Solutions to Exercise 2A

1 **A**

When $n = 1$,

$$\begin{aligned} \text{RHS} &= 1^2 \\ &= 1 \\ &= \text{LHS} \end{aligned}$$

so the statement is true for $n = 1$.

B

Suppose that $k \geq 1$ is a positive integer for which the statement is true.

That is, suppose $1 + 3 + 5 + \cdots + (2k - 1) = k^2$ $\quad(**)$

We prove the statement for $n = k + 1$.

That is, we prove $1 + 3 + 5 + \cdots + (2k - 1) + (2(k + 1) - 1) = (k + 1)^2$

$$\begin{aligned} \text{LHS} &= 1 + 3 + 5 + \cdots + (2k - 1) + (2(k + 1) - 1) \\ &= 1 + 3 + 5 + \cdots + (2k - 1) + (2k + 2 - 1) \\ &= 1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) \\ &= k^2 + (2k + 1) && \text{by the induction hypothesis } (**) \\ &= k^2 + 2k + 1 \\ &= (k + 1)^2 \\ &= \text{RHS} \end{aligned}$$

C

It follows from parts **A** and **B** by mathematical induction, that:

$1 + 3 + 5 + \cdots + (2n - 1) = n^2$ for all integers $n \geq 1$.

2a **A**

When $n = 1$,

$$\begin{aligned} \text{RHS} &= \frac{1}{2}(1)(1 + 1) \\ &= 1 \\ &= \text{LHS} \end{aligned}$$

so the statement is true for $n = 1$.

B

Suppose that $k \geq 1$ is a positive integer for which the statement is true.

That is, suppose $1 + 2 + 3 + \cdots + k = \frac{1}{2}k(k+1)$ $\qquad (**)$

We prove the statement for $n = k + 1$.

That is, we prove $1 + 2 + 3 + \cdots + k + (k+1) = \frac{1}{2}(k+1)\big((k+1)+1\big)$

$$\begin{aligned}
\text{LHS} &= 1 + 2 + 3 + \cdots + k + (k+1) \\
&= \tfrac{1}{2}k(k+1) + (k+1) && \text{by the induction hypothesis } (**) \\
&= (k+1)\left(\frac{1}{2}k + 1\right) \\
&= \frac{1}{2}(k+1)(k+2) \\
&= \frac{1}{2}(k+1)\big((k+1)+1\big) \\
&= \text{RHS}
\end{aligned}$$

C

It follows from parts **A** and **B** by mathematical induction, that:

$1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n+1)$ for all integers $n \geq 1$.

2b **A**

When $n = 1$,

$$\begin{aligned}
\text{RHS} &= 2^1 - 1 \\
&= 1 \\
&= \text{LHS}
\end{aligned}$$

so the statement is true for $n = 1$.

B

Suppose that $k \geq 1$ is a positive integer for which the statement is true.

That is, suppose $1 + 2 + 2^2 + \cdots + 2^{k-1} = 2^k - 1$ $\qquad (**)$

We prove the statement for $n = k + 1$.

That is, we prove $1 + 2 + 2^2 + \cdots + 2^{k-1} + 2^{(k+1)-1} = 2^{k+1} - 1$

$$\begin{aligned}
\text{LHS} &= 1 + 2 + 2^2 + \cdots + 2^{k-1} + 2^{(k+1)-1} \\
&= 2^k - 1 + 2^{(k+1)-1} && \text{by the induction hypothesis } (**) \\
&= 2^k + 2^k - 1 \\
&= 2 \times 2^k - 1 \\
&= 2^{k+1} - 1 \\
&= \text{RHS}
\end{aligned}$$

C

It follows from parts **A** and **B** by mathematical induction, that:

$1 + 2 + 2^2 + \cdots + 2^{k-1} + 2^{n-1} = 2^n - 1$ for all integers $n \geq 1$.

2c **A**

When $n = 1$,

$$\begin{aligned}\text{RHS} &= \frac{1}{4}(5^1 - 1)\\ &= \frac{1}{4}(5 - 1)\\ &= \frac{4}{4}\\ &= 1\\ &= \text{LHS}\end{aligned}$$

so the statement is true for $n = 1$.

B

Suppose that $k \geq 1$ is a positive integer for which the statement is true.

That is, suppose $1 + 5 + 5^2 + \cdots + 5^{k-1} = \frac{1}{4}(5^k - 1)$ $\qquad (**)$

We prove the statement for $n = k + 1$.

That is, we prove $1 + 5 + 5^2 + \cdots + 5^{k-1} + 5^{(k+1)-1} = \frac{1}{4}(5^{k+1} - 1)$

$$\begin{aligned}\text{LHS} &= 1 + 5 + 5^2 + \cdots + 5^{k-1} + 5^{(k+1)-1}\\ &= \tfrac{1}{4}(5^k - 1) + 5^{(k+1)-1} \qquad \text{by the induction hypothesis } (**)\\ &= \frac{1}{4}\left(5^k - 1 + 4 \times 5^{(k+1)-1}\right)\\ &= \frac{1}{4}(5^k - 1 + 4 \times 5^k)\\ &= \frac{1}{4}(5 \times 5^k - 1)\\ &= \frac{1}{4}(5^{k+1} - 1)\\ &= \text{RHS}\end{aligned}$$

C

It follows from parts **A** and **B** by mathematical induction, that:

$1 + 5 + 5^2 + \cdots + 5^{n-1} = \frac{1}{4}(5^n - 1)$ for all integers $n \geq 1$.

2d **A**

When $n = 1$,

$$\begin{aligned} \text{RHS} &= \frac{1}{3}(1)(1+1)(1+2) \\ &= \frac{1}{3} \times 2 \times 3 \\ &= 1 \times 2 \\ &= \text{LHS} \end{aligned}$$

so the statement is true for $n = 1$.

B

Suppose that $k \geq 1$ is a positive integer for which the statement is true.

That is, suppose $1 \times 2 + 2 \times 3 + 3 \times 4 + \cdots + k(k+1) = \frac{1}{3}k(k+1)(k+2)$ $(**)$

We prove the statement for $n = k + 1$.

That is, we prove $1 \times 2 + 2 \times 3 + 3 \times 4 + \cdots + k(k+1) + (k+1)\big((k+1)+1\big)$

$$= \frac{1}{3}(k+1)\big((k+1)+1\big)\big((k+1)+2\big)$$

$$\begin{aligned} \text{LHS} &= 1 \times 2 + 2 \times 3 + 3 \times 4 + \cdots + k(k+1) + (k+1)\big((k+1)+1\big) \\ &= \frac{1}{3}k(k+1)(k+2) + (k+1)\big((k+1)+1\big) \end{aligned}$$

by the induction hypothesis $(**)$

$$\begin{aligned} &= \frac{1}{3}k(k+1)(k+2) + (k+1)(k+2) \\ &= (k+1)(k+2)\left(\frac{1}{3}k + 1\right) \\ &= \frac{1}{3}(k+1)(k+2)(k+3) \\ &= \frac{1}{3}(k+1)\big((k+1)+1\big)\big((k+1)+2\big) \\ &= \text{RHS} \end{aligned}$$

C

It follows from parts **A** and **B** by mathematical induction, that:

$1 \times 2 + 2 \times 3 + 3 \times 4 + \cdots + n(n+1) = \frac{1}{3}n(n+1)(n+2)$ for all integers $n \geq 1$.

2e **A**

When $n = 1$,

$$\begin{aligned} \text{RHS} &= \frac{1}{6}(1)(1+1)(2+7) \\ &= \frac{1}{6} \times 1 \times 2 \times 9 \\ &= \frac{18}{6} \\ &= 3 \\ &= 1 \times 3 \\ &= \text{LHS} \end{aligned}$$

so the statement is true for $n = 1$.

B

Suppose that $k \geq 1$ is a positive integer for which the statement is true.

That is, suppose:

$$1 \times 3 + 2 \times 4 + 3 \times 5 + \cdots + k(k+2) = \frac{1}{6}k(k+1)(2k+7) \qquad (**)$$

We prove the statement for $n = k + 1$.

That is, we prove $1 \times 3 + 2 \times 4 + 3 \times 5 + \cdots + k(k+2) + (k+1)\big((k+1)+2\big)$

$$= \frac{1}{6}(k+1)\big((k+1)+1\big)(2(k+1)+7)$$

$$\begin{aligned} \text{LHS} &= 1 \times 2 + 2 \times 3 + 3 \times 4 + \cdots + k(k+2) + (k+1)\big((k+1)+2\big) \\ &= \frac{1}{6}k(k+1)(2k+7) + (k+1)\big((k+1)+2\big) \end{aligned}$$

by the induction hypothesis $(**)$

$$\begin{aligned} &= \frac{1}{6}k(k+1)(2k+7) + (k+1)(k+3) \\ &= (k+1)\left(\frac{1}{6}k(2k+7) + k + 3\right) \\ &= \frac{1}{6}(k+1)(k(2k+7) + 6k + 18) \\ &= \frac{1}{6}(k+1)(2k^2 + 7k + 6k + 18) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{6}(k+1)(2k^2+13k+18) \\
&= \frac{1}{6}(k+1)(k+2)(2k+9) \\
&= \frac{1}{6}(k+1)\big((k+1)+1\big)(2(k+1)+7) \\
&= \text{RHS}
\end{aligned}
$$

C

It follows from parts **A** and **B** by mathematical induction, that:

$1 \times 3 + 2 \times 4 + 3 \times 5 + \cdots + n(n+2) = \frac{1}{6}n(n+1)(2n+7)$ for all integers $n \geq 1$.

2f **A**

When $n = 1$,

$$
\begin{aligned}
\text{RHS} &= \frac{1}{6}(1)(1+1)(2(1)+1) \\
&= \frac{1}{6} \times 1 \times 2 \times 3 \\
&= 1 \\
&= \text{LHS}
\end{aligned}
$$

so the statement is true for $n = 1$.

B

Suppose that $k \geq 1$ is a positive integer for which the statement is true.

That is, suppose $1^2 + 2^2 + 3^2 + \cdots + k^2 = \frac{1}{6}k(k+1)(2k+1)$ $\qquad (**)$

We prove the statement for $n = k + 1$.

That is, we prove $1^2 + 2^2 + 3^2 + \cdots + k^2 + (k+1)^2$

$$= \frac{1}{6}(k+1)\big((k+1)+1\big)(2(k+1)+1)$$

$$
\begin{aligned}
\text{LHS} &= 1^2 + 2^2 + 3^2 + \cdots + k^2 + (k+1)^2 \\
&= \frac{1}{6}k(k+1)(2k+1) + (k+1)^2 \qquad \text{by the induction hypothesis } (**) \\
&= (k+1) \times \left(\frac{1}{6}k(2k+1) + (k+1)\right) \\
&= \frac{1}{6}(k+1)(k(2k+1) + 6k + 6) \\
&= \frac{1}{6}(k+1)(2k^2 + k + 6k + 6)
\end{aligned}
$$

$$= \frac{1}{6}(k+1)(2k^2+7k+6)$$
$$= \frac{1}{6}(k+1)(k+2)(2k+3)$$
$$= \frac{1}{6}(k+1)\big((k+1)+1\big)(2(k+1)+1)$$
$$= \text{RHS}$$

C

It follows from parts **A** and **B** by mathematical induction, that:

$1^2+2^2+3^2+\cdots+n^2 = \frac{1}{6}n(n+1)(2n+1)$ for all integers $n \ge 1$.

2g **A**

When $n = 1$,

$$\text{RHS} = \frac{1}{3}(1)(2(1)-1)(2(1)+1)$$
$$= \frac{1}{3}\times 1\times 1\times 3$$
$$= 1$$
$$= \text{LHS}$$

so the statement is true for $n = 1$.

B

Suppose that $k \ge 1$ is a positive integer for which the statement is true.

That is, suppose $1^2+3^2+5^2+\cdots+(2k-1)^2 = \frac{1}{3}k(2k-1)(2k+1)$ $\quad(**)$

We prove the statement for $n = k+1$.

That is, we prove $1^2+3^2+5^2+\cdots+(2k-1)^2+(2(k+1)-1)^2$

$$= \frac{1}{3}(k+1)(2(k+1)-1)(2(k+1)+1)$$

$$\text{LHS} = 1^2+3^2+5^2+\cdots+(2k-1)^2+(2(k+1)-1)^2$$
$$= \frac{1}{3}k(2k-1)(2k+1)+(2(k+1)-1)^2 \quad \text{by the induction hypothesis } (**)$$
$$= \frac{1}{3}k(2k-1)(2k+1)+(2k+2-1)^2$$
$$= \frac{1}{3}k(2k-1)(2k+1)+(2k+1)^2$$
$$= (2k+1)\left(\frac{1}{3}k(2k-1)+(2k+1)\right)$$

$$\begin{aligned}
&= \frac{1}{3}(2k+1)\big(k(2k-1)+3(2k+1)\big)\\
&= \frac{1}{3}(2k+1)(2k^2-k+6k+3)\\
&= \frac{1}{3}(2k+1)(2k^2+5k+3)\\
&= \frac{1}{3}(2k+1)(k+1)(2k+3)\\
&= \frac{1}{3}(k+1)(2k+1)(2k+3)\\
&= \frac{1}{3}(k+1)(2(k+1)-1)(2(k+1)+1)\\
&= \text{RHS}
\end{aligned}$$

C

It follows from parts **A** and **B** by mathematical induction, that:

$1^2+3^2+5^2+\cdots+(2n-1)^2=\frac{1}{3}n(2n-1)(2n+1)$ for all integers $n \geq 1$.

2h **A**

When $n=1$,

$$\begin{aligned}
\text{RHS} &= \frac{1}{1+1}\\
&= \frac{1}{2}\\
&= \frac{1}{1\times 2}\\
&= \text{LHS}
\end{aligned}$$

so the statement is true for $n=1$.

B

Suppose that $k \geq 1$ is a positive integer for which the statement is true.

That is, suppose $\frac{1}{1\times2}+\frac{1}{2\times3}+\frac{1}{3\times4}+..+\frac{1}{k(k+1)}=\frac{k}{k+1}$ $\quad(**)$

We prove the statement for $n=k+1$.

That is, we prove $\frac{1}{1\times2}+\frac{1}{2\times3}+\frac{1}{3\times4}+..+\frac{1}{k(k+1)}+\frac{1}{(k+1)((k+1)+1)}=\frac{k+1}{(k+1)+1}$

$$\begin{aligned}
\text{LHS} &= \frac{1}{1\times2}+\frac{1}{2\times3}+\frac{1}{3\times4}+\ldots+\frac{1}{k(k+1)}+\frac{1}{(k+1)((k+1)+1)}\\
&= \frac{k}{k+1}+\frac{1}{(k+1)((k+1)+1)} \qquad \text{by the induction hypothesis } (**)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \\
&= \frac{k(k+2)}{(k+1)(k+2)} + \frac{1}{(k+1)(k+2)} \\
&= \frac{k^2+2k}{(k+1)(k+2)} + \frac{1}{(k+1)(k+2)} \\
&= \frac{k^2+2k+1}{(k+1)(k+2)} \\
&= \frac{(k+1)^2}{(k+1)(k+2)} \\
&= \frac{k+1}{k+2} \\
&= \text{RHS}
\end{aligned}
$$

C

It follows from parts **A** and **B** by mathematical induction, that:

$\frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + .. + \frac{1}{n(n+1)} = \frac{n}{n+1}$ for all integers $n \geq 1$.

2i **A**

When $n = 1$,

$$
\begin{aligned}
\text{RHS} &= \frac{1}{2(1)+1} \\
&= \frac{1}{3} \\
&= \frac{1}{1\times 3} \\
&= \text{LHS}
\end{aligned}
$$

so the statement is true for $n = 1$.

B

Suppose that $k \geq 1$ is a positive integer for which the statement is true.

That is, suppose $\frac{1}{1\times 3} + \frac{1}{3\times 5} + \frac{1}{5\times 7} + \cdots + \frac{1}{(2k-1)(2k+1)} = \frac{k}{2k+1}$ (**)

We prove the statement for $n = k + 1$.

That is, we prove

$\frac{1}{1\times 3} + \frac{1}{3\times 5} + \frac{1}{5\times 7} + \cdots + \frac{1}{(2k-1)(2k+1)} + \frac{1}{(2(k+1)-1)(2(k+1)+1)} = \frac{(k+1)}{2(k+1)+1}$

$$\begin{aligned}\text{LHS} &= \frac{1}{1\times 3}+\frac{1}{3\times 5}+\frac{1}{5\times 7}+\cdots+\frac{1}{(2k-1)(2k+1)}\\ &\quad +\frac{1}{(2(k+1)-1)(2(k+1)+1)}\\ &= \frac{k}{2k+1}+\frac{1}{(2(k+1)-1)(2(k+1)+1)} \qquad \text{by the induction hypothesis } (**)\\ &= \frac{k}{2k+1}+\frac{1}{(2k+1)(2k+3)}\\ &= \frac{k(2k+3)}{(2k+1)(2k+3)}+\frac{1}{(2k+1)(2k+3)}\\ &= \frac{2k^2+3k}{(2k+1)(2k+3)}+\frac{1}{(2k+1)(2k+3)}\\ &= \frac{2k^2+3k+1}{(2k+1)(2k+3)}\\ &= \frac{(2k+1)(k+1)}{(2k+1)(2k+3)}\\ &= \frac{k+1}{2k+3}\\ &= \frac{k+1}{2(k+1)+1}\\ &= \text{RHS}\end{aligned}$$

C

It follows from parts **A** and **B** by mathematical induction, that:

$\frac{1}{1\times 3}+\frac{1}{3\times 5}+\frac{1}{5\times 7}+\cdots+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}$ for all integers $n \geq 1$.

3h $\lim\limits_{n\to\infty}\frac{n}{n+1}=\lim\limits_{n\to\infty}\frac{1}{1+\frac{1}{n}}=\frac{1}{1+0}=\frac{1}{1}=1$

3i $\lim\limits_{n\to\infty}\frac{n}{2n+1}=\lim\limits_{n\to\infty}\frac{1}{2+\frac{1}{n}}=\frac{1}{2+0}=\frac{1}{2}$

4a **A**

When $n = 1$,

$$\begin{aligned}\text{RHS} &= \frac{1}{12}1(1+1)(1+2)(3+1)\\ &= \frac{1}{12}(1)(2)(3)(4)\\ &= \frac{1}{12}(24)\end{aligned}$$

$= 2$
$= 1^2 \times 2$
$= \text{LHS}$

so the statement is true for $n = 1$.

B

Suppose that $k \geq 1$ is a positive integer for which the statement is true.

That is, suppose $1^2 \times 2 + 2^2 \times 3 + 3^2 \times 4 + \cdots + k^2(k+1)$

$$= \tfrac{1}{12}k(k+1)(k+2)(3k+1) \qquad (**)$$

We prove the statement for $n = k + 1$.

That is, we prove

$$1^2 \times 2 + 2^2 \times 3 + 3^2 \times 4 + \cdots + (k)^2(k+1) + (k+1)^2\big((k+1)+1\big)$$

$$= \tfrac{1}{12}(k+1)((k+1)+1)((k+1)+2)(3(k+1)+1)$$

$$\begin{aligned}
\text{LHS} &= 1^2 \times 2 + 2^2 \times 3 + 3^2 \times 4 + \cdots + (k)^2(k+1) + (k+1)^2\big((k+1)+1\big) \\
&= \tfrac{1}{12}k(k+1)(k+2)(3k+1) + (k+1)^2\big((k+1)+1\big)
\end{aligned}$$

by the induction hypothesis $(**)$

$$\begin{aligned}
&= \frac{1}{12}k(k+1)(k+2)(3k+1) + (k+1)^2(k+2) \\
&= (k+1)(k+2)\left(\frac{1}{12}k(3k+1) + k + 1\right) \\
&= \frac{1}{12}(k+1)(k+2)(k(3k+1) + 12k + 12) \\
&= \frac{1}{12}(k+1)(k+2)(3k^2 + k + 12k + 12) \\
&= \frac{1}{12}(k+1)(k+2)(3k^2 + 13k + 12) \\
&= \frac{1}{12}(k+1)(k+2)(k+3)(3k+4) \\
&= \frac{1}{12}(k+1)((k+1)+1)((k+1)+2)(3(k+1)+1) \\
&= \text{RHS}
\end{aligned}$$

C

It follows from parts **A** and **B** by mathematical induction, that:

$1^2 \times 2 + 2^2 \times 3 + 3^2 \times 4 + \cdots + n^2(n+1) = \tfrac{1}{12}n(n+1)(n+2)(3n+1)$ for all integers $n \geq 1$.

4b **A**

When $n = 1$,

$$\begin{aligned}\text{RHS} &= \frac{1}{12}1(1+1)(1+2)(3+5)\\ &= \frac{1}{12}(1)(2)(3)(8)\\ &= \frac{1}{12}(48)\\ &= 4\\ &= 1 \times 2^2\\ &= \text{LHS}\end{aligned}$$

so the statement is true for $n = 1$.

B

Suppose that $k \geq 1$ is a positive integer for which the statement is true.

That is, suppose $1 \times 2^2 + 2 \times 3^2 + 3 \times 4^2 + \cdots + k(k+1)^2$

$$= \tfrac{1}{12}k(k+1)(k+2)(3k+5) \qquad (**)$$

We prove the statement for $n = k + 1$.

That is, we prove

$$1 \times 2^2 + 2 \times 3^2 + 3 \times 4^2 + \cdots + k(k+1)^2 + (k+1)\big((k+1)+1\big)^2$$

$$= \tfrac{1}{12}(k+1)((k+1)+1)((k+1)+2)(3(k+1)+5)$$

$$\begin{aligned}\text{LHS} &= 1 \times 2^2 + 2 \times 3^2 + 3 \times 4^2 + \cdots + k(k+1)^2 + (k+1)((k+1)+1)^2\\ &= \tfrac{1}{12}k(k+1)(k+2)(3k+5) \ + (k+1)((k+1)+1)^2\end{aligned}$$

by the induction hypothesis $(**)$

$$\begin{aligned}&= \frac{1}{12}k(k+1)(k+2)(3k+5) \ + (k+1)(k+2)^2\\ &= (k+1)(k+2)\left(\frac{1}{12}k(3k+5) + (k+2)\right)\\ &= \frac{1}{12}(k+1)(k+2)\big(k(3k+5) + 12(k+2)\big)\\ &= \frac{1}{12}(k+1)(k+2)(3k^2 + 5k + 12k + 24)\\ &= \frac{1}{12}(k+1)(k+2)(3k^2 + 17k + 24)\\ &= \frac{1}{12}(k+1)(k+2)(k+3)(3k+8)\end{aligned}$$

$$= \frac{1}{12}(k+1)((k+1)+1)((k+1)+2)(3(k+1)+5)$$
$$= \text{RHS}$$

C

It follows from parts **A** and **B** by mathematical induction, that:

$1 \times 2^2 + 2 \times 3^2 + 3 \times 4^2 + \cdots + n(n+1)^2 = \frac{1}{12}n(n+1)(n+2)(3n+5)$ for all integers $n \geq 1$.

4c **A**

When $n = 1$,

$$\begin{aligned}\text{RHS} &= 1 \times 2^1\\ &= 2\\ &= 2 \times 1\\ &= 2 \times 2^0\\ &= \text{LHS}\end{aligned}$$

so the statement is true for $n = 1$.

B

Suppose that $k \geq 1$ is a positive integer for which the statement is true.

That is, suppose $2 \times 2^0 + 3 \times 2^1 + 4 \times 2^2 + \cdots + (k+1) \times 2^{k-1}$

$$= k \times 2^k \qquad (**)$$

We prove the statement for $n = k + 1$.

That is, we prove

$$2 \times 2^0 + 3 \times 2^1 + 4 \times 2^2 + \cdots + (k+1) \times 2^{k-1} + \big((k+1)+1\big) \times 2^{(k+1)-1}$$
$$= (k+1) \times 2^{(k+1)}$$

$$\begin{aligned}\text{LHS} &= 2 \times 2^0 + 3 \times 2^1 + 4 \times 2^2 + \cdots + (k+1) \times 2^{k-1} + ((k+1)+1) \times 2^{(k+1)-1}\\ &= k \times 2^k + ((k+1)+1) \times 2^{(k+1)-1} \qquad \text{by the induction hypothesis } (**)\\ &= k \times 2^k + ((k+1)+1) \times 2^k\\ &= k \times 2^k + (k+2) \times 2^k\\ &= (k+k+2) \times 2^k\\ &= (2k+2) \times 2^k\\ &= (k+1) \times 2 \times 2^k\\ &= (k+1) \times 2^{k+1}\\ &= \text{RHS}\end{aligned}$$

C

It follows from parts **A** and **B** by mathematical induction, that:

$2 \times 2^0 + 3 \times 2^1 + 4 \times 2^2 + \cdots + (n+1) \times 2^{n-1} = n \times 2^n$ for all integers $n \geq 1$.

5a **A**

When $n = 1$,

$$\begin{aligned} \text{RHS} &= (1+1)! - 1 \\ &= 2! - 1 \\ &= 2 - 1 \\ &= 1 \\ &= 1 \times 1! \\ &= \text{LHS} \end{aligned}$$

so the statement is true for $n = 1$.

B

Suppose that $k \geq 1$ is a positive integer for which the statement is true.

That is, suppose $1 \times 1! + 2 \times 2! + 3 \times 3! + k \times k! = (k+1)! - 1$ $\quad (**)$

We prove the statement for $n = k + 1$.

That is, we prove $1 \times 1! + 2 \times 2! + 3 \times 3! + k \times k! + (k+1) \times (k+1)!$
$= (k+2)! - 1$

$$\begin{aligned} \text{LHS} &= 1 \times 1! + 2 \times 2! + 3 \times 3! + k \times k! + (k+1) \times (k+1)! \\ &= (k+1)! - 1 + (k+1) \times (k+1)! \quad \text{by the induction hypothesis } (**) \\ &= (k+1)! + (k+1) \times (k+1)! - 1 \\ &= (k+1)!\,(1 + k + 1) - 1 \\ &= (k+1)!\,(k+2) - 1 \\ &= (k+2)! - 1 \\ &= \text{RHS} \end{aligned}$$

C

It follows from parts **A** and **B** by mathematical induction, that:

$1 \times 1! + 2 \times 2! + 3 \times 3! + n \times n! = (n+1)! - 1$ for all integers $n \geq 1$.

5b **A**

When $n = 1$,

$$\begin{aligned} \text{RHS} &= 1(1+1)! \\ &= 2! \end{aligned}$$

$$\begin{aligned}&= 2\\ &= 2 \times 1!\\ &= \text{LHS}\end{aligned}$$

so the statement is true for $n = 1$.

B

Suppose that $k \geq 1$ is a positive integer for which the statement is true.

That is, suppose $2 \times 1! + 5 \times 2! + 10 \times 3! + \cdots + (k^2 + 1)k! = k(k+1)! \qquad (**)$

We prove the statement for $n = k + 1$.

That is, we prove

$$2 \times 1! + 5 \times 2! + 10 \times 3! + \cdots + (k^2 + 1)k! + ((k+1)^2 + 1)(k+1)! = (k+1)(k+2)!$$

$$\begin{aligned}\text{LHS} &= 2 \times 1! + 5 \times 2! + 10 \times 3! + \cdots + (k^2 + 1)k! + ((k+1)^2 + 1)(k+1)!\\ &= k(k+1)! + ((k+1)^2 + 1)(k+1)! \qquad \text{by the induction hypothesis } (**)\\ &= (k+1)!\big(k + ((k+1)^2 + 1)\big)\\ &= (k+1)!\big(k + (k^2 + 2k + 1 + 1)\big)\\ &= (k+1)!\,(k^2 + 3k + 2)\\ &= (k+1)!\,(k+1)(k+2)\\ &= (k+1)(k+2)(k+1)!\\ &= (k+1)(k+2)!\\ &= \text{RHS}\end{aligned}$$

C

It follows from parts **A** and **B** by mathematical induction, that:

$2 \times 1! + 5 \times 2! + 10 \times 3! + \cdots + (n^2 + 1)n! = n(n+1)!$ for all integers $n \geq 1$.

5c **A**

When $n = 1$,

$$\begin{aligned}\text{RHS} &= 1 - \frac{1}{(1+1)!}\\ &= 1 - \frac{1}{2!}\\ &= 1 - \frac{1}{2}\\ &= \frac{1}{2}\\ &= \frac{1}{2!}\end{aligned}$$

$= \text{LHS}$

so the statement is true for $n = 1$.

B

Suppose that $k \geq 1$ is a positive integer for which the statement is true.

That is, suppose $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{k}{(k+1)!} = 1 - \frac{1}{(k+1)!}$ $\quad (**)$

We prove the statement for $n = k + 1$.

That is, we prove $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{k}{(k+1)!} + \frac{k+1}{(k+2)!} = 1 - \frac{1}{(k+2)!}$

$$\begin{aligned}
\text{LHS} &= \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{k}{(k+1)!} + \frac{k+1}{(k+2)!} \\
&= 1 - \frac{1}{(k+1)!} + \frac{k+2}{(k+2)!} \qquad \text{by the induction hypothesis } (**) \\
&= 1 - \frac{k+2}{(k+2)(k+1)!} + \frac{k+1}{(k+2)!} \\
&= 1 - \frac{k+2}{(k+2)!} + \frac{k+1}{(k+2)!} \\
&= 1 - \frac{1}{(k+2)!} \\
&= \text{RHS}
\end{aligned}$$

C

It follows from parts **A** and **B** by mathematical induction, that:

$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$ for all integers $n \geq 1$.

6a Suppose that $k \geq 1$ is a positive integer for which the statement is true.

That is, suppose $1 + 3 + 5 + \cdots + (2k - 1) = (k)^2 + 2$ $\quad (**)$

We prove the statement for $n = k + 1$.

That is, we prove $1 + 3 + 5 + \cdots + (2k - 1) + (2(k + 1) - 1) = (k + 1)^2 + 2$

$$\begin{aligned}
\text{LHS} &= 1 + 3 + 5 + \cdots + (2k - 1) + (2(k + 1) - 1) \\
&= k^2 + 2 + (2(k + 1) - 1) \qquad \text{by the induction hypothesis } (**) \\
&= k^2 + 2 + (2k + 2 - 1) \\
&= k^2 + 2 + 2k + 1 \\
&= k^2 + 2k + 3 \\
&= k^2 + 2k + 1 + 2 \\
&= (k + 1)^2 + 2 \\
&= \text{RHS}
\end{aligned}$$

6b When $n = 1$,

$$\begin{aligned}\text{RHS} &= 1^2 + 2\\ &= 3\\ &\neq \text{LHS}\end{aligned}$$

so the statement is **not** true for $n = 1$.

7a **A**

When $n = 1$,

$$\begin{aligned}\text{RHS} &= 1(1+1)+1\\ &= 2+1\\ &= 3\\ &= \text{LHS}\end{aligned}$$

so the statement is true for $n = 1$.

B

Suppose that $k \geq 1$ is a positive integer for which the statement is true.

That is, suppose $3 + 6 + 9 + \cdots + 3k = k(k+1) + 1$ $\quad (**)$

We prove the statement for $n = k + 1$.

That is, we prove $3 + 6 + 9 + \cdots + 3k + 3(k+1) = (k+1)((k+1)+1) + 1$

$$\begin{aligned}\text{LHS} &= 3 + 6 + 9 + \cdots + 3k + 3(k+1)\\ &= k(k+1) + 1 + 3(k+1) \quad \text{by the induction hypothesis } (**)\\ &= k(k+1) + 3(k+1) + 1\\ &= (k+3)(k+1) + 1\\ &= k^2 + 4k + 3 + 1\\ &= k^2 + 4k + 4\end{aligned}$$

But:

$$\begin{aligned}\text{RHS} &= (k+1)\big((k+1)+1\big) + 1\\ &= (k+1)(k+2) + 1\\ &= k^2 + 3k + 2 + 1\\ &= k^2 + 3k + 3\\ &\neq \text{LHS}\end{aligned}$$

So the proof breaks down.

7b If it is true for $n = k$, it does not follow that it is true for $n = k + 1$.

8a $P(-1) = 4(-1)^3 + 18(-1)^2 + 23(-1) + 9 = 0$, hence $n + 1$ is a factor.

$$P(n) = 4n^3 + 18n^2 + 23n + 9$$
$$= (n + 1)(4n^2 + 14n + 9)$$

8b **A**

When $n = 1$,

$$\begin{aligned}\text{RHS} &= \frac{1}{3}(1)(4 + 6 - 1)\\ &= \frac{9}{3}\\ &= 3\\ &= 1 \times 3\\ &= \text{LHS}\end{aligned}$$

so the statement is true for $n = 1$.

B

Suppose that $k \geq 1$ is a positive integer for which the statement is true.

That is, suppose $1 \times 3 + 3 \times 5 + 5 \times 7 + \cdots + (2k - 1)(2k + 1)$

$$= \tfrac{1}{3}k(4k^2 + 6k - 1) \qquad (**)$$

We prove the statement for $n = k + 1$.

That is, we prove

$$1 \times 3 + 3 \times 5 + 5 \times 7 + \cdots + (2k - 1)(2k + 1) + (2(k + 1) - 1)(2(k + 1) + 1)$$
$$= \tfrac{1}{3}(k + 1)(4(k + 1)^2 + 6(k + 1) - 1)$$

$$\begin{aligned}\text{LHS} &= 1 \times 3 + 3 \times 5 + 5 \times 7 + \cdots + (2k - 1)(2k + 1)\\ &\qquad + (2(k + 1) - 1)(2(k + 1) + 1)\\ &= \tfrac{1}{3}k(4k^2 + 6k - 1) + (2(k + 1) - 1)(2(k + 1) + 1)\end{aligned}$$

by the induction hypothesis $(**)$

$$\begin{aligned}&= \frac{1}{3}k(4k^2 + 6k - 1) + (2k + 2 - 1)(2k + 2 + 1)\\ &= \frac{1}{3}k(4k^2 + 6k - 1) + (2k + 1)(2k + 3)\\ &= \frac{1}{3}\big(k(4k^2 + 6k - 1) + 3(2k + 1)(2k + 3)\big)\\ &= \frac{1}{3}\big(k(4k^2 + 6k - 1) + 3(4k^2 + 8k + 3)\big)\end{aligned}$$

$$= \frac{1}{3}(4k^3 + 6k^2 - k + 12k^2 + 24k + 9)$$
$$= \frac{1}{3}(4k^3 + 18k^2 + 23k + 9)$$
$$= \tfrac{1}{3}(k + 1)(4k^2 + 14k + 9) \qquad \text{using answer to question 8a}$$
$$= \frac{1}{3}(k + 1)(4k^2 + 8k + 4 + 6k + 6 - 1)$$
$$= \frac{1}{3}(k + 1)(4(k^2 + 2k + 1) + 6(k + 1) - 1)$$
$$= \tfrac{1}{3}(k + 1)(4(k + 1)^2 + 6(k + 1) - 1)$$
$$= \text{RHS}$$

C

It follows from parts **A** and **B** by mathematical induction, that:

$1 \times 3 + 3 \times 5 + 5 \times 7 + \cdots + (2n - 1)(2n + 1) = \tfrac{1}{3}n(4n^2 + 6n - 1)$ for all integers $n \geq 1$.

9 **A**

When $n = 1$,

$$\text{RHS} = \frac{1}{6}(1)(1 + 1)(1 + 2)$$
$$= \frac{1}{6}(1)(2)(3)$$
$$= 1$$
$$= \text{LHS}$$

so the statement is true for $n = 1$.

B

Suppose that $k \geq 1$ is a positive integer for which the statement is true.

That is, suppose $1 + (1 + 2) + (1 + 2 + 3) + \cdots + (1 + 2 + 3 + \cdots + k)$

$$= \tfrac{1}{6}k(k + 1)(k + 2) \qquad (**)$$

We prove the statement for $n = k + 1$.

That is, we prove $1 + (1 + 2) + (1 + 2 + 3) + \cdots + (1 + 2 + 3 + \cdots + k) + \big(1 + 2 + 3 + \cdots + k + (k + 1)\big) = \tfrac{1}{6}(k + 1)\big((k + 1) + 1\big)\big((k + 1) + 2\big)$

$$\text{LHS} = 1 + (1 + 2) + (1 + 2 + 3) + \cdots + (1 + 2 + 3 + \cdots + k)$$
$$+\big(1 + 2 + 3 + \cdots + k + (k + 1)\big)$$

$$= \tfrac{1}{6}k(k+1)(k+2) + \big(1+2+3+\cdots+k+(k+1)\big)$$

by the induction hypothesis $(**)$

$$= \frac{1}{6}k(k+1)(k+2) + (1+2+3+\cdots+k) + (k+1)$$
$$= \tfrac{1}{6}k(k+1)(k+2) + \tfrac{1}{2}k(k+1) + (k+1) \qquad \text{from Question 2a}$$
$$= (k+1)\left(\frac{1}{6}(k)(k+2) + \frac{1}{2}k + 1\right)$$
$$= \frac{1}{6}(k+1)\big((k)(k+2) + 3k + 6\big)$$
$$= \frac{1}{6}(k+1)(k^2 + 2k + 3k + 6)$$
$$= \frac{1}{6}(k+1)(k^2 + 5k + 6)$$
$$= \frac{1}{6}(k+1)(k+2)(k+3)$$
$$= \frac{1}{6}(k+1)\big((k+1)+1\big)\big((k+1)+2\big)$$
$$= \text{RHS}$$

C

It follows from parts **A** and **B** by mathematical induction, that:

$1 + (1+2) + (1+2+3) + \cdots + (1+2+3+\cdots+n) = \tfrac{1}{6}n(n+1)(n+2)$ for all integers $n \ge 1$.

10 **A**

When $n = 1$,

$$\begin{aligned} \text{RHS} &= 2^1(1) \\ &= 2 \\ &= (1+1) \\ &= \text{LHS} \end{aligned}$$

so the statement is true for $n = 1$.

B

Suppose that $k \ge 1$ is a positive integer for which the statement is true.

That is, suppose $(k+1)(k+2)(k+3) \times \ldots \times 2k$

$$= 2^k(1 \times 3 \times 5 \times \ldots \times (2k-1)) \qquad (**)$$

We prove the statement for $n = k + 1$.

That is, we prove $\big((k+1)+1\big)\big((k+1)+2\big)\big((k+1)+3\big)\times \ldots \times 2(k+1)$

$$= 2^{k+1}(1\times 3\times 5\times \ldots \times (2k-1)(2(k+1)-1)) \qquad (**)$$

$$\begin{aligned}\text{LHS} &= \big((k+1)+1\big)\big((k+1)+2\big)\big((k+1)+3\big)\times \ldots 2k\times(2k+1)\times 2(k+1)\\ &= (k+2)(k+3)\times \ldots \times 2k\times(2k+1)\times 2(k+1)\\ &= 2(k+1)(k+2)(k+3)\times \ldots \times 2k\times(2k+1)\\ &= 2\times 2^k\big(1\times 3\times 5\times \ldots \times (2k-1)\big)\times(2k+1)\end{aligned}$$

by the induction hypothesis $(**)$,

$$\begin{aligned}&= 2^{k+1}\big(1\times 3\times 5\times \ldots \times (2k-1)\times(2k+1)\big)\\ &= 2^{k+1}\big(1\times 3\times 5\times \ldots \times (2k-1)\times(2(k+1)-1)\big)\\ &= \text{RHS}\end{aligned}$$

C

It follows from parts **A** and **B** by mathematical induction, that:

$(n+1)(n+2)(n+3)\times \ldots \times 2n = 2^n(1\times 3\times 5\times \ldots \times (2n-1))$ for all integers $n\geq 1$.

11a **A**

When $n=1$,

$$\begin{aligned}\text{RHS} &= \frac{1}{4}(1-1)(1)(1+1)(1+2)\\ &= 0\end{aligned}$$

$$\begin{aligned}\text{LHS} &= \sum_{r=1}^{1}(r^3-r)\\ &= 1^3-1\\ &= 0\\ &= \text{RHS}\end{aligned}$$

so the statement is true for $n=1$.

B

Suppose that $k\geq 1$ is a positive integer for which the statement is true.

That is, suppose $\sum_{r=1}^{k}(r^3-r) = \frac{1}{4}(k-1)(k)(k+1)(k+2) \qquad (**)$

We prove the statement for $n=k+1$.

That is, we prove $\sum_{r=1}^{k+1}(r^3 - r)$

$$= \frac{1}{4}\big((k+1)-1\big)\big((k+1)\big)\big((k+1)+1\big)\big((k+1)+2\big)$$

$$\begin{aligned}
\text{LHS} &= \sum_{r=1}^{k+1}(r^3 - r) \\
&= (k+1)^3 - (k+1) + \sum_{r=1}^{k}(r^3 - r) \\
&= (k+1)^3 - (k+1) + \tfrac{1}{4}(k-1)(k)(k+1)(k+2)
\end{aligned}$$

by the induction hypothesis $(**)$,

$$\begin{aligned}
&= (k+1)\left[(k+1)^2 - 1 + \frac{1}{4}(k-1)(k)(k+2)\right] \\
&= \frac{1}{4}(k+1)[4(k+1)^2 - 4 + (k-1)(k)(k+2)] \\
&= \frac{1}{4}(k+1)[4(k+1)^2 - 4 + (k^2 - k)(k+2)] \\
&= \frac{1}{4}(k+1)[4(k^2 + 2k + 1) - 4 + k^3 + k^2 - 2k] \\
&= \frac{1}{4}(k+1)[k^3 + 5k^2 + 6k] \\
&= \frac{1}{4}(k+1)k(k^2 + 5k + 6) \\
&= \frac{1}{4}(k+1)k(k+2)(k+3) \\
&= \frac{1}{4}k(k+1)(k+2)(k+3) \\
&= \frac{1}{4}\big((k+1)-1\big)\big((k+1)\big)\big((k+1)+1\big)\big((k+1)+2\big) \\
&= \text{RHS}
\end{aligned}$$

C

It follows from parts **A** and **B** by mathematical induction, that:

$\sum_{r=1}^{n}(r^3 - r) = \frac{1}{4}(n-1)(n)(n+1)(n+2)$ for all integers $n \geq 1$.

11b **A**

When $n = 1$,

$$\begin{aligned}
\text{RHS} &= \frac{1}{2}(1)^3(1+1)^3 \\
&= \frac{8}{2}
\end{aligned}$$

$$
\begin{aligned}
&= 4
\end{aligned}
$$

$$
\begin{aligned}
\text{LHS} &= \sum_{r=1}^{1}(3r^5 + r^3) \\
&= 3 + 1 \\
&= 4 \\
&= \text{RHS}
\end{aligned}
$$

so the statement is true for $n = 1$.

B

Suppose that $k \geq 1$ is a positive integer for which the statement is true.

That is, suppose $\sum_{r=1}^{k}(3r^5 + r^3) = \frac{1}{2}k^3(k+1)^3$ $(**)$

We prove the statement for $n = k + 1$.

That is, we prove $\sum_{r=1}^{k+1}(3r^5 + r^3) = \frac{1}{2}(k+1)^3(k+2)^3$

$$
\begin{aligned}
\text{LHS} &= \sum_{r=1}^{k+1}(3r^5 + r^3) \\
&= 3(k+1)^5 + (k+1)^3 + \sum_{r=1}^{k}(3r^5 + r^3) \\
&= 3(k+1)^5 + (k+1)^3 + \frac{1}{2}k^3(k+1)^3 \qquad \text{by the induction hypothesis } (**) \\
&= (k+1)^3\left[3(k+1)^2 + 1 + \frac{1}{2}k^3\right] \\
&= \frac{1}{2}(k+1)^3[6(k^2 + 2k + 1) + 2 + k^3] \\
&= \frac{1}{2}(k+1)^3[k^3 + 6k^2 + 12k + 8] \\
&= \frac{1}{2}(k+1)^3(k+2)^3 \\
&= \text{RHS}
\end{aligned}
$$

C

It follows from parts **A** and **B** by mathematical induction, that:

$\sum_{r=1}^{n}(3r^5 + r^3) = \frac{1}{2}n^3(n+1)^3$ for all integers $n \geq 1$.

11c **A**

When $n = 1$,

$$\begin{aligned}\text{RHS} &= (1 - 2 + 3) \times 2^{1+1} - 6\\ &= 2 \times 4 - 6\\ &= 2\end{aligned}$$

$$\begin{aligned}\text{LHS} &= \sum_{r=1}^{1} r^2 \times 2^r\\ &= 1 \times 2\\ &= 2\\ &= \text{RHS}\end{aligned}$$

so the statement is true for $n = 1$.

B

Suppose that $k \geq 1$ is a positive integer for which the statement is true.

That is, suppose $\sum_{r=1}^{k} r^2 \times 2^r = (k^2 - 2k + 3) \times 2^{k+1} - 6 \qquad (**)$

We prove the statement for $n = k + 1$.

That is, we prove $\sum_{r=1}^{k+1} r^2 \times 2^r = ((k+1)^2 - 2(k+1) + 3) \times 2^{(k+1)+1} - 6$

$$\begin{aligned}\text{LHS} &= \sum_{r=1}^{k+1} r^2 \times 2^r\\ &= (k+1)^2 \times 2^{k+1} + \sum_{r=1}^{k} r^2 \times 2^r\\ &= (k+1)^2 \times 2^{k+1} + (k^2 - 2k + 3) \times 2^{k+1} - 6\end{aligned}$$

by the induction hypothesis $(**)$,

$$\begin{aligned}&= 2^{k+1}[(k+1)^2 + (k^2 - 2k + 3)] - 6\\ &= 2^{k+1}[k^2 + 2k + 1 + k^2 - 2k + 3] - 6\\ &= 2^{k+1}[2k^2 + 4] - 6\\ &= 2(k^2 + 2) \times 2^k - 6\\ &= (k^2 + 2) \times 2^{k+2} - 6\\ &= (k^2 + 2k + 1 - 2k - 2 + 3) \times 2^{k+2} - 6\\ &= ((k+1)^2 - 2(k+1) + 3) \times 2^{(k+1)+1} - 6\\ &= \text{RHS}\end{aligned}$$

C

It follows from parts **A** and **B** by mathematical induction, that:

$\sum_{r=1}^{n} r^2 \times 2^r = (n^2 - 2n + 3) \times 2^{n+1} - 6$ for all integers $n \geq 1$.

12 **A**

When $n = 1$,

$$\begin{aligned}\text{RHS} &= 1 \times H(1)\\ &= 1 \times 1\\ &= 1\\ &= \text{LHS}\end{aligned}$$

so the statement is true for $n = 1$.

B

Suppose that $k \geq 1$ is a positive integer for which the statement is true.

That is, suppose $k + H(1) + H(2) + \cdots + H(k-1) = k \times H(k)$ $\quad (**)$

We prove the statement for $n = k + 1$.

That is, we prove $(k+1) + H(1) + H(2) + \cdots + H(k-1) + H(k)$

$$= (k+1) \times H(k+1)$$

$$\begin{aligned}\text{LHS} &= (k+1) + H(1) + H(2) + \cdots + H(k-1) + H(k)\\ &= \big(k + H(1) + H(2) + \cdots + H(k-1)\big) + \big(1 + H(k)\big)\\ &= k \times H(k) + \big(1 + H(k)\big) \qquad \text{by the induction hypothesis } (**)\\ &= k \times H(k) + 1 + H(k)\\ &= 1 + (k+1)H(k)\\ &= 1 + (k+1)\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k}\right)\\ &= (k+1)\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} + \frac{1}{k+1}\right)\\ &= (k+1) \times H(k+1)\\ &= \text{RHS}\end{aligned}$$

C

It follows from parts **A** and **B** by mathematical induction, that:

$n + H(1) + H(2) + \cdots + H(n-1) = n \times H(n)$ for all integers $n \geq 1$.

13a

$$\begin{aligned}\text{LHS} &= \frac{\cos\alpha - \cos\alpha\cos 2\beta + \sin\alpha\sin 2\beta}{2\sin\beta}\\ &= \frac{\cos\alpha - \cos\alpha\,(1-2\sin^2\beta) + \sin\alpha \times 2\sin\beta\cos\beta}{2\sin\beta}\\ &= \frac{2\cos\alpha\sin^2\beta + 2\sin\alpha\sin\beta\cos\beta}{2\sin\beta}\\ &= \cos\alpha\sin\beta + \sin\alpha\cos\beta\\ &= \sin(\alpha+\beta)\\ &= \text{RHS}\end{aligned}$$

13b **A**

When $n = 1$,

$$\begin{aligned}\text{RHS} &= \frac{1-\cos 2\theta}{2\sin\theta}\\ &= \frac{1-(1-2\sin^2\theta)}{2\sin\theta}\\ &= \frac{2\sin^2\theta}{2\sin\theta}\\ &= \sin\theta\\ &= \text{LHS}\end{aligned}$$

so the statement is true for $n = 1$.

B

Suppose that $k \geq 1$ is a positive integer for which the statement is true.

That is, suppose $\sin\theta + \sin 3\theta + \cdots + \sin(2k-1)\theta = \frac{1-\cos 2k\theta}{2\sin\theta}$ $(**)$

We prove the statement for $n = k + 1$.

That is, we prove

$$\sin\theta + \sin 3\theta + \cdots + \sin(2k-1)\theta + \sin(2k+1)\theta = \frac{1-\cos 2(k+1)\theta}{2\sin\theta}$$

$$\begin{aligned}\text{LHS} &= \sin\theta + \sin 3\theta + \cdots + \sin(2k-1)\theta + \sin(2k+1)\theta\\ &= \frac{1-\cos 2k\theta}{2\sin\theta} + \sin(2k+1)\theta && \text{by the induction hypothesis } (**),\\ &= \frac{1-\cos 2k\theta + 2\sin\theta\sin(2k+1)\theta}{2\sin\theta}\\ &= \frac{1-\cos 2k\theta + (\cos 2k\theta - \cos(2k\theta+2\theta))}{2\sin\theta} && \text{using part a with } \alpha = 2k\theta \text{ and } \beta = \theta\end{aligned}$$

$$= \frac{1 - \cos(2k\theta + 2\theta)}{2\sin\theta}$$
$$= \frac{1 - \cos(2(k+1)\theta)}{2\sin\theta}$$
$$= \text{RHS}$$

C

It follows from parts **A** and **B** by mathematical induction, that:

$\sin\theta + \sin 3\theta + \cdots + \sin(2n-1)\theta = \frac{1-\cos 2n\theta}{2\sin\theta}$ for all integers $n \geq 1$.

Solutions to Exercise 2B

1 **A**

When $n = 1$, $7^n - 1 = 7^1 - 1 = 7 - 1 = 6$ which is divisible by 6

so the statement is true for $n = 1$.

B

Suppose that $k \geq 1$ is a positive integer for which the statement is true.

That is, suppose $7^k - 1 = 6m$, for some integer m. $(**)$

We prove the statement for $n = k + 1$.

That is, we prove $7^{k+1} - 1$ is divisible by 6.

$$\begin{aligned} 7^{k+1} - 1 &= 7 \times 7^k - 1 \\ &= 7 \times 7^k - 7 + 6 \\ &= 7(7^k - 1) + 6 \\ &= 7(6m) + 6 \qquad \text{by the induction hypothesis } (**) \\ &= 6(7m + 1), \text{ which is divisible by 6 as required.} \end{aligned}$$

C

It follows from parts **A** and **B** by mathematical induction that the statement is true for all positive integers n.

2a **A**

When $n = 1$, $5^n - 1 = 5^1 - 1 = 5 - 1 = 4$ which is divisible by 4

so the statement is true for $n = 1$.

B

Suppose that $k \geq 1$ is a positive integer for which the statement is true.

That is, suppose $5^k - 1 = 4m$, for some integer m. $(**)$

We prove the statement for $n = k + 1$.

That is, we prove $5^{k+1} - 1$ is divisible by 4.

$$\begin{aligned} 5^{k+1} - 1 &= 5 \times 5^k - 1 \\ &= 5 \times 5^k - 5 + 4 \\ &= 5(5^k - 1) + 4 \\ &= 5(4m) + 4 \qquad \text{by the induction hypothesis } (**) \\ &= 4(5m + 1), \text{ which is divisible by 4 as required.} \end{aligned}$$

C

It follows from parts **A** and **B** by mathematical induction that the statement is true for all integers $n \geq 1$.

2b **A**

When $n = 1$, $9^n + 3 = 9^1 + 3 = 12 = 2 \times 6$ which is divisible by 6

so the statement is true for $n = 1$.

B

Suppose that $k \geq 1$ is a positive integer for which the statement is true.

That is, suppose $9^k + 3 = 6m$, for some integer m. $(**)$

We prove the statement for $n = k + 1$.

That is, we prove $9^{k+1} + 3$ is divisible by 6.

$$\begin{aligned} 9^{k+1} + 3 &= 9 \times 9^k + 3 \\ &= 9(9^k + 3) - 9 \times 3 + 3 \\ &= 9(9^k + 3) - 27 + 3 \\ &= 9(6m) - 24 \quad \text{by the induction hypothesis } (**) \\ &= 6(9m - 4), \text{ which is divisible by 6 as required.} \end{aligned}$$

C

It follows from parts **A** and **B** by mathematical induction that the statement is true for all integers $n \geq 1$.

2c **A**

When $n = 1$, $3^2 + 7 = 9 + 7 = 16 = 8 \times 2$ which is divisible by 8

so the statement is true for $n = 1$.

B

Suppose that $k \geq 1$ is a positive integer for which the statement is true.

That is, suppose $3^{2k} + 7 = 8m$, for some integer m. $(**)$

We prove the statement for $n = k + 1$.

That is, we prove $3^{2(k+1)} + 7$ is divisible by 8.

$$\begin{aligned} 3^{2(k+1)} + 7 &= 3^{2k+2} + 7 \\ &= 3^2 \times 3^{2k} + 7 \end{aligned}$$

$$
\begin{aligned}
&= 9 \times 3^{2k} + 7 \\
&= 9 \times (3^{2k} + 7) - 7 \times 9 + 7 \\
&= 9 \times (8m) - 56 \qquad \text{by the induction hypothesis } (**) \\
&= 8 \times 9m - 8 \times 7 \\
&= 8(9m - 7)\text{, which is divisible by 8 as required.}
\end{aligned}
$$

C

It follows from parts **A** and **B** by mathematical induction that the statement is true for all integers $n \geq 1$.

2d **A**

When $n = 1$, $5^2 - 1 = 25 - 1 = 24$ which is divisible by 24

so the statement is true for $n = 1$.

B

Suppose that $k \geq 1$ is a positive integer for which the statement is true.

That is, suppose $5^{2k} - 1 = 24m$, for some integer m. $(**)$

We prove the statement for $n = k + 1$.

That is, we prove $5^{2(k+1)} - 1$ is divisible by 24.

$$
\begin{aligned}
5^{2(k+1)} - 1 &= 5^{2k+2} - 1 \\
&= 5^2 \times 5^{2k} - 1 \\
&= 25 \times 5^{2k} - 1 \\
&= 25 \times (5^{2k} - 1) + 25 - 1 \\
&= 25 \times (24m) + 24 \qquad \text{by the induction hypothesis } (**) \\
&= 24(25m + 1)\text{, which is divisible by 24 as required.}
\end{aligned}
$$

C

It follows from parts **A** and **B** by mathematical induction that the statement is true for all integers $n \geq 1$.

3a

n	0	1	2	3	4
$11^n - 1$	0	10	120	1330	14 640

From this we can hypothesise that the expression will always be divisible by 10.

3b **A**

When $n = 1, 11^0 - 1 = 1 - 1 = 0$ which is divisible by 10

so the statement is true for $n = 1$.

B

Suppose that $k \geq 1$ is a positive integer for which the statement is true.

That is, suppose , $11^k - 1 = 10m$, for some integer m. $\quad (**)$

We prove the statement for $n = k + 1$.

That is, we prove $11^{k+1} - 1$ is divisible by 10.

$$\begin{aligned}11^{k+1} - 1 &= 11 \times 11^k - 1\\ &= 11(11^k - 1) + 11 - 1\\ &= 11(10m) + 10 \qquad \text{by the induction hypothesis } (**)\\ &= 10(11m + 1), \text{ which is divisible by 10 as required.}\end{aligned}$$

C

It follows from parts **A** and **B** by mathematical induction that the statement is true for all integers $n \geq 0$.

4a **A**

When $n = 0, 0^3 + 2(0) = 0 + 0 = 0$ which is divisible by 3

so the statement is true for $n = 0$.

B

Suppose that $k \geq 0$ is an integer for which the statement is true.

That is, suppose $k^3 + 2k = 3m$, for some integer m. $\quad (**)$

We prove the statement for $n = k + 1$.

That is, we prove $(k + 1)^3 + 2(k + 1)$ is divisible by 3.

$(k + 1)^3 + 2(k + 1)$

$$\begin{aligned}&= (k + 1)((k + 1)^2 + 2)\\ &= (k + 1)(k^2 + 2k + 1 + 2)\\ &= (k + 1)(k^2 + 2k + 3)\\ &= k^3 + 2k^2 + 3k + k^2 + 2k + 3\\ &= k^3 + 3k^2 + 5k + 3\\ &= (k^3 + 2k) + 3k^2 + 3k + 3\end{aligned}$$

$= 3m + 3k^2 + 3k + 3$ by the induction hypothesis $(**)$
$= 3(m + k^2 + k + 1)$ which is divisible by 3.

C

It follows from parts **A** and **B** by mathematical induction that the statement is true for all integers $n \geq 0$.

4b **A**

When $n = 0$, $8^0 - 7(0) + 6 = 1 + 0 + 6 = 7$ which is divisible by 7

so the statement is true for $n = 0$.

B

Suppose that $k \geq 0$ is an integer for which the statement is true.

That is suppose , $8^k - 7k + 6 = 7m$, for some integer m. $(**)$

We prove the statement for $n = k + 1$.

That is, we prove $8^{k+1} - 7(k + 1) + 6$ is divisible by 7.

$8^{k+1} - 7(k + 1) + 6$
$= 8 \times 8^k - 7k - 7 + 6$
$= 8 \times 8^k - 7k - 1$
$= 8(8^k - 7k + 6) + 56k - 48 - 7k - 1$
$= 8(8^k - 7k + 6) + 49k - 49$
$= 8(7m) + 7(7k - 7)$ by the induction hypothesis $(**)$
$= 7(8m + 7k - 7)$ which is divisible by 7.

C

It follows from parts **A** and **B** by mathematical induction that the statement is true for all integers $n \geq 0$.

4c **A**

When $n = 0$, $9(9^0 - 1) - 8(0) = 9 \times 0 = 0$ which is divisible by 64

so the statement is true for $n = 0$.

B

Suppose that $k \geq 0$ is an integer for which the statement is true.

That is, suppose $9(9^k - 1) - 8k = 64m$, for some integer m.

Note that rearranging this gives $9(9^k - 1) = 64m + 8k$. $(**)$

We prove the statement for $n = k + 1$.

That is, we prove $9(9^{k+1} - 1) - 8(k + 1)$ is divisible by 64.

$9(9^{k+1} - 1) - 8(k + 1)$
$= 9(9 \times 9^k - 1) - 8(k + 1)$
$= 81 \times 9^k - 9 - 8k - 8$
$= 81 \times 9^k - 17 - 8k$
$= 81 \times 9^k - 81 + 64 - 8k$
$= 81(9^k - 1) + 64 - 8k$
$= 9 \times 9(9^k - 1) + 64 - 8k$
$= 9(64m + 8k) + 64 - 8k$ by the induction hypothesis $(**)$
$= 9 \times 64m + 9 \times 8k + 64 - 8k$
$= 9 \times 64m + 8 \times 8k + 64$
$= 9 \times 64m + 64k + 64$
$= 64(9m + 1k + 1)$ which is divisible by 64

C

It follows from parts **A** and **B** by mathematical induction that the statement is true for all integers $n \geq 0$.

5a **A**

When $n = 0$, $5^0 + 2 \times 11^0 = 1 + 2 = 3$ which is divisible by 3

so the statement is true for $n = 0$.

B

Suppose that $k \geq 0$ is an integer for which the statement is true.

That is, suppose $5^k + 2 \times 11^k = 3m$, for some integer m.

Note that rearranging this gives $5^k = 3m - 2 \times 11^k$. $(**)$

We prove the statement for $n = k + 1$.

That is, we prove $5^{k+1} + 2 \times 11^{k+1}$ is divisible by 3.

$5^{k+1} + 2 \times 11^{k+1}$
$= 5 \times 5^k + 2 \times 11^{k+1}$
$= 5 \times (3m - 2 \times 11^k) + 2 \times 11^{k+1}$ by the induction hypothesis $(**)$
$= 15m - 10 \times 11^k + 2 \times 11 \times 11^k$
$= 15m - 10 \times 11^k + 22 \times 11^k$
$= 15m + 12 \times 11^k$
$= 3(5m + 4 \times 11^k)$ which is divisible by 3.

C

It follows from parts **A** and **B** by mathematical induction that the statement is true for all integers $n \geq 0$.

5b **A**

When $n = 0$, $3^{3(0)} + 2^{0+2} = 1 + 2^2 = 1 + 4 = 5$ which is divisible by 5

so the statement is true for $n = 0$.

B

Suppose that $k \geq 0$ is an integer for which the statement is true.

That is, suppose $3^{3k} + 2^{k+2} = 5m$, for some integer m.

Note that rearranging this gives $3^{3k} = 5m - 2^{k+2}$. $(**)$

We prove the statement for $n = k + 1$.

That is, we prove $3^{3(k+1)} + 2^{(k+1)+2}$ is divisible by 5.

$3^{3(k+1)} + 2^{(k+1)+2}$
$= 3^{3k+3} + 2^{k+3}$
$= 3^3 \times 3^{3k} + 2 \times 2^{k+2}$
$= 27 \times 3^{3k} + 2 \times 2^{k+2}$
$= 27(5m - 2^{k+2}) + 2 \times 2^{k+2}$ by the induction hypothesis $(**)$
$= 27 \times 5m - 27 \times 2^{k+2} + 2 \times 2^{k+2}$
$= 27 \times 5m - 25 \times 2^{k+2}$
$= 5(27 \times m - 5 \times 2^{k+2})$ which is divisible by 5.

C

It follows from parts **A** and **B** by mathematical induction that the statement is true for all integers $n \geq 0$.

5c **A**

When $n = 0$, $11^{0+2} + 12^{0+1} = 11^2 + 12 = 133$ which is divisible by 133.

So, the statement is true for $n = 0$.

B

Suppose that $k \geq 0$ is an integer for which the statement is true

That is, suppose $11^{k+2} + 12^{2k+1} = 133m$, for some integer m.

Note that rearranging this gives $11^{k+2} = 133m - 12^{2k+1}$. $(**)$

We prove the statement for $n = k + 1$.

That is, we prove $11^{(k+1)+2} + 12^{2(k+1)+1}$ is divisible by 133.

$11^{(k+1)+2} + 12^{2(k+1)+1}$
$= 11^{k+3} + 12^{2k+3}$
$= 11 \times 11^{k+2} + 12^2 \times 12^{2k+1}$
$= 11 \times (133m - 12^{2k+1}) + 12^2 \times 12^{2k+1}$ by the induction hypothesis $(**)$
$= 11 \times 133m - 11 \times 12^{2k+1} + 12^2 \times 12^{2k+1}$
$= 11 \times 133m + (12^2 - 11)12^{2k+1}$
$= 11 \times 133m + 133 \times 12^{2k+1}$
$= 133(11m + 12^{2k+1})$ which is divisible by 133.

C

It follows from parts **A** and **B** by mathematical induction that the statement is true for all integers $n \geq 0$.

6 **A**

When $n = 1$, $x^1 - 1 = x - 1$ which is divisible by $x - 1$.

So, the statement is true for $n = 1$.

B

Suppose that $k \geq 1$ is a positive integer for which the statement is true.

That is, suppose $x^k - 1 = m(x - 1)$, for some integer m. $(**)$

We prove the statement for $n = k + 1$.

That is, we prove $x^{k+1} - 1$ is divisible by $x - 1$.

$x^{k+1} - 1$
$= x \times x^k - 1$
$= x(x^k - 1) + x - 1$
$= xm(x - 1) + (x - 1)$ by the induction hypothesis $(**)$
$= (x - 1)(xm + 1)$,which is divisible by $x - 1$

C

It follows from parts **A** and **B** by mathematical induction that the statement is true for all integers $n \geq 1$.

7 Suppose that $k \geq 1$ is a positive integer for which the statement is true.

That is, suppose $8k^2 + 14 = 4m$, for some integer m. $(**)$

We prove the statement for $n = k + 1$.

That is, we prove $8(k+1)^2 + 14$ is divisible by 4.

$8(k+1)^2 + 14$

$= 8(k^2 + 2k + 1) + 14$
$= 8k^2 + 16k + 8 + 14$
$= (8k^2 + 14) + 16k + 8$
$= 4m + 16k + 8$ by the induction hypothesis $(**)$
$= 4(m + 4k + 2)$ which is divisible by 4.

Now we show that $8n^2 + 14$ is never divisible by 4 if n is an integer.

Firstly, we show by induction that $8n^2 + 14 = 4a + 2$ for some integer a.

When $n = 1$, $8(1) + 14 = 22 = 4 \times 5 + 2$ which is in the form $4a + 2$.

So, the statement is true for $n = 1$.

Suppose that $k \geq 1$ is a positive integer for which the statement is true.

That is, suppose $8k^2 + 14 = 4m + 2$, for some integer m. $(**)$

We prove the statement for $n = k + 1$.

That is, we prove $8(k+1)^2 + 14 = 4a + 2$ for some integer a.

$8(k+1)^2 + 14$

$= 8(k^2 + 2k + 1) + 14$
$= 8k^2 + 16k + 8 + 14$
$= (8k^2 + 14) + 16k + 8$
$= 4m + 2 + 16k + 8$ by the induction hypothesis $(**)$
$= 4(m + 4k + 2) + 2$ which is in the form $4a + 2$ where a is an integer

$4a + 2$ always has remainder 2 upon division by 4 and hence is never divisible by 4 (assuming that n was a whole number).

This shows that the first step of the proof, showing true for a base case such as $n = 0$ or $n = 1$, is necessary.

8a Show that the statement is true for $n = 0$.

$f(0) = 0 - 0 + 17 = 17$ which is prime.

Suppose that $k \geq 0$ is an integer for which the statement is true.

That is, suppose $k^2 - k + 17 = P$, where P is a prime. $(**)$

We attempt to prove the statement for $n = k + 1$.

That is, we attempt to prove $(k+1)^2 - (k+1) + 17$ is prime.

$(k+1)^2 - (k+1) + 17$
$= k^2 + 2k + 1 - k - 1 + 17$
$= (k^2 - k + 17) + 2k$
$= P + 2k$ by the induction hypothesis $(**)$

And now we are stuck!

Hence, we require the step showing that the statement is true for $n = k + 1$ given that it is true for any $n = k$. Below we list the first 17 outputs of the function. Note that the first 16 are prime.

n	0	1	2	3	4	5	6	7	8	9
$f(n)$	17	17	19	23	29	37	47	59	73	89

n	10	11	12	13	14	15	16	17
$f(n)$	107	127	149	173	199	227	257	289

Note that $289 = 17^2$ and hence it is not prime.

8b Show that the statement is true for $n = 0$.

$f(0) = 0 + 0 + 41 = 41$ which is prime.

Suppose that $k \geq 0$ is an integer for which the statement is true.

That is, suppose $k^2 + k + 41 = P$, where P is a prime. $(**)$

We attempt to prove the statement for $n = k + 1$.

That is, we attempt to prove $(k+1)^2 + (k+1) + 41$ is prime.

$(k+1)^2 + (k+1) + 41$
$= k^2 + 2k + 1 + k + 1 + 41$
$= (k^2 + k + 41) + 2k + 2$

$= P + 2k + 2$ by the induction hypothesis $(**)$

And now we are stuck!

Hence, we require the step showing that the statement is true for $n = k + 1$ given that it is true for any $n = k$.

Note that $f(40) = 1681 = 41^2$ and hence is not prime, thus providing us with a counter-example.

9 **A**

When $n = 0$, $3^{2^0} - 1 = 3^1 - 1 = 3 - 1 = 2$ which is divisible by 2^1

so the statement is true for $n = 0$.

B

Suppose that $k \geq 0$ is an integer for which the statement is true.

That is, suppose $3^{2^k} - 1 = 2^{k+1}m$, for some positive integer m.

Note that rearranging this gives $3^{2^k} = 2^{k+1}m + 1$. $(**)$

We prove the statement for $n = k + 1$.

That is, we prove $3^{2^{k+1}} - 1$ is divisible by 2^{k+2}.

$3^{2^{k+1}} - 1$
$= 3^{2\times 2^k} - 1$
$= \left(3^{2^k}\right)^2 - 1$
$= (2^{k+1}m + 1)^2 - 1$ by the induction hypothesis $(**)$
$= 2^{2k+2}m^2 + 2^{k+2}m + 1 - 1$
$= 2^{k+2}(2^k m^2 + m)$, which is divisible by 2^{k+2}.

C

It follows from parts **A** and **B** by mathematical induction that the statement is true for all integers $n \geq 0$.

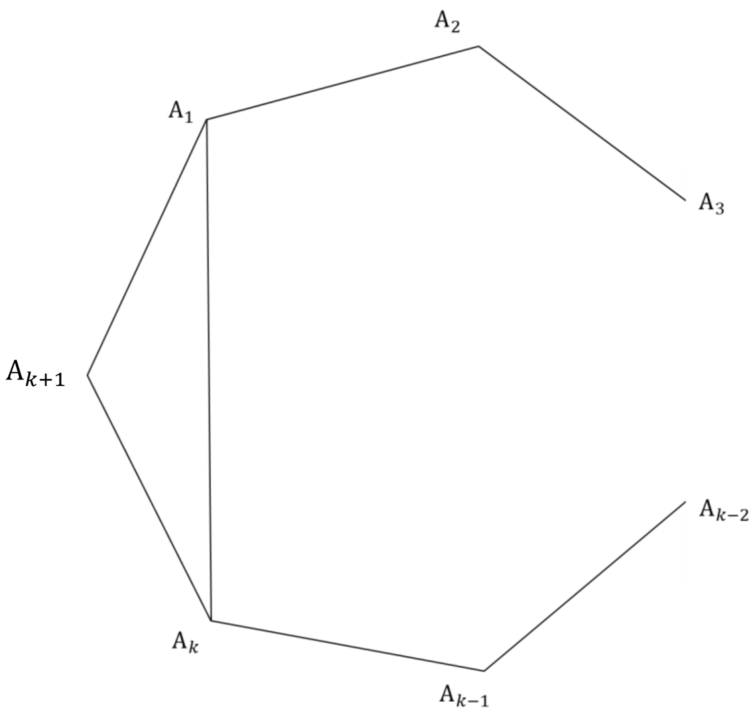
10 **A**

When $n = 3$, $(3 - 2) \times 180° = 180°$ which is the angle sum of a triangle or 1 straight angle so the statement is true for $n = 3$.

B

Assume that a k-gon, where $k \geq 3$, has angle sum $(k-2) \times 180°$. $(**)$

Prove that a $(k+1)$-gon has angle sum $(k-1) \times 180°$.



The angle sum of the $(k+1)$-gon $A_1A_2 \ldots A_kA_{k+1}$
$=$ the angle sum of the k-gon $A_1A_2 \ldots A_k$ + the angle sum of $\Delta A_1A_kA_{k+1}$

$= (k-2) \times 180° + 180°$ by the induction hypothesis $(**)$

$= (k-1) \times 180°$ as required.

C

It follows from parts **A** and **B** by mathematical induction that the statement is true for all integers $n \geq 3$.

11 **A**

A 0-member set is the empty set which has $2^0 = 1$ subset (itself) so the result is true for $n = 0$.

B

Assume that a k-member set has 2^k subsets. $(**)$

Prove that a $(k+1)$-member set has 2^{k+1} subsets.

Suppose we have a k-member set, and we add a new member.

Then each subset of the k-member set (there are 2^k of these by $(**)$) is also a subset of the $(k+1)$-member set, and we get 2^k new subsets when we add the new member to each of the previous 2^k subsets.

So the resulting number of subsets for the $(k+1)$-member set is

$2^k + 2^k = 2 \times 2^k = 2^{k+1}$ as required.

C

It follows from parts **A** and **B** by mathematical induction that the statement is true for all integers $n \geq 0$.

12 **A**

$\frac{d}{dx}(x^1) = 1 = 1x^0$ so the result is true for $n = 1$.

B

Assume that $\frac{d}{dx}(x^k) = kx^{k-1}$. $(**)$

Prove that $\frac{d}{dx}(x^{k+1}) = (k+1)x^k$.

$$\begin{aligned}
\text{LHS} &= \frac{d}{dx}(x \times x^k) \\
&= x \times \tfrac{d}{dx}(x^k) + x^k \times \tfrac{d}{dx}(x) && \text{by the product rule} \\
&= x \times kx^{k-1} + x^k \times 1 && \text{by the induction hypothesis } (**) \\
&= kx^k + x^k \\
&= (k+1)x^k \\
&= \text{RHS}
\end{aligned}$$

C

It follows from parts **A** and **B** by mathematical induction that the statement is true for all integers $n \geq 1$.

13a **A**

When $n = 0, n^2 + 2n = 0^2 + 2(0) = 0$ which is a multiple of 8

so the result is true for $n = 0$.

B

Assume that the result is true for $n = k$, where k is even.

That is, assume that $k^2 + 2k = 8m$, where m is a positive integer. $(**)$

Prove the result is true for $n = k + 2$.

That is, prove that $(k + 2)^2 + 2(k + 2)$ is a multiple of 8.

$(k + 2)^2 + 2(k + 2)$
$= k^2 + 4k + 4 + 2k + 4$
$= (k^2 + 2k) + 4k + 8$
$= 8m + 4k + 8$ by the induction hypothesis $(**)$
$= 8m + 4 \times 2l + 8$ (k is even, so $k = 2l$ for some integer l)
$= 8(m + l + 1)$ which is a multiple of 8.

C

It follows from parts **A** and **B** by mathematical induction that the statement is true for all even integers $n \geq 0$.

13b **A**

When $n = 1, 3^1 + 7^1 = 10$ which is divisible by 10

so the result is true for $n = 1$.

B

Assume that the result is true for $n = k$, where k is odd.

That is, assume that $3^k + 7^k = 10m$, where m is a positive integer.

This can be rearranged as $3^k = 10m - 7^k$. $(**)$

Prove the result is true for $n = k + 2$.

That is, prove that $3^{k+2} + 7^{k+2}$ is a multiple of 10.

$3^{k+2} + 7^{k+2}$
$= 3^2 \times 3^k + 7^2 \times 7^k$
$= 9 \times 3^k + 49 \times 7^k$
$= 9 \times (10m - 7^k) + 49 \times 7^k$ by the induction hypothesis $(**)$
$= 90m - 9 \times 7^k + 49 \times 7^k$

$= 90m + 40 \times 7^k$
$= 10(9m + 4 \times 7^k)$ which is divisible by 10.

C

It follows from parts **A** and **B** by mathematical induction that the statement is true for all odd integers $n \geq 1$.

Solutions to Chapter 2 Review

1a **A**

When $n = 1$,

$$\begin{aligned} \text{RHS} &= 1(2(1) - 1) \\ &= 1 \\ &= \text{LHS} \end{aligned}$$

so the statement is true for $n = 1$.

B

Suppose that $k \geq 1$ is a positive integer for which the statement is true.

That is, suppose $1 + 5 + 9 + \cdots + (4k - 3) = k(2k - 1)$ $\quad (**)$

We prove the statement for $n = k + 1$.

That is, we prove

$$1 + 5 + 9 + \cdots + (4k - 3) + (4(k + 1) - 3) = (k + 1)(2(k + 1) - 1)$$

$$\begin{aligned} \text{LHS} &= 1 + 5 + 9 + \cdots + (4k - 3) + (4(k + 1) - 3) \\ &= 1 + 5 + 9 + \cdots + (4k - 3) + (4k + 4 - 3) \\ &= 1 + 5 + 9 + \cdots + (4k - 3) + (4k + 1) \\ &= k(2k - 1) + (4k + 1) \qquad \text{by the induction hypothesis } (**) \\ &= 2k^2 - k + 4k + 1 \\ &= 2k^2 + 3k + 1 \\ &= (k + 1)(2k + 1) \\ &= (k + 1)(2(k + 1) - 1) \\ &= \text{RHS} \end{aligned}$$

C

It follows from parts **A** and **B** by mathematical induction, that:

$1 + 5 + 9 + \cdots + (4n - 3) = n(2n - 1)$ for all integers $n \geq 1$.

1b **A**

When $n = 1$,

$$\begin{aligned} \text{RHS} &= \frac{1}{6}(7^1 - 1) \\ &= \frac{6}{6} \\ &= 1 \\ &= \text{LHS} \end{aligned}$$

so the statement is true for $n = 1$.

B

Suppose that $k \geq 1$ is a positive integer for which the statement is true.

That is, suppose $1 + 7 + 7^2 + \cdots + 7^{k-1} = \frac{1}{6}(7^k - 1)$ $\qquad (**)$

We prove the statement for $n = k + 1$.

That is, we prove $1 + 7 + 7^2 + \cdots + 7^{k-1} + 7^{k+1-1} = \frac{1}{6}(7^{k+1} - 1)$

$$\begin{aligned}
\text{LHS} &= 1 + 7 + 7^2 + \cdots + 7^{k-1} + 7^{k+1-1} \\
&= 1 + 7 + 7^2 + \cdots + 7^{k-1} + 7^k \\
&= \tfrac{1}{6}(7^k - 1) + 7^k \qquad \text{by the induction hypothesis } (**) \\
&= \frac{1}{6}(7^k - 1 + 6 \times 7^k) \\
&= \frac{1}{6}(7 \times 7^k - 1) \\
&= \frac{1}{6}(7^{k+1} - 1) \\
&= \text{RHS}
\end{aligned}$$

C

It follows from parts **A** and **B** by mathematical induction, that:

$1 + 7 + 7^2 + \cdots + 7^{n-1} = \frac{1}{6}(7^n - 1)$ for all integers $n \geq 1$.

1c **A**

When $n = 1$,

$$\begin{aligned}
\text{RHS} &= \frac{1}{6}(1)(1 + 1)(2 \times 1 + 13) \\
&= \frac{1}{6}(1)(2)(15) \\
&= 5 \\
&= \text{LHS}
\end{aligned}$$

so the statement is true for $n = 1$.

B

Suppose that $k \geq 1$ is a positive integer for which the statement is true.

That is, suppose $1 \times 5 + 2 \times 6 + 3 \times 7 + \cdots + k(k+4)$

$$= \tfrac{1}{6}k(k+1)(2k+13) \qquad (**)$$

We prove the statement for $n = k + 1$.

That is, we prove $1 \times 5 + 2 \times 6 + 3 \times 7 + \cdots + k(k+4) + (k+1)\big((k+1)+4\big)$

$$= \tfrac{1}{6}(k+1)((k+1)+1)(2(k+1)+13)$$

$$\begin{aligned}
\text{LHS} &= 1 \times 5 + 2 \times 6 + 3 \times 7 + \cdots + k(k+4) + (k+1)((k+1)+4) \\
&= \tfrac{1}{6}k(k+1)(2k+13) + (k+1)((k+1)+4)
\end{aligned}$$

by the induction hypothesis $(**)$

$$\begin{aligned}
&= \frac{1}{6}(k+1)\big[k(2k+13) + 6\big((k+1)+4\big)\big] \\
&= \frac{1}{6}(k+1)[k(2k+13) + 6(k+5)] \\
&= \frac{1}{6}(k+1)[2k^2 + 13k + 6k + 30] \\
&= \frac{1}{6}(k+1)[2k^2 + 19k + 30] \\
&= \frac{1}{6}(k+1)(k+2)(2k+15) \\
&= \frac{1}{6}(k+1)\big((k+1)+1\big)(2(k+1)+13) \\
&= \text{RHS}
\end{aligned}$$

C

It follows from parts **A** and **B** by mathematical induction, that:

$1 \times 5 + 2 \times 6 + 3 \times 7 + \cdots + n(n+4) = \tfrac{1}{6}n(n+1)(2n+13)$ for all integers $n \geq 1$.

1d **A**

When $n = 1$,

$$\begin{aligned}
\text{RHS} &= \frac{1}{2(1+2)} \\
&= \frac{1}{2(3)}
\end{aligned}$$

$$\begin{aligned}&= \frac{1}{6}\\ &= \text{LHS}\end{aligned}$$

so the statement is true for $n = 1$.

B

Suppose that $k \geq 1$ is a positive integer for which the statement is true.

That is, suppose $\frac{1}{2\times3} + \frac{1}{3\times4} + \frac{1}{4\times5} + \cdots + \frac{1}{(k+1)(k+2)} = \frac{k}{2(k+2)}$ $\quad(**)$

We prove the statement for $n = k + 1$.

That is, we prove $\frac{1}{2\times3} + \frac{1}{3\times4} + \frac{1}{4\times5} + \cdots + \frac{1}{(k+1)(k+2)} + \frac{1}{((k+1)+1)((k+1)+2)} = \frac{(k+1)}{2((k+1)+2)}$

$$\begin{aligned}\text{LHS} &= \frac{1}{2\times3} + \frac{1}{3\times4} + \frac{1}{4\times5} + \cdots + \frac{1}{(k+1)(k+2)} + \frac{1}{((k+1)+1)((k+1)+2)}\\
&= \frac{k}{2(k+2)} + \frac{1}{((k+1)+1)((k+1)+2)} \qquad \text{by the induction hypothesis } (**)\\
&= \frac{k}{2(k+2)} + \frac{2}{2((k+1)+1)((k+1)+2)}\\
&= \frac{k}{2(k+2)} + \frac{2}{2(k+2)(k+3)}\\
&= \frac{k(k+3)}{2(k+2)(k+3)} + \frac{2}{2(k+2)(k+3)}\\
&= \frac{k^2+3k+2}{2(k+2)(k+3)}\\
&= \frac{(k+1)(k+2)}{2(k+2)(k+3)}\\
&= \frac{(k+1)}{2(k+3)}\\
&= \frac{(k+1)}{2((k+1)+2)}\\
&= \text{RHS}\end{aligned}$$

C

It follows from parts **A** and **B** by mathematical induction, that:

$\frac{1}{2\times3} + \frac{1}{3\times4} + \frac{1}{4\times5} + \cdots + \frac{1}{(n+1)(n+2)} = \frac{n}{2(n+2)}$ for all integers $n \geq 1$.

1e **A**

When $n = 1$,

$$\begin{aligned}\text{RHS} &= 2 - \frac{1+2}{2^1}\\ &= 2 - \frac{3}{2}\\ &= \frac{1}{2}\\ &= \text{LHS}\end{aligned}$$

so the statement is true for $n = 1$.

B

Suppose that $k \geq 1$ is a positive integer for which the statement is true.

That is, suppose $\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \cdots + \frac{k}{2^k} = 2 - \frac{k+2}{2^k}$ $\qquad (**)$

We prove the statement for $n = k + 1$.

That is, we prove $\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \cdots + \frac{k}{2^k} + \frac{k+1}{2^{k+1}} = 2 - \frac{k+3}{2^{k+1}}$

$$\begin{aligned}\text{LHS} &= \frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \cdots + \frac{k}{2^k} + \frac{k+1}{2^{k+1}}\\ &= 2 - \frac{k+2}{2^k} + \frac{k+1}{2^{k+1}} \qquad \text{by the induction hypothesis } (**)\\ &= 2 - \frac{2(k+2) - (k+1)}{2^{k+1}}\\ &= 2 - \frac{2k + 4 - k - 1}{2^{k+1}}\\ &= 2 - \frac{k+3}{2^{k+1}}\\ &= \text{RHS}\end{aligned}$$

C

It follows from parts **A** and **B** by mathematical induction, that:

$\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \cdots + \frac{n}{2^n} = 2 - \frac{n+2}{2^n}$ for all integers $n \geq 1$.

2a **A**

When $n = 1$, $7^{2-1} + 5 = 7 + 5 = 12$ which is divisible by 12

so the statement is true for $n = 1$.

B

Suppose that $k \geq 1$ is a positive integer for which the statement is true.

That is, suppose $7^{2k-1} + 5 = 12m$, for some integer m.

Note that rearranging this gives $7^{2k-1} = 12m - 5$. $(**)$

We prove the statement for $n = k + 1$.

That is, we prove $7^{2(k+1)-1} + 5$ is divisible by 12.

$7^{2(k+1)-1} + 5$
$= 7^{2k+2-1} + 5$
$= 7^{2k+1} + 5$
$= 7^2 \times 7^{2k-1} + 5$
$= 7^2(12m - 5) + 5$ by the induction hypothesis $(**)$
$= 49(12m - 5) + 5$
$= 49 \times 12m - 5 \times 49 + 5 \times 1$
$= 49 \times 12m + 5(1 - 49)$
$= 49 \times 12m - 5 \times 12 \times 4$
$= 12(49m - 20)$ which is divisible by 12.

C

It follows from parts **A** and **B** by mathematical induction that the statement is true for all integers $n \geq 1$.

2b **A**

When $n = 0$, $2^0 + 6(0) - 1 = 1 + 0 - 1 = 0$ which is divisible by 9

so the statement is true for $n = 0$.

B

Suppose that $k \geq 0$ is an integer for which the statement is true.

That is, suppose $2^{2k} + 6k - 1 = 9m$, for some integer m.

Note that rearranging this gives $2^{2k} = 9m - 6k + 1$. $(**)$

We prove the statement for $n = k + 1$.

That is, we prove $2^{2(k+1)} + 6(k + 1) - 1$ is divisible by 9.

$2^{2(k+1)} + 6(k + 1) - 1$
$= 2^{2k+2} + 6k + 6 - 1$
$= 2^2 \times 2^{2k} + 6k + 5$

$= 2^2 \times (9m - 6k + 1) + 6k + 5$ by the induction hypothesis $(**)$
$= 36m - 24k + 4 + 6k + 5$
$= 36m - 18k + 9$
$= 9(4m - 2k + 1)$ which is divisible by 9.

C

It follows from parts **A** and **B** by mathematical induction that the statement is true for all integers $n \geq 0$.

2c **A**

When $n = 1$, $2^4 + 5^1 = 16 + 5 = 21$ which is divisible by 21.

so the statement is true for $n = 1$.

B

Suppose that $k \geq 1$ is a positive integer for which the statement is true.

That is, suppose $2^{2k+2} + 5^{2k-1} = 21m$, for some integer m.

Note that rearranging this gives $2^{2k+2} = 21m - 5^{2k-1}$. $(**)$

We prove the statement for $n = k + 1$.

That is, we prove $2^{2(k+1)+2} + 5^{2(k+1)-1}$ is divisible by 21.

$2^{2(k+1)+2} + 5^{2(k+1)-1}$
$= 2^{2k+2+2} + 5^{2k-1+2}$
$= 2^2 \times 2^{2k+2} + 5^2 \times 5^{2k-1}$
$= 2^2 \times (21m - 5^{2k-1}) + 5^2 \times 5^{2k-1}$ by the induction hypothesis $(**)$
$= 4 \times (21m - 5^{2k-1}) + 5^2 \times 5^{2k-1}$
$= 4 \times 21m - 4 \times 5^{2k-1} + 5^2 \times 5^{2k-1}$
$= 4 \times 21m - 4 \times 5^{2k-1} + 25 \times 5^{2k-1}$
$= 4 \times 21m + 21 \times 5^{2k-1}$
$= 21(4m + 5^{2k-1})$ which is divisible by 21.

C

It follows from parts **A** and **B** by mathematical induction that the statement is true for all integers $n \geq 1$.

2d **A**

When $n = 0, 0 + 1^3 + 2^3 = 0 + 1 + 8 = 9$ which is divisible by 9

so the statement is true for $n = 0$.

B

Suppose that $k \geq 0$ is an integer for which the statement is true.

That is, suppose $k^3 + (k+1)^3 + (k+2)^3 = 9m$, for some integer m.

Note that rearranging this gives $(k+1)^3 + (k+2)^3 = 9m - k^3$ $\quad (**)$

We prove the statement for $n = k + 1$.

That is, we prove $(k+1)^3 + ((k+1)+1)^3 + ((k+1)+2)^3$ is divisible by 9.

$(k+1)^3 + ((k+1)+1)^3 + ((k+1)+2)^3$
$= (k+1)^3 + (k+2)^3 + (k+3)^3$
$= 9m - k^3 + (k+3)^3$ by the induction hypothesis $(**)$
$= 9m - k^3 + k^3 + 9k^2 + 27k + 27$
$= 9m + 9k^2 + 27k + 27$
$= 9(m + k^2 + 3k + 3)$ which is divisible by 9.

C

It follows from parts **A** and **B** by mathematical induction that the statement is true for all integers $n \geq 0$.

3a

n	0	1	2	3
$2^{3n} - 3^n$	0	5	55	485

The expression is always divisible by 5 for all whole numbers $n \geq 0$.

3b **A**

When $n = 0$, $2^0 - 3^0 = 1 - 1 = 0$ which is divisible by 5

so the statement is true for $n = 0$.

B

Suppose that $k \geq 0$ is an integer for which the statement is true.

That is, suppose $2^{3k} - 3^k = 5m$, for some integer m.

Note that rearranging this gives $2^{3k} = 5m + 3^k$. $\quad (**)$

We prove the statement for $n = k + 1$.

That is, we prove $2^{3(k+1)} - 3^{k+1}$ is divisible by 5.

$2^{3(k+1)} - 3^{k+1}$
$= 2^{3k+3} - 3^{k+1}$
$= 2^3 \times 2^{3k} - 3 \times 3^k$
$= 2^3 \times (5m + 3^k) - 3 \times 3^k$ by the induction hypothesis $(**)$
$= 8 \times (5m + 3^k) - 3 \times 3^k$
$= 5 \times 8m + 8 \times 3^k - 3 \times 3^k$
$= 5 \times 8m + 5 \times 3^k$
$= 5(8m + 3^k)$ which is divisible by 5.

C

It follows from parts **A** and **B** by mathematical induction that the statement is true for all whole numbers $n \geq 0$.

4a **A**

When $n = 1$,

$$\begin{aligned}\text{RHS} &= (1+1)! - 1\\ &= 2! - 1\\ &= 2 - 1\\ &= 1\end{aligned}$$

$$\begin{aligned}\text{LHS} &= \sum_{r=1}^{1} r \times r!\\ &= 1 \times 1!\\ &= 1\\ &= \text{RHS}\end{aligned}$$

so the statement is true for $n = 1$.

B

Suppose that $k \geq 1$ is a positive integer for which the statement is true.

That is, suppose $\sum_{r=1}^{k} r \times r! = (k+1)! - 1$ $(**)$

We prove the statement for $n = k + 1$.

That is, we prove $\sum_{r=1}^{k+1} r \times r! = ((k+1)+1)! - 1$.

$$\text{LHS} = \sum_{r=1}^{k+1} r \times r!$$

$$= \sum_{r=1}^{k} r \times r! + (k+1) \times (k+1)!$$
$$= (k+1)! - 1 + (k+1) \times (k+1)! \quad \text{by the induction hypothesis } (**),$$
$$= (k+1)! + (k+1) \times (k+1)! - 1$$
$$= (1 + (k+1))(k+1)! - 1$$
$$= (k+2)(k+1)! - 1$$
$$= (k+2)! - 1$$
$$= ((k+1)+1)! - 1$$
$$= \text{RHS}$$

C

It follows from parts **A** and **B** by mathematical induction, that:

$\sum_{r=1}^{n} r \times r! = (n+1)! - 1$ for all integers $n \geq 1$.

4b **A**

When $n = 1$,

$$\text{RHS} = 1 - \frac{1}{1!}$$
$$= 1 - 1$$
$$= 0$$
$$\text{LHS} = \sum_{r=1}^{1} \frac{r-1}{r!}$$
$$= \frac{1-1}{1!}$$
$$= 0$$
$$= \text{RHS}$$

so the statement is true for $n = 1$.

B

Suppose that $k \geq 1$ is a positive integer for which the statement is true.

That is, suppose $\sum_{r=1}^{k} \frac{r-1}{r!} = 1 - \frac{1}{k!}$ $\qquad (**)$

We prove the statement for $n = k + 1$.

That is, we prove $\sum_{r=1}^{k+1} \frac{r-1}{r!} = 1 - \frac{1}{(k+1)!}$.

$$\text{LHS} = \sum_{r=1}^{k+1} \frac{r-1}{r!}$$

$$= \sum_{r=1}^{k} \frac{r-1}{r!} + \frac{(k+1)-1}{(k+1)!}$$

$= 1 - \frac{1}{k!} + \frac{(k+1)-1}{(k+1)!}$ by the induction hypothesis $(**)$,

$$= 1 - \frac{k+1}{(k+1)!} + \frac{(k+1)-1}{(k+1)!}$$

$$= 1 - \frac{k+1-k}{(k+1)!}$$

$$= 1 - \frac{1}{(k+1)!}$$

$= \text{RHS}$

C

It follows from parts **A** and **B** by mathematical induction, that:

$\sum_{r=1}^{n} \frac{r-1}{r!} = 1 - \frac{1}{n!}$ for all integers $n \geq 1$.

The limiting sum of the series in part b is:

$$\lim_{n\to\infty}\left(1 - \frac{1}{n!}\right)$$

$= 1$

5 **A**

When $n = 1$,

$$\text{RHS} = \frac{1}{2}(1)(6-3-1)$$

$$= \frac{2}{2}$$

$= 1$

$\text{LHS} = 1^2$

$= 1$

$= \text{RHS}$

so the statement is true for $n = 1$.

B

Suppose that $k \geq 1$ is a positive integer for which the statement is true.

That is, suppose $1^2 + 4^2 + 7^2 + \cdots + (3k-2)^2 = \frac{1}{2}k(6k^2 - 3k - 1) \quad (**)$

We prove the statement for $n = k + 1$.

That is, we prove $1^2 + 4^2 + 7^2 + \cdots + (3k-2)^2 + (3(k+1)-2)^2$

$$= \frac{1}{2}(k+1)(6(k+1)^2 - 3(k+1) - 1)$$

$$\begin{aligned}
\text{LHS} &= 1^2 + 4^2 + 7^2 + \cdots + (3k-2)^2 + (3(k+1)-2)^2 \\
&= \tfrac{1}{2}k(6k^2 - 3k - 1) + (3(k+1)-2)^2 \quad \text{by the induction hypothesis } (**) \\
&= \frac{1}{2}k(6k^2 - 3k - 1) + (3k + 3 - 2)^2 \\
&= \frac{1}{2}k(6k^2 - 3k - 1) + (3k + 1)^2 \\
&= \frac{1}{2}(6k^3 - 3k^2 - k) + 9k^2 + 6k + 1 \\
&= \frac{1}{2}(6k^3 - 3k^2 - k + 18k^2 + 12k + 2) \\
&= \frac{1}{2}(6k^3 + 15k^2 + 11k + 2) \\
&= \frac{1}{2}(k+1)(6k^2 + 9k + 2)
\end{aligned}$$

$$\begin{aligned}
\text{RHS} &= \frac{1}{2}(k+1)(6(k+1)^2 - 3(k+1) - 1) \\
&= \frac{1}{2}(k+1)(6(k^2 + 2k + 1) - 3k - 3 - 1) \\
&= \frac{1}{2}(k+1)(6k^2 + 12k + 6 - 3k - 4) \\
&= \frac{1}{2}(k+1)(6k^2 + 9k + 2) \\
&= \text{LHS}
\end{aligned}$$

C

It follows from parts **A** and **B** by mathematical induction, that:

$1^2 + 4^2 + 7^2 + \cdots + (3n-2)^2 = \frac{1}{2}n(6n^2 - 3n - 1)$ for all integers $n \geq 1$.

6 **A**

When $n = 1$,

$$\begin{aligned}
\text{RHS} &= 1^2(2-1) \\
&= 1 \\
&= 1^3 \\
&= \text{LHS}
\end{aligned}$$

so the statement is true for $n = 1$.

B

Suppose that $k \geq 1$ is a positive integer for which the statement is true.

That is, suppose $1^3 + 3^3 + 5^3 + \cdots + (2k-1)^3 = k^2(2k^2 - 1)$ $\qquad (**)$

We prove the statement for $n = k + 1$.

That is, we prove $1^3 + 3^3 + 5^3 + \cdots + (2k-1)^3 + (2(k+1)-1)^3$

$$= (k+1)^2(2(k+1)^2 - 1)$$

$$\begin{aligned}
\text{LHS} &= 1^3 + 3^3 + 5^3 + \cdots + (2k-1)^3 + (2(k+1)-1)^3 \\
&= k^2(2k^2-1) + (2(k+1)-1)^3 \qquad \text{by the induction hypothesis } (**) \\
&= k^2(2k^2-1) + (2k+2-1)^3 \\
&= k^2(2k^2-1) + (2k+1)^3 \\
&= k^2(2k^2-1) + 8k^3 + 12k^2 + 6k + 1 \\
&= 2k^4 - k^2 + 8k^3 + 12k^2 + 6k + 1 \\
&= 2k^4 + 8k^3 + 11k^2 + 6k + 1 \\
&= (k+1)^2(2k^2 + 4k + 1) \\
&= (k+1)^2(2(k^2 + 2k + 1) - 1) \\
&= (k+1)^2(2(k+1)^2 - 1) \\
&= \text{RHS}
\end{aligned}$$

C

It follows from parts **A** and **B** by mathematical induction, that:

$1^3 + 3^3 + 5^3 + \cdots + (2n-1)^3 = n^2(2n^2 - 1)$ for all integers $n \geq 1$.