Solutions to Exercise 4A

1a $f'(x) > 0$ at the points A, G and I because the slope of the tangent line is positive at A, G and I.

1b $f'(x) < 0$ at the points C and E because the slope of the tangent line is negative at C and E.

1c $f'(x) = 0$ at the points B, D, F and H because the slope of the tangent line is zero at the points B, D, F and H.

2a $y = -5x + 2$

$\frac{dy}{dx} = -5$

Hence, $\frac{dy}{dx} < 0$ for all $x \in \mathbb{R}$.

Therefore, $y = -5x + 2$ is decreasing for all x.

2b $y = x + 7$

$\frac{dy}{dx} = 1$

Hence, $\frac{dy}{dx} > 0$ for all $x \in \mathbb{R}$.

Therefore, $y = x + 7$ is increasing for all x.

2c $y = x^3$

$\frac{dy}{dx} = 3x^2$

Hence, $\frac{dy}{dx} > 0$ for all x in $\mathbb{R}$ except 0 and $\frac{dy}{dx} = 0$ for $x = 0$.

Therefore, $y = x^3$ is increasing in $\mathbb{R}\backslash\{0\}$ and stationary at $x = 0$.

2d $\frac{dy}{dx} = 6x$ and $\frac{dy}{dx} = 0$ when $6x = 0$ or $x = 0$

The sign table for $\frac{dy}{dx}$ is shown below.

x	-1	0	$+1$
$\frac{dy}{dx}$	$-$	0	$+$
y	\	Minimum turning point	/

Therefore, y is stationary at $x = 0$ and increasing for $x > 0$.

2e $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$ and $\frac{dy}{dx}$ is never zero.

The sign table for $\frac{dy}{dx}$ is shown below.

x	$x > 0$
$\frac{dy}{dx}$	$+$
y	/

Therefore, y is increasing for $x > 0$.

2f $\frac{dy}{dx} = -2 \times x^{-3} = -\frac{2}{x^3}$ and $\frac{dy}{dx}$ is never zero.

The sign table for $\frac{dy}{dx}$ is shown below.

x	-1	0	$+1$
$\frac{dy}{dx}$	$+$	undefined	$-$
y	/	undefined	\

Therefore, y is undefined at $x = 0$, increasing for $x < 0$ and decreasing for $x > 0$.

3a $f'(x) = 4 - 2x$

3b i The sign table for $f'(x)$ is shown below.

x	1	2	3
$f'(x)$	$+$	0	$-$
y	/	Maximum turning point	\

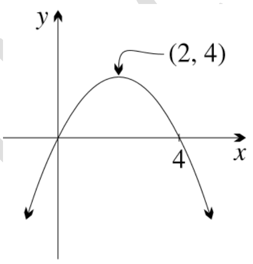
$f'(x) > 0$ when $x < 2$

3b ii $f'(x) < 0$ when $x > 2$ (as shown in 3bi)

3b iii $f'(x) = 0$ when $4 - 2x = 0$ or $x = 2$

3c $f(2) = 4 \times (2) - (2)^2 = 4$ when $x = 2$. Therefore, $(2, 4)$ is the maximum turning point.

The graph of $f(x)$ is shown below.



4a $f'(x) = 3x^2 - 6x$

4b i The sign table for $f'(x)$ is shown below.

x	-1	0	1	2	3
$f'(x)$	$+$	0	$-$	0	$+$
y	/	Maximum turning point	\	Minimum turning point	/

$f'(x) > 0$ when $x < 0$ or $x > 2$

4b ii $f'(x) < 0$ when $0 < x < 2$ (as shown in 4bi)

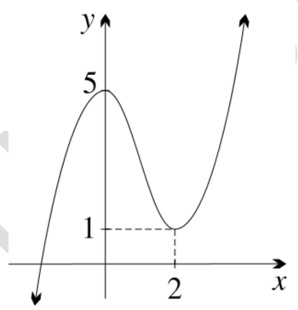
4b iii $f'(x) = 0$ when $3x^2 - 6x = 0$

$$3x(x-2) = 0$$

$$x = 0 \text{ or } x = 2$$

4c $f(0) = (0)^3 - 3(0)^2 + 5 = 5$ when $x = 0$. Therefore, $(0, 5)$ is the maximum turning point.

$f(2) = (2)^3 - 3(2)^2 + 5 = 1$ when $x = 2$. Therefore, $(2, 1)$ is the minimum turning point.

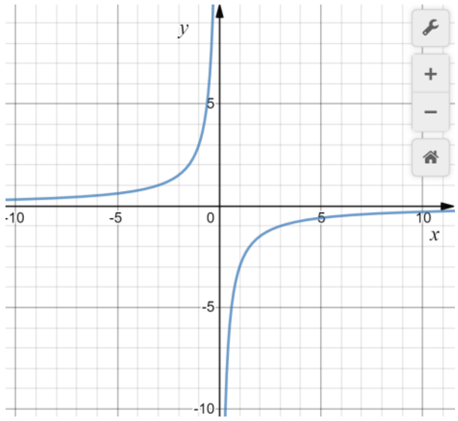


5a $f'(x) = -3 \times (-1) \times x^{-2} = \frac{3}{x^2}$

$f'(x) > 0$ for all x. Therefore, $f(x)$ is increasing for all x in its domain.

5b The function is not continuous at $x = 0$,

$f(x) > 0$ for $x < 0$ and $f(x) < 0$ for $x > 0$ as shown in the below graph of $f(x)$.



6a $y = x^2 - 4x + 1$

$y' = 2x - 4$

$y' > 0$ when $2x - 4 > 0$ or $x > 2$

Therefore, $y = x^2 - 4x + 1$ is increasing when $x > 2$.

6b $y = 7 - 6x - x^2$

$y' = -6 - 2x$

$y' > 0$ when $-6 - 2x > 0$ or $-6 > 2x$ or $x < -3$

Therefore, $y = 7 - 6x - x^2$ is increasing when $x < -3$.

6c $y = 2x^3 - 6x$

$y' = 6x^2 - 6$

$y' > 0$ when $6x^2 - 6 > 0$

$6x^2 > 6$

$x^2 > 1$

or $x < -1$ or $x > 1$

Therefore, $y = x^2 - 4x + 1$ is increasing when $x < -1$ or $x > 1$.

6d $y = x^3 - 3x^2 + 7$

$y' = 3x^2 - 6x$

$y' > 0$ when $3x^2 - 6x > 0$

$3x(x-2) > 0$

or $x < 0$ or $x > 2$

Therefore, $y = x^2 - 4x + 1$ is increasing when $x < 0$ or $x > 2$.

7a $\frac{dy}{dx} = 3x^2 - 2x - 1 = (3x+1)(x-1) = 0$ when $x = -\frac{1}{3}$ or $x = 1$

x	-1	$-\frac{1}{3}$	0	1	2
$\frac{dy}{dx}$	$+$	0	$-$	0	$+$
y	/	Maximum turning point	\	Minimum turning point	/

As shown on the above table, y is decreasing for $-\frac{1}{3} < x < 1$

7b $\frac{dy}{dx} = 3x^2 - 6x - 24 = 3(x+2)(x-4) = 0$ when $x = -2$ or $x = 4$

x	-3	-2	0	4	5
$\frac{dy}{dx}$	$+$	0	$-$	0	$+$
y	/	Maximum turning point	\	Minimum turning point	/

As shown on the above table, y is increasing for $x < -2$ or $x > 4$

8a $f(x) = \frac{1}{3}x^3 + x^2 + 5x + 7$ then $f'(x) = x^2 + 2x + 5$

8b $f'(x) = x^2 + 2x + 5 - 4 + 4$

$= x^2 + 2x + 1 + 4$

$= (x+1)^2 + 4$

Since $f'(x) > 0$ for all x, $f(x)$ is increasing for all x

8c $f(-3) = \frac{1}{3}(-3)^3 + (-3)^2 + 5 \times (-3) + 7 = -9 + 9 - 15 + 7 = -8$

$f(0) = \frac{1}{3}(0)^3 + (0)^2 + 5 \times (0) + 7 = 7$

$f(-3) = -8, f(0) = 7$ and $f(x)$ is increasing for all x.

Hence the curve crosses the x-axis exactly once between $x = -3$ and $x = 0$ and nowhere else.

9a $f(x) = \frac{2x}{x-3}$ where $x \neq 3$

$$f'(x) = \frac{2 \times (x-3) - (2x)}{(x-3)^2}$$

$$= \frac{2x - 6 - 2x}{(x-3)^2}$$

$$= -\frac{6}{(x-3)^2}$$

Since $(x-3)^2 > 0, f'(x) < 0$ for all $x \neq 3$

9b $f(x) = \frac{x^3}{x^2+1}$

$$f'(x) = \frac{3x^2(x^2+1) - x^3(2x)}{(x^2+1)^2}$$

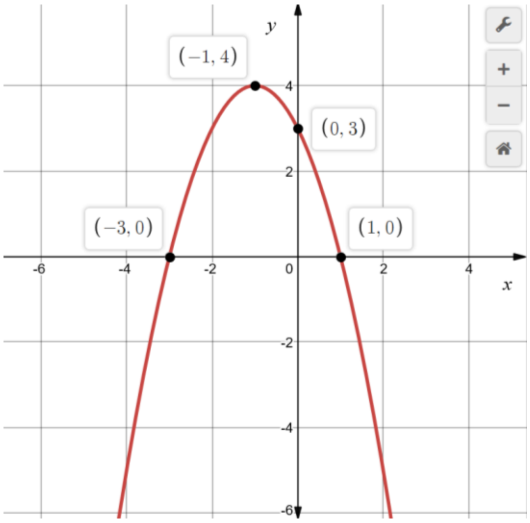
$$f'(x) = \frac{x^2(x^2+3)}{(x^2+1)^2}$$

$f'(x) > 0$ for all $x \neq 0$ and $f'(0) = 0$. Therefore, $f(x) = \frac{x^3}{x^2+1}$ is increasing for $x \neq 0$ and stationary for $x = 0$.

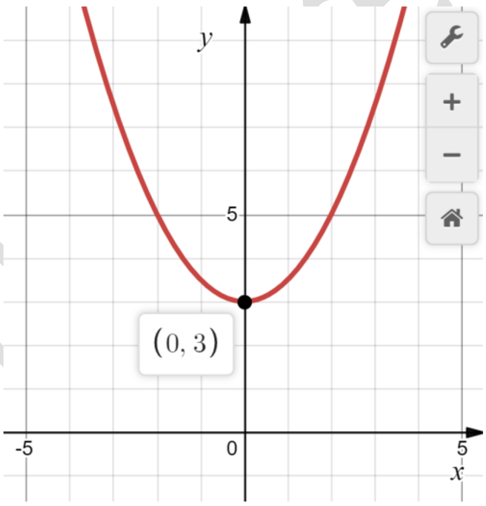
10a Since $f(1) = 0$ and $f(-3) = 0$, the x-intercepts are $(1, 0)$ and $(-3, 0)$.

Since $f'(-1) = 0$, and $f'(x) > 0$ when $x < -1$ and $f'(x) < 0$ when $x > -1$, there is a maximum turning point at $x = -1$.

Therefore, a possible graph is:



10b Since $f'(0) = 0$, $f'(x) < 0$ when $x < 0$ and $f'(x) > 0$ when $x > 0$ there is a minimum turning point at $x = 0$. There is no x-intercept because $f(x) > 0$ for all x. Therefore, a possible graph is:



10c Since $f(3) = 0$, $(3, 0)$ is an x-intercept. Since $f(x)$ is odd, $f(-x) = -f(x)$.

Hence, $f(-3) = -f(3)$. Therefore, $f(-3) = 0$ and $(-3, 0)$ is an x-intercept.

Since $f'(1) = 0$, and $f'(x) > 0$ when $x > 1$ and $f'(x) < 0$ when $0 \leq x < 1$,

there is a minimum turning point at $x = 1$. Therefore, there is a maximum turning point at $x = -1$.

11a III (If the function is a parabola then the first derivative is a linear function.)

11b I (As $x \to 0^-$, $f(x) \to \infty$ and so does the first derivative.)

11c IV (The function has a stationary point at $x = 0$ and the function has stationary point of inflection at $x = -a$ and $x = a$ where $a > 0$.)

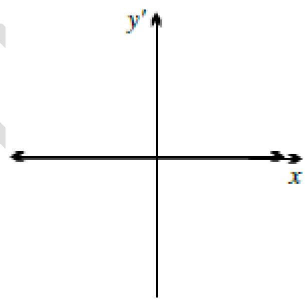
11d II (The function has a stationary point of inflection at $x = 0$.)

12a From the graph of the function, (a horizontal line), we see that as x changes, y remains constant.

The function can be expressed in the form $f(x) = b$.

Hence $f'(x) = 0$ over its domain.

So the graph of $y = f'(x)$ i.e. $y = 0$ is shown below.

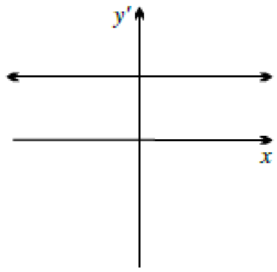


12b From the graph of the function (a straight line with positive gradient), we see that as x increases, y increases.

The function can be expressed in the form $f(x) = mx + b$, where $m > 0$.

Hence $f'(x) = m$ over its domain.

So the graph of $y = f'(x)$ i.e. $f'(x) = m$ where $m > 0$ is shown below.

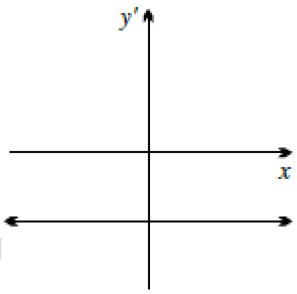


12c From the graph of the function (a straight line with negative gradient), we see that as x increases, y decreases.

The function can be expressed in the form $f(x) = mx + b$, where $m < 0$.

Hence $f'(x) = m$ over its domain.

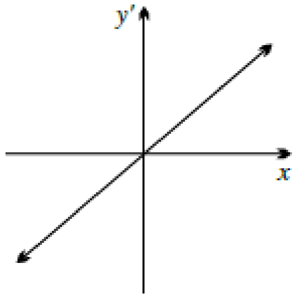
So the graph of $y = f'(x)$ i.e. $f'(x) = m$ where $m < 0$ is shown below.



12d The graph of the function is decreasing for $x < 0$, is stationary at $x = 0$ and is increasing for $x > 0$.

$f'(x) < 0$ for $x < 0$, $f'(x) = 0$ at $x = 0$ and $f'(x) > 0$ for $x > 0$

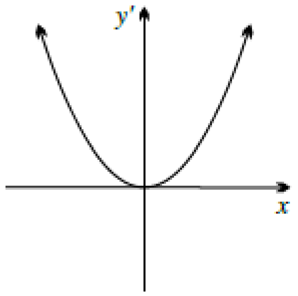
So the graph of $y = f'(x)$ is shown below.



12e The graph of the function is increasing for $x<0$, is stationary at $x=0$ and increasing for $x>0$.

$f'(x)>0$ for $x<0$, $f'(x)=0$ at $x=0$ and $f'(x)>0$ for $x>0$

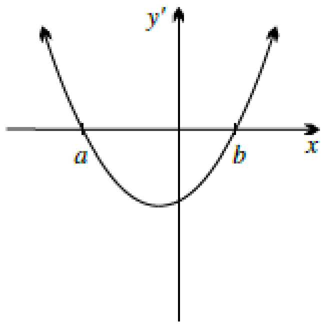
So the graph of $y=f'(x)$ is shown below.



12f The graph of the function is increasing for $x<a$, is stationary at $x=a$, is decreasing for $a<x<b$, is stationary at $x=b$ and increasing for $x>b$.

$f'(x)>0$ for $x<a$, $f'(x)=0$ at $x=a$, $f'(x)<0$ for $a<x<b$, $f'(x)=0$ at $x=b$ and $f'(x)>0$ for $x>b$

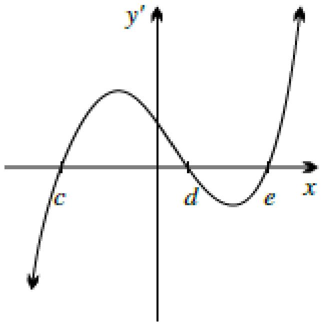
So the graph of $y=f'(x)$ is shown below.



12g The graph of the function is decreasing for $x<c$, is stationary at $x=c$, is increasing for $c<x<d$, is stationary at $x=d$, is decreasing for $d<x<e$, is stationary at $x=e$ and increasing for $x>e$.

$f'(x)<0$ for $x<c$, $f'(x)=0$ at $x=c$, $f'(x)>0$ for $c<x<d$, $f'(x)=0$ at $x=d$, $f'(x)<0$ for $d<x<e$, $f'(x)=0$ at $x=e$ and $f'(x)>0$ for $x>e$

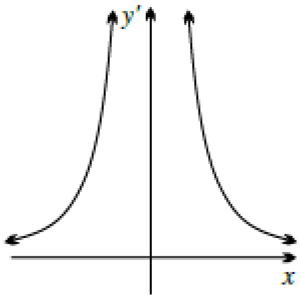
So the graph of $y=f'(x)$ is shown below.



12h The graph of the function is increasing for $x<0$ and increasing for $x>0$.

$f'(x)>0$ for $x<0$ and $f'(x)>0$ for $x>0$

So the graph of $y=f'(x)$ is shown below.



13

$$\frac{dy}{dx}=\frac{2x\times(2x^2+x+1)-x^2\times(4x+1)}{(2x^2+x+1)^2}$$

$$=\frac{4x^3+2x^2+2x-4x^3-x^2}{(2x^2+x+1)^2}$$

$$= \frac{x^2 + 2x}{(2x^2 + x + 1)^2}$$

$$= \frac{x(x + 2)}{(2x^2 + x + 1)^2}$$

$\frac{dy}{dx} = 0$ when $\frac{x(x+2)}{(2x^2+x+1)^2} = 0$ or $x = 0$ or $x = -2$

x	-3	-2	-1	0	1
$\frac{dy}{dx}$	$+$	0	$-$	0	$+$
y	/	Maximum turning point	\	Minimum turning point	/

As shown on the above table, y is decreasing for $-2 < x < 0$

14a i $f'(x) = \frac{-2x \times (x^2+1) - (1-x^2) \times 2x}{(x^2+1)^2} = \frac{-2x^3 - 2x - 2x + 2x^3}{(x^2+1)^2} = \frac{-4x}{(x^2+1)^2}$

14a ii $f(0) = \frac{1-(0)^2}{(0)^2+1} = 1$

14a iii $f(-x) = \frac{1-(-x)^2}{(-x)^2+1} = \frac{1-x^2}{x^2+1} = f(x)$. Since $f(-x) = f(x)$, $f(x)$ is an even function.

14b $f'(x) = 0$ when $x = 0$

x	-1	0	1
$f'(x)$	$+$	0	$-$
y	/	Maximum turning point	\

As shown in the above table, $(0, 1)$ is a maximum turning point and since $\lim\limits_{x\to\pm\infty}\frac{1-x^2}{x^2+1}=-1$, $f(x)\leq 1$ for all $x\in\mathbb{R}$.

15a $f'(x)=0$ when $x=0$, $x=-2$ or $x=1$. Therefore, the graph of $f'(x)$ cuts the x-axis at $(-2,0)$, $(0,0)$ and $(1,0)$

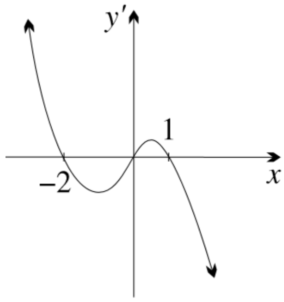
$f'(-3)=-(-3)\times\big((-3)+2\big)\times\big((-3)-1\big)=12.$

Therefore, $f'(x)>0$ for $x<-2$

$f'(x)<0$ for $-2<x<0$

$f'(x)>0$ for $0<x<1$

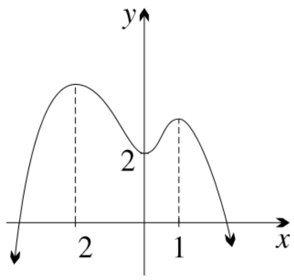
$f'(x)<0$ for $x>1$ as shown in the below graph.



15b

x	-3	-2	-1	0	$\frac{1}{2}$	1	2
$f'(x)$	$+$	0	$-$	0	$+$	0	$-$
$f(x)$	/	Maximum turning point	\	Minimum turning point	/	Maximum turning point	\

As shown in the above table, the graph of $f(x)$ is,



Solutions to Exercise 4B

1a $y = x^2 - 6x + 8$

$\frac{dy}{dx} = 2x - 6$

$\frac{dy}{dx} = 0$ when $2x - 6 = 0$ or $x = 3$

Therefore, $x = 3$ is the x-coordinate of the stationary point of y.

1b $y = x^2 + 4x + 3$

$\frac{dy}{dx} = 2x + 4$

$\frac{dy}{dx} = 0$ when $2x + 4 = 0$ or $x = -2$

Therefore, $x = -2$ is the x-coordinate of the stationary point of y.

1c $y = x^3 - 3x$

$\frac{dy}{dx} = 3x^2 - 3$

$\frac{dy}{dx} = 0$ when $3x^2 - 3 = 0$

$x^2 = 1$

$x = -1$ or $x = 1$

Therefore, $x = -1$ and $x = 1$ are the x-coordinates of the stationary points of y.

2a $y = x^2 - 4x + 7$

$\frac{dy}{dx} = 2x - 4$

$\frac{dy}{dx} = 0$ when $2x - 4 = 0$ or $x = 2$

Therefore, $x = 2$ is the x-coordinate of the stationary point.

For $x = 2$, $y = 2^2 - 4 \times 2 + 7 = 3$.

Therefore, the coordinates of the stationary point are $(2, 3)$.

2b $y = x^2 - 8x + 16$

$\frac{dy}{dx} = 2x - 8$

$\frac{dy}{dx} = 0$ when $2x - 8 = 0$ or $x = 4$

Therefore, $x = 4$ is the x-coordinate of the stationary point.

For $x = 4$, $y = 4^2 - 8 \times 4 + 16 = 0$.

Therefore, the coordinates of the stationary point are $(4, 0)$.

2c $y = 3x^2 - 6x + 1$

$\frac{dy}{dx} = 6x - 6$

$\frac{dy}{dx} = 0$ when $6x - 6 = 0$ or $x = 1$

Therefore, $x = 1$ is the x-coordinate of the stationary point.

For $x = 1$, $y = 3 \times 1^2 - 6 \times 1 + 1 = -2$.

Therefore, the coordinates of the stationary point are $(1, -2)$.

2d $y = -x^2 + 2x - 1$

$\frac{dy}{dx} = -2x + 2$

$\frac{dy}{dx} = 0$ when $-2x + 2 = 0$ or $x = 1$

Therefore, $x = 1$ is the x-coordinate of the stationary point.

For $x = 1$, $y = -1^2 + 2 \times 1 - 1 = 0$.

Therefore, the coordinates of the stationary point are $(1, 0)$.

2e $y = x^3 - 3x^2$

$\frac{dy}{dx} = 3x^2 - 6x$

$\frac{dy}{dx} = 3x(x - 2)$

$\frac{dy}{dx} = 0$ when $3x(x - 2) = 0$. Hence, $\frac{dy}{dx} = 0$ when $x = 0$ or $x = 2$.

Therefore, $x = 0$ and $x = 2$ are the x-coordinates of the stationary points.

For $x = 0, y = 0^3 - 3 \times 0^2 = 0$.

Therefore, one of the stationary points is at $(0, 0)$.

For $x = 2, y = 2^3 - 3 \times 2^2 = -4$.

Therefore, the other stationary point is at $(2, -4)$.

2f $y = x^4 - 4x + 1$

$\frac{dy}{dx} = 4x^3 - 4$

$\frac{dy}{dx} = 4(x^3 - 1)$

$\frac{dy}{dx} = 0$ when $4(x^3 - 1) = 0$ or $x = 1$.

Therefore, $x = 1$ is the x-coordinate of the stationary point.

For $x = 1, y = 1^4 - 4 \times 1 + 1 = -2$.

Therefore, the coordinates of the stationary point are $(1, -2)$.

3a $y = x^2 - 4x + 3$ or $y = (x - 1)(x - 3)$

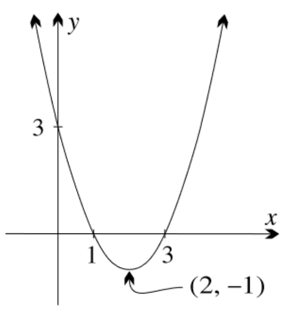
x-intercepts: 1, 3

y-intercept: 3

$\frac{dy}{dx} = 2x - 4$

$\frac{dy}{dx} = 0$ when $2x - 4 = 0$ or $x = 2$.

x	1	2	3
y	0	-1	0
slope	-2	0	2



3b $y = 12 + 4x - x^2$ or $y = -(x+2)(x-6)$

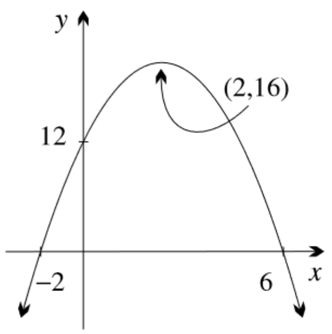
x-intercepts: $-2, 6$

y-intercept: 12

$\frac{dy}{dx} = 4 - 2x$

$\frac{dy}{dx} = 0$ when $4 - 2x = 0$ or $x = 2$.

x	1	2	3
y	15	16	15
slope	2	0	-2



3c $y = x^2 + 6x + 8$ or $y = (x+2)(x+4)$

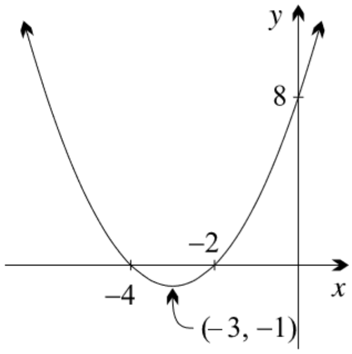
x-intercepts: $-4, -2$

y-intercept: 8

$\frac{dy}{dx} = 2x + 6$

$\frac{dy}{dx} = 0$ when $2x + 6 = 0$ or $x = -3$.

x	-4	-3	-2
y	0	-1	0
slope	-2	0	2



3d $y = 15 - 2x - x^2$ or $y = -(x+5)(x-3)$

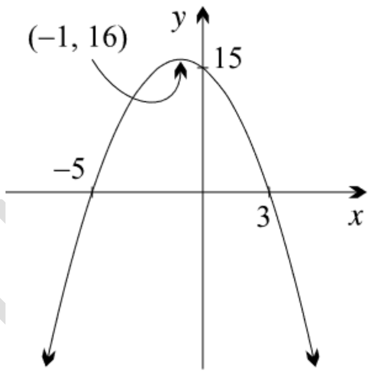
x-intercepts: $-5, 3$

y-intercept: 15

$\frac{dy}{dx} = -2 - 2x$

$\frac{dy}{dx} = 0$ when $-2 - 2x = 0$ or $x = -1$.

x	-2	-1	0
y	15	16	15
slope	2	0	-2



4a $f(x) = x^2 - 2x - 3$

$f'(x) = 2x - 2$

$f'(1) = 2 \times 1 - 2 = 0.$

Therefore, there is a stationary point at $x = 1$.

x	0	1	2
$f'(x)$	\	0	/

Therefore, the stationary point at $x = 1$ is a local minimum.

4b $f(x) = 15 + 2x - x^2$

$f'(x) = 2 - 2x$

$f'(1) = 2 - 2 \times 1 = 0.$

Therefore, there is a stationary point at $x = 1$.

x	0	1	2
$f'(x)$	/	0	\

Therefore, the stationary point at $x = 1$ is a local maximum.

4c $f(x) = x^3 + 3x^2 - 9x + 2$

$f'(x) = 3x^2 + 6x - 9$

$f'(x) = 3(x^2 + 2x - 3)$

$f'(x) = 3(x - 1)(x + 3)$

$f'(x) = 0$ when $x = 1$ and $x = -3$.

Hence, there are stationary points at both $x = 1$ and $x = -3$.

x	-4	-3	0	1	2
$f'(x)$	/	0	\	0	/

Therefore, the stationary point at $x = 1$ is a local minimum.

4d $f(x) = x^3 - 3x^2 + 3x + 1$

$f'(x) = 3x^2 - 6x + 3$

$f'(x) = 3(x - 1)^2$

$f'(x) = 0$ when $x = 1$.

Therefore, there is a stationary point at $x = 1$.

x	0	1	2
$f'(x)$	/	0	/

Therefore, the stationary point at $x = 1$ is a stationary or horizontal point of inflection.

5a $y = x^2 + 4x - 12$ or $y = (x + 6)(x - 2)$

x-intercepts: $-6, 2$

y-intercept: -12

$\frac{dy}{dx} = 2x + 4$

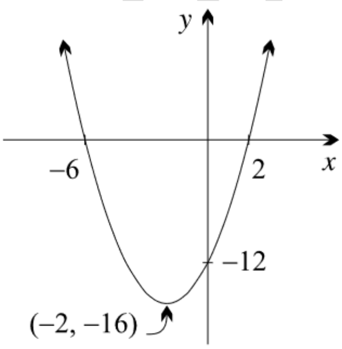
$\frac{dy}{dx} = 0$ when $2x + 4 = 0$ or $x = -2$.

Therefore, there is a stationary point at $x = -2$.

When $x = -2, y = (-2)^2 + 4 \times (-2) - 12 = -16$

x	-3	-2	0
$f'(x)$	\	0	/

Therefore, the stationary point at $(-2, -16)$ is a local minimum.



5b $y = 5 - 4x - x^2$ or $y = -(x+5)(x-1)$

x-intercepts: $-5, 1$

y-intercept: 5

$\frac{dy}{dx} = -4 - 2x$

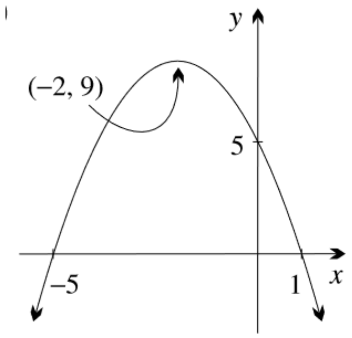
$\frac{dy}{dx} = 0$ when $-4 - 2x = 0$ or $x = -2$.

Therefore, there is a stationary point at $x = -2$.

When $x = -2, y = 5 - 4 \times (-2) - (-2)^2 = 9$

x	-3	-2	0
$f'(x)$	/	0	\

Therefore, the stationary point at $(-2, 9)$ is a local maximum.



6a $y = x^3 - 3x^2$ then $\frac{dy}{dx} = 3x^2 - 6x = 3x(x-2)$

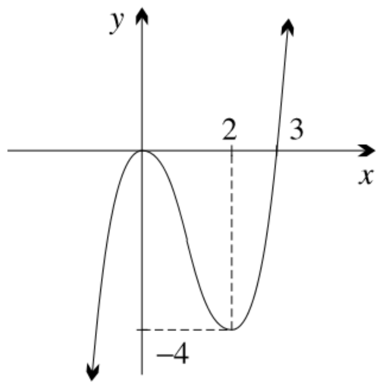
6b $3x(x-2) = 0$ when $x = 0$ or $x = 2$

x	-1	0	1	2	3
$\frac{dy}{dx}$	+	0	−	0	+
y	/	Maximum turning point	\	Minimum turning point	/

$y = (0)^3 - 3(0)^2 = 0$ when $x = 0$ and $y = (2)^3 - 3(2)^2 = -4$ when $x = 2$

Therefore, there is a maximum turning point at $(0, 0)$ and a minimum turning point at $(2, -4)$.

6c $y = x^3 - 3x^2 = 0$ when $x = 0$ or $x = 3$. So $(0, 0)$ and $(3, 0)$ are the x-intercepts.



7a $y = 12x - x^3$

$y' = 12 - 3x^2$

$y' = 3(4 - x^2)$

$y' = 3(2 - x)(2 + x)$

7b $y' = 0$ when $3(2 - x)(2 + x) = 0$. So $y' = 0$ when $x = -2$ or $x = 2$.

When $x = -2$, $y = -16$ and when $x = 2$, $y = 16$.

x	-3	-2	0	2	3
y	-9	-16	0	16	9
y'	\	0	/	0	\

Therefore, there is a maximum turning point at $(2, 16)$ and a minimum turning point at $(-2, -16)$.

7c $y = 12x - x^3$

$y = x(12 - x^2)$

$y = x(2\sqrt{3} + x)(2\sqrt{3} - x)$

x-intercepts: $-2\sqrt{3}, 2\sqrt{3}$

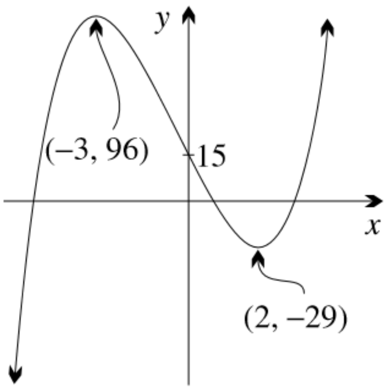
y-intercept: 0

8a $y = 2x^3 + 3x^2 - 36x + 15$ and the y-intercept is $(0, 15)$

$y' = 6x^2 + 6x - 36 = 6(x + 3)(x - 2)$

$6(x + 3)(x - 2) = 0$ when $x = -3$ and $x = 2$

x	-4	-3	0	2	3
y	79	96	15	-29	-12
y'	$+$	0	$-$	0	$+$
y	/	Maximum turning point	\	Minimum turning point	/

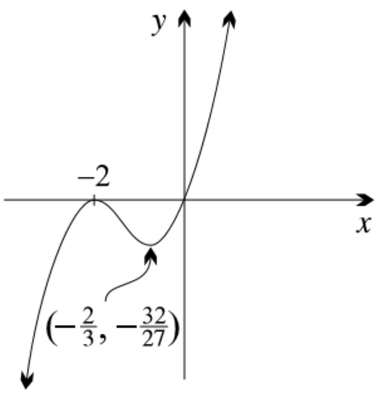


8b $y = x^3 + 4x^2 + 4x = x(x + 2)^2$, x-intercepts are $(0, 0)$ and $(-2, 0)$ and the y-intercept is $(0, 0)$.

$y' = 3x^2 + 8x + 4 = (3x + 2)(x + 2)$

$(3x + 2)(x + 2) = 0$ when $x = -\frac{2}{3}$ and $x = -2$

x	-3	-2	-1	$-\frac{2}{3}$	0
y	-3	0	-1	$-\frac{32}{27}$	0
y'	$+$	0	$-$	0	$+$
y	/	Maximum turning point	\	Minimum turning point	/

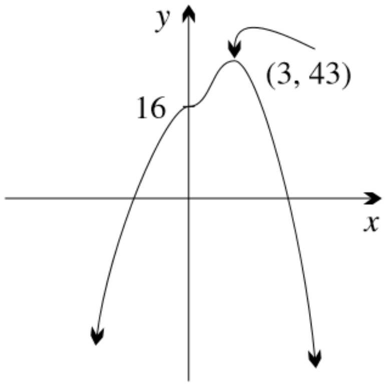


8c $y = 16 + 4x^3 - x^4$, and the y-intercept is $(0, 16)$.

$y' = 12x^2 - 4x^3 = 4x^2(3 - x)$

$4x^2(3 - x) = 0$ when $x = 0$ and $x = 3$

x	-1	0	1	3	4
y	11	16	19	43	16
y'	$+$	0	$+$	0	$-$
y	/	Stationary point of inflection	/	Maximum turning point	\

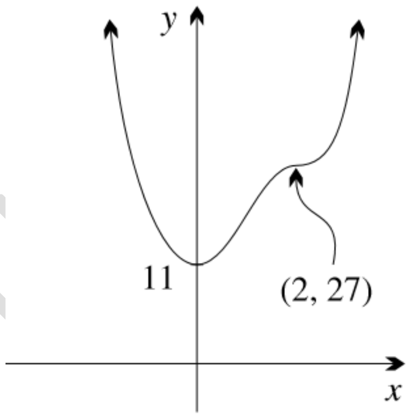


8d $y = 3x^4 - 16x^3 + 24x^2 + 11$, and the y-intercept is $(0, 11)$.

$y' = 12x^3 - 48x^2 + 48x = 12x(x^2 - 4x + 4) = 12x(x - 2)^2$

$4x^2(3 - x) = 0$ when $x = 0$ and $x = 3$

x	-1	0	1	2	3
y	48	11	22	27	38
y'	$-$	0	$+$	0	$+$
y	\	Minimum turning point	/	Stationary point of inflection	/



9a $y = x(x-2)^3$

$y' = 1 \times (x-2)^3 + x \times 3(x-2)^2 \times 1$

$y' = (x-2)^3 + 3x(x-2)^2$

$y' = (x-2)^2[(x-2) + 3x]$

$y' = 2(2x-1)(x-2)^2$

9b $y' = 0$ when $2(2x-1)(x-2)^2 = 0$. So $y' = 0$ when $x = \frac{1}{2}$ or $x = 2$.

When $x = \frac{1}{2}, y = -\frac{27}{16}$ and when $x = 2, y = 0$.

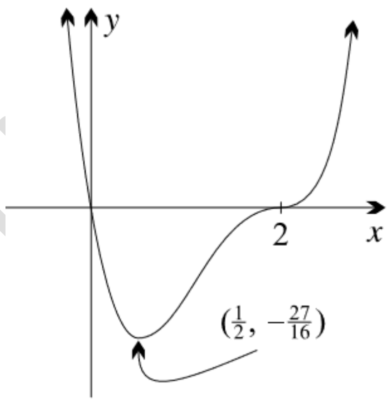
x	0	$\frac{1}{2}$	1	2	3
y'	\	0	/	0	/

Therefore, $\left(\frac{1}{2}, -\frac{27}{16}\right)$ is a minimum turning point and $(2, 0)$ is a stationary point of inflection.

9c $y = x(x-2)^3$

x-intercepts: 0, 2

y-intercept: 0



10a $y = x^2(x-4)^2$ then $\frac{dy}{dx} = 2x \times (x-4)^2 + x^2 \times 2(x-4)$

$$= 2x(x-4)\big((x-4)+x\big)$$

$$= 4x(x-4)(x-2)$$

and $(0, 0)$ and $(4, 0)$ are x-intercepts.

10b $4x(x-4)(x-2) = 0$ when $x = 0$ or $x = 2$ or $x = 4$

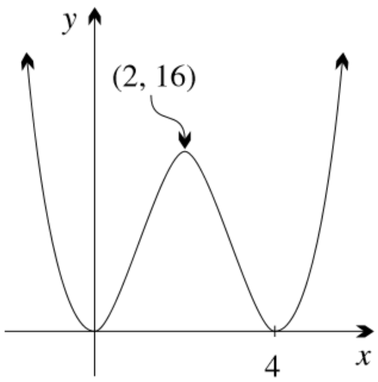
x	-1	0	1	2	3	4	5
y'	$-$	0	$+$	0	$-$	0	$+$
y	\	Minimum turning point	/	0	\	Maximum turning point	/

$y = (0)^2\big((0)-4\big)^2 = 0$ when $x = 0$, so $(0, 0)$ is the minimum turning point.

$y = (2)^2\big((2)-4\big)^2 = 16$ when $x = 2$, so $(2, 16)$ is the maximum turning point.

$y = (4)^2\big((4)-4\big)^2 = 0$ when $x = 4$, so $(4, 0)$ is the minimum turning point.

10c



11a $y = (x-5)^2(2x+1)$

$y' = 2(x-5)(2x+1) + (x-5)^2 \times 2$

$y' = (x-5)[(4x+2)+(2x-10)]$

$y' = (x-5)(6x-8)$

$y' = 2(x-5)(3x-4)$

11b $\quad y' = 0$ when $x = 5$ or $x = \frac{4}{3}$

When $x = \frac{4}{3}, y = 49\frac{8}{27}$ and when $x = 5, y = 0$.

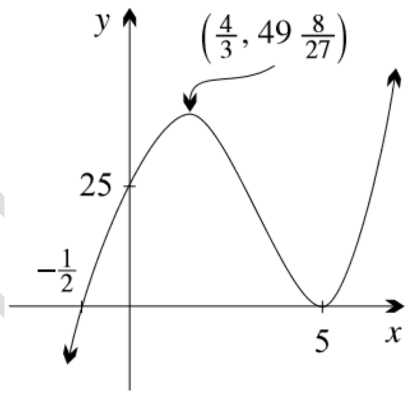
x	1	$\frac{4}{3}$	2	5	6
y'	/	0	\	0	/

Therefore, $\left(\frac{4}{3}, 49\frac{8}{27}\right)$ is a maximum turning point and $(5, 0)$ is a minimum turning point.

11c $\quad y = (x-5)^2(2x+1)$

x-intercepts: $-\frac{1}{2}, 5$

y-intercept: 25



12a $\quad y = x^2 + ax - 15$ then $y' = 2x + a$

$2x + a = 0$ when $x = 4$ then $2 \times 4 + a = 0$ and $a = -8$

12b $y = x^2 + ax + 7$ then $y' = 2x + a$

$2x + a = 0$ when $x = -1$ then $2 \times -1 + a = 0$ and $a = 2$

13a $f(x) = ax^2 + 4x + c$

$f'(x) = 2ax + 4$

If $(-1, 1)$ is a turning point, then $f'(-1) = 0$ and $2a \times (-1) + 4 = 0$

Therefore, $a = 2$.

If $a = 2$, then $f(x) = 2x^2 + 4x + c$

Since $(-1, 1)$ is a point on $f(x) = 2x^2 + 4x + c$,

$f(-1) = 2(-1)^2 + 4 \times (-1) + c = 1$

Therefore, $c = 3$.

13b $y = x^3 + bx^2 + cx + 5$

$y' = 3x^2 + 2bx + c$

If there are stationary points at $x = -2$ and $x = 4$, then $y' = 0$ for $x = -2$ and $x = 4$.

Thus, $3(-2)^2 + 2b \times (-2) + c = 0$ and $3(4)^2 + 2b \times (4) + c = 0$

$4b - c = 12$ and $8b + c = -48$

Adding the two equations gives $12b = -36$ or $b = -3$.

Substituting $b = -3$ into $8b + c = -48$ gives $-24 + c = -48$, so $c = -24$.

Therefore, $b = -3$ and $c = -24$.

14a $y = ax^2 + bx + c$ then $y' = 2ax + b$

If the function passes through the point, $(1, 4)$ then $a(1)^2 + b \times (1) + c = 4$

Therefore, $a + b + c = 4$

If the function passes through the point, $(-1, 6)$ then $a(-1)^2 + b \times (-1) + c = 6$

Therefore, $a - b + c = 6$

If the slope is zero at $x = -\frac{1}{2}$ then $y' = 2a \times \left(-\frac{1}{2}\right) + b = 0$

Therefore, $-a + b = 0$

14b Since $-a+b=0, a=b$

Hence, $2a+c=4$ and $c=6$ (substitute a for b in the first two equations)

So $2a+6=4,\ a=b=-1$ and $c=6$.

15a $y=ax^2+bx+c$ passes through the origin. So $a(0)^2+b\times 0+c=0$.

Therefore, $c=0$.

15b $y=ax^2+bx+c$

$\frac{dy}{dx}=2ax+b$

Slope of the line $y=2x$ is 2 for all $x\in\mathbb{R}$ and since $y=2x$ is tangent to the curve

$y=ax^2+bx+c$ at the origin, $\frac{dy}{dx}=2ax+b=2$ for $x=0$.

Hence, $2a\times 0+b=2$ and $b=2$.

15c $\frac{dy}{dx}=2ax+b$ and there is a maximum turning point at $x=1$, then $2a(1)+b=0$

Therefore, $2a+b=0$.

$b=2$ (from 15b) and $2a+b=0$, then $2a+2=0$. Therefore, $a=-1$.

16 Given $y=ax^3+bx^2+cx+d$

$$\frac{dy}{dx}=3ax^2+2bx+c$$

The function has a relative maximum at $(-2,27)$. So when $x=-2$, $\frac{dy}{dx}=0$.

$$3a(-2)^2+2b(-2)+c=0$$

Hence $12a-4b+c=0$.

The function has a relative minimum at $(1,0)$. So when $x=-2$, $\frac{dy}{dx}=0$.

$$3a(1)^2+2b(1)+c=0$$

Hence $3a+2b+c=0$.

$(1,0)$ lies on the curve and so when $x=1, y=0$

$a(1)^3+b(1)^2+c(1)+d=0$

Hence $a+b+c+d=0$.

$(-2,27)$ lies on the curve and so when $x=-2, y=27$

$a(-2)^3+b(-2)^2+c(-2)+d=27$

Hence $-8a+4b-2c+d=27$.

By subtracting we obtain:

$9a-3b+3c=-27$

$3a+2b+c=0 \qquad (1)$

$12a-4b+c=0 \qquad (2)$

$9a-3b+3c=-27 \qquad (3)$

(2) – (1) gives:

$9a-6b=0 \qquad (4)$

$3\times(2)-(3)$ gives:

$27a-9b=27 \qquad (5)$

$(5)-3\times(4)$ gives:

$9b=27 \Rightarrow b=3$

Substituting $b=3$ into (4) and solving $9a-18=0$ for a we obtain $a=2$.

Substituting $a=2$ and $b=3$ into (1) and solving $c+12=0$ for c we obtain $c=-12$.

So $a=2, b=3$ and $c=-12$.

Substituting $a=2$, $b=3$ and $c=-12$ into $a+b+c+d=0$ and solving $d-7=0$ for d we obtain $d=7$.

17a Given $f(x)=\dfrac{3x}{x^2+1}$

Applying the quotient rule on $f(x)=\dfrac{3x}{x^2+1}$:

Let $u=3x$ and $v=x^2+1$.

Then $u'=3$ and $v'=2x$.

$$\begin{aligned}f'(x)&=\frac{vu'-uv'}{v^2}\\&=\frac{3(x^2+1)-(3x)(2x)}{(x^2+1)^2}\\&=\frac{3x^2+3-6x^2}{(x^2+1)^2}\\&=\frac{3-3x^2}{(x^2+1)^2}\end{aligned}$$

So $f'(x)=\dfrac{3(1-x)(1+x)}{(x^2+1)^2}$.

17b There are stationary points where $f'(x)=0$.

$3(1-x)(1+x)=0 \Rightarrow x=\pm1$

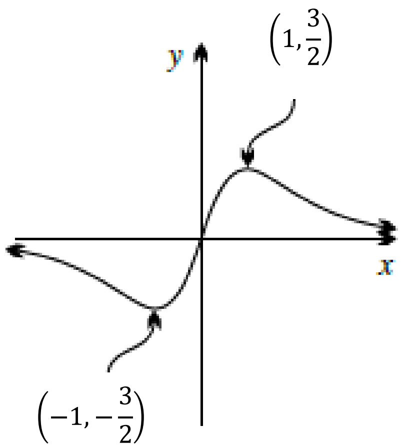
So there are stationary points at $x=-1$ and $x=1$.

x	-2	-1	0	1	2
$f'(x)$	$-\frac{9}{25}$	0	3	0	$-\frac{9}{25}$
slope	\	–	/	–	\

When $x=-1$, $y=-\dfrac{3}{2}$ and when $x=1$, $y=\dfrac{3}{2}$.

Hence $\left(-1,-\dfrac{3}{2}\right)$ is a minimum turning point and $\left(1,\dfrac{3}{2}\right)$ is a maximum turning point.

17c



17d i The line $y = c$ where $c > \frac{3}{2}$ does not intersect the graph of $y = f(x)$.

Hence, for $c > \frac{3}{2}$, the equation has no roots.

17d ii The line $y = \frac{3}{2}$ touches the graph of $y = f(x)$ at the maximum turning point.

Hence, for $c = \frac{3}{2}$, the equation has one root.

17d iii The line $y = c$ where $0 < c < \frac{3}{2}$ intersects the graph of $y = f(x)$ at two points.

Hence, for $0 < c < \frac{3}{2}$, the equation has two roots.

17d iv The line $y = 0$ touches the graph of $y = f(x)$ at the origin.

Hence, for $c = 0$, the equation has one root.

18a To answer this question as framed, it is best to graph polynomial functions that have only even powers of x. Derivatives of such polynomial functions will therefore have only odd powers of x.

So $f(-x)=f(x)$ and $f'(-x)=-f'(x)$.

To formally prove that the derivative of an even function is odd, we would need to prove that if f is even then $f'(-x)=-f'(x)$.

18b To answer this question as framed, it is best to graph polynomial functions that have only odd powers of x. Derivatives of such polynomial functions will therefore have only even powers of x.

So $f(-x)=-f(x)$ and $f'(-x)=f'(x)$.

To formally prove that the derivative of an odd function is even, we would need to prove that if f is odd then $f'(-x)=f'(x)$.

18c If f is an even polynomial function and hence has only even powers of x, then $f(-x)=f(x)$. Derivatives of f will therefore have only odd powers of x and so $f'(-x)=-f'(x)$.

If f is an odd polynomial function and hence has only even powers of x, then $f(-x)=-f(x)$. Derivatives of f will therefore have only even powers of x and so $f'(-x)=f'(x)$.

Solutions to Exercise 4C

1a A maximum turning point, B minimum turning point.

1b C is a minimum point.

1c D is a horizontal point of inflection, because the graph is stationary at D and concavity changes before and after D. E is a maximum turning point.

1d F and H are minimum turning points and G is a maximum point.

1e I is a minimum point.

1f J is a horizontal point of inflection, because it is a stationary point and the concavity changes before and after D. K is a minimum turning point and L is a maximum turning point.

2a $y' = x(x-3)^2 = 0$ when $x = 0$ or $x = 3$. Hence, y has stationary points at $x = 0$ and $x = 3$. $y'' = 3(x-3)(x-1) = 0$ when $x = 1$ or $x = 3$. Hence, there are points of inflection at $x = 1$ and $x = 3$. Therefore, there is a turning point at $x = 0$ and a horizontal point of inflection at $x = 3$ as shown in the below table.

x	-1	0	1	3	4
y'	$-$	0	$+$	0	$+$
y	\	Minimum turning point	/	Horizontal point of inflection	/

2b $y' = (x+2)^3(x-4) = 0$ when $x = -2$ or $x = 4$. Hence, there are stationary points at $x = -2$ and $x = 4$. $y'' = 2(x+2)^2(2x-5) = 0$ when $x = -2$ or $x = \frac{5}{2}$.

Hence, there is a point of inflection at $x = \frac{5}{2}$ but not at $x = -2$,because $x = -2$ is a double root and concavity is the same when $x < -2$ and $-2 < x < \frac{5}{2}$.

Therefore, y has a turning point at $x = -2$ and $x = 4$ as shown in the below table.

x	-3	-2	0	4	5
y'	$+$	0	$-$	0	$+$
y	/	Maximum turning point	\	Minimum turning point	/

2c $y' = \frac{x}{x-1} = 0$ when $x = 0$ and undefined when $x = 1$. Hence, y has a stationary point at $x = 1$.

$y'' = \frac{-1}{(x-1)^2} < 0$ for $x \in \mathbb{R} - \{1\}$

x	-1	0	$\frac{1}{2}$	1	2
y'	$+$	0	$-$	undefined	$+$
y	/	Maximum turning point	\	undefined	/

As shown in the above table, y has a maximum turning point at $x = 0$ and a discontinuity at $x = 1$.

2d $y' = \frac{x^2}{x-1} = 0$ when $x = 0$ and y' is undefined when $x = 1$. Hence, y has a stationary point at $x = 0$ and not defined at $x = 1$.

$y'' = \frac{x(x-2)}{(x-1)^2} = 0$ when $x = 0$ and $x = 2$. Hence, y has an inflection point at $x = 0$ and $x = 2$.

x	-1	0	$\frac{1}{2}$	1	2
y'	$-$	0	$-$	undefined	$+$
y	\	Horizontal point of inflection	\	undefined	/

As shown in the above table, y has a horizontal point of inflection at $x = 0$ and has a discontinuity at $x = 1$.

2e $y' = \frac{x}{(x-1)^2} = 0$ when $x = 0$ and y' is undefined when $x = 1$. Hence, y has a stationary point at $x = 0$ and has a discontinuity at $x = 1$.

$y'' = \frac{-(x+1)}{(x-1)^3} = 0$ when $x = -1$. Hence, y has an inflection point at $x = -1$.

x	-1	0	$\frac{1}{2}$	1	2
y'	$-$	0	$+$	undefined	$+$
y	\	Minimum turning point	/	undefined	/

As shown in the above table, y has a minimum turning point at $x = 0$ and has a discontinuity at $x = 1$.

2f $y' = \frac{x^2}{(x-1)^3} = 0$ when $x = 0$ and y' is undefined when $x = 1$. Hence, y has a stationary point at $x = 0$ and has a discontinuity at $x = 1$.

$y'' = \frac{-x(x+2)}{(x-1)^4} = 0$ when $x = -2$ or $x = 0$. Hence, y has inflection points at $x = -2$ and $x = 0$.

x	-1	0	$\frac{1}{2}$	1	2
y'	$-$	0	$-$	undefined	$+$
y	\	Horizontal point of inflection	\	undefined	/

As shown in the above table, y has a horizontal point of inflection at $x = 0$ and has a discontinuity at $x = 1$.

2g $y' = x - \frac{1}{x} = \frac{x^2-1}{x} = 0$ when $x = -1$ or $x = 1$. Hence, y has stationary points at $x = -1$ and $x = 1$. y' is undefined when $x = 0$ then y has a discontinuity at $x = 0$.

$y'' = \frac{x^2+1}{x^2} > 0$ for all x.

x	-3	-1	-1	0	$\frac{1}{2}$	1	2
y'	$-$	0	$+$	undefined	$-$	0	$+$
y	\	Minimum turning point	/	undefined	\	Minimum turning point	/

As shown in the above table, y has minimum turning points at $x = -1$ and at $x = 1$, and has a discontinuity at $x = 0$.

2h $y' = \sqrt{x} - \frac{1}{\sqrt{x}} = \frac{x-1}{\sqrt{x}} = 0$ when $x = 1$ and y' is undefined when $x \leq 0$.

Therefore, y has a stationary point at $x = 1$ and is undefined when $x < 0$.

$y'' = \frac{x+1}{2\sqrt{x^3}} = 0$ when $x = -1$. Hence, y has an inflection point at $x = -1$.

x	$\frac{1}{2}$	1	2
y'	$-$	0	$+$
y	\	Minimum turning point	/

As shown in the above table, y has a minimum turning point at $x = 1$ and is undefined when $x < 0$.

2i $y' = \frac{2-x}{\sqrt{2+x} \times (1-x)^3} = 0$ when $x = 2$. Therefore, y has a stationary point at $x = 2$ and is undefined when $x \leq -2$ and when $x = 1$.

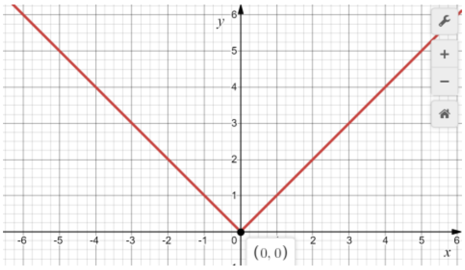
x	-2	-1	1	$\frac{3}{2}$	2	3
y'	undefined	$+$	undefined	$-$	0	$+$
y	undefined	/	undefined	\	Minimum turning point	/

As shown in the above table, y has a minimum turning point at $x = 2$ and is discontinuous at $x = -2$ and $x = 1$.

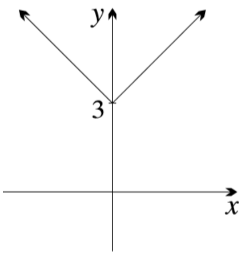
3a

Let $g = |x|$ be a function defined on real numbers.

When the graph of $g = |x|$ (shown below)



is translated 3 units upwards along the y-axis, the graph of $y = |x| + 3$ is obtained (shown below).

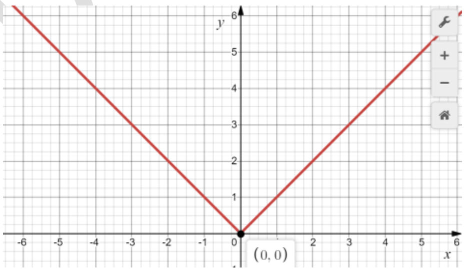


3b $y' = 1$ when $x > 0$ and $y' = -1$ when $x < 0$

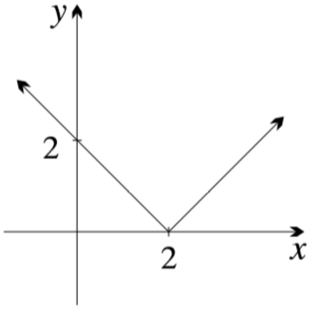
3c y' is not defined at $(0, 3)$, because it is a sharp corner and $y' = 1$ when $x > 0$ and $y' = -1$ when $x < 0$

4a Let $g = |x|$ be a function defined on real numbers.

When the graph of $g = |x|$ (shown below)



is translated 2 units towards right along the x-axis, the graph of $y = |x - 2|$ is obtained (shown below).



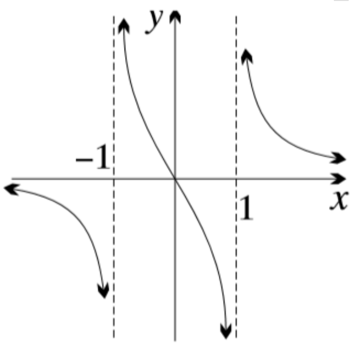
4b $y' = 1$ when $x > 2$ and $y' = -1$ when $x < 2$

4c y' is not defined at $(2, 0)$, because it is a sharp corner and $y' = 1$ when $x > 2$ and $y' = -1$ when $x < 2$

5a $y' = \frac{1\times(x^2-1)-x\times(2x)}{(x^2-1)^2} = \frac{x^2-1-2x^2}{(x^2-1)^2} = \frac{-(x^2+1)}{(x^2-1)^2}$ then $y' < 0$ for all x. Thus, the function y is decreasing for all values of x where it is defined.

$(x^2 - 1)^2 = 0$ when $x^2 - 1 = 0$ or when $x = -1$ or $x = 1$

Therefore, y' has vertical asymptotes at $x = -1$ and $x = 1$



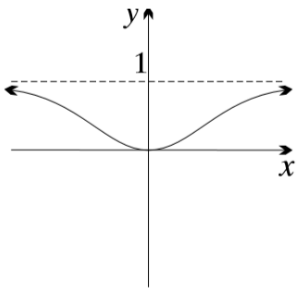
5b $y' = \frac{2x \times (1+x^2) - x^2 \times (2x)}{(1+x^2)^2} = \frac{2x + 2x^3 - 2x^3}{(1+x^2)^2} = \frac{2x}{(1+x^2)^2}$ then $y' < 0$ when $x < 0$ and $y' > 0$ when $x > 0$. Therefore, y is decreasing when $x < 0$ and increasing when $x > 0$.

x	-1	0	1
y'	$-$	0	$+$
y	\	Minimum turning point	/

$f(0) = 0$ then the minimum turning point is $(0, 0)$

$(1 + x^2)^2$ is never zero. Therefore, there are no vertical asymptotes.

Since $\lim_{x \to \pm\infty} \frac{x^2}{1+x^2} = 1$, there is a horizontal asymptote at $y = 1$.



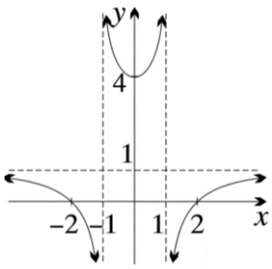
5c $y' = \frac{2x \times (x^2-1) - (x^2-4) \times (2x)}{(x^2-1)^2} = \frac{2x^3 - 2x - 2x^3 + 8x}{(x^2-1)^2} = \frac{6x}{(x^2-1)^2} = 0$ when $x = 0$. Therefore, y has a stationary point at $x = 0$. $f(0) = 4$, then the stationary point is $(0, 4)$.

$(x^2 - 1)^2 = 0$ when $x = -1$ or $x = 1$. Therefore, there are vertical asymptotes at $x = -1$ and $x = 1$.

Since $\lim_{x \to \pm\infty} y = 1$ there is a horizontal asymptote at $y = 1$.

x	-2	-1	$\frac{1}{2}$	0	$\frac{1}{2}$	1	2
y'	$-$	undefined	$-$	0	$+$	undefined	$+$
y	\	Vertical asymptote	\		/	Vertical asymptote	/

				Minimum turning point			



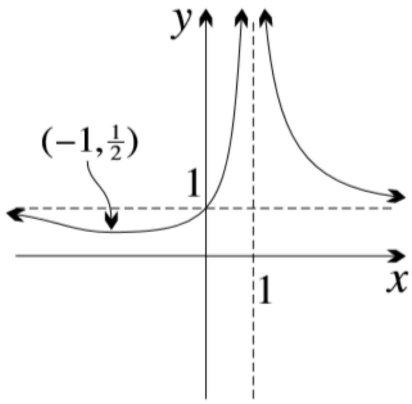
5d $y' = \frac{2x\times(x-1)^2-(x^2+1)\times(2x-2)}{((x-1)^2)^2} = \frac{2x^3-4x+2x-2x^3+2x^2+2x-2}{(x-1)^4} = \frac{-2(x+1)}{(x-1)^3} = 0$ when

$x = -1$. Therefore, there is a stationary point at $x = -1$.

x	-2	-1	0	1	2
y'	$-$	0	$+$	undefined	$-$
y	\	Minimum turning point	/	Vertical asymptote	\

$f(-1) = \frac{1}{2}$ then the minimum turning point is $\left(-1, \frac{1}{2}\right)$ and the y-intercept is $(0, 1)$ because $f(0) = 1$. Since $(x-1)^3 = 0$ when $x = 1$, there is a vertical asymptote at $x = 1$.

Since $\lim_{x\to\pm\infty} y = 1$ there is a horizontal asymptote at $y = 1$.



6a $f'(x) = \frac{1}{5}(x-2)^{-\frac{4}{5}} = \frac{1}{5 \times \sqrt[5]{(x-2)^4}}$

6b $f'(x)$ is never zero.

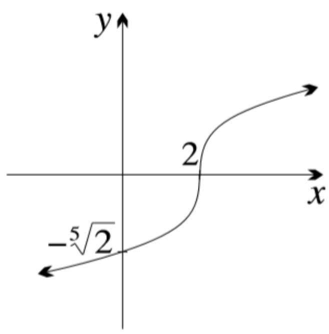
$f(2) = ((2)-2)^{\frac{1}{5}} = 0$ then the function passes through $(2, 0)$ and $f'(2)$ is undefined. Therefore, there is a vertical tangent at $x = 2$

6c Table of slopes:

x	0	2	3
y'	$+$	undefined	$+$
y	/	Vertical tangent	/

Graph:

$f(0) = \left((0) - 2\right)^{\frac{1}{5}} = -\sqrt[5]{2}$. The y-intercept is $\left(0, -\sqrt[5]{2}\right)$



7a $\quad f'(x) = \frac{2}{3}(x-1)^{-\frac{1}{3}} = \frac{2}{3 \times \sqrt[3]{x-1}}$

7b $\quad f'(x)$ is never zero.

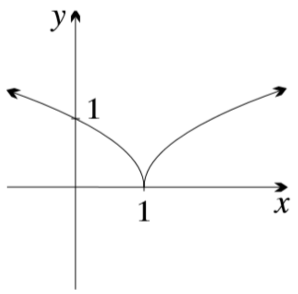
$f(1) = \left((1) - 1\right)^{\frac{1}{5}} = 0$ then the function passes through $(1, 0)$ and $f'(1)$ is undefined. Therefore, there is a vertical tangent at $x = 1$

7c $\quad$ Table of slopes:

x	0	1	2
y'	$-$	undefined	$+$
y	\	Vertical tangent	/

Graph:

$f(0) = \left((0) - 1\right)^{\frac{2}{3}} = 1$. The y-intercept is $(0, 1)$



8a The domain is $\mathbb{R} - \{0\}$ (All the real numbers except zero), because the function is not defined at $x = 0$.

8b $y = x + \frac{1}{x} = \frac{x^2+1}{x}$

$\frac{dy}{dx} = \frac{2x \times x - (x^2+1) \times 1}{x^2} = \frac{x^2-1}{x^2}$

$\frac{dy}{dx} = 0$ when $\frac{x^2-1}{x^2} = 0$ or $x^2 - 1 = 0$ which is satisfied when $x = -1$ or $x = 1$

And $\frac{dy}{dx}$ is undefined when $x = 0$. Therefore, $\frac{dy}{dx}$ is not continuous at $x = 0$.

8c Table of slopes:

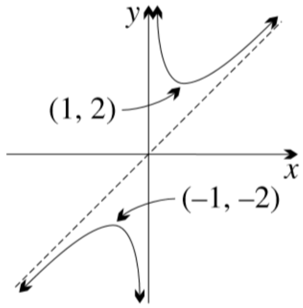
x	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2
y'	$+$	0	$-$	undefined	$-$	0	$+$
y	/	Maximum turning point	\	Vertical Asymptote	\	Minimum turning point	/

8d $\lim_{x \to \pm\infty} \frac{1}{x} = 0$ (As x gets larger, $\frac{1}{x}$ converges to zero)

The vertical asymptote is $x = 0$.

The oblique asymptote is $y = x$ because $y - x \to 0$ as $|x| \to \infty$.

8e



9a $f(x) = \sqrt{x} + \frac{1}{\sqrt{x}}$ and $f(x)$ is not defined for $x \leq 0$.

Therefore, the domain of $f(x)$ is $\mathbb{R}^+$ or $x > 0$ (All positive real numbers).

9b $f(x) = \sqrt{x} + \frac{1}{\sqrt{x}} = \frac{x+1}{\sqrt{x}}$ then

$$f'(x) = \frac{1\times\sqrt{x} - (x+1)\times\frac{1}{2\sqrt{x}}}{x} = \frac{\sqrt{x} - \frac{x+1}{2\sqrt{x}}}{x} = \frac{\frac{2x-x-1}{2\sqrt{x}}}{x} = \frac{x-1}{2x\sqrt{x}}$$

$f'(x) = 0$ when $\frac{x-1}{2x\sqrt{x}} = 0$ or when $x = 1$

9c Table of slopes:

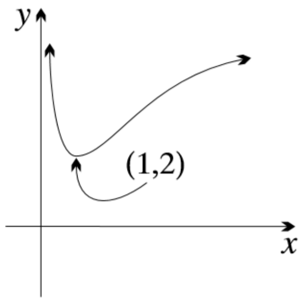
x	0	1	2
y'	$-$	0	$+$
y	\	Minimum turning point	/

$f(1) = \sqrt{1} + \frac{1}{\sqrt{1}} = 2$. Therefore, the minimum turning point is $(1, 2)$

9d $\lim_{x\to\infty} \sqrt{x} + \frac{1}{\sqrt{x}} = \infty$. Therefore as x gets larger, $f(x)$ gets larger.

$\lim_{x\to\infty} \frac{x-1}{2x\sqrt{x}} = 0$. Therefore as x gets larger, $f'(x)$ converges to zero.

9e



10a $y = x - \frac{1}{x} = \frac{x^2-1}{x}$ then y has a vertical asymptote at $x = 0$

$y = 0$ when $x = 1$ or $x = -1$.

Therefore, $(-1, 0)$ and $(1, 0)$ are the x-intercepts of y.

$\frac{dy}{dx} = \frac{2x \times x - (x^2-1) \times 1}{x^2} = \frac{x^2+1}{x^2}$

$\frac{dy}{dx}$ is never zero. Therefore, the graph of y does not have any stationary points.

$\frac{dy}{dx}$ is undefined when $x = 0$. Therefore, $\frac{dy}{dx}$ is not continuous at $x = 0$.

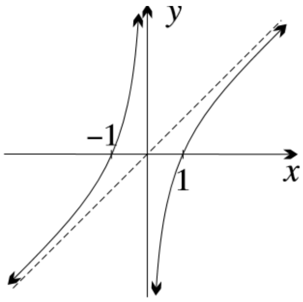
Table of slopes:

x	-1	0	1
y'	$+$	undefined	$+$
y	/	Vertical Asymptote	/

$\lim_{x \to \pm\infty} \frac{1}{x} = 0$ (As x gets larger, $\frac{1}{x}$ converges to zero)

The vertical asymptote is $x = 0$.

The oblique asymptote is $y = x$ because $y - x \to 0$ as $|x| \to \infty$.



10b $y = x^2 + \frac{1}{x^2} = \frac{x^4+1}{x^2}$ then y is never zero. Therefore, the graph of y does not have an x-intercept.

$\frac{dy}{dx} = \frac{4x^3 \times x^2 - (x^4+1) \times 2x}{x^4} = \frac{4x^5 - 2x^5 - 2x}{x^4} = \frac{2x^5 - 2x}{x^4} = \frac{2x(x^4-1)}{x^4} = \frac{2(x^4-1)}{x^3}$

$\frac{dy}{dx} = 0$ when $\frac{2(x^4-1)}{x^3} = 0$ or $x^4 - 1 = 0$ which is satisfied when $x = -1$ or $x = 1$

And $\frac{dy}{dx}$ is undefined when $x = 0$. Therefore, $\frac{dy}{dx}$ is not continuous at $x = 0$.

Table of slopes:

x	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2
y'	$-$	0	$+$	undefined	$-$	0	$+$
y	\	Minimum turning point	/	Vertical Asymptote	\	Minimum turning point	/

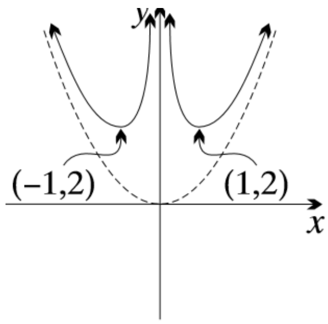
The vertical asymptote is $x = 0$, because y is undefined at $x = 0$.

$y = 2$ when $x = -1$ and $x = 1$. Hence, $(-1, 2)$ and $(1, 2)$ are the minimum turning points.

$\lim_{x \to \pm\infty} \frac{1}{x^2} = 0$ (As x gets larger, $\frac{1}{x^2}$ converges to zero)

The oblique asymptote is $y = x^2$ because $y - x^2 \to 0$ as $|x| \to \infty$.

Therefore, the graph of $y = x^2 + \frac{1}{x^2}$ is,



11a The domain of y is $x \geq 0$ because $\sqrt{x}$ is not defined when $x < 0$ and $y = 0$ when $x = 0$. Therefore, $(0, 0)$ is both the x- and y-intercept of y.

y does not have a vertical asymptote because $\sqrt{9 + x^2}$ is never zero.

$\lim_{x \to \infty} y = 0$ because the denominator of y increases faster than its numerator as x gets larger. Therefore, y has a horizontal asymptote at $x = 0$.

11b $y = \frac{\sqrt{x}}{\sqrt{9+x^2}} = \left(\frac{x}{x^2+9}\right)^{\frac{1}{2}}$ then

$$\frac{dy}{dx} = \frac{1}{2} \times \left(\frac{x}{x^2+9}\right)^{-\frac{1}{2}} \times \frac{1\times(x^2+9) - x \times 2x}{(x^2+9)^2}$$

$$\frac{dy}{dx} = \frac{1}{2} \times \left(\frac{x^2+9}{x}\right)^{\frac{1}{2}} \times \frac{9-x^2}{(x^2+9)^2}$$

$$\frac{dy}{dx} = \frac{\sqrt{x^2+9}}{2\sqrt{x}} \times \frac{9-x^2}{(x^2+9)^2}$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}\times(x^2+9)^{-\frac{1}{2}}} \times \frac{9-x^2}{(x^2+9)^2}$$

$$\frac{dy}{dx} = \frac{9-x^2}{2\sqrt{x}\times(x^2+9)^{\frac{3}{2}}}$$

$$\frac{dy}{dx} = \frac{(3-x)(3+x)}{2\sqrt{x}\times(x^2+9)^{\frac{3}{2}}}$$

11c $\frac{dy}{dx} = 0$ when $\frac{(3-x)(3+x)}{2\sqrt{x}\times(x^2+9)^{\frac{3}{2}}} = 0$ or when $x = -3$ or $x = 3$

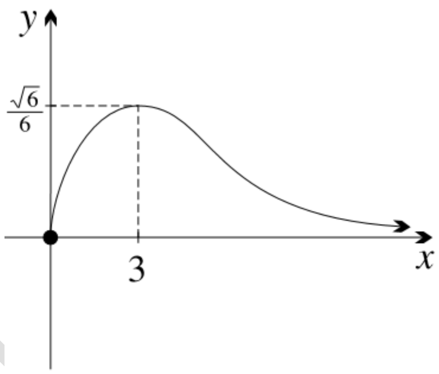
$\frac{dy}{dx}$ is not defined when $x \leq 0$ and $y = \frac{\sqrt{6}}{6}$ when $x = 3$.

Therefore, $\left(3, \frac{\sqrt{6}}{6}\right)$ is the maximum turning point.

The table of slopes is as shown below.

x	0	3	4
y'	$+$	0	$-$
y	/	Maximum turning point	\

11d $y \to 0$ and $\frac{dy}{dx} \to \infty$ as $x \to 0^+$. Therefore, the curve emerges almost vertically from the origin.



12a $y = x^{\frac{1}{2}} - x^{\frac{3}{2}}$ then $y' = \frac{1}{2}x^{-\frac{1}{2}} - \frac{3}{2}x^{\frac{1}{2}} = \frac{1}{2\sqrt{x}} - \frac{3\sqrt{x}}{2} = \frac{\sqrt{x}-3x\sqrt{x}}{2x} = \frac{\sqrt{x}(1-3x)}{2x}$

Hence, $y' = 0$ when $\frac{\sqrt{x}(1-3x)}{2x} = 0$ or $1 - 3x = 0$ or $x = \frac{1}{3}$

$(x \neq 0$ because y' is undefined at $x = 0)$

The coordinates of the point where y' is zero are $\left(\frac{1}{3}, \frac{2\sqrt{3}}{9}\right)$

because $y = \left(\frac{1}{3}\right)^{\frac{1}{2}} - \left(\frac{1}{3}\right)^{\frac{3}{2}} = \frac{1}{\sqrt{3}} - \frac{1}{3\sqrt{3}} = \frac{2}{3\sqrt{3}} = \frac{2\sqrt{3}}{9}$

12b The graph of $y^2 = x(1-x)^2$ can be sketched by first sketching the graph of $y_1 = \sqrt{x(1-x)^2}$ and then $y_2 = -\sqrt{x(1-x)^2}$ on the same set of axes.

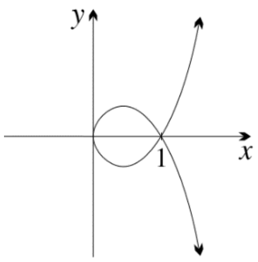
The x-intercepts of both y_1 and y_2 are $(0, 0)$ and $(1, 0)$ because $y_1 = y_2 = 0$ when $x = 0$ and $x = 1$.

$y_1{}' = \sqrt{x} + \frac{x-1}{2\sqrt{x}} > 0$ when $x > 1$. Therefore, y_1 is increasing when $x > 1$

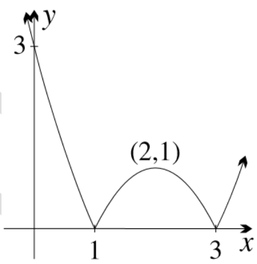
$y_1{}' = \sqrt{x} + \frac{x-1}{2\sqrt{x}} = 0$ when $x = \frac{1}{3}$.

Therefore, y_1 has a minimum turning point at $x = \frac{1}{3}$.

Since y_2 is the reflection of y_1 in the x-axis, the graph of $y^2 = x(1-x)^2$ is:

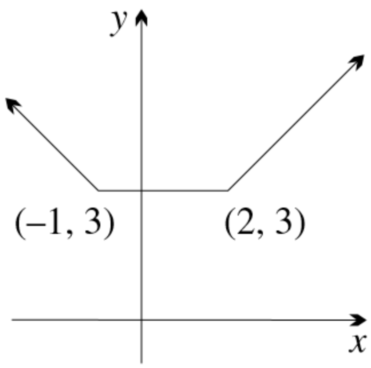


13a To sketch the graph of $f(x)$, sketch the graph of $g(x) = (x-1)(x-3)$ and then fold the part of the function below the x-axis such that $g(x)$ is never zero.



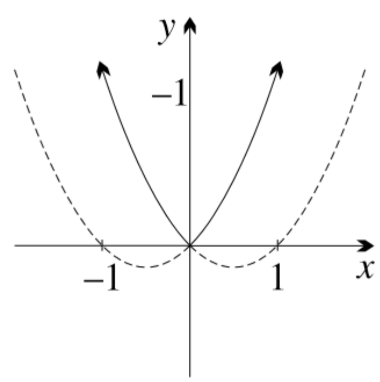
13b To sketch the graph of $y = |x - 2| + |x + 1|$ construct a sign-table as shown below (the critical points are $x = 2$ and $x = -1$)

x	$x < -1$	-1	$-1 < x < 2$	2	$x > 2$
y	$-2x + 1$	3	3	3	$2x - 1$



13c To sketch the graph of $y = x^2 + |x|$ construct a sign-table as shown below (the critical point is $x = 0$)

x	$x < 0$	0	$x > 0$
y	$x^2 - x$	3	$x^2 + x$



Solutions to Exercise 4D

1a $y = x^3$

$y' = 3x^2$

$y'' = 6x$

$y''' = 6$

1b $y = x^{10}$

$y' = 10x^9$

$y'' = 90x^8$

$y''' = 720x^7$

1c $y = x^7$

$y' = 7x^6$

$y'' = 42x^5$

$y''' = 210x^4$

1d $y = x^2$

$y' = 2x$

$y'' = 2$

$y''' = 0$

1e $y = 2x^4$

$y' = 8x^3$

$y'' = 24x^2$

$y''' = 48x$

1f $y = 3x^5$

$y' = 15x^4$

$y'' = 60x^3$

$y''' = 180x^2$

1g $y = 4 - 3x$

$y' = -3$

$y'' = 0$

$y''' = 0$

1h $y = x^2 - 3x$

$y' = 2x - 3$

$y'' = 2$

$y''' = 0$

1i $y = 4x^3 - x^2$

$y' = 12x^2 - 2x$

$y'' = 24x - 2$

$y''' = 24$

1j $y = 4x^5 + 2x^3$

$y' = 20x^4 + 6x^2$

$y'' = 80x^3 + 12x$

$y''' = 240x^2 + 12$

2a $y = x(x + 3)$

$y' = 1 \times (x + 3) + x \times 1$

$= 2x + 3$

$y'' = 2$

2b $y = x^2(x-4)$

$y' = 2x \times (x-4) + x^2 \times 1$

$= 3x^2 - 8x$

$y'' = 6x - 8$

2c $y = (x-2)(x+1)$

$y' = 1 \times (x+1) + (x-2) \times 1$

$= 2x - 1$

$y'' = 2$

2d $y = (3x+2)(x-5)$

$y' = 3 \times (x-5) + (3x+2) \times 1$

$= 6x - 13$

$y'' = 6$

2e $y = 3x^2(2x^3 - 3x^2)$

$y' = 6x \times (2x^3 - 3x^2) + 3x^2 \times (6x^2 - 6x)$

$= 12x^4 - 18x^3 + 18x^4 - 18x^3$

$= 30x^4 - 36x^3$

$y'' = 120x^3 - 108x^2$

2f $y = 4x^3(x^5 + 2x^2)$

$y' = 12x^2 \times (x^5 + 2x^2) + 4x^3 \times (5x^4 + 4x)$

$= 12x^7 + 24x^4 + 20x^7 + 16x^4$

$= 32x^7 + 40x^4$

$y'' = 224x^6 + 160x^3$

3a $y = x^{0.3}$

$y' = 0.3x^{-0.7}$

$y'' = -0.21x^{-1.7}$

$y''' = 0.357x^{-2.7}$

3b $y = x^{-1}$

$y' = -x^{-2}$

$= -\frac{1}{x^2}$

$y'' = 2x^{-3}$

$= \frac{2}{x^3}$

$y''' = -6x^{-4}$

$= -\frac{6}{x^4}$

3c $y = x^{-2}$

$y' = -2x^{-3}$

$= -\frac{2}{x^3}$

$y'' = 6x^{-4}$

$= \frac{6}{x^4}$

$y''' = -24x^{-5}$

$= -\frac{24}{x^5}$

3d $y = 5x^{-3}$

$y' = -15x^{-4}$

$= -\frac{15}{x^4}$

$y'' = 60x^{-5}$

$= \frac{60}{x^5}$

$y''' = -300x^{-6}$

$= -\frac{300}{x^6}$

3e $y = x^2 + x^{-1}$

$y' = 2x - x^{-2}$

$= 2x - \frac{1}{x^2}$

$y'' = 2 + 2x^{-3}$

$= 2 + \frac{2}{x^3}$

$y''' = -6x^{-4}$

$= -\frac{6}{x^4}$

4a $f(x) = x^{-3}$

$f'(x) = -3x^{-4} = -\frac{3}{x^4}$

$f''(x) = 12x^{-5} = \frac{12}{x^5}$

4b $f(x) = x^{-4}$

$f'(x) = -4x^{-5} = -\frac{4}{x^5}$

$f''(x) = 20x^{-6} = \frac{20}{x^6}$

4c $f(x) = 3x^{-2}$

$f'(x) = -6x^{-3} = -\frac{6}{x^3}$

$f''(x) = 18x^{-4} = \frac{18}{x^4}$

4d $f(x) = 2x^{-3}$

$f'(x) = -6x^{-4} = -\frac{6}{x^4}$

$f''(x) = 24x^{-5} = \frac{24}{x^5}$

5a $y = (x+1)^2$

$y' = 2(x+1)$

$= 2x + 2$

$y'' = 2$

5b $y = (3x-5)^3$

$y' = 3(3x-5)^2 \times 3$

$= 9(3x-5)^2$

$y'' = 18(3x-5) \times 3$

$= 54(3x-5)$

5c $y = (1-4x)^2$

$y' = 2(1-4x) \times (-4)$

$= -8(1-4x)$

$= 32x - 8$

$y'' = 32$

5d $y = (8-x)^{11}$

$y' = 11(8-x)^{10} \times (-1)$

$= -11(8-x)^{10}$

$y'' = -110(8-x)^9 \times (-1)$

$= 110(8-x)^9$

6a $y = (x+2)^{-1}$

$y' = -(x+2)^{-2}$

$= -\dfrac{1}{(x+2)^2}$

$y'' = 2(x+2)^{-3}$

$= \dfrac{2}{(x+2)^3}$

6b $y = (3-x)^{-2}$

$y' = -2(3-x)^{-3} \times (-1)$

$= \dfrac{2}{(3-x)^3}$

$y'' = -6(3-x)^{-4} \times (-1)$

$= \dfrac{6}{(3-x)^4}$

6c $y = (5x+4)^{-3}$

$y' = -3(5x+4)^{-4} \times (5)$

$= -\dfrac{15}{(5x+4)^4}$

$y'' = 60(5x+4)^{-5} \times (5)$

$= \dfrac{300}{(5x+4)^5}$

6d $y = 2(4-3x)^{-2}$

$y' = -4(4-3x)^{-3} \times (-3)$

$= \dfrac{12}{(4-3x)^3}$

$y'' = -36(4-3x)^{-4} \times (-3)$

$= \dfrac{108}{(4-3x)^4}$

7a $f(x) = \sqrt{x}$

$= x^{\frac{1}{2}}$

$f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$

$= \frac{1}{2\sqrt{x}}$

$f''(x) = -\frac{1}{4}x^{-\frac{3}{2}}$

$= -\frac{1}{4x\sqrt{x}}$

7b $f(x) = \sqrt[3]{x}$

$= x^{\frac{1}{3}}$

$f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$

$f''(x) = -\frac{2}{9}x^{-\frac{5}{3}}$

7c $f(x) = x\sqrt{x}$

$= x^{\frac{3}{2}}$

$f'(x) = \frac{3}{2}x^{\frac{1}{2}}$

$= \frac{3\sqrt{x}}{2}$

$f''(x) = \frac{3}{4}x^{-\frac{1}{2}}$

$= \frac{3}{4\sqrt{x}}$

7d $f(x) = \frac{1}{\sqrt{x}}$

$= x^{-\frac{1}{2}}$

$f'(x) = -\frac{1}{2}x^{-\frac{3}{2}}$

$= -\frac{1}{2x\sqrt{x}}$

$f''(x) = \frac{3}{4}x^{-\frac{5}{2}}$

$= \dfrac{3}{4x^2\sqrt{x}}$

7e $f(x) = \sqrt{x+2}$

$= (x+2)^{\frac{1}{2}}$

$f'(x) = \frac{1}{2}(x+2)^{-\frac{1}{2}} \times 1$

$= \dfrac{1}{2\sqrt{x+2}}$

$f''(x) = -\frac{1}{4}(x+2)^{-\frac{3}{2}} \times 1$

$= -\dfrac{1}{4(x+2)^{\frac{3}{2}}}$

7f $f(x) = \sqrt{1-4x}$

$= (1-4x)^{\frac{1}{2}}$

$f'(x) = \frac{1}{2}(1-4x)^{-\frac{1}{2}} \times (-4)$

$= -\dfrac{2}{\sqrt{1-4x}}$

$f''(x) = (1-4x)^{-\frac{3}{2}} \times (-4)$

$= -\dfrac{4}{(1-4x)^{\frac{3}{2}}}$

8a $f'(x) = 3x^2 + 6x + 5$

$f''(x) = 6x + 6$

8b i $f'(0) = 3(0)^2 + 6(0) + 5 = 5$

8b ii $f'(1) = 3(1)^2 + 6(1) + 5 = 14$

8b iii $f''(0) = 6 \times (0) + 6 = 6$

8b iv $f''(1) = 6 \times (1) + 6 = 12$

9a i $f(x) = 3x + x^3$

$f'(x) = 3 + 3x^2$

$f'(2) = 3 + 3 \times 2^2 = 15$

9a ii $f(x) = 3x + x^3$

$f'(x) = 3 + 3x^2$

$f''(x) = 6x$

$f''(2) = 6 \times 2 = 12$

9a iii $f(x) = 3x + x^3$

$f'(x) = 3 + 3x^2$

$f''(x) = 6x$

$f'''(x) = 6$

$f'''(2) = 6$

9a iv $f(x) = 3x + x^3$

$f'(x) = 3 + 3x^2$

$f''(x) = 6x$

$f'''(x) = 6$

$f''''(x) = 0$

$f''''(2) = 0$

9b i $f(x) = (2x - 3)^4$

$f'(x) = 4(2x - 3)^3 \times 2$

$f'(x) = 8(2x - 3)^3$

$f'(1) = 8(2 \times 1 - 3)^3 = -8$

9b ii $f''(x) = 24(2x - 3)^2 \times 2$

$f''(x) = 48(2x - 3)^2$

$f''(1) = 48(2 \times 1 - 3)^2 = 48$

9b iii $f'''(x) = 96(2x - 3) \times 2$

$f'''(1) = 192(2 \times 1 - 3) = -192$

9b iv $f''''(x) = 192 \times 2 = 384$

$f''''(1) = 384$

10a $y = \frac{x}{x+1}$

$$y' = \frac{(x+1) - x}{(x+1)^2}$$

$$= \frac{1}{(x+1)^2}$$

$$= (x+1)^{-2}$$

$$y'' = -2(x+1)^{-3}$$

$$= \frac{-2}{(x+1)^3}$$

10b $y = \frac{x-1}{2x+5}$

$$y' = \frac{(2x+5) - (x-1) \times 2}{(2x+5)^2}$$

$$= \frac{7}{(2x+5)^2}$$

$= 7(2x+5)^{-2}$

$y'' = -14(2x+5)^{-3} \times 2$

$= \dfrac{-28}{(2x+5)^3}$

11 $f(x) = x(x-1)^4$

$f'(x) = 1 \times (x-1)^4 + x \times 4(x-1)^3 \times 1$

$= (x-1)^4 + 4x(x-1)^3$

$= (x-1)^3(5x-1)$

$f''(x) = 3(x-1)^2 \times 1 \times (5x-1) + (x-1)^3 \times 5$

$= (x-1)^2 \times (15x-3) + 5(x-1)^3$

$= (x-1)^2[(15x-3) + 5(x-1)]$

$= (x-1)^2(20x-8)$

$= 4(x-1)^2(5x-2)$

12a $y' = 4x^3 - 12x$

$y'' = 12x^2 - 12$

$12x^2 - 12 = 0$ when $x^2 = 1$

Therefore, $y'' = 0$ when $x = -1$ or $x = 1$.

12b $y = x^3 + x^2 - 5x + 7$

$y' = 3x^2 + 2x - 5$

$y'' = 6x + 2$

$6x + 2 = 0$ when $x = -\frac{1}{3}$

Therefore, $y'' = 0$ when $= -\frac{1}{3}$.

13a Let $y = x^n$.

$$y' = nx^{n-1}$$

$$y'' = n(n-1)x^{n-2}$$

$$y''' = n(n-1)(n-2)x^{n-3}$$

13b Continuing the pattern from part a until $n-k=0$ where k is the order of the derivative we obtain:

$$y^{(k)} = n(n-1)(n-2)\ldots(n-k+1)x^{n-k}$$

Setting $n=k$ we obtain:

$$\begin{aligned} y^{(n)} &= \left(n(n-1)(n-2)\ldots 1\right)x^0 \\ &= n(n-1)(n-2)\ldots 1 \end{aligned}$$

The $(n+1)^{\text{st}}$ derivative of x^n is:

$$\begin{aligned} y^{(n+1)} &= \frac{d}{dx}\left(n(n-1)(n-2)\ldots 1\right) \\ &= 0 \end{aligned}$$

14a

$$\frac{d}{dx}\left(x\frac{dy}{dx}\right)$$

$$= \frac{d}{dx}\left(x(6x+7)\right)$$

$$= \frac{d}{dx}(6x^2+7x)$$

$$= 12x + 7$$

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx}$$

$$= x(6) + (6x+7)$$

$$= 12x + 7$$

Therefore, $\frac{d}{dx}\left(x\frac{dy}{dx}\right) = x\frac{d^2y}{dx^2} + \frac{dy}{dx}$

14b

$$\frac{d}{dx}\left(y\frac{dy}{dx}\right)$$

$$= \frac{d}{dx}\left((2x-1)^4(4 \times (2x-1)^3 \times 2)\right)$$

$$= \frac{d}{dx}(8(2x-1)^7)$$

$$= 56 \times (2x-1)^6 \times 2 = 112(2x-1)^6$$

$$y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2$$

$$= (2x-1)^4(24(2x-1)^2 \times 2) + (8(2x-1)^3)^2$$

$$= 48(2x-1)^6 + (8(2x-1)^3)^2$$

$$= 48(2x-1)^6 + 64(2x-1)^6$$

$$= 112(2x-1)^6$$

Therefore, $\frac{d}{dx}\left(y\frac{dy}{dx}\right) = y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2$

14c $y = 2x^2 - \frac{3}{\sqrt{x}} = 2x^2 - 3x^{-\frac{1}{2}}$ then $\frac{dy}{dx} = 4x + \frac{3}{2}x^{-\frac{3}{2}}$ and $\frac{d^2y}{dx^2} = 4 - \frac{9}{4}x^{-\frac{5}{2}}$

Hence, $2x^2\frac{d^2y}{dx^2} = 2x^2\left(4 - \frac{9}{4}x^{-\frac{5}{2}}\right) = 8x^2 - \frac{9}{2}x^{-\frac{1}{2}}$ and

$x\frac{dy}{dx} + 2y = x\left(4x + \frac{3}{2}x^{-\frac{3}{2}}\right) + 2\left(2x^2 - 3x^{-\frac{1}{2}}\right) = 8x^2 - \frac{9}{2}x^{-\frac{1}{2}}$

Therefore, $2x^2\frac{d^2y}{dx^2} = x\frac{dy}{dx} + 2y$

15 $y = x^a + x^{-b}$ then $y' = ax^{a-1} - bx^{-b-1}$ and

$y'' = a(a-1)x^{a-2} + b(b+1)x^{-b-2}$

Hence,

$x^2y'' + 2xy' = x^2(a(a-1)x^{a-2} + b(b+1)x^{-b-2}) + 2x(ax^{a-1} - bx^{-b-1})$

$$= a(a-1)x^a + b(b+1)x^{-b} + 2ax^a - 2bx^{-b}$$
$$= (a^2+a)x^a + (b^2-b)x^{-b}$$

$(a^2+a)x^a + (b^2-b)x^{-b} = 12(x^a + x^{-b})$ when $a^2 + a = 12$ and $b^2 - b = 12$

Or when $a^2 + a - 12 = 0$ and $b^2 - b - 12 = 0$.

Therefore, $a = -4$ or $a = 3$ and $b = -3$ or $b = 4$.

A possible combination can be $a = 3$ and $b = 4$.

Solutions to Exercise 4E

1a

Point	A	B	C	D	E	F	G	H	I
y'	0	+	0	−	0	−	0	+	0
y''	+	0	−	0	0	0	+	0	0

2a $f(x) = x^3 - 3x^2$

$f'(x) = 3x^2 - 6x$

$f''(x) = 6x - 6$

For $x = 0$, $f''(0) = -6$ and $-6 < 0$. Therefore, $f''(x)$ is concave down at $x = 0$.

2b $f(x) = x^3 + 4x^2 - 5x + 7$

$f'(x) = 3x^2 + 8x - 5$

$f''(x) = 6x + 8$

For $x = 0$, $f''(0) = 8$ and $8 > 0$. Therefore, $f''(x)$ is concave up at $x = 0$.

2c $f(x) = x^4 + 2x^2 - 3$

$f'(x) = 4x^3 + 4x$

$f''(x) = 12x^2 + 4$

For $x = 0$, $f''(0) = 4$ and $4 > 0$. Therefore, $f''(x)$ is concave up at $x = 0$.

2d $f(x) = 6x - 7x^2 - 8x^4$

$f'(x) = 6 - 14x - 32x^3$

$f''(x) = -14 - 96x^2$

For $x = 0$, $f''(0) = -14$ and $-14 < 0$.

Therefore, $f''(x)$ is concave down at $x = 0$.

3a $f(x) = x^2 - 4x + 4$

$f'(x) = 2x - 4$

For $x = 2$, $f'(2) = 2 \times 0 - 4 = 0$. So $f(x)$ has a stationary point at $x = 2$.

$f''(x) = 2$ and since $f''(x) > 0$ for all values of x, $f(x)$ is concave up for all values of x. Therefore, the stationary point at $x = 2$ is a local minimum.

3b $f(x) = 5 + 4x - x^2$

$f'(x) = 4 - 2x$

For $x = 2$, $f'(2) = 4 - 2 \times 2 = 0$. So $f(x)$ has a stationary point at $x = 2$.

$f''(x) = -2$ and since $f''(x) < 0$ for all values of x, $f(x)$ is concave down for all values of x. Therefore, the stationary point at $x = 2$ is a local maximum.

3c $f(x) = x^3 - 12x$

$f'(x) = 3x^2 - 12$

For $x = 2$, $f'(2) = 3 \times 2^2 - 12 = 0$. So $f(x)$ has a stationary point at $x = 2$.

$f''(x) = 6x$

For $x = 2$, $f''(2) = 6 \times 2 = 12$. Since $f''(2) > 0$, $f(x)$ is concave up at $x = 2$.

Therefore, the stationary point at $x = 2$ is a local minimum.

3d $f(x) = 2x^3 - 3x^2 - 12x + 5$

$f'(x) = 6x^2 - 6x - 12$

For $x = 2$, $f'(2) = 6 \times 2^2 - 6 \times 2 - 12 = 0$. So $f(x)$ has a stationary point at $x = 2$.

$f''(x) = 12x - 6$

For $x = 2$, $f''(2) = 12 \times 2 - 6 = 18$. Since $f''(2) > 0$, $f(x)$ is concave up at $x = 2$. Therefore, the stationary point at $x = 2$ is a local minimum.

4a $y = x^2 - 3x + 7$

$\frac{dy}{dx} = 2x - 3$

$\frac{d^2y}{dx^2} = 2$

$\frac{d^2y}{dx^2} > 0$ for all values of x. Therefore, the curve $y = x^2 - 3x + 7$ is concave up for all values of x.

4b $y = -3x^2 + 2x - 4$

$\frac{dy}{dx} = -6x + 2$

$\frac{d^2y}{dx^2} = -6$

$\frac{d^2y}{dx^2} < 0$ for all values of x. Therefore, the curve $y = -3x^2 + 2x - 4$ is concave down for all values of x.

5a $y = x^3 - 3x^2 - 5x + 2$

$\frac{dy}{dx} = 3x^2 - 6x - 5$

$\frac{d^2y}{dx^2} = 6x - 6$

5b i $\frac{d^2y}{dx^2} = 0$ when $6x - 6 = 0$ or $x = 1$.

$\frac{d^2y}{dx^2} > 0$ when $6x - 6 > 0$ or $6x > 6$ or $x > 1$

Therefore, $y = x^3 - 3x^2 - 5x + 2$ is concave up for $x > 1$.

5b ii $\frac{d^2y}{dx^2} < 0$ when $6x - 6 < 0$ or $6x < 6$ or $x < 1$

Therefore, $y = x^3 - 3x^2 - 5x + 2$ is concave down for $x < 1$.

6a $\frac{dy}{dx} = 3x^2 - 2x - 5x$

$\frac{d^2y}{dx^2} = 6x - 2$

6b i $\frac{d^2y}{dx^2} = 6x - 2$ and $6x - 2 = 0$ when $x = 3$

$\frac{d^2y}{dx^2} > 0$ when $6x - 2 > 0$ or $6x > 2$ or $x > \frac{1}{3}$.

Therefore, y is concave up when $x > \frac{1}{3}$.

6b ii $\frac{d^2y}{dx^2} < 0$ when $x < 3$. Therefore, y is concave down when $x < 3$.

7 $y'' = 3x^3(x+3)^2(x-2)$

$y'' = 0$ when $3x^3(x+3)^2(x-2) = 0$,

$x = 0$, $x = -3$ or $x = 2$.

As shown in the below table, $y'' > 0$ for $x < -3$ and $-3 < x < 0$.

Hence, $x = -3$ is not an inflection point.

x	-4	-3	-1	0	1	2	3
y''	$+$	0	$+$	0	$-$	0	$+$

The x-coordinates of the points of inflection are $x = 0$ and $x = 2$.

8a $f(x) = x^3 - 3x$

$f'(x) = 3x^2 - 3 = 3(x^2 - 1) = 3(x-1)(x+1)$

$f''(x) = 6x$

8b $3(x-1)(x+1) = 0$ when $x = -1$ and $x = 1$

$f(-1) = (-1)^3 - 3 \times (-1) = 2$. Therefore, $(-1, 2)$ is a stationary point.

$f(1) = (1)^3 - 3 \times (1) = -2$. Therefore, $(1, -2)$ is a stationary point.

8c $f''(-1) = 6 \times (-1) = -6$ so $f''(-1) < 0$.

Therefore, $(-1, 2)$ is a local maximum turning point.

$f''(1) = 6 \times 1 = 6$ so $f''(1) > 0$.

Therefore, $(1, -2)$ is a local minimum turning point.

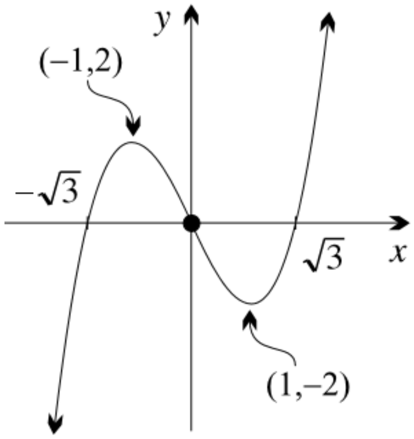
8d $f''(x) = 6x = 0$ when $x = 0$

$f(0) = (0)^3 - 3 \times 0 = 0$ so $(0, 0)$ is the point of inflection.

8e $f(x) = x^3 - 3x$

$x^3 - 3x = 0$ when $x\left(x - \sqrt{3}\right)\left(x + \sqrt{3}\right) = 0$

Therefore, $(0, 0)$, $\left(\sqrt{3}, 0\right)$ and $\left(-\sqrt{3}, 0\right)$ are the x-intercepts.



9a $f(x) = x^3 - 6x^2 - 15x + 1$

$f'(x) = 3x^2 - 12x - 15$

$= 3(x^2 - 4x - 5)$

$= 3(x - 5)(x + 1)$

$f''(x) = 6x - 12$

$= 6(x - 2)$

9b When $f'(x) = 0$, $3(x-5)(x+1) = 0$ so $x = 5$ or $x = -1$.

Therefore, there is a stationary point at both $x = 5$ and $x = -1$.

$f(5) = 5^3 - 6 \times 5^2 - 15 \times 5 + 1 = -99$

$f(-1) = (-1)^3 - 6 \times (-1)^2 - 15 \times (-1) + 1 = 9$

At $x = 5$, $f''(5) = 6(5-2) = 18$.

Since $f''(5) > 0$, $f(x)$ is concave up at $x = 5$ and the stationary point at $(5, -99)$ is a local minimum.

At $x = -1$, $f''(-1) = 6(-1-2) = -18$.

Since $f''(-1) < 0$, $f(x)$ is concave down at $x = -1$ and the stationary point at $(-1, 9)$ is a local maximum.

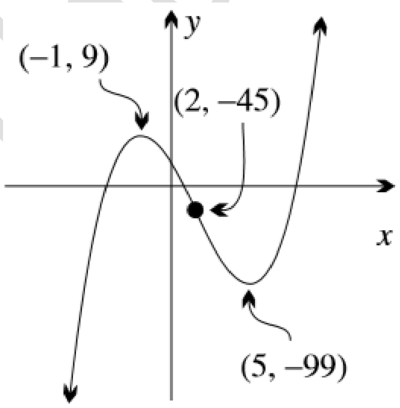
9c $f''(x) = 0$ when $6(x-2) = 0$ or when $x = 2$

$f(2) = 2^3 - 6 \times 2^2 - 15 \times 2 + 1 = -45$

x	1	2	3
$f''(x)$	$-$	0	$+$

$f(x)$ is concave down for $x < 2$ and concave up for $x > 2$. Therefore, there is a point of inflection at $(2, -45)$.

9d



10a $f'(x) = 3x^2 - 6x - 9 = 3(x^2 - 2x - 3) = 3(x-3)(x+1)$

$f''(x) = 6x - 6 = 6(x-1)$

10b $3(x-3)(x+1) = 0$ when $x = -1$ and $x = 3$

$f(-1) = (-1)^3 - 3 \times (-1)^2 - 9 \times (-1) + 11 = 16$

$f(3) = 3^3 - 3 \times 3^2 - 9 \times 3 + 11 = -16$

Therefore, $(-1, 16)$ and $(3, -16)$ are stationary points.

$f''(-1) = 6 \times (-1) - 6 = -12$ so $f''(-1) < 0$.

Therefore, $(-1, 16)$ is a maximum turning point.

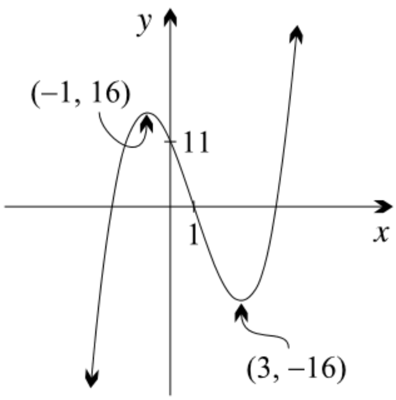
$f''(3) = 6 \times 3 - 6 = 12$ so $f''(3) > 0$.

Therefore, $(3, -16)$ is a minimum turning point.

10c $f''(x) = 6x - 6 = 0$ when $x = 1$ and $f(1) = 0$

Therefore, $(1, 0)$ is an inflection point.

10d



Since $f(0) = 11$, the y-intercept is $(0, 11)$.

11a $y = 3 + 4x^3 - x^4$

$y' = 12x^2 - 4x^3$

$= 4x^2(3 - x)$

$$y'' = 24x - 12x^2$$
$$= 12x(2 - x)$$

11b $y' = 0$ when $4x^2(3 - x) = 0$ or when $x = 0$ or $x = 3$

x	-1	0	1	3	4
y'	/	0	/	0	\

$f(x)$ is increasing for $x < 0$ and $0 < x < 3$.

Therefore, there is a stationary point of inflection at $x = 0$.

$f(x)$ is increasing for $0 < x < 3$ and decreasing for $x > 3$.

Therefore, there is a local maximum at $x = 3$.

For $x = 3$ $y = 3 + 4 \times 3^3 - 3^4 = 30$

The local maximum point is at $(3, 30)$.

11c $y'' = 0$ when $12x(2 - x) = 0$ or when $x = 0$ or $x = 2$

x	-1	0	1	2	3
y''	$-$	0	$+$	0	$-$

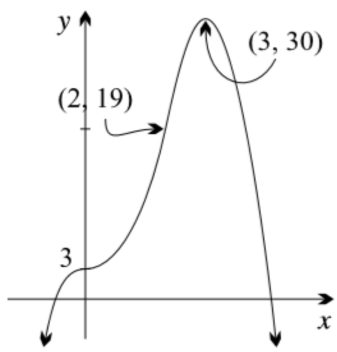
Therefore, there is a point of inflection at $x = 0$ and $x = 2$

$f(0) = 3 + 4 \times 0^3 - 0^4 = 3$

$f(2) = 3 + 4 \times 2^3 - 2^4 = 19$

Stationary points of inflection are at $(0, 3)$ and $(2, 19)$.

11d



12a $y' = 6x^2 - 6x - 12 = 6(x^2 - x - 2) = 6(x-2)(x+1)$

$6(x-2)(x+1) = 0$ when $x = -1$ and $x = 2$

x	-2	-1	0	2	3
y'	$+$	0	$-$	0	$+$
slope	/	Maximum turning point	\	Minimum turning point	/

y is increasing when $y' > 0$ or when $x < -1$ and $x > 2$

12b y is decreasing when $y' < 0$ or when $-1 < x < 2$ (refer to the table in 12a)

12c $y'' = 12x - 6$ and $12x - 6 = 0$ when $x = \frac{1}{2}$

$y'' > 0$ when $12x - 6 > 0$ or $12x > 6$ or when $x > \frac{1}{2}$

12d $y'' < 0$ when $12x - 6 < 0$ or $12x < 6$ or when $x < \frac{1}{2}$

13a $y = x^3 + 3x^2 - 72x + 14$

$y' = 3x^2 + 6x - 72$

$= 3(x^2 + 2x - 24)$

$= 3(x+6)(x-4)$

$y'' = 6x + 6$

$= 6(x+1)$

13b $y'' = 0$ when $6(x+1) = 0$ or $x = -1$

Therefore, there is a point of inflection at $x = -1$.

When $x = -1, y = (-1)^3 + 3(-1)^2 - 72 \times (-1) + 14 = 88$.

Therefore the point of inflection is at $(-1, 88)$.

13c The gradient of the tangent at $x = -1$ is

$f'(-1) = 3(-1)^2 + 6 \times (-1) - 72 = -75$

13d The equation of the tangent at $(-1, 88)$ is

$y - 88 = -75\big(x - (-1)\big)$

$y = -75x + 13$ or $75x + y - 13 = 0$

14a $f'(x) = 3x^2$ and $f''(x) = 6x$

$g'(x) = 4x^3$ and $g''(x) = 12x^2$

14b $f''(0) = 6 \times 0 = 0$

$g''(x) = 12 \times 0^2 = 0$

No, we cannot determine the nature of the stationary points from this calculation as $f''(0) = g''(x) = 0$.

14c

x	-1	0	1
$f'(x)$	$+$	0	$+$
$f''(x)$	$-$	0	$+$
f	/	Stationary point of inflection	/
$g'(x)$	$-$	0	$+$
$g''(x)$	$+$	0	$+$
g	\	Minimum turning point	/

f has a stationary point of inflection at $x = 0$ and g has a minimum turning point at $x = 0$.

15a $y = x^3 - ax^2 + 3x - 4$

$y' = 3x^2 - 2ax + 3$

$y'' = 6x - 2a$

If $y'' = 0$ when $x = 2$, then $6 \times 2 - 2a = 0$ or $2a = 12$

Therefore $a = 6$.

15b $y = x^3 + 2ax^2 + 3x - 4$

$y' = 3x^2 + 4ax + 3$

$y'' = 6x + 4a$

When $x = -1, y'' > 0$

$6 \times (-1) + 4a > 0$

$4a > 6$

$a > \dfrac{3}{2}$

Therefore y'' is concave up when $a > 1\frac{1}{2}$.

15c $y = x^4 + ax^3 + bx^2$

$y' = 4x^3 + 3ax^2 + 2bx$

$y'' = 12x^2 + 6ax + 2b$

If y has a point of inflection at $(2, 0)$ then when $x = 2, y'' = 0$. Hence:

$12 \times 2^2 + 6a \times 2 + 2b = 0$

$48 + 12a + 2b = 0$

$12a + 2b = -48$

$6a + b = -24 \qquad (1)$

Also when $x = 2, y = 0$

$2^4 + a \times 2^3 + b \times 2^2 = 0$

$16 + 8a + 4b = 0$

$8a + 4b = -16$

$2a + b = -4 \qquad (2)$

(1) − (2) gives:

$4a = -20$

$a = -5$

Substituting $a = -5$ into (2) gives:

$-10 + b = -4$

$b = 6$

15d $y = x^4 + ax^3 - x^2$

$y' = 4x^3 + 3ax^2 - 2x$

$y'' = 12x^2 + 6ax - 2$

When $x = 1, y'' > 0$

$12 \times 1^2 + 6a \times 1 - 2 > 0$

$12 + 6a - 2 > 0$

$6a > -10$

$a > -\frac{5}{3}$

So y'' is concave up when $a > -\frac{5}{3}$.

y is increasing when $y' > 0$

$4 \times 1^3 + 3a \times 1^2 - 2 \times 1 > 0$

$4 + 3a - 2 > 0$

$3a > -2$

$a > -\frac{2}{3}$

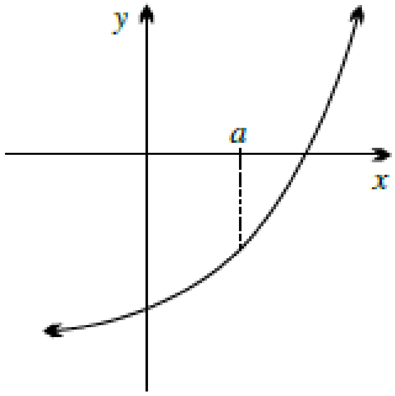
So y' is increasing when $a > -\frac{2}{3}$.

Therefore, y is concave up and increasing when $a > -\frac{2}{3}$.

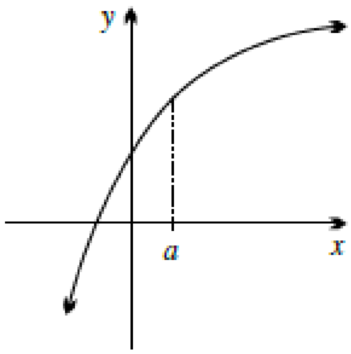
16a Increasing, because $f'(x) > 0$ in the domain of f.

16b Since $f'(x)$ is decreasing in the domain of f, f is concave down in its domain.

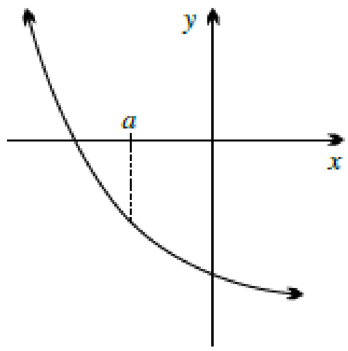
17a If $f'(a) > 0$ and $f''(a) > 0$, then the continuous function $f(x)$ about $x = a$ is increasing and concave up about $x = a$.



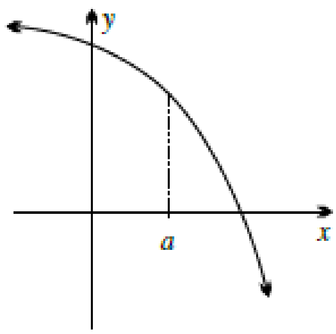
17b If $f'(a) > 0$ and $f''(a) < 0$, then the continuous function $f(x)$ about $x = a$ is increasing and concave down about $x = a$.



17c If $f'(a) < 0$ and $f''(a) > 0$, then the continuous function $f(x)$ about $x = a$ is decreasing and concave up about $x = a$.



17d If $f'(a) < 0$ and $f''(a) < 0$, then the continuous function $f(x)$ about $x = a$ is decreasing and concave down about $x = a$.



18a $y = \frac{1}{3}x^3 - 3x^2 + 11x - 9$

$y' = x^2 - 6x + 11$

$= (x^2 - 6x + 9) - 9 + 11$

$= (x - 3)^2 + 2$

So $y' \geq 2$ for all real x. Hence the equation $y' = 0$ has no solutions and so the graph of the function has no stationary points.

18b $y' = x^2 - 6x + 11$

$y'' = 2x - 6$

$y'' = 0$ when $2x - 6 = 0$ or $x = 3$

So there is a point of inflection at $x = 3$.

When $x = 3, y = \frac{1}{3} \times 3^3 - 3 \times 3^2 + 11 \times 3 - 9 = 6$

x	2	3	4
y''	–2	0	2
concavity	down		up

So $(3,6)$ is a point of inflection.

The graph is concave down for $x < 3$ and concave up for $x > 3$.

18c The graph has one x-intercept because the function is continuous and increasing for all real x.

19a $y = \frac{x+2}{x-3}$ then $y' = \frac{1\times(x-3)-(x+2)\times 1}{(x-3)^2} = \frac{x-3-x-2}{(x-3)^2} = \frac{-5}{(x-3)^2}$

$y'' = \frac{0\times(x-3)^2-(-5)\times 2\times(x-3)}{(x-3)^4} = \frac{10x-30}{(x-3)^4} = \frac{10(x-3)}{(x-3)^4} = \frac{10}{(x-3)^3}$

19b The sign table of $(x-3)^3$ is shown below

x	$x < 3$	3	$x > 3$
$(x-3)^3$	$-$	0	$+$

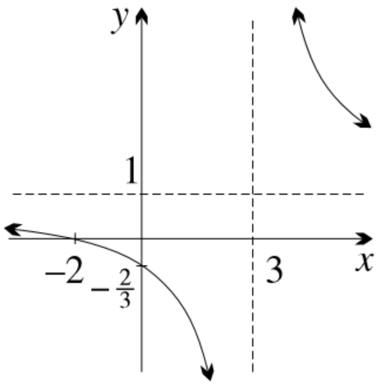
Therefore, $y = \frac{x+2}{x-3}$ is concave up when $x > 3$ and concave down when $x < 3$.

19c Since $x - 3 = 0$ when $x = 3$, the graph of $y = \frac{x+2}{x-3}$ has a vertical asymptote at

$x = 3$ and since $\lim\limits_{x\to\pm\infty} \frac{x+2}{x-3} = 1$, y has a horizontal asymptote at $y = 1$.

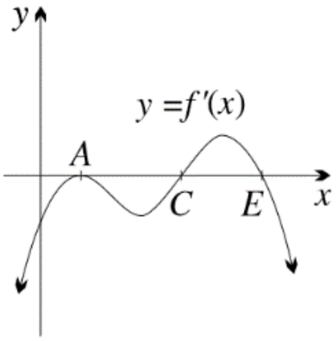
Since $f(-2) = \frac{-2+2}{-2-3} = 0$, the graph cuts the x-axis at $(-2, 0)$ and the y-intercept

is $\left(0, -\frac{2}{3}\right)$ as $f(0) = \frac{0+2}{0-3} = -\frac{2}{3}$



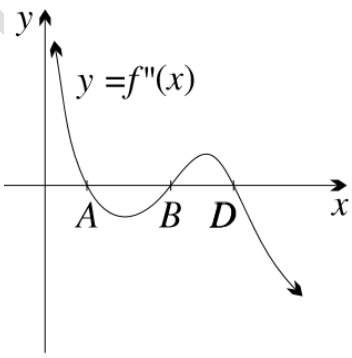
20a $f'(x) = 0$ at A, C and E

$f(x)$	decreasing	A	decreasing	C	increasing	E	decreasing
$f'(x)$	$-$	0	$-$	0	$+$	0	$-$



20b $f''(x) = 0$ at A, B and D

$f(x)$	Concave up	A	Concave down	B	Concave up	D	Concave down
$f''(x)$	$+$	0	$-$	0	$+$	0	$-$



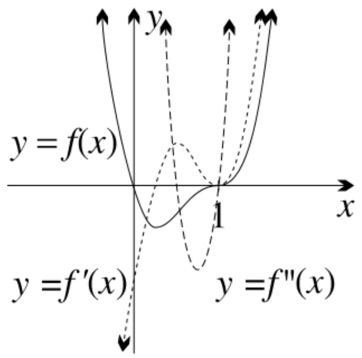
21a $f'(x) = 1 \times (x-1)^3 + x \times 3 \times (x-1)^2 = (x-1)^2(4x-1)$

$f''(x) = 2 \times (x-1) \times (4x-1) + (x-1)^2 \times 4 = 6(x-1)(2x-1)$

21b $f(x)$ has a minimum turning point where $f'(x)$ cuts the x-axis and a stationary point of inflection where $f'(x)$ touches and $f''(x)$ cuts the x-axis.

$f'(x) < 0$ when $f(x)$ is decreasing and $f'(x) > 0$ when $f(x)$ is increasing.

$f(x)$ is concave up when $f''(x) > 0$ and concave down when $f''(x) < 0$



22 $y = ax^3 + bx^2 + cx + d$

$(0, 5)$ lies on the curve and so when $x = 0, y = 5$

Hence $d = 5$.

The curve has a turning point at $(0, 5)$. So when $x = 0, y' = 0$.

$y' = 3ax^2 + 2bx + c$

Hence $c = 0$.

$(-1, 0)$ lies on the curve and so when $x = -1, y = 0$

$-a + b = -5$ (1)

The curve has a point of inflection at $x = \frac{1}{2}$. So when $x = \frac{1}{2}$, $y'' = 0$.

$y'' = 6ax + 2b$

$3a + 2b = 0$ (2)

$3 \times (1) + (2)$ gives $5b = -15$ and so $b = -3$

Substituting $b = -3$ into (2) and solving $3a - 6 = 0$ for a we obtain $a = 2$.

So $a = 2, b = -3, c = 0$ and $d = 5$.

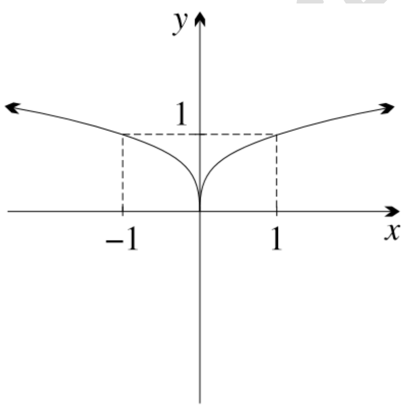
23a i $y = x^{\frac{2}{3}}$ then $y' = \frac{2}{3}x^{-\frac{1}{3}}$. $y' > 0$ when $x > 0$. Therefore, $y = x^{\frac{2}{3}}$ is increasing when $x > 0$.

23a ii $y' < 0$ when $x < 0$. Therefore, $y = x^{\frac{2}{3}}$ is decreasing when $x < 0$.

23a iii $y'' = -\frac{2}{9}x^{-\frac{4}{3}} < 0$ for all $x \in \mathbb{R} - \{0\}$. Therefore, $y = x^{\frac{2}{3}}$ is never concave up in its domain.

23a iv $y'' = -\frac{2}{9}x^{-\frac{4}{3}} < 0$ for all $x \in \mathbb{R} - \{0\}$. Therefore, $y = x^{\frac{2}{3}}$ is concave down In its domain.

23b $y = (0)^{\frac{2}{3}} = 0$ then the x- and y-intercept is $(0, 0)$.



Solutions to Exercise 4F

1a $y = 0$ when $6x^2 - x^3 = 0$ or $x^2(6 - x) = 0$.

Hence, $y = 0$ when $x = 0$ or $x = 6$

Therefore, the point A is at $(6, 0)$.

1b $y = 6x^2 - x^3$

$y' = 12x - 3x^2$ and $y' = 0$ when $12x - 3x^2 = 0$ or $3x(4 - x) = 0$

Hence, $y' = 0$ when $x = 0$ or $x = 4$.

Therefore, both $x = 0$ and $x = 4$ are stationary points, and $x = 4$ is the x-coordinate of the point B.

When $x = 4, y = 6(4)^2 - (4)^3 = 32$. Therefore, B is at $(4, 32)$.

1c $y'' = 12 - 6x$

$y'' = 0$ when $12 - 6x = 0$ or $x = 2$. Hence, there is an inflection point at $x = 2$.

When $x = 2, y = 6(2)^2 - (2)^3 = 16$. Therefore, C is at $(2, 16)$.

2a $f'(x) = 0$ when $x = -1$ or $x = 2$.

2b $f''(x) = 0$ at the point of inflection. Without the rule of the function, this is difficult to locate but it looks to be at $x = 0$.

2c $f(x)$ is increasing when $-1 < x < 2$.

2d Using the answer from part b, $f''(x) > 0$ when $x < 0$.

3a $y = f(x) = 27x - x^3$

$$\begin{aligned} f(-x) &= 27(-x) - (-x)^3 \\ &= -27x + x^3 \\ &= -(27x - x^3) \end{aligned}$$

$= -f(x)$

$y = f(x) = 27x - x^3$ is an odd function because $f(-x) = -f(x)$.

Since y is an odd function, its graph has point symmetry in the origin.

3b $y' = 27 - 3x^2$

$= 3(9 - x^2)$

$y'' = -6x$

3c $y' = 0$ when $3(9 - x^2) = 0$ or $x = \pm 3$

When $x = -3,\ y = 27 \times (-3) - (-3)^3 = -54$

When $x = 3,\ y = 27 \times 3 - 3^3 = 54$

Therefore $(-3,\ -54)$ and $(3, 54)$ are stationary points.

$f''(-3) = -6 \times (-3) = 18$ and $18 > 0$.

Therefore, y is concave up at $x = -3$ and $(-3,\ -54)$ is a local minimum point.

$f''(3) = -6 \times 3 = -18$ and $-18 < 0$.

Therefore, y is concave down at $x = 3$ and $(3,\ 54)$ is a local maximum point.

3d

x	-1	0	1
y''	$+$	0	$-$

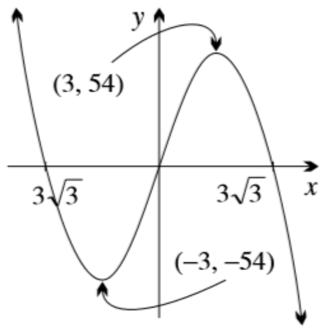
$y'' > 0$ for $x < 0$ and $y'' < 0$ for $x > 0$.

Therefore, there is a point of inflection at $x = 0$.

3e $y' = 27 - 3(0)^2 = 27$ when $x = 0$. Therefore, the gradient at the point of inflection is 27.

3f $y = x(27 - x^2) = 0$ when $x = 0$ or $x = \pm 3\sqrt{3}$

Therefore, $(0,0)$, $\left(-3\sqrt{3},0\right)$ and $\left(3\sqrt{3},0\right)$ are the x-intercepts and the graph of y is shown below.



4a $f(x) = 2x^3 - 3x^2 + 5$ then $f'(x) = 6x^2 - 6x = 6x(x-1)$

$f''(x) = 12x - 6 = 6(2x-1)$

4b $f'(x) = 6x(x-1) = 0$ when $x = 0$ or $x = 1$.

Therefore, there are stationary points at $x = 0$ and $x = 1$.

$f(0) = 2(0)^3 - 3(0)^2 + 5 = 5$ and $f(1) = 2(1)^3 - 3(1)^2 + 5 = 4$

Therefore, $(0,5)$ and $(1,4)$ are the stationary points.

$f''(0) = 6(2 \times 0 - 1) = -6$ and $f''(0) < 0$. Hence the function is concave down at $x = 0$ and therefore, has a maximum turning point at $(0,5)$

$f''(1) = 6(2 \times 1 - 1) = 6$ and $f''(0) > 0$. Hence the function is concave up at $x = 1$ and therefore, has a minimum turning point at $(1,4)$

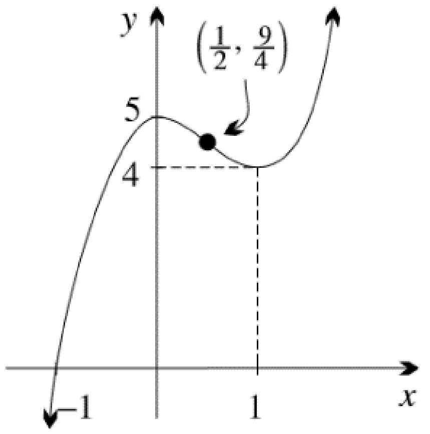
4c $f''(x) = 6(2x-1) = 0$ when $x = \frac{1}{2}$.

Therefore, there is a point of inflection at $x = \frac{1}{2}$.

$f'\left(\frac{1}{2}\right) = 6 \times \frac{1}{2} \times \left(\frac{1}{2} - 1\right) = -\frac{3}{2}$

At the point of inflection, the gradient is $-\frac{3}{2}$.

4d



$f\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right)^3 - 3\left(\frac{1}{2}\right)^2 + 5 = \frac{9}{4}$. Therefore, the point of inflection is at $\left(\frac{1}{2}, \frac{9}{4}\right)$.

$f(-1) = 2(-1)^3 - 3(-1)^2 + 5 = -2 - 3 + 5 = 0$.

Therefore, $(-1, 0)$ is the x-intercept. Refer to 6b for the coordinates of the turning points.

5a $y = x(x-6)^2$

$y = 0$ when $x = 0$ and $x = 6$. Therefore, $(0, 0)$ and $(6, 0)$ are the x-intercepts.

$y' = (x-6)^2 + x \times 2(x-6)$

$= 3x^2 - 24x + 36$

$= 3(x-6)(x-2)$

$y' = 0$ when $x = 2$ or $x = 6$.

$y = 32$ when $x = 2$ and $y = 0$ when $x = 6$

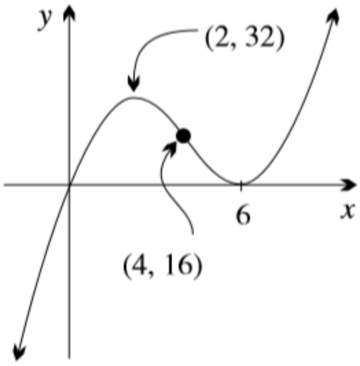
Therefore, $(2, 32)$ and $(6, 0)$ are stationary points.

$y'' = 6x - 24$

$y'' = 0$ when $6x - 24 = 0$ or $x = 4$.

$y = 16$ when $x = 4$. Therefore, $(4, 16)$ is a point of inflection.

Since $y'' < 0$ when $x = 2$ and $y'' > 0$ when $x = 6$, $(2, 32)$ is a local maximum and $(6, 0)$ is a local minimum point.



5b $y = x^3 - 3x^2 - 24x + 5$

$y' = 3x^2 - 6x - 24$

$= 3(x - 4)(x + 2)$

$y' = 0$ when $x = -2$ or $x = 4$.

$y = 33$ when $x = -2$ and $y = -75$ when $x = 4$

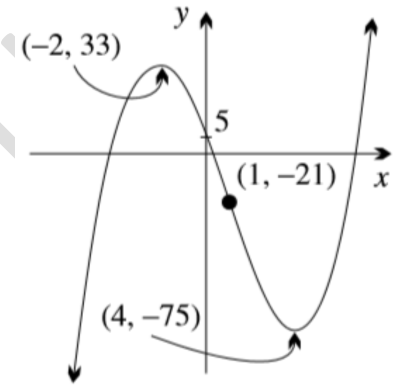
Therefore $(-2, 33)$ and $(4, -75)$ are stationary points.

$y'' = 6x - 6$

$y'' = 0$ when $6x - 6 = 0$ or $x = 1$.

$y = -21$ when $x = 1$. Therefore, $(1, -21)$ is a point of inflection.

Since $y'' < 0$ when $x = -2$ and $y'' > 0$ when $x = 4$, $(-2, 33)$ is a local maximum and $(4, -75)$ is a local minimum point.



6a $y = 12x^3 - 3x^4 + 11$

$y' = 36x^2 - 12x^3 = 12x^2(3 - x)$

$y'' = 72x - 36x^2 = 36x(2 - x)$

6b $y' = 12x^2(3 - x) = 0$ when $x = 0$ or $x = 3$. Therefore, there are stationary points at $x = 0$ and $x = 3$.

When $x = 0$, $y = 12(0)^3 - 3(0)^4 + 11 = 11$

When $x = 3$, $y = 12(3)^3 - 3(3)^4 + 11 = 92$

Therefore $(0, 11)$ and $(3, 92)$ are the stationary points.

6c When $x = 3$, $y'' = 72 \times (3) - 36(3)^2 = -108$.

Since $y'' < 0$ when $x = 3$, the function is concave down at $x = 3$. Therefore, the point $(3, 92)$ is a maximum turning point.

This method fails for $x = 0$ because $y'' = 72 \times (0) - 36(0)^2 = 0$ when $x = 0$.

6d

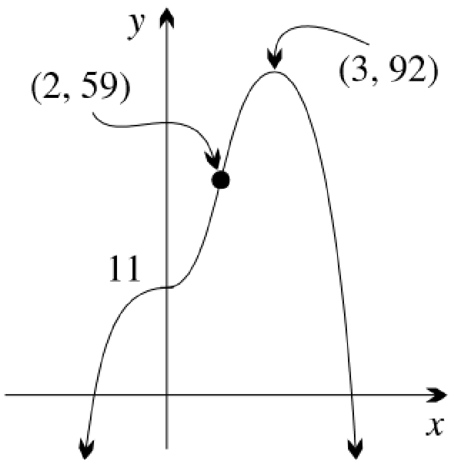
x	-1	0	1	3	4
y'	$+$	0	$+$	0	$-$
y	/	Stationary point of inflection	/	Maximum turning point	\

6e $y'' = 36x(2 - x) = 0$ when $x = 0$ or $x = 2$.

Therefore there are inflection points at $x = 0$ and $x = 2$.

When $x = 2$, $y = 12(2)^3 - 3(2)^4 + 11 = 59$

6f



7 $y = x^4 - 16x^3 + 72x^2 + 10$

$y' = 4x^3 - 48x^2 + 144x$

$= 4x(x^2 - 12x + 36)$

$= 4x(x - 6)^2$

$y' = 0$ when $x = 0$ or $x = 6$

$y = 10$ when $x = 0$ and $y = 442$ when $x = 6$

Therefore $(0, 10)$ and $(6, 442)$ are stationary points.

x	-1	0	1	6	7
y'	\	0	/	0	/

$y' < 0$ when $x < 0$ and $y' > 0$ when $0 < x < 6$. Therefore, $(0, 10)$ is a local minimum point.

$y' > 0$ when $0 < x < 6$ and $y' > 0$ when $x > 6$. Therefore, $(6, 442)$ is a stationary point of inflection.

$y'' = 12x^2 - 96x + 144$

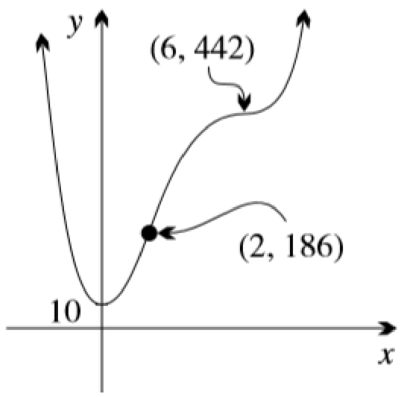
$= 12(x^2 - 8x + 12)$

$= 12(x - 2)(x - 6)$

$y'' = 0$ when $12(x - 2)(x - 6) = 0$ or $x = 2$ or $x = 6$.

x	1	2	3	6	7
y''	$+$	0	$-$	0	$+$

When $x = 2, y = -186$. Therefore, $(2, -186)$ is a point of inflection.



8a $f(x) = \dfrac{1}{x^2 - 4} = (x^2 - 4)^{-1}$

$f'(x) = -1 \times (x^2 - 4)^{-2} \times 2x$

$= -\dfrac{2x}{(x^2 - 4)^2}$

8b There are stationary points where $f'(x) = 0$.

$-\frac{2x}{(x^2-4)^2} = 0$ when $-2x = 0$ *or* $x = 0$

So there is a stationary point at $x = 0$ and discontinuities at $x = \pm 2$.

x	$=1$	0	1
$f'(x)$	$\frac{2}{9}$	3	$-\frac{2}{9}$
slope	/	–	\

When $x = 0$, $y = -\dfrac{1}{4}$.

Hence $\left(0,-\frac{1}{4}\right)$ is a maximum turning point.

8c If the function is even, then $f(-x)=f(x)$.

$$\begin{aligned} f(-x) &= \frac{1}{(-x)^2-4} \\ &= \frac{1}{x^2-4} \\ &= f(x) \end{aligned}$$

Hence the function is even.

The graph has line symmetry in the y- axis.

8d Vertical asymptotes occur for $x^2-4=0$.

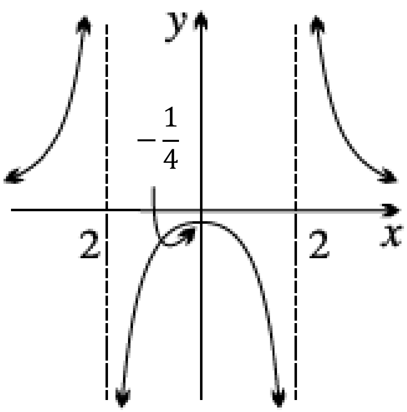
So the (vertical) asymptotes are $x=\pm 2$.

The domain is $x\neq\pm 2$.

8e As $x\to\pm\infty, f(x)\to 0$.

So the horizontal asymptote is $y=0$.

8f



8g The range is $y>0$ or $y\leq-\frac{1}{4}$.

9a $f(x) = \dfrac{x}{x^2 - 4}$

Applying the quotient rule on $f(x) = \dfrac{x}{x^2 - 4}$:

Let $u = x$ and $v = x^2 - 4$.

Then $u' = 1$ and $v' = 2x$.

$$\begin{aligned} f'(x) &= \frac{vu' - uv'}{v^2} \\ &= \frac{(1)(x^2 - 4) - (x)(2x)}{(x^2 - 4)^2} \\ &= \frac{-(x^2 + 4)}{(x^2 - 4)^2} \end{aligned}$$

So $f'(x) = -\dfrac{x^2 + 4}{(x^2 - 4)^2}$.

9b $f'(x) = -\dfrac{x^2 + 4}{(x^2 - 4)^2}$

$f'(x) = 0$ when $x^2 + 4 = 0$

So $f'(x) \le -4$ for all real x. Hence the equation $f'(x) = 0$ has no solutions and so the curve $y = f(x)$ has no stationary points.

Further, as $f'(x) \le -4$ for all real x i.e. $f'(x)$ is negative for all values of x then $f(x)$ is decreasing for all values of x.

Hence the curve is always decreasing.

9c $f''(x) = \dfrac{2x^3 + 24x}{(x^2 - 4)^3}$

Solving $f''(x) = 0$ for x we obtain:

$2x^3 + 24x = 0 \Rightarrow 2x(x^2 + 12) = 0$

$x^2+12=0$ has no real solutions and so $x=0$.

So there is a point of inflection at $x=0$.

x	-1	0	1
$f''(x)$	$-\frac{26}{27}$	0	$\frac{26}{27}$
concavity	down		up

So $(0,0)$ is a point of inflection.

The gradient of the tangent at $(0,0)$ is given by $f'(0)$.

$$f'(0)=-\frac{1}{4}$$

So the gradient of the tangent at $(0,0)$ is $-\frac{1}{4}$.

9d The domain is $x\neq\pm2$.

Vertical asymptotes occur for $x^2-4=0$.

So the (vertical) asymptotes are $x=\pm2$.

9e As $x\to\pm\infty$, $f(x)\to0$.

So the horizontal asymptote is $y=0$.

9f If the function is odd, then $f(-x)=-f(x)$.

$$\begin{aligned}f(-x)&=\frac{(-x)}{(-x)^2-4}\\&=-\frac{x}{x^2-4}\\&=-f(x)\end{aligned}$$

Hence the function is odd.

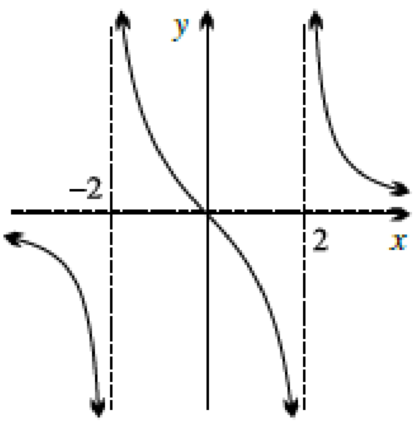
The graph has point symmetry in the origin.

9g When $x = 0, y = 0$. The function has discontinuities at $x = \pm 2$.

x	–3	–2	–1	0	1	2	3
$f(x)$	$-\frac{3}{5}$	undef	$\frac{1}{3}$	0	$-\frac{1}{3}$	undef	$\frac{3}{5}$

Hence $f(x)$ is positive for $-2 < x < 0$ or $x > 2$ and negative for $x < -2$ or $0 < x < 2$.

9h



9i The range is all real values of y.

10a $f(x) = \frac{x^2}{1+x^2}$ then $f'(x) = \frac{2x\times(1+x^2)-x^2\times(2x)}{(1+x^2)^2} = \frac{2x}{(1+x^2)^2}$ and

$$f''(x) = \frac{2\times(1+x^2)^2-2x\times\left(2\times(1+x^2)\times(2x)\right)}{(1+x^2)^4} = \frac{(1+x^2)(2+2x^2-8x^2)}{(1+x^2)^4} = \frac{2-6x^2}{(1+x^2)^3}$$

10b $f'(x) = 0$ when $\frac{2x}{(1+x^2)^2} = 0$ or $x = 0$, and $f(0) = 0$. Therefore, there is a stationary point at $(0, 0)$.

x		0	
$f'(x)$	$-$	0	$+$

Hence, $(0, 0)$ is a minimum turning point.

10c $f''(x) = 0$ when $\frac{2-6x^2}{(1+x^2)^3} = 0$ or

$2 - 6x^2 = 0$

$x^2 = \frac{1}{3}$

$x = -\frac{1}{\sqrt{3}}$ or $x = \frac{1}{\sqrt{3}}$

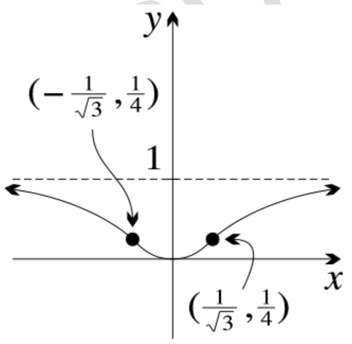
$f\left(-\frac{1}{\sqrt{3}}\right) = f\left(\frac{1}{\sqrt{3}}\right) = \frac{1}{4}$

Therefore, there are points of inflection at $\left(-\frac{1}{\sqrt{3}}, \frac{1}{4}\right)$ and $\left(\frac{1}{\sqrt{3}}, \frac{1}{4}\right)$.

10d $\lim_{x\to\pm\infty} \frac{x^2}{1+x^2} = 1$ and $f(1) = \frac{1^2}{1+1^2} = \frac{1}{2}$

Therefore, there is a horizontal asymptote at $\left(1, \frac{2}{2}\right)$

10e



11a $f(x) = \frac{4x}{x^2+9}$ then $f'(x) = \frac{4\times(x^2+9)-(4x)(2x)}{(x^2+9)^2} = \frac{36-4x^2}{(x^2+9)^2}$ and

$$f''(x) = \frac{(-8x)\times(x^2+9)^2-(36-4x^2)\times2\times(x^2+9)(2x)}{(x^2+9)^4} = \frac{(4x)(x^2+9)(-2x^2-18-36+4x^2)}{(x^2+9)^4} = \frac{8x^3-216x}{(x^2+9)^3}$$

11b $f'(x) = 0$ when $\frac{36-4x^2}{(x^2+9)^2} = 0$ or when $x = -3$ or $x = 3$.

$f(-3) = \frac{4\times(-3)}{(-3)^2+9} = -\frac{12}{18} = -\frac{2}{3}$ and $f(3) = \frac{4\times(3)}{(3)^2+9} = \frac{2}{3}$

Therefore, $f(x)$ has stationary points at $\left(-3, -\frac{2}{3}\right)$ and $\left(3, \frac{2}{3}\right)$

x		-3		3	
$f'(x)$	$-$	0	$+$	0	$-$

Hence, $\left(-3, -\frac{2}{3}\right)$ is a minimum turning point and $\left(3, \frac{2}{3}\right)$ is a maximum turning point.

11c $f''(x) = 0$ when $\frac{8x^3-216x}{(x^2+9)^3} = 0$ or when $8x(x^2 - 27) = 0$

Or when $x = 0, x = -3\sqrt{3}$ or $x = 3\sqrt{3}$.

$f(0) = \frac{4\times(0)}{(0)^2+9} = 0$

$f(-3\sqrt{3}) = \frac{4\times(-3\sqrt{3})}{(-3\sqrt{3})^2+9} = \frac{-12\sqrt{3}}{36} = -\frac{\sqrt{3}}{3}$

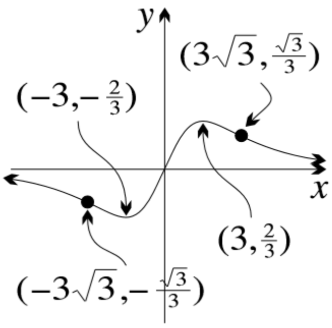
$f(3\sqrt{3}) = \frac{4\times(3\sqrt{3})}{(3\sqrt{3})^2+9} = \frac{12\sqrt{3}}{36} = \frac{\sqrt{3}}{3}$

Therefore, $f(x)$ has points of inflection at $(0, 0)$, $(-3\sqrt{3},)$ and $(3\sqrt{3},)$

11d $\lim_{x\to\pm\infty} \frac{4x}{x^2+9} = 0$

Therefore, $f(x)$ has a horizontal asymptote at $y = 0$.

11e



12 $y = x^5 - 15x^3$ then $y' = 5x^4 - 45x^2$ and $y'' = 20x^3 - 90x$

$y' = 5x^4 - 45x^2 = 0$ when $5x^2(x^2 - 9) = 0$

Then there are stationary points at $x = 0$ and $x = \pm 3$

$y = (0)^5 - 15(0)^3 = 0$

$y = (-3)^5 - 15(-3)^3 = 162$

$y = (3)^5 - 15(3)^3 = -162$

Therefore, $(0, 0)$, $(-3, 162)$ and $(3, -162)$ are the stationary points.

x		-3		0		3	
y'	$+$	0	$-$	0	$-$	0	$+$

Hence, $(-3, 162)$ is a maximum turning point, $(3, -162)$ is a minimum turning point and $(0, 0)$ is a stationary point.

$y'' = 20x^3 - 90x = 0$ when $10x(2x^2 - 9) = 0$

Then there are points of inflection at $x = 0$, $x = -\frac{3}{\sqrt{2}}$ and $x = \frac{3}{\sqrt{2}}$

$y = (0)^5 - 15(0)^3 = 0$

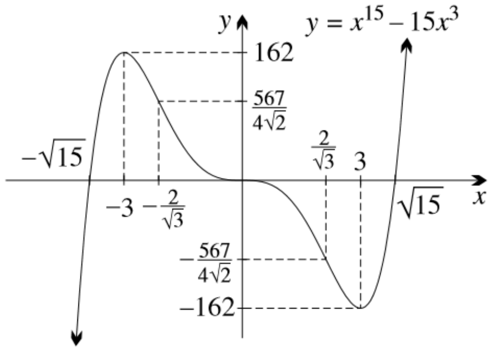
$y = \left(-\frac{3}{\sqrt{2}}\right)^5 - 15\left(-\frac{3}{\sqrt{2}}\right)^3 = \frac{567}{4\sqrt{2}}$

$y = \left(\frac{3}{\sqrt{2}}\right)^5 - 15\left(\frac{3}{\sqrt{2}}\right)^3 = -\frac{567}{4\sqrt{2}}$

Therefore, $(0, 0)$ is a horizontal point of inflection (because it is a stationary point), and $\left(-\frac{3}{\sqrt{2}}, \frac{567}{4\sqrt{2}}\right)$ and $\left(\frac{3}{\sqrt{2}}, -\frac{567}{4\sqrt{2}}\right)$ are the points of inflection.

$\lim\limits_{x\to\pm\infty}(x^5 - 15x^3) = \pm\infty$ (no horizontal asymptote)

The graph is,



13a $y = \frac{x^2-x-2}{x^2}$ then $y' = \frac{(2x-1)x^2-(x^2-x-2)(2x)}{x^4} = \frac{x+4}{x^3}$ and

$y'' = \frac{1\times x^3-(x+4)(3x^2)}{(x^3)^2} = \frac{-2(x+6)}{x^4}$

$y = \frac{x^2-x-2}{x^2} = \frac{(x+1)(x-2)}{x^2} = 0$ when $x = -1$ or $x = 2$. Therefore, the x-intercepts are $(1-, 0)$ and $(2, 0)$

$y' = \frac{x+4}{x^3} = 0$ when $x + 4 = 0$ or $x = -4$

Then there is a stationary point at $x = -4$

$y = \frac{(-4)^2-(-4)-2}{(-4)^2} = \frac{9}{8}$

x		-4	
y'	$+$	0	$-$

Therefore, $\left(-4, \frac{9}{8}\right)$ is a maximum turning point.

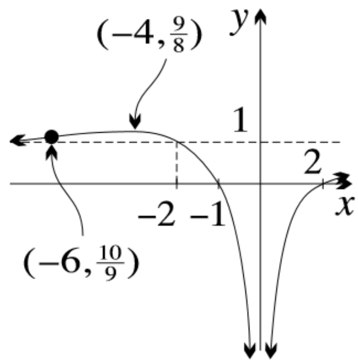
Since y is not continuous at $x = 0$ and $\lim_{x \to \pm 0} \left(\frac{x^2-x-2}{x^2}\right) = -\infty$. Therefore, $x = 0$ is a vertical asymptote and $y \to -\infty$ $x \to 0$.

$y'' = \frac{-2(x+6)}{x^4} = 0$ when $x = -6$ and $y = \frac{(-6)^2-(-6)-2}{(-6)^2} = \frac{40}{36} = \frac{10}{9}$

Therefore, $\left(-6, \frac{10}{9}\right)$ is the point of inflection.

$\lim_{x \to \pm\infty} \left(\frac{x^2-x-2}{x^2}\right) = 1$. Therefore, there is a horizontal asymptote at $y = 1$

The graph is,



13b $y = \frac{x^2-2x}{(x+2)^2}$ then $y' = \frac{(2x-2)(x+2)^2-(x^2-2x)\times 2\times(x+2)}{((x+2)^2)^2} = \frac{2(3x-2)}{(x+2)^3}$ and

$y'' = \frac{6\times(x+2)^3-2(3x-2)(3(x+2)^2)}{((x+2)^3)^2} = \frac{-12(x-2)}{(x+2)^4}$

$y = \frac{x^2-2x}{(x+2)^2} = \frac{x(x-2)}{x^2} = 0$ when $x = 0$ or $x = 2$. Therefore, the x-intercepts are $(0, 0)$ and $(2, 0)$

$y' = \frac{2(3x-2)}{(x+2)^3} = 0$ when $3x - 2 = 0$ or $x = \frac{2}{3}$

Then there is a stationary point at $x = \frac{2}{3}$

$y = \frac{\left(\frac{2}{3}\right)^2 - 2\left(\frac{2}{3}\right)}{\left(\left(\frac{2}{3}\right)+2\right)^2} = -\frac{1}{8}$

x		$\frac{2}{3}$	
y'	$-$	0	$+$

Therefore, $\left(\frac{2}{3}, -\frac{1}{8}\right)$ is a minimum turning point.

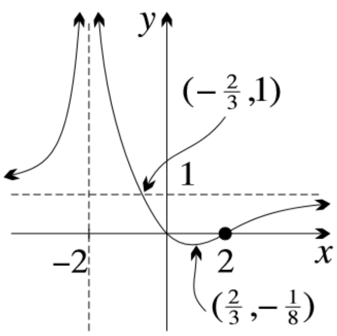
Since y is not continuous at $x = -2$ and $\lim_{x \to -2} \left(\frac{x^2-2x}{(x+2)^2}\right) = \infty$. Therefore, $x = -2$ is a vertical asymptote and $y \to \infty$ $x \to -2$.

$y'' = \frac{-12(x-2)}{(x+2)^4} = 0$ when $x = 2$ and $y = \frac{(2)^2-2(2)}{((2)+2)^2} = \frac{0}{16} = 0$

Therefore, $(2, 0)$ is the point of inflection.

$\lim_{x \to \pm\infty} \left(\frac{x^2-2x}{(x+2)^2}\right) = 1$. Therefore, there is a horizontal asymptote at $y = 1$

The graph is,



14a $f(x) = (x+5)(x-1)$ then $f'(x) = 1 \times (x-1) + (x+5) \times 1 = 2x + 4$

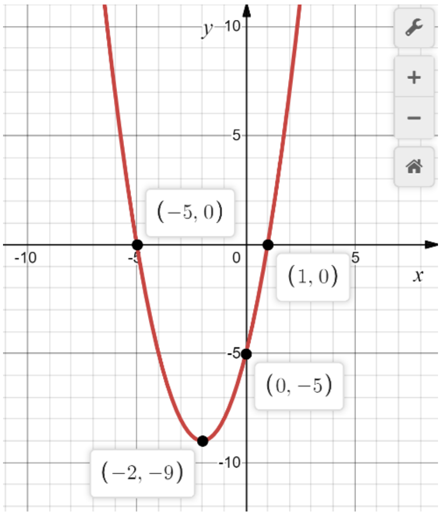
$f(x) = (x+5)(x-1) = 0$ when $x = -5$ and $x = 1$. Therefore, $(-5, 0)$ and $(1, 0)$ are the x-intercepts.

$f'(x) = 2x + 4 = 0$ when $x = -2$.

x		-2	
y'	$-$	0	$+$

And $f(-2) = \big((-2) + 5\big)\big((-2) - 1\big) = -9$

Therefore, $(-2, -9)$ is a minimum turning point.



14b $f(x) = \frac{1}{f(x)}$ when $f^2(x) - 1 = 0$ or

$(x+5)^2(x-1)^2 - 1 = 0$

$x^4 + 8x^3 + 6x^2 - 40x + 24 = 0$ when

$x = -2 + 2\sqrt{2}$, $x = -2 - 2\sqrt{2}$, $x = -2 + \sqrt{10}$ or $x = -2 - \sqrt{10}$

$f(-2 + 2\sqrt{2}) = \left((-2 + 2\sqrt{2}) + 5\right)\left((-2 + 2\sqrt{2}) - 1\right) = -1$

$f(-2 - 2\sqrt{2}) = \left((-2 - 2\sqrt{2}) + 5\right)\left((-2 - 2\sqrt{2}) - 1\right) = -1$

$f(-2 + \sqrt{10}) = \left((-2 + \sqrt{10}) + 5\right)\left((-2 + \sqrt{10}) - 1\right) = 1$

$f(-2 - \sqrt{10}) = \left((-2 - \sqrt{10}) + 5\right)\left((-2 - \sqrt{10}) - 1\right) = 1$

Therefore, these functions intersect at the points:

$(-2 + 2\sqrt{2}, -1)$, $(-2 - 2\sqrt{2}, -1)$, $(-2 + \sqrt{10}, 1)$ and $(-2 - \sqrt{10}, 1)$

14c $\frac{1}{f(x)} = \frac{1}{(x+5)(x-1)} = \frac{1}{x^2+4x-5}$ then $\frac{d\left(\frac{1}{f(x)}\right)}{dx} = \frac{0\times(x^2+4x-5)-1\times(2x+4)}{\left((x+5)(x-1)\right)^2} = \frac{-2(x+2)}{(x+5)^2(x-1)^2}$

$\frac{d\left(\frac{1}{f(x)}\right)}{dx} = 0$ when $\frac{-2(x+2)}{(x+5)^2(x-1)^2} = 0$ or $x = -2$. Therefore, there is a stationary point at $x = -2$.

x		-2	
$\frac{d\left(\frac{1}{f(x)}\right)}{dx}$	$+$	0	$-$

Hence, $\frac{1}{f(x)}$ is increasing when $x < -2$,whereas $f(x)$ is decreasing when $x < -2$ (14a)

Hence, $\frac{1}{f(x)}$ is decreasing when $x > -2$,whereas $f(x)$ is increasing when $x > -2$ (14a)

14d $\frac{1}{f(x)}$ has a stationary point at $x = -2$ (14c) and $\frac{1}{f(x)} = \frac{1}{((-2)+5)((-2)-1)} = -\frac{1}{9}$

Therefore, the stationary point is $\left(-2, -\frac{1}{9}\right)$ and it is a maximum turning point.

14e $\frac{1}{f(x)}$ has vertical asymptotes at $x = -5$ and $x = 1$

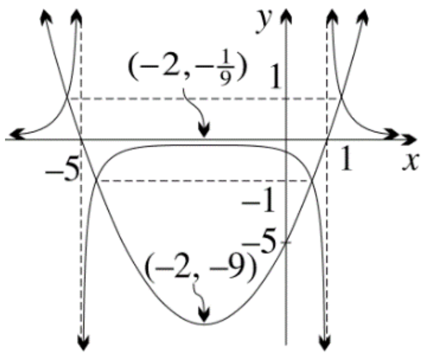
$\frac{d^2\left(\frac{1}{f(x)}\right)}{dx^2} = \frac{6(x^2+4x+7)}{(x-1)^3(x+5)^3}$

Sign table of $\frac{d^2\left(\frac{1}{f(x)}\right)}{dx^2}$ is:

x		-5		1	
$\frac{d^2\left(\frac{1}{f(x)}\right)}{dx^2}$	$+$	0	$-$	0	$+$

Therefore, $\frac{1}{f(x)}$ is concave up when $x < -5$ and $x > 1$,

and concave down when $-5 < x < 1$


y
x
$(-2, -\frac{1}{9})$
1
–5
1
–1
–5
(–2, –9)

Solutions to Exercise 4G

1a A is a local maximum

B is a local minimum

1b C is a global maximum

D is a local minimum

E is a local maximum

F is a global minimum

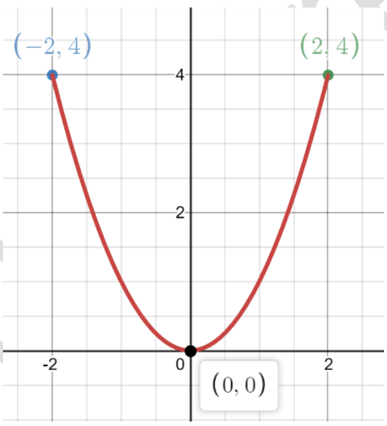
1c G is a global maximum

H is a horizontal point of inflection

1d I is a horizontal point of inflection

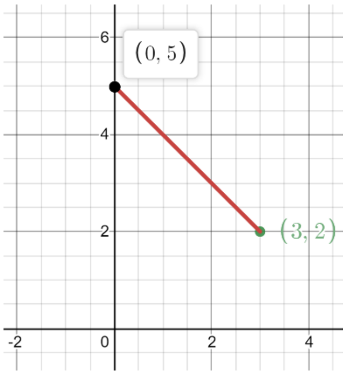
J is a global minimum

2a The graph of $y = x^2$ for $-2 < x < 2$ is shown below:



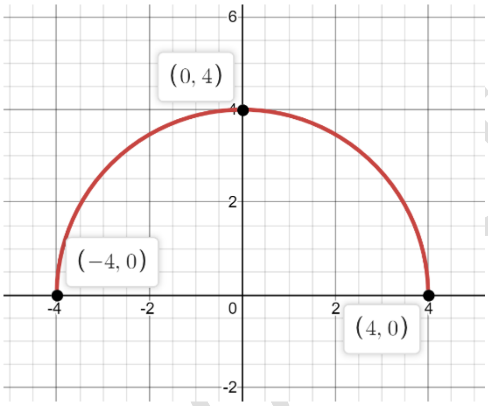
The global minimum is 0 at $x = 0$ and the global maximum is 4 at both $x = -2$ and $x = 2$.

2b The graph of $y = 5 - x$ for $0 < x < 3$ is shown below.



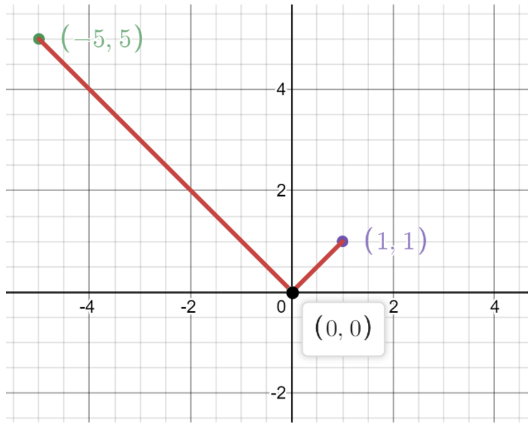
The global minimum is 2 at $x = 3$ and the global maximum is 5 at $x = 0$.

2c The graph of $y = \sqrt{16 - x^2}$ for $-4 < x < 4$ is shown below.



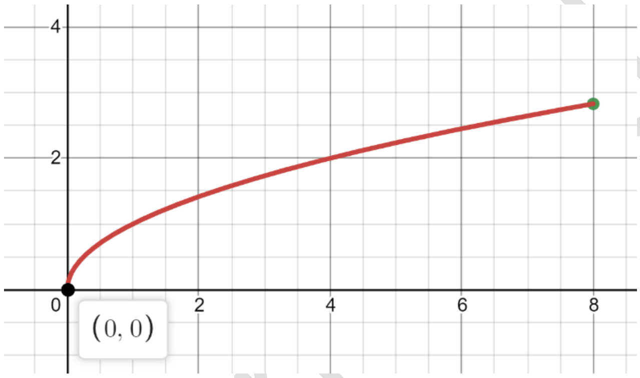
The global minimum is 0 at both $x = -4$ and at $x = 4$. The global maximum is 4 at $x = 0$.

2d The graph of $y = |x|$ for $-5 < x < 1$ is shown below.



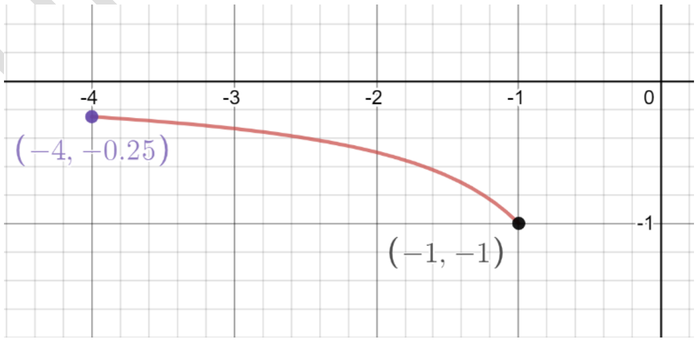
The global minimum is 0 at $x = 0$. The global maximum is 5 at $x = -5$.

2e The graph of $y = \sqrt{x}$ for $0 < x < 8$ is shown below.



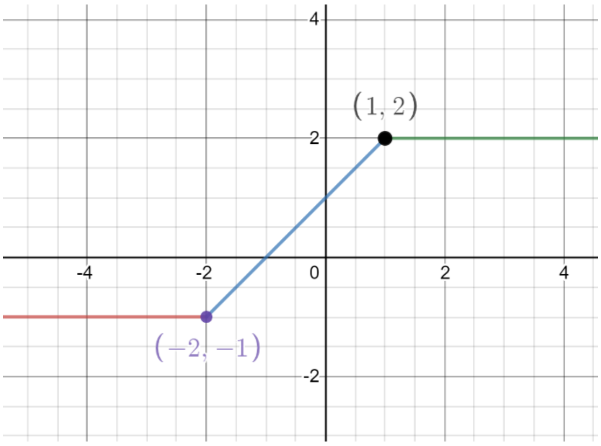
The global minimum is 0 at $x = 0$. The global maximum is $2\sqrt{2}$ at $x = 8$.

2f The graph of $y = \frac{1}{x}$ for $-4 < x < -1$ is shown below.



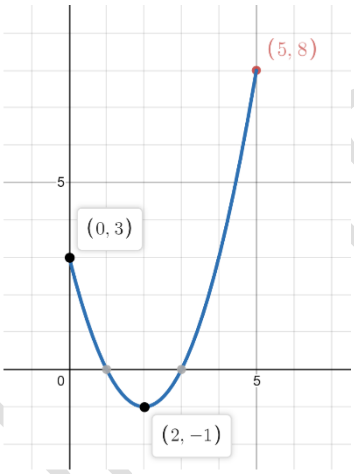
The global minimum is -1 at $x = -1$. The global maximum is $-\frac{1}{4}$ at $x = -4$.

2g The graph of $y = \begin{cases} -1, & x < -2 \\ x+1, & -2 \le x < 1 \\ 2, & x \ge 1 \end{cases}$ in its specified domain is shown below.



The global minimum is -1 for $x \le -2$. The global maximum is 2 for $x \ge 1$.

3a The graph of $y = x^2 - 4x + 3$ for $0 < x < 5$ is shown below.



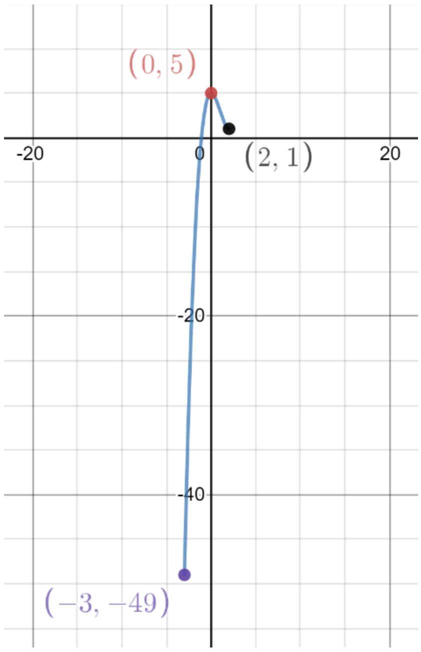
$y' = 2x - 4$

$y' = 0$ when $2x - 4 = 0$ or $x = 2$.

When $x = 2, y = -1$. Therefore $(2, -1)$ is the stationary point and the absolute minimum is -1 at $x = 2$.

The absolute maximum is 8 at $x = 5$.

3b The graph of $y = x^3 - 3x^2 + 5$ for $-3 < x < 2$ is shown below.



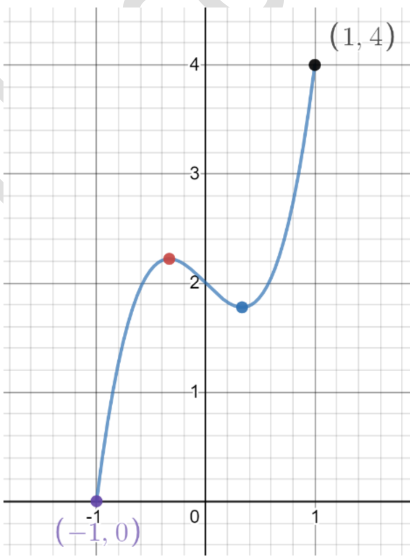
$y' = 3x^2 - 6x = x(3x - 6)$

$y' = 0$ when $x(3x - 6) = 0$ or $x = 0$ or $x = 2$.

When $x = 0$, $y = 5$. Therefore, $(0, 5)$ is a stationary point and the absolute maximum is 5 at $x = 0$.

The absolute minimum is -49 at $x = -3$.

3c The graph of $y = 3x^3 - x + 2$ for $-1 < x < 1$ is shown below.



$y' = 9x^2 - 1 = 9\left(x - \frac{1}{3}\right)\left(x + \frac{1}{3}\right)$

$y' = 0$ when $9\left(x - \frac{1}{3}\right)\left(x + \frac{1}{3}\right) = 0$ or when $x = -\frac{1}{3}$ or $x = \frac{1}{3}$.

There are stationary points at $x = -\frac{1}{3}$ and $x = \frac{1}{3}$.

x	-1	$-\frac{1}{3}$	0	$\frac{1}{3}$	1
y'	/	0	\	0	/

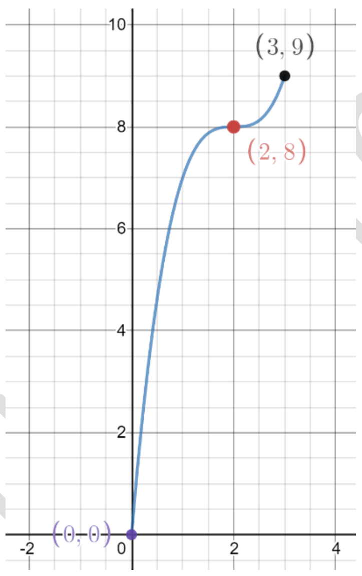
When $x = -\frac{1}{3}$, $y = \frac{20}{9}$. Therefore $\left(-\frac{1}{3}, \frac{20}{9}\right)$ is a local maximum point.

When $x = \frac{1}{3}$, $y = \frac{16}{9}$. Therefore $\left(\frac{1}{3}, \frac{16}{9}\right)$ is a local minimum point.

The absolute minimum is 0 at $x = -1$.

The absolute maximum is 4 at $x = 1$.

3d The graph of $y = x^3 - 6x^2 + 12x$ in its specified domain is shown below.



$y' = 3x^2 - 12x + 12 = 3(x - 2)^2$

$y' = 0$ when $3(x - 2)^2 = 0$ or $x = 2$. Thus, there is a stationary point at $x = 2$

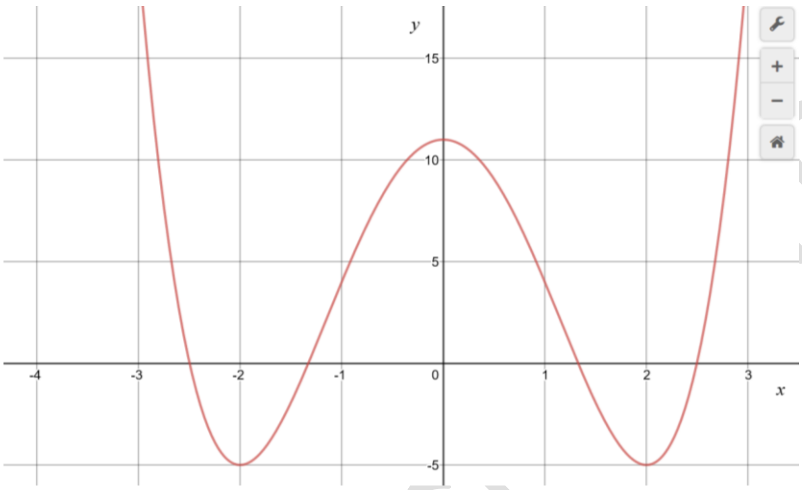
x	1	2	3
y'	/	0	/

$y' > 0$ for both $x < 2$ and $x > 2$, and $y = 8$ when $x = 2$. Therefore, $(2, 8)$ is a stationary point of inflection.

The absolute minimum is 0 at $x = 0$.

The absolute maximum is 9 at $x = 3$.

4 The graph of $y = x^4 - 8x^2 + 11$ is shown below.



4a $y = x^4 - 8x^2 + 11$ for $1 \le x \le 3$

$y' = 4x^3 - 16x$

There are stationary points where $y' = 0$.

$4x^3 - 16x = 0 \Rightarrow x = -2, 0, 2$

As $1 \le x \le 3$, there is a stationary point at $x = 2$.

x	1	2	3
$f'(x)$	-12	0	60
slope	\	–	/

When $x = 2$, $y = -5$. Hence $(2, -5)$ is a local minimum turning point.

Substituting the boundaries: when $x = 1, y = 4$ and when $x = 3, y = 20$.

So the global minimum is -5 at $x=2$ and the global maximum is 20 at $x=3$.

There is a local maximum of 4 at $x = 1$. (Note: this is not a local maximum turning point.)

4b $\quad y=x^4-8x^2+11$ for $-4\le x\le 1$

$y'=4x^3-16x$

There are stationary points where $y'=0$.

$4x^3-16x=0 \Rightarrow x=-2,0,2$

As $-4\le x\le 1$, there are stationary points at $x=-2$ and $x=0$.

x	-3	-2	-1	0	1
$f'(x)$	-60	0	12	0	-12
slope	\	–	/	–	\

When $x=-2$, $y=-5$ and when $x=0$, $y=11$.

Hence $(-2,-5)$ is a local minimum turning point and $(0,11)$ is a local maximum turning point.

Substituting the boundaries: when $x=-4, y=139$ and when $x=1, y=4$.

So the global minimum is -5 at $x=-2$ and the global maximum is 139 at $x=-4$.

There is a local minimum of 4 at $x = 1$. (Note: this is not a local minimum turning point.)

4c $\quad y=x^4-8x^2+11$ for $-1\le x\le 0$

There are no stationary points for $-1\le x\le 0$.

Substituting the boundaries: when $x=-1, y=4$ and when $x=0, y=11$.

So the global minimum is 4 at $x=-1$ and the global maximum is 11 at $x=0$.

Solutions to Exercise 4H

1a If $P = xy$ and $2x + y = 12$, then rearrange $2x + y = 12$ to obtain $y = 12 - 2x$ and substitute in P:

$P = x(12 - 2x)$

$P = 12x - 2x^2$

1b $\frac{dP}{dx} = 12 - 4x$ and $\frac{dP}{dx} = 0$ when $12 - 4x = 0$ or $x = 3$.

x	0	3	4
$\frac{dP}{dx}$	/	0	\

Therefore, there is a stationary point at $x = 3$ and it is a global maximum because $P = 12x - 2x^2$ is a parabola. Thus, the value of x that maximises P is 3.

1c When $x = 3, P = 12(3) - 2(3)^2 = 18$.

Therefore, the maximum value of P is 18.

2a If $Q = x^2 + y^2$ and $x + y = 8$, then rearrange $x + y = 8$ to make y the subject of the formula and substitute it in Q.

$y = 8 - x$

$Q = x^2 + (8 - x)^2$

$Q = 2x^2 - 16x + 64$

2b $\frac{dQ}{dx} = 4x - 16$ and $\frac{dQ}{dx} = 0$ when $4x - 16 = 0$ or $x = 4$.

x	0	4	5
$\frac{dQ}{dx}$	\	0	/

Therefore, there is a stationary point at $x = 4$ and it is a global minimum because the graph of $Q = 2x^2 - 16x + 64$ is a parabola. So the value of x that minimises Q is 4.

2c When $x = 4, Q = 2(4)^2 - 16(4) + 64 = 32$.

Therefore, the minimum value of Q is 32.

3 $\frac{dV}{dt} = 8t - 3t^2$

$\frac{dV}{dt} = 0$ when $8t - 3t^2 = 0$ or $t(8 - 3t) = 0$.

Hence, there is a stationary point at $t = 0$ and $= \frac{8}{3}$.

t	-1	0	1	$\frac{8}{3}$	3
$\frac{dV}{dt}$	\	0	/	0	\

So there is a local minimum point at $t = 0$ and a local maximum at $t = \frac{8}{3}$.

Therefore, the quantity of the vitamins in the patient's body is at its maximum when $t = \frac{8}{3}$ hours (or 2 hours and 40 minutes).

4a Let the length of the side parallel to the wall be y metres.

Then the perimeter is $40 = 2x + y$. Hence, $y = 40 - 2x$

4b $A = x \times y$ and $y = 40 - 2x$, then:

$A = x \times (40 - 2x)$

$\quad = 40x - 2x^2$

4c $\frac{dA}{dx} = 40 - 4x$

$\frac{dA}{dx} = 0$ when $40 - 4x = 0$ or $x = 10$

x	0	10	11
$\frac{dA}{dx}$	/	0	\

So there is a local maximum point at $x = 10$ and the value of x that maximises the area of the garden bed is 10 metres.

4d The maximum area is when $x = 10$:

$A = 40 \times (10) - 2(10)^2$

$= 200\ \text{m}^2$

5a Let the width of the rectangle be y cm. Then $36 = x \times y$ and $y = \frac{36}{x}$ cm.

5b $P = 2x + 2y$ and $= \frac{36}{x}$, then $P = 2x + \frac{72}{x}$.

5c $P = 2x + 72 \times x^{-1}$

$\frac{dP}{dx} = 2 - 72 \times x^{-2}$

$\frac{dP}{dx} = 2 - \frac{72}{x^2}$

$\frac{dP}{dx} = 0$ when $2 - \frac{72}{x^2} = 0$ or $x^2 = 36$. Hence, $x = -6$ or $x = 6$.

t	-7	-6	0	6	7
$\frac{dV}{dt}$	/	0	\	0	/

There is a local minimum point at $x = 6$ so the minimum value of P occurs when $x = 6$.

5d The minimum possible perimeter is $P = 2 \times 6 + \frac{72}{6} = 24$ cm.

6a Area $\Delta ACD = \frac{1}{2} \times \text{base} \times \text{height}$

$\frac{1}{2} \times 2y \times 2x = 1200$

$y \times 2x = 1200$

$y = \dfrac{1200}{2x}$

$y = \dfrac{600}{x}$

The total length of fencing:

$L = 2x + 3y$

$= 2x + 3 \times \dfrac{600}{x}$

$= 2x + \dfrac{1800}{x}$

6b $L' = 2 - 1800x^{-2}$

$= \frac{2x^2 - 1800}{x^2}$

$\frac{2x^2-1800}{x^2} = 0$ when

$2x^2 - 1800 = 0$

$x^2 = 900$

$x = \pm 30$ (though $x = 30$ as $x > 0$)

x	-40	-30	10	30	40
L'	$+$	0	$-$	0	$+$
L	/	Maximum turning point	\	Minimum turning point	/

Since L has a minimum turning point at $x = 30$, the least possible length of fencing can be obtained when $x = 30$ m and $y = \frac{600}{x} = \frac{600}{30} = 20$ m.

7a The frame has three h-metre-long and four w-metre-long sticks.

Thus, $3h + 4w = 12$

$4w = 12 - 3h$

$w = \frac{1}{4}(12 - 3h)$

7b $A = h \times w$

$A = h \times \frac{1}{4}(12 - 3h)$

$= h \times \left(3 - \frac{3h}{4}\right)$

$= 3h - \frac{3h^2}{4}$

7c $\frac{dA}{dh} = 3 - \frac{6h}{4}$

$= 3 - \frac{3h}{2}$

$\frac{dA}{dh} = 0$ when $3 - \frac{3h}{2} = 0$ or $h = 2$

h	0	2	3
$\frac{dA}{dh}$	/	0	\

Hence, the area of the frame is maximised when $h = 2$.

Therefore, $h = 2$ and $w = \frac{1}{4}(12 - 3 \times 2) = \frac{3}{2}$.

8a Since one square is formed using a piece of wire that has length x, the length of one side of this square is $\frac{x}{4}$. The other square will be formed using a piece of wire that has length $10 - x$, so the length of one side of this square is $\frac{10-x}{4}$.

8b The side length of one square is $\frac{x}{4}$ and the other square is $\frac{10-x}{4}$.

For the combined area:

$$A = \left(\frac{x}{4}\right)^2 + \left(\frac{10 - x}{4}\right)^2$$

$$= \frac{x^2 + (10 - x)^2}{16}$$

$$= \frac{x^2 + 100 - 20x + x^2}{16}$$

$$= \frac{2x^2 - 20x + 100}{16}$$

$$= \frac{1}{8}(x^2 - 10x + 50)$$

8c $\frac{dA}{dx} = \frac{1}{8}(2x - 10)$

$= \frac{1}{4}(x - 5)$

$\frac{dA}{dx} = 0$ when $\frac{1}{4}(x - 5) = 0$ or when $x = 5$

Therefore, there is a stationary point at $x = 5$.

Since $\frac{d^2A}{dx^2} = \frac{1}{4} > 0$ for all x, the function is concave up.

Hence, the value of x that minimises A is 5.

8d When $x = 5$, $A = \frac{1}{8}(5^2 - 10 \times 5 + 50) = \frac{25}{8}$.

Therefore, $\frac{25}{8}$ cm^2 is the least possible combined area.

9a $R = x \times \left(47 - \frac{1}{3}x\right) = x\left(47 - \frac{1}{3}x\right)$

9b $P = R - C$

$= x\left(47 - \frac{1}{3}x\right) - \left(\frac{1}{5}x^2 + 15x + 10\right)$

$= 47x - \frac{1}{3}x^2 - \frac{1}{5}x^2 - 15x - 10$

$= -\frac{8}{15}x^2 + 32x - 10$

9c $\frac{dP}{dx} = -\frac{16}{15}x + 32$

$\frac{dP}{dx} = 0$ when $-\frac{16}{15}x + 32 = 0$ or $x = 30$

x	0	30	31
$\frac{dP}{dx}$	/	0	\

There is a local maximum point at $x = 30$. Hence, 30 telescopes should be made daily to maximise the profit.

10a The area of the base is x^2 and the area of one lateral face is $x \times h$. Therefore,

$S = x^2 + 4xh$

10b If $V = 32$ and $V =$ area of base $\times$ height,

$32 = x^2 \times h$

$h = \frac{32}{x^2}$

Therefore:

$S = x^2 + 4x \times \frac{32}{x^2}$

$= x^2 + \frac{128}{x}$

10c $\frac{dS}{dx} = 2x - \frac{128}{x^2} = \frac{2x^3-128}{x^2}$

$\frac{dS}{dx} = \frac{2x^3-128}{x^2} = 0$ when

$2x^3 - 128 = 0$

$x^3 = 64$

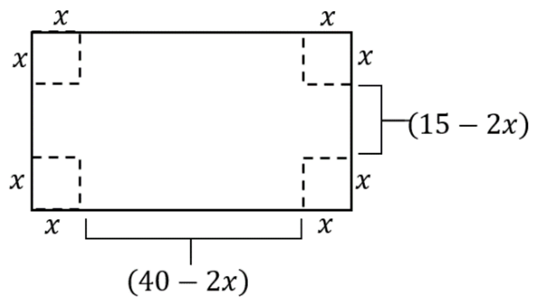
$x = 4$

Since $\frac{dS}{dx} < 0$ when $x < 4$ and $\frac{dS}{dx} > 0$ when $x > 4$, there is a minimum turning point at $x = 4$.

When $x = 4$, $h = \frac{32}{4^2} = 2$

So the dimensions of the minimum surface area are 4 cm by 4 cm by 2 cm.

11a After cutting squares of side length x cm from the corners of a rectangular sheet, we obtain the following diagram.



Length of the box (l) is $(40 - 2x)$ cm.

Width (w) of the box is $(15 - 2x)$ cm.

Height of the box (h) is x cm.

11b Volume (V) of the box is:

$V = l \times w \times h$

$V = (40 - 2x) \times (15 - 2x) \times x$

$= x(600 - 110x + 4x^2)$

$= 600x - 110x^2 + 4x^3$

11c $\frac{dV}{dx} = 600 - 220x + 12x^2$

$= 4(150 - 55x + 3x^2)$

$= 4(3x - 10)(x - 15)$

$\frac{dV}{dx} = 0$ when $4(3x - 10)(x - 15) = 0$.

Hence, there are stationary points at $x = \frac{10}{3}$ and $x = 15$.

x	0	$\frac{10}{3}$	5	15	16
$\frac{dV}{dx}$	/	0	\	0	/

Therefore, there is a local maximum point at $x = \frac{10}{3}$ and so the value of x that maximises the volume of the box is $\frac{10}{3}$.

12a $w^2 + d^2 = 48^2$

$d^2 = 48^2 - w^2$

$d^2 = 2304 - w^2$

Substituting into $s = kwd^2$ for $k > 0$ gives:

$s = kw(2304 - w^2)$

12b $s = 2304kw - kw^3$

$\frac{ds}{dw} = 2304k - 3kw^2$

$\frac{ds}{dw} = 0$ when $2304k - 3kw^2 = 0$

$2304k = 3kw^2$

$w^2 = 768$

$w = 16\sqrt{3}$ as $w > 0$

$\frac{d^2s}{dw^2} = -6kw$

For $k > 0$, $\frac{d^2s}{dw^2} < 0$ when $w = 16\sqrt{3}$ cm. Therefore, there is a local maximum at $w = 16\sqrt{3}$ and the width of the strongest rectangular beam that can be cut from the log is $16\sqrt{3}$ cm.

Substituting $w = 16\sqrt{3}$ in $d^2 = 2304 - w^2$ gives:

$d^2 = 2304 - \left(16\sqrt{3}\right)^2$

$d^2 = 2304 - 768$

$d^2 = 1536$

$d = 16\sqrt{6}$ cm

Dimensions are width $16\sqrt{3}$ cm and depth $16\sqrt{6}$ cm.

13a Let V be the volume of the box and $V = xyh$.

Let P be the perimeter of the base and $P = 2x + 2y$.

$P = 40$ and so $x + y = 20$

Let A be the surface area of the box and $A = 2xy + 2xh + 2yh$.

$A = 300$ and so $300 = 2xy + 2xh + 2yh$

$$\begin{aligned} 150 &= xy + xh + yh \\ &= xy + h(x+y) \\ &= xy + 20h \end{aligned}$$

Substituting $xy = 150 - 20h$ into $V = xyh$ we obtain $V = 150h - 20h^2$.

13b $V = 150h - 20h^2$ where $0 < h < 7.5$

$V' = 150 - 40h$

Solving $V' = 0$ for h we obtain $h = 3.75$.

$V'' = -40(< 0)$

Hence the global maximum occurs at $h = 3.75$.

So $150 = xy + 75 \Rightarrow xy = 75$ and $x + y = 20 \Rightarrow y = 20 - x$.

Substituting $y = 20 - x$ into $xy = 75$ gives $x(20 - x) = 75$.

$$\begin{aligned} x^2 - 20x + 75 &= 0 \\ (x-5)(x-15) &= 0 \\ x &= 5, 15 \end{aligned}$$

So $y = 15, 5$.

Hence the dimensions of the box are 15 cm by 5 cm by 3.75 cm.

14a $\dfrac{x}{x+4} = \dfrac{6}{y+6}$

$x(y + 6) = 6(x + 4)$

$xy + 6x = 6x + 24$

So $xy = 24$.

14b $A=\frac{1}{2}bh$

$$\begin{aligned}A&=\frac{1}{2}(x+4)(y+6)\\&=\frac{1}{2}(xy+6x+4y+24)\\&=\frac{1}{2}(24+6x+4y+24)\\&=\frac{1}{2}(48+6x+4y)\\&=24+3x+2y\end{aligned}$$

$xy=24 \Rightarrow y=\frac{24}{x}$

So $A=24+3x+\frac{48}{x}$.

14c $A=24+3x+\frac{48}{x}$

$\frac{dA}{dx}=3-\frac{48}{x^2}$

$\frac{dA}{dx}=0$ when:

$$\begin{aligned}3-\frac{48}{x^2}&=0\\3x^2-48&=0\\3\left(x^2-16\right)&=0\\x&=4\ (x>0)\end{aligned}$$

$\frac{d^2A}{dx^2}=\frac{96}{x^3}$

When $x=4$, $\frac{d^2A}{dx^2}=\frac{96}{64}(>0)$.

Hence the stationary point is a global minimum in the domain $x>0$.

Substituting $x=4$ into $A=24+3x+\frac{48}{x}$ we obtain $A=24+12+12=48$.

Hence the minimum possible area of ΔTSU is 48 cm^2.

15a Base area$= \pi r^2 = \pi x^2$ m^2 and cost of manufacturing the base is $a \times \pi x^2$ dollars

Curved area$= 2\pi r \times h = 2\pi x \times h$ m^2 and cost of manufacturing the curved area is $b \times 2\pi x \times h$ dollars.

Therefore, $c = \pi\, x^2 a + 2\pi\, xhb$

15b $V = \pi r^2 \times h = \pi x^2 \times h$ and $h = \frac{c-\pi\, x^2 a}{2\pi\, xb}$

Therefore, $V = x \times \frac{c-\pi\, x^2 a}{2b} = \frac{x}{2b}(c - \pi\, x^2 a)$

15c $\frac{dV}{dx} = \frac{c-3\pi\, x^2 a}{2b} = 0$ or when $c - 3\pi\, x^2 a = 0$ or $x = \sqrt{\frac{c}{3\pi a}}$

x		$\sqrt{\frac{c}{3\pi a}}$	
$\frac{dV}{dx}$	$+$	0	$-$

Therefore, V has a maximum at $x = \sqrt{\frac{c}{3\pi a}}$

Cost of base when $x = \sqrt{\frac{c}{3\pi a}}$ is $a \times \pi x^2 = a \times \pi \times \frac{c}{3\pi a} = \frac{c}{3}$

16a Let the area of the page be A.

Area of printed material $= 80$ cm^2

$(y-2-2)(x-1-1)=80$

So $(y-4)(x-2)=80$.

$$xy-2y-4x+8=80$$
$$y(x-2)=4(x-2)+80$$
$$\frac{y(x-2)}{x-2}=\frac{4(x-2)}{x-2}+\frac{80}{x-2}$$

Hence $y = 4 + \dfrac{80}{x-2}$.

16b Substituting $y = 4 + \dfrac{80}{x-2}$ into $A = xy$ we obtain:

$$\begin{aligned} A &= x\left(4 + \frac{80}{x-2}\right) \\ &= \frac{x(4x - 8 + 80)}{x-2} \\ &= \frac{x(4x+72)}{x-2} \end{aligned}$$

So $A = \dfrac{4x^2 + 72x}{x-2}$.

16c $A = \dfrac{4x^2 + 72x}{x-2}$

Applying the quotient rule on $A = \dfrac{4x^2 + 72x}{x-2}$:

Let $u = 4x^2 + 72x$ and $v = x - 2$.

Then $u' = 8x + 72$ and $v' = 1$.

$$\begin{aligned} \frac{dA}{dx} &= \frac{vu' - uv'}{v^2} \\ &= \frac{(x-2)(8x+72) - (4x^2 + 72x)}{(x-2)^2} \\ &= \frac{8x^2 + 56x - 144 - 4x^2 - 72x}{(x-2)^2} \\ &= \frac{4x^2 - 16x - 144}{(x-2)^2} \end{aligned}$$

So $\dfrac{dA}{dx} = \dfrac{4(x^2 - 4x - 36)}{(x-2)^2}$.

16d $\frac{dA}{dx}=\frac{4\left(x^2-4x-36\right)}{(x-2)^2}$

$\frac{dA}{dx}=0$ when:

$4\left(x^2-4x-36\right)=0$

$$\begin{aligned}x&=\frac{4\pm\sqrt{(-4)^2-4\times1\times(-36)}}{2}\\&=\frac{4\pm\sqrt{160}}{2}\\&=\frac{4+4\sqrt{10}}{2}\quad(x>0)\\&=2\left(1+\sqrt{10}\right)\end{aligned}$$

x	8	$2\left(1+\sqrt{10}\right)$	9
$\frac{dA}{dx}$	$-\frac{4}{9}$	0	$\frac{36}{49}$
slope	\	–	/

The stationary point is a global minimum in the domain $x>0$.

Substituting $x=2\left(1+\sqrt{10}\right)$ into $y=4+\frac{80}{x-2}$ we obtain:

$$\begin{aligned}y&=4+\frac{80}{2+2\sqrt{10}-2}\\&=4+\frac{40}{\sqrt{10}}\times\frac{\sqrt{10}}{\sqrt{10}}\\&=4+4\sqrt{10}\\&=4\left(1+\sqrt{10}\right)\end{aligned}$$

So the dimensions for the page in order to use the least amount of paper is $2\left(\sqrt{10}+1\right)$ cm by $4\left(\sqrt{10}+1\right)$ cm.

17a Because $\text{time} = \dfrac{\text{distance}}{\text{speed}}$, the time for the trip is $\dfrac{250}{v}$ hours.

Hence the total cost is:

$$\begin{aligned} C &= (\text{cost per hour}) \times (\text{time for the trip}) \\ &= (6400 + v^2) \times \frac{250}{v} \end{aligned}$$

So the cost of the trip, in cents, is $C = 250\left(\dfrac{6400}{v} + v\right)$.

17b $C = 250\left(\dfrac{6400}{v} + v\right)$ where $v > 0$

$$\begin{aligned} \frac{dC}{dv} &= 250\left(-\frac{6400}{v^2} + 1\right) \\ &= \frac{250(v^2 - 6400)}{v^2} \\ &= \frac{250(v-80)(v+80)}{v^2} \end{aligned}$$

So $\dfrac{dC}{dv}$ has a single zero at $v = 80$ in the domain $v > 0$.

$\dfrac{d^2C}{dv^2} = \dfrac{250 \times 12800}{v^3}$ which is positive for all $v > 0$

So $v = 80$ gives a global minimum in the domain $v > 0$.

So the speed at which the cost of the journey is minimised is 80 km/h.

17c When $v = 80$, $C = 250\left(\dfrac{6400}{80} + 80\right) = 40000$ (cents).

So the minimum cost of the journey is $\$400$.

18a $I_C = \dfrac{W}{x^2} + \dfrac{2W}{(30-x)^2}$ (Since P is between the light sources, its distance from the second light source is $30 - x$ metres)

18b $\frac{d(I_C)}{dx} = W \times \frac{d\left(\frac{1}{x^2}+\frac{2}{(30-x)^2}\right)}{dx} = W \times \left(\frac{-4}{(x-30)^3} - \frac{2}{x^3}\right) = 0$

When $x = \frac{30}{\sqrt[3]{2}+1} \doteqdot 13.27$

19a Using Pythagoras' Theorem, the distance rowed is $\sqrt{6^2+x^2}$ and the rowing speed is 8 km/h.

The distance run is $20-x$ and the running speed is 10 km/h.

Using $\text{time} = \frac{\text{distance}}{\text{speed}}$ with T denoting the total time taken:

$$T = \frac{\sqrt{36+x^2}}{8} + \frac{20-x}{10}$$

So $T = \frac{1}{8}\sqrt{36+x^2} + \frac{1}{10}(20-x)$.

19b $T = \frac{1}{8}\sqrt{36+x^2} + \frac{1}{10}(20-x)$

$$\frac{dT}{dx} = \frac{x}{8\sqrt{36+x^2}} - \frac{1}{10}$$

$\frac{dT}{dx} = 0$ when:

$$\frac{x}{8\sqrt{36+x^2}} - \frac{1}{10} = 0$$

$$10x = 8\sqrt{36+x^2}$$

$$100x^2 = 64\left(36+x^2\right)$$

$$100x^2 - 64x^2 = 64\times 36$$

$$36x^2 - 64\times 36 = 0$$

$$36\left(x^2-64\right) = 0$$

$$x = 8 \ (0 \le x \le 20)$$

x	7	8	9
$\frac{dT}{dx}$	–0.005...	0	0.004...
slope	\	–	/

So T has a local minimum at $x=8$.

20 $\frac{1}{v}+\frac{1}{u}=\frac{1}{f}$

$\frac{u+v}{uv}=\frac{1}{f}$

$u+v=\frac{uv}{f}$ ... (1) (distance between the object and the image)

And $\frac{1}{v}=\frac{1}{f}-\frac{1}{u}$

$\frac{1}{v}=\frac{u-f}{fu}$

$v=\frac{fu}{u-f}$ ... (2)

From (1) and (2), $u+v=\frac{u\left(\frac{fu}{u-f}\right)}{f}=\frac{u^2}{u-f}$

$\frac{d(u+v)}{du}=\frac{d\left(\frac{u^2}{u-f}\right)}{du}=\frac{2u\times(u-f)-u^2\times 1}{(u-f)^2}=\frac{u^2-2uf}{(u-f)^2}=0$ when $u^2-2uf=0$

Or $u=2f$.

u		$2f$	
$\frac{d(u+v)}{du}$	$-$	0	$+$

Hence, $u=2f$ is a minimum turning point and the distance between the object and the image is minimum when $u=2f$. Thus, the minimum distance is,

$u+v=\frac{u^2}{u-f}=\frac{(2f)^2}{(2f)-f}=4f$.

21 Let t_1 be the time taken in air, then $t_1=\frac{d_1}{v_1}$

Let t_2 be the time taken in water, then $t_2=\frac{d_2}{v_2}$

Using Pythagoras' Theorem:

$d_1=\sqrt{a^2+x^2}$ and $d_2=\sqrt{b^2+(c-x)^2}$

So $t_1 = \frac{\sqrt{a^2+x^2}}{v_1}$ and $t_2 = \frac{\sqrt{b^2+(c-x)^2}}{v_2}$

$t_1 + t_2$

$$= \frac{\sqrt{a^2+x^2}}{v_1} + \frac{\sqrt{b^2+(c-x)^2}}{v_2}$$

For minimum t, $\frac{d}{dx}(t_1 + t_2) = 0$

$$\frac{d}{dx}\left(\frac{\sqrt{a^2+x^2}}{v_1} + \frac{\sqrt{b^2+(c-x)^2}}{v_2}\right) = 0$$

$$\frac{\frac{1}{2}(a^2+x^2)^{-\frac{1}{2}} \times 2x}{v_1} + \frac{\frac{1}{2}(b^2+(c-x)^2)^{-\frac{1}{2}} \times 2(c-x) \times (-1)}{v_2} = 0$$

$$\frac{x}{v_1\sqrt{a^2+x^2}} + \frac{-(c-x)}{v_2\sqrt{b^2+(c-x)^2}} = 0$$

$$\frac{x}{v_1\sqrt{a^2+x^2}} = \frac{c-x}{v_2\sqrt{b^2+(c-x)^2}}$$

Using trigonometry:

$\sin\theta_1 = \frac{x}{d_1} = \frac{x}{\sqrt{a^2+x^2}}$ and $\sin\theta_2 = \frac{c-x}{d_2} = \frac{c-x}{\sqrt{b^2+(c-x)^2}}$

Substituting for $\frac{x}{\sqrt{a^2+x^2}}$ and $\frac{c-x}{\sqrt{b^2+(c-x)^2}}$ in

$$\frac{x}{v_1\sqrt{a^2+x^2}} = \frac{c-x}{v_2\sqrt{b^2+(c-x)^2}}$$

gives

$$\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}$$

Solutions to Exercise 4I

1a $h + 2\pi r = 10$ then $h = 10 - 2\pi r$

1b $V = \pi r^2 \times h = \pi r^2 \times (10 - 2\pi r)$

1c $\frac{dV}{dr} = \frac{d(10\pi r^2 - 2\pi^2 r^3)}{dr} = 20\pi r - 6\pi^2 r^2 = r(20\pi - 6\pi^2 r)$

Therefore, V has stationary points when $r = 0$ or $20\pi - 6\pi^2 r = 0$ or $r = \frac{10}{3\pi}$

r		0		$\frac{10}{3\pi}$	
$\frac{dV}{dr}$	$-$	0	$+$	0	$-$

Therefore, $r = \frac{10}{3\pi}$ is a maximum turning point.

1d Hence, the maximum volume is:

$$V = \pi\left(\frac{10}{3\pi}\right)^2 \times \left(10 - 2\pi\left(\frac{10}{3\pi}\right)\right) = \frac{1000}{27\pi}$$

2a Surface area$= 2\pi r \times h + 2\pi r^2 = 60\pi$

$$h = \frac{60\pi - 2\pi r^2}{2\pi r} = \frac{30 - r^2}{r}$$

2b $V = \pi r^2 \times h = \pi r^2 \times \frac{30 - r^2}{r} = \pi r(30 - r^2)$

2c $\frac{dV}{dr} = \frac{d(30\pi r - \pi r^3)}{dr} = 30\pi - 3\pi r^2 = 0$ when $r^2 = 10$ or $r = \pm\sqrt{10}$

r		$-\sqrt{10}$		$\sqrt{10}$	
$\frac{dV}{dr}$	$-$	0	$+$	0	$-$

Therefore, V has a maximum turning point at $r = \sqrt{10}$.

Hence, the maximum volume is: $V = \pi(\sqrt{10})\left(30 - (\sqrt{10})^2\right) = 20\sqrt{10}\pi$ cm³

3a $S = \pi {r_1}^2 + \pi {r_2}^2$ and $r_1 + r_2 = k$

Hence, $S = \pi {r_1}^2 + \pi(k - r_1)^2$

3b $\frac{dS}{r_1} = 2\pi r_1 - 2\pi(k - r_1) = 0$

When $2\pi(r_1 - r_2) = 0$ or $r_1 = r_2$

4a $\frac{\theta}{360} = \frac{L-2r}{2\pi r}$ (the ratio of θ to one revolution is equal to the ratio of the sector arc to the circumference of the circle) then $\theta = \frac{L-2r}{r} = \frac{L}{r} - 2$

4b $A = \pi r^2 \times \frac{\frac{L}{r}-2}{2\pi} = \frac{r(L-2r)}{2}$ is maximum when

$\frac{dA}{dr} = \frac{d\left(\frac{r(L-2r)}{2}\right)}{dr} = 0$

$\frac{d\left(\frac{Lr}{2}-r^2\right)}{dr} = \frac{L}{2} - 2r = 0$ or $L = 4r$ or $r = \frac{1}{4}L$

5a $\Delta ABC \sim \Delta ADE$ because $\angle ABC = \angle ADE = 90°$ (because $BC \parallel DE$) and $\angle ACB = \angle AED$ (because $BC \parallel DE$ and AE is a straight line)

5b Since $\Delta ABC \sim \Delta ADE, \frac{BC}{DE} = \frac{AB}{AD}$

Therefore, $\frac{r}{12} = \frac{40-h}{40}$ then $\frac{40r}{12} = 40 - h$ and $h = 40 - \frac{10}{3}r$

5c $V = \pi r^2 \times h = \pi r^2 \times \left(40 - \frac{10}{3}r\right) = 40\pi r^2 - \frac{10}{3}\pi r^3$

5d $\frac{dV}{dr} = \frac{d\left(40\pi r^2 - \frac{10}{3}\pi r^3\right)}{dr} = 80\pi r - 10\pi r^2 = 10\pi r(8 - r) = 0$ when

$r = 0$ or $r = 8$

The value of r for which V is maximised is $r = 8$.

6 Let the perimeter of a rectangle be k units, width x units and length y units.

Then $k = 2x + 2y$ and $y = \frac{k-2x}{2}$

Hence. The area is $A(x) = x \times \frac{k-2x}{2}$ and $A'(x) = x \times \frac{k-2x}{2} = \frac{k}{2} - 2x = 0$ when

$x = \frac{\frac{k}{2}}{2} = \frac{k}{4}$

x		$\frac{k}{4}$	
$A'(x)$	$+$	0	$-$

Therefore, the area is maximum when $x = \frac{k}{4}$. Hence, when $y = \frac{k-2\times\frac{k}{4}}{2} = \frac{k}{4}$ or,

when the shape is a square.

7a $R^2 = \left(\frac{h}{2}\right)^2 + r^2$ then $r^2 = R^2 - \frac{1}{4}h^2$

7b $V = \pi r^2 \times h = \pi\left(R^2 - \frac{1}{4}h^2\right) \times h = \frac{\pi}{4}h(4R^2 - h^2)$

7c $\frac{dV}{dh} = \frac{d\left(\frac{\pi}{4}(4R^2h - h^3)\right)}{dh} = \frac{\pi}{4}(4R^2 - 3h^2) = 0$ when $4R^2 - 3h^2 = 0$ or $h = \sqrt{\frac{4R^2}{3}} = \frac{2\sqrt{3}}{3}R$

Therefore, the volume of the cylinder is maximised when $h = \frac{2\sqrt{3}}{3}R$

7d

$$\frac{V_{\text{sphere}}}{V_{\text{cylinder}}}$$

$$= \frac{\frac{4}{3}\pi R^3}{\pi \times r^2 \times h}$$

$$= \frac{\frac{4}{3}\pi R^3}{\pi \times \left(R^2 - \left(\frac{2\sqrt{3}}{3}R\right)^2\right) \times \frac{2\sqrt{3}}{3}R}$$

(when $h = \frac{2\sqrt{3}}{3}R$, $r^2 = R^2 - \left(\frac{2\sqrt{3}}{6}R\right)^2$)

$$= \frac{\frac{4}{3}R^2}{\left(R^2 - \left(\frac{2\sqrt{3}}{6}R\right)^2\right) \times \frac{2\sqrt{3}}{3}}$$

$$= \frac{4}{3} \times \frac{3}{2\sqrt{3}} \times \frac{R^2}{R^2 - \left(\frac{2\sqrt{3}}{6}R\right)^2}$$

$$= \frac{2}{\sqrt{3}} \times \frac{R^2}{\frac{2}{3} \times R^2}$$

$$= \frac{2}{\sqrt{3}} \times \frac{3}{2}$$

$$= \frac{\sqrt{3}}{1}$$

Therefore, the ratio of the volume of the sphere to the maximum volume of the cylinder is $\sqrt{3} : 1$.

8a $S = \pi r^2 + 2\pi r \times h$ then $h = \frac{S - \pi r^2}{2\pi r}$

8b $V = \pi r^2 \times h = \pi r^2 \times \left(\frac{S - \pi r^2}{2\pi r}\right) = \frac{1}{2} \times r \times (S - \pi r^2) = \frac{1}{2}(Sr - \pi r^3)$

$\frac{dV}{dr} = \frac{1}{2}(S - 3\pi r^2) = 0$ when $S - 3\pi r^2 = 0$ or $r = \sqrt{\frac{S}{3\pi}}$

Or when $h = \frac{S-\pi r^2}{2\pi r} = r$ (because $S - \pi r^2 = 2\pi r^2$, $S = 3\pi r^2$, $r = \sqrt{\frac{S}{3\pi}}$)

9 $V_{\text{cone}} = \frac{\pi r^2 \times h}{3}$ where r and h are the radius and height of the cone.

Since $\left(\frac{h}{2}\right)^2 + r^2 = R^2$, $h^2 = 4(R^2 - r^2)$ and $h = 2\sqrt{R^2 - r^2}$

Hence, $V_{\text{cone}} = \frac{\pi r^2 \times 2\sqrt{R^2-r^2}}{3} = \frac{2\pi}{3}\sqrt{R^2 r^4 - r^6}$

$\frac{d\,(\text{V}_{\text{cone}})}{dr} = \frac{d\left(\frac{2\pi}{3}\sqrt{R^2r^4-r^6}\right)}{dr} = \frac{2\pi}{3} \times \frac{4R^2r^3-6r^5}{2\sqrt{R^2r^4-r^6}} = \frac{\pi}{3} \times \frac{2r^3(2R^2-3r^2)}{\sqrt{R^2r^4-r^6}} = 0$ when $r = 0$ or

$2R^2 - 3r^2 = 0$ or $r^2 = \frac{2R^2}{3}$

When the volume of the cone is maximised, $r^2 = \frac{2R^2}{3}$.

So $\frac{h^2}{R^2} = \frac{4(R^2-r^2)}{R^2} = \frac{4\left(R^2-\frac{2R^2}{3}\right)}{R^2}$

Therefore, $\frac{h^2}{R^2} = \frac{4}{3}$ and $\frac{h}{R} = \frac{2}{\sqrt{3}}$ and the ratio of h to R is $2{:}\sqrt{3}$.

10a $A_{\text{rectangle}} = \text{width} \times \text{height} = x \times y$ and $r^2 = x^2 + y^2$ and $x^2 = r^2 - y^2$

Then $A_{\text{rectangle}} = y\sqrt{r^2 - y^2}$

10b $\frac{d(A_{\text{rectangle}})}{dy} = \frac{d\left(\sqrt{y^2r^2-y^4}\right)}{dy} = \frac{2r^2y-4y^3}{2\sqrt{y^2r^2-y^4}} = \frac{r^2y-2y^3}{y\sqrt{r^2-y^2}} = \frac{r^2-2y^2}{\sqrt{r^2-y^2}} = 0$

when $r^2 - 2y^2 = 0$ or when $r^2 = 2y^2$.

Hence, the area of the rectangle is maximum when $y^2 = \frac{r^2}{2}$.

Therefore, the maximum area is:

$$A_{\text{rectangle}} = y\sqrt{r^2 - y^2} = \sqrt{\frac{r^2}{2}} \times \sqrt{r^2 - \frac{r^2}{2}} = \frac{1}{2}r^2$$

11a $\left(\frac{h}{2}\right)^2 + r^2 = R^2$ then $\frac{h}{2} = \sqrt{R^2 - r^2}$ and $h = 2\sqrt{R^2 - r^2}$

11b $S = 2\pi r \times h = 2\pi r \times 2\sqrt{R^2 - r^2} = 4\pi r\sqrt{R^2 - r^2}$

11c

$$\frac{dS}{dr} = \frac{d\left(4\pi r\sqrt{R^2 - r^2}\right)}{dr}$$
$$= \frac{d\left(4\pi\sqrt{R^2r^2 - r^4}\right)}{dr}$$
$$= 4\pi \times \frac{2R^2r - 4r^3}{2\sqrt{R^2r^2 - r^4}}$$
$$= 4\pi \times \frac{R^2r - 2r^3}{r\sqrt{R^2 - r^2}}$$
$$= 4\pi \times \frac{R^2 - 2r^2}{\sqrt{R^2 - r^2}}$$

$\frac{dS}{dr} = 0$ when $R^2 - 2r^2 = 0$ or $r^2 = \frac{R^2}{2}$. Therefore, the cylinder has maximum surface area when $r^2 = \frac{R^2}{2}$ and the maximum surface area is

$S = 4\pi \times \sqrt{\frac{R^2}{2}} \times \sqrt{R^2 - \frac{R^2}{2}} = 2\pi R^2$

12 $V = \pi r^2 \times h$ then $h = \frac{V}{\pi r^2}$

$S = 2\pi r \times h + 2\pi r^2 = 2\pi r \times \frac{V}{\pi r^2} + 2\pi r^2 = 2V\frac{1}{r} + 2\pi r^2$

$\frac{dS}{dr} = 4\pi r - \frac{2V}{r^2} = \frac{4\pi r^3 - 2V}{r^2} = 0$ when $4\pi r^3 - 2V = 0$ or $V = 2\pi r^3$

Hence, $h = \frac{V}{\pi r^2} = \frac{2\pi r^3}{\pi r^2} = 2r$ then $\frac{r}{h} = \frac{1}{2}$ when the surface area is minimised and the $r : h$ ratio is $1 : 2$.

13 $V = \frac{1}{3}\pi r^2\sqrt{s^2 - r^2}$ where $s^2 = h^2 + r^2$, $h^2 = s^2 - r^2$ and $h = \sqrt{s^2 - r^2}$

$S = \pi rs$ then $s = \frac{S}{\pi r}$. Therefore, $V = \frac{1}{3}\pi r^2\sqrt{\left(\frac{S}{\pi r}\right)^2 - r^2}$

$$\frac{dV}{dr} = \frac{d\left(\frac{1}{3}\pi r^2\sqrt{\left(\frac{S}{\pi r}\right)^2 - r^2}\right)}{dr} = \frac{\sqrt{S^2 - \pi^2 r^4}}{3} - \frac{2\pi^2 r^4}{3\sqrt{S^2 - \pi^2 r^4}}$$

$\frac{dV}{dr} = 0$ when $r^2 = \frac{S}{\sqrt{3}\,\pi}$, $h = \sqrt{\left(\frac{S}{\pi r}\right)^2 - \left(\frac{S}{\sqrt{3}\,\pi}\right)}$ and $h^2 = \frac{S\sqrt{3}}{\pi} - \left(\frac{S}{\sqrt{3}\,\pi}\right) = \frac{2S}{\sqrt{3}\,\pi}$

$\frac{h^2}{r^2} = \frac{\frac{2S}{\sqrt{3}\,\pi}}{\frac{S}{\sqrt{3}\,\pi}} = 2$ then $\frac{h}{r} = \sqrt{2}$.

Therefore, $h : r = \sqrt{2} : 1$ when the volume is maximised.

14 Let one of the equal sides of the triangle be a and one of the equal base angles be θ. Then the area of the triangle is

$A(\theta) = \frac{1}{2} \times b \times h = \frac{1}{2} \times (2a\cos(\theta) \times (a\sin\theta) = a^2 \cos\theta\ \sin\theta$ ...(1)

And Area$= ra + \frac{rb}{2} = ra + ra\cos\theta$ ... (2)

$a^2 \cos\theta\ \sin\theta = ra + ra\cos\theta$ ((1) and (2) solved together)

$a = \frac{r + r\cos\theta}{\cos\theta\sin\theta}$

Hence, $A(\theta) = a^2\cos\theta\ \sin\theta = \left(\frac{r+r\cos\theta}{\cos\theta\sin\theta}\right)^2 \cos\theta\ \sin\theta$

$A(\theta) = \frac{(r+r\cos\theta)^2}{\cos\theta\sin\theta}$

When $\frac{dA(\theta)}{d\theta} = 0$ is solved, $\theta = 60^\circ$ is the stationary point where the area is minimum.

And when $\theta = 60^\circ$, the triangle is an equilateral triangle with area $\frac{a^2\sqrt{3}}{4}$ and height $\frac{a\sqrt{3}}{2}$ which is equal to $3r$. Therefore, $r = \frac{a\sqrt{3}}{6}$ then $a = \frac{6r}{\sqrt{3}}$ and the minimum area is $\frac{a^2\sqrt{3}}{4} = \frac{\left(\frac{6r}{\sqrt{3}}\right)^2\sqrt{3}}{4} = 3\sqrt{3}r^2$

Solutions to Exercise 4J

1a Let $\frac{dy}{dx} = x^6$

Then $y = \frac{1}{7}x^7 + C$, for some constant C.

1b Let $\frac{dy}{dx} = x^3$

Then $y = \frac{1}{4}x^4 + C$, for some constant C.

1c Let $\frac{dy}{dx} = x^{10}$

Then $y = \frac{1}{11}x^{11} + C$, for some constant C.

1d Let $\frac{dy}{dx} = 3x$

Then $y = \frac{3}{2}x^2 + C$, for some constant C.

1e Let $\frac{dy}{dx} = 5$

Then $y = 5x + C$, for some constant C.

1f Let $\frac{dy}{dx} = 5x^9$

Then $= \frac{5}{10}x^{10} + C$, for some constant C.

$y = \frac{1}{2}x^{10} + C$, for some constant C.

1g Let $\frac{dy}{dx} = 21x^6$

Then $= \frac{21}{7}x^7 + C$, for some constant C

$y = 3x^7 + C$, for some constant C.

1h Let $\frac{dy}{dx} = 0$

Then $y = C$, for some constant C.

2a Let $\frac{dy}{dx} = x^2 + x^4$

Then $= \frac{1}{3}x^3 + \frac{1}{5}x^5 + C$, for some constant C.

2b Let $\frac{dy}{dx} = 4x^3 - 5x^4$

Then $= x^4 - x^5 + C$, for some constant C.

2c Let $\frac{dy}{dx} = 2x^2 + 5x^7$

Then $= \frac{2}{3}x^3 + \frac{5}{8}x^8 + C$, for some constant C.

2d Let $\frac{dy}{dx} = x^2 - x + 1$

Then $= \frac{1}{3}x^3 - \frac{1}{2}x^2 + x + C$, for some constant C.

2e Let $\frac{dy}{dx} = 3 - 4x + 16x^7$

Then $= 3x - 2x^2 + 2x^8 + C$, for some constant C.

2f Let $\frac{dy}{dx} = 3x^2 - 4x^3 - 5x^4$

Then $= x^3 - x^4 - x^5 + C$, for some constant C.

3a Let $\frac{dy}{dx} = x(x-3) = x^2 - 3x$

Then $= \frac{1}{3}x^3 - \frac{3}{2}x^2 + C$, for some constant C.

3b Let $\frac{dy}{dx} = (x+1)(x-2) = x^2 - x - 2$

Then $= \frac{1}{3}x^3 - \frac{1}{2}x^2 - 2x + C$, for some constant C.

3c Let $\frac{dy}{dx} = (3x-1)(x+4) = 3x^2 + 11x - 4$

Then $= x^3 + \frac{11}{2}x^2 - 4x + C$, for some constant C.

3d Let $\frac{dy}{dx} = x^2(5x^3 - 4x) = 5x^5 - 4x^3$

Then $= \frac{5}{6}x^6 - x^4 + C$, for some constant C.

3e Let $\frac{dy}{dx} = 2x^3(4x^4 + 1) = 8x^7 + 2x^3$

Then $= x^8 + \frac{1}{2}x^4 + C$, for some constant C.

3f Let $\frac{dy}{dx} = (x-3)(1+x^2) = x + x^3 - 3 - 3x^2$

Then $y = \frac{1}{2}x^2 + \frac{1}{4}x^4 - 3x - x^3 + C$, for some constant C.

4a i $y' = 2x + 3$

$y = x^2 + 3x + C$, for some constant C.

If $y = 3$ when $x = 0$, then $3 = 0^2 + 3 \times 0 + C$.

Hence, $C = 3$

Therefore, $y = x^2 + 3x + 3$

4a ii $y' = 2x + 3$

$y = x^2 + 3x + C$, for some constant C.

If $y = 8$ when $x = 1$, then $8 = 1^2 + 3 \times 1 + C$.

Hence, $C = 4$

Therefore, $y = x^2 + 3x + 4$

4b i $y' = 9x^2 + 4$

$y = 3x^3 + 4x + C$, for some constant C.

If $y = 1$ when $x = 0$, then $1 = 3 \times 0^3 + 4 \times 0 + C$.

Hence, $C = 1$

Therefore, $y = 3x^3 + 4x + 1$

4b ii $y' = 9x^2 + 4$

$y = 3x^3 + 4x + C$, for some constant C.

If $y = 5$ when $x = 1$, then $5 = 3 \times 1^3 + 4 \times 1 + C$.

Hence, $C = -2$

Therefore, $y = 3x^3 + 4x - 2$

4c i $y' = 3x^2 - 4x + 7$

$y = x^3 - 2x^2 + 7x + C$, for some constant C.

If $y = 0$ when $x = 0$, then $0 = 0^3 - 2 \times 0^2 + 7 \times 0 + C$.

Hence, $C = 0$

Therefore, $y = x^3 - 2x^2 + 7x$

4c ii $y' = 3x^2 - 4x + 7$

$y = x^3 - 2x^2 + 7x + C$, for some constant C.

If $y = -1$ when $x = 1$, then $-1 = 1^3 - 2 \times 1^2 + 7 \times 1 + C$.

Hence, $C = -7$

Therefore, $y = x^3 - 2x^2 + 7x - 7$

5a Let $\frac{dy}{dx} = \frac{1}{x^2} = x^{-2}$

Then $y = \frac{x^{-1}}{-1} + C$, for some constant C

$= -x^{-1} + C$

$= -\frac{1}{x} + C$

5b Let $\frac{dy}{dx} = \frac{1}{x^3} = x^{-3}$

Then $= \frac{x^{-2}}{-2} + C$, for some constant C

$= -\frac{1}{2x^2} + C$

5c Let $\frac{dy}{dx} = -\frac{2}{x^3} = -2x^{-3}$

Then $= \frac{-2x^{-2}}{-2} + C$, for some constant C

$= x^{-2} + C$

$= \frac{1}{x^2} + C$

5d Let $\frac{dy}{dx} = -\frac{3}{x^4} = -3x^{-4}$

Then $y = \frac{-3x^{-3}}{-3} + C$, for some constant C

$= x^{-3} + C$

$= \frac{1}{x^3} + C$

5e Let $\frac{dy}{dx} = \frac{1}{x^2} - \frac{1}{x^3} = x^{-2} - x^{-3}$

Then $y = \frac{x^{-1}}{-1} - \frac{x^{-2}}{-2} + C$, for some constant C

$= -\frac{1}{x} + \frac{1}{2x^2} + C$

6a Let $\frac{dy}{dx} = \sqrt{x} = x^{\frac{1}{2}}$

Then $y = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + C$, for some constant C

$= \frac{2}{3}x^{\frac{3}{2}} + C$

6b Let $\frac{dy}{dx} = \frac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$

Then $y = \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + C$, for some constant C

$= 2\sqrt{x} + C$

6c Let $\frac{dy}{dx} = \sqrt[3]{x} = x^{\frac{1}{3}}$

Then $y = \frac{x^{\frac{4}{3}}}{\frac{4}{3}} + C$, for some constant C

$= \frac{3}{4}x^{\frac{4}{3}} + C$

6d Let $\frac{dy}{dx} = \sqrt[3]{x} = x^{\frac{1}{3}}$

Then $y = \frac{x^{\frac{4}{3}}}{\frac{4}{3}} + C$, for some constant C

$= \frac{3}{4}x^{\frac{4}{3}} + C$

6e Let $\frac{dy}{dx} = \sqrt[5]{x^3} = x^{\frac{3}{5}}$

Then $y = \frac{x^{\frac{8}{5}}}{\frac{8}{5}} + C$, for some constant C

$= \frac{5}{8}x^{\frac{8}{5}} + C$

7a $\frac{dy}{dx} = \sqrt{x} = x^{\frac{1}{2}}$

Then $y = \frac{2}{3}x^{\frac{3}{2}} + C$, for some constant C.

If $y = 1$ when $x = 0$, then $1 = \frac{2}{3} \times 0^{\frac{3}{2}} + C$

Hence, $C = 1$

Therefore, $y = \frac{2}{3}x^{\frac{3}{2}} + 1$

7b Let $\frac{dy}{dx} = \sqrt{x} = x^{\frac{1}{2}}$

Then $= \frac{2}{3}x^{\frac{3}{2}} + C$, for some constant C.

If $y = 2$ when $x = 9$, then $2 = \frac{2}{3} \times 9^{\frac{3}{2}} + C$

Hence, $C = -16$

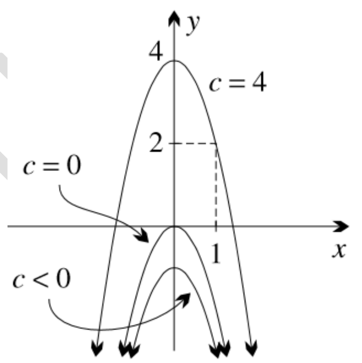
Therefore, $y = \frac{3}{2}x^{\frac{3}{2}} - 16$

8a $\frac{dy}{dx} = -4x$

$y = -2x^2 + c$, for some constant c

Some of the family of curves of $y = -2x^2 + c$ are shown below.

For example, when $c = 0, y = -2x^2$.



If $y = -2x^2 + c$ passes through $A(1, 2)$ then $2 = -2 \times 1^2 + c$ so $c = 4$

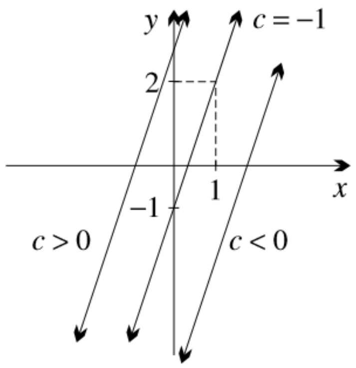
and $y = -2x^2 + 4$.

8b $\frac{dy}{dx} = 3$

$y = 3x + c$, for some constant c.

Some of the family of curves of $y = 3x + c$ are shown below.

For example, when $c = -1, y = 3x - 1$.



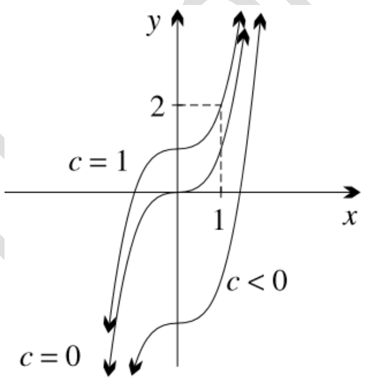
If $y = 3x + c$ passes through $A(1, 2)$ then $2 = 3 \times 1 + c$ so $c = -1$ and $y = 3x - 1$.

8c $\frac{dy}{dx} = 3x^2$

$y = x^3 + c$, for some constant c

Some of the family of curves of $y = x^3 + c$ are shown below.

For example, when $c = 0, y = x^3$.



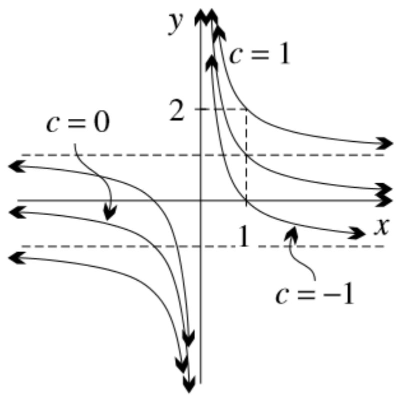
If $y = x^3 + c$ passes through $A(1, 2)$ then $2 = 1^3 + c$ so $c = 1$ and $y = x^3 + 1$.

8d $\frac{dy}{dx} = -\frac{1}{x^2} = -x^{-2}$

$y = x^{-1} + c$ or $y = \frac{1}{x} + c$, for some constant c

Some of the family of curves of $y = \frac{1}{x} + c$ are shown below.

For example, when $c = 0, y = \frac{1}{x}$.



If $y = \frac{1}{x} + c$ passes through $A(1, 2)$ then $2 = \frac{1}{1} + c$ so $c = 1$ and $y = \frac{1}{x} + 1$.

9a Let $\frac{dy}{dx} = (x + 1)^3$

Then $y = \frac{(x+1)^4}{4} + C$, for some constant C

$= \frac{1}{4}(x + 1)^4 + C$

9b Let $\frac{dy}{dx} = (x - 2)^5$

Then $y = \frac{(x-2)^6}{6} + C$, for some constant C

$= \frac{1}{6}(x - 2)^6 + C$

9c Let $\frac{dy}{dx} = (x + 5)^2$

Then $y = \frac{(x+5)^3}{3} + C$, for some constant C

$= \frac{1}{3}(x + 5)^3 + C$

9d Let $\frac{dy}{dx} = (2x + 3)^4$

Then $y = \frac{(2x+3)^5}{2\times5} + C$, for some constant C

$= \frac{1}{10}(2x + 3)^5 + C$

9e Let $\frac{dy}{dx} = (3x - 4)^6$

Then $y = \frac{(3x-4)^7}{3\times7} + C$, for some constant C

$= \frac{1}{21}(3x - 4)^7 + C$

9f Let $\frac{dy}{dx} = (5x - 1)^3$

Then $y = \frac{(5x-1)^4}{5\times4} + C$, for some constant C

$= \frac{1}{20}(5x - 1)^4 + C$

9g Let $\frac{dy}{dx} = (1 - x)^3$

Then $y = \frac{(1-x)^4}{-1\times4} + C$, for some constant C

$= -\frac{1}{4}(1 - x)^4 + C$

9h Let $\frac{dy}{dx} = (1 - 7x)^3$

Then $y = \frac{(1-7x)^4}{-7\times4} + C$, for some constant C

$= -\frac{1}{28}(1 - x)^4 + C$

9i Let $\frac{dy}{dx} = \frac{1}{(x-2)^4} = (x-2)^{-4}$

Then $y = \frac{(x-2)^{-3}}{-3} + C$, for some constant C

$$= -\frac{1}{3}(x-2)^{-3} + C$$

$$= -\frac{1}{3(x-2)^3} + C$$

9j Let $\frac{dy}{dx} = \frac{1}{(1-x)^{10}} = (1-x)^{-10}$

Then $y = \frac{(1-x)^{-9}}{-1\times-9} + C$, for some constant C

$$= \frac{1}{9}(1-x)^{-9} + C$$

$$= \frac{1}{9(1-x)^9} + C$$

10a Let $y' = \sqrt{x+1} = (x+1)^{\frac{1}{2}}$

Then $y = \frac{(x+1)^{\frac{3}{2}}}{\frac{3}{2}} + C$, for some constant C

$$= \frac{2}{3}(x+1)^{\frac{3}{2}} + C$$

10b Let $y' = \sqrt{x-5} = (x-5)^{\frac{1}{2}}$

Then $y = \frac{(x-5)^{\frac{3}{2}}}{\frac{3}{2}} + C$, for some constant C

$$= \frac{2}{3}(x-5)^{\frac{3}{2}} + C$$

10c Let $y' = \sqrt{1-x} = (1-x)^{\frac{1}{2}}$

Then $y = \frac{(1-x)^{\frac{3}{2}}}{-1\times\frac{3}{2}} + C$, for some constant C

$= -\frac{2}{3}(1-x)^{\frac{3}{2}} + C$

10d Let $y' = \sqrt{2x-7} = (2x-7)^{\frac{1}{2}}$

Then $y = \frac{(2x-7)^{\frac{3}{2}}}{2\times\frac{3}{2}} + C$, for some constant C

$= \frac{1}{3}(2x-7)^{\frac{3}{2}} + C$

10e Let $y' = \sqrt{3x-4} = (3x-4)^{\frac{1}{2}}$

Then $y = \frac{(3x-4)^{\frac{3}{2}}}{3\times\frac{3}{2}} + C$, for some constant C

$= \frac{2}{9}(3x-4)^{\frac{3}{2}} + C$

11a $y' = (x-1)^4$

Then $y = \frac{1}{5}(x-1)^5 + C$, for some constant C.

If $y = 0$ when $x = 1$,

$0 = \frac{1}{5}(1-1)^5 + C$

$C = 0$

Therefore, $y = \frac{1}{5}(x-1)^5$

11b $y' = (2x+1)^3$

Then $y = \frac{1}{8}(2x+1)^4 + C$, for some constant C.

If $y = -1$ when $x = 0$,

$-1 = \frac{1}{8}(2 \times 0 + 1)^4 + C$

$C = -\frac{9}{8}$

Therefore, $y = \frac{1}{8}(2x+1)^4 - \frac{9}{8}$

11c $y' = \sqrt{2x+1} = (2x+1)^{\frac{1}{2}}$

Then $y = \frac{1}{3}(2x+1)^{\frac{3}{2}} + C$, for some constant C.

If $y = \frac{1}{3}$ when $x = 0$,

$\frac{1}{3} = \frac{1}{3}(2 \times 0 + 1)^{\frac{3}{2}} + C$

$C = 0$

Therefore, $y = \frac{1}{3}(2x+1)^{\frac{3}{2}}$

12a $\frac{dy}{dt} = 3x^4 - x^3 + 1$

$y = \frac{3x^5}{5} - \frac{x^4}{4} + x + C$, for some constant C

If the curve passes through the origin,

$0 = \frac{3 \times 0^5}{5} - \frac{0^4}{4} + 0 + C$

$C = 0$

Therefore, $y = \frac{3}{5}x^5 - \frac{1}{4}x^4 + x$

12b $\frac{dy}{dt} = 2 + 3x^2 - x^3$

$y = 2x + x^3 - \frac{x^4}{4} + C$, for some constant C

If the curve passes through the point $(2, 6)$,

$6 = 2 \times 2 + 2^3 - \frac{2^4}{4} + C$

$6 = 8 + C$

$C = -2$

Therefore, $y = -\frac{1}{4}x^4 + x^3 + 2x - 2$

12c $y' = (2 - 5x)^3$

$y = \frac{(2-5x)^4}{-5\times4} + C$, for some constant C

$= -\frac{1}{20}(2 - 5x)^4 + C$

If the curve passes through the point $\left(\frac{1}{5}, 1\right)$,

$1 = -\frac{1}{20}\left(2 - 5 \times \frac{1}{5}\right)^4 + C$

$1 = -\frac{1}{20} + C$

$C = \frac{21}{20}$

Therefore, $y = -\frac{1}{20}(2 - 5x)^4 + \frac{21}{20}$

13 $\frac{dy}{dt} = 8t^3 - 6t^2 + 5$

$y = 2t^4 - 2t^3 + 5t + C$, for some constant C

If $y = 4$ when $t = 0$,

$4 = 2 \times 0^4 - 2 \times 0^3 + 5 \times 0 + C$

$C = 4$

Therefore, $y = 2t^4 - 2t^3 + 5t + 4$

When $t = 2$, $y = 2 \times 2^4 - 2 \times 2^3 + 5 \times 2 + 4 = 30$

14 This rule can't be used when $n=-1$ because when $n=-1$, the rule gives the primitive of x^{-1} as $\frac{x^0}{0}$, which is undefined.

15 $y''=6x+4$ and when $x=1, y'=2$ and $y=4$

$y'=3x^2+4x+C$, for some constant C

When $x=1, y'=2$, so we obtain:

$2=3+4+C$

$C=-5$

Hence $y'=3x^2+4x-5$.

$y=x^3+2x^2-5x+D$, for some constant D

When $x=1, y=4$, so we obtain:

$4=1+2-5+D$

$D=6$

Hence $y=x^3+2x^2-5x+6$.

16a $f''(x)=2x-10$, where $f'(3)=20$ and $f(3)=-34$

$f'(x)=x^2-10x+C$, for some constant C

$f'(3)=20$ and so we obtain:

$20=9-30+C$

$C=41$

So $f'(x)=x^2-10x+41$.

16b $f'(x)=x^2-10x+41$ and $f(3)=-34$

$f(x)=\frac{1}{3}x^3-5x^2+41x+D$, for some constant D

$f(3)=-34$ and so we obtain:

$-34 = 9 - 45 + 123 + D$

$D = -121$

So $f(x) = \frac{1}{3}x^3 - 5x^2 + 41x - 121$.

$f(0) = -121$ and so the graph cuts the y-axis at $(0, -121)$

17 $y'' = 8 - 6x$ and the curve passes through the points $(1, 6)$ and $(-1, 8)$.

$y' = 8x - 3x^2 + C$, for some constant C

$y = 4x^2 - x^3 + Cx + D$, for some constant D

$(1, 6)$ lies on the curve and so when $x = 1, y = 6$

$6 = 4 - 1 + C + D$

$C + D = 3 \qquad (1)$

$(-1, 8)$ lies on the curve and so when $x = -1, y = 8$

$8 = 4 + 1 - C + D$

$-C + D = 3 \qquad (2)$

(1) + (2) gives $2D = 6$ and so $D = 3$

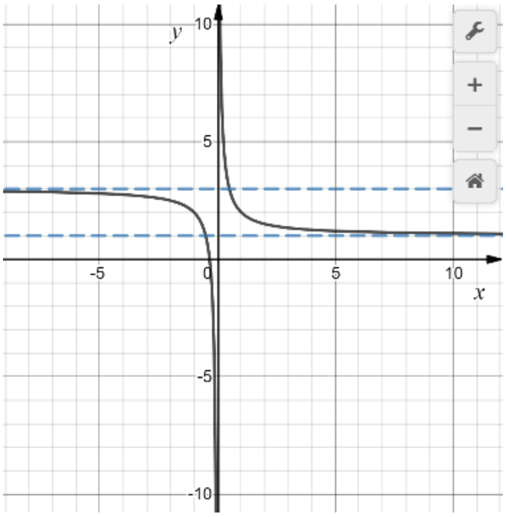
Substituting $D = 3$ into (2) we obtain $C = 0$.

So $y = -x^3 + 4x^2 + 3$.

18 $\int -\frac{1}{x^2}\,dx = \int -x^{-2}\,dx = x^{-1} + c = \frac{1}{x} + c = f(x)$

If $f(1) = 2$ then $\frac{1}{1} + c = 2$ and $c = 1$ when $x > 0$

If $f(-1) = 2$ then $\frac{1}{-1} + c = 2$ and $c = 3$ when $x < 0$



19a $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ then

$P'(x) = na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + a_1$

$P''(x) = (n-1)na_n x^{n-2} + (n-2)(n-1)a_{n-1}x^{n-3} + \cdots + 2a_2$

The nth degree of the derivative has the term:

$\big(n-(n-1)\big) \ldots (n-1)na_n x^{n-n}$

Hence, the n^{th} degree of the derivative is a constant which is not zero (does not vanish). Therefore, if we find the $(n+1)^{th}$ derivative of $P(x)$, it will be equal to zero (the polynomial vanishes).

19b $P(x) = a_k x^k + a_{k-1}x^{k-1} + \cdots + a_1 x + a_0$

If we find the k^{th} derivative we will see that it is a constant and if we continue finding the $(k+1)^{th}$ derivative, we will observe that it is equal to zero like in 19a. Therefore, for any given polynomial, if $(n+1)^{th}$ term vanishes but not the n^{th} derivative, then we can say that the degree of the polynomial is n.

Solutions to Chapter review

1 In the diagram, assume that B, D and G are stationary points, C and F are inflection points and E is both a stationary point and an inflection point.

1a $f'(x) > 0$ at points C and H because the slope of the tangent line is positive at these points (the function $f(x)$ is increasing at these points).

1b $f'(x) < 0$ at points A and F because the slope of the tangent line is negative at these points (the function $f(x)$ is decreasing at these points).

1c $f'(x) = 0$ at points B, D, E and G because the slope of the tangent line is zero at these points.

1d $f''(x) > 0$ at points B and G because the curve is concave up.

1e $f''(x) < 0$ at point D because the curve is concave down.

1f $f''(x) = 0$ at points C, E and F.

2a $f(x) = x^3 - x^2 - x - 7$

$f'(x) = 3x^2 - 2x - 1$

2b i $f'(0) = 3 \times 0^2 - 2 \times 0 - 1 = -1$. Since $f'(0) < 0$ when $x = 0$, $f(x)$ is decreasing at $x = 0$.

2b ii $f'(1) = 3 \times 1^2 - 2 \times 1 - 1 = 0$. Since $f'(0) = 0$ when $x = 1$, $f(x)$ is stationary at $x = 1$.

2b iii $f'(-1) = 3 \times (-1)^2 - 2 \times (-1) - 1 = 4$. Since $f'(-1) > 0$ when $x = -1$, $f(x)$ is increasing at $x = -1$.

2b iv $f'(3) = 3 \times 3^2 - 2 \times 3 - 1 = 20$. Since $f'(3) > 0$ when $x = 3$, $f(x)$ is increasing at $x = 3$.

3a $f(x) = x^2 - 4x + 3$

$f'(x) = 2x - 4$

3b i $f'(x) > 0$ when $2x - 4 > 0$ or $x > 2$. Hence, $f(x)$ is increasing when $x > 2$.

3b ii $f'(x) < 0$ when $2x - 4 < 0$ or $x < 2$. Hence, $f(x)$ is decreasing when $x < 2$.

3b iii $f'(x) = 0$ when $2x - 4 = 0$ or $x = 2$. Hence, $f(x)$ is stationary when $x = 2$.

4a $f(x) = x^3$

$f'(x) = 3x^2$

$f'(1) = 3 \times 1^2 = 3$

Since $f'(x) > 0$ when $x = 1$, $f(x)$ is increasing at $x = 1$.

4b $f(x) = (x + 2)(x - 3) = x^2 - x - 6$

$f'(x) = 2x - 1$

$f'(1) = 2 \times 1 - 1 = 1$

Since $f'(x) > 0$ when $x = 1$, $f(x)$ is increasing at $x = 1$.

4c $f(x) = (x - 1)^5$

$f'(x) = 5(x - 1)^4$

$f'(1) = 5(1 - 1)^4 = 0$

Since $f'(x) = 0$ when $x = 1$, $f(x)$ is stationary at $x = 1$.

4d $f(x) = \frac{x+1}{x-3}$

Using the quotient rule:

$$f'(x) = \frac{(x-3)(1) - (x+1)(1)}{(x-3)^2}$$

$$= \frac{x - 3 - x - 1}{(x-3)^2}$$

$$= \frac{-4}{(x-3)^2}$$

$f'(1) = \frac{-4}{(1-3)^2} = -1$

Since $f'(x) < 0$ when $x = 1$, $f(x)$ is decreasing at $x = 1$.

5a $y = x^7$

$y' = 7x^6$

$y'' = 42x^5$

5b $y = x^3 - 4x^2$

$y' = 3x^2 - 8x$

$y'' = 6x - 8$

5c $y = (x-2)^5$

$y' = 5(x-2)^4$

$y'' = 20(x-2)^3$

5d $y = \frac{1}{x} = x^{-1}$

$y' = -x^{-2} = -\frac{1}{x^2}$

$y'' = 2x^{-3} = \frac{2}{x^3}$

6a $f(x) = x^3 - 2x^2 + 4x - 5$

$f'(x) = 3x^2 - 4x + 4$

$f''(x) = 6x - 4$

$f''(1) = 6 \times 1 - 4 = 2$

Since $f''(1) > 0$, the curve is concave up at $x = 1$.

6b $f(x) = 6 - 2x^3 - x^4$

$f'(x) = -6x^2 - 4x^3$

$f''(x) = -12x - 12x^2$

$f''(1) = -12 \times 1 - 12 \times 1^2 = -24$

Since $f''(1) < 0$, the curve is concave down at $x = 1$.

7a $f(x) = 2x^3 - 3x^2 + 6x - 1$

$f'(x) = 6x^2 - 6x + 6$

$f''(x) = 12x - 6$

7b i $f''(x) = 0$ when $12x - 6 = 0$ or $x = \frac{1}{2}$

x	0	$\frac{1}{2}$	1
$f''(x)$	$-$	0	$+$

$f''(x) > 0$ when $x > \frac{1}{2}$. Therefore, $f(x)$ is concave up when $x > \frac{1}{2}$.

7b ii $f''(x) < 0$ when $x < \frac{1}{2}$. Therefore, $f(x)$ is concave down when $x < \frac{1}{2}$.

8a $y = x^3 - 6x^2 + 9x - 11$

$y' = 3x^2 - 12x + 9 = 3(x - 1)(x - 3)$

$y'' = 6x - 12 = 6(x - 2)$

$y' = 0$ when $x = 1$ and 3

$y'' = 0$ when $x = 2$

x	0	1	2	3	4
y'	$+$	0	$-$	0	$+$
y''	$-$	$-$	0	$+$	$+$

From the table, y is increasing when $x < 1$ and $x > 3$.

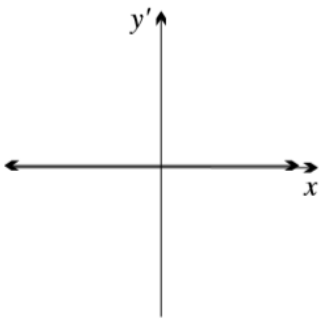
8b From the table, y is decreasing when $1 < x < 3$.

8c From the table, y is concave up when $x > 2$.

8d From the table, y is concave down when $x < 2$

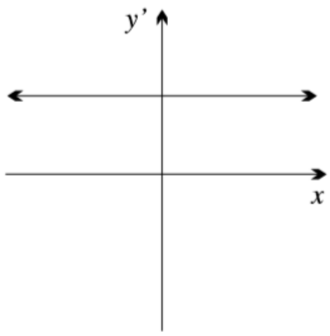
9a $f(x)$ is constant for all $x \in \mathbb{R}$

Therefore, $f'(x) = 0$ for all $x \in \mathbb{R}$ and the graph of $f'(x)$ is:



9b $f(x)$ is increasing at a constant rate for all $x \in \mathbb{R}$

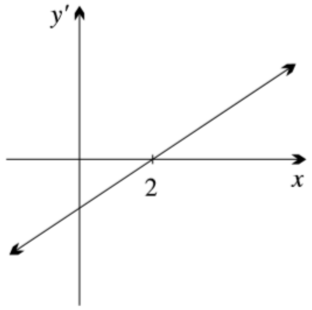
Therefore, the graph of $f'(x)$ should look like this:



9c $f(x)$ is decreasing when $x < 2$ and increasing when $x > 2$

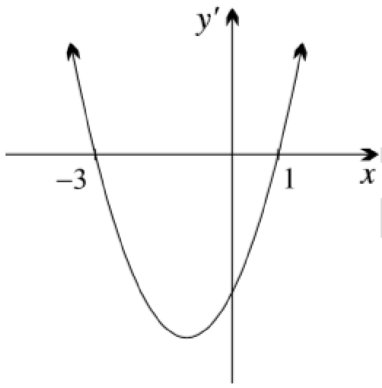
Therefore, $f'(x) < 0$ when $x < 2$ and $f'(x) > 0$ when $x > 2$.

Hence, the graph of $f'(x)$ should look like this:



9d $f(x)$ is increasing when $x < -3$, decreasing when $-3 < x < 1$ and increasing when $x > 1$. Therefore, $f'(x) > 0$ when $x < -3$, $f'(x) < 0$ when $-3 < x < 1$ and $f'(x) > 0$ when $x > 1$.

Hence, the graph of $f'(x)$ should look like this:



10a $y = x^3 - 6x^2 + 9x - 11$

$y' = 3x^2 + 2x - 1 = (x + 1)(3x - 1)$

$(x + 1)(3x - 1) = 0$ when $x = -1$ or $x = \frac{1}{3}$

Therefore, there are stationary points at $x = -1$ and $x = \frac{1}{3}$.

When $x = -1, y = 3$ and when $= \frac{1}{3}, y = \frac{49}{27}$.

So the stationary points are $(-1, 3)$ and $\left(\frac{1}{3}, \frac{49}{27}\right)$; that is, $P(-1, 3)$ and $Q\left(\frac{1}{3}, \frac{49}{27}\right)$.

10b $y'' = 6x + 2 = 2(3x + 1)$

$y'' > 0$ when $2(3x + 1) > 0$

$3x + 1 > 0$

$x > -\frac{1}{3}$

y is concave up when $x > -\frac{1}{3}$

10c When the functions $y_1 = x^3 + x^2 - x + 2$ and $y_2 = k$ are graphed on the same coordinate plane, they have three intersection points when $\frac{49}{27} < k < 3$.

Therefore, $x^3 + x^2 - x + 2 = k$ has three distinct solutions when $\frac{49}{27} < k < 3$.

11a $y = x^2 - 6x - 7$

$y = (x - 7)(x + 1)$ and $y = 0$ for $x = -1$ or $x = 7$

Hence, $(-1, 0)$ and $(7, 0)$ are x-intercepts.

When $x = 0$, $y = -7$ so $(0, -7)$ is the y-intercept.

$y' = 2x - 6$ and $y' = 0$ when $2x - 6 = 0$ or $x = 3$.

Hence, there is a stationary point at $x = 3$.

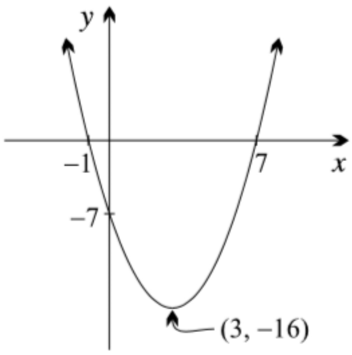
x	0	3	4
y'	\	0	/

When $x = 3, y = 3^2 - 6 \times 3 - 7 = -16$.

Therefore, $(3, -16)$ is a minimum turning point.

$y'' = 2$. Hence, $y'' > 0$ for all $x \in \mathbb{R}$.

Therefore, $y = x^2 - 6x - 7$ does not have an inflection point.



11b $y = x^3 - 6x^2 + 8$

When $x = 0$, $y = 8$ so $(0, 8)$ is the y-intercept.

$y' = 3x^2 - 12x$ and $y' = 0$ when $3x(x - 4) = 0$ or when $x = 0$ or $x = 4$.

Hence, there is a stationary point at $x = 0$ and $x = 4$.

x	-1	0	1	4	5
y'	/	0	\	0	/

When $x = 0$, $y = 0^3 - 6 \times 0^2 + 8 = 8$.

Therefore, $(0, 8)$ is a maximum turning point.

When $x = 4$, $y = 4^3 - 6 \times 4^2 + 8 = -24$.

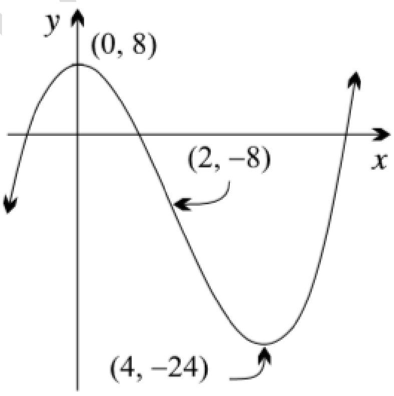
Therefore, $(4, -24)$ is a minimum turning point.

$y'' = 6x - 12$ and $y'' = 0$ when $6x - 12 = 0$ or $x = 2$.

Hence there is an inflection point at $x = 2$.

When $x = 2$, $y = 2^3 - 6 \times 2^2 + 8 = -8$.

Therefore, $(2, -8)$ is an inflection point.



11c $y = 2x^3 - 3x^2 - 12x + 1$

When $x = 0$, $y = 1$ so $(0, 1)$ is the y-intercept.

$y' = 6x^2 - 6x - 12$ and $y' = 0$ when $6(x-2)(x+1) = 0$ or when $x = -1$ or $x = 2$.

Hence, there is a stationary point at $x = -1$ and $x = 2$.

x	-2	-1	0	2	3
y'	/	0	\	0	/

When $x = -1$, $y = 2 \times (-1)^3 - 3 \times (-1)^2 - 12 \times (-1) + 1 = 8$.

Therefore, $(-1, 8)$ is a maximum turning point.

When $x = 2$, $y = 2 \times 2^3 - 3 \times 2^2 - 12 \times 2 + 1 = -19$.

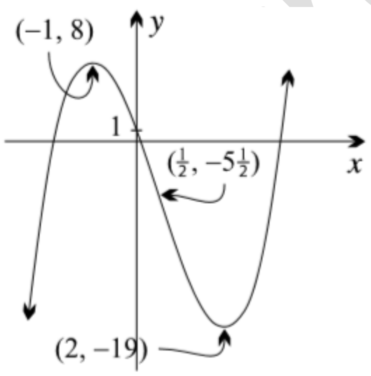
Therefore, $(2, -19)$ is a minimum turning point.

$y'' = 12x - 6$ and $y'' = 0$ when $12x - 6 = 0$ or $x = \frac{1}{2}$.

Hence there is an inflection point at $x = \frac{1}{2}$.

When $x = \frac{1}{2}$, $y = 2 \times \left(\frac{1}{2}\right)^3 - 3 \times \left(\frac{1}{2}\right)^2 - 12 \times \left(\frac{1}{2}\right) + 1 = -5\frac{1}{2}$.

Therefore, $\left(\frac{1}{2}, -5\frac{1}{2}\right)$ is an inflection point.



12a $y = x^3 - 3x^2 - 9x + 11$

$y' = 3x^2 - 6x - 9 = 3(x-3)(x+1)$

$3(x-3)(x+1) = 0$ when $x = -1$ or $x = 3$. Therefore, there are stationary

points at $x = -1$ or $x = 3$.

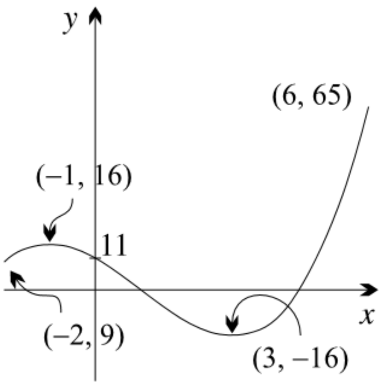
When $x = -1,\ y = 16$ and when $x = 3, y = -16$.

Thus, the stationary points are $(-1, 16)$ and $(3, -16)$.

x		-1		3	
y'	$+$	0	$-$	0	$+$

Therefore, $(-1, 16)$ is a maximum turning point and $(3, -16)$ is a minimum turning point.

The y-intercept is $(0, 11)$.



12b Looking at the boundary points:

When $x = 6,\ y = 65$ so 65 is the global maximum.

When $x = -2, y = 9$, which is larger than $f(3) = -16$ so -16 is the global minimum.

13a $y = x^2 - ax + 9$

$y' = 2x - a$

$y' = 0$ when $2x - a = 0$ or $x = \frac{a}{2}$

If the tangent to $y = x^2 - ax + 9$ is horizontal at $x = -1$, then $x = \frac{a}{2} = -1$.

Therefore, $a = -2$.

13b $y = ax^2 + bx + 3$

Since the point $(-1, 0)$ is on the graph of y,

$a \times (-1)^2 + b \times (-1) + 3 = 0$

$a - b + 3 = 0$

$a = b - 3 \qquad (1)$

$y' = 2ax + b$

$y' = 0$ when $2ax + b = 0$ or $x = -\frac{b}{2a}$.

Hence, there is a stationary point at $x = -\frac{b}{2a}$.

If $(-1, 0)$ is a turning point, then $-1 = -\frac{b}{2a}$.

$2a - b = 0 \qquad (2)$

Substituting (1) into (2) gives:

$2(b - 3) - b = 0$

$2b - 6 - b = 0$

$b = 6$

and $a = 6 - 3 = 3$

Therefore, $a = 3$ and $b = 6$.

14a $y = x^4 - 4x^3 + 7$

$y' = 4x^3 - 12x^2$

$y'' = 12x^2 - 24x = 12x(x - 2)$

$12x(x - 2) = 0$ when $x = 0$ or $x = 2$

Therefore, y has inflection points at $x = 0$ and $x = 2$.

When $x = 2, y = -9$, so $(2, -9)$ is a point of inflection.

14b When $x = 2$, $y' = 4 \times 2^3 - 12 \times 2^2 = 32 - 48 = -16$.

Therefore, the slope at $(2, -9)$ is -16.

14c Using $y - y_1 = m(x - x_1)$ when $m = -16$ and $(x_1, y_1) = (2, -9)$ gives:

$y - (-9) = -16(x - 2)$

$y + 9 = -16x + 32$

$16x + y - 23 = 0$

Therefore, $16x + y - 23 = 0$ is the equation of the tangent at $(2, -9)$.

15a $S = 175 + 18t^2 - t^4$ for $0 \le t \le 5$

When $t = 0$, $S = 175 + 18 \times 0^2 - 0^4 = 175$

Therefore, the initial number of students that are logged on is 175.

15b When $t = 5$, $S = 175 + 18 \times 5^2 - 5^4 = 0$

Therefore, the number of students that are logged on at the end of the five hours is 0.

15c $S' = 36t - 4t^3$

$= 4t(9 - t^2)$

$= 4t(3 - t)(3 + t)$

$S' = 0$ when $4t(3 - t)(3 + t) = 0$ or when $t = -3$ or $t = 3$.

t	0	3	4
S'	/	0	\

(The values for $t < 0$ are not included in the table as they are not in the domain of S.)

There is a local maximum at $t = 3$.

When $t = 3$, $S = 175 + 18 \times 3^2 - 3^4 = 256$

Therefore, the maximum number of students logged onto the website is 256.

16a Volume = length × width × height

where length is $(16 - 2x)$ cm, width is $(6 - 2x)$ cm and height is x cm.

$V = (16 - 2x) \times (6 - 2x) \times x$

$= x(4x^2 - 44x + 96)$

$= 4x^3 - 44x^2 + 96x$

16b $\frac{dV}{dx} = 12x^2 - 88x + 96$

$= 4(3x^2 - 22x + 24)$

$= 4(3x - 4)(x - 6)$

$4(3x - 4)(x - 6) = 0$ when $x = \frac{4}{3}$ or $x = 6$

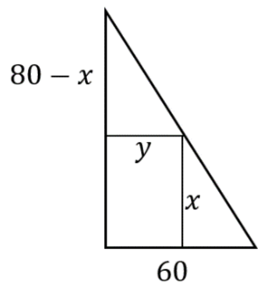
x		$\frac{4}{3}$		6	
V'	+	0	−	0	+

Thus, there is a local maximum at $x = \frac{4}{3}$.

When $x = \frac{4}{3}$, $V = 4 \times \left(\frac{4}{3}\right)^3 - 44 \times \left(\frac{4}{3}\right)^2 + 96 \times \left(\frac{4}{3}\right) = \frac{1600}{27}$

Therefore, the maximum volume is $\frac{1600}{27}$ cm^3.

17a



$\frac{y}{60} = \frac{80-x}{80}$ then $y = \frac{60}{80}(80 - x) = \frac{3}{4}(80 - x)$

17b $A = x \times y = x \times \frac{3}{4}(80 - x) = \frac{3}{4}(80x - x^2)$

$\frac{dA}{dx} = \frac{3}{4}(80 - 2x) = 0$ when $x = 40$. Therefore, the area of the rectangle is the maximum when $x = 40$ cm and $y = \frac{3}{4}(80 - x) = 30$ cm.

18a $V = \frac{1}{3}\pi r^2 h$ and $r + h = 12$

Then the height is $h = 12 - r$ and the volume is:

$V = \frac{1}{3}\pi r^2(12 - r\,)$

$V = 4\pi r^2 - \frac{1}{3}\pi r^3$

18b $V = 4\pi r^2 - \frac{1}{3}\pi r^3$

$V' = 8\pi r - \pi r^2$

$V' = 0$ when $8\pi r - \pi r^2 = 0$ or $\pi r(8 - r) = 0$

Hence, $V' = 0$ when $r = 0$ or $r = 8$.

x	0	1	8	9
y'	0	/	0	\

(The values of $r < 0$ are not included in the table as r cannot be negative)

There is a local maximum at $r = 8$.

Therefore, the radius that yields the maximum volume is 8 m.

19a Let $\frac{dy}{dx} = x^7$

$y = \frac{x^8}{8} + C$, for some constant C

19b Let $\frac{dy}{dx} = 2x$

$y = 2 \times \frac{x^2}{2} + C$, for some constant C

$= x^2 + C$

19c Let $\frac{dy}{dx} = 4$

$y = 4x + C$, for some constant C

19d Let $\frac{dy}{dx} = 10x^4$

$y = 10 \times \frac{x^5}{5} + C$, for some constant C

$= 2x^5 + C$

19e Let $\frac{dy}{dx} = 8x + 3x^2 - 4x^3$

$y = 8 \times \frac{x^2}{2} + 3 \times \frac{x^3}{3} - 4 \times \frac{x^4}{4} + C$, for some constant C

$= 4x^2 + x^3 - x^4 + C$

20a Let $\frac{dy}{dx} = 3x(x-2) = 3x^2 - 6x$

$y = 3 \times \frac{x^3}{3} - 6 \times \frac{x^2}{2} + C$, for some constant C

$= x^3 - 3x^2 + C$

20b $\frac{dy}{dx} = (x+1)(x-5) = x^2 - 4x - 5$

$y = \frac{x^3}{3} - 4 \times \frac{x^2}{2} - 5x + C$, for some constant C

$= \frac{x^3}{3} - 2x^2 - 5x + C$

20c $\frac{dy}{dx} = (2x-3)^2 = 4x^2 - 12x + 9$

$y = \frac{4}{3}x^3 - 6x^2 + 9x + C$, for some constant C

21a $\frac{dy}{dx} = (x+1)^5$

$y = \frac{1}{6}(x+1)^6 + C$, for some constant C

21b $\frac{dy}{dx} = (x-4)^7$

$y = \frac{1}{8}(x-4)^8 + C$, for some constant C

21c $\frac{dy}{dx} = (2x-1)^3$

$y = \frac{1}{2\times4}(2x-1)^4 + C$, for some constant C

$= \frac{1}{8}(2x-1)^4 + C$

22a $\frac{dy}{dx} = \frac{1}{x^2} = x^{-2}$

$y = -x^{-1} + C$, for some constant C

$= -\frac{1}{x} + C$

22b $\frac{dy}{dx} = \sqrt{x} = x^{\frac{1}{2}}$

$y = \frac{2}{3}x^{\frac{3}{2}} + C$, for some constant C

23 $f'(x) = 3x^2 - 4x + 1$ then $f(x) = x^3 - 2x^2 + x + C$

If $(2, 5)$ is on the graph of $f(x)$ then:

$f(2) = 2^3 - 2\times 2^2 + 2 + C = 5$

$8 - 8 + 2 + C = 5$

$C = 3$

So $f(x) = x^3 - 2x^2 + x + 3$.

24 $f'(x) = 4x - 3$

$f(x) = 2x^2 - 3x + C$, for some constant C.

If $f(2) = 7$, $f(2) = 2\times 2^2 - 3\times 2 + C = 7$

$8 - 6 + C = 7$

$C = 5$

Hence, $f(x) = 2x^2 - 3x + 5$

and $f(4) = 2 \times 4^2 - 3 \times 4 + 5 = 25$

25 $f(x) = \frac{1}{x^2-x-2}$

$f(-x) = \frac{1}{(-x)^2-(-x)-2} = \frac{1}{x^2+x-2}$

$-f(x) = \frac{-1}{x^2-x-2} = \frac{1}{-x^2+x+2}$

Since $f(x) \neq f(-x)$, $f(x)$ is not an even function.

Since $f(-x) \neq -f(x)$, $f(x)$ is not an odd function.

Therefore, $f(x)$ is neither an even nor an odd function.

25b $f'(x) = \frac{0\times(x^2-x-2)-1\times(2x-1)}{(x^2-x-2)^2} = -\frac{2x-1}{(x^2-x-2)^2} = \frac{1-2x}{(x^2-x-2)^2}$

25c $f'(x) = 0$ when $\frac{1-2x}{(x^2-x-2)^2} = 0$, $1 - 2x = 0$ or $x = \frac{1}{2}$

$f\left(\frac{1}{2}\right) = \frac{1}{\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)-2} = -\frac{4}{9}$

Therefore, $f(x)$ has a stationary point at $\left(\frac{1}{2}, -\frac{4}{9}\right)$

$f''(x) = \frac{6(x^2-x+1)}{(x^2-x-2)^3}$ and the sign table of $f''(x)$ is:

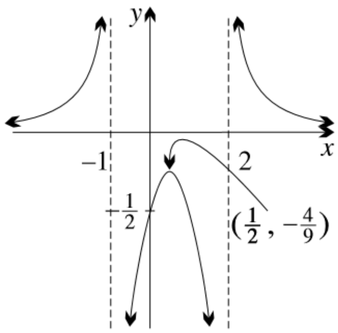
x		-2		1	
$f''(x)$	$+$	undefined	$-$	undefined	$+$

Therefore, $\left(\frac{1}{2}, -\frac{4}{9}\right)$ is a maximum turning point.

25d $x^2 - x - 2 = (x-2)(x+1) = 0$ then $f(x)$ has vertical asymptotes at $x = 2$ and $x = -1$.

25e $\lim_{x\to\pm\infty} \frac{1}{x^2-x-2} = 0$. Therefore, $f(x)$ has a vertical asymptote at $y = 0$.

25f



26a $y = \frac{x^2-1}{x^2-4} = 0$ when $x^2 - 1 = 0$, $x = -1$ or $x = 1$. Therefore, y cuts the x-axis at $(-1, 0)$ and $(1, 0)$.

$y = \frac{(0)^2-1}{(0)^2-4} = \frac{1}{4}$ when $x = 0$. Therefore, y cuts the y-axis at $\left(0, \frac{1}{4}\right)$

26b $x^2 - 4 = 0$ when $x = -2$ or $x = 2$.

Hence, y is undefined when $x = -2$ or $x = 2$. Therefore, the graph of y has vertical asymptotes at $x = -2$ and $x = 2$.

26c $y = \frac{x^2-1}{x^2-4}$ then $y' = \frac{(2x)(x^2-4)-(x^2-1)(2x)}{(x^2-4)^2} = -\frac{6x}{(x^2-4)^2}$

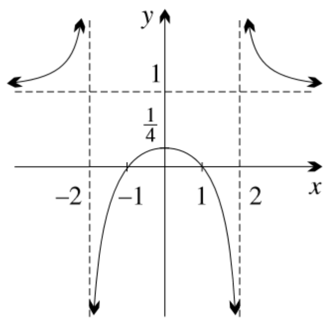
26d $y' = -\frac{6x}{(x^2-4)^2} = 0$ when $x = 0$. Therefore there is a stationary point at $x = 0$.

x		0	
y''	$+$	0	$-$

$y = \frac{(0)^2-1}{(0)^2-4} = \frac{1}{4}$ when $x = 0$. Therefore, the stationary point $\left(0, \frac{1}{4}\right)$ is a maximum turning point.

26e $f(x) = f(-x) = \frac{(-x)^2-1}{(-x)^2-4} = \frac{x^2-1}{x^2-4}$. Therefore, $f(x)$ is an even function.

26f $\lim_{x\to\pm\infty} \frac{x^2-1}{x^2-4} = 1$. Therefore, y has a horizontal asymptote at $y = 1$.



27a $S = 3x \times 4 + x \times 4 + h \times 4 = 16x + 4h$

27b $V = B \times h = 3x \times x \times h = 3x^2h$. Hence, $4374 = 3x^2h$ and $h = \frac{4374}{3x^2} = \frac{1458}{x^2}$

Therefore, $S = 16x + 4h = 16x + 4 \times \frac{1458}{x^2} = 16x + \frac{5832}{x^2}$

27c $S' = 16 - \frac{11664}{x^3} = \frac{(16x^3-11664)}{x^3} = 0$ when $x^3 = 729$ or $x = 9$.

x		0		9	
S'	$+$	undefined	$-$	0	$+$

Hence there is a minimum turning point at $x = 9$ and

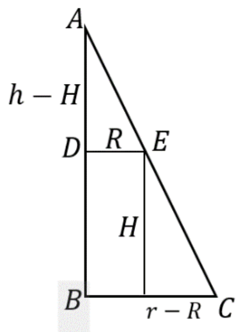
The minimum amount is $S = 16 \times 9 + \frac{5832}{9^2} = 216$ m and the dimensions are

Width: 9 m

Length: 27 m

Height: $216 = 16 \times 9 + 4h$ (When S is minimum) then, $h = 18$ m

28a $\Delta ABC \sim \Delta ADE$. Therefore, $\frac{r-R}{r} = \frac{H}{h}$ and $H = \frac{h(r-R)}{r}$



28b $V = \pi R^2 \times H = \pi R^2 \times \frac{h(r-R)}{r} = \frac{\pi R^2 \times h(r-R)}{r}$

28c

$$\frac{dV}{dR} = \frac{d\left(\frac{\pi R^2 \times h(r-R)}{r}\right)}{dR}$$

$$= \frac{d\left(\pi h R^2 - \frac{1}{r}\pi h R^3\right)}{dR}$$

$$= 2\pi h R - \frac{3\pi}{r} h R^2$$

$$= \pi R h \left(2 - \frac{3R}{r}\right)$$

$\frac{dV}{dR} = 0$, when $R = 0$, $2 - \frac{3R}{r} = 0$ or $R = \frac{2}{3}r$

Therefore, when $R = \frac{2}{3}r$, the maximum volume is:

$$V = \frac{\pi\left(\frac{2}{3}r\right)^2 \times h\left(r - \frac{2}{3}r\right)}{r}$$

$$= \frac{\pi h \frac{4r^2}{9} \times \left(\frac{r}{3}\right)}{r}$$

$$= \frac{4}{27}\pi r^2 h$$

29a $P = 12 = 2a + b$ (where b is the base and a is the length of the equal sides)

$b = 12 - 2a$ and $h^2 = a^2 - \left(\frac{b}{2}\right)^2$.

Then, $h = \sqrt{\frac{4a^2 - b^2}{4}} = \sqrt{\frac{4a^2 - (12-2a)^2}{4}} = \sqrt{\frac{4a^2 - (144 - 48a + 4a^2)}{4}}$

$= \sqrt{\frac{48a - 144}{4}} = \sqrt{12a - 36} = 2\sqrt{3a - 9}$

Hence, $A = \frac{1}{2} \times b \times h = \frac{1}{2} \times (12 - 2a) \times 2\sqrt{3a - 9} = (12 - 2a) \times \sqrt{3a - 9}$

Or $A = 2\sqrt{3}(6 - a)\sqrt{a - 3}$

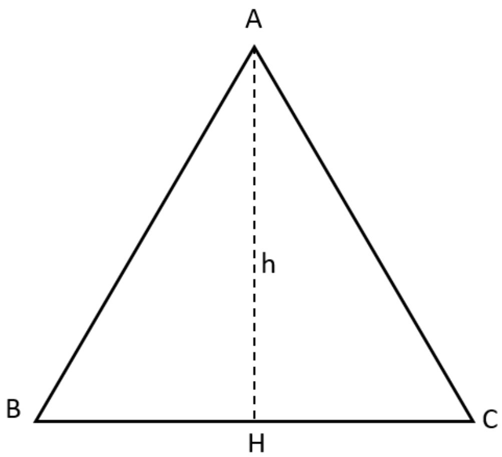
$A' = -2\sqrt{3(a - 3)} - \frac{(a-6)\sqrt{3}}{\sqrt{a-3}}$ and $A' = 0$ when $a = 4$.

Therefore, the maximum area is $A = 2\sqrt{3}\big(6 - (4)\big)\sqrt{(4) - 3} = 4\sqrt{3}$ cm²

29b Assume that the figure below is an isosceles triangle

where $AB = AC = a$ and $BC = b$, the perimeter (P) of ΔABC is $P = 2a + b$, and

the height is h.



Then the area (A) of the triangle is

$A = \frac{\text{base} \times \text{height}}{2} = \frac{1}{2}(P - 2a)\sqrt{a^2 - \left(\frac{P-2a}{2}\right)^2}$ and

$\frac{dA}{da} = -1 \times \sqrt{a^2 - \left(\frac{P-2a}{2}\right)^2} + \frac{1}{2}(P - 2a)\frac{p}{2\sqrt{a^2 - \left(\frac{P-2a}{2}\right)^2}} = 0$ when $a = \frac{P}{3}$

Since $P = 2a + b$ and $a = \frac{P}{3}$ when the area of the triangle is maximised,

$b = P - 2a = P - \frac{2P}{3} = \frac{P}{3}$ and $a = b = \frac{P}{3}$

Therefore, the triangle is an equilateral triangle.