Solutions to Exercise 7A

1a

angle size in radians	1	0.5	0.2	0.1	0.08
$\sin x$	0.841 471	0.479 426	0.198 669	0.099 833	0.079 915
$\dfrac{\sin x}{x}$	0.841 471	0.958 851	0.993 347	0.998 334	0.998 934
$\tan x$	1.557 408	0.546 302	0.202 71	0.100 335	0.080 171
$\dfrac{\tan x}{x}$	1.557 408	1.092 605	1.013 55	1.003 347	1.002 139
$\cos x$	0.540 302	0.877 583	0.980 067	0.995 004	0.996 802

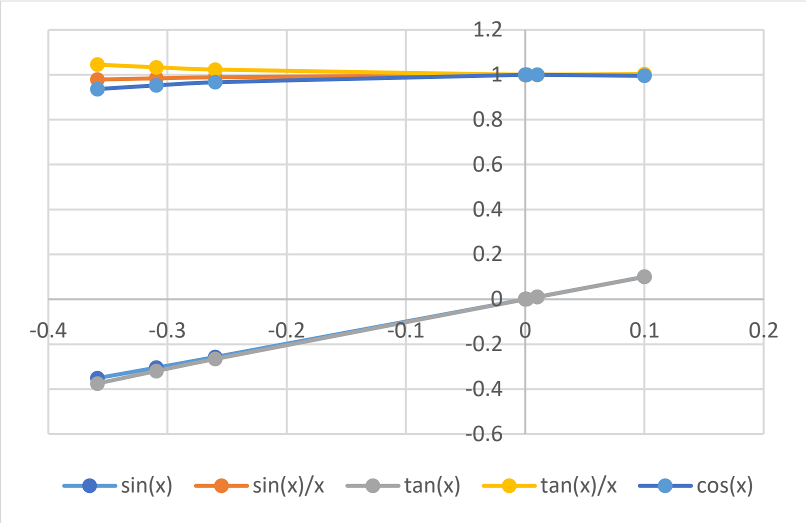
angle size in radians	0.05	0.02	0.01	0.005	0.002
$\sin x$	0.049 979	0.019 999	0.01	0.005	0.002
$\dfrac{\sin x}{x}$	0.999 583	0.999 933	0.999 983	0.999 996	0.999 999
$\tan x$	0.050 042	0.020 003	0.01	0.005	0.002
$\dfrac{\tan x}{x}$	1.000 834	1.000 133	1.000 033	1.000 008	1.000 001
$\cos x$	0.998 75	0.9998	0.999 95	0.999 988	0.999 998

1b By observation of the table, we see that as $x \to 0$, $\frac{\sin x}{x} \to 1$ and $\frac{\tan x}{x} \to 1$.

Refer to page 312 of the textbook for a more rigorous proof of these results.

2 Answers will vary. An example of the graph that could be produced is shown below.

Uncorrected proofs



Uncorrected proofs

3a $2° = \frac{\pi}{180°} \times 2°$ radians $= \frac{\pi}{90}$ radians

3b Since we know that as $x \to 0, \frac{\sin x}{x} \to 1$, we also know that as $x \to 0$, $\sin x \to x$ and hence it follows that for small values of x, $\sin x \doteqdot x$.

When $x = \frac{\pi}{90}$ radians, x is small so $\sin \frac{\pi}{90} \doteqdot \frac{\pi}{90}$.

We know from above that $2° = \frac{\pi}{90}$ radians, hence $\sin 2° \doteqdot \frac{\pi}{90}$.

3c From above we know that $\sin 2° \doteqdot \frac{\pi}{90}$.

Substituting in $\pi = 3.142$ gives

$\sin 2° \doteqdot \frac{3.142}{90} = \frac{0.3142}{9} \doteqdot 0.0349$

4a

angle size in degrees	$60°$	$30°$	$10°$	$5°$	$2°$	$1°$
angle size in radians	$\frac{\pi}{3}$	$\frac{\pi}{6}$	$\frac{\pi}{18}$	$\frac{\pi}{36}$	$\frac{\pi}{90}$	$\frac{\pi}{180}$
$\sin x$	0.8660	0.5	0.1736	0.087 16	0.034 90	0.017 45
$\frac{\sin x}{x}$	0.8270	0.9549	0.9949	0.9987	0.9998	0.9999

$\tan x$	1.732	0.5774	0.1763	0.087 49	0.034 92	0.017 46
$\dfrac{\tan x}{x}$	1.654	1.103	1.010	1.003	1.000	1.000
$\cos x$	0.5000	0.8660	0.9848	0.9962	0.9994	0.9998

angle size in degrees	$20'$	$5'$	$1'$	$30''$	$1''$
angle size in radians	$\dfrac{\pi}{540}$	$\dfrac{\pi}{2160}$	$\dfrac{\pi}{10\,800}$	$\dfrac{\pi}{21\,600}$	$\dfrac{\pi}{648\,000}$
$\sin x$	0.005 818	0.001 454	0.000 290 9	0.000 145 4	0.000 004 848
$\dfrac{\sin x}{x}$	1.000	1.000	1.000	1.000	1.000
$\tan x$	0.005 818	0.001 454	0.000 290 9	0.000 145 4	0.000 004 848
$\dfrac{\tan x}{x}$	1.000	1.000	1.000	1.000	1.000
$\cos x$	1.000	1.000	1.000	1.000	1.000

4b $\sin x < x < \tan x$

4c i From the table, we can see that:

$$\lim_{x\to 0}\frac{\sin x}{x} = 1$$

4c ii From the table, we can see that:

$$\lim_{x\to 0}\frac{\tan x}{x} = 1$$

4d $x \le 0.0774$ (correct to four decimal places) or $x \le 4°26'$

5 An example of the values one may obtain from a spreadsheet are shown. Note that the sheet will round some values to 1 when the angle becomes very small.

Uncorrected proofs

angle size in radians	0.1	0.01	0.001	0.0001	0.000 01	0.000 001	0.000 000 1
$\frac{\sin x}{x}$	0.998 334	0.999 983	1	1	1	1	1
$\frac{\tan x}{x}$	1.003 347	1.000 033	1	1	1	1	1
$\cos x$	0.995 004	0.999 95	1	1	1	1	1

6a $\lim\limits_{x\to 0}\frac{\sin x}{x} = 1$ (One of the Fundamental Limits)

6b $\lim\limits_{x\to 0}\frac{\sin 2x}{x}$

$= \lim\limits_{x\to 0}\frac{\sin 2x}{x}\times\frac{2}{2}$

$= \lim\limits_{x\to 0}\frac{\sin 2x}{2x}\times 2$

Let, $y = 2x$

As, $x \to 0$, $2x \to 0$ and hence, $y \to 0$

Then, $\lim\limits_{x\to 0} 2x = \lim\limits_{y\to 0} y = 0$

Therefore, by substituting, $y = 2x$

$\lim\limits_{x\to 0}\frac{\sin 2x}{2x}\times 2$

$= \lim\limits_{y\to 0}\frac{\sin y}{y}\times 2$

We know that, $\lim\limits_{x\to 0}\frac{\sin x}{x} = 1$, hence,

$\lim\limits_{x\to 0}\frac{\sin 2x}{x} = 1\times 2 = 2$

6c $\lim\limits_{x\to 0}\frac{\sin x}{2x}$

$= \frac{1}{2}\times\lim\limits_{x\to 0}\frac{\sin x}{x}$

We know that, $\lim\limits_{x\to 0}\frac{\sin x}{x} = 1$, hence,

Uncorrected proofs

$\lim_{x\to 0}\frac{\sin x}{2x} = \frac{1}{2}\times 1 = \frac{1}{2}$

6d $\lim_{x\to 0}\frac{\sin 3x}{2x}$

$= \lim_{x\to 0}\frac{\sin 3x}{2x}\times\frac{3}{3}$

$= \lim_{x\to 0}\frac{\sin 3x}{3x}\times\frac{3}{2}$

Let, $y = 3x$

As, $x \to 0$, $3x \to 0$ and hence, $y \to 0$

Then, $\lim_{x\to 0} 3x = \lim_{y\to 0} y = 0$

Therefore, by substituting, $y = 3x$

$\lim_{x\to 0}\frac{\sin 3x}{3x}\times\frac{3}{2}$

$= \lim_{y\to 0}\frac{\sin y}{y}\times\frac{3}{2}$

We know that, $\lim_{x\to 0}\frac{\sin x}{x} = 1$, hence,

$\lim_{x\to 0}\frac{\sin 3x}{2x} = 1\times\frac{3}{2} = \frac{3}{2}$

6e $\lim_{x\to 0}\frac{5x}{\sin 3x}$

$= 5\times\lim_{x\to 0}\frac{x}{\sin 3x}$

$= 5\times\lim_{x\to 0}\frac{x}{\sin 3x}\times\frac{3}{3}$

$= \frac{5}{3}\times\lim_{x\to 0}\frac{3x}{\sin 3x}$

$= \frac{5}{3}\times\lim_{x\to 0}\frac{1}{\frac{\sin 3x}{3x}}$

Let, $y = 3x$

As, $x \to 0$, $3x \to 0$ and hence, $y = 0$

Then, $\lim_{x\to 0} 3x = \lim_{y\to 0} y = 0$

Therefore, by substituting, $y = 3x$

Uncorrected proofs

$\frac{5}{3} \times \lim_{x \to 0} \frac{1}{\frac{\sin 3x}{3x}}$

$= \frac{5}{3} \times \lim_{x \to 0} \frac{1}{\frac{\sin y}{y}}$

We know that, $\lim_{x \to 0} \frac{\sin x}{x} = 1$, hence,

$\lim_{x \to 0} \frac{5x}{\sin 3x} = \frac{5}{3} \times 1 = \frac{5}{3}$

6f $\lim_{x \to 0} \frac{\sin 3x + \sin 5x}{x}$

$= \lim_{x \to 0} \frac{\sin 3x}{x} + \lim_{x \to 0} \frac{\sin 5x}{x}$

$= 3 \lim_{x \to 0} \frac{\sin 3x}{3x} + 5 \lim_{x \to 0} \frac{\sin 5x}{5x}$

$= 3(1) + 5(1)$

$= 8$

7

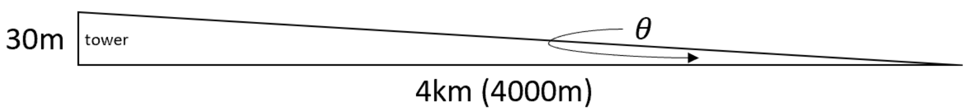
$\sin x = \frac{\text{vertical distance}}{\text{hypotenuse}}$

$\sin 5° = \frac{\text{vertical distance}}{1}$

$\text{vertical distance} = \sin 5°$

$\text{vertical distance} = \sin \frac{\pi}{36} \doteqdot \frac{\pi}{36}\ \text{km} = \frac{1000\pi}{36}\ \text{m} \doteqdot 87\ \text{m}$

8



As can be seen from the diagram, $\tan \theta = \frac{30}{4000}$.

Hence, as this is a small angle,

$\theta \doteqdot \frac{30}{4000}$ radians

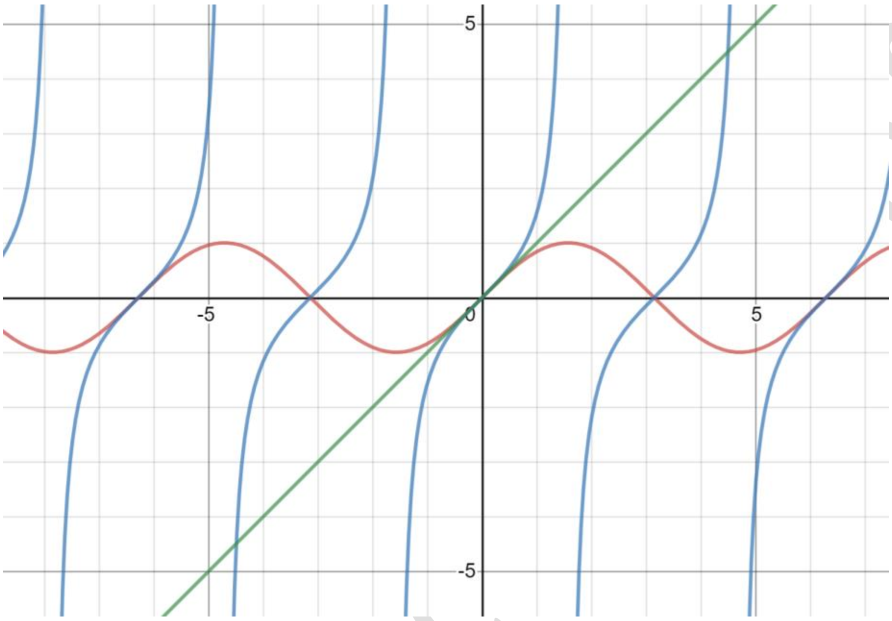
Uncorrected proofs

$$= \frac{30}{4000} \times \frac{180^\circ}{\pi}$$

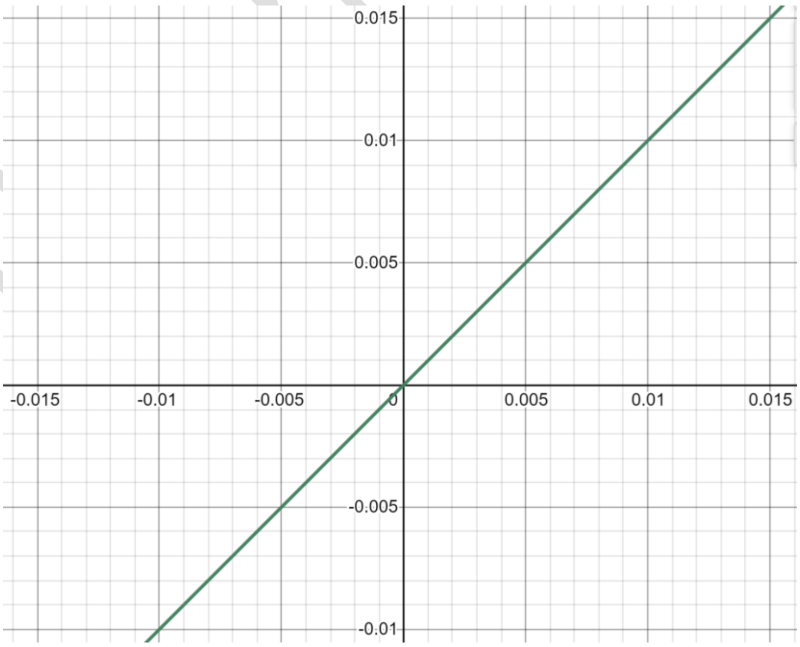
$$\doteqdot 0.4297^\circ$$

$$\doteqdot 26'$$

9 Zoomed out

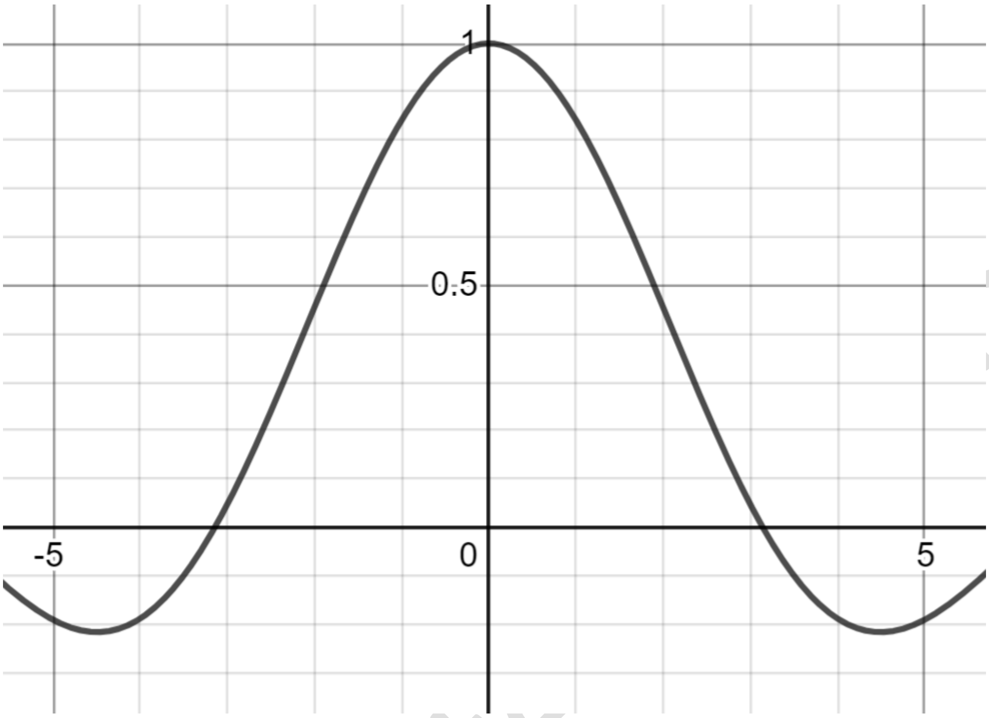


Zoomed in

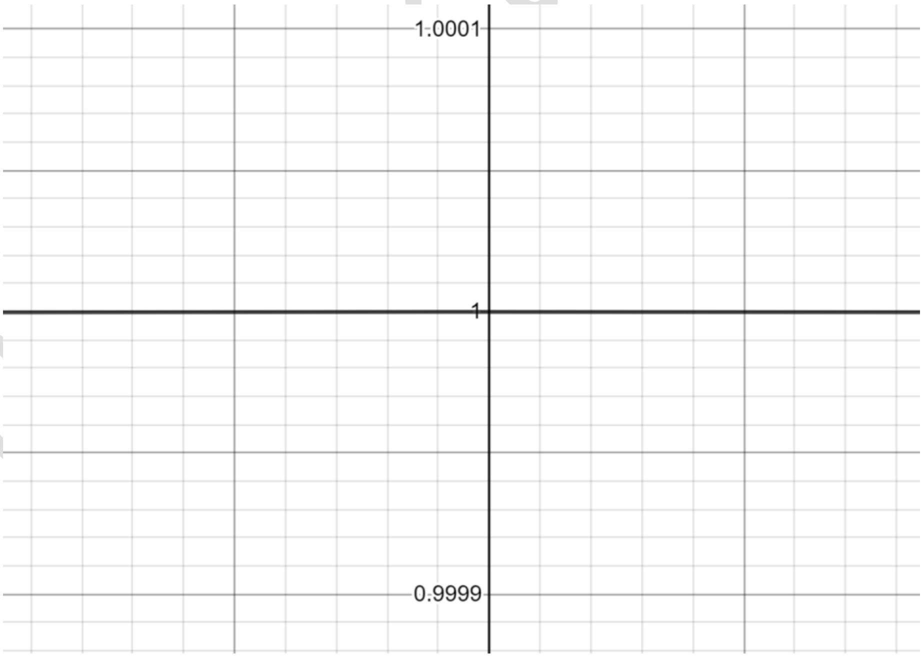


10 For $y = \frac{\sin x}{x}$

Zoomed out

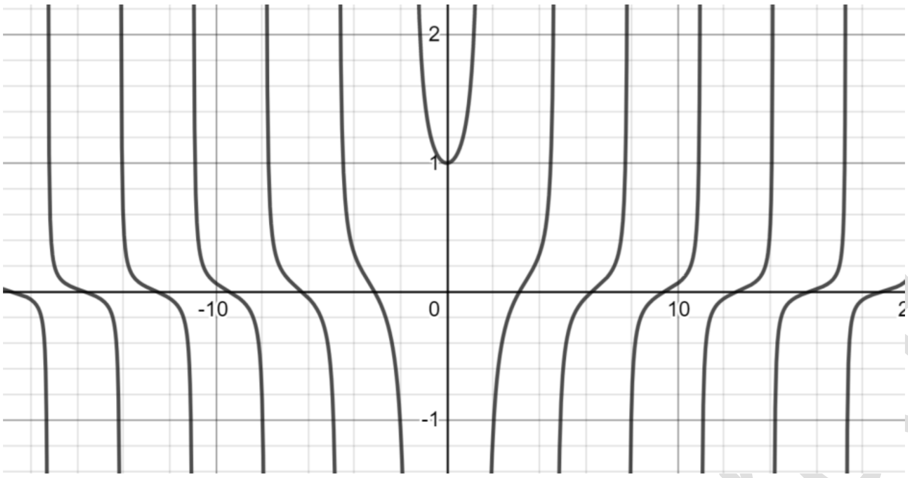


Zoomed in

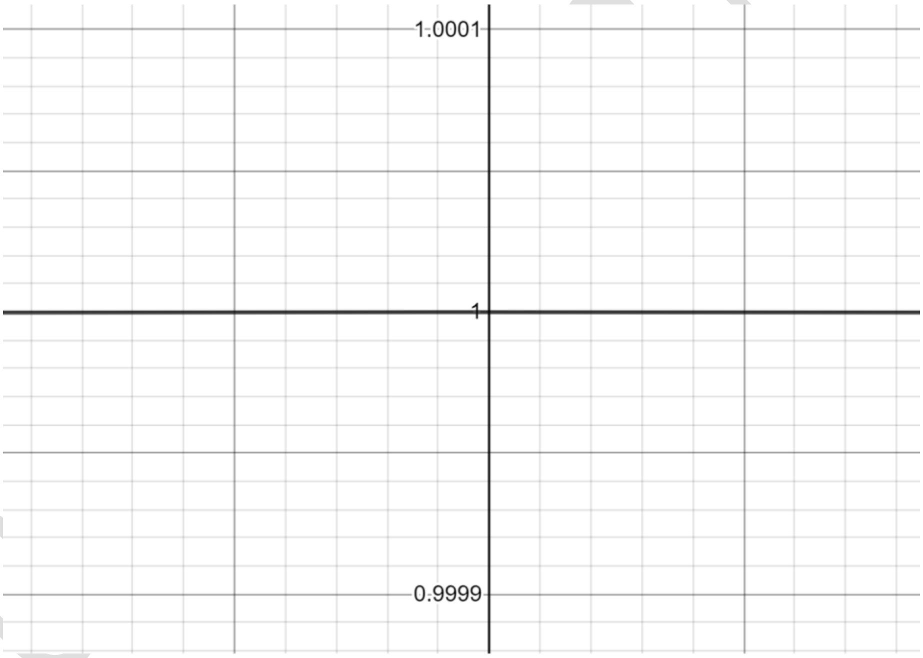


For $y = \frac{\tan x}{x}$

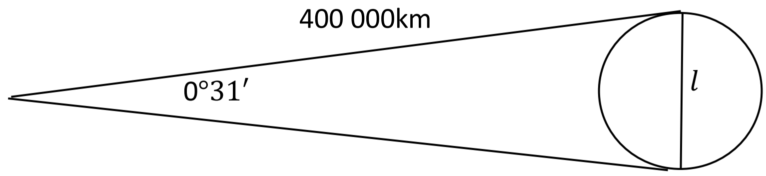
Zoomed out



Zoomed in



11



Firstly, converting $0°31'$ to radians gives $0°31' \times \frac{\pi}{180°} = 0.009\ 017\ 534\ 46\ldots$

Using the formula $l = r\theta$, as the diameter of the moon is approximately equal to the arc of a circle whose centre is the point of observation, gives
$l = 400\ 000 \times 0.009\ 017\ 534\ 46\ldots = 3607.013\ldots$ km which is 3600 km to the nearest 100 km.

12 The total side length of a polygon with 300 sides will be approximately equal to that of a circle with radius 60 cm.

The circumference of a circle with radius 60 cm is

$2\pi r = 2\pi \times 60 = 120\pi = 376.99$ cm.

Hence, as the polygon is regular and all its sides are of the same length, in order to have the total side length approximately 376.99 cm long, each individual side must be $\frac{376.99}{300} \doteqdot 1.26$ cm long.

13a $AB = \sqrt{a^2 + b^2 - 2ab\cos C}$

$= \sqrt{r^2 + r^2 - 2(r)(r)\cos x}$

$= \sqrt{2r^2(1 - \cos x)}$

Hence $AB^2 = 2r^2(1 - \cos x)$.

The formula for arc length is $l = r\theta$, hence applying this for AB gives $AB = rx$.

13b If the chord and arc are approximately equal for small angles then

$AB_{\text{chord}} \doteqdot AB_{\text{arc}}$

$AB_{\text{chord}}^2 \doteqdot AB_{\text{arc}}^2$

$2r^2(1 - \cos x) \doteqdot (rx)^2$

Uncorrected proofs

$2r^2(1-\cos x) \doteqdot r^2x^2$

$2(1-\cos x) \doteqdot x^2$

$1-\cos x \doteqdot \dfrac{x^2}{2}$

$\cos x \doteqdot 1-\dfrac{x^2}{2}$

The arc is longer than the chord, so

$AB_{\text{chord}} \doteqdot AB_{\text{arc}}$

$AB^2_{\text{chord}} \doteqdot AB^2_{\text{arc}}$

$2r^2(1-\cos x) < (rx)^2$

$2r^2(1-\cos x) < r^2x^2$

$2(1-\cos x) < x^2$

$1-\cos x < \dfrac{x^2}{2}$

$\cos x > 1-\dfrac{x^2}{2}$

Hence $\cos x$ is larger than the approximation.

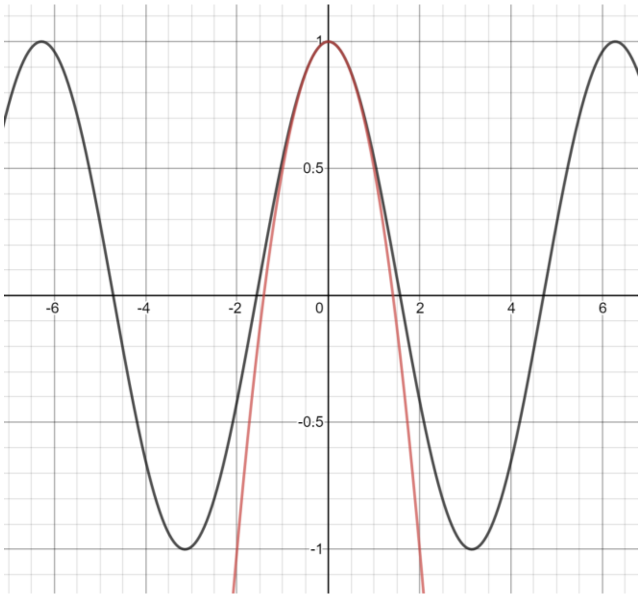
13c

	1°	10°	20°	30°
$\cos x$	0.999 847 695	0.984 808	0.939 693	0.866 025
$1-\frac{1}{2}x^2$	0.999 847 691	0.984 769	0.939 077	0.862 922
Error	3.8662×10^{-9}	3.86×10^{-4}	0.000 616	0.003 103

The approximation is highly accurate at 1° and becomes less accurate as the size of the angle increases.

Uncorrected proofs

14



As discussed in the previous question the arc is longer than the chord, so $\cos x$ is larger than the approximation. We can also see this as we zoom in on the curve.

Note that in the graph above we can see that the red graph of $y = 1 - \frac{1}{2}x^2$ is always below the black graph of $y = \cos x$ and hence $y = \cos x$ is larger.

Uncorrected proofs

15a Using compound angle formula:

$\sin(A - B) = \sin A \cos B - \cos A \sin B$

15b $\sin(\theta - x)$

$= \sin\theta \cos x - \cos\theta \sin x$ (using compound angle formula)

x is a small angle, hence

$\sin x \doteqdot x$ and $\cos x \doteqdot 1$

Therefore,

$\sin(\theta - x)$

$= \sin\theta \cos x - \cos\theta \sin x$

$= \sin\theta \times 1 - \cos\theta \times x$

$= \sin\theta - x\cos\theta$

as required

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15c $29°57' = \left(29 + \frac{57}{60}\right)° = \frac{599}{20}° = \frac{599\pi}{3600}$ radians

This is $\frac{\pi}{3600}$ away from $\frac{\pi}{6}$. $\frac{\pi}{3600}$ is a very small angle.

Let $x = \frac{\pi}{3600}$ and $\theta = \frac{\pi}{6}$

$$\sin\left(\frac{\pi}{6} - \frac{\pi}{3600}\right) = \sin\left(\frac{\pi}{6}\right) - \frac{\pi}{3600}\cos\left(\frac{\pi}{6}\right)$$

$$\sin\left(\frac{\pi}{6} - \frac{\pi}{3600}\right) = \frac{1}{2} - \frac{\pi}{3600} \times \frac{\sqrt{3}}{2}$$

$$\sin\frac{599\pi}{3600} \doteqdot \frac{3600}{7200} - \frac{\pi\sqrt{3}}{7200}$$

$$\sin 29°57' \doteqdot \frac{3600 - \sqrt{3}\pi}{7200}$$

15d Using a calculator the left hand side evaluates to $\sin 29°57' = 0.49924406 \ldots$ whilst the right hand side evaluates to $\frac{3600-\sqrt{3}\pi}{7200} = 0.4992442503 \ldots$ from this we see that the approximation is accurate to 6 decimal places.

15e In radians, $29° = \frac{\pi}{180} \times 29 = \frac{29\pi}{180}$.

Note that $\frac{\pi}{6} - \frac{29\pi}{180} = \frac{\pi}{180}$

Hence, using the formula from part b with $\theta = \frac{\pi}{6}$ and $x = \frac{\pi}{180}$

$$\sin\left(\frac{\pi}{6} - \frac{\pi}{180}\right) \doteqdot \sin\left(\frac{\pi}{6}\right) - \frac{\pi}{180}\cos\left(\frac{\pi}{6}\right)$$

$$\sin\left(\frac{29\pi}{180}\right) \doteqdot \sin\left(\frac{\pi}{6}\right) - \frac{\pi}{180}\cos\left(\frac{\pi}{6}\right)$$

$$\sin(29°) \doteqdot \frac{1}{2} - \frac{\pi\sqrt{3}}{360}$$

$$\sin(29°) \doteqdot \frac{180 - \pi\sqrt{3}}{360}$$

The left hand side evaluates to $\sin(29°) = 0.484\,809\,620\,2 \ldots$ and the right hand side evaluates to $\frac{180-\pi\sqrt{3}}{360} = 0.484\,885\,005\,3 \ldots$ so this approximation is accurate to four decimal places.

Uncorrected proofs

$\cos 31°$

$$= \sin(90° - 31°)$$
$$= \sin 59°$$
$$= \sin\left(\frac{59\pi}{180}\right)$$
$$= \sin\left(\frac{60\pi}{180} - \frac{\pi}{180}\right)$$

Using the formula from part b with $\theta = \frac{60\pi}{180} = \frac{\pi}{3}$ and $x = \frac{\pi}{180}$

$\cos 31°$

$$\doteqdot \sin\frac{\pi}{3} - \left(\frac{\pi}{180}\right)\cos\frac{\pi}{3}$$
$$= \frac{\sqrt{3}}{2} - \frac{\pi}{360}$$
$$= \frac{180\sqrt{3} - \pi}{360}$$

$\cos 31° = 0.857\ 167\ 300\ 7 \ldots$ and $\frac{180\sqrt{3}-\pi}{360} = 0.857\ 298\ 757\ 5 \ldots$ so this is accurate to three decimal places.

$\tan 61°$

$$= \frac{\sin 61°}{\cos 61°}$$
$$= \frac{\sin 61°}{\sin 90° - 61°}$$
$$= \frac{\sin 61°}{\sin 29°}$$
$$= \frac{\sin\left(\frac{61\pi}{180}\right)}{\sin 29°}$$
$$= \frac{\sin\left(\frac{60\pi}{180} + \frac{\pi}{180}\right)}{\sin 29°}$$

Using the formula from part b with $\theta = \frac{60\pi}{180} = \frac{\pi}{3}$ and $x = -\frac{\pi}{180}$

$\tan 61°$

$$\doteqdot \frac{\sin\left(\frac{\pi}{3}\right) - -\frac{\pi}{180}\cos\left(\frac{\pi}{3}\right)}{\sin 29°}$$
$$= \frac{\sin\left(\frac{\pi}{3}\right) + \frac{\pi}{180}\cos\left(\frac{\pi}{3}\right)}{\sin 29°}$$

Uncorrected proofs

$$= \frac{\frac{\sqrt{3}}{2} + \frac{\pi}{180} \times \frac{1}{2}}{\sin 29°}$$

Now using the result obtained for $\sin 29°$

$\tan 61°$

$$\doteq \frac{\frac{\sqrt{3}}{2} + \frac{\pi}{180} \times \frac{1}{2}}{\frac{180 - \pi\sqrt{3}}{360}}$$
$$= \frac{\left(\frac{\sqrt{3}}{2} + \frac{\pi}{180} \times \frac{1}{2}\right) \times 360}{\frac{180 - \pi\sqrt{3}}{360} \times 360}$$
$$= \frac{180\sqrt{3} + \pi}{180 - \pi\sqrt{3}}$$

$\tan 61° = 1.804\ 047\ 755 \ldots$ and $\frac{180\sqrt{3}+\pi}{180-\pi\sqrt{3}} = 1.804\ 040\ 217 \ldots$ so this is accurate to five decimal places.

$\cot 59°$

$$= \frac{\cos 59°}{\sin 59°}$$
$$= \frac{\sin(90° - 59°)}{\sin 59°}$$
$$= \frac{\sin 31°}{\sin 59°}$$
$$= \frac{\sin\left(\frac{31\pi}{180}\right)}{\sin\left(\frac{59\pi}{180}\right)}$$
$$= \frac{\sin\left(\frac{30\pi}{180} + \frac{\pi}{180}\right)}{\sin\left(\frac{60\pi}{180} - \frac{\pi}{180}\right)}$$
$$= \frac{\sin\left(\frac{\pi}{6} + \frac{\pi}{180}\right)}{\sin\left(\frac{\pi}{3} - \frac{\pi}{180}\right)}$$
$$\doteq \frac{\sin\frac{\pi}{6} + \frac{\pi}{180}\cos\frac{\pi}{6}}{\sin\frac{\pi}{3} - \frac{\pi}{180}\cos\frac{\pi}{3}}$$
$$= \frac{\frac{1}{2} + \frac{\pi\sqrt{3}}{360}}{\frac{\sqrt{3}}{2} - \frac{\pi}{360}}$$

$$= \frac{180 + \pi\sqrt{3}}{180\sqrt{3} - \pi} \qquad \text{(multiplying by 360 on top and bottom)}$$

$\cot 59° = 0.600\ 860\ 619\ \ldots$ and $\frac{180+\pi\sqrt{3}}{180\sqrt{3}-\pi} = 0.600\ 858\ 207\ 5\ \ldots$ so this is accurate to four decimal places.

$$\sin 46° = \sin\frac{46\pi}{180}$$
$$= \sin\left(\frac{45\pi}{180} + \frac{\pi}{180}\right)$$
$$= \sin\left(\frac{\pi}{4} + \frac{\pi}{180}\right)$$
$$\doteqdot \sin\frac{\pi}{4} + \frac{\pi}{180}\cos\frac{\pi}{4}$$
$$\doteqdot \frac{1}{\sqrt{2}} + \frac{\pi}{180\sqrt{2}}$$
$$\doteqdot \frac{180 + \pi}{180\sqrt{2}}$$

$\sin 46° = 0.719\ 339\ 800\ 3\ \ldots$ and $\frac{180+\pi}{180\sqrt{2}} = 0.719\ 448\ 122\ 7$ so this is accurate to three decimal places.

Uncorrected proofs

Solutions to Exercise 7B

1a $y = \sin x$

$$\frac{dy}{dx} = \cos x$$

1b $y = \cos x$

$$\frac{dy}{dx} = -\sin x$$

1c

$y = \tan x$

$$\frac{dy}{dx} = \sec^2 x$$

1d $y = 2\sin x$

$$\frac{dy}{dx} = 2\cos x$$

1e $y = \sin 2x$

$y = \sin u \quad \text{where } u = 2x$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= \cos u \times 2 \\ &= \cos 2x \times 2 \\ &= 2\cos 2x\end{aligned}$$

1f $y = 3\cos x$

$$\frac{dy}{dx} = -3\sin x$$

Uncorrected proofs

1g $y = \cos 3x$

$y = \cos u \quad \text{where } u = 3x$

$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

$= -\sin u \times 3$

$= -\sin 3x \times 3$

$= -3 \sin 3x$

1h $y = \tan 4x$

$y = \tan u \quad \text{where } u = 4x$

$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

$= \sec^2 u \times 4$

$= 4 \sec^2 4x$

1i $y = 4 \tan x$

$\dfrac{dy}{dx} = 4 \sec^2 x$

1j $y = 2 \sin 3x$

$y = 2 \sin u \quad \text{where } u = 3x$

$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

$= 2 \cos u \times 3$

$= 2 \cos 3x \times 3$

$= 6 \cos 3x$

1k $y = 2 \tan 2x$

$y = 2 \tan u \quad \text{where } u = 2x$

$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

Uncorrected proofs

$= 2\sec^2 u \times 2$

$= 2\sec^2 2x \times 2$

$= 4\sec^2 2x$

1l $y = 4\cos 2x$

$y = 4\cos u$ where $u = 2x$

$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

$= -4\sin u \times 2$

$= -4\sin 2x \times 2$

$= -8\sin 2x$

1m $y = -\sin 2x$

$y = -\sin u$ where $u = 2x$

$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

$= -\cos u \times 2$

$= -\cos 2x \times 2$

$= -2\cos 2x$

1n $y = -\cos 2x$

$y = -\cos u$ where $u = 2x$

$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

$= \sin u \times 2$

$= \sin 2x \times 2$

$= 2\sin 2x$

Uncorrected proofs

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1o $y = -\tan 2x$

$y = -\tan u \quad \text{where } u = 2x$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= -\sec^2 u \times 2$$

$$= -\sec^2 2x \times 2$$

$$= -2\sec^2 2x$$

1p $y = \tan\frac{1}{2}x$

$y = \tan u \quad \text{where } u = \frac{1}{2}x$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= \sec^2 u \times \frac{1}{2}$$

$$= \frac{1}{2}\sec^2 \frac{1}{2}x$$

1q $y = \cos\frac{1}{2}x$

$y = \cos u \quad \text{where } u = \frac{1}{2}x$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= -\sin u \times \frac{1}{2}$$

$$= -\sin\frac{1}{2}x \times \frac{1}{2}$$

$$= -\frac{1}{2}\sin\frac{1}{2}x$$

Uncorrected proofs

1r $y = \sin\frac{x}{2}$

$y = \sin u \quad \text{where } u = \frac{x}{2}$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= \cos u \times \frac{1}{2}$$

$$= \cos\frac{x}{2} \times \frac{1}{2}$$

$$= \frac{1}{2}\cos\frac{x}{2}$$

1s $y = 5\tan\frac{1}{5}x$

$y = 5\tan u \quad \text{where } u = \frac{1}{2}x$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= 5\sec^2 u \times \frac{1}{5}$$

$$= \sec^2\frac{1}{5}x$$

1t

$$y = 6\cos\frac{x}{3}$$

$y = 6\cos u \quad \text{where } u = \frac{x}{3}$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= 6 \times (-\sin u) \times \frac{1}{3}$$

$$= 6 \times (-\sin\frac{x}{3}) \times \frac{1}{3}$$

$$= -2\sin\frac{x}{3}$$

Uncorrected proofs

1u $y = 12\sin\frac{x}{4}$

$y = 12\sin u \quad \text{where } u = \frac{x}{4}$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= 12\cos u \times \frac{1}{4}$$

$$= 12\cos\frac{x}{4} \times \frac{1}{4}$$

$$= 3\cos\frac{x}{4}$$

2a $y = \sin 2\pi x$

$$\frac{dy}{dx} = \cos 2\pi x \times \frac{d}{dx}(2\pi x)$$

$$= \cos 2\pi x \times 2\pi$$

$$= 2\pi\cos 2\pi x$$

2b

$$y = \tan\frac{\pi}{2}x$$

$$\frac{dy}{dx} = \sec^2\frac{\pi}{2}x \times \frac{d}{dx}\left(\frac{\pi}{2}x\right)$$

$$= \sec^2\frac{\pi}{2}x \times \frac{\pi}{2}$$

$$= \frac{\pi}{2}\sec^2\frac{\pi}{2}x$$

2c $y = 3\sin x + \cos 5x$

$$\frac{dy}{dx} = 3\cos x \times \frac{d}{dx}(x) - \sin 5x \times \frac{d}{dx}(5x)$$

$$= 3\cos x \times 1 - \sin 5x \times 5$$

$$= 3\cos x - 5\sin 5x$$

Uncorrected proofs

2d $y = 4\sin \pi x + 3\cos \pi x$

$$\frac{dy}{dx} = 4\cos \pi x \times \frac{d}{dx}(\pi x) + 3 \times (-\sin \pi x) \times \frac{d}{dx}(\pi x)$$

$$= 4\cos \pi x \times \pi + 3 \times (-\sin \pi x) \times \pi$$

$$= 4\pi \cos \pi x - 3\pi \sin \pi x$$

2e $y = \sin(2x - 1)$

$$\frac{dy}{dx} = \cos(2x - 1) \times \frac{d}{dx}(2x - 1)$$

$$= \cos(2x - 1) \times 2$$

$$= 2\cos(2x - 1)$$

2f $y = \tan(1 + 3x)$

$$\frac{dy}{dx} = \sec^2(1 + 3x) \times \frac{d}{dx}(1 + 3x)$$

$$= \sec^2(1 + 3x) \times 3$$

$$= 3\sec^2(1 + 3x)$$

2g $y = 2\cos(1 - x)$

$$\frac{dy}{dx} = 2 \times (-\sin(1 - x)) \times \frac{d}{dx}(1 - x)$$

$$= 2 \times (-\sin(1 - x)) \times (-1)$$

$$= 2\sin(1 - x)$$

2h $y = \cos(5x + 4)$

$$\frac{dy}{dx} = -\sin(5x + 4) \times \frac{d}{dx}(5x + 4)$$

$$= -\sin(5x + 4) \times 5$$

$$= -5\sin(5x + 4)$$

Uncorrected proofs

2i

$$y = 7\sin(2-3x)$$
$$\frac{dy}{dx} = 7\cos(2-3x) \times \frac{d}{dx}(2-3x)$$
$$= 7\cos(2-3x) \times (-3)$$
$$= -21\cos(2-3x)$$

2j

$$y = 10\tan(10-x)$$
$$\frac{dy}{dx} = 10\sec^2(10-x) \times \frac{d}{dx}(10-x)$$
$$= 10\sec^2(10-x) \times (-1)$$
$$= -10\sec^2(10-x)$$

2k

$$y = 6\sin\left(\frac{x+1}{2}\right)$$
$$= 6\sin\left(\frac{x}{2}+\frac{1}{2}\right)$$
$$\frac{dy}{dx} = 6\cos\left(\frac{x}{2}+\frac{1}{2}\right) \times \frac{d}{dx}\left(\frac{x}{2}+\frac{1}{2}\right)$$
$$= 6\cos\left(\frac{x}{2}+\frac{1}{2}\right) \times \frac{1}{2}$$
$$= 3\cos\left(\frac{x}{2}+\frac{1}{2}\right)$$
$$= 3\cos\left(\frac{x+1}{2}\right)$$

2l

$$y = 15\cos\left(\frac{2x+1}{5}\right)$$
$$= 15\cos\left(\frac{2x}{5}+\frac{1}{5}\right)$$
$$\frac{dy}{dx} = 15\left(-\sin\left(\frac{2x}{5}+\frac{1}{5}\right)\right) \times \frac{d}{dx}\left(\frac{2x}{5}+\frac{1}{5}\right)$$
$$= 15\left(-\sin\left(\frac{2x}{5}+\frac{1}{5}\right)\right) \times \frac{2}{5}$$
$$= -15\left(\frac{2}{5}\right)\sin\left(\frac{2x}{5}+\frac{1}{5}\right)$$
$$= -6\sin\left(\frac{2x+1}{5}\right)$$

Uncorrected proofs

3a

$$\begin{aligned}
y &= \sin 2x \\
\frac{dy}{dx} &= 2\cos 2x \\
\frac{d^2y}{dx^2} &= 2\times -2\sin 2x \\
&= -4\sin 2x \\
\frac{d^3y}{dx^3} &= -4\times 2\cos 2x \\
&= -8\cos 2x \\
\frac{d^4y}{dx^4} &= -8\times -2\sin 2x \\
&= 16\sin 2x
\end{aligned}$$

The amplitude for each of these expressions is given by the coefficient of each trigonometric term and hence are: 2, 4, 8 and 16.

3b

$$\begin{aligned}
y &= \cos 10x \\
\frac{dy}{dx} &= -10\sin 10x \\
\frac{d^2y}{dx^2} &= -10\times 10\cos 10x \\
&= -100\cos 10x \\
\frac{d^3y}{dx^3} &= -100\times -10\sin 10x \\
&= 1000\sin 10x \\
\frac{d^4y}{dx^4} &= 1000\times 10\cos 10x \\
&= 10\,000\cos 10x
\end{aligned}$$

Uncorrected proofs

3c

$$
\begin{aligned}
y &= \sin\frac{1}{2}x \\
\frac{dy}{dx} &= \frac{1}{2}\cos\frac{1}{2}x \\
\frac{d^2y}{dx^2} &= \frac{1}{2}\times -\frac{1}{2}\sin\frac{1}{2}x \\
&= -\frac{1}{4}\sin\frac{1}{2}x \\
\frac{d^3y}{dx^3} &= -\frac{1}{4}\times\frac{1}{2}\cos 2x \\
&= -\frac{1}{8}\cos\frac{1}{2}x \\
\frac{d^4y}{dx^4} &= -\frac{1}{8}\times -\frac{1}{2}\sin\frac{1}{2}x \\
&= \frac{1}{16}\sin\frac{1}{2}x
\end{aligned}
$$

3d

$$
\begin{aligned}
y &= \cos\frac{1}{3}x \\
\frac{dy}{dx} &= -\frac{1}{3}\sin\frac{1}{3}x \\
\frac{d^2y}{dx^2} &= -\frac{1}{3}\times\frac{1}{3}\cos\frac{1}{3}x \\
&= -\frac{1}{9}\cos\frac{1}{3}x \\
\frac{d^3y}{dx^3} &= -\frac{1}{9}\times -\frac{1}{3}\sin\frac{1}{3}x \\
&= \frac{1}{27}\sin\frac{1}{3}x \\
\frac{d^4y}{dx^4} &= \frac{1}{27}\times\frac{1}{3}\cos\frac{1}{3}x \\
&= \frac{1}{81}\cos\frac{1}{3}x
\end{aligned}
$$

The amplitude for each of these expressions is given by the coefficient of each trigonometric term and hence are: $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}$

Uncorrected proofs

4 $f'(x) = -2\sin 2x$

4a

$$\begin{aligned} f'(0) &= -2\sin(0) \\ &= 0 \end{aligned}$$

4b

$$\begin{aligned} f'\left(\frac{\pi}{12}\right) &= -2\sin\left(\frac{\pi}{6}\right) \\ &= -2\times\frac{1}{2} \\ &= -1 \end{aligned}$$

4c

$$\begin{aligned} f'\left(\frac{\pi}{6}\right) &= -2\sin\left(\frac{\pi}{3}\right) \\ &= -2\times\frac{\sqrt{3}}{2} \\ &= -\sqrt{3} \end{aligned}$$

4d

$$\begin{aligned} f'\left(\frac{\pi}{4}\right) &= -2\sin\left(\frac{\pi}{2}\right) \\ &= -2\times 1 \\ &= -2 \end{aligned}$$

5

$$f(x) = \sin u, \text{ where } u = \left(\frac{1}{4}x + \frac{\pi}{2}\right)$$

$$\begin{aligned} f'(x) &= \cos u \times \frac{1}{4} \\ &= \frac{1}{4}\cos\left(\frac{1}{4}x + \frac{\pi}{2}\right) \end{aligned}$$

Uncorrected proofs

5a

$$\begin{aligned} f'(0) &= \frac{1}{4}\cos\left(\frac{\pi}{2}\right) \\ &= 0 \end{aligned}$$

5b

$$\begin{aligned} f'(2\pi) &= \frac{1}{4}\cos\left(\frac{\pi}{2}+\frac{\pi}{2}\right) \\ &= \frac{1}{4}\cos(\pi) \\ &= -\frac{1}{4} \end{aligned}$$

5c

$$\begin{aligned} f'(-\pi) &= \frac{1}{4}\cos\left(-\frac{\pi}{4}+\frac{\pi}{2}\right) \\ &= \frac{1}{4}\cos\left(\frac{\pi}{4}\right) \\ &= \frac{1}{8}\sqrt{2} \end{aligned}$$

5d

$$\begin{aligned} f'(\pi) &= \frac{1}{4}\cos\left(\frac{\pi}{4}+\frac{\pi}{2}\right) \\ &= \frac{1}{4}\cos\left(\frac{3\pi}{4}\right) \\ &= -\frac{1}{8}\sqrt{2} \end{aligned}$$

Uncorrected proofs

6a

$$y = x\sin x$$
$$u = x, \frac{du}{dx} = 1$$
$$v = \sin x, \frac{dv}{dx} = \cos x$$
$$\frac{dy}{dx} = \sin x \times 1 + x \times \cos x$$
$$= x\cos x + \sin x$$

6b

$$y = 2x\tan 2x$$
$$u = 2x, \frac{du}{dx} = 2$$
$$v = \tan 2x, \frac{dv}{dx} = 2\sec^2 2x$$
$$\frac{dy}{dx} = \tan 2x \times 2 + 2x \times 2\sec^2 2x$$
$$= 2\tan 2x + 4x\sec^2 2x$$
$$= 2\left(\tan 2x + 2x\sec^2 2x\right)$$

6c

$$y = x^2\cos 2x$$
$$u = x^2, \frac{du}{dx} = 2x$$
$$v = \cos 2x, \frac{dv}{dx} = -2\sin 2x$$
$$\frac{dy}{dx} = \cos 2x \times 2x + x^2 \times -2\sin 2x$$
$$= 2x\cos 2x - 2x^2\sin 2x$$
$$= 2x\left(\cos 2x - x\sin 2x\right)$$

Uncorrected proofs

6d

$$y = x^3 \sin 3x$$
$$u = x^3, \frac{du}{dx} = 3x^2$$
$$v = \sin 3x, \frac{dv}{dx} = 3\cos 3x$$
$$\frac{dy}{dx} = \sin 3x \times 3x^2 + x^3 \times 3\cos 3x$$
$$= 3x^2 \sin 3x + 3x^3 \cos 3x$$
$$= 3x^2 \left(\sin 3x + x \cos 3x\right)$$

7a

$$y = \frac{\sin x}{x}$$

If $y = \frac{u}{v}$ then

$$u = \sin x, \frac{du}{dx} = \cos x$$
$$v = x, \frac{dv}{dx} = 1$$
$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$
$$= \frac{x \times \cos x - \sin x \times 1}{x^2}$$
$$= \frac{x \cos x - \sin x}{x^2}$$

7b

$$y = \frac{\cos x}{x}$$

If $y = \frac{u}{v}$ then

$$u = \cos x, \frac{du}{dx} = -\sin x$$
$$v = x, \frac{dv}{dx} = 1$$

Uncorrected proofs

$$\begin{aligned}\frac{dy}{dx} &= \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \\ &= \frac{x \times (-\sin x) - \cos x \times 1}{x^2} \\ &= \frac{-x\sin x - \cos x}{x^2}\end{aligned}$$

7c

$$y = \frac{x^2}{\cos x}$$

If $y = \frac{u}{v}$ then

$u = x^2, \frac{du}{dx} = 2x$

$v = \cos x, \frac{dv}{dx} = -\sin x$

$$\begin{aligned}\frac{dy}{dx} &= \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \\ &= \frac{(\cos x) \times (2x) - x^2 \times (-\sin x)}{(\cos^2 x)} \\ &= \frac{2x\cos x + x^2 \sin x}{\cos^2 x} \\ &= \frac{x(2\cos x + x\sin x)}{\cos^2 x}\end{aligned}$$

7d

$$y = \frac{x}{1 + \sin x}$$

If $y = \frac{u}{v}$ then

$u = x, \frac{du}{dx} = 1$

$v = 1 + \sin x, \frac{dv}{dx} = \cos x$

$$\begin{aligned}\frac{dy}{dx} &= \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \\ &= \frac{(1 + \sin x) \times 1 - x \times \cos x}{(1 + \sin x)^2}\end{aligned}$$

Uncorrected proofs

$$= \frac{1 + \sin x - x\cos x}{(1 + \sin x)^2}$$

8a $y = \sin(x^2)$

$$\frac{dy}{dx} = \cos(x^2) \times \frac{d}{dx}(x^2)$$

$$= \cos(x^2) \times 2x$$

$$= 2x\cos(x^2)$$

8b $y = \sin(1 - x^2)$

$$\frac{dy}{dx} = \cos(1 - x^2) \times \frac{d}{dx}(1 - x^2)$$

$$= \cos(1 - x^2) \times (-2x)$$

$$= -2x\cos(1 - x^2)$$

8c $y = \cos(x^3 + 1)$

$$\frac{dy}{dx} = (-\sin(x^3 + 1)) \times \frac{d}{dx}(x^3 + 1)$$

$$= (-\sin(x^3 + 1)) \times 3x^2$$

$$= -3x^2\sin(x^3 + 1)$$

8d $y = \sin\frac{1}{x} = \sin(x^{-1})$

$$\frac{dy}{dx} = \cos\frac{1}{x} \times \frac{d}{dx}(x^{-1})$$

$$= \cos\frac{1}{x} \times (-x^{-2})$$

$$= -\frac{1}{x^2}\cos\frac{1}{x}$$

Uncorrected proofs

8e $y = \cos^2 x = (\cos x)^2$

$$\frac{dy}{dx} = 2\cos x \times \frac{d}{dx}(\cos x)$$

$$= 2\cos x \times (-\sin x)$$

$$= -\sin x \times 2\cos x$$

$$= -2\cos x \sin x$$

8f $y = \sin^3 x = (\sin x)^3$

$$\frac{dy}{dx} = (3\sin^2 x) \times \frac{d}{dx}(\sin x)$$

$$= (3\sin^2 x) \times \cos x$$

$$= 3\sin^2 x \cos x$$

8g $y = \tan^2 x = (\tan x)^2$

$$\frac{dy}{dx} = (2\tan x) \times \frac{d}{dx}(\tan x)$$

$$= (2\tan x) \times \sec^2 x$$

$$= 2\tan x \sec^2 x$$

8h $y = \tan\sqrt{x} = \tan\left(x^{\frac{1}{2}}\right)$

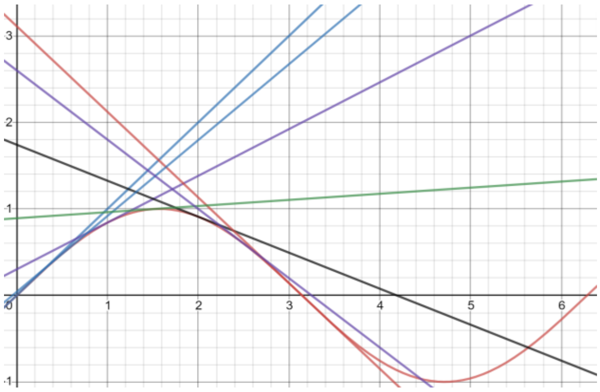
$$\frac{dy}{dx} = \left(\sec^2\sqrt{x}\right) \times \frac{d}{dx}\left(x^{\frac{1}{2}}\right)$$

$$= \left(\sec^2\sqrt{x}\right) \times \frac{1}{2}x^{-\frac{1}{2}}$$

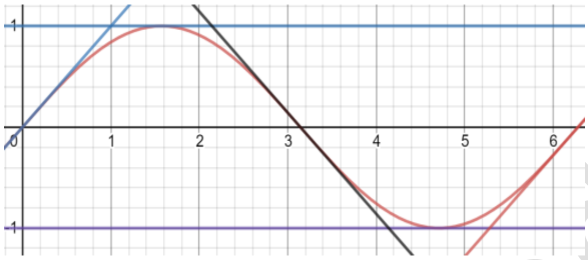
$$= \frac{1}{2\sqrt{x}}\sec^2\sqrt{x}$$

Uncorrected proofs

9a Drawing the tangents where $x = 0, 0.5, 1, 1.5,\ 2,\ 2.5, 3$



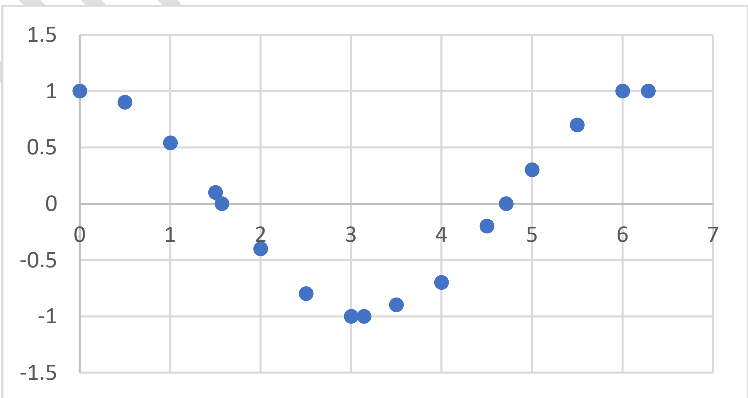
Drawing the tangents where $x = \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$



9b

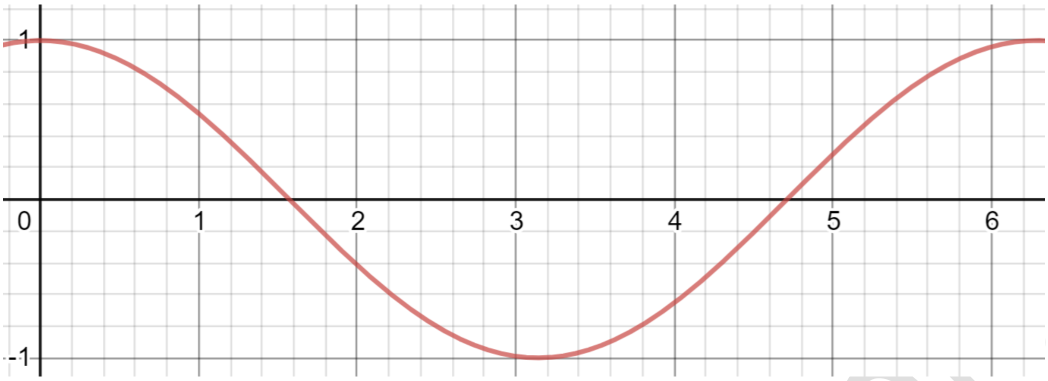
x	0	0.5	1	1.5	$\frac{\pi}{2}$	2	2.5	3	π	3.5	4	4.5	$\frac{3\pi}{2}$	5	5.5	6	2π
$f'(x)$	1	0.9	0.5	0.1	0	-0.4	-0.8	-1	-1	-0.9	-0.7	-0.2	0	0.3	0.7	1	1

9c Plotting each of the points found in part b gives:



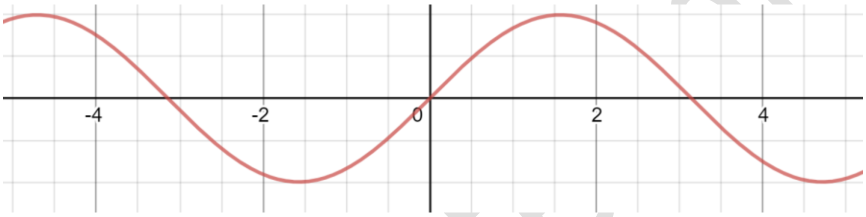
Uncorrected proofs

9d We can see that if we fit a continuous curve to the points plotted (in part c) it appears as below.

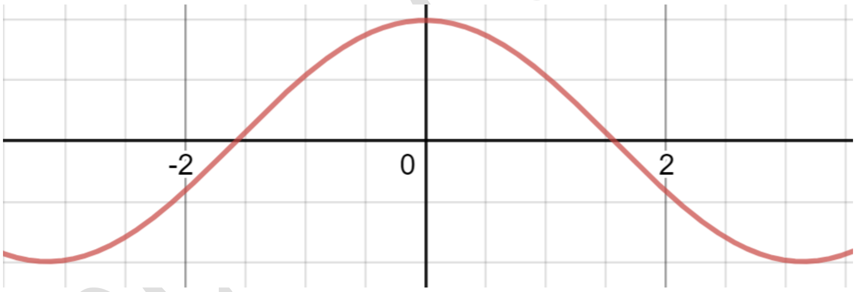


By inspection, this is the graph of $y = \cos x$.

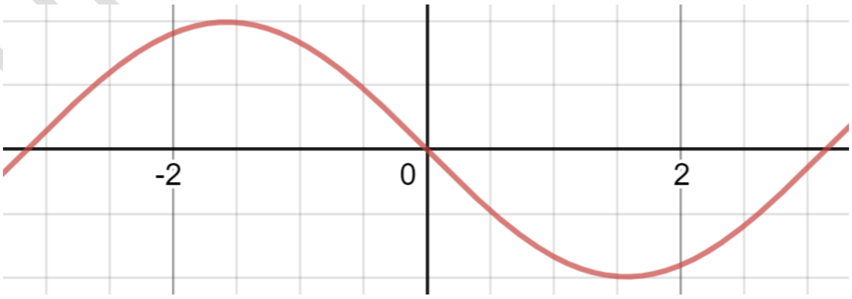
10 $y = \sin x$



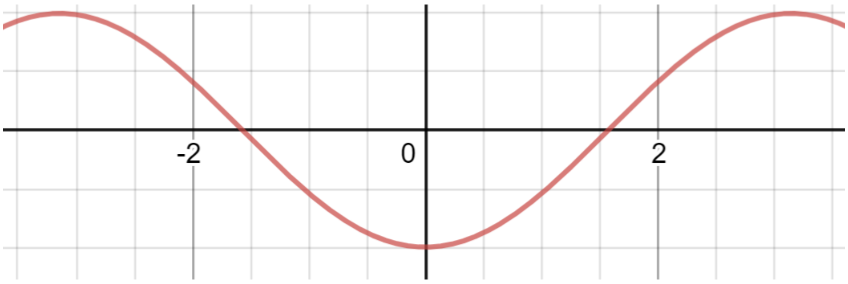
$y' = \cos x$



$y'' = -\sin x$



$y''' = -\cos x$



11a $f(x) = e^{\tan x}$

$f'(x) = e^{\tan x} \times \dfrac{d}{dx}(\tan x)$

$= e^{\tan x} \sec^2 x$

11b $f(x) = e^{\sin 2x}$

$f'(x) = e^{\sin 2x} \times \frac{d}{dx}(\sin 2x)$

$= e^{\sin 2x} \times 2\cos 2x$
$= 2e^{\sin 2x} \cos 2x$

11c $f(x) = \sin(e^{2x})$

$f'(x) = \cos(e^{2x}) \times \frac{d}{dx}(e^{2x})$

$= \cos(e^{2x}) \times 2e^{2x}$
$= 2e^{2x} \cos(e^{2x})$

11d $f(x) = \log_e(\cos x)$

$f'(x) = \frac{1}{\cos x} \times \frac{d}{dx}(\cos x)$

$= \dfrac{1}{\cos x} \times (-\sin x)$

$= -\dfrac{\sin x}{\cos x}$

$= -\tan x$

Uncorrected proofs

11e $f(x) = \log_e(\sin x)$

$$f'(x) = \frac{1}{\sin x} \times \frac{d}{dx}(\sin x)$$

$$= \frac{1}{\sin x} \times \cos x$$

$$= \frac{\cos x}{\sin x}$$

$$= \cot x$$

11f $f(x) = \log_e(\cos 4x)$

$$f'(x) = \frac{1}{\cos 4x} \times \frac{d}{dx}(\cos 4x)$$

$$= \frac{1}{\cos 4x} \times (-4 \sin 4x)$$

$$= -\frac{4 \sin 4x}{\cos 4x}$$

$$= -4 \tan 4x$$

12a $y = \sin x \cos x$

$$\frac{dy}{dx} = \cos x \times \frac{d}{dx}(\sin x) + \sin x \frac{d}{dx}(\cos x)$$

$$= \cos x \cos x + \sin x\,(-\sin x)$$

$$= \cos x \cos x - \sin x \sin x$$

$$= \cos^2 x - \sin^2 x$$

12b $y = \sin^2 7x = (\sin 7x)^2$

$$\frac{dy}{dx} = 2 \sin 7x \times \frac{d}{dx}(\sin 7x)$$

$$= 2 \sin 7x \times 7 \cos 7x$$

$$= 14 \sin 7x \cos 7x$$

Uncorrected proofs

12c $y = \cos^5 3x = (\cos 3x)^5$

$$\frac{dy}{dx} = 5(\cos 3x)^4 \times \frac{d}{dx}(\cos 3x)$$

$$= 5\cos^4 3x \times (-3\sin 3x)$$

$$= -15\cos^4 3x \sin 3x$$

12d $y = (1 - \cos 3x)^3$

$$\frac{dy}{dx} = 3(1 - \cos 3x)^2 \times \frac{d}{dx}(1 - \cos 3x)$$

$$= 3(1 - \cos 3x)^2 \times (3\sin 3x)$$

$$= 9\sin 3x\,(1 - \cos 3x)^2$$

12e $y = \sin 2x \sin 4x$

$$\frac{dy}{dx} = \sin 4x \times \frac{d}{dx}(\sin 2x) + \sin 2x \times \frac{d}{dx}(\sin 4x)$$

$$= \sin 4x \times 2\cos 2x + \sin 2x \times 4\cos 4x$$

$$= 2\sin 4x \cos 2x + 4\sin 2x \cos 4x$$

$$= 2(\cos 2x \sin 4x + 2\sin 2x \cos 4x)$$

12f $y = \tan^3(5x - 4) = (\tan(5x - 4))^3$

$$\frac{dy}{dx} = 3(\tan(5x - 4))^2 \times \frac{d}{dx}(\tan(5x - 4))$$

$$= 3\tan^2(5x - 4) \times 5\sec^2(5x - 4)$$

$$= 15\tan^2(5x - 4)\sec^2(5x - 4)$$

13a

$$f(x) = \frac{1}{1 + \sin x} = (1 + \sin x)^{-1}$$

$$f'(x) = -1(1 + \sin x)^{-2} \times \frac{d}{dx}(1 + \sin x)$$
$$= -(1 + \sin x)^{-2} \times \cos x$$

Uncorrected proofs

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$$= \frac{-\cos x}{(1+\sin x)^2}$$

13b

$$f(x) = \frac{\sin x}{1+\cos x}$$

If $f(x) = \frac{u}{v}$ then

$u = \sin x$

$$\frac{du}{dx} = \cos x$$

$v = 1 + \cos x$

$$\frac{dv}{dx} = -\sin x$$

Using the quotient rule:

$$f'(x) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{(1+\cos x) \times \cos x - \sin x \times (-\sin x)}{(1+\cos x)^2}$$

$$= \frac{\cos x + \cos^2 x + \sin^2 x}{(1+\cos x)^2}$$

$$= \frac{\cos x + 1}{(1+\cos x)^2}$$

$$= \frac{1+\cos x}{(1+\cos x)^2}$$

$$= \frac{1}{1+\cos x}$$

13c

$$f(x) = \frac{1-\sin x}{\cos x}$$

If $f(x) = \frac{u}{v}$ then

$u = 1 - \sin x$

$$\frac{du}{dx} = -\cos x$$

$v = \cos x$

Uncorrected proofs

$$\frac{dv}{dx} = -\sin x$$

Using the quotient rule:

$$\begin{aligned} f'(x) &= \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \\ &= \frac{\cos x\,(-\cos x) - (1 - \sin x)(-\sin x)}{\cos^2 x} \\ &= \frac{-\cos^2 x + \sin x - \sin^2 x}{(1 - \sin^2 x)} \\ &= \frac{\sin x - (\cos^2 x + \sin^2 x)}{(1 - \sin x)(1 + \sin x)} \\ &= \frac{\sin x - 1}{(1 - \sin x)(1 + \sin x)} \\ &= \frac{-(1 - \sin x)}{(1 - \sin x)(1 + \sin x)} \\ &= \frac{-1}{1 + \sin x} \end{aligned}$$

13d

$$f(x) = \frac{\cos x}{\cos x + \sin x}$$

If $f(x) = \frac{u}{v}$ then

$$u = \cos x$$

$$\frac{du}{dx} = -\sin x$$

$$v = \cos x + \sin x$$

$$\frac{dv}{dx} = -\sin x + \cos x$$

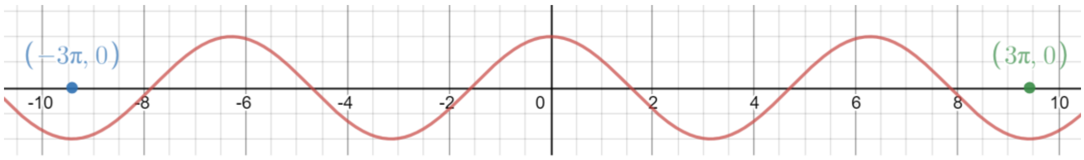
Using the quotient rule:

$$\begin{aligned} f'(x) &= \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \\ &= \frac{(\cos x + \sin x)(-\sin x) - (\cos x)(-\sin x + \cos x)}{(\cos x + \sin x)^2} \\ &= \frac{-\sin x\cos x - \sin^2 x + \sin x\cos x - \cos^2 x}{(\cos x + \sin x)^2} \end{aligned}$$

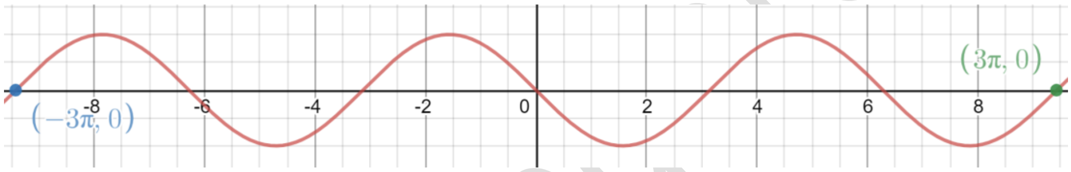
Uncorrected proofs

$$= \frac{-(\sin^2 x + \cos^2 x)}{(\cos x + \sin x)^2}$$
$$= \frac{-1}{(\cos x + \sin x)^2}$$

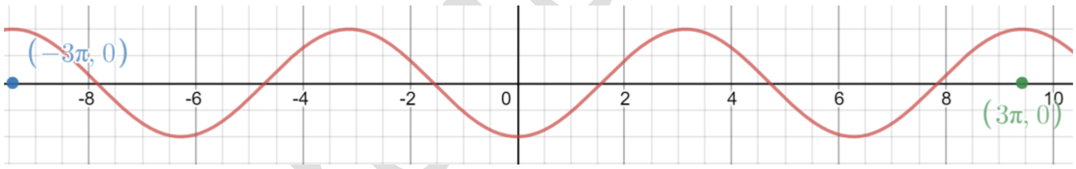
14a $y = \cos x$ for $-3\pi \le x \le 3\pi$



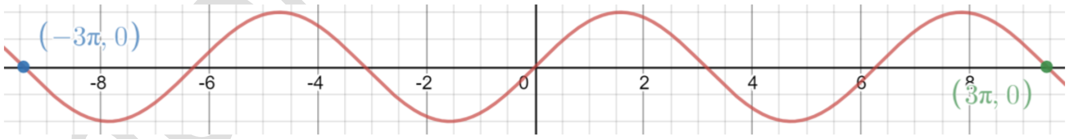
14b $y' = -\sin x$



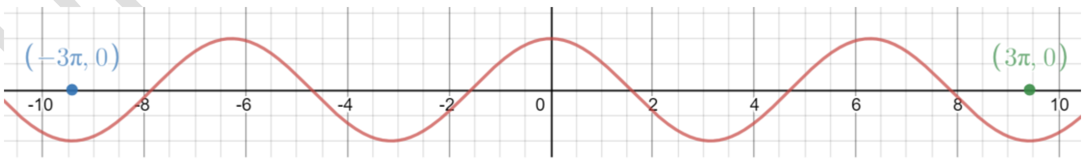
$y'' = -\cos x$



$y''' = \sin x$



$y''' = \cos x$



14c i The graphs are reflections of each other in the x-axis.

14c ii The graphs are identical.

15 See question 14a, b.

The graphs of the successive differentiation of $\sin x$ are shown on page 321 of the textbook.

16a $y = e^x \sin x$

$$y' = \sin x \times \frac{d}{dx}(e^x) + e^x \times \frac{d}{dx}(\sin x)$$
$$= \sin x \times e^x + e^x \times \cos x$$
$$= e^x \sin x + e^x \cos x$$

$$y'' = \sin x \times \frac{d}{dx}(e^x) + e^x \times \frac{d}{dx}(\sin x) + \cos x \times \frac{d}{dx}(e^x) + e^x \times \frac{d}{dx}(\cos x)$$
$$= \sin x \times e^x + e^x \times \cos x + \cos x \times e^x + e^x \times (-\sin x)$$
$$= e^x \sin x + e^x \cos x + e^x \cos x - e^x \sin x$$
$$= 2e^x \cos x$$

$y'' - 2y' + 2y$

$$= 2e^x \cos x - 2(e^x \sin x + e^x \cos x) + 2(e^x \sin x)$$
$$= 2e^x \cos x - 2e^x \sin x - 2e^x \cos x + 2e^x \sin x$$
$$= 0 \quad \text{as required}$$

16b $y = e^{-x} \cos x$

$$y' = \cos x \times \frac{d}{dx}(e^{-x}) + e^{-x} \times \frac{d}{dx}(\cos x)$$
$$= \cos x \times (-e^{-x}) + e^{-x} \times (-\sin x)$$
$$= -e^{-x} \cos x - e^{-x} \sin x$$

$$y'' = -\left(\cos x \times \frac{d}{dx}(e^{-x}) + e^{-x} \times \frac{d}{dx}(\cos x)\right) - \left(\sin x \times \frac{d}{dx}(e^{-x}) + e^{-x} \times \frac{d}{dx}(\sin x)\right)$$
$$= -(-e^{-x} \cos x - e^{-x} \sin x) - (-e^{-x} \sin x + e^{-x} \cos x)$$
$$= e^{-x} \cos x + e^{-x} \sin x + e^{-x} \sin x - e^{-x} \cos x$$
$$= 2e^{-x} \sin x$$

$y'' + 2y' + 2y$

$$= 2e^{-x} \sin x + 2(-e^{-x} \cos x - e^{-x} \sin x) + 2(e^{-x} \cos x)$$
$$= 2e^{-x} \sin x - 2e^{-x} \cos x - 2e^{-x} \sin x + 2e^{-x} \cos x$$
$$= 0 \quad \text{as required}$$

Uncorrected proofs

17a $y = \frac{1}{3}\tan^3 x - \tan x + x$

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(\frac{1}{3}\tan^3 x\right) - \frac{d}{dx}(\tan x) + \frac{d}{dx}(x)\\ &= \frac{1}{3}\frac{d}{dx}(\tan^3 x) - \frac{d}{dx}(\tan x) + \frac{d}{dx}(x)\\ &= \frac{1}{3} \times 3 \times \tan^2 x \times \frac{d}{dx}(\tan x) - \sec^2 x + 1\\ &= \frac{1}{3} \times 3 \times \tan^2 x \times \sec^2 x - \sec^2 x + 1\\ &= \tan^2 x \sec^2 x - \sec^2 x + 1 \qquad \text{as required}\end{aligned}$$

17b

$$\frac{dy}{dx} = \tan^2 x \sec^2 x - \sec^2 x + 1$$

Using the identity $\sec^2 x = 1 + \tan^2 x$:

$$\begin{aligned}\frac{dy}{dx} &= \tan^2 x\,(1 + \tan^2 x) - (1 + \tan^2 x) + 1\\ &= \tan^2 x + \tan^4 x - 1 - \tan^2 x + 1\\ &= \tan^4 x \qquad \text{as required}\end{aligned}$$

18a $\log_b\left(\frac{P}{Q}\right) = \log_b P - \log_b Q$

18b

$$f(x) = \log_e\left(\frac{1 + \sin x}{\cos x}\right) = \log_e(1 + \sin x) - \log_e(\cos x)$$

$$\begin{aligned}f'(x) &= \frac{1}{1 + \sin x} \times \frac{d}{dx}(1 + \sin x) - \frac{1}{\cos x} \times \frac{d}{dx}(\cos x)\\ &= \frac{1}{1 + \sin x} \times \cos x - \frac{1}{\cos x} \times (-\sin x)\\ &= \frac{\cos x}{1 + \sin x} + \frac{\sin x}{\cos x}\\ &= \frac{\cos^2 x}{\cos x\,(1 + \sin x)} + \frac{\sin x\,(1 + \sin x)}{\cos x\,(1 + \sin x)}\\ &= \frac{\cos^2 x}{\cos x\,(1 + \sin x)} + \frac{\sin x + \sin^2 x}{\cos x\,(1 + \sin x)}\end{aligned}$$

Uncorrected proofs

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$$= \frac{\cos^2 x + \sin^2 x + \sin x}{\cos x\,(1 + \sin x)}$$
$$= \frac{1 + \sin x}{\cos x\,(1 + \sin x)}$$
$$= \frac{1}{\cos x}$$
$$= \sec x$$

19a

$$\frac{d}{dx}(\sec x)$$
$$= \frac{d}{dx}((\cos x)^{-1})$$
$$= -1(\cos x)^{-2} \times \frac{d}{dx}(\cos x)$$
$$= -\frac{1}{\cos^2 x} \times (-\sin x)$$
$$= \frac{1}{\cos x} \times \frac{\sin x}{\cos x}$$
$$= \sec x \tan x$$

19b

$$\frac{d}{dx}(\operatorname{cosec} x)$$
$$= \frac{d}{dx}((\sin x)^{-1})$$
$$= -1(\sin x)^{-2} \times \frac{d}{dx}(\sin x)$$
$$= -\frac{1}{\sin^2 x} \times \cos x$$
$$= -\frac{1}{\sin x} \times \frac{\cos x}{\sin x}$$
$$= -\operatorname{cosec} x \cot x$$

Uncorrected proofs

19c

$$\frac{d}{dx}(\cot x)$$

$$= \frac{d}{dx}((\tan x)^{-1})$$
$$= -1(\tan x)^{-2} \times \frac{d}{dx}(\tan x)$$
$$= -\frac{1}{\tan^2 x} \times \sec^2 x$$
$$= -\frac{\cos^2 x}{\sin^2 x} \times \frac{1}{\cos^2 x}$$
$$= -\frac{1}{\sin^2 x}$$
$$= -\text{cosec}^2 x$$

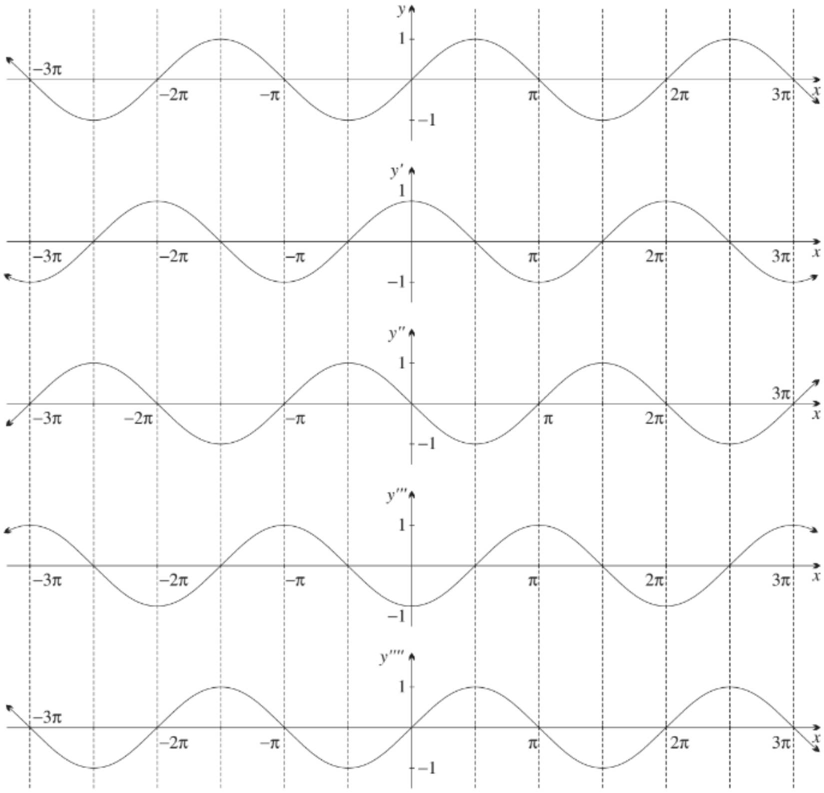
20

$$\frac{d}{dx}\left(\frac{1}{5}\sin^5 x - \frac{1}{7}\sin^7 x\right)$$
$$= \left(\frac{5}{5}\sin^4 x\right) \times \frac{d}{dx}(\sin x) - \left(\frac{7}{7}\sin^6 x\right) \times \frac{d}{dx}(\sin x)$$
$$= \sin^4 x \times \cos x - \sin^6 x \times \cos x$$
$$= \sin^4 x \cos x - \sin^6 x \cos x$$
$$= \sin^4 x \cos x - \sin^4 x \cos x \sin^2 x$$
$$= \sin^4 x \cos x - \sin^4 x \cos x\,(1 - \cos^2 x)$$
$$= \sin^4 x \cos x - \sin^4 x \cos x + \sin^4 x \cos^3 x$$
$$= \sin^4 x \cos^3 x$$

Uncorrected proofs

21a i $y = \sin x$

Then, $\frac{d}{dx}(\sin x) = \cos x$



It shows the first four derivatives of $y = \sin x$

Each application of differentiation shifts the wave left $\frac{\pi}{2}$ which is the quarter of 2π. Thus, the differentiation advances the phase by $\frac{\pi}{2}$ hence, the first derivative is

$\frac{d}{dx}(\sin x) = \cos x = \sin(x + \frac{\pi}{2})$

21a ii Each application of differentiation shifts the wave left $\frac{\pi}{2}$ which is the quarter of 2π. Thus, the differentiation advances the phase by $\frac{\pi}{2}$ hence, the second derivative will shift by 2 times of $\frac{\pi}{2}$, which is advances the phase by π

$\frac{d^2y}{dx^2} = \frac{d^2}{dx^2}(\sin x) = -\sin x = \sin(x + \pi)$

Uncorrected proofs

21a iii Each application of differentiation shifts the wave left $\frac{\pi}{2}$ which is the quarter of 2π. Thus, the differentiation advances the phase by $\frac{\pi}{2}$ hence, the second derivative will shift by 3 times of $\frac{\pi}{2}$, which is advances the phase by $\frac{3\pi}{2}$

$$\frac{d^3y}{dx^3} = \frac{d^3}{dx^3}(\sin x) = \sin(x + \frac{3\pi}{2})$$

21b By observation of the results in question 21a, the expression is

$$\frac{d^ny}{dx^n} = \sin(x + \frac{n\pi}{2})$$

22a LHS

$$= \frac{1}{2}(\sin((m+n)x) + \sin((m-n)x))$$
$$= \frac{1}{2}(\sin(mx+nx) + \sin(mx-nx))$$
$$= \frac{1}{2}(\sin mx \cos nx + \cos mx \sin nx + \sin mx \cos(-nx) + \cos(mx)\sin(-nx))$$
$$= \frac{1}{2}(\sin mx \cos nx + \cos mx \sin nx + \sin mx \cos nx - \cos mx \sin nx)$$
$$= \frac{1}{2}(2\sin mx \cos nx)$$
$$= \sin mx \cos nx$$

22b $\frac{d}{dx}(\sin mx \times \cos nx)$

$$= \frac{d}{dx}\left[\frac{1}{2}[(\sin(m+n)x + \sin(m-n)x)]\right]$$
$$= \frac{1}{2}[(m+n)\cos\big((m+n)x\big) + (m-n)\cos\big((m-n)x\big)]$$

22c LHS

$$= \frac{1}{2}(\cos(m+n)x + \cos(m-n)x)$$
$$= \frac{1}{2}(\cos(mx+nx) + \cos(mx-nx))$$
$$= \frac{1}{2}(\cos mx \cos nx - \sin mx \sin nx + \cos mx \cos(-nx) - \sin mx \sin(-nx))$$
$$= \frac{1}{2}[\cos mx \cos nx - \sin mx \sin nx + \cos mx \cos nx - - \sin mx \sin nx]$$
$$= \frac{1}{2}[\cos mx \cos nx - \sin mx \sin nx + \cos mx \cos nx + \sin mx \sin nx]$$

Uncorrected proofs

$$= \frac{1}{2}[2\cos mx \cos nx]$$
$$= \cos mx \cos nx$$

$$\frac{d}{dx}(\cos mx \cos nx)$$
$$= \frac{d}{dx}\left[\frac{1}{2}(\cos(m+n)x + \cos(m-n)x)\right]$$
$$= \frac{1}{2}(-(m+n)\sin(m+n)x - (m-n)\sin(m-n)x)$$
$$= -\frac{1}{2}[(m+n)\sin(m+n)x + (m-n)\sin(m-n)x]$$

23 $y = e^{-x}(\cos 2x + \sin 2x)$

$$y' = (\cos 2x + \sin 2x) \times \frac{d}{dx}(e^{-x}) + e^{-x} \times \frac{d}{dx}(\cos 2x + \sin 2x)$$
$$= (\cos 2x + \sin 2x) \times (-e^{-x}) + e^{-x} \times \left(\frac{d}{dx}(\cos 2x) + \frac{d}{dx}(\sin 2x)\right)$$
$$= (\cos 2x + \sin 2x) \times (-e^{-x}) + e^{-x}(-2\sin 2x + 2\cos 2x)$$
$$= -e^{-x}\cos 2x - e^{-x}\sin 2x - 2e^{-x}\sin 2x + 2e^{-x}\cos 2x$$
$$= e^{-x}\cos 2x - 3e^{-x}\sin 2x$$

$$y'' = \left(\cos 2x \times \frac{d}{dx}(e^{-x}) + e^{-x} \times \frac{d}{dx}(\cos 2x)\right)$$
$$-3\left(\sin 2x \times \frac{d}{dx}(e^{-x}) + e^{-x} \times \frac{d}{dx}(\sin 2x)\right)$$
$$= (-e^{-x}\cos 2x - 2e^{-x}\sin 2x) - 3(-e^{-x}\sin 2x + 2e^{-x}\cos 2x)$$
$$= -e^{-x}\cos 2x - 2e^{-x}\sin 2x + 3e^{-x}\sin 2x - 6e^{-x}\cos 2x$$
$$= -7e^{-x}\cos 2x + e^{-x}\sin 2x$$

$$y'' + 2y' + 5y$$
$$= -7e^{-x}\cos 2x + e^{-x}\sin 2x + 2(e^{-x}\cos 2x - 3e^{-x}\sin 2x)$$
$$+5(e^{-x}\cos 2x + e^{-x}\sin 2x)$$
$$= -7e^{-x}\cos 2x + e^{-x}\sin 2x + 2e^{-x}\cos 2x - 6e^{-x}\sin 2x$$
$$+5e^{-x}\cos 2x + 5e^{-x}\sin 2x$$
$= 0$ as required

Uncorrected proofs

24a $y = \ln(\tan 2x)$

$\text{LHS} = \frac{d}{dx}\ln(\tan 2x)$

Let, $u = \tan 2x$

Hence, $\frac{d}{dx}\ln(\tan 2x)$ and using chain rule

$= \left(\frac{d}{du}\ln u\right) \times \frac{d}{dx}\tan 2x$

$= \frac{1}{u} \times 2\sec^2 2x$

$= \frac{1}{\tan 2x} \times 2\sec^2 2x$

$= \frac{1}{\frac{\sin 2x}{\cos 2x}} \times 2\sec 2x \times \sec 2x$

$= \frac{\cos 2x}{\sin 2x} \times 2\sec 2x \times \frac{1}{\cos 2x}$

$= \frac{1}{\sin 2x} \times 2\sec 2x$

$= 2 \times \frac{1}{\sin 2x} \times \sec 2x$

$= 2 \times \sec 2x \times \operatorname{cosec} 2x$

$= \text{RHS}$

Hence, proved.

24b $y = \ln\left(\frac{\sqrt{2}-\cos x}{\sqrt{2}+\cos x}\right)$

$\text{LHS} = \frac{d}{dx}\ln\left(\frac{\sqrt{2}-\cos x}{\sqrt{2}+\cos x}\right)$

Let $u = \frac{\sqrt{2}-\cos x}{\sqrt{2}+\cos x}$

Hence, $\frac{d}{du}\ln\left(\frac{\sqrt{2}-\cos x}{\sqrt{2}+\cos x}\right)$

$= \left(\frac{d}{du}\ln u\right)$

$= \frac{1}{u}$

$= \frac{1}{\frac{\sqrt{2}-\cos x}{\sqrt{2}+\cos x}}$

$= \frac{\sqrt{2}+\cos x}{\sqrt{2}-\cos x}$

Uncorrected proofs

and,

$\frac{du}{dx} = \frac{d}{dx}\left(\frac{\sqrt{2}-\cos x}{\sqrt{2}+\cos x}\right)$

$= \frac{(\sqrt{2}+\cos x)\times\frac{d}{dx}(\sqrt{2}-\cos x)-(\sqrt{2}-\cos x)\times\frac{d}{dx}(\sqrt{2}+\cos x)}{(\sqrt{2}+\cos x)^2}$

$= \frac{(\sqrt{2}+\cos x)\times-(-\sin x)-(\sqrt{2}-\cos x)\times(-\sin x)}{(\sqrt{2}+\cos x)^2}$

$= \frac{(\sqrt{2}+\cos x)\times\sin x-(\sqrt{2}-\cos x)\times(-\sin x)}{(\sqrt{2}+\cos x)^2}$

$= \frac{\sqrt{2}\sin x+\sin x\cos x+\sqrt{2}\sin x-\sin x\cos x}{(\sqrt{2}+\cos x)^2}$

$= \frac{2\sqrt{2}\sin x}{(\sqrt{2}+\cos x)^2}$

Thus, by the chain rule

$\frac{d}{dx}\ln\left(\frac{\sqrt{2}-\cos x}{\sqrt{2}+\cos x}\right)$

$= \left(\frac{d}{du}\ln u\right)\frac{du}{dx}$

$= \frac{\sqrt{2}+\cos x}{\sqrt{2}-\cos x}\times\frac{2\sqrt{2}\sin x}{(\sqrt{2}+\cos x)^2}$

$= \frac{2\sqrt{2}\sin x}{(\sqrt{2}-\cos x)\times(\sqrt{2}+\cos x)}$

$= \frac{2\sqrt{2}\sin x}{2-\cos^2 x}$

$= \frac{2\sqrt{2}\sin x}{1+1-\cos^2 x}$

$= \frac{2\sqrt{2}\sin x}{1+\sin^2 x}$

$=$ RHS

Hence, proved.

25 Differentiation by first principles formula is:

$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

Applying this formula to the function $f(x) = \cos x$ gives:

$\frac{d}{dx}(\cos x) = \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h}$

Uncorrected proofs

Using the difference of cosines identity:

$\cos P - \cos Q = -2\sin\frac{1}{2}(P+Q)\sin\frac{1}{2}(P-Q)$

Then, letting $P = x + h$ and $Q = x$ we have $P + Q = 2x + h$ and $P - Q = h$

$$\frac{d}{dx}(\cos x) = \lim_{h\to 0}\frac{\cos(x+h) - \cos x}{h}$$

$$= \lim_{h\to 0}\frac{-2\sin\frac{1}{2}(2x+h)\sin\frac{1}{2}h}{h}$$

$$= \lim_{h\to 0} -2\sin\left(x+\frac{1}{2}h\right)\frac{\sin\frac{1}{2}h}{h}$$

$$= \lim_{h\to 0} -\sin\left(x+\frac{1}{2}h\right)\frac{\sin\frac{1}{2}h}{\frac{1}{2}h}$$

$$= \left(-\sin\left(x+\frac{1}{2}(0)\right)\right)\times 1 \text{ (because } \lim_{h\to 0}\frac{\sin\frac{1}{2}h}{\frac{1}{2}h} = 1\text{)}$$

$$= -\sin x$$

Uncorrected proofs

Solutions to Exercise 7C

1a $y = \sin x$

$\dfrac{dy}{dx} = \cos x$

When $x = 0, \frac{dy}{dx} = \cos 0 = 1$

Gradient of the tangent to $y = \sin x$ at $x = 0$ is 1.

1b $y = \cos x$

$\dfrac{dy}{dx} = -\sin x$

When $x = \frac{\pi}{2}, \frac{dy}{dx} = -\sin\frac{\pi}{2} = -1$

Gradient of the tangent to $y = \cos x$ at $x = \frac{\pi}{2}$ is -1.

1c $y = \sin x$

$\dfrac{dy}{dx} = \cos x$

When $x = \frac{\pi}{3}, \frac{dy}{dx} = \cos\frac{\pi}{3} = \frac{1}{2}$

Gradient of the tangent to $y = \sin x$ at $x = \frac{\pi}{3}$ is $\frac{1}{2}$.

1d $y = \cos x$

$\dfrac{dy}{dx} = -\sin x$

When $x = \frac{\pi}{6}, \frac{dy}{dx} = -\sin\frac{\pi}{6} = -\frac{1}{2}$

Gradient of the tangent to $y = \cos x$ at $x = \frac{\pi}{6}$ is $-\frac{1}{2}$.

1e $y = \sin x$

$\dfrac{dy}{dx} = \cos x$

When $x = \frac{\pi}{4}, \frac{dy}{dx} = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

Gradient of the tangent to $y = \sin x$ at $x = \frac{\pi}{4}$ is $\frac{1}{\sqrt{2}}$.

Uncorrected proofs

1f $y = \tan x$

$\dfrac{dy}{dx} = \sec^2 x$

When $x = 0, \frac{dy}{dx} = \sec^2 0 = 1$

Gradient of the tangent to $y = \tan x$ at $x = 0$ is 1.

1g $y = \tan x$

$\dfrac{dy}{dx} = \sec^2 x$

When $x = \frac{\pi}{4}, \frac{dy}{dx} = \sec^2 \frac{\pi}{4} = 2$

Gradient of the tangent to $y = \tan x$ at $x = \frac{\pi}{4}$ is 2.

1h $y = \cos 2x$

$\dfrac{dy}{dx} = -2\sin 2x$

When $x = \frac{\pi}{4}, \frac{dy}{dx} = -2\sin\frac{\pi}{2} = -2 \times 1 = -2$

Gradient of the tangent to $y = \cos 2x$ at $x = \frac{\pi}{4}$ is -2.

1i $y = -\cos\frac{1}{2}x$

$\dfrac{dy}{dx} = \dfrac{1}{2}\sin\dfrac{1}{2}x$

When $x = \frac{2\pi}{3}, \frac{dy}{dx} = \frac{1}{2}\sin\frac{\pi}{3} = \frac{1}{2} \times \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4}$

Gradient of the tangent to $y = -\cos\frac{1}{2}x$ at $x = \frac{2\pi}{3}$ is $\frac{\sqrt{3}}{4}$.

1j $y = \sin\frac{x}{2}$

$\dfrac{dy}{dx} = \dfrac{1}{2}\cos\dfrac{x}{2}$

When $x = \frac{2\pi}{3}, \frac{dy}{dx} = \frac{1}{2}\cos\frac{\pi}{3} = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

Gradient of the tangent to $y = \sin\frac{x}{2}$ at $x = \frac{2\pi}{3}$ is $\frac{1}{4}$.

Uncorrected proofs

1k $y = \tan 2x$

$\dfrac{dy}{dx} = 2\sec^2 2x$

When $x = \frac{\pi}{6}, \frac{dy}{dx} = 2\sec^2\frac{\pi}{3} = 2 \times 4 = 8$

Gradient of the tangent to $y = \tan 2x$ at $x = \frac{\pi}{6}$ is 8.

1l $y = \sin 2x$

$\dfrac{dy}{dx} = 2\cos 2x$

When $x = \frac{\pi}{12}, \frac{dy}{dx} = 2\cos\frac{\pi}{6} = 2 \times \frac{\sqrt{3}}{2} = \sqrt{3}$

Gradient of the tangent to $y = \sin 2x$ at $x = \frac{\pi}{12}$ is $\sqrt{3}$.

2a For $y = \sin x$

$\dfrac{dy}{dx} = \cos x$

When $x = 0$, the gradient of $\sin x$ is $\frac{dy}{dx} = \cos 0 = 1$.

Thus the equation of the tangent at $(0, 0)$ is

$y - y_1 = m(x - x_1)$

$y - 0 = 1(x - 0)$

$y = x$

Hence $y = x$ is a tangent to $y = \sin x$ at $(0, 0)$.

2b For $y = \tan x$

$\dfrac{dy}{dx} = \sec^2 x$

When $x = 0$, the gradient of $\tan x$ is $\frac{dy}{dx} = \sec^2 0 = 1$.

Thus, the equation of the tangent is

$y - y_1 = m(x - x_1)$

$y - 0 = 1(x - 0)$

$y = x$

Uncorrected proofs

Hence $y = x$ is a tangent to $y = \tan x$ at $(0, 0)$.

2c For $y = \cos x$

$$\frac{dy}{dx} = -\sin x$$

When $x = \frac{\pi}{2}$, the gradient of $\cos x$ is $\frac{dy}{dx} = -\sin\left(\frac{\pi}{2}\right) = -1$.

Thus the equation of the tangent at $(\frac{\pi}{2}, 0)$ is

$$y - y_1 = m(x - x_1)$$

$$y - 0 = -1\left(x - \frac{\pi}{2}\right)$$

$$y = -x + \frac{\pi}{2}$$

Hence $y = \frac{\pi}{2} - x$ is a tangent to $y = \cos x$ at $(\frac{\pi}{2}, 0)$.

3a For $y = \sin x$

$$\frac{dy}{dx} = \cos x$$

When $x = \pi$, the gradient of $\sin x$ is $\frac{dy}{dx} = \cos\pi = -1$.

Thus, the equation of the tangent at $(\pi, 0)$ is

$$y - y_1 = m(x - x_1)$$

$$y - 0 = -1(x - \pi)$$

$$y = -x + \pi$$

3b For $y = \tan x$

$$\frac{dy}{dx} = \sec^2 x$$

When $x = \frac{\pi}{4}$, the gradient of $\tan x$ is $\frac{dy}{dx} = \sec^2\frac{\pi}{4} = 2$.

Thus, the equation of the tangent at $\left(\frac{\pi}{4}, 1\right)$ is

$$y - y_1 = m(x - x_1)$$

Uncorrected proofs

$y - 1 = 2\left(x - \frac{\pi}{4}\right)$

$y = 2x - \frac{\pi}{2} + 1$

$2x - y = \frac{\pi}{2} - 1$

3c For $y = \cos x$

$\frac{dy}{dx} = -\sin x$

When $x = \frac{\pi}{6}$, the gradient of $\cos x$ is $\frac{dy}{dx} = -\sin\frac{\pi}{6} = -\frac{1}{2}$.

Thus, the equation of the tangent at $\left(\frac{\pi}{6}, \frac{\sqrt{3}}{2}\right)$ is

$y - y_1 = m(x - x_1)$

$y - \frac{\sqrt{3}}{2} = -\frac{1}{2}\left(x - \frac{\pi}{6}\right)$

$y = -\frac{x}{2} + \frac{\pi}{12} + \frac{\sqrt{3}}{2}$

$\frac{x}{2} + y = \frac{\pi}{12} + \frac{\sqrt{3}}{2}$

$x + 2y = \frac{\pi}{6} + \sqrt{3}$

3d For $y = \cos 2x$

$\frac{dy}{dx} = -2\sin 2x$

When $x = \frac{\pi}{4}$, the gradient of $\cos 2x$ is $\frac{dy}{dx} = -2\sin\frac{\pi}{2} = -2 \times 1 = -2$.

Thus, the equation of the tangent at $\left(\frac{\pi}{4}, 0\right)$ is

$y - y_1 = m(x - x_1)$

$y - 0 = -2\left(x - \frac{\pi}{4}\right)$

$y = -2x + \frac{\pi}{2}$

Uncorrected proofs

3e For $y = \sin 2x$

$$\frac{dy}{dx} = 2\cos 2x$$

When $x = \frac{\pi}{3}$, the gradient of $\sin 2x$ is $\frac{dy}{dx} = 2\cos\frac{2\pi}{3} = 2 \times -\frac{1}{2} = -1$.

Thus, the equation of the tangent at $\left(\frac{\pi}{3}, \frac{\sqrt{3}}{2}\right)$ is

$$y - y_1 = m(x - x_1)$$

$$y - \frac{\sqrt{3}}{2} = -1\left(x - \frac{\pi}{3}\right)$$

$$y = -x + \frac{\pi}{3} + \frac{\sqrt{3}}{2}$$

$$x + y = \frac{\pi}{3} + \frac{\sqrt{3}}{2}$$

3f For $y = x\sin x$

$$\frac{dy}{dx} = \sin x \times \frac{d}{dx}(x) + x \times \frac{d}{dx}(\sin x)$$
$$= \sin x + x\cos x$$

When $x = \pi$, the gradient of $x\sin x$ is $\frac{dy}{dx} = \sin\pi + \pi\cos\pi = 0 + \pi \times -1 = -\pi$.

Thus, the equation of the tangent at $(\pi, 0)$ is

$$y - y_1 = m(x - x_1)$$

$$y - 0 = -\pi(x - \pi)$$

$$y = -\pi x + \pi^2$$

4a For $y = 2\sin x$

$$\frac{dy}{dx} = 2\cos x$$

The gradient is zero when $\frac{dy}{dx} = 0$.

$$2\cos x = 0$$

$$\cos x = 0$$

$$x = \frac{\pi}{2}, \frac{3\pi}{2}$$

Uncorrected proofs

4b For $y = 2\sin x - x$

$$\frac{dy}{dx} = 2\cos x - 1$$

The gradient is zero when $\frac{dy}{dx} = 0$.

$$2\cos x - 1 = 0$$

$$\cos x = \frac{1}{2}$$

$$x = \frac{\pi}{3}, \frac{5\pi}{3}$$

4c For $y = 2\cos x + x$

$$\frac{dy}{dx} = -2\sin x + 1$$

The gradient is zero when $\frac{dy}{dx} = 0$.

$$-2\sin x + 1 = 0$$

$$2\sin x = 1$$

$$\sin x = \frac{1}{2}$$

$$x = \frac{\pi}{6}, \frac{5\pi}{6}$$

4d For $y = 2\sin x + \sqrt{3}x$

$$\frac{dy}{dx} = 2\cos x + \sqrt{3}$$

The gradient is zero when $\frac{dy}{dx} = 0$.

$$2\cos x + \sqrt{3} = 0$$

$$2\cos x = -\sqrt{3}$$

$$\cos x = -\frac{\sqrt{3}}{2}$$

$$x = \frac{5\pi}{6}, \frac{7\pi}{6}$$

Uncorrected proofs

5a For $y = 2\sin x - \cos 2x$

$$\frac{dy}{dx} = 2\cos x + 2\sin 2x$$

When $x = \frac{\pi}{6}$, the gradient of $2\sin x - \cos 2x$ is

$$\frac{dy}{dx} = 2\cos\frac{\pi}{6} + 2\sin\frac{2\pi}{6} = \frac{2\sqrt{3}}{2} + \frac{2\sqrt{3}}{2} = 2\sqrt{3}$$

Thus, the equation of the tangent at $\left(\frac{\pi}{6}, \frac{1}{2}\right)$ is

$$y - y_1 = m(x - x_1)$$

$$y - \frac{1}{2} = 2\sqrt{3}\left(x - \frac{\pi}{6}\right)$$

$$y - \frac{1}{2} = 2\sqrt{3}x - \frac{\pi\sqrt{3}}{3}$$

$$2\sqrt{3}x - y = \frac{1}{3}\pi\sqrt{3} - \frac{1}{2}$$

5b The gradient of the normal is $m_{norm} = -\frac{1}{\left(\frac{dy}{dx}\right)} = -\frac{1}{2\sqrt{3}}$

Thus, the equation of the normal at $\left(\frac{\pi}{6}, \frac{1}{2}\right)$ is

$$y - y_1 = m(x - x_1)$$

$$y - \frac{1}{2} = -\frac{1}{2\sqrt{3}}\left(x - \frac{\pi}{6}\right)$$

$$y - \frac{1}{2} = -\frac{1}{2\sqrt{3}}x + \frac{\pi}{12\sqrt{3}}$$

Multiplying by $2\sqrt{3}$ gives:

$$2\sqrt{3}y - \sqrt{3} = -x + \frac{\pi}{6}$$

$$x + 2\sqrt{3}y = \frac{\pi}{6} + \sqrt{3}$$

Uncorrected proofs

6a $y = \sin^2 x$

$$y' = \frac{d}{dx}(\sin^2 x)$$

$$= 2\sin x \times \frac{d}{dx}(\sin x)$$

$$= 2\sin x \times \cos x$$

$$= 2\sin x \cos x$$

6b When $x = \frac{\pi}{4}$,

$$m_{\text{tangent}} = y' = 2\sin\frac{\pi}{4}\cos\frac{\pi}{4} = 2\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) = 1$$

$$m_{\text{norm}} = -\frac{1}{y'} = -1$$

6c Note that when $x = \frac{\pi}{4}$, $y = \sin^2\frac{\pi}{4} = \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$

Tangent:

$$y - y_1 = m(x - x_1)$$

$$y - \frac{1}{2} = 1\left(x - \frac{\pi}{4}\right)$$

$$y = x - \frac{\pi}{4} + \frac{1}{2}$$

$$x - y = \frac{\pi}{4} - \frac{1}{2}$$

Normal:

$$y - y_1 = m(x - x_1)$$

$$y - \frac{1}{2} = -1\left(x - \frac{\pi}{4}\right)$$

$$y = -x + \frac{\pi}{4} + \frac{1}{2}$$

$$x + y = \frac{\pi}{4} + \frac{1}{2}$$

Uncorrected proofs

6d For the tangent:

When $y = 0$,

$$x - 0 = \frac{\pi}{4} - \frac{1}{2}$$

$$x = \frac{\pi}{4} - \frac{1}{2}$$

Hence $P(\frac{\pi}{4} - \frac{1}{2}, 0)$.

For the normal:

When $x = 0$,

$$0 + y = \frac{\pi}{4} + \frac{1}{2}$$

$$y = \frac{\pi}{4} + \frac{1}{2}$$

Hence $Q\left(0, \frac{\pi}{4} + \frac{1}{2}\right)$.

Thus the triangle has base length $\left(\frac{\pi}{4} - \frac{1}{2}\right)$ and height $\left(\frac{\pi}{4} + \frac{1}{2}\right)$ so

$$A = \frac{1}{2}\left(\frac{\pi}{4} - \frac{1}{2}\right)\left(\frac{\pi}{4} + \frac{1}{2}\right)$$

$$= \frac{1}{2}\left(\frac{\pi - 2}{4}\right)\left(\frac{\pi + 2}{4}\right)$$

$$= \frac{1}{32}(\pi - 2)(\pi + 2)$$

$$= \frac{1}{32}(\pi^2 - 4)\ \mathrm{u}^2$$

7a $y = e^{\sin x}$

$$y' = \frac{d}{dx}\left(e^{\sin x}\right)$$

$$= e^{\sin x} \times \frac{d}{dx}(\sin x)$$

$$= e^{\sin x} \times \cos x$$

$$= \cos x\, e^{\sin x}$$

Uncorrected proofs

7b The tangent is horizontal when $y' = 0$. This is when

$\cos x\, e^{\sin x} = 0$

As $e^{\sin x} \neq 0$ for all x, we conclude that we must have

$\cos x = 0$

$x = \frac{\pi}{2}, \frac{3\pi}{2}$ for $x \in [0, 2\pi]$

8a $y = e^{\cos x}$

$$y' = \frac{d}{dx}(e^{\cos x})$$

$$= e^{\cos x} \times \frac{d}{dx}(\cos x)$$

$$= e^{\cos x} \times (-\sin x)$$

$$= -\sin x\, e^{\cos x}$$

8b The tangent is horizontal when $y' = 0$. This is when

$-\sin x\, e^{\cos x} = 0$

As $e^{\cos x} \neq 0$, we conclude that we must have

$\sin x = 0$

$x = 0, \pi, 2\pi$ for $x \in [0, 2\pi]$

9a $y = \cos x + \sqrt{3} \sin x$

$y' = -\sin x + \sqrt{3} \cos x$

$y'' = -\cos x - \sqrt{3} \sin x$

9b Stationary points occur when $y' = 0$. This is when

$-\sin x + \sqrt{3} \cos x = 0$

$\sin x = \sqrt{3} \cos x$

$\tan x = \sqrt{3}$

$x = \frac{\pi}{3}, \frac{4\pi}{3}$ for $x \in [0, 2\pi]$

Uncorrected proofs

When $x = \frac{\pi}{3}$, $y = 2$ and $y'' = -2 < 0$. Hence the curve is concave down at this point and thus there is a maximum turning point at $(\frac{\pi}{3}, 2)$.

When $x = \frac{4\pi}{3}$, $y = -2$ and $y'' = 2 > 0$. Hence the curve is concave up at this point and thus there is a minimum turning point at $(\frac{4\pi}{3}, -2)$.

9c Points of inflection occur when $y'' = 0$. This is when

$-\cos x - \sqrt{3}\sin x = 0$

$\sqrt{3}\sin x = -\cos x$

$\frac{\sin x}{\cos x} = -\frac{1}{\sqrt{3}}$

$\tan x = -\frac{1}{\sqrt{3}}$

$x = \frac{5\pi}{6}, \frac{11\pi}{6}$ for $x \in [0, 2\pi]$

When $x = \frac{5\pi}{6}$, $y = 0$ and when $x = \frac{11\pi}{6}$, $y = 0$.

Thus, there are possible points of inflection at $(\frac{5\pi}{6}, 0)$ and $(\frac{11\pi}{6}, 0)$.

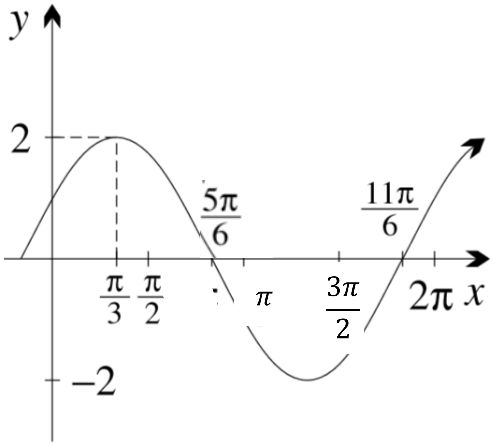
Furthermore, we test y'' on either side of these points to confirm that the concavity changes.

x	0	$\frac{5\pi}{6}$	π	$\frac{11\pi}{6}$	2π
y''	-1	0	1	0	-1

As we can see the concavity changes on either side of these points and hence they must both be points of inflection.

Uncorrected proofs

9d A sketch of $y = \cos x + \sqrt{3}\sin x$ is shown below.



10a $y = x + \sin x$

$y' = 1 + \cos x$

$y'' = 0 - \sin x$

Hence

$y'' = -\sin x$ as required

10b Stationary points occur when $y' = 0$. This is when

$1 + \cos x = 0$

$\cos x = -1$

$x = -\pi, \pi$ for $x \in (-2\pi, 2\pi)$

When $x = \pi, y = \pi$ and $y'' = -\sin \pi = 0$.

When $x = -\pi, y = -\pi$ and $y'' = -\sin(-\pi) = 0$.

So $(-\pi, -\pi)$ and (π, π) are possible horizontal points of inflection.

Testing the derivative to confirm this

x	-2π	$-\pi$	0	π	2π
y'	2	0	2	0	2

Thus the derivative is positive either side of the stationary points and hence the stationary points are points of inflection.

Uncorrected proofs

10c Points of inflection occur when $y'' = 0$. This is when

$-\sin x = 0$

$\sin x = 0$

$x = -\pi, 0, \pi \quad$ for $x \in (-2\pi, 2\pi)$

We have already seen that there are horizontal points of inflection at $(-\pi, -\pi)$ and (π, π).

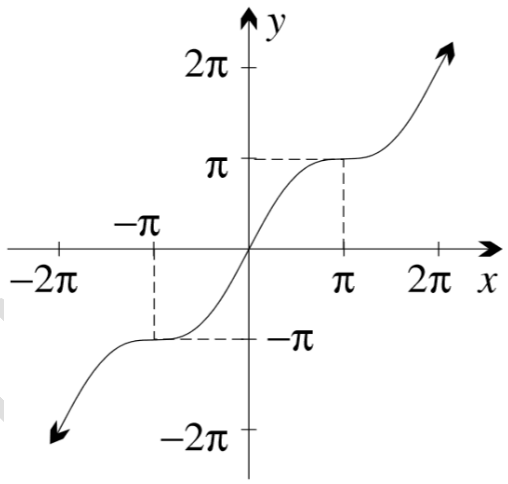
When $x = 0$, $y = 0$ and $y'' = 0$. Hence a possible point of inflection at $(0, 0)$.

We can test y'' on either side of these three points to confirm that the concavity changes.

x	$-\frac{3\pi}{2}$	$-\pi$	$-\frac{\pi}{2}$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
y''	-1	0	1	0	1	0	-1

As we can see the concavity changes on either side of these points and hence, they must be points of inflection.

10d A sketch of $y = x + \sin x$ is shown below.



11

$y = 2\sin x + x$

$y' = 2\cos x + 1$

$y'' = -2\sin x$

$y' = 0$ for stationary points:

$2\cos x + 1 = 0$

$2\cos x = -1$

$\cos x = -\frac{1}{2}$

$x = \frac{2\pi}{3}, \frac{4\pi}{3} \quad \text{for } x \in [0, 2\pi]$

When $x = \frac{2\pi}{3}$, $y = \frac{2\pi}{3} + \sqrt{3}$, and $y'' < 0$. Hence the curve is concave down and this is a maximum turning point.

When $x = \frac{4\pi}{3}$, $y = \frac{4\pi}{3} - \sqrt{3}$, and $y'' > 0$. Hence the curve is concave up and this is a minimum turning point.

Hence there is a maximum turning point at $(\frac{2\pi}{3}, \frac{2\pi}{3} + \sqrt{3})$ and a minimum turning point at $(\frac{4\pi}{3}, \frac{4\pi}{3} - \sqrt{3})$.

$y'' = 0$ for points of inflection:

$-2\sin x = 0$

$x = 0, \pi, 2\pi \quad \text{for } x \in [0, 2\pi]$

When $x = 0$, $y = 0$

When $x = \pi, y = \pi$

When $x = 2\pi, y = 2\pi$

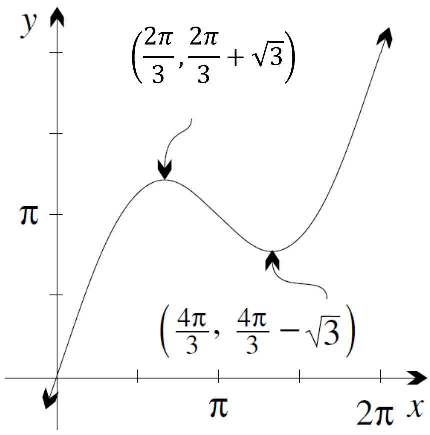
Possible points of inflection at $(0, 0)$, (π, π) and $(2\pi, 2\pi)$.

We can test y'' on either side of these three points to see if the concavity changes.

x	$-\frac{\pi}{2}$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$
y''	2	0	-2	0	2	0	-2

Uncorrected proofs

As we can see the concavity changes on either side of these points and hence, they must be points of inflection.



12 The area of the triangle is given by $A = \frac{1}{2}ab\sin C = \frac{1}{2}(10)(10)\sin\theta = 50\sin\theta$

Thus $\frac{dA}{d\theta} = 50\cos\theta$

The angle is increasing at $3° = 3 \times \frac{\pi}{180} = \frac{\pi}{60}$ radians per minute. Hence $\frac{d\theta}{dt} = \frac{\pi}{60}$.

Now, $\frac{dA}{dt} = \frac{dA}{d\theta} \times \frac{d\theta}{dt}$

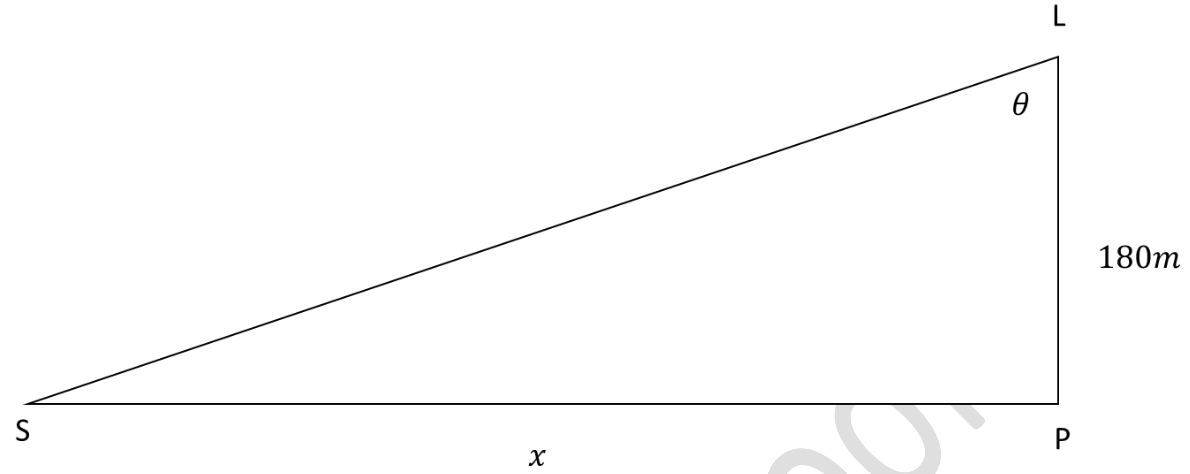
$= 50\cos\theta \times \frac{\pi}{60}$

When $\theta = 30° = 30 \times \frac{\pi}{180} = \frac{\pi}{6}$ radians

$\frac{dA}{dt} = \left(50\cos\frac{\pi}{6}\right) \times \frac{\pi}{60} = \frac{5\sqrt{3}\pi}{12}$ cm² per minute

Uncorrected proofs

13



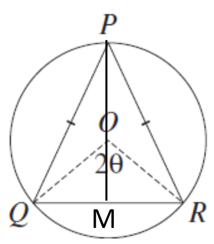
Let S be a point along the shore xm from P where the light from the lighthouse is shining. We aim to find $\frac{dx}{dt}$ when $x = 300$. From the diagram it follows that $\tan\theta = \frac{x}{180}$ and hence $x = 180\tan\theta$. Thus $\frac{dx}{d\theta} = 180\sec^2\theta$. Note that one revolution is 2π radians. Hence the light rotates at $\frac{2\pi}{10} = \frac{\pi}{5}$ rad/sec so $\frac{d\theta}{dt} = \frac{\pi}{5}$

Hence, $\frac{dx}{dt} = \frac{dx}{d\theta} \times \frac{d\theta}{dt} = 180\sec^2\theta \times \frac{\pi}{5} = 36\pi\sec^2\theta$ m/s

When $x = 300$, $\tan\theta = \frac{300}{180}$ and $\sec^2\theta = 1 + \tan^2\theta = 1 + \left(\frac{300}{180}\right)^2$.

Thus, it follows that $\frac{dx}{dt} = 36\pi\left(1 + \left(\frac{300}{180}\right)^2\right) = 136\pi$ m/s at a point 300 m along the shore from the point .

14a



$$\sin\theta = \frac{QM}{OQ} = \frac{QM}{1} = QM$$

$$\cos\theta = \frac{OM}{OR} = \frac{OM}{1} = OM$$

14b

$$\begin{aligned} A &= \frac{1}{2}bh \\ &= \frac{1}{2}QR \times PM \\ &= \frac{1}{2}(2QM)(PO + OM) \\ &= \frac{1}{2}2\sin\theta\,(1+\cos\theta) \\ &= \sin\theta\,(1+\cos\theta) \end{aligned}$$

14c $A = \sin\theta\,(1+\cos\theta)$

$= \sin\theta + \sin\theta\cos\theta$

$= \sin\theta + \dfrac{1}{2}\sin 2\theta$

$\dfrac{dA}{d\theta} = \cos\theta + \cos 2\theta$

$= \cos\theta + 2\cos^2\theta - 1$

$= (2\cos\theta - 1)(\cos\theta + 1)$

For maxima/minima, $\frac{dA}{d\theta} = 0$.

$(2\cos\theta - 1)(\cos\theta + 1) = 0$

$2\cos\theta - 1 = 0$ or $\cos\theta + 1 = 0$

For our angles to be acute, $\cos\theta > 0$.

So $\cos\theta = -1$ or $\theta = \pi$ is not a solution. Hence:

$2\cos\theta - 1 = 0$

$2\cos\theta = 1$

$\cos\theta = \dfrac{1}{2}$

$\theta = \dfrac{\pi}{3}$

$\dfrac{d^2A}{d\theta^2} = -\sin\theta - 2\sin 2\theta$

When $\theta = \frac{\pi}{3}, \frac{d^2A}{d\theta^2} < 0$, hence the curve is concave down and this is a maximum.

Uncorrected proofs

Hence the maximum area occurs when $\theta = \frac{\pi}{3}$.

Note that the triangle is equilateral as:

$\angle QOR = 2\theta = \frac{2\pi}{3}$ and $\angle POQ = \angle POR = \frac{1}{2}(2\pi - \angle QOR) = \frac{1}{2}\left(2\pi - \frac{2\pi}{3}\right) = \frac{2\pi}{3}$.
Hence all vertices of the triangle are evenly spaced from one another around the unit circle and hence they must be the vertices of an equilateral triangle.

15a

$$\frac{d}{d\theta}\left(\frac{2 - \sin\theta}{\cos\theta}\right)$$

$$= \frac{\cos\theta \times \frac{d}{d\theta}(2 - \sin\theta) - (2 - \sin\theta)\frac{d}{d\theta}(\cos\theta)}{\cos^2\theta}$$
$$= \frac{\cos\theta \times (-\cos\theta) - (2 - \sin\theta)(-\sin\theta)}{\cos^2\theta}$$
$$= \frac{-\cos^2\theta + 2\sin\theta - \sin^2\theta}{\cos^2\theta}$$
$$= \frac{-(\cos^2\theta + \sin^2\theta) + 2\sin\theta}{\cos^2\theta}$$
$$= \frac{-1 + 2\sin\theta}{\cos^2\theta}$$
$$= \frac{2\sin\theta - 1}{\cos^2\theta}$$

15b Local maximum and minimum values occur when:

$$\frac{d}{d\theta}\left(\frac{2 - \sin\theta}{\cos\theta}\right) = 0$$

$$\frac{2\sin\theta - 1}{\cos^2\theta} = 0$$

$\cos\theta \neq 0$, that is, $\theta \neq \frac{\pi}{2}$ or $\frac{3\pi}{2}$

However, we are considering the interval $0 \leq \theta \leq \frac{\pi}{4}$, so $\cos\theta \neq 0$. Hence:

$2\sin\theta - 1 = 0$

$$\sin\theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{6}$$

Uncorrected proofs

Now we find the values of local maxima/minima and the endpoints:

When $\theta = 0, \frac{2-\sin 0}{\cos 0} = \frac{2-0}{1} = 2$

When $\theta = \frac{\pi}{6}, \frac{2-\sin\frac{\pi}{6}}{\cos\frac{\pi}{6}} = \frac{2-\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{3}{2} \times \frac{2}{\sqrt{3}} = \sqrt{3}$

When $\theta = \frac{\pi}{4}, \frac{2-\sin\frac{\pi}{4}}{\cos\frac{\pi}{4}} = \frac{2-\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} = \sqrt{2}\left(2 - \frac{1}{\sqrt{2}}\right) = 2\sqrt{2} - 1$

Hence the maximum value is 2 when $\theta = 0$ and the minimum value is $\sqrt{3}$ when $\theta = \frac{\pi}{6}$.

16a $y = 2\sin x + \cos 2x$

$y' = 2\cos x - 2\sin 2x$

$y'' = -2\sin x - 4\cos 2x$

16b When $\sin x = \frac{1}{2}$ and $\cos x = 0$

Then,
$y' = 2\cos x - 2\sin 2x$

$y' = 2\cos x - 2 \times 2\sin x\cos x$, using the formula, $\sin 2x = 2\sin x\cos x$

$y' = 2\cos x - 4\sin x\cos x$

$y' = 2\cos x\,(1 - 2\sin x)$

Therefore if $\cos x = 0$ or $\sin x = \frac{1}{2}$, then $y' = 0$.

Hence, proved

16c Stationary points occur when $y' = 0$. This is when:

$2\cos x - 2\sin 2x = 0$

$-2\sin 2x = -2\cos x$

$2\sin 2x = 2\cos x$

$\sin 2x = \cos x$

Using the formula, $\sin 2x = 2\sin x\cos x$,

$2\sin x\cos x = \cos x$

Uncorrected proofs

$2\sin x\cos x - \cos x = 0$

$\cos x\,(2\sin x - 1) = 0$

$2\sin x = 1$ or $\cos x = 0$

For $\sin x = \frac{1}{2}$

$x = \frac{\pi}{6}, \frac{5\pi}{6}$ for $x \in [-\pi, \pi]$

When $x = \frac{\pi}{6}$, $y = \frac{3}{2}$ and $y'' = -3 < 0$. Hence the curve is concave down at this point and thus there is a maximum turning point at $(\frac{\pi}{6}, \frac{3}{2})$.

When $x = \frac{5\pi}{6}$, $y = \frac{3}{2}$ and $y'' = -1 < 0$. Hence the curve is concave down at this point and thus there is a maximum turning point at $(\frac{5\pi}{6}, \frac{3}{2})$.

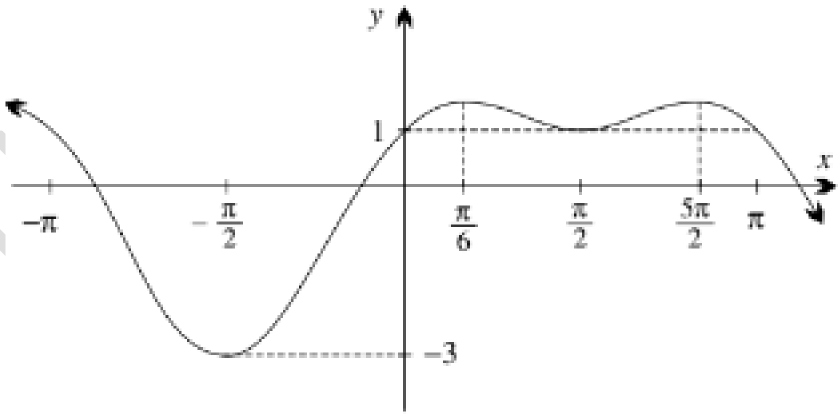
For $\cos x = 0$

$x = -\frac{\pi}{2}, \frac{\pi}{2}$ for $x \in [-\pi, \pi]$

When $x = -\frac{\pi}{2}$, $y = -3$ and $y'' = 6 > 0$. Hence the curve is concave up at this point and thus there is a minimum turning point at $(-\frac{\pi}{2}, -3)$.

When $x = \frac{\pi}{2}$, $y = 1$ and $y'' = 2 > 0$. Hence the curve is concave up at this point and thus there is a minimum turning point at $(\frac{\pi}{2}, 1)$.

16d Sketch of the curve is shown as below



Uncorrected proofs

17a $y = e^{-x} \cos x$

$y' = e^{-x} \dfrac{d}{dx}(\cos x) + \cos x \dfrac{d}{dx}(e^{-x})$

$= e^{-x} \times -\sin x + \cos x \times -e^{-x}$

$= -e^{-x} \sin x - \cos x \, e^{-x}$

$= -e^{-x}(\sin x + \cos x)$

$y'' = -e^{-x} \frac{d}{dx}(\sin x + \cos x) + (\sin x + \cos x)\frac{d}{dx}(-e^{-x})$

$= -e^{-x} \times \left(\frac{d}{dx}(\sin x) + \frac{d}{dx}(\cos x)\right) + (\sin x + \cos x)\frac{d}{dx}(-e^{-x})$

$= -e^{-x} \times (\cos x - \sin x) + (\sin x + \cos x) \times e^{-x}$

$= -e^{-x} \cos x + e^{-x} \sin x + e^{-x} \sin x + e^{-x} \cos x$

$= 2e^{-x} \sin x$

17b Stationary points occur when $y' = 0$. This is when

$-e^{-x}(\sin x + \cos x) \ = 0$

$\sin x + \cos x = 0$ since $-e^{-x}$ is always less than 0.

$\sin x = -\cos x$

$\dfrac{\sin x}{\cos x} = -1$

$\tan x = -1$

$x = -\frac{\pi}{4}, \frac{3\pi}{4}$ for $x \in [-\pi, \pi]$

When $x = -\frac{\pi}{4}$, $y = \frac{1}{\sqrt{2}} e^{\frac{\pi}{4}}$ and $y'' = -\sqrt{2}e^{\frac{\pi}{4}} < 0$. Hence the curve is concave down at this point and thus there is a maximum turning point at $\left(-\frac{\pi}{4}, \frac{1}{\sqrt{2}} e^{\frac{\pi}{4}}\right)$.

When $x = \frac{3\pi}{4}$, $y = -\frac{1}{\sqrt{2}} e^{-\frac{3\pi}{4}}$ and $y'' = \sqrt{2}e^{-\frac{3\pi}{4}} > 0$. Hence the curve is concave up at this point and thus there is a minimum turning point at $\left(\frac{3\pi}{4}, -\frac{1}{\sqrt{2}} e^{-\frac{3\pi}{4}}\right)$.

17c Points of inflection occur when $y'' = 0$. This is when

$2e^{-x} \sin x = 0$

$\sin x = 0$ (since $e^{-x} > 0$ for all x)

Uncorrected proofs

$x = -\pi, 0, \pi \qquad$ for $x \in [-\pi, \pi]$

When $x = -\pi, y = -e^{\pi}$, when $x = 0, y = 1$ and when $x = \pi, y = -e^{-\pi}$

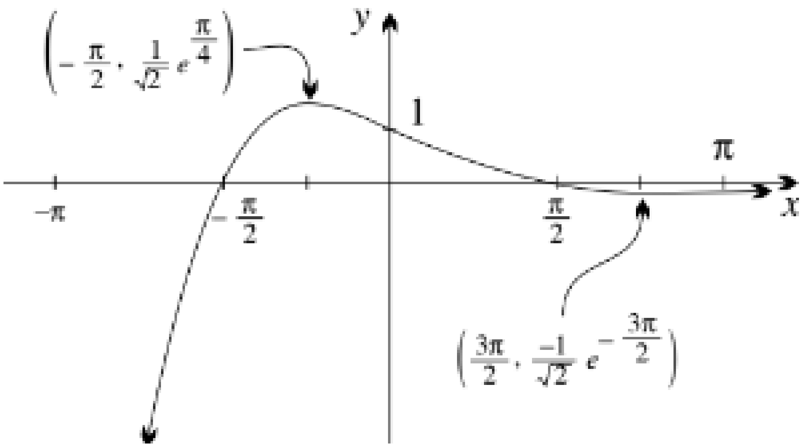
Thus, there are possible points of inflection at $(-\pi, -e^{\pi}), (0,1)$ and $(\pi, -e^{-\pi})$.

Confirming that these are points of inflection

x	$-\pi$	0	π
y'	e^{π}	-1	$e^{-\pi}$

Since these points are not stationary points, and hence not local maxima or minima, they must be points of inflection.

17d Sketch of the curve is shown as below



18a The angle of inclination is $\pi - a$ and so $m = \tan(\pi - a) = -\tan a$

18b The line is of form $y = mx + b$, since $m = -\tan\alpha$, $y = -x\tan\alpha + b$. Substituting in $(2, 1)$,

$1 = -2\tan\alpha + b$

$b = 1 + 2\tan\alpha$

Thus $y = -x\tan\alpha + 1 + 2\tan\alpha$

The x-intercept occurs when $y = 0$

$0 = -x\tan\alpha + 1 + 2\tan\alpha$

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$x \tan\alpha = 1 + 2\tan\alpha$

$$x = \frac{1}{\tan\alpha} + 2$$

Thus the x-intercept is $P = \left(\frac{1}{\tan\alpha} + 2, 0\right)$

The y-intercept is when $x = 0$

$y = 0 + 1 + 2\tan\alpha = 1 + 2\tan\alpha$

Thus the y-intercept is

$Q = (0, 2\tan\alpha + 1)$

18c

$$\begin{aligned} A &= \frac{1}{2}bh \\ &= \frac{1}{2}(2\tan\alpha + 1)\left(\frac{1}{\tan\alpha} + 2\right) \\ &= \frac{1}{2}(2\tan\alpha + 1)\left(\frac{1 + 2\tan\alpha}{\tan\alpha}\right) \\ &= \frac{1}{2}(2\tan\alpha + 1)\left(\frac{1 + 2\tan\alpha}{\tan\alpha}\right) \\ &= \frac{(2\tan\alpha + 1)^2}{2\tan\alpha} \end{aligned}$$

18d

Let $\tan\alpha = x$

$$\frac{dA}{dx} = \frac{(2x-1)(2x+1)}{2x^2}$$

For maximum area, $\frac{dA}{dx} = 0$

Therefore $(2x - 1) = 0$ or $(2x + 1) = 0$

Therefore $x = \frac{1}{2}$ or $x = -\frac{1}{2}$.

Since $x = \tan\alpha\,, x > 0$ as $0 < \alpha < \frac{\pi}{2}$

Therefore maximum when $\tan\alpha = \frac{1}{2}$.

Uncorrected proofs

19a For $y = \tan x$

$$\frac{dy}{dx} = \sec^2 x$$

When $x = 0$, the gradient of $\tan x$ is $\frac{dy}{dx} = \sec^2 0 = 1$.

Thus, the equation of the tangent at $(0, 0)$ is

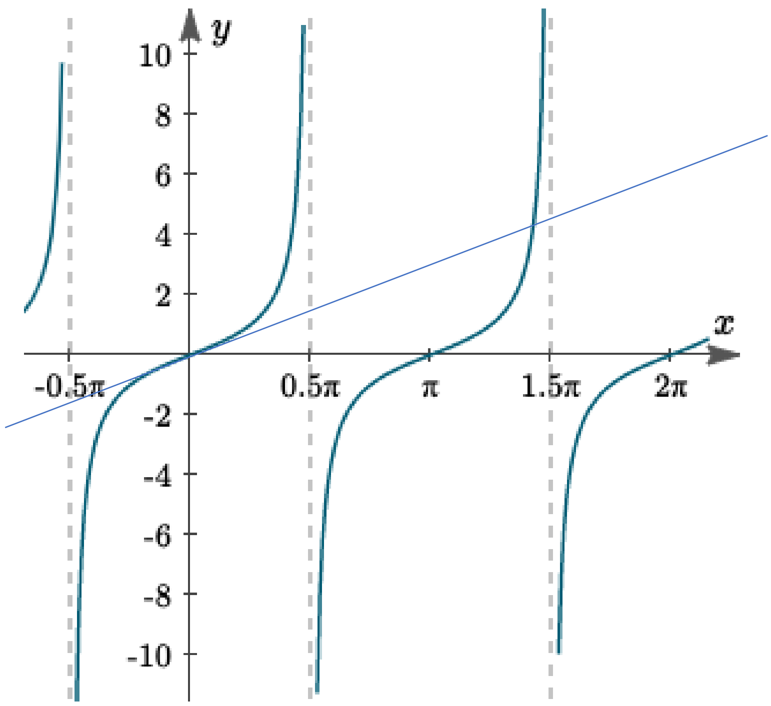
$y - y_1 = m(x - x_1)$

$y - 0 = 1(x - 0)$

$y = x$

Hence $y = x$ is a tangent to $y = \tan x$ at $(0, 0)$.

19b Sketch of $y = \tan x$ and $y = x$ is given below



For the given domain, $0 < x < \frac{\pi}{2}$, the graph of $y = \tan x$ is above the graph of $y = x$

Hence, $\tan x > x$

19c $f(x) = \frac{\sin x}{x}$

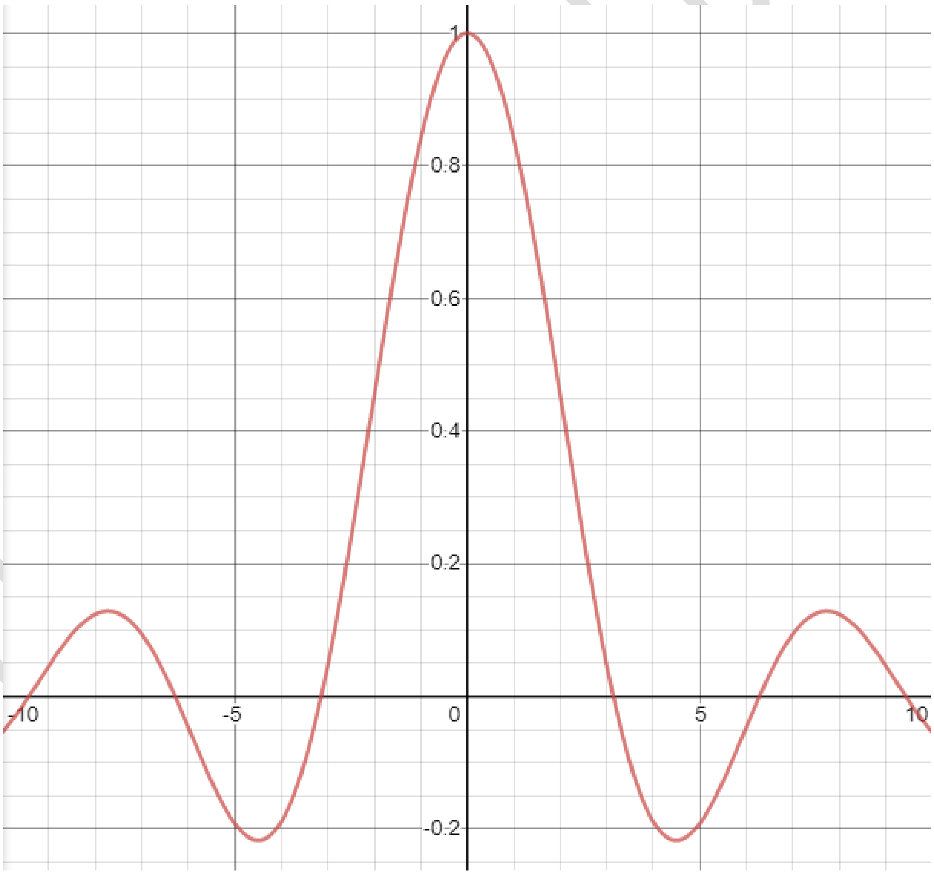
$$f'(x) = \frac{x \times \frac{d}{dx}(\sin x) - \sin x \times \frac{d}{dx}(x)}{x^2}$$

$$= \frac{x \times \cos x - \sin x \times 1}{x^2}$$

$$= \frac{x \cos x - \sin x}{x^2}$$

Since $\tan x > x$ for the given domain, it follows that $\frac{\sin x}{\cos x} < x$. For the given domain $\cos x > 0$, hence $\sin x > x \cos x$ and in turn $x \cos x - \sin x < 0$. Since $x^2 > 0$ for $x \neq 0$, we have that $\frac{x \cos x - \sin x}{x^2} < 0$ and hence $f'(x) < 0$.

19d Sketch of $f(x) = \frac{\sin x}{x}$



As we can see, $f(x) > \frac{2}{\pi}$ at the given domain hence, $\sin x > \frac{2x}{\pi}$.

20a $y = \sin^2 x + \cos x$

$y' = 2\sin x \cos x - \sin x$

$y' = \sin x(2\cos x - 1)$

$y'' = (2\cos x - 1) \times \frac{d}{dx}(\sin x) + \sin x \times \frac{d}{dx}(2\cos x - 1)$

$= (2\cos x - 1) \times \cos x + \sin x \times (-2\sin x)$

$= 2\cos^2 x - \cos x - 2\sin^2 x$

$= 2(\cos^2 x - \sin^2 x) - \cos x$

Stationary points occur when $y' = 0$. This is when:

$\sin x(2\cos x - 1) = 0$

$\sin x = 0$ and $2\cos x - 1 = 0$

$\sin x = 0$ and $\cos x = \frac{1}{2}$

$x = 0, \frac{\pi}{3}, \frac{5\pi}{3}, \pi, 2\pi \quad$ for $x \in [0, 2\pi]$

When $x = 0$, $y = 1$ and $y'' = 1 > 0$. Hence the curve is concave up at this point and thus there is a minimum turning point at $(0, 1)$.

When $x = \frac{\pi}{3}$, $y = \frac{5}{4}$ and $y'' = -\frac{3}{4} < 0$. Hence the curve is concave down at this point and thus there is a maximum turning point at $(\frac{\pi}{3}, \frac{5}{4})$.

When $x = \frac{5\pi}{3}$, $y = \frac{5}{4}$ and $y'' = -\frac{3}{4} < 0$. Hence the curve is concave down at this point and thus there is a maximum turning point at $(\frac{5\pi}{3}, \frac{5}{4})$.

When $x = \pi$, $y = -1$ and $y'' = 1 > 0$. Hence the curve is concave up at this point and thus there is a minimum turning point at $(\pi, -1)$.

When $x = 2\pi$, $y = 1$ and $y'' = 1 > 0$. Hence the curve is concave up at this point and thus there is a minimum turning point at $(2\pi, 1)$.

The x-intercepts occur when $y = 0$, this is when $\sin^2 x + \cos x = 0$

$1 - \cos^2 x + \cos x = 0$

$\cos^2 x - \cos x - 1 = 0$

Using the quadratic formula

$$\cos x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2}$$

Uncorrected proofs

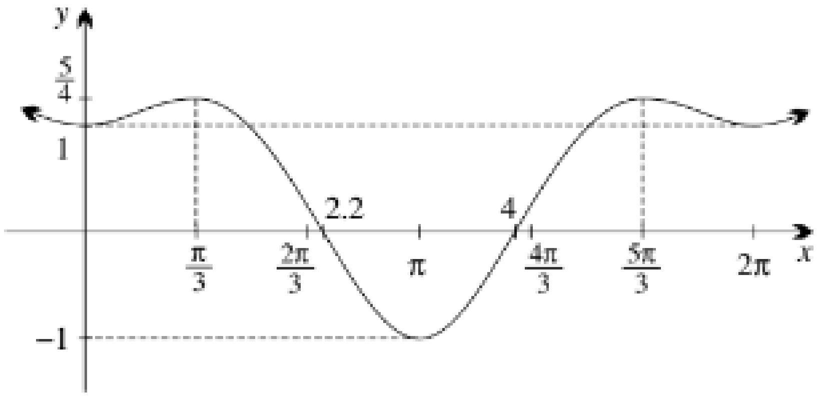
$= \dfrac{1 \pm \sqrt{5}}{2}$

Hence

$x = \cos^{-1}\left(\frac{1-\sqrt{5}}{2}\right)$ due to domain $[-1, 1]$.

Note: $\cos(-x + 2\pi) = \cos(x)$

Therefore another solution is $\frac{-(1-\sqrt{5})}{2} + 2$.



20b $y = \sin^3 x \cos x$

$y' = \cos x \times \dfrac{d}{dx}(\sin^3 x) + \sin^3 x \times \dfrac{d}{dx}(\cos x)$

$y' = \cos x(3\sin^2 x \cos x) + \sin^3 x(-\sin x)$

$= 3\sin^2 x\cos^2 x - \sin^4 x$

$y'' = 3[\cos^2 x \times \dfrac{d}{dx}(\sin^2 x) + sin^2 x \times \dfrac{d}{dx}(\cos^2 x)] - \dfrac{d}{dx}(\sin^4 x)$

$= 3(\cos^2 x \times 2\sin x \cos x + \sin^2 x \times -2\sin x \cos x) - 4\sin^3 x \cos x$

$= 3(2\cos^3 x \sin x - 2\sin^3 x \cos x) - 4\sin^3 x \cos x$

$= 6\cos^3 x \sin x - 10\sin^3 x \cos x$

Stationary points occur when $y' = 0$. This is when

$3\sin^2 x\cos^2 x - \sin^4 x = 0$

$\sin^2 x(3\cos^2 x - \sin^2 x) = 0$

$\sin^2 x = 0$ and $(3\cos^2 x - \sin^2 x) = 0$

$\sin^2 x = 0$ and $\tan^2 x = 3$

$\sin x = 0$ and $\tan x = \pm\sqrt{3}$

$x = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}, \pi, 2\pi \quad$ for $x \in [0, 2\pi]$

When $x = 0$, $y = 0$ and $y'' = 0$. Hence the curve is a stationary point of inflection at this point $(0, 0)$.

When $x = \frac{\pi}{3}$, $y = \frac{3\sqrt{3}}{16}$ and $y'' = -\frac{3\sqrt{3}}{2} < 0$. Hence the curve is concave down at this point and thus there is a maximum turning point at $(\frac{\pi}{3}, \frac{3\sqrt{3}}{16})$.

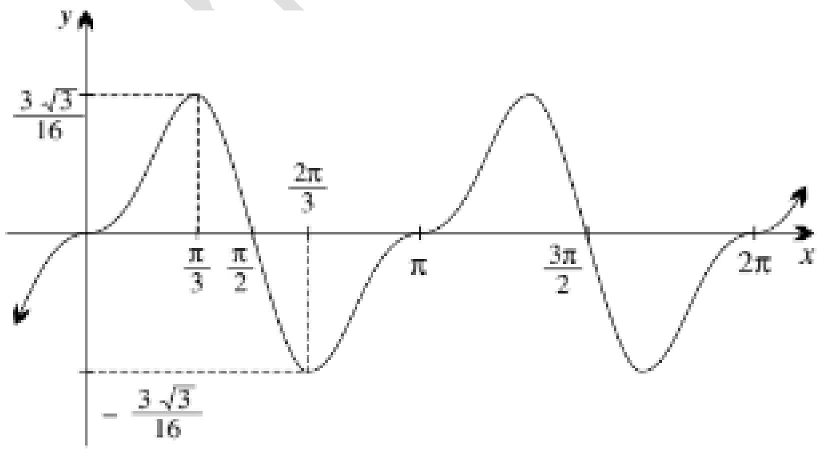
When $x = \frac{2\pi}{3}$, $y = -\frac{3\sqrt{3}}{16}$ and $y'' = \frac{3\sqrt{3}}{2} > 0$. Hence the curve is concave up at this point and thus there is a minimum turning point at $(\frac{2\pi}{3}, -\frac{3\sqrt{3}}{16})$.

When $x = \frac{4\pi}{3}$, $y = \frac{3\sqrt{3}}{16}$ and $y'' = -\frac{3\sqrt{3}}{2} < 0$. Hence the curve is concave down at this point and thus there is a maximum turning point at $(\frac{4\pi}{3}, \frac{3\sqrt{3}}{16})$.

When $x = \frac{5\pi}{3}$, $y = -\frac{3\sqrt{3}}{16}$ and $y'' = \frac{3\sqrt{3}}{2} > 0$. Hence the curve is concave up at this point and thus there is a minimum turning point at $(\frac{5\pi}{3}, -\frac{3\sqrt{3}}{16})$.

When $x = \pi$, $y = 0$ and $y'' = 0$. Hence the curve is a stationary point of inflection at this point $(\pi, 0)$.

When $x = 2\pi$, $y = 0$ and $y'' = 0$. Hence the curve is a stationary point of inflection at this point $(2\pi, 0)$.



20c $y = \tan^2 x - 2\tan x$

$$y' = \frac{d}{dx}(\tan^2 x) - \frac{d}{dx}(2\tan x)$$

$$y' = 2\tan x \cdot \frac{d}{dx}(\tan x) - 2\sec^2 x$$
$$y' = 2\tan x \sec^2 x - 2\sec^2 x$$

$$y' = 2\sec^2 x(\tan x - 1)$$

Stationary points occur when $y' = 0$, that is when:

$2\sec^2 x\,(\tan x - 1) = 0$

$\sec x = 0$ or $\tan x = 1$

However, $\sec x \neq 0$ for all x, thus the only stationary points are at $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$ where $\tan x = 1$.

$$y'' = \frac{d}{dx}(2\sec^2 x)(\tan x - 1) + 2\sec^2 x \frac{d}{dx}(\tan x - 1)$$

$$= \frac{d}{dx}((2\cos\ x)^{-2})(\tan x - 1) + 2\sec^4 x$$

$$= -4(-\sin x)(\cos x)^{-3}(\tan x - 1) + 2\sec^4 x$$

$$= 4\sin x \sec^3 x \tan x - 4\sin x \sec^3 x + 2\sec^4 x$$

$$= 4\frac{\sin^2 x}{\cos^4 x} - 4\frac{\sin x}{\cos^3 x} + \frac{2}{\cos^4 x}$$

$$= \sec^4 x\,(4\sin^2 x - 4\sin x\cos x + 2)$$

When $x = \frac{\pi}{4}$, $y = -1$, $y'' = 8 > 0$ hence the curve is concave up and this is a minimum turning point.

When $x = \frac{5\pi}{4}$, $y = -1$, $y'' = 8 > 0$ hence the curve is concave up and this is a minimum turning point.

Thus, there are minima at $\left(\frac{\pi}{4}, -1\right)$ and $\left(\frac{5\pi}{4}, -1\right)$.

Since $\tan x$ is undefined at $x = \frac{\pi}{2}$ and $\frac{3\pi}{2}$, the curve is also undefined at these values of x.

The x-intercepts occur when $y = 0$, this is when $\tan^2 x - 2\tan x = 0$

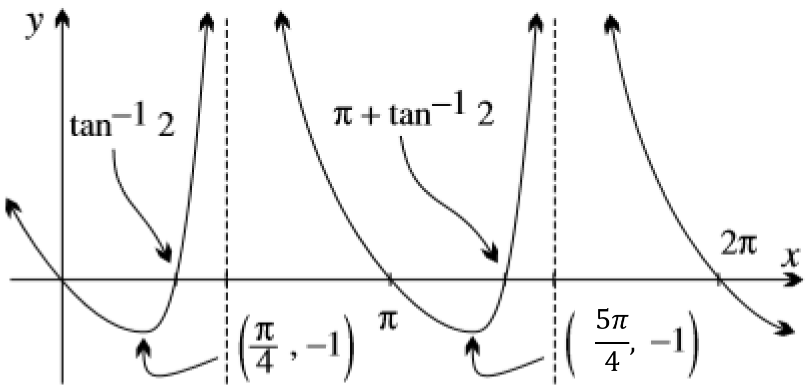
$\tan x\,(\tan x - 2) = 0$

Uncorrected proofs

$\tan x = 0$ or $\tan x = 2$

Hence the x-intercepts are when $x = 0, \pi, 2\pi$ (for $\tan x = 0$) and $x = \tan^{-1} 2\,, \pi + \tan^{-1} 2$ for ($\tan x = 2$).

Drawing the graph gives



21a $f(x) = \frac{\sin x}{x}$

As one cannot divide by zero, the domain is $x \neq 0$

$$\begin{aligned} f(-x) &= \frac{\sin(-x)}{-x} \\ &= \frac{-\sin x}{-x} \\ &= \frac{\sin x}{x} \\ &= f(x) \end{aligned}$$

Hence the function is even.

$$\begin{aligned} &\lim_{x\to\infty} |f(x)| \\ &= \lim_{x\to\infty} \left|\frac{\sin x}{x}\right| \\ &\leq \lim_{x\to\infty} \left|\frac{1}{x}\right| \\ &= 0 \end{aligned}$$

Hence, $\lim_{x\to\infty} |f(x)| \leq 0$ and since absolute values are non-negative $\lim_{x\to\infty} |f(x)| \geq 0$. Thus it follows that $\lim_{x\to\infty} |f(x)| = 0$ and thus $\lim_{x\to\infty} f(x) = \pm 0 = 0$.

The zeroes of $f(x)$ occur when $f(x) = 0$, this is when $\frac{\sin x}{x} = 0$ and is thus when $\sin x = 0$ for $x \neq 0$. This is all values $x = \lambda\pi$ where λ is a nonzero integer.

Uncorrected proofs

21b

$$f(x) = \frac{\sin x}{x}$$

$$f'(x) = \frac{x \times \frac{d}{dx}(\sin x) - \sin x \times \frac{dx}{dx}}{x^2}$$

$$= \frac{x \times \cos x - \sin x \times 1}{x^2}$$

$$= \frac{x \cos x - \sin x}{x^2}$$

The tangent is horizontal when $y' = 0$. This is when:

$$\frac{x \cos x - \sin x}{x^2} = 0$$

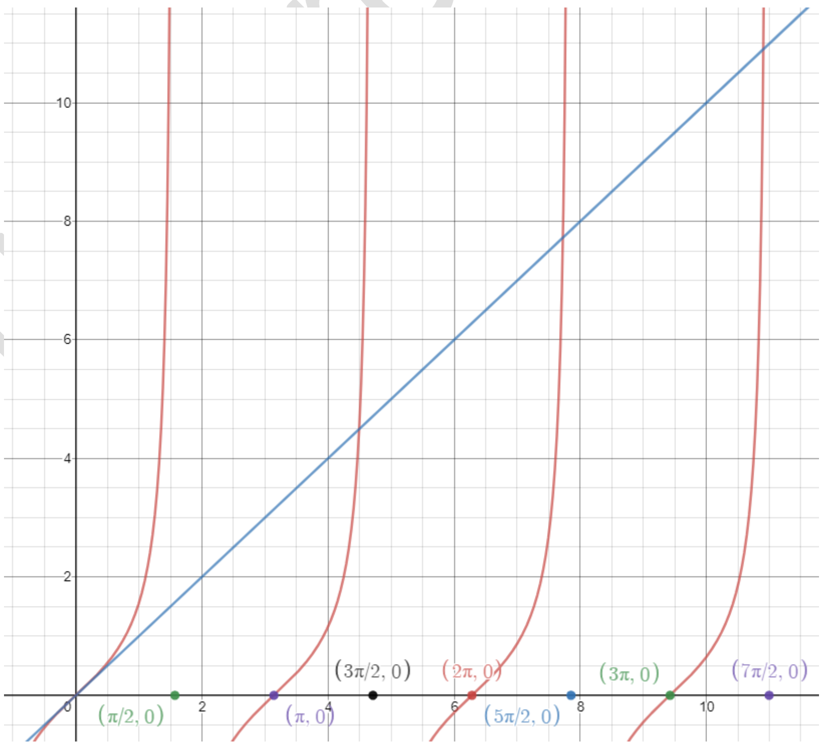
As $x \neq 0$, we conclude that we must have:

$x \cos x - \sin x = 0$

$\sin x = x \cos x$

$\tan x = x$

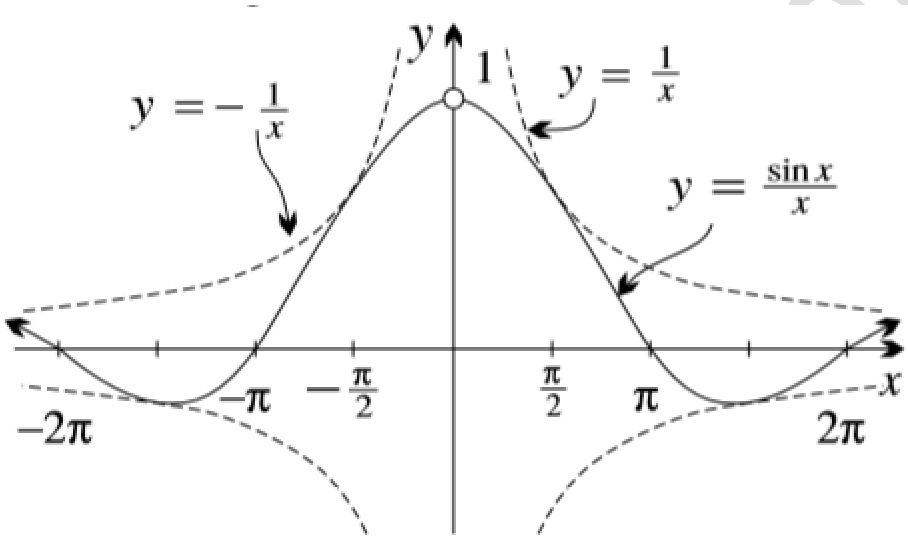
21c Since the stationary points occur when $\tan x = x$, the stationary points occur when the two graphs (as drawn below) intersect.



By observation, the graph of $y = x$ crosses the graph of $y = \tan x$ just to the left of $x = \frac{3\pi}{2}$, of $x = \frac{5\pi}{2}$ and of $x = \frac{7\pi}{2}$. Using the calculator, the three turning points of $y = f(x)$ are approximately $(1.43\pi, -0.217)$, $(2.46\pi, 0.128)$ and $(3.47\pi, -0.091)$.

21d Note that since division by zero is undefined, there is an open circle when $x = 0$. Also note that since $-1 \le \sin x \le 1$ it must be the case that:

$$-\frac{1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}$$



Uncorrected proofs

Solutions to Exercise 7D

1a

$$\int \sec^2 x \, dx$$

$$= \tan x + C$$

1b

$$\int \cos x \, dx$$

$$= \sin x + C$$

1c

$$\int \sin x \, dx$$

$$= -\cos x + C$$

1d

$$\int -\sin x \, dx$$

$$= \cos x + C$$

1e

$$\int 2\cos x \, dx$$

$$= 2\int \cos x \, dx$$

$$= 2\sin x + C$$

Uncorrected proofs

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1f

$$\int \cos 2x \, dx$$

$$= \frac{1}{2}\sin 2x + C$$

1g

$$\int \frac{1}{2}\cos x \, dx$$

$$= \frac{1}{2}\int \cos x \, dx$$

$$= \frac{1}{2}\sin x + C$$

1h

$$\int \cos \frac{1}{2}x \, dx$$

$$= 2\sin \frac{1}{2}x + C$$

1i

$$\int \sin 2x \, dx$$

$$= -\frac{1}{2}\cos 2x + C$$

1j

$$\int \sec^2 5x \, dx$$

$$= \frac{1}{5}\tan 5x + C$$

Uncorrected proofs

1k

$$\int \cos 3x \, dx$$

$$= \frac{1}{3}\sin 3x + C$$

1l

$$\int \sec^2 \frac{1}{3}x \, dx$$

$$= 3\tan\frac{1}{3}x + C$$

1m

$$\int \sin\frac{x}{2} \, dx$$

$$= -2\cos\frac{x}{2} + C$$

1n

$$\int -\cos\frac{1}{5}x \, dx$$

$$= -\int \cos\frac{1}{5}x \, dx$$

$$= -5\sin\frac{1}{5}x + C$$

1o

$$\int -4\sin 2x \, dx$$

$$= -4\int \sin 2x \, dx$$

$$= -4 \times -\frac{1}{2}\cos 2x + C$$

$$= 2\cos 2x + C$$

Uncorrected proofs

1p

$$\int \frac{1}{4}\sin\frac{1}{4}x\,dx$$

$$= \frac{1}{4}\int \sin\frac{1}{4}x\,dx$$

$$= \frac{1}{4} \times -4\cos\frac{1}{4}x + C$$

$$= -\cos\frac{1}{4}x + C$$

1q

$$\int 12\sec^2\frac{1}{3}x\,dx$$

$$= 12\int \sec^2\frac{1}{3}x\,dx$$

$$= 12 \times 3\tan\frac{1}{3}x + C$$

$$= 36\tan\frac{1}{3}x + C$$

1r

$$\int 2\cos\frac{x}{3}\,dx$$

$$= 2\int \cos\frac{x}{3}\,dx$$

$$= 2 \times 3\sin\frac{x}{3} + C$$

$$= 6\sin\frac{x}{3} + C$$

Uncorrected proofs

2a

$$\int_0^{\frac{\pi}{2}} \cos x \, dx$$

$$= [\sin x]_0^{\frac{\pi}{2}}$$

$$= \sin \frac{\pi}{2} - \sin 0$$

$$= 1 - 0$$

$$= 1$$

2b

$$\int_0^{\frac{\pi}{6}} \cos x \, dx$$

$$= [\sin x]_0^{\frac{\pi}{6}}$$

$$= \sin \frac{\pi}{6} - \sin 0$$

$$= \frac{1}{2} - 0$$

$$= \frac{1}{2}$$

2c

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin x \, dx$$

$$= [-\cos x]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= -[\cos x]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= -\left(\cos \frac{\pi}{2} - \cos \frac{\pi}{4}\right)$$

$$= -\left(0 - \frac{1}{\sqrt{2}}\right)$$

Uncorrected proofs

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$$= \frac{1}{\sqrt{2}}$$

2d

$$\int_0^{\frac{\pi}{3}} \sec^2 x \, dx$$

$$= [\tan x]_0^{\frac{\pi}{3}}$$

$$= \tan\frac{\pi}{3} - \tan 0$$

$$= \sqrt{3} - 0$$

$$= \sqrt{3}$$

2e

$$\int_0^{\frac{\pi}{4}} 2\cos 2x \, dx$$

$$= 2\int_0^{\frac{\pi}{4}} \cos 2x \, dx$$

$$= 2 \times \frac{1}{2}[\sin 2x]_0^{\frac{\pi}{4}}$$

$$= [\sin 2x]_0^{\frac{\pi}{4}}$$

$$= \sin\frac{\pi}{2} - \sin 0$$

$$= 1 - 0$$

$$= 1$$

2f

$$\int_0^{\frac{\pi}{3}} \sin 2x \, dx$$

$$= \left[-\frac{1}{2}\cos 2x\right]_0^{\frac{\pi}{3}}$$

$$= -\frac{1}{2}[\cos 2x]_0^{\frac{\pi}{3}}$$

Uncorrected proofs

$$= -\frac{1}{2}\left(\cos\frac{2\pi}{3} - \cos 0\right)$$
$$= -\frac{1}{2}\left(-\frac{1}{2} - 1\right)$$
$$= -\frac{1}{2}\left(-\frac{3}{2}\right)$$
$$= \frac{3}{4}$$

2g

$$\int_0^{\frac{\pi}{2}} \sec^2\left(\frac{1}{2}x\right) dx$$

$$= \left[2\tan\left(\frac{1}{2}x\right)\right]_0^{\frac{\pi}{2}}$$

$$= 2\left[\tan\left(\frac{1}{2}x\right)\right]_0^{\frac{\pi}{2}}$$

$$= 2\left(\tan\frac{\pi}{4} - \tan 0\right)$$

$$= 2(1 - 0)$$

$$= 2$$

2h

$$\int_{\frac{\pi}{3}}^{\pi} \cos\left(\frac{1}{2}x\right) dx$$

$$= 2\left[\sin\left(\frac{1}{2}x\right)\right]_{\frac{\pi}{3}}^{\pi}$$

$$= 2\left(\sin\frac{\pi}{2} - \sin\frac{\pi}{6}\right)$$

$$= 2\left(1 - \frac{1}{2}\right)$$

$$= 1$$

Uncorrected proofs

2i

$$\int_0^{\pi}(2\sin x - \sin 2x)dx$$
$$= \left[-2\cos x - \left(-\frac{1}{2}\cos 2x\right)\right]_0^{\pi}$$
$$= \left[-2\cos x + \frac{1}{2}\cos 2x\right]_0^{\pi}$$
$$= \left(-2\cos\pi + \frac{1}{2}\cos 2\pi\right) - \left(-2\cos 0 + \frac{1}{2}\cos 0\right)$$
$$= \left(-2(-1) + \frac{1}{2}(1)\right) - \left(-2(1) + \frac{1}{2}(1)\right)$$
$$= 2 + \frac{1}{2} + 2 - \frac{1}{2}$$
$$= 4$$

3a Since $\frac{dy}{dx} = \sin x$, it follows from integration, that $y = -\cos x + C$.

Substituting $(0, 0)$ into the equation then gives

$0 = -\cos 0 + C$

$0 = -1 + C$

$C = 1$

Hence if the curve passes through the origin,

$y = -\cos x + 1 = 1 - \cos x$

3b Since $\frac{dy}{dx} = \cos x - 2\sin 2x$, it follows from integration, that

$y = \sin x + \cos 2x + C$

Substituting $(0, 0)$ into the equation then gives

$0 = \sin 0 + \cos 0 + C$

$0 = 0 + 1 + C$

$C = -1$

Hence if the curve passes through the origin,

$y = \sin x + \cos 2x - 1$

Uncorrected proofs

3c Since $\frac{dy}{dx} = \sin x + \cos x$, it follows from integration that, $y = -\cos x + \sin x + C$. Substituting $(\pi, -2)$ into the equation then gives

$-2 = -\cos\pi + \sin\pi + C$

$-2 = -(-1) + 0 + C$

$C = -3$

Hence if the curve passes through the point $(\pi, -2)$,

$y = -\cos x + \sin x - 3$

4a There are 200 squares measuring 0.1 units × 0.1 units in the region under the curve between 0 and π.

Hence the total area is $200 \times 0.01 = 2$ square units and thus

$\int_0^{\pi} \sin x \, dx = 2$ as required

4b i

$$\int_0^{\frac{\pi}{4}} \sin x \, dx$$

$\doteqdot 0.3$

4b ii

$$\int_0^{\frac{\pi}{2}} \sin x \, dx$$

$$= \frac{1}{2}\int_0^{\pi} \sin x \, dx$$

$$= \frac{1}{2} \times 2$$

$= 1$

4b iii

$$\int_0^{\frac{3\pi}{4}} \sin x \, dx$$

Uncorrected proofs

$$= 2\int_0^{\frac{\pi}{2}} \sin x\,dx - \int_0^{\frac{\pi}{4}} \sin x\,dx$$

$$\doteqdot 2 \times 1 - 0.3$$

$$= 1.7$$

4b iv

$$\int_0^{\frac{5\pi}{4}} \sin x\,dx$$

$$= \int_0^{\pi} \sin x\,dx + \int_{\pi}^{\frac{5\pi}{4}} \sin x\,dx$$

$$= \int_0^{\pi} \sin x\,dx - \int_0^{\frac{\pi}{4}} \sin x\,dx$$

$$\doteqdot 2 - 0.3$$

$$= 1.7$$

4b v

$$\int_0^{\frac{3\pi}{2}} \sin x\,dx$$

$$= \int_0^{\pi} \sin x\,dx + \int_{\pi}^{\frac{3\pi}{2}} \sin x\,dx$$

$$= \int_0^{\pi} \sin x\,dx - \int_0^{\frac{\pi}{2}} \sin x\,dx$$

$$= 2 - 1$$

$$= 1$$

Uncorrected proofs

4b vi

$$\int_0^{\frac{7\pi}{4}} \sin x \, dx$$

$$= \int_0^{\pi} \sin x \, dx + \int_{\pi}^{\frac{7\pi}{4}} \sin x \, dx$$

$$= \int_0^{\pi} \sin x \, dx - \int_0^{\frac{3\pi}{4}} \sin x \, dx$$

$$\doteqdot 2 - 1.7$$

$$= 0.3$$

4c i

$$\int_0^{\frac{\pi}{4}} \sin x \, dx$$

$$= -[\cos x]_0^{\frac{\pi}{4}}$$

$$= -\left(\cos\frac{\pi}{4} - \cos 0\right)$$

$$= -\frac{1}{\sqrt{2}} + 1$$

$$\doteqdot 0.3$$

4c ii

$$\int_0^{\frac{\pi}{2}} \sin x \, dx$$

$$= -[\cos x]_0^{\frac{\pi}{2}}$$

$$= -\left(\cos\frac{\pi}{2} - \cos 0\right)$$

$$= 0 + 1$$

$$= 1$$

Uncorrected proofs

4c iii

$$\int_0^{\frac{3\pi}{4}} \sin x\, dx$$

$$= -[\cos x]_0^{\frac{3\pi}{4}}$$

$$= -\left(\cos\frac{3\pi}{4} - \cos 0\right)$$

$$= -\left(-\frac{1}{\sqrt{2}} - 1\right)$$

$$\doteqdot 1.7$$

4c iv

$$\int_0^{\frac{5\pi}{4}} \sin x\, dx$$

$$= -[\cos x]_0^{\frac{5\pi}{4}}$$

$$= -\left(\cos\frac{5\pi}{4} - \cos 0\right)$$

$$= -\left(-\frac{1}{\sqrt{2}} - 1\right)$$

$$\doteqdot 1.7$$

4c v

$$\int_0^{\frac{3\pi}{2}} \sin x\, dx$$

$$= -[\cos x]_0^{\frac{3\pi}{2}}$$

$$= -\left(\cos\frac{3\pi}{2} - \cos 0\right)$$

$$= -(0 - 1)$$

$$= 1$$

Uncorrected proofs

4c vi

$$\int_0^{\frac{7\pi}{4}} \sin x \, dx$$

$$= -[\cos x]_0^{\frac{7\pi}{4}}$$

$$= -\left(\cos\frac{7\pi}{4} - \cos 0\right)$$

$$= -\left(\frac{1}{\sqrt{2}} - 1\right)$$

$$\doteqdot 0.3$$

5 Answers will vary.

6a

$$\int \cos(x+2)\, dx$$

$$= \sin(x+2) + C$$

6b

$$\int \cos(2x+1)\, dx$$

$$= \frac{1}{2}\sin(2x+1) + C$$

6c

$$\int \sin(x+2)\, dx$$

$$= -\cos(x+2) + C$$

Uncorrected proofs

6d

$$\int \sin(2x+1)\ dx$$

$$= -\frac{1}{2}\cos(2x+1) + C$$

6e

$$\int \cos(3x-2)\ dx$$

$$= \frac{1}{3}\sin(3x-2) + C$$

6f

$$\int \sin(7-5x)\ dx$$

$$= -\frac{1}{5} \times -\cos(7-5x) + C$$

$$= \frac{1}{5}\cos(7-5x) + C$$

6g

$$\int \sec^2(4-x)\ dx$$

$$= -\tan(4-x) + C$$

6h

$$\int \sec^2\left(\frac{1-x}{3}\right)\ dx$$

$$= \int \sec^2\left(\frac{1}{3}-\frac{x}{3}\right)\ dx$$

$$= -3\tan\left(\frac{1}{3}-\frac{x}{3}\right) + C$$

$$= -3\tan\left(\frac{1-x}{3}\right) + C$$

Uncorrected proofs

6i

$$\int \sin\left(\frac{1-x}{3}\right)\,dx$$

$$= \int \sin\left(\frac{1}{3}-\frac{x}{3}\right)dx$$

$$= -3 \times -\cos\left(\frac{1}{3}-\frac{x}{3}\right)+C$$

$$= 3\cos\left(\frac{1-x}{3}\right)+C$$

7a

$$\int \left(6\cos 3x - 4\sin\frac{1}{2}x\right)dx$$

$$= \int (6\cos 3x)\,dx - \int \left(4\sin\frac{1}{2}x\right)dx$$

$$= 6 \times \frac{1}{3}\sin 3x - \left(4 \times -2\cos\frac{1}{2}x\right)+C$$

$$= 2\sin 3x - \left(-8\cos\frac{1}{2}x\right)+C$$

$$= 2\sin 3x + 8\cos\frac{1}{2}x + C$$

7b

$$\int \left(8\sec^2 2x - 10\cos\frac{1}{4}x + 12\sin\frac{1}{3}x\right)\,dx$$

$$= \int (8\sec^2 2x)\,dx + \int \left(-10\cos\frac{1}{4}x\right)dx + \int \left(12\sin\frac{1}{3}x\right)dx$$

$$= 8 \times \frac{1}{2}\tan 2x + \left(-10 \times 4\sin\frac{1}{4}x\right) + 12 \times 3 \times -\cos\frac{1}{3}x + C$$

$$= 4\tan 2x - 40\sin\frac{1}{4}x - 36\cos\frac{1}{3}x + C$$

8a $f'(x) = \pi\cos\pi x$

$$f(x) = \int \pi\cos\pi x\,dx$$

$$= \pi \times \frac{1}{\pi}\sin\pi x + C$$

$$= \sin\pi x + C$$

Uncorrected proofs

As $f(0) = 0$,

$\sin 0 + C = 0$

$C = 0$

Hence $f(x) = \sin \pi x$ and

$f\left(\frac{1}{3}\right) = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$

8b $f'(x) = \cos \pi x$

$f(x) = \int \cos \pi x \, dx$

$= \frac{1}{\pi} \sin \pi x + C$

As $f(0) = \frac{1}{2\pi}$,

$\frac{1}{\pi} \sin 0 + C = \frac{1}{2\pi}$

$C = \frac{1}{2\pi}$

Hence

$f(x) = \frac{1}{\pi} \sin \pi x + \frac{1}{2\pi}$

$f\left(\frac{1}{6}\right) = \frac{1}{\pi} \sin \frac{\pi}{6} + \frac{1}{2\pi}$

$= \frac{1}{\pi} \times \frac{1}{2} + \frac{1}{2\pi}$

$= \frac{1}{2\pi} + \frac{1}{2\pi}$

$= \frac{2}{2\pi}$

$= \frac{1}{\pi}$

Uncorrected proofs

8c $f''(x) = 18\cos 3x$

$f'(x) = \int 18\cos 3x\, dx$

$= 18 \times \frac{1}{3}\sin 3x + C$

$= 6\sin 3x + C$

As $f'(0) = 1$,

$6\sin 0 + C = 1$

$C = 1$

Hence

$f'(x) = 6\sin 3x + 1$

$f(x) = \int (6\sin 3x + 1)\, dx$

$= 6 \times -\frac{1}{3}\cos 3x + x + C$

$= -2\cos 3x + x + C$

As $f\left(\frac{\pi}{2}\right) = 1$,

$-2\cos\frac{3\pi}{2} + \frac{\pi}{2} + C = 1$

$C = 1 + 2 \times 0 - \frac{\pi}{2}$

$C = 1 - \frac{\pi}{2}$

Hence

$f(x) = -2\cos 3x + x + 1 - \frac{\pi}{2}$

9a

$\int a\sin(ax + b)\ dx$

$= a \times -\frac{1}{a}\cos(ax + b) + C$

$= -\cos(ax + b) + C$

Uncorrected proofs

9b

$$\int \pi^2 \cos \pi x \ dx$$

$$= \pi^2 \times \frac{1}{\pi} \sin \pi x + C$$

$$= \pi \sin \pi x + C$$

9c

$$\int \frac{1}{u} \sec^2(v + ux) \ dx$$

$$= \frac{1}{u} \times \frac{1}{u} \tan(v + ux) + C$$

$$= \frac{1}{u^2} \tan(v + ux) + C$$

9d

$$\int \frac{a}{\cos^2 ax} \ dx$$

$$= \int a \sec^2 ax \ dx$$

$$= a \times \frac{1}{a} \tan ax + C$$

$$= \tan ax + C$$

10a $1 + \tan^2 x = \sec^2 x$

$$\int \tan^2 x \, dx$$

$$= \int (\sec^2 x - 1) \, dx$$

$$= \int \sec^2 x \, dx - \int 1 \, dx$$

$$= \tan x - x + C$$

Uncorrected proofs

10b $1-\sin^2 x=\cos^2 x$

$$\int_0^{\frac{\pi}{3}} \frac{2}{1-\sin^2 x}\,dx$$

$$=\int_0^{\frac{\pi}{3}} \frac{2}{\cos^2 x}\,dx$$

$$=\int_0^{\frac{\pi}{3}} 2\sec^2 x\,dx$$

$$=2[\tan x]_0^{\frac{\pi}{3}}$$

$$=2\left(\tan\frac{\pi}{3}-\tan 0\right)$$

$$=2\left(\sqrt{3}-0\right)$$

$$=2\sqrt{3}$$

11a $\int \frac{f'(x)}{f(x)}\,dx=\ln|f(x)|+c$ (from the table of standard integrals)

11b Note that if $f(x)=1+\sin x$ then $f'(x)=\cos x$ hence

$$\int_0^{\frac{\pi}{6}} \frac{\cos x}{1+\sin x}\,dx$$

$$=[\ln|1+\sin x|]_0^{\frac{\pi}{6}}$$

$$=\ln\left(1+\sin\frac{\pi}{6}\right)-\ln(1+\sin 0)$$

$$=\ln\left(1+\frac{1}{2}\right)-\ln(1)$$

$$=\ln\left(\frac{3}{2}\right)-0$$

$$\doteqdot 0.4$$

Uncorrected proofs

12a Note that if $f(x) = \cos x$ then $f'(x) = -\sin x$ hence

$$\int_0^{\frac{\pi}{4}} \tan x \, dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos x} dx$$

$$= -\int_0^{\frac{\pi}{4}} \frac{-\sin x}{\cos x} dx$$

$$= -[\ln|\cos x|]_0^{\frac{\pi}{4}}$$

$$= -\left(\ln\left(\cos\frac{\pi}{4}\right) - \ln(\cos 0)\right)$$

$$= -\ln\left(\frac{1}{\sqrt{2}}\right) + \ln 1$$

$$= -\ln\left(2^{-\frac{1}{2}}\right) + 0$$

$$= -\left(-\frac{1}{2}\right)\ln 2$$

$$= \frac{1}{2}\ln 2$$

12b Note that if $f(x) = \sin x$ then $f'(x) = \cos x$ hence

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cot x \ dx$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos x}{\sin x} dx$$

$$= [\ln|\sin x|]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= \ln\left(\sin\frac{\pi}{2}\right) - \ln\left(\sin\frac{\pi}{6}\right)$$

$$= \ln 1 - \ln\frac{1}{2}$$

$$= \ln 1 - \ln(2^{-1})$$

$$= 0 - (-1) \times \ln 2$$

Uncorrected proofs

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$= 0 + \ln 2$

$= \ln 2$

13a

$\frac{d}{dx}(\sin^5 x)$

$= 5\sin^4 x \times \frac{d}{dx}(\sin x)$

$= 5\sin^4 x \times \cos x$

$= 5\sin^4 x \cos x$

$\int \sin^4 x \cos x \, dx$

$= \frac{1}{5}\int 5\sin^4 x \cos x \, dx$

$= \frac{1}{5}\int \frac{d}{dx}(\sin^5 x) \, dx$

$= \frac{1}{5}\sin^5 x + C$

13b

$\frac{d}{dx}(\tan^3 x)$

$= 3\tan^2 x \times \frac{d}{dx}(\tan x)$

$= 3\tan^2 x \sec^2 x$

$\int \tan^2 x \sec^2 x \, dx$

$= \frac{1}{3}\int 3\tan^2 x \sec^2 x \, dx$

$= \frac{1}{3}\int \frac{d}{dx}(\tan^3 x) \, dx$

$= \frac{1}{3}\tan^3 x + C$

Uncorrected proofs

14a

$$\frac{d}{dx}\left(e^{\sin x}\right)$$

$$= e^{\sin x} \times \frac{d}{dx}(\sin x)$$

$$= e^{\sin x} \times \cos x$$

$$= \cos x \, e^{\sin x}$$

$$\int_0^{\frac{\pi}{2}} \cos x \, e^{\sin x} \, dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{d}{dx}\left(e^{\sin x}\right) dx$$

$$= \left[e^{\sin x}\right]_0^{\frac{\pi}{2}}$$

$$= e^{\sin\frac{\pi}{2}} - e^{\sin 0}$$

$$= e^1 - e^0$$

$$= e - 1$$

14b

$$\frac{d}{dx}\left(e^{\tan x}\right)$$

$$= e^{\tan x} \times \frac{d}{dx}(\tan x)$$

$$= e^{\tan x} \times \sec^2 x$$

$$= \sec^2 x \, e^{\tan x}$$

$$\int_0^{\frac{\pi}{4}} \sec^2 x \, e^{\tan x} \, dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{d}{dx}\left(e^{\tan x}\right) dx$$

$$= \left[e^{\tan x}\right]_0^{\frac{\pi}{4}}$$

$$= e^{\tan\frac{\pi}{4}} - e^{\tan 0}$$

Uncorrected proofs

$= e^1 - e^0$

$= e - 1$

15a

$\frac{d}{dx}(\sin x - x\cos x)$

$= \frac{d}{dx}(\sin x) - \frac{d}{dx}(x\cos x)$

$= \frac{d}{dx}(\sin x) - \left(\cos x \times \frac{d}{dx}(x) + x\frac{d}{dx}(\cos x)\right)$

$= \cos x - (\cos x - x\sin x)$

$= x\sin x$

$\int_0^{\frac{\pi}{2}} x\sin x\, dx$

$= \int_0^{\frac{\pi}{2}} \frac{d}{dx}(\sin x - x\cos x)\, dx$

$= [\sin x - x\cos x]_0^{\frac{\pi}{2}}$

$= \left(\sin\frac{\pi}{2} - \frac{\pi}{2}\cos\frac{\pi}{2}\right) - (\sin 0 - 0)$

$= 1 - 0 - 0 + 0$

$= 1$

15b

$\frac{d}{dx}\left(\frac{1}{3}\cos^3 x - \cos x\right)$

$= \frac{d}{dx}\left(\frac{1}{3}\cos^3 x\right) - \frac{d}{dx}(\cos x)$

$= \left(\frac{1}{3} \times 3\cos^2 x\right) \times \frac{d}{dx}(\cos x) - (-\sin x)$

$= \cos^2 x \times (-\sin x) + \sin x$

$= -\sin x\cos^2 x + \sin x$

$= -\sin x\,(1 - \sin^2 x) + \sin x$

$= -\sin x + \sin^3 x + \sin x$

$= \sin^3 x$

Uncorrected proofs

$$\int_0^{\frac{\pi}{3}} \sin^3 x$$

$$= \int_0^{\frac{\pi}{3}} \frac{d}{dx}\left(\frac{1}{3}\cos^3 x - \cos x\right)\, dx$$

$$= \left[\frac{1}{3}\cos^3 x - \cos x\right]_0^{\frac{\pi}{3}}$$

$$= \left(\frac{1}{3}\cos^3 \frac{\pi}{3} - \cos\frac{\pi}{3}\right) - \left(\frac{1}{3}\cos^3 0 - \cos 0\right)$$

$$= \left(\frac{1}{3} \times \frac{1}{8} - \frac{1}{2}\right) - \left(\frac{1}{3} \times 1 - 1\right)$$

$$= \left(\frac{1}{24} - \frac{1}{2}\right) - \left(\frac{1}{3} - 1\right)$$

$$= \frac{5}{24}$$

16a

$$\int_0^{\pi} \sin x \cos^8 x\, dx$$

Let, $f(x) = \cos x$

$f'(x) = -\sin x$

Then,

$$\int_0^{\pi} \sin x cos^8 x\, dx$$

$$= \left[-\frac{1}{9}\cos^9 x\right]_0^{\pi}$$

$$= \left(-\frac{1}{9} \times -1\right) - \left(-\frac{1}{9} \times 1\right)$$

$$= \frac{1}{9} + \frac{1}{9} = \frac{2}{9}$$

Uncorrected proofs

16b

$$\int_0^{\frac{\pi}{2}} \sin x \cos^8 x \, dx$$

Let, $f(x) = \cos x$

$f'(x) = -\sin x$

Then,

$$\int_0^{\frac{\pi}{2}} \sin x \cos^n x \, dx$$

$$= \left[-\frac{1}{n+1}\cos^{n+1} x\right]_0^{\frac{\pi}{2}}$$

$$= \left(-\frac{1}{n+1} \times 0\right) - \left(-\frac{1}{n+1} \times 1\right)$$

$$= \frac{1}{n+1}$$

16c

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x \sin^7 x \, dx$$

Let, $f(x) = \sin x$

$f'(x) = \cos x$

Then,

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x \sin^7 x \, dx$$

$$= \left[\frac{1}{8}\sin^8 x\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \left(\frac{1}{8} \times 1\right) - \left(\frac{1}{8} \times 1\right)$$

$$= \frac{1}{8} - \frac{1}{8}$$

$$= 0$$

Uncorrected proofs

16d

$$\int_0^{\frac{\pi}{6}} \cos x \sin^n x \, dx$$

Let, $f(x) = \sin x$

$f'(x) = \cos x$

Then,

$$\int_0^{\frac{\pi}{6}} \cos x \sin^n x \, dx$$

$$= \left[\frac{1}{n+1} \sin^{n+1} x\right]_0^{\frac{\pi}{6}}$$

$$= \left(\frac{1}{n+1} \times \left(\frac{1}{2}\right)^{n+1}\right) - \left(\frac{1}{n+1} \times 0\right)$$

$$= \frac{1}{2^{n+1}(n+1)}$$

16e

$$\int_0^{\frac{\pi}{3}} \sec^2 x \tan^7 x \, dx$$

Let, $f(x) = \tan x$

$f'(x) = \sec^2 x$

Then,

$$\int_0^{\frac{\pi}{3}} \sec^2 x \tan^7 x \, dx$$

$$= \left[\frac{1}{8} \tan^8 x\right]_0^{\frac{\pi}{3}}$$

$$= \left(\frac{1}{8} \times 81\right) - \left(\frac{1}{8} \times 0\right)$$

$$= \frac{81}{8}$$

$$= 10\frac{1}{8}$$

Uncorrected proofs

16f

$$\int_0^{\frac{\pi}{4}} \sec^2 x \tan^n x \, dx$$

Let, $f(x) = \tan x$

$f'(x) = \sec^2 x$

Then,

$$\int_0^{\frac{\pi}{4}} \sec^2 x \tan^n x \, dx$$

$$= \left[\frac{1}{n+1}\tan^{n+1} x\right]_0^{\frac{\pi}{4}}$$

$$= \left(\frac{1}{n+1} \times 1\right) - \left(\frac{1}{n+1} \times 0\right)$$

$$= \frac{1}{n+1}$$

17a

$$\int \sin x \cos x \, dx$$

First, let, $f(x) = \sin x$

$f'(x) = \cos x$

Then, as per the reverse chain rule,

$$\int \sin x \cos x \, dx$$

$$= \frac{1}{2}\sin^2 x + C$$

Again,

$$\int \sin x \cos x \, dx$$

$= \int \frac{1}{2} \sin 2x \, dx$ as per the identity, $\sin 2x = 2 \sin x \cos x$

$$= \frac{1}{2} \int \sin 2x \, dx$$

$$= \frac{1}{2} \times -\frac{1}{2} \cos 2x$$

Uncorrected proofs

$= -\frac{1}{4}\cos 2x + D$

17b

$\sin^2 x = \frac{1}{2}(1 - \cos 2x) = \frac{1}{2} - \frac{1}{2}\cos 2x$ (as $\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x$)

So, $\frac{1}{2}\sin^2 x + C$

$= \frac{1}{2}(\frac{1}{2} - \frac{1}{2}\cos 2x) + C$

$= \frac{1}{4} - \frac{1}{4}\cos 2x + C$

$= -\frac{1}{4}\cos 2x + (C + \frac{1}{4})$

$= -\frac{1}{4}\cos 2x + D$

18

$\frac{d}{dx}(x \sin 2x)$

$= \sin 2x \frac{d}{dx}(x) + x\frac{d}{dx}(\sin 2x)$

$= \sin 2x \times 1 + x \times 2\cos 2x$
$= \sin 2x + 2x\cos 2x$

As $\sin 2x + 2x\cos 2x = \frac{d}{dx}(x \sin 2x)$

$2x\cos 2x = \frac{d}{dx}(x\sin 2x) + \sin 2x$

$x\cos 2x = \frac{1}{2}\frac{d}{dx}(x\sin 2x) + \frac{1}{2}\sin 2x$

Hence,

$\int_0^{\frac{\pi}{4}} x\cos 2x\, dx$

$= \int_0^{\frac{\pi}{4}} (\frac{1}{2}\frac{d}{dx}(x\sin 2x) + \frac{1}{2}\sin 2x)dx$

$= \int_0^{\frac{\pi}{4}} \frac{1}{2}\frac{d}{dx}(x\sin 2x)dx + \frac{1}{2}\int_0^{\frac{\pi}{4}} \sin 2x\, dx$

Uncorrected proofs

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}} \frac{d}{dx}(x\sin 2x)dx + \frac{1}{2}\int_0^{\frac{\pi}{4}} \sin 2x\, dx$$

$$= \frac{1}{2}[x\sin 2x]_0^{\frac{\pi}{4}} + \frac{1}{2}\left[-\frac{1}{2}\cos 2x\right]_0^{\frac{\pi}{4}}$$

$$= \tfrac{1}{2}[x\sin 2x]_0^{\frac{\pi}{4}} - \tfrac{1}{4}[\cos 2x]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{2}\left(\frac{\pi}{4}\times 1 - 0\right) - \frac{1}{4}(1-0)$$

$$= \frac{\pi}{8} - \frac{1}{4}$$

$$= \frac{\pi - 2}{8}$$

19a

$$\frac{d}{dx}(\tan^3 x)$$

$$= 3\tan^2 x \times \frac{d}{dx}(\tan x)$$

$$= 3\tan^2 x \sec^2 x$$

$= 3[(\sec^2 x - 1)\sec^2 x]$ using the identity, $\sec^2 x - \tan^2 x = 1$

$$= 3(\sec^4 x - \sec^2 x)$$

19b As $3(\sec^4 x - \sec^2 x) = \frac{d}{dx}(\tan^3 x)$

$$3\sec^4 x - 3\sec^2 x = \tfrac{d}{dx}(\tan^3 x)$$

$$3\sec^4 x = \tfrac{d}{dx}(\tan^3 x) + 3\sec^2 x$$

$$\sec^4 x = \tfrac{1}{3}\tfrac{d}{dx}(\tan^3 x) + \sec^2 x$$

Hence,

$$\int_0^{\frac{\pi}{4}} \sec^4 x\, dx$$

$$= \int_0^{\frac{\pi}{4}} (\frac{1}{3}\frac{d}{dx}(\tan^3 x) + \sec^2 x)dx$$

Uncorrected proofs

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$$= \int_0^{\frac{\pi}{4}} \frac{1}{3}\frac{d}{dx}(\tan^3 x)dx + \int_0^{\frac{\pi}{4}} \sec^2 x\, dx$$

$$= \frac{1}{3}\int_0^{\frac{\pi}{4}} \frac{d}{dx}(\tan^3 x)dx + \int_0^{\frac{\pi}{4}} \sec^2 x\, dx$$

$$= \frac{1}{3}(\tan^3 x)_0^{\frac{\pi}{4}} + (\tan x)_0^{\frac{\pi}{4}}$$

$$= \frac{1}{3}(1-0) + (1-0)$$

$$= \frac{1}{3} + 1$$

$$= \frac{4}{3}$$

20a $\sin(A+B) + \sin(A-B)$

Using the identity $\sin(A \pm B) = \sin A \cos B \pm \sin B \cos A$

$\sin(A+B) + \sin(A-B)$

$= \sin A \cos B + \sin B \cos A + \sin A \cos B - \sin B \cos A$

$= \sin A \cos B + \sin A \cos B$

$= 2 \sin A \cos B$

Hence, proved

20b i Using the proof from part a and letting $A = 3x$ and $B = 2x$:

$$\int_0^{\frac{\pi}{2}} 2 \sin 3x \cos 2x\, dx$$

$$= \int_0^{\frac{\pi}{2}} (\sin(3x + 2x) + \sin(3x - 2x))dx$$

$$= \int_0^{\frac{\pi}{2}} (\sin 5x + \sin x)dx$$

$$= \int_0^{\frac{\pi}{2}} \sin 5x\, dx + \int_0^{\frac{\pi}{2}} \sin x\, dx$$

Uncorrected proofs

$$= \left[-\frac{1}{5}\cos 5x\right]_0^{\frac{\pi}{2}} + [-\cos x]_0^{\frac{\pi}{2}}$$

$$= \left[-\tfrac{1}{5}\cos 5x\right]_0^{\frac{\pi}{2}} - [\cos x]_0^{\frac{\pi}{2}}$$

$$= \left(0 - \left(-\frac{1}{5}\right)\right) - (-1)$$

$$= \frac{1}{5} + 1$$

$$= \frac{6}{5}$$

20b ii Using the proof from part a and letting $A = 3x$ and $B = 4x$:

$$\int_0^{\pi} \sin 3x \cos 4x \, dx$$

$$= \int_0^{\pi} \frac{1}{2}(\sin(3x + 4x) + \sin(3x - 4x))dx$$

$$= \int_0^{\pi} \frac{1}{2}(\sin 7x + \sin(-x))\, dx$$

$$= \int_0^{\pi} \frac{1}{2}(\sin 7x - \sin x)dx$$

$$= \frac{1}{2}\int_0^{\pi} \sin 7x \, dx - \frac{1}{2}\int_0^{\pi} \sin x \, dx$$

$$= \frac{1}{2}\left[-\frac{1}{7}\cos 7x\right]_0^{\pi} - \frac{1}{2}[-\cos x]_0^{\pi}$$

$$= \frac{1}{2}\left[-\frac{1}{7}\cos 7x\right]_0^{\pi} + \frac{1}{2}[\cos x]_0^{\pi}$$

$$= \frac{1}{2}\left(\frac{1}{7} + \frac{1}{7}\right) + \frac{1}{2}(-1 - 1)$$

$$= \frac{1}{2} \times \frac{2}{7} + \frac{1}{2} \times -2$$

$$= \frac{1}{7} - 1$$

$$= -\frac{6}{7}$$

Uncorrected proofs

20c i The primitive is

$$F(x) = \int \sin mx \cos nx \, dx$$
$$= \int \frac{1}{2}\sin\big((m-n)x\big) + \frac{1}{2}\sin\big((m+n)x\big)\,dx$$
$$= -\frac{1}{2(m-n)}\cos\big((m-n)x\big) - \frac{1}{2(m+n)}\cos\big((m+n)x\big) + C$$

Hence

$$\int_{-\pi}^{\pi} \sin mx \cos nx \, dx$$
$$= F(\pi) - F(-\pi)$$
$$= \left(-\frac{1}{2(m-n)}\cos\big((m-n)\pi\big) - \frac{1}{2(m+n)}\cos\big((m+n)\pi\big)\right)$$
$$-\left(-\frac{1}{2(m-n)}\cos\big((m-n)(-\pi)\big) - \frac{1}{2(m+n)}\cos\big((m+n)(-\pi)\big)\right)$$
$$= \left(-\frac{1}{2(m-n)}\cos\big((m-n)\pi\big) - \frac{1}{2(m+n)}\cos\big((m+n)\pi\big)\right)$$
$$-\left(-\frac{1}{2(m-n)}\cos\big((m-n)\pi\big) - \frac{1}{2(m+n)}\cos\big((m+n)\pi\big)\right)$$
$$= 0$$

20c ii Let $f(x) = \sin mx \cos nx$. It follows that

$$f(-x) = \sin m(-x) \cos n(-x)$$
$$= \sin m(-x) \cos nx$$
$$= -\sin mx \cos nx$$
$$= -f(x)$$

This function is odd. For any odd function $f(x)$, $\int_{-a}^{a} f(x)\,dx = 0$. Hence $\int_{-\pi}^{\pi} \sin mx \cos nx \, dx = 0$.

21 $A(2\sin x + \cos x) + B(2\cos x - \sin x) = 7\sin x + 11\cos x$

$(2A - B)\sin x + (A + 2B)\cos x = 7\sin x + 11\cos x$

Equating coefficients gives:

$2A - B = 7$ (1)

$A + 2B = 11$ (2)

$5A = 25$ (2) + 2 × (1)

Uncorrected proofs

$A = 5$

$5B = 15 \qquad 2 \times (2) - (1)$

$B = 3$

21b

$$\int_0^{\frac{\pi}{2}} \frac{7\sin x + 11\cos x}{2\sin x + \cos x}\,dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{5(2\sin x + \cos x) + 3(2\cos x - \sin x)}{(2\sin x + \cos x)}\,dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{5(2\sin x + \cos x) + 3(2\cos x - \sin x)}{(2\sin x + \cos x)}\,dx$$

$$= \int_0^{\frac{\pi}{2}} 5 + 3\left(\frac{2\cos x - \sin x}{2\sin x + \cos x}\right)dx$$

$$= [5x + 3\ln(2\sin x + \cos x)]_0^{\frac{\pi}{2}}$$

$$= \left[5\left(\frac{\pi}{2}\right) + 3\ln\left(2\sin\left(\frac{\pi}{2}\right) + \cos\left(\frac{\pi}{2}\right)\right)\right] - [5(0) + 3\ln(2\sin 0 + \cos 0)]$$

$$= \left[5\left(\frac{\pi}{2}\right) + 3\ln(2)\right] - [5(0) + 3\ln(1)]$$

$$= \frac{5\pi}{2} + 3\ln 2$$

$$= \frac{1}{2}(5\pi + 6\ln 2)$$

22a $\quad \cos t \le 1$

$$\int_0^x \cos t\,dt \le \int_0^x 1\,dt$$

$[\sin t]_0^x \le [1]_0^x$

$\sin x - \sin 0 \le x - 0$

$\sin x \le x$

Uncorrected proofs

22b $\sin t \leq t$

$$\int_0^x \sin t\,dt \leq \int_0^x t\,dt$$

$$[-\cos t]_0^x \leq \left[\frac{t^2}{2!}\right]_0^x$$

$$[\cos t]_0^x \geq -\left[\frac{t^2}{2!}\right]_0^x$$

$$\cos x - \cos 0 \geq -\frac{x^2}{2!} + \frac{0^2}{2!}$$

$$\cos x \geq 1 - \frac{x^2}{2!}$$

22c

$$\int_0^x \cos t\,dt \geq \int_0^x 1 - \frac{t^2}{2!}\,dt$$

$$[\sin t]_0^x \geq \left[t - \frac{t^3}{3!}\right]_0^x$$

$$\sin x - \sin 0 \geq x - \frac{x^3}{3!} - (0 - 0)$$

$$\sin x \geq x - \frac{x^3}{3!}$$

$$\int_0^x \sin t\,dt \geq \int_0^x t - \frac{t^3}{3!}\,dt$$

$$[-\cos t]_0^x \geq \left[\frac{t^2}{2!} - \frac{t^4}{4!}\right]_0^x$$

$$[\cos t]_0^x \leq \left[-\frac{t^2}{2!} + \frac{t^4}{4!}\right]_0^x$$

$$\cos x - \cos 0 \leq -\frac{x^2}{2!} + \frac{x^4}{4!}$$

$$\cos x - 1 \leq -\frac{x^2}{2!} + \frac{x^4}{4!}$$

Uncorrected proofs

$$\cos x \leq 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$

22d We have shown this is true in the case where $n = 1$ above.

Assume true for $n = k$.

$$\sin x \leq x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{x^{4k+1}}{(4k+1)!} \leq \sin x + \frac{x^{4k+3}}{(4k+3)!}$$

This implies that:

$$\sin t \leq t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots + \frac{t^{4k+1}}{(4k+1)!} \leq \sin t + \frac{t^{4k+3}}{(4k+3)!}$$

Integrating from 0 to x gives:

$$\int_0^x \sin t \, dt \leq \int_0^x t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots + \frac{t^{4k+1}}{(4k+1)!} dt \leq \int_0^x \sin t + \frac{t^{4k+3}}{(4k+3)!} dt$$

$$[-\cos t]_0^x \leq \left[\frac{t^2}{2} - \frac{t^4}{4!} + \frac{t^6}{6!} - \frac{t^8}{8!} + \cdots + \frac{t^{4k+2}}{(4k+2)!}\right]_0^x \leq \left[-\cos t + \frac{t^{4k+4}}{(4k+4)!}\right]_0^x$$

$$-[\cos x - 1] \leq \frac{x^2}{2} - \frac{x^4}{4!} + \frac{x^6}{6!} - \frac{x^8}{8!} + \cdots + \frac{x^{4k+2}}{(4k+2)!}$$
$$\leq -\cos x + \frac{x^{4k+4}}{(4k+4)!} - [-1 + 0]$$

$$1 - \cos x \leq \frac{x^2}{2} - \frac{x^4}{4!} + \frac{x^6}{6!} - \frac{x^8}{8!} + \cdots + \frac{x^{4k+2}}{(4k+2)!} \leq 1 - \cos x + \frac{x^{4k+4}}{(4k+4)!}$$

Substituting in t for x

$$1 - \cos t \leq \frac{t^2}{2} - \frac{t^4}{4!} + \frac{t^6}{6!} - \frac{t^8}{8!} + \cdots + \frac{t^{4k+2}}{(4k+2)!} \leq 1 - \cos t + \frac{t^{4k+4}}{(4k+4)!}$$

Integrating again gives

$$\int_0^x 1 - \cos t \, dt \leq \int_0^x \frac{t^2}{2} - \frac{t^4}{4!} + \frac{t^6}{6!} - \frac{t^8}{8!} + \cdots + \frac{t^{4k+2}}{(4k+2)!} dt$$
$$\leq \int_0^x 1 - \cos t + \frac{t^{4k+4}}{(4k+4)!} dt$$

$$[t - \sin t]_0^x \leq \left[\frac{t^3}{3!} - \frac{t^5}{5!} + \frac{t^7}{7!} + \cdots - \frac{t^{4k+1}}{(4k+1)!} + \frac{t^{4k+3}}{(4k+3)!}\right]_0^x$$
$$\leq \left[t - \sin t + \frac{t^{4k+5}}{(4k+5)!}\right]_0^x$$

Uncorrected proofs

$$x - \sin x \le \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots - \frac{x^{4k+1}}{(4k+1)!} + \frac{x^{4k+3}}{(4k+3)!} \le x - \sin x + \frac{x^{4k+5}}{(4k+5)!}$$

$$-\sin x \le -x + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \cdots - \frac{x^{4k+1}}{(4k+1)!} + \frac{x^{4k+3}}{(4k+3)!} \le -\sin x + \frac{x^{4k+5}}{(4k+5)!}$$

This implies that

$$-\sin t \le -t + \frac{t^3}{3!} - \frac{t^5}{5!} + \frac{t^7}{7!} - \cdots - \frac{t^{4k+1}}{(4k+1)!} + \frac{t^{4k+3}}{(4k+3)!} \le \sin t + \frac{t^{4k+5}}{(4k+5)!}$$

Integrating from 0 to x gives

$$\int_0^x -\sin t\,dt \le \int_0^x -t + \frac{t^3}{3!} - \frac{t^5}{5!} + \frac{t^7}{7!} - \cdots - \frac{t^{4k+1}}{(4k+1)!} + \frac{t^{4k+3}}{(4k+3)!}\,dt$$
$$\le \int_0^x \sin t - \frac{t^{4k+5}}{(4k+5)!}\,dt$$

$$[\cos t]_0^x \le \left[-\frac{t^2}{2} + \frac{t^4}{4!} - \frac{t^6}{6!} + \frac{t^8}{8!} - \cdots - \frac{t^{4k+2}}{(4k+2)!} + \frac{t^{4k+4}}{(4k+4)!}\right]_0^x$$
$$\le \left[\cos t + \frac{t^{4k+6}}{(4k+6)!}\right]_0^x$$

$$[\cos x - 1] \le -\frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots - \frac{x^{4k+2}}{(4k+2)!} + \frac{x^{4k+4}}{(4k+4)!}$$
$$\le \cos x + \frac{x^{4k+6}}{(4k+6)!} - [1 + 0]$$

$$\cos x \le 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots - \frac{x^{4k+2}}{(4k+2)!} + \frac{x^{4k+4}}{(4k+4)!} \le \cos x + \frac{x^{4k+6}}{(4k+6)!}$$

Substituting in t for x

$$\cos t \le 1 - \frac{t^2}{2} + \frac{t^4}{4!} - \frac{t^6}{6!} + \frac{t^8}{8!} - \cdots - \frac{t^{4k+2}}{(4k+2)!} + \frac{t^{4k+4}}{(4k+4)!} \le \cos t + \frac{t^{4k+6}}{(4k+6)!}$$

Integrating again gives

$$\int_0^x \cos t\,dt \le 1 - \frac{t^2}{2} + \frac{t^4}{4!} - \frac{t^6}{6!} + \frac{t^8}{8!} - \cdots - \frac{t^{4k+2}}{(4k+2)!} + \frac{t^{4k+4}}{(4k+4)!}\,dt$$
$$\le \int_0^x \cos t + \frac{t^{4k+6}}{(4k+6)!}\,dt$$

$$[\sin t]_0^x \le \left[\frac{t^3}{3!} - \frac{t^5}{5!} + \frac{t^7}{7!} + \cdots - \frac{t^{4k+3}}{(4k+3)!} + \frac{t^{4k+5}}{(4k+5)!}\right]_0^x \le \left[\sin t + \frac{t^{4k+7}}{(4k+5)!}\right]_0^x$$

Uncorrected proofs

$$\sin x \le x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{x^{4k+5}}{(4k+5)!} \le \sin x + \frac{x^{4k+7}}{(4k+7)!}$$

$$\sin x \le x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{x^{4(k+1)+1}}{(4(k+1)+1)!} \le \sin x + \frac{x^{4(k+1)+3}}{(4(k+1)+3)!}$$

Hence we have shown that the statement holds when $n = k + 1$ and thus by the principle of mathematical induction must be true for all positive values of n.

Now $\lim\limits_{n\to\infty} \frac{x^{4k+3}}{(4k+3)!} = 0$, this can be argued for positive x as follows, since $x > 0$, $4x + 3 > x$, and $\frac{x}{4x+3} < 1$ hence $\lim\limits_{n\to\infty} \frac{x^{4k+3}}{(4k+3)!} = \lim\limits_{n\to\infty} \left(\frac{x^{4x+3}}{(4x+3)!} \times \frac{x}{4x+4} \times \frac{x}{4x+5} \times \ldots \times \frac{x}{4n+3}\right)$

$$\le \lim_{n\to\infty} \left(\frac{x^{4x+3}}{(4x+3)!} \times \frac{x}{4x+3} \times \frac{x}{4x+3} \times \ldots\right)$$

$$= \lim_{n\to\infty} \left(\frac{x^{4x+3}}{(4x+3)!} \times \left(\frac{x}{4x+3}\right)^{4(n-x)}\right)$$

$$= \left(\frac{x^{4x+3}}{(4x+3)!}\right) \lim_{n\to\infty} \left(\left(\frac{x}{4x+3}\right)^{4(n-x)}\right)$$

$$= \left(\frac{x^{4x+3}}{(4x+3)!}\right)(0) \text{ (as } 0 < \frac{x}{4x+3} < 1)$$

$$= 0$$

This means that

$$\lim_{n\to\infty} \sin x \le \lim_{n\to\infty} x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{x^{4n+1}}{(4n+1)!} \le \lim_{n\to\infty} \left(\sin x + \frac{x^{4n+3}}{(4n+3)!}\right)$$

$$\lim_{n\to\infty} \sin x \le \lim_{n\to\infty} x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{x^{4n+1}}{(4n+1)!} \le \lim_{n\to\infty} (\sin x + 0)$$

$$\sin x \le \lim_{n\to\infty} x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{x^{4n+1}}{(4n+1)!} \le \sin x$$

$$\sin x \le x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \le \sin x$$

And hence it must be the case that

$x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$ converges with limit to $\sin x$

Uncorrected proofs

22e Firstly, we show by induction that

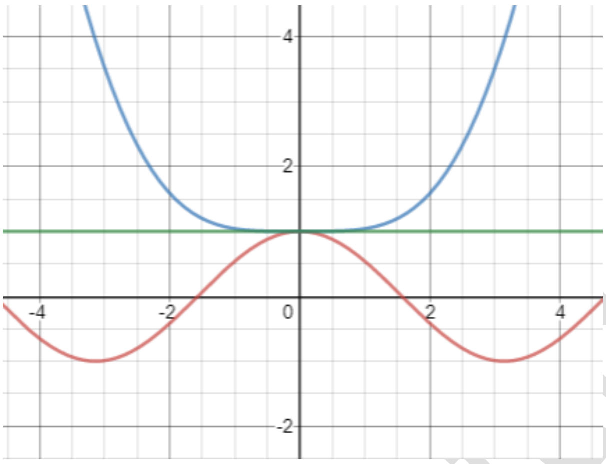
$$\cos x \leq 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{6^5}{6!} + \cdots + \frac{x^{4n}}{(4n)!} \leq \cos x + \frac{x^{4n+2}}{(4n+2)!}$$

Holds for all values of n

In the case where $n = 0$

$$\cos x \leq 1 \leq \cos x + \frac{x^2}{2!}$$

This can be observed to be true graphically



Now we assume that this statement is true in the case where $n = k$

$$\cos x \leq 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{x^{4k}}{(4k)!} \leq \cos x + \frac{x^{4k+2}}{4k+2}$$

Substitute in $t = x$

$$\cos t \leq 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \cdots + \frac{t^{4k}}{(4k)!} \leq \cos t + \frac{t^{4k+2}}{4k+2}$$

Now integrating from 0 to x gives

$$\int_0^x \cos t \, dt \leq \int_0^x 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \cdots + \frac{t^{4k}}{(4k)!} dt \leq \int_0^x \cos t + \frac{t^{4k+2}}{4k+2} dt$$

$$\sin x \leq x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{x^{4k+1}}{(4k+1)!} \leq \sin x + \frac{x^{4k+3}}{(4k+3)!}$$

We have already shown in part d that this then gives that

$$\cos x \leq 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots - \frac{x^{4k+2}}{(4k+2)!} + \frac{x^{4k+4}}{(4k+4)!} \leq \cos x + \frac{x^{4k+6}}{(4k+6)!}$$

Uncorrected proofs

$$\cos x \le 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots - \frac{x^{4k+2}}{(4k+2)!} + \frac{x^{4(k+1)}}{\big(4(k+1)\big)!}$$
$$\le \cos x + \frac{x^{4(k+1)+2}}{(4(k+1)+2)!}$$

Hence, we have shown that this statement is true in the case where $n = k + 1$ and thus must be true for all positive n by the principle of mathematical induction.

Similarly, to part 22d, $\lim_{n\to\infty} \frac{x^{4(k+1)+2}}{(4(k+1)+2)!} = 0$

Thus

$$\lim_{n\to\infty} \cos x \le \lim_{n\to\infty} 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots - \frac{x^{4k+2}}{(4k+2)!} + \frac{x^{4(k+1)}}{\big(4(k+1)\big)!}$$
$$\le \lim_{n\to\infty} \left(\cos x + \frac{x^{4(k+1)+2}}{(4(k+1)+2)!}\right)$$

$$\lim_{n\to\infty} \cos x \le \lim_{n\to\infty} 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots - \frac{x^{4k+2}}{(4k+2)!} + \frac{x^{4(k+1)}}{\big(4(k+1)\big)!}$$
$$\le \lim_{n\to\infty} \cos x + 0$$

$$\lim_{n\to\infty} \cos x \le \lim_{n\to\infty} 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots - \frac{x^{4k+2}}{(4k+2)!} + \frac{x^{4(k+1)}}{\big(4(k+1)\big)!}$$
$$\le \lim_{n\to\infty} \cos x$$

$$\cos x \le \lim_{n\to\infty} 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots - \frac{x^{4k+2}}{(4k+2)!} + \frac{x^{4(k+1)}}{\big(4(k+1)\big)!} \le \cos x$$

$$\cos x \le 1 - \frac{x^2}{2} + \frac{x^4}{4!} + \cdots \le \cos x$$

Thus $1 - \frac{x^2}{2} + \frac{x^4}{4!} + \cdots$ converges with limit to $\cos x$.

22f Let $f(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + \cdots$ and $g(x) = \cos x$

$$f(-x) = 1 - \frac{(-x)^2}{2} + \frac{(-x)^4}{4!} + \cdots$$
$$= 1 - \frac{x^2}{2} + \frac{x^4}{4!} + \cdots$$
$$= f(x)$$

$$g(-x) = \cos(-x) = \cos x = g(x)$$

Uncorrected proofs

Now since $f(x)$ converges with limit to $g(x)$,

$f(-x)$ converges with limit to $g(-x)$ which implies that the result holds for negative values.

This means that $1-\frac{x^2}{2}+\frac{x^4}{4!}+\cdots$ converges with limit to $\cos x$ for both positive and negative values of x.

Let $f(x)=x-\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots$ and $g(x)=\sin x$

$$f(-x)=(-x)-\frac{(-x)^3}{3!}+\frac{(-x)^5}{5!}+\cdots$$
$$=-(x-\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots)$$
$$=-f(x)$$

$$g(-x)=\sin(-x)$$
$$=-\sin x$$
$$=-g(x)$$

Now since $g(x)$ converges with limit to $f(x)$, $-g(x)$ converges with limit to $-f(x)$, and thus $g(-x)$ converges with limit to $f(-x)$.

This means that the result $x-\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots$ converges with limit to $\sin x$ for negative values of x.

Uncorrected proofs

Solutions to Exercise 7E

1a Area between the curve and the x-axis

$$= \int_0^{\frac{\pi}{2}} \cos x \; dx$$

$$= [\sin x]_0^{\frac{\pi}{2}}$$

$$= \sin\frac{\pi}{2} - \sin 0$$

$$= 1 - 0$$

$$= 1 \text{ square unit}$$

1b Area between the curve and the x-axis

$$= \int_0^{\frac{\pi}{6}} \cos x \; dx$$

$$= [\sin x]_0^{\frac{\pi}{6}}$$

$$= \sin\frac{\pi}{6} - \sin 0$$

$$= \frac{1}{2} - 0$$

$$= \tfrac{1}{2} \text{ square unit}$$

2a Area between the curve and the x-axis

$$= \int_0^{\frac{\pi}{4}} \sec^2 x \; dx$$

$$= [\tan x]_0^{\frac{\pi}{4}}$$

$$= \tan\frac{\pi}{4} - \tan 0$$

$$= 1 - 0$$

$$= 1 \text{ square unit}$$

Uncorrected proofs

2b Area between the curve and the x-axis

$$= \int_0^{\frac{\pi}{3}} \sec^2 x \ dx$$

$$= [\tan x]_0^{\frac{\pi}{3}}$$

$$= \tan\frac{\pi}{3} - \tan 0$$

$$= \sqrt{3} - 0$$

$$= \sqrt{3} \text{ square units}$$

3a Area between the curve and the x-axis

$$= \int_0^{\frac{\pi}{4}} \sin x \ dx$$

$$= -[\cos x]_0^{\frac{\pi}{4}}$$

$$= -\left(\cos\frac{\pi}{4} - \cos 0\right)$$

$$= -\left(\frac{1}{\sqrt{2}} - 1\right)$$

$$= \left(1 - \tfrac{1}{\sqrt{2}}\right) \text{ square units}$$

3b Area between the curve and the x-axis

$$= \int_0^{\frac{\pi}{6}} \sin x \ dx$$

$$= -[\cos x]_0^{\frac{\pi}{6}}$$

$$= -\left(\cos\frac{\pi}{6} - \cos 0\right)$$

$$= -\left(\frac{\sqrt{3}}{2} - 1\right)$$

$$= \left(1 - \tfrac{\sqrt{3}}{2}\right) \text{ square units}$$

Uncorrected proofs

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4a Area between the curve and the x-axis

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin x \, dx$$

$$= -[\cos x]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= -\left(\cos\frac{\pi}{2} - \cos\frac{\pi}{6}\right)$$

$$= -\left(0 - \frac{\sqrt{3}}{2}\right)$$

$$= \frac{\sqrt{3}}{2} \text{ square units}$$

4b Area between the curve and the x-axis

$$= \left|\int_{\frac{2\pi}{3}}^{\pi} \cos x \, dx\right|$$

$$= \left|[\sin x]_{\frac{2\pi}{3}}^{\pi}\right|$$

$$= \left|\sin\pi - \sin\frac{2\pi}{3}\right|$$

$$= \left|0 - \frac{\sqrt{3}}{2}\right|$$

$$= \frac{\sqrt{3}}{2} \text{ square units}$$

Note that we must take the absolute value of the integral as the curve is below the x–axis.

5a Required region is above the x-axis.

Area between the curve and the x-axis

$$= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin x \, dx$$

$$= -[\cos x]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$$

Uncorrected proofs

$= -\left(\cos\frac{\pi}{2} - \cos\frac{\pi}{3}\right)$

$= -\left(0 - \frac{1}{2}\right)$

$= \frac{1}{2}$ square unit

5b Required region is above the x-axis.

Area between the curve and the x-axis

$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin 2x \ dx$

$= -\frac{1}{2}[\cos 2x]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$

$= -\frac{1}{2}\left(\cos\pi - \cos\frac{\pi}{2}\right)$

$= -\frac{1}{2}(-1 - 0)$

$= \frac{1}{2}$ square unit

5c Required region is above the x-axis.

Area between the curve and the x-axis

$= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \cos x \ dx$

$= [\sin x]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$

$= \sin\frac{\pi}{2} - \sin\frac{\pi}{3}$

$= 1 - \frac{\sqrt{3}}{2}$

$= \frac{1}{2}(2 - \sqrt{3})$ square units

Uncorrected proofs

5d Required region is above the x-axis.

Area between the curve and the x-axis

$$= \int_{\frac{\pi}{12}}^{\frac{\pi}{6}} \cos 3x \ dx$$

$$= \frac{1}{3}[\sin 3x]_{\frac{\pi}{12}}^{\frac{\pi}{6}}$$

$$= \frac{1}{3}\left(\sin\frac{\pi}{2} - \sin\frac{\pi}{4}\right)$$

$$= \frac{1}{3}\left(1 - \frac{1}{\sqrt{2}}\right)$$

$$= \frac{1}{6}\left(2 - \frac{2}{\sqrt{2}}\right)$$

$$= \tfrac{1}{6}(2 - \sqrt{2}) \text{ square units}$$

5e Required region is above the x-axis.

Area between the curve and the x-axis

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec^2 x \ dx$$

$$= [\tan x]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$= \tan\frac{\pi}{3} - \tan\frac{\pi}{6}$$

$$= \sqrt{3} - \frac{1}{\sqrt{3}}$$

$$= \frac{1}{3}\left(3\sqrt{3} - \frac{3}{\sqrt{3}}\right)$$

$$= \frac{1}{3}(3\sqrt{3} - \sqrt{3})$$

$$= \frac{1}{3}(2\sqrt{3})$$

$$= \tfrac{2}{3}\sqrt{3} \text{ square units}$$

Uncorrected proofs

5f Required region is above the x-axis.

Area between the curve and the x-axis

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sec^2 \frac{1}{2}x \ dx$$

$$= 2\left[\tan\frac{1}{2}x\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= 2\left(\tan\frac{\pi}{4} - \tan\left(-\frac{\pi}{4}\right)\right)$$

$$= 2(1-(-1))$$

$$= 4 \ \text{square units}$$

6a Area of shaded region

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\sin x - \cos x)dx$$

$$= [-\cos x - \sin x]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \left(-\cos\frac{\pi}{2} - \sin\frac{\pi}{2}\right) - \left(-\cos\frac{\pi}{4} - \sin\frac{\pi}{4}\right)$$

$$= (0-1) - \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right)$$

$$= -1 + \frac{2}{\sqrt{2}}$$

$$= \frac{2}{\sqrt{2}} - 1$$

$$= \left(\sqrt{2}-1\right) \ \text{square units}$$

6b Area of shaded region

$$= \int_{0}^{\frac{\pi}{3}} (\sin 2x - \sin x)dx$$

$$= \left[-\frac{1}{2}\cos 2x + \cos x\right]_{0}^{\frac{\pi}{3}}$$

$$= \left(-\frac{1}{2}\cos\frac{2\pi}{3} + \cos\frac{\pi}{3}\right) - \left(-\frac{1}{2}\cos 0 + \cos 0\right)$$

Uncorrected proofs

$$= \left(-\frac{1}{2} \times -\frac{1}{2} + \frac{1}{2}\right) - \left(-\frac{1}{2} \times 1 + 1\right)$$

$$= \left(\frac{1}{4} + \frac{1}{2}\right) - \left(-\frac{1}{2} + 1\right)$$

$$= \frac{1}{4} \text{ square units}$$

6c Area of shaded region

$$= \int_0^{\frac{\pi}{2}} (x - \sin x) dx$$

$$= \left[\frac{x^2}{2} + \cos x\right]_0^{\frac{\pi}{2}}$$

$$= \left(\frac{\left(\frac{\pi}{2}\right)^2}{2} + \cos\frac{\pi}{2}\right) - \left(\frac{0^2}{2} + \cos 0\right)$$

$$= \frac{\pi^2}{8} + 0 - 0 - 1$$

$$= \left(\frac{\pi^2}{8} - 1\right) \text{ square units}$$

6d Area of shaded region

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 - \cos x) dx$$

$$= [x - \sin x]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \left(\frac{\pi}{2} - \sin\frac{\pi}{2}\right) - \left(-\frac{\pi}{2} - \sin\left(-\frac{\pi}{2}\right)\right)$$

$$= \left(\frac{\pi}{2} - 1\right) - \left(-\frac{\pi}{2} + 1\right)$$

$$= (\pi - 2) \text{ square units}$$

Uncorrected proofs

7a Note that between 0 and $\frac{\pi}{4}$ the area is bounded by the curve $y = \sin x$ and the x-axis and then between $\frac{\pi}{4}$ and $\frac{\pi}{2}$ the curve is bounded by $y = \cos x$ and the x-axis.

Area of shaded region

$$= \int_0^{\frac{\pi}{4}} \sin x \, dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos x \, dx$$

$$= [-\cos x]_0^{\frac{\pi}{4}} + [\sin x]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \left(-\cos\frac{\pi}{4} + \cos 0\right) + \left(\sin\frac{\pi}{2} - \sin\frac{\pi}{4}\right)$$

$$= \left(-\frac{1}{\sqrt{2}} + 1\right) + \left(1 - \frac{1}{\sqrt{2}}\right)$$

$$= 2 - \frac{2}{\sqrt{2}}$$

$$= \left(2 - \sqrt{2}\right) \text{ square units}$$

7b Note that between -1 and 0 the area is bounded by the curve $y = x + 1$ and the x-axis and then between 0 and $\frac{\pi}{2}$ the curve is bounded by $y = \cos x$ and the x-axis.

Area of shaded region

$$= \int_{-1}^{0} (x + 1) \, dx + \int_0^{\frac{\pi}{2}} \cos x \, dx$$

$$= \left[\frac{x^2}{2} + x\right]_{-1}^{0} + [\sin x]_0^{\frac{\pi}{2}}$$

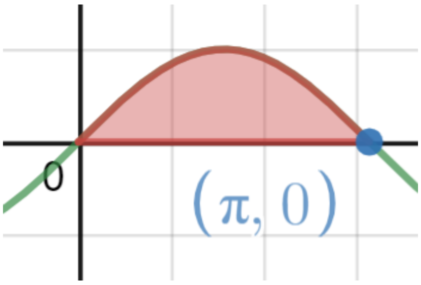
$$= \left(0 + 0 - \left(\frac{(-1)^2}{2} + (-1)\right)\right) + \left(\sin\frac{\pi}{2} - \sin 0\right)$$

$$= \left(-\frac{1}{2} + 1\right) + (1 - 0)$$

$$= 1\frac{1}{2} \text{ square units}$$

Uncorrected proofs

8a Graph of $y = \sin x$ is shown below.



Based on the graph, the required area

$= \int_0^{\pi} \sin x \, dx$

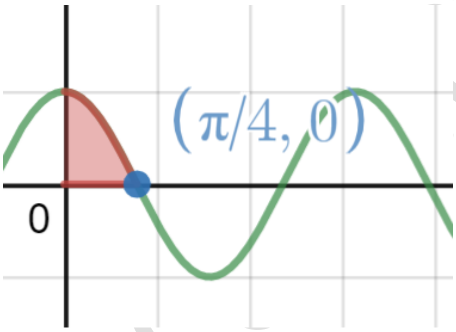
$= [-\cos x]_0^{\pi}$

$= -\cos \pi - (-\cos 0)$

$= -(-1) + 1$

$= 2$ square units

8b Graph of $y = \cos 2x$ is shown below.



Based on the graph, the required area

$= \int_0^{\frac{\pi}{4}} \cos 2x \, dx$

$= \frac{1}{2}[\sin 2x]_0^{\frac{\pi}{4}}$

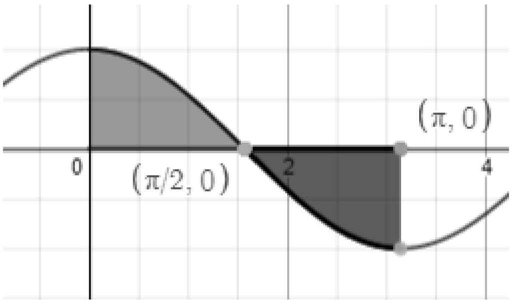
$= \frac{1}{2}\left(\sin \frac{\pi}{2} - 0\right)$

$= \frac{1}{2}(1 - 0)$

$= \frac{1}{2}$ square unit

Uncorrected proofs

9a Graph of $y = \cos x$ is shown below.



Using symmetry, the area from $x = 0$ to $x = \frac{\pi}{2}$ is the same as the area from $x = \frac{\pi}{2}$ to $x = \pi$. We can use the positive signed area from $x = 0$ to $x = \frac{\pi}{2}$ in our calculation.

Area of required region

$$= 2 \times \int_0^{\frac{\pi}{2}} \cos x \, dx$$

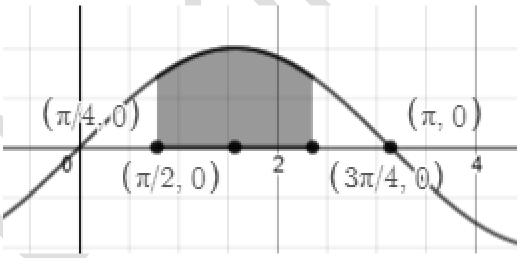
$$= 2[\sin x]_0^{\frac{\pi}{2}}$$

$$= 2\left(\sin\frac{\pi}{2} - \sin 0\right)$$

$$= 2(1 - 0)$$

$$= 2 \text{ square units}$$

9b Graph of $y = \sin x$ is shown below.



Since the required area is above the x-axis, we can calculate the area from $x = \frac{\pi}{4}$ to $x = \frac{3\pi}{4}$.

Area of required region

$$= \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \sin x \, dx$$

$= [-\cos x]_{\frac{\pi}{4}}^{\frac{3\pi}{4}}$

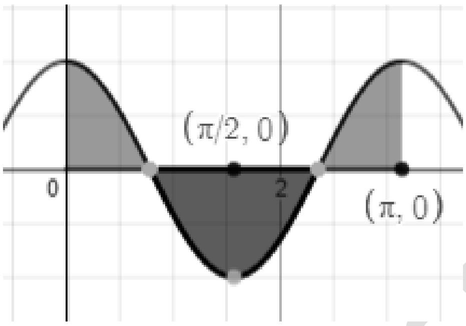
$= -\cos\frac{3\pi}{4} + \cos\frac{\pi}{4}$

$= -\left(-\frac{1}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}}$

$= \frac{2}{\sqrt{2}}$

$= \sqrt{2}$ square units

9c Graph of $y = \cos 2x$ is shown below.



Using symmetry, the area from $x = 0$ to $x = \frac{\pi}{4}$ is the same as the area from $x = \frac{\pi}{4}$ to $x = \frac{\pi}{2}$ and the area from $x = \frac{\pi}{2}$ to $x = \frac{3\pi}{4}$ and the area from $x = \frac{3\pi}{4}$ to $x = \pi$. We can use the positive signed area from $x = 0$ to $x = \frac{\pi}{4}$ in our calculation.

Area of required region

$= 4 \times \int_0^{\frac{\pi}{4}} \cos 2x \, dx$

$= 4 \times \frac{1}{2}\int_0^{\frac{\pi}{4}} 2\cos 2x \, dx$

$= 2[\sin 2x]_0^{\frac{\pi}{4}}$

$= 2\left(\sin\frac{\pi}{2} - \sin 0\right)$

$= 2(1 - 0)$

$= 2$ square units

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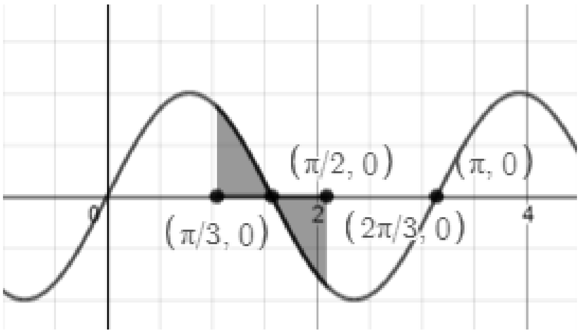
9d Graph of $y = \sin 2x$ is shown below.



Using symmetry, the area from $x = \frac{\pi}{3}$ to $x = \frac{\pi}{2}$ is the same as the area from $x = \frac{\pi}{2}$ to $x = \frac{2\pi}{3}$. We can use the positive signed area from $x = \frac{\pi}{3}$ to $x = \frac{\pi}{2}$ in our calculation.

Area of required region

$$= 2\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin 2x\, dx$$

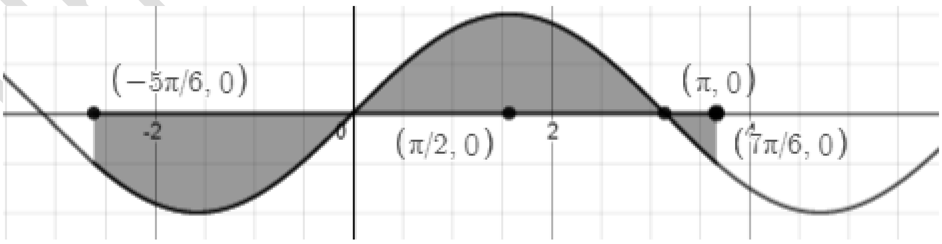
$$= 2 \times -\frac{1}{2}[\cos 2x]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$$

$$= -\left(\cos \pi - \cos \frac{2\pi}{3}\right)$$

$$= -\left(-1 - \left(-\frac{1}{2}\right)\right)$$

$$= \frac{1}{2} \text{ square units}$$

9e Graph of $y = \sin x$ is shown below.



Using symmetry, the area from $x = -\frac{5\pi}{6}$ to $x = 0$ plus the area from $x = \pi$ to $x = \frac{7\pi}{6}$ is the same as the area from $x = 0$ to $x = \pi$. We can use the positive signed area from $x = 0$ to $x = \pi$ in our calculation.

Uncorrected proofs

Area of required region

$= 2\int_0^{\pi} \sin x \, dx$

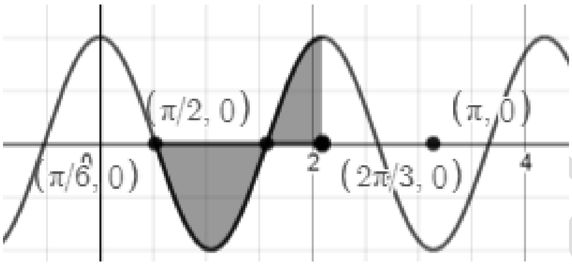
$= -2[\cos x]_0^{\pi}$

$= -2(\cos \pi - \cos 0)$

$= -2(-1 - 1)$

$= 4$ square units

9f Graph of $y = \cos 3x$ is shown below.



Using symmetry, the area from $x = \frac{\pi}{6}$ to $x = \frac{\pi}{3}$ is the same as the area from $x = \frac{\pi}{3}$ to $x = \frac{\pi}{2}$ and the area from $x = \frac{\pi}{2}$ to $x = \frac{2\pi}{3}$. We can use the positive signed area from $x = \frac{\pi}{2}$ to $x = \frac{2\pi}{3}$ in our calculation.

Area of required region

$= 3 \times \int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} \cos 3x \, dx$

$= 3 \times \frac{1}{3}[\sin 3x]_{\frac{\pi}{2}}^{\frac{2\pi}{3}}$

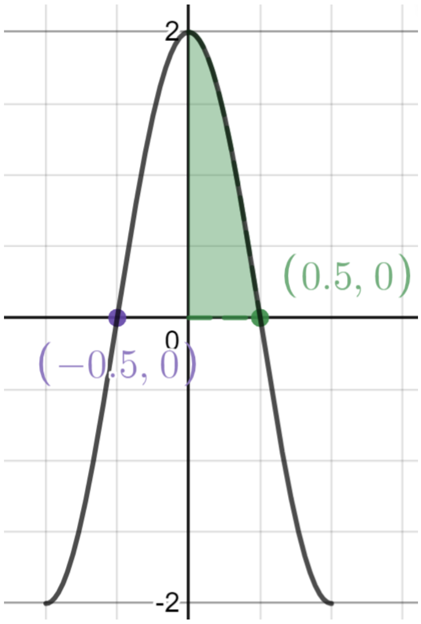
$= \sin 2\pi - \sin \frac{3\pi}{2}$

$= 0 - (-1)$

$= 1$ square unit

Uncorrected proofs

10a Graph of $y = 2\cos \pi x$ is shown below.



10b By observation of the graph, the total area between the two intercepts will be equal to two times that of the area shaded in green.

Area of required region

$$= 2 \times \int_0^{\frac{1}{2}} 2\cos \pi x \, dx$$

$$= 2\left[\frac{2}{\pi}\sin \pi x\right]_0^{\frac{1}{2}}$$

$$= \frac{4}{\pi}\left(\sin\frac{\pi}{2} - \sin 0\right)$$

$$= \frac{4}{\pi}(1-0)$$

$= \frac{4}{\pi}$ square units

Uncorrected proofs

11 Area of window

$$= \int_{-1}^{1} 3\cos\left(\frac{\pi}{2}x\right)dx$$

$$= 3\left[\frac{2}{\pi}\sin\frac{\pi}{2}x\right]_{-1}^{1}$$

$$= \frac{6}{\pi}\sin\left(\frac{\pi}{2}\right) - \frac{6}{\pi}\sin\left(-\frac{\pi}{2}\right)$$

$$= \frac{6}{\pi}\times 1 - \frac{6}{\pi}\times(-1)$$

$$= \frac{12}{\pi}$$

$$\doteqdot 3.8\text{ m}^2$$

12 Area of enclosed region

$$= 2\int_{0}^{\pi}\big(x - (x - \sin x)\big)\,dx$$

$$= 2\int_{0}^{\pi}\sin x\,dx$$

$$= 2[-\cos x]_0^{\pi}$$

$$= 2(-\cos\pi - (-\cos 0))$$

$$= 2(1+1)$$

$$= 4\text{ square units}$$

13 Area of region R

$$= \int_{0}^{\frac{\pi}{3}}\tan x\,dx$$

$$= \int_{0}^{\frac{\pi}{3}}\frac{\sin x}{\cos x}dx$$

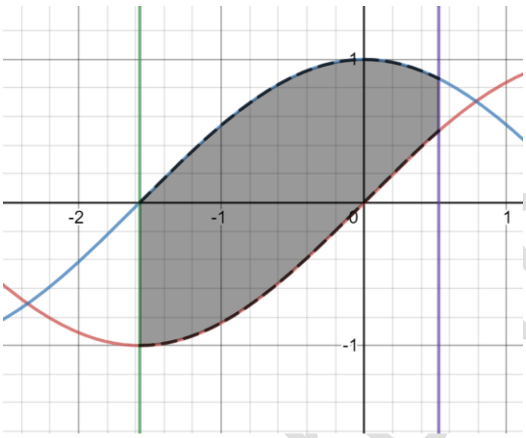
$$= \int_{0}^{\frac{\pi}{3}}\frac{-\frac{d}{dx}(\cos x)}{\cos x}dx$$

$$= -[\ln|\cos x|]_0^{\frac{\pi}{3}}$$

Uncorrected proofs

$= -\left(\ln\left|\cos\frac{\pi}{3}\right| - \ln|\cos 0|\right)$
$= -\left(\ln\left|\frac{1}{2}\right| - \ln 1\right)$
$= -\left(\ln\frac{1}{2} - 0\right)$
$= -\ln\frac{1}{2}$
$= -\ln 2^{-1}$
$= -(-\ln 2)$
$= \ln 2$ square units

14a Graph of $y = \sin x$ (red line) and $y = \cos x$ (blue line) is shown below. Vertical line $x = -\frac{\pi}{2}$ is shown in green and vertical line $x = \frac{\pi}{6}$ is shown in purple.



14b Area of required region

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{6}} (\cos x - \sin x)dx$

$= [\sin x + \cos x]_{-\frac{\pi}{2}}^{\frac{\pi}{6}}$

$= \left(\sin\frac{\pi}{6} + \cos\frac{\pi}{6}\right) - \left(\sin\left(-\frac{\pi}{2}\right) + \cos\left(-\frac{\pi}{2}\right)\right)$

$= \left(\frac{1}{2} + \frac{\sqrt{3}}{2}\right) - (-1 + 0)$

$= \frac{3}{2} + \frac{\sqrt{3}}{2}$

Uncorrected proofs

$= \frac{1}{2}(3 + \sqrt{3})$ square units

15 When $x = -\frac{\pi}{2}$

$y = \sin\left(-\frac{\pi}{2}\right) = -1$

$y = \cos 2\left(-\frac{\pi}{2}\right) = \cos(-\pi) = -1$

Hence the two curves meet at $\left(-\frac{\pi}{2}, -1\right)$.

When $x = \frac{\pi}{6}$

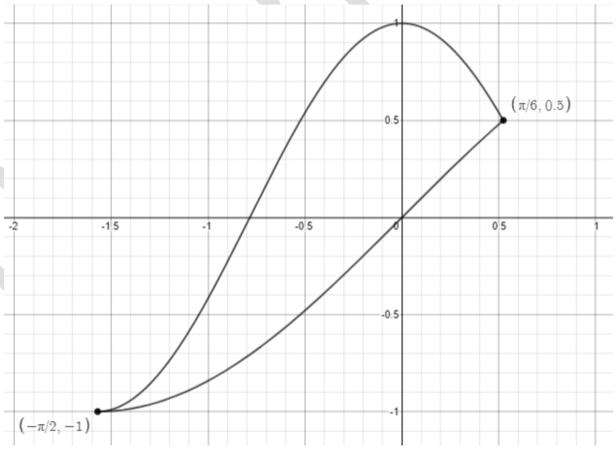
$y = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$

$y = \cos 2\left(\frac{\pi}{6}\right) = \cos\frac{\pi}{3} = \frac{1}{2}$

Hence the two curves meet at $\left(\frac{\pi}{6}, \frac{1}{2}\right)$.

Hence we have shown by substitution that the curves meet at $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{6}$ as they have the same values of y for each of the values of x and hence must pass through the same point.

15b Graph of $y = \cos 2x$ (top curve) and $y = \sin x$ (bottom curve) shown below.



15c Area of required region

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{6}} (\cos 2x - \sin x)dx$$

$$= \left[\frac{1}{2}\sin 2x + \cos x\right]_{-\frac{\pi}{2}}^{\frac{\pi}{6}}$$

$$= \left(\frac{1}{2}\sin\frac{\pi}{3} + \cos\frac{\pi}{6}\right) - \left(\frac{1}{2}\sin(-\pi) + \cos\left(-\frac{\pi}{2}\right)\right)$$

$$= \left(\frac{1}{2} \times \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\right) - \left(\frac{1}{2} \times 0 + 0\right)$$

$$= \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{2}$$

$$= \frac{\sqrt{3}}{4} + \frac{2\sqrt{3}}{4}$$

$$= \frac{3\sqrt{3}}{4}$$

$= \frac{3}{4}\sqrt{3}$ square units

16a LHS $= \sqrt{2}\sin(x + \frac{\pi}{4})$

Use the identity, $\sin(A + B) = \sin A \cos B + \cos A \sin B$

$$\sqrt{2}\sin(x + \tfrac{\pi}{4})$$

$$= \sqrt{2}\left(\sin x \times \cos\tfrac{\pi}{4} + \cos x \times \sin\tfrac{\pi}{4}\right)$$

$$= \sqrt{2}\left(\sin x \times \tfrac{1}{\sqrt{2}} + \cos x \times \tfrac{1}{\sqrt{2}}\right)$$

$$= \sqrt{2}\left(\tfrac{1}{\sqrt{2}}\sin x + \tfrac{1}{\sqrt{2}}\cos x\right)$$

$$= \sqrt{2} \times \tfrac{1}{\sqrt{2}}\sin x + \sqrt{2} \times \tfrac{1}{\sqrt{2}}\cos x$$

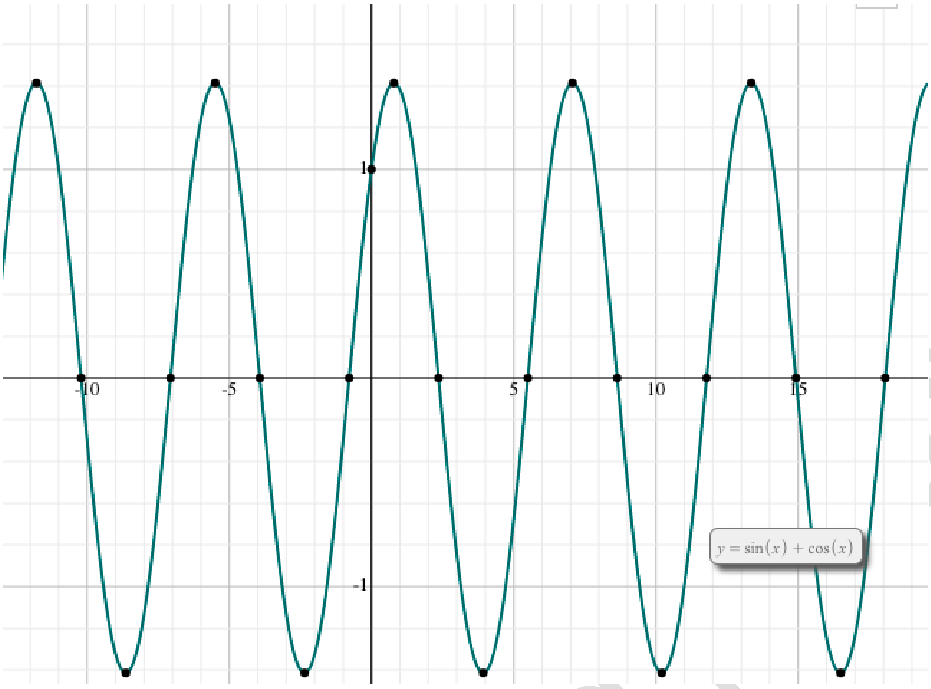
$$= \sin x + \cos x$$

$= \text{RHS}$

Hence, proved.

Uncorrected proofs

16b



Area under one arc is clearly seen between the x-coordinate $-\frac{\pi}{4}$ and $\frac{3\pi}{4}$.

Area

$$= \int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} (\sin x + \cos x)\, dx$$

$$= [-\cos x + \sin x]_{-\frac{\pi}{4}}^{\frac{3\pi}{4}}$$

$$= \left(-\cos\frac{3\pi}{4} + \sin\frac{3\pi}{4}\right) - \left(-\cos\left(-\frac{\pi}{4}\right) + \sin\left(-\frac{\pi}{4}\right)\right)$$

$$= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) - \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right)$$

$$= \left(\frac{2}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}$$

$$= \frac{4}{\sqrt{2}}$$

$= 2\sqrt{2}$ square units

Uncorrected proofs

17a i For positive integers n:

$$\int_0^{2\pi} \sin nx\, dx$$

$$= \left[-\frac{1}{n}\cos nx\right]_0^{2\pi}$$

$$= -\frac{1}{n}(\cos 2n\pi - \cos 0)$$

$$= -\frac{1}{n}(1 - 1)$$

$$= 0$$

17a ii For positive integers n:

$$\int_0^{2\pi} \cos nx\, dx$$

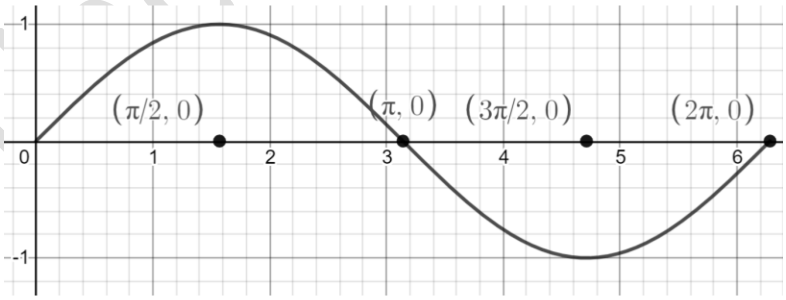
$$= \frac{1}{n}[\sin nx]_0^{2\pi}$$

$$= \frac{1}{n}(\sin 2n\pi - \sin 0)$$

$$= \frac{1}{n}(0 - 0)$$

$$= 0$$

17b i Graph of $y = \sin x$ is shown below.



Using symmetry:

Area of required region

$$= 2 \times \int_0^{\pi} \sin x\, dx$$

$= 2[-\cos x]_0^{\pi}$

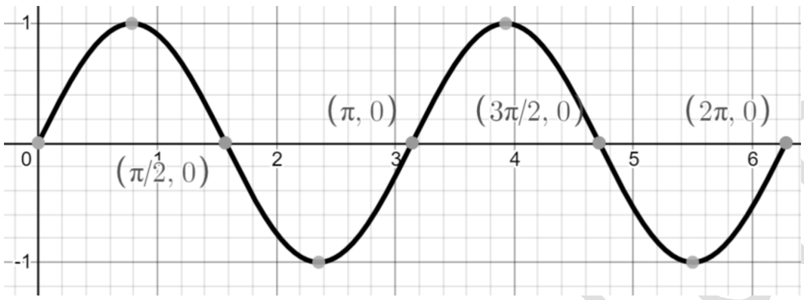
$= -2[\cos x]_0^{\pi}$

$= -2(\cos \pi - \cos 0)$

$= -2(-1 - 1)$

$= 4$ square units

17b ii Graph of $y = \sin 2x$ is shown below.



Using symmetry:

Area of required region

$= 4 \times \int_0^{\frac{\pi}{2}} \sin 2x \, dx$

$= 4\left[-\frac{1}{2}\cos 2x\right]_0^{\frac{\pi}{2}}$

$= -2[\cos 2x]_0^{\frac{\pi}{2}}$

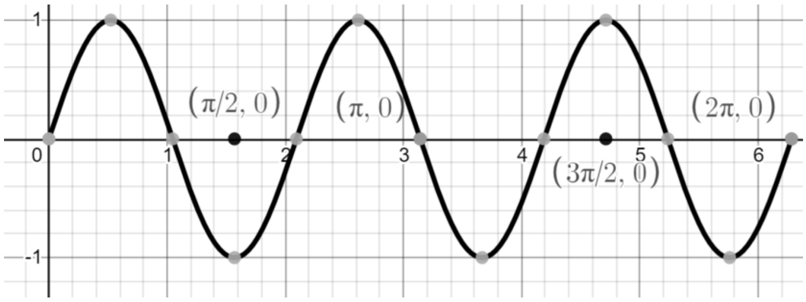
$= -2(\cos \pi - \cos 0)$

$= -2(-1 - 1)$

$= 4$ square units

Uncorrected Proofs

17b iii Graph of $y = \sin 3x$ is shown below.



Using symmetry:

Area of required region

$= 6 \times \int_0^{\frac{\pi}{3}} \sin 3x \, dx$

$= 6\left[-\frac{1}{3}\cos 3x\right]_0^{\frac{\pi}{3}}$

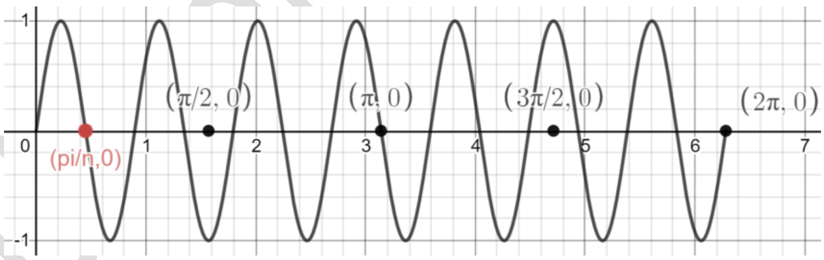
$= -2[\cos 3x]_0^{\frac{\pi}{3}}$

$= -2(\cos \pi - \cos 0)$

$= -2(-1 - 1)$

$= 4$ square units

17b iv



Using symmetry and pattern observed from parts i, ii and iii:

Area of required region

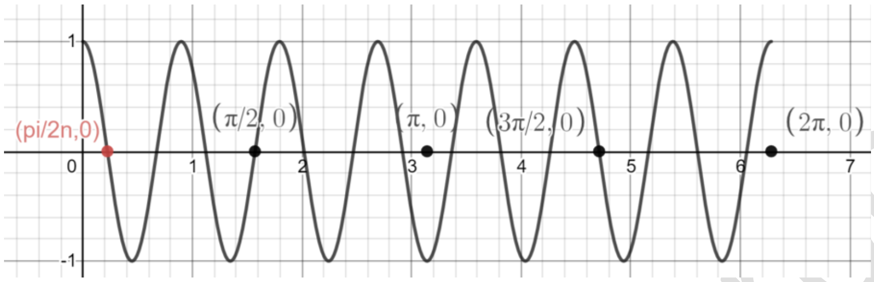
$= 2n \times \int_0^{\frac{\pi}{n}} \sin nx \, dx$

$= 2n\left[-\frac{1}{n}\cos nx\right]_0^{\frac{\pi}{n}}$

Uncorrected proofs

$= -2[\cos nx]_0^{\frac{\pi}{n}}$
$= -2[\cos \pi - \cos 0]$
$= -2[-1 - 1]$
$= 4$ square units

17b v



Using symmetry:

Area of required region

$= 4n \times \int_0^{\frac{\pi}{2n}} \cos nx \, dx$

$= 4n \left[\frac{1}{n} \sin nx\right]_0^{\frac{\pi}{2n}}$

$= 4[\sin nx]_0^{\frac{\pi}{2n}}$

$= 4\left[\sin \frac{\pi}{2} - \sin 0\right]$

$= 4[1 - 0]$

$= 4$ square units

18a As n is a positive integer, we know that $\cos 2\pi n = 1$, so

$\int_0^n (1 + \sin 2\pi x) \; dx$

$= \left[x - \frac{1}{2\pi} \cos 2\pi x\right]_0^n$

$= \left(n - \frac{1}{2\pi} \cos 2\pi n\right) - \left(0 - \frac{1}{2\pi} \cos 0\right)$

$= \left(n - \frac{1}{2\pi} \times 1\right) - \left(0 - \frac{1}{2\pi} \times 1\right)$

$= n - \frac{1}{2\pi} + \frac{1}{2\pi}$

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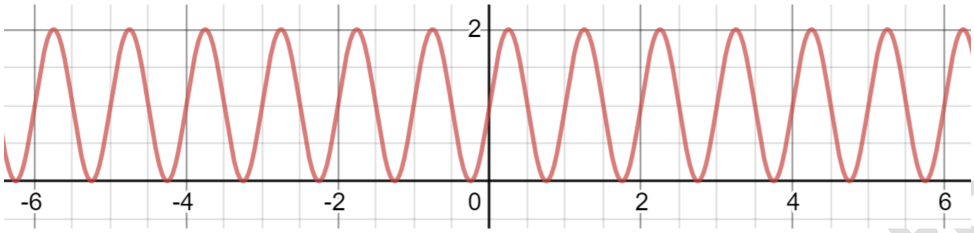
Uncorrected proofs

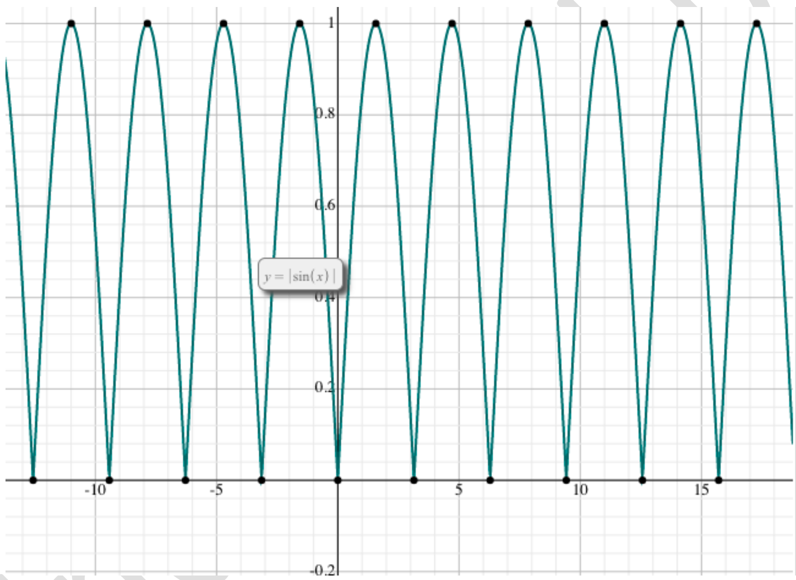
$= n$

18b



The curve is below $y = 1$ just as much as it is above $y = 1$, so the area is equal to the area of a rectangle n units long and 1 unit high.

19



Since each arc is the same over the 6π distance, we only need to calculate the integral from $x = 0$ to $x = \pi$ and multiply by 6.

$6\int_0^{\pi} \sin x \, dx$

$= 6[-\cos x]_0^{\pi}$

$= 6[1 - (-1)]$

$= 12$

20a Given the fact, $\sin x < x < \tan x$ for $0 < x < \frac{\pi}{2}$

We understand that, $x^2 > 0$,

Hence, $x^2 \sin x < x^3 < x^2 \tan x$ for $0 < x < \frac{\pi}{2}$

20b $x^2 \sin x \le x^3 \le x^2 \tan x$

For $0 < x < \frac{\pi}{2}$, and hence it must be true for $0 < x < \frac{\pi}{4}$ so integrating over this interval gives

$$\int_0^{\frac{\pi}{4}} x^2 \sin x \, dx \le \int_0^{\frac{\pi}{4}} x^3 \, dx \le \int_0^{\frac{\pi}{4}} x^2 \tan x \, dx$$

$$\int_0^{\frac{\pi}{4}} x^2 \sin x \, dx \le \left[\frac{x^4}{4}\right]_0^{\frac{\pi}{4}} \le \int_0^{\frac{\pi}{4}} x^2 \tan x \, dx$$

$$\int_0^{\frac{\pi}{4}} x^2 \sin x \, dx \le \frac{\left(\frac{\pi}{4}\right)^4}{4} \le \int_0^{\frac{\pi}{4}} x^2 \tan x \, dx$$

$$\int_0^{\frac{\pi}{4}} x^2 \sin x \, dx \le \frac{\pi^4}{4^5} \le \int_0^{\frac{\pi}{4}} x^2 \tan x \, dx$$

21a

$$y = \frac{1}{1 + \sin x}$$

$$y' = \frac{d}{dx}\left(\frac{1}{1 + \sin x}\right)$$

$$y' = \frac{(1 + \sin x)\frac{d}{dx}(1) - 1\frac{d}{dx}(1 + \sin x)}{(1 + \sin x)^2}$$

$$y' = \frac{0 - 1 \times (0 + \cos x)}{(1 + \sin x)^2}$$

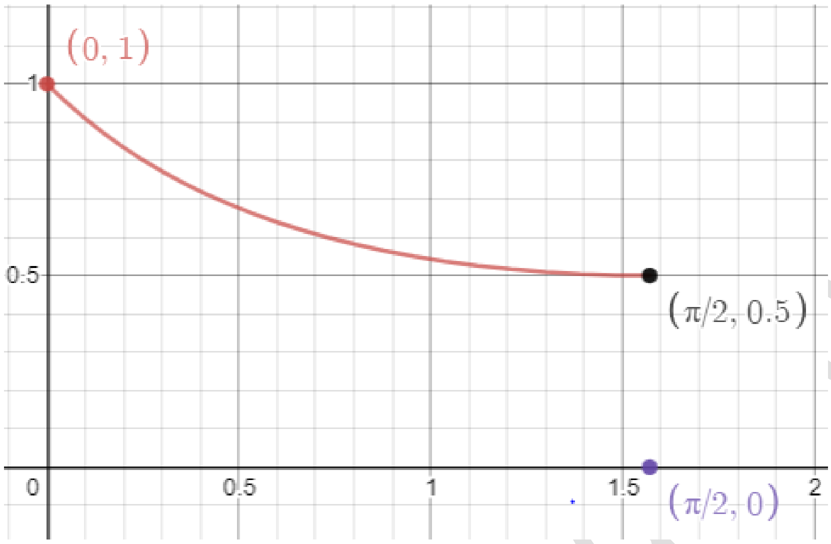
$$y' = \frac{-\cos x}{(1 + \sin x)^2}$$

Hence, proved

Uncorrected proofs

21b In the given domain for $0 < x < \frac{\pi}{2}$, $\cos x$ and $(1 + \sin x)^2$ both are positive hence, y' is negative which is why, $y = \frac{1}{1+\sin x}$ is decreasing.

21c



$\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin x} dx$ represents the area underneath the curve. The rectangle of width $\frac{\pi}{2}$ and height $\frac{1}{2}$ underestimates this area, whilst the rectangle of witdh $\frac{\pi}{2}$ and height 1 overestimates this area. Thus it follows that

$$\frac{\pi}{2} \times \frac{1}{2} < \int_0^{\frac{\pi}{2}} \frac{1}{1 + \sin x} dx < \frac{\pi}{2} \times 1$$

$$\frac{\pi}{4} < \int_0^{\frac{\pi}{2}} \frac{1}{1 + \sin x} dx < \frac{\pi}{2}$$

22a $\int_{-4\pi}^{4\pi} \sin 3x \, dx$

$\sin 3x$ is an odd integrand in the domain, hence

$\int_{-4\pi}^{4\pi} \sin 3x \, dx = 0$

22b $\int_{-2\pi}^{2\pi} \cos^2 x \sin^3 x dx$

$\cos^2 x \sin^3 x$ is an odd integrand in the domain, hence

$\int_{-2\pi}^{2\pi} \cos^2 x \sin^3 x dx = 0$

Uncorrected proofs

22c $\int_{-\frac{5\pi}{2}}^{\frac{5\pi}{2}} \cos x \, dx$

$\cos x$ is an even integrand in the domain, hence

$\int_{-\frac{5\pi}{2}}^{\frac{5\pi}{2}} \cos x \, dx$

$= 2\int_0^{\frac{5\pi}{2}} \cos x \, dx$

$= 2(\sin x)_0^{\frac{5\pi}{2}}$

$= 2(\sin\frac{5\pi}{2} - \sin 0)$

$= 2(1 - 0) = 2$

22d $\int_{-\pi}^{\pi} \sec^2 \frac{1}{3} x dx$

$\sec^2 \frac{1}{3} x$ is an even integrand in the domain, hence

$\int_{-\pi}^{\pi} \sec^2 \frac{1}{3} x dx$

$= 2\int_0^{\pi} \sec^2 \frac{1}{3} x dx$

$= 2 \times 3(\tan\frac{1}{3}x)_0^{\pi}$

$= 6(\tan\frac{\pi}{3} - \tan 0)$

$= 6(\sqrt{3} - 0)$

$= 6\sqrt{3}$

22e $\int_{-\pi}^{\pi}(3 + 2x + \sin x) \, dx$

In the given domain, 3 is an even integrand while $2x$ and $\sin x$ are odd integrand

Hence,

$\int_{-\pi}^{\pi}(3 + 2x + \sin x) \, dx$, is written as

$\int_{-\pi}^{\pi} 3 \, dx$

$= 2\int_0^{\pi} 3 \, dx$

$= 2 \times 3[x]_0^{\pi}$

Uncorrected proofs

$= 6(\pi - 0)$

$= 6\pi$

22f $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin 2x + \cos 3x + 3x^2)\, dx$

In the given domain, $\sin 2x$ is an odd integrand while $\cos 3x$ and $3x^2$ are even integrand

Hence,

$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin 2x + \cos 3x + 3x^2)\, dx$, is written as

$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos 3x + 3x^2)\, dx$

$= 2\int_0^{\frac{\pi}{2}} (\cos 3x + 3x^2)\ dx$

$= 2\int_0^{\frac{\pi}{2}} (\cos 3x) dx) + 2\int_0^{\frac{\pi}{2}} (3x^2) dx$

$= 2\left[(\frac{1}{3}\sin 3x)_0^{\frac{\pi}{2}}\right] + 2\left[(x^3)_0^{\frac{\pi}{2}}\right]$

$= 2(\frac{1}{3}\sin\frac{3\pi}{2} - \frac{1}{3}\sin 0) + 2\left(\frac{\pi^3}{8} - 0\right)$

$= 2\left(\frac{1}{3} \times -1 - 0\right) + \frac{\pi^3}{4}$

$= -\frac{2}{3} + \frac{\pi^3}{4}$

23a

$$\frac{d}{dx}\left(-\frac{1}{2}e^{-x}(\sin x + \cos x)\right)$$

$$= \frac{d}{dx}\left(-\frac{1}{2}e^{-x}\right)(\sin x + \cos x) + \left(-\frac{1}{2}e^{-x}\right)\frac{d}{dx}\big((\sin x + \cos x)\big)$$

$$= \frac{1}{2}e^{-x}(\sin x + \cos x) + \left(-\frac{1}{2}e^{-x}\right)(\cos x - \sin x)$$

$$= \frac{1}{2}e^{-x}(\sin x + \cos x + \sin x - \cos x)$$

$$= e^{-x}\sin x$$

Uncorrected proofs

23b

$$\int_0^N e^{-x}\sin x\,dx$$

$$= \int_0^N \frac{d}{dx}\left(-\frac{1}{2}e^{-x}(\sin x + \cos x)\right)dx$$

$$= \left[-\frac{1}{2}e^{-x}(\sin x + \cos x)\right]_0^N$$

$$= \left[-\frac{1}{2}e^{-N}(\sin N + \cos N)\right] - \left[-\frac{1}{2}e^{-0}(\sin 0 + \cos 0)\right]$$

$$= \left[-\frac{1}{2}e^{-N}(\sin N + \cos N)\right] - \left[-\frac{1}{2}(1)\right]$$

$$= \frac{1}{2} - \frac{1}{2}e^{-N}(\sin N + \cos N)$$

$$\int_0^\infty e^{-x}\sin x\,dx$$

$$= \lim_{N\to\infty}\int_0^N e^{-x}\sin x\,dx$$

$$= \lim_{N\to\infty}\left(\frac{1}{2} - \frac{1}{2}e^{-N}(\sin N + \cos N)\right)$$

$$= \lim_{N\to\infty}\left(\frac{1}{2} - \frac{1}{2}(0)(\sin N + \cos N)\right)$$

$$= \frac{1}{2}$$

23c

$$\int_0^\pi e^{-x}\sin x\,dx$$

$$= \left[-\frac{1}{2}e^{-x}(\sin x + \cos x)\right]_0^\pi$$

$$= \left[-\frac{1}{2}e^{-\pi}(\sin\pi + \cos\pi)\right] - \left[-\frac{1}{2}e^{-0}(\sin 0 + \cos 0)\right]$$

$$= \left[-\frac{1}{2}e^{-\pi}(\sin\pi + \cos\pi)\right] - \left[-\frac{1}{2}e^{-0}(1)\right]$$

$$= \frac{1}{2} - \frac{1}{2}e^{-\pi}(-1)$$

$$= \frac{1}{2}(1 + e^{-\pi})$$

Uncorrected proofs

$$\int_{2\pi}^{3\pi} e^{-x}\sin x\,dx$$

$$= \left[-\frac{1}{2}e^{-x}(\sin x + \cos x)\right]_{\pi}^{3\pi}$$
$$= \left[-\frac{1}{2}e^{-3\pi}(\sin 3\pi + \cos 3\pi)\right] - \left[-\frac{1}{2}e^{-2\pi}(\sin 2\pi + \cos 2\pi)\right]$$
$$= \left[-\frac{1}{2}e^{-3\pi}(-1)\right] - \left[-\frac{1}{2}e^{-2\pi}(1)\right]$$
$$= \frac{1}{2}(e^{-2\pi} + e^{-3\pi})$$

Thus the limiting sum is

$$\frac{1}{2}(1+e^{-\pi}) + \frac{1}{2}(e^{-2\pi}+e^{-3\pi}) + \frac{1}{2}(e^{-4\pi}+e^{-5\pi}) + \cdots$$
$$= \left[\frac{1}{2}(1+e^{-\pi})\right] + e^{-2\pi}\left[\frac{1}{2}(1+e^{-\pi})\right] + e^{-4\pi}\left[\frac{1}{2}(1+e^{-\pi})\right] + \cdots$$

This is a GP with $a = \left[\frac{1}{2}(1+e^{-\pi})\right]$ and $r = e^{-2\pi}$ so the limiting sum is

$$\frac{a}{1-r} = \frac{\frac{1}{2}(1+e^{-\pi})}{1-e^{-2\pi}} = \frac{\frac{1}{2}(1+e^{-\pi})}{(1-e^{-\pi})(1+e^{-\pi})} = \frac{\frac{1}{2}}{1-e^{-\pi}} = \frac{e^{\pi}}{2(e^{\pi}-1)}$$

23d

$$\int_{\pi}^{2\pi} e^{-x}\sin x\,dx$$

$$= \left[-\frac{1}{2}e^{-x}(\sin x + \cos x)\right]_{\pi}^{2\pi}$$
$$= \left[-\frac{1}{2}e^{-2\pi}(\sin 2\pi + \cos 2\pi)\right] - \left[-\frac{1}{2}e^{-\pi}(\sin \pi + \cos \pi)\right]$$
$$= \left[-\frac{1}{2}e^{-2\pi}(1)\right] - \left[-\frac{1}{2}e^{-\pi}(-1)\right]$$
$$= -\frac{1}{2}(e^{-\pi} + e^{-2\pi})$$

$$\int_{3\pi}^{4\pi} e^{-x}\sin x\,dx$$

$$= \left[-\frac{1}{2}e^{-x}(\sin x + \cos x)\right]_{3\pi}^{4\pi}$$

Uncorrected proofs

$$= \left[-\frac{1}{2}e^{-4\pi}(\sin 4\pi + \cos 4\pi)\right] - \left[-\frac{1}{2}e^{-3\pi}(\sin 3\pi + \cos 3\pi)\right]$$
$$= \left[-\frac{1}{2}e^{-4\pi}(1)\right] - \left[-\frac{1}{2}e^{-3\pi}(-1)\right]$$
$$= -\frac{e^{-2\pi}}{2}(e^{-\pi} + e^{-2\pi})$$

Thus the limiting sum is

$$= \left[-\frac{1}{2}(e^{-\pi} + e^{-2\pi})\right] + e^{-2\pi}\left[-\frac{1}{2}(e^{-\pi} + e^{-2\pi})\right] + e^{-4\pi}\left[-\frac{1}{2}(e^{-\pi} + e^{-2\pi})\right] + \cdots$$

This is a GP with $a = \left[-\frac{1}{2}(e^{-\pi} + e^{-2\pi})\right]$ and $r = e^{-2\pi}$ so the limiting sum is

$$\frac{a}{1-r}$$
$$= \frac{-\frac{1}{2}(e^{-\pi} + e^{-2\pi})}{1 - e^{-2\pi}}$$
$$= \frac{-\frac{e^{-\pi}}{2}(1 + e^{-\pi})}{(1 - e^{-\pi})(1 + e^{-\pi})}$$
$$= \frac{-\frac{e^{-\pi}}{2}}{(1 - e^{-\pi})}$$
$$= \frac{e^{-\pi}}{2(e^{-\pi} - 1)}$$
$$= \frac{1}{2(1 - e^{\pi})}$$

Hence the area underneath the curve is $\left|\frac{1}{2(1-e^{\pi})}\right| = \frac{1}{2(e^{\pi}-1)}$

Hence the total area contained is $\frac{e^{\pi}}{2(e^{\pi}-1)} + \frac{1}{2(e^{\pi}-1)} = \frac{e^{\pi}}{2(e^{\pi}-1)}$

Now, the integral in part b is given by the area above the curve minus the area below the curve, that is

$$\int_0^{\infty} e^{-x} \sin x \, dx$$
$$= \frac{e^{\pi}}{2(e^{\pi} - 1)} - \frac{1}{2(e^{\pi} - 1)}$$
$$= \frac{(e^{\pi} - 1)}{2(e^{\pi} - 1)}$$
$$= \frac{1}{2}$$

Thus confirming the result from part b.

Uncorrected proofs

Solutions to Chapter review

1a

$$\frac{d}{dx}(5\sin x)$$

$$= 5 \times \frac{d}{dx}(\sin x)$$

$$= 5 \times \cos x$$

$$= 5\cos x$$

1b

$$\frac{d}{dx}(\sin 5x)$$

$$= 5 \times \cos 5x$$

$$= 5\cos 5x$$

1c

$$\frac{d}{dx}(5\cos 5x)$$

$$= 5 \times \frac{d}{dx}(\cos 5x)$$

$$= 5 \times (-5\sin 5x)$$

$$= -25\sin 5x$$

1d

$$\frac{d}{dx}(\tan(5x-4))$$

$$= \sec^2(5x-4) \times \frac{d}{dx}(5x-4)$$

$$= \sec^2(5x-4) \times 5$$

$$= 5\sec^2(5x-4)$$

Uncorrected proofs

1e

$$\frac{d}{dx}(x\sin 5x)$$
$$= \sin 5x \times \frac{d}{dx}(x) + x \times \frac{d}{dx}(\sin 5x)$$
$$= \sin 5x \times 1 + x \times 5\cos 5x$$
$$= \sin 5x + 5x\cos 5x$$

1f Using the quotient rule, if $y = \frac{u}{v}$ then

$$u = \cos 5x$$

$$\frac{du}{dx} = -5\sin 5x$$

$$v = x$$

$$\frac{dv}{dx} = 1$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$
$$= \frac{x \times (-5\sin 5x) - \cos 5x \times (1)}{x^2}$$
$$= \frac{-5x\sin 5x - \cos 5x}{x^2}$$

1g

$$\frac{d}{dx}(\sin^5 x)$$

$$= 5\sin^4 x \times \frac{d}{dx}(\sin x)$$

$$= 5\sin^4 x \times \cos x$$

$$= 5\sin^4 x\cos x$$

Uncorrected proofs

1h

$$\frac{d}{dx}(\tan x^5)$$

$$= \sec^2 x^5 \times \frac{d}{dx}(x^5)$$

$$= \sec^2 x^5 \times 5x^4$$

$$= 5x^4 \sec^2 x^5$$

1i

$$\frac{d}{dx}(e^{\cos 5x})$$

$$= e^{\cos 5x} \times \frac{d}{dx}(\cos 5x)$$

$$= e^{\cos 5x} \times -5\sin 5x$$

$$= -5\sin 5x\, e^{\cos 5x}$$

1j

$$\frac{d}{dx}(\ln(\sin 5x))$$

$$= \frac{1}{\sin 5x} \times \frac{d}{dx}(\sin 5x)$$

$$= \frac{1}{\sin 5x} \times 5\cos 5x$$

$$= \frac{5\cos 5x}{\sin 5x}$$

$$= 5\cot 5x$$

2 For $y = \cos 2x$

$$\frac{dy}{dx} = -2\sin 2x$$

Hence when $x = \frac{\pi}{3}$, the gradient of $\cos 2x$ is $-2\sin\frac{2\pi}{3} = -2 \times \frac{\sqrt{3}}{2} - \sqrt{3}$.

Uncorrected proofs

3a For $y = \tan x$

When $x = \frac{\pi}{3}$, $y = \tan\frac{\pi}{3} = \sqrt{3}$

$$\frac{dy}{dx} = \sec^2 x$$

Hence when $x = \frac{\pi}{3}$, the gradient of $\tan x$ is $\sec^2\frac{\pi}{3} = 4$

Thus, the equation of the tangent at $\left(\frac{\pi}{3}, \sqrt{3}\right)$ is

$$y - y_1 = m(x - x_1)$$

$$y - \sqrt{3} = 4\left(x - \frac{\pi}{3}\right)$$

$$y = 4x - \frac{4\pi}{3} + \sqrt{3}$$

3b For $y = x\cos x$

When $x = \frac{\pi}{2}$, $y = \frac{\pi}{2}\cos\frac{\pi}{2} = 0$

$$\frac{dy}{dx} = \cos x\frac{d}{dx}(x) + x\frac{d}{dx}(\cos x)$$

$$= \cos x - x\sin x$$

Hence when $x = \frac{\pi}{2}$, the gradient of $x\cos x$ is

$$\cos\frac{\pi}{2} - \frac{\pi}{2}\sin\frac{\pi}{2}$$

$$= 0 - \frac{\pi}{2} \times 1$$

$$= -\frac{\pi}{2}$$

Thus, the equation of the tangent at $\left(\frac{\pi}{2}, 0\right)$ is

$$y - y_1 = m(x - x_1)$$

$$y - 0 = -\frac{\pi}{2}\left(x - \frac{\pi}{2}\right)$$

$$y = -\frac{\pi}{2}x + \frac{\pi^2}{4}$$

Uncorrected proofs

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4a $y = x + \cos x$

$\frac{dy}{dx} = 1 - \sin x$

The stationary points occur when $\frac{dy}{dx} = 0$

$1 - \sin x = 0$

$\sin x = 1$

Thus the solution is $x = \frac{\pi}{2}$ for $0 \le x \le 2\pi$.

4b $y = \sin x - \cos x$

$\frac{dy}{dx} = \cos x - (-\sin x)$

$= \cos x + \sin x$

The stationary points occur when $\frac{dy}{dx} = 0$

$\cos x + \sin x = 0$

$\sin x = -\cos x$

$\frac{\sin x}{\cos x} = -1$

$\tan x = -1$

Thus the solutions are $x = \frac{3\pi}{4}, \frac{7\pi}{4}$ for $0 \le x \le 2\pi$

5a

$\int 4\cos x\, dx$

$= 4\sin x + C$

5b

$\int \sin 4x\, dx$

$= -\frac{1}{4}\int (-4\sin 4x)\, dx$

$= -\frac{1}{4} \times \cos 4x + C$

Uncorrected proofs

$$= -\frac{1}{4}\cos 4x + C$$

5c

$$\int \sec^2 \frac{1}{4}x\,dx$$

$$= 4\int \frac{1}{4}\sec^2 \frac{1}{4}x\,dx$$

$$= 4 \times \tan \frac{1}{4}x + C$$

$$= 4\tan \frac{1}{4}x + C$$

6a

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sec^2 x\,dx$$

$$= [\tan x]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$

$$= \tan\frac{\pi}{3} - \tan\frac{\pi}{4}$$

$$= \sqrt{3} - 1$$

6b

$$\int_0^{\frac{\pi}{4}} \cos 2x\,dx$$

$$= \frac{1}{2}[\sin 2x]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{2}\left[\sin\frac{\pi}{2} - \sin 0\right]$$

$$= \frac{1}{2}[1 - 0]$$

$$= \frac{1}{2}$$

Uncorrected proofs

6c

$$\int_0^{\frac{1}{3}} \pi \sin \pi x \, dx$$

$$= \pi \times \left[-\frac{1}{\pi} \cos \pi x\right]_0^{\frac{1}{3}}$$

$$= -[\cos \pi x]_0^{\frac{1}{3}}$$

$$= -\left(\cos \frac{\pi}{3} - \cos 0\right)$$

$$= -\left(\frac{1}{2} - 1\right)$$

$$= \frac{1}{2}$$

7

$$\int_0^{\frac{1}{4}} \sin 3x \, dx$$

$$= -\frac{1}{3} [\cos 3x]_0^{\frac{1}{4}}$$

$$= -\frac{1}{3}\left(\cos \frac{3}{4} - \cos 0\right)$$

$$= -\frac{1}{3}\left(\cos \frac{3}{4} - 1\right)$$

$$\doteqdot 0.089$$

8 $y' = \cos \frac{1}{2} x$

$$y = 2 \sin \frac{1}{2} x + C$$

Substituting $(\pi, 1)$ into the equation for y gives

$$1 = 2 \sin \frac{\pi}{2} + C$$

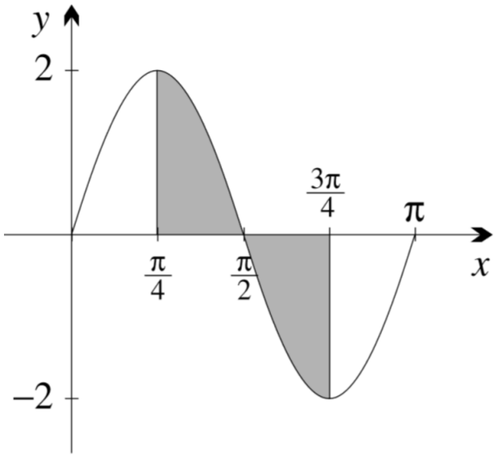
$$1 = 2 + C$$

$$C = -1$$

Uncorrected proofs

Hence the equation of the function is $y = 2\sin\frac{1}{2}x - 1$.

9a Graph of $y = 2\sin 2x$ is shown below. (Take graph of $y = \sin x$ and then dilate it by a factor of 2 from the x–axis and a factor of $\frac{1}{2}$ from the y–axis.)



9b The area between the curve and x–axis from $x = \frac{\pi}{4}$ to $x = \frac{\pi}{2}$ is equal to that between the curve and x–axis from $x = \frac{\pi}{2}$ to $x = \frac{3\pi}{4}$. We can calculate the total area using twice the positive signed area from $x = \frac{\pi}{4}$ to $x = \frac{\pi}{2}$.

$$A = 2 \times \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 2\sin 2x\, dx$$

$$= 2 \times [-\cos 2x]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= -2\left(\cos\pi - \cos\frac{\pi}{2}\right)$$

$$= -2(-1 - 0)$$

$$= 2 \text{ u}^2$$

10 The required area is equal to the area between the line $y = \cos x$ and the x–axis from $x = 0$ to $x = \frac{\pi}{2}$ less the area between the line $y = \cos 2x$ and the x–axis from $x = 0$ and $x = \frac{\pi}{4}$.

Area of required region

Uncorrected proofs

$$= \int_0^{\frac{\pi}{2}} \cos x \, dx - \int_0^{\frac{\pi}{4}} \cos 2x \, dx$$

$$= [\sin x]_0^{\frac{\pi}{2}} - \frac{1}{2}[\sin 2x]_0^{\frac{\pi}{4}}$$

$$= \left(\sin\frac{\pi}{2} - \sin 0\right) - \frac{1}{2}\left[\sin\frac{\pi}{2} - \sin 0\right]$$

$$= 1 - 0 - \frac{1}{2} \times 1 + 0$$

$$= 1 - \frac{1}{2}$$

$$= \frac{1}{2} \text{ u}^2$$

10b Area of required region

$$= \int_0^{\frac{2\pi}{3}} (\cos x - \cos 2x) \, dx$$

$$= \left[\sin x - \frac{1}{2}\sin 2x\right]_0^{\frac{2\pi}{3}}$$

$$= \left(\sin\frac{2\pi}{3} - \frac{1}{2}\sin\frac{4\pi}{3}\right) - \left(\sin 0 - \frac{1}{2}\sin 0\right)$$

$$= \frac{\sqrt{3}}{2} - \frac{1}{2} \times -\frac{\sqrt{3}}{2} - 0 + 0$$

$$= \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{4}$$

$$= \frac{2\sqrt{3}}{4} + \frac{\sqrt{3}}{4}$$

$$= \frac{3\sqrt{3}}{4} \text{ square units}$$

11a $\tan x = \frac{\sin x}{\cos x}$

11b Note that $\frac{d}{dx}(\cos x) = -\sin x$.

Area of required region

$$= \int_0^{\frac{\pi}{4}} \tan x \, dx$$

Uncorrected proofs

$= \int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos x} dx$

$= \int_0^{\frac{\pi}{4}} \frac{-\frac{d}{dx}(\cos x)}{\cos x} dx$

$= -[\ln|\cos x|]_0^{\frac{\pi}{4}}$

$= -\left(\ln\left|\cos\frac{\pi}{4}\right| - \ln|\cos 0|\right)$

$= -\ln\frac{1}{\sqrt{2}} + \ln 1$

$= -\ln\left(2^{-\frac{1}{2}}\right) + 0$

$= -\left(-\frac{1}{2}\right)\ln 2 + 0$

$= \frac{1}{2}\ln 2$ square units

12a To find the x-intercepts, put $y = 0$

Then, $2\cos x + \sin 2x = 0$

Use the identity, $\sin 2x = 2\sin x\cos x$

$2\cos x + 2\sin x\cos x = 0$

$2\cos x\,(1 + \sin x) = 0$

$2\cos x = 0$ and $1 + \sin x = 0$

$\cos x = 0$ and $\sin x = -1$

According to the given domain, $-\pi \le x \le \pi$

$\cos x = 0$ and $\sin x = -1$

$x = \frac{\pi}{2}, -\frac{\pi}{2}$

Thus the x-intercepts are $\left(\frac{\pi}{2}, 0\right)$ and $\left(-\frac{\pi}{2}, 0\right)$.

The y-intercept occurs when, $x = 0$

$y = 2\cos x + \sin 2x$

$= 2\cos 0 + \sin 0$

Uncorrected proofs

$= 2 \times 1 = 2$

Thus the y-intercept is $(0, 2)$.

12b Stationary points occur when $y' = 0$. This is when

$y' = -2\sin x + 2\cos 2x = 0$

Solving this gives

$2\cos 2x = 2\sin x$

$2(\cos^2 x - \sin^2 x) = 2\sin x$

$2(1 - \sin^2 x - \sin^2 x) = 2\sin x$

$(1 - \sin^2 x - \sin^2 x) = \sin x$

$1 - 2\sin^2 x = \sin x$

$2\sin^2 x + \sin x - 1 = 0$

$(2\sin x - 1)(\sin x + 1) = 0$

$\sin x = \frac{1}{2}$ or $\sin x = -1$ so $x = -\frac{\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$. Substituting this back into the equation for y gives the stationary points to be $(-\frac{\pi}{2}, 0)$, $\left(\frac{\pi}{6}, \frac{3\sqrt{3}}{2}\right)$ and $\left(\frac{5\pi}{6}, -\frac{3\sqrt{3}}{3}\right)$.

Now we find the nature of these stationary points, $y'' = -2\cos x - 4\sin 2x$.

When $y = -\frac{\pi}{2}$, $y'' = 0$ (possible point of inflection)

When $y = \frac{\pi}{6}$, $y'' - 3\sqrt{3} < 0$ (maximum)

When $y = \frac{5\pi}{6}$, $y'' = 3\sqrt{3} > 0$ (minimum)

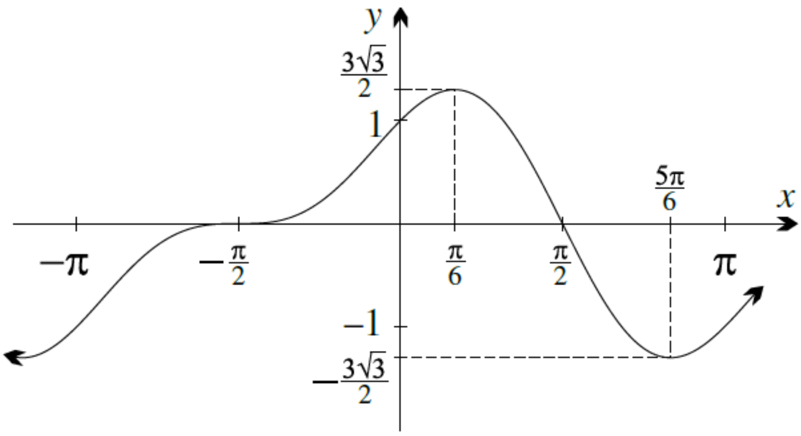
Confirming that there is a point of inflection at $y = -\frac{\pi}{2}$

x	$-\frac{3\pi}{4}$	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$
y'	$\sqrt{2}$	0	$\sqrt{2}$

So the gradient does not change and it is indeed a point of inflection, hence we obtain that $(-\frac{\pi}{2}, 0)$ is a point of inflection $\left(\frac{\pi}{6}, \frac{3\sqrt{3}}{2}\right)$ is a maximum turning point and $\left(\frac{5\pi}{6}, -\frac{3\sqrt{3}}{3}\right)$ is a minimum turning point.

Uncorrected proofs

12c



13a $y = e^x \sin x$

$$y' = e^x \frac{d}{dx}(\sin x) + \sin x \frac{d}{dx}(e^x)$$

$$= e^x \times \cos x + \sin x \times e^x$$

$$= e^x \cos x + \sin x\, e^x$$

$$= e^x(\sin x + \cos x)$$

$$y'' = e^x \frac{d}{dx}(\sin x + \cos x) + (\sin x + \cos x)\frac{d}{dx}(e^x)$$

$$= e^x \times \left(\frac{d}{dx}(\sin x) + \frac{d}{dx}(\cos x)\right) + (\sin x + \cos x)\frac{d}{dx}(e^x)$$

$$= e^x \times (\cos x - \sin x) + (\sin x + \cos x) \times e^x$$

$$= e^x \cos x - e^x \sin x + e^x \sin x + e^x \cos x$$

$$= 2e^x \cos x$$

13b Stationary points occur when $y' = 0$. This is when

$$e^x(\sin x + \cos x) = 0$$

$$\sin x + \cos x = 0$$

$$\sin x = -\cos x$$

$$\frac{\sin x}{\cos x} = -1$$

$$\tan x = -1$$

$x = -\frac{\pi}{4}, \frac{3\pi}{4} \quad$ for $x \in [-\pi, \pi]$

When $x = -\frac{\pi}{4}$, $y = -\frac{1}{\sqrt{2}}e^{-\frac{\pi}{4}}$ and $y'' = \sqrt{2}e^{-\frac{\pi}{4}} > 0$. Hence the curve is concave up at this point and thus there is a minimum turning point at $\left(-\frac{\pi}{4}, -\frac{1}{\sqrt{2}}e^{-\frac{\pi}{4}}\right)$.

When $x = \frac{3\pi}{4}$, $y = \frac{1}{\sqrt{2}}e^{\frac{3\pi}{4}}$ and $y'' = -\sqrt{2}e^{\frac{3\pi}{4}} < 0$. Hence the curve is concave down at this point and thus there is a maximum turning point at $\left(\frac{3\pi}{4}, \frac{1}{\sqrt{2}}e^{\frac{3\pi}{4}}\right)$.

13c Points of inflection occur when $y'' = 0$. This is when

$2e^x \cos x = 0$

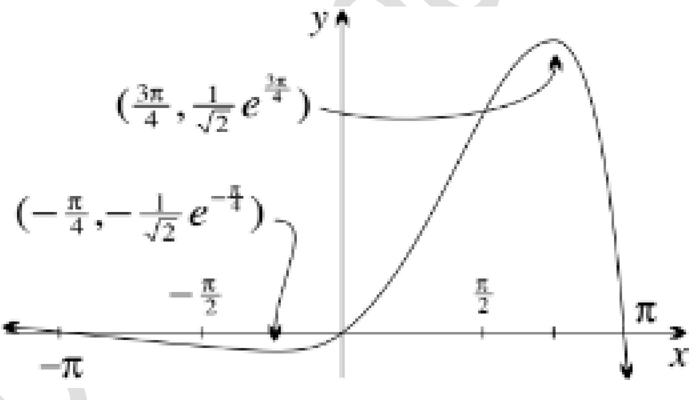
$\cos x = 0$

$x = -\frac{\pi}{2}, \frac{\pi}{2} \quad$ for $x \in [-\pi, \pi]$

When $x = -\frac{\pi}{2}$, $y = -e^{-\frac{\pi}{2}}$, and when $x = \frac{\pi}{2}$, $y = e^{\frac{\pi}{2}}$

Thus, there are possible points of inflection at $\left(-\frac{\pi}{2}, -e^{-\frac{\pi}{2}}\right)$, and $\left(\frac{\pi}{2}, e^{\frac{\pi}{2}}\right)$.

13d Sketch of the curve is shown as below



14a

$$\frac{d\theta}{dt} = 0.1$$

$$A = \frac{1}{2}r^2\theta$$
$$= \frac{1}{2}6^2\theta$$

Uncorrected proofs

$$= \frac{36}{2}\theta$$
$$= 18\theta$$

$$\frac{dA}{d\theta} = 18$$

$$\frac{dA}{dt} = \frac{dA}{d\theta} \times \frac{d\theta}{dt}$$
$$= 18 \times 0.1$$
$$= 1.8 \text{ cm}^2/\text{min}$$

14b

$$A = \frac{1}{2}r^2 \sin\theta$$
$$= \frac{1}{2}6^2 \sin\theta$$
$$= 18\sin\theta$$

$$\frac{dA}{d\theta} = 18\cos\theta$$

$$\frac{dA}{dt} = \frac{dA}{d\theta} \times \frac{d\theta}{dt}$$
$$= 18\cos\theta \times 0.1$$
$$= 1.8\cos\theta$$

When $\theta = \frac{\pi}{4}$

$$\frac{dA}{dt} = 1.8\cos\frac{\pi}{4}$$
$$= \frac{9\sqrt{2}}{10}\text{ cm}^2/\text{min}$$

14c

$$A = \frac{1}{2}r^2(\theta - \sin\theta)$$

$$A = \frac{1}{2}6^2(\theta - \sin\theta)$$

Uncorrected proofs

$A = 18(\theta - \sin\theta)$

$\frac{dA}{d\theta} = 18(1 - \cos\theta)$

$\frac{d\theta}{dt} = 0.1$

$\frac{dA}{dt} = \frac{dA}{d\theta} \times \frac{d\theta}{dt}$

$= 18(1 - \cos\theta)0.1$

$= 1.8(1 - \cos\theta)$

Maximum rate of increase when $\frac{d^2A}{dt^2} = 0$

$\frac{d^2A}{dt^2} = 1.8\sin\theta \times \frac{d\theta}{dt}$

$\frac{d^2A}{dt^2} = 1.8\sin\theta \times 0.1$

$\frac{d^2A}{dt^2} = 0.18\sin\theta$

Therefore $0.18\sin\theta = 0$

$\theta = 0$

15a $\cos\alpha = \frac{SP}{PQ} = \frac{SP}{d}$ hence $SP = d\cos\alpha$

Thus the area of ΔSPT is

$A = \frac{1}{2}SP \times ST\sin\alpha$

$= \frac{1}{2}SP \times SP \times \sin\alpha$

$= \frac{1}{2}d\cos\alpha \times d\cos\alpha \times \sin\alpha$

$= \frac{1}{2}d^2\cos^2\alpha\sin\alpha$

Uncorrected proofs

15b $\frac{dA}{d\alpha} = \frac{1}{2}d^2(-2\sin\alpha\cos\alpha)\sin\alpha + \frac{1}{2}d^2\cos^2\alpha\,(\cos\alpha)$

$$= \frac{1}{2}d^2\cos\alpha\,(-2\sin^2\alpha + \cos^2\alpha)$$

$$= \frac{1}{2}d^2\cos\alpha\,(-2(1-\cos^2\alpha) + \cos^2\alpha)$$

$$= \frac{1}{2}d^2\cos\alpha\,(3\cos^2\alpha - 2)$$

For stationary points

$$\frac{dA}{d\alpha} = 0$$

$$\frac{1}{2}d^2\cos\alpha\,(3\cos^2\alpha - 2) = 0$$

$\cos\alpha = 0$ or $\cos\alpha = \pm\sqrt{\frac{2}{3}}$

Noting that $0 \le \alpha < \frac{\pi}{2}$, $\alpha = \cos^{-1}\sqrt{\frac{2}{3}}$ is the only stationary point in the range. The endpoints in this range both give an area of zero. Substituting in $\alpha = \cos^{-1}\sqrt{\frac{2}{3}}$ gives

$$A = \frac{1}{2}d^2\cos^2\left(\cos^{-1}\sqrt{\frac{2}{3}}\right)\sin\left(\cos^{-1}\sqrt{\frac{2}{3}}\right)$$

$$= \frac{1}{2}d^2\left(\frac{2}{3}\right)\left(\sqrt{1-\frac{2}{3}}\right)$$

$$= \frac{d^2}{3\sqrt{3}}$$

$$= \frac{1}{9}d^2\sqrt{3}$$

16a

$\int e^{2x}\cos e^{2x}\,dx$

Let, $u = 2x$, $\frac{du}{dx} = \frac{d}{dx}(2x) = 2$ hence, $dx = \frac{1}{2}du$

Now using substitution,

$\int e^{2x}\cos e^{2x}\,dx$

$= \int e^{u}\cos e^{u}\,\frac{1}{2}du$

Uncorrected proofs

$= \frac{1}{2}\int e^u \cos e^u \, du$

Let, $v = e^u$, $\frac{dv}{du} = \frac{d}{du}(e^u) = e^u$ hence, $dv = e^u du$

$= \frac{1}{2}\int e^u \cos e^u \, du$

$= \frac{1}{2}\int \cos v \, dv$

$= \frac{1}{2}\sin v + C$

$= \frac{1}{2}\sin e^u + C$

$= \frac{1}{2}\sin e^{2x} + C$

16b $\int \frac{\sin e^{-2x}}{e^{2x}} dx$

Let, $u = -2x$, $\frac{du}{dx} = \frac{d}{dx}(-2x) = -2$ hence, $dx = -\frac{1}{2}du$

Now using substitution,

$\int \frac{\sin e^{-2x}}{e^{2x}} dx$

$= \int -\frac{\sin e^u}{2e^{-u}} du$

$= -\frac{1}{2}\int \frac{\sin e^u}{e^{-u}} du$

$= -\frac{1}{2}\int e^u \sin e^u \, du$

Let, $v = e^u$, $\frac{dv}{du} = \frac{d}{du}(e^u) = e^u$ hence, $dv = e^u du$

$= -\frac{1}{2}\int e^u \sin e^u \, du$

$= -\frac{1}{2}\int \sin v \, dv$

$= \frac{1}{2}\cos v + C$

$= \frac{1}{2}\cos e^u + C$

$= \frac{1}{2}\cos e^{-2x} + C$

Uncorrected proofs

16c

$\int \frac{\sec^2 x}{3\tan x+1} dx$

$= \int \frac{\frac{1}{\cos^2 x}}{3\tan x+1} dx$

$= \int \frac{1}{\cos^2 x(3\tan x+1)} dx$

Let $u = 3\tan x + 1$, $\frac{du}{dx} = \frac{d}{dx}(3\tan x + 1) = 3\sec^2 x$ hence, $dx = \frac{1}{3}\sec^2 x\ du$

$dx = \frac{1}{3}\sec^2 x\ du$

$= \frac{1}{3\cos^2 x}\ du$

$= \int \frac{1}{3u}\ du$

$= \frac{1}{3}\int \frac{1}{u}\ du$

$= \frac{1}{3}\log_e u + C$

$= \frac{1}{3}\log_e|3\tan x + 1| + C$

16d $\int \frac{3\sin x}{4+5\cos x} dx$

Let, $u = 4 + 5\cos x$, $\frac{du}{dx} = \frac{d}{dx}(4 + 5\cos x) = -5\sin x$ hence, $dx = -\frac{1}{5}\sin x\ du$

$\int \frac{3\sin x}{4+5\cos x} dx$

$= \int -\frac{3}{5u} du$

$= -\frac{3}{5}\int \frac{1}{u} du$

$= -\frac{3}{5}\log_e u + C$

$= -\frac{3}{5}\log_e|4 + 5\cos x| + C$

16e $\int \frac{1-\cos^3 x}{1-\sin^2 x} dx$

$= \int \left(\frac{1}{1-\sin^2 x} - \frac{\cos^3 x}{1-\sin^2 x}\right) dx$

Uncorrected proofs

$= \int \frac{1}{1-\sin^2 x} dx - \int \frac{\cos^3 x}{1-\sin^2 x} dx$

$= \int \frac{1}{\cos^2 x} dx - \int \frac{\cos^3 x}{\cos^2 x} dx$

$= \int \sec^2 x\, dx - \int \cos x\, dx$

$= \tan x - \sin x + C$

16f $\int_0^{\pi} \sin x \cos^2 x$

Let $f(x) = \cos x$

$f'(x) = -\sin x$

Then,

$\int_0^{\pi} \sin x \cos^2 x\, dx$

$= \left[-\frac{1}{3}\cos^3 x\right]_0^{\pi}$

$= -\frac{1}{3} \times -1 - (-\frac{1}{3} \times 1)$

$= \frac{1}{3} + \frac{1}{3}$

$= \frac{2}{3}$

17a LHS

$= \tan^3 x$

$= \tan x \times \tan^2 x$

$= \tan x(\sec^2 x - 1)$

$= \tan x \sec^2 x - \tan x$

$= \text{RHS}$

Hence, proved.

Uncorrected proofs

17b i $\int \tan^3 x \, dx$

$= \int \tan x \, (\sec^2 x - 1) dx$
$= \int \tan x \sec^2 x \, dx - \int \tan x \, dx$

Let $u = \tan x$, $du = \sec^2 x \, dx$

$\int \tan^3 x \, dx$

$= \int u \, du - \int \dfrac{\sin x}{\cos x} dx$
$= \dfrac{1}{2} u^2 - -\log_e(\cos x) + C$
$= \dfrac{1}{2} \tan^2 x + \log_e(\cos x) + C$

17b ii $\int \tan^5 x \, dx$

$= \int \tan^3 x \tan^2 x \, dx$
$= \int \tan^3 x \, (\sec^2 x - 1) \, dx$
$= \int \tan^3 x \sec^2 x \, dx - \int \tan^3 x \, dx$
$= \int \tan^3 x \sec^2 x \, dx - \left(\dfrac{1}{2} \tan^2 x + \log_e(\cos x)\right)$

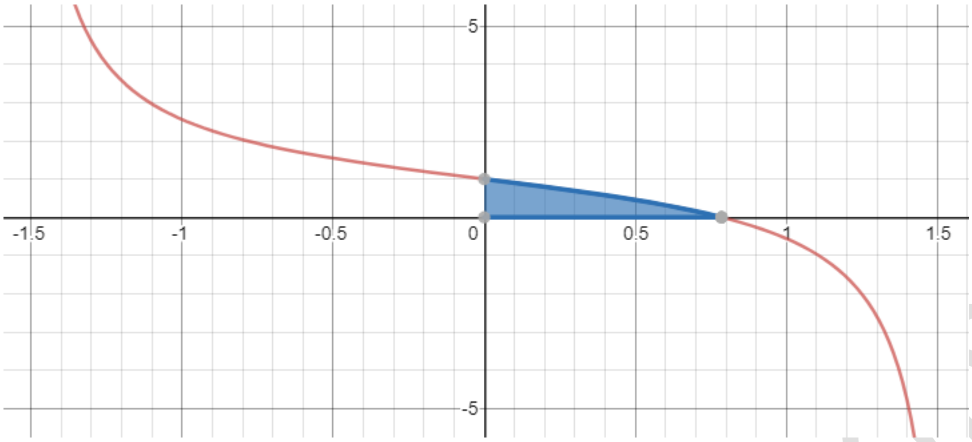
Let $u = \tan x$, $du = \sec^2 x \, dx$

$\int \tan^5 x \, dx$

$= \int u^3 du - \left(\dfrac{1}{2} \tan^2 x + \log_e(\cos x)\right)$
$= \dfrac{u^4}{4} - \dfrac{1}{2} \tan^2 x - \log_e(\cos x) + C$
$= \dfrac{\tan^4 x}{4} - \dfrac{1}{2} \tan^2 x - \log_e(\cos x) + C$

Uncorrected proofs

18a $y = 1 - \tan x$



18b

The curve intersects the axis when $y = 0$, that is when $1 - \tan x = 0$ and hence when $\tan x = 1$ and thus when $x = \frac{\pi}{4}$.

$$\int_0^{\frac{\pi}{4}} 1 - \tan x \, dx = \int_0^{\frac{\pi}{4}} 1 - \frac{\sin x}{\cos x} dx$$

$$= [x + \ln(\cos x)]_0^{\frac{\pi}{4}}$$

$$= \left[\frac{\pi}{4} + \ln\left(\cos\frac{\pi}{4}\right)\right] - [0 + \ln(\cos 0)]$$

$$= \left[\frac{\pi}{4} + \ln\left(\cos\frac{\pi}{4}\right)\right] - [0 + \ln(1)]$$

$$= \left[\frac{\pi}{4} + \ln\left(\cos\frac{\pi}{4}\right)\right] - [0 + 0]$$

$$= \left[\frac{\pi}{4} + \ln\left(\frac{1}{\sqrt{2}}\right)\right] - [0 + 0]$$

$$= \left(\frac{\pi}{4} - \frac{1}{2}\ln 2\right) \text{ square units}$$

19a LHS

$= \cos(A + B) + \cos(A - B)$

$= \cos A \cos B - \sin A \sin B + \cos A \cos B + \sin A \sin B$

(using the identity $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$)

$= \cos A \cos B + \cos A \cos B$

$= 2 \cos A \cos B$

$=$ RHS

Uncorrected proofs

Hence, proved.

19b i

$\int_0^{\frac{\pi}{2}} 2\cos 3x \cos 2x\, dx$

Let, $A = 3x$ and $B = 2x$, using the formula from 19a

$$\int_0^{\frac{\pi}{2}} 2\cos 3x \cos 2x\, dx$$

$$= \int_0^{\frac{\pi}{2}} \cos(3x + 2x) + \cos(3x - 2x)\, dx$$

$$= \int_0^{\frac{\pi}{2}} (\cos 5x + \cos x) dx$$

$$= \int_0^{\frac{\pi}{2}} \cos 5x\, dx + \int_0^{\frac{\pi}{2}} \cos x\, dx$$

$$= \left[\frac{1}{5}\sin 5x\right]_0^{\frac{\pi}{2}} + [\sin x]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{5} \times 1 - 0 + 1 - 0$$

$$= \frac{1}{5} + 1$$

$$= \frac{6}{5}$$

19b ii

$$\int_0^{\pi} \cos 3x \cos 4x\, dx$$

Let, $A = 3x$ and $B = 4x$, Putting in the above section 20a proof

$$= \int_0^{\pi} \left(\frac{1}{2}\cos(3x + 4x) + \cos(3x - 4x)\right) dx$$

$$= \int_0^{\pi} \frac{1}{2}(\cos 7x + \cos(-x))\, dx$$

Uncorrected proofs

$$= \int_0^\pi \frac{1}{2}(\cos 7x + \cos x)dx$$

$$= \frac{1}{2}\int_0^\pi \cos 7x\, dx + \frac{1}{2}\int_0^\pi \cos x\, dx$$

$$= \frac{1}{2}\left[\frac{1}{7}\sin 7x\right]_0^\pi + \frac{1}{2}[\sin x]_0^\pi$$

$$= 0$$

19c i

$$\int \cos mx \cos nx\, dx = \int \frac{1}{2}\left[\cos\left((m-n)x\right) + \cos\left((m+n)x\right)\right]dx$$
$$= \frac{1}{2}\left[\frac{1}{m-n}\sin\left((m-n)x\right) + \frac{1}{m+n}\sin\left((m+n)x\right)\right] + c$$

19c ii When $m = n$

$$\int \cos nx \cos nx\, dx$$

$$= \int \frac{1}{2}\left[\cos\left((n-n)x\right) + \cos\left((n+n)x\right)\right]dx$$

$$= \int \frac{1}{2}\left[\cos(0) + \cos\left((n+n)x\right)\right]dx$$

$$= \int \frac{1}{2} + \frac{1}{2}\cos(2nx)\, dx$$

$$= \frac{1}{2}x + \frac{1}{4n}\sin(2nx) + C$$

19d

When $m \neq n$

$$\int_{-\pi}^{\pi} \cos mx \cos nx\, dx$$

$$= \left[\frac{1}{2}\left[\frac{1}{m-n}\sin\left((m-n)x\right) + \frac{1}{m+n}\sin\left((m+n)x\right)\right]\right]_{-\pi}^{\pi}$$
$$= \frac{1}{2}\left[\frac{1}{m-n}\sin\left((m-n)\pi\right) + \frac{1}{m+n}\sin\left((m+n)\pi\right)\right]$$

Uncorrected proofs

$$-\frac{1}{2}\left[\frac{1}{m-n}\sin\big((m-n)(-\pi)\big)+\frac{1}{m+n}\sin\big((m+n)(-\pi)\big)\right]$$
$$=\frac{1}{2}\left[\frac{1}{m-n}0+\frac{1}{m+n}0\right]-\frac{1}{2}\left[\frac{1}{m-n}0+\frac{1}{m+n}0\right]$$
$$=0$$

When $m = n$

$$\int_{-\pi}^{\pi}\cos nx\cos nx\,dx$$

$$=\int_{-\pi}^{\pi}\cos^2 nx\,dx$$
$$=\int_{-\pi}^{\pi}\frac{1}{2}[1+\cos 2nx]\,dx$$
$$=\left[\frac{1}{2}\left(x+\frac{1}{2n}\sin 2nx\right)\right]_{-\pi}^{\pi}$$
$$=\left[\frac{1}{2}\left(\pi+\frac{1}{2n}\sin 2n\pi\right)\right]-\left[\frac{1}{2}\left((-\pi)+\frac{1}{2n}\sin 2n(-\pi)\right)\right]$$
$$=\left[\tfrac{1}{2}(\pi+0)\right]-\left[\tfrac{1}{2}\big((-\pi)+0\big)\right]$$
$$=\pi$$

Uncorrected proofs