Solutions to Exercise 5A Foundation questions

1 yz-plane: $x = 0$

xz-plane: $y = 0$

xy-plane: $z = 0$

2a $(-2,\ 3, 1)$ lies in the second octant.

2b $(2,\ 3, -1)$ lies in the fifth octant.

2c $(2, -3, 1)$ lies in the fourth octant.

2d $(-2,\ 3, -1)$ lies in the sixth octant.

2e $(2, -3, -1)$ lies in the eighth octant.

2f $(-2, -3, 1)$ lies in the third octant.

3a $(3, 2, 5 - 6) = (3, 2, -1)$

3b $(3 - 8, 2, 5) = (-5, 2, 5)$

3c $(3, 2 + 10, 5) = (3, 12, 5)$

3d $(3 + 5\ , 2, 5 + 7) = (8, 2, 12)$

3e $(3, 2 - 3, 5 - 4) = (3, -1, 1)$

3f $(3, 2, 5 \times -1) = (3, 2, -5)$

3g $(3 \times -1, 2, 5) = (-3, 2, 5)$

3h $(3, 2 \times -1, 5) = (3, -2, 5)$

3i $(3, 2 \times -1, 5 \times -1) = (3, -2, -5)$

4a

$A = (2, 0, 0)$

$B = (2, 2, 0)$

$C = (2, 2, 2)$

$D = (2, 0, 2)$

$O = (0, 0, 0)$

$P = (0, 2, 0)$

$Q = (0, 2, 2)$

$R = (0, 0, 2)$

4b For a right-angled triangle, $a^2 + b^2 = c^2$

So for any diagonal on a face:

$2^2 + 2^2 = c^2$

$c = \sqrt{8}$

$\quad = 2\sqrt{2}$

4c For a right-angled triangle, $a^2 + b^2 = c^2$

In the case of the space diagonal, $a = 2$ and $b = 2\sqrt{2}$

So

$2^2 + \overline{2\sqrt{2}}^2 = c^2$

$c = \sqrt{12}$

$\quad = 2\sqrt{3}$

4d The plane $OABP$ has the equation: $z = 0$

The plane $RDCQ$ has the equation: $z = 2$

The plane $ABCD$ has the equation: $x = 2$

The plane $OPQR$ has the equation: $x = 0$

The plane $OADR$ has the equation: $y = 0$

The plane $PBCQ$ has the equation: $y = 2$

5a $C = (2, 4, 3)$

$A = (2, 0, 0)$

$B = (2, 4, 0)$

$D = (2, 0, 3)$

$P = (0, 4, 0)$

$Q = (0, 4, 3)$

$R = (0, 0, 3)$

5b For a right-angled triangle, $a^2 + b^2 = c^2$

So for any diagonal on a face:

$2^2 + 4^2 = c^2$

$c = \sqrt{20}$

$= 2\sqrt{5}$

5c For a right-angled triangle $a^2 + b^2 = c^2$

In the case of the space diagonal, $a = 3$ and $b = 2\sqrt{5}$

So

$3^2 + \left(2\sqrt{5}\right)^2 = c^2$

$c = \sqrt{29}$

5d The plane $OABP$ has the equation: $z = 0$

The plane $RDCQ$ has the equation: $z = 3$

The plane $ABCD$ has the equation: $x = 2$

The plane $OPQR$ has the equation: $x = 0$

The plane $OADR$ has the equation: $y = 0$

The plane $PBCQ$ has the equation: $y = 4$

6a

$$\text{Area} = \frac{1}{2} \times \text{base} \times \text{height}$$

$$\text{Area} = \frac{1}{2} \times 3 \times 4$$

$$= 6 \text{ square units}$$

6b

$$\text{Volume} = \frac{1}{3} \times \text{base} \times \text{height}$$

$$= \frac{1}{3} \times 6 \times 5$$

$$= 10 \text{ cubic units}$$

Solutions to Exercise 5A Development questions

7a

$$AB^2 = (B_x - A_x)^2 + \left(B_y - A_y\right)^2 + (B_z - A_z)^2$$

The three points are:

$\mathrm{O} = (0,\ 0,\ 0)$

$\mathrm{A} = (2,\ 6,\ 3)$

$\mathrm{B} = (-3,\ 5, -8)$

$$\left|\overrightarrow{OA}\right|^2 = (2-0)^2 + (6-0)^2 + (3-0)^2$$

$$= 4 + 36 + 9$$

$$= 49$$

$$\left|\overrightarrow{OA}\right| = \sqrt{49}$$

$$= 7$$

$$\left|\overrightarrow{OB}\right|^2 = (-3-0)^2 + (5-0)^2 + (-8-0)^2$$

$$= 9 + 25 + 64$$

$$= 98$$

$$\left|\overrightarrow{OB}\right| = \sqrt{98}$$

$$= 7\sqrt{2}$$

$$\left|\overrightarrow{AB}\right|^2 = (-3-2)^2 + (5-6)^2 + (-8-3)^2$$

$$= 25 + 1 + 121$$

$$= 147$$

$$\left|\overrightarrow{AB}\right| = \sqrt{147}$$

$$= 7\sqrt{3}$$

7b

$$\left|\overrightarrow{OA}\right| = 7$$

$$\left|\overrightarrow{OA}\right|^2 = 49$$

$|\overrightarrow{OB}| = 7\sqrt{2}$

$|\overrightarrow{OB}|^2 = 98$

$|\overrightarrow{AB}| = 7\sqrt{3}$

$|\overrightarrow{AB}|^2 = 147$

For a right-angled triangle, Pythagoras's theorem should hold; that is:

$|\overrightarrow{OA}|^2 + |\overrightarrow{OB}|^2 = |\overrightarrow{AB}|^2$

$$\begin{aligned}\text{LHS} &= |\overrightarrow{OA}|^2 + |\overrightarrow{OB}|^2\\ &= 49 + 98\\ &= 147\\ &= AB^2\\ &= \text{RHS}\end{aligned}$$

Pythagoras's theorem is satisfied so the angle opposite the hypotenuse is right angled. That is, $\angle A\mathrm{OB} = 90°$.

8

$\mathrm{A} = (3, -1, -3)$

$\mathrm{B} = (1, -5, -7)$

$\mathrm{C} = (-1, 3, 3)$

$\mathrm{M} = \left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}, \dfrac{z_1 + z_2}{2}\right)$

Midpoint of the line AB:

$$\begin{aligned}\mathrm{M} &= \left(\frac{3+1}{2}, \frac{-1-5}{2}, \frac{-3+7}{2}\right)\\ &= (2, -3, 2)\end{aligned}$$

Midpoint of the line AC:

$$\begin{aligned}\mathrm{N} &= \left(\frac{3-1}{2}, \frac{-1+3}{2}, \frac{-3+3}{2}\right)\\ &= (1, 1, 0)\end{aligned}$$

For MN:

$$\left|\overrightarrow{MN}\right|^2 = (N_x - M_x)^2 + \left(N_y - M_y\right)^2 + (N_z - M_z)^2$$

$$= (1-2)^2 + \left(1-(-3)\right)^2 + (0-2)^2$$

$$= 1 + 16 + 4$$

$$= 21$$

$$\left|\overrightarrow{MN}\right| = \sqrt{21}$$

For BC:

$$\left|\overrightarrow{BC}\right|^2 = (-1-1)^2 + \left(3-(-5)\right)^2 + (3-7)^2$$

$$= 4 + 64 + 16$$

$$= 84$$

$$\left|\overrightarrow{BC}\right| = \sqrt{84}$$

$$= 2\sqrt{21}$$

$$= 2\overrightarrow{MN}$$

$$\left|\overrightarrow{MN}\right| = \frac{1}{2}\left|\overrightarrow{BC}\right|$$

9a

$$\text{P} = (-6,\ -8,\ 14)$$

$$\text{Q} = (-10,\ 20,\ 22)$$

$$\text{M} = \left(\frac{-10-6}{2},\ \frac{20-8}{2}, \frac{22+14}{2}\right)$$

$$= \left(\frac{-16}{2},\ \frac{12}{2}, \frac{36}{2}\right)$$

$$= (-8,\ 6,\ 18)$$

9b

$$\text{X} = \frac{1}{2}(P + M)$$

$$= \frac{1}{2}(-14, -2,\ 32)$$

$$= (-7, -1, 16)$$

$$\text{Y} = \frac{1}{2}(Q + M)$$

$$= \frac{1}{2}(-18,\ 26,\ 40)$$

$$= (-9,\ 13,\ 20)$$

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P$(1,\ 0,\ 0)$

Q$(-3,\ -1,\ 1)$

R$(-2,\ 3,\ 4)$

C$(-1,\ 1,\ 2)$

If the lengths of CP, CQ and CR are equal, the points P, Q and R are shown to be on the surface of a sphere centred at C:

$$\left|\overrightarrow{CP}\right|^2 = \big(1 - (-1)\big)^2 + (0 - 1)^2 + (0 - 2)^2$$

$$\left|\overrightarrow{CP}\right| = \sqrt{4 + 1 + 4}$$

$$= 3$$

$$\left|\overrightarrow{CQ}\right|^2 = \big(-3 - (-1)\big)^2 + (-1 - 1)^2 + (1 - 2)^2$$

$$\left|\overrightarrow{CQ}\right| = \sqrt{4 + 4 + 1}$$

$$= 3$$

$$\left|\overrightarrow{CR}\right|^2 = \big(-2 - (-1)\big)^2 + (3 - 1)^2 + (4 - 2)^2$$

$$\left|\overrightarrow{CR}\right| = \sqrt{1 + 4 + 4}$$

$$= 3$$

So the lengths $\left|\overrightarrow{CP}\right| = \left|\overrightarrow{CQ}\right| = \left|\overrightarrow{CR}\right|$ and the conditions are satisfied.

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So the length between $(x,\ x + 5,\ x - 2)$ and $(1,\ 0,\ -1)$ will be:

$$\sqrt{(x - 1)^2 + (x + 5)^2 + (x - 1)^2}$$

$$= \sqrt{x^2 - 2x + 1 + x^2 + 10x + 25 + x^2 - 2x + 1}$$

$= \sqrt{3x^2 + 6x + 27}$

We know

$\sqrt{3x^2 + 6x + 27} = 2\sqrt{6}$

$3x^2 + 6\text{x} + 27 = 24$

$3x^2 + 6\text{x} + 3 = 0$

$x^2 + 2\text{x} + 1 = 0$

$(x + 1)^2 = 0$

$x = -1$

12a

$\text{A} = (4,\ 2,\ 6)$

$\text{B} = (-2,\ 0,\ 2)$

$\text{C} = (10,\ -2,\ 4)$

In order for the triangle to be an isosceles triangle, two sides must be of equal length:

$\left|\overrightarrow{\text{AB}}\right|^2 = (-6)^2 + (-2)^2 + (-4)^2$

$= 36 + 4 + 16$

$= 56$

$\left|\overrightarrow{\text{AB}}\right| = \sqrt{56}$

$= 2\sqrt{14}$

$\left|\overrightarrow{\text{AC}}\right|^2 = 6^2 + (-4)^2 + (-2)^2$

$= 36 + 16 + 4$

$= 56$

$\left|\overrightarrow{\text{AC}}\right| = \sqrt{56}$

$= 2\sqrt{14}$

$\left|\overrightarrow{\text{BC}}\right|^2 = 12^2 + (-2)^2 + 2^2$

$= 144 + 4 + 4$

$= 152$

$|\overrightarrow{\mathrm{BC}}| = \sqrt{152}$

$= 2\sqrt{38}$

$|\overrightarrow{AB}| = |\overrightarrow{AC}|$

$= 2\sqrt{14}$

So as exactly two sides of the triangle have the same length,

ΔABC is an isosceles triangle.

12b

The area of the triangle is:

$\frac{1}{2}\text{base} \times \text{height}$

The base of the triangle is:

$|\overrightarrow{BC}| = 2\sqrt{38}$

The height of the triangle can be found as the length from A to the middle point between B and C, M:

$\mathrm{B}(-2,\ 0,\ 2)$

$\mathrm{C}(10, -2, 4)$

$\mathrm{M} = \left(\frac{10-2}{2}, \frac{-2+0}{2}, \frac{4+2}{2}\right)$

$= \left(-\frac{8}{2}, -\frac{2}{2}, \frac{6}{2}\right)$

$= (4, -1, 3)$

$\overrightarrow{AM} = (4-4, 2-(-1), 3-6)$

$= (0, 3, -3)$

$|\overrightarrow{AM}|^2 = 0^2 + (-3)^2 + (-3)^2$

$|\overrightarrow{AM}| = \sqrt{18}$

$= 3\sqrt{2}$

So:

$\text{Area} = \frac{1}{2}|\overrightarrow{BC}| \times |\overrightarrow{\mathrm{AM}}|$

$$= \frac{1}{2}\left(2\sqrt{38}\right) \times 3\sqrt{2}$$

$$= \sqrt{38} \times 3\sqrt{2}$$

$$= 6\sqrt{19} \text{ units}^2$$

13a

For the plane:

$3x + 4y + 6z = 12$

The intersecting plane on the xy-plane will occur when $z = 0$.

$3x + 4y = 12$

13b

The intersecting plane on the xz-plane will occur when $y = 0$.

$3x + 6z = 12$

The intersecting plane on the yz-plane will occur when $x = 0$.

$4y + 6z = 12$

13c

A line, as they fit the form: $y = mx + c$

13d

The intersection of two non-parallel planes will be a line.

Solutions to Exercise 5A Enrichment questions

14 In $\triangle PRO$,

$PO^2 = a^2 + b^2 + c^2, PR^2 = b^2 + c^2, OR^2 = a^2$

So, $PO^2 = PR^2 + OR^2$

So, $\angle PRO = 90^\circ$ (Converse of Pythagoras)

The proofs that PSO and PTO are right angles are very similar.

15a z can take any real value.

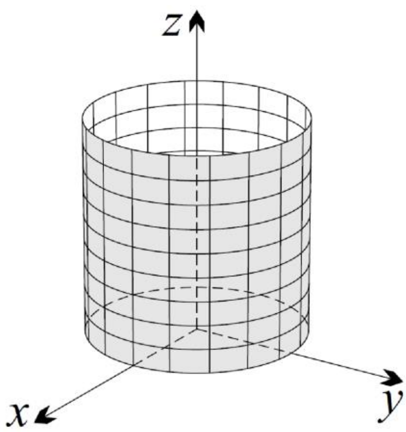
15b The horizontal plane $z = k$, for any real value of k, intersects the cylinder $x^2 + y^2 = 4$ in the horizontal circle $x^2 + y^2 = 4$.

So, the intersection is the circle with centre $(0,0,k)$ and radius 2 in the plane $z = k$.

15c It is the curved surface of a cylinder of infinite height with radius 2 units.

The z-axis is its axis of symmetry.

15d



16a $z = x^2 + y^2 \geq 0$ for all $x, y \in R$, since a square is never negative.

16b The circle $x^2 + y^2 = k$, where $k > 0$, has centre $(0,0,k)$ and radius $\sqrt{k}$.

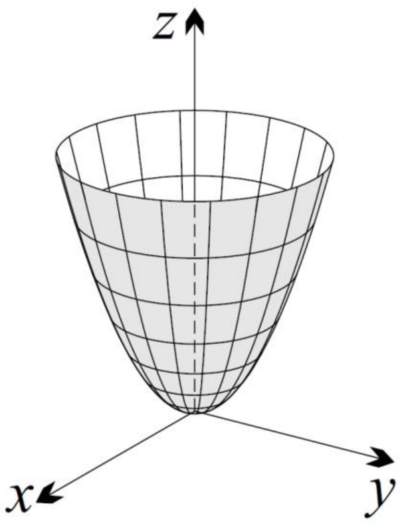
If $k = 0$, then the intersection is the origin.

If $k > 0$, then there is no intersection.

16c The xz-plane has equation $y = 0$.

Hence, the intersection is the parabola defined by $z = x^2$ and $y = 0$.

16d



Solutions to Exercise 5B Foundation questions

1a i $P(2,-3,5)$

$$\overrightarrow{OP} = \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}$$

1a ii $P(2,-3,5)$

$$\overrightarrow{OP} = 2\underset{\sim}{i} - 3\underset{\sim}{j} + 5\underset{\sim}{k}$$

1b i $P(-4,0,13)$

$$\overrightarrow{OP} = \begin{bmatrix} -4 \\ 0 \\ 13 \end{bmatrix}$$

1b ii $P(-4,0,13)$

$$\overrightarrow{OP} = -4\underset{\sim}{i} + 13\underset{\sim}{k}$$

1c i $P(a,-2a,-3a)$

$$\overrightarrow{OP} = \begin{bmatrix} a \\ -2a \\ -3a \end{bmatrix}$$

$$= a\begin{bmatrix} 1 \\ -2 \\ -3 \end{bmatrix}$$

1c ii $P(a,-2a,-3a)$

$$\overrightarrow{OP} = a\underset{\sim}{i} - 2a\underset{\sim}{j} - 3a\underset{\sim}{k}$$

2a $\underset{\sim}{a} = 4\underset{\sim}{i} - 3\underset{\sim}{k}$

$$|\underset{\sim}{a}|^2 = 4^2 + (-3)^2$$

$$|\underset{\sim}{a}|^2 = 16 + 9$$

$$|\underset{\sim}{a}|^2 = 25$$

$|\underset{\sim}{a}| = 5$

$\hat{\underset{\sim}{a}} = \frac{1}{|\underset{\sim}{a}|}\underset{\sim}{a}$

So

$\hat{\underset{\sim}{a}} = \frac{1}{5}\begin{bmatrix} 4 \\ -3 \end{bmatrix}$

$\hat{\underset{\sim}{a}} = \begin{bmatrix} \frac{4}{5} \\ -\frac{3}{5} \end{bmatrix}$

2b $\underset{\sim}{a} = \underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k}$

$|\underset{\sim}{a}|^2 = 1^2 + 2^2 + (-2)^2$

$|\underset{\sim}{a}|^2 = 1 + 4 + 4$

$|\underset{\sim}{a}|^2 = 9$

$|\underset{\sim}{a}| = 3$

$\hat{\underset{\sim}{a}} = \frac{1}{|\underset{\sim}{a}|}\underset{\sim}{a}$

So

$\hat{\underset{\sim}{a}} = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ -\frac{2}{3} \end{bmatrix}$

3a $\underset{\sim}{v} = -\underset{\sim}{i} - 4\underset{\sim}{j} + \underset{\sim}{k}$

$|\underset{\sim}{v}|^2 = (-1)^2 + (-4)^2 + 1^2$

$|\underset{\sim}{v}|^2 = 1 + 16 + 1$

$|\underset{\sim}{v}|^2 = 18$

$|\underset{\sim}{v}| = \sqrt{18}$

$|\underset{\sim}{v}| = 3\sqrt{2}$

$$\hat{\underset{\sim}{v}} = \frac{1}{|\underset{\sim}{v}|}\underset{\sim}{v}$$

So

$$\hat{\underset{\sim}{v}} = \frac{1}{3\sqrt{2}}\begin{bmatrix}-1\\-4\\1\end{bmatrix}$$

$$\hat{\underset{\sim}{v}} = \frac{\sqrt{2}}{6}\begin{bmatrix}-1\\-4\\1\end{bmatrix}$$

$$\hat{\underset{\sim}{v}} = \frac{1}{6}\begin{bmatrix}-\sqrt{2}\\-4\sqrt{2}\\\sqrt{2}\end{bmatrix}$$

3b $\underset{\sim}{v} = 5\underset{\sim}{i} + 3\underset{\sim}{j} - 4\underset{\sim}{k}$

$$|\underset{\sim}{v}|^2 = 5^2 + 3^2 + (-4)^2$$

$$|\underset{\sim}{v}|^2 = 25 + 9 + 16$$

$$|\underset{\sim}{v}|^2 = 50$$

$$|\underset{\sim}{v}| = \sqrt{50}$$

$$|\underset{\sim}{v}| = 5\sqrt{2}$$

$$\hat{\underset{\sim}{v}} = \frac{1}{|\underset{\sim}{v}|}\underset{\sim}{v}$$

So

$$\hat{\underset{\sim}{v}} = \frac{1}{5\sqrt{2}}\begin{bmatrix}5\\3\\-4\end{bmatrix}$$

$$\hat{\underset{\sim}{v}} = \frac{\sqrt{2}}{10}\begin{bmatrix}5\\3\\-4\end{bmatrix}$$

$$\hat{\underset{\sim}{v}} = \frac{1}{10}\begin{bmatrix}5\sqrt{2}\\3\sqrt{2}\\-4\sqrt{2}\end{bmatrix}$$

4a $\underset{\sim}{p} = \begin{bmatrix} 4 \\ -2 \\ 7 \end{bmatrix}$

$\underset{\sim}{q} = \begin{bmatrix} -3 \\ -6 \\ 9 \end{bmatrix}$

$$\begin{aligned} 2\underset{\sim}{p} + \underset{\sim}{q} &= 2\begin{bmatrix} 4 \\ -2 \\ 7 \end{bmatrix} + \begin{bmatrix} -3 \\ -6 \\ 9 \end{bmatrix} \\ &= \begin{bmatrix} 8 \\ -4 \\ 14 \end{bmatrix} + \begin{bmatrix} -3 \\ -6 \\ 9 \end{bmatrix} \\ &= \begin{bmatrix} 5 \\ -10 \\ 23 \end{bmatrix} \end{aligned}$$

4b $\underset{\sim}{p} = \begin{bmatrix} 4 \\ -2 \\ 7 \end{bmatrix}$

$\underset{\sim}{q} = \begin{bmatrix} -3 \\ -6 \\ 9 \end{bmatrix}$

$$\begin{aligned} \left|2\underset{\sim}{p} + \underset{\sim}{q}\right| &= \left|2\begin{bmatrix} 4 \\ -2 \\ 7 \end{bmatrix} + \begin{bmatrix} -3 \\ -6 \\ 9 \end{bmatrix}\right| \\ &= \left|\begin{bmatrix} 8 \\ -4 \\ 14 \end{bmatrix} + \begin{bmatrix} -3 \\ -6 \\ 9 \end{bmatrix}\right| \\ &= \left|\begin{bmatrix} 5 \\ -10 \\ 23 \end{bmatrix}\right| \\ &= \sqrt{5^2 + (-10)^2 + 23^2} \\ &= \sqrt{25 + 100 + 529} \\ &= \sqrt{654} \end{aligned}$$

4c $\underset{\sim}{p} = \begin{bmatrix} 4 \\ -2 \\ 7 \end{bmatrix}$

$\underset{\sim}{q} = \begin{bmatrix} -3 \\ -6 \\ 9 \end{bmatrix}$

$$\begin{aligned} \underset{\sim}{p} - 5\underset{\sim}{q} &= \begin{bmatrix} 4 \\ -2 \\ 7 \end{bmatrix} - 5\begin{bmatrix} -3 \\ -6 \\ 9 \end{bmatrix} \\ &= \begin{bmatrix} 4 \\ -2 \\ 7 \end{bmatrix} + \begin{bmatrix} 15 \\ 30 \\ -45 \end{bmatrix} \\ &= \begin{bmatrix} 19 \\ 28 \\ -38 \end{bmatrix} \end{aligned}$$

4d $\underset{\sim}{p} = \begin{bmatrix} 4 \\ -2 \\ 7 \end{bmatrix}$

$\underset{\sim}{q} = \begin{bmatrix} -3 \\ -6 \\ 9 \end{bmatrix}$

$$\begin{aligned} \left|\underset{\sim}{p} - 5\underset{\sim}{q}\right| &= \left|\begin{bmatrix} 4 \\ -2 \\ 7 \end{bmatrix} - 5\begin{bmatrix} -3 \\ -6 \\ 9 \end{bmatrix}\right| \\ &= \left|\begin{bmatrix} 4 \\ -2 \\ 7 \end{bmatrix} + \begin{bmatrix} 15 \\ 30 \\ -45 \end{bmatrix}\right| \\ &= \left|\begin{bmatrix} 19 \\ 28 \\ -38 \end{bmatrix}\right| \\ &= \sqrt{19^2 + 28^2 + (-38)^2} \\ &= \sqrt{361 + 784 + 1444} \\ &= \sqrt{2589} \end{aligned}$$

5a $\underset{\sim}{p} = 2\underset{\sim}{i} + 7\underset{\sim}{j} - \underset{\sim}{k}$

$\underset{\sim}{q} = 5\underset{\sim}{i} - 5\underset{\sim}{j} + 3\underset{\sim}{k}$

$\overrightarrow{PQ} = \underset{\sim}{q} - \underset{\sim}{p}$

$= (5\underset{\sim}{i} - 5\underset{\sim}{j} + 3\underset{\sim}{k}) - (2\underset{\sim}{i} + 7\underset{\sim}{j} - \underset{\sim}{k})$

$= 3\underset{\sim}{i} - 12\underset{\sim}{j} + 4\underset{\sim}{k}$

5b $\underset{\sim}{p} = 2\underset{\sim}{i} + 7\underset{\sim}{j} - \underset{\sim}{k}$

$\underset{\sim}{q} = 5\underset{\sim}{i} - 5\underset{\sim}{j} + 3\underset{\sim}{k}$

$\overrightarrow{QP} = \underset{\sim}{p} - \underset{\sim}{q}$

$= (2\underset{\sim}{i} + 7\underset{\sim}{j} - \underset{\sim}{k}) - (5\underset{\sim}{i} - 5\underset{\sim}{j} + 3\underset{\sim}{k})$

$= -3\underset{\sim}{i} + 12\underset{\sim}{j} - 4\underset{\sim}{k}$

5c We know from question 5a that $\overrightarrow{PQ} = 3\underset{\sim}{i} - 12\underset{\sim}{j} + 4\underset{\sim}{k}$.

$\left|\overrightarrow{PQ}\right|^2 = 3^2 + (-12)^2 + 4^2$

$\left|\overrightarrow{PQ}\right|^2 = 9 + 144 + 16$

$\left|\overrightarrow{PQ}\right|^2 = 169$

$\left|\overrightarrow{PQ}\right| = 13$

6a $\overrightarrow{OA} = \begin{bmatrix} 6 \\ 0 \\ -3 \end{bmatrix}$

$\overrightarrow{OB} = \begin{bmatrix} -2 \\ -3 \\ -1 \end{bmatrix}$

$\overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB}$

$= \begin{bmatrix} 6 \\ 0 \\ -3 \end{bmatrix} - \begin{bmatrix} -2 \\ -3 \\ -1 \end{bmatrix}$

$$= \begin{bmatrix} 8 \\ 3 \\ -2 \end{bmatrix}$$

6b $\overrightarrow{OA} = \begin{bmatrix} 6 \\ 0 \\ -3 \end{bmatrix}$

$$\overrightarrow{OB} = \begin{bmatrix} -2 \\ -3 \\ -1 \end{bmatrix}$$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$= \begin{bmatrix} -2 \\ -3 \\ -1 \end{bmatrix} - \begin{bmatrix} 6 \\ 0 \\ -3 \end{bmatrix}$$

$$= \begin{bmatrix} -8 \\ -3 \\ 2 \end{bmatrix}$$

6c We know from question 6b that $\overrightarrow{AB} = \begin{bmatrix} -8 \\ -3 \\ 2 \end{bmatrix}$.

$$\left|\overrightarrow{AB}\right|^2 = (-8)^2 + (-3)^2 + 2^2$$

$$\left|\overrightarrow{AB}\right|^2 = 64 + 9 + 4$$

$$\left|\overrightarrow{AB}\right|^2 = 77$$

$$\left|\overrightarrow{AB}\right|^2 = \sqrt{77}$$

Solutions to Exercise 5B Development questions

7

$$\underset{\sim}{a} = \begin{bmatrix} 3 \\ 5 \\ -1 \end{bmatrix}$$

$$\underset{\sim}{b} = \begin{bmatrix} -2 \\ -4 \\ 4 \end{bmatrix}$$

$$\lambda_1 \underset{\sim}{a} + \lambda_2 \underset{\sim}{b} = \begin{bmatrix} 14 \\ 26 \\ -18 \end{bmatrix}$$

$$\lambda_1 \begin{bmatrix} 3 \\ 5 \\ -1 \end{bmatrix} + \lambda_2 \begin{bmatrix} -2 \\ -4 \\ 4 \end{bmatrix} = \begin{bmatrix} 14 \\ 26 \\ -18 \end{bmatrix}$$

$$3\lambda_1 - 2\lambda_2 = 14 \qquad (1)$$

$$-\lambda_1 + 4\lambda_2 = -18$$

$$\lambda_1 = 4\lambda_2 + 18 \qquad (2)$$

Substituting (2) back into equation (1) then solving for λ_2 we get:

$$3(4\lambda_2 + 18) - 2\lambda_2 = 14$$

$$12\lambda_2 + 54 - 2\lambda_2 = 14$$

Hence,

$$10\lambda_2 = -40$$

$$\lambda_2 = -4$$

$$\lambda_1 = 4\lambda_2 + 18$$

$$= -16 + 18$$

$$= 2$$

8

$$\underset{\sim}{a} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$$

$$\underset{\sim}{b} = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$$

$$\underset{\sim}{c} = \begin{bmatrix} 4 \\ 3 \\ -2 \end{bmatrix}$$

$$\lambda_1 \underset{\sim}{a} + \lambda_2 \underset{\sim}{b} + \lambda_3 \underset{\sim}{c} = \begin{bmatrix} -7 \\ -14 \\ 7 \end{bmatrix}$$

$$\lambda_1 \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} + \lambda_3 \begin{bmatrix} 4 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -7 \\ -14 \\ 7 \end{bmatrix}$$

$$-\lambda_1 + 4\lambda_3 = -7$$

$$\lambda_1 = 7 + 4\lambda_3 \qquad (1)$$

$$2\lambda_1 - 2\lambda_2 + 3\lambda_3 = -14 \qquad (2)$$

$$\lambda_2 - 2\lambda_3 = 7$$

$$\lambda_2 = 7 + 2\lambda_3 \qquad (3)$$

Substituting (1) and (3) back into equation (2) then solving for λ_3 we get:

$$2(7 + 4\lambda_3) - 2(7 + 2\lambda_3) + 3\lambda_3 = -14$$

$$14 + 8\lambda_3 - 14 - 4\lambda_3 + 3\lambda_3 = -14$$

$$7\lambda_3 = -14$$

$$\lambda_3 = -2$$

Solving for λ_1:

$$\lambda_1 = 7 + 4(-2)$$

$$= -1$$

Solving for λ_2:

$$\lambda_2 = 7 + 2(-2)$$

$$= 3$$

9a

$$A = \begin{bmatrix} -1 \\ 4 \\ -3 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 \\ 8 \\ -7 \end{bmatrix}$$

$$\overrightarrow{AB} = \begin{bmatrix} 0-(-1) \\ 2-4 \\ 1-(-3) \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$$

$$\overrightarrow{CD} = \begin{bmatrix} 0-3 \\ 8-2 \\ -7-5 \end{bmatrix}$$

$$= \begin{bmatrix} -3 \\ 6 \\ -12 \end{bmatrix}$$

If $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are parallel, there exists a value whereby $\overrightarrow{AB} = a\overrightarrow{CD}$.

$$\begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} = a\begin{bmatrix} -3 \\ 6 \\ -12 \end{bmatrix}$$

This holds true if $a = -3$, thus $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are parallel.

9b

$$A = \begin{bmatrix} -1 \\ 4 \\ -3 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 \\ 8 \\ -7 \end{bmatrix}$$

$$\overrightarrow{BC} = \begin{bmatrix} 3-0 \\ 2-2 \\ 5-1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}$$

$$\overrightarrow{AD} = \begin{bmatrix} 0-(-1) \\ 8-4 \\ -7-(-3) \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 4 \\ -4 \end{bmatrix}$$

If $\overrightarrow{BC}$ and $\overrightarrow{AD}$ are parallel, there exists a value whereby $\overrightarrow{BC} = a\overrightarrow{AD}$.

$$\begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix} = a \begin{bmatrix} 1 \\ 4 \\ -4 \end{bmatrix}$$

There is no value for a to satisfy this equation, thus $\overrightarrow{BC}$ and $\overrightarrow{AD}$ are not parallel

10

$A(-2, -1, 0)$

$B(0, 5, -2)$

$C(4, 17, -6)$

$$\overrightarrow{AB} = \begin{bmatrix} 2 \\ 6 \\ -2 \end{bmatrix}$$

$$\overrightarrow{BC} = \begin{bmatrix} 4 \\ 12 \\ -4 \end{bmatrix}$$

For the points A, B and C to be colinear the vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$ must be parallel.
If $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are parallel, there exists a value whereby $\overrightarrow{AB} = a\overrightarrow{BC}$.

So $\begin{bmatrix} 2 \\ 6 \\ -2 \end{bmatrix} = a \begin{bmatrix} 4 \\ 12 \\ -4 \end{bmatrix}$, which is satisfied for $a = \frac{1}{2}$.

$\overrightarrow{BC} = 2\overrightarrow{AB}$

Thus points A, B and C are collinear.

11

$A(5,4,7)$

$B(7,-1,-4)$

$C(-1,-3,-5)$

$D(-3,2,6)$

In order for $ABCD$ to be a parallelogram the vectors $\overrightarrow{AB}, \overrightarrow{CD}$ and $\overrightarrow{BC}, \overrightarrow{AD}$ will be parallel and the diagonals bisect one another.

$$\overrightarrow{AB} = \begin{bmatrix} 2 \\ -5 \\ -11 \end{bmatrix}$$

$$\overrightarrow{CD} = \begin{bmatrix} -2 \\ 5 \\ 11 \end{bmatrix}$$

If $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are parallel, there exists a value, a whereby $\overrightarrow{AB} = a\overrightarrow{CD}$.

$\begin{bmatrix} 2 \\ -5 \\ -11 \end{bmatrix} = a \begin{bmatrix} -2 \\ 5 \\ 11 \end{bmatrix}$, which is satisfied for $a = -1$.

So $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are parallel (and equal in length since $a = -1$).

$$\overrightarrow{BC} = \begin{bmatrix} -8 \\ -2 \\ -1 \end{bmatrix}$$

$$\overrightarrow{AD} = \begin{bmatrix} -8 \\ -2 \\ -1 \end{bmatrix}$$

If $\overrightarrow{BC}$ and $\overrightarrow{AD}$ are parallel, there exists a value b whereby $\overrightarrow{BC} = b\overrightarrow{AD}$.

$\begin{bmatrix} -8 \\ -2 \\ -1 \end{bmatrix} = b \begin{bmatrix} -8 \\ -2 \\ -1 \end{bmatrix}$, which is satisfied for $b = 1$.

So $\overrightarrow{BC}$ and $\overrightarrow{AD}$ are parallel (and equal in length as $b = 1$).

We can show that the diagonals bisect one another if the midpoints of AC and BD are the same.

$$M_{AC} = \frac{1}{2}\begin{bmatrix} 5-1 \\ 4-3 \\ 7-5 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 0.5 \\ 1 \end{bmatrix}$$

$$M_{BD} = \frac{1}{2}\begin{bmatrix} 7-3 \\ -1+2 \\ -4+6 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 0.5 \\ 1 \end{bmatrix}$$

$M_{AC} = M_{BD}$

Since the opposite sides are parallel and the diagonals bisect each other, the points $ABCD$ form a parallelogram.

12

$A = 3\underset{\sim}{i} - 8\underset{\sim}{j} - 2\underset{\sim}{k}$

$B = 2\underset{\sim}{i} + 4\underset{\sim}{j} + 5\underset{\sim}{k}$

$C = -2\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}$

$D = A + \overrightarrow{AD}$

As it is a parallelogram, $\overrightarrow{AD}, \overrightarrow{BC}$ are parallel and so, $\overrightarrow{AD} = \overrightarrow{BC}$.

$D = A + \overrightarrow{BC}$

$= A + (C - B)$

$= 3\underset{\sim}{i} - 8\underset{\sim}{j} - 2\underset{\sim}{k} + \left((-2\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}) - (2\underset{\sim}{i} + 4\underset{\sim}{j} + 5\underset{\sim}{k})\right)$

$= 3\underset{\sim}{i} - 8\underset{\sim}{j} - 2\underset{\sim}{k} - 4\underset{\sim}{i} - 6\underset{\sim}{j} - 4\underset{\sim}{k}$

$= -\underset{\sim}{i} - 14\underset{\sim}{j} - 6\underset{\sim}{k}$

13

$A(2, 1, 3)$

Find the magnitude of $\overrightarrow{OA}$.

$\left|\overrightarrow{OA}\right|^2 = 2^2 + 1^2 + 3^2$

$= 14$

$\left|\overrightarrow{OA}\right| = \sqrt{14}$

Let B, C and D be the values of A on the x-, y- and z-axis respectively so:

$B(2, 0, 0)$

$C(0, 1, 0)$

$D(0, 0, 3)$

For the angle between A and the x-axis:

$|\overrightarrow{OB}| = 2$

$$\begin{aligned}\angle AOB &= \cos^{-1}\left(\frac{2}{\sqrt{14}}\right)\\ &= 57.689°\\ &\simeq 58°\end{aligned}$$

For the angle between A and the y-axis:

$|\overrightarrow{OC}| = 1$

$$\begin{aligned}\angle AOC &= \cos^{-1}\left(\frac{1}{\sqrt{14}}\right)\\ &= 74.499°\\ &\simeq 74°\end{aligned}$$

For the angle between A and the z-axis:

$|\overrightarrow{OD}| = 3$

$$\begin{aligned}\angle AOD &= \cos^{-1}\left(\frac{3}{\sqrt{14}}\right)\\ &= 36.699°\\ &\simeq 37°\end{aligned}$$

14a

$$\underset{\sim}{a} = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$$

$$\underset{\sim}{b} = \begin{bmatrix} 5 \\ 5 \\ -8 \end{bmatrix}$$

$$\underset{\sim}{p} = \frac{1}{k+\ell}(\ell\underset{\sim}{a} + k\underset{\sim}{b})$$

$k = 1$

$\ell = 2$

$$\underset{\sim}{p} = \frac{1}{1+2}\left(2\begin{bmatrix}2\\-1\\-2\end{bmatrix} + \begin{bmatrix}5\\5\\-8\end{bmatrix}\right)$$

$$= \frac{1}{3}\begin{bmatrix}9\\3\\-12\end{bmatrix}$$

$$= \begin{bmatrix}3\\1\\-4\end{bmatrix}$$

14b

If P divides $\overrightarrow{AB}$ externally $k = -1$ or $\ell = -2$

So using $k = -1, \ell = 2$:

$$\underset{\sim}{p} = \frac{1}{k+\ell}(\ell\underset{\sim}{a} + k\underset{\sim}{b})$$

$$= \frac{1}{-1+2}\left(2\begin{bmatrix}2\\-1\\-2\end{bmatrix} - \begin{bmatrix}5\\5\\-8\end{bmatrix}\right)$$

$$= \begin{bmatrix}-1\\-7\\4\end{bmatrix}$$

15a

$\underset{\sim}{a} = -4\underset{\sim}{i} - 3\underset{\sim}{j} + 5\underset{\sim}{k}$

$\underset{\sim}{b} = 6\underset{\sim}{i} - 8\underset{\sim}{j} + 10\underset{\sim}{k}$

$$\underset{\sim}{p} = \frac{1}{k+\ell}(\ell\underset{\sim}{a} + k\underset{\sim}{b})$$

$k = 2$

$\ell = 3$

$$\underset{\sim}{p} = \frac{1}{2+3}\left(3(-4\underset{\sim}{i} - 3\underset{\sim}{j} + 5\underset{\sim}{k}) + 2(6\underset{\sim}{i} - 8\underset{\sim}{j} + 10\underset{\sim}{k})\right)$$

$$= \frac{1}{5}\left(-12\underset{\sim}{i} - 9\underset{\sim}{j} + 15\underset{\sim}{k} + 12\underset{\sim}{i} - 16\underset{\sim}{j} + 20\underset{\sim}{k}\right)$$

$$= \frac{1}{5}\left(-25\underset{\sim}{j} + 35\underset{\sim}{k}\right)$$

$$= -5\underset{\sim}{j} + 7\underset{\sim}{k}$$

15b

If P divides $\overrightarrow{AB}$ externally, $k = -2$ or $\ell = -3$.

So using $k = -2, \ell = 3$:

$\underset{\sim}{a} = -4\underset{\sim}{i} - 3\underset{\sim}{j} + 5\underset{\sim}{k}$

$\underset{\sim}{b} = 6\underset{\sim}{i} - 8\underset{\sim}{j} + 10\underset{\sim}{k}$

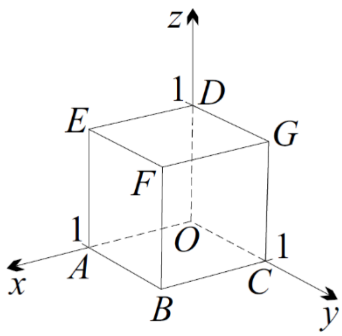
$$\underset{\sim}{p} = \frac{1}{k+\ell}\left(\ell\underset{\sim}{a} + k\underset{\sim}{b}\right)$$

$$= \frac{1}{-2+3}\left(3\left(-4\underset{\sim}{i} - 3\underset{\sim}{j} + 5\underset{\sim}{k}\right) - 2\left(6\underset{\sim}{i} - 8\underset{\sim}{j} + 10\underset{\sim}{k}\right)\right)$$

$$= \left(-12\underset{\sim}{i} - 9\underset{\sim}{j} + 15\underset{\sim}{k} - 12\underset{\sim}{i} + 16\underset{\sim}{j} - 20\underset{\sim}{k}\right)$$

$$= -24\underset{\sim}{i} + 7\underset{\sim}{j} - 5\underset{\sim}{k}$$

16a



$A = (1, 0, 0)$

$G = (0, 1, 1)$

$\overrightarrow{AG} = (0-1)\underset{\sim}{i} + (1-0)\underset{\sim}{j} + (1-0)\underset{\sim}{k}$

$= -\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$

16b

$\overrightarrow{AG} = -\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$

$\left|\overrightarrow{AG}\right|^2 = (-1)^2 + 1^2 + \underset{\sim}{1}^2$

$\left|\overrightarrow{AG}\right| = \sqrt{3}$

16c

$$\overrightarrow{OG} = \underset{\sim}{j} + \underset{\sim}{k}$$

$$\overrightarrow{OB} = \underset{\sim}{i} + \underset{\sim}{j}$$

$$\overrightarrow{OH} = \frac{1}{2}(\overrightarrow{OG} + \overrightarrow{OB})$$

$$= \frac{1}{2}(\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{j} + \underset{\sim}{k})$$

$$= \frac{1}{2}\underset{\sim}{i} + \underset{\sim}{j} + \frac{1}{2}\underset{\sim}{k}$$

$$\left|\overrightarrow{OH}\right|^2 = \left(\frac{1}{2}\right)^2 + 1^2 + \left(\frac{1}{2}\right)^2$$

$$= \frac{6}{4}$$

$$\left|\overrightarrow{OH}\right| = \frac{\sqrt{6}}{2}$$

17a

We want to show that $(\lambda_1 + \lambda_2)\underset{\sim}{a} = \lambda_1\underset{\sim}{a} + \lambda_2\underset{\sim}{a}$, where $\lambda_1, \lambda_2 \epsilon \mathbb{R}$

$$\underset{\sim}{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$$\text{LHS} = (\lambda_1 + \lambda_2)\underset{\sim}{a}$$

$$= (\lambda_1 + \lambda_2)\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 a_1 + \lambda_2 a_1 \\ \lambda_1 a_2 + \lambda_2 a_2 \\ \lambda_1 a_3 + \lambda_3 a_1 \end{bmatrix}$$

$$= \lambda_1\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \lambda_2\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$$= \lambda_1\underset{\sim}{a} + \lambda_2\underset{\sim}{a}$$

$$= \text{RHS}$$

17b

We want to show that $\lambda(\underset{\sim}{a} + \underset{\sim}{b}) = \lambda\underset{\sim}{a} + \lambda\underset{\sim}{b}$, where $\lambda \epsilon \mathbb{R}$

$$\underset{\sim}{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$$\underset{\sim}{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\begin{aligned}
\text{LHS} &= \lambda(\underset{\sim}{a} + \underset{\sim}{b}) \\
&= \lambda\left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}\right) \\
&= \lambda \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{bmatrix} \\
&= \begin{bmatrix} \lambda a_1 + \lambda b_1 \\ \lambda a_2 + \lambda b_2 \\ \lambda a_3 + \lambda b_3 \end{bmatrix} \\
&= \lambda \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \lambda \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \\
&= \lambda\underset{\sim}{a} + \lambda\underset{\sim}{b} \\
&= \text{RHS}
\end{aligned}$$

Solutions to Exercise 5B Enrichment questions

18a The vector equation is:

$$\lambda_1\begin{bmatrix}1\\1\\1\end{bmatrix}+\lambda_2\begin{bmatrix}1\\2\\0\end{bmatrix}+\lambda_3\begin{bmatrix}0\\-1\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

From this vector equation we obtain the simultaneous equations:

$\lambda_1+\lambda_2=0$ (1)

$\lambda_1+2\lambda_2-\lambda_3=0$ (2)

$\lambda_1+\lambda_3=0$ (3)

If we let $\lambda_1=k$ it follows from (1) and (3) that $\lambda_2=\lambda_3=-k$.

Clearly k cam be non-zero, so there are infinitely many non-trivial solutions for $\lambda_1,\lambda_2,\lambda_3$, (e.g., $1,-1,-1$).

In fact, any non-zero value of k provides us with the non-trivial solution,

$\lambda_1=k,\ \lambda_2=-k,\ \lambda_3=-k.$

Hence, the set of vectors $\underline{a},\underline{b},\underline{c}$ is linearly dependent, since $k\underline{a}-k\underline{b}-k\underline{c}=\underline{0}$ for all $k\in R$.

18b The vector equation is:

$$\lambda_1\begin{bmatrix}1\\1\\1\end{bmatrix}+\lambda_2\begin{bmatrix}1\\2\\0\end{bmatrix}+\lambda_3\begin{bmatrix}0\\-1\\2\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

From this vector equation we obtain the simultaneous equations:

$\lambda_1+\lambda_2=0$ (1)

$\lambda_1+2\lambda_2-\lambda_3=0$ (2)

$\lambda_1+2\lambda_3=0$ (3)

$2\times(1)-(2)$: $\lambda_1+\lambda_3=0$ (4)

$(3)-(4)$: $\lambda_3=0$

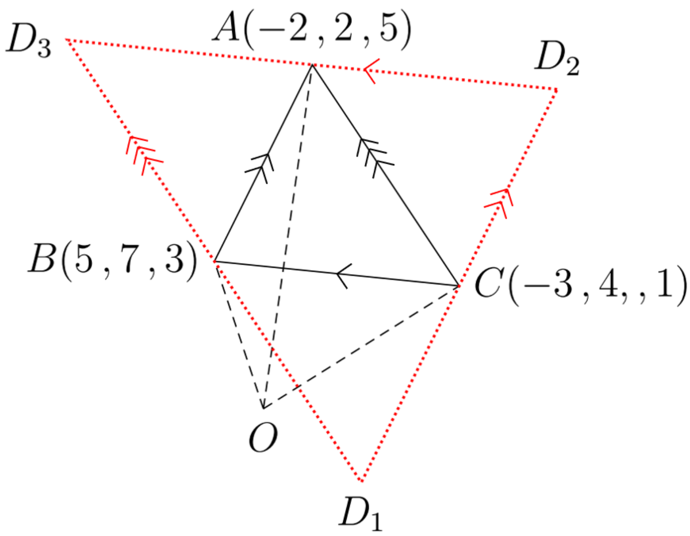
So, $\lambda_1=\lambda_2=\lambda_3=0$ is the only solution.

Hence, the set of vectors is linearly independent.

19 In the plane containing A, B and C, there are three possible locations of D.

These are the points D_1, D_2 and D_3 shown below.

In the diagram a line is drawn through each vertex of $\triangle ABC$ parallel to the opposite side.



Note that $\Delta ABC \ ||| \ \Delta D_1D_2D_3$ with enlargement factor of 2.

(Look at the parallelograms.)

Let $\overrightarrow{OA} = \underset{\sim}{a}$, $\overrightarrow{OB} = \underset{\sim}{b}$ and $\overrightarrow{OC} = \underset{\sim}{c}$.

Then,

$\overrightarrow{OD_1}$	$\overrightarrow{OD_2}$	$\overrightarrow{OD_3}$
$= \overrightarrow{OB} + \overrightarrow{BD_1}$	$= \overrightarrow{OC} + \overrightarrow{CD_2}$	$= \overrightarrow{OA} + \overrightarrow{AD_3}$
$= \overrightarrow{OB} + \overrightarrow{AC}$	$= \overrightarrow{OC} + \overrightarrow{BA}$	$= \overrightarrow{OA} + \overrightarrow{CB}$
$= \underset{\sim}{b} + \underset{\sim}{c} - \underset{\sim}{a}$	$= \underset{\sim}{c} + \underset{\sim}{a} - \underset{\sim}{b}$	$= \underset{\sim}{a} + \underset{\sim}{b} - \underset{\sim}{c}$
Hence,	Hence,	Hence,
$D_1 = (4, 9, -1)$	$D_2 = (-10, -1, 3)$	$D_3 = (6, 5, 7)$

20 $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are non-zero and non-parallel with,

$\lambda\underset{\sim}{a} + \mu\underset{\sim}{b} = l\underset{\sim}{a} + m\underset{\sim}{b}$ (1)

Rearranging (1),

$(\lambda - l)\underset{\sim}{a} + (\mu - m)\underset{\sim}{b} = \underset{\sim}{0}$ (2)

Since $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are non-zero and non-parallel, it follows that $\underset{\sim}{b}$ is not a scalar multiple of $\underset{\sim}{a}$.

So, from question 18, $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are linearly independent.

Hence, the only solution to equation (2) is the trivial solution

$\lambda - l = \mu - m = 0$.

Thus, $\lambda = l$ and $\mu = m$.

Solutions to Exercise 5C Foundation questions

1a $\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}| \cos\theta$

$|\underset{\sim}{a}| = 4$

$|\underset{\sim}{b}| = 6$

$\theta = 45°$

So

$\underset{\sim}{a} \cdot \underset{\sim}{b} = 4 \times 6 \times \cos 45°$

$= 24 \times \frac{\sqrt{2}}{2}$

$= 12\sqrt{2}$

1b $\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}| \cos\theta$

$|\underset{\sim}{a}| = 5$

$|\underset{\sim}{b}| = 8$

$\theta = 120°$

So

$\underset{\sim}{a} \cdot \underset{\sim}{b} = 5 \times 8 \times \cos 120°$

$= 40 \times -\frac{1}{2}$

$= -20$

2a $\underset{\sim}{a} = 3\underset{\sim}{i} - \underset{\sim}{j} + 5\underset{\sim}{k}$

$\underset{\sim}{b} = 2\underset{\sim}{i} + 6\underset{\sim}{j} + \underset{\sim}{k}$

$\underset{\sim}{a} \cdot \underset{\sim}{b} = (3 \times 2) + (-1 \times 6) + (5 \times 1)$

$= 6 - 6 + 5$

$= 5$

2b $\underset{\sim}{a} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$

$\underset{\sim}{b} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$

$\underset{\sim}{a} \cdot \underset{\sim}{b} = (x_1 \times x_2) + (y_1 \times y_2) + (z_1 \times z_2)$

$= x_1x_2 + y_1y_2 + z_1z_2$

2c $\underset{\sim}{a} = a_1\underset{\sim}{i} + a_2\underset{\sim}{j} + a_3\underset{\sim}{k}$

$\underset{\sim}{b} = b_1\underset{\sim}{i} + b_2\underset{\sim}{j} + b_3\underset{\sim}{k}$

$\underset{\sim}{a} \cdot \underset{\sim}{b} = (a_1 \times b_1) + (a_2 \times b_2) + (a_3 \times b_3)$

$= a_1b_1 + a_2b_2 + a_3b_3$

3 $\underset{\sim}{a} = a_1\underset{\sim}{i} + a_2\underset{\sim}{j} + a_3\underset{\sim}{k}$

$|\underset{\sim}{a}|^2 = a_1^2 + a_2^2 + a_3^2$

For $\underset{\sim}{a} \cdot \underset{\sim}{a} = |\underset{\sim}{a}|^2$

$\text{LHS} = \underset{\sim}{a} \cdot \underset{\sim}{a}$

$= a_1^2 + a_2^2 + a_3^2$

$= |\underset{\sim}{a}|^2$

$= \text{RHS}$

Thus $\underset{\sim}{a} \cdot \underset{\sim}{a} = |\underset{\sim}{a}|^2$

4a $\underset{\sim}{a} = 2\underset{\sim}{i} - 7\underset{\sim}{j} + 3\underset{\sim}{k}$

$\underset{\sim}{b} = -4\underset{\sim}{i} + \underset{\sim}{j} + 5\underset{\sim}{k}$

$\underset{\sim}{a} \cdot \underset{\sim}{b} = (2 \times -4) + (-7 \times 1) + (3 \times 5)$

$= -8 - 7 + 15$

$= 0$

4b As $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$ this means that the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is 90°, as

$\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}| \cos\theta$ and both $|\underset{\sim}{a}|$ and $|\underset{\sim}{b}|$ are non-zero.

So we can conclude that $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are perpendicular.

5 $\underset{\sim}{a} = \begin{bmatrix} 13 \\ 23 \\ 7 \end{bmatrix}$

$\underset{\sim}{b} = \begin{bmatrix} 2 \\ 1 \\ -7 \end{bmatrix}$

$\underset{\sim}{c} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$

If $\underset{\sim}{a} \perp \underset{\sim}{b}$ and $\underset{\sim}{a} \perp \underset{\sim}{c}$, then $\underset{\sim}{a} \cdot \underset{\sim}{b} = \underset{\sim}{a} \cdot \underset{\sim}{c} = 0$ and vice versa.

For $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$

$$\begin{aligned} \text{LHS} &= \underset{\sim}{a} \cdot \underset{\sim}{b} \\ &= (13 \times 2) + (23 \times 1) + (7 \times -7) \\ &= 26 + 23 - 49 \\ &= 0 \\ &= \text{RHS} \end{aligned}$$

For $\underset{\sim}{a} \cdot \underset{\sim}{c} = 0$

$$\begin{aligned} \text{LHS} &= \underset{\sim}{a} \cdot \underset{\sim}{c} \\ &= (13 \times 3) + (23 \times -2) + (7 \times 1) \\ &= 39 - 46 + 7 \\ &= 0 \\ &= \text{RHS} \end{aligned}$$

Thus $\underset{\sim}{a}$ is perpendicular to both $\underset{\sim}{b}$ and $\underset{\sim}{c}$

6a $\underset{\sim}{a} = \begin{bmatrix} -3 \\ 9 \\ 6 \end{bmatrix}$

$$\begin{aligned} \underset{\sim}{a} \cdot \underset{\sim}{a} &= (-3 \times -3) + (9 \times 9) + (6 \times 6) \\ &= 9 + 81 + 36 \\ &= 126 \end{aligned}$$

6b $\underset{\sim}{b} = \begin{bmatrix} 8 \\ 4 \\ -10 \end{bmatrix}$

$$\begin{aligned} 2\underset{\sim}{b} \cdot \underset{\sim}{b} &= 2\big((8 \times 8) + (4 \times 4) + (-10 \times -10)\big) \\ &= 2(64 + 16 + 100) \\ &= 2 \times 180 \\ &= 360 \end{aligned}$$

6c $\underset{\sim}{a} = \begin{bmatrix} -3 \\ 9 \\ 6 \end{bmatrix}$

$\underset{\sim}{b} = \begin{bmatrix} 8 \\ 4 \\ -10 \end{bmatrix}$

$$\begin{aligned} \underset{\sim}{a} \cdot \underset{\sim}{b} &= (-3 \times 8) + (9 \times 4) + (6 \times -10) \\ &= -24 + 36 - 60 \\ &= -48 \end{aligned}$$

6d $\underset{\sim}{a} = \begin{bmatrix} -3 \\ 9 \\ 6 \end{bmatrix}$

$\underset{\sim}{b} = \begin{bmatrix} 8 \\ 4 \\ -10 \end{bmatrix}$

$$\underset{\sim}{a} \cdot (\underset{\sim}{a} + \underset{\sim}{b}) = \begin{bmatrix} -3 \\ 9 \\ 6 \end{bmatrix} \cdot \left(\begin{bmatrix} -3 \\ 9 \\ 6 \end{bmatrix} + \begin{bmatrix} 8 \\ 4 \\ -10 \end{bmatrix} \right)$$

$$= \begin{bmatrix} -3 \\ 9 \\ 6 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 13 \\ -4 \end{bmatrix}$$

$$= (-3 \times 5) + (9 \times 13) + (6 \times -4)$$

$$= -15 + 117 - 24$$

$$= 78$$

Solutions to Exercise 5C Development questions

7a

$$\underset{\sim}{a} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \underset{\sim}{b} = \begin{bmatrix} 2 \\ 6 \\ -3 \end{bmatrix}$$

$|\underset{\sim}{a}|^2 = 1^2 + 2^2 + 2^2$

$= 9$

$|\underset{\sim}{a}| = \sqrt{9}$

$= 3$

$|\underset{\sim}{b}|^2 = 2^2 + 6^2 + (-3)^2$

$= 49$

$|\underset{\sim}{b}| = \sqrt{49}$

$= 7$

$\underset{\sim}{a} \cdot \underset{\sim}{b} = (1 \times 2) + (2 \times 6) + (2 \times -3)$

$= 2 + 12 \pm 6$

$= 8$

Substituting the associated values into the inequation:

$-21 \leq 8 \leq 21$

So the Cauchy-Schwarz inequality, $-|\underset{\sim}{a}||\underset{\sim}{b}| \leq \underset{\sim}{a} \cdot \underset{\sim}{b} \leq |\underset{\sim}{a}||\underset{\sim}{b}|$ is satisfied.

7b

$\underset{\sim}{a} = -\underset{\sim}{i} + 3\underset{\sim}{j}$

$\underset{\sim}{b} = -6j + 2\underset{\sim}{k}$

$|\underset{\sim}{a}|^2 = 1^2 + 3^2$

$= 10$

$|\underset{\sim}{a}| = \sqrt{10}$

$|\underset{\sim}{b}|^2 = (-6)^2 + 2^2$

$= 40$

$|\underset{\sim}{b}| = \sqrt{40}$

$= 2\sqrt{10}$

$|\underset{\sim}{a}||\underset{\sim}{b}| = 2\sqrt{10} \times \sqrt{10}$

$= 20$

$\underset{\sim}{a} \cdot \underset{\sim}{b} = (-1 \times 0) + (-6 \times 3) + (0 \times 2)$

$= -18$

Substituting the associated values into the inequation:

$-20 \leq -18 \leq 20$

So the Cauchy-Schwarz inequality, $-|\underset{\sim}{a}||\underset{\sim}{b}| \leq \underset{\sim}{a} \cdot \underset{\sim}{b} \leq |\underset{\sim}{a}||\underset{\sim}{b}|$ is satisfied.

8a

$\underset{\sim}{a} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \underset{\sim}{b} = \begin{bmatrix} 2 \\ 6 \\ -3 \end{bmatrix}$

$|\underset{\sim}{a}|^2 = 1^2 + 2^2 + 2^2$

$= 9$

$|\underset{\sim}{a}| = \sqrt{9}$

$= 3$

$|\underset{\sim}{b}|^2 = 2^2 + 6^2 + (-3)^2$

$= 49$

$|\underset{\sim}{b}| = \sqrt{49}$

$= 7$

$|\underset{\sim}{a} + \underset{\sim}{b}|^2 = (1 + 2)^2 + (2 + 6)^2 + (2 - 3)^2$

$= 9 + 64 + 1$

$= 74$

$|\underset{\sim}{a} + \underset{\sim}{b}| = \sqrt{74}$

$\big||\underset{\sim}{a}| - |\underset{\sim}{b}|\big| \leq |\underset{\sim}{a} + \underset{\sim}{b}| \leq |\underset{\sim}{a}| + |\underset{\sim}{b}|$

Substituting we get:

$4 \leq \sqrt{74} \leq 10$

So the triangle inequality holds.

8b

$\underset{\sim}{a} = -\underset{\sim}{i} + 3\underset{\sim}{j}$

$\underset{\sim}{b} = -6j + 2\underset{\sim}{k}$

$|\underset{\sim}{a}| = \sqrt{10}$

$|\underset{\sim}{b}| = 2\sqrt{10}$

$|\underset{\sim}{a} + \underset{\sim}{b}|^2 = (-1+0)^2 + (3-6)^2 + (0+2)^2$

$= 1 + 9 + 4$

$= 14$

$|\underset{\sim}{a} + \underset{\sim}{b}| = \sqrt{14}$

$\big||\underset{\sim}{a}| - |\underset{\sim}{b}|\big| \leq |\underset{\sim}{a} + \underset{\sim}{b}| \leq |\underset{\sim}{a}| + |\underset{\sim}{b}|$

Substituting we get:

$\left|\sqrt{10} - 2\sqrt{10}\right| \leq \sqrt{14} \leq \sqrt{10} + 2\sqrt{10}$

$\sqrt{10} \leq \sqrt{14} \leq 3\sqrt{10}$

So the triangle inequality holds.

9

$\underset{\sim}{a} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}, \underset{\sim}{b} = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix}, \underset{\sim}{c} = \begin{bmatrix} -2 \\ 9 \\ -5 \end{bmatrix}, \underset{\sim}{d} = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}$

$\overrightarrow{AB} = \begin{bmatrix} 4-2 \\ 1-3 \\ 3-5 \end{bmatrix}$

$= \begin{bmatrix} 2 \\ -2 \\ -2 \end{bmatrix}$

$\overrightarrow{CD} = \begin{bmatrix} -3-(-2) \\ 1-9 \\ 2-(-5) \end{bmatrix}$

$= \begin{bmatrix} -1 \\ -8 \\ 7 \end{bmatrix}$

$$\overrightarrow{AB} \cdot \overrightarrow{CD} = \begin{bmatrix} 2 \\ -2 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ -8 \\ 7 \end{bmatrix}$$

$$= -2 + 16 - 14$$

$$= 0$$

So as $\overrightarrow{AB} \cdot \overrightarrow{CD} = 0$, $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are perpendicular.

10a

$$\underset{\sim}{a} = \begin{bmatrix} 2 \\ -2 \\ -5 \end{bmatrix}, \underset{\sim}{b} = \begin{bmatrix} 3 \\ \lambda \\ -2 \end{bmatrix}$$

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = \begin{bmatrix} 3 \\ \lambda \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -2 \\ -5 \end{bmatrix}$$

$$= (2 \times 3) + (-2 \times \lambda) + (-5 \times -2)$$

$$= -2\lambda + 16$$

If $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are perpendicular, then $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$

So

$$-2\lambda + 16 = 0$$

$$-2\lambda = -16$$

$$\lambda = 8$$

10b

$$\underset{\sim}{a} = \begin{bmatrix} -4 \\ \lambda + 3 \\ 2 \end{bmatrix}, \underset{\sim}{b} = \begin{bmatrix} \lambda \\ 5 \\ -\lambda^2 \end{bmatrix}$$

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = \begin{bmatrix} -4 \\ \lambda + 3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} \lambda \\ 5 \\ -\lambda^2 \end{bmatrix}$$

$$= (-4 \times \lambda) + \big((\lambda + 3) \times 5\big) + (2 \times -\lambda^2)$$

$$= -2\lambda^2 + \lambda + 15$$

If $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are perpendicular, then $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$

Hence,

$-2\lambda^2 + \lambda + 15 = 0$

$2\lambda^2 - \lambda - 15 = 0$

$(2\lambda + 5)(\lambda - 3) = 0$

$\lambda = -\dfrac{5}{2}$ or $\lambda = 3$

11

Let $\underset{\sim}{a} = \underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$ and $\underset{\sim}{b} = 2\underset{\sim}{i} + \underset{\sim}{j} - 3\underset{\sim}{k}$

For a vector $\underset{\sim}{c}$ which is perpendicular to $\underset{\sim}{a}$ and $\underset{\sim}{b}$, $\underset{\sim}{b} \cdot \underset{\sim}{c} = \underset{\sim}{a} \cdot \underset{\sim}{c} = 0$

$\underset{\sim}{c} = \lambda_1 \underset{\sim}{i} + \lambda_2 \underset{\sim}{j} + \lambda_3 \underset{\sim}{k}$

$\underset{\sim}{b} \cdot \underset{\sim}{c} = 2\lambda_1 + \lambda_2 - 3\lambda_3$

$0 = 2\lambda_1 + \lambda_2 - 3\lambda_3 \qquad (1)$

$\underset{\sim}{a} \cdot \underset{\sim}{c} = \lambda_1 - \lambda_2 + 2\lambda_3$

$0 = \lambda_1 - \lambda_2 + 2\lambda_3 \qquad (2)$

Eliminating λ_2 by equating adding (1) and (2)

$0 = 3\lambda_1 - \lambda_3$

$\lambda_3 = 3\lambda_1 \qquad (3)$

Now substituting (3) into (2)

$0 = \lambda_1 - \lambda_2 + 2(3\lambda_1)$

$\lambda_2 = 7\lambda_1$

So $\underset{\sim}{c}$ is a vector of the form $\lambda_1\left(\underset{\sim}{i} + 7\underset{\sim}{j} + 3\underset{\sim}{k}\right)$ where $\lambda_1 \in R$

One such vector is $\underset{\sim}{i} + 7\underset{\sim}{j} + 3\underset{\sim}{k}$.

12

$\underset{\sim}{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \underset{\sim}{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

For $\underset{\sim}{a} \cdot (\lambda \underset{\sim}{b}) = \lambda(\underset{\sim}{a} \cdot \underset{\sim}{b})$

$\text{LHS} = \underset{\sim}{a} \cdot (\lambda \underset{\sim}{b})$

$$= \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \cdot \left(\lambda \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \right)$$

$$= \lambda a_1 b_1 + \lambda a_2 b_2 + \lambda a_3 b_3$$

$$= \lambda(a_1 b_1 + a_2 b_2 + a_3 b_3)$$

$$= \lambda(\underset{\sim}{a} \cdot \underset{\sim}{b})$$

$$= \text{RHS}$$

So the equation is satisfied.

13

$$\underset{\sim}{a} = a_1 \underset{\sim}{i} + a_2 \underset{\sim}{j} + a_3 \underset{\sim}{k}$$

$$\underset{\sim}{b} = b_1 \underset{\sim}{i} + b_2 \underset{\sim}{j} + b_3 \underset{\sim}{k}$$

$$\underset{\sim}{c} = c_1 \underset{\sim}{i} + c_2 \underset{\sim}{j} + c_3 \underset{\sim}{k}$$

$$\underset{\sim}{a} \cdot (\underset{\sim}{b} + \underset{\sim}{c}) = \underset{\sim}{a} \cdot \underset{\sim}{b} + \underset{\sim}{a} \cdot \underset{\sim}{c}$$

$$\text{LHS} = \underset{\sim}{a} \cdot (\underset{\sim}{b} + \underset{\sim}{c})$$

$$= (a_1 \underset{\sim}{i} + a_2 \underset{\sim}{j} + a_3 \underset{\sim}{k}) \cdot \left((b_1 + c_1)\underset{\sim}{i} + (b_2 + c_2)\underset{\sim}{j} + (b_3 + c_3)\underset{\sim}{k}\right)$$

$$= (a_1 b_1 + a_1 c_1 + a_2 b_2 + a_2 c_2 + a_3 b_3 + a_3 c_3)$$

$$= (a_1 b_1 + a_2 b_2 + a_3 b_3) + (a_1 c_1 + a_2 c_2 + a_3 c_3)$$

$$= \underset{\sim}{a} \cdot \underset{\sim}{b} + \underset{\sim}{a} \cdot \underset{\sim}{c}$$

$$= \text{RHS}$$

So the equation is satisfied.

14a

A vector is a unit vector if its magnitude is equal to 1

$$\underset{\sim}{u} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$|\underset{\sim}{u}|^2 = \frac{1}{2}(1 + 1)$$

$$= 1$$

$|\underset{\sim}{u}| = 1$

So $\underset{\sim}{u}$ is a unit vector.

$\underset{\sim}{v} = \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\\0\end{bmatrix}$

$|\underset{\sim}{v}|^2 = \frac{1}{2}(1+1)$

$= 1$

$|\underset{\sim}{v}| = 1$

So $\underset{\sim}{v}$ is a unit vector.

$\underset{\sim}{w} = \begin{bmatrix}0\\0\\1\end{bmatrix}$

$|\underset{\sim}{w}|^2 = 1$

$|\underset{\sim}{w}| = 1$

So $\underset{\sim}{w}$ is a unit vector.

$\underset{\sim}{u} \cdot \underset{\sim}{w} = \frac{1}{\sqrt{2}}(0+0+0)$

$= 0$

$\underset{\sim}{u} \cdot \underset{\sim}{v} = \frac{1}{2}(-1+1+0)$

$= 0$

$\underset{\sim}{v} \cdot \underset{\sim}{w} = \frac{1}{\sqrt{2}}(0+0+0)$

$= 0$

So $\underset{\sim}{u}$, $\underset{\sim}{v}$ and $\underset{\sim}{w}$ are perpendicular to each other.

Thus $\underset{\sim}{u}$, $\underset{\sim}{v}$ and $\underset{\sim}{w}$ are orthonormal.

14b

A vector is a unit vector if its magnitude is equal to 1.

$\underset{\sim}{u} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\end{bmatrix}$

$$|\underset{\sim}{u}|^2 = \frac{1}{2}(1+1)$$

$$= 1$$

$$|\underset{\sim}{u}| = 1$$

So $\underset{\sim}{u}$ is a unit vector.

$$\underset{\sim}{v} = \frac{1}{2\sqrt{2}}\begin{bmatrix} 1 \\ \sqrt{6} \\ -1 \end{bmatrix}$$

$$|\underset{\sim}{v}|^2 = \frac{1}{8}(1+6+1)$$

$$= 1$$

$$|\underset{\sim}{v}| = 1$$

So $\underset{\sim}{v}$ is a unit vector.

$$\underset{\sim}{w} = \frac{1}{2\sqrt{2}}\begin{bmatrix} -\sqrt{3} \\ \sqrt{2} \\ \sqrt{3} \end{bmatrix}$$

$$|\underset{\sim}{w}|^2 = \frac{1}{8}(3+2+3)$$

$$= 1$$

$$|\underset{\sim}{w}| = 1$$

So $\underset{\sim}{w}$ is a unit vector.

$$\underset{\sim}{u} \cdot \underset{\sim}{w} = \frac{1}{4}\left(-\sqrt{3}+0+\sqrt{3}\right)$$

$$= 0$$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = \frac{1}{4}(1+0-1)$$

$$= 0$$

$$\underset{\sim}{v} \cdot \underset{\sim}{w} = \frac{1}{8}\left(-\sqrt{3}+\sqrt{2}\sqrt{6}-\sqrt{3}\right)$$

$$= \frac{1}{8}\left(-\sqrt{3}+2\sqrt{3}-\sqrt{3}\right)$$

$$= 0$$

So $\underset{\sim}{u}$, $\underset{\sim}{v}$ and $\underset{\sim}{w}$ are perpendicular to each other.

Thus $\underset{\sim}{u}$, $\underset{\sim}{v}$ and $\underset{\sim}{w}$ are orthonormal.

15a

$\underset{\sim}{a} \cdot (\underset{\sim}{b} + \underset{\sim}{c}) = \underset{\sim}{b} \cdot (\underset{\sim}{a} - \underset{\sim}{c})$

Using the distributive law:

$\underset{\sim}{a} \cdot (\underset{\sim}{b} + \underset{\sim}{c}) = \underset{\sim}{a} \cdot \underset{\sim}{b} + \underset{\sim}{a} \cdot \underset{\sim}{c}$

$\underset{\sim}{a} \cdot \underset{\sim}{b} + \underset{\sim}{a} \cdot \underset{\sim}{c} = \underset{\sim}{b} \cdot (\underset{\sim}{a} - \underset{\sim}{c})$

$\underset{\sim}{a} \cdot \underset{\sim}{c} = \underset{\sim}{b} \cdot (\underset{\sim}{a} - \underset{\sim}{c}) - \underset{\sim}{a} \cdot \underset{\sim}{b}$

For $\underset{\sim}{c} \cdot (\underset{\sim}{a} + \underset{\sim}{b}) = 0$

LHS $= \underset{\sim}{c} \cdot (\underset{\sim}{a} + \underset{\sim}{b})$

$= \underset{\sim}{c} \cdot \underset{\sim}{a} + \underset{\sim}{c} \cdot \underset{\sim}{b}$

$= \underset{\sim}{b} \cdot (\underset{\sim}{a} - \underset{\sim}{c}) - \underset{\sim}{a} \cdot \underset{\sim}{b} + \underset{\sim}{c} \cdot \underset{\sim}{b}$

$= \underset{\sim}{b} \cdot \underset{\sim}{a} - \underset{\sim}{b} \cdot \underset{\sim}{c} - \underset{\sim}{a} \cdot \underset{\sim}{b} + \underset{\sim}{c} \cdot \underset{\sim}{b}$

$= 0$

$=$ RHS

So the equation is satisfied.

15b

$(\underset{\sim}{a} \cdot \underset{\sim}{b})\underset{\sim}{c} = (\underset{\sim}{b} \cdot \underset{\sim}{c})\underset{\sim}{a}$

If $\underset{\sim}{a}$ is parallel to $\underset{\sim}{c}$, then $\underset{\sim}{a} = \lambda \underset{\sim}{c}$, where $\lambda \epsilon \mathbb{R}$

LHS $= (\underset{\sim}{a} \cdot \underset{\sim}{b})\underset{\sim}{c}$

$= (\lambda \underset{\sim}{c} \cdot \underset{\sim}{b})\underset{\sim}{c}$

$= (\underset{\sim}{c} \cdot \underset{\sim}{b})\lambda \underset{\sim}{c}$

$= (\underset{\sim}{c} \cdot \underset{\sim}{b})\underset{\sim}{a}$

$=$ RHS

So the equation is satisfied.

If $\underset{\sim}{b}$ is perpendicular to a and $\underset{\sim}{c}$, then $\underset{\sim}{b} \cdot \underset{\sim}{c} = \underset{\sim}{a} \cdot \underset{\sim}{b} = 0$

LHS $= (\underset{\sim}{a} \cdot \underset{\sim}{b})\underset{\sim}{c}$

$= (\underset{\sim}{b} \cdot \underset{\sim}{c})\underset{\sim}{c}$

$= \text{RHS}$

So the equation is satisfied.

16a

$$\underset{\sim}{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \underset{\sim}{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

If $\underset{\sim}{a} + \underset{\sim}{b}$ and $\underset{\sim}{a} - \underset{\sim}{b}$ are perpendicular then $(\underset{\sim}{a} + \underset{\sim}{b}) \cdot (\underset{\sim}{a} - \underset{\sim}{b}) = 0$

$$0 = \left(\begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{bmatrix}\right) \cdot \left(\begin{bmatrix} a_1 - b_1 \\ a_2 - b_2 \\ a_3 - b_3 \end{bmatrix}\right)$$

$$0 = (a_1 + b_1)(a_1 - b_1) + (a_2 + b_2)(a_2 - b_2) + (a_3 + b_3)(a_3 - b_3)$$

$$0 = (a_1^2 - b_1^2) + (a_2^2 - b_2^2) + (a_3^2 - b_3^2)$$

$$a_1^2 + a_2^2 + a_3^2 = b_1^2 + b_2^2 + b_3^2$$

$$|\underset{\sim}{a}|^2 = |\underset{\sim}{b}|^2$$

Hence $|\underset{\sim}{a}| = |\underset{\sim}{b}|$.

16b

$$\underset{\sim}{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \underset{\sim}{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$|(\underset{\sim}{a} + \underset{\sim}{b})| = |(\underset{\sim}{a} - \underset{\sim}{b})|$$

$$\left\|\begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{bmatrix}\right\| = \left\|\begin{bmatrix} a_1 - b_1 \\ a_2 - b_2 \\ a_3 - b_3 \end{bmatrix}\right\|$$

$$\sqrt{(a_1 + b_1)^2 + (a_2 + b_2)^2 + (a_3 + b_3)^2} = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2}$$

Raising both sides to the power of 2 gives:

$$(a_1 + b_1)^2 + (a_2 + b_2)^2 + (a_3 + b_3)^2 = (a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2$$

$$a_1^2 + 2a_1b_1 + b_1^2 + a_2^2 + 2a_2b_2 + b_2^2 + a_3^2 + 2a_3b_3 + b_3^2$$
$$= a_1^2 - 2a_1b_1 + b_1^2 + a_2^2 - 2a_2b_2 + b_2^2 + a_3^2 - 2a_3b_3 + b_3^2$$

$$4a_1b_1 + 4a_2b_2 + 4a_3b_3 = 0$$

$$a_1b_1 + a_2b_2 + a_3b_3 = 0$$

For $\underset{\sim}{a}$ and $\underset{\sim}{b}$ to be perpendicular $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$

$$\begin{aligned} \text{LHS} &= \underset{\sim}{a} \cdot \underset{\sim}{b} \\ &= a_1 b_1 + a_2 b_2 + a_3 b_3 \\ &= 0 \\ &= \text{RHS} \end{aligned}$$

So the equation is satisfied.

17

$$\underset{\sim}{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \underset{\sim}{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}, \underset{\sim}{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

Firstly, $\overrightarrow{AB} \perp \overrightarrow{OC}$ implies that

$$\overrightarrow{AB} \cdot \overrightarrow{OC} = 0$$

$$(\underset{\sim}{b} - \underset{\sim}{a}) \cdot \underset{\sim}{c} = 0$$

$$\underset{\sim}{b} \cdot \underset{\sim}{c} = \underset{\sim}{a} \cdot \underset{\sim}{c} \qquad (1)$$

Likewise, $\overrightarrow{BC} \perp \overrightarrow{OA}$ implies that

$$\overrightarrow{BC} \cdot \overrightarrow{OA} = 0$$

$$(\underset{\sim}{c} - \underset{\sim}{b}) \cdot \underset{\sim}{a} = 0$$

$$\underset{\sim}{c} \cdot \underset{\sim}{a} = \underset{\sim}{b} \cdot \underset{\sim}{a} \qquad (2)$$

Therefore, consider

$$\begin{aligned} \overrightarrow{AC} \cdot \overrightarrow{OB} &= (\underset{\sim}{c} - \underset{\sim}{a}) \cdot \underset{\sim}{b} \\ &= \underset{\sim}{c} \cdot \underset{\sim}{b} - \underset{\sim}{a} \cdot \underset{\sim}{b} \\ &= \underset{\sim}{b} \cdot \underset{\sim}{c} - \underset{\sim}{b} \cdot \underset{\sim}{a} && \text{(commutative law)} \\ &= \underset{\sim}{a} \cdot \underset{\sim}{c} - \underset{\sim}{c} \cdot \underset{\sim}{a} && \text{(from (1) and (2))} \\ &= 0 && \text{(commutative law)} \end{aligned}$$

Hence, $\overrightarrow{AC} \perp \overrightarrow{OB}$.

18

$|\underset{\sim}{a}| = 2, |\underset{\sim}{b}| = 3$, and $\underset{\sim}{a} \cdot \underset{\sim}{b} = 5$

$|\underset{\sim}{a} + \underset{\sim}{b}|^2 = |\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2 + 2\underset{\sim}{a} \cdot \underset{\sim}{b}$

$$\begin{aligned} |\underset{\sim}{a} + \underset{\sim}{b}| &= \sqrt{|\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2 + 2\underset{\sim}{a} \cdot \underset{\sim}{b}} \\ &= \sqrt{2^2 + 3^2 + 2(5)} \\ &= \sqrt{23} \end{aligned}$$

19

$|\underset{\sim}{u}| = 2\sqrt{2}, |\underset{\sim}{v}| = 2\sqrt{3}$, and $\underset{\sim}{u} \cdot \underset{\sim}{v} = -4$

$|\underset{\sim}{u} - \underset{\sim}{v}|^2 = |\underset{\sim}{u}|^2 + |\underset{\sim}{v}|^2 - 2\underset{\sim}{u} \cdot \underset{\sim}{v}$

$$\begin{aligned} |\underset{\sim}{u} - \underset{\sim}{v}| &= \sqrt{|\underset{\sim}{u}|^2 + |\underset{\sim}{v}|^2 - 2\underset{\sim}{u} \cdot \underset{\sim}{v}} \\ &= \sqrt{\left(2\sqrt{2}\right)^2 + \left(2\sqrt{3}\right)^2 - 2(-4)} \\ &= \sqrt{8 + 12 + 8} \\ &= \sqrt{28} \\ &= 2\sqrt{7} \end{aligned}$$

20

$\left||\underset{\sim}{a}| - |\underset{\sim}{b}|\right| \le |\underset{\sim}{a} + \underset{\sim}{b}| \le |\underset{\sim}{a}| + |\underset{\sim}{b}|$

Squaring the middle term gives:

$$\begin{aligned} |\underset{\sim}{a} + \underset{\sim}{b}|^2 &= (\underset{\sim}{a} + \underset{\sim}{b}) \cdot (\underset{\sim}{a} + \underset{\sim}{b}) \\ &= |\underset{\sim}{a}|^2 + 2|\underset{\sim}{a}||\underset{\sim}{b}| + |\underset{\sim}{b}|^2 \end{aligned}$$

Rearranging for $-2|\underset{\sim}{a}||\underset{\sim}{b}|$ gives:

$-2|\underset{\sim}{a}||\underset{\sim}{b}| = -|\underset{\sim}{a} + \underset{\sim}{b}|^2 + |\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2$

We know from the Cauchy-Schwarz inequality that:

$-|\underset{\sim}{a}||\underset{\sim}{b}| \le \underset{\sim}{a} \cdot \underset{\sim}{b} \le |\underset{\sim}{a}||\underset{\sim}{b}|$

So

$-|\underset{\sim}{a} + \underset{\sim}{b}|^2 + |\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2 \le 2|\underset{\sim}{a}||\underset{\sim}{b}|$

$$|\underset{\sim}{a}|^2 - 2|\underset{\sim}{a}||\underset{\sim}{b}| + |\underset{\sim}{b}|^2 \le |\underset{\sim}{a} + \underset{\sim}{b}|^2$$

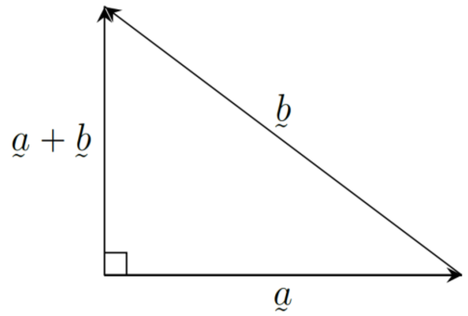
$$(|\underset{\sim}{a}| - |\underset{\sim}{b}|)^2 \le |\underset{\sim}{a} + \underset{\sim}{b}|^2$$

Hence

$$\big||\underset{\sim}{a}| - |\underset{\sim}{b}|\big| \le |\underset{\sim}{a} + \underset{\sim}{b}|$$

Solutions to Exercise 5C Enrichment questions

21



Given $|\underline{b}| = \sqrt{2}|\underline{a}|$,

Then $|\underline{b}|^2 = 2|\underline{a}|^2$

So $\underline{b} \cdot \underline{b} = 2\underline{a} \cdot \underline{a}$ (1)

Since $\underline{a} + \underline{b}$ is perpendicular to $\underline{a}$,

$(\underline{a} + \underline{b}) \cdot \underline{a} = 0$

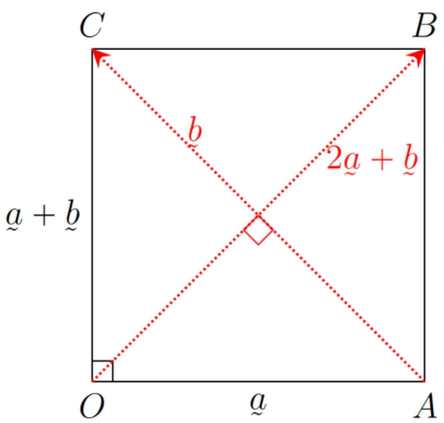
$\underline{a} \cdot \underline{a} + \underline{a} \cdot \underline{b} = 0$

$\frac{1}{2}\underline{b} \cdot \underline{b} + \underline{a} \cdot \underline{b} = 0$ (from (1))

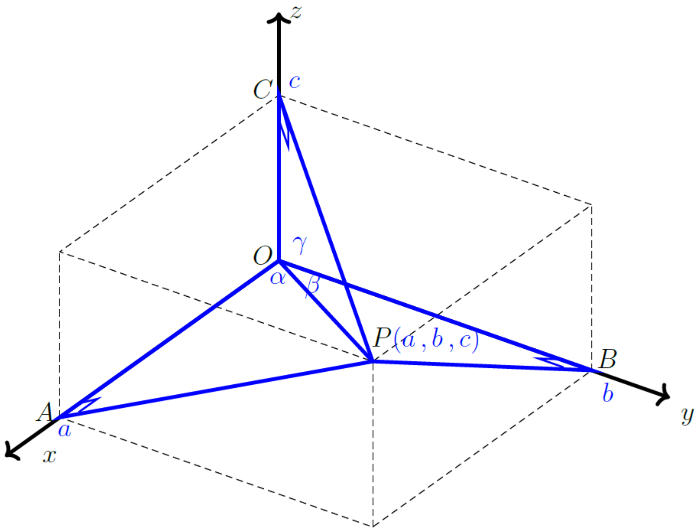
$(2\underline{a} + \underline{b}) \cdot \underline{b} = 0$

Hence $2\underline{a} + \underline{b}$ is perpendicular to $\underline{b}$.

Note: $\underline{a}$ and $\underline{a} + \underline{b}$ represent the sides of a square.



22



OP is the hypotenuse of each of the right-angled triangles OAP, OBP, OCP.

By basic trigonometry,

$\cos\alpha = \frac{a}{|\overrightarrow{OP}|}$, $\cos\beta = \frac{b}{|\overrightarrow{OP}|}$ $\cos\gamma = \frac{c}{|\overrightarrow{OP}|}$

Hence,

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma$

$= \frac{a^2+b^2+c^2}{|\overrightarrow{OP}|^2}$

$= \frac{|\overrightarrow{OP}|^2}{|\overrightarrow{OP}|^2}$

$= 1$

Solutions to Exercise 5D Foundation questions

1a $\quad \underset{\sim}{a} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

$\underset{\sim}{b} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$

$\underset{\sim}{a} \cdot \underset{\sim}{b} = (1 \times 2) + (2 \times 1) + (1 \times -1)$

$= 2 + 2 - 1$

$= 3$

1b $\quad \underset{\sim}{a} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

$\underset{\sim}{b} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$

$|\underset{\sim}{a}|^2 = 1^2 + 2^2 + 1^2$

$= 6$

$|\underset{\sim}{a}| = \sqrt{6}$

$|\underset{\sim}{b}|^2 = 2^2 + 1^2 + (-1)^2$

$= 6$

$|\underset{\sim}{b}| = \sqrt{6}$

1c $\quad \underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}| \cos \theta$

$\underset{\sim}{a} \cdot \underset{\sim}{b} = 3$

$|\underset{\sim}{a}| = \sqrt{6}$

$|\underset{\sim}{b}| = \sqrt{6}$

So

$\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}| \cos \theta$

$3 = \sqrt{6} \times \sqrt{6} \times \cos \theta$

$\theta = \cos^{-1}\frac{1}{2}$

$\theta = 60°$

$\theta = \frac{\pi}{3}$

2a $\quad \underset{\sim}{a} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$

$|\underset{\sim}{a}|^2 = 2^2 + 0^2 + 0^2$

$|\underset{\sim}{a}|^2 = 4$

$|\underset{\sim}{a}| = 2$

$\underset{\sim}{b} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$

$|\underset{\sim}{b}|^2 = 2^2 + 1^2 + (-2)^2$

$|\underset{\sim}{b}|^2 = 9$

$|\underset{\sim}{b}| = 3$

$\underset{\sim}{a} \cdot \underset{\sim}{b} = (2 \times 2) + (0 \times 1) + (0 \times -2)$

$\underset{\sim}{a} \cdot \underset{\sim}{b} = 4$

$\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}| \cos\theta$

$\cos\theta = \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}$

$= \frac{4}{2 \times 3}$

$= \frac{2}{3}$

2b $\quad \underset{\sim}{a} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$

$|\underset{\sim}{a}|^2 = 1^2 + (-1)^2 + (-1)^2$

$|\underset{\sim}{a}|^2 = 3$

$|\underset{\sim}{a}| = \sqrt{3}$

$\underset{\sim}{b} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$

$|\underset{\sim}{b}|^2 = 2^2 + 1^2 + (-1)^2$

$|\underset{\sim}{b}|^2 = 6$

$|\underset{\sim}{b}| = \sqrt{6}$

$\underset{\sim}{a} \cdot \underset{\sim}{b} = (1 \times 2) + (-1 \times 1) + (-1 \times -1)$

$\underset{\sim}{a} \cdot \underset{\sim}{b} = 2 - 1 + 1$

$= 2$

$\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}| \cos\theta$

$\cos\theta = \dfrac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}$

$= \dfrac{2}{\sqrt{3} \times \sqrt{6}}$

$= \dfrac{2}{\sqrt{18}}$

$= \dfrac{2}{3\sqrt{2}}$

$= \dfrac{\sqrt{2}}{3}$

3 $\underset{\sim}{a} = \begin{bmatrix} 3 \\ -2 \\ -3 \end{bmatrix}$

$|\underset{\sim}{a}|^2 = 3^2 + (-2)^2 + (-3)^2$

$|\underset{\sim}{a}|^2 = 9 + 4 + 9$

$|\underset{\sim}{a}|^2 = 22$

$|\underset{\sim}{a}| = \sqrt{22}$

$\underset{\sim}{b} = \begin{bmatrix} -1 \\ 3 \\ -4 \end{bmatrix}$

$|\underset{\sim}{b}|^2 = (-1)^2 + 3^2 + (-4)^2$

$|\underset{\sim}{b}|^2 = 1 + 9 + 16$

$|\underset{\sim}{b}|^2 = 26$

$|\underset{\sim}{b}| = \sqrt{26}$

$\underset{\sim}{a} \cdot \underset{\sim}{b} = (3 \times -1) + (-2 \times 3) + (-3 \times -4)$

$\underset{\sim}{a} \cdot \underset{\sim}{b} = -3 - 6 + 12$

$= 3$

$\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}| \cos\theta$

$\cos\theta = \dfrac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}$

So for

$\cos\theta = \dfrac{3}{2\sqrt{143}}$

$\text{LHS} = \cos\theta$

$= \dfrac{3}{\sqrt{22} \times \sqrt{26}}$

$= \dfrac{3}{2\sqrt{143}}$

$= \text{RHS}$

4a $\underset{\sim}{v_1} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

$|\underset{\sim}{v_1}|^2 = 3^2 + 2^2 + 1^2$

$|\underset{\sim}{v_1}|^2 = 14$

$|\underset{\sim}{v_1}| = \sqrt{14}$

$\underset{\sim}{v_2} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$|\underset{\sim}{v_2}|^2 = 1^2 + 2^2 + 3^2$

$|\underset{\sim}{v_2}|^2 = 14$

$|\underset{\sim}{v_2}| = \sqrt{14}$

$\underset{\sim}{v_1} \cdot \underset{\sim}{v_2} = (3 \times 1) + (2 \times 2) + (1 \times 3)$

$\underset{\sim}{v_1} \cdot \underset{\sim}{v_2} = 3 + 4 + 3$

$$= 10$$

$$\underset{\sim}{v_1} \cdot \underset{\sim}{v_2} = |\underset{\sim}{v_1}||\underset{\sim}{v_2}| \cos\theta$$

$$\cos\theta = \frac{\underset{\sim}{v_1} \cdot \underset{\sim}{v_2}}{|\underset{\sim}{v_1}||\underset{\sim}{v_2}|}$$

$$\cos\theta = \frac{10}{\sqrt{14} \times \sqrt{14}}$$

$$\cos\theta = \frac{5}{7}$$

$$\theta = \cos^{-1}\frac{5}{7}$$

$$\theta = 44.42 \ldots°$$

$$\theta \doteqdot 44°$$

4b $\underset{\sim}{v_1} = \begin{bmatrix} 5 \\ 3 \\ -1 \end{bmatrix}$

$$|\underset{\sim}{v_1}|^2 = 5^2 + 3^2 + (-1)^2$$

$$|\underset{\sim}{v_1}|^2 = 25 + 9 + 1$$

$$|\underset{\sim}{v_1}|^2 = 35$$

$$|\underset{\sim}{v_1}| = \sqrt{35}$$

$$\underset{\sim}{v_2} = \begin{bmatrix} -2 \\ 2 \\ -6 \end{bmatrix}$$

$$|\underset{\sim}{v_2}|^2 = (-2)^2 + 2^2 + (-6)^2$$

$$|\underset{\sim}{v_2}|^2 = 4 + 4 + 36$$

$$|\underset{\sim}{v_2}| = \sqrt{44}$$

$$\underset{\sim}{v_1} \cdot \underset{\sim}{v_2} = (5 \times -2) + (3 \times 2) + (-1 \times -6)$$

$$\underset{\sim}{v_1} \cdot \underset{\sim}{v_2} = -10 + 6 + 6$$

$$= 2$$

$$\underset{\sim}{v_1} \cdot \underset{\sim}{v_2} = |\underset{\sim}{v_1}||\underset{\sim}{v_2}| \cos\theta$$

$$\cos\theta = \frac{\underset{\sim}{v_1} \cdot \underset{\sim}{v_2}}{|\underset{\sim}{v_1}||\underset{\sim}{v_2}|}$$

$$\cos\theta = \frac{2}{\sqrt{35} \times \sqrt{44}}$$

$$\cos\theta = \frac{1}{\sqrt{35} \times \sqrt{11}}$$

$$\theta = \cos^{-1}\frac{1}{\sqrt{385}}$$

$$\theta = 87.08\ldots°$$

$$\theta \doteqdot 87°$$

5 $$\underset{\sim}{a} = \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix}$$

$$\underset{\sim}{\hat{\imath}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\underset{\sim}{\hat{\jmath}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\underset{\sim}{\hat{k}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$proj_{\underset{\sim}{\hat{\imath}}}\underset{\sim}{a} = \left(\frac{\underset{\sim}{a} \cdot \underset{\sim}{\hat{\imath}}}{\underset{\sim}{\hat{\imath}} \cdot \underset{\sim}{\hat{\imath}}}\right)\underset{\sim}{\hat{\imath}}$$

So for $proj_{\underset{\sim}{\hat{\imath}}}\underset{\sim}{a} = 3\underset{\sim}{\imath}$

$$\text{LHS} = proj_{\underset{\sim}{\hat{\imath}}}\underset{\sim}{a}$$

$$= \left(\frac{\begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}\right)\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= 3\underset{\sim}{\imath}$$

$$= \text{RHS}$$

$$proj_{\underset{\sim}{j}}\underset{\sim}{a} = \left(\frac{\underset{\sim}{a} \cdot \underset{\sim}{\hat{j}}}{\underset{\sim}{\hat{j}} \cdot \underset{\sim}{\hat{j}}}\right)\underset{\sim}{\hat{j}}$$

So for $proj_{\underset{\sim}{\hat{j}}}\underset{\sim}{a} = -2\underset{\sim}{j}$

LHS $= proj_{\underset{\sim}{\hat{j}}}\underset{\sim}{a}$

$$= \left(\frac{\begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}\right)\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$= -2\underset{\sim}{j}$

$=$ RHS

$$proj_{\underset{\sim}{k}}\underset{\sim}{a} = \left(\frac{\underset{\sim}{a} \cdot \underset{\sim}{\hat{k}}}{\underset{\sim}{\hat{k}} \cdot \underset{\sim}{\hat{k}}}\right)\underset{\sim}{\hat{k}}$$

So for $proj_{\underset{\sim}{\hat{k}}}\underset{\sim}{a} = 5\underset{\sim}{k}$

LHS $= proj_{\underset{\sim}{\hat{k}}}\underset{\sim}{a}$

$$= \left(\frac{\begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}\right)\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$= 5\underset{\sim}{k}$

$=$ RHS

Solutions to Exercise 5D Development questions

6a

$$\underset{\sim}{a} = \underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}$$

$$\underset{\sim}{b} = 2\underset{\sim}{i} - 2\underset{\sim}{j} - \underset{\sim}{k}$$

$$\begin{aligned}\text{proj}_{\underset{\sim}{b}}\underset{\sim}{a} &= \frac{\underset{\sim}{b}\cdot\underset{\sim}{a}}{\underset{\sim}{b}\cdot\underset{\sim}{b}}\underset{\sim}{b} \\ &= \frac{2-2+1}{4+4+1}\underset{\sim}{b} \\ &= \frac{1}{9}\underset{\sim}{b} \\ &= \frac{2}{9}\underset{\sim}{i} - \frac{2}{9}\underset{\sim}{j} - \frac{1}{9}\underset{\sim}{k}\end{aligned}$$

6b

$$\underset{\sim}{a} = \begin{bmatrix}3\\2\\2\end{bmatrix}, \underset{\sim}{b} = \begin{bmatrix}4\\1\\-1\end{bmatrix}$$

$$\begin{aligned}\text{proj}_{\underset{\sim}{b}}\underset{\sim}{a} &= \frac{\underset{\sim}{b}\cdot\underset{\sim}{a}}{\underset{\sim}{b}\cdot\underset{\sim}{b}}\underset{\sim}{b} \\ &= \frac{12+2-2}{16+1+1}\underset{\sim}{b} \\ &= \frac{2}{3}\underset{\sim}{b} \\ &= \begin{bmatrix}\frac{8}{3}\\ \frac{2}{3}\\ -\frac{2}{3}\end{bmatrix}\end{aligned}$$

7a

$$\text{a} = 2\underset{\sim}{i} + 3\underset{\sim}{j} - 2\underset{\sim}{k}$$

$$\underset{\sim}{b} = 4\underset{\sim}{i} - 2\underset{\sim}{j} + 5\underset{\sim}{k}$$

$$\text{proj}_{\underset{\sim}{b}}\underset{\sim}{a} = |\underset{\sim}{a}|\cos\theta$$

$$= |\underset{\sim}{a}|\left(\frac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}\right)$$

$$= \frac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{|\underset{\sim}{b}|}$$

$|\underset{\sim}{b}|^2 = 16 + 4 + 25$

$= 45$

$|\underset{\sim}{b}| = \sqrt{45}$

$= 3\sqrt{5}$

Hence,

$$\text{proj}_{\underset{\sim}{b}}\underset{\sim}{a} = \frac{8-6-10}{3\sqrt{5}}$$

$$= -\frac{8}{3\sqrt{5}}$$

So the length of the projection is $\frac{8}{3\sqrt{5}}$.

7b

$$\text{a} = \begin{bmatrix}1\\1\\3\end{bmatrix}, \underset{\sim}{b} = \begin{bmatrix}8\\4\\1\end{bmatrix}$$

$\text{proj}_{\underset{\sim}{b}}\underset{\sim}{a} = |\underset{\sim}{a}|\cos\theta$

$$= |\underset{\sim}{a}|\left(\frac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}\right)$$

$$= \frac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{|\underset{\sim}{b}|}$$

$|\underset{\sim}{b}|^2 = 64 + 16 + 1$

$= 81$

$|\underset{\sim}{b}| = \sqrt{81}$

$= 9$

Hence,

$$proj_{\underset{\sim}{b}}\underset{\sim}{a} = \frac{8+4+3}{9}$$

$$= \frac{15}{9}$$

$$= \frac{5}{3}$$

So the length of the projection is $\frac{5}{3}$.

8a

$A = (2,\ 7,\ -12)$

$B = (-1,\ 5,\ -5)$

$C = (4,\ 1,\ -4)$

$$\overrightarrow{BA} = \big(2 - (-1)\big)\underset{\sim}{i} + (7 - 5)\underset{\sim}{j} + \big(-12 - (-5)\big)\underset{\sim}{k}$$

$$= 3\underset{\sim}{i} + 2\underset{\sim}{j} - 7\underset{\sim}{k}$$

$$\overrightarrow{BC} = \big(4 - (-1)\big)\underset{\sim}{i} + (1 - 5)\underset{\sim}{j} + \big(-4 - (-5)\big)\underset{\sim}{k}$$

$$= 5\underset{\sim}{i} - 4\underset{\sim}{j} + \underset{\sim}{k}$$

8b

$$\overrightarrow{BA} = 3\underset{\sim}{i} + 2\underset{\sim}{j} - 7\underset{\sim}{k}$$

$$\overrightarrow{BC} = 5\underset{\sim}{i} - 4\underset{\sim}{j} + \underset{\sim}{k}$$

$$\angle ABC = \cos^{-1}\left(\frac{\overrightarrow{BA} \cdot \overrightarrow{BC}}{\left|\overrightarrow{BA}\right|\left|\overrightarrow{BC}\right|}\right)$$

$$\overrightarrow{BA} \cdot \overrightarrow{BC} = 15 - 8 - 7$$

$$= 0$$

Hence,

$$\angle ABC = \cos^{-1} 0$$

$$= 90^\circ$$

8c

$\overrightarrow{AC} = \overrightarrow{BC} - \overrightarrow{BA}$

$= 2\underset{\sim}{\imath} - 6\underset{\sim}{\jmath} + 8\underset{\sim}{k}$

$|\overrightarrow{BA}| = \sqrt{9 + 4 + 49}$

$= \sqrt{62}$

$|\overrightarrow{BC}| = \sqrt{25 + 16 + 1}$

$= \sqrt{42}$

$|\overrightarrow{AC}| = \sqrt{4 + 36 + 64}$

$= \sqrt{104}$

$= 2\sqrt{26}$

Pythagoras' theorem: $|\overrightarrow{BA}|^2 + |\overrightarrow{BC}|^2 = |\overrightarrow{AC}|^2$

$\text{LHS} = |\overrightarrow{BA}|^2 + |\overrightarrow{BC}|^2$

$= (\sqrt{62})^2 + (\sqrt{42})^2$

$= 62 + 42$

$= 104$

$= |\overrightarrow{AC}|^2$

$= \text{RHS}$

So the equation is satisfied.

9a

$\text{A} = (3,\ -3,\ 1)$

$\text{B} = (-2,\ 1,\ 2)$

$\text{C} = (4,\ 0,\ -1)$

$\overrightarrow{AB} = \begin{bmatrix} -2-3 \\ 1-(-3) \\ 2-1 \end{bmatrix}$

$= \begin{bmatrix} -5 \\ 4 \\ 1 \end{bmatrix}$

$$\overrightarrow{AC} = \begin{bmatrix} 4-3 \\ 0-(-3) \\ -1-1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$$

9b

$$\overrightarrow{BA} = \begin{bmatrix} -5 \\ 4 \\ 1 \end{bmatrix}$$

$$\overrightarrow{BC} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$$

$$\angle ABC = \cos^{-1}\left(\frac{\overrightarrow{BA}\cdot\overrightarrow{BC}}{|\overrightarrow{BA}||\overrightarrow{BC}|}\right)$$

Hence,

$$\angle ABC = \cos^{-1}\left(\frac{-5+12-2}{\sqrt{25+16+1}\times\sqrt{1+9+4}}\right)$$

$$= \cos^{-1}\left(\frac{5}{\sqrt{42}\times\sqrt{14}}\right)$$

$$= \cos^{-1}\left(\frac{5}{14\sqrt{3}}\right)$$

$$\simeq 78°$$

10

$$\mathrm{P} = (-4,\ -1,\ 6)$$

$$\mathrm{Q} = (-5,\ 3,\ 4)$$

$$\mathrm{R} = (-3,\ 4,\ -7)$$

$$\overrightarrow{QP} = \underset{\sim}{i} - 4\underset{\sim}{j} + 2\underset{\sim}{k}$$

$$\overrightarrow{QR} = 2\underset{\sim}{i} + \underset{\sim}{j} - 11\underset{\sim}{k}$$

$$\overrightarrow{QP}\cdot\overrightarrow{QR} = |\overrightarrow{QP}||\overrightarrow{QR}|\cos\theta \quad (\text{where } \theta = \angle PQR)$$

$$\overrightarrow{QP}\cdot\overrightarrow{QR} = 2 - 4 - 22$$

$= -24$

$\left|\overrightarrow{QP}\right|^2 = 1^2 + (-4)^2 + (2)^2$

$= 21$

$\left|\overrightarrow{QP}\right| = \sqrt{21}$

$\left|\overrightarrow{QR}\right|^2 = 2^2 + 1^2 + (-11)^2$

$= 126$

$\left|\overrightarrow{QR}\right| = \sqrt{126}$

$= 3\sqrt{14}$

Hence,

$\angle P\text{QR} = \cos^{-1}\left(\dfrac{\overrightarrow{QP} \cdot \overrightarrow{QR}}{\left|\overrightarrow{QP}\right|\left|\overrightarrow{QR}\right|}\right)$

$= \cos^{-1}\left(\dfrac{-24}{\sqrt{21} \times 3\sqrt{14}}\right)$

$= \cos^{-1}\left(\dfrac{-24}{21\sqrt{6}}\right)$

$= \cos^{-1}\left(\dfrac{-8}{7\sqrt{6}}\right)$

$= 117.811\ldots°$

$\simeq 117°49'$

11a

$\text{A} = (1,\ 0,\ -1)$

$\text{B} = (1,\ 1,\ 1)$

$\text{C} = (0,\ 1,\ -1)$

$\overrightarrow{CB} = \underset{\sim}{i} + 2\underset{\sim}{k}$

$\overrightarrow{CA} = \underset{\sim}{i} - \underset{\sim}{j}$

$\overrightarrow{CB} \cdot \overrightarrow{CA} = 1 + 0 + 0$

$= 1$

$$\left|\overrightarrow{CB}\right|^2 = 1^2 + 0^2 + 2^2$$

$$= 5$$

$$\left|\overrightarrow{CB}\right| = \sqrt{5}$$

$$\left|\overrightarrow{CA}\right|^2 = 1^2 + (-1)^2 + 0^2$$

$$= 2$$

$$\left|\overrightarrow{CA}\right| = \sqrt{2}$$

$$\cos\angle ACB = \frac{\overrightarrow{CB}\cdot\overrightarrow{CA}}{\left|\overrightarrow{CB}\right|\left|\overrightarrow{CA}\right|}$$

$$= \frac{1}{\sqrt{5}\times\sqrt{2}}$$

$$= \frac{1}{\sqrt{10}}$$

11b

$$\text{Area} = \frac{1}{2}\text{ab}\sin C$$

For $\Delta A\text{BC}$,

$$\text{Area} = \frac{1}{2}\left|\overrightarrow{CA}\right|\left|\overrightarrow{CB}\right|\sin\angle ACB$$

Since $\cos\angle ACB = \frac{1}{\sqrt{10}}$, $\sin\angle ACB = \frac{3}{\sqrt{10}}$ (using Pythagoras' theorem)

Also, $\left|\overrightarrow{CB}\right| = \sqrt{5}$ and $\left|\overrightarrow{CA}\right| = \sqrt{2}$

$$\text{Area} = \frac{1}{2}\times\sqrt{2}\times\sqrt{5}\times\frac{3}{\sqrt{10}}$$

$$= \frac{3}{2} \text{ square units}$$

12a

$$\text{P} = (-4,\ 3,\ -1)$$

$$\text{A} = (3,\ 2,\ 1)$$

$$\text{B} = (0,\ -4,\ 1)$$

$$\overrightarrow{AP} = \begin{bmatrix} -4-3 \\ 3-2 \\ -1-1 \end{bmatrix}$$

$$= \begin{bmatrix} -7 \\ 1 \\ -2 \end{bmatrix}$$

$$\overrightarrow{AB} = \begin{bmatrix} 0-3 \\ -4-2 \\ 1-1 \end{bmatrix}$$

$$= \begin{bmatrix} -3 \\ -6 \\ 0 \end{bmatrix}$$

12b

$$\overrightarrow{AP} = \underset{\sim}{p}$$

$$= \begin{bmatrix} -7 \\ 1 \\ -2 \end{bmatrix}$$

$$\overrightarrow{AB} = \underset{\sim}{b}$$

$$= \begin{bmatrix} -3 \\ -6 \\ 0 \end{bmatrix}$$

$$\text{proj}_{\underset{\sim}{b}}\, \underset{\sim}{p} = \frac{\underset{\sim}{b} \cdot \underset{\sim}{p}}{\underset{\sim}{b} \cdot \underset{\sim}{b}} \underset{\sim}{b}$$

$$= \frac{(-3 \times -7) + (-6 \times 1) + (0 \times -2)}{(-3 \times -3) + (-6 \times -6) + (0 \times 0)} \times \begin{bmatrix} -3 \\ -6 \\ 0 \end{bmatrix}$$

$$= \frac{21-6}{9+36} \times \begin{bmatrix} -3 \\ -6 \\ 0 \end{bmatrix}$$

$$= \frac{15}{45} \times \begin{bmatrix} -3 \\ -6 \\ 0 \end{bmatrix}$$

$$= \frac{1}{3} \times \begin{bmatrix} -3 \\ -6 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix}$$

12c

$$d = \left|\text{proj}_{\underset{\sim}{b}}\, \underset{\sim}{p} - \underset{\sim}{p}\right|$$

$$\text{proj}_{\underset{\sim}{b}}\, \underset{\sim}{p} = \begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix}$$

$$\underset{\sim}{p} = \begin{bmatrix} -7 \\ 1 \\ -2 \end{bmatrix}$$

$$\text{proj}_{\underset{\sim}{b}}\, \underset{\sim}{p} - \underset{\sim}{p} = \begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix} - \begin{bmatrix} -7 \\ 1 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ -3 \\ 2 \end{bmatrix}$$

$$d^2 = 6^2 + (-3)^2 + 2^2$$

$$= 36 + 9 + 4$$

$$= 49$$

$$d = 7 \text{ units}$$

13a

$$P = (3,\ -2,\ 1)$$

$$A = (1,\ -11,\ -4)$$

$$B = (9,\ 3,\ 8)$$

$$\overrightarrow{AP} = \underset{\sim}{p}$$

$$= \begin{bmatrix} 3-1 \\ -2-(-11) \\ 1-(-4) \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 9 \\ 5 \end{bmatrix}$$

$$\overrightarrow{AB} = \underset{\sim}{b}$$

$$= \begin{bmatrix} 9-1 \\ 3-(-11) \\ 8-(-4) \end{bmatrix}$$

$$= \begin{bmatrix} 8 \\ 14 \\ 12 \end{bmatrix}$$

$$\text{proj}_{\underset{\sim}{b}}\, \underset{\sim}{p} = \frac{\underset{\sim}{b} \cdot \underset{\sim}{p}}{\underset{\sim}{b} \cdot \underset{\sim}{b}} \underset{\sim}{b}$$

$$= \frac{(8 \times 2) + (14 \times 9) + (12 \times 5)}{(8 \times 8) + (14 \times 14) + (12 \times 12)} \times \begin{bmatrix} 8 \\ 14 \\ 12 \end{bmatrix}$$

$$= \frac{16 + 126 + 60}{64 + 196 + 144} \times \begin{bmatrix} 8 \\ 14 \\ 12 \end{bmatrix}$$

$$= \frac{202}{404} \times 2 \begin{bmatrix} 4 \\ 7 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ 7 \\ 6 \end{bmatrix}$$

$$\text{d} = \left|\text{proj}_{\underset{\sim}{b}}\, \underset{\sim}{p} - \underset{\sim}{p}\right|$$

$$\text{proj}_{\underset{\sim}{b}}\, \underset{\sim}{p} - \underset{\sim}{p} = \begin{bmatrix} 4 \\ 7 \\ 6 \end{bmatrix} - \begin{bmatrix} 2 \\ 9 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$$d^2 = (2^2 + (-2)^2 + 1^2)$$

$$= 9$$

$$\text{d} = 3 \text{ units}$$

13b

$$\text{P} = (0,\ 0,\ 3)$$

$$\text{A} = (1,\ 2,\ 1)$$

$$\text{B} = (4,\ 0,\ 0)$$

$$\overrightarrow{AP} = \underset{\sim}{p}$$

$$= \begin{bmatrix} 0 - 1 \\ 0 - 2 \\ 3 - 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix}$$

$$\overrightarrow{AB} = \underset{\sim}{b}$$

$$= \begin{bmatrix} 4-1 \\ 0-2 \\ 0-1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}$$

$$\text{proj}_{\underset{\sim}{b}}\, \underset{\sim}{p} = \frac{\underset{\sim}{b} \cdot \underset{\sim}{p}}{\underset{\sim}{b} \cdot \underset{\sim}{b}} \underset{\sim}{b}$$

$$= \frac{(3 \times -1) + (-2 \times -2) + (-1 \times 2)}{(3 \times 3) + (-2 \times -2) + (-1 \times -1)} \times \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}$$

$$= \frac{-3 + 4 - 2}{9 + 4 + 1} \times \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}$$

$$= -\frac{1}{14} \times \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}$$

$$\text{d} = \left|\text{proj}_{\underset{\sim}{b}}\, \underset{\sim}{p} - \underset{\sim}{p}\right|$$

$$\text{proj}_{\underset{\sim}{b}}\, \underset{\sim}{p} - \underset{\sim}{p} = -\frac{1}{14} \times \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix} - \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix}$$

$$= \frac{1}{14} \begin{bmatrix} 11 \\ 30 \\ -27 \end{bmatrix}$$

$$d^2 = \left(\frac{1}{14}\right)^2 (11^2 + 30^2 + (-27)^2)$$

$$= \frac{1}{196}(121 + 900 + 729)$$

$$= \frac{1750}{196}$$

$$= \frac{125}{14}$$

$$\text{d} = \sqrt{\frac{125}{14}}$$

$= \frac{5\sqrt{70}}{14}$ units

14

$\overrightarrow{AG} = (0 - \text{a})\underset{\sim}{i} + (a - 0)\underset{\sim}{j} + (a - 0)\underset{\sim}{k}$

$= -\text{a}\underset{\sim}{i} + a\underset{\sim}{j} + a\underset{\sim}{k}$

$\overrightarrow{CE} = (\text{a} - 0)\underset{\sim}{i} + (0 - a)\underset{\sim}{j} + (a - 0)\underset{\sim}{k}$

$= \text{a}\underset{\sim}{i} - a\underset{\sim}{j} + a\underset{\sim}{k}$

To find the acute angle we use $\overrightarrow{EC}$.

$\overrightarrow{EC} = -\overrightarrow{CE}$

$= -\text{a}\underset{\sim}{i} + a\underset{\sim}{j} - a\underset{\sim}{k}$

$\overrightarrow{AG} \cdot \overrightarrow{EC} = a^2 + a^2 - a^2$

$= a^2$

$\left|\overrightarrow{AG}\right|^2 = a^2 + (-a)^2 + a^2$

$= 3a^2$

$\left|\overrightarrow{AG}\right| = \sqrt{3a^2}$

$= \sqrt{3}\text{a}$

$\left|\overrightarrow{EC}\right|^2 = (-a)^2 + a^2 + (-a)^2$

$= 3a^2$

$\left|\overrightarrow{EC}\right| = \sqrt{3a^2}$

$= \sqrt{3}\text{a}$

$\cos\theta = \frac{\overrightarrow{AG} \cdot \overrightarrow{EC}}{\left|\overrightarrow{AG}\right|\left|\overrightarrow{EC}\right|}$

$= \frac{a^2}{\sqrt{3}a \times \sqrt{3}a}$

$= \frac{a^2}{3a^2}$

So $\theta = \arccos\left(\frac{1}{3}\right)$

15

$O = (0,\ 0,\ 0)$

$A = (1, 0, 0)$

$B = (1,\ 2,\ 0)$

$C = (0, 2, 0)$

$D = (0,\ 0,\ 3)$

$E = (1, 0, 3)$

$F = (1,\ 2,\ 3)$

$G = (0, 2, 3)$

For $\overrightarrow{OF}$ and $\overrightarrow{BD}$,

$\overrightarrow{OF} = (1-0)\underset{\sim}{i} + (2-0)\underset{\sim}{j} + (3-0)\underset{\sim}{k}$

$\quad = \underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}$

$\overrightarrow{BD} = (0-1)\underset{\sim}{i} + (0-2)\underset{\sim}{j} + (3-0)\underset{\sim}{k}$

$\quad = -\underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{k}$

$\overrightarrow{OF} \cdot \overrightarrow{BD} = -1 - 4 + 9$

$\quad = 4$

$\left|\overrightarrow{OF}\right|^2 = 1^2 + 2^2 + 3^2$

$\quad = 14$

$\left|\overrightarrow{OF}\right| = \sqrt{14}$

$\left|\overrightarrow{BD}\right|^2 = (-1)^2 + (-2)^2 + 3^2$

$\quad = 14$

$\left|\overrightarrow{BD}\right| = \sqrt{14}$

$\cos\theta = \dfrac{\overrightarrow{OF} \cdot \overrightarrow{BD}}{\left|\overrightarrow{OF}\right|\left|\overrightarrow{BD}\right|}$

$$= \frac{4}{\sqrt{14} \times \sqrt{14}}$$

$$= \frac{2}{7}$$

For $\overrightarrow{OF}$ and $\overrightarrow{CE}$,

From previous part, $\overrightarrow{OF} = \underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}$ and $|\overrightarrow{OF}| = \sqrt{14}$

$\overrightarrow{CE} = (1-0)\underset{\sim}{i} + (0-2)\underset{\sim}{j} + (3-0)\underset{\sim}{k}$

$= \underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{k}$

$\overrightarrow{OF} \cdot \overrightarrow{CE} = 1 - 4 + 9$

$= 6$

$|\overrightarrow{CE}|^2 = (1)^2 + (-2)^2 + 3^2$

$= 14$

$|\overrightarrow{CE}| = \sqrt{14}$

$$\cos\theta = \frac{\overrightarrow{OF} \cdot \overrightarrow{CE}}{|\overrightarrow{OF}||\overrightarrow{CE}|}$$

$$= \frac{6}{\sqrt{14} \times \sqrt{14}}$$

$$= \frac{3}{7}$$

For $\overrightarrow{OF}$ and $\overrightarrow{AG}$,

From previous part, $\overrightarrow{OF} = \underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}$ and $|\overrightarrow{OF}| = \sqrt{14}$

$\overrightarrow{AG} = (0-1)\underset{\sim}{i} + (2-0)\underset{\sim}{j} + (3-0)\underset{\sim}{k}$

$= -\underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}$

$\overrightarrow{OF} \cdot \overrightarrow{AG} = -1 + 4 + 9$

$= 12$

$|\overrightarrow{AG}|^2 = (-1)^2 + 2^2 + 3^2$

$$= 14$$

$$\left|\overrightarrow{AG}\right| = \sqrt{14}$$

$$\cos\theta = \frac{\overrightarrow{OE}\cdot\overrightarrow{AD}}{\left|\overrightarrow{OE}\right|\left|\overrightarrow{AD}\right|}$$

$$= \frac{8}{\sqrt{14}\times\sqrt{14}}$$

$$= \frac{6}{7}$$

16

Let a $= |AD| = |DC| = |BD|$

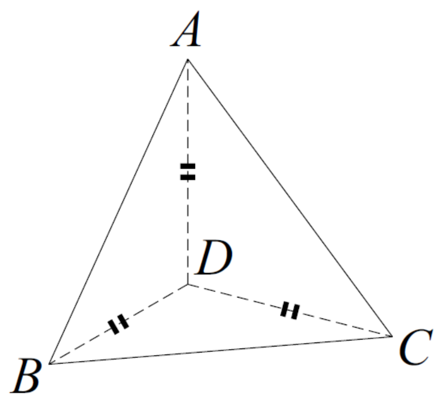
A = (0, 0, a)

B = (0, a, 0)

C = (a, 0, 0)

D = (0, 0, 0)

M is the midpoint between B and C.



$$\text{M} = \left(\frac{0+a}{2}, \frac{a+0}{2}, \frac{0+0}{2}\right)$$

$$= \left(\frac{a}{2}, \frac{a}{2}, 0\right)$$

We want to find the angle made by $\overrightarrow{MD}$ and $\overrightarrow{MA}$.

$$\overrightarrow{MD} = \left(0-\frac{a}{2}\right)\underset{\sim}{i} + \left(0-\frac{a}{2}\right)\underset{\sim}{j} + (0-0)\underset{\sim}{k}$$

$$= -\frac{a}{2}\underset{\sim}{i} - \frac{a}{2}\underset{\sim}{j}$$

$$\overrightarrow{MA} = (0,\ 0,\ a) - \left(\frac{a}{2}, \frac{a}{2}, 0\right)$$

$$= -\frac{a}{2}\underset{\sim}{i} - \frac{a}{2}\underset{\sim}{j} + a\underset{\sim}{k}$$

$$\overrightarrow{MD}\cdot\overrightarrow{MA} = \left(-\frac{a}{2}\times-\frac{a}{2}\right) + \left(-\frac{a}{2}\times-\frac{a}{2}\right) + (a\times 0)$$

$$= \frac{2a^2}{4}$$

$$= \frac{a^2}{2}$$

$$\left|\overrightarrow{MD}\right|^2 = \left(-\frac{a}{2}\right)^2 + \left(-\frac{a}{2}\right)^2$$

$$= \frac{a^2}{4} + \frac{a^2}{4}$$

$$= \frac{a^2}{2}$$

$$\left|\overrightarrow{MD}\right| = \frac{a}{\sqrt{2}}$$

$$\left|\overrightarrow{MA}\right|^2 = \left(-\frac{a}{2}\right)^2 + \left(-\frac{a}{2}\right)^2 + a^2$$

$$= \frac{3a^2}{2}$$

$$\left|\overrightarrow{MA}\right| = \sqrt{\frac{3}{2}}\mathrm{a}$$

$$\cos \angle AMD = \frac{\overrightarrow{MD} \cdot \overrightarrow{MA}}{\left|\overrightarrow{MD}\right|\left|\overrightarrow{MA}\right|}$$

$$= \frac{\left(\frac{a^2}{2}\right)}{\frac{a}{\sqrt{2}} \times \sqrt{\frac{3}{2}}a}$$

$$= \frac{\sqrt{2}}{2} \times \sqrt{\frac{2}{3}}$$

$$= \frac{1}{\sqrt{3}}$$

$$\angle A\mathrm{MD} = \cos^{-1}\frac{1}{\sqrt{3}}$$

17a

$$\overrightarrow{OA} = -5\underset{\sim}{i} + 22\underset{\sim}{j} + 5\underset{\sim}{k}$$

$$\overrightarrow{OB} = \underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}$$

$\overrightarrow{OC} = 4\underset{\sim}{i} + 3\underset{\sim}{j} + 2\underset{\sim}{k}$

$\overrightarrow{OD} = -\underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k}$

$\overrightarrow{BC} = (4-1)\underset{\sim}{i} + (3-2)\underset{\sim}{j} + (2-3)\underset{\sim}{k}$

$= 3\underset{\sim}{i} + \underset{\sim}{j} - 1\underset{\sim}{k}$

$\overrightarrow{BA} = (-5-1)\underset{\sim}{i} + (22-2)\underset{\sim}{j} + (5-3)\underset{\sim}{k}$

$= -6\underset{\sim}{i} + 20\underset{\sim}{j} + 2\underset{\sim}{k}$

$\overrightarrow{BD} = (-1-1)\underset{\sim}{i} + (2-2)\underset{\sim}{j} + (-3-3)\underset{\sim}{k}$

$= -2\underset{\sim}{i} - 6\underset{\sim}{k}$

$\overrightarrow{BC} \cdot \overrightarrow{BD} = (3 \times -2) + (1 \times 0) + (-1 \times -6)$

$= -6 + 6$

$= 0$

$$\cos \angle CBD = \frac{\overrightarrow{BC} \cdot \overrightarrow{BD}}{\left|\overrightarrow{BD}\right|\left|\overrightarrow{BC}\right|}$$

Hence,

$\angle C\mathrm{BD} = \cos^{-1} 0$

$\angle C\mathrm{BD} = 90°$

17b

$\overrightarrow{OA} = -5\underset{\sim}{i} + 22\underset{\sim}{j} + 5\underset{\sim}{k}$

$\overrightarrow{OB} = \underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}$

$\overrightarrow{OC} = 4\underset{\sim}{i} + 3\underset{\sim}{j} + 2\underset{\sim}{k}$

$\overrightarrow{OD} = -\underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k}$

$\overrightarrow{AB} = \big(1-(-5)\big)\underset{\sim}{i} + (2-22)\underset{\sim}{j} + (3-5)\underset{\sim}{k}$

$= 6\underset{\sim}{i} - 20\underset{\sim}{j} - 2\underset{\sim}{k}$

$\overrightarrow{BC} = (4-1)\underset{\sim}{i} + (3-2)\underset{\sim}{j} + (2-3)\underset{\sim}{k}$

$= 3\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}$

$\overrightarrow{BD} = (-1-1)\underset{\sim}{i} + (2-2)\underset{\sim}{j} + (-3-3)\underset{\sim}{k}$

$= -2\underset{\sim}{i} - 6\underset{\sim}{k}$

For $\overrightarrow{AB}$ to be perpendicular to $\overrightarrow{BC}$ and $\overrightarrow{BD}$,

$\overrightarrow{AB} \cdot \overrightarrow{BC} = 0$ and $\overrightarrow{AB} \cdot \overrightarrow{BD} = 0$.

$\overrightarrow{AB} \cdot \overrightarrow{BC} = (6 \times 3) + (-20 \times 1) + (-2 \times -1)$

$= 18 - 20 + 2$

$= 0$

So $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are perpendicular.

$\overrightarrow{AB} \cdot \overrightarrow{BD} = (6 \times -2) + (-20 \times 0) + (-2 \times -6)$

$= -12 + 0 + 12$

$= 0$

So $\overrightarrow{AB}$ and $\overrightarrow{BD}$ are perpendicular.

17c

The volume of a tetrahedron is:

$\text{V} = \dfrac{1}{3} \times X \times Z$

X = Area of the base of the pyramid

Z = Height of the pyramid

We know that $\overrightarrow{AB}$ and $\overrightarrow{BD}$ are perpendicular.

So the lengths $|\overrightarrow{AB}|$, $|\overrightarrow{BD}|$ and $|\overrightarrow{DA}|$ form a right-angled triangle.

$\overrightarrow{AB} = 6\underset{\sim}{i} - 20\underset{\sim}{j} - 2\underset{\sim}{k}$

$|\overrightarrow{AB}|^2 = 6^2 + (-20)^2 + (-2)^2$

$= 36 + 400 + 4$

$= 440$

$|\overrightarrow{AB}| = \sqrt{440}$

$= 2\sqrt{110}$

$\overrightarrow{BD} = -2\underset{\sim}{i} - 6\underset{\sim}{k}$

$\left|\overrightarrow{BD}\right|^2 = (-2)^2 + (-6)^2$

$= 4 + 36$

$= 40$

$\left|\overrightarrow{BD}\right| = \sqrt{40}$

$= 2\sqrt{10}$

So the area of the base will be

$X = \frac{1}{2} \times 2\sqrt{110} \times 2\sqrt{10}$

$= 20\sqrt{11}$ square units

Since $\angle C\text{BD} = 90°$, and $\Delta A\text{BD}$ is perpendicular to $\overrightarrow{BC}$, the height Z will be $\left|\overrightarrow{BC}\right|$.

$\overrightarrow{OB} = \underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}$

$\overrightarrow{OC} = 4\underset{\sim}{i} + 3\underset{\sim}{j} + 2\underset{\sim}{k}$

So $\overrightarrow{BC} = 3\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}$

$\left|\overrightarrow{BC}\right|^2 = 3^2 + 1^2 + (-1)^2$

$= 9 + 1 + 1$

$= 11$

$\left|\overrightarrow{BC}\right| = Z = \sqrt{11}$

So the volume of the tetrahedron is

$V = \frac{1}{3} \times X \times Z$

$= \frac{1}{3} \times 20\sqrt{11}\sqrt{11}$

$= \frac{20}{3} \times 11$

$= \frac{220}{3}$ cubic units

18a

$\overrightarrow{OA} = 2\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}$

$\overrightarrow{OB} = 6\underset{\sim}{i} - 3\underset{\sim}{j} + 2\underset{\sim}{k}$

$\overrightarrow{AB} = 4\underset{\sim}{i} - 4\underset{\sim}{j} + 4\underset{\sim}{k}$

We want to show that: $\overrightarrow{OP} = (2 + 4\lambda)\underset{\sim}{i} + (1 - 4\lambda)\underset{\sim}{j} + (4\lambda - 2)\underset{\sim}{k}$

$$\begin{aligned}\overrightarrow{OP} &= \overrightarrow{OA} + \lambda\overrightarrow{AB} \\ &= \left(2\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}\right) + \lambda\left(4\underset{\sim}{i} - 4\underset{\sim}{j} + 4\underset{\sim}{k}\right) \\ &= (2 + 4\lambda)\underset{\sim}{i} + (1 - 4\lambda)\underset{\sim}{j} + (4\lambda - 2)\underset{\sim}{k}\end{aligned}$$

18b

For $\overrightarrow{OP}$ and $\overrightarrow{AB}$ to be perpendicular: $\overrightarrow{OP} \cdot \overrightarrow{AB} = 0$

$$\begin{aligned}\overrightarrow{OP} \cdot \overrightarrow{AB} &= \left((2 + 4\lambda)\underset{\sim}{i} + (1 - 4\lambda)\underset{\sim}{j} + (4\lambda - 2)\underset{\sim}{k}\right) \cdot \left(4\underset{\sim}{i} - 4\underset{\sim}{j} + 4\underset{\sim}{k}\right) \\ &= 4(2 + 4\lambda) + (-4)(1 - 4\lambda) + 4(4\lambda - 2) \\ &= 16\lambda + 8 + 16\lambda - 4 + 16\lambda - 8 \\ &= 48\lambda - 4 \\ &= 0\end{aligned}$$

$48\lambda = 4$

$$\lambda = \frac{1}{12}$$

18c

$\angle A\mathrm{OP} = \angle B\mathrm{OP}$

$$\cos^{-1}\left(\frac{\overrightarrow{OA} \cdot \overrightarrow{OP}}{|\overrightarrow{OA}||\overrightarrow{OP}|}\right) = \cos^{-1}\left(\frac{\overrightarrow{OB} \cdot \overrightarrow{OP}}{|\overrightarrow{OB}||\overrightarrow{OP}|}\right)$$

$$\frac{\overrightarrow{OA} \cdot \overrightarrow{OP}}{|\overrightarrow{OA}|} = \frac{\overrightarrow{OB} \cdot \overrightarrow{OP}}{|\overrightarrow{OB}|}$$

Note that:

$$\overrightarrow{OA} \cdot \overrightarrow{OP} = \left(2\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}\right) \cdot \left((2 + 4\lambda)\underset{\sim}{i} + (1 - 4\lambda)\underset{\sim}{j} + (4\lambda - 2)\underset{\sim}{k}\right)$$

$$= 2(2 + 4\lambda) + (1 - 4\lambda) - 2(4\lambda - 2)$$

$$= 4 + 8\lambda + 1 - 4\lambda - 8\lambda + 4$$

$$= -4\lambda + 9$$

$$\overrightarrow{OB} \cdot \overrightarrow{OP} = \left(6\underset{\sim}{i} - 3\underset{\sim}{j} + 2\underset{\sim}{k}\right) \cdot \left((2 + 4\lambda)\underset{\sim}{i} + (1 - 4\lambda)\underset{\sim}{j} + (4\lambda - 2)\underset{\sim}{k}\right)$$

$$= 6(2 + 4\lambda) - 3(1 - 4\lambda) + 2(4\lambda - 2)$$

$$= 12 + 24\lambda - 3 + 12\lambda + 8\lambda - 4$$

$$= 44\lambda + 5$$

$$\left|\overrightarrow{OA}\right|^2 = 2^2 + 1^2 + (-2)^2$$

$$= 9$$

$$\left|\overrightarrow{OA}\right| = 3$$

$$\left|\overrightarrow{OB}\right|^2 = 6^2 + (-3)^2 + 2^2$$

$$= 49$$

$$\left|\overrightarrow{OB}\right| = 7$$

Substituting gives:

$$\frac{-4\lambda + 9}{3} = \frac{44\lambda + 5}{7}$$

$$-28\lambda + 63 = 132\lambda + 15$$

$$-160\lambda = -48$$

$$\lambda = \frac{3}{10}$$

19

$$\theta = \cos^{-1} \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}$$

$$= \cos^{-1} \frac{4}{21}$$

Therefore,

$$\cos^{-1}\frac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}=\cos^{-1}\frac{4}{21}$$

$$\frac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}=\frac{4}{21}$$

$$\underset{\sim}{a}=\begin{bmatrix}6\\-2\\3\end{bmatrix}, \underset{\sim}{b}=\begin{bmatrix}-2\\-4\\\lambda\end{bmatrix}$$

$$\underset{\sim}{a}\cdot\underset{\sim}{b}=\begin{bmatrix}-2\\-4\\\lambda\end{bmatrix}\cdot\begin{bmatrix}6\\-2\\3\end{bmatrix}$$

$$=(-12)+(8)+(3\lambda)$$

$$=3\lambda-4$$

$$|\underset{\sim}{a}|^2=6^2+(-2)^2+3^2$$

$$=36+4+9$$

$$=49$$

$$|\underset{\sim}{a}|=7$$

$$|\underset{\sim}{b}|^2=(-2)^2+(-4)^2+\lambda^2$$

$$=4+16+\lambda^2$$

$$=20+\lambda^2$$

$$|\underset{\sim}{b}|=\sqrt{20+\lambda^2}$$

Substituting, and solving for λ

$$\frac{(3\lambda-4)}{7\sqrt{20+\lambda^2}}=\frac{4}{21}$$

$$\sqrt{20+\lambda^2}=\frac{21}{28}(3\lambda-4)$$

$$\sqrt{20+\lambda^2}=\frac{9}{4}\lambda-3$$

$$20+\lambda^2=\left(\frac{9}{4}\lambda-3\right)^2$$

$$20+\lambda^2=\frac{81}{16}\lambda^2-\frac{54}{4}\lambda+9$$

$$0=\frac{81}{16}\lambda^2-\lambda^2-\frac{54}{4}\lambda-11$$

$$0 = \frac{65}{16}\lambda^2 - \frac{54}{4}\lambda - 11$$

$$0 = \frac{65}{16}\lambda^2 - \frac{27}{2}\lambda - 11$$

$$0 = 65\lambda^2 - 216\lambda - 176$$

Factorising the quadratic gives:

$0 = (65\lambda + 44)(\lambda - 4)$

$\lambda = -\dfrac{44}{65}$ or 4

Alternatively, using the quadratic formula:

$$\lambda = \frac{216 \pm \sqrt{(-216)^2 - 4 \times 65 \times -176}}{2 \times 65}$$

$$= \frac{216 \pm \sqrt{92\,416}}{130}$$

$$= \frac{216 \pm 304}{130}$$

$$= \frac{520}{130} \text{ or } -\frac{88}{130}$$

$$= 4 \text{ or } -\frac{44}{65}$$

20a

For A, B and P to be collinear, $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{p}$ all lie on one line.

This is true if $\overrightarrow{AP}$ is parallel to $\overrightarrow{BP}$:

$$\overrightarrow{AP} = \underset{\sim}{p} - \underset{\sim}{a}$$

$$= \lambda\underset{\sim}{a} + (1-\lambda)\underset{\sim}{b} - \underset{\sim}{a}$$

$$= (\lambda - 1)\underset{\sim}{a} + (1-\lambda)\underset{\sim}{b}$$

$$= (\lambda - 1)(\underset{\sim}{a} - \underset{\sim}{b})$$

$$\overrightarrow{BP} = \underset{\sim}{p} - \underset{\sim}{b}$$

$$= \lambda\underset{\sim}{a} + (1-\lambda)\underset{\sim}{b} - \underset{\sim}{b}$$

$$= \lambda\underset{\sim}{a} - \lambda\underset{\sim}{b}$$

$= \lambda(\underset{\sim}{a} - \underset{\sim}{b})$

Since $\overrightarrow{AP}$ and $\overrightarrow{BP}$ are both multiples of $(\underset{\sim}{a} - \underset{\sim}{b})$, they are parallel.

Therefore A, B and P must be collinear.

20b

$\underset{\sim}{a} = \underset{\sim}{i} + \underset{\sim}{j}$

$\underset{\sim}{b} = 4\underset{\sim}{i} - 2\underset{\sim}{j} + 6\underset{\sim}{k}$

$\underset{\sim}{p} = \lambda\underset{\sim}{a} + (1-\lambda)\underset{\sim}{b}$

$\overrightarrow{OA} = \underset{\sim}{i} + \underset{\sim}{j}$

$\overrightarrow{OP} = \lambda\underset{\sim}{a} + (1-\lambda)\underset{\sim}{b}$

$= (\underset{\sim}{i} + \underset{\sim}{j})\lambda + (1-\lambda)(4\underset{\sim}{i} - 2\underset{\sim}{j} + 6\underset{\sim}{k})$

$= \lambda\underset{\sim}{i} + \lambda\underset{\sim}{j} + 4\underset{\sim}{i} - 2\underset{\sim}{j} + 6\underset{\sim}{k} - 4\lambda\underset{\sim}{i} + 2\lambda\underset{\sim}{j} - 6\lambda\underset{\sim}{k}$

$= (4-3\lambda)\underset{\sim}{i} + (3\lambda-2)\underset{\sim}{j} + 6(1-\lambda)\underset{\sim}{k}$

$|\overrightarrow{OA}|^2 = 1 + 1$

$= 2$

$|\overrightarrow{OA}| = \sqrt{2}$

$|\overrightarrow{OP}|^2 = (4-3\lambda)^2 + (3\lambda-2)^2 + 36(1-\lambda)^2$

$= 16 - 24\lambda + 9\lambda^2 + 9\lambda^2 - 12\lambda + 4 + 36 - 72\lambda + 36\lambda^2$

$= 54\lambda^2 - 108\lambda + 56$

$|\overrightarrow{OP}| = \sqrt{2} \times \sqrt{(27\lambda^2 - 54\lambda + 28)}$

$\overrightarrow{OA} \cdot \overrightarrow{OP} = 4 - 3\lambda + 3\lambda - 2$

$= 2$

When $\angle AOP = 60°$,

$$\cos 60° = \frac{\overrightarrow{OA} \cdot \overrightarrow{OP}}{|\overrightarrow{OA}||\overrightarrow{OP}|}$$

$$\frac{1}{2} = \frac{2}{\sqrt{2} \times \sqrt{2} \times \sqrt{(27\lambda^2 - 54\lambda + 28)}}$$

$$2 = \sqrt{(27\lambda^2 - 54\lambda + 28)}$$

$$4 = 27\lambda^2 - 54\lambda + 28$$

$$0 = 27\lambda^2 - 54\lambda + 24$$

$$0 = 9\lambda^2 - 18\lambda + 8$$

$$0 = (3\lambda - 2)(3\lambda - 4)$$

$$\lambda = \frac{2}{3}, \text{ or } \lambda = \frac{4}{3}$$

Solutions to Exercise 5D Enrichment questions

21a $\overrightarrow{CB} = -2\underset{\sim}{i} - 6\underset{\sim}{j} + 6\underset{\sim}{k}$

$\overrightarrow{CA} = 4\underset{\sim}{i} + 12\underset{\sim}{j}$

So, $\overrightarrow{CB} \cdot \overrightarrow{CA} = -80$

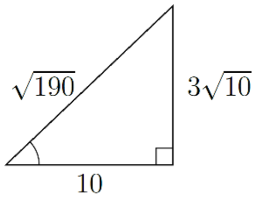
$\cos \angle ACB$

$= \frac{\overrightarrow{CB}.\overrightarrow{CA}}{|CB||CA|}$

$= \frac{-80}{2\sqrt{19} \cdot 4\sqrt{10}}$

$= \frac{-80}{8\sqrt{190}}$

$= \frac{-10}{\sqrt{190}}$, (so $\angle ACB$ is obtuse)



$\sin \angle ACB = \frac{3\sqrt{10}}{\sqrt{190}} = \frac{3}{\sqrt{19}}$

Area $\triangle ABC$

$= \frac{1}{2}|CB||CA| \sin \angle ACB$

$= \frac{1}{2} \cdot 2\sqrt{19} \cdot 4\sqrt{10} \cdot \frac{3}{\sqrt{19}}$

$= 12\sqrt{10}$ u^2

21a [Alternative solution that avoids fractions.]

Area $\triangle ABC$

$= \frac{1}{2}|CA||CB| \sin \angle ACB$

$= \frac{1}{2}|\overrightarrow{CA}||\overrightarrow{CB}|.\sqrt{1 - \cos^2 \angle ACB}$

$= \frac{1}{2}\sqrt{|\overrightarrow{CA}|^2|\overrightarrow{CB}|^2 - (|\overrightarrow{CA}||\overrightarrow{CB}|\cos\angle ACB)^2}$

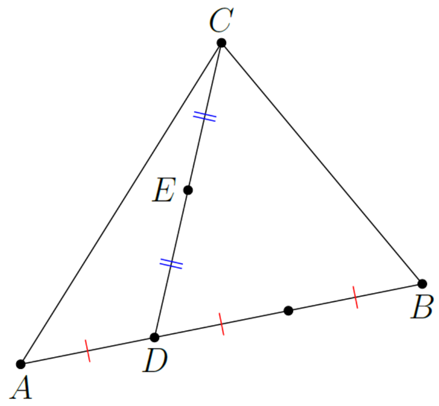
$= \frac{1}{2}\sqrt{(\overrightarrow{CA}.\overrightarrow{CA})(\overrightarrow{CB}.\overrightarrow{CB}) - (\overrightarrow{CA}.\overrightarrow{CB})^2}$

$= \frac{1}{2}\sqrt{160 \times 76 - 80^2}$

$= \frac{1}{2}\sqrt{5760}$

$= 12\sqrt{10}\ \text{u}^2$, as before.

21b $\quad AD: AB = 1:3$



$\overrightarrow{AB} = -6\underset{\sim}{i} - 18\underset{\sim}{j} + 6\underset{\sim}{k}$

$\overrightarrow{AD} = -2\underset{\sim}{i} - 6\underset{\sim}{j} + 2\underset{\sim}{k}$

Hence,

$\overrightarrow{OD}$

$= \overrightarrow{OA} + \overrightarrow{AD}$

$= \left(9\underset{\sim}{i} + 7\underset{\sim}{j} - \underset{\sim}{k}\right) + \left(-2\underset{\sim}{i} - 6\underset{\sim}{j} + 2k\right)$

$= \left(7\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}\right)$

$\overrightarrow{DC} = -2\underset{\sim}{i} - 6\underset{\sim}{j} - 2\underset{\sim}{k}$

$\overrightarrow{DE} = -\underset{\sim}{i} - 3\underset{\sim}{j} - \underset{\sim}{k}$

Hence,

$\overrightarrow{OE}$

$= \overrightarrow{OD} + \overrightarrow{DE}$

$= 6\underset{\sim}{i} - 2\underset{\sim}{j}$

21c $\overrightarrow{OE}.\overrightarrow{AB}$

$= \left(6\underset{\sim}{i} - 2\underset{\sim}{j}\right).\left(-6\underset{\sim}{i} - 18\underset{\sim}{j} + 6\underset{\sim}{k}\right)$

$= -36 + 36$

$= 0$

$\overrightarrow{OE}.\overrightarrow{AC}$

$= \left(6\underset{\sim}{i} - 2\underset{\sim}{j}\right).\left(-4\underset{\sim}{i} - 12\underset{\sim}{j}\right)$

$= -24 + 24$

$= 0$

Hence, $\overrightarrow{OE}$ is perpendicular to both $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

Also, $\overrightarrow{OE}$ is perpendicular to the plane ABC.

21d V

$= \frac{1}{3} \times$ area of base $\times \perp$ height

$= \frac{1}{3} \times$ area of $\triangle\ ABC \times \left|\overrightarrow{OE}\right|$

$= \frac{1}{3} \times 12\sqrt{10} \times 2\sqrt{10}$

$= 80\ \ \text{u}^3$

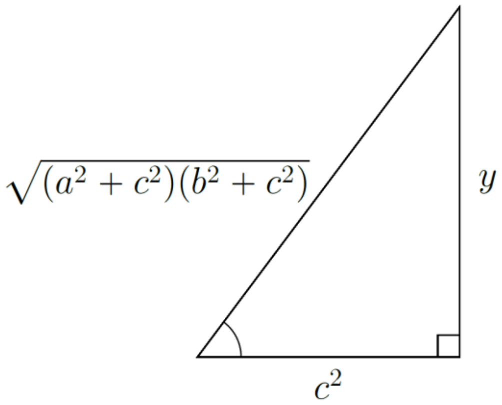
22 $|AOB| = \frac{ab}{2}$, $|AOC| = \frac{ac}{2}$ $|BOC| = \frac{bc}{2}$

$\cos \angle ACB$

$= \frac{\overrightarrow{CA}.\overrightarrow{CB}}{\left|\overrightarrow{CA}\right|\left|\overrightarrow{CB}\right|}$

$= \frac{\begin{bmatrix} a \\ 0 \\ -c \end{bmatrix}.\begin{bmatrix} 0 \\ b \\ -c \end{bmatrix}}{\sqrt{a^2+c^2}.\sqrt{b^2+c^2}}$

$= \frac{c^2}{\sqrt{(a^2+c^2)(b^2+c^2)}}$



$y^2 + c^4 = a^2b^2 + a^2c^2 + b^2c^2 + c^4$

So, $y = \sqrt{a^2b^2 + a^2c^2 + b^2c^2}$

$|ABC|$

$= \frac{1}{2}|\overrightarrow{CA}||\overrightarrow{CB}|.\sin\angle ACB$

$= \frac{1}{2}\cancel{\sqrt{a^2+c^2}.\sqrt{b^2+c^2}}.\frac{\sqrt{a^2b^2+a^2c^2+b^2c^2}}{\cancel{\sqrt{(a^2+c^2)(b^2+c^2)}}}$

$= \frac{1}{2}\sqrt{a^2b^2 + a^2c^2 + b^2c^2}$

Hence,

$|AOB|^2 + |BOC|^2 + |COA|^2$

$= \frac{1}{4}a^2b^2 + \frac{1}{4}a^2c^2 + \frac{1}{4}b^2c^2$

$= \frac{1}{4}(a^2b^2 + a^2c^2 + b^2c^2)$

$= |ABC|^2$

Solutions to Exercise 5E Foundation questions

1 For a quadrilateral $OABC$, where the diagonals OB and AC bisect each other

Let M be the point where the diagonals intersect one another.

Since M bisects both diagonals:

$$\overrightarrow{OM} = \frac{\overrightarrow{OB}}{2} = \overrightarrow{MB}$$

$$\overrightarrow{AM} = \frac{\overrightarrow{AC}}{2} = \overrightarrow{MC}$$

Now we consider two opposite sides of the quadrilateral:

$$\begin{aligned}\overrightarrow{OA} &= \overrightarrow{OM} + \overrightarrow{MA}\\ &= \overrightarrow{OM} - \overrightarrow{AM}\\ &= \frac{\overrightarrow{OB}}{2} - \frac{\overrightarrow{AC}}{2}\end{aligned}$$

$$\begin{aligned}\overrightarrow{CB} &= \overrightarrow{CM} + \overrightarrow{MB}\\ &= -\overrightarrow{MC} + \overrightarrow{MB}\\ &= -\frac{\overrightarrow{AC}}{2} + -\frac{\overrightarrow{OB}}{2}\\ &= \overrightarrow{OA}\end{aligned}$$

Therefore sides OA and CB are parallel and equal. This is sufficient to prove that $OABC$ is a parallelogram.

2a $\underset{\sim}{a} \cdot \underset{\sim}{c} = 0$ as they are perpendicular.

2b $\overrightarrow{OB} = \underset{\sim}{a} + \underset{\sim}{c}$

$\overrightarrow{AC} = \underset{\sim}{c} - \underset{\sim}{a}$

2c Since the diagonals are perpendicular, $\overrightarrow{OB} \cdot \overrightarrow{AC} = 0$

Therefore

$(\underset{\sim}{a} + \underset{\sim}{c}) \cdot (\underset{\sim}{c} - \underset{\sim}{a}) = 0$

Expanding:

$\underset{\sim}{a} \cdot \underset{\sim}{c} - \underset{\sim}{a} \cdot \underset{\sim}{a} + \underset{\sim}{c} \cdot \underset{\sim}{c} - \underset{\sim}{c} \cdot \underset{\sim}{a} = 0$

From part a:

$\underset{\sim}{a} \cdot \underset{\sim}{c} = \underset{\sim}{c} \cdot \underset{\sim}{a} = 0$

Therefore

$\underset{\sim}{c} \cdot \underset{\sim}{c} - \underset{\sim}{a} \cdot \underset{\sim}{a} = 0$

$|\underset{\sim}{c}|^2 - |\underset{\sim}{a}|^2 = 0$

$|\underset{\sim}{c}|^2 = |\underset{\sim}{a}|^2$

$|\underset{\sim}{c}| = |\underset{\sim}{a}|$

Therefore $OABC$ is a rectangle whose sides are equal, making it a square.

3a $\overrightarrow{OA} = \underset{\sim}{a}$

$\overrightarrow{OB} = \underset{\sim}{b}$

$\overrightarrow{OM} = \underset{\sim}{m}$

$|\overrightarrow{OA}| = |\overrightarrow{OB}|$ as both $|\overrightarrow{OA}|$ and $|\overrightarrow{OB}|$ are the radius of the circle.

For $\underset{\sim}{a} \cdot \underset{\sim}{a} = \underset{\sim}{b} \cdot \underset{\sim}{b}$

$$\begin{aligned} \text{LHS} &= \underset{\sim}{a} \cdot \underset{\sim}{a} \\ &= |\overrightarrow{OA}|^2 \\ &= |\overrightarrow{OB}|^2 \\ &= \underset{\sim}{b} \cdot \underset{\sim}{b} \\ &= \text{RHS} \end{aligned}$$

Thus $\underset{\sim}{a} \cdot \underset{\sim}{a} = \underset{\sim}{b} \cdot \underset{\sim}{b}$.

3b $\overrightarrow{OA} = \underset{\sim}{a}$

$\overrightarrow{OB} = \underset{\sim}{b}$

$\overrightarrow{OM} = \underset{\sim}{m}$

Therefore:

$\overrightarrow{AM} = \underset{\sim}{m} - \underset{\sim}{a}$

$(\underset{\sim}{m} - \underset{\sim}{a}) \cdot (\underset{\sim}{m} - \underset{\sim}{a}) = \left|\overrightarrow{AM}\right|^2$

Similarly:

$\overrightarrow{BM} = \underset{\sim}{m} - \underset{\sim}{b}$

$(\underset{\sim}{m} - \underset{\sim}{b}) \cdot (\underset{\sim}{m} - \underset{\sim}{b}) = \left|\overrightarrow{BM}\right|^2$

Because M bisects AB, we know that $\left|\overrightarrow{AM}\right| = \left|\overrightarrow{BM}\right|$.

Therefore $\left|\overrightarrow{AM}\right|^2 = \left|\overrightarrow{BM}\right|^2$

Therefore $(\underset{\sim}{m} - \underset{\sim}{a}) \cdot (\underset{\sim}{m} - \underset{\sim}{a}) = (\underset{\sim}{m} - \underset{\sim}{b}) \cdot (\underset{\sim}{m} - \underset{\sim}{b})$.

3c For $\overrightarrow{OM} \perp \overrightarrow{AB}$, it is sufficient to prove that $\overrightarrow{OM} \cdot \overrightarrow{AB} = 0$

From part b:

$(\underset{\sim}{m} - \underset{\sim}{a}) \cdot (\underset{\sim}{m} - \underset{\sim}{a}) = (\underset{\sim}{m} - \underset{\sim}{b}) \cdot (\underset{\sim}{m} - \underset{\sim}{b})$

$\underset{\sim}{m} \cdot \underset{\sim}{m} - 2\underset{\sim}{m} \cdot \underset{\sim}{a} + \underset{\sim}{a} \cdot \underset{\sim}{a} = \underset{\sim}{m} \cdot \underset{\sim}{m} - 2\underset{\sim}{m} \cdot \underset{\sim}{b} + \underset{\sim}{b} \cdot \underset{\sim}{b}$

$-2\underset{\sim}{m} \cdot \underset{\sim}{a} = -2\underset{\sim}{m} \cdot \underset{\sim}{b}$ (using the fact that $\underset{\sim}{a} \cdot \underset{\sim}{a} = \underset{\sim}{b} \cdot \underset{\sim}{b}$)

$\underset{\sim}{m} \cdot \underset{\sim}{a} = \underset{\sim}{m} \cdot \underset{\sim}{b}$

$\overrightarrow{OM} = \underset{\sim}{m}$ and $\overrightarrow{AB} = \underset{\sim}{b} - \underset{\sim}{a}$

$\overrightarrow{OM} \cdot \overrightarrow{AB}$

$= \underset{\sim}{m} \cdot (\underset{\sim}{b} - \underset{\sim}{a})$

$= \underset{\sim}{m} \cdot \underset{\sim}{b} - \underset{\sim}{m} \cdot \underset{\sim}{a}$

$= 0$

Therefore $\overrightarrow{OM} \perp \overrightarrow{AB}$.

4 The converse result is that if M lies on the chord AB and $\overrightarrow{OM} \perp \overrightarrow{AB}$, then M bisects AB.

We have:

$\overrightarrow{OA} = \underset{\sim}{a}$

$\overrightarrow{OB} = \underset{\sim}{b}$

$\overrightarrow{OM} = \underset{\sim}{m}$

$\underset{\sim}{a} \cdot \underset{\sim}{a} = \underset{\sim}{b} \cdot \underset{\sim}{b}$

and since $\overrightarrow{OM} \perp \overrightarrow{AB}$, we have

$\overrightarrow{OM} \cdot \overrightarrow{AB} = 0$

Therefore

$\underset{\sim}{m} \cdot (\underset{\sim}{b} - \underset{\sim}{a}) = 0$

$\underset{\sim}{m} \cdot \underset{\sim}{b} = \underset{\sim}{m} \cdot \underset{\sim}{a}$

$(\underset{\sim}{m} - \underset{\sim}{a}) \cdot (\underset{\sim}{m} - \underset{\sim}{a})$

$= \underset{\sim}{m} \cdot \underset{\sim}{m} - 2\underset{\sim}{m} \cdot \underset{\sim}{a} + \underset{\sim}{a} \cdot \underset{\sim}{a}$

$= \underset{\sim}{m} \cdot \underset{\sim}{m} - 2\underset{\sim}{m} \cdot \underset{\sim}{b} + \underset{\sim}{b} \cdot \underset{\sim}{b}$ (using $\underset{\sim}{m} \cdot \underset{\sim}{b} = \underset{\sim}{m} \cdot \underset{\sim}{a}$ and $\underset{\sim}{a} \cdot \underset{\sim}{a} = \underset{\sim}{b} \cdot \underset{\sim}{b}$)

$= (\underset{\sim}{m} - \underset{\sim}{b}) \cdot (\underset{\sim}{m} - \underset{\sim}{b})$

Therefore

$(\underset{\sim}{m} - \underset{\sim}{a}) \cdot (\underset{\sim}{m} - \underset{\sim}{a}) = (\underset{\sim}{m} - \underset{\sim}{b}) \cdot (\underset{\sim}{m} - \underset{\sim}{b})$

$|(\underset{\sim}{m} - \underset{\sim}{a})|^2 = |(\underset{\sim}{m} - \underset{\sim}{b})|^2$

$|\overrightarrow{AM}| = |\overrightarrow{BM}|$

Therefore M bisects AB.

Solutions to Exercise 5E Development questions

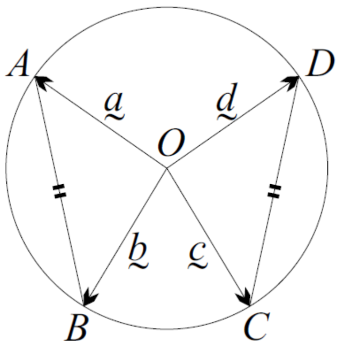
5a

For $|\underset{\sim}{b} - \underset{\sim}{a}|^2 = |\underset{\sim}{d} - \underset{\sim}{c}|^2$

Let

$\underset{\sim}{b} - \underset{\sim}{a} = \overrightarrow{AB}$

$\underset{\sim}{d} - \underset{\sim}{c} = \overrightarrow{DC}$



From the diagram,

$|\overrightarrow{AB}| = |\overrightarrow{DC}|$

Hence,

$$\begin{aligned}\text{LHS} &= |\underset{\sim}{b} - \underset{\sim}{a}|^2 \\ &= |\overrightarrow{AB}|^2 \\ &= |\overrightarrow{DC}|^2 \\ &= |\underset{\sim}{d} - \underset{\sim}{c}|^2 \\ &= \text{RHS}\end{aligned}$$

So the equation is satisfied.

5b

For $\underset{\sim}{a} \cdot \underset{\sim}{b} = \underset{\sim}{d} \cdot \underset{\sim}{c}$

Using the result from part a,

$$|\underset{\sim}{b} - \underset{\sim}{a}|^2 = |\underset{\sim}{d} - \underset{\sim}{c}|^2$$

$$|\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2 - 2\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{d}|^2 + |\underset{\sim}{c}|^2 - 2\underset{\sim}{d} \cdot \underset{\sim}{c}$$

$$2\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2 - |\underset{\sim}{d}|^2 - |\underset{\sim}{c}|^2 + 2\underset{\sim}{d} \cdot \underset{\sim}{c}$$

But since they are radial,

$$\begin{aligned}|\underset{\sim}{a}| &= |\underset{\sim}{b}| \\ &= |\underset{\sim}{c}| \\ &= |\underset{\sim}{d}|\end{aligned}$$

Hence,

$2\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}|^2 + |\underset{\sim}{a}|^2 - |\underset{\sim}{a}|^2 - |\underset{\sim}{a}|^2 + 2\underset{\sim}{d} \cdot \underset{\sim}{c}$

$\underset{\sim}{a} \cdot \underset{\sim}{b} = \underset{\sim}{d} \cdot \underset{\sim}{c}$

So the equation is satisfied.

5c

For $\angle AOB = \angle COD$

$\text{LHS} = \angle AOB$

$= \cos^{-1} \dfrac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}$

$= \cos^{-1} \dfrac{\underset{\sim}{c} \cdot \underset{\sim}{d}}{|\underset{\sim}{c}||\underset{\sim}{d}|}$

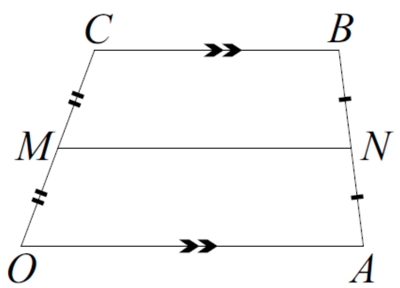
$= \angle COD$

$= \text{RHS}$

So the equation is satisfied.

6a

$\overrightarrow{MN} = \dfrac{1}{2}(\underset{\sim}{a} + \underset{\sim}{b} - \underset{\sim}{c})$



Let

$\overrightarrow{ON} = \dfrac{1}{2}(\underset{\sim}{b} + \underset{\sim}{a})$

$\overrightarrow{OM} = \dfrac{1}{2}\underset{\sim}{c}$

Hence,

$LHS = \overrightarrow{MN}$

$= \overrightarrow{ON} - \overrightarrow{OM}$

$= \dfrac{1}{2}(\underset{\sim}{b} + \underset{\sim}{a}) - \dfrac{1}{2}\underset{\sim}{c}$

$= \dfrac{1}{2}(\underset{\sim}{b} + \underset{\sim}{a} - \underset{\sim}{c})$

$= \text{RHS}$

So the equation is satisfied.

6b

For $\underset{\sim}{b} - \underset{\sim}{c} = k\underset{\sim}{a}$

As $\overrightarrow{CB}$ and $\overrightarrow{OA}$ are parallel there must exist a constant value k whereby

$\overrightarrow{CB} = k\overrightarrow{OA}$

Where

$\overrightarrow{CB} = \underset{\sim}{b} - \underset{\sim}{c}$

$\overrightarrow{OA} = \underset{\sim}{a}$

Hence,

$\underset{\sim}{b} - \underset{\sim}{c} = k\underset{\sim}{a}$

So the equation is satisfied

6c

$$\overrightarrow{MN} = \frac{1}{2}(\underset{\sim}{a} + \underset{\sim}{b} - \underset{\sim}{c})$$

$$= \frac{1}{2}(\underset{\sim}{a} + k\underset{\sim}{a})$$

$$= \frac{1}{2}\underset{\sim}{a}(1 + k)$$

This means there exists a constant value $k' = \frac{1}{2}(1 + k)$, for which

$\overrightarrow{MN} = k'\overrightarrow{OA}$

Therefore, $\overrightarrow{MN}$ is parallel to $\overrightarrow{OA}$ and by extension $\overrightarrow{CB}$.

7

For the shape $MNPR$,

Let

$\overrightarrow{OA} = \underset{\sim}{a}$

$\overrightarrow{OB} = \underset{\sim}{b}$

$\overrightarrow{OC} = \underset{\sim}{c}$

Because M is the midpoint of OA:

$$\overrightarrow{OM} = \frac{1}{2}\left(\overrightarrow{OA}\right)$$

$$= \frac{1}{2}\underset{\sim}{a}$$

Because N is the midpoint of AB:

$$\overrightarrow{ON} = \frac{1}{2}\left(\overrightarrow{OB} + \overrightarrow{OA}\right)$$

$$= \frac{1}{2}(\underset{\sim}{b} + \underset{\sim}{a})$$

Because P is the midpoint of BC:

$$\overrightarrow{OP} = \frac{1}{2}\left(\overrightarrow{OC} + \overrightarrow{OB}\right)$$

$$= \frac{1}{2}(\underset{\sim}{c} + \underset{\sim}{b})$$

Because R is the midpoint of CO:

$$\overrightarrow{OR} = \frac{1}{2}\left(\overrightarrow{OC}\right)$$

$$= \frac{1}{2}\underset{\sim}{c}$$

For $MNPR$ to be a parallelogram, both pairs of opposite sides are parallel and equal in length such that $\overrightarrow{MN} = \overrightarrow{RP}$ and $\overrightarrow{NP} = \overrightarrow{MR}$.

$$\overrightarrow{MN} = \overrightarrow{ON} - \overrightarrow{OM}$$

$$= \frac{1}{2}(\underset{\sim}{b} + \underset{\sim}{a}) - \frac{1}{2}\underset{\sim}{a}$$

$$= \frac{1}{2}\underset{\sim}{b}$$

$$\overrightarrow{RP} = \overrightarrow{OP} - \overrightarrow{OR}$$

$$= \frac{1}{2}(\underset{\sim}{c} + \underset{\sim}{b}) - \frac{1}{2}\underset{\sim}{c}$$

$$= \frac{1}{2}\underset{\sim}{b}$$

$$= \overrightarrow{MN}$$

Likewise,

$$\overrightarrow{NP} = \overrightarrow{OP} - \overrightarrow{ON}$$

$$= \frac{1}{2}(\underset{\sim}{c} + \underset{\sim}{b}) - \frac{1}{2}(\underset{\sim}{b} + \underset{\sim}{a})$$

$$= \frac{1}{2}(\underset{\sim}{c} - \underset{\sim}{a})$$

$$\overrightarrow{MR} = \overrightarrow{OR} - \overrightarrow{OM}$$

$$= \frac{1}{2}\underset{\sim}{c} - \frac{1}{2}\underset{\sim}{a}$$

$$= \frac{1}{2}(\underset{\sim}{c} - \underset{\sim}{a})$$

$$= \overrightarrow{NP}$$

Therefore, $MNPR$ is a parallelogram

8a

We want to find $\angle FOD$.

$$\angle FOD = \cos^{-1}\frac{\overrightarrow{OF}\cdot\overrightarrow{OD}}{|\overrightarrow{OF}||\overrightarrow{OD}|}$$

Where

$$\overrightarrow{OD} = \begin{bmatrix} 0 \\ a \\ a \end{bmatrix}$$

$$\overrightarrow{OF} = \begin{bmatrix} a \\ 0 \\ a \end{bmatrix}$$

Now,

$$|\overrightarrow{OD}|^2 = a^2 + a^2$$

$$|\overrightarrow{OD}| = \sqrt{2}a$$

$$|\overrightarrow{OF}|^2 = a^2 + a^2$$

$$|\overrightarrow{OF}| = \sqrt{2}a$$

$$\overrightarrow{OF}\cdot\overrightarrow{OD} = (0 \times a) + (a \times 0) + (a \times a)$$

$$= a^2$$

Hence,

$$\angle FOD = \cos^{-1}\left(\frac{a^2}{\sqrt{2}a \times \sqrt{2}a}\right)$$

$$= \cos^{-1}\left(\frac{1}{2}\right)$$

$$= 60°$$

8b

The triangle $\angle FOD$ is an equilateral triangle as all interior angles of the triangle are 60°.

8c

The triangular pyramid $OBDF$ is a regular tetrahedron, as each face of the tetrahedron is an equilateral triangle.

8d

We want to find $\angle FXD$.

$$\angle FXD = \cos^{-1}\frac{\overrightarrow{XF} \cdot \overrightarrow{XD}}{|\overrightarrow{XF}||\overrightarrow{XD}|}$$

Where

$$\overrightarrow{OX} = \begin{bmatrix}\frac{1}{2}a \\ \frac{1}{2}a \\ \frac{1}{2}a\end{bmatrix}$$

$$\overrightarrow{XF} = \begin{bmatrix}a \\ 0 \\ a\end{bmatrix} - \begin{bmatrix}\frac{1}{2}a \\ \frac{1}{2}a \\ \frac{1}{2}a\end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}a \\ -\frac{1}{2}a \\ \frac{1}{2}a \end{bmatrix}$$

$$\overrightarrow{XD} = \begin{bmatrix} -\frac{1}{2}a \\ \frac{1}{2}a \\ \frac{1}{2}a \end{bmatrix}$$

Now,

$$\left|\overrightarrow{XD}\right|^2 = \frac{1}{4}a^2 + \frac{1}{4}a^2 + \frac{1}{4}a^2$$

$$= \frac{3}{4}a^2$$

$$\left|\overrightarrow{XD}\right| = \frac{\sqrt{3}}{2}a$$

$$\left|\overrightarrow{XF}\right|^2 = \frac{1}{4}a^2 + \frac{1}{4}a^2 + \frac{1}{4}a^2$$

$$= \frac{3}{4}a^2$$

$$\left|\overrightarrow{XF}\right| = \frac{\sqrt{3}}{2}a$$

$$\overrightarrow{XF} \cdot \overrightarrow{XD} = \left(-\frac{1}{2}a \times \frac{1}{2}a\right) + \left(-\frac{1}{2}a \times \frac{1}{2}a\right) + \left(\frac{1}{2}a \times \frac{1}{2}a\right)$$

$$= -\frac{1}{4}a^2$$

Hence,

$$\angle FXD = \cos^{-1}\left(\frac{-\frac{1}{4}a^2}{\frac{\sqrt{3}}{2}a \times \frac{\sqrt{3}}{2}a}\right)$$

$$= \cos^{-1}\left(-\frac{1}{3}\right)$$

$$= 109.471\ldots^\circ$$

$$\approx 109^\circ\ 28'$$

9a

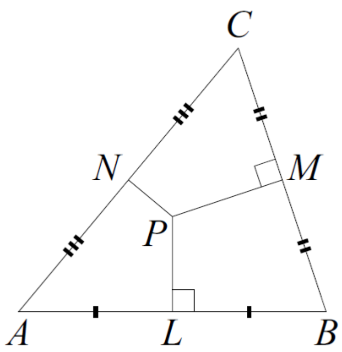
$\overrightarrow{PN} = \overrightarrow{BN} - \overrightarrow{BP}$

Note that

$\overrightarrow{AB} = \underset{\sim}{u}$

$\overrightarrow{BC} = \underset{\sim}{v}$

$\overrightarrow{PL} = \underset{\sim}{w}$



First calculate $\overrightarrow{BN}$ and $\overrightarrow{BP}$:

$$\begin{aligned}\overrightarrow{BN} &= \frac{1}{2}\left(\overrightarrow{BC} + \overrightarrow{BA}\right)\\ &= \frac{1}{2}\left(\overrightarrow{BC} - \overrightarrow{AB}\right)\\ &= \frac{1}{2}(\underset{\sim}{v} - \underset{\sim}{u})\end{aligned}$$

$$\begin{aligned}\overrightarrow{BP} &= \overrightarrow{BL} + \overrightarrow{LP}\\ &= -\overrightarrow{LB} - \overrightarrow{PL}\\ &= -\frac{1}{2}\overrightarrow{AB} - \overrightarrow{PL}\\ &= -\frac{1}{2}\underset{\sim}{u} - \underset{\sim}{w}\end{aligned}$$

Hence,

$$\begin{aligned}\overrightarrow{PN} &= \overrightarrow{BN} - \overrightarrow{BP}\\ &= \frac{1}{2}(\underset{\sim}{v} - \underset{\sim}{u}) - \left(-\frac{1}{2}\underset{\sim}{u} - \underset{\sim}{w}\right)\\ &= \frac{1}{2}\underset{\sim}{v} - \frac{1}{2}\underset{\sim}{u} + \frac{1}{2}\underset{\sim}{u} + \underset{\sim}{w}\\ &= \frac{1}{2}\underset{\sim}{v} + \underset{\sim}{w}\end{aligned}$$

9b

For $\overrightarrow{PN} \cdot \overrightarrow{AC} = 0$

Because $\overrightarrow{AB} \perp \overrightarrow{PL}$,

$\overrightarrow{AB} \cdot \overrightarrow{PL} = 0$

$\underset{\sim}{u} \cdot \underset{\sim}{w} = 0$

Likewise, because $\overrightarrow{BC} \perp \overrightarrow{MP}$,

$$\overrightarrow{BC} \cdot \overrightarrow{MP} = 0$$

$$\overrightarrow{BC} \cdot \left(\overrightarrow{MB} + \overrightarrow{BP}\right) = 0$$

$$\overrightarrow{BC} \cdot \left(-\frac{1}{2}\overrightarrow{BC} + \overrightarrow{BP}\right) = 0$$

$$\underset{\sim}{v} \cdot \left(-\frac{1}{2}\underset{\sim}{v} - \frac{1}{2}\underset{\sim}{u} - \underset{\sim}{w}\right) = 0$$

$$-\frac{1}{2}\underset{\sim}{v} \cdot \underset{\sim}{v} - \frac{1}{2}\underset{\sim}{v} \cdot \underset{\sim}{u} - \underset{\sim}{v} \cdot \underset{\sim}{w} = 0$$

$$\frac{1}{2}\underset{\sim}{v} \cdot \underset{\sim}{v} + \frac{1}{2}\underset{\sim}{v} \cdot \underset{\sim}{u} + \underset{\sim}{v} \cdot \underset{\sim}{w} = 0$$

For the triangle ΔABC to be concurrent, $\overrightarrow{PN} \perp \overrightarrow{AC}$ and $\overrightarrow{PN} \cdot \overrightarrow{AC} = 0$.

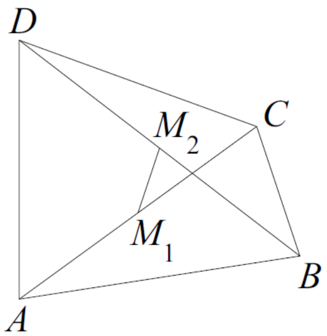
$$\begin{aligned}
\text{LHS} &= \overrightarrow{PN} \cdot \overrightarrow{AC} \\
&= \overrightarrow{PN} \cdot (\overrightarrow{AB} + \overrightarrow{BC}) \\
&= \left(\frac{1}{2}\underset{\sim}{v} + \underset{\sim}{w}\right) \cdot (\underset{\sim}{u} + \underset{\sim}{v}) \\
&= \frac{1}{2}\underset{\sim}{v} \cdot \underset{\sim}{u} + \underset{\sim}{w} \cdot \underset{\sim}{u} + \frac{1}{2}\underset{\sim}{v} \cdot \underset{\sim}{v} + \underset{\sim}{w} \cdot \underset{\sim}{v} \\
&= (\underset{\sim}{w} \cdot \underset{\sim}{u}) + \left(\frac{1}{2}\underset{\sim}{v} \cdot \underset{\sim}{v} + \frac{1}{2}\underset{\sim}{v} \cdot \underset{\sim}{u} + \underset{\sim}{w} \cdot \underset{\sim}{v}\right) \\
&= 0 \\
&= \text{RHS}
\end{aligned}$$

Thus, the perpendicular bisectors of the sides of ΔABC are concurrent.

10

For $|\overrightarrow{AB}|^2 + |\overrightarrow{BC}|^2 + |\overrightarrow{CD}|^2 + |\overrightarrow{DA}|^2 = |\overrightarrow{AC}|^2 + |\overrightarrow{BD}|^2 + 4|\overrightarrow{M_1M_2}|^2$

Note that:



$$|\overrightarrow{AB}|^2 = \overrightarrow{AB} \cdot \overrightarrow{AB}$$

$$= \underset{\sim}{b} \cdot \underset{\sim}{b}$$

$$|\overrightarrow{BC}|^2 = \overrightarrow{BC} \cdot \overrightarrow{BC}$$

$$= (\overrightarrow{AC} - \overrightarrow{AB}) \cdot (\overrightarrow{AC} - \overrightarrow{AB})$$

$$= (\underset{\sim}{c} - \underset{\sim}{b}) \cdot (\underset{\sim}{c} - \underset{\sim}{b})$$

$$= \underset{\sim}{c} \cdot \underset{\sim}{c} - 2\underset{\sim}{c} \cdot \underset{\sim}{b} + \underset{\sim}{b} \cdot \underset{\sim}{b}$$

$$|\overrightarrow{CD}|^2 = \overrightarrow{CD} \cdot \overrightarrow{CD}$$

$$= (\overrightarrow{AD} - \overrightarrow{AC}) \cdot (\overrightarrow{AD} - \overrightarrow{AC})$$

$$= (\underset{\sim}{d} - \underset{\sim}{c}) \cdot (\underset{\sim}{d} - \underset{\sim}{c})$$

$$= \underset{\sim}{d} \cdot \underset{\sim}{d} - 2\underset{\sim}{d} \cdot \underset{\sim}{c} + \underset{\sim}{c} \cdot \underset{\sim}{c}$$

$$|\overrightarrow{DA}|^2 = |\overrightarrow{AD}|^2$$

$$= \overrightarrow{AD} \cdot \overrightarrow{AD}$$

$$= \underset{\sim}{d} \cdot \underset{\sim}{d}$$

$$|\overrightarrow{AC}|^2 = \overrightarrow{AC} \cdot \overrightarrow{AC}$$

$$= \underset{\sim}{c} \cdot \underset{\sim}{c}$$

$$|\overrightarrow{BD}|^2 = \overrightarrow{BD} \cdot \overrightarrow{BD}$$

$$= (\overrightarrow{AD} - \overrightarrow{AB}) \cdot (\overrightarrow{AD} - \overrightarrow{AB})$$

$$= (\underset{\sim}{d} - \underset{\sim}{b}) \cdot (\underset{\sim}{d} - \underset{\sim}{b})$$

$$= \underset{\sim}{d} \cdot \underset{\sim}{d} - 2\underset{\sim}{d} \cdot \underset{\sim}{b} + \underset{\sim}{b} \cdot \underset{\sim}{b}$$

For M_1 and M_2 where M_1 is the centre of AC and M_2 is the centre of BD,

$$\overrightarrow{AM_1} = \frac{1}{2}\overrightarrow{AC}$$

$$= \frac{1}{2}\underset{\sim}{c}$$

$\overrightarrow{AM_2} = \frac{1}{2}(\overrightarrow{AD} + \overrightarrow{AB})$

$= \frac{1}{2}(\underset{\sim}{d} + \underset{\sim}{b})$

$\overrightarrow{M_1M_2} = \overrightarrow{AM_2} - \overrightarrow{AM_1}$

$= \frac{1}{2}(\underset{\sim}{d} + \underset{\sim}{b}) - \frac{1}{2}\underset{\sim}{c}$

$= \frac{1}{2}(\underset{\sim}{d} + \underset{\sim}{b} - \underset{\sim}{c})$

$\left|\overrightarrow{M_1M_2}\right|^2 = \overrightarrow{M_1M_2} \cdot \overrightarrow{M_1M_2}$

$= \frac{1}{2}(\underset{\sim}{d} + \underset{\sim}{b} - \underset{\sim}{c}) \cdot \frac{1}{2}(\underset{\sim}{d} + \underset{\sim}{b} - \underset{\sim}{c})$

$= \frac{1}{4}(\underset{\sim}{d} \cdot \underset{\sim}{d} + \underset{\sim}{d} \cdot \underset{\sim}{b} - \underset{\sim}{d} \cdot \underset{\sim}{c} + \underset{\sim}{b} \cdot \underset{\sim}{d} + \underset{\sim}{b} \cdot \underset{\sim}{b} - \underset{\sim}{b} \cdot \underset{\sim}{c} - \underset{\sim}{c} \cdot \underset{\sim}{d} - \underset{\sim}{c} \cdot \underset{\sim}{b} + \underset{\sim}{c} \cdot \underset{\sim}{c})$

$= \frac{1}{4}(\underset{\sim}{b} \cdot \underset{\sim}{b} + \underset{\sim}{c} \cdot \underset{\sim}{c} + \underset{\sim}{d} \cdot \underset{\sim}{d} + 2\underset{\sim}{d} \cdot \underset{\sim}{b} - 2\underset{\sim}{b} \cdot \underset{\sim}{c} - 2\underset{\sim}{d} \cdot \underset{\sim}{c})$

$4\left|\overrightarrow{M_1M_2}\right|^2 = \underset{\sim}{b} \cdot \underset{\sim}{b} + \underset{\sim}{c} \cdot \underset{\sim}{c} + \underset{\sim}{d} \cdot \underset{\sim}{d} + 2\underset{\sim}{d} \cdot \underset{\sim}{b} - 2\underset{\sim}{b} \cdot \underset{\sim}{c} - 2\underset{\sim}{d} \cdot \underset{\sim}{c}$

Hence,

$\text{LHS} = \left|\overrightarrow{AB}\right|^2 + \left|\overrightarrow{BC}\right|^2 + \left|\overrightarrow{CD}\right|^2 + \left|\overrightarrow{DA}\right|^2$

$= \underset{\sim}{b} \cdot \underset{\sim}{b} + (\underset{\sim}{c} \cdot \underset{\sim}{c} - 2\underset{\sim}{c} \cdot \underset{\sim}{b} + \underset{\sim}{b} \cdot \underset{\sim}{b}) + (\underset{\sim}{d} \cdot \underset{\sim}{d} - 2\underset{\sim}{d} \cdot \underset{\sim}{c} + \underset{\sim}{c} \cdot \underset{\sim}{c}) + \underset{\sim}{d} \cdot \underset{\sim}{d}$

$= (\underset{\sim}{c} \cdot \underset{\sim}{c}) + (\underset{\sim}{d} \cdot \underset{\sim}{d} + \underset{\sim}{b} \cdot \underset{\sim}{b}) + (\underset{\sim}{b} \cdot \underset{\sim}{b} + \underset{\sim}{c} \cdot \underset{\sim}{c} + \underset{\sim}{d} \cdot \underset{\sim}{d} - 2\underset{\sim}{c} \cdot \underset{\sim}{b} - 2\underset{\sim}{d} \cdot \underset{\sim}{c})$
$+ (2\underset{\sim}{d} \cdot \underset{\sim}{b} - 2\underset{\sim}{d} \cdot \underset{\sim}{b})$

$= (\underset{\sim}{c} \cdot \underset{\sim}{c}) + (\underset{\sim}{d} \cdot \underset{\sim}{d} + -2\underset{\sim}{d} \cdot \underset{\sim}{b} + \underset{\sim}{b} \cdot \underset{\sim}{b})$
$+ (\underset{\sim}{b} \cdot \underset{\sim}{b} + \underset{\sim}{c} \cdot \underset{\sim}{c} + \underset{\sim}{d} \cdot \underset{\sim}{d} + 2\underset{\sim}{d} \cdot \underset{\sim}{b} - 2\underset{\sim}{c} \cdot \underset{\sim}{b} - 2\underset{\sim}{d} \cdot \underset{\sim}{c})$

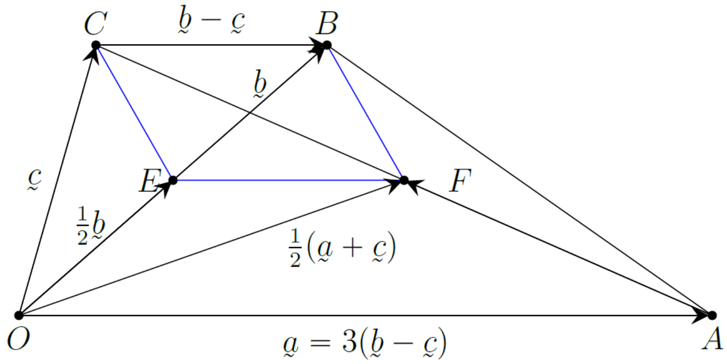
$= \left|\overrightarrow{AC}\right|^2 + \left|\overrightarrow{BD}\right|^2 + 4\left|\overrightarrow{M_1M_2}\right|^2$

$= \text{RHS}$

So the equation is satisfied.

Solutions to Exercise 5E Enrichment questions

11



Let $\overrightarrow{OA} = \underline{a}$, $\overrightarrow{OB} = \underline{b}$, $\overrightarrow{OC} = \underline{c}$

Then, $\overrightarrow{CB} = \underline{b} - \underline{c}$ and so, $\overrightarrow{OA} = \underline{a} = 3(\underline{b} - \underline{c})$

$\overrightarrow{OE} = \frac{1}{2}\overrightarrow{OB} = \frac{1}{2}\underline{b}$

$\overrightarrow{OF}$

$= \overrightarrow{OA} + \overrightarrow{AF}$

$= \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AC}$

$= \underline{a} + \frac{1}{2}(\underline{c} - \underline{a})$

$= \frac{1}{2}(\underline{a} + \underline{c})$

Hence,

$\overrightarrow{EF}$

$= \overrightarrow{OF} - \overrightarrow{OE}$

$= \frac{1}{2}(\underline{a} + \underline{c} - \underline{b})$

$= \frac{1}{2}(3(\underline{b} - \underline{c}) + \underline{c} - \underline{b})$

$= \frac{1}{2}(2\underline{b} - 2\underline{c})$

$= \underline{b} - \underline{c}$

$= \overrightarrow{CB}$

Hence, $EFBC$ is a parallelogram. (Opposite sides CB and EF are parallel and equal.)

12 Let $OABC$ be a quadrilateral with vertices,

$O(0,0,0)$, $A(a_1,a_2,a_3)$, $B(b_1,b_2,b_3)$, $C(c_1,c_2,c_3)$

$$\overrightarrow{OA}=\begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix},\quad \overrightarrow{AB}=\begin{bmatrix}b_1-a_1\\b_2-a_2\\b_3-a_3\end{bmatrix},\quad \overrightarrow{BC}=\begin{bmatrix}c_1-b_1\\c_2-b_2\\c_3-b_3\end{bmatrix},\quad \overrightarrow{CO}=\begin{bmatrix}-c_1\\-c_2\\-c_3\end{bmatrix}$$

Hence, the sum of the squares of the sides is,

$$\left|\overrightarrow{OA}\right|^2+\left|\overrightarrow{AB}\right|^2+\left|\overrightarrow{BC}\right|^2+\left|\overrightarrow{CO}\right|^2$$

$$=a_1{}^2+a_2{}^2+a_3{}^2+(b_1-a_1)^2+(b_2-a_2)^2+(b_3-a_3)^2+(c_1-b_1)^2$$

$$+(c_2-b_2)^2+(c_3-b_3)^2+c_1{}^2+c_2{}^2+c_3{}^2$$

$$=2\left(a_1{}^2+a_2{}^2+a_3{}^2+b_1{}^2+b_2{}^2+b_3{}^2+c_1{}^2+c_2{}^2+c_3{}^2\right)$$

$$-2(a_1b_1+a_2b_2+a_3b_3+b_1c_1+b_2c_2+b_3c_3) \qquad (1)$$

The diagonals are OB and AC, where $\overrightarrow{OB}=\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$ and $\overrightarrow{AC}=\begin{bmatrix}c_1-a_1\\c_2-a_2\\c_3-a_3\end{bmatrix}$

Hence, the sum of the squares of the diagonals is:

$$\left|\overrightarrow{OB}\right|^2+\left|\overrightarrow{AC}\right|^2$$

$$=b_1{}^2+b_2{}^2+b_3{}^2+(c_1-a_1)^2+(c_2-a_2)^2+(c_3-a_3)^2$$

$$=\left(a_1{}^2+a_2{}^2+a_3{}^2+b_1{}^2+b_2{}^2+b_3{}^2+c_1{}^2+c_2{}^2+c_3{}^2\right)$$

$$-2(a_1c_1+a_2c_2+a_3c_3) \qquad (2)$$

Let the midpoints of OB and AC be M and N, respectively.

Then $\overrightarrow{OM}=\frac{1}{2}\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$ and $\overrightarrow{ON}=\overrightarrow{OA}+\frac{1}{2}\overrightarrow{AC}=\frac{1}{2}\begin{bmatrix}c_1+a_1\\c_2+a_2\\c_3+a_3\end{bmatrix}$

Hence, $\overrightarrow{MN}=\frac{1}{2}\begin{bmatrix}a_1-b_1+c_1\\a_2-b_2+c_2\\a_3-b_3+c_3\end{bmatrix}$

So, 4 times the square of the distance between the midpoints of the diagonals is:

$$\left(\overrightarrow{MN}\right)^2$$

$$=4\times\frac{1}{4}\left((a_1-b_1+c_1)^2+(a_2-b_2+c_2)^2+(a_3-b_3+c_3)^2\right)$$

$$=\left(a_1{}^2+a_2{}^2+a_3{}^2+b_1{}^2+b_2{}^2+b_3{}^2+c_1{}^2+c_2{}^2+c_3{}^2\right)$$

$$-2(a_1b_1 + a_2b_2 + a_3b_3 + b_1c_1 + b_2c_2 + b_3c_3 - a_1c_1 - a_2c_2 - a_3c_3) \qquad (3)$$

Now to prove the theorem:

equation (1) = equation (2) + equation (3).

Hence, the result is proven.

12 [Alternatively, questions 10 and 12 are essentially the same and can be done as follows.]

Using the diagram in question 10, translate the figure so that A coincides with the origin.

Let, $\overrightarrow{AB} = \overrightarrow{OB} = \underline{b}$, $\quad \overrightarrow{AC} = \overrightarrow{OC} = \underline{c}$, $\quad \overrightarrow{AD} = \overrightarrow{OD} = \underline{d}$

So, $\overrightarrow{AM_1} = \overrightarrow{OM_1} = \frac{1}{2}\underline{c}$ and $\overrightarrow{AM_2} = \overrightarrow{OM_2} = \frac{1}{2}(\underline{b} + \underline{d})$, and thus,

$\overrightarrow{M_1M_2} = \frac{1}{2}(\underline{b} + \underline{d} - \underline{c})$.

Hence, LHS

$= |\underline{b}|^2 + |\underline{c} - \underline{b}|^2 + |\underline{d} - \underline{c}|^2 + |\underline{d}|^2$

$= \underline{b}.\underline{b} + (\underline{c} - \underline{b}).(\underline{c} - \underline{b}) + (\underline{d} - \underline{c}).(\underline{d} - \underline{c}) + \underline{d}.\underline{d}$

$= \underline{b}.\underline{b} + \underline{c}.\underline{c} - 2\underline{b}.\underline{c} + \underline{b}.\underline{b} + \underline{d}.\underline{d} - 2\underline{c}.\underline{d} + \underline{c}.\underline{c} + \underline{d}.\underline{d}$

$= 2(\underline{b}.\underline{b} + \underline{c}.\underline{c} + \underline{d}.\underline{d} - \underline{b}.\underline{c} - \underline{c}.\underline{d})$

RHS

$= |\underline{c}|^2 + |\underline{d} - \underline{b}|^2 + 4.\left|\frac{1}{2}(\underline{b} + \underline{d} - \underline{c})\right|^2$

$= \underline{c}.\underline{c} + (\underline{d} - \underline{b}).(\underline{d} - \underline{b}) + 4.\frac{1}{4}(\underline{b} + \underline{d} - \underline{c}).(\underline{b} + \underline{d} - \underline{c})$

$= \underline{c}.\underline{c} + \underline{d}.\underline{d} - \cancel{2\underline{b}.\underline{d}} + \underline{b}.\underline{b} + \underline{b}.\underline{b} + \cancel{\underline{b}.\underline{d}} - \underline{b}.\underline{c} + \cancel{\underline{b}.\underline{d}} + \underline{d}.\underline{d} - \underline{c}.\underline{d} - \underline{b}.\underline{c}$

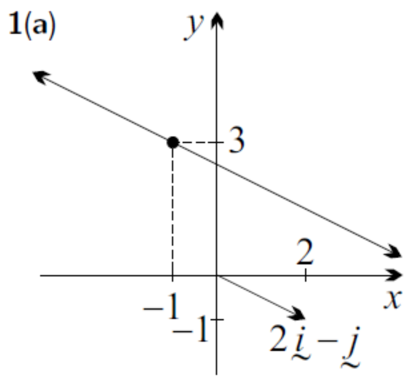
$\qquad - \underline{c}.\underline{d} + \underline{c}.\underline{c}$

$= 2(\underline{b}.\underline{b} + \underline{c}.\underline{c} + \underline{d}.\underline{d} - \underline{b}.\underline{c} - \underline{c}.\underline{d})$

$=$ LHS

Solutions to Exercise 5F Foundation questions

1a The line intersects the point $A = (-1, 3)$ and is in the direction $\underset{\sim}{b} = 2\underset{\sim}{i} - \underset{\sim}{j}$.



1b $A = (-1, 3)$

$\underset{\sim}{a} = -\underset{\sim}{i} + 3\underset{\sim}{j}\underset{\sim}{b} = 2\underset{\sim}{i} - \underset{\sim}{j}$

$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$

$\underset{\sim}{r} = (-\underset{\sim}{i} + 3\underset{\sim}{j}) + \lambda(2\underset{\sim}{i} - \underset{\sim}{j}), \lambda \in \mathbb{R}$

1c $\underset{\sim}{r} = (-\underset{\sim}{i} + 3\underset{\sim}{j}) + \lambda(2\underset{\sim}{i} - \underset{\sim}{j})$

$m = -\frac{1}{2}$

1d $\underset{\sim}{r} = (-\underset{\sim}{i} + 3\underset{\sim}{j}) + \lambda(2\underset{\sim}{i} - \underset{\sim}{j})$

So

$x = 2\lambda - 1$

$y = 3 - \lambda$

$x = 2(3 - y) - 1$

$3 - y = \frac{1}{2}x + \frac{1}{2}$

$y = -\frac{1}{2}x + \frac{5}{2}$

2a $\underset{\sim}{a} = x\underset{\sim}{i} + y\underset{\sim}{j}$

$x = 3$

$y = \frac{2}{3}(3) - 4$

$y = -2$

so

$\underset{\sim}{a} = 3\underset{\sim}{i} - 2\underset{\sim}{j}$

2b $\underset{\sim}{b} = b_1\underset{\sim}{i} + b_2\underset{\sim}{j}$

$m = \frac{b_2}{b_1}$

$m = \frac{2}{3}$

$\underset{\sim}{b} = 3\underset{\sim}{i} + 2\underset{\sim}{j}$

2c $\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$

$\underset{\sim}{a} = 3\underset{\sim}{i} - 2\underset{\sim}{j}$

$\underset{\sim}{b} = 3\underset{\sim}{i} + 2\underset{\sim}{j}$

$\underset{\sim}{r} = 3\underset{\sim}{i} - 2\underset{\sim}{j} + \lambda(3\underset{\sim}{i} + 2\underset{\sim}{j}), \lambda \in \mathbb{R}$

3a i For $Ax + By = 0$,

directional vector $\underset{\sim}{b}$ is:

$\underset{\sim}{b} = \begin{bmatrix} -B \\ A \end{bmatrix}$ or $\underset{\sim}{b} = \begin{bmatrix} B \\ -A \end{bmatrix}$

So for

$x - 3y + 12 = 0$

$\underset{\sim}{b} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

3a ii $x - 3y + 12 = 0$

For the x-intercept:

$x - 3(0) + 12 = 0$

$x = -12$

x-intercept $= (-12, 0)$

So the position vector will be:

$\underset{\sim}{a} = \begin{bmatrix} -12 \\ 0 \end{bmatrix}$

For the y-intercept:

$0 - 3y + 12 = 0$

$-3y = -12$

$y = 4$

y-intercept $= (0, 4)$

So the position vector will be:

$\underset{\sim}{a} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$

3a iii $\underset{\sim}{r} = \underset{\sim}{a} + \lambda \underset{\sim}{b}$

Using the y-intercept,

$\underset{\sim}{a} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$

$\underset{\sim}{b} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

So

$\underset{\sim}{r} = \begin{bmatrix} 0 \\ 4 \end{bmatrix} + \lambda \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

3b i $x + 3y = 6$

$x + 3y - 6 = 0$

So for the directional vector

$\underset{\sim}{b} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$

x-intercept is

$x + 3(0) - 6 = 0$

$x = 6$

$x\text{-intercept} = (6, 0)$

So the position vector will be:

$\underset{\sim}{a} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$

So

$\underset{\sim}{r} = \begin{bmatrix} 6 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} -3 \\ 1 \end{bmatrix}$

3b ii $y = 3$

$y - 3 = 0$

So for the directional vector

$\underset{\sim}{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

y-intercept is

$y = 3$

$y\text{-intercept} = (3, 0)$

So the position vector will be:

$\underset{\sim}{a} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$

So

$\underset{\sim}{r} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

3b iii $x = -5$

$x + 5 = 0$

So for the directional vector

$\underset{\sim}{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

x-intercept is $x = -5$

$x\text{-intercept} = (-5, 0)$

So the position vector will be:

$\underset{\sim}{a} = \begin{bmatrix} -5 \\ 0 \end{bmatrix}$

So

$\underset{\sim}{r} = \begin{bmatrix} -5 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

4a $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 \\ 5 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

$x = -3 + \lambda$

$y = 5 + 4\lambda$

So

$\lambda = x + 3$

Substituting this back into the equation for y gives:

$y = 5 + 4(x + 3)$

$y = 5 + 4x + 12$

$y = 4x + 17$

4b $\underset{\sim}{r} = 5\underset{\sim}{i} + 2\underset{\sim}{j} + \lambda(-2\underset{\sim}{i} + 3\underset{\sim}{j})$

$x = 5 - 2\lambda$

$y = 2 + 3\lambda$

$\frac{y - 2}{3} = \lambda$

Substituting this back into the equation for x gives:

$x = 5 - 2\left(\frac{y - 2}{3}\right)$

$x - 5 = \frac{4 - 2y}{3}$

$3x - 15 = 4 - 2y$

So

$3x + 2y = 19$

5a $\underset{\sim}{r} = \begin{bmatrix} -4 \\ 2 \end{bmatrix} + \lambda \begin{bmatrix} 3 \\ -5 \end{bmatrix}$

For $(2, -8)$

$\begin{bmatrix} 2 \\ -8 \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \end{bmatrix} + \lambda \begin{bmatrix} 3 \\ -5 \end{bmatrix}$

For x-equation:

$2 = -4 + 3\lambda$

$6 = 3\lambda$

$2 = \lambda$

For y-equation:

$-8 = 2 - 5\lambda$

$10 = 5\lambda$

$2 = \lambda$

As $\lambda = 2$ for both equations, the point $(2, -8)$ lies on the line $\underset{\sim}{r}$.

5b $\underset{\sim}{r} = \begin{bmatrix} -4 \\ 2 \end{bmatrix} + \lambda \begin{bmatrix} 3 \\ -5 \end{bmatrix}$

For $(-13, 17)$

$\begin{bmatrix} -13 \\ 17 \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \end{bmatrix} + \lambda \begin{bmatrix} 3 \\ -5 \end{bmatrix}$

For x-equation:

$-13 = -4 + 3\lambda$

$-9 = 3\lambda$

$-3 = \lambda$

For y-equation:

$17 = 2 - 5\lambda$

$-15 = 5\lambda$

$-3 = \lambda$

As $\lambda = -3$ for both equations, the point $(-13, 17)$ lies on the line $\underset{\sim}{r}$.

5c $\underset{\sim}{r} = \begin{bmatrix} -4 \\ 2 \end{bmatrix} + \lambda \begin{bmatrix} 3 \\ -5 \end{bmatrix}$

For $(8, -20)$

$\begin{bmatrix} 8 \\ -20 \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \end{bmatrix} + \lambda \begin{bmatrix} 3 \\ -5 \end{bmatrix}$

For x-equation:

$8 = -4 + 3\lambda$

$12 = 3\lambda$

$4 = \lambda$

For y-equation:

$-20 = 2 - 5\lambda$

$22 = 5\lambda$

$\frac{22}{5} = \lambda$

As there are different values for λ, the point $(8,20)$ does not lie on the line $\underset{\sim}{r}$.

6a $P(7, 0, -5)$

$\underset{\sim}{a} = 7\underset{\sim}{i} - 5\underset{\sim}{k}$

$\underset{\sim}{b} = -4\underset{\sim}{i} - 6\underset{\sim}{j} + 9\underset{\sim}{k}$

$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$

$\underset{\sim}{r} = 7\underset{\sim}{i} - 5\underset{\sim}{k} + \lambda\left(-4\underset{\sim}{i} - 6\underset{\sim}{j} + 9\underset{\sim}{k}\right)$

6b $P(3, 4, 5)$

$\underset{\sim}{a} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$

$\underset{\sim}{b} = \begin{bmatrix} -6 \\ -7 \\ -8 \end{bmatrix}$

$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$

$\underset{\sim}{r} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} + \lambda \begin{bmatrix} -6 \\ -7 \\ -8 \end{bmatrix}$

7a $P(3, -2, -4)$

$\underset{\sim}{r} = 2\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k} + \lambda(5\underset{\sim}{i} - 3\underset{\sim}{j} - \underset{\sim}{k})$

$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$

$\underset{\sim}{a} = 3\underset{\sim}{i} - 2\underset{\sim}{j} - 4\underset{\sim}{k}$

$\underset{\sim}{b} = 5\underset{\sim}{i} - 3\underset{\sim}{j} - \underset{\sim}{k}$

So

$\underset{\sim}{r} = 3\underset{\sim}{i} - 2\underset{\sim}{j} - 4\underset{\sim}{k} + \lambda(5\underset{\sim}{i} - 3\underset{\sim}{j} - \underset{\sim}{k})$

7b $P(-1, -1, 2)$

$\underset{\sim}{r} = \frac{1}{3}\underset{\sim}{i} - \frac{1}{3}\underset{\sim}{j} - \underset{\sim}{k} + \lambda\left(\frac{1}{6}\underset{\sim}{i} + \frac{1}{3}\underset{\sim}{j} + \frac{1}{2}\underset{\sim}{k}\right)$

$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$

$\underset{\sim}{a} = -1\underset{\sim}{i} - 1\underset{\sim}{j} + 2\underset{\sim}{k}$

$\underset{\sim}{b} = \frac{1}{6}\underset{\sim}{i} + \frac{1}{3}\underset{\sim}{j} + \frac{1}{2}\underset{\sim}{k}$

We multiply the directional vector by a number without altering the direction it represents so:

$\underset{\sim}{b} = \frac{6}{6}\underset{\sim}{i} + \frac{6}{3}\underset{\sim}{j} + \frac{6}{2}\underset{\sim}{k}$

$\underset{\sim}{b} = \underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}$

So

$\underset{\sim}{r} = -\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k} + \lambda(\underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k})$

8a $\underset{\sim}{r} = \begin{bmatrix} 4 \\ -7 \\ -1 \end{bmatrix} + \lambda\begin{bmatrix} -2 \\ 3 \\ -6 \end{bmatrix}$

$P = (8, -13, 11)$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -7 \\ -1 \end{bmatrix} + \lambda\begin{bmatrix} -2 \\ 3 \\ -6 \end{bmatrix}$

For x-equation:

$8 = 4 - 2\lambda$

$-4 = 2\lambda$

$\lambda = -2$

For y-equation:

$-13 = -7 + 3\lambda$

$-6 = 3\lambda$

$\lambda = -2$

For z-equation:

$11 = -1 - 6\lambda$

$12 = -6\lambda$

$\lambda = -2$

As $\lambda = -2$ for all three equations, the point P lies on the line $\underset{\sim}{r}$.

8b $\underset{\sim}{r} = \begin{bmatrix} 4 \\ -7 \\ -1 \end{bmatrix} + \lambda \begin{bmatrix} -2 \\ 3 \\ -6 \end{bmatrix}$

$P = (-4, 5, -25)$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -7 \\ -1 \end{bmatrix} + \lambda \begin{bmatrix} -2 \\ 3 \\ -6 \end{bmatrix}$

For x-equation:

$-4 = 4 - 2\lambda$
$8 = 2\lambda$
$\lambda = 4$

For y-equation:

$5 = -7 + 3\lambda$

$12 = 3\lambda$

$\lambda = 4$

For z-equation:

$-25 = -1 - 6\lambda$

$24 = 6\lambda$

$\lambda = 4$

As $\lambda = 4$ for all three equations, the point P lies on the line $\underset{\sim}{r}$.

Solutions to Exercise 5F Development questions

9a i $x + 2y - 4 = 0$

$y = 2 - \frac{1}{2}x$

For $x = 0, y = 2$

$(0, 2)$

For $x = 2, y = 1$

$(2, 1)$

The directional vector, $\underset{\sim}{a}$ for this line is:

$\underset{\sim}{a} = \begin{bmatrix} 2-0 \\ 1-2 \end{bmatrix}$

$\underset{\sim}{a} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

9a ii $\underset{\sim}{a} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

The vector, $\underset{\sim}{n}$, perpendicular to $\underset{\sim}{a}$, is:

$\underset{\sim}{n} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

9a iii The vector equation for the perpendicular line through $(2, -3)$ is

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

9b $x - y + 3 = 0$

$y = x + 3$

For $x = 0, y = 3$

$(0, 3)$

For $x = 1, y = 4$

$(1, 4)$

So the directional vector for this equation is:

$\underset{\sim}{a} = \begin{bmatrix} 1-0 \\ 4-3 \end{bmatrix}$

$\underset{\sim}{a} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

The vector, $\underset{\sim}{n}$, perpendicular to $\underset{\sim}{a}$, is:

$\underset{\sim}{n} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

The vector equation for the perpendicular line through $(1, -2)$ is

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

10 $\underset{\sim}{r} = \underset{\sim}{a} + \lambda \underset{\sim}{b}$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ m \end{bmatrix}$

This gives:

$x = x_1 + \lambda$

$x - x_1 = \lambda$

$\dfrac{y - y_1}{m} = \lambda$

$\dfrac{y - y_1}{m} = x - x_1$

So

$y - y_1 = m(x - x_1)$

11a $A(4, 3)$

$B(6, 0)$

$\overrightarrow{AB} = \begin{bmatrix} 6-4 \\ 0-3 \end{bmatrix}$

$= \begin{bmatrix} 2 \\ -3 \end{bmatrix}$

$\underset{\sim}{r} = 4\underset{\sim}{i} + 3\underset{\sim}{j} + \lambda(2\underset{\sim}{i} - 3\underset{\sim}{j})$

11b $A(-7, 5)$

$B(-13, -8)$

$$\overrightarrow{AB} = \begin{bmatrix} -13-(-7) \\ -8-5 \end{bmatrix}$$

$$= \begin{bmatrix} -6 \\ -13 \end{bmatrix}$$

$$\underset{\sim}{r} = -7\underset{\sim}{i} + 5\underset{\sim}{j} - \lambda(6\underset{\sim}{i} + 13\underset{\sim}{j})$$

which is equivalent to:

$$\underset{\sim}{r} = -7\underset{\sim}{i} + 5\underset{\sim}{j} + \lambda(6\underset{\sim}{i} + 13\underset{\sim}{j})$$

12a $P(-1, 3, 1)$

$Q(2, 4, 5)$

$$\overrightarrow{PQ} = \begin{bmatrix} 2-(-1) \\ 4-3 \\ 5-1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}$$

$$\underset{\sim}{r} = -\underset{\sim}{i} + 3\underset{\sim}{j} + \underset{\sim}{k} + \lambda(3\underset{\sim}{i} + \underset{\sim}{j} + 4\underset{\sim}{k})$$

12b $P(7, -11, 14)$

$Q(17, 9, -16)$

$$\overrightarrow{PQ} = \begin{bmatrix} 17-7 \\ 9-(-11) \\ -16-14 \end{bmatrix}$$

$$= \begin{bmatrix} 10 \\ 20 \\ -30 \end{bmatrix}$$

$$\underset{\sim}{r} = 7\underset{\sim}{i} - 11\underset{\sim}{j} + 14\underset{\sim}{k} + \lambda(10\underset{\sim}{i} + 20\underset{\sim}{j} - 30\underset{\sim}{k})$$

13a $A(1, -2)$

$B(5, 4)$

$$\overrightarrow{AB} = \begin{bmatrix} 5-1 \\ 4-(-2) \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} + \lambda \begin{bmatrix} 4 \\ 6 \end{bmatrix}, 0 \leq \lambda \leq 1$

13b $A(-1, 1, -2)$

$B(2, 3, -1)$

$\overrightarrow{AB} = \begin{bmatrix} 2-(-1) \\ 3-1 \\ -1-(-2) \end{bmatrix}$

$= \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} + \lambda \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, 0 \leq \lambda \leq 1$

14 $\underset{\sim}{r}_1 = \begin{bmatrix} -4 \\ 3 \\ -1 \end{bmatrix} + \lambda \begin{bmatrix} 6 \\ -15 \\ -24 \end{bmatrix}$

$\underset{\sim}{r}_2 = \begin{bmatrix} 2 \\ -5 \\ -4 \end{bmatrix} + \mu \begin{bmatrix} -4 \\ 10 \\ 16 \end{bmatrix}$

If $\underset{\sim}{r}_1$ and $\underset{\sim}{r}_2$ are parallel then there is a value a, that is the ratio between the directional vectors of $\underset{\sim}{r}_1$ and $\underset{\sim}{r}_2$ so

$\begin{bmatrix} 6 \\ -15 \\ -24 \end{bmatrix} = a \begin{bmatrix} -4 \\ 10 \\ 16 \end{bmatrix}$

$6 = -4a$

$a = -\dfrac{3}{2}$

$\text{LHS} = \begin{bmatrix} 6 \\ -15 \\ -24 \end{bmatrix}$

$= -\dfrac{3}{2}\begin{bmatrix} -4 \\ 10 \\ 16 \end{bmatrix}$

$= \text{RHS}$

Since LHS = RHS, $\underset{\sim}{r}_1$ and $\underset{\sim}{r}_2$ are parallel.

15a $\underset{\sim}{r}_1 = \begin{bmatrix} 4 \\ 8 \\ 3 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

$\underset{\sim}{r}_2 = \begin{bmatrix} 7 \\ 6 \\ 5 \end{bmatrix} + \mu \begin{bmatrix} 6 \\ 4 \\ 5 \end{bmatrix}$

$\begin{bmatrix} 4 \\ 8 \\ 3 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 6 \\ 5 \end{bmatrix} + \mu \begin{bmatrix} 6 \\ 4 \\ 5 \end{bmatrix}$

Equating the first two components and solving for λ:

$4 + \lambda = 7 + 6\mu$

$\lambda = 3 + 6\mu$

Equating the second two components and substituting for λ:

$8 + 2\lambda = 6 + 4\mu$

$2\lambda = -2 + 4\mu$

$2(3 + 6\mu) = -2 + 4\mu$

$6 + 12\mu = -2 + 4\mu$

$8\mu = -8$

$\mu = -1$

Hence $\lambda = -3$.

So the point of intersection is:

$\begin{bmatrix} 7 \\ 6 \\ 5 \end{bmatrix} + (-1) \begin{bmatrix} 6 \\ 4 \\ 5 \end{bmatrix} = (1, 2, 0)$

As a check, substituting $\lambda = -3$ into $\underset{\sim}{r}_1$ gives the same point.

15b $\underset{\sim}{r}_1 = \begin{bmatrix} 7 \\ -3 \\ 8 \end{bmatrix} + \lambda \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$

$\underset{\sim}{r}_2 = \begin{bmatrix} -2 \\ 1 \\ 10 \end{bmatrix} + \mu \begin{bmatrix} 5 \\ -3 \\ -4 \end{bmatrix}$

$\begin{bmatrix} 7 \\ -3 \\ 8 \end{bmatrix} + \lambda \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 10 \end{bmatrix} + \mu \begin{bmatrix} 5 \\ -3 \\ -4 \end{bmatrix}$

Equating the second two components and solving for λ:

$-3-\lambda = 1-3\mu$

$\lambda = 3\mu - 4$

Equating the first two components and substituting for λ:

$7+4\lambda = -2+5\mu$

$4\lambda = -9+5\mu$

$4(3\mu-4) = -9+5\mu$

$12\mu - 16 = -9+5\mu$

$7\mu = 7$

$\mu = 1$

Hence $\lambda = -1$.

So the point of intersection is:

$$\begin{bmatrix}-2\\1\\10\end{bmatrix} + \begin{bmatrix}5\\-3\\-4\end{bmatrix} = (3,-2,6)$$

As a check, substituting $\lambda = -1$ into $\underset{\sim}{r}_1$ gives the same point.

16 $\underset{\sim}{r}_1 = \begin{bmatrix}1\\0\\-1\end{bmatrix} + \lambda\begin{bmatrix}2\\-1\\1\end{bmatrix}$

$\underset{\sim}{r}_2 = \begin{bmatrix}1\\1\\0\end{bmatrix} + \mu\begin{bmatrix}-4\\3\\-3\end{bmatrix}$

$$\begin{bmatrix}1\\0\\-1\end{bmatrix} + \lambda\begin{bmatrix}2\\-1\\1\end{bmatrix} = \begin{bmatrix}1\\1\\0\end{bmatrix} + \mu\begin{bmatrix}-4\\3\\-3\end{bmatrix}$$

Equating the first two components and solving for λ:

$1+2\lambda = 1-4\mu$

$\lambda = -2\mu$

Equating the second two components and substituting for λ:

$-\lambda = 1+3\mu$

$2\mu = 1+3\mu$

$\mu = -1$

Hence

$\lambda = 2$

Equating the third two components and substituting for λ and μ:

$-1 + \lambda = -3\mu$

$\text{LHS} = -1 + 2 = 1$ and

$\text{RHS} = 3$

$\text{LHS} \neq \text{RHS}$

As the simultaneous equations are inconsistent the lines do not intersect.

These lines are not parallel as there is no value for a at which $v_1 = a\underset{\sim}{v}_2$

where v_1 and v_2 is the directional vectors for $\underset{\sim}{r}_1$ and $\underset{\sim}{r}_2$ respectively and a is a real number.

So $\underset{\sim}{r}_1$ and $\underset{\sim}{r}_2$ are skew.

17a $\underset{\sim}{v}_1 = \begin{bmatrix} 3 \\ -2 \\ 3 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

$\underset{\sim}{v}_2 = \begin{bmatrix} -2 \\ -2 \\ 4 \end{bmatrix} + \mu \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$

$$\begin{bmatrix} 3 \\ -2 \\ 3 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ 4 \end{bmatrix} + \mu \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$$

Equating the first components and solving for μ:

$3 + 2\lambda = -2 + \mu$

$\mu = 2\lambda + 5$

Equating the second components and substituting for μ:

$-2 - \lambda = -2 + 2\mu$

$-2 - \lambda = -2 + 2(2\lambda + 5)$

$-\lambda = 4\lambda + 10$

$-5\lambda = 10$

$\lambda = -2$

Hence

$\mu = 2(-2) + 5$

$\mu = 1$

Using the third components and substituting for λ and μ:

$3 + \lambda = 4 - 3\mu$

$\text{LHS} = 3 - 2 = 1$ and

$\text{RHS} = 4 - 3 = 1$

So $\underset{\sim}{v}_1$ and $\underset{\sim}{v}_2$ intersect.

$\underset{\sim}{v}_1 = \begin{bmatrix} 3 \\ -2 \\ 3 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 3 \end{bmatrix} + (-2) \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

So $\underset{\sim}{v}_1$ intersects $\underset{\sim}{v}_2$ at $(-1, 0, 1)$ or $-\underset{\sim}{i} + \underset{\sim}{k}$.

17b $\underset{\sim}{v}_1 = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$

$\underset{\sim}{v}_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + \mu \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$

$\begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + \mu \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$

Using the first two components and solving for μ:

$3 + 2\lambda = 2 - \mu$

$\mu = -1 - 2\lambda$

Using the second two components and substituting for μ:

$1 + \lambda = -1 + 2\mu$

$1 + \lambda = -1 + 2(-1 - 2\lambda)$

$4 = -5\lambda$

$\lambda = -\frac{4}{5}$

Hence

$\mu = -1 - 2\left(-\frac{4}{5}\right)$

$\mu = \frac{3}{5}$

Using the third two components and substituting for λ and μ:

$4 - \lambda = 1 + 3\mu$

$\text{LHS} = 4 - \left(-\frac{4}{5}\right) = \frac{16}{5}$ and

$\text{RHS} = 1 + 3\left(\frac{3}{5}\right) = \frac{14}{5}$

$\text{LHS} \neq \text{RHS}$

As the simultaneous equations are inconsistent, the lines do not intersect.

So $\underset{\sim}{r}_1$ and $\underset{\sim}{r}_2$ are skew.

18a For the points $(2, 0, 1)$ and $(-1, 3, 4)$

the directional vector $\underset{\sim}{v}_1$ will be:

$\underset{\sim}{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix}$

$\underset{\sim}{v}_1 = \begin{bmatrix} 3 \\ -3 \\ -3 \end{bmatrix}$

So a vector for the line will be:

$\underset{\sim}{r}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} 3 \\ -3 \\ -3 \end{bmatrix}$

For the points $(-1, 3, 0)$ and $(4, -2, 5)$,

the directional vector $\underset{\sim}{v}_2$ will be:

$\underset{\sim}{v}_2 = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix} - \begin{bmatrix} 4 \\ -2 \\ 5 \end{bmatrix}$

$$\underset{\sim}{v}_2 = \begin{bmatrix} -5 \\ 5 \\ -5 \end{bmatrix}$$

So a vector for the line will be:

$$\underset{\sim}{r}_2 = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix} + \mu \begin{bmatrix} -5 \\ 5 \\ -5 \end{bmatrix}$$

So $\underset{\sim}{r}_1$ and $\underset{\sim}{r}_2$ will intersect for $\underset{\sim}{r}_1 = \underset{\sim}{r}_2$.

$$\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} 3 \\ -3 \\ -3 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix} + \mu \begin{bmatrix} -5 \\ 5 \\ -5 \end{bmatrix}$$

Using the first two components and solving for μ:

$$2 + 3\lambda = -1 - 5\mu$$

$$\mu = -\frac{3}{5} - \frac{3}{5}\lambda$$

Using the third two components and substituting for μ:

$$1 - 3\lambda = -5\mu$$

$$1 - 3\lambda = -5\left(-\frac{3}{5} - \frac{3}{5}\lambda\right)$$

$$1 - 3\lambda = 3 + 3\lambda$$

$$-2 = 6\lambda$$

$$\lambda = -\frac{1}{3}$$

Hence

$$\mu = -\frac{3}{5} - \frac{3}{5}\lambda$$

$$\mu = -\frac{3}{5} - \frac{3}{5}\left(-\frac{1}{3}\right)$$

$$\mu = -\frac{2}{5}$$

So substituting λ in $\underset{\sim}{r}_1$,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 3 \\ -3 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

So $\underset{\sim}{r}_1$ and $\underset{\sim}{r}_2$ intersect at $(1, 1, 2)$.

18b In order to find the angle between the two vectors, we can use the directional vectors of $\underset{\sim}{r}_1$ and $\underset{\sim}{r}_2$.

$$\underset{\sim}{v}_1 = \begin{bmatrix} 3 \\ -3 \\ -3 \end{bmatrix}$$

$$\underset{\sim}{v}_2 = \begin{bmatrix} -5 \\ 5 \\ -5 \end{bmatrix}$$

$$\theta = \cos^{-1} \frac{\underset{\sim}{v}_1 \cdot \underset{\sim}{v}_2}{|\underset{\sim}{v}_1||v_2|}$$

$$v_1 \cdot \underset{\sim}{v}_2 = \begin{bmatrix} 3 \\ -3 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} -5 \\ 5 \\ -5 \end{bmatrix}$$

$$= (3 \times -5) + (-3 \times 5) + (-3 \times -5)$$

$$= -15$$

$$|\underset{\sim}{v}_1|^2 = (3)^2 + (-3)^2 + (-3)^2 = 27$$

$$|\underset{\sim}{v}_1| = \sqrt{27} = 3\sqrt{3}$$

$$|\underset{\sim}{v}_2|^2 = (-5)^2 + 5^2 + (-5)^2 = 75$$

$$|\underset{\sim}{v}_2| = \sqrt{75} = 5\sqrt{3}$$

So

$$\theta = \cos^{-1} \frac{\underset{\sim}{v}_1 \cdot \underset{\sim}{v}_2}{|\underset{\sim}{v}_1||\underset{\sim}{v}_2|}$$

$$\theta = \cos^{-1}\left(-\frac{15}{3\sqrt{3} \times 5\sqrt{3}}\right)$$

$$\theta = \cos^{-1}\left(-\frac{1}{3}\right)$$

For the acute angle,

$$\theta = \cos^{-1}\left(\frac{1}{3}\right)$$

$$\theta \doteqdot 70.5°$$

19

$$\underset{\sim}{r}_1 = \begin{bmatrix} 2 \\ 9 \\ 13 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\underset{\sim}{r}_2 = \begin{bmatrix} a \\ 7 \\ -2 \end{bmatrix} + \mu \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix}$$

So if $\underset{\sim}{r}_1$ and $\underset{\sim}{r}_2$ intersect then, $\underset{\sim}{r}_1 = \underset{\sim}{r}_2$.

$$\begin{bmatrix} 2 \\ 9 \\ 13 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} a \\ 7 \\ -2 \end{bmatrix} + \mu \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix}$$

Equating the first two components and solving for a:

$2 + \lambda = a - \mu$

$a = 2 + \lambda + \mu$

Equating the third two components and solving for λ:

$13 + 3\lambda = -2 - 3\mu$

$3\lambda = -15 - 3\mu$

$\lambda = -5 - \mu$

Equating the second two components and substituting for λ:

$9 + 2\lambda = 7 + 2\mu$

$9 + 2(-5 - \mu) = 7 + 2\mu$

$-10 - 2\mu = -2 + 2\mu$

$-4\mu = 8$

$\mu = -2$

Hence

$\lambda = -5 - \mu$

$\lambda = -5 - (-2)$

$\lambda = -3$

So

$a = 2 + \lambda + \mu$

$a = 2 - 3 - 2$

$a = -3$

20a $\underset{\sim}{r} = \begin{bmatrix} 0 \\ -4 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ -4 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Let $\lambda = 1$, so

$A = (0, -4) + (1,2)$

$A = (1, -2)$

Let $\lambda = 2$, so

$B = (0, -4) + (2,4)$

$B = (2,0)$

20b Using A and B from part a:

$P = (-2, 3)$

$A = (1, -2)$

$B = (2, 0)$

$\overrightarrow{AP} = \begin{bmatrix} -2 - 1 \\ 3 - (-2) \end{bmatrix}$

$\overrightarrow{AP} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$

$\overrightarrow{AB} = \begin{bmatrix} 2 - 1 \\ 0 - (-2) \end{bmatrix}$

$\overrightarrow{AB} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

20c $\overrightarrow{AB} = \underset{\sim}{b}$

$\underset{\sim}{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\overrightarrow{AP} = \underset{\sim}{p}$

$\underset{\sim}{p} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$

$proj_{\underset{\sim}{b}} \underset{\sim}{p} = \dfrac{\underset{\sim}{b} \cdot \underset{\sim}{p}}{|\underset{\sim}{b}|^2} \underset{\sim}{b}$

$$proj_{\underset{\sim}{b}}\underset{\sim}{p} = \frac{(1 \times -3) + (2 \times 5)}{1^2 + 2^2}\begin{bmatrix}1\\2\end{bmatrix}$$

$$proj_{\underset{\sim}{b}}\underset{\sim}{p} = \frac{7}{5}\begin{bmatrix}1\\2\end{bmatrix}$$

$$proj_{\underset{\sim}{b}}\underset{\sim}{p} = \frac{1}{5}\begin{bmatrix}7\\14\end{bmatrix}$$

20d The perpendicular distance d from the point P to the line is:

$$d = \left|proj_{\underset{\sim}{b}}\underset{\sim}{p} - \underset{\sim}{p}\right|$$

$$proj_{\underset{\sim}{b}}\underset{\sim}{p} = \frac{1}{5}\begin{bmatrix}7\\14\end{bmatrix}$$

$$\underset{\sim}{p} = \begin{bmatrix}-3\\5\end{bmatrix}$$

So

$$d = \left|proj_{\underset{\sim}{b}}\underset{\sim}{p} - \underset{\sim}{p}\right|$$

$$d = \left|\frac{1}{5}\begin{bmatrix}7\\14\end{bmatrix} - \begin{bmatrix}-3\\5\end{bmatrix}\right|$$

$$d = \left|\begin{bmatrix}\frac{22}{5}\\-\frac{11}{5}\end{bmatrix}\right|$$

$$d^2 = \left(\frac{22}{5}\right)^2 + \left(-\frac{11}{5}\right)^2$$

$$d^2 = \frac{484}{25} + \frac{121}{25}$$

$$d = \sqrt{\frac{121}{5}}$$

$$d = \frac{11\sqrt{5}}{5} \text{ units}$$

21a

$$\underset{\sim}{r} = \begin{bmatrix}-1\\1\\0\end{bmatrix} + \lambda\begin{bmatrix}1\\0\\2\end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

Let $\lambda = 0$, so

$A = (-1,1,0)$

Let $\lambda = 1$, so

$B = (-1,1,0) + (1,0,2)$

$B = (0,1,2)$

21b Using A and B from part a:

$P = (1, -1\ ,1)$

$A = (-1, 1, 0)$

$B = (0, 1, 2)$

$$\overrightarrow{AP} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} - \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$\overrightarrow{AP} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$$\overrightarrow{AB} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$\overrightarrow{AB} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

21c $\overrightarrow{AB} = \underset{\sim}{b}$

$$\underset{\sim}{b} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$\overrightarrow{AP} = \underset{\sim}{p}$

$$\underset{\sim}{p} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$$proj_{\underset{\sim}{b}}\underset{\sim}{p} = \frac{\underset{\sim}{b} \cdot \underset{\sim}{p}}{|\underset{\sim}{b}|^2}\underset{\sim}{b}$$

$$proj_{\underset{\sim}{b}}\underset{\sim}{p} = \frac{(1 \times 2) + (0 \times -2) + (2 \times 1)}{1^2 + 0^2 + 2^2}\begin{bmatrix}1\\0\\2\end{bmatrix}$$

$$proj_{\underset{\sim}{b}}\underset{\sim}{p} = \frac{4}{5}\begin{bmatrix}1\\0\\2\end{bmatrix}$$

$$proj_{\underset{\sim}{b}}\underset{\sim}{p} = \frac{1}{5}\begin{bmatrix}4\\0\\8\end{bmatrix}$$

21d The perpendicular distance d from the point P to the line is:

$$d = \left|proj_{\underset{\sim}{b}}\underset{\sim}{p} - \underset{\sim}{p}\right|$$

$$proj_{\underset{\sim}{b}}\underset{\sim}{p} = \frac{1}{5}\begin{bmatrix}4\\0\\8\end{bmatrix}$$

$$\underset{\sim}{p} = \begin{bmatrix}2\\-2\\1\end{bmatrix}$$

So

$$d = \left|proj_{\underset{\sim}{b}}\underset{\sim}{p} - \underset{\sim}{p}\right|$$

$$d = \left|\frac{1}{5}\begin{bmatrix}4\\0\\8\end{bmatrix} - \begin{bmatrix}2\\-2\\1\end{bmatrix}\right|$$

$$d = \left|\begin{bmatrix}-\frac{6}{5}\\2\\\frac{3}{5}\end{bmatrix}\right|$$

$$d^2 = \left(-\frac{6}{5}\right)^2 + 2^2 + \left(\frac{3}{5}\right)^2$$

$$d^2 = \frac{36}{25} + 4 + \frac{9}{25}$$

$$d = \sqrt{\frac{145}{25}}$$

$$d = \frac{\sqrt{145}}{5} \text{ units}$$

22a Two lines are parallel if there is value a at which $\underset{\sim}{n}_1 = a\underset{\sim}{n}_2$

where $\underset{\sim}{n}_1$ and $\underset{\sim}{n}_2$ are the directional vectors for $\underset{\sim}{v}_1$ and $\underset{\sim}{v}_2$ respectively and a is a real number.

In this case the directional vectors $\underset{\sim}{n}_1 = \underset{\sim}{n}_2$ so the lines must be parallel.

22b For the line $\underset{\sim}{v}_2$

where $\underset{\sim}{v}_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + \mu \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$

A point on the line $\underset{\sim}{v}_2$ is:

$P = (1, -2, 1)$

The position vector for P, $\underset{\sim}{c}$ will be

$\underset{\sim}{c} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ or $\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}$

22c A point A on the line $\underset{\sim}{v}_1$ is $(2,1,-2)$.

A point P on the line $\underset{\sim}{v}_2$ is $(1,-2,1)$ from Question 22b.

$\overrightarrow{AP} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$

$\underset{\sim}{p} = \begin{bmatrix} -1 \\ -3 \\ 3 \end{bmatrix}$

The direction of lines $\underset{\sim}{v}_1$ and $\underset{\sim}{v}_1$ is

$\underset{\sim}{b} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$

The perpendicular distance d from the point P to the line $\underset{\sim}{v}_2$ is

$d = \left|proj_{\underset{\sim}{b}}\underset{\sim}{p} - \underset{\sim}{p}\right|$

$proj_{\underset{\sim}{b}}\, \underset{\sim}{p} = \dfrac{\underset{\sim}{b} \cdot \underset{\sim}{p}}{|\underset{\sim}{b}|^2}\underset{\sim}{b}$

Note that

$|\underset{\sim}{b}|^2 = 1^2 + (-2)^2 + 3^2 = 14$

$\underset{\sim}{b} \cdot \underset{\sim}{p} = (-1) + 6 + 9 = 14$

$proj_{\underset{\sim}{b}}\, \underset{\sim}{p} = \left(\frac{14}{14}\right) \underset{\sim}{b} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$

Hence,

$d = \left|proj_{\underset{\sim}{b}} \underset{\sim}{p} - \underset{\sim}{p}\right|$

$= \left|\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} - \begin{bmatrix} -1 \\ -3 \\ 3 \end{bmatrix}\right|$

$= \left|\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}\right|$

$d^2 = 2^2 + 1^2 + 0^2$

$= 5$

$d = \sqrt{5}$ units

23a If $ABCD$ is a rhombus then all sides will be equal and opposing sides will be parallel.

$\underset{\sim}{a} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$

$\underset{\sim}{b} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$\underset{\sim}{c} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$\overrightarrow{AB} = (\underset{\sim}{a} + \underset{\sim}{b}) - (\underset{\sim}{a})$

$\overrightarrow{AB} = \underset{\sim}{b}$

$\overrightarrow{AB} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$\overrightarrow{BC} = (\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c}) - (\underset{\sim}{a} + \underset{\sim}{b})$

$\overrightarrow{BC} = \underset{\sim}{c}$

$\overrightarrow{BC} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$\overrightarrow{CD} = \underset{\sim}{b}$

$\overrightarrow{CD} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$\overrightarrow{CD} = (\underset{\sim}{a} + \underset{\sim}{c}) - (\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c})$

$\overrightarrow{DA} = (\underset{\sim}{a}) - (\underset{\sim}{a} + \underset{\sim}{c})$

$\overrightarrow{DA} = -\underset{\sim}{c}$

$\overrightarrow{DA} = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$

So in order for the opposing sides to be parallel, for real numbers a and b,

$\overrightarrow{AB} = a\overrightarrow{CD}$ and $\overrightarrow{BC} = b\overrightarrow{DA}$

These hold for $a = 1$ and $b = -1$

If all sides are equal in length, then:

$\left|\overrightarrow{AB}\right|^2 = \left|\overrightarrow{BC}\right|^2 = \left|\overrightarrow{CD}\right|^2 = \left|\overrightarrow{DA}\right|^2$

We already know that

$\left|\overrightarrow{AB}\right|^2 = \left|\overrightarrow{CD}\right|^2$ and $\left|\overrightarrow{BC}\right|^2 = \left|\overrightarrow{DA}\right|^2$

So we need to see if $\left|\overrightarrow{BC}\right|^2 = \left|\overrightarrow{CD}\right|^2$

$\left|\overrightarrow{BC}\right|^2 = 2^2 + 3^2$

$\left|\overrightarrow{BC}\right|^2 = \left|\overrightarrow{CD}\right|^2$

So as $ABCD$ has all sides equal in length and opposite sides are parallel,

$ABCD$ is a rhombus.

23b The bisector of angle $\angle ABC$ is BD.

$\overrightarrow{OB} = \underset{\sim}{a} + \underset{\sim}{b}$

$\overrightarrow{OB} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

$\overrightarrow{OD} = \underset{\sim}{a} + \underset{\sim}{c}$

$\overrightarrow{OD} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\underset{\sim}{r}_{BD} = \overrightarrow{OB} + \lambda(\overrightarrow{OD} - \overrightarrow{OB})$

$\underset{\sim}{r}_{BD} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \lambda\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \end{bmatrix}\right)$

$\underset{\sim}{r}_{BD} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \lambda\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

As the bisector of angle $\angle ABC$ is perpendicular to $\angle BAD$,

the vector equation for AC will be

$$\underset{\sim}{r}_{AC} = \overrightarrow{OA} + \lambda \begin{bmatrix} -1 \\ (-1) \times 1 \end{bmatrix}$$

$$\underset{\sim}{r}_{AC} = \begin{bmatrix} -1 \\ -2 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

24a For $\overrightarrow{OM}$

$$\overrightarrow{OA} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

$$\overrightarrow{OC} = \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$$

$$\overrightarrow{OM} = \frac{1}{2}\left(\overrightarrow{OA} + \overrightarrow{OC}\right)$$

$$\overrightarrow{OM} = \frac{1}{2}\left(\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}\right)$$

$$\overrightarrow{OM} = \begin{bmatrix} 2 \\ \frac{1}{2} \\ \frac{3}{2} \end{bmatrix}$$

24b For a vector equation for the line BD

$$\underset{\sim}{r}_{BD} = \overrightarrow{OB} + \lambda\left(\overrightarrow{OD} - \overrightarrow{OB}\right)$$

$$\overrightarrow{OB} = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$$

$$\overrightarrow{OD} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

$$\underset{\sim}{r}_{BD} = \overrightarrow{OB} + \lambda\left(\overrightarrow{OD} - \overrightarrow{OB}\right)$$

$$\underset{\sim}{r}_{BD} = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} + \lambda\left(\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}\right)$$

$$\underset{\sim}{r}_{BD} = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix}$$

24c Solving for λ:

$$\underset{\sim}{r}_{BD} = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix}$$

$$z = 2\lambda$$

$$\lambda = \frac{z}{2}$$

$$M = \left(2, \frac{1}{2}, \frac{3}{2}\right)$$

So

$$\lambda = \frac{3}{2} \times \frac{1}{2}$$

$$\lambda = \frac{3}{4}$$

Substituting this back into $\underset{\sim}{r}_{BD}$ should be equal to M if M is on the line:

$$\begin{aligned} \text{LHS} &= \underset{\sim}{r}_{BD} \\ &= \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} + \frac{3}{4} \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 \\ \frac{3}{2} \\ \frac{3}{2} \end{bmatrix} \\ &= \begin{bmatrix} 2 \\ \frac{1}{2} \\ \frac{3}{2} \end{bmatrix} \\ &= \overrightarrow{OM} \\ &= \text{RHS} \end{aligned}$$

So M lies on the line $\underset{\sim}{r}_{BD}$.

24d $BM: MD$

$\overrightarrow{OB} = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$

$\overrightarrow{OD} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$

$\overrightarrow{OM} = \begin{bmatrix} 2 \\ \frac{1}{2} \\ \frac{3}{2} \end{bmatrix}$

$\overrightarrow{BM} = \left(\begin{bmatrix} 2 \\ \frac{1}{2} \\ \frac{3}{2} \end{bmatrix} - \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} \right)$

$\overrightarrow{BM} = \begin{bmatrix} 3 \\ \frac{3}{2} \\ \frac{3}{2} \end{bmatrix}$

$\overrightarrow{MD} = \left(\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ \frac{1}{2} \\ \frac{3}{2} \end{bmatrix} \right)$

$\overrightarrow{MD} = \begin{bmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$

$BM: MD = \begin{bmatrix} 3 \\ \frac{3}{2} \\ \frac{3}{2} \end{bmatrix} : \begin{bmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$

$= 3: 1$

25a The interval of the line segment AB.

25b The ray with endpoint B in the direction of $\underset{\sim}{b} - \underset{\sim}{a}$.

25b The ray with endpoint A in the direction of $\underset{\sim}{a} - \underset{\sim}{b}$.

Solutions to Exercise 5F Enrichment questions

26a P is the variable point $(-2+\lambda, 1, 2-\lambda)$, and so,

$$\overrightarrow{AP} = \begin{bmatrix} -3+\lambda \\ 0 \\ 1-\lambda \end{bmatrix}$$

$$\left|\overrightarrow{AP}\right|^2$$

$$= (\lambda-3)^2 + (1-\lambda)^2$$

$$= 2\lambda^2 - 8\lambda + 10$$

$$\left|\overrightarrow{AP}\right|$$

$$= \sqrt{2\lambda^2 - 8\lambda + 10}$$

26b $\left|\overrightarrow{AP}\right|$

$$= \sqrt{2(\lambda^2 - 4\lambda + 4) + 10 - 8}$$

$$= \sqrt{2(\lambda-2)^2 + 2}$$

The minimum distance from A to ℓ is $\sqrt{2}$ units, and this occurs when $\lambda = 2$.

26c When $\lambda = 2$, $\overrightarrow{AP} = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}$.

The direction of ℓ is $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

$$\begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = -1 + 0 + 1 = 0$$

So, $\overrightarrow{AP}$ is $\perp$ to the direction of ℓ.

Hence, the minimum distance is the perpendicular distance.

26d The dot product of $\overrightarrow{AP}$ and the direction of vector ℓ is:

$$\begin{bmatrix} -3+\lambda \\ 0 \\ 1-\lambda \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$= -3 + \lambda - 1 + \lambda$

$= 2\lambda - 4$

For the minimum (i.e., the perpendicular) distance,

$2\lambda - 4 = 0$

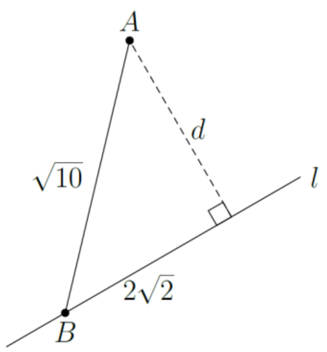
$\lambda = 2$

When $\lambda = 2$, $\overrightarrow{AP} = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}$ and $|\overrightarrow{AP}| = \sqrt{(-1)^2 + (0)^2 + (-1)^2} = \sqrt{2}$

26e Suppose we let B be the point on ℓ corresponding to $\lambda = 0$.

So, B is the point $(-2,1,2)$.

Then $\overrightarrow{BA} = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}$ and, hence, $|\overrightarrow{BA}| = \sqrt{10}$.



$proj_{\ell}\overrightarrow{BA}$

$= \dfrac{\left|\overrightarrow{BA} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right|}{\left\|\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right\|}$

$= \frac{3+1}{\sqrt{2}}$

$= 2\sqrt{2}$

Finally, by Pythagoras,

$d^2 = \left(\sqrt{10}\right)^2 - \left(2\sqrt{2}\right)^2 = 2$

Hence, the minimum distance from A to ℓ is $\sqrt{2}$.

Solutions to Exercise 5G Foundation questions

1a $c = (6, -9)$

$\underset{\sim}{c} = \begin{bmatrix} 6 \\ -9 \end{bmatrix}$

$r = 2\sqrt{7}$

For a circle centred on (a, b)

$(x - a)^2 + (y - b)^2 = r^2$

So

$(x - 6)^2 + (y + 9)^2 = \left(2\sqrt{7}\right)^2$

$(x - 6)^2 + (y + 9)^2 = 28$

1b $c = (6, -9)$

$\underset{\sim}{c} = \begin{bmatrix} 6 \\ -9 \end{bmatrix}$

$r = 2\sqrt{7}$

$|\underset{\sim}{r} - \underset{\sim}{c}| = r$

so

$\left|\underset{\sim}{r} - \begin{bmatrix} 6 \\ -9 \end{bmatrix}\right| = 2\sqrt{7}$

1c For $|\underset{\sim}{a}| = |\underset{\sim}{b}| = r^2$, where $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$

$\underset{\sim}{r} - \underset{\sim}{c} = \underset{\sim}{a} \cos\theta + \underset{\sim}{b} \sin\theta$

$\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 6 \\ -9 \end{bmatrix} = \begin{bmatrix} 2\sqrt{7} \\ 0 \end{bmatrix} \cos\theta + \begin{bmatrix} 0 \\ 2\sqrt{7} \end{bmatrix} \sin\theta$

So the two parametric equations are:

$x = 6 + 2\sqrt{7}\cos\theta$

$y = 2\sqrt{7}\sin\theta - 9$

2a $c = (-2, 7, -4)$

$\underset{\sim}{c} = \begin{bmatrix} -2 \\ 7 \\ -4 \end{bmatrix}$

$r = 9$

$(x+2)^2 + (y-7)^2 + (z+4)^2 = 81$

2b $|\underset{\sim}{r} - \underset{\sim}{c}| = r$

$\left|\underset{\sim}{r} - \begin{bmatrix} -2 \\ 7 \\ -4 \end{bmatrix}\right| = 9$

3a $\left|\underset{\sim}{r} - \begin{bmatrix} -5 \\ -10 \end{bmatrix}\right| = 3\sqrt{5}$

$(x+5)^2 + (y+10)^2 = \left(3\sqrt{5}\right)^2$

$(x+5)^2 + (y+10)^2 = 45$

3b $\left|\underset{\sim}{r} - \begin{bmatrix} 3 \\ -1 \\ 8 \end{bmatrix}\right| = 11$

$(x-3)^2 + (y+1)^2 + (z-8)^2 = 11^2$

$(x-3)^2 + (y+1)^2 + (z-8)^2 = 121$

4 $x = 5 + 2\sqrt{2}\cos\theta$

$y = 2\sqrt{2}\sin\theta - 3$

For $|\underset{\sim}{a}| = |\underset{\sim}{b}| = r^2$, where $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$

$\underset{\sim}{r} - \underset{\sim}{c} = \underset{\sim}{a}\cos\theta + \underset{\sim}{b}\sin\theta$

$x - 5 = 2\sqrt{2}\cos\theta$

$y - (-3) = 2\sqrt{2}\sin\theta$

So the vector equation will be:

$\left|\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 5 \\ -3 \end{bmatrix}\right| = 2\sqrt{2}$

The Cartesian equation is:

$(x-5)^2+(y+3)^2=\left(2\sqrt{2}\right)^2$

$(x-5)^2+(y+3)^2=8$

5a $x^2+y^2-6x+8y=0$

$(x^2-6x)+(y^2+8y)=0$

Completing the square gives

$(x^2-6x+9)-9+(y^2+8y+16)-16=0$

$(x-3)^2-9+(y+4)^2-16=0$

$(x-3)^2+(y+4)^2=25$

So

$r=5$

$\underset{\sim}{c}=\begin{bmatrix}3\\-4\end{bmatrix}$

$|\underset{\sim}{r}-\underset{\sim}{c}|=r$

So

$\left|\underset{\sim}{r}-\begin{bmatrix}3\\-4\end{bmatrix}\right|=5$

5b $x^2+y^2+z^2+x-2y-5z=0$

$(x^2+x)+(y^2-2y)+(z^2-5z)=0$

Completing the square gives

$\left(x^2+x+\frac{1}{4}\right)-\frac{1}{4}+(y^2-2y+1)-1+\left(z^2-5z+\frac{25}{4}\right)-\frac{25}{4}=0$

$\left(x+\frac{1}{2}\right)^2+(y-1)^2+\left(z-\frac{5}{2}\right)^2=\frac{30}{4}$

So

$\underset{\sim}{c}=\begin{bmatrix}-\frac{1}{2}\\1\\\frac{5}{2}\end{bmatrix}$

$$r^2 = \frac{30}{4}$$

$$r = \frac{\sqrt{30}}{2}$$

$$|\underset{\sim}{r} - \underset{\sim}{c}| = r$$

So

$$\left|\underset{\sim}{r} - \begin{bmatrix} -\frac{1}{2} \\ 1 \\ \frac{5}{2} \end{bmatrix}\right| = \frac{\sqrt{30}}{2}$$

6 $P(8, -5, 2)$

$$\left|\underset{\sim}{r} - \begin{bmatrix} 5 \\ -3 \\ -4 \end{bmatrix}\right| = 7$$

$$\left|\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} 5 \\ -3 \\ -4 \end{bmatrix}\right| = 7$$

For the point P:

$$\left|\begin{bmatrix} 8 \\ -5 \\ 2 \end{bmatrix} - \begin{bmatrix} 5 \\ -3 \\ -4 \end{bmatrix}\right|$$

$$= \left|\begin{bmatrix} 3 \\ -2 \\ 6 \end{bmatrix}\right|$$

$$= \sqrt{3^2 + (-2)^2 + 6^2}$$

$$= \sqrt{9 + 4 + 36}$$

$$= \sqrt{49}$$

$$= 7$$

Thus point P is a point on the surface of the sphere.

7 $A(-4,-5,6)$

$\left|\underset{\sim}{r}-\begin{bmatrix}-2\\4\\-1\end{bmatrix}\right| = 3\sqrt{15}$

$\left|\begin{bmatrix}x\\y\\z\end{bmatrix}-\begin{bmatrix}-2\\4\\-1\end{bmatrix}\right| = 3\sqrt{15}$

For the point A:

$\left|\begin{bmatrix}-4\\-5\\6\end{bmatrix}-\begin{bmatrix}-2\\4\\-1\end{bmatrix}\right|$

$= \left|\begin{bmatrix}-2\\-9\\7\end{bmatrix}\right|$

$= \sqrt{(-2)^2+(-9)^2+7^2}$

$= \sqrt{4+81+49}$

$= \sqrt{134}$

Since $\sqrt{134} < 3\sqrt{15}$, point A lies inside the circle.

8 $\left(\underset{\sim}{r}-(2\underset{\sim}{i}+\underset{\sim}{j}-\underset{\sim}{k})\right)\cdot\left(\underset{\sim}{r}-(2\underset{\sim}{i}+\underset{\sim}{j}-\underset{\sim}{k})\right) = 20$

$\begin{bmatrix}x-2\\y-1\\z+1\end{bmatrix}\cdot\begin{bmatrix}x-2\\y-1\\z+1\end{bmatrix} = 20$

$(x-2)^2+(y-1)^2+(z+1)^2 = 20$

$r^2 = 20$

$r = \sqrt{20}$
$r = 2\sqrt{5}$

So the centre of the circle is $(2,1,-1)$

The radius of the circle is $2\sqrt{5}$.

9a $\underset{\sim}{r}(t) = (2\cos t + 1)\underset{\sim}{i} + (2\sin t - 1)\underset{\sim}{j}$

$x = 2\cos t + 1$

$y = 2\sin t - 1$

9b $x = 2\cos t + 1$

$\cos t = \dfrac{x-1}{2}$

$y = 2\sin t - 1$

$\sin t = \dfrac{y+1}{2}$

Since $\sin^2 t + \cos^2 t = 1$,

$\left(\dfrac{x-1}{2}\right)^2 + \left(\dfrac{y+1}{2}\right)^2 = 1$

$(x-1)^2 + (y+1)^2 = 4$

Solutions to Exercise 5G Development questions

10a $\underset{\sim}{r}(t) = (t-2)\underset{\sim}{i} + (t^2-2)\underset{\sim}{j}$, for $t \geq 0$

$x = t - 2$

$t = x + 2$

$y = t^2 - 2$

$y = (x+2)^2 - 2$

10b Since $t \geq 0$ and $x = t - 2$,

$x \geq 0 - 2$

$x \geq -2$

So domain is $[-2, \infty)$.

10c Graph of $y = (x+2)^2 - 2$ for $x \geq -2$ shown below.

x-intercept: when $y = 0$,

$(x+2)^2 - 2 = 0$

$(x+2)^2 = 2$

$x + 2 = \pm\sqrt{2}$

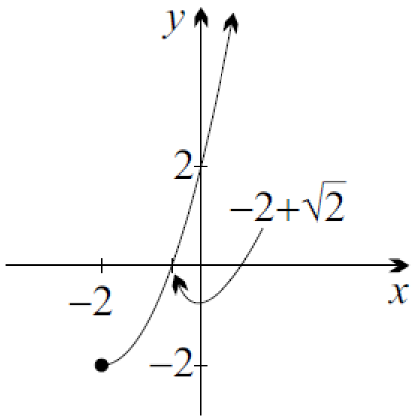
$x = -2 \pm \sqrt{2}$

But $x \geq -2$, so x-intercept is $-2 + \sqrt{2}$.

y-intercept: when $x = 0$,

$y = (0+2)^2 - 2 = 2$

So y-intercept is 2.



11a $\underset{\sim}{a} = 3\underset{\sim}{i} - \underset{\sim}{j}$

For a circle with radius r, centred on the origin its vector equation will be:

$r = |\underset{\sim}{a}|$

$|\underset{\sim}{r}|^2 = 3^2 + (-1)^2$

$|\underset{\sim}{r}| = \sqrt{10}$

11b $\underset{\sim}{a} = 3\underset{\sim}{i} - \underset{\sim}{j}$

The tangent of the circle at point $\acute{a}$ is perpendicular to the radius.

So $(\underset{\sim}{r} - \underset{\sim}{a}) \cdot \underset{\sim}{a} = 0$

$\left(\underset{\sim}{r} - (3\underset{\sim}{i} - \underset{\sim}{j})\right) \cdot (3\underset{\sim}{i} - \underset{\sim}{j}) = 0$

11c $\underset{\sim}{a} = 3\underset{\sim}{i} - \underset{\sim}{j}$

$r = \sqrt{10}$

$(\underset{\sim}{r} - \underset{\sim}{a}) \cdot \underset{\sim}{a} = 0$

$\underset{\sim}{r} \cdot \underset{\sim}{a} - \underset{\sim}{a} \cdot \underset{\sim}{a} = 0$

$(x\underset{\sim}{i} + y\underset{\sim}{j}) \cdot (3\underset{\sim}{i} - \underset{\sim}{j}) + 10 = 0$

$(3x - y) - 10 = 0$

So the Cartesian equation for the tangent is:

$y = 3x - 10$

12a $\underset{\sim}{r} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \lambda \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

So the parametric equations are:

$x = 3\lambda + 1$

$y = 2\lambda - 1$

12b $\left|\underset{\sim}{r} - \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right| = \sqrt{13}$

$\underset{\sim}{r} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \lambda \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$\left|\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \lambda \begin{bmatrix} 3 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right| = \sqrt{13}$

$\left|\lambda \begin{bmatrix} 3 \\ 2 \end{bmatrix}\right| = \sqrt{13}$

$9\lambda^2 + 4\lambda^2 = 13$

$\lambda^2 = 1$

$\lambda = \pm 1$

Substituting $\lambda = \pm 1$ and solving for x and y:

$A = (3(1) + 1, 2(1) - 1)$

$A = (4, 1)$

$B = (3(-1) + 1, 2(-1) - 1)$

$B = (-2, -3)$

13 Let d be the distance between the centres of the spheres. Then

$$\begin{aligned} d^2 &= \big(5 - (-3)\big)^2 + \big((-6) - 2\big)^2 + (3 - 7)^2 \\ &= 8^2 + (-8)^2 + (-4)^2 \\ &= 64 + 64 + 16 \\ &= 144 \end{aligned}$$

$d = 12$

Since the sum of the radii is $7 + 5 = 12 = d$, the spheres must touch each other at a single point, otherwise their centres would be closer.

14a The circle of intersection will be perpendicular to the axis the spheres are centred on as both spheres centre on the z-axis, one at $(0, 0, 0)$ and the other at $(0, 0, 5)$.

The sphere of intersection between the two spheres will be parallel to the xy-plane.

14b $|\underset{\sim}{r}| = 3$

$$\left|\underset{\sim}{r} - \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix}\right| = 4$$

$$\left|\begin{bmatrix} x \\ y \\ z-5 \end{bmatrix}\right| = 4$$

$x^2 + y^2 + z^2 = 9$ (0)

$y^2 = 9 - x^2 - z^2$ (1)

$x^2 + y^2 + (z-5)^2 = 16$ (2)

Substituting (1) into (2) gives us:

$x^2 + (9 - x^2 - z^2) + (z-5)^2 = 16$

$9 - z^2 + (z-5)^2 = 16$

$9 - z^2 + z^2 - 10z + 25 = 16$

$-18 = -10z$

$z = \dfrac{9}{5}$

Substituting this equation into (1):

$x^2 + y^2 + \left(\dfrac{9}{5}\right)^2 = 9$

$x^2 + y^2 = 9 - \dfrac{81}{25}$

$x^2 + y^2 = \dfrac{144}{25}$

As $x^2 + y^2 = r^2$

$r = \frac{12}{5}$

The intersecting circle is centred on $(0, 0, z)$. So, the intersecting circle is centred on:

$C = \left(0, 0, \frac{9}{5}\right)$

15 $(x-3)^2 + (y+4)^2 + (z+2)^2 = 81$

$\left|\underset{\sim}{r} - \begin{bmatrix} 3 \\ -4 \\ -2 \end{bmatrix}\right| = 81$

$\underset{\sim}{r} = \begin{bmatrix} -3 \\ 16 \\ -9 \end{bmatrix} + \lambda \begin{bmatrix} 7 \\ -12 \\ 3 \end{bmatrix}$

$\left|\begin{bmatrix} -3 \\ 16 \\ -9 \end{bmatrix} + \lambda \begin{bmatrix} 7 \\ -12 \\ 3 \end{bmatrix} - \begin{bmatrix} 3 \\ -4 \\ -2 \end{bmatrix}\right| = 81$

$\left|\begin{bmatrix} -6+7\lambda \\ 20-12\lambda \\ -7+3\lambda \end{bmatrix}\right| = 81$

$(-6+7\lambda)^2 + (20-12\lambda)^2 + (-7+3\lambda)^2 = 81$

$36 - 84\lambda + 49\lambda^2 + 400 - 480\lambda + 144\lambda^2 + 49 - 42\lambda + 9\lambda^2 = 81$

$202\lambda^2 - 606\lambda + 404 = 0$

$\lambda^2 - 3\lambda + 2 = 0$

$(\lambda - 1)(\lambda - 2) = 0$

So the line intersects with the sphere for the values of λ where

$\lambda = 1, 2$

Substituting these values back into the line equation $\underset{\sim}{r}$:

For $\lambda = 1$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ 16 \\ -9 \end{bmatrix} + \begin{bmatrix} 7 \\ -12 \\ 3 \end{bmatrix}$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ -6 \end{bmatrix}$

For $\lambda = 2$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ 16 \\ -9 \end{bmatrix} + 2\begin{bmatrix} 7 \\ -12 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11 \\ -8 \\ -3 \end{bmatrix}$$

The line intersects with the sphere at points:

$(4, 4, -6)$ and $(11, -8, -3)$

16a $\underset{\sim}{r} = \begin{bmatrix} -2 \\ 3 \\ 4 \end{bmatrix} + \lambda\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ 4 \end{bmatrix} + \lambda\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$$

$x = 3\lambda - 2$

$y = 4\lambda + 3$

$z = 5\lambda + 4$

16b Substituting the parametric equations into the equation for the plane:

$x = 3\lambda - 2$

$y = 4\lambda + 3$

$z = 5\lambda + 4$

$2x + 4y - z = 55$

$2(3\lambda - 2) + 4(4\lambda + 3) - (5\lambda + 4) = 55$

$6\lambda - 4 + 16\lambda + 12 - 5\lambda - 4 = 55$

$17\lambda + 4 = 55$

$\lambda = 3$

Substituting λ into $\underset{\sim}{r}$ will give us the point of intersection.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ 4 \end{bmatrix} + 3\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 15 \\ 19 \end{bmatrix}$$

So the point of intersection between the line and the plane is:

$(7, 15, 19)$

17a $\quad \underset{\sim}{r} = \begin{bmatrix} \frac{1}{2}(e^t + e^{-t}) \\ \frac{1}{2}(e^t + e^{-t}) \end{bmatrix}$

$$x = \frac{1}{2}(e^t + e^{-t})$$

$$x^2 = \frac{1}{4}(e^t + e^{-t})^2$$

$$x^2 = \frac{1}{4}(e^{t+t} + e^{t-t} + e^{-t+t} + e^{-t-t})$$

$$x^2 = \frac{1}{4}(e^{2t} + 2 + e^{-2t})$$

$$x^2 - \frac{1}{2} = \frac{1}{4}(e^{2t} + e^{-2t})$$

$$y = \frac{1}{2}(e^t - e^{-t})$$

$$y^2 = \frac{1}{4}(e^t - e^{-t})^2$$

$$y^2 = \frac{1}{4}(e^{t+t} - e^{t-t} - e^{-t+t} + e^{-t-t})$$

$$y^2 = \frac{1}{4}(e^{2t} + e^{-2t} - 2)$$

$$y^2 + \frac{1}{2} = \frac{1}{4}(e^{2t} + e^{-2t})$$

Equating the LHS for x and y gives:

$$x^2 - \frac{1}{2} = y^2 + \frac{1}{2}$$

$$x^2 - y^2 = 1$$

17b $\underset{\sim}{r} = \begin{bmatrix} 2\sin t \\ 2\sin t\tan t \end{bmatrix}$

$x = 2\sin t$

We know that, $\cos^2 t + \sin^2 t = 1$

So

$x^2 = 4\sin^2 t$

$x^2 = 4 - 4\cos^2 t$

$\dfrac{4 - x^2}{4} = \cos^2 t$

$\tan t = \dfrac{\sin t}{\cos t}$

$y = 2\sin t\tan t$

$y = \dfrac{2\sin^2 t}{\cos t}$

$y^2 = 4\sin^4 t \times \dfrac{1}{\cos^2 t}$

$y^2 = \dfrac{1}{4}x^4 \times \dfrac{1}{\left(\dfrac{4 - x^2}{4}\right)}$

$y^2 = \dfrac{x^4}{(4 - x^2)}$

$y = \pm\dfrac{x^2}{\sqrt{(4 - x^2)}}$

Solutions to Exercise 5G Enrichment questions

18a Possible direction vectors are,

$$\overrightarrow{AB} = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} \quad \text{and} \quad \overrightarrow{AC} = \begin{bmatrix} 2 \\ 5 \\ -2 \end{bmatrix}$$

Hence, a possible equation for $\mathcal{P}$ is,

$$\underset{\sim}{r} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} + \mu \begin{bmatrix} 2 \\ 5 \\ -2 \end{bmatrix}, \text{where } \lambda, \mu \in R$$

(There is an infinite number of correct answers to part a. In each case, the corresponding cartesian equation is the same.)

18b A set of parametric equations of $\mathcal{P}$ is:

$x = 1 + \lambda + 2\mu \qquad (1)$

$y = -1 + 4\lambda + 5\mu \qquad (2)$

$z = \lambda - 2\mu \qquad (3)$

(1) – (3):

$x - z = 1 + 4\mu$

$\therefore \mu = \frac{1}{4}(x - z - 1)$

Substituting μ into (3)

$z = \lambda - \frac{1}{2}(x - z - 1)$

$\lambda = \frac{1}{2}(x - z - 1) + z$

$\lambda = \frac{1}{2}(x + z - 1)$

Substituting λ and μ into (2)

$y = -1 + 2(x + z - 1) + \frac{5}{4}(x - z - 1)$

$4y = -4 + 8x + 8z - 8 + 5x - 5z - 5$

$13x - 4y + 3z = 17$

19 We want a parametric vector equation of the plane.

Introduce 2 parameters, λ and μ letting $y = \lambda$ and $z = \mu$.

Then, $ax + b\lambda + c\mu = d$

So, $x = \frac{1}{a}(d - b\lambda - c\mu)$

So, a parametric vector equation is,

$$\underset{\sim}{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{a}(d - b\lambda - c\mu) \\ \lambda \\ \mu \end{bmatrix}$$

i.e., $\underset{\sim}{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{d}{a} \\ 0 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} \frac{-b}{a} \\ 1 \\ 0 \end{bmatrix} + \mu \begin{bmatrix} \frac{-c}{a} \\ 0 \\ 1 \end{bmatrix}$

Hence, two non-parallel direction vectors are:

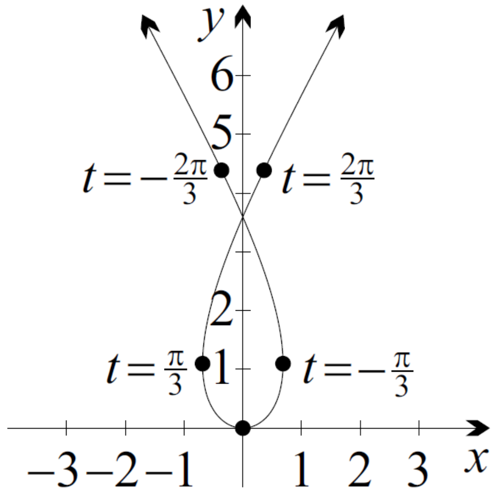
$\begin{bmatrix} \frac{-b}{a} \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} \frac{-c}{a} \\ 0 \\ 1 \end{bmatrix}$

Now, $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} \frac{-b}{a} \\ 0 \\ 1 \end{bmatrix} = -b + b = 0$

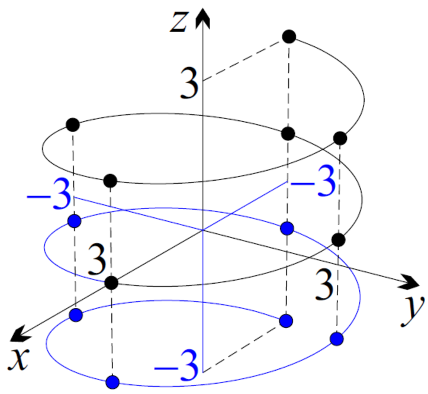
And, $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} \frac{-c}{a} \\ 0 \\ 1 \end{bmatrix} = -c + c = 0$

Hence, the vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is perpendicular to the plane.

20a



20b



$t < 0$: blue, $t \geq 0$: black

Solutions to Exercise 5H Chapter review

1 $\underset{\sim}{a} = 6\underset{\sim}{i} - 3\underset{\sim}{j} + 2\underset{\sim}{k}$

$|\underset{\sim}{a}|^2 = 6^2 + (-3)^2 + 2^2$

$|\underset{\sim}{a}|^2 = 36 + 9 + 4$

$|\underset{\sim}{a}|^2 = 49$

$|\underset{\sim}{a}| = 7$

$\hat{\underset{\sim}{a}} = \dfrac{\underset{\sim}{a}}{|\underset{\sim}{a}|}$

$= \dfrac{1}{7}\left(6\underset{\sim}{i} - 3\underset{\sim}{j} + 2\underset{\sim}{k}\right)$

$= \dfrac{6}{7}\underset{\sim}{i} - \dfrac{3}{7}\underset{\sim}{j} + \dfrac{2}{7}\underset{\sim}{k}$

2a $A = 3\underset{\sim}{i} - 1\underset{\sim}{j} - 6\underset{\sim}{k}$

$B = -2\underset{\sim}{i} - 5\underset{\sim}{j} + 1\underset{\sim}{k}$

$\overrightarrow{AB} = -2\underset{\sim}{i} - 5\underset{\sim}{j} + 1\underset{\sim}{k} - \left(3\underset{\sim}{i} - 1\underset{\sim}{j} - 6\underset{\sim}{k}\right)$

$= -5\underset{\sim}{i} - 4\underset{\sim}{j} + 7\underset{\sim}{k}$

2b $\overrightarrow{BA} = -\overrightarrow{AB}$

$= 5\underset{\sim}{i} + 4\underset{\sim}{j} - 7\underset{\sim}{k}$

2c Distance $AB = |\overrightarrow{AB}|$

$\overrightarrow{AB} = -5\underset{\sim}{i} - 4\underset{\sim}{j} + 7\underset{\sim}{k}$

$\left|\overrightarrow{AB}\right|^2 = (-5)^2 + (-4)^2 + 7^2$

$\left|\overrightarrow{AB}\right|^2 = 25 + 16 + 49$

$\left|\overrightarrow{AB}\right| = \sqrt{90} = 3\sqrt{10}$

3 $A = (6, 12, 7)$

$B = (10, 2, -15)$

$C = (-4, 1, 5)$

$D = (-2, -4, -6)$

$\overrightarrow{AB} = (10 - 6)\underset{\sim}{i} + (2 - 12)\underset{\sim}{j} + (-15 - 7)\underset{\sim}{k}$

$= 4\underset{\sim}{i} - 10\underset{\sim}{j} - 22\underset{\sim}{k}$

$\overrightarrow{CD} = (-2 - (-4))\underset{\sim}{i} + (-4 - 1)\underset{\sim}{j} + (-6 - 5)\underset{\sim}{k}$

$= 2\underset{\sim}{i} - 5\underset{\sim}{j} - 11\underset{\sim}{k}$

If $\overrightarrow{AB}||\overrightarrow{CD}$, then $\overrightarrow{AB} = a\overrightarrow{CD}$, for $a \in \mathbb{R}$

$\overrightarrow{AB} = a\overrightarrow{CD}$

This holds for $a = 2$.

So $\overrightarrow{AB}$ is parallel to $\overrightarrow{CD}$.

4 $A = (2, 3, -1)$

$B = (5, -1, 1)$

$C = (-4, 11, -5)$

For A, B and C to be collinear $\overrightarrow{AB}||\overrightarrow{BC}$

$\overrightarrow{AB} = (5 - 2)\underset{\sim}{i} + (-1 - 3)\underset{\sim}{j} + (1 - (-1))\underset{\sim}{k}$

$= 3\underset{\sim}{i} - 4\underset{\sim}{j} + 2\underset{\sim}{k}$

$\overrightarrow{BC} = (-4 - 5)\underset{\sim}{i} + (11 - (-1))\underset{\sim}{j} + (-5 - 1)\underset{\sim}{k}$

$= -9\underset{\sim}{i} + 12\underset{\sim}{j} - 6\underset{\sim}{k}$

$= -3\overrightarrow{AB}$

Since $\overrightarrow{AB} = a\overrightarrow{BC}$, for $a \in \mathbb{R}$,

A, B and C are collinear.

5a $\underset{\sim}{a} = \begin{bmatrix} 4 \\ -3 \\ 5 \end{bmatrix}$

$\underset{\sim}{a} \cdot \underset{\sim}{a} = |\underset{\sim}{a}|^2$

$$= \begin{bmatrix} 4 \\ -3 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -3 \\ 5 \end{bmatrix}$$

$$= 16 + 9 + 25$$

$$= 50$$

5b $\underset{\sim}{b} = \begin{bmatrix} 6 \\ 2 \\ -2 \end{bmatrix}$

$\underset{\sim}{b} \cdot \underset{\sim}{b} = |\underset{\sim}{b}|^2$

$$= \begin{bmatrix} 6 \\ 2 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 2 \\ -2 \end{bmatrix}$$

$$= 36 + 4 + 4$$

$$= 44$$

5c $\underset{\sim}{a} = \begin{bmatrix} 4 \\ -3 \\ 5 \end{bmatrix}$

$\underset{\sim}{b} = \begin{bmatrix} 6 \\ 2 \\ -2 \end{bmatrix}$

$\underset{\sim}{a} \cdot \underset{\sim}{b} = \begin{bmatrix} 4 \\ -3 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 2 \\ -2 \end{bmatrix}$

$$= 24 - 6 - 10$$

$$= 8$$

5d $\underset{\sim}{a} = \begin{bmatrix} 4 \\ -3 \\ 5 \end{bmatrix}$

$\underset{\sim}{b} = \begin{bmatrix} 6 \\ 2 \\ -2 \end{bmatrix}$

$(\underset{\sim}{a} + \underset{\sim}{b}) \cdot (\underset{\sim}{a} + \underset{\sim}{b})$

$= |\underset{\sim}{a}^2| + 2\underset{\sim}{a} \cdot \underset{\sim}{b} + |\underset{\sim}{b}^2|$

$= 50 + 16 + 44$

$= 110$

6 $\quad \underset{\sim}{a} = \begin{bmatrix} -1 \\ 4 \\ 5 \end{bmatrix}$

$\underset{\sim}{b} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

$\underset{\sim}{c} = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$

$\underset{\sim}{d} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$

$\overrightarrow{AB} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - \begin{bmatrix} -1 \\ 4 \\ 5 \end{bmatrix}$

$= \begin{bmatrix} 2 \\ -4 \\ -3 \end{bmatrix}$

$\overrightarrow{CD} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} - \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$

$= \begin{bmatrix} 5 \\ 1 \\ 2 \end{bmatrix}$

$\overrightarrow{AB}$ and $\overrightarrow{CD}$ are perpendicular if:

$\overrightarrow{AB} \cdot \overrightarrow{CD} = 0$

$\text{LHS} = \overrightarrow{AB} \cdot \overrightarrow{CD}$

$= \begin{bmatrix} 2 \\ -4 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 1 \\ 2 \end{bmatrix}$

$= 10 - 4 - 6$

$= 0$

$= \text{RHS}$

So $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are perpendicular.

7 $\underset{\sim}{a} = (\lambda + 4)\underset{\sim}{i} + 2\underset{\sim}{j} + 4\underset{\sim}{k}$

$\underset{\sim}{b} = 2\underset{\sim}{i} + (\lambda - 4)\underset{\sim}{j} + \underset{\sim}{k}$

$\underset{\sim}{a} \cdot \underset{\sim}{b}$

$= ((\lambda + 4)\underset{\sim}{i} + 2\underset{\sim}{j} + 4\underset{\sim}{k}) \cdot (2\underset{\sim}{i} + (\lambda - 4)\underset{\sim}{j} + \underset{\sim}{k})$

$= 2(\lambda + 4) + 2(\lambda - 4) + 4$

$= 2\lambda + 8 + 2\lambda - 8 + 4$

$= 4\lambda + 4$

For $\underset{\sim}{a}$ and $\underset{\sim}{b}$ to be perpendicular, $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$.

$4\lambda + 4 = 0$

$\lambda = -1$

8 $\underset{\sim}{a} = \begin{bmatrix} -2 \\ -3 \\ 2 \end{bmatrix}$

$\underset{\sim}{b} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

$|\underset{\sim}{a}|^2 = 4 + 9 + 4$

$|\underset{\sim}{a}|^2 = 4 + 9 + 4$

$= 17$

$|\underset{\sim}{a}| = \sqrt{17}$

$|\underset{\sim}{b}|^2 = 1 + 4 + 1$

$= 6$

$|\underset{\sim}{b}| = \sqrt{6}$

$\underset{\sim}{a} \cdot \underset{\sim}{b} = \begin{bmatrix} -2 \\ -3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

$= -2 + 6 + 2$

$= 6$

$\cos\theta = \dfrac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}$

$$= \frac{6}{\sqrt{102}}$$

9 $\underset{\sim}{a} = 2\underset{\sim}{i} + \underset{\sim}{j} - 3\underset{\sim}{k}$

$\underset{\sim}{b} = 4\underset{\sim}{i} - 3\underset{\sim}{j} - 2\underset{\sim}{k}$

$$Proj_{\underset{\sim}{b}}\underset{\sim}{a} = \frac{\underset{\sim}{b} \cdot \underset{\sim}{a}}{\underset{\sim}{b} \cdot \underset{\sim}{b}}\underset{\sim}{b}$$

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = (2\underset{\sim}{i} + \underset{\sim}{j} - 3\underset{\sim}{k}) \cdot (4\underset{\sim}{i} - 3\underset{\sim}{j} - 2\underset{\sim}{k})$$

$$= 8 - 3 + 6$$

$$= 11$$

$$\underset{\sim}{b} \cdot \underset{\sim}{b} = \left(4\underset{\sim}{i} - 3\underset{\sim}{j} - 2\underset{\sim}{k}\right) \cdot \left(4\underset{\sim}{i} - 3\underset{\sim}{j} - 2\underset{\sim}{k}\right)$$

$$= 16 + 9 + 4$$

$$= 29$$

$$Proj_{\underset{\sim}{b}}\underset{\sim}{a} = \frac{\underset{\sim}{b} \cdot \underset{\sim}{a}}{\underset{\sim}{b} \cdot \underset{\sim}{b}}\underset{\sim}{b}$$

$$= \frac{11}{29}(4\underset{\sim}{i} - 3\underset{\sim}{j} - 2\underset{\sim}{k})$$

$$= \frac{44}{29}\underset{\sim}{i} - \frac{33}{29}\underset{\sim}{j} - \frac{22}{29}\underset{\sim}{k}$$

10a $P = (2, 3, 1)$

$A = (1, 0, -2)$

$B = (0, -1, 1)$

$$\overrightarrow{AP} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}$$

$$\overrightarrow{AB} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}$$

10b $\overrightarrow{AP} = \underset{\sim}{p}$

$$= \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}$$

$\overrightarrow{AB} = \underset{\sim}{b}$

$$= \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}$$

$$\underset{\sim}{b} \cdot \underset{\sim}{p} = \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}$$

$$= -1 - 3 + 9$$

$$= 5$$

$$\underset{\sim}{b} \cdot \underset{\sim}{b} = \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}$$

$$= 1 + 1 + 9$$

$$= 11$$

$$Proj_{\underset{\sim}{b}} \underset{\sim}{p} = \frac{\underset{\sim}{b} \cdot \underset{\sim}{p}}{\underset{\sim}{b} \cdot \underset{\sim}{b}} \underset{\sim}{b}$$

$$= \frac{5}{11} \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}$$

10c $\underset{\sim}{p} = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}$

$$d = \left| Proj_{\underset{\sim}{b}} \underset{\sim}{p} - \underset{\sim}{p} \right| = \left| \frac{\underset{\sim}{b} \cdot \underset{\sim}{p}}{\underset{\sim}{b} \cdot \underset{\sim}{b}} \underset{\sim}{b} - \underset{\sim}{p} \right|$$

$$Proj_{\underset{\sim}{b}} \underset{\sim}{p} = \frac{5}{11} \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}$$

$$d = \left| \frac{5}{11} \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix} \right|$$

$$= \left\| \begin{bmatrix} -\frac{5}{11} \\ -\frac{5}{11} \\ \frac{15}{11} \end{bmatrix} - \begin{bmatrix} \frac{11}{11} \\ \frac{33}{11} \\ \frac{33}{11} \end{bmatrix} \right\|$$

$$= \left\| \begin{bmatrix} -\frac{16}{11} \\ -\frac{38}{11} \\ -\frac{18}{11} \end{bmatrix} \right\|$$

$$= \sqrt{\left(-\frac{16}{11}\right)^2 + \left(-\frac{38}{11}\right)^2 + \left(-\frac{18}{11}\right)^2}$$

$$= \sqrt{\frac{256}{121} + \frac{1444}{121} + \frac{324}{121}}$$

$$= \frac{\sqrt{2024}}{11}$$

$$= \frac{2\sqrt{506}}{11} \text{ units}$$

11 $X = (-5, 7, 3)$

$Y = (5, -2\ 6)$

$Z = (3, -5, -4)$

$$\overrightarrow{YX} = \begin{bmatrix} -5-5 \\ 7-(-2) \\ 3-6 \end{bmatrix} = \begin{bmatrix} -10 \\ 9 \\ -3 \end{bmatrix}$$

$$\overrightarrow{YZ} = \begin{bmatrix} 3-5 \\ -5-(-2) \\ -4-6 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \\ -10 \end{bmatrix}$$

$$\overrightarrow{YX} \cdot \overrightarrow{YZ} = \begin{bmatrix} -10 \\ 9 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ -3 \\ -10 \end{bmatrix}$$

$$\overrightarrow{XY} \cdot \overrightarrow{YZ} = (-10 \times -2) + (9 \times -3) + (-3 \times -10)$$

$$= 20 - 27 + 30$$

$= 23$

$\left|\overrightarrow{XY}\right|^2 = (-10)^2 + 9^2 + (-3)^2$

$= 100 + 81 + 9$

$= 190$

$\left|\overrightarrow{XY}\right| = \sqrt{190}$

$\left|\overrightarrow{YZ}\right|^2 = (-2)^2 + (-3)^2 + (-10)^2$

$= 4 + 9 + 100$

$= 113$

$\left|\overrightarrow{YZ}\right| = \sqrt{113}$

$$\cos \angle XYZ = \frac{\overrightarrow{XY} \cdot \overrightarrow{YZ}}{\left|\overrightarrow{XY}\right|\left|\overrightarrow{YZ}\right|}$$

$$\cos \angle XYZ = \frac{23}{\sqrt{190} \times \sqrt{113}}$$

$\angle XYZ \doteqdot 81°$

12a For a rhombus with verticies A, B, C, D with respective position vectors $\underset{\sim}{a}, \underset{\sim}{b}, \underset{\sim}{c}, \underset{\sim}{d}$ and diagonals of length a and b:

$$\underset{\sim}{a} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\underset{\sim}{b} = \begin{bmatrix} \frac{1}{2}a \\ \frac{1}{2}b \end{bmatrix}$$

$$\underset{\sim}{c} = \begin{bmatrix} a \\ 0 \end{bmatrix}$$

$$\underset{\sim}{d} = \begin{bmatrix} \frac{1}{2}a \\ -\frac{1}{2}b \end{bmatrix}$$

The midpoint of $AB = M$

$$\overrightarrow{OM} = \frac{1}{2}\left(\begin{bmatrix} \frac{1}{2}a \\ \frac{1}{2}b \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right)$$

$$= \begin{bmatrix} \frac{1}{4}a \\ \frac{1}{4}b \end{bmatrix}$$

The midpoint of $BC = N$

$$\overrightarrow{ON} = \frac{1}{2}\left(\begin{bmatrix} a \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{2}a \\ \frac{1}{2}b \end{bmatrix}\right)$$

$$= \begin{bmatrix} \frac{3}{4}a \\ \frac{1}{4}b \end{bmatrix}$$

The midpoint of $CD = P$

$$\overrightarrow{OP} = \frac{1}{2}\left(\begin{bmatrix} \frac{1}{2}a \\ -\frac{1}{2}b \end{bmatrix} + \begin{bmatrix} a \\ 0 \end{bmatrix}\right)$$

$$= \begin{bmatrix} \frac{3}{4}a \\ -\frac{1}{4}b \end{bmatrix}$$

The midpoint of $DA = Q$

$$\overrightarrow{OQ} = \frac{1}{2}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{2}a \\ -\frac{1}{2}b \end{bmatrix}\right)$$

$$= \begin{bmatrix} \frac{1}{4}a \\ -\frac{1}{4}b \end{bmatrix}$$

So the sides of the quadrilateral $MNPQ$ will be:

$$\overrightarrow{MN} = \begin{bmatrix} \frac{3}{4}a \\ \frac{1}{4}b \end{bmatrix} - \begin{bmatrix} \frac{1}{4}a \\ \frac{1}{4}b \end{bmatrix} = \begin{bmatrix} \frac{1}{2}a \\ 0 \end{bmatrix}$$

$$\overrightarrow{NP} = \begin{bmatrix} \frac{3}{4}a \\ -\frac{1}{4}b \end{bmatrix} - \begin{bmatrix} \frac{3}{4}a \\ \frac{1}{4}b \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{2}b \end{bmatrix}$$

$$\overrightarrow{PQ}=\begin{bmatrix}\frac{1}{4}a\\-\frac{1}{4}b\end{bmatrix}-\begin{bmatrix}\frac{3}{4}a\\-\frac{1}{4}b\end{bmatrix}=\begin{bmatrix}-\frac{1}{2}a\\0\end{bmatrix}$$

$$\overrightarrow{QM}=\begin{bmatrix}\frac{1}{4}a\\\frac{1}{4}b\end{bmatrix}-\begin{bmatrix}\frac{1}{4}a\\-\frac{1}{4}b\end{bmatrix}=\begin{bmatrix}0\\\frac{1}{2}b\end{bmatrix}$$

For a rectangle, opposite sides are equal in length and parallel, and adjacent sides are at right-angles.

$\overrightarrow{MN}=-\overrightarrow{PQ}$

$\overrightarrow{NP}=-\overrightarrow{QM}$

So $\overrightarrow{MN}$ and$\overrightarrow{PQ}$ are equal in length and parallel and $\overrightarrow{NP}$ and $\overrightarrow{QM}$ are equal in length and parallel.

For adjacent sides $\overrightarrow{MN}$ and $\overrightarrow{NP}$ to be perpendicular, $\overrightarrow{MN}.\overrightarrow{NP}=0$:

$$\begin{aligned}\overrightarrow{MN}.\overrightarrow{NP}&=\begin{bmatrix}\frac{1}{2}a\\0\end{bmatrix}.\begin{bmatrix}0\\-\frac{1}{2}b\end{bmatrix}\\&=0\end{aligned}$$

So $\overrightarrow{MN}$ and $\overrightarrow{NP}$ are perpendicular.

Therefore opposite sides are parallel and equal in length and adjacent sides are at right angles. Hence $NPQM$ is a rectangle.

12b For a rectangle with vertices A, B, C, D with respective position vectors $\underset{\sim}{a}, \underset{\sim}{b}, \underset{\sim}{c}, \underset{\sim}{d}$ and length a and width b:

$\underset{\sim}{a}=\begin{bmatrix}0\\0\end{bmatrix}$

$\underset{\sim}{b}=\begin{bmatrix}0\\b\end{bmatrix}$

$\underset{\sim}{c}=\begin{bmatrix}a\\b\end{bmatrix}$

$\underset{\sim}{d}=\begin{bmatrix}a\\0\end{bmatrix}$

The midpoint of $AB=M$

$$\overrightarrow{OM}=\frac{1}{2}\left(\begin{bmatrix}0\\b\end{bmatrix}+\begin{bmatrix}0\\0\end{bmatrix}\right)=\begin{bmatrix}0\\\frac{1}{2}b\end{bmatrix}$$

The midpoint of $BC = N$

$$\overrightarrow{ON}=\frac{1}{2}\left(\begin{bmatrix}a\\b\end{bmatrix}+\begin{bmatrix}0\\b\end{bmatrix}\right)=\begin{bmatrix}\frac{1}{2}a\\b\end{bmatrix}$$

The midpoint of $CD = P$

$$\overrightarrow{OP}=\frac{1}{2}\left(\begin{bmatrix}a\\0\end{bmatrix}+\begin{bmatrix}a\\b\end{bmatrix}\right)=\begin{bmatrix}a\\\frac{1}{2}b\end{bmatrix}$$

The midpoint of $DA = Q$

$$\overrightarrow{OQ}=\frac{1}{2}\left(\begin{bmatrix}0\\0\end{bmatrix}+\begin{bmatrix}a\\0\end{bmatrix}\right)=\begin{bmatrix}\frac{1}{2}a\\0\end{bmatrix}$$

So the sides of the quadrilateral $MNPQ$ will be:

$$\overrightarrow{MN}=\begin{bmatrix}\frac{1}{2}a\\b\end{bmatrix}-\begin{bmatrix}0\\\frac{1}{2}b\end{bmatrix}=\begin{bmatrix}\frac{1}{2}a\\\frac{1}{2}b\end{bmatrix}$$

$$\overrightarrow{NP}=\begin{bmatrix}a\\\frac{1}{2}b\end{bmatrix}-\begin{bmatrix}\frac{1}{2}a\\b\end{bmatrix}=\begin{bmatrix}\frac{1}{2}a\\-\frac{1}{2}b\end{bmatrix}$$

$$\overrightarrow{PQ}=\begin{bmatrix}\frac{1}{2}a\\0\end{bmatrix}-\begin{bmatrix}a\\\frac{1}{2}b\end{bmatrix}=\begin{bmatrix}-\frac{1}{2}a\\-\frac{1}{2}b\end{bmatrix}$$

$$\overrightarrow{QM}=\begin{bmatrix}0\\\frac{1}{2}b\end{bmatrix}-\begin{bmatrix}\frac{1}{2}a\\0\end{bmatrix}=\begin{bmatrix}-\frac{1}{2}a\\\frac{1}{2}b\end{bmatrix}$$

For a rhombus all sides are equal in length and opposite sides are parallel.

$\overrightarrow{MN}=-\overrightarrow{PQ}$

$\overrightarrow{NP}=-\overrightarrow{QM}$

So $\overrightarrow{MN}$ and $\overrightarrow{PQ}$ are parallel (and equal in length) and $\overrightarrow{NP}$ and $\overrightarrow{QM}$ are parallel (and equal in length).

$$|\overrightarrow{MN}| = \sqrt{\left(\frac{1}{2}a\right)^2 + \left(\frac{1}{2}b\right)^2}$$

$$= \sqrt{\frac{1}{4}a^2 + \frac{1}{4}b^2}$$

$$= \frac{1}{2}\sqrt{a^2 + b^2}$$

Similarly,

$$|\overrightarrow{NP}| = \sqrt{\left(\frac{1}{2}a\right)^2 + \left(-\frac{1}{2}b\right)^2}$$

$$= \sqrt{\frac{1}{4}a^2 + \frac{1}{4}b^2}$$

$$= \frac{1}{2}\sqrt{a^2 + b^2}$$

So all sides are equal in length.

Hence $MNPQ$ is a rhombus.

13 $y = 2x + 3$

Let $x = \lambda$, so $y = 2\lambda + 3$. Therefore,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \lambda \\ 2\lambda + 3 \end{bmatrix}$$

Hence,

$$\underset{\sim}{r} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} + \begin{bmatrix} \lambda \\ 2\lambda \end{bmatrix}$$

$$\underset{\sim}{r} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

14 $\underset{\sim}{r} = \begin{bmatrix} 2 \\ -4 \end{bmatrix} + \lambda \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$x = 2 + 3\lambda \quad (1)$

$y = -4 + \lambda$

$\lambda = y + 4$

Substituting for λ in (1):

$x = 2 + 3(y + 4)$

$x = 2 + 3y + 12$

$x - 3y = 14$

15 $\underset{\sim}{r} = \begin{bmatrix} -6 \\ 4 \\ 3 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$

$x = -6 + 2\lambda$ (1)

$y = 4 + \lambda$ (2)

$z = 3 - 2\lambda$ (3)

ℓ intersects the xy plane at $z = 0$. Substituting this into (3) to find λ,

$0 = 3 - 2\lambda$

$\lambda = \frac{3}{2}$

Substitute into (1) and (2) for x and y:

$x = -6 + 2\left(\frac{3}{2}\right) = -3$

$y = 4 + \left(\frac{3}{2}\right) = \frac{11}{2}$

So, the intersection with xy plane is $(-3, \frac{11}{2}, 0)$.

Similarly, for the yz plane $x = 0$, giving $\lambda = 3$ from (1).

Then $y = 7$ and $z = -3$.

So, the intersection with xy plane is $(0, 7, -3)$.

Similarly, for the xz plane $y = 0$, giving $\lambda = -4$ from (2).

Then $x = -14$ and $z = 11$.

So, the intersection with xy plane is $(-14, 0, 11)$.

16a $\underset{\sim}{r} = \begin{bmatrix} 6 \\ -4 \\ -3 \end{bmatrix} + \lambda \begin{bmatrix} -5 \\ 2 \\ 7 \end{bmatrix}$

$x = 6 - 5\lambda$

So $\lambda = \dfrac{6 - x}{5}$ (1)

$y = -4 + 2\lambda$

So $\lambda = \dfrac{y + 4}{2}$ (2)

$z = -3 + 7\lambda$

So $\lambda = \dfrac{z + 3}{7}$ (3)

Consider the point $(-4, 0, 13)$. If it lies on the line then the values for λ should be the same.

Substituting $x = -4$ into (1):

$\lambda = \dfrac{6 - (-4)}{5} = 2$

Substituting $y = 0$ into (2):

$\lambda = \dfrac{0 + 4}{2} = 2$

Substituting $z = 13$ into (3):

$\lambda = \dfrac{13 + 3}{7} = \dfrac{16}{7}$

As all the values for λ are not the same, $(-4, 0, 13)$ is not a point on the line $\underset{\sim}{r}$.

16b $\underset{\sim}{r} = \begin{bmatrix} 6 \\ -4 \\ -3 \end{bmatrix} + \lambda \begin{bmatrix} -5 \\ 2 \\ 7 \end{bmatrix}$

$x = 6 - 5\lambda$

So $\lambda = \dfrac{6 - x}{5}$ (1)

$y = -4 + 2\lambda$

So $\lambda = \dfrac{y + 4}{2}$ (2)

$z = -3 + 7\lambda$

So $\lambda = \frac{z+3}{7}$ (3)

Consider the point $(16, -8, -17)$. If it lies on the line then the values for λ should be the same.

Substituting $x = 16$ into (1):

$\lambda = \frac{6-16}{5} = -2$

Substituting $y = -8$ into (2):

$\lambda = \frac{-8+4}{2} = -2$

Substituting $z = -17$ into (3):

$\lambda = \frac{-17+3}{7} = -2$

As all the values for λ are the same, $(16, -8, -17)$ is a point on the line $\underset{\sim}{r}$.

17 $\underset{\sim}{p} = \underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}$

$Q = (2, -1, 2)$

$\underset{\sim}{q} = 2\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$

$\overrightarrow{PQ} = (2-1)\underset{\sim}{i} + (-1-1)\underset{\sim}{j} + (2-(-1))\underset{\sim}{k}$

$\overrightarrow{PQ} = \underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{k}$

$\underset{\sim}{r} = \underset{\sim}{p} + \lambda\overrightarrow{PQ}$

$\underset{\sim}{r} = \underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k} + \lambda(\underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{k})$

18a $\underset{\sim}{r}_1 = \begin{bmatrix} 6 \\ 5 \\ 3 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$

$\underset{\sim}{r}_2 = \begin{bmatrix} -3 \\ 7 \\ 2 \end{bmatrix} + \mu \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix}$

For a point of intersection, $\underset{\sim}{r}_1 = \underset{\sim}{r}_2$:

$\begin{bmatrix} 6 \\ 5 \\ 3 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -3 \\ 7 \\ 2 \end{bmatrix} + \mu \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix}$

$6 + 2\lambda = -3 + 5\mu$

$\lambda = -\frac{9}{2} + \frac{5}{2}\mu$

$5 + \lambda = 7 - 4\mu$

Substituting λ we get

$5 + \left(-\frac{9}{2} + \frac{5}{2}\mu\right) = 7 - 4\mu$

$\frac{1}{2} - 7 = -4\mu - \frac{5}{2}\mu$

$-\frac{13}{2} = -\frac{13}{2}\mu$

$\mu = 1$

For λ

$\lambda = -\frac{9}{2} + \frac{5}{2}(1)$

$\lambda = -2$

So:

$x = 6 + 2\lambda$

$x = 6 + 2(-2)$

$x = 2$

$y = 5 + \lambda$

$y = 5 + (-2)$

$y = 3$

$z = 3 + 4\lambda$

$z = 3 + 4(-2)$

$z = -5$

So $\underset{\sim}{r}_1$ intersects $\underset{\sim}{r}_2$ at point:

$(2, 3, -5)$

NOTE: since x, y terms were used to find λ, μ, we should verify that z value is consistent.

Thus also $z = 2 - 7\mu = 2 - 7 = -5$.

18b $\underset{\sim}{r}_1 = -7\underset{\sim}{i} - 1\underset{\sim}{j} + 7\underset{\sim}{k} + \lambda(2\underset{\sim}{i} + 3\underset{\sim}{j} - 4\underset{\sim}{k})$

$\underset{\sim}{r}_2 = 9\underset{\sim}{i} - 4\underset{\sim}{j} - 16\underset{\sim}{k} + \mu(4\underset{\sim}{i} - 3\underset{\sim}{j} - 5\underset{\sim}{k})$

For a point of intersection, $\underset{\sim}{r}_1 = \underset{\sim}{r}_2$:

$-7\underset{\sim}{i} - \underset{\sim}{j} + 7\underset{\sim}{k} + \lambda\left(2\underset{\sim}{i} + 3\underset{\sim}{j} - 4\underset{\sim}{k}\right) = 9\underset{\sim}{i} - 4\underset{\sim}{j} - 16\underset{\sim}{k} + \mu(4\underset{\sim}{i} - 3\underset{\sim}{j} - 5\underset{\sim}{k})$

$-7 + 2\lambda = 9 + 4\mu$

$2\lambda = 16 + 4\mu$

$\lambda = 8 + 2\mu$

$-1 + 3\lambda = -4 - 3\mu$

Substituting λ we get

$-1 + 3(8 + 2\mu) = -4 - 3\mu$

$27 = -9\mu$

$\mu = -3$

For λ

$\lambda = 8 + 2(-3)$

$\lambda = 2$

So:

$x = -7 + 2\lambda$

$x = -7 + 2(2)$

$x = -3$

$y = -1 + 3\lambda$

$y = -1 + 3(2)$

$y = 5$

$z = 7 - 4\lambda$

$z = 7 - 4(2)$

$z = 13$

So $\underset{\sim}{r}_1$ intersects $\underset{\sim}{r}_2$ at point:

$(-3, 5, -1)$

NOTE: since x, y terms were used to find λ, μ, we should verify that z value is consistent.

Thus also $z = -16 - 5\mu = -16 + 15 = -1$.

19a $\underset{\sim}{c} = 3\underset{\sim}{i} - 4\underset{\sim}{j} + 2\underset{\sim}{k}$

$r = \sqrt{7}$

$|\underset{\sim}{r} - \underset{\sim}{c}|^2 = r^2$

$(x-3)^2 + (y+4)^2 + (z-2)^2 = 7$

19b $\underset{\sim}{c} = 3\underset{\sim}{i} + 4\underset{\sim}{j} + 2\underset{\sim}{k}$

$r = \sqrt{7}$

$|\underset{\sim}{r} - \underset{\sim}{c}| = r$

$$\left|\underset{\sim}{r} - \begin{bmatrix} 3 \\ -4 \\ 2 \end{bmatrix}\right| = \sqrt{7}$$

20 $P = (5, -1, 4)$

$$\underset{\sim}{p} = \begin{bmatrix} 5 \\ -1 \\ 4 \end{bmatrix}$$

$$\left|\underset{\sim}{r} - \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}\right| = 7$$

$r = 7$

$$\underset{\sim}{c} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$$

$\overrightarrow{CP} = \underset{\sim}{p} - \underset{\sim}{c}$

$= \begin{bmatrix} 5 \\ -1 \\ 4 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$

$= \begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix}$

$\left|\overrightarrow{CP}\right|^2 = 3^2 + (-4)^2 + 5^2$

$\left|\overrightarrow{CP}\right|^2 = 9 + 16 + 25$

$\left|\overrightarrow{CP}\right|^2 = 50$

$\overrightarrow{CP} = \sqrt{50} > 7$

So as $\overrightarrow{CP} > r$ the point $(5, 1, 4)$ lies outside of the sphere $\left|\underset{\sim}{r} - \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}\right| = 7$.

21 $x^2 + y^2 + z^2 - 4x - 10y + 12z + 41 = 0$

$(x^2 - 4x) + (y^2 - 10y) + (z^2 + 12z) = -41$

$(x^2 - 4x + 4) + (y^2 - 10y + 25) + (z^2 + 12z + 36) = 24$

$(x-2)^2 + (y-5)^2 + (z+6)^2 = 24$

Centre is $(2, 5, -6)$

$r^2 = 24$

$r = \sqrt{24}$

$= 2\sqrt{6}$

Radius is $2\sqrt{6}$ units.

So vector equation is

$\left|\underset{\sim}{r} - \begin{bmatrix} 2 \\ 5 \\ -6 \end{bmatrix}\right| = 2\sqrt{6}$

22 $(x+2)^2 + (y-3)^2 + (z+4)^2 = 125$

$\left|\underset{\sim}{r} - \begin{bmatrix} -2 \\ 3 \\ -4 \end{bmatrix}\right| = 125$

$$\underset{\sim}{r} = \begin{bmatrix} 4 \\ -5 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ 13 \\ -11 \end{bmatrix}$$

$$\left\| \begin{bmatrix} 4 \\ -5 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ 13 \\ -11 \end{bmatrix} - \begin{bmatrix} -2 \\ 3 \\ -4 \end{bmatrix} \right\| = 125$$

$$\left\| \begin{bmatrix} 2\lambda + 6 \\ 13\lambda - 8 \\ -11\lambda + 5 \end{bmatrix} \right\| = 125$$

$$(2\lambda + 6)^2 + (13\lambda - 8)^2 + (-11\lambda + 5)^2 = 125$$

$$4\lambda^2 + 24\lambda + 36 + 169\lambda^2 - 208\lambda + 64 + 121\lambda^2 - 110\lambda + 25 = 125$$

$$294\lambda^2 - 294\lambda + 125 = 125$$

$$294\lambda^2 - 294\lambda = 0$$

$$\lambda(\lambda - 1) = 0$$

So the line intersects with the sphere for the values of λ where

$$\lambda = 0, 1$$

Substituting these values back into the line equation gives:

$$\underset{\sim}{r} = \begin{bmatrix} 4 \\ -5 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ 13 \\ -11 \end{bmatrix}$$

For $\lambda = 0$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -5 \\ 1 \end{bmatrix}$$

For $\lambda = 1$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -5 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 13 \\ -11 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \\ -10 \end{bmatrix}$$

The line intersects with the sphere at points:

$(4, -5, 1)$ and $(6, 8, -10)$

23a

$$\underset{\sim}{r} = (2t)\underset{\sim}{i} + \left(\frac{2}{1+t^2}\right)\underset{\sim}{j}$$

$$x = 2t$$

$$t = \frac{x}{2}$$

$$y = \frac{2}{1+t^2}$$

$$y = \frac{2}{1+\left(\frac{x}{2}\right)^2}$$

$$y = \frac{8}{4+x^2}$$

23b

$$\underset{\sim}{r} = \left(\frac{2t}{1+t^2}\right)\underset{\sim}{i} + \left(\frac{1-t^2}{1+t^2}\right)\underset{\sim}{j}$$

$$|\underset{\sim}{r}|^2 = \underset{\sim}{r}\cdot\underset{\sim}{r}$$

$$|\underset{\sim}{r}|^2 = \left(\frac{2t}{1+t^2}\right)^2 + \left(\frac{1-t^2}{1+t^2}\right)^2$$

$$= \frac{4t^2}{(1+t^2)^2} + \frac{(1-t^2)^2}{(1+t^2)^2}$$

$$= \frac{1}{(1+t^2)^2}(4t^2 + 1 - 2t^2 + t^4)$$

$$= \frac{(1+t^2)^2}{(1+t^2)^2}$$

$$= 1$$

So we have a circle with radius 1.

$x^2 + y^2 = 1$ where $y \neq -1$

Note: the constraint $y \neq -1$ comes from the fact

$$y = \frac{1-t^2}{1+t^2} = \frac{-1-t^2+2}{1+t^2} = -1 + \frac{2}{1+t^2}$$

So, for any t, $\frac{2}{1+t^2} > 0$, so $y > -1$

23c $\underset{\sim}{r} = \sin t\, \underset{\sim}{i} + \sin 2t\, \underset{\sim}{j}$

$x = \sin t$

$t = \sin^{-1} x$

$y = \sin 2t$

$\quad = 2\sin t \times \cos t$

$\quad = 2\sin(\sin^{-1} x) \times \cos(\sin^{-1} x)$

$y = \pm 2x\sqrt{1 - x^2}$ (as $\sin x$ and $\cos x$ can be both positive and negative)