{15} will be given by adding the values for each of the instalments, hence

$A_{15} = (1500 \times 1.07) + (1500 \times 1.07^2) + \cdots + (1500 \times 1.07^{15})$

3b $A_{15} = \frac{a(r^n-1)}{r-1} = \frac{(1500\times1.07)(1.07^{15}-1)}{1.07-1} = \$40\ 332$

4a i On the target date, the first instalment will have been compounded 24 times, hence the value of the first instalment is $250 \times (1 + \frac{0.06}{12})^{24} = 250 \times 1.005^{24}$.

4a ii On the target date, the second instalment will have been compounded 23 times, hence the value of the second instalment is $250 \times (1 + \frac{0.06}{12})^{23} = 250 \times 1.005^{23}$.

4a iii On the target date, the last instalment will have been compounded 1 time, hence the value of the last instalment is $250 \times (1 + \frac{0.06}{12})^{1} = 250 \times 1.005$.

4a iv A series for A_{24}, given by adding the value of each instalment is

$A_{24} = 250 \times 1.005 + 250 \times 1.005^2 + \cdots + 250 \times 1.005^{24}$

4b The above series is a GP with $a = 250 \times 1.005$, $r = 1.005$ and with 24 terms. Hence

$$A_{24} = \frac{a(r^n-1)}{r-1} = \frac{250 \times 1.005(1.005^{24}-1)}{1.005-1} = \$6390$$

5a i At the end of 25 years, the first instalment will have been compounded 25 times, hence the value of the first instalment is $3000 \times (1 + 0.065)^{25} = 3000 \times 1.065^{25}$.

5a ii At the end of 25 years, the first instalment will have been compounded 24 times, hence the value of the first instalment is $3000 \times (1 + 0.065)^{24} = 3000 \times 1.065^{24}$.

5a iii At the end of 25 years, the last instalment will have been compounded 1 time, hence the value of the last instalment is $3000 \times (1 + 0.065)^{1} = 3000 \times 1.065$.

5a iv A series for A_{25}, given by adding the value of each instalment is

$A_{25} = 3000 \times 1.065 + 3000 \times 1.065^2 + \cdots 3000 \times 1.065^{25}$

5b The above series is a GP with $a = 3000 \times 1.065$, $r = 1.065$ and with 25 terms. Hence

$$A_{25} = \frac{a(r^n - 1)}{r - 1} = \frac{3000 \times 1.065(1.065^{25} - 1)}{1.065 - 1}$$

5c The value after 25 years will be

$$A_{25} = \frac{a(r^n - 1)}{r - 1} = \frac{3000 \times 1.065(1.065^{25} - 1)}{1.065 - 1} = \$188\ 146$$

The total amount contributed is $3000 \times 25 = \$75\ 000$

6a At the end of 20 years, the first instalment will have been compounded 20 times, hence the value of the first instalment is $12\ 000 \times (1 + 0.09)^{20} = 12\ 000 \times 1.09^{20}$.

At the end of 20 years, the second instalment will have been compounded 19 times, hence the value of the second instalment is $12\ 000 \times (1 + 0.09)^{19} = 12\ 000 \times 1.09^{19}$.

At the end of 20 years, the last instalment will have been compounded 1 time, hence the value of the last instalment is $12\ 000 \times (1 + 0.09)^{1} = 12\ 000 \times 1.09$.

From this we can see that adding all contributions together, we will get

$$A_{20} = 12\ 000 \times 1.09 + 12\ 000 \times 1.09^2 + \cdots + 12\ 000 \times 1.09^{20}$$

This is a GP with $a = 12\ 000 \times 1.09$, $r = 1.09$ and 20 terms, hence

$$\begin{aligned} A_{20} &= S_{20} \\ &= \frac{a(r^n - 1)}{r - 1} \\ &= \frac{12\ 000 \times 1.09(1.09^{20} - 1)}{1.09 - 1} \\ &= \frac{12\ 000 \times 1.09(1.09^{20} - 1)}{0.09} \end{aligned}$$

6b $669 174.36

6c Zoya's total contributions are $12\ 000 \times 20 = \$240000$, hence this exceeds her total contributions by $669\ 174.36 - 240000 = \$429\ 174.36$.

6d At the end of 20 years, the first instalment will have been compounded 20 times, hence the value of the first instalment is $M \times (1 + 0.09)^{20} = M \times 1.09^{20}$.

At the end of 20 years, the second instalment will have been compounded 19 times, hence the value of the second instalment is $M \times (1 + 0.09)^{19} = M \times 1.09^{19}$.

At the end of 20 years, the last instalment will have been compounded 1 time, hence the value of the last instalment is $M \times (1 + 0.09)^1 = M \times 1.09$.

From this we can see that adding all contributions together, we will get

$A_{20} = M \times 1.09 + M \times 1.09^2 + \cdots + M \times 1.09^{20}$

This is a GP with $a = M \times 1.09$, $r = 1.09$ and 20 terms, hence

$$\begin{aligned} A_{20} &= S_{20} \\ &= \frac{a(r^n - 1)}{r - 1} \\ &= \frac{M \times 1.09(1.09^{20} - 1)}{1.09 - 1} \\ &= \frac{M \times 1.09(1.09^{20} - 1)}{0.09} \end{aligned}$$

6e To have \$1 000 000 at the end of 20 years

$A_{20} = 1\,000\,000$

$$\frac{M \times 1.09(1.09^{20} - 1)}{0.09} = 1\,000\,000$$

$$M = 1\,000\,000 \div \frac{1.09(1.09^{20} - 1)}{0.09} = \$17\,932.55$$

7a At the end of n years, the first instalment will have been compounded n times, hence the value of the first instalment is $M \times (1 + 0.075)^n = M \times 1.075^n$.

At the end of n years, the second instalment will have been compounded $n - 1$ times, hence the value of the second instalment is $M \times (1 + 0.075)^{n-1} = M \times 1.075^{n-1}$.

At the end of n years, the last instalment will have been compounded 1 time, hence the value of the last instalment is $M \times (1 + 0.075)^1 = M \times 1.075$.

From this we can see that adding all contributions together, we will get

$A_n = M \times 1.075 + M \times 1.075^2 + M \times 1.075^n$

This is a GP with $a = M \times 1.075$, $r = 1.075$ and n terms, hence

$$A_n = S_n$$
$$= \frac{a(r^n - 1)}{r - 1}$$
$$= \frac{M \times 1.075(1.075^n - 1)}{1.075}$$
$$= \frac{M \times 1.075(1.075^n - 1)}{0.075}$$

7b In order to have \$1 500 000 in 25 years' time,

$$A_{20} = 1\,500\,000$$

$$\frac{M \times 1.075(1.075^{25} - 1)}{0.075} = 1\,500\,000$$

$M = 1\,500\,000 \div \frac{1.075(1.075^{25}-1)}{0.075} =$ \$20 526.52 to the nearest cent

7c i In order to have superannuation more than \$750 000

$$A_n > 750\,000$$

$$\frac{M \times 1.075(1.075^n - 1)}{0.075} > 750\,000$$

$$\frac{20\,526.52 \times 1.075(1.075^n - 1)}{0.075} > 750\,000$$

$$1.075^n - 1 > \frac{750\,000 \times 0.075}{20\,526.52 \times 1.075}$$

$$1.075^n > \frac{750\,000 \times 0.075}{20\,526.52 \times 1.075} + 1$$

as required

7c ii By use of a calculator

$$\frac{750\,000 \times 0.075}{20\,526.52 \times 1.075} + 1 = 3.5492$$

Hence

$$1.075^n > 3.5492$$

7c iii $1.075^n > 3.5492$

$$n > \frac{\ln 3.5492}{\ln 1.075}$$

$$n > 17.52$$

The smallest integer solution to this is $n = 18$.

8a The person makes 20 investments of \$10 000 each. Hence the total investment made is $20 \times 10\,000 = \$200\,000$.

8b At the beginning of 2040, the first instalment will have been compounded 20 times, hence the value of the first instalment is

$$10\,000 \times (1 + 0.1)^{20} = 10\,000 \times 1.1^{20} = \$67\,275.$$

8c At the beginning of 2040, the first instalment will have been compounded 20 times, hence the value of the first instalment is $10\,000 \times (1 + 0.1)^{20} = 10\,000 \times 1.1^{20}$.

At the beginning of 2040, the second instalment will have been compounded 19 times, hence the value of the second instalment is $10\,000 \times (1 + 0.1)^{19} = 10\,000 \times 1.1^{19}$.

At the beginning of 2040, the last instalment will have been compounded 1 time, hence the value of the last instalment is $10\,000 \times (1 + 0.1)^1 = 10\,000 \times 1.1$.

From this we can see that adding all contributions together, we will get

$$A_{20} = 10\,000 \times 1.1 + 10\,000 \times 1.1^2 + 10\,000 \times 1.1^{20}$$

This is a GP with $a = 10\,000 \times 1.1$, $r = 1.1$ and 20 terms, hence

$$\begin{aligned} A_{20} &= S_{20} \\ &= \frac{a(r^n - 1)}{r - 1} \\ &= \frac{10\,000 \times 1.1(1.1^{20} - 1)}{1.1 - 1} \\ &= \frac{10\,000 \times 1.1(1.1^{20} - 1)}{0.1} \\ &= 630\,024.9944 \\ &= \$630\,025 \text{ (to the nearest dollar)} \end{aligned}$$

8d i Similarly to above

$$\begin{aligned} A_n &= S_n \\ &= \frac{a(r^n - 1)}{r - 1} \\ &= \frac{10\,000 \times 1.1(1.1^n - 1)}{1.1 - 1} \\ &= \frac{10\,000 \times 1.1(1.1^n - 1)}{0.1} \\ &= 100\,000 \times 1.1 \times (1.1^n - 1) \end{aligned}$$

8d ii The target is reached when

$$A_n > 1\,000\,000$$

$$100\,000 \times 1.1 \times (1.1^n - 1) > 1\,000\,000$$

$$1.1 \times (1.1^n - 1) > 10$$

$$(1.1^n - 1) > \frac{10}{1.1}$$

$$1.1^n > \frac{10}{1.1} + 1$$

8d iii

$$1.1^n > \frac{10}{1.1} + 1$$

$$n > \frac{\ln\left(\frac{10}{1.1} + 1\right)}{\ln 1.1}$$

$$n > 24.25$$

The smallest integer solution is $n = 25$ and hence it will take 25 years for the superannuation to be worth \$1 000 000

8e For a contribution of M, following the same method as above, we obtain

$$\begin{aligned} A_n &= S_n \\ &= \frac{a(r^n - 1)}{r - 1} \\ &= \frac{M \times 1.1(1.1^n - 1)}{1.1 - 1} \end{aligned}$$

$$= \frac{M \times 1.1(1.1^n - 1)}{0.1}$$
$$= 10M \times 1.1 \times (1.1^n - 1)$$
$$= 11M(1.1^n - 1)$$

Now, if $A_{20} > 1\,000\,000$, then

$11M(1.1^{20} - 1) > 1\,000\,000$

$$M > \frac{1\,000\,000}{11(1.1^{20} - 1)}$$

$M > 15872.29$

Hence the monthly contribution needs to be $M \doteqdot \$15\,872$

9a $18 \times 20 = \$360$

9b The values of the investments form a GP with $a = 20$ and $r = 1.095$. Note the first deposit occurs on Jane's "0" birthday, so there are 19 deposits. Hence, the total amount is

A_{19}

$$= S_{19}$$
$$= \frac{a(r^n - 1)}{r - 1}$$
$$= \frac{20(1.095^{19} - 1)}{1.095 - 1}$$
$$= \frac{20(1.095^{19} - 1)}{0.095}$$
$$= \$970.27$$

10a The first investment will be worth 5000×1.08^5, the second will be 5000×1.08^4 and so on. The most recent will be 5000×1.08^1. This forms a GP of 5 terms with $a = 5000 \times 1.08$ and $r = 1.08$. Hence, the total payout will be

A_5

$$= S_5$$
$$= \frac{a(r^n - 1)}{r - 1}$$
$$= \frac{5000 \times 1.08^1(1.08^5 - 1)}{1.08 - 1}$$

$= \dfrac{5000 \times 1.08^1(1.08^5 - 1)}{0.08}$
$\doteqdot \$31\ 680$ (to the nearest dollar)

10b The first investment will be worth 5000×1.08^{25}, the second will be 5000×1.08^{24} and so on. The most recent will be 5000×1.08^1. This forms a GP of 25 terms with $a = 5000 \times 1.08$ and $r = 1.08$. Hence, the total payout will be

A_{25}

$= S_{25}$
$= \dfrac{a(r^n - 1)}{r - 1}$
$= \dfrac{5000 \times 1.08^1(1.08^{25} - 1)}{1.08 - 1}$
$= \dfrac{5000 \times 1.08^1(1.08^{25} - 1)}{0.08}$
$\doteqdot \$394\ 772$ (to the nearest dollar)

10c The first investment will be worth 5000×1.08^{40}, the second will be 5000×1.08^{39} and so on. The most recent will be 5000×1.08^1. This forms a GP of 5 terms with $a = 5000 \times 1.08$ and $r = 1.08$. Hence, the total payout will be

A_{40}

$= S_{40}$
$= \dfrac{a(r^n - 1)}{r - 1}$
$= \dfrac{5000 \times 1.08^1(1.08^{40} - 1)}{1.08 - 1}$
$= \dfrac{5000 \times 1.08^1(1.08^{40} - 1)}{0.08}$
$\doteqdot \$1\ 398\ 905$ (to the nearest dollar)

11a The first payment will be cost 20 000, the second payment will cost $20\ 000 \times 1.045$ and so on. The 6th payment will cost $20\ 000 \times 1.045^5$. This forms a GP of 6 terms with $a = 20\ 000$ and $r = 1.045$. Hence, the total payout will be

A_6

$= S_6$
$= \dfrac{a(r^n - 1)}{r - 1}$

$= \dfrac{20\,000(1.045^6 - 1)}{1.045 - 1}$
$= \dfrac{20\,000(1.045^6 - 1)}{0.045}$
$\doteqdot \$134\,338$ (to the nearest dollar)

11b The first payment will be cost 20 000, the second payment will cost $20\,000 \times 1.045$ and so on. The 12th payment will cost $20\,000 \times 1.045^{11}$.

This forms a GP of 12 terms with $a = 20\,000$ and $r = 1.045$. Hence, the total payout will be

A_{12}

$= S_{12}$
$= \dfrac{a(r^n - 1)}{r - 1}$
$= \dfrac{20\,000(1.045^{12} - 1)}{1.045 - 1}$
$= \dfrac{20\,000(1.045^{12} - 1)}{0.045}$
$\doteqdot \$309\,281$ (to the nearest dollar)

12 Let M be the annual premium

$A_1 = 1.125 \times M$

$A_2 = 1.125(M + A_1) = 1.125(M + 1.125 \times M) = 1.125M + 1.125^2M$

Similarly, it follows that

$A_n = 1.125M + 1.125^2M + \cdots 1.125^nM$

The terms in this sequence form a GP with $a = 1.125M$ and $r = 1.125$. Hence

$$A_n = \frac{a(r^n - 1)}{r - 1} = \frac{1.125M\ (1.125^{\mathrm{n}} - 1)}{1.125 - 1} = \frac{1.125M\ (1.125^{\mathrm{n}} - 1)}{0.125}$$

In order to have \$500 000 after 25 years

$A_{25} = 500\,000$

$$\frac{1.125M(1.125^{25} - 1)}{0.125} = 500\,000$$

$$M = 500\,000 \div \frac{1.125(1.125^{25} - 1)}{0.125} = \$3086$$

13a At the end of the first year the value of the policy will be

$A_1 = 1.09 \times 500$

At the end of the second year the value will be

$$\begin{aligned} A_2 &= 1.09 \times (500 + A_1) = 1.09 \times (500 + 1.09 \times 500) \\ &= 1.09 \times 500 + 1.09^2 \times 500 \end{aligned}$$

Similarly, the value at the end of the nth year will be

$A_n = 1.09 \times 500 + 1.09^2 \times 500 + \cdots + 1.09^n \times 500$

Each term in the sum form a GP with $a = 1.09 \times 500$ and $r = 1.09$. Hence

$$A_n = \frac{a(r^n - 1)}{r - 1} = \frac{1.09 \times 500(1.09^n - 1)}{1.09 - 1} = \frac{1.09 \times 500(1.09^n - 1)}{0.09}$$

So the payout, which occurs 45 years after the initial investment will be

$$A_{45} = \frac{1.09 \times 500(1.09^{45} - 1)}{0.09} = \$286\,593$$

13b i $A_{34} = \frac{1.09 \times 500(1.09^{34} - 1)}{0.09} = \$107\,355$

13b ii $A = \$107\,355 + 0.25(\$286\,593 - \$107\,355) = \$152\,165$

14a At the end of the first year the value of the fund will be

$A_1 = 1.06 \times 2000$

At the end of the second year the value will be

$$\begin{aligned} A_2 &= 1.06 \times (2000 + A_1) = 1.06 \times (2000 + 1.06 \times 2000) \\ &= 1.06 \times 2000 + 1.06^2 \times 500 \end{aligned}$$

Similarly, the value at the end of the nth year will be

$A_n = 1.06 \times 2000 + 1.06^2 \times 2000 + \cdots + 1.06^n \times 2000$

Each term in the sum form a GP with $a = 1.06 \times 2000$ and $r = 1.06$. Hence

$$A_n = \frac{a(r^n - 1)}{r - 1} = \frac{1.06 \times 2000(1.06^n - 1)}{1.06 - 1} = \frac{1.06 \times 2000(1.06^n - 1)}{0.06}$$

Hence the value after 10 years will be

$$A_{10} = \frac{1.06 \times 2000(1.06^{10} - 1)}{0.06} = \$27\,943.29$$

14b The fund will reach \$70 000 when $A_n = 70\,000$ this is when

$$\frac{1.06 \times 2000(1.06^n - 1)}{0.06} = 70\,000$$

$$(1.06^n - 1) = 70\,000 \div \frac{1.06 \times 2000}{0.06}$$

$$1.06^n - 1 = 1.981130275 \ldots$$

$$1.06^n = 2.981130275 \ldots$$

$$n \doteqdot \frac{\ln 2.9811}{\ln 1.06}$$

$$n \doteqdot \frac{\ln 2.9811}{\ln 1.06} \doteqdot 18.75$$

Now note that $n = 18$ denotes the end of the 18th year. Hence $n \doteqdot 18.75$ will be during the 19th year. The fund will reach \$70 000 during the 19th year.

15a 18

15b This is the same value that was obtained in 7c by use of logarithms.

15c By trial and error, you should obtain 25

16 Refer to the answers for questions 3 – 11

17a \$M was deposited at the start of the first month and it is then compounded at the end for the month at a rate of $r = 0.01$. Thus $A_1 = M \times (1 + 0.01) = M \times 1.01$.

17b At the start of the 2nd month, there is A_1 left from the previous month and a further \$M added. At the end for the 2nd month all of this money is then compounded at a rate of $r = 0.01$. Thus $A_2 = 1.002(M + A_1)$.

At the start of the $(n+1)$th month, there is A_n left from the previous month and a further \$M added. At the end for the $(n+1)$th month all of this money is then compounded at a rate of $r = 0.01$. Thus $A_{n+1} = 1.002(M + A_n)$.

17c $A_1 = M \times 1.01$

$A_2 = 1.01 \times (M + 1.01 \times M) = 1.01 \times M + 1.01^2 \times M$

$A_3 = 1.01 \times (M + A_2)$
$= 1.01 \times (M + 1.01 \times M + 1.01^2 \times M)$
$= 1.01 \times M + 1.01^2 \times M + 1.01^3 \times M$

$A_n = 1.01 \times M + 1.01^2 \times M + 1.01^3 \times M + \cdots + 1.01^n \times M$

17d The terms of A_n form a GP with $a = 1.01 \times M$ and $r = 1.01$.

Thus $A_n = \frac{a(r^n-1)}{r-1} = \frac{1.01M\times(1.01^n-1)}{1.01-1} = \frac{1.01M\times(1.01^n-1)}{0.01} = 101M(1.01^n - 1)$

17e After 3 years, 36 months have passed. This means that there will be

$A_{36} = 101(100)(1.01^{36} - 1) = \$4350.76$

17f At the end of 5 years, $5 \times 12 = 60$ months have passed. Thus we require

$A_{60} = 30\ 000$

$101M(1.01^{60} - 1) = 30\ 000$

$$M = \frac{30\ 000}{101(1.01^{60} - 1)} = \$363.70$$

18a \$100 was deposited at the start of the first week and it is then compounded at the end for the week at a rate of $r = \frac{0.104}{52} = 0.002$.

Thus $A_1 = 100 \times (1 + 0.002) = 100 \times 1.002$.

At the start of the $(n+1)$thweek, there is A_n left from the previous week and a further \$100 added. At the end for the $(n+1)$th week all of this money is then compounded at a rate of $r = \frac{0.104}{52} = 0.002$. Thus $A_{n+1} = 1.002(100 + A_n)$.

18b $A_2 = 1.002 \times (100 + 1.002 \times 100) = 1.002 \times 100 + 1.002^2 \times 100$

$A_3 = 1.002 \times (100 + A_2)$
$= 1.002 \times (100 + 1.002 \times 100 + 1.002^2 \times 100)$
$= 1.002 \times 100 + 1.002^2 \times 100 + 1.002^3 \times 100$

$A_n = 1.002 \times 100 + 1.002^2 \times 100 + 1.002^3 \times 100 + \cdots + 1.002^n \times 100$

18c The terms of A_n form a GP with $a = 1.002 \times 100$ and $r = 1.002$.

Thus A_n

$= \dfrac{a(r^n - 1)}{r - 1}$

$= \dfrac{1.002 \times 100 \times (1.002^n - 1)}{1.002 - 1}$

$= \dfrac{1.002 \times 100 \times (1.002^n - 1)}{0.002}$

$= 50\ 100 \times (1.002^n - 1)$

18d The couple has \$100 000 when

$100\ 000 < \mathrm{A_n}$

$100\ 000 < 50\ 100 \times (1.002^n - 1)$

$1.996 < 1.002^n - 1$

$1.002^n > 2.996$

$n > \dfrac{\ln 2.996}{\ln 1.002} = 549.19$

Hence it will take 550 weeks for the couple to have \$100 000.

Solutions to Exercise 14E

1a $A = P(1+R)^n = 501(1+0.1)^4 = \$733.51$

1b i $A = P(1+R)^n = 158.05(1+0.1)^3 = \$210.36$

1b ii $A = P(1+R)^n = 158.05(1+0.1)^2 = \$191.24$

1b iii $A = P(1+R)^n = 158.05(1+0.1)^1 = \$173.86$

1b iv $A = P(1+R)^n = 158.05(1+0.1)^0 = \$158.05$

1b v $A_{\text{repayment}} = \$210.36 + \$191.24 + \$173.86 + \$158.05 = \$733.51 = A_{\text{loan}}$

1c i $\$158.05, \$173.86, \$191.24, \$210.36$

These terms form a GP with common ratio 1.1 as

$$\frac{173.86}{158.05} = \frac{191.24}{173.86} = \frac{210.36}{191.24} = 1.1$$

1c ii $a = 158.05, r = 1.1$ and $n = 4$

1c iii

$$\begin{aligned}
&A_4 \\
&= S_4 \\
&= \frac{a(r^n - 1)}{r - 1} \\
&= \frac{158.05(1.1^4 - 1)}{1.1 - 1} \\
&= \frac{158.05(1.1^4 - 1)}{0.1} \\
&\doteqdot \$733.51
\end{aligned}$$

This is the same as in part b v.

2a $A = P(1+R)^n = 5600(1+0.05)^5 = \$7147.18$

2b i $A = P(1+R)^n = 1293.46(1+0.05)^4 = \$1572.21$

2b ii $A = P(1+R)^n = 1293.46(1+0.05)^3 = \$1497.34$

2b iii Third instalment: $A = P(1+R)^n = 1293.46(1+0.05)^2 = \$1426.04$

Fourth instalment: $A = P(1+R)^n = 1293.46(1+0.05)^1 = \$1358.13$

Fifth instalment: $A = P(1+R)^n = 1293.46(1+0.05)^0 = \$1293.46$

2b iv $A_{\text{repayment}}$
$= \$1572.21 + \$1497.34 + \$1426.04 + \$1358.13 + \$1293.46$
$= \$7147.18$
$= A_{\text{loan}}$

2c i $\$1293.46, \$1358.13, \$1426.04, \$1497.34, \$1572.21$

These terms form a GP with common ratio 1.05 as

$$\frac{1358.13}{1293.46} = \frac{1426.04}{1358.13} = \frac{1497.34}{1426.04} = \frac{1572.21}{1497.34} = 1.05$$

2c ii $a = 1293.46, r = 1.05$ and $n = 5$

2c iii A_5
$$= S_5$$
$$= \frac{a(r^n - 1)}{r - 1}$$
$$= \frac{1293.46(1.05^5 - 1)}{1.05 - 1}$$
$$= \frac{1293.46(1.01^5 - 1)}{0.05}$$
$$\doteqdot \$7147.18$$

This is the same as in part b iv.

3a i $A = P(1+R)^n = 15\,000(1+0.07)^{15} = 15\,000(1.07)^{15}$

3a ii $A = P(1+R)^n = 1646.92(1+0.07)^{14} = 1646.92(1.07)^{14}$

3a iii $A = P(1+R)^n = 1646.92(1+0.07)^{13} = 1646.92(1.07)^{13}$

3a iv $A = P(1+R)^n = 1646.92(1+0.07)^{1} = 1646.92(1.07)^{1} = 1646.92(1.07)$

3a v $A = P(1+R)^n = 1646.92(1+0.07)^{0} = 1646.92$

3a vi
$$\begin{aligned} A_{15} &= A_{\text{loan}} - A_{\text{repaid}} \\ &= 15\,000(1.07)^{15} \\ &\quad -(1646.92 + 1646.92(1.07) + \cdots + 1646.92(1.07)^{13} + 1646.92(1.07)^{14}) \end{aligned}$$

3b A_{repaid} forms a GP with $a = 1646.92$, $r = 1.07$ and 15 terms, hence

A_{repaid}

$$\begin{aligned} &= S_{15} \\ &= \frac{a(r^n-1)}{r-1} \\ &= \frac{1646.92(1.07^{15}-1)}{1.07-1} \\ &= \frac{1646.92(1.07^{15}-1)}{0.07} \end{aligned}$$

Thus

$$\begin{aligned} A_{15} &= A_{\text{loan}} - A_{\text{repaid}} \\ &= 15\,000(1.07)^{15} - \frac{1646.92(1.07^{15}-1)}{0.07} \end{aligned}$$

3c $A_{15} = \$0$, hence the loan has been repaid

4a i $A = P(1+R)^n = 100\,000(1+0.005)^{20\times 12} = 100\,000 \times 1.005^{240}$

4a ii $A = P(1+R)^n = M(1+0.005)^{239} = M \times 1.005^{239}$

4a iii Second: $A = P(1+R)^n = M(1+0.005)^{238} = M \times 1.005^{238}$, Last: M

4a iv
$$\begin{aligned} A_{240} &= A_{\text{loan}} - A_{\text{repaid}} \\ &= 100\,000 \times 1.005^{240} - (M \times 1.005 + M \times 1.005^2 + \cdots + M \times 1.005^{239}) \end{aligned}$$

4b A_{repaid} forms a GP with $a = M, r = 1.005$ and 240 terms, hence

A_{repaid}

$$\begin{aligned} &= S_{240} \\ &= \frac{a(r^n - 1)}{r - 1} \\ &= \frac{M(1.005^{240} - 1)}{1.005 - 1} \\ &= \frac{M(1.005^{240} - 1)}{0.005} \end{aligned}$$

Thus

$$\begin{aligned} A_{240} &= A_{\text{loan}} - A_{\text{repaid}} \\ &= 100\,000 \times 1.005^{240} - \frac{M(1.005^{240} - 1)}{0.005} \end{aligned}$$

4c This is when the loan is repaid.

4d $A_{240} = 0$

$$100\,000 \times 1.005^{240} - \frac{M(1.005^{240} - 1)}{0.005} = 0$$

$$100\,000 \times 1.005^{240} = \frac{M(1.005^{240} - 1)}{0.005}$$

$M = (100\,000 \times 1.005^{240}) \div \frac{(1.005^{240}-1)}{0.005} = \$716.43$ (to the nearest cent)

4e $240 \times 716.43 = \$171\,943.20$

5a i $A = P(1+R)^n = 10\,000(1+0.015)^{60} = 10\,000 \times 1.015^{60}$

5a ii $A = P(1+R)^n = M(1+0.015)^{59} = M \times 1.015^{59}$

5a iii Second: $A = P(1+R)^n = M(1+0.015)^{58} = M \times 1.015^{58}$, Last: M

5a iv $A_{60} = A_{\text{loan}} - A_{\text{repaid}}$
$= 10\,000 \times 1.015^{60} - (M + 1.015M + 1.015^2 M + \cdots + 1.015^{59} M)$

5b A_{repaid} forms a GP with $a = M, r = 1.015$ and 60 terms, hence

A_{repaid}

$$= S_{60}$$
$$= \frac{a(r^n - 1)}{r - 1}$$
$$= \frac{M(1.015^{60} - 1)}{1.015 - 1}$$
$$= \frac{M(1.015^{60} - 1)}{0.015}$$

Thus

$A_{60} = A_{\text{loan}} - A_{\text{repaid}}$
$$= 10\,000 \times 1.015^{60} - \frac{M(1.015^{60} - 1)}{0.015}$$

But, since the loan is payed off after 60 months, $A_{60} = 0$ so

$$0 = 10\,000 \times 1.015^{60} - \frac{M(1.015^{60} - 1)}{0.015}$$

5c

$$0 = 10\,000 \times 1.015^{60} - \frac{M(1.015^{60} - 1)}{0.015}$$

$$10\,000 \times 1.015^{60} = \frac{M(1.015^{60} - 1)}{0.015}$$

$$M = 10\,000 \times 1.015^{60} \div \frac{(1.015^{60} - 1)}{0.015} = \$254 \text{ (to the nearest dollar)}$$

6a $A_{180} = A_{\text{loan}} - A_{\text{repaid}}$

$$= 165\,000 \times 1.0075^{180} - (1700 + 1700 \times 1.0075 + \cdots + 1700 \times 1.0075^{179})$$

6b A_{repaid} forms a GP with $a = 1700$, $r = 1.0075$ and 180 terms, hence

A_{repaid}

$$= S_{180}$$
$$= \frac{a(r^n - 1)}{r - 1}$$
$$= \frac{1700(1.0075^{180} - 1)}{1.075 - 1}$$
$$= \frac{1700(1.0075^{180} - 1)}{0.0075}$$

Thus

$A_{180} = A_{\text{loan}} - A_{\text{repaid}}$

$$= 165\,000 \times 1.0075^{180} - \frac{1700(1.0075^{180} - 1)}{0.0075}$$

6c $A_{180} = -\$10\,012.67$, hence more than the required amount has been repayed. Thus the loan was repaid in less than 15 years (as this is much larger than the value of a single instalment).

7a $A_n = A_{\text{loan}} - A_{\text{repaid}}$

$$= 250\,000 \times 1.006^n - (2000 + 2000 \times 1.006 + 2000 \times 1.006^2 + \cdots + 2000 \times 1.006^{n-1})$$

7b A_{repaid} forms a GP with $a = 2000$, $r = 1.006$ and n terms, hence

A_{repaid}

$$= S_n$$
$$= \frac{a(r^n - 1)}{r - 1}$$
$$= \frac{2000(1.006^n - 1)}{1.006 - 1}$$
$$= \frac{2000(1.006^n - 1)}{0.006}$$

Thus

$$A_n = A_{\text{loan}} - A_{\text{repaid}}$$
$$= 250\,000 \times 1.006^n - \frac{2000(1.006^n - 1)}{0.006}$$

7c $A_{10\times 12} = A_{120} = \$162498$ (to the nearest dollar) which is more than half.

7d $A_{240} = -\$16\,881$ (to the nearest dollar). Hence, as this is larger than the value of an instalment, the loan is paid out in less than 20 years.

7e $A_n = 0$ for the loan to be paid

$$250\,000 \times 1.006^n - \frac{2000(1.006^n - 1)}{0.006} = 0$$

$$1500 \times 1.006^n = 2000(1.006^n - 1)$$

$$1500 \times 1.006^n = 2000 \times 1.006^n - 2000$$

$$500 \times 1.006^n = 2000$$

$$1.006^n = 4$$

Hence

$$\log 1.006^n = \log 4$$

$$n \log 1.006 = \log 4$$

$$n = \frac{\log 4}{\log 1.006}$$

7f $n = \frac{\log 4}{\log 1.006} = 231.74\ldots$

The smallest integer solution is hence 232 (we cannot round down, otherwise the loan will not be paid off). Thus the loan is paid off $240 - 232 = 8$ months early.

8a $A_n = A_{\text{loan}} - A_{\text{repaid}}$

$$= 500\,000\left(1 + \frac{0.0525}{12}\right)^n$$

$$-\left(10\,000+10\,000\left(1+\frac{0.0525}{12}\right)+10\,000\left(1+\frac{0.0525}{12}\right)^{2}+10\,000\left(1+\frac{0.0525}{12}\right)^{n-1}\right)$$

A_{repaid} forms a GP with $a = 10\,000$, $r = \left(1+\frac{0.0525}{12}\right)$ and n terms, hence

A_{repaid}

$= S_{180}$

$= \dfrac{a(r^n-1)}{r-1}$

$= \dfrac{10\,000\left(\left(1+\frac{0.0525}{12}\right)^n-1\right)}{\left(1+\frac{0.0525}{12}\right)-1}$

Thus

$A_n = A_{\text{loan}} - A_{\text{repaid}}$

$$= 500\,000\left(1+\frac{0.0525}{12}\right)^n - \frac{10\,000\left(\left(1+\frac{0.0525}{12}\right)^n-1\right)}{\left(1+\frac{0.0525}{12}\right)-1}$$

$$= 500\,000\times 1.004375^n - \frac{10\,000(1.004375^n-1)}{1.004375-1}$$

$$= 500\,000\times 1.004375^n - \frac{10\,000(1.004375^n-1)}{0.004375}$$

8b The loan is paid off when

$A_n = 0$

$500\,000\times 1.004375^n - \dfrac{10000(1.004375^n-1)}{0.004375} = 0$

$500\,000\times 1.004375^n = \dfrac{10000(1.004375^n-1)}{0.004375}$

$2187.50\times 1.004375^n = 10000(1.004375^n-1)$

$10\,000 = (10\,000-2187.50)\times 1.004375^n$

$10\,000 = (10\,000-2187.50)\times 1.004375^n$

$1.004375^n \times 7812.50 = 10\,000$

$1.004375^n = 1.28$

8c $1.004375^n = 1.28$

$\ln 1.004375^n = \ln 1.28$

$n \ln 1.004375 = \ln 1.28$

$n = \dfrac{\ln 1.28}{\ln 1.004375} = 56.55$

Hence rounding up gives 57 months. However the final repayment will only be \$5490.41.

9a The loan is repaid in 25 years which is $25 \times 12 = 300$ months.

Hence the total amount owing after 300 months must be zero so $A_{300} = 0$.

9b $A_{300} = A_{\text{loan}} - A_{\text{repaid}}$

$$= 180\,000\left(1+\frac{0.066}{12}\right)^{300} - \left(M + M\left(1+\frac{0.066}{12}\right) + \cdots + M\left(1+\frac{0.066}{12}\right)^{299}\right)$$

A_{repaid} forms a GP with $a = M$, $r = \left(1+\frac{0.066}{12}\right)$ and 300 terms, hence

A_{repaid}

$= S_{300}$

$= \dfrac{a(r^n - 1)}{r-1}$

$$= \frac{M\left(\left(1+\frac{0.066}{12}\right)^{300} - 1\right)}{\left(1+\frac{0.066}{12}\right) - 1}$$

Thus

$A_{300} = A_{\text{loan}} - A_{\text{repaid}}$

$$= 180\,000\left(1+\frac{0.066}{12}\right)^{300} - \frac{M\left(\left(1+\frac{0.066}{12}\right)^{300} - 1\right)}{\left(1+\frac{0.066}{12}\right) - 1}$$

$$= 180\,000 \times 1.0055^{300} - \frac{M(1.0055^{300} - 1)}{1.0055 - 1}$$

$$= 180\,000 \times 1.0055^{300} - \frac{M(1.0055^{300} - 1)}{0.0055}$$

9c Since $A_{300} = 0$

$$180\,000 \times 1.0055^{300} - \frac{M(1.0055^{300} - 1)}{0.0055} = 0$$

$$180\,000 \times 1.0055^{300} = \frac{M(1.0055^{300} - 1)}{0.0055}$$

$$M = 180\,000 \times 1.0055^{300} \div \frac{(1.0055^{300} - 1)}{0.0055} = \$1226.64 \text{ (to the nearest cent)}$$

9d $300 \times 1226.64 = \$367\,993$ (to the nearest dollar)

9e $I = \text{total repaid} - \text{total borrowed} = 367\,993 - 180\,000 = \$187\,993$

$I = PRn$

$187\,993 = 180\,000(R)(25)$

$$R = \frac{187\,993}{180\,000 \times 25} \doteqdot 0.042$$

Hence the simple interest rate would be 4.2% per annum.

10a
$$\begin{aligned} A_{300} &= A_{\text{loan}} - A_{\text{repaid}} \\ &= 15\,000\left(1 + \frac{0.135}{12}\right)^{5\times 12} \\ &\quad - \left(M + M\left(1 + \frac{0.135}{12}\right) + \cdots + M\left(1 + \frac{0.135}{12}\right)^{5\times 12 - 1}\right) \\ &= 15\,000\left(1 + \frac{0.135}{12}\right)^{60} - \left(M + M\left(1 + \frac{0.135}{12}\right) + \cdots + M\left(1 + \frac{0.135}{12}\right)^{59}\right) \end{aligned}$$

A_{repaid} forms a GP with $a = M, r = \left(1 + \frac{0.135}{12}\right)$ and 60 terms, hence

A_{repaid}

$$= S_{60}$$
$$= \frac{a(r^n - 1)}{r - 1}$$
$$= \frac{M\left(\left(1 + \frac{0.135}{12}\right)^{60} - 1\right)}{\left(1 + \frac{0.135}{12}\right) - 1}$$

Thus

$A_{60} = A_{\text{loan}} - A_{\text{repaid}}$

$$= 15\,000\left(1+\frac{0.135}{12}\right)^{60} - \frac{M\left(\left(1+\frac{0.135}{12}\right)^{60}-1\right)}{\left(1+\frac{0.135}{12}\right)-1}$$

$$= 15\,000 \times 1.01125^{60} - \frac{M(1.01125^{60}-1)}{1.01125-1}$$

Since $A_{60} = 0$,

$0 = 15\,000 \times 1.01125^{60} - \frac{M(1.01125^{60}-1)}{1.01125-1}$ as required.

10b In order to pay back the loan

$A_{60} = 0$

$$15\,000 \times 1.01125^{60} - \frac{M(1.01125^{60}-1)}{0.01125} = 0$$

$$15\,000 \times 1.01125^{60} = \frac{M(1.01125^{60}-1)}{0.01125}$$

$M = 15\,000 \times 1.01125^{60} \div \frac{(1.01125^{60}-1)}{0.01125} = \$345$ (to the nearest dollar)

11a

$$A_{2\times5} = 30\,000\left(1+\frac{0.133}{2}\right)^{2\times5} - \left(M + M\left(1+\frac{0.133}{2}\right) + \cdots + M\left(1+\frac{0.133}{2}\right)^{9}\right)$$

Noting that the series is a GP with $a = M, r = \left(1+\frac{0.133}{2}\right)$ and 10 terms

$$A_{10} = 30\,000\left(1+\frac{0.133}{2}\right)^{10} - \left(\frac{M\left(\left(1+\frac{0.133}{2}\right)^{10}-1\right)}{\left(1+\frac{0.133}{2}\right)-1}\right)$$

$$= 30\,000 \times 1.0665^{10} - \left(\frac{M(1.0665^{10}-1)}{1.0665-1}\right)$$

$$= 30\,000 \times 1.0665^{10} - \left(\frac{M(1.0665^{10}-1)}{0.0665}\right)$$

In order to have the loan paid off

$A_{10} = 0$

Hence

$$30\,000 \times 1.0665^{10} - \left(\frac{M(1.0665^{10} - 1)}{0.0665}\right) = 0$$

$$30\,000 \times 1.0665^{10} = \left(\frac{M(1.0665^{10} - 1)}{0.0665}\right)$$

$$M(1.0665^{10} - 1) = 30\,000 \times 1.0665^{10} \times 0.0665$$

$$M = \frac{30\,000 \times 1.0665^{10} \times 0.0665}{(1.0665^{10} - 1)} = \$4202 \text{ (to the nearest dollar)}$$

11b $A_{10} = 30\,000\left(1 + \frac{0.133}{2}\right)^{10} - \left(\frac{4202\left(\left(1+\frac{0.133}{2}\right)^{10} - 1\right)}{\left(1+\frac{0.133}{2}\right) - 1}\right) = \$6.56$

11c Each instalment is approximately 48 cents short because of rounding.

12a

$$A_{300} = P \times \left(1 + \frac{0.075}{12}\right)^{300} - \left(1600 + 1600 \times \left(1 + \frac{0.075}{12}\right) + \cdots 1600 \times \left(1 + \frac{0.075}{12}\right)^{299}\right)$$

$$= P \times 1.00625^{300} - (1600 + 1600 \times 1.00625 + \cdots 1600 \times 1.00625^{299})$$

Noting that the series is a GP with $a = 1600$, $r = 1.00625$ and 300 terms

$$A_{300} = P \times (1 + 0.00625)^{300} - \frac{1600(1.00625^{300} - 1)}{1.00625 - 1}$$

$$= P \times (1.00625)^{300} - \frac{1600(1.00625^{300} - 1)}{0.00625}$$

12b In order to be able to pay off the loan whilst obtaining the maximum amount possible, we have $A_{300} = 0$

$$P \times 1.00625^{300} - \frac{1600(1.00625^{300} - 1)}{0.00625} = 0$$

$$P \times 1.00625^{300} = \frac{1600(1.00625^{300} - 1)}{0.00625}$$

$P = \frac{1600(1.00625^{300}-1)}{0.00625} \div 1.00625^{300} = \$216\ 511$ (to the nearest dollar)

13a $1500 + 1500 \times \left(1 - \frac{0.23}{12}\right) = \$2915.90$

13b $\$3000 - \$2915.90 = \$84.10$

14a Noting that the initial loan has interest compounding at a rate of $\frac{0.06}{12}$, and noting that the first repayment is made at the end of the first month. The amount owing at the end of the first month will be.

$$A_1 = 170\ 000 \times \left(1 + \frac{0.06}{12}\right) - 1650$$

Now, at the end of the second month, interest will have accumulated on the remaining amount owing and another repayment is made. This gives

$$A_2 = \left(1 + \frac{0.06}{12}\right) \times A_1 - 1650$$
$$= \left(1 + \frac{0.06}{12}\right) \times \left(170\ 000 \times \left(1 + \frac{0.06}{12}\right) - 1650\right) - 1650$$
$$= 170\ 000 \times \left(1 + \frac{0.06}{12}\right)^2 - 1650\left(1 + \left(1 + \frac{0.06}{12}\right)\right)$$

Similarly, at the end of n months, the amount owing will be

$$A_n = 170\ 000 \times \left(1 + \frac{0.06}{12}\right)^n - 1650\left(1 + \left(1 + \frac{0.06}{12}\right) + \cdots + \left(1 + \frac{0.06}{12}\right)^{n-1}\right)$$

Noting that the terms in the right hand brackets form a GP with $a = 1, r = \left(1 + \frac{0.06}{12}\right)$ and containing n terms. The amount owing may be written as

$$A_n = 170\ 000 \times \left(1 + \frac{0.06}{12}\right)^n - 1650\left(\frac{\left(\left(1 + \frac{0.06}{12}\right)^n - 1\right)}{\left(1 + \frac{0.06}{12}\right) - 1}\right)$$

Hence after 1 year (12 months) the amount owing is

$$A_n = 170\ 000 \times 1.005^{12} - 1650\left(\frac{(1.005^{12} - 1)}{1.005 - 1}\right) = \$160\ 131.55$$

14b Similarly, if we treat \$160131.55 as the principal for the remaining 14 years, then for the amount owing n months after the first year

$$A_n = 160\,131.55 \times \left(1+\frac{0.085}{12}\right)^n - 1650\left(\frac{\left(\left(1+\frac{0.085}{12}\right)^n - 1\right)}{\left(1+\frac{0.085}{12}\right)-1}\right)$$

$$A_{168} = 160\,131.55 \times \left(1+\frac{0.085}{12}\right)^{168} - 1650\left(\frac{\left(\left(1+\frac{0.085}{12}\right)^{168} - 1\right)}{\left(1+\frac{0.085}{12}\right)-1}\right) = -\$5388.19$$

After 14 years the amount owing will be -5388.19. As this number is less than zero this means the couple will have paid off the loan in time. Hence they can afford to agree to the loan contract.

15a Noting that the initial superannuation has interest compounding at a rate of R, and noting that the first payment is made at the end of the first month. The amount remaining at the end of the first month will be.

$$B_1 = P - M$$

Now, at the end of the second month, interest will have accumulated on the remaining amount left and another payment is made. This gives

$$B_2 = R \times B_1 - M = (1+R) \times (P - M) - M = P \times (1+R) - M(1 + (1+R))$$

Similarly, at the end of n months, the amount remaining will be

$$B_n = P \times (1+R)^{n-1} - M(1 + (1+R) + (1+R)^2 + \cdots + (1+R)^{n-1})$$

Noting that the terms in the right hand brackets form a GP with $a = 1, r = 1 + R$ and containing n terms. The amount owing may be written as

$$B_n = P \times (1+R)^{n-1} - M\left(\frac{a(r^n - 1)}{r - 1}\right)$$

$$= P \times (1+R)^{n-1} - M\left(\frac{((1+R)^n - 1)}{1 + R - 1}\right)$$

$$= P \times (1+R)^{n-1} - M\left(\frac{((1+R)^n - 1)}{R}\right)$$

15b The payments run out after 20 years. This is $20 \times 12 = 240$ months.

Hence $B_{240} = 0$

15c Noting that $P = 300\ 000$ and that $R = \frac{0.055}{12}$

$B_{240} = 0$

$$300\ 000 \times \left(1 + \frac{0.055}{12}\right)^{240-1} - M\left(\frac{\left(\left(1+\frac{0.055}{12}\right)^{240} - 1\right)}{\frac{0.055}{12}}\right) = 0$$

$$M\left(\frac{\left(\left(1+\frac{0.055}{12}\right)^{240} - 1\right)}{\frac{0.055}{12}}\right) = 300\ 000 \times \left(1 + \frac{0.055}{12}\right)^{239}$$

$$M = 300\ 000 \times \left(1 + \frac{0.055}{12}\right)^{240-1} \div \left(\frac{\left(\left(1+\frac{0.055}{12}\right)^{240} - 1\right)}{\frac{0.055}{12}}\right) = \$2054.25$$

16 Noting that the initial loan has interest compounding at a rate of $\frac{0.12}{12} = 0.01$, and noting that the first repayment is made at the end of the first sixth months. The amount owing at the end of the first sixth months.

$A_1 = 500\ 000 \times 1.01^6 - M$

Note, as this loan is compounding monthly, then we must raise 1.1 to the power of 6 after the first 6 months.

Now, at the end of the second sixth month period, interest will have accumulated on the remaining amount owing and another repayment is made. This gives

$$\begin{aligned} A_2 &= 1.01^6 A_1 - M \\ &= 1.01^6(500\ 000 \times 1.01^6 - M) - M \\ &= 1.01^{12} \times 500\ 000 - M(1 + 1.01^6) \end{aligned}$$

Similarly, at the end of n 6 month periods, the amount owing will be

$$A_n = 1.01^{6n} \times 500\ 000 - M(1 + 1.01^6 + 1.01^{12} + \cdots + 1.01^{6n-6})$$

Noting that the terms in the right hand brackets form a GP with $a = 1, r = 1.01^6$ and containing n terms. The amount owing may be written as

$$A_n = 1.01^{6n} \times 500\ 000 - M\left(\frac{((1.01^6)^n - 1)}{(1.01^6) - 1}\right)$$

If this is to be paid off after 20 such instalments then

$A_{20} = 0$

$$1.01^{120} \times 500\ 000 - M\left(\frac{((1.01^6)^{20} - 1)}{(1.01^6) - 1}\right) = 0$$

$$1.01^{120} \times 500\,000 = M\left(\frac{((1.01^6)^{20} - 1)}{(1.01^6) - 1}\right)$$

$$M = 1.01^{120} \times 500\,000 \div \left(\frac{((1.01^6)^{20} - 1)}{(1.01^6) - 1}\right) = \$44\,131.77$$

17a $A_2 - A_1 = \$7846.68$ whereas $A_{55} - A_{55} = \$9889.36$.

So we see that the balance decreases more quickly towards the end of the loan.

17b $A_{57} = -4509.585864$ is the first term less than (or equal to) 0

17c This is the same as the answer in question 8

17d 8 months

18a Noting that the initial loan has interest compounding at a rate of $\frac{0.06}{12} = 0.005$, and noting that the first repayment is made at the end of the first month. The amount owing at the end of the first month will be.

$$A_1 = 1.005P - M$$

18b Now, at the end of the second month, interest will have accumulated on the remaining amount owing and another repayment is made. This gives

$$A_2 = 1.005A_1 - M = 1.005(1.005P - M) - M = 1.005^2P - M(1 + 1.005)$$

Similarly, at the end of n months, the amount owing will be the amount remaining in the previous month, with added interest and then the monthly repayment subtracted off

$$A_{n+1} = 1.005A_n - M$$

18c $A_{n+1} = 1.005A_n - M$

Applying this recursively gives

$$A_2 = 1.005^2P - M(1 + 1.005)$$

$A_3 = 1.005^3P - M(1 + 1.005 + 1.005^2)$

$A_n = 1.005^nP - M(1 + 1.005 + 1.005^2 + \cdots + 1.005^{n-1})$

18d Noting that the terms in the right hand brackets form a GP with $a = 1, r = 1.005$ and containing n terms. The amount owing may be written as

$$A_n = 1.005^nP - M\left(\frac{a(r^n - 1)}{r - 1}\right) = 1.005^nP - M\left(\frac{(1.005^n - 1)}{1.005 - 1}\right)$$

$$= 1.005^nP - M\left(\frac{(1.005^n - 1)}{0.005}\right)$$

$$= 1.005^nP - 200M(1.005^n - 1)$$

18e To be paid off in 20 years

$A_{20\times12} = 0$

$1.005^{240}P - 200M(1.005^{240} - 1) = 0$

$1.005^{240}P = 200M(1.005^{240} - 1)$

$$M = \frac{1.005^{240}P}{200(1.005^{240} - 1)} = \frac{1.005^{240}(150\ 000)}{200(1.005^{240} - 1)} = \$1074.65$$

18f With each instalment \$1000

$A_{240} = 1.005^{240}(150\ 000) - 200(1000)(1.005^{240} - 1) = \$34\ 489.78$

19a Noting that the initial loan has interest compounding at a rate of $\frac{0.096}{12} = 0.008$, and noting that the first repayment is made at the end of the first month. The amount owing at the end of the first month will be.

$A_1 = 1.008P - M$

Similarly, at the end of n months, the amount owing will be the amount remaining in the previous month, with added interest and then the monthly repayment subtracted off

$A_{n+1} = 1.008A_n - M$

19b Applying the above result recursively gives

$$A_n = 1.008^n P - M(1 + 1.008 + 1.008^2 + \cdots + 1.008^{n-1})$$

so

$$A_2 = 1.008^2 P - M(1 + 1.008)$$

$$A_3 = 1.008^3 P - M(1 + 1.008 + 1.008^2)$$

19c Noting that the terms in the right hand brackets form a GP with $a = 1, r = 1.008$ and containing n terms. The amount owing may be written as

$$A_n = 1.008^n P - M\left(\frac{a(r^n - 1)}{r - 1}\right) = 1.008^n P - M\left(\frac{(1.008^n - 1)}{1.008 - 1}\right)$$

$$= 1.008^n P - M\left(\frac{(1.008^n - 1)}{0.008}\right)$$

$$= 1.008^n P - 125M(1.008^n - 1)$$

19d In order to have $A_{25\times12} = 0$

$$1.008^{300} P - 125M(1.008^{300} - 1) = 0$$

$$1.008^{300} P = 125M(1.008^{300} - 1)$$

$$P = \frac{125M(1.008^{300} - 1)}{1.008^{300}} = \frac{125(1200)(1.008^{300} - 1)}{1.008^{300}} = \$136\ 262$$

19e In order to have $A_n = 0$

$$1.008^n P - 125M(1.008^n - 1) = 0$$

$$1.008^n (100\ 000) - 125(1000)(1.008^n - 1) = 0$$

$$(100\ 000 - 125 \times 1000)1.008^n = -125 \times 1000$$

$$1.008^n = \frac{-125 \times 1000}{100\ 000 - 125 \times 1000} = \frac{-125000}{-25000} = 5$$

$$n = \frac{\ln 5}{\ln 1.008} = 201.9834$$

Hence it will take 202 months.

Solutions to Chapter review

1a $T_2 - T_1 = 44 - 31 = 13$

$T_3 - T_2 = 57 - 44 = 13$

Hence all terms have the same common difference of 13. Thus this is an AP with $a = 31$ and $d = 13$.

1b For an AP,

$T_n = a + (n - 1)d$
$= 31 + (n - 1) \times 13$
$= 31 + 13(n - 1)$
$= 31 + 13n - 13$
$= 13n + 18$

To find the number of terms we solve the equation

$T_n = 226$

$13n + 18 = 226$

$13n = 208$

$n = 16$

Hence there are 16 terms in the sequence

1c $S_n = \frac{n}{2}(a + l) = \frac{16}{2}(31 + 226) = 2056$

2a

$$\frac{T_3}{T_2} = \frac{6}{12} = \frac{1}{2}$$

$$\frac{T_2}{T_1} = \frac{12}{24} = \frac{1}{2}$$

Hence all terms have the same common ratio so this is a GP with $r = \frac{1}{2}$ and $a = 24$.

2b $|r| = \frac{1}{2} < 1$ and hence there is a limiting sum

2c

$$S_\infty = \frac{a}{1-r} = \frac{24}{1-\frac{1}{2}} = \frac{24}{\left(\frac{1}{2}\right)} = 2(24) = 48$$

$$S_n = \frac{a(r^n - 1)}{r - 1} = \frac{24\left(\left(\frac{1}{2}\right)^n - 1\right)}{\frac{1}{2} - 1} = \frac{24\left(\left(\frac{1}{2}\right)^n - 1\right)}{-\frac{1}{2}} = 48\left(1 - \left(\frac{1}{2}\right)^n\right)$$

Hence

$$S_{10} = 48\left(1 - \left(\frac{1}{2}\right)^{10}\right) \doteqdot 47.953125 \ldots \doteqdot 48.0 = S_\infty \text{ (to 3 significant figures)}$$

3a $2^n > 2000$

$$n > \frac{\ln 2000}{\ln 2}$$

$n > 10.97$

Hence the smallest integer solution is $n = 11$

3b $1.08^n > 2000$

$$n > \frac{\ln 2000}{\ln 1.08}$$

$n > 98.76$

Hence the smallest integer solution is $n = 99$

3c $0.98^n < 0.01$

$$n > \frac{\ln 0.01}{\ln 0.98}$$

$n > 227.95$

Hence the smallest integer solution is $n = 228$

3d

$\left(\frac{1}{2}\right)^n < 0.0001$

$n > \frac{\ln 0.0001}{\ln\left(\frac{1}{2}\right)}$

$n > 13.29$

Hence the smallest integer solution is $n = 14$

4 The volume flowing through the well is given by a GP with $a = 900$ and $r = \frac{29}{30}$.

Hence the total volume will be given by the limiting sum

$S_\infty = \frac{a}{1-r} = \frac{900}{1-\frac{29}{30}} = \frac{900}{\frac{1}{30}} = 30(900) = 27\,000$ litres

5 By the annual company profits form GP as a 14% increase per annum means that each year the profits will be multiplied by 1.14 of the previous. That the profits between each year has a common ratio of $r = 1.14$. Hence

$$T_n = ar^n = a1.14^n$$

The profit will have increased by 2000% when $T_n > 21a$ (when they are 21 × their initial value). Solving this equation gives

$a1.14^n > 21a$

$1.14^n > 21$

$n > \frac{\ln 21}{\ln 1.14}$

$n > 23.24$

The smallest integer solution to this is $n = 24$.

6a By definition her salary is a GP as a 4% increase means that each year her salary will be the previous year's salary multiplied by 1.04. That the salary between each year has a common ratio of $r = 1.04$.

6b As this is a GP with $a = 35\ 000$ and $r = 1.04$ her annual salary will be given by

$T_n = ar^{n-1} = 35\ 000(1.04)^{n-1}$

And her total earnings will be given by

$$S_n = \frac{a(r^n - 1)}{r - 1} = \frac{35\ 000((1.04)^n - 1)}{1.04 - 1} = 875\ 000((1.04)^n - 1)$$

Hence after 10 years her annual salary will be $T_{10} = \$49\ 816$ and her total earnings will be $S_{10} = \$420\ 214$.

7a As the salary is increasing by the same amount of \$4000 each year, it will be an AP with $a = \$47\ 000$ and $d = \$4000$. Hence

$T_n = a + (n-1)d$
$= 47\ 000 + (n-1) \times 4000$
$= 47\ 000 + 4000n - 4000$
$= 4000n + 43\ 000$

7b In order to be at least twice the salary of 2004, Darko's salary must satisfy

$T_n > 94\ 000$

$4000n + 43\ 000 > 94\ 000$

$4000n > 51\ 000$

$n > 12.75$

Hence it the smallest integer is $n = 13$. This is 13 years after 2004 and hence would be the year 2017.

8 Her salary is a GP with $a = 53\ 000$ and $r = 1.03$. Hence the salary after n years is given by

$T_n = ar^{n-1} = 53\ 000(1.03)^{n-1}$

Her salary will be twice the original salary when

$T_n > 106\ 000$

$53\ 000(1.03)^{n-1} > 106\ 000$

$(1.03)^{n-1} > 2$

$$n - 1 > \frac{\ln 2}{\ln 1.03}$$

$n > \dfrac{\ln 2}{\ln 1.03} + 1$

$n > 24.45$

Hence her salary will be twice the original salary during the 25th year (after 2005) which is in 2030.

9a $A = P(1+R)^n = 12\,000\left(1 + \frac{0.0525}{12}\right)^{12\times5} = \$15\,593.19$

9b $I = A - P = 15\,593.19 - 12\,000 = \$3593.19$

9c In order for simple interest to yield the same interest, we must have

$I = PRn$

$3593.19 = 12\,000R(5)$

$R = \frac{3593.19}{12\,000\times5} \doteqdot 0.0599 = 5.99\%$ (to 3 significant figures)

10a $A = P(1-R)^n = 25\,000(1-0.12)^4 = 25\,000(0.88)^4 \doteqdot \$14\,992$

10b The average loss is given by $\frac{25\,000-14\,992}{4} = \$2502$ per year

10c Letting P be the value of the new car, we have

$A = P(1-R)^n$

$25\,000 = P(0.88)^4$

$P = \dfrac{25\,000}{(0.88)^4} \doteqdot \$41\,688$

10d The average loss is given by $\frac{41\,688-25\,000}{4} = \$4172$ per year

11a $A_{15} = 8000\times1.075 + 8000\times1.075^2 + \cdots + 8000\times1.075^{15}$

This is a GP with $a = 8000 \times 1.075$, $r = 1.075$ and 15 terms, hence

$$A_{15} = \frac{a(r^n - 1)}{r - 1}$$
$$= \frac{8000 \times 1.075 \times (1.075^{15} - 1)}{1.075 - 1}$$
$$= \frac{8000 \times 1.075 \times (1.075^{15} - 1)}{0.075}$$

11b $A_{15} = \$224\,617.94$

11c $\$224\,617.94 - \$8000 \times 15 = \$104\,617.94$

11d $A_{17} = \frac{8000 \times 1.075 \times (1.075^{17} - 1)}{0.075} = \$227\,419.10$ and the contributions were $17 \times 8000 = \$136\,000.00$.

Hence the value is more than double that of the contributions.

12a $A_n = M \times 1.066 + M \times 1.066^2 + \cdots + M \times 1.066^n$

This is a GP with $a = M \times 1.066$, $r = 1.066$ and n terms, hence

$$A_n = \frac{a(r^n - 1)}{r - 1}$$
$$= \frac{M \times 1.066 \times (1.066^n - 1)}{1.066 - 1}$$
$$= \frac{M \times 1.066 \times (1.066^n - 1)}{0.066}$$

12b To have 500 000 in 25 years time, $A_{25} = 500\,000$

$500\,000 = \frac{M \times 1.066 \times (1.066^{25} - 1)}{0.066}$

$M = 500\,000 \div \frac{1.066 \times (1.066^{25} - 1)}{0.066} = \$7852.46$ (to the nearest cent)

13a $A_{180} = A_{\text{loan}} - A_{\text{repaid}}$

$$= 159\,000\left(1 + \frac{0.0675}{12}\right)^{15\times 12} - \left(1415 + 1415\left(1 + \frac{0.0675}{12}\right) + \cdots + 1415\left(1 + \frac{0.0675}{12}\right)^{15\times 12-1}\right)$$

$$= 159\,000 \times 1.005625^{180} - (1415 + 1415 \times 1.005625 + \cdots + 1415 \times 1.005625^{179})$$

13b A_{repaid} forms a GP with $a = 1415$, $r = 1.005625$ and 180 terms, hence

A_{repaid}

$= S_{180}$

$= \dfrac{a(r^n - 1)}{r - 1}$

$= \dfrac{1416(1.005625^{180} - 1)}{1.005625 - 1}$

Thus

$A_{180} = A_{\text{loan}} - A_{\text{repaid}}$

$$= 159\,000 \times 1.005625^{180} - \frac{1416(1.005625^{180} - 1)}{1.005625 - 1}$$

$$= 159\,000 \times 1.005625^{180} - \frac{1416(1.005625^{180} - 1)}{0.005625}$$

13c $A_{180} = \$-2479.44$, hence the loan is actually paid out in less than 15 years.

13d With a monthly repayment of M

$$A_{180} = 159\,000 \times 1.005625^{180} - \frac{M(1.005625^{60} - 1)}{0.005625}$$

Hence, to pay off completely in 15 years

$A_{180} = 0$

$$159\,000 \times 1.005625^{180} - \frac{M(1.005625^{60} - 1)}{0.005625} = 0$$

$$159\,000 \times 1.005625^{180} = \frac{M(1.005625^{60} - 1)}{0.005625}$$

$M = 159\,000 \times 1.005625^{180} \div \frac{(1.005625^{60} - 1)}{0.005625} = \$1407.01$ (to the nearest cent)

14a $A_N = A_{\text{loan}} - A_{\text{repaid}}$

$$= 1\,700\,000\left(1+\frac{0.045}{12}\right)^n - \left(18\,000 + 18\,000\left(1+\frac{0.045}{12}\right) + \cdots + 18\,000\left(1+\frac{0.045}{12}\right)^{n-1}\right)$$

$$= 1\,700\,000 \times 1.00375^n - (18\,000 + 18\,000 \times 1.00375 + \cdots + 18\,000 \times 1.00375^{n-1})$$

14b A_{repaid} forms a GP with $a = 18\,000$, $r = 1.00375$ and n terms, hence

A_{repaid}

$= S_{180}$

$= \dfrac{a(r^n - 1)}{r - 1}$

$= \dfrac{18\,000(1.00375^n - 1)}{1.00375 - 1}$

Thus

$A_n = A_{\text{loan}} - A_{\text{repaid}}$

$$= 1\,700\,000 \times 1.00375^n - \frac{18\,000(1.00375^n - 1)}{1.00375 - 1}$$

$$= 1\,700\,000 \times 1.00375^n - \frac{18\,000(1.00375^n - 1)}{0.00375}$$

14c $A_{5\times 12} = A_{60} = \$919\,433$, which is more than half

14d $A_{120} = -\$57677.61$, hence the loan is actually paid out in less than 10 years

14e When $A_n = 0$

$$1\,700\,000 \times 1.00375^n - \frac{18\,000(1.00375^n - 1)}{0.00375} = 0$$

$$1\,700\,000 \times 1.00375^n = \frac{18\,000(1.00375^n - 1)}{0.00375}$$

$$6375 \times 1.00375^n = 18\,000(1.00375^n - 1)$$

$$6375 \times 1.00375^n = 18\,000 \times 1.00375^n - 18\,000$$

$$11625 \times 1.00375^n = 18000$$

$1.00375^n = 1.5484$

$\log_{10} 1.00375^n = \log_{10} 1.5484$

$n \log_{10} 1.00375 = \log_{10} 1.5484$

$n = \dfrac{\log_{10} 1.5484}{\log_{10} 1.00375}$

14f

$n = \dfrac{\log_{10} 1.5484}{\log_{10} 1.00375} = 116.81$

Hence the loan can be paid off after 117 months which is 3 months early.