Solutions to Exercise 16A

1 A $(0.2 + 0.3 + 0.3 + 0.2 = 1)$ and C $(0.15 + 0.2 + 0.4 + 0.25 = 1)$

2a The sum of apex numbers can vary between 2 and 8.

x	2	3	4	5	6	7	8
$P(X = x)$	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{3}{16}$	$\frac{4}{16}$	$\frac{3}{16}$	$\frac{2}{16}$	$\frac{1}{16}$

$P(X = 2) = \frac{1}{4} \times \frac{1}{4} = \frac{1}{16}$ (1, 1)

$P(X = 3) = \frac{1}{4} \times \frac{1}{4} + \frac{1}{4} \times \frac{1}{4} = \frac{2}{16}$ (1, 2 or 2, 1)

$P(X = 4) = \frac{1}{4} \times \frac{1}{4} + \frac{1}{4} \times \frac{1}{4} + \frac{1}{4} \times \frac{1}{4} = \frac{3}{16}$ (1, 3 or 2, 2 or 3, 1)

$P(X = 5) = \frac{1}{4} \times \frac{1}{4} + \frac{1}{4} \times \frac{1}{4} + \frac{1}{4} \times \frac{1}{4} + \frac{1}{4} \times \frac{1}{4} = \frac{4}{16}$ (1, 4 or 4, 1 or 2, 3 or 3, 2)

$P(X = 6) = \frac{1}{4} \times \frac{1}{4} + \frac{1}{4} \times \frac{1}{4} + \frac{1}{4} \times \frac{1}{4} = \frac{3}{16}$ (2, 4 or 3, 3 or 4, 2)

$P(X = 7) = \frac{1}{4} \times \frac{1}{4} + \frac{1}{4} \times \frac{1}{4} = \frac{2}{16}$ (3, 4 or 4, 3)

$P(X = 8) = \frac{1}{4} \times \frac{1}{4} = \frac{1}{16}$ (4, 4)

2b i $P(X < 5)$

$= P(X = 2) + P(X = 3) + P(X = 4)$

$= \frac{1}{16} + \frac{2}{16} + \frac{3}{16}$

$= \frac{3}{8}$

2b ii $P(X > 7)$

$= P(X = 8)$

$= \frac{1}{16}$

2b iii $P(X < 2) = 0$ (The sum cannot be less than 2.)

2b iv $P(X \leq 10) = 1$

(The sum is always less than 10 because the sum can be 2, 3, 4, 5, 6, 7, 8.)

2c i $P(X < 4)$

$= P(X = 2) + P(X = 3)$

$= \frac{1}{16} + \frac{2}{16}$

$= \frac{3}{16}$

2c ii $P(X \text{ is odd})$

$= P(X = 3) \text{ or } P(X = 5) \text{ or } P(X = 7)$

$= \frac{2}{16} + \frac{4}{16} + \frac{2}{16}$

$= \frac{1}{2}$

2c iii $P(X \neq 2)$

$= 1 - P(X = 2)$

$= 1 - \frac{1}{16}$

$= \frac{15}{16}$

2c iv $P(X \geq 6)$

$= P(X = 6) + P(X = 7) + P(X = 8)$

$= \frac{3}{16} + \frac{2}{16} + \frac{1}{16}$

$= \frac{3}{8}$

3a

score x	1	2	3	4	5	Total
f_r	0.1	0.2	0.45	0.15	0.1	1
xf_r	0.1	0.4	1.35	0.6	0.5	2.95
x^2f_r	0.1	0.8	4.05	2.4	2.5	9.85

To fill the row of f_r , multiply each score by its frequency and write the sum in the last cell.

To fill the row of $x^2 f_r$, multiply the square of each score by its frequency and write the sum in the last cell.

3b The sum of probabilities of scores is 1.

3c Sum of row 3 is 2.95.

$$\bar{x} = \sum x f_r$$
$$= 2.95$$

3d $\bar{x}$ is a measure of the centre of the data set.

3e Sum of row 4 is 9.85.

$$s^2 = \sum x^2 f_r - (\bar{x})^2$$
$$= 9.85 - (2.95)^2$$
$$= 1.1475$$
$$\doteqdot 1.15$$

3f

$$s = \sqrt{s^2}$$
$$= \sqrt{1.1475}$$
$$\doteqdot 1.07$$

3g Sample standard deviation is a measure of the spread of the dataset.

3h $\bar{x}$ is the sample mean which is an estimate of the mean, $\mu = E(X)$.

s is the standard deviation of the sample mean which is an estimate of the standard deviation σ.

3i Since the sample mean is $\bar{x} = 2.95$,

an estimate of the sum after 100 throws is $100 \times 2.95 = 295$.

4a

Score x	3	4	5	6	7	Total
Relative frequency, f_r	0.04	0.21	0.35	0.25	0.15	1
$x\,f_r$	0.12	0.84	1.75	1.50	1.05	5.26
$x^2 f_r$	0.36	3.36	8.75	9.00	7.35	28.82

$\bar{x} = 5.26$

$$s^2 = \sum x^2 f_r - (\bar{x})^2$$

$$= 28.82 - (5.26)^2$$

$$= 1.1524$$

Therefore,

$$s = \sqrt{1.1524}$$

$$\doteqdot 1.07$$

4b The centre of the data is about 2.3 units greater but the spread is about the same, according to the standard deviation.

5a Total frequency is 20. Hence, the median is the average of the 10th score and the 11th score when the scores are in ascending order.

$$\text{Median} = \frac{3+4}{2} = 3.5$$

Mode is the score that appears the most. Therefore, the mode is 4.

5b

score x	1	2	3	4	5	6	Total
frequency f	2	4	4	8	2	0	10
$P(X = x)$	0.1	0.2	0.2	0.4	0.1	0	1
$x \times P(x)$	0.1	0.4	0.6	1.6	0.5	0	3.2
$x^2 \times P(x)$	0.1	0.8	1.8	6.4	2.5	0	11.6

5c Expected value is the sum of the 4th row. Therefore, $E(X) = 3.2$.

5d Variance, $\text{Var}(X)$, is

$$\begin{aligned}\sigma^2 &= \sum x^2 P(X = x) - \mu^2 \\ &= 11.6 - (3.2)^2 \\ &= 1.36\end{aligned}$$

5e Standard deviation, σ, is

$$\begin{aligned}\sigma &= \sqrt{\sigma^2} \\ &= \sqrt{1.36} \\ &\doteqdot 1.17\end{aligned}$$

5f It is usual to expect that for a quiz (covering recent work and including short easy questions) the marks will be high. These marks don't look impressive.

5g Since each score should be multiplied by 5, the expected value will be

$E(X) = 3.2 \times 5 = 16$ and the variance will be $\text{Var}\,(X) = 1.36 \times 5^2 = 34$.

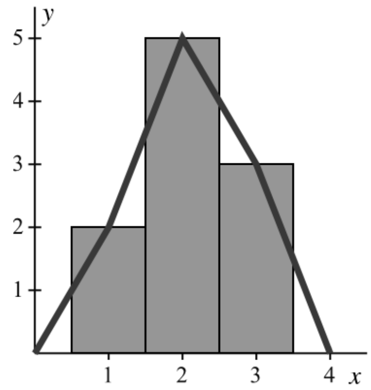
Therefore,

$$\sigma = \sqrt{\text{Var}\,(X)}$$

$$= \sqrt{34}$$

$$\doteqdot 5.83$$

6a i



6a ii Total area

$= 1 \times 2 + 1 \times 5 + 1 \times 3$

$= 10$

6a iii Total area under the frequency polygon

$$= \frac{2 \times 1}{2} + \frac{2 + 5}{2} \times 1 + \frac{3 + 5}{2} \times 1 + \frac{3 \times 1}{2}$$

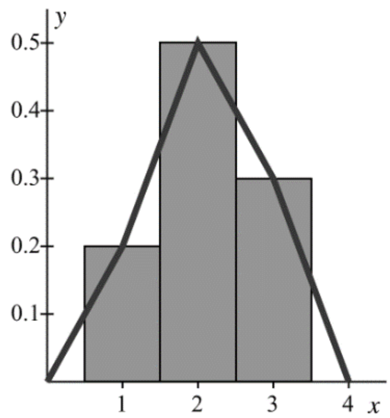
$= 10$

6a iv Both areas are the same and equal to the total frequency, that is, the number of scores.

6b i

score x	1	2	3
frequency f	2	5	3
relative frequency f_r	0.2	0.5	0.3

6b ii



6b iii Total area

$= 1 \times 0.2 + 1 \times 0.5 + 1 \times 0.3$

$= 1$

6b iv Total area under the frequency polygon

$$= \frac{0.2 \times 1}{2} + \frac{0.2 + 0.5}{2} \times 1 + \frac{0.3 + 0.5}{2} \times 1 + \frac{0.3 \times 1}{2}$$

$= 1$

6b v Both areas are the same and equal to the total 1, that is, the sum of the relative frequencies. (This will only happen when the rectangles have width 1.)

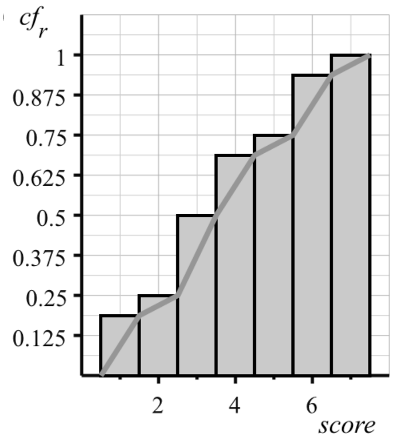
6b vi The relative frequencies are estimates of the probabilities. Note that both add to 1, both are non-negative, and both measure the chance that a random value will lie within the given rectangle of the histogram. A relative frequency is the experimental probability of an outcome, and is an estimate of the theoretical probability.

7a

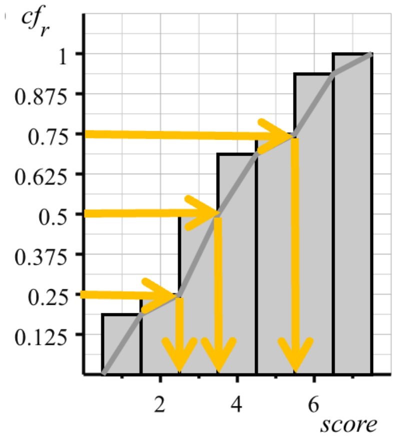
x	1	2	3	4	5	6	7	Total
f	3	1	4	3	1	3	1	16
f_r	$\frac{3}{16}$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{3}{16}$	$\frac{1}{16}$	$\frac{3}{16}$	$\frac{1}{16}$	1
cf	3	4	8	11	12	15	16	—
cf_r	$\frac{3}{16}$	$\frac{4}{16}$	$\frac{8}{16}$	$\frac{11}{16}$	$\frac{12}{16}$	$\frac{15}{16}$	1	—

7b

Score x	1	2	3	4	5	6	7
cf_r	0.1875	0.25	0.5	0.6875	0.75	0.9375	1



7c As shown in the below histogram, $Q_1 = 2.5$, $Q_2 = 3.5$, $Q_3 = 5.5$.

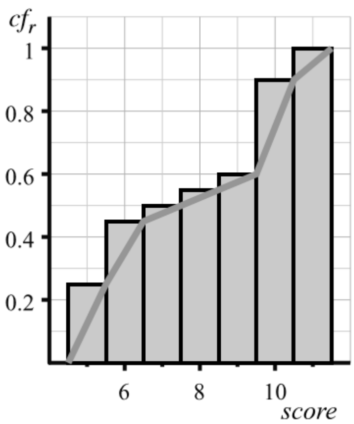


8a

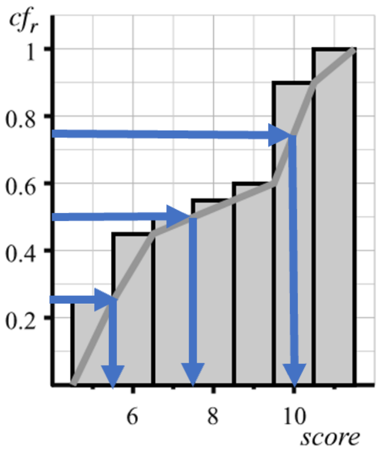
x	5	6	7	8	9	10	11	Total
f	5	4	1	1	1	6	2	20
f_r	$\frac{5}{20}$	$\frac{4}{20}$	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{6}{20}$	$\frac{2}{20}$	1
cf	5	9	10	11	12	15	20	—
cf_r	$\frac{5}{20}$	$\frac{9}{20}$	$\frac{10}{20}$	$\frac{11}{20}$	$\frac{12}{20}$	$\frac{18}{20}$	1	—

8b

Score x	5	6	7	8	9	10	11
cf_r	0.25	0.45	0.5	0.55	0.60	0.90	1



8c As shown in the below histogram, $Q_1 = 5.5$, $Q_2 = 7.5$, $Q_3 = 10$.



9a The relative frequency of the households who have no cars is 0.25.

Therefore, $\frac{1}{4}$ of the households have no cars.

9b The relative frequency of the households who have 1 car is $0.5 = \frac{1}{2}$.

Therefore, $\frac{1}{4} + \frac{1}{2} = \frac{3}{4}$ of the households have at most one car.

9c $P(\text{a household chosen at random has 3 cars}) = 0.1$

(The ratio of people who have 3 cars and the population.)

9d $P(\text{a household chosen at random has 3 cars}) = 0.1$.

Then, 10% of the population has 3 cars.

$P(\text{a household chosen at random has 4 cars}) = 0.025$.

Then, 2.5% of the population has 4 cars.

Therefore, 12.5% of the households have 3 or more cars. Hence, the town planners will not recommend additional on-street parking.

9e

x	0	1	2	3	4
$P(X = x)$	0.25	0.5	0.125	0.10	0.025

9f Sum of the probabilities is $0.25 + 0.5 + 0.125 + 0.1 + 0.025 = 1$

Since the probabilities of individual events are the y-coordinates of the points on the relative frequency polygon, they are the heights of the rectangles with widths 1 unit. Therefore, the sum of the areas of the rectangles in the relative frequency histogram is $0.25 + 0.5 + 0.125 + 0.1 + 0.025 = 1$ which is equal to the sum of the probabilities.

9g The triangles cut off above the polygon fit into the spaces below the polygon.

9h This is an average, and is best understood by saying that for a large sample of n houses, we would expect them to have about $1.15n$ cars between them — see the next part.

9i 115 cars. We are assuming that streets in the suburb are uniform with respect to car ownership. Streets closer to train stations may manage with fewer cars because people catch the train to work, more affluent streets may own more cars, people may adjust car ownership to allow for availability of off-street or on-street parking.

9j

x	0	1	2	3	4
$P(X \le x)$	0.25	0.75	0.875	0.975	1

$P(X \le 0) = 0.25$, $P(X \le 1) = 0.25 + 0.5 = 0.75$

$P(X \le 2) = 0.25 + 0.5 + 0.125 = 0.875$

$P(X \leq 3) = 0.25 + 0.5 + 0.125 + 0.10 = 0.975$

$P(X \leq 4) = 0.25 + 0.5 + 0.125 + 0.10 + 0.025 = 1$

9k The data shown in 9j agree with the graph below because

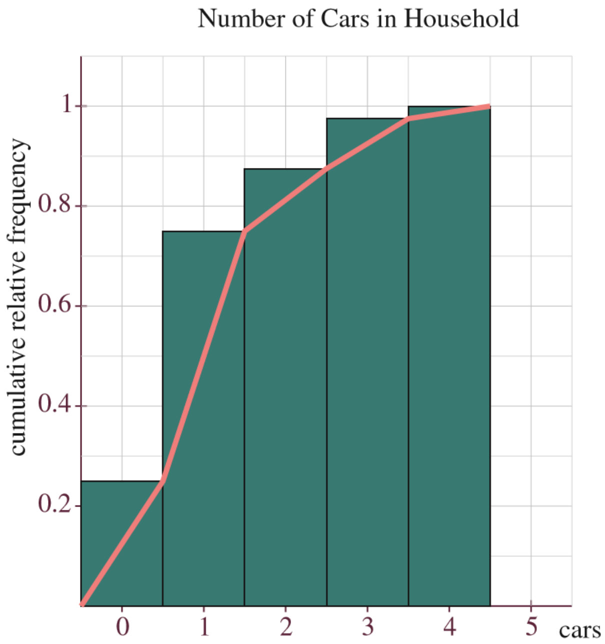
0.25 is the height of the rectangle representing 0 cars.

0.75 is the height of the rectangle representing 1 car.

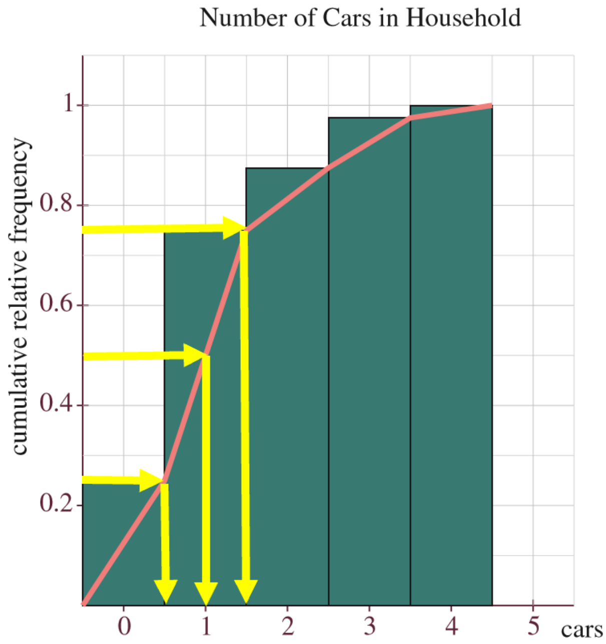
0.875 is the height of the rectangle representing 2 cars.

0.975 is the height of the rectangle representing 3 cars.

1 is the height of the rectangle representing 4 cars.

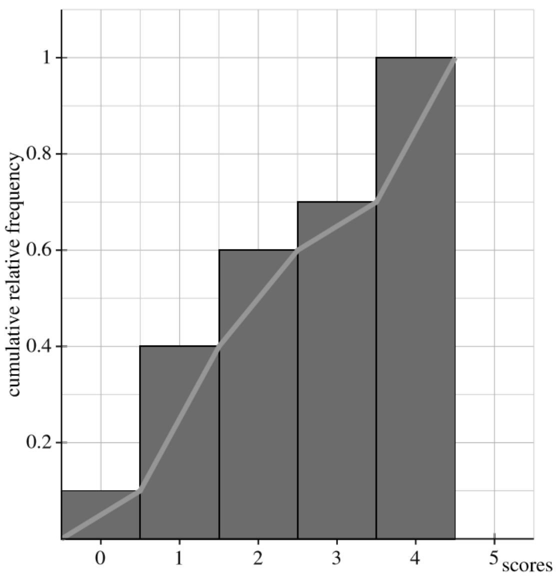


9l As shown in the histogram, $Q_1 = 0.5$, $Q_2 = 1$, $Q_3 = 1.5$

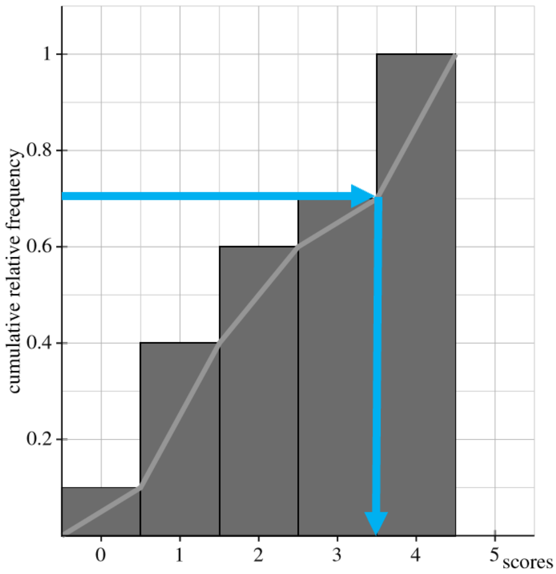


10a

Score x	0	1	2	3	4
cf_r	0.1	0.4	0.6	0.7	1

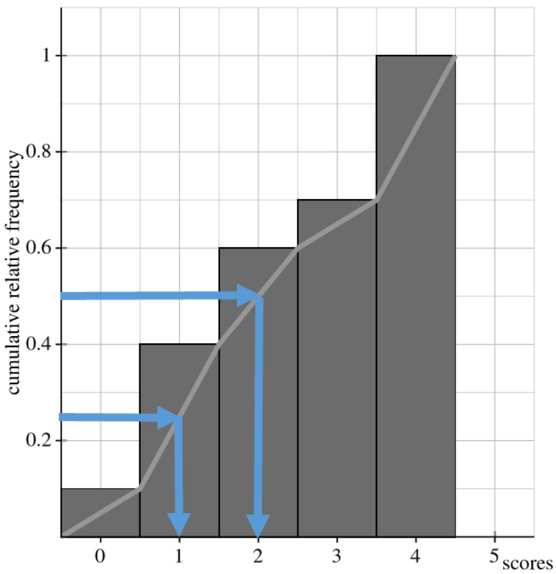


10b



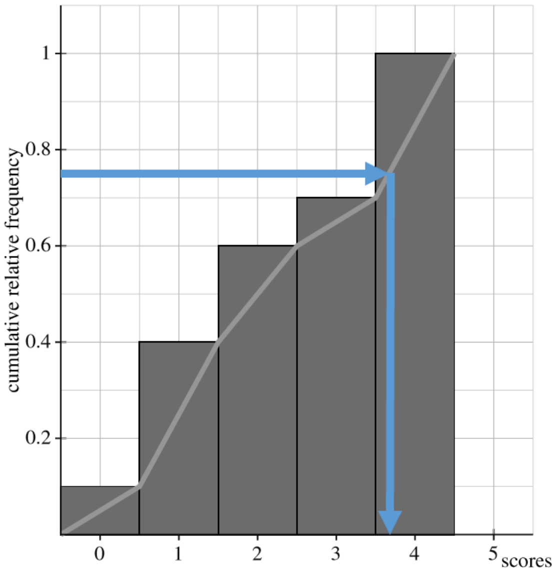
3.5 is the estimated 70th percentile.

10c



$Q_1 = 1$ and $Q_2 = 2$

10d



$Q_3 \doteqdot 3.7$

10e $3.5 + 0.05 \times \frac{4.5 - 3.5}{1 - 0.7} = 3.67$. Therefore, they agree.

11a Since the total number of people who purchased an item is 100, the median is the amount the 50th person spent. Therefore, the median is \$3.50. The mode is the cost of the item that was sold the most. Therefore, the mode is also \$3.50.

11b

spent	0–1	1–2	2–3	3–4	4–5	Total
cc x	0.50	1.50	2.50	3.50	4.50	—
f	20	5	15	40	20	100
f_r	0.20	0.05	0.15	0.40	0.20	1

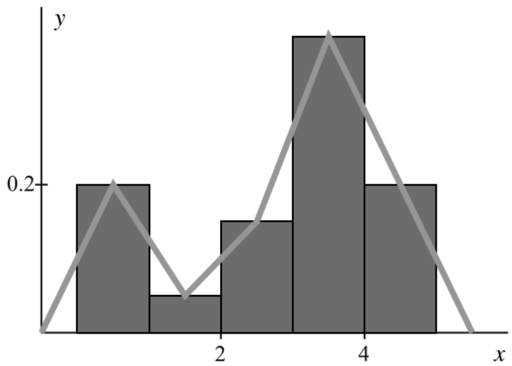
11c $E(X) = 0.5 \times 0.20 + 1.5 \times 0.05 + 2.5 \times 0.15 + 3.5 \times 0.40 + 4.5 \times 0.20 = 2.85$

Expected value is \$2.85.

$$\begin{aligned}\text{Var}(X) &= \textstyle\sum x^2 \times P(x) - \mu^2 \\ &= (0.5)^2 \times 0.20 + (1.5)^2 \times 0.05 + (2.5)^2 \times 0.15 + (3.5)^2 \times 0.40 \\ &\qquad + (4.5)^2 \times 0.20 - (2.85)^2 \\ &= 1.9275\end{aligned}$$

(Standard deviation $= \sqrt{1.9275} \doteqdot 1.39$)

11d



11e Since the data is representing the population, the relative frequencies in the table are the probabilities of the corresponding events happening.

$P(X = 0.5) = 0.2$

$P(X = 1.5) = 0.05$

$P(X = 2.5) = 0.15$

$P(X = 3.5) = 0.4$

$P(X = 4.5) = 0.2$

11f The sum of the probabilities in part e is 1. This represents the area of the relative frequency polygon, or the area under the frequency polygon bounded by the x-axis (they are the same). This only happens because the rectangles have width 1.

11g i $\Pr(X \le 3) = 0.2 + 0.05 + 0.15 = 0.4$

$\Pr(X = 3.5) = 0.2 + 0.05 + 0.15 = 0.4$

Since $\Pr(X \le 3) = \Pr(X = 3.5)$, these events are equally likely.

11g ii $\Pr(X = 0.5) = 0.2$ and $\Pr(X = 3.5) = 0.4$.

Therefore, the amount spent is more likely $\$3 - \$4$.

11h $E(Y) = 2.5 \times 0.20 + 3.5 \times 0.05 + 4.5 \times 0.15 + 5.5 \times 0.40 + 6.5 \times 0.20 = \$4.85$

$E(Y)$ represents the expected amount of dollars that may be spent.

$$\begin{aligned}\mathrm{Var}(X) &= \sum x^2 \times P(x) - \mu^2 \\ &= (2.5)^2 \times 0.20 + (3.5)^2 \times 0.05 + (4.5)^2 \times 0.15 + (5.5)^2 \times 0.40 \\ &\qquad + (6.5)^2 \times 0.20 - (4.85)^2 \\ &= 1.9275\end{aligned}$$

The variance is the same.

12a The histogram covers 40 grid rectangles.

12b i $0.3 \times 0.5 = 0.15$

12b ii Counting grid rectangles, this is $12 \div 40 = 0.3$

12c i The height of each rectangle measures the relative frequency per degree, and the heights are respectively 0.1 and 0.2. Thus it is twice as likely to be (in the interval) 20°C.

12c ii In the class 19.25-19.75.

12d i This seems to be a similar calculation to part a, except now we are looking at the area under the polygon, not the histogram.

The area is a rectangle with size $1 \times 0.1 = 0.1$.

12d ii Area of trapezium from 19–19.5

$= \frac{1}{2} \times \frac{1}{2} \times (0.1 + 0.3)$

$= 0.1$

Area of trapezium from 19.5–20.5

$= \frac{1}{2} \times 1 \times (0.3 + 0.1)$

$= 0.2$

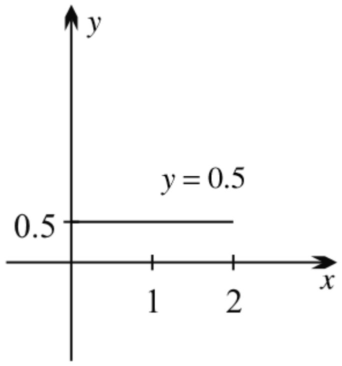
Thus the total area is 0.3.

12e First, the histogram only records the maximum daily temperature. Secondly, it recorded 20 consecutive days, but there will be natural variation over the year, and even within a season.

Solutions to Exercise 16B

Let C be a constant.

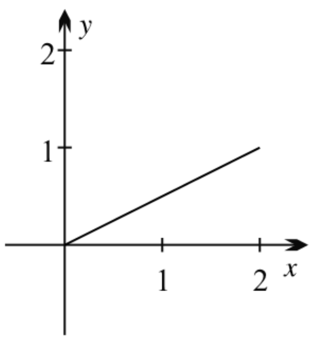
1a i $f(x) > 0$ for all $0 \le x \le 2$.



1a ii

$$\int_0^2 \frac{1}{2}\,dx = \left[\frac{x}{2}\right]_0^2 = 1 - 0 = 1$$

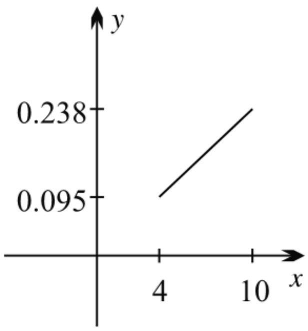
1b i $f(x) > 0$ for all $0 \le x \le 2$.



1b ii

$$\int_0^2 \frac{1}{2}x\,dx = \left[\frac{x^2}{4}\right]_0^2 = 1 - 0 = 1$$

1c i $f(x) > 0$ for all $4 \leq x \leq 10$.



1c ii

$$\int_4^{10} \frac{1}{42} x\, dx = \left[\frac{x^2}{84}\right]_4^{10} = \frac{100}{84} - \frac{16}{84} = 1$$

2a

$$\int_0^1 3x^2\, dx = [x^3]_0^1 = 1 \text{ and } f(x) \geq 0 \text{ where } 0 \leq x \leq 1.$$

Hence, $f(x)$ is a probability density function where $0 \leq x \leq 1$.

Since $f(x)$ is increasing in the interval $[0, 1]$ its mode is $x = 1$.

2b

$$\int_1^5 \frac{1}{4} x\, dx = \left[\frac{x^2}{8}\right]_1^5 = \frac{25}{8} - \frac{1}{8} = 3$$

Hence, $f(x)$ is not a probability density function where $1 \leq x \leq 5$.

2c

$$\int_0^3 \frac{4 - 2x}{3}\, dx = \left[\frac{4x - x^2}{3}\right]_0^3 = \frac{3}{3} - \frac{0}{3} = 1$$

However, $f(x) < 0$ when $x > 2$. Therefore, $f(x)$ is not a probability density function.

2d

$$\int_0^1 (n+1)\,x^n\,dx = [x^{n+1}]_0^1 = 1^{n+1} - 0 = 1$$

If $n \geq 0$ then $f(x) \geq 0$ for all x in its domain and since t is increasing in the interval $[0, 1]$ its mode is $x = 1$.

2e $\int_0^{\pi} \frac{1}{2}\sin(x)\,dx = \left[-\frac{1}{2}\cos(x)\right]_0^{\pi}$

$$= \left(\frac{1}{2}\right) - \left(-\frac{1}{2}\right) = 1 \text{ and } f(x) \geq 0 \text{ where } 0 \leq x \leq \pi.$$

Hence, $f(x)$ is a probability density function and its mode is $x = \frac{\pi}{2}$ as $\sin(x)$ has a maximum turning point at $x = \frac{\pi}{2}$.

2f

$$\int_0^2 \frac{1}{12}(3x^2 + 2x)\,dx$$

$$= \left[\frac{1}{12}(x^3 + x^2)\right]_0^2$$

$$= \left(\frac{1}{12} \times (8 + 4)\right) - \left(\frac{1}{12}(0)\right)$$

$$= 1$$

and $f(x) = \frac{1}{12}x(3x + 2)$

Since $f(x)$ is a parabola and has roots at $x = -\frac{2}{3}$, $x = 0$ and is concave up, $f(x) \geq 0$ where $0 \leq x \leq 2$. Therefore, $f(x)$ is a probability density function and its mode is $x = 2$.

3a

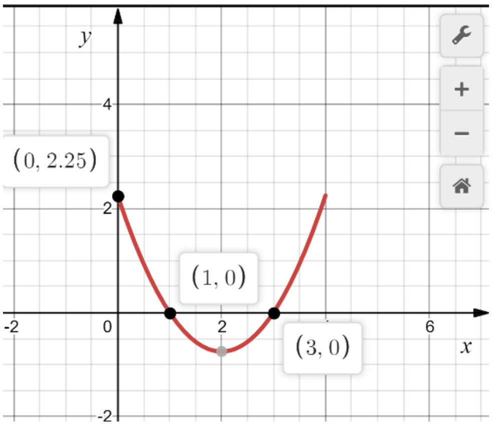
$$\int_0^4 \frac{3}{4}(x^2 - 4x + 3)\, dx$$

$$= \frac{3}{4}\left[\left(\frac{x^3}{3} - 2x^2 + 3x\right)\right]_0^4$$

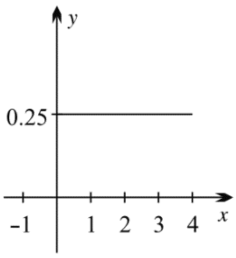
$$= \frac{3}{4}\left(\frac{64}{3} - 32 + 12\right)$$

$$= 1$$

3b Since $f(x) < 0$ where $1 < x < 3$, $f(x)$ is not a probability density function.



4a



4b $f(x) \geq 0$ where $0 \leq x \leq 4$ and $\int_0^4 0.25\, dx = [0.25x]_0^4 = 1$. Therefore, $f(x)$ is a probability density function.

4c i $P(0 \le X \le 1)$

$= \int_0^1 0.25\, dx$

$= [0.25x]_0^1$

$= 0.25 \times 1 - 0.25 \times 0$

$= 0.25$

4c ii $P(1 \le X \le 3)$

$= \int_1^3 0.25\, dx$

$= [0.25x]_1^3$

$= 0.25 \times 3 - 0.25 \times 1$

$= 0.5$

4c iii $P(X \le 2)$

$= \int_0^2 0.25\, dx$

$= [0.25x]_0^2$

$= 0.25 \times 2 - 0.25 \times 0$

$= 0.5$

4c iv $P(X = 2)$

$= \int_2^2 0.25\, dx$

$= [0.25x]_2^2$

$= 0.25 \times 2 - 0.25 \times 2$

$= 0$

4c v $P(X \le 3)$

$= \int_0^3 0.25\,dx$

$= [0.25x]_0^3$

$= 0.25 \times 3 - 0.25 \times 0$

$= 0.75$

4c vi $P(X \ge 1) = 1 - P(X < 1)$

$= 1 - \int_0^1 0.25\,dx$

$= 1 - [0.25x]_0^1$

$= 1 - (0.25 - 0)$

$= 0.75$

4d $P(2 \le X \le 3)$

$= \int_2^3 0.25\,dx$

$= [0.25x]_2^3$

$= 0.25 \times 3 - 0.25 \times 2$

$= 0.25$ and

$P(X \le 3) - P(X \le 2) = \int_0^3 0.25\,dx - \int_0^2 0.25\,dx$

$= [0.25x]_0^3 - [0.25x]_0^2$

$= 0.75 - 0.5$

$= 0.25$

Therefore, $P(2 \le X \le 3) = P(X \le 3) - P(X \le 2)$.

5a $F(x) = \int \frac{1}{32}x\,dx = \frac{1}{64}x^2 + C$ and $F(8) = \frac{1}{64}(8)^2 + C = 1$. Then, $C = 0$.

Therefore, $F(x) = \frac{1}{64}x^2$

5b $F(x) = \int \frac{3}{16}x^2\,dx = \frac{1}{16}x^3 + C$ and $F(2) = \frac{1}{16}(2)^3 + C = 1$.

Then, $C = \frac{1}{2}$. Therefore, $F(x) = \frac{1}{16}x^3 + \frac{1}{2} = \frac{1}{16}(x^3 + 8)$

5c $F(x) = \int \frac{3}{2}(1 - x^2)\,dx = \frac{3}{2}\left(x - \frac{x^3}{3}\right) + C$ and $F(1) = \frac{3}{2}\left(1 - \frac{1}{3}\right) + C = 1$.

Then, $C = 0$. Therefore, $F(x) = \frac{3}{2}\left(x - \frac{x^3}{3}\right) = \frac{x}{2}(3 - x^2)$

5d $F(x) = \int \frac{1}{e}(e^x + 1)\,dx = \frac{1}{e}(e^x + x) + C$ and $F(1) = \frac{1}{e}(e + 1) + C = 1$.

Then, $C = -\frac{1}{e}$.

Therefore, $F(x) = \frac{1}{e}(e^x + x) - \frac{1}{e}$

$= \frac{1}{e}(e^x + x - 1)$

6a For 5a:

$F(x) = \frac{1}{64}x^2 = 0.5$ when $x^2 = 32$ or $x = 4\sqrt{2}$. Therefore, the median is $x = 4\sqrt{2}$.

For 5b:

$F(x) = \frac{1}{16}(x^3 + 8) = 0.5$ when $x = 0$. Therefore, the median is $x = 0$.

6b For 5a:

$F(x) = \frac{1}{64}x^2 = 0.25$ when $x^2 = 16$ or $x = 4$. Therefore, the Q_1 is $x = 4$.

$F(x) = \frac{1}{64}x^2 = 0.75$ when $x^2 = 48$ or $x = 4\sqrt{3}$. Therefore, the Q_3 is $x = 4\sqrt{3}$.

For 5b:

$F(x) = \frac{1}{16}(x^3 + 8) = 0.25$ when $x = -\sqrt[3]{4}$. Therefore, the Q_1 is $x = -\sqrt[3]{4}$.

$F(x) = \frac{1}{16}(x^3 + 8) = 0.75$ when $x = \sqrt[3]{4}$. Therefore, the Q_3 is $x = \sqrt[3]{4}$.

7a The area Bud wanders is:

(Area of the property)−(area of the house)

$= 15 \times 20 - (13 \times 10)$

$= 170\text{ m}^2$

The area directly to the right of the house is: $10 \times 2 = 20\text{ m}^2$

$P(\text{neighbours will be stressed}) = \frac{20}{170} \doteqdot 0.12$ or 12%

7b $P(\text{Jack's mother will be stressed})$

$= \dfrac{\pi \times 2^2}{170}$

$\doteqdot 0.07$ or 7%

7c $P(\text{Sally will be stressed})$

$= 0.12 + 0.07$

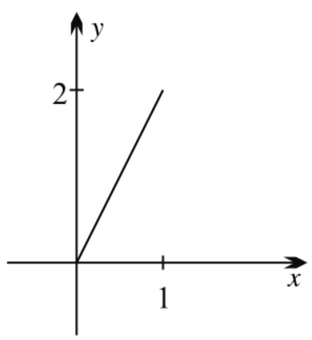
$= 0.19$ or 19%

7d $P(\text{Jack's father will be stressed})$

$= 1 - 0.19$

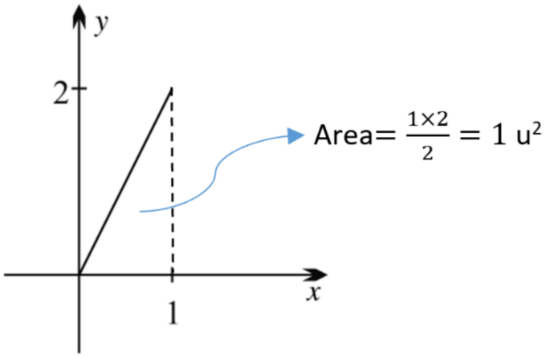
$= 0.81$ or 81%

8a



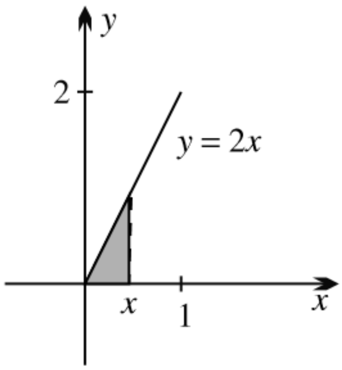
$f(x) \geq 0$ for $0 \leq x \leq 1$.

8b



$$\int_0^1 2x\,dx = [x^2]_0^1 = 1^2 - 0^2 = 1 \text{ square unit}$$

8c i



$F(x) = \int 2x\,dx = x^2 + C$ and $F(1) = 0$. Then, $C = 0$ and $F(x) = x^2$

Therefore, $P(X \leq x) = x^2$

8c ii $P(X \leq x)$

$$= \int_0^x 2t\,dt$$

$$= [t^2]_0^x$$

$$= x^2$$

8d $x^2 = 0.25$ when $x = 0.5$. Therefore, $Q_1 = \frac{1}{2}$

$x^2 = 0.5 = \frac{1}{2}$ when $x = \frac{1}{\sqrt{2}}$. Therefore, $Q_2 = \frac{1}{\sqrt{2}}$

$x^2 = 0.75 = \frac{3}{4}$ when $x = \frac{\sqrt{3}}{2}$. Therefore, $Q_3 = \frac{\sqrt{3}}{2}$

9a If $y = cx^4$ then $\int_0^3 cx^4\,dx = 1$

$c\left[\frac{x^5}{5}\right]_0^3 = c \times \frac{3^5}{5} = 1$

$c = \frac{5}{243}$

9b If $y = c$ then $\int_0^6 c\,dx = 1$

$c[x]_0^6 = 6c = 1$

$c = \frac{1}{6}$

9c If $y = c$ then $\int_{-5}^5 c\,dx = 1$

$c[x]_{-5}^5 = 5c - (-5c) = 1$

$10c = 1$

$c = \frac{1}{10}$

9d If $f(x) = \frac{8}{3}(1 - x)$ then

$$\frac{8}{3}\int_0^c (1 - x)\,dx = 1$$

$$\frac{8}{3}\left[x - \frac{x^2}{2}\right]_0^c = \frac{8}{3}\left(c - \frac{c^2}{2}\right) = 1$$

$$\frac{2c - c^2}{2} = \frac{3}{8}$$

$$c^2 - 2c = \frac{3}{-4}$$

$-4c^2 + 8c - 3 = 0$

$(2c - 1)(3 - 2c) = 0$

Therefore, $c = \frac{1}{2}$ or $c = \frac{3}{2}$

10a The function $f(x) \geq 0$ for all x in the domain and the area under the graph of the function in the interval $0 \leq x \leq 5$ is: $0.125 \times 2 + 0.25 \times 3 = 1$. Therefore, $f(x)$ is a probability density function.

10b

x	0	1	2	3	4	5
$P(X \leq x)$	0	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1

$P(X \leq 1) = \frac{1}{8}$ (The area enclosed by the axes, the graph and the line $x = 1$)

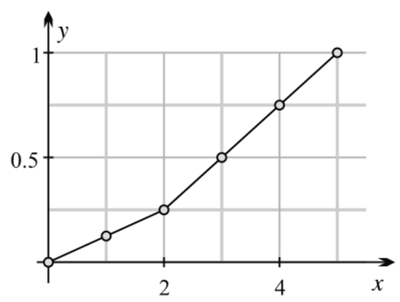
$P(X \leq 2) = \frac{1}{4}$ (The area enclosed by the axes, the graph and the line $x = 2$)

$P(X \leq 3) = \frac{1}{2}$ (The area enclosed by the axes, the graph and the line $x = 3$)

$P(X \leq 4) = \frac{3}{4}$ (The area enclosed by the axes, the graph and the line $x = 4$)

$P(X \leq 5) = 1$ (The area enclosed by the axes, the graph and the line $x = 5$)

10c

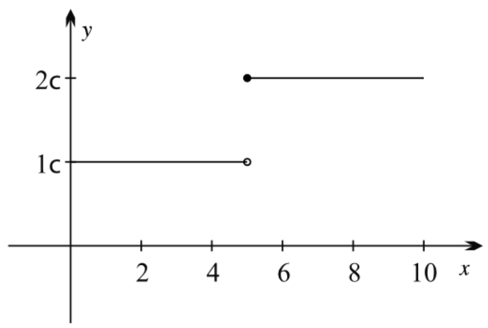


10d $F(x) = \begin{cases} \frac{1}{8}x & , 0 \le x < 2 \\ \frac{1}{4}x - \frac{1}{4}, & 2 \le x \le 5 \end{cases}$

The line graph in the interval $0 \le x < 2$ is $\frac{1}{8}x$.

The line graph in the interval $2 \le x \le 5$ is $\frac{1}{4}x - \frac{1}{4}$.

11a



11b $\int_0^5 c\,dx + \int_5^{10} 2c\,dx = 1$

$$[cx]_0^5 + [2cx]_5^{10} = 1$$

$$5c + (20c - 10c) = 1$$

$$15c = 1$$

$$c = \frac{1}{15}$$

11c $F(x) = \begin{cases} cx & , 0 \le x < 5 \\ 2cx - 5c & , 5 \le x \le 10 \end{cases}$

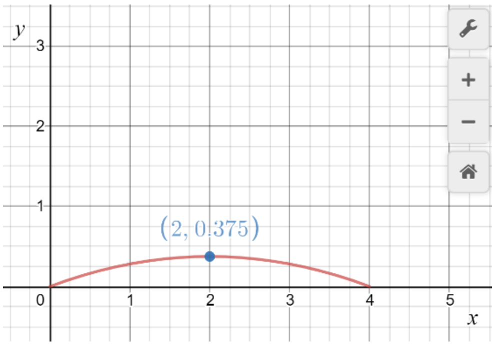
11d $P(1 < X < 7) = F(7) - F(1)$

$$= (14c - 5c) - (c)$$

$$= 8c$$

$$= \frac{8}{15}$$

12a The probability density function $f(x) = \frac{3}{32}x(4-x)$ where $0 \le x \le 4$ is shown below.



The mode of $f(x)$ is $x = 2$ (where the maximum value is).

12b $\int_0^4 \frac{3}{32}x(4-x)\,dx = \frac{3}{32}\int_0^4 (4x - x^2)\,dx$

$= \frac{3}{32}\left[2x^2 - \frac{x^3}{3}\right]_0^4$

$= \frac{3}{32}\left[\left(2(4)^2 - \frac{(4)^3}{3}\right) - \left(2(0)^2 - \frac{(0)^3}{3}\right)\right]$

$= \frac{3}{32}\left[\left(32 - \frac{64}{3}\right) - (0)\right]$

$= \frac{3}{32} \times 32 - \frac{3}{32} \times \frac{64}{3}$

$= 3 - 2$

$= 1$

12c Since the function is symmetric about the line $x = 2$,

the area $\int_0^2 \frac{3}{32}x(4-x)\,dx = P(X \le 2) = 0.5$

12d $P(X \le 1)$

$= \int_0^1 \frac{3}{32}x(4-x)\,dx$

$= \frac{3}{32}\left[2x^2 - \frac{x^3}{3}\right]_0^1$

$$= \frac{3}{32}\left[\left(2(1)^2 - \frac{(1)^3}{3}\right) - \left(2(0)^2 - \frac{(0)^3}{3}\right)\right]$$

$$= \frac{3}{32}\left[\left(2 - \frac{1}{3}\right)\right]$$

$$= \frac{5}{32}$$

$P(X > 1)$

$$= \int_1^4 \frac{3}{32}x(4 - x)\,dx$$

$$= \frac{3}{32}\left[2x^2 - \frac{x^3}{3}\right]_1^4$$

$$= \frac{3}{32}\left[\left(2(4)^2 - \frac{(4)^3}{3}\right) - \left(2(1)^2 - \frac{(1)^3}{3}\right)\right]$$

$$= \frac{3}{32}\left[\left(32 - \frac{64}{3}\right) - \left(2 - \frac{1}{3}\right)\right]$$

$$= \frac{3}{32}\left[\frac{32}{3} - \frac{5}{3}\right]$$

$$= \frac{27}{32}$$

$P(X \leq 1) + P(X > 1) = \frac{5}{32} + \frac{27}{32} = 1$ because they are complementary events.

12e $P(X \leq 0.5)$

$$= \int_0^{0.5} \frac{3}{32} x(4-x)\, dx$$

$$= \frac{3}{32}\left[2x^2 - \frac{x^3}{3}\right]_0^{0.5}$$

$$= \frac{3}{32}\left[\left(2(0.5)^2 - \frac{(0.5)^3}{3}\right) - \left(2(0)^2 - \frac{(0)^3}{3}\right)\right]$$

$$= \frac{3}{32}\left[\left(\frac{1}{2} - \frac{1}{24}\right) - (0)\right]$$

$$= \frac{3}{32} \times \frac{11}{24}$$

$$= \frac{11}{256}$$

$P(X \geq 3.5)$

$$= \int_{3.5}^{4} \frac{3}{32} x(4-x)\, dx$$

$$= \frac{3}{32}\left[2x^2 - \frac{x^3}{3}\right]_{3.5}^{4}$$

$$= \frac{3}{32}\left[\left(2(4)^2 - \frac{(4)^3}{3}\right) - \left(2(3.5)^2 - \frac{(3.5)^3}{3}\right)\right]$$

$$= \frac{3}{32}\left[\left(32 - \frac{64}{3}\right) - \left(\frac{49}{2} - \frac{343}{24}\right)\right]$$

$$= \frac{3}{32}\left[\frac{32}{3} - \frac{245}{24}\right]$$

$$= \frac{3}{32} \times \frac{11}{24}$$

$$= \frac{11}{256}$$

$P(X \leq 0.5) = P(X \geq 3.5)$

12f $F(x)$

$$= \int_0^x f(t)\, dt$$

$$= \int_0^x \frac{3}{32} t(4-t)\, dt$$

$$= \frac{3}{32}\left[2t^2 - \frac{t^3}{3}\right]_0^x$$

$$= \frac{3}{32}\left[\left(2(x)^2 - \frac{(x)^3}{3}\right) - (0)\right]_0^x$$

$$= \frac{3}{32}\left(\frac{6x^2 - x^3}{3}\right)$$

$$= \frac{1}{32}x^2(6-x)$$

12g i $P(X < 1.5)$

$$= \int_0^{1.5} \frac{3}{32} x(4-x)\, dx$$

$$= \frac{3}{32}\left[2x^2 - \frac{x^3}{3}\right]_0^{1.5}$$

$$= \frac{3}{32}\left[\left(2(1.5)^2 - \frac{(1.5)^3}{3}\right) - \left(2(0)^2 - \frac{(0)^3}{3}\right)\right]$$

$$= \frac{3}{32}\left[\left(\frac{9}{2} - \frac{27}{24}\right) - (0)\right]$$

$$= \frac{3}{32} \times \frac{81}{24}$$

$$= \frac{81}{256}$$

12g ii $P(1 < X < 1.5)$

$$= \int_0^{1.5} \frac{3}{32} x(4-x)\,dx$$

$$= \frac{3}{32}\left[2x^2 - \frac{x^3}{3}\right]_1^{1.5}$$

$$= \frac{3}{32}\left[\left(2(1.5)^2 - \frac{(1.5)^3}{3}\right) - \left(2(1)^2 - \frac{(1)^3}{3}\right)\right]$$

$$= \frac{3}{32}\left[\left(\frac{9}{2} - \frac{27}{24}\right) - \left(\frac{5}{3}\right)\right]$$

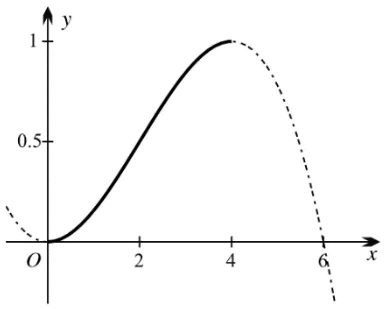
$$= \frac{3}{32} \times \frac{41}{24}$$

$$= \frac{41}{256}$$

$$P(X < 1.5) - P(X < 1) = \frac{81}{256} - \frac{5}{32} = \frac{41}{256}$$

Therefore, $P(1 < X < 1.5) = P(X < 1.5) - P(X < 1)$

12h



12i $P(X < 2)$

$$= \int_0^{2} \frac{3}{32} x(4-x)\,dx$$

$$= \frac{3}{32}\left[2x^2 - \frac{x^3}{3}\right]_0^{2}$$

$$= \frac{3}{32}\left[\left(2(2)^2 - \frac{(2)^3}{3}\right) - \left(2(0)^2 - \frac{(0)^3}{3}\right)\right]$$

$$= \frac{3}{32}\left[\left(8 - \frac{8}{3}\right) - (0)\right]$$

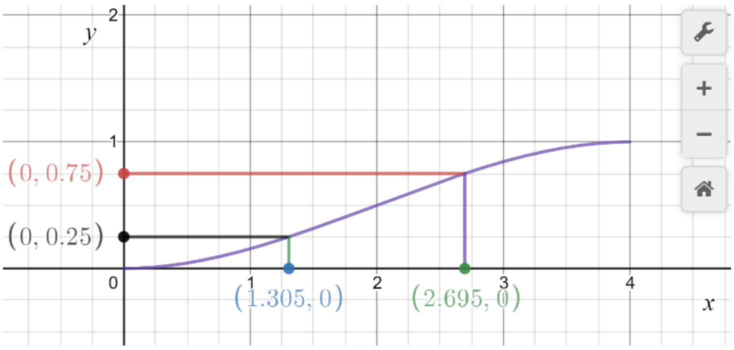
$$= \frac{3}{32} \times \frac{16}{3}$$

$$= \frac{1}{2}$$

$P(X < 2)$ is half of $P(X \leq 4)$.

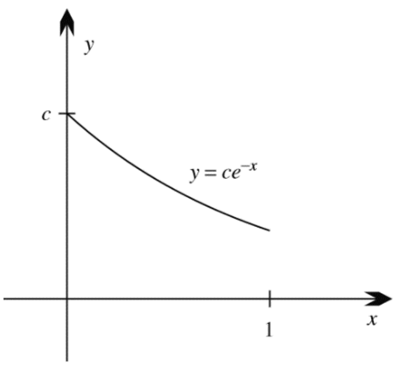
Therefore, 50% of the data lie to the left of the line $x = 2$.

12j



$Q_1 \doteqdot 1.3$ and $Q_2 \doteqdot 2.7$

13a Assuming $c > 0$, the graph of $f(x) = ce^{-x}$ is as follows.



13b If $f(x)$ is a probability density function where $0 \le x \le 1$ then $\int_0^1 f(x)\,dx = 1$

Hence, $\int_0^1 ce^{-x}\,dx = [-ce^{-x}]_0^1 = 1$

Then $(-ce^{-1}) - (-ce^{-0}) = 1$ and $-\frac{c}{e} + c = 1$

Therefore, $c = \frac{e}{e-1}$

13c $F(x) = \int_0^x ce^{-t}\,dt = [-ce^{-t}]_0^x$

$$= [-ce^{-x} - (-ce^{-0})]$$

$$= c - ce^{-x}$$

$$= c(1 - e^{-x})$$

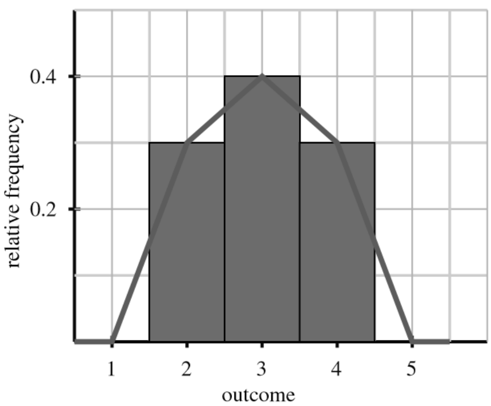
$$= \frac{e}{e-1}(1 - e^{-x})$$

13d To find Q_1, solve $\frac{e}{e-1}(1 - e^{-x}) = 0.25$, then $x = \ln\frac{4e}{3e+1}$ or $x \doteqdot 0.17$

To find Q_2, solve $\frac{e}{e-1}(1 - e^{-x}) = 0.5$, then $x = \ln\frac{2e}{e+1}$ or $x \doteqdot 0.38$

To find Q_3, solve $\frac{e}{e-1}(1 - e^{-x}) = 0.75$, then $x = \ln\frac{4e}{e+3}$ or $x \doteqdot 0.64$

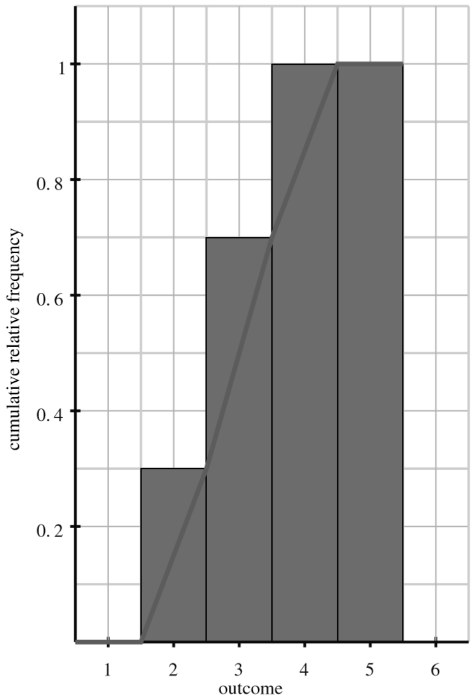
14a



14b Area of histogram$= 0.3 \times 1 + 0.4 \times 1 + 0.3 \times 1 = 1$

Area of relative frequency polygon$= \frac{0.3\times1}{2} + \frac{0.3+0.4}{2} \times 1 + \frac{0.4+0.3}{2} \times 1 + \frac{0.3\times1}{2} = 1$

14c



14d $Q_1 = 2.3$, $Q_2 = 3$, $Q_3 = 3.7$

14e i $\int_1^5 \frac{3}{32}(x-1)(5-x)dx$

$$= \frac{3}{32}\int_1^5 (-x^2+6x-5)\,dx$$

$$= -\frac{3}{32}\left[\frac{x^3}{3} - 3x^2 + 5x\right]_1^5$$

$$= -\frac{3}{32}\left[\left(\frac{(5)^3}{3} - 3(5)^2 + 5(5)\right) - \left(\frac{(1)^3}{3} - 3(1)^2 + 5(1)\right)\right]$$

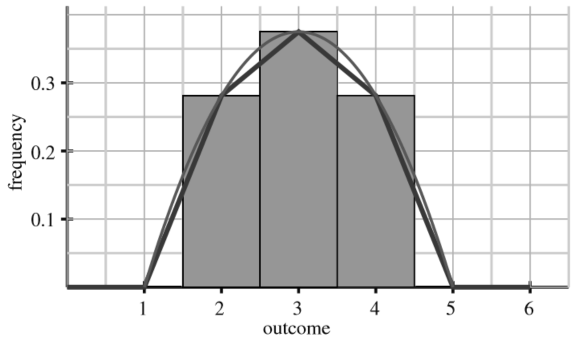
$$= -\frac{3}{32} \times -\frac{32}{3}$$

$$= 1$$

Therefore, $f(x)$ is a probability density function.

14e ii

x	1	2	3	4	5
$f(x)$	0	0.28	0.375	0.28	0



14e iii

$$F(x) = \int_1^x \frac{3}{32}(t-1)(5-t)dt$$

$$= \frac{3}{32}\int_1^x (-t^2+6t-5)\,dt$$

$$= -\frac{3}{32}\left[\frac{t^3}{3} - 3t^2 + 5t\right]_1^x$$

$$= -\frac{3}{32}\left[\left(\frac{(x)^3}{3} - 3(x)^2 + 5(x)\right) - \left(\frac{(1)^3}{3} - 3(1)^2 + 5(1)\right)\right]$$

$$= \frac{1}{32}(-x^3 + 9x^2 - 15x + 7)$$

14e iv $Q_1 = 2.3$ and

$$F(2.3) = \frac{1}{32}(-(2.3)^3 + 9(2.3)^2 - 15(2.3) + 7)$$

$$= \frac{1}{32} \times 7.943$$

$$\doteqdot 0.25$$

$Q_2 = 3$ and

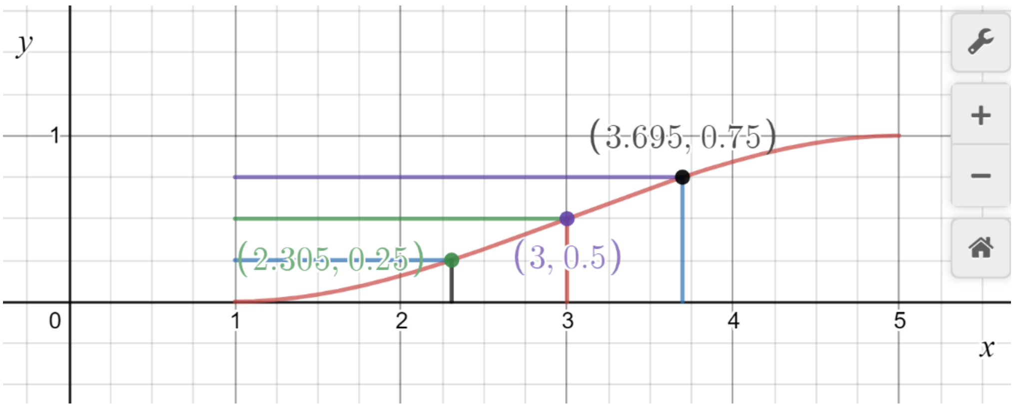
$$F(3) = \frac{1}{32}(-(3)^3 + 9(3)^2 - 15(3) + 7)$$
$$= \frac{1}{32} \times 16$$
$$\doteqdot 0.5$$

$Q_3 = 3.7$ and

$$F(3.7) = \frac{1}{32}(-(3.7)^3 + 9(3.7)^2 - 15(3.7) + 7)$$
$$= \frac{1}{32} \times 24.057$$
$$\doteqdot 0.75$$

All the probabilities are close to the estimates.

14e v



$Q_1 = 2.3$, $Q_2 = 3$ and $Q_3 = 3.7$ still seem good approximations.

15a

Clearly $f(x) > 0$, since it is the square of a nonzero number.

$$\int_1^{\infty} \frac{1}{x^2}\,dx = \left[\tfrac{-1}{x}\right]_1^{\infty}$$
$$= 0 - \frac{-1}{1}$$
$$= 1$$

15b

$$\begin{aligned} F(x) &= \int_1^x \frac{1}{t^2}\,dt \\ &= \left[\tfrac{-1}{t}\right]_1^x \\ &= \frac{-1}{x} - \frac{-1}{1} \\ &= 1 - \frac{1}{x} \end{aligned}$$

15c

Note that as $x \to \infty$, $\frac{1}{x} \to 0$ and hence $F(x) \to 1$. Thus the total probability is 1. This agrees with part (a).

15d

To find Q_1:

$$\begin{aligned} F(Q_1) &= \tfrac{1}{4} \\ 1 - \frac{1}{Q_1} &= \tfrac{1}{4} \\ \frac{1}{Q_1} &= \tfrac{3}{4} \\ Q_1 &= \tfrac{4}{3} \end{aligned}$$

To find Q_2:

$$\begin{aligned} F(Q_2) &= \tfrac{1}{2} \\ 1 - \frac{1}{Q_2} &= \tfrac{1}{2} \\ \frac{1}{Q_2} &= \tfrac{1}{2} \\ Q_2 &= 2 \end{aligned}$$

To find Q_3:

$$\begin{aligned} F(Q_3) &= \tfrac{3}{4} \\ 1 - \frac{1}{Q_3} &= \tfrac{3}{4} \\ \frac{1}{Q_3} &= \tfrac{1}{4} \\ Q_3 &= 4 \end{aligned}$$

16a

It is a property of the exponential function e^{-x} that it is always positive (this is evident from the graph and follows from the definition of powers).

$$\begin{aligned}\int_0^\infty e^{-x}\,dx &= \left[-e^{-x}\right]_0^\infty \\ &= 0-(-1) \\ &= 1\end{aligned}$$

16b

$$\begin{aligned}F(x) &= \int_0^x e^{-t}\,dt \\ &= \left[-e^{-t}\right]_0^x \\ &= -e^{-x}-(-1) \\ &= 1-e^{-x}\end{aligned}$$

16c

To find Q_1:

$$\begin{aligned}F(Q_1) &= \tfrac{1}{4} \\ 1-e^{-Q_1} &= \tfrac{1}{4} \\ e^{-Q_1} &= \tfrac{3}{4} \\ -Q_1 &= \ln\tfrac{3}{4} \\ Q_1 &= \ln\tfrac{4}{3}\end{aligned}$$

To find Q_2:

$$\begin{aligned}F(Q_2) &= \tfrac{1}{2} \\ 1-e^{-Q_2} &= \tfrac{1}{2} \\ e^{-Q_2} &= \tfrac{1}{2} \\ -Q_2 &= \ln\tfrac{1}{2} \\ Q_2 &= \ln 2\end{aligned}$$

To find Q_3:

$$\begin{aligned}F(Q_3) &= \tfrac{3}{4} \\ 1-e^{-Q_3} &= \tfrac{3}{4} \\ e^{-Q_3} &= \tfrac{1}{4} \\ -Q_3 &= \ln\tfrac{1}{4} \\ Q_3 &= \ln 4\end{aligned}$$

17a Unit square centred on $(0.5, 0.5)$.

Area of square $= 1 \times 1 = 1$ square unit

Circle centred on $(0.5, 0.5)$.

Area of circle with radius 0.5 unit $= \pi \times \left(\frac{1}{2}\right)^2 = \frac{\pi}{4}$ square units

Hence, ratio of the areas $= \frac{\frac{\pi}{4}}{1} = \frac{\pi}{4}$

17b i This returns the square of the distance from a random point in the square to the centre of the square and circle.

17b ii If the point is inside the circle, the value is 1, since inside the circle the condition (RAND()-0.5)^2+(RAND()-0.5)^2<0.25 is true.

17b iii If the point is outside the circle, the value is 0, since outside the circle the condition (RAND()-0.5)^2+(RAND()-0.5)^2<0.25 is false.

17c The code measures the relative frequency of points lying inside the circle, that is, the probability that the point will lie inside the circle. The value in cell C1 should approach π.

Solutions to Exercise 16C

1a The function is never negative and $\int_0^{10} \frac{1}{10}\, dx = \left[\frac{x}{10}\right]_0^{10} = 1$. Therefore, $f(x)$ is a valid probability density function.

1b $E(X) = \int_a^b x\, f(x) dx$ then

$$= \int_0^{10} x \times \frac{1}{10} dx$$

$$= \left[\frac{x^2}{20}\right]_0^{10}$$

$$= 5$$

1c Yes, 5 is in the centre of this distribution interval $[0, 10]$.

1d $\text{Var}(X) = \int_a^b (x-\mu)^2 f(x) dx$ then

$$= \int_0^{10} (x-5)^2 \times \frac{1}{10} dx$$

$$= \frac{1}{10}\left[\frac{x^3}{3} - 5x^2 + 25x\right]_0^{10}$$

$$= \frac{1}{10}\left[\left(\frac{(10)^3}{3} - 5(10)^2 + 25(10)\right) - \left(\frac{(0)^3}{3} - 5(0)^2 + 25(0)\right)\right]$$

$$= \frac{1}{10} \times \frac{250}{3}$$

$$= \frac{25}{3}$$

Therefore, $\sigma = \sqrt{\frac{25}{3}} = 2.886\ 75 \ldots \doteqdot 2.9$

1e $\text{Var}(X) = \int_a^b x^2 f(x) dx - \mu^2$ then

$$= \int_0^{10} x^2 \times \frac{1}{10} dx - 5^2$$

$$= \left[\frac{x^3}{30}\right]_0^{10} - 25$$

$= \frac{100}{3} - 25$

$= \frac{25}{3}$

This answer agrees with the previous result.

2a

x	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$F(x)$	2	5	1	3	10	2	3	2	2	7	1	3	8	4	1	2	8	5	6	8

2b $\mu = 4.15$

$\sigma^2 = 7.4275 \doteqdot 7.4$

$\sigma = 2.725\,344 \ldots \doteqdot 2.7$

2c The mean and standard deviation agree with the results in question 1.

2d If it was impossible to get a random 10, yes it would affect the validity of the model and the results.

3a The function is never negative and $\int_{-1}^{1} \frac{3}{2}x^2 dx = \left[\frac{x^3}{2}\right]_{-1}^{1} = 1$. Therefore, $f(x)$ is a valid probability density function.

3b $E(X) = \int_a^b x\, f(x) dx$ then

$= \int_{-1}^{1} x \times \frac{3}{2}x^2 dx$

$= \frac{3}{2}\left[\frac{x^4}{4}\right]_{-1}^{1}$

$= 0$

3c $\quad \text{Var}(X) = \int_a^b (x-\mu)^2 f(x)dx$ then

$$= \int_{-1}^{1} (x-0)^2 \times \frac{3}{2}x^2 dx$$

$$= \frac{3}{2}\left[\frac{x^5}{5}\right]_{-1}^{1}$$

$$= \frac{3}{5}$$

$$\sigma = \sqrt{\frac{3}{5}} = \frac{\sqrt{15}}{5}$$

3d

$$\int_{0-\frac{\sqrt{15}}{5}}^{0+\frac{\sqrt{15}}{5}} \frac{3}{2}x^2\, dx$$

$$= \left[\frac{x^3}{2}\right]_{-\frac{\sqrt{15}}{5}}^{\frac{\sqrt{15}}{5}}$$

$$= \frac{3\sqrt{15}}{25}$$

$$\doteqdot 0.46 \text{ or } 46\%$$

4a i $\quad$ The function is never negative and $\int_0^1 2x\, dx = [x^2]_0^1 = 1$. Therefore, $f(x)$ is a valid probability density function.

4a ii $\quad E(X) = \int_a^b x\, f(x)dx$ then

$$= \int_0^1 x \times 2x\, dx$$

$$= \frac{2}{3}[x^3]_0^1$$

$$= \frac{2}{3}$$

4a iii $\text{Var}(X) = \int_a^b (x-\mu)^2 f(x)dx$ then

$$= \int_0^1 \left(x - \frac{2}{3}\right)^2 \times 2x dx$$

$$= \left[\frac{x^2(9x^2 - 16x + 8)}{18}\right]_0^1$$

$$= \frac{1}{18}$$

$$\sigma = \sqrt{\frac{1}{18}} = \frac{1}{3\sqrt{2}} = \frac{\sqrt{2}}{6}$$

4a iv

$$\int_{\frac{2}{3}-\frac{\sqrt{2}}{6}}^{\frac{2}{3}+\frac{\sqrt{2}}{6}} 2x\, dx$$

$$= [x^2]_{\frac{2}{3}-\frac{\sqrt{2}}{6}}^{\frac{2}{3}+\frac{\sqrt{2}}{6}}$$

$$= \frac{4\sqrt{2}}{9}$$

$$\doteqdot 0.63 \text{ or } 63\%$$

4b i The function is never negative and $\int_{-1}^{1} |x|\, dx = 2\int_0^1 x\, dx = 2\left[\frac{x^2}{2}\right]_0^1 = 1$. Therefore, $f(x)$ is a valid probability density function.

4b ii $E(X) = \int_a^b x\, f(x)dx$ then

$$= \int_{-1}^{1} x \times |x|\, dx$$

$$= 0$$

4b iii $\text{Var}(X) = \int_a^b (x-\mu)^2 f(x)dx$ then

$$= \int_{-1}^{1} (x-0)^2 \times |x| dx$$

$= \int_0^1 x^3\,dx + \int_{-1}^0 -x^3\,dx$

$= \left[\frac{x^4}{4}\right]_0^1 + \left[\frac{-x^4}{4}\right]_{-1}^0$

$= \frac{1}{2}$

$\sigma = \sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2}$

4b iv

$\int_{0-\frac{\sqrt{2}}{2}}^{0+\frac{\sqrt{2}}{2}} |x|\,dx$

$= 2\left[\frac{x^2}{2}\right]_0^{\frac{\sqrt{2}}{2}}$

$= \frac{1}{2}$

$= 0.5$ or 50%

4c i The function is never negative and $\int_0^4 \frac{3}{64}x^2\,dx = \left[\frac{x^3}{64}\right]_0^4 = 1$. Therefore, $f(x)$ is a valid probability density function.

4c ii $E(X) = \int_a^b x\,f(x)dx$ then

$= \int_0^4 x \times \frac{3}{64}x^2\,dx$

$= \frac{3}{64 \times 4}[x^4]_0^4$

$= 3$

4c iii $\text{Var}(X) = \int_a^b (x-\mu)^2 f(x)dx$ then

$$= \int_0^4 (x-3)^2 \times \frac{3}{64}x^2 dx$$

$$= \left[\frac{3x^3(2x^2 - 15x + 30)}{640}\right]_0^4$$

$$= \frac{3}{5}$$

$$\sigma = \sqrt{\frac{3}{5}} = \frac{\sqrt{15}}{5}$$

4c iv

$$\int_{3-\frac{\sqrt{15}}{5}}^{3+\frac{\sqrt{15}}{5}} \frac{3}{64}x^2\, dx$$

$$= \left[\frac{1}{64}x^3\right]_{3\frac{\sqrt{15}}{5}}^{3+\frac{\sqrt{15}}{5}}$$

$$= \frac{69\sqrt{15}}{400}$$

$$\doteqdot 0.668 \text{ or } 67\%$$

5a $\int_0^c \frac{1}{c} dx = \left[\frac{x}{c}\right]_0^c = \frac{c}{c} - \frac{0}{c} = 1$ and the function is never negative. Therefore, $f(x)$ is a valid probability density function.

5b $E(X)$

$$= \int_0^c \frac{1}{c} \times x\, dx$$

$$= \left[\frac{x^2}{2c}\right]_0^c$$

$$= \frac{(c)^2}{2c} - \frac{(0)^2}{2c}$$

$= \frac{c}{2}$

Yes, for a measure of centre of this uniform distribution, the mean is as expected.

5c Var$(X) = \int_a^b (x-\mu)^2 f(x)dx$ then

$= \int_0^c \left(x - \frac{c}{2}\right)^2 \times \frac{1}{c} dx$

$= \frac{1}{c}\left[\left(\frac{x^3}{3} - c\frac{x^2}{2} + \frac{c^2}{4}x\right)\right]_0^c$

$= \frac{1}{c}\left[\left(\frac{(c)^3}{3} - c\frac{(c)^2}{2} + \frac{c^2}{4}(c)\right) - \left(\frac{(0)^3}{3} - c\frac{(0)^2}{2} + \frac{c^2}{4}(0)\right)\right]$

$= \frac{1}{c} \times \frac{c^3}{12}$

$= \frac{c^2}{12}$

5d The answer agrees for this special case with $c = 10$.

5e $E(aX+b) = aE(X) + b$ then $E(X) = \frac{c}{2} + h$

and Var$(aX+b) = a^2$ Var(X) then Var(X) is unchanged, where $a = 1$ and $b = c$

5f Substitute $h + c = k$ in the previous result: $E(X) = \frac{k+h}{2}$ and Var$(X) = \frac{(k-h)^2}{12}$

6a $\int_0^2 \frac{1}{8} dx + \int_2^5 \frac{1}{4} dx = \left[\frac{x}{8}\right]_0^2 + \left[\frac{x}{4}\right]_2^5 = \frac{2}{8} + \left(\frac{5}{4} - \frac{2}{4}\right) = 1$ and the function is never negative. Therefore, $f(x)$ is a valid probability density function.

6b $E(X)$

$= \int_0^2 \frac{1}{8} \times x\, dx + \int_2^5 \frac{1}{4} \times x\, dx$

$= \left[\frac{x^2}{16}\right]_0^2 + \left[\frac{x^2}{8}\right]_2^5$

$= \frac{4}{16} + \left(\frac{25}{8} - \frac{4}{8}\right)$

$= \frac{23}{8}$

$\text{Var}(X) = \int_a^b (x-\mu)^2 f(x)dx$ then

$= \int_0^2 \left(x - \frac{23}{8}\right)^2 \times \frac{1}{8}\, dx + \int_2^5 \left(x - \frac{23}{8}\right)^2 \times \frac{1}{4}\, dx$

$= \frac{1}{8}\left[\frac{\left(x-\frac{23}{8}\right)^3}{3}\right]_0^2 + \frac{1}{4}\left[\frac{\left(x-\frac{23}{8}\right)^3}{3}\right]_2^5$

$= \frac{1}{24}\left(\left(2-\frac{23}{8}\right)^3 - \left(0-\frac{23}{8}\right)^3\right) + \frac{1}{12}\left(\left(5-\frac{23}{8}\right)^3 - \left(2-\frac{23}{8}\right)^3\right)$

$= \frac{1}{24}\left(-\frac{343}{512} + 23\frac{391}{512}\right) + \frac{1}{12}\left(9\frac{305}{512} + \frac{343}{512}\right)$

$= \frac{1}{24} \times 23\frac{3}{32} + \frac{1}{12} \times 10\frac{17}{64}$

$= \frac{739}{768} + \frac{219}{256}$

$= \frac{349}{192}$

$= 1.817\,708\ldots$

$\doteqdot 1.82$

7

$$\text{LHS} = \int_a^b x^2 f(x)\,dx - \int_a^b 2\mu x f(x)\,dx + \int_a^b \mu^2 f(x)\,dx.$$

By the definition of a PDF,

$\text{Term } 3 = \mu^2 \int_a^b f(x)\,dx = \mu^2.$

By the formula for the mean,

$\text{Term } 2 = -2\mu \int_a^b x f(x)\,dx = -2\mu^2.$

8a

Since the path is a polygon, we only need check the equations given agree where the intervals join.

When $0 \le x \le 1$: $\frac{2}{10}x = 0, \frac{2}{10}$ at $x = 0, 1$
When $1 \le x \le 2$: $\frac{1}{10}(3x - 1) = \frac{2}{10}, \frac{5}{10}$ at $x = 1, 2$
When $2 \le x \le 3$: $\frac{1}{10}(-2x + 9) = \frac{5}{10}, \frac{3}{10}$ at $x = 2, 3$
When $3 \le x \le 4$: $\frac{1}{10}(-3x + 12) = \frac{3}{10}, 0$ at $x = 3, 4$
The graph is continuous on the interval $0 \le x \le 4$ and the values agree with the graph drawn previously in Q6 (or the tabulated relative frequencies).

8b

$$\begin{aligned}\mathrm{E}(X) &= \int_0^1 \tfrac{2}{10}x^2\,dx + \int_1^2 \tfrac{1}{10}(3x^2 - x)\,dx + \int_2^3 \tfrac{1}{10}(-2x^2 + 9x)\,dx + \int_3^4 \tfrac{1}{10}(-3x^2 + 12x)\,dx \\ &= \left[\tfrac{2}{30}x^3\right]_0^1 + \left[\tfrac{1}{10}(x^3 - \tfrac{1}{2}x^2)\right]_1^2 + \left[\tfrac{1}{10}(-\tfrac{2}{3}x^3 + \tfrac{9}{2}x^2)\right]_2^3 + \left[\tfrac{1}{10}(-x^3 + 6x^2)\right]_3^4 \\ &= \left(\tfrac{2}{30} - 0\right) + \left(\tfrac{6}{10} - \tfrac{1}{20}\right) + \left(\tfrac{45}{20} - \tfrac{38}{30}\right) + \left(\tfrac{32}{10} - \tfrac{27}{10}\right) \\ &= 2.1\end{aligned}$$

8c

$E(X) = \frac{1}{n}\sum xf = \frac{1}{10}(2 + 10 + 9) = 2.1$. This answer agrees.

8d

Not only do both satisfy the condition that the area under the curve is 1, but they give the same result for the expected value.

9a

$f(x) \ge 0$ since $f(x)$ is a square (or since $x \ge 0$).

$$\begin{aligned}\text{Further, } \int_1^\infty 3x^{-4}\,dx &= \left[-x^{-3}\right]_1^\infty \\ &= (0) - (-1) \\ &= 1\end{aligned}$$

Hence $f(x)$ is a valid pdf.

9b

$$\begin{aligned}
\mathrm{E}(X) &= \int_1^\infty x\,f(x)\,dx \\
&= \int_1^\infty 3x^{-3}\,dx \\
&= \left[-\tfrac{3}{2}x^{-2}\right]_1^\infty \\
&= (0) - (-\tfrac{3}{2}) \\
&= \tfrac{3}{2}
\end{aligned}$$

$$\begin{aligned}
\mathrm{E}(X^2) &= \int_1^\infty x^2 f(x)\,dx \\
&= \int_1^\infty 3x^{-2}\,dx \\
&= \left[-3x^{-1}\right]_1^\infty \\
&= (0) - (-3) \\
&= 3
\end{aligned}$$

$$\begin{aligned}
\mathrm{Var}(X) &= \mathrm{E}(X^2) - \mathrm{E}(X)^2 \\
&= 3 - \tfrac{9}{4} \\
&= \tfrac{3}{4}
\end{aligned}$$

9c i

$$\begin{aligned}
P(X \le 4) &= \int_1^4 3x^{-4}\,dx \\
&= \left[-x^{-3}\right]_1^4 \\
&= (-\tfrac{1}{4^3}) - (-1) \\
&= 1 - \tfrac{1}{4^3}
\end{aligned}$$

9c ii

$$\begin{aligned}
P(X \ge 2) &= \int_2^\infty 3x^{-4}\,dx \\
&= \left[-x^{-3}\right]_2^\infty \\
&= (0) - (-2^{-3}) \\
&= \tfrac{1}{8}
\end{aligned}$$

9c iii

$$\begin{aligned}P(2 \le X \le 5) &= \int_2^5 3x^{-4}\,dx \\ &= \left[-x^{-3}\right]_2^5 \\ &= \left(-\tfrac{1}{125}\right) - \left(-\tfrac{1}{8}\right) \\ &= \tfrac{117}{1000}\end{aligned}$$

9d

$$\begin{aligned}F(x) &= \int_1^x 3t^{-4}\,dt \\ &= \left[-t^{-3}\right]_1^x \\ &= \left(-\tfrac{1}{x^3}\right) - (-1) \\ &= 1 - \tfrac{1}{x^3}\end{aligned}$$

10a

$$\frac{d}{dx}xe^{-x} = e^{-x} - xe^{-x},$$

$$\begin{aligned}\text{so} \qquad xe^{-x} &= \int e^{-x}\,dx - \int xe^{-x}\,dx \\ &= -e^{-x} - \int xe^{-x}\,dx\end{aligned}$$

$$\text{thus} \int xe^{-x}\,dx = -e^{-x} - xe^{-x}$$

10b

$$\begin{aligned}\mathrm{E}(X) &= \left[-e^{-x} - xe^{-x}\right]_0^\infty \\ &= (0) - (-1) \\ &= 1\end{aligned}$$

This integral requires the idea of *dominance*, i.e. that $xe^{-x} \to 0$ as $x \to \infty$, because e^{-x} goes to 0 much faster than x goes to infinity.

10c

The derivative is $(2xe^{-x} - x^2e^{-x}) + (2e^{-x} - 2xe^{-x}) - 2e^{-x} = -x^2e^{-x}$,

so $\int x^2e^{-x}\,dx = -x^2e^{-x} - 2xe^{-x} - 2e^{-x} + C$

10d

$$\begin{aligned}\mathrm{E}(X^2) &= \int_0^\infty x^2e^{-x}\,dx \\ &= \left[-x^2e^{-x} - 2xe^{-x} - 2e^{-x}\right]_0^\infty \\ &= (0-0-0)-(0-0-2) \\ &= 2\end{aligned}$$

$$\begin{aligned}\text{Hence } \mathrm{Var}(X) &= \mathrm{E}(X^2) - E(X)^2 \\ &= 2 - 1^2 \\ &= 1\end{aligned}$$

11a Consider

$$\frac{1}{x} - \frac{1}{x+1}$$

Writing with a common denominator gives

$$\frac{1}{x} - \frac{1}{x+1}$$

$$= \frac{1(x+1) - 1(x)}{x(x+1)}$$

$$= \frac{x+1-x}{x(x+1)}$$

$$= \frac{1}{x(x+1)}$$

Hence

$$f(x) = \frac{1}{\ln 2}\left(\frac{1}{x} - \frac{1}{x+1}\right)$$

becomes

$$f(x) = \frac{1}{\ln 2}\left(\frac{1}{x(x+1)}\right)$$

or

$$f(x) = \frac{1}{\ln 2} \times \frac{1}{x(x+1)}$$

11b For $x \geq 1$,

$$\int f(x)\,dx$$

$$= \int \frac{1}{\ln 2} \times \frac{1}{x(x+1)}\,dx$$

$$= \int \frac{1}{\ln 2}\left(\frac{1}{x} - \frac{1}{x+1}\right)dx$$

$$= \frac{1}{\ln 2}\int \left(\frac{1}{x} - \frac{1}{x+1}\right)dx$$

$$= \frac{1}{\ln 2}(\ln x - \ln(x+1))$$

$$= \frac{1}{\ln 2} \times \ln\left(\frac{x}{x+1}\right)$$

11c

$$\int_1^\infty f(x)\,dx$$

$$= \int_1^\infty \frac{1}{\ln 2}\left(\frac{1}{x} - \frac{1}{x+1}\right)dx$$

$$= \left[\frac{1}{\ln 2} \times \ln\left(\frac{x}{x+1}\right)\right]_1^\infty$$

$$= \frac{1}{\ln 2}\left[\ln\left(\frac{x}{x+1}\right)\right]_1^\infty$$

$$= \frac{1}{\ln 2}\left(\ln 1 - \ln\frac{1}{2}\right)$$

$$= \frac{1}{\ln 2}(0 + \ln 2)$$

$= \dfrac{\ln 2}{\ln 2}$

$= 1$

Hence $f(x)$ is a valid PDF in the domain $[1, \infty)$.

11d E(X)

$= \displaystyle\int_1^{\infty} xf(x)\, dx$

$= \displaystyle\int_1^{\infty} x \times \frac{1}{\ln 2}\left(\frac{1}{x} - \frac{1}{x+1}\right) dx$

$= \displaystyle\frac{1}{\ln 2}\int_1^{\infty} \left(1 - \frac{x}{x+1}\right) dx$

$= \displaystyle\frac{1}{\ln 2}\int_1^{\infty} \frac{(x+1) - x}{x+1}\, dx$

$= \displaystyle\frac{1}{\ln 2}\int_1^{\infty} \frac{1}{x+1}\, dx$

$= \dfrac{1}{\ln 2}[\ln(x+1)]_1^{\infty}$

$= \dfrac{1}{\ln 2}(\ln \infty - \ln 2)$

which is undefined

Hence E(X) does not exist for this PDF.

12a

$\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx$

$= [\tan^{-1} x]_{-\infty}^{\infty}$

$= \tan^{-1} \infty - \tan^{-1}(-\infty)$

$= \dfrac{\pi}{2} - \left(-\dfrac{\pi}{2}\right)$

$= \pi$

12b From part a:

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \pi$$

$$\int_{-\infty}^{\infty} f(x)\,dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\pi(1+x^2)}\,dx$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$$

$$= \frac{1}{\pi} \times \pi$$

$$= 1$$

Hence $f(x)$ is a valid PDF in the domain $(-\infty, \infty)$.

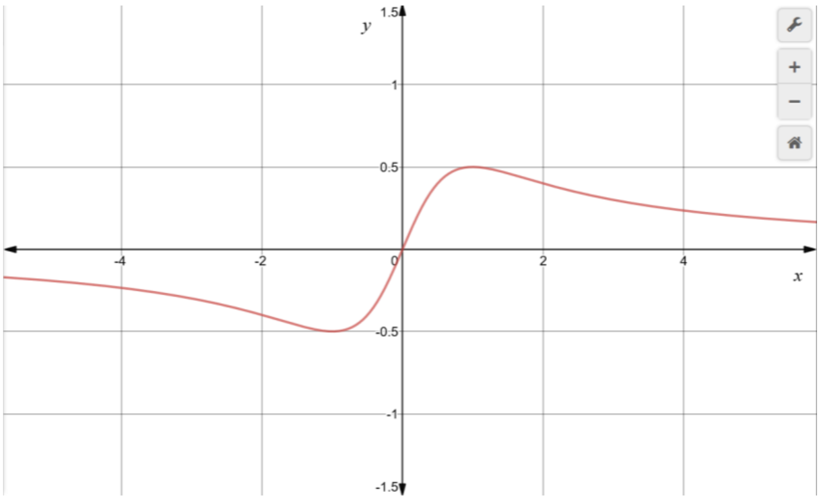
12c $\mathrm{E}(X)$

$$= \int_{-\infty}^{\infty} x f(x)\,dx$$

$$= \int_{-\infty}^{\infty} x \times \frac{1}{\pi(1+x^2)}\,dx$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \left(\frac{x}{1+x^2}\right)dx$$

Consider the graph of $y = \frac{x}{1+x^2}$ shown below. Note the symmetry of this odd function.



The integral from $-\infty$ to 0 is negative and the integral from 0 to ∞ is positive. Hence the integral over the symmetric domain $(-\infty, \infty)$ is 0.

This is clear from the graph above where the equal areas (half below and half above the x-axis) cancel.

That is, $\mathrm{E}(x) = 0$.

12d $\mathrm{E}(x) = \mu = 0$

$\mathrm{Var}(X)$

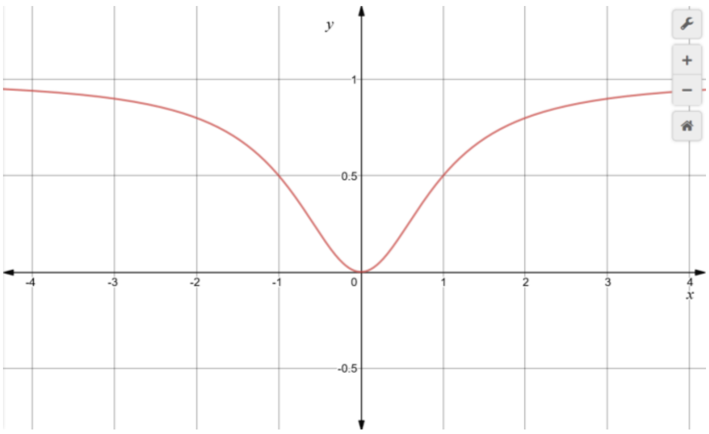
$$= \int_{-\infty}^{\infty} x^2 f(x)\, dx - \mu^2$$

$$= \int_{-\infty}^{\infty} x^2 f(x)\, dx$$

$$= \int_{-\infty}^{\infty} x^2 \times \frac{1}{\pi(1+x^2)} dx$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x^2}{1+x^2} dx$$

Consider the graph of $y = \frac{x^2}{1+x^2}$ shown below. Note the symmetry.



Hence

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x^2}{1+x^2} dx$$

$$= \frac{2}{\pi} \int_{0}^{\infty} \frac{x^2}{1+x^2} dx$$

$$= \frac{2}{\pi}\int_0^\infty \left(1 - \frac{1}{1+x^2}\right) dx$$

$$= \frac{2}{\pi}\left[x - \tan^{-1} x\right]_0^\infty$$

which is infinite since $x \to \infty$ and $\tan^{-1} x \to \frac{\pi}{2}$ as $x \to \infty$

Hence the variance is not defined.

Solutions to Exercise 16D

1a $P(Z \leq 0) = 0.5000$

1b $P(Z \leq 1) = 0.8413$

1c $P(Z \leq 2) = 0.9772$

1d $P(Z \leq 1.5) = 0.9332$

1e $P(Z \leq 0.4) = 0.6554$

1f $P(Z \leq 2.3) = 0.9893$

1g $P(Z \leq 1.2) = 0.8849$

1h $P(Z \leq 5) = 1.0000$

2 The total area under the curve is 1, so the areas of regions to the right and left of $z = a$ add to 1. This identity is true for any probability distribution.

2a
$$\begin{aligned} P(Z > 0) &= 1 - P(Z \leq 0) \\ &= 1 - 0.5 \\ &= 0.5 \end{aligned}$$

2b
$$\begin{aligned} P(Z > 1) &= 1 - P(Z \leq 1) \\ &= 1 - 0.8413 \\ &= 0.1587 \end{aligned}$$

2c $P(Z > 2) = 1 - P(Z \le 2)$

$= 1 - 0.9772$

$= 0.0228$

2d $P(Z > 2.4) = 1 - P(Z \le 2.4)$

$= 1 - 0.9918$

$= 0.0082$

2e $P(Z > 1.3) = 1 - P(Z \le 1.3)$

$= 1 - 0.9032$

$= 0.0968$

2f $P(Z > 0.7) = 1 - P(Z \le 0.7)$

$= 1 - 0.7580$

$= 0.2420$

2g $P(Z \ge 1.6) = 1 - P(Z \le 1.6)$

$= 1 - 0.9452$

$= 0.0548$

2h $P(Z > 8) = 1 - P(Z \le 8)$

$= 1 - 1$

$= 0$

3a From the even symmetry of the graph,

$P(Z < -a) = P(Z > a) = 1 - P(Z \le a)$.

(The result also holds for a ≤ 0, but this is not useful to us.) This result is certainly not true for all probability distributions.

3b i $P(Z < -1.2) = 1 - P(Z \leq 1.2)$

$= 1 - 0.8849$

$= 0.1151$

3b ii $P(Z < -2.3) = 1 - P(Z \leq 2.3)$

$= 1 - 0.9893$

$= 0.0107$

3b iii $P(Z < -0.2) = 1 - P(Z \leq 0.2)$

$= 1 - 0.5793$

$= 0.4207$

3b iv $P(Z < -3.2) = 1 - P(Z \leq 3.2)$

$= 1 - 0.8849$

$= 0.1151$

3b v $P(Z < -5) = 1 - P(Z \leq 5)$

$= 1 - 0.9993$

$= 0.0007$

3b vi $P(Z \leq -0.7) = 1 - P(Z < 0.7)$

$= 1 - 0.7580$

$= 0.2420$

3b vii $P(Z < -1.6) = 1 - P(Z \leq 1.6)$

$= 1 - 0.9452$

$= 0.0548$

3b viii $P(Z \leq -1.4) = 1 - P(Z < 1.4)$

$= 1 - 0.9192$

$= 0.0808$

3b ix $P(Z < -0) = 1 - P(Z \leq 0)$

$= 1 - 0.5$

$= 0.5000$

4a $\mu = 0$ then the mean is at the centre of the normal distribution, which is symmetrical, half the data is to the left and half the data is to the right of the mean. Therefore, $P(X \leq 0) = \frac{1}{2} = 0.5$.

4b i $P(0 < Z \leq 1.3) = P(Z \leq 1.3) - P(Z < 0)$

$= 0.9032 - 0.5$

$= 0.4032$

4b ii $P(0 < Z \leq 2.4) = P(Z \leq 2.4) - P(Z < 0)$

$= 0.9918 - 0.5$

$= 0.4918$

4b iii $P(0 < Z \leq 0.7) = P(Z \leq 0.7) - P(Z < 0)$

$= 0.7580 - 0.5$

$= 0.2580$

4b iv $P(-2.4 \leq Z < 0) = P(Z < 0) - P(Z \leq -2.4)$

$= P(Z < 0) - \big(1 - P(Z < 2.4)\big)$

$= 0.5 - (1 - 0.9918)$

$= 0.4918$

4b v $P(-1.1 \leq Z < 0) = P(Z < 0) - P(Z \leq -1.1)$

$= P(Z < 0) - \big(1 - P(Z < 1.1)\big)$

$= 0.5 - (1 - 0.8643)$

$= 0.3643$

4b vi $P(-0.7 \leq Z < 0) = P(Z < 0) - P(Z \leq -0.7)$

$= P(Z < 0) - \big(1 - P(Z < 0.7)\big)$

$= 0.5 - (1 - 0.7580)$

$= 0.2580$

4b vii $P(0 < Z \leq 1.6) = P(Z \leq 1.6) - P(Z < 0)$

$= 0.9452 - 0.5$

$= 0.4452$

4b viii $P(-1.3 \leq Z \leq 0) = P(Z \leq 0) - P(Z \leq -1.3)$

$= P(Z < 0) - \big(1 - P(Z < 1.3)\big)$

$= 0.5 - (1 - 0.9032)$

$= 0.4032$

4b ix $P(0 < Z \leq 5) = P(Z \leq 5) - P(Z < 0)$

$= 1 - 0.5$

$= 0.5000$

4c i $P(-1.3 \leq Z < 1.3) = 2 \times (P(Z < 1.3) - P(Z < 0)$

$= 2 \times (0.9032 - 0.5)$

$= 2 \times 0.4032$

$= 0.8064$

4c ii $P(-2.4 \le Z < 2.4) = 2 \times (P(Z < 2.4) - P(Z < 0)$

$= 2 \times (0.9918 - 0.5)$

$= 2 \times 0.4918$

$= 0.9836$

4c iii $P(-0.8 \le Z < 0.8) = 2 \times (P(Z < 0.8) - P(Z < 0)$

$= 2 \times (0.7881 - 0.5)$

$= 2 \times 0.2881$

$= 0.5762$

4c iv $P(-2.9 \le Z < 2.9) = 2 \times (P(Z < 2.9) - P(Z < 0)$

$= 2 \times (0.9981 - 0.5)$

$= 2 \times 0.4981$

$= 0.9962$

4c v $P(-0.4 \le Z < 0.4) = 2 \times (P(Z < 0.4) - P(Z < 0)$

$= 2 \times (0.6554 - 0.5)$

$= 2 \times 0.1554$

$= 0.3108$

4c vi $P(-1.5 \le Z < 1.5) = 2 \times (P(Z < 1.5) - P(Z < 0)$

$= 2 \times (0.9332 - 0.5)$

$= 2 \times 0.4332$

$= 0.8664$

5 $P(Z \le 2) = P(Z < 2)$ parts a and e

$P(Z \le -1) = P(Z \ge 1)$ parts b and g

$P(Z \le 1.2) = P(Z > -1.2)$ parts c and h

$P(Z = 4) = P(Z = 2.3)$ parts d and f

6 $P(Z \le 5) = P(Z < 5)$ parts a and c

$P(Z > -1.7) = P(Z < 1.7)$ parts b and g

$P(Z \ge 2) = P(Z \le -2)$ parts d and f

$P(Z = 3) = P(Z = 1.2)$ parts e and h

7a Since $b > a$, the area from $-\infty$ to b includes the area from $-\infty$ to a. The difference of these two areas, $P(Z \le b) - P(Z < a)$, is therefore $P(a \le Z \le b)$.

7b i
$$\begin{aligned} P(1.2 \le Z < 1.5) &= P(Z < 1.5) - P(Z < 1.2) \\ &= 0.9332 - 0.8849 \\ &= 0.0483 \end{aligned}$$

7b ii
$$\begin{aligned} P(0.2 \le Z < 2.3) &= P(Z < 2.3) - P(Z < 0.2) \\ &= 0.9893 - 0.5793 \\ &= 0.4100 \end{aligned}$$

7b iii
$$\begin{aligned} P(0.6 \le Z < 1.7) &= P(Z < 1.7) - P(Z < 0.6) \\ &= 0.9554 - 0.7257 \\ &= 0.2297 \end{aligned}$$

7b iv
$$\begin{aligned} P(-2 \le Z < -1.2) &= P(Z < -1.2) - P(Z < -2) \\ &= [1 - P(Z \le 1.2)] - [1 - P(Z \le 2)] \\ &= (1 - 0.8849) - (1 - 0.9772) \\ &= 0.1151 - 0.0228 \\ &= 0.0923 \end{aligned}$$

7b v $P(-4 \leq Z < -0.2) = P(Z < -0.2) - P(Z < -4)$

$= [1 - P(Z \leq 0.2)] - [1 - P(Z \leq 4)]$

$= (1 - 0.5793) - (1 - 1)$

$= 0.4207$

7b vi $P(-2.7 \leq Z < -1) = P(Z < -1) - P(Z < -2.7)$

$= [1 - P(Z \leq 1)] - [1 - P(Z \leq 2.7)]$

$= (1 - 0.8413) - (1 - 0.9965)$

$= 0.1587 - 0.0035$

$= 0.1552$

7c i $P(-1.5 \leq Z < 2.2) = P(Z < 2.2) - P(Z < -1.5)$

$= P(Z < 2.2) - [1 - P(Z \leq 1.5)]$

$= 0.9861 - (1 - 0.9332)$

$= 0.9861 - 0.0668$

$= 0.9193$

7c ii $P(-0.9 \leq Z < 1.2) = P(Z < 1.2) - P(Z < -0.9)$

$= P(Z < 1.2) - [1 - P(Z \leq 0.9)]$

$= 0.8849 - (1 - 0.8159)$

$= 0.8849 - 0.1841$

$= 0.7008$

7c iii $P(-2.9 \leq Z < 1.3) = P(Z < 1.3) - P(Z < -2.9)$

$= P(Z < 1.3) - [1 - P(Z \leq 2.9)]$

$= 0.9032 - (1 - 0.9981)$

$= 0.9032 - 0.0019$

$= 0.9013$

8a $P(Z < 0) = 0.5$

8b $P(Z = 4) = 0$

8c $P(Z > 1.8) = 1 - P(Z \leq 1.8)$

$= 1 - 0.9641$

$= 0.0359$

8d $P(Z \leq 1.2) = 0.8849$

8e $P(Z \geq 1.2) = 1 - P(Z < 1.2)$

$= 1 - 0.8849$

$= 0.1151$

8f $P(0 \leq Z \leq 1.2) = P(Z \leq 1.2) - P(Z < 0)$

$= 0.8849 - 0.5$

$= 0.3849$

8g $P(Z \leq -1.8) = 1 - P(Z < 1.8)$

$= 1 - 0.9641$

$= 0.0359$

8h $P(Z \geq -1.2) = P(Z \leq 1.2)$

$= 0.8849$

8i $P(1.2 \leq Z \leq 1.8) = P(Z \leq 1.8) - P(Z < 1.2)$

$= 0.9641 - 0.8849$

$= 0.0792$

8j $P(-1.8 \leq Z \leq 1.2) = P(Z \leq 1.2) - P(Z < -1.8)$

$= P(Z \leq 1.2) - [1 - P(Z \leq 1.8)]$

$= 0.8849 - (1 - 0.9641)$

$= 0.8849 - 0.0359$

$= 0.8490$

9a $P(Z \leq 1.3) = 0.9032$

9b $P(Z = 2.4) = 0$

9c $P(Z > 0.4) = 1 - P(Z \leq 0.4)$

$= 1 - 0.6554$

$= 0.3446$

9d $P(Z \leq 1.7) = 0.9554$

9e $P(Z \geq -1.3) = P(Z \leq 1.3)$

$= 0.9032$

9f $P(0 \leq Z \leq 1.5) = 0.9332 - 0.5$

$= 0.4332$

9g $P(Z \leq -0.8) = 1 - P(Z < 0.8)$

$= 1 - 0.7881$

$= 0.2119$

9h $P(Z \geq 0.2) = 1 - P(Z < 0.2)$

$= 1 - 0.5793$

$= 0.4207$

9i $$\begin{aligned} P(1.1 \le Z \le 1.5) &= P(Z \le 1.5) - P(Z < 1.1) \\ &= 0.9332 - 0.8643 \\ &= 0.0689 \end{aligned}$$

9j $$\begin{aligned} P(-1.3 \le Z \le 2.2) &= P(Z \le 2.2) - P(Z < -1.3) \\ &= P(Z \le 2.2) - [1 - P(Z \le 1.3)] \\ &= 0.9861 - (1 - 0.9032) \\ &= 0.9861 - 0.0968 \\ &= 0.8893 \end{aligned}$$

10a $$\begin{aligned} P(Z \le 1.2 \text{ or } Z \ge 1.8) &= P(Z \le 1.2\ ) + P(Z \ge 1.8) \\ &= 0.8849 + [1 - P(Z < 1.8)] \\ &= 0.8849 + 1 - 0.9641 \\ &= 0.9208 \end{aligned}$$

10b $$\begin{aligned} P(Z \le 1.8 \text{ and } Z \ge 1.2) &= P(1.2 \le Z \le 1.8) \\ &= P(Z \le 1.8) - P(Z < 1.2) \\ &= 0.9641 - 0.8849 \\ &= 0.0792 \end{aligned}$$

10c $$\begin{aligned} P(Z \le 0.2 \text{ or } Z \ge 1.6) &= P(Z \le 0.2\ ) + P(Z \ge 1.6) \\ &= 0.5793 + [1 - P(Z < 1.6)] \\ &= 0.5793 + 1 - 0.9452 \\ &= 0.6341 \end{aligned}$$

10d $P(Z \leq 2.4 \text{ and } Z \geq 1.7) = P(1.7 \leq Z \leq 2.4)$

$= P(Z \leq 2.4) - P(Z < 1.7)$

$= 0.9918 - 0.9554$

$= 0.0364$

11 The answers are the same when questions are solved using technology.

12a $P(Z \leq 0) = 50\%$ (because $z = 0$ is the axis of symmetry)

12b $P(Z \leq 1)$

$= P(Z \leq 0) + \dfrac{P(-1 \leq Z \leq 1)}{2}$

$= 50\% + \frac{68\%}{2}$

$= 84\%$

12c $P(Z \leq 2)$

$= P(Z \leq 0) + \dfrac{P(-2 \leq Z \leq 2)}{2}$

$= 50\% + \frac{95\%}{2}$

$= 97.5\%$

(Note the inaccuracy here, from the tables, it should be 97.72%)

12d $P(Z < -1)$

$= P(Z \leq 0) - \dfrac{P(-1 \leq Z \leq 1)}{2}$

$= 50\% - \frac{68\%}{2}$

$= 16\%$

12e $P(0 \le Z \le 3)$

$= \dfrac{P(-3 \le Z \le 3)}{2}$

$= \dfrac{99.7\%}{2}$

$= 49.85\%$

12f $P(0 \le Z < 1)$

$= \dfrac{P(-1 \le Z \le 1)}{2}$

$= \dfrac{68\%}{2}$

$= 34\%$

12g $P(-2 \le Z \le 0)$

$= \dfrac{P(-2 \le Z \le 2)}{2}$

$= \dfrac{95\%}{2}$

$= 47.5\%$

12h $P(-3 < Z \le -2)$

$= \dfrac{P(-3 \le Z \le 3)}{2} - \dfrac{P(-2 \le Z \le 2)}{2}$

$= \dfrac{99.7\%}{2} - \dfrac{95}{2}\%$

$= 49.85\% - 47.5\%$

$= 2.35\%$

12i $P(-1 \le Z \le 1) = 68\%$

12j $P(-3 < Z \le 1)$

$= \dfrac{P(-3 \le Z \le 3)}{2} + \dfrac{P(-1 \le Z \le 1)}{2}$

$= \dfrac{99.7\%}{2} + \dfrac{68\%}{2}$

$= 49.85\% + 34\%$

$= 83.85\%$

12k $P(-2 \le Z < 1)$

$= \dfrac{P(-2 \le Z \le 2)}{2} + \dfrac{P(-1 \le Z \le 1)}{2}$

$= \dfrac{95\%}{2} + \dfrac{68\%}{2}$

$= 47.5\% + 34\%$

$= 81.5\%$

12l $P(-2 \le Z \le 7)$

$= \dfrac{P(-2 \le Z \le 2)}{2} + P(Z \ge 0)$

$= \frac{95\%}{2} + 50\%$

$= 47.5\% + 50\%$

$= 97.5\%$

13a $b = 1$ (because 68% of the data is between $-1 \le Z \le 1$)

13b $P(0 \le Z \le b) = 0.475$ then $b = 2$ because

$P(0 \le Z \le 2)$

$= \dfrac{P(-2 \le Z \le 2)}{2}$

$= \dfrac{95\%}{2}$

$= 47.5\%$

$= 0.475$

13c $P(Z \geq b) = 84\%$ then $b = -1$ because $P(Z \geq -1) = P(Z \leq 1)$ and

$P(Z \leq 1)$

$= P(Z \leq 0) + \dfrac{P(-1 \leq Z \leq 1)}{2}$

$= 50\% + \frac{68\%}{2}$

$= 84\%$

13d $P(-2b \leq Z \leq b) = 0.815$ then $b = 1$ because

$P(-2 \leq Z < 1)$

$= \dfrac{P(-2 \leq Z \leq 2)}{2} + \dfrac{P(-1 \leq Z \leq 1)}{2}$

$= \dfrac{95\%}{2} + \dfrac{68\%}{2}$

$= 47.5\% + 34\%$

$= 81.5\%$

$= 0.815$

13e $P(-3b \leq Z \leq 3b) = 0.997$ then $b = 1$ because

$P(-3 < Z \leq 3) = 99.7\% = 0.997$

13f $P(Z^2 \leq b) = 0.95$ then $b = 4$ because if $P(Z^2 \leq b)$ then $P(-\sqrt{b} \leq Z \leq \sqrt{b})$ and

$P\left(-\sqrt{4} \leq Z \leq \sqrt{4}\right) = P(-2 \leq Z \leq 2) = 95\% = 0.95$

14a $P(Z < a) = 0.7257$ then $a = 0.6$

14b $P(Z \leq a) = 0.9893$ then $a = 2.3$

14c $P(Z < -a) = 0.1151$ then

$P(Z < -a) = 1 - P(Z \leq a) = 0.1151$

Hence, $P(Z \leq a) = 1 - 0.1151$

$P(Z \leq a) = 0.8849$

Therefore, $a = 1.2$

14d $P(Z < a) = 0.2119$

$1 - 0.2119 = 0.7881$ and $P(Z \leq 0.8) = 0.7881$

If $P(Z \leq 0.8) = 1 - P(Z < a)$ then $a = -0.8$

14e $P(-a \leq Z < a) = 0.7286$ then $\frac{P(-a \leq Z < a)}{2} = P(0 < Z < a) = 0.3643$

Since $P(0 \leq Z) + P(0 < Z < a) = 0.5 + 0.3643 = 0.8643$

and $P(Z \leq 1.1) = 0.8643$, $a = 1.1$

14f $P(-a < Z \leq a) = 0.9906$ then $\frac{P(-a < Z \leq a)}{2} = P(0 < Z \leq a) = 0.4953$

Since $P(0 \leq Z) + P(0 < Z \leq a) = 0.5 + 0.4953 = 0.9953$

and $P(Z \leq 2.6) = 0.9953$, $a = 2.6$.

15a i $P(-1 < Z < 1) \doteqdot 68\%$

15a ii $P(Z < 2) \doteqdot 97.5\%$

15a iii $P(Z < -3 \text{ or } Z > 3) = 0.3\%$

15b Let a be the distance from the target where $P(-a \leq Z \leq a) = 50\%$.

Then $P(0 < Z \leq a) = 25\%$. Since $P(Z \leq 0) = 50\%$,

$P(0 < Z \leq a) + P(Z \leq 0) = 25\% + 50\% = 75\% = P(Z \leq a)$

If $P(Z \leq a) = 75\%$ then $a = 0.67449 \doteqdot 0.7$ cm.

16 Mathematically, $P(Z = a) = \int_a^a f(x)\,dx$, which is an area of zero width.

Practically, this represents the probability of getting a value exactly $Z = a$ for a continuous distribution, for example a height of exactly 1.7142435345345 … metres. In a continuous distribution, all such probabilities are zero.

17a i The domain is all real numbers, because $e^{-\frac{1}{2}z^2}$ is defined for all $z \in \mathbb{R}$.

17a ii $\phi(z) = \phi(-z)$. Therefore, $\phi(z)$ is an even function.

17a iii $\phi(z)$ is symmetric about $x = 0$.

17a iv $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\,dz = 1$

17a v $z = -1$ and $z = 1$

17a vi $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$ then $\phi(0) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}0^2} = \frac{1}{\sqrt{2\pi}}$

The maximum turning point is $(0, \frac{1}{\sqrt{2\pi}})$ when $z = 0$.

17a vii There are no z-intercepts because the function has a horizontal asymptote at $y = 0$.

17b i The mean is 0.

17b ii The mode is 0.

17b iii The median is 0.

17b iv The standard deviation is 1.

17c $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$

18a $f(x) = e^{-\frac{1}{2}x^2}$ and $f(-x) = e^{-\frac{1}{2}(-x)^2} = e^{-\frac{1}{2}x^2}$. Therefore, $f(x) = f(-x)$ and $f(x)$ is an even function.

18b $f(x) = e^{-\frac{1}{2}x^2}$ then $f'(x) = \left(-\frac{1}{2} \times 2 \times x\right) e^{-\frac{1}{2}x^2} \times \ln e = -xe^{-\frac{1}{2}x^2}$

and $f''(x) = (-1) \times e^{-\frac{1}{2}x^2} + (-x) \times \left(-\frac{1}{2} \times 2 \times x\right) e^{-\frac{1}{2}x^2} \times \ln e$

$= -e^{-\frac{1}{2}x^2} + x^2 e^{-\frac{1}{2}x^2}$

$= (x^2 - 1)e^{-\frac{1}{2}x^2}$

18c The x-coordinates of the stationary points are the solutions of the equation:

$f'(x) = 0$. Thus, $-xe^{-\frac{1}{2}x^2} = 0$ when $x = 0$.

x	-1	0	1
$f'(x)$	$+$	0	$-$
nature	/	Local maximum	\

$f(0) = e^{-\frac{1}{2}0^2} = 1$. Therefore, the local maximum turning point is: $(0, 1)$

18d The x-coordinates of the inflection points are the solutions of the equation:

$f''(x) = 0$. Thus, $(x^2 - 1)e^{-\frac{1}{2}x^2} = 0$ when $x = -1$ or $x = 1$

$f(-1) = e^{-\frac{1}{2}(-1)^2} = \frac{1}{\sqrt{e}}$ and $f(1) = e^{-\frac{1}{2}(1)^2} = \frac{1}{\sqrt{e}}$

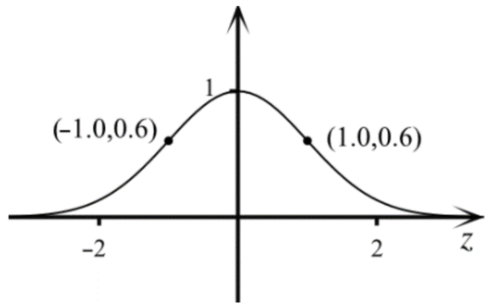
Therefore, the inflection points are: $\left(-1, \frac{1}{\sqrt{e}}\right)$ and $\left(1, \frac{1}{\sqrt{e}}\right)$

(or $(-1,\ e^{-05})$ and $(1,\ e^{-05})$)

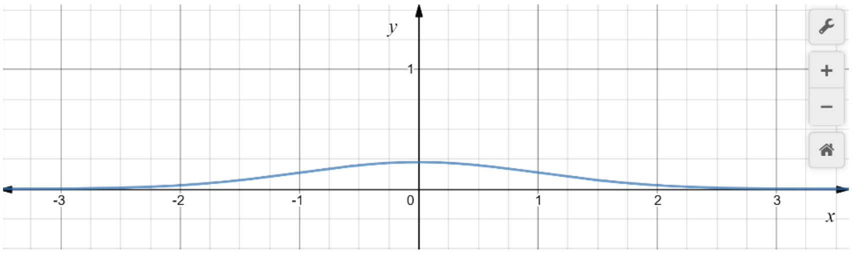
18e Since $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$ has a horizontal asymptote at $y = 0$,

$f(x) \to 0$ as $x \to \infty$ and as $x \to -\infty$

18f



18g



19a i $P(0 \le Z \le 1) = 0.3401$

19a ii $P(-1 \le Z \le Z) = 0.6802$

19a iii The graph is concave up on $[0, 1]$ and the concavity changes at the point of inflection at $z = 1$. Thus, the polygonal path of the trapezoidal rule will lie below the exact curve.

19a iv This is good agreement with the empirical rule (68) and the table (0.6826).

19b i $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$ and

$\phi(-2) = \phi(2) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(-2)^2} = 0.053\ 991$

$\phi(-1) = \phi(1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(-1)^2} = 0.241\ 971$ and $\phi(0) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}0^2} = 0.398\ 942$

Calculating the area using the trapezoid rule,

$A = 2 \times \left(\frac{0.241971 - 0.053991}{2} \times 1 + \frac{0.398942 - 0.241971}{2} \times 1\right)$

$= 2 \times (0.09399 + 0.078486) = 2 \times 0.194618 = 0.344951$

19b i $P(-2 < Z < 2) = 2 \times 0.4750 = 0.95$

19b ii $P(-3 < Z < 3) = 2 \times 0.4981 = 0.9962$

20a

$E(Z) = \int_{-\infty}^{\infty} z \times \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\, dz$, which is the integral of an odd function on a symmetric domain, so

$E(Z) = 0$.

20b

$$\frac{d}{dz}\left(ze^{-\frac{1}{2}z^2}\right) = 1 \times e^{-\frac{1}{2}z^2} + z \times -ze^{-\frac{1}{2}z^2}$$

$$ze^{-\frac{1}{2}z^2} = \int e^{-\frac{1}{2}z^2}\, dz - \int z \times ze^{-\frac{1}{2}z^2}\, dz$$

$$\left[ze^{-\frac{1}{2}z^2}\right]_{-\infty}^{\infty} = \int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2}\, dz - \int_{-\infty}^{\infty} z^2 e^{-\frac{1}{2}z^2}\, dz$$

The LHS is 0, so

$$\int_{-\infty}^{\infty} z^2 e^{-\frac{1}{2}z^2}\, dz = \int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2}\, dz$$

and $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^2 e^{-\frac{1}{2}z^2}\, dz = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2}\, dz$

$= 1$

Thus we have shown that $E(Z^2) = 1$.

20c

Using the previous part,

$$\begin{aligned}\mathrm{VAR}(Z) &= E(Z^2) - E(Z)^2 \\ &= 1 - 0 \\ &= 1\end{aligned}$$

Solutions to Exercise 16E

1a $z = \frac{x-\mu}{\sigma} = \frac{5-4}{1} = 1$, 1 SD above

1b $z = \frac{x-\mu}{\sigma} = \frac{7-13}{3} = -2$, 2 SD below

1c $z = \frac{x-\mu}{\sigma} = \frac{0.75-0.5}{0.25} = 1$, 1 SD above

1d $z = \frac{x-\mu}{\sigma} = \frac{-5-1}{3} = -2$, 2 SD below

1e $z = \frac{x-\mu}{\sigma} = \frac{120-114}{1.2} = 5$, 5 SD above

1f $z = \frac{x-\mu}{\sigma} = \frac{2.20-2.35}{0.05} = -3$, 3 SD below

2a i $z = \frac{x-\mu}{\sigma} = \frac{60-50}{4} = +2.5$

2a ii $z = \frac{x-\mu}{\sigma} = \frac{375-450}{25} = -3$

2a iii $z = \frac{x-\mu}{\sigma} = \frac{3.85-3.19}{0.12} = +5.5$

2a iv $z = \frac{x-\mu}{\sigma} = \frac{25-23}{8} = +0.25$

2b i iii is furthest from the mean (the mean is $z = 0$)

2b ii i, iii, iv are above the mean (the mean is $z = 0$)

2b iii ii is below the mean (the mean is $z = 0$)

2b iv iv is within 2 SD from the mean (the mean is $z = 0$)

2b v ii and iii are not within the middle 68% of the data (the middle 68% is within ± 1 SD distance to the mean, which is $z = 0$)

3a $z = \frac{x-\mu}{\sigma} = \frac{5-4}{2} = 0.5$ then $P(X \le 5) = P(Z \le 0.5)$

3b $z = \frac{x-\mu}{\sigma} = \frac{4.5-4}{2} = 0.25$ then $P(X > 4.5) = P(Z > 0.25)$

3c $z = \frac{x-\mu}{\sigma} = \frac{2-4}{2} = -1$ then $P(X \le 2) = P(Z \le -1)$

3d $z = \frac{x-\mu}{\sigma} = \frac{1-4}{2} = -1.5$ then $P(X \ge 1) = P(Z \ge -1.5)$

3e $z = \frac{x-\mu}{\sigma} = \frac{0-4}{2} = -2$ and $z = \frac{x-\mu}{\sigma} = \frac{3-4}{2} = -0.5$ then

$P(0 \le X \le 3) = P(-2 \le Z \le -0.5)$

3f $z = \frac{x-\mu}{\sigma} = \frac{0.5-4}{2} = -1.75$ and $z = \frac{x-\mu}{\sigma} = \frac{4.5-4}{2} = 0.25$ then

$P(0.5 \le X \le 4.5) = P(-1.75 \le Z \le 0.25)$

4a $z = \frac{x-\mu}{\sigma} = \frac{5-5}{2} = 0$ then

$P(X \ge 5)$

$= P(Z \ge 0)$

$= 0.5$

4b $z = \frac{x-\mu}{\sigma} = \frac{3-5}{2} = -1$ and $z = \frac{x-\mu}{\sigma} = \frac{7-5}{2} = 1$ then

$P(3 \le X \le 7)$

$= P(-1 \le Z \le 1)$

$= 0.68$

4c $z = \frac{x-\mu}{\sigma} = \frac{9-5}{2} = 2$ then

$P(X \le 9)$

$= P(Z \le 2)$

$= P(Z \le 0) + \dfrac{P(-2 \le Z \le 2)}{2}$

$= 0.5 + \dfrac{0.95}{2}$

$= 0.975$

4d $z = \frac{x-\mu}{\sigma} = \frac{1-5}{2} = -2$ then

$P(X \ge 1)$

$= P(Z \ge -2)$

$= \dfrac{P(-2 \le Z \le 2)}{2} + P(Z \ge 0)$

$= \dfrac{0.95}{2} + 0.5$

$= 0.975$

4e $z = \frac{x-\mu}{\sigma} = \frac{-1-5}{2} = -3$ and $z = \frac{x-\mu}{\sigma} = \frac{7-5}{2} = 1$ then

$P(-1 \le X \le 7)$

$= P(-3 \le Z \le 1)$

$= \dfrac{P(-3 \le Z \le 3)}{2} + \dfrac{P(-1 \le Z \le 1)}{2}$

$= \dfrac{0.997}{2} + \dfrac{0.68}{2}$

$= 0.8385$

4f $z = \frac{x-\mu}{\sigma} = \frac{1-5}{2} = -2$ and $z = \frac{x-\mu}{\sigma} = \frac{3-5}{2} = -1$ then

$P(1 \leq X \leq 3)$

$= P(-2 \leq Z \leq -1)$

$= \frac{P(-2 \leq Z \leq 2)}{2} - \frac{P(-1 \leq Z \leq 1)}{2}$

$= \frac{0.95}{2} - \frac{0.68}{2}$

$= 0.135$

5a $z = \frac{x-\mu}{\sigma} = \frac{10-12}{2} = -1$ and $z = \frac{x-\mu}{\sigma} = \frac{18-12}{2} = 3$ then

$P(10 \leq X \leq 18)$

$= P(-1 \leq Z \leq 3)$

$= \frac{P(-1 \leq Z \leq 1)}{2} + \frac{P(-3 \leq Z \leq 3)}{2}$

$= \frac{0.68}{2} + \frac{0.997}{2}$

$= 0.8385$

5b $z = \frac{x-\mu}{\sigma} = \frac{42-37}{5} = 1$ then

$P(X \geq 42)$

$= P(Z \geq 1)$

$= 1 - P(Z < 1)$

$= 1 - \left(P(Z \leq 0) + \frac{P(-1 \leq Z \leq 1)}{2}\right)$

$= 1 - \left(0.5 + \frac{0.68}{2}\right)$

$= 1 - 0.84$

$= 0.16$

5c $z = \frac{x-\mu}{\sigma} = \frac{4.5-4}{0.25} = 2$ then

$P(X \geq 4.5)$

$= P(Z \geq 2)$

$= 1 - P(Z < 2)$

$= 1 - \left(P(Z \leq 0) + \frac{P(-2 \leq Z \leq 2)}{2}\right)$

$= 1 - \left(0.5 + \frac{0.95}{2}\right)$

$= 1 - 0.975$

$= 0.025$

6a $z = \frac{x-\mu}{\sigma} = \frac{3-5}{0.8} = -2.5$ and $z = \frac{x-\mu}{\sigma} = \frac{7-5}{0.8} = 2.5$ then

$P(3 \leq X \leq 7)$

$= P(-2.5 \leq Z \leq 2.5)$

$= 0.9876$

6b $z = \frac{x-\mu}{\sigma} = \frac{20-4}{10} = 1.6$ then

$P(X \geq 20)$

$= P(Z \geq 1.6)$

$= 1 - P(Z < 1.6)$

$= 1 - 0.945201$

$= 0.0548$

6c $z = \frac{x-\mu}{\sigma} = \frac{8-12}{5} = -0.8$ then

$P(X \leq 8)$

$= P(Z \leq -0.8)$

$= 1 - P(Z < 0.8)$

$= 1 - 0.788145$

$= 0.2119$

6d $z = \frac{x-\mu}{\sigma} = \frac{-39-0}{30} = -1.3$ then

$P(X \geq -39)$

$= P(Z \geq -1.3)$

$= P(Z \leq 1.3)$

$= 0.9032$

6e $z = \frac{x-\mu}{\sigma} = \frac{36-20}{10} = 1.6$ then

$P(X < 36)$

$= P(Z < 1.6)$

$= 0.9452$

6f $z = \frac{x-\mu}{\sigma} = \frac{3-8}{2} = -2.5$ and $z = \frac{x-\mu}{\sigma} = \frac{5-8}{2} = -1.5$ then

$P(3 < X \leq 5)$

$= P(-2.5 < Z \leq -1.5)$

$= 0.0606$

7a The score is above the mean.

7b The score is below the mean.

7c The score is equal to the mean.

8a $73 - 8 \leq X \leq 73 + 8 = 65 \leq X \leq 81$

Therefore, the data values which lie within one standard deviation of the mean are 69 and 80.

8b $73 - 3 \times 8 \leq X \leq 73 + 3 \times 8 = 49 \leq X \leq 97$

Therefore, the data values which lie within three standard deviation of the mean are $69, 80, 95, 50, 90, 52, 45$.

8c $X < 73 - 2 \times 8$

$X < 57$

Therefore, the data values which lie more than two standard deviations below the mean are $43, 45, 50, 52$.

8d $X > 73 + 2.5 \times 8$

$X > 93$

Therefore, the data values which lie more than two and a half standard deviations above the mean are 95 and 98.

8e It doesn't look very normal ('bell shaped')

Here is the stem-and-leaf plot of the data:

4	3	5	
5	0	2	
6	9		
7			
8	0		
9	0	5	8

9a i English test result: $z = \frac{x-\mu}{\sigma} = \frac{90-65}{10} = 2.5$

Maths test result: $z = \frac{x-\mu}{\sigma} = \frac{92-62}{15} = 2$

Student A's English test result is much better than the average compared to their mathematics test result.

9a ii English test result: $z = \frac{x-\mu}{\sigma} = \frac{57-65}{10} = -0.8$

Maths test result: $z = \frac{x-\mu}{\sigma} = \frac{53-62}{15} = -0.04$

Student B's English test result is further below the average than their mathematics test result. Therefore, their mathematics test result is better than their English test result (compared with the class average).

9a iii English test result: $z = \frac{x-\mu}{\sigma} = \frac{80-65}{10} = 1.5$

Maths test result: $z = \frac{x-\mu}{\sigma} = \frac{77-62}{15} = 1$

Student C's English test result is much better than the average compared to their mathematics test result.

9b $z = \frac{x-\mu}{\sigma} = \frac{95-62}{15} = 2.2$

$P(Z > 2.2)$

$= 1 - P(Z \leq 2.2)$

$= 1 - 0.986\,097$

$= 0.013\,903$

$\doteqdot 1.4\%$

9c The mathematics mean of 62 is 0.3 English standard deviations below the English mean 65%.

$P(Z > -0.3)$

$= P(Z < 0.3)$

$= 0.6179$

$\doteqdot 0.62$

10a About 408 scores (68% of 600) will lie within one SD from the mean, that is, in $[40, 60]$. About 570 scores (95% of 600) will lie within two SDs from the mean, that is, in $[30, 70]$. About 598 scores (99.7% of 600) will lie within three SDs from the mean, that is, in $[20, 80]$.

10b i $P(X \leq 55) = 0.691462$ and $0.691462 \times 600 = 415$

Therefore, approximately 415 scores will lie in the interval $[-\infty, 55]$.

10b ii $P(35 \leq X \leq 50) = 0.433193$ and $0.433193 \times 600 = 260$

Therefore, approximately 260 scores will lie in the interval $[35, 50]$.

10b iii $P(38 \leq X \leq 62) = 0.769861$ and $0.769861 \times 600 = 462$

Therefore, approximately 462 scores will lie in the interval $[38, 62]$.

10c x that is not in the interval $\mu - 2.70\sigma \leq x \leq \mu + 2.70\sigma$ is an outlier. Hence, $50 - 2.7 \times 10 \leq x \leq 50 + 2.7 \times 10 = 23 \leq x \leq 77$ and all the x values that are out of this interval are outliers.

$P(23 \leq X \leq 77) = 0.993066$ and $1 - P(23 \leq X \leq 77) = 0.006934$.

Therefore, $0.006934 \times 600 = 4$ scores are expected to be outliers.

11a i Using the formula, z-score $= \frac{x-\mu}{\sigma}$, or by inspection, we have the z-scores:

Assessment 1: $z = \frac{50-60}{10} = -1$

Assessment 2: $z = \frac{53-65}{8} = -1.5$

Assessment 3: $z = \frac{67-75}{4} = -2$

11a ii The average of $-1, -1.5, -2$ is -1.5. Thus, his average deviation from the mean is 1.5 standard deviations below the mean.

11a iii $63 - 1.5 \times 12 = 45$

11a iv Some assessments may be harder than others – simply averaging his other results takes no account of this.

11a v Jack may perform better in certain types of assessments, for example, in Biology lab experiments, or he may perform better at certain times of the year. For example, his results may improve towards the end of the year. This method does not allow for these effects.

11b Assessment 1: $z = \frac{64-60}{10} = 0.4$

Assessment 2: $z = \frac{70-65}{8} = 0.625$

Assessment 3: $z = \frac{79-75}{4} = 1$

Average of 0.4, 0.625 and 1 is 0.675.

Jill's estimate is $63 + 0.675 \times 12 = 71.1$.

Solutions to Exercise 16F

1a $z = \frac{x-\mu}{\sigma} = \frac{50-70}{10} = -2$

$P(Z > -2)$

$= P(Z \leq 2)$

$= P(Z \leq 0) + \dfrac{P(-2 \leq Z \leq 2)}{2}$

$= 0.5 + \dfrac{0.95}{2}$

$= 0.975$ or 97.5%

1b $z = \frac{x-\mu}{\sigma} = \frac{80-70}{10} = 1$

$P(Z \leq 1)$

$= P(Z \leq 0) + \dfrac{P(-1 \leq Z \leq 1)}{2}$

$= 0.5 + \dfrac{0.68}{2}$

$= 0.84$ or 84%

2a $z = \frac{x-\mu}{\sigma} = \frac{95-68}{9} = 3$

$P(Z > 3)$

$= 1 - P(Z \leq 3)$

$= 1 - \left[P(Z \leq 0) + \dfrac{P(-3 \leq Z \leq 3)}{2}\right]$

$= 1 - \left(0.5 + \dfrac{0.997}{2}\right)$

$= 0.0015$

Therefore, $0.0015 \times 2000 = 3$ students

2b $z = \frac{x-\mu}{\sigma} = \frac{50-68}{9} = -2$

$P(Z < -2)$

$= 1 - P(Z \le 2)$

$= 1 - \left[P(Z \le 0) + \frac{P(-2 \le Z \le 2)}{2}\right]$

$= 1 - \left(0.5 + \frac{0.95}{2}\right)$

$= 1 - 0.975$

$= 0.025$

Therefore, $0.025 \times 2000 = 50$ students

2c $z = \frac{x-\mu}{\sigma} = \frac{59-68}{9} = -1$ and $z = \frac{x-\mu}{\sigma} = \frac{86-68}{9} = 2$

$P(-1 < Z < 2)$

$= \frac{P(-1 \le Z \le 1)}{2} + \frac{P(-2 \le Z \le 2)}{2}$

$= \frac{0.68}{2} + \frac{0.95}{2}$

$= 0.815$

Therefore, $0.815 \times 2000 = 1630$ students

3a The range of two standard deviations of the mean:

$\mu - 2\sigma < x < \mu + 2\sigma$

$= 2 - 2 \times 0.1 < x < 2 + 2 \times 0.1$

$= 1.8 < x < 2.2$

The screws with sizes below 1.8 cm are considered undersized.

$z = \frac{x-\mu}{\sigma} = \frac{1.8-2}{0.1} = -2$

$P(X < 1.8)$

$= P(Z < -2)$

$= 1 - P(Z \le 2)$

$= 1 - \left[P(Z \leq 0) + \frac{P(-2 \leq Z \leq 2)}{2}\right]$

$= 1 - \left(0.5 + \frac{0.95}{2}\right)$

$= 1 - 0.975$

$= 0.025$ or 2.5%

3b $z = \frac{x-\mu}{\sigma} = \frac{2.3-2}{0.1} = 3$

Thus, $P(X > 2.3) = P(Z > 3)$

$P(Z > 3) = 1 - P(Z \leq 3)$

$= 1 - \left[P(Z \leq 0) + \frac{P(-3 \leq Z \leq 3)}{2}\right]$

$= 1 - \left(0.5 + \frac{0.997}{2}\right)$

$= 1 - 0.975$

$= 0.0015$

Therefore, $0.0015 \times 2400 = 3.6$ screws (perhaps round to 4)

4 $z = \frac{x-\mu}{\sigma} = \frac{72-68}{2} = 2$ and $z = \frac{x-\mu}{\sigma} = \frac{64-68}{2} = -2$

$P(Z < -2)$ or $P(Z > 72)$

$= 1 - P(-2 < Z < 2)$

$= 1 - 0.95$

$= 0.05$

Therefore, 5% are discarded.

5a $z = \frac{x-\mu}{\sigma} = \frac{140-98}{15} = 2.8$

$P(Z > 2.8)$

$= 1 - P(Z \leq 2.8)$

$= 1 - 0.997445$

$= 0.002555$

Therefore, approximately 0.26% of the population are expected to be geniuses.

5b Using 0.26% from part a:

$0.0026 \times 25\,000\,000 = 65\,000$

Therefore, approximately 65 000 people are expected to be geniuses.

6 $z = \frac{x-\mu}{\sigma} = \frac{240-219}{41} = 0.512195 \doteqdot 0.5$

$P(Z > 0.5)$

$= 1 - P(Z \leq 0.5)$

$= 1 - 0.691\,462$

$= 0.308\,538$

Therefore, approximately 31% of the population have high cholesterol levels.

7a If $P(Z < z) = 90\%$ then $z = 1.28155 \doteqdot 1.28$

7b Since $= z \times \sigma + \mu$,

$x = 1.28155 \times 7.5 + 176 = 185.612 \doteqdot 186$ cm

7c i Interpolate between 1.6 and 1.7

7c ii $P(Z < z) = 99.7\%$ when $z = 2.74778$

Since $= z \times \sigma + \mu$,

$x = 2.74778 \times 7.5 + 176 = 196.608 \doteqdot 197$cm

8 $P(Z < z) = 95\%$ when $z = 1.64485 \doteqdot 1.6$

Hence, $P(Z > -1.64485) = 95\%$

Since $= z \times \sigma + \mu$,

$x = -1.64485 \times 2 + 500 = 496.71 \doteqdot 496.7$ g

9a $z = \frac{x-\mu}{\sigma} = \frac{274-266}{16} = 0.5$

$P(X < 274)$

$= P(Z < 0.5)$

$= 0.6915$

$\doteqdot 69\%$

9b i $266 - 7 = 259$ days

$z = \frac{x-\mu}{\sigma} = \frac{259-266}{16} = -0.4375$

$P(X < 259)$

$= P(Z < -0.4375)$

$= 1 - P(Z \leq 0.4375)$

$= 0.669\,126$

$= 0.330\,874$

$\doteqdot 33\%$

9b ii $266 + 7 = 273$ days

$z = \frac{x-\mu}{\sigma} = \frac{273-266}{16} = 0.4375$

$P(X > 273)$

$= P(Z > 0.4375)$

$= 1 - P(Z \leq 0.4375)$

$= 1 - 0.669\,126$

$= 0.330\,874$

$\doteqdot 33\%$

10a $z = \frac{x-\mu}{\sigma} = \frac{60-71}{9} \doteqdot -1.2$

$P(X < 60)$

$\doteqdot P(Z < -1.2)$

$= 1 - P(Z \leq 1.2)$

$= 1 - 0.8849$

$= 0.1151$

$\doteqdot 12\%$

10b $z = \frac{x-\mu}{\sigma} = \frac{100-71}{9} \doteqdot 3.2$

$P(X > 100)$

$\doteqdot P(Z > 3.2)$

$= 1 - P(Z \leq 3.2)$

$= 1 - 0.9993$

$= 0.0007$

$= 0.07\%$

10c i $z = \frac{x-\mu}{\sigma} = \frac{60-76}{9.5} \doteqdot -1.7$

$P(X < 60)$

$\doteqdot P(Z < -1.7)$

$= 1 - P(Z \leq 1.7)$

$= 1 - 0.9554$

$= 0.0446$

$\doteqdot 4.5\%$

10c ii $z = \frac{x-\mu}{\sigma} = \frac{100-76}{9.5} \doteqdot 2.5$

$P(X > 100)$

$\doteqdot P(Z > 2.5)$

$= 1 - P(Z \leq 2.5)$

$= 1 - 0.9938$

$= 0.0062$

$\doteqdot 0.6\%$

11

Method 1 (Algebraic): $P(X > 100) = 97.7$, so transforming to standard normal:

$$P(Z > \frac{100-\mu}{\sigma}) = 97.7$$

$$P(Z < \frac{\mu-100}{\sigma}) = 97.7$$

$$\frac{\mu-100}{\sigma} = 2 \quad \text{(by reading the tables in reverse)}$$

$$\mu = 100 + 2\sigma$$

$P(X < 115) = 69.1$, so transforming to standard normal:

$$P(Z < \frac{115-\mu}{\sigma}) = 69.1$$

$$\frac{115-\mu}{\sigma} = 0.5 \quad \text{(by reading the tables in reverse)}$$

$$115 - \mu = 0.5\sigma$$

$$\mu = 115 - 0.5\sigma$$

Solving these equations simultaneously, we have $\mu + 4\mu = 100 + 460$, hence $\mu = 112$.

Method 2 (Intuitive approach): Using the standard normal tables in reverse, data above 2σ below the mean corresponds to the probability $P(X > 100) = 97.7$. Data below 0.5σ above the mean corresponds to the probability $P(X < 115) = 69.1$. Thus 2.5 standard deviations is $115 - 100 = 15$ g, so 1 standard deviation is 6 g. The mean weight is 112 g (half a standard deviation below 115 g).

Solutions to Exercise 16G

1a $\mu = 7.5$ and $\sigma = 2.5$

1b

Class centre	0.5	1.5	2.5	3.5	4.5	5.5	6.5	7.5
frequency	2	5	24	46	88	116	138	158
Relative frequency	2	7	31	77	165	281	419	577

Class centre	8.5	9.5	10.5	11.5	12.5	13.5	14.5
frequency	144	113	78	47	27	10	4
Relative frequency	721	834	912	959	986	996	1000

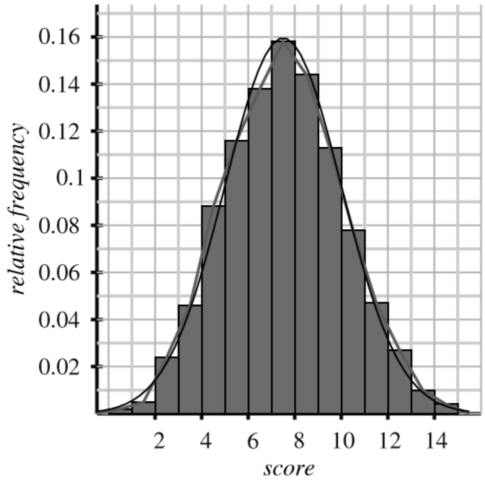
$$E(X) = \sum xP(x) = \frac{0.5\times2+1.5\times5+2.5\times24+\cdots+14.5\times4}{1000} = \frac{7533}{1000} = 7.533 \doteqdot 7.5$$

$$\begin{aligned}\text{Var}(X) &= E(X^2) - E(X)^2 \\ &= \sum x^2P(x) - E(X)^2 \\ &= \frac{63\,034}{1000} - 7.533^2 \\ &= 6.287\,911\end{aligned}$$

Therefore, $\sigma = \sqrt{\text{Var}(X)} = \sqrt{6.287\,911} = 2.50757\ldots \doteqdot 2.5$

Yes, the answers agree with the previous results.

1c



1d Either perform the experiment more than 1000 times, or average more than three random numbers at each stage.

4c The mean should be about 5 and the standard deviation about 1.6.

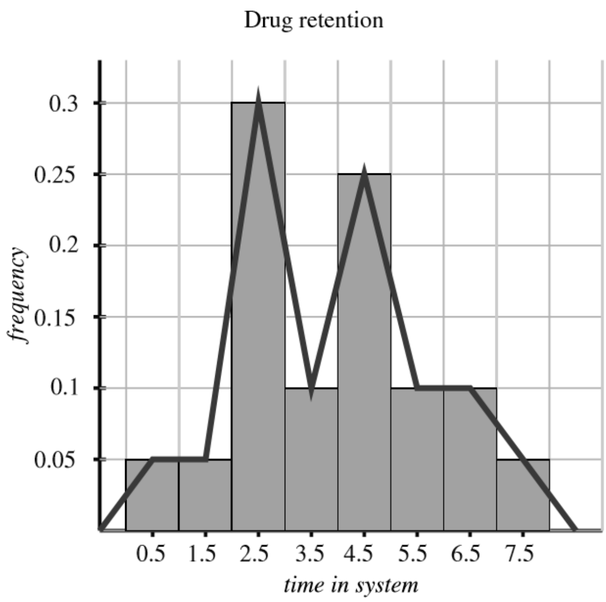
8d The maximum point on the curve has coordinates $\left(0, \frac{1}{\sqrt{2\pi}}\right)$.

Solutions to Chapter review

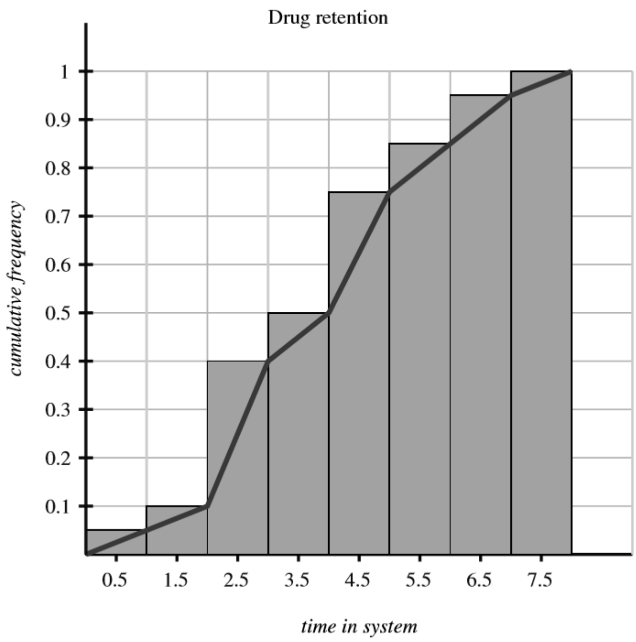
1a

x	0–1	1–2	2–3	3–4	4–5	5–6	6–7	7–8
cc	0.5	1.5	2.5	3.5	4.5	5.5	6.5	7.5
f	1	1	6	2	5	2	2	1
cf	1	2	8	10	15	17	19	20
f_r	0.05	0.05	0.3	0.1	0.25	0.1	0.1	0.05
cf_r	0.05	0.1	0.4	0.5	0.75	0.85	0.95	1

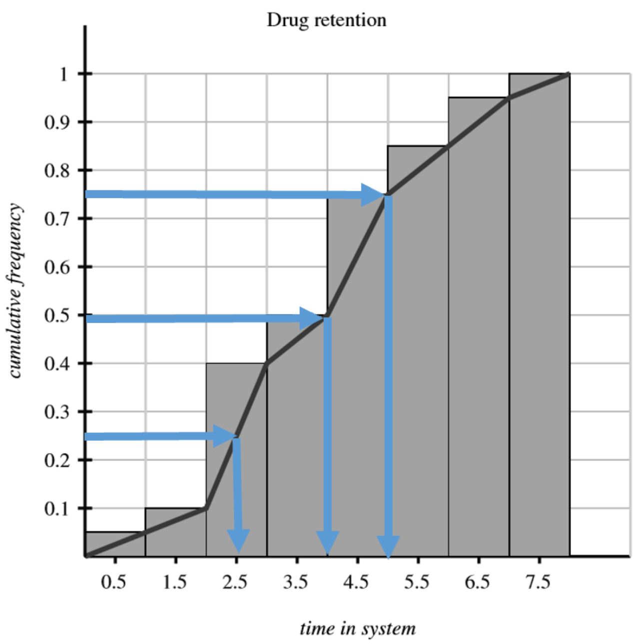
1b



1c



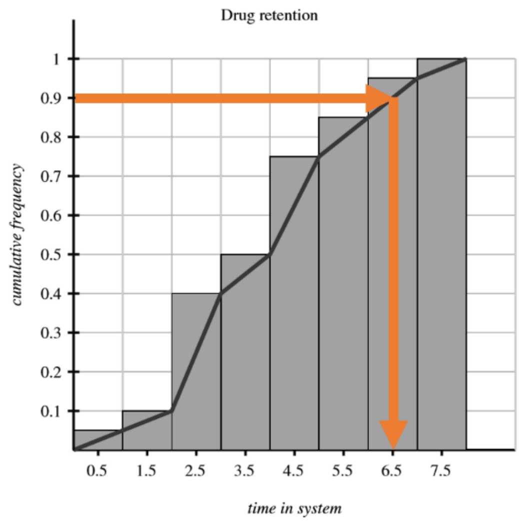
1d



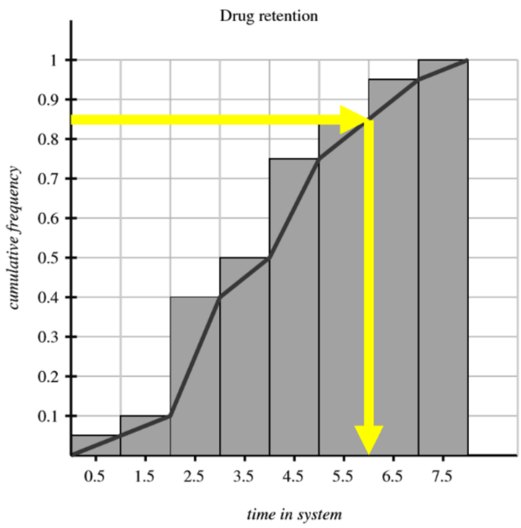
1d i $\quad Q_2 = 4.0$

1d ii $\quad Q_1 = 2.5$ and $Q_3 = 5.0$

1d iii As shown below, the ninth decile is 6.5.



1d iv As shown below, the eighty-fifth percentile is 6.0.



1e This was only a preliminary experiment, and a larger dataset may resolve the unusual outcomes. It may be worth investigating any common links between patients falling in the two intervals associated with the two modes — perhaps different sexes react differently to the drug, perhaps it was administered differently, or perhaps the two groups are behaving differently after medication, for example, changing their levels of follow-up exercise or their food intake.

2a False, it joins to the right end.

2b False. The area under the relative frequency polygon is 1 if the rectangles each have width 1.

2c True.

2d True.

2e True.

2f The empirical rule says 99.7% and only applies to a normal distribution, so false in general.

3a $f(x)$ is never negative and $\int_{-10}^{10} \frac{1}{20}\, dx = \left[\frac{x}{20}\right]_{-10}^{10}$

$$= \frac{10}{20} - \left(-\frac{10}{20}\right)$$

$$= 1.$$

Therefore, $f(x)$ is a probability density function.

3b A uniform probability distribution with a uniform probability density function.

3c $\int_{-10}^{10} x \times \frac{1}{20}\, dx$

$$= \left[\frac{x^2}{40}\right]_{-10}^{10}$$

$$= \frac{100}{40} - \frac{100}{40}$$

$$= 0$$

3d Var(X)

$= \int_a^b x^2 f(x) - E(x)^2$

$= \int_{-10}^{10} x^2 \frac{1}{20} dx - 0^2$

$= \left[\frac{x^3}{60}\right]_{-10}^{10}$

$= \frac{1000}{60} - \left(-\frac{1000}{60}\right)$

$= \frac{100}{3}$

$\sigma = \sqrt{\text{Var}(X)} = \sqrt{\frac{100}{3}} = \frac{10}{\sqrt{3}} = \frac{10\sqrt{3}}{3}$

4a $f(x)$ is never negative in its domain and

$\int_0^2 \frac{3}{16}(4 - x^2)dx$

$= \left[\frac{3x}{4} - \frac{x^3}{16}\right]_0^2$

$= \left(\frac{3}{2} - \frac{1}{2}\right) - \left(\frac{0}{4} - \frac{0}{16}\right)$

$= 1$

Therefore, $f(x)$ is a probability density function.

4b $F(x) = \int_0^x \frac{3}{16}(4 - t^2)dt$

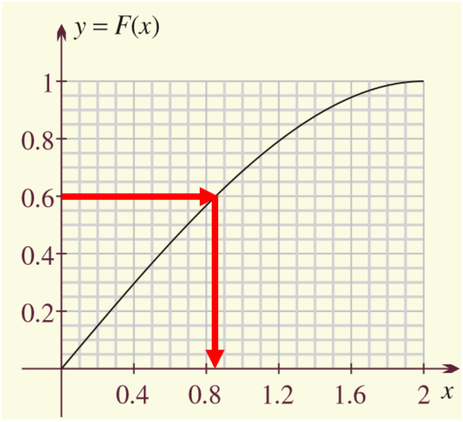
$= \left[\frac{3}{16}\left(4t - \frac{t^3}{3}\right)\right]_0^x$

$= \frac{3}{4}x - \frac{1}{16}x^3$

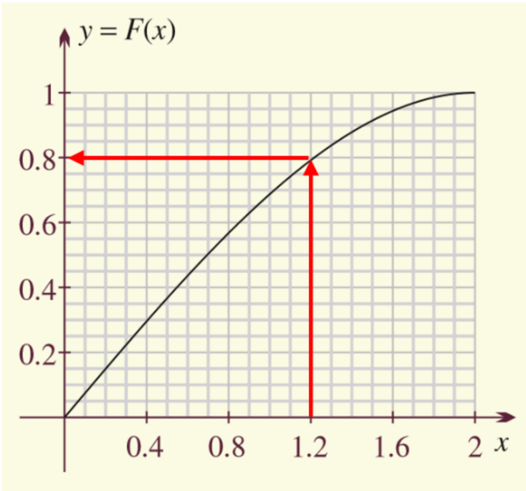
$= \frac{1}{16}x(12 - x^2)$ where $0 \le x \le 2$

4c i $Q_1 = 0.34, Q_2 = 0.7, Q_3 = 1.1$

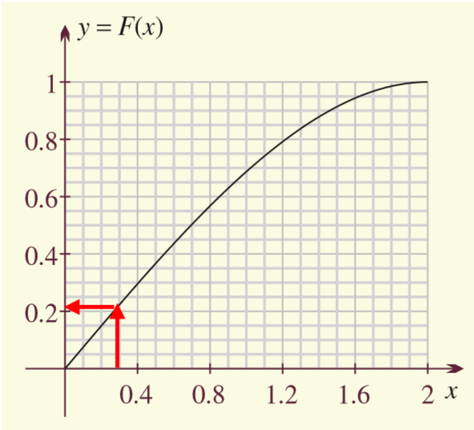
4c ii The 6th decile is 0.85 as shown below.



4c iii $P(X \leq 1.2) = 0.8$ as shown below.



4c iv $P(X \geq 0.3) = 1 - P(X < 0.3)$ and $P(X < 0.3) = 0.22$ is shown below.



Therefore, $P(X \geq 0.3) = 1 - 0.22 = 0.78$

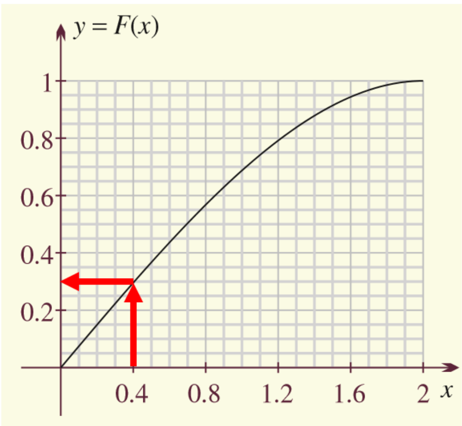
4c v $P(0.2 \leq X \leq 0.4) = P(X \leq 0.4) - P(X \leq 0.2)$

$= 0.3 - 0.15$

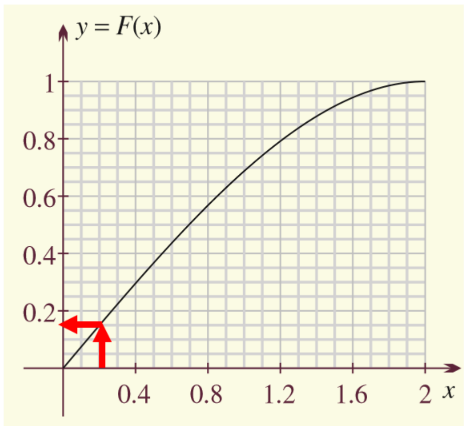
$= 0.15$

$P(X \leq 0.4)$ and $P(X \leq 0.2)$ are shown below.

$P(X \leq 0.4)$



$P(X \leq 0.2)$



5a $P(Z < 0) = 0.5$

5b $P(Z < 1.3) = 0.9032$

5c $$\begin{aligned} P(-1.8 < Z < 1.8) &= P(Z < 1.8) - P(Z \leq -1.8) \\ &= P(Z < 1.8) - [1 - P(Z < 1.8)] \\ &= 2 \times P(Z < 1.8) - 1 \\ &= 2 \times 0.9641 - 1 \\ &= 0.9282 \end{aligned}$$

5d $$\begin{aligned} P(Z > 0.5) &= 1 - P(Z \leq 0.5) \\ &= 1 - 0.6915 \\ &= 0.3085 \end{aligned}$$

5e $$\begin{aligned} P(Z < -0.2) &= 1 - P(Z \leq 0.2) \\ &= 1 - 0.5793 \\ &= 0.4207 \end{aligned}$$

5f $$\begin{aligned} P(-0.1 < Z < 1.2) &= P(Z < 1.2) - P(Z \leq -0.1) \\ &= P(Z < 1.2) - [1 - P(Z < 0.1)] \\ &= 0.8849 - (1 - 0.5398) \\ &= 0.8849 - 0.4602 \\ &= 0.4247 \end{aligned}$$

6a $z = \frac{x-\mu}{\sigma} = \frac{16-10}{3} = 2$

$$P(X \le 16) = P(Z \le 2)$$
$$= P(Z < 0) + \frac{P(-2 \le Z \le 2)}{2}$$
$$= 0.5 + \frac{0.95}{2}$$
$$= 0.975 \text{ or } 97.5\%$$

6b $z = \frac{x-\mu}{\sigma} = \frac{3.5-5}{1.5} = -1$

$$P(X \ge 3.5) = P(Z \ge -1)$$
$$= P(Z < 0) + \frac{P(-1 \le Z \le 1)}{2}$$
$$= 0.5 + \frac{0.68}{2}$$
$$= 0.84 \text{ or } 84\%$$

6c $z = \frac{x-\mu}{\sigma} = \frac{1.85-2}{0.15} = -1$ and $z = \frac{x-\mu}{\sigma} = \frac{2.3-2}{0.15} = 2$

$$P(1.85 \le X \le 2.3) = P(-1 \le Z \le 2)$$
$$= \frac{P(-1 \le Z \le 1)}{2} + \frac{P(-2 \le Z \le 2)}{2}$$
$$= \frac{0.68}{2} + \frac{0.95}{2}$$
$$= 0.34 + 0.475$$
$$= 0.815 \text{ or } 81.5\%$$

6d $z = \frac{x-\mu}{\sigma} = \frac{13.65-15}{0.45} = -3$ and $z = \frac{x-\mu}{\sigma} = \frac{14.1-15}{0.45} = -2$

$$P(13.65 \le X \le 14.1) = P(-3 \le Z \le -2)$$
$$= \frac{P(-3 \le Z \le 3)}{2} - \frac{P(-2 \le Z \le 2)}{2}$$
$$= \frac{0.997}{2} - \frac{0.95}{2}$$

$= 0.4985 - 0.475$

$= 0.0235$ or 2.35%

7a $z = \frac{x-\mu}{\sigma} = \frac{22.5-20}{5} = 0.5$

Hence, $P(X \le 22.5) = P(Z \le 0.5)$

$= 0.6915$

7b $z = \frac{x-\mu}{\sigma} = \frac{62-50}{10} = 1.2$

Hence, $P(X \ge 62) = P(Z \ge 1.2)$

$= 1 - P(Z < 1.2)$

$= 1 - 0.8849$

$= 0.1151$

7c $z = \frac{x-\mu}{\sigma} = \frac{3.96-4}{0.2} = -0.2$ and $z = \frac{x-\mu}{\sigma} = \frac{4.3-4}{0.2} = 1.5$

Hence, $P(3.96 \le X \le 4.3) = P(-0.2 \le Z \le 1.5)$

$= P(Z \le 1.5) - [1 - P(Z \le 0.2)]$

$= 0.9332 - (1 - 0.5793)$

$= 0.9332 - 0.4207$

$= 0.5125$

7d $z = \frac{x-\mu}{\sigma} = \frac{6.79-5.75}{1.3} = 0.8$ and $z = \frac{x-\mu}{\sigma} = \frac{8.09-5.75}{1.3} = 1.8$

Hence, $P(6.79 \le X \le 8.09) = P(0.8 \le Z \le 1.8)$

$= P(Z \le 1.8) - P(Z \le 0.8)$

$= 0.9641 - 0.7881$

$= 0.176$

8a The expected life of washing machines is $6 \times 12 + 4 = 76$ months.

$P(\text{more than 8 years}) = P(\text{more than 96 months}) = P(X > 96)$

$z = \frac{x-\mu}{\sigma} = \frac{96-76}{15} \doteqdot 1.3$

$P(X > 96) \doteqdot P(Z > 1.3)$

$= 1 - P(Z \leq 1.3)$

$= 1 - 0.9032$

$= 0.0968$ or 9.7%

8b $P(\text{less than 5 years}) = P(\text{less than 60 months}) = P(X < 60)$

$z = \frac{x-\mu}{\sigma} = \frac{60-76}{15} \doteqdot -1.07$

$P(X < 5) \doteqdot P(Z < -1.07)$

$= P(Z > 1.07)$

$= 1 - P(Z \leq 1.07)$

$= 1 - 0.858$ (using online calculator)

$= 0.142$ or 14.2%

This is probably an unacceptable risk for the manufacturer, and they should increase the mean life, or decrease the standard deviation, or adjust the length of their advertised warranty.