Solutions to Exercise 1A Foundation questions

1a i^2
$= i^{4(0)+2}$
$= -1$

1b i^4
$= i^{4(1)}$
$= 1$

1c i^7
$= i^{4(1)+3}$
$= -i$

1d i^{13}
$= i^{4(3)+1}$
$= i$

1e i^{29}
$= i^{4(7)+1}$
$= i$

1f i^{2010}
$= i^{4(502)+2}$
$= -1$

1g $i^3 + i^4 + i^5$

$= i^{4(0)+3} + i^{4(1)} + i^{4(1)+1}$

$= -i + 1 + i$
$= 1$

1h $i^7 + i^{16} + i^{21} + i^{22}$

$i^{4(1)+3} + i^{4(4)} + i^{4(5)+1} + i^{4(5)+2}$

$= -i + 1 + i - 1$
$= 0$

2a $\overline{2i}$
$= \overline{2} \times \overline{i}$
$= 2(-i)$
$= -2i$

2b $\overline{3 + i}$
$= \overline{3} + \overline{i}$
$= 3 - i$

2c $\overline{1 - i}$
$= \overline{1} - \overline{i}$
$= 1 - (-i)$
$= 1 + i$

2d $\overline{5 - 3i}$
$= \overline{5} + \overline{-3i}$
$= \overline{5} + \overline{-3} \times \overline{i}$
$= 5 - 3\overline{i}$
$= 5 - 3(-i)$
$= 5 + 3i$

2e $\overline{-3 + 2i}$
$= \overline{-3} + \overline{2i}$
$= \overline{-3} + \overline{2} \times \overline{i}$
$= -3 + 2(-i)$
$= -3 - 2i$

3a $(7+3i)+(5-5i)$
$= 7+3i+5-5i$
$= (7+5)+(3-5)i$
$= 12-2i$

3b $(-8+6i)+(2-4i)$
$= -8+6i+2-4i$
$= (-8+2)+(6-4)i$
$= -6+2i$

3c $(4-2i)-(3-7i)$
$= 4-2i-3+7i$
$= (4-3)+(-2+7)i$
$= 1+5i$

3d $(3-5i)-(-4+6i)$
$= 3-5i+4-6i$
$= (3+4)+(-5-6)i$
$= 7-11i$

4a $(4+5i)i$
$= 4i+5i^2$
$= 4i-5$
$= -5+4i$

4b $(1+2i)(3-i)$
$= 3-i+6i-2i^2$
$= 3-i+6i+2$
$= 5+5i$

4c $(3+2i)(4-i)$
$= 12-3i+8i-2i^2$
$= 12-3i+8i+2$
$= 14+5i$

4d $(-7+5i)(8-6i)$
$= -56 + 42i + 40i - 30i^2$
$= -56 + 42i + 40i + 30$
$= -26 + 82i$

4e $(5+i)^2$
$= 25 + 10i + i^2$
$= 25 + 10i - 1$
$= 24 + 10i$

4f $(2-3i)^2$
$= 4 - 12i + 9i^2$
$= 4 - 12i - 9$
$= -5 - 12i$

4g $(2+i)^3$
$= 2^3 + 3 \times 2^2 i + 3 \times 2i^2 + i^3$
$= 8 + 12i - 6 - i$
$= 2 + 11i$

4h $(1-i)^4$
$= ((1-i)^2)^2$
$= (1 - 2i + i^2)^2$
$= (1 - 2i - 1)^2$
$= (-2i)^2$
$= 4i^2$
$= -4$

4i $(3-i)^4$
$= ((3-i)^2)^2$
$= (9 - 6i + i^2)^2$
$= (9 - 6i - 1)^2$
$= (8 - 6i)^2$
$= 64 - 96i + 36i^2$
$= 64 - 96i - 36$
$= 28 - 96i$

5a $(1+2i)(1-2i)$
$= 1^2 - (2i)^2$
$= 1 - 4i^2$
$= 1 + 4$
$= 5$

5b $(4+i)(4-i)$
$= 4^2 - i^2$
$= 16 + 1$
$= 17$

5c $(5+2i)(5-2i)$
$= 5^2 - (2i)^2$
$= 25 - 4i^2$
$= 25 + 4$
$= 29$

5d $(-4-7i)(-4+7i)$
$= (-4)^2 - (7i)^2$
$= 16 - 49i^2$
$= 16 + 49$
$= 65$

6a

$\frac{1}{i}$
$= \frac{i}{i \times i}$
$= \frac{i}{-1}$
$= -i$

6b

$$\frac{2+i}{i}$$
$$=\frac{(2+i)i}{i\times i}$$
$$=\frac{2i+i^2}{i^2}$$
$$=\frac{2i-1}{-1}$$
$$=-2i+1$$
$$=1-2i$$

6c

$$\frac{5-i}{1-i}$$
$$=\frac{(5-i)(1+i)}{(1-i)(1+i)}$$
$$=\frac{5+5i-i-i^2}{1-i^2}$$
$$=\frac{5+5i-i+1}{1+1}$$
$$=\frac{6+4i}{2}$$
$$=3+2i$$

6d

$$\frac{6-7i}{4+i}$$
$$=\frac{(6-7i)(4-i)}{(4+i)(4-i)}$$
$$=\frac{24-6i-28i+7i^2}{4^2-i^2}$$
$$=\frac{24-6i-28i-7}{16+1}$$
$$=\frac{17-34i}{17}$$
$$=1-2i$$

6e

$$\frac{-11+13i}{5+2i}$$
$$=\frac{(-11+13i)(5-2i)}{(5+2i)(5-2i)}$$
$$=\frac{-55+22i+65i-26i^2}{5^2-(2i)^2}$$
$$=\frac{-55+22i+65i-26i^2}{5^2-4i^2}$$
$$=\frac{-55+22i+65i+26}{25+4}$$
$$=\frac{-29+87i}{29}$$
$$=-1+3i$$

6f

$$\frac{(1+i)^2}{3-i}$$
$$=\frac{1+2i+i^2}{3-i}$$
$$=\frac{1+2i-1}{3-i}$$
$$=\frac{2i}{3-i}$$
$$=\frac{2i(3+i)}{(3-i)(3+i)}$$
$$=\frac{6i+2i^2}{9-i^2}$$
$$=\frac{6i-2}{9+1}$$
$$=\frac{6i-2}{10}$$
$$=\frac{3}{5}i-\frac{1}{5}$$
$$=-\frac{1}{5}+\frac{3}{5}i$$

7a $\overline{(iz)}$
$= \overline{\left(i(1+2i)\right)}$
$= \overline{i+2i^2}$
$= \overline{i-2}$
$= \overline{i} - \overline{2}$
$= -i-2$
$= -2-i$

7b $w + \overline{z}$
$= (3-i) + \overline{(1+2i)}$
$= 3 - i + (1-2i)$
$= 4-3i$

7c $2z + iw$
$= 2(1+2i) + i(3-i)$
$= 2 + 4i + 3i - i^2$
$= 2 + 4i + 3i + 1$
$= 3 + 7i$

7d $\mathrm{Im}(5i - z)$
$= \mathrm{Im}\left(5i - (1+2i)\right)$
$= \mathrm{Im}(5i - 1 - 2i)$
$= \mathrm{Im}(3i - 1)$
$= 3$

7e z^2
$= (1+2i)^2$
$= 1 + 4i + (2i)^2$
$= 1 + 4i + 4i^2$
$= 1 + 4i - 4$
$= -3 + 4i$

8a $\overline{z} - w$
$= \overline{8+i} - (2-3i)$
$= 8 - i - 2 + 3i$
$= 6 + 2i$

8b $\mathrm{Im}(3iz + 2w)$

$= \mathrm{Im}\big(3i(8 + i) + 2(2 - 3i)\big)$

$= \mathrm{Im}\big(24i + 3i^2 + 2(2 - 3i)\big)$

$= \mathrm{Im}(24i - 3 + 4 - 6i)$

$= \mathrm{Im}(18i + 1)$

$= 18$

8c zw

$= (8 + i)(2 - 3i)$

$= 16 - 24i + 2i - 3i^2$

$= 16 - 24i + 2i + 3$

$= 19 - 22i$

8d $65 \div z$

$= \dfrac{65}{z}$

$= \dfrac{65}{8 + i}$

$= \dfrac{65(8 - i)}{(8 + i)(8 - i)}$

$= \dfrac{65(8 - i)}{8^2 - i^2}$

$= \dfrac{65(8 - i)}{64 + 1}$

$= \dfrac{65(8 - i)}{65}$

$= 8 - i$

8e

$$\frac{z}{w}$$
$$= \frac{8+i}{2-3i}$$
$$= \frac{(8+i)(2+3i)}{(2-3i)(2+3i)}$$
$$= \frac{16+24i+2i+3i^2}{2^2-(3i)^2}$$
$$= \frac{16+24i+2i-3}{2^2-9i^2}$$
$$= \frac{16+24i+2i-3}{4+9}$$
$$= \frac{16+26i-3}{4+9}$$
$$= \frac{13+26i}{13}$$
$$= 1+2i$$

9a

$$-zw$$
$$= -(2-i)(-5-12i)$$
$$= (2-i)(5+12i)$$
$$= 10+24i-5i-12i^2$$
$$= 10+24i-5i+12$$
$$= 22+19i$$

9b

$$(1+i)\overline{z}-w$$
$$= (1+i)\overline{(2-i)}-(-5-12i)$$
$$= (1+i)(2+i)-(-5-12i)$$
$$= 2+2i+i+i^2+5+12i$$
$$= 2+2i+i-1+5+12i$$
$$= 6+15i$$

9c

$$\frac{10}{\overline{\overline{z}}}$$

$$= \frac{10}{\overline{\overline{2-i}}}$$

$$= \frac{10}{2+i}$$

$$= \frac{10(2-i)}{(2-i)(2+i)}$$

$$= \frac{20-10i}{4-i^2}$$

$$= \frac{20-10i}{4+1}$$

$$= \frac{20-10i}{5}$$

$$= 4-2i$$

9d

$$\frac{w}{2-3i}$$

$$= \frac{-5-12i}{2-3i}$$

$$= \frac{(-5-12i)(2+3i)}{(2-3i)(2+3i)}$$

$$= \frac{-10-15i-24i-36i^2}{2^2-(3i)^2}$$

$$= \frac{-10-15i-24i+36}{4+9}$$

$$= \frac{26-39i}{13}$$

$$= 2-3i$$

9e $\mathrm{Re}\big((1+4i)z\big)$

$$= \mathrm{Re}\big((1+4i)(2-i)\big)$$

$$= \mathrm{Re}(2-i+8i-4i^2)$$

$$= \mathrm{Re}(2-i+8i+4)$$

$$= \mathrm{Re}(6+7i)$$

$$= 6$$

Solutions to Exercise 1A Development questions

10a $(x + yi)(2 - 3i) = -13i$

$2x - 3xi + 2yi - 3yi^2 = -13i$

$2x - 3xi + 2yi - 3y(-1) = -13i$

$2x - 3xi + 2yi + 3y = -13i$

$(2x + 3y) + (-3x + 2y)i = -13i$

Equating real and imaginary parts,

$2x + 3y = 0$ (1)

$-3x + 2y = -13$ (2)

$3 \times (1) + 2 \times (2)$ gives:

$13y = -26$

$y = -2$

(1) becomes:

$2x - 6 = 0$

$x = 3$

Thus, $x = 3$ and $y = -2$.

10b $(1 + 4i)(x + yi) = 6 + 7i$

$x + yi + 4xi + 4yi^2 = 6 + 7i$

$x + yi + 4xi - 4y = 6 + 7i$

$(x - 4y) + (4x + y)i = 6 + 7i$

Equating real and imaginary parts,

$x - 4y = 6$ (1)

$4x + y = 7$ (2)

$4 \times (2) + (1)$ gives:

$17x = 34$

$x = 2$

(2) becomes:

$8 + y = 7$

$y = -1$

Thus, $x = 2$ and $y = -1$.

10c $(1 + i)x + (2 - 3i)y = 10$

$(x + 2y) + (x - 3y)i = 10$

Equating real and imaginary parts,

$x + 2y = 10 \qquad (1)$

$x - 3y = 0 \qquad (2)$

$(1) - (2)$ gives:

$5y = 10$

$y = 2$

(2) becomes:

$x - 6 = 0$

$x = 6$

Thus, $x = 6$ and $y = 2$.

10d $x(1 + 2i) + y(2 - i) = 4 + 5i$

$(x + 2y) + (2x - y)i = 4 + 5i$

Equating real and imaginary parts,

$x + 2y = 4 \qquad (1)$

$2x - y = 5 \qquad (2)$

$(1) + 2 \times (2)$ gives:

$5x = 14$

$x = \frac{14}{5}$

(2) becomes:

$\frac{28}{5} - y = 5$

$y = \frac{3}{5}$

Thus $x = \frac{14}{5}$ and $y = \frac{3}{5}$.

10e

$\frac{x}{2+i} + \frac{y}{2+3i} = 4 + i$

$\frac{x(2+3i) + y(2+i)}{(2+i)(2+3i)} = 4 + i$

$\frac{2x + 3xi + 2y + yi}{4 + 6i + 2i - 3} = 4 + i$

$\frac{2x + 3xi + 2y + yi}{1 + 8i} = 4 + i$

$2x + 3xi + 2y + yi = (4 + i)(1 + 8i)$

$2x + 3xi + 2y + yi = 4 + 32i + i - 8$

$(2x + 2y) + (3x + y)i = -4 + 33i$

Equating real and imaginary parts,

$2x + 2y = -4$ or

$x + y = -2 \quad (1)$

$3x + y = 33 \quad (2)$

$(2) - (1)$ gives:

$2x = 35$

$x = \frac{35}{2}$

(1) becomes:

$\frac{35}{2} + y = -2$

$y = -\frac{39}{2}$

Thus, $x = \frac{35}{2}$ and $y = -\frac{39}{2}$.

11a

$$\frac{1}{1+i}+\frac{2}{1+2i}$$

$$=\frac{1(1-i)}{(1+i)(1-i)}+\frac{2(1-2i)}{(1+2i)(1-2i)}$$

$$=\frac{1-i}{1^2-i^2}+\frac{2-4i}{1-4i^2}$$

$$=\frac{1-i}{1+1}+\frac{2-4i}{1+4}$$

$$=\frac{1}{2}-\frac{i}{2}+\frac{2}{5}-\frac{4}{5}i$$

$$=\left(\frac{1}{2}+\frac{2}{5}\right)+\left(-\frac{1}{2}-\frac{4}{5}\right)i$$

$$=\frac{5+4}{10}+\frac{-5-8}{10}i$$

$$=\frac{9}{10}-\frac{13}{10}i$$

11b

$$\frac{1+i\sqrt{3}}{2}+\frac{2}{1+i\sqrt{3}}$$

$$=\frac{1+i\sqrt{3}}{2}+\frac{2}{1+i\sqrt{3}}\times\frac{1-i\sqrt{3}}{1-i\sqrt{3}}$$

$$=\frac{1+i\sqrt{3}}{2}+\frac{2-2\sqrt{3}i}{1+3}$$

$$=\frac{1+i\sqrt{3}}{2}+\frac{2-2\sqrt{3}i}{4}$$

$$=\frac{1+i\sqrt{3}+1-i\sqrt{3}}{2}$$

$$=\frac{2}{2}$$

$$=1$$

11c

$$\frac{3+2i}{2-5i}+\frac{3-2i}{2+5i}$$

$$=\frac{(3+2i)(2+5i)+(3-2i)(2-5i)}{(2+5i)(2-5i)}$$

$$=\frac{6+15i+4i-10+6-15i-4i-10}{4+25}$$

$$=\frac{12-20+19i-19i}{29}$$

$$=-\frac{8}{29}$$

11d

$$\frac{-8+5i}{-2-4i}-\frac{3+8i}{1+2i}$$

$$=\frac{-8+5i}{-2(1+2i)}-\frac{3+8i}{1+2i}$$

$$=\frac{-8+5i}{-2(1+2i)}-\frac{-2(3+8i)}{-2(1+2i)}$$

$$=\frac{-8+5i+2(3+8i)}{-2(1+2i)}$$

$$=\frac{-8+5i+6+16i}{-2(1+2i)}$$

$$=\frac{-2+21i}{-2(1+2i)}$$

$$=\frac{-2+21i}{-2(1+2i)}\times\frac{1-2i}{1-2i}$$

$$=\frac{-2+4i+21i-42i^2}{-2(1-4i^2)}$$

$$=\frac{-2+25i+42}{-2(5)}$$

$$=\frac{40+25i}{-10}$$

$$=-4-\frac{5}{2}i$$

12a $\overline{z+w}$

$= \overline{x+\iota y+a+\iota b}$

$= \overline{(a+x)+(b+y)\iota}$

$= (a+x)-(b+y)i$

$= (x-\iota y)+(a-bi)$

$= \bar{z}+\bar{w}$

12b $\overline{z-w}$

$= \overline{x+\iota y-a-\iota b}$

$= \overline{(-a+x)+(-b+y)\iota}$

$= (-a+x)-(-b+y)i$

$= (x-yi)-(a-bi)$

$= \bar{z}-\bar{w}$

12c $\overline{zw}$

$= \overline{(x+\iota y)(a+\iota b)}$

$= \overline{xa+bx\iota+ay\iota+by\iota^2}$

$= \overline{xa+bx\iota+ay\iota-by}$

$= \overline{(xa-by)+(bx+ay)\iota}$

$= (xa-by)-(bx+ay)i$

$= xa-bxi-ayi-by$

$= xa-bxi-ayi+byi^2$

$= (x-iy)(a-bi)$

$= \bar{z}\,\bar{w}$

12d Consider the case where $a=x$ and $b=y$, we then have that $z=w$. The result in part c then gives that $\overline{zz}=\bar{z}\bar{z}$ and hence $\overline{z^2}=(\bar{z})^2$.

12e

$$\overline{\left(\frac{1}{z}\right)}$$

$$= \overline{\left(\frac{1}{z} \times \frac{\bar{z}}{\bar{z}}\right)}$$

$$= \overline{\left(\frac{\bar{z}}{z\bar{z}}\right)}$$

$$= \overline{\left(\frac{\bar{z}}{|z|^2}\right)}$$

$$= \frac{1}{|z|^2}\bar{\bar{z}}$$

$$= \frac{1}{|z|^2}\overline{\overline{(x + \iota y)}}$$

$$= \frac{1}{|z|^2}\ \overline{x - \iota y}$$

$$= \frac{1}{|z|^2}(x + iy)$$

$$= \frac{z}{|z|^2}$$

$$= \frac{z}{z\bar{z}}$$

$$= \frac{1}{\bar{z}}$$

12f

$$\overline{\left(\frac{z}{w}\right)}$$

$$= \overline{\left(\frac{x + \iota y}{a + \iota b}\right)}$$

$$= \overline{\left(\frac{(x + \iota y)(a - \iota b)}{(a + \iota b)(a - \iota b)}\right)}$$

$$= \overline{\left(\frac{ax - bx\iota + ay\iota - by\iota^2}{a^2 + b^2}\right)}$$

$$= \overline{\left(\frac{(ax + by) + (ay - bx)\iota}{a^2 + b^2}\right)}$$

$$= \overline{\left(\frac{(ax+by)}{a^2+b^2} + \frac{(ay-bx)}{a^2+b^2}\imath\right)}$$

$$= \frac{(ax+by)}{a^2+b^2} - \frac{(ay-bx)}{a^2+b^2}i$$

$$= \frac{ax+by-ayi+bxi}{a^2+b^2}$$

$$= \frac{ax-ayi+bxi-byi^2}{a^2+b^2}$$

$$= \frac{(x-iy)(a+ib)}{(a+ib)(a-ib)}$$

$$= \frac{x-iy}{a-ib}$$

$$= \frac{\bar{z}}{\overline{w}}$$

13a $z + \bar{z}$

$= a + ib + \overline{a + \imath b}$

$= a + ib + a - ib$

$= 2a$

Since a is real, $2a$ and hence $z + \bar{z}$ must also be real.

13b $z - \bar{z}$

$= a + ib - \left(\overline{a + \imath b}\right)$

$= a + ib - (a - ib)$

$= a + ib - a + ib$

$= 2bi$

Since b is real, $2bi$ and hence $z - \bar{z}$ must be imaginary.

13c $z^2 + (\bar{z})^2$

$= (a + ib)^2 + \left(\overline{a + \imath b}\right)^2$

$= (a + ib)^2 + (a - ib)^2$

$= a^2 + 2abi + b^2i^2 + a^2 - 2abi + b^2i^2$

$= 2a^2 + 2b^2i^2$

$= 2a^2 - 2b^2$

Since a and b are real, $2a^2 - 2b^2$ and hence $z^2 + (\bar{z})^2$ must be real.

13d $z\bar{z}$

$= (a + ib)\overline{a + ib}$

$= (a + ib)(a - ib)$

$= a^2 - i^2b^2$

$= a^2 + b^2$

Since a and b are real, $a^2 \geq 0$ and $b^2 \geq 0$ so $a^2 + b^2 \geq 0$. This in turn means that $z\bar{z} \geq 0$. Hence $z\bar{z}$ is real and positive.

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$\dfrac{z}{z - i}$

$= \dfrac{a + bi}{a + bi - i}$

$= \dfrac{a + bi}{a + (b - 1)i}$

$= \dfrac{(a + bi)(a - (b - 1)i)}{(a + (b - 1)i)(a - (b - 1)i)}$

$= \dfrac{(a + bi)(a - (b - 1)i)}{a^2 - (b - 1)^2i^2}$

$= \dfrac{a^2 - a(b - 1)i + abi - b(b - 1)i^2}{a^2 + (b - 1)^2}$

$= \dfrac{a^2 - abi + ai + abi + b(b - 1)}{a^2 + (b - 1)^2}$

$= \dfrac{a^2 + ai + b^2 - b}{a^2 + (b - 1)^2}$

$= \dfrac{a^2 + b^2 - b}{a^2 + (b - 1)^2} + \dfrac{a}{a^2 + (b - 1)^2}i$

If $\frac{z}{z-i}$ is real, then the imaginary component must be equal to zero, this means that

$$\frac{a}{a^2+(b-1)^2}=0$$

$$a=0$$

Hence, $z=0+bi=bi$. Thus z is either imaginary or zero.

15 z^2

$=(a+bi)^2$

$=a^2+2abi+b^2i^2$

$=a^2+2abi-b^2$

$\bar{z}^2$

$=(a-bi)^2$

$=a^2-2abi+b^2i^2$

$=a^2-2abi-b^2$

If $z^2=\bar{z}^2$ then

$$a^2+2abi-b^2=a^2-2abi-b^2$$

$$4abi=0$$

Hence $a=0$ or $b=0$. This means that either z is purely real or purely imaginary.

16a z^{-1}

$$=\frac{1}{z}$$

$$=\frac{1}{x+iy}$$

$$=\frac{(x-iy)}{(x-iy)(x+iy)}$$

$$=\frac{x-iy}{x^2-i^2y^2}$$

$$=\frac{x-iy}{x^2+y^2}$$

16b z^{-2}

$= (z^{-1})^2$

$= \left(\dfrac{x - iy}{x^2 + y^2}\right)^2$

$= \dfrac{x^2 - 2ixy + i^2y^2}{(x^2 + y^2)^2}$

$= \dfrac{x^2 - y^2 - 2ixy}{(x^2 + y^2)^2}$

16c

$\dfrac{z - 1}{z + 1}$

$= \dfrac{x - 1 + iy}{x + 1 + iy}$

$= \dfrac{(x - 1 + iy)(x + 1 - iy)}{(x + 1)^2 + y^2}$

$= \dfrac{x^2 + x - ixy - x - 1 + iy + ixy + iy + y^2}{(x + 1)^2 + y^2}$

$= \dfrac{x^2 + y^2 - 1 + 2iy}{(x + 1)^2 + y^2}$

Solutions to Exercise 1A Enrichment questions

17 Let $z = x + iy$ and $w = a + ib$.

$z + w = (a + x) + i(b + y)$

$\quad = (a + x) \quad$ (since $z + w$ is real)

Thus

$$b + y = 0 \tag{1}$$

$zw = (x + iy) \times (a + ib)$

$\quad = (ax - by) + i(ay + bx)$

$\quad = (ax - by) \quad$ (since zw is real)

Thus

$$ay + bx = 0 \tag{2}$$

From (1), $y = -b$, substituted in (2) gives,

$-ab + bx = 0$

or

$$b(x - a) = 0 \tag{3}$$

In (3), either $b \neq 0$ so that $x = a$

and so $z = x + iy$, $w = x - iy$

i.e. $z = \bar{w}$

or in (3), $b = 0$, so $z = x$ and $w = a$ (from (1), $y = 0$ if $b = 0$)

i.e. $\text{Im}(z) = 0 = \text{Im}(w)$

18 $\dfrac{1}{1-z} = \dfrac{1-\bar{z}}{(1-z)(1-\bar{z})}$

$= \dfrac{1-\bar{z}}{1-(z+\bar{z})+z\bar{z}}$

$= \dfrac{1-2\cos\theta+2i\sin\theta}{1-4\cos\theta+4\cos^2\theta+4\sin^2\theta}$ $\quad (z+\bar{z}=2\mathrm{Re}(z)$ and $z\bar{z}=(\mathrm{Re}(z))^2+(\mathrm{Im}(z))^2)$

$= \dfrac{(1-2\cos\theta)+2i\sin\theta}{5-4\cos\theta}$ $\quad (\cos^2\theta+\sin^2\theta=1)$

So $\mathrm{Re}\left(\dfrac{1}{1-z}\right) = \dfrac{1-2\cos\theta}{5-4\cos\theta}$

19 $\dfrac{1+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta} = \dfrac{(1+\sin\theta+i\cos\theta)^2}{(1+\sin\theta)^2+(\cos\theta)^2}$

$= \dfrac{1+2\sin\theta+\sin^2\theta-\cos^2\theta+i2(1+\sin\theta)\cos\theta}{1+2\sin\theta+\sin^2\theta+\cos^2\theta}$

$= \dfrac{2(1+\sin\theta)(\sin\theta+i\cos\theta)}{2(1+\sin\theta)}$

$= \sin\theta+i\cos\theta$

20 $\dfrac{2}{1+z} = \dfrac{2(1+\bar{z})}{(1+z)(1+\bar{z})}$

$= \dfrac{2(1+\bar{z})}{1+(z+\bar{z})+z\bar{z}}$

$= \dfrac{2(1+\cos\theta-i\sin\theta)}{1+2\cos\theta+\cos^2\theta+\sin^2\theta}$ $\quad (z+\bar{z}=2\mathrm{Re}(z)$ and $z\bar{z}=(\mathrm{Re}(z))^2+(\mathrm{Im}(z))^2)$

$= \dfrac{2(1+\cos\theta)-i\sin\theta}{2(1+\cos\theta)}$ $\quad (\cos^2\theta+\sin^2\theta=1)$

$= 1-\dfrac{i\sin\theta}{1+\cos\theta}$

now $\dfrac{\sin\theta}{1+\cos\theta} = \dfrac{\frac{2t}{1+t^2}}{1+\frac{1-t^2}{1+t^2}}$ $\quad$ where $t=\tan\dfrac{\theta}{2}$

$= \dfrac{2t}{1+t^2+1-t^2}$

$= \dfrac{2t}{2}$

$= t$

So $\dfrac{2}{1+z} = 1-it$

Solutions to Exercise 1B Foundation questions

1a $z^2 + 9 = 0$

$z^2 - 9i^2 = 0$

$z^2 - (3i)^2 = 0$

$(z - 3i)(z + 3i) = 0$

$z = \pm 3i$

1b $(z - 2)^2 + 16 = 0$

$(z - 2)^2 - 16i^2 = 0$

$(z - 2)^2 - (4i)^2 = 0$

$(z - 2 - 4i)(z - 2 + 4i) = 0$

$z = 2 \pm 4i$

1c $z^2 + 2z + 5 = 0$

$z^2 + 2z + 1 + 4 = 0$

$(z + 1)^2 + 4 = 0$

$(z + 1)^2 - 4i^2 = 0$

$(z + 1)^2 - (2i)^2 = 0$

$(z + 1 - 2i)(z + 1 + 2i) = 0$

$z = -1 \pm 2i$

1d $z^2 - 6z + 10 = 0$

$z^2 - 6z + 9 + 1 = 0$

$(z - 3)^2 + 1 = 0$

$(z - 3)^2 - i^2 = 0$

$(z - 3 - i)(z - 3 + i) = 0$

$z = 3 \pm i$

1e $16z^2 - 16z + 5 = 0$

$16z^2 - 16z + 4 + 1 = 0$

$(4z - 2)^2 + 1 = 0$

$(4z - 2)^2 - i^2 = 0$

$(4z - 2 - i)(4z - 2 + i) = 0$

$4z = 2 \pm i$

$z = \frac{1}{2} \pm \frac{1}{4}i$

1f $4z^2 + 12z + 25 = 0$

$4z^2 + 12z + 9 + 16 = 0$

$(2z + 3)^2 + 16 = 0$

$(2z + 3)^2 - 16i^2 = 0$

$(2z + 3)^2 - (4i)^2 = 0$

$(2z + 3 + 4i)(2z + 3 - 4i) = 0$

$2z = -3 \pm 4i$

$z = -\frac{3}{2} \pm 2i$

2a $z^2 + 36$
$= z^2 - 36i^2$

$= z^2 - (6i)^2$

$= (z - 6i)(z + 6i)$

2b $z^2 + 8$

$= z^2 - 8i^2$
$= z^2 - \left(2\sqrt{2}i\right)^2$
$= \left(z - 2\sqrt{2}i\right)\left(z + 2\sqrt{2}i\right)$

2c $z^2 - 2z + 10$
$= z^2 - 2z + 1 + 9$
$= (z-1)^2 - 9i^2$
$= (z-1)^2 - (3i)^2$
$= (z - 1 - 3i)(z - 1 + 3i)$

2d $z^2 + 4z + 5$
$= z^2 + 4z + 4 + 1$
$= (z+2)^2 - i^2$
$= (z + 2 - i)(z + 2 + i)$

2e $z^2 - 6z + 14$
$= z^2 - 6z + 9 + 5$
$= (z-3)^2 - 5i^2$
$= (z-3)^2 - \left(\sqrt{5}i\right)^2$
$= (z - 3 - \sqrt{5}i)(z - 3 + \sqrt{5}i)$

2f $z^2 + z + 1$
$= z^2 + z + \frac{1}{4} + \frac{3}{4}$
$= \left(z + \frac{1}{2}\right)^2 - \frac{3}{4}i^2$
$= \left(z + \frac{1}{2}\right)^2 - \left(\frac{\sqrt{3}}{2}i\right)^2$
$= \left(z + \frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\left(z + \frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$

3a $\left(z - i\sqrt{2}\right)\left(z + i\sqrt{2}\right) = 0$
$z^2 - \left(i\sqrt{2}\right)^2 = 0$
$z^2 + 2 = 0$

3b $\left(z-(1-i)\right)\left(z-(1+i)\right)=0$

$z^2-(1-i+1+i)z+(1-i)(1+i)=0$

$z^2-2z+(1^2-i^2)=0$

$z^2-2z+2=0$

3c $(z+1-2i)(z+1+2i)=0$

$z^2+(1-2i+1+2i)z+(1-2i)(1+2i)=0$

$z^2+2z+(1^2-4i^2)=0$

$z^2+2z+(1+4)=0$

$z^2+2z+5=0$

3d $\left(z-\left(2-i\sqrt{3}\right)\right)\left(z-\left(2+i\sqrt{3}\right)\right)=0$

$z^2-\left(2+i\sqrt{3}+2-i\sqrt{3}\right)z+\left(2-i\sqrt{3}\right)\left(2+i\sqrt{3}\right)=0$

$z^2-4z+\left(2^2-\left(i\sqrt{3}\right)^2\right)=0$

$z^2-4z+(2^2+3)=0$

$z^2-4z+7=0$

4a Let $(x+iy)^2=2i$ where x and y are real.

$x^2+2ixy-y^2=2i$

$x^2-y^2+2ixy=2i$

Equating real and imaginary parts gives:

$x^2-y^2=0$

$2xy=2$

By inspection of the term, $x=1$ and $y=1$ or $x=-1$ and $y=-1$.

So, the roots are $\pm(1+i)$.

4b Let $(x + iy)^2 = 3 + 4i$ where x and y are real.

$x^2 + 2ixy - y^2 = 3 + 4i$

$x^2 - y^2 + 2ixy = 3 + 4i$

Equating real and imaginary parts gives:

$x^2 - y^2 = 3$ and

$2xy = 4$

By inspection, $x = 2$ and $y = 1$ or $x = -2$ and $y = -1$.

So, the roots are $\pm(2 + i)$.

4c Let $(x + iy)^2 = -8 - 6i$ where x and y are real.

$x^2 + 2ixy - y^2 = -8 - 6i$

$x^2 - y^2 + 2ixy = -8 - 6i$

Equating real and imaginary parts gives:

$x^2 - y^2 = -8$ and

$2xy = -6$

By inspection, $x = -1$ and $y = 3$ or $x = 1$ and $y = -3$.

So, the roots are $\pm(-1 + 3i)$.

4d Let $(x + iy)^2 = 35 + 12i$ where x and y are real.

$x^2 + 2ixy - y^2 = 35 + 12i$

$x^2 - y^2 + 2ixy = 35 + 12i$

Equating real and imaginary parts gives:

$x^2 - y^2 = 35$ and

$2xy = 12$

By inspection, $x = 6$ and $y = 1$ or $x = -6$ and $y = -1$.

So, the roots are $\pm(6 + i)$.

4e Let $(x + iy)^2 = -5 + 12i$ where x and y are real.

$x^2 + 2ixy - y^2 = -5 + 12i$

$x^2 - y^2 + 2ixy = -5 + 12i$

Equating real and imaginary parts gives:

$x^2 - y^2 = -5$ and

$2xy = 12$

By inspection, $x = 2$ and $y = 3$ or $x = -2$ and $y = -3$.

So, the roots are $\pm(2 + 3i)$.

4f Let $(x + iy)^2 = 24 - 10i$ where x and y are real.

$x^2 + 2ixy - y^2 = 24 - 10i$

$x^2 - y^2 + 2ixy = 24 - 10i$

Equating real and imaginary parts gives:

$x^2 - y^2 = 24$ and

$2xy = -10$

By inspection, $x = 5$ and $y = -1$ or $x = -5$ and $y = +1$.

So, the roots are $\pm(5 - i)$.

4g Let $(x + iy)^2 = -15 - 8i$ where x and y are real.

$x^2 + 2ixy - y^2 = -15 - 8i$

$x^2 - y^2 + 2ixy = -15 - 8i$

Equating real and imaginary parts gives:

$x^2 - y^2 = -15$ and

$2xy = -8$

By inspection, $x = 1$ and $y = -4$ or $x = -1$ and $y = 4$.

So, the roots are $\pm(1 - 4i)$.

4h Let $(x + iy)^2 = 9 - 40i$ where x and y are real

$x^2 + 2ixy - y^2 = 9 - 40i$

$x^2 - y^2 + 2ixy = 9 - 40i$

Equating real and imaginary parts gives:

$x^2 - y^2 = 9$ and

$2xy = -40$

By inspection, $x = 5$ and $y = -4$ or $x = -5$ and $y = 4$.

So, the roots are $\pm(5 - 4i)$.

Solutions to Exercise 1B Development questions

5a Let $z = a + bi$ be a square root of $-3 - 4i$

$$\begin{aligned} z^2 &= (a + bi)^2 \\ &= a^2 + 2abi + b^2 i^2 \\ &= a^2 - b^2 + 2abi \end{aligned}$$

Thus $a^2 - b^2 + 2abi = -3 - 4i$.

Equating real parts,

$$a^2 - b^2 = -3 \qquad (1)$$

Equating imaginary parts,

$$2ab = -4$$

$$b = -\frac{2}{a} \qquad (2)$$

Substituting (2) into (1):

$$a^2 - \left(-\frac{2}{a}\right)^2 = -3$$

$$a^2 - \frac{4}{a^2} = -3$$

$$a^4 - 4 = -3a^2$$

$$a^4 + 3a^2 - 4 = 0$$

$$(a^2 + 4)(a^2 - 1) = 0$$

Hence $a = \pm 1$ (as a is real)

When $a = 1, b = -2$ and $z = 1 - 2i$.

When $a = -1, b = 2$ and $z = -1 + 2i$.

5b $z^2 - 3z + (3 + i) = 0$

$$\left(z - \tfrac{3}{2}\right)^2 - \tfrac{9}{4} + 3 + i = 0 \qquad \text{(Completing the square)}$$

$$\left(z - \frac{3}{2}\right)^2 - \frac{9}{4} + \frac{12 + 4i}{4} = 0$$

$$\left(z - \frac{3}{2}\right)^2 + \frac{3 + 4i}{4} = 0$$

$$\left(z-\frac{3}{2}\right)^2=-\frac{3+4i}{4}$$

$$\left(z-\frac{3}{2}\right)^2=\frac{-3-4i}{4}$$

$$z-\frac{3}{2}=\pm\frac{\sqrt{-3-4i}}{2}$$

$$z=\frac{3}{2}\pm\frac{\sqrt{-3-4i}}{2}$$

Now, using the square roots found in part (a),

$$z=\frac{3}{2}\pm\frac{(1-2i)}{2}$$

$$z=\frac{3}{2}\pm\left(\frac{1}{2}-i\right)$$

$$z=2-i,1+i$$

6a Let $z=a+bi$ be a square root of $-8+6i$.

$$\begin{aligned}z^2&=(a+bi)^2\\&=a^2+2abi+b^2i^2\\&=a^2-b^2+2abi\end{aligned}$$

Thus $a^2-b^2+2abi=-8+6i$.

Equating real parts,

$$a^2-b^2=-8 \qquad (1)$$

Equating imaginary parts,

$$2ab=6$$

$$b=\frac{3}{a} \qquad (2)$$

Substituting (2) into (1):

$$a^2-\left(\frac{3}{a}\right)^2=-8$$

$$a^2-\frac{9}{a^2}=-8$$

$$a^4-9=-8a^2$$

$$a^4+8a^2-9=0$$

$(a^2+9)(a^2-1)=0$

Hence $a=\pm 1$ (as a is real)

When $a=1, b=3$ and $z=1+3i$.

When $a=-1, b=-3$ and $z=-1-3i$.

6b $\Delta=\left(-(7-i)\right)^2-4(1)(14-5i)$

$=49-14i+i^2-56+20i$

$=49-14i-1-56+20i$

$=-8+6i$

$\lambda^2=-8+6i$

But from part a) we have,

$\lambda=\pm(1+3i)$

Hence,

$$z=\frac{-b-\lambda}{2a} \text{ or } \frac{-b+\lambda}{2a}$$

$$z=\frac{-\left(-(7-i)\right)\pm(1+3i)}{2(1)}$$

$z=4+i$ or $3-2i$

7a Using the method in Box 12 to solve $z^2-z+(1+i)=0$:

1. $\Delta=b^2-4ac$
   $=(-1)^2-4(1)(1+i)$
   $=1-4(1+i)$
   $=1-4-4i$
   $=-3-4i$
2. $\lambda^2=-3-4i$ so, from 5a), $\lambda=\pm(1-2i)$
3. Thus $z=\frac{-(-1)-(1-2i)}{2(1)}$ or $\frac{-(-1)+(1-2i)}{2(1)}$

$$z=\frac{2i}{2(1)} \text{ or } \frac{2-2i}{2(1)}$$

$z=i$ or $1-i$

7b Using the method in Box 12 to solve $z^2 + 3z + (4 + 6i) = 0$:

1. $\Delta = b^2 - 4ac$
   $= (3)^2 - 4(1)(4 + 6i)$
   $= 9 - 16 - 24i$
   $= -7 - 24i$

2. $\lambda^2 = -7 - 24i$

   Let $\lambda = a + bi$ be a square root of $-7 - 24i$

   $\lambda^2 = (a + bi)^2$
   $= a^2 + 2abi + b^2i^2$
   $= a^2 - b^2 + 2abi$

   Thus $a^2 - b^2 + 2abi = -7 - 24i$.

   Equating real parts,

   $a^2 - b^2 = -7 \qquad (1)$

   Equating imaginary parts,

   $2ab = -24$

   $$b = -\frac{12}{a} \qquad (2)$$

   Substituting (2) into (1):

   $$a^2 - \left(-\frac{12}{a}\right)^2 = -7$$

   $$a^2 - \frac{144}{a^2} = -7$$

   $a^4 - 144 = -7a^2$

   $a^4 + 7a^2 - 144 = 0$

   $(a^2 - 9)(a^2 + 16) = 0$

   $a = \pm 3$ (as a is real)

   From (2) when $a = 3$, $b = -4$ and when $a = -3$, $b = 4$.

   This gives $\lambda = \pm(3 - 4i)$

3. Thus $z = \frac{-(3)+(3-4i)}{2(1)}$ or $\frac{-(3)-(3-4i)}{2(1)}$

   $$z = \frac{-4i}{2(1)} \text{ or } \frac{-6 + 4i}{2(1)}$$

   $z = -2i$ or $-3 + 2i$

7c Using the method in Box 12 to solve $z^2 - 6z + (9 - 2i) = 0$:

1. $$\begin{aligned}\Delta &= b^2 - 4ac \\ &= (-6)^2 - 4(1)(9 - 2i) \\ &= 36 - 36 + 8i \\ &= 8i\end{aligned}$$

2. $\lambda^2 = 8i$

   Let $\lambda = a + bi$ be a square root of $8i$.

   $$\begin{aligned}\lambda^2 &= (a + bi)^2 \\ &= a^2 + 2abi + b^2 i^2 \\ &= a^2 - b^2 + 2abi\end{aligned}$$

   Thus $a^2 - b^2 + 2abi = 8i$.

   Equating real parts,

   $a^2 - b^2 = 0 \qquad (1)$

   Equating imaginary parts,

   $2ab = 8$

   $b = \dfrac{4}{a} \qquad (2)$

   Substituting (2) into (1):

   $a^2 - \left(\dfrac{4}{a}\right)^2 = 0$

   $a^2 - \dfrac{16}{a^2} = 0$

   $a^4 - 16 = 0$

   $(a^2 - 4)(a^2 + 4) = 0$

   $a = \pm 2$ (as a is real)

   From (2) when $a = 2, b = 2$ and when $a = -2, b = -2$.

   This gives $\lambda = \pm 2(1 + i)$.

3. Thus $z = \frac{-(-6)+2(1+i)}{2(1)}$ or $\frac{-(-6)-2(1+i)}{2(1)}$

   $z = -\frac{8+2i}{2(1)}$ or $\frac{4-2i}{2(1)}$

   $z = 2 - i$ or $4 + i$

7d Using the method in Box 12 to solve $(1+i)z^2 + z - 5 = 0$:

1. $\Delta = b^2 - 4ac$
   $= 1 - 4(1+i)(-5)$
   $= 1 - (-20 - 20i)$
   $= 21 + 20i$

2. $\lambda^2 = 21 + 20i$

   Let $\lambda = a + bi$ be a square root of $21 + 20i$.

   $\lambda^2 = (a+bi)^2$
   $= a^2 + 2abi + b^2i^2$
   $= a^2 - b^2 + 2abi$

   Thus $a^2 - b^2 + 2abi = 21 + 20i$.

   Equating real parts,

   $a^2 - b^2 = 21 \qquad (1)$

   Equating imaginary parts,

   $2ab = 20$

   $b = \dfrac{10}{a} \qquad (2)$

   Substituting (2) into (1):

   $a^2 - \left(\dfrac{10}{a}\right)^2 = 21$

   $a^2 - \dfrac{100}{a^2} = 21$

   $a^4 - 100 = 21a^2$
   $a^4 - 21a^2 - 100 = 0$

   $(a^2 - 25)(a^2 + 4) = 0$

   $a^2 = 25$ or -4

   Hence $a = \pm 5$

   When $a = 5$, $b = \frac{10}{5} = 2$.

   When $a = -5$, $b = \frac{10}{(-5)} = -2$.

   This gives, $\lambda = \pm(5 + 2i)$.

3. Thus $z = \frac{-(1)+5+2i}{2(1+i)}$ or $\frac{-(1)-(5+2i)}{2(1+i)}$

   $z = \frac{2+i}{1+i}$ or $\frac{-3-i}{1+i}$

   $z = \frac{(2+i)(1-i)}{(1-i)(1+i)}$ or $\frac{(-3-i)(1-i)}{(1-i)(1+i)}$

   $z = \frac{1}{2}(3-i)$ or $z = -2+i$

7e Using the method in Box 12 to solve $z^2 + (2+i) - 13(1-i) = 0$:

1. $\Delta = b^2 - 4ac$
   $= (2+i)^2 - 4(1)\big(-13(1-i)\big)$
   $= 4 + 4i + i^2 + 52(1-i)$
   $= 4 + 4i - 1 + 52 - 52i$
   $= 55 - 48i$

2. $\lambda^2 = 55 - 48i$

   Let $\lambda = a + bi$ be a square root of $55 - 48i$.

   $\lambda^2 = (a+bi)^2$
   $= a^2 + 2abi + b^2i^2$
   $= a^2 - b^2 + 2abi$

   Thus $a^2 - b^2 + 2abi = 55 - 48i$.

   Equating real parts,

   $a^2 - b^2 = 55$ (1)

   Equating imaginary parts,

   $2ab = -48$

   $b = \dfrac{-24}{a}$ (2)

   Substituting (2) into (1):

   $a^2 - \left(-\dfrac{24}{a}\right)^2 = 55$

   $a^2 - \dfrac{576}{a^2} = 55$

   $a^4 - 576 = 55a^2$

$a^2 - 55a^2 - 576 = 0$

$a^2 = \frac{-(-55) \pm \sqrt{(-55)^2 - 4(1)(-576)}}{2}$

$a^2 = \frac{55 \pm \sqrt{5329}}{2}$

$a^2 = \frac{55 \pm 73}{2}$

$a^2 = 64$ or -9

Hence $a = \pm 8$

When $a = 8, b = \frac{-24}{8} = -3$.

When $a = -8, b = \frac{-24}{-8} = 3$.

This gives, $\lambda = \pm(8 - 3i)$.

3. Thus $z = \frac{-(2+i)+(8-3i)}{2(1)}$ or $\frac{-(2+i)-(8-3i)}{2(1)}$

$z = \frac{6-4i}{2(1)}$ or $\frac{-10+2i}{2(1)}$

$z = 3 - 2i$ or $-5 + i$

7f Using the method in Box 12 to solve $iz^2 - 2(1 + i) + 10 = 0$:

1. $\Delta = b^2 - 4ac$
$= \left(-2(1+i)\right)^2 - 4(i)(10)$
$= 4(1 + 2i + i^2) - 40i$
$= 4(1 + 2i - 1) - 40i$
$= 8i - 40i$
$= -32i$

2. $\lambda^2 = -32i$

Let $\lambda = a + bi$ be a square root of $-32i$

$\lambda^2 = (a + bi)^2$
$= a^2 + 2abi + b^2i^2$
$= a^2 - b^2 + 2abi$

Thus $a^2 - b^2 + 2abi = -32i$.

Equating real parts,

$a^2 - b^2 = 0 \qquad (1)$

Equating imaginary parts,

$2ab = -32$

$b = \dfrac{-16}{a} \qquad (2)$

Substituting (2) into (1):

$a^2 - \left(-\dfrac{16}{a}\right)^2 = 0$

$a^2 - \dfrac{16^2}{a^2} = 0$

$a^4 - 16^2 = 0$

$(a^2 - 16)(a^2 + 16) = 0$

Hence $a = \pm 4$

When $a = 4, b = -\frac{16}{4} = -4$.

When $a = -4, b = \frac{-16}{-4} = 4$.

This gives, $\lambda = \pm 4(1 - i)$.

3. Thus $z = \frac{2(1+i)-(4-4i)}{2(i)}$ or $\frac{2(1+i)+(4-4i)}{2(i)}$

$z = \frac{-2+6i}{2i}$ or $\frac{6-2i}{2i}$

$z = \frac{-2i+6i^2}{2i^2}$ or $\frac{6i-2i^2}{2i^2}$

$z = -\frac{-2i-6}{2}$ or $-\frac{6i+2}{2}$

$z = 3 + i$ or $-1 - 3i$

8a $z^2 + wz + (1 + i) = 0$

Since i is a root of the equation,

$i^2 + wi + (1 + i) = 0$

$-1 + wi + (1 + i) = 0$

$wi + i = 0$

$wi = -i$

$w = -1$

8b Since a and b are real, the conjugate $3 + 2i$ must also be a root of the equation.

The sum of the roots is $-\frac{a}{1}$, hence

$3 + 2i + 3 - 2i = -a$

$6 = -a$

$a = -6$

The products of the roots is $\frac{b}{1} = b$, hence

$(3 + 2i)(3 - 2i) = b$

$9 - 4i^2 = b$

$b = 9 + 4$

$b = 13$

8c Since $1 - 2i$ is a root of the equation, we can find k by substituting it into the equation.

$(1 - 2i)^2 - (3 + i)(1 - 2i) + k = 0$

$1 - 4i + 4i^2 - (3 - 6i + i - 2i^2) + k = 0$

$1 - 4i - 4 - (3 - 5i + 2) + k = 0$

$-3 - 4i - (5 - 5i) + k = 0$

$-8 + i + k = 0$

$k = 8 - i$

Now, the sum of the roots is equal to $-\left(-\frac{3+i}{1}\right) = 3 + i$, hence, if our second root is α

$1 - 2i + \alpha = 3 + i$

$\alpha = 2 + 3i$

9 $z\bar{z} = 5$

$z = \dfrac{5}{\bar{z}}$

$z^2 = \dfrac{5z}{\bar{z}}$

$\dfrac{z^2}{5} = \dfrac{z}{\bar{z}}$

$\dfrac{z}{\bar{z}} = \dfrac{1}{5}(3 + 4i)$

$\dfrac{z^2}{5} = \dfrac{1}{5}(3 + 4i)$

$z^2 = 3 + 4i$

Let $z = a + bi$.

$z^2 = a^2 - b^2 + 2abi$

Hence $a^2 - b^2 + 2abi = 3 + 4i$

Equating real parts,

$a^2 - b^2 = 3 \qquad (1)$

Equating imaginary parts,

$2ab = 4$

$b = \dfrac{2}{a} \qquad (2)$

Substituting (2) in (1):

$a^2 - \left(\dfrac{2}{a}\right)^2 = 3$

$a^2 - \dfrac{4}{a^2} = 3$

$a^4 - 4 = 3a^2$

$a^4 - 3a^2 - 4 = 0$

$(a^2 - 4)(a^2 + 1) = 0$

As a is real, $a^2 \geq 0$ and hence $a = \pm 2$.

When $a = 2$, $b = 1$ and when $a = -2$, $b = -1$. Thus $z = \pm(2 + i)$.

10a $z^2 - 2z\cos\theta + 1 = 0$

$z^2 - 2z\cos\theta + \cos^2\theta - \cos^2\theta + 1 = 0$

$(z - \cos\theta)^2 - \cos^2\theta + 1 = 0$

$(z - \cos\theta)^2 + 1 - \cos^2\theta = 0$

$(z - \cos\theta)^2 + \sin^2\theta = 0$

$(z - \cos\theta)^2 = -\sin^2\theta$

$(z - \cos\theta)^2 = (i\sin\theta)^2$

$(z - \cos\theta) = \pm i\sin\theta$

$z = \cos\theta \pm i\sin\theta$

$z = \operatorname{cis}(\pm\theta)$

10b $\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$

Substituting $z = \operatorname{cis}\theta$ into the right-hand side of the equation gives

$\frac{1}{2}\left(z + \frac{1}{z}\right)$

$= \frac{1}{2}\left(\operatorname{cis}\theta + \frac{1}{\operatorname{cis}\theta}\right)$

$= \frac{1}{2}\left(\operatorname{cis}\theta + \operatorname{cis}(-\theta)\right)$ (using De Moivres theorem)

$= \frac{1}{2}(\cos\theta + i\sin\theta + \cos\theta - i\sin\theta)$

$= \frac{1}{2}(2\cos\theta)$

$= \cos\theta$

Substituting $z = \operatorname{cis}(-\theta)$ into the right-hand side of the equation gives

$\frac{1}{2}\left(z + \frac{1}{z}\right)$

$= \frac{1}{2}\left(\operatorname{cis}(-\theta) + \frac{1}{\operatorname{cis}(-\theta)}\right)$

$= \frac{1}{2}(\text{cis}(-\theta) + \text{cis}\,\theta)$ (using De Moivres theorem)

$$= \frac{1}{2}(\text{cis}(-\theta) + \text{cis}\,\theta)$$

$$= \frac{1}{2}(\cos\theta - i\sin\theta + \cos\theta + i\sin\theta)$$

$$= \frac{1}{2}(2\cos\theta)$$

$= \cos\theta$

Hence the equation holds true for both results found in part **a.**

11a $z^3 = -1$

$z^3 + 1 = 0$

$z^3 + 1^3 = 0$

$(z+1)(z^2 - z(1) + 1^2) = 0$

$(z+1)(z^2 - z + 1) = 0$

Hence one root is $z = -1$, and using the quadratic formula, we find the roots of

$z^2 - z + 1 = 0$

$$z = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(1)}}{2}$$

$$= \frac{1 \pm \sqrt{1-4}}{2}$$

$$= \frac{1 \pm \sqrt{3}i}{2}$$

So, the roots are $z = -1, \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$.

11b $z^3 + i = 0$

$z^3 - i^3 = 0$

$(z - i)(z^2 + iz + i^2) = 0$

$z = i$ or $z^2 + iz - 1 = 0$

Using the quadratic formula,

$$z = \frac{-i \pm \sqrt{i^2 - 4(1)(-1)}}{2}$$

$$= \frac{-i \pm \sqrt{-1 + 4}}{2}$$

$$= \frac{-i \pm \sqrt{3}}{2}$$

Hence the roots are $z = i$ or $z = \pm\frac{\sqrt{3}}{2} - \frac{1}{2}i$.

12a Since ω is a root of the equation, it must satisfy the equation when substituted into the above equation. Thus $a\omega^2 + b\omega + c = 0$.

12b $\overline{a\omega^2 + b\omega + c} = 0$

$\overline{a\omega^2} + \overline{b\omega} + c = 0$

$a\overline{\omega^2} + b\overline{\omega} + \overline{\omega} = 0$ (since a and b are real)

$a(\overline{\omega})^2 + b\overline{\omega} + \overline{\omega} = 0$

12c When a quadratic has real coefficients, for any complex root, the conjugate must also be a root.

13a The conjugate $\overline{\alpha}$ must also be a root since the polynomial has real coefficients.

13b The equation $z^2 - 2\mathrm{Re}(\alpha)z + \alpha\bar{\alpha} = 0$ has real coefficients.

This is because 1, $-2\mathrm{Re}(z)$ and $\alpha\bar{\alpha} = |a|^2$ are all real.

Substituting α into the left-hand side of the equation gives

$\alpha^2 - 2\mathrm{Re}(\alpha)\alpha + \alpha\bar{\alpha}$

$= \alpha(\alpha - 2\mathrm{Re}(\alpha) + \bar{\alpha})$

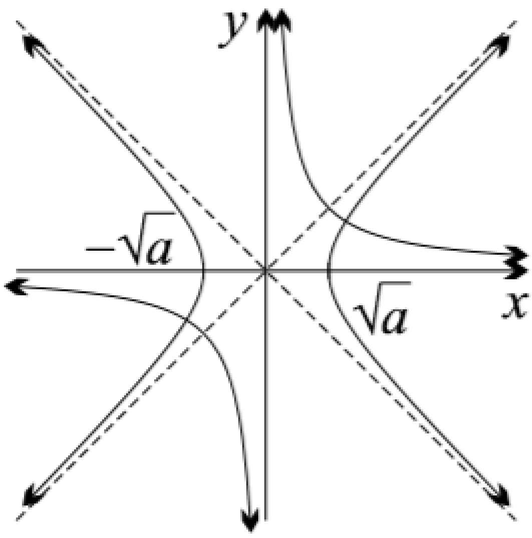
$= \alpha(\alpha + \bar{\alpha} - 2\mathrm{Re}(\alpha))$

$= \alpha(2\mathrm{Re}(\alpha) - 2\mathrm{Re}(\alpha))$

$= \alpha(0)$

$= 0$

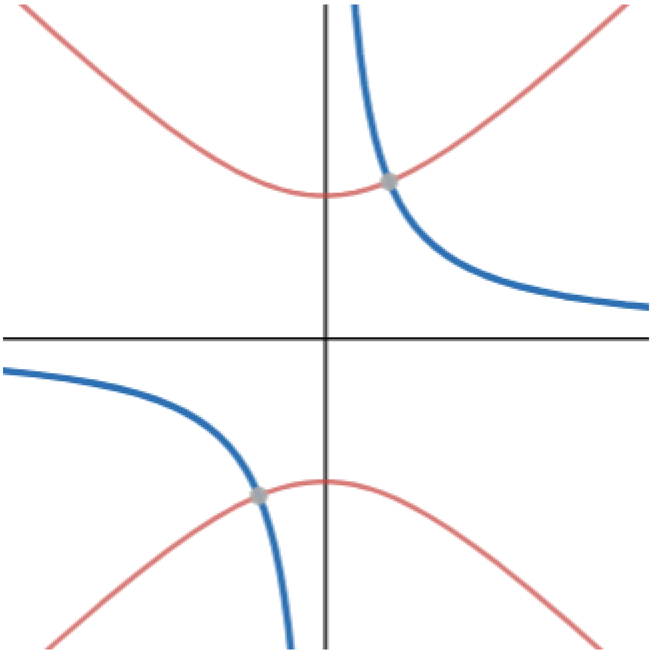
14ai $x^2 - y^2 = a$ is shown by the curve that intersects the x-axis, and $2xy = b$ is given by the curve in the 1st and 3rd quadrants.



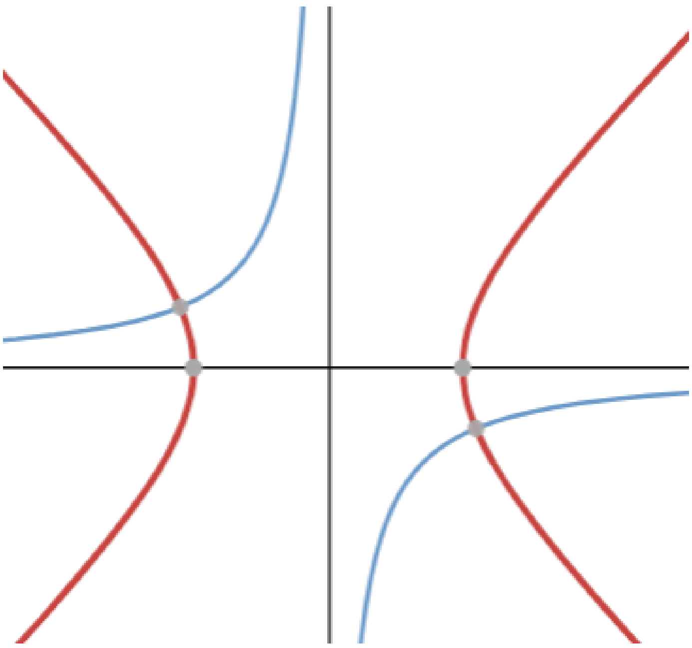
14aii Since the two graphs have two distinct points of intersection, we can infer that the set of equations $x^2 - y^2 = a$ and $2xy = b$ will have two distinct sets of solutions. Each set of solutions corresponds to a unique value of $z = a + ib$ and hence a distinct square root.

14b Below we sketch each of the possible cases. In each case note that there are always exactly two solutions.

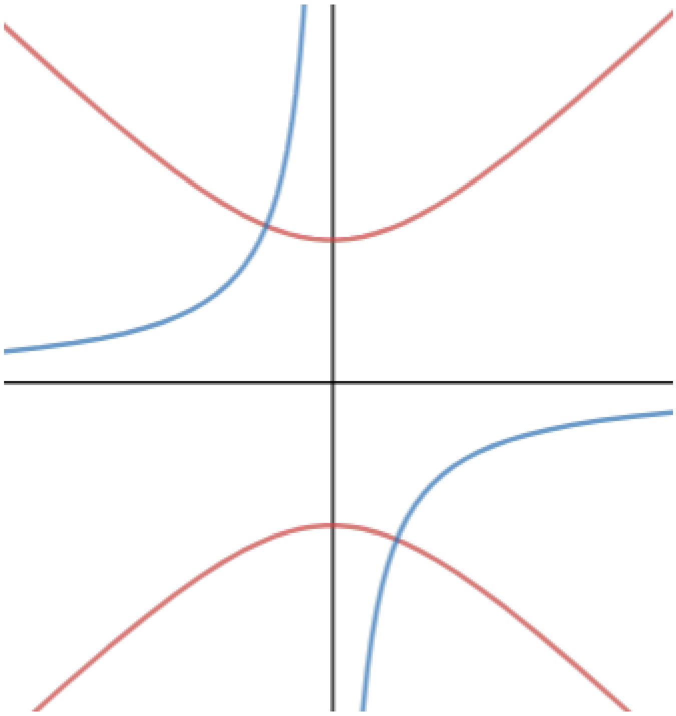
$a < 0, b > 0$



$a > 0, b < 0$



$a < 0, b < 0$



15a Let the square root of $-i$ be $z = a + ib$.

Using the result in Box 13,

$a^2 - b^2 = 0$

$a^2 = b^2$

$a = \pm b \qquad (1)$

$ab = -\frac{1}{2} \qquad (2)$

Substituting (1) into (2):

$$\pm a^2 = -\frac{1}{2}$$

Since a is real, $a^2 > 0$ and hence $a^2 = \frac{1}{2}$ so $a = \pm\frac{1}{\sqrt{2}}$.

Now, using (2):

When $a = \frac{1}{\sqrt{2}}$, $b = -\frac{1}{\sqrt{2}}$ and hence $z = \frac{1}{\sqrt{2}}(1 - i)$.

When $a = -\frac{1}{\sqrt{2}}$, $b = \frac{1}{\sqrt{2}}$ and hence $z = -\frac{1}{\sqrt{2}}(1 - i)$.

15b Let the square root of $-6 + 8i$ be $z = a + ib$.

Using the result in Box 13,

$a^2 - b^2 = -6$ (1)

$ab = \frac{1}{2}(8)$ (2)

Substituting (2) into (1):

$$a^2 - \left(\frac{4}{a}\right)^2 = -6$$

$$a^2 - \frac{16}{a^2} = -6$$

$$a^4 - 16 = -6a^2$$

$$a^4 + 6a^2 - 16 = 0$$

$$(a^2 - 2)(a^2 + 8) = 0$$

Since a is real, $a^2 > 0$ and hence $a^2 = 2$ so $a = \pm\sqrt{2}$.

Now, using (2):

When $a = \sqrt{2}$, $b = 2\sqrt{2}$ and hence $z = \sqrt{2}(1 + 2i)$.

When $a = -\sqrt{2}$, $b = -2\sqrt{2}$ and hence $z = -\sqrt{2}(1 + 2i)$.

15c Let the square root of $2 + 2i\sqrt{3}$ be $z = a + ib$

Using the result in Box 13,

$a^2 - b^2 = 2$ (1)

$ab = \sqrt{3}$ (2)

Substituting (2) into (1):

$$a^2 - \left(\frac{\sqrt{3}}{a}\right)^2 = 2$$

$$a^2 - \frac{3}{a^2} = 2$$

$$a^4 - 3 = 2a^2$$

$$a^4 - 2a^2 - 3 = 0$$

$$(a^2 - 3)(a^2 + 1) = 0$$

Since a is real, $a^2 > 0$ and hence $a^2 = 3$ so $a = \pm\sqrt{3}$.

Now, using (2):

When $a = \sqrt{3}$, $b = 1$ and hence $z = (\sqrt{3} + i)$.

When $a = -\sqrt{3}$, $b = -1$ and hence $z = -(\sqrt{3} + i)$.

15d Let the square root of $10 - 24i$ be $z = a + ib$

Using the result in Box 13,

$a^2 - b^2 = 10 \quad (1)$

$ab = -12 \qquad (2)$

Substituting (2) into (1):

$$a^2 - \left(\frac{-12}{a}\right)^2 = 10$$

$$a^2 - \frac{144}{a^2} = 10$$

$$a^4 - 144 = 10a^2$$

$$a^4 - 10a^2 - 144 = 0$$

$$(a^2 - 18)(a^2 + 8) = 0$$

$$a^2 = 18 \text{ or} - 8$$

Since a is real, $a^2 > 0$ and hence $a^2 = 18$ so $a = \pm 3\sqrt{2}$.

Now, using (2):

When $a = 3\sqrt{2}$, $b = -2\sqrt{2}$ and hence $z = \sqrt{2}(3 - 2i)$.

When $a = -3\sqrt{2}$, $b = 2\sqrt{2}$ and hence $z = -\sqrt{2}(3 - 2i)$.

15e Let the square root of $2-4i$ be $z = a + ib$.

Using the result in Box 13,

$a^2 - b^2 = 2 \quad (1)$

$ab = -2 \qquad (2)$

Substituting (2) into (1):

$$a^2 - \left(\frac{-2}{a}\right)^2 = 2$$

$$a^2 - \frac{4}{a^2} = 2$$

$$a^4 - 4 = 2a^2$$

$$a^4 - 2a^2 - 4 = 0$$

$$a^2 = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-4)}}{2}$$

$$a^2 = \frac{2 \pm \sqrt{20}}{2}$$

$$a^2 = \frac{2 \pm 2\sqrt{5}}{2}$$

$$a^2 = 1 \pm \sqrt{5}$$

Since a is real, $a^2 > 0$ and hence $a^2 = 1 + \sqrt{5}$ so $a = \pm\sqrt{\sqrt{5}+1}$.

Now, using (2):

When $a = \sqrt{\sqrt{5}+1}$,

$$\begin{aligned} b &= -\frac{2}{\sqrt{\sqrt{5}+1}} \\ &= \frac{-2\sqrt{\sqrt{5}-1}}{\sqrt{(\sqrt{5}-1)(\sqrt{5}+1)}} \\ &= \frac{-2\sqrt{\sqrt{5}-1}}{\sqrt{5-1}} \\ &= \frac{-2\sqrt{\sqrt{5}-1}}{2} \\ &= -\sqrt{\sqrt{5}-1} \end{aligned}$$

and hence $z = (\sqrt{\sqrt{5}+1} - i\sqrt{\sqrt{5}-1})$

When $a = -\sqrt{\sqrt{5}+1}$,

$$b = - -\frac{2}{\sqrt{\sqrt{5}+1}}$$

$$= \frac{2\sqrt{\sqrt{5}-1}}{\sqrt{(\sqrt{5}-1)(\sqrt{5}+1)}}$$

$$= \frac{2\sqrt{\sqrt{5}-1}}{\sqrt{5-1}}$$

$$= \frac{2\sqrt{\sqrt{5}-1}}{2}$$

$$= \sqrt{\sqrt{5}-1}$$

and hence $z = -(\sqrt{\sqrt{5}+1} - i\sqrt{\sqrt{5}-1})$

16a $z^2 + (4+2i)z + (1+2i) = 0$

$\Delta = b^2 - 4ac$

$= (4+2i)^2 - 4(1)(1+2i)$

$= 16 + 16i + 4i^2 - 4 - 8i$

$= 12 + 8i + 4i^2$

$= 8 + 8i$

Let $\lambda = a + bi$ be a square root of Δ.

Using Box 13,

$a^2 - b^2 = 8$ (1)

$ab = 4$ (2)

Substituting (2) into (1):

$$a^2 - \left(\frac{4}{a}\right)^2 = 8$$

$$a^2 - \frac{16}{a^2} = 8$$

$a^4 - 16 = 8a^2$

$a^4 - 8a^2 - 16 = 0$

$$a^2 = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(1)(-16)}}{2}$$

$$a^2 = \frac{8 \pm \sqrt{128}}{2}$$

$$a^2 = \frac{8 \pm 8\sqrt{2}}{2}$$

$$a^2 = 4 \pm 4\sqrt{2}$$

Since $a^2 \geq 0$ as a is real, it follows that $a^2 = 4 + 4\sqrt{2}$. Thus

$$a = \pm\sqrt{4 + 4\sqrt{2}} = \pm 2\sqrt{1 + \sqrt{2}}$$

When $a = 2\sqrt{1 + \sqrt{2}}$,

$$\begin{aligned} b &= \frac{4}{2\sqrt{1+\sqrt{2}}} \\ &= \frac{2}{\sqrt{\sqrt{2}+1}} \\ &= \frac{2\sqrt{\sqrt{2}-1}}{\sqrt{\sqrt{2}+1}\sqrt{\sqrt{2}-1}} \\ &= \frac{2\sqrt{\sqrt{2}-1}}{\sqrt{2-1}} \\ &= 2\sqrt{\sqrt{2}-1} \end{aligned}$$

When $a = -2\sqrt{1 + \sqrt{2}}$,

$$\begin{aligned} b &= -\frac{4}{2\sqrt{1+\sqrt{2}}} \\ &= -\frac{2}{\sqrt{\sqrt{2}+1}} \\ &= -\frac{2\sqrt{\sqrt{2}-1}}{\sqrt{\sqrt{2}+1}\sqrt{\sqrt{2}-1}} \\ &= -\frac{2\sqrt{\sqrt{2}-1}}{\sqrt{2-1}} \\ &= -2\sqrt{\sqrt{2}-1} \end{aligned}$$

Hence $\lambda = \pm\left(2\sqrt{\sqrt{2}+1} - i2\sqrt{\sqrt{2}-1}\right)$

Hence the roots of the equation are

$$z = \frac{-b \pm \lambda}{2a}$$

$$z = \frac{-(4+2i) \pm \left(2\sqrt{\sqrt{2}+1} - i2\sqrt{\sqrt{2}-1}\right)}{2(1)}$$

$$z = -2 - i \pm \left(\sqrt{\sqrt{2}+1} - i\sqrt{\sqrt{2}-1}\right)$$

16b $z^2 - 2(1+i)z + (2+6i) = 0$

$\Delta = b^2 - 4ac$

$= \left(-2(1+i)\right)^2 - 4(1)(2+6i)$

$= 4(1+2i-1) - 8 - 24i$

$= 8i - 8 - 24i$

$= -8 - 16i$

Let $\lambda = a + bi$ be a square root of Δ.

Using Box 13,

$a^2 - b^2 = -8$ (1)

$ab = -8$ (2)

Substituting (2) into (1):

$$a^2 - \left(-\frac{8}{a}\right)^2 = -8$$

$$a^2 - \frac{64}{a^2} = -8$$

$a^4 - 64 = -8a^2$

$a^4 + 8a^2 - 64 = 0$

$$a^2 = \frac{-8 \pm \sqrt{(-8)^2 - 4(1)(-64)}}{2}$$

$$a^2 = \frac{-8 \pm \sqrt{320}}{2}$$

$$a^2 = \frac{-8 \pm 8\sqrt{5}}{2}$$

$$a^2 = -4 \pm 4\sqrt{5}$$

Since a is real $a^2 \geq 0$ it follows that $a^2 = -4 + 4\sqrt{5}$ thus

$$a = \pm 2\sqrt{-1 + \sqrt{5}}$$

When $a = 2\sqrt{-1 + \sqrt{5}}$,

$$\begin{aligned} b &= \frac{-8}{2\sqrt{-1+\sqrt{5}}} \\ &= -\frac{4}{\sqrt{\sqrt{5}-1}} \\ &= -\frac{4\sqrt{\sqrt{5}+1}}{\sqrt{\sqrt{5}-1}\sqrt{\sqrt{5}+1}} \\ &= -\frac{4\sqrt{\sqrt{5}+1}}{\sqrt{5-1}} \\ &= -\frac{4\sqrt{\sqrt{5}+1}}{2} \\ &= -2\sqrt{\sqrt{5}+1} \end{aligned}$$

When $a = -2\sqrt{-1 + \sqrt{5}}$,

$$\begin{aligned} b &= -\frac{-8}{2\sqrt{-1+\sqrt{5}}} \\ &= \frac{4}{\sqrt{\sqrt{5}-1}} \\ &= \frac{4\sqrt{\sqrt{5}+1}}{\sqrt{\sqrt{5}-1}\sqrt{\sqrt{5}+1}} \\ &= \frac{4\sqrt{\sqrt{5}+1}}{\sqrt{5-1}} \\ &= \frac{4\sqrt{\sqrt{5}+1}}{2} \\ &= 2\sqrt{\sqrt{5}+1} \end{aligned}$$

Thus, the roots of the equation are

$$z = \frac{-(-2(1+i)) \pm \left(2\sqrt{\sqrt{5}-1} - 2i\sqrt{\sqrt{5}+1}\right)}{2}$$

$$z = 1 + i \pm \left(\sqrt{\sqrt{5}-1} - i\sqrt{\sqrt{5}+1}\right)$$

16c $z^2 + 2(1 - i\sqrt{3})z + 2 + 2i\sqrt{3} = 0$

$\Delta = b^2 - 4ac$

$= \left(2(1 - i\sqrt{3})\right)^2 - 4(1)(2 + 2i\sqrt{3})$

$= 4(1 - 2i\sqrt{3} - 3) - 8 - 8i\sqrt{3}$

$= 4(-2i\sqrt{3} - 2) - 8 - 8i\sqrt{3}$

$= -8i\sqrt{3} - 8 - 8 - 8i\sqrt{3}$

$= -16 - 16i\sqrt{3}$

Let $\lambda = a + bi$ be a square root of Δ.

Using Box 13,

$a^2 - b^2 = -16$ (1)

$ab = -8\sqrt{3}$ (2)

Substituting (2) into (1):

$$a^2 - \left(-\frac{8\sqrt{3}}{a}\right)^2 = -16$$

$$a^2 - \frac{192}{a^2} = -16$$

$a^4 - 192 = -16a^2$

$a^4 + 16a^2 - 192 = 0$

$$a^2 = \frac{-(16) \pm \sqrt{(-16)^2 - 4(1)(-192)}}{2}$$

$$a^2 = \frac{-(16) \pm \sqrt{1024}}{2}$$

$$a^2 = \frac{-16 \pm 32}{2}$$

$a^2 = 8 \qquad \text{(since } a^2 > 0\text{)}$

$a = \pm 2\sqrt{2}$

When $a = 2\sqrt{2}$,

$$\begin{aligned} b &= -\frac{8\sqrt{3}}{2\sqrt{2}} \\ &= -\frac{4\sqrt{3}}{\sqrt{2}} \\ &= -\frac{4\sqrt{6}}{2} \\ &= -2\sqrt{6} \end{aligned}$$

When $a = -2\sqrt{2}$,

$$\begin{aligned} b &= -\left(-\frac{8\sqrt{3}}{2\sqrt{2}}\right) \\ &= \frac{4\sqrt{3}}{\sqrt{2}} \\ &= \frac{4\sqrt{6}}{2} \\ &= 2\sqrt{6} \end{aligned}$$

So $\lambda = \pm\left(2\sqrt{2} - 2\sqrt{6}i\right)$

Hence the roots of the equation are

$$\begin{aligned} z &= \frac{-2\left(1 - i\sqrt{3}\right) \pm \left(2\sqrt{2} - 2\sqrt{6}i\right)}{2(1)} \\ &= -1 + i\sqrt{3} \pm \left(\sqrt{2} - i\sqrt{6}\right) \end{aligned}$$

16d $\quad z^2 + (1 - i)z + (i - 1) = 0$

$$\begin{aligned} \Delta &= b^2 - 4ac \\ &= (1 - i)^2 - 4(1)(i - 1) \\ &= (1 - 2i - 1) - 4i + 4 \\ &= 4 - 6i \end{aligned}$$

Let $\lambda = a + bi$ be a square root of Δ.

Using Box 13,

$a^2 - b^2 = 4 \quad (1)$

$ab = -3 \qquad (2)$

Substituting (2) into (1):

$$a^2 - \left(\frac{-3}{a}\right)^2 = 4$$

$$a^2 - \frac{9}{a^2} = 4$$

$$a^4 - 9 = 4a^2$$

$$a^4 - 4a^2 - 9 = 0$$

$$a^2 = \frac{4 \pm \sqrt{4^2 - 4(1)(-9)}}{2(1)}$$

$$a^2 = \frac{4 \pm \sqrt{52}}{2}$$

$$a^2 = \frac{4 \pm 2\sqrt{13}}{2}$$

$$a^2 = 2 \pm \sqrt{13}$$

Since $a^2 \geq 0$, $a^2 = 2 + \sqrt{13}$

$$a = \pm\sqrt{\sqrt{13} + 2}$$

When $a = \sqrt{\sqrt{13} + 2}$,

$$\begin{aligned} b &= \frac{-3}{\sqrt{\sqrt{13} + 2}} \\ &= \frac{-3\sqrt{\sqrt{13} - 2}}{\sqrt{\sqrt{13} - 2}\sqrt{\sqrt{13} + 2}} \\ &= \frac{-3\sqrt{\sqrt{13} - 2}}{\sqrt{13 - 4}} \\ &= -\sqrt{\sqrt{13} - 2} \end{aligned}$$

When $a = -\sqrt{\sqrt{13} + 2}$,

$$\begin{aligned} b &= -\frac{-3}{\sqrt{\sqrt{13} - 2}} \\ &= -\frac{-3\sqrt{\sqrt{13} - 2}}{\sqrt{\sqrt{13} - 2}\sqrt{\sqrt{13} + 2}} \end{aligned}$$

$$= \frac{3\sqrt{\sqrt{13} - 2}}{\sqrt{13 - 4}}$$

$$= \sqrt{\sqrt{13} - 2}$$

So $\lambda = \pm\left(\sqrt{\sqrt{13} + 2} - i\sqrt{\sqrt{13} - 2}\right)$

Hence the roots of the equation are

$$z = \frac{-(1 - i) \pm \left(\sqrt{\sqrt{13} + 2} - i\sqrt{\sqrt{13} - 2}\right)}{2(1)}$$

$$= \frac{1}{2}\left(-(1 - i) \pm \left(\sqrt{\sqrt{13} + 2} - i\sqrt{\sqrt{13} - 2}\right)\right)$$

$$= \frac{1}{2}\left(-1 + i \pm \left(\sqrt{\sqrt{13} + 2} - i\sqrt{\sqrt{13} - 2}\right)\right)$$

Solutions to Exercise 1B Enrichment questions

17 **a** Since α is a root, $\alpha^3 - 1 = 0$

taking conjugates, $(\overline{\alpha^3}) - 1 = 0$

so $(\bar{\alpha})^3 - 1 = 0$

that is, $\bar{\alpha}$ is also a root

since there are only two complex roots, $\beta = \bar{\alpha}$

b $(\alpha\bar{\alpha})^3 = \alpha^3(\bar{\alpha})^3$

$= 1$

and $\alpha\bar{\alpha}$ is real

hence $\alpha\bar{\alpha} = 1$

$= \alpha^3$

thus $\bar{\alpha} = \alpha^2 \quad (\alpha \neq 0)$

also $\alpha\bar{\alpha} = (\bar{\alpha})^3$

$\alpha = (\bar{\alpha})^2 \quad (\bar{\alpha} \neq 0)$

c $\alpha^3 - 1 = 0$

so $(\alpha - 1)(\alpha^2 + \alpha + 1) = 0$

thus $(\alpha^2 + \alpha + 1) = 0 \quad (\alpha \neq 1)$

d Let $m \in \mathbb{N}$

When $n = 3m$, the sum is

$(1+\alpha+\alpha^2)+\alpha^3(1+\alpha+\alpha^2)+\alpha^6(1+\alpha+\alpha^2)+\cdots+\alpha^{3(m-1)}(1+\alpha+\alpha^2)$

$= 0+\alpha^3\times 0+\alpha^6\times 0+\cdots+\alpha^{3(m-1)}\times 0$

$= 0 \quad (*)$

When $n = 3m+1$, the sum is

$(1+\alpha+\alpha^2)+\alpha^3(1+\alpha+\alpha^2)+\alpha^6(1+\alpha+\alpha^2)+\cdots+\alpha^{3(m-1)}(1+\alpha+\alpha^2)+\alpha^{3m}$

$= 0+\alpha^{3m} \quad (\text{by } *)$

$= (\alpha^3)^m \quad (\alpha^3 = 1)$

$= 1$

When $n = 3m+2$, the sum is

$(1+\alpha+\alpha^2)+\alpha^3(1+\alpha+\alpha^2)+\alpha^6(1+\alpha+\alpha^2)+\cdots+\alpha^{3(m-1)}(1+\alpha+\alpha^2)+\alpha^{3m}+\alpha^{3m+1}$

$= 0+(\alpha^3)^m(1+\alpha) \quad (\text{by } *)$

$= 1+\alpha \quad (\alpha^3 = 1)$

$= -\alpha^2 \quad (1+\alpha+\alpha^2 = 0)$

18 **a** From sums and products of roots

$\alpha+\beta = -\dfrac{b}{a}$, which is real

Hence

$$\operatorname{Im}(\alpha+\beta) = 0 \qquad \text{(i)}$$

$\alpha\beta = \dfrac{c}{a}$, which is real

Hence

$$\operatorname{Im}(\alpha\beta) = 0 \qquad \text{(ii)}$$

b From **a** (i), $y + v = 0$

and $y = -v$

From **a** (ii), $uy + xv = 0$

so $uy - xy = 0$

$y(u - x) = 0$

and $y \neq 0$ because $b^2 - 4ac < 0$ (not zero)

hence $u = x$

thus $\alpha = x + iy$

$\beta = u + iv$

$= x - iy$

$= \bar{\alpha}$

19 Using the results in boxes 13 and 14:

$x^2 + y^2 = \sqrt{a^2 - b^2}$ (Box 14)

$$\left.\begin{aligned} x^2 - y^2 &= a \\ 2xy &= b \end{aligned}\right\} \text{(Box 13)}$$

adding and subtracting

$2x^2 = a + \sqrt{a^2 + b^2}$ $\quad 2y^2 = \sqrt{a^2 + b^2} - a$

$x^2 = \frac{1}{2}(a + \sqrt{a^2 + b^2})$ $\quad y^2 = \frac{1}{2}(\sqrt{a^2 + b^2} - a)$

take the positive square root (it does not matter which here)

$x = \sqrt{\frac{1}{2}(\sqrt{a^2 + b^2} + a)}$

the sign of y is then the sign of b, which is the sign of $\frac{b}{|b|}$ for $b \neq 0$

$y = \frac{b}{|b|}\sqrt{\frac{1}{2}(\sqrt{a^2 + b^2} - a)}$ so the two square roots are $x + iy$ and $-(x + iy)$, hence

$x + iy = \pm\left(\sqrt{\frac{1}{2}(\sqrt{a^2 + b^2} + a)} + i\frac{b}{|b|}\sqrt{\frac{1}{2}(\sqrt{a^2 + b^2} - a)}\right)$

Note that if $b = 0$,then $x = \pm\sqrt{a}$ $\quad (y = 0)$

Solutions to Exercise 1C Foundation questions

1a $2 = 2 + 0i$

In Cartesian form, this is $(2, 0)$.

1b $i = 0 + i$

In Cartesian form, this is $(0, 1)$.

1c $-3 + 5i$

In Cartesian form, this is $(-3, 5)$.

1d $\overline{2 + 2i} = 2 - 2i$

In Cartesian form, this is $(2, -2)$.

1e $-5(1 + i) = -5 - 5i$

In Cartesian form, this is $(-5, -5)$.

1f $(2 + i)i = 2i + i^2 = 2i - 1 = -1 + 2i$

In Cartesian form, this is $(-1, 2)$.

2a Complex number that represents $(-3, 0)$ is

$-3 + 0i = -3$

2b Complex number that represents $(0, 3)$ is

$0 + 3i = 3i$

2c Complex number that represents $(7, -5)$ is

$7 - 5i$

2d Complex number that represents (a, b) is

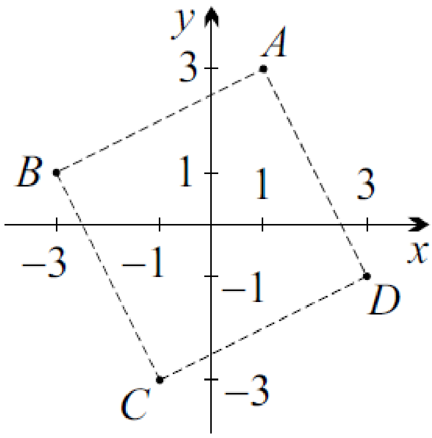
$a + bi$

3a Point A: $1 + 3i$

Point B: $i(1 + 3i) = i + 3i^2 = i - 3 = -3 + i$

Point C: $i^2(1 + 3i) = -1(1 + 3i) = -1 - 3i$

Point D: $i^3(1 + 3i) = -i(1 + 3i) = -i - 3i^2 = -i + 3 = 3 - i$



3b A square

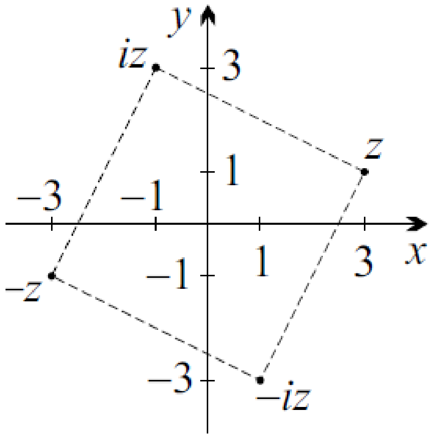
3c An anticlockwise rotation of 90° about the origin.

4a $z = 3 + i$

$iz = i(3 + i) = 3i + i^2 = 3i - 1 = -1 + 3i$

$-z = -(3 + i) = -3 - i$

$-iz = -i(3 + i) = -3i - i^2 = -3i + 1 = 1 - 3i$



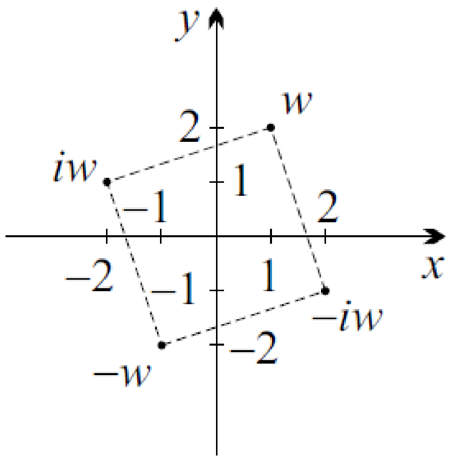
The points form a square.

4b $w = 1 + 2i$

$iw = i(1 + 2i) = i + 2i^2 = i - 2 = -2 + i$

$-w = -(1 + 2i) = -1 - 2i$

$-iw = -i(1 + 2i) = -i - 2i^2 = -i + 2 = 2 - i$



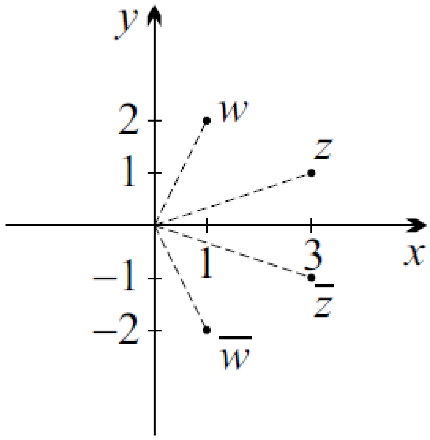
The points form a square.

4c $z = 3 + i$

$\bar{z} = \overline{3 + i} = 3 - i$

$w = 1 + 2i$

$\overline{w} = \overline{1 + 2i} = 1 - 2i$

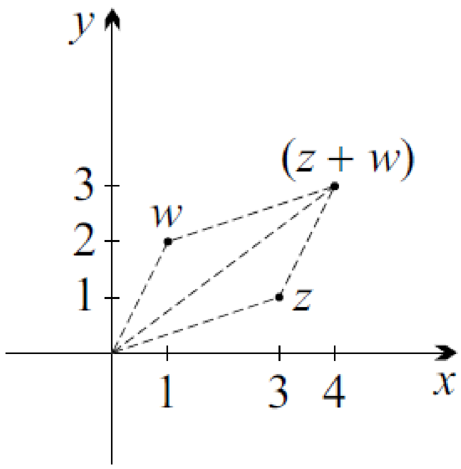


Conjugate pairs are reflections in the real axis.

4d $z = 3 + i$

$w = 1 + 2i$

$z + w = 3 + i + 1 + 2i = 4 + 3i$

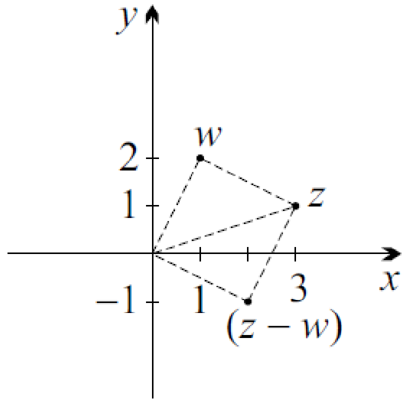


With O at the origin, the points form a parallelogram.

4e $z = 3 + i$

$w = 1 + 2i$

$z - w = 3 + i - (1 + 2i) = 2 - i$

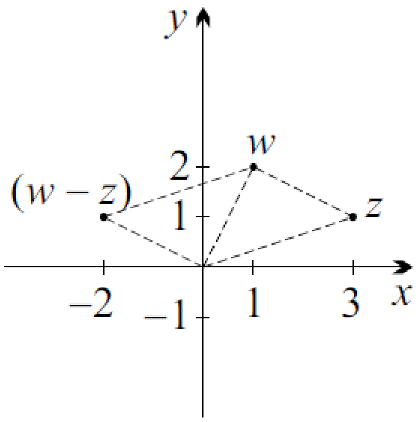


With O at the origin, the points form a parallelogram. (For this particular choice of z and w, that parallelogram happens to be a square; for other values of z and w, the points will still form a parallelogram, but not necessarily a square.)

4f $z = 3 + i$

$w = 1 + 2i$

$w - z = (1 + 2i) - (3 + i) = -2 + i$



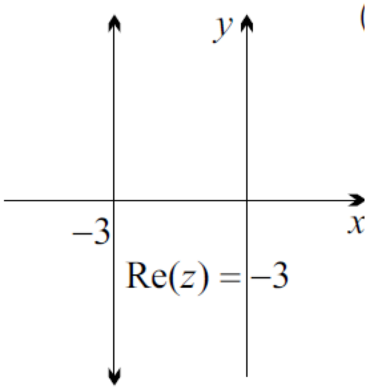
With O at the origin, the points form a parallelogram.

5 For the following section, let $z = x + iy$ where x and y are real.

5a $\mathrm{Re}(z) = -3$

$\mathrm{Re}(x + iy) = -3$

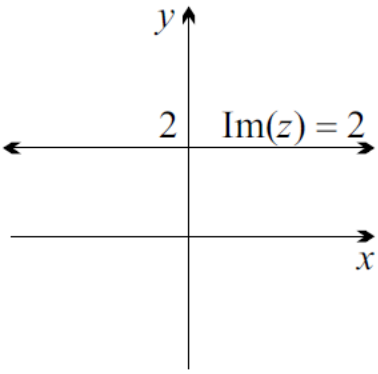
$x = -3$



5b $\text{Im}(z) = 2$

$\text{Im}(x + iy) = 2$

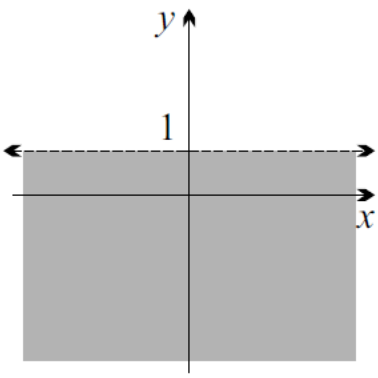
$y = 2$



5c $\text{Im}(z) < 1$

$\text{Im}(x + iy) < 1$

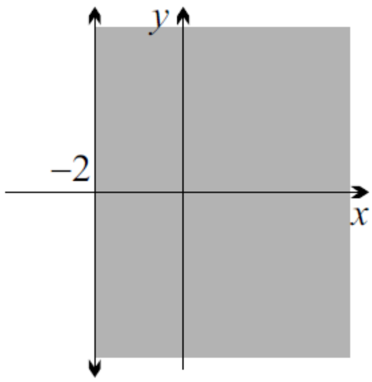
$y < 1$



5d $\mathrm{Re}(z) \geq -2$

$\mathrm{Re}(x + iy) \geq -2$

$x \geq -2$

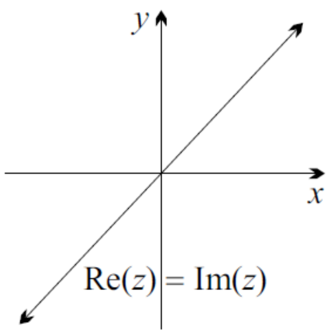


5e $\mathrm{Re}(z) = \mathrm{Im}(z)$

$\mathrm{Re}(x + iy) = \mathrm{Im}(x + iy)$

$x = y$

$y = x$

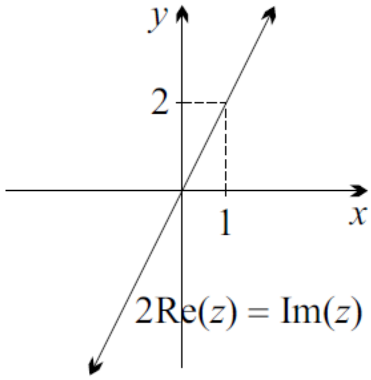


5f $2\mathrm{Re}(z) = \mathrm{Im}(z)$

$2\mathrm{Re}(x + iy) = \mathrm{Im}(x + iy)$

$2x = y$

$y = 2x$

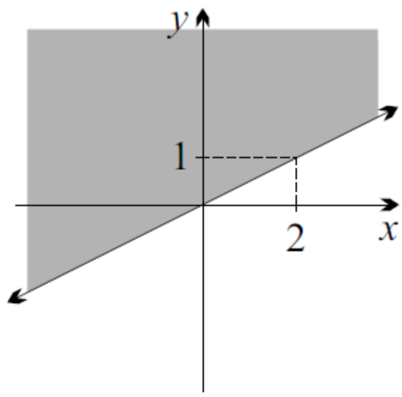


5g $\mathrm{Re}(z) \leq 2\mathrm{Im}(z)$

$\mathrm{Re}(x + iy) \leq 2\mathrm{Im}(x + iy)$

$x \leq 2y$

$y \geq \frac{1}{2}x$



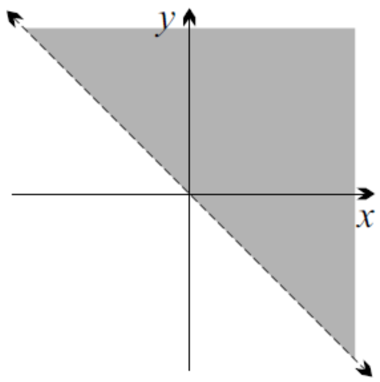
5h $\mathrm{Re}(z) > -\mathrm{Im}(z)$

$\mathrm{Re}(x + iy) > -\mathrm{Im}(x + iy)$

$x > -y$

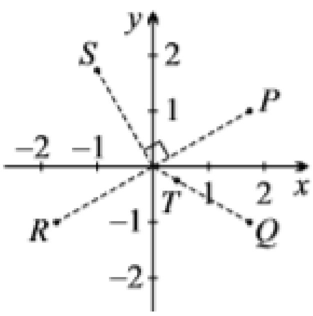
$-y < x$

$y > -x$



Solutions to Exercise 1C Development questions

6 P will be 2 units from the origin making an angle of $\frac{\pi}{6}$ with the x-axis. Taking the conjugate of a complex number makes the imaginary part negative and leaves the real part unaffected. Hence, we make the y-component of z negative. This is equivalent to reflecting P about the x-axis to obtain Q. Taking the negative of a complex number is equivalent to a 180° rotation or a reflection about the x and then y-axes, hence this gives the position of R. Multiplying by i is equivalent to a rotation by $\frac{\pi}{2}$ about the origin. Noting that $\frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{1}{|z|^2}\bar{z}$, T must have the same angle as Q. Since $|z| > 1$, $\frac{1}{|z|^2} < 1$ and hence T will be closer to the origin than Q.

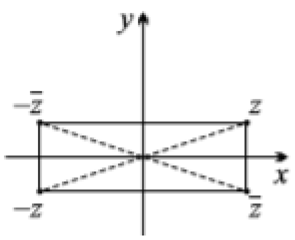


7 Let $z = x + iy$

$$\begin{aligned} -\bar{z} &= -(x - iy) \\ &= -x + iy \end{aligned}$$

Hence $-\bar{z}$ is equivalent to the transformation $(x, y) \to (-x, y)$ which is a reflection of the point representing z in the y-axis.

8a



Let $z = x + iy$ so $\bar{z} = x - iy$, $-z = -x - iy$ and $-\bar{z} = -x + iy$

Consider the diagonal between $-z$ and z. It has length

$$\sqrt{(x-(-x))^2 + \left(y-(-y)\right)^2}$$

$= \sqrt{(2x)^2 + (2y)^2}$
$= 2\sqrt{x^2 + y^2}$

The midpoint is

$$\left(\frac{-x + x}{2}, \frac{-y + y}{2}\right) = (0, 0)$$

Now, the diagonal between $-\bar{z}$ and $\bar{z}$. It has length

$$\sqrt{(x - (-x))^2 + (-y - y)^2}$$

$= \sqrt{(2x)^2 + (2y)^2}$
$= 2\sqrt{x^2 + y^2}$

The midpoint is

$$\left(\frac{-x + x}{2}, \frac{y - y}{2}\right) = (0, 0)$$

Thus, both diagonals have the same length, and as they have the same midpoint they must bisect each other. Hence the points form a rectangle.

8b Geometry of opposites: Since z and $-z$ are rotations of each other by π about the origin the midpoint of the two points must be the origin. Similarly, the midpoint of $\bar{z}$ and $-\bar{z}$ must be the origin. Thus, the two diagonals bisect one another.

Geometry of conjugates: Since z and $\bar{z}$ are reflections of one another about the real axis they must have the same length. Similarly, $-z$ and $-\bar{z}$ must have a common length. Hence it follows that both diagonals are of the same length.

Thus, the diagonals are equal and bisect one another so the points form a rectangle.

9 Note that these results follow immediately from the result in the book as z being real and z being imaginary are just special cases of z being complex.

9a Consider x where x is real, this lies on the x-axis. Multiplying by i gives ix. This lies on the y-axis. If $x > 0$ then x and ix are on the positive x- and y-axes respectively. Hence this is a rotation by $\frac{\pi}{2}$. If $x < 0$ then x and ix are on the negative x- and y-axes respectively. Hence this is a rotation by $\frac{\pi}{2}$.

9b Consider ix where x is real, this lies on the y-axis.

Multiplying by i gives $i^2x = -x$ this lies on the x-axis.

If $x > 0$ then ix and $-x$ are on the positive y and negative x-axes respectively. Hence this is a rotation by $\frac{\pi}{2}$.

If $x < 0$ then ix and $-x$ are on the negative y and positive x-axes respectively. Hence this is a rotation by $\frac{\pi}{2}$.

10a

$$m_{OA} = \frac{b-0}{a-0} = \frac{b}{a}$$

$$\begin{aligned} w &= i(a+ib) \\ &= ai - b \\ &= -b + ai \end{aligned}$$

$$\begin{aligned} m_{OB} &= \frac{a-0}{-b-0} \\ &= -\frac{a}{b} \\ &= -\frac{1}{m_{OA}} \end{aligned}$$

Hence it follows that $OA \perp OB$.

10b $OA = \sqrt{a^2 + b^2}$

$$\begin{aligned} OB &= \sqrt{a^2 + (-b)^2} \\ &= \sqrt{a^2 + b^2} \\ &= OA \end{aligned}$$

10c It has a right angle and two sides are of the same length. Hence it is right-isosceles triangle.

11

$$\frac{1}{z} - \frac{1}{\bar{z}} = i$$

Multiplying through by $z\bar{z}$ (note that $z \neq 0$),

$\bar{z} - z = iz\bar{z}$

Let $z = x + iy$.

$(x - iy) - (x + iy) = i|z|^2$

$-2iy = i|z|^2$

$-2y = |z|^2$

$-2y = (x^2 + y^2)$

$x^2 + y^2 + 2y = 0$

$x^2 + y^2 + 2y + 1 = 1$

$x^2 + (y + 1)^2 = 1$

This is the circle with centre $(0, -1)$ and radius 1 unit, omitting the origin.

12 $\mathrm{Re}\left(\frac{z-6}{z}\right) = 0$

$$\mathrm{Re}\left(1 - \frac{6}{z}\right) = 0$$

$$\mathrm{Re}\left(1 - \frac{6\bar{z}}{z\bar{z}}\right) = 0$$

$$\mathrm{Re}\left(1 - \frac{6\bar{z}}{|z|^2}\right) = 0$$

Let $z = x + iy$

$$\mathrm{Re}\left(1 - \frac{6}{|z|^2}(x - iy)\right) = 0$$

$$\mathrm{Re}\left(1 - \frac{6}{|z|^2}x + i\frac{6y}{|z|^2}\right) = 0$$

$$1 - \frac{6}{|z|^2}x = 0$$

$6x = |z|^2$ (note $z \neq 0$)

$6x = x^2 + y^2$

$x^2 - 6x + y^2 = 0$

$x^2 - 6x + 9 + y^2 = 9$

$(x - 3)^2 + y^2 = 9$

This is the circle with centre $(3, 0)$ and radius 3 units, omitting the origin.

13 $(z - 2)\overline{(z - 2)} = 9$

$(z - 2)(\bar{z} - \bar{2}) = 9$

$(z - 2)(\bar{z} - 2) = 9$

$z\bar{z} - 2z - 2\bar{z} + 4 = 9$

$|z|^2 - 2(z + \bar{z}) = 5$

Let $z = x + iy$

$x^2 + y^2 - 2(x + iy + x - iy) = 5$

$x^2 + y^2 - 4x = 5$

$x^2 - 4x + 4 + y^2 = 9$

$(x - 2)^2 + y^2 = 3$

This is the equation of a circle with centre $(2, 0)$ and radius $\sqrt{3}$ units.

14 $z\bar{z} = \big(\mathrm{Re}(z - 1 + 3i)\big)^2$

$|z|^2 = \big(\mathrm{Re}(z - 1 + 3i)\big)^2$

Let $z = x + iy$

$x^2 + y^2 = \big(\mathrm{Re}(x + iy - 1 + 3i)\big)^2$

$x^2 + y^2 = (x - 1)^2$

$x^2 + y^2 = x^2 - 2x + 1$

$2x = 1 - y^2$

$x = \dfrac{1}{2}(1 - y^2)$

This is the parabola with focus the origin and directrix $x = 1$.

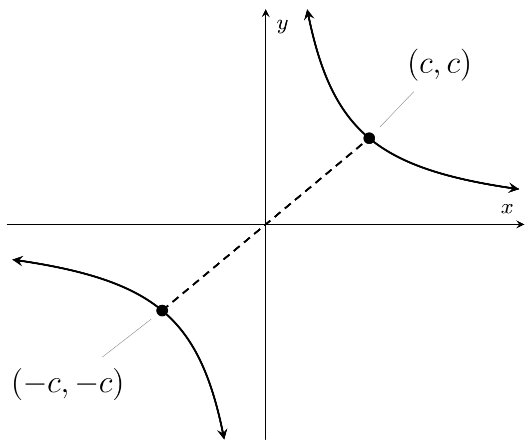
Solutions to Exercise 1C Enrichment questions

15 **a** $\text{Im}(z^2) = 2xy$

so $2xy = 2c^2$

$xy = c^2$

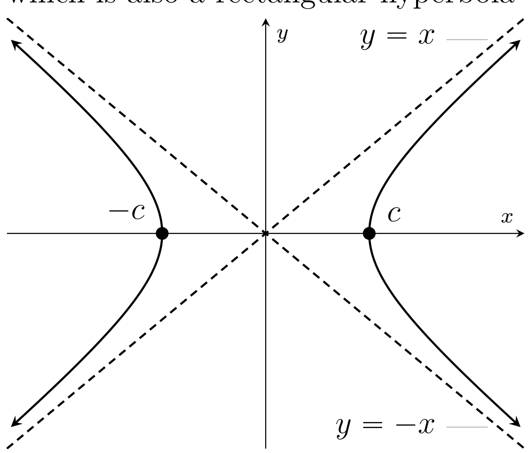
which is a rectangular hyperbola through (c, c) and $(-c, -c)$



b $\text{Re}(z^2) = x^2 - y^2$

so $x^2 - y^2 = c^2$

which is also a rectangular hyperbola with asymptotes $y = \pm x$ and x-intercepts at $\pm c$

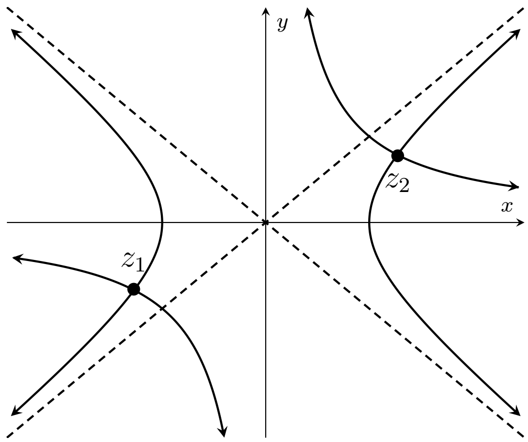


Note that if $z = x + iy$ and $z^2 = c^2 + 2ic^2$, then equating real and imaginary parts give

so $x^2 - y^2 = c^2$ (real part)

and $xy = c^2$ (imaginary part)

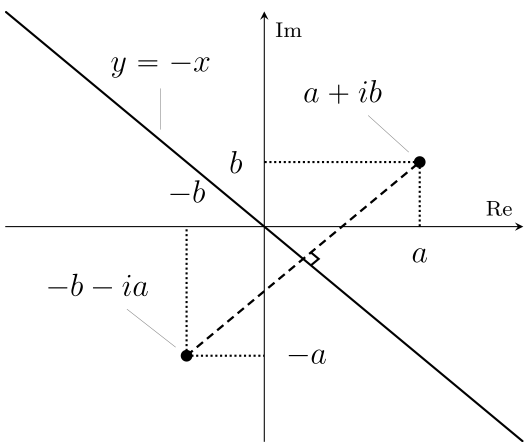
Hence the solution is the intersection of these two curves. Thus there are always two, in opposite quadrants, labelled z_1 and z_2 in the graph.



16 Let $z = a + ib$ then

$$
\begin{aligned}
-i\bar{z} &= -i(a - ib) \\
&= -ai + i^2 b \\
&= -b - ai
\end{aligned}
$$

which is the result of swapping a with $-b$ and b with $-a$. This is a reflection in $y = -x$.



17 Let $z = x + iy$

$$
\begin{aligned}
\frac{1}{z} &= \frac{\bar{z}}{z\bar{z}} \\
&= \frac{x - iy}{x^2 + y^2}
\end{aligned}
$$

The sign of the imaginary part has changed, so there is a reflection in the x-axis. Both real and imaginary parts have been multiplied by $\dfrac{1}{x^2 + y^2}$ so there is an enlargement centred on the origin, with a factor $\dfrac{1}{x^2 + y^2}$.

Solutions to Exercise 1D Foundation questions

1a $\quad z = 3 = 3 + 0i$

$$|z| = \sqrt{3^2 + 0^2}$$
$$= 3$$

1b $\quad z = -5i = 0 - 5i$

$$|z| = \sqrt{0^2 + (-5)^2}$$
$$= 5$$

1c $\quad z = 1 - i$

$$|z| = \sqrt{1^2 + (-1)^2}$$
$$= \sqrt{2}$$

1d $\quad z = -\sqrt{3} - i$

$$|z| = \sqrt{\left(-\sqrt{3}\right)^2 + (-1)^2}$$
$$= \sqrt{4}$$
$$= 2$$

1e $\quad z = -3 + 4i$

$$|z| = \sqrt{(-3)^2 + 4^2}$$
$$= \sqrt{9 + 16}$$
$$= \sqrt{25}$$
$$= 5$$

1f $z = 15 + 8i$

$|z| = \sqrt{15^2 + 8^2}$

$= \sqrt{225 + 64}$

$= \sqrt{289}$

$= 17$

2 Note that $-\pi < \text{Arg}(z) \leq \pi$.

2a $z = -2$ lies along the negative x-axis, hence $\text{Arg}(z) = \pi$

2b $z = 4i$ lies along the positive y-axis, hence $\text{Arg}(z) = \frac{\pi}{2}$

2c $z = 2 - 2i$ (fourth quadrant)

$\text{Arg}(z)$

$= \tan^{-1}\left(\frac{-2}{2}\right)$

$= \tan^{-1}(-1)$

$= -\frac{\pi}{4}$

2d $z = 1 + \sqrt{3}i$ (first quadrant)

$\text{Arg}(z)$

$= \tan^{-1}\left(\frac{\sqrt{3}}{1}\right)$

$= \tan^{-1}\sqrt{3}$

$= \frac{\pi}{3}$

2e $z = -3 + 3i$ (second quadrant)

Arg(z)

$= \tan^{-1}\left(\frac{3}{-3}\right)$

$= \tan^{-1}(-1)$

$= \pi - \frac{\pi}{4}$

$= \frac{3\pi}{4}$

2f $z = -\sqrt{3} - i$ (third quadrant)

Arg(z)

$= \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$= -\pi + \frac{\pi}{6}$

$= -\frac{5\pi}{6}$

3a For $2i = 0 + 2i$:

$r = \sqrt{0^2 + 2^2} = 2$

This lies along the positive y-axis, hence $\theta = \frac{\pi}{2}$.

The complex number is $2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$ or $2 \operatorname{cis} \frac{\pi}{2}$.

3b For $-4 = -4 + 0i$:

$r = \sqrt{(-4)^2 + 0^2} = 4$

This lies along the negative x-axis, hence $\theta = \pi$.

The complex number is $4(\cos\pi + i\sin\pi)$ or $4 \operatorname{cis} \pi$.

3c For $1 + i$ (in first quadrant):

$$r = \sqrt{1^2 + 1^2}$$

$$= \sqrt{2}$$

$$\theta = \tan^{-1}\left(\frac{1}{1}\right)$$

$$= \tan^{-1} 1$$

$$= \frac{\pi}{4}$$

The complex number is $\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$ or $\sqrt{2}\text{ cis }\frac{\pi}{4}$.

3d For $\sqrt{3} - i$ (in fourth quadrant):

$$r = \sqrt{\left(\sqrt{3}\right)^2 + (-1)^2}$$

$$= \sqrt{3 + 1}$$

$$= \sqrt{4}$$

$$= 2$$

$$\theta = \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$$

$$= -\frac{\pi}{6}$$

The complex number is $2\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$ or $2\text{ cis }\left(-\frac{\pi}{6}\right)$.

3e For $-1+\sqrt{3}i$ (in second quadrant):

$$r = \sqrt{(-1)^2 + \left(\sqrt{3}\right)^2}$$

$$= \sqrt{1+3}$$

$$= \sqrt{4}$$

$$= 2$$

$$\theta = \tan^{-1}\left(\frac{\sqrt{3}}{-1}\right)$$

$$= \tan^{-1}\left(-\sqrt{3}\right)$$

$$= \pi - \frac{\pi}{3}$$

$$= \frac{2\pi}{3}$$

The complex number is $2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$ or $2\,\text{cis}\,\frac{2\pi}{3}$.

3f For $-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i$ (in third quadrant):

$$r = \sqrt{\left(-\frac{1}{\sqrt{2}}\right)^2 + \left(-\frac{1}{\sqrt{2}}\right)^2}$$

$$= \sqrt{\frac{1}{2}+\frac{1}{2}}$$

$$= \sqrt{1}$$

$$= 1$$

$$\theta = \tan^{-1}\left(\frac{-\frac{1}{\sqrt{2}}}{-\frac{1}{\sqrt{2}}}\right)$$

$$= -\pi + \tan^{-1} 1$$

$$= -\pi + \frac{\pi}{4}$$

$$= -\frac{3\pi}{4}$$

The complex number is $\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)$ or $\operatorname{cis}\left(-\frac{3\pi}{4}\right)$.

4a For $3 + 4i$ (in first quadrant):

$$r = \sqrt{3^2 + 4^2}$$

$$= \sqrt{25}$$

$$= 5$$

$$\theta = \tan^{-1}\frac{4}{3}$$

$$\doteqdot 0.93$$

The complex number is $5(\cos(0.93) + i\sin(0.93))$ or $5 \operatorname{cis}(0.93)$.

4b For $12 - 5i$ (in fourth quadrant):

$$r = \sqrt{12^2 + (-5)^2}$$
$$= \sqrt{169}$$
$$= 13$$

$$\theta = \tan^{-1}\left(-\frac{5}{12}\right)$$
$$= -\tan^{-1}\left(\frac{5}{12}\right)$$
$$\doteqdot -0.39$$

The complex number is $13(\cos(-0.39) + i\sin(-0.39))$ or $13 \text{ cis } (-0.39)$.

4c For $-2 + i$ (in second quadrant):

$$r = \sqrt{2^2 + (-1)^2}$$
$$= \sqrt{5}$$

$$\theta = \tan^{-1}\left(-\frac{1}{2}\right)$$
$$= \pi - \tan^{-1}\left(\frac{1}{2}\right)$$
$$\doteqdot 2.68$$

The complex number is $\sqrt{5}(\cos(2.68) + i\sin(2.68))$ or $\sqrt{5} \text{ cis } (2.68)$.

4d For $-1-3i$ (in third quadrant):

$r=\sqrt{(-1)^2+(-3)^2}$

$=\sqrt{1+9}$

$=\sqrt{10}$

$\theta=\tan^{-1}\left(\frac{-3}{-1}\right)$

$=-\pi+\tan^{-1}3$

$\doteqdot -1.89$

The complex number is $\sqrt{10}(\cos(-1.89)+i\sin(-1.89))$ or $\sqrt{10}\operatorname{cis}(-1.89)$.

5a $3\operatorname{cis}0$

$=3(\cos 0+i\sin 0)$

$=3(1+0i)$

$=3$

5b $5\operatorname{cis}\left(-\frac{\pi}{2}\right)$

$=5\left(\cos\left(-\frac{\pi}{2}\right)+i\sin\left(-\frac{\pi}{2}\right)\right)$

$=5(0+(-1)i)$

$=-5i$

5c $4\operatorname{cis}\frac{\pi}{4}$

$=4\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$

$=4\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)$

$=\frac{4}{\sqrt{2}}+\frac{4}{\sqrt{2}}i$

$=2\sqrt{2}+2\sqrt{2}i$

5d $6 \text{ cis}\left(-\frac{\pi}{6}\right)$

$$= 6\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$$

$$= 6\left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)$$

$$= 3\sqrt{3} - 3i$$

5e $2 \text{ cis } \frac{3\pi}{4}$

$$= 2\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$

$$= 2\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)$$

$$= -\frac{2}{\sqrt{2}} + \frac{2}{\sqrt{2}}i$$

$$= -\sqrt{2} + \sqrt{2}i$$

5f $2 \text{ cis}\left(-\frac{2\pi}{3}\right)$

$$= 2\left(\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right)$$

$$= 2\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$$

$$= -1 - \sqrt{3}i$$

6a $z = 1 - i$ (in fourth quadrant):

$r = \sqrt{(1)^2 + (-1)^2}$

$= \sqrt{1 + 1}$

$= \sqrt{2}$

$\theta = \tan^{-1}\left(\frac{-1}{1}\right)$

$= \tan^{-1}(-1)$

$= -\frac{\pi}{4}$

$z = \sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)$

6b $\bar{z} = 1 + i$ (in first quadrant):

$r = \sqrt{(1)^2 + (1)^2}$

$= \sqrt{1 + 1}$

$= \sqrt{2}$

$\theta = \tan^{-1}\left(\frac{1}{1}\right)$

$= \tan^{-1} 1$

$= \frac{\pi}{4}$

$\bar{z} = \sqrt{2}\,\text{cis}\,\frac{\pi}{4}$

Alternatively:

$\bar{z} = \overline{\sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)}$

$= \sqrt{2}\,\text{cis}\left(-\left(-\frac{\pi}{4}\right)\right)$

$= \sqrt{2}\,\text{cis}\,\frac{\pi}{4}$

6c $-z = -1 + i$ (in second quadrant):

$$r = \sqrt{(-1)^2 + (1)^2}$$

$$= \sqrt{1 + 1}$$

$$= \sqrt{2}$$

$$\theta = \tan^{-1}\left(\frac{1}{-1}\right)$$

$$= \pi - \frac{\pi}{4}$$

$$= \frac{3\pi}{4}$$

$$-z = \sqrt{2}\ \mathrm{cis}\ \frac{3\pi}{4}$$

6d $iz = i(1 - i) = 1 + i$ (in first quadrant)

$$r = \sqrt{(1)^2 + (1)^2}$$

$$= \sqrt{1 + 1}$$

$$= \sqrt{2}$$

$$\theta = \tan^{-1}\left(\frac{1}{1}\right)$$

$$= \frac{\pi}{4}$$

$$iz = \sqrt{2}\ \mathrm{cis}\ \frac{\pi}{4}$$

6e $z^2 = (1 - i)^2$

$= 1 - 2i + i^2$

$= -2i$

$r = \sqrt{(0)^2 + (-2)^2}$

$= \sqrt{4}$

$= 2$

$\theta = -\frac{\pi}{2}$ since $-2i$ lies on the negative y-axis

$z^2 = 2 \text{ cis } \left(-\frac{\pi}{2}\right)$

6f $(\bar{z})^{-1} = (1+i)^{-1}$

$$= \frac{1}{1+i}$$

$$= \frac{1}{1+i} \times \frac{1-i}{1-i}$$

$$= \frac{1-i}{1-i^2}$$

$$= \frac{1-i}{2} \qquad \text{(in fourth quadrant)}$$

$$r = \sqrt{\left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2}$$

$$= \sqrt{\frac{1}{4} + \frac{1}{4}}$$

$$= \sqrt{\frac{1}{2}}$$

$$= \frac{1}{\sqrt{2}}$$

$$\theta = \tan^{-1}\left(\frac{-\frac{1}{2}}{\frac{1}{2}}\right)$$

$$\theta = \tan^{-1}(-1)$$

$$= -\frac{\pi}{4}$$

$$(\bar{z})^{-1} = \frac{1}{\sqrt{2}} \operatorname{cis}\left(-\frac{\pi}{4}\right)$$

7a

$$5 \operatorname{cis} \frac{\pi}{12} \times 2 \operatorname{cis} \frac{\pi}{4}$$

$$= (5 \times 2) \operatorname{cis}\left(\frac{\pi}{12} + \frac{\pi}{4}\right)$$

$$= 10 \operatorname{cis} \frac{\pi}{3}$$

7b $3 \text{ cis } \theta \times 3 \text{ cis } 2\theta$

$= (3 \times 3) \text{ cis}(\theta + 2\theta)$

$= 9 \text{ cis } 3\theta$

7c

$6 \text{ cis } \dfrac{\pi}{2} \div 3 \text{ cis } \dfrac{\pi}{6}$

$= (6 \div 3) \text{ cis}\left(\dfrac{\pi}{2} - \dfrac{\pi}{6}\right)$

$= 2 \text{ cis } \dfrac{\pi}{3}$

7d

$\dfrac{3 \text{ cis } 5\alpha}{2 \text{ cis } 4\alpha}$

$= \dfrac{3}{2} \text{ cis}(5\alpha - 4\alpha)$

$= \dfrac{3}{2} \text{cis } \alpha$

7e

$\left(4 \text{ cis } \dfrac{\pi}{5}\right)^2$

$= 4^2 \text{cis}\left(\dfrac{\pi}{5} \times 2\right)$

$= 16 \text{ cis } \dfrac{2\pi}{5}$

7f

$\left(2 \text{ cis } \dfrac{2\pi}{7}\right)^3$

$= 2^3 \text{ cis}\left(3 \times \dfrac{2\pi}{7}\right)$

$= 8 \text{ cis } \dfrac{6\pi}{7}$

8a $|z - w|$

$= |1 + 3i - (-1 + i)|$

$= |2 + 2i|$

$= \sqrt{2^2 + 2^2}$

$= \sqrt{4 + 4}$

$= 2\sqrt{2}$

8b $|z - w|$

$= |(1 - i) - (4 + 2i)|$

$= |-3 - 3i|$

$= \sqrt{(-3)^2 + (-3)^2}$

$= \sqrt{9 + 9}$

$= 3\sqrt{2}$

8c $|z - w|$

$= \left|\left(4 - 2i\sqrt{3}\right) - \left(1 + i\sqrt{3}\right)\right|$

$= \left|3 - 3i\sqrt{3}\right|$

$= \sqrt{3^2 + \left(3\sqrt{3}\right)^2}$

$= \sqrt{36}$

$= 6$

8d $|z - w|$

$= \left|\left(3 + 3i\sqrt{3}\right) - \left(-3 + i\sqrt{3}\right)\right|$

$= \left|6 + 2i\sqrt{3}\right|$

$= \sqrt{6^2 + \left(2\sqrt{3}\right)^2}$

$= \sqrt{48}$

$= 4\sqrt{3}$

8e $|z - w|$

$= |(2 + i) - (-1 - 3i)|$

$= |3 + 4i|$

$= \sqrt{3^2 + 4^2}$

$= \sqrt{25}$

$= 5$

8f $|z - w|$

$= |(-2 - i) - (-1 + i)|$

$= |-1 - 2i|$

$= \sqrt{1^2 + 2^2}$

$= \sqrt{5}$

9a $\text{Arg}(z - w)$

$= \text{Arg}(2 + 2i)$ (in first quadrant)

$= \tan^{-1}\left(\frac{2}{2}\right)$

$= \frac{\pi}{4}$

9b $\text{Arg}(z - w)$

$= \text{Arg}(-3 - 3i)$ (in third quadrant)

$= \tan^{-1}\left(\frac{-3}{-3}\right)$

$= -\pi + \tan^{-1} 1$

$= -\pi + \frac{\pi}{4}$

$= -\frac{3\pi}{4}$

9c $\operatorname{Arg}(z - w)$

$= \operatorname{Arg}(3 - 3i\sqrt{3})$ (in fourth quadrant)

$= \tan^{-1}\left(\frac{-3\sqrt{3}}{3}\right)$

$= \tan^{-1}(-\sqrt{3})$

$= -\frac{\pi}{3}$

9d $\operatorname{Arg}(z - w)$

$= \operatorname{Arg}(6 + 2i\sqrt{3})$ (in first quadrant)

$= \tan^{-1}\left(\frac{2\sqrt{3}}{6}\right)$

$= \tan^{-1}\left(\frac{\sqrt{3}}{3}\right)$

$= \frac{\pi}{6}$

9e $\operatorname{Arg}(z - w)$

$= \operatorname{Arg}(3 + 4i)$ (in first quadrant)

$= \tan^{-1}\left(\frac{4}{3}\right)$

$\doteqdot 0.93$

9f $\operatorname{Arg}(z - w)$

$= \operatorname{Arg}(-1 - 2i)$ (in third quadrant)

$= \tan^{-1}\left(\frac{-2}{-1}\right)$

$= -\pi + \tan^{-1} 2$

$\doteqdot -2.03$

10 Multiplying a complex number by w produces a rotation of θ radians about the origin, so $w = \text{cis}\,\theta$.

10a

$$\begin{aligned} w &= \text{cis}\,\frac{\pi}{2} \\ &= \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} \\ &= 0 + i \times 1 \\ &= i \end{aligned}$$

10b

$$\begin{aligned} w &= \text{cis}\,\pi \\ &= \cos\pi + i\sin\pi \\ &= -1 + i \times 0 \\ &= -1 \end{aligned}$$

10c

$$\begin{aligned} w &= \text{cis}\,\frac{\pi}{3} \\ &= \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} \\ &= \frac{1}{2} + i \times \frac{\sqrt{3}}{2} \\ &= \frac{1}{2}\left(1 + i\sqrt{3}\right) \end{aligned}$$

10d

$$w = \operatorname{cis} \frac{3\pi}{4}$$

$$= \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}$$

$$= -\frac{1}{\sqrt{2}} + i \times \frac{1}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}}(-1 + i)$$

10e

$$w = \operatorname{cis} \frac{5\pi}{6}$$

$$= \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}$$

$$= -\frac{\sqrt{3}}{2} + i \times \frac{1}{2}$$

$$= \frac{1}{2}\left(-\sqrt{3} + i\right)$$

10f

$$w = \operatorname{cis}\left(-\frac{\pi}{2}\right)$$

$$= \cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)$$

$$= 0 + i \times -1$$

$$= -i$$

10g

$$w = \operatorname{cis}\left(-\frac{\pi}{4}\right)$$

$$= \cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)$$

$$= \frac{1}{\sqrt{2}} + i \times -\frac{1}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}}(1 - i)$$

10h

$$w = \operatorname{cis}\left(-\frac{2\pi}{3}\right)$$

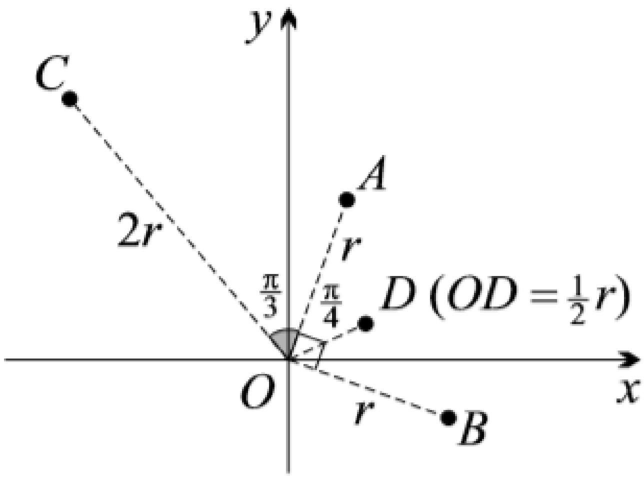
$$= \cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)$$

$$= -\frac{1}{2} + i \times -\frac{\sqrt{3}}{2}$$

$$= -\frac{1}{2}(1 + i\sqrt{3})$$

Solutions to Exercise 1D Development questions

11 Since $0 < \arg z < \frac{\pi}{2}$, we know that A must be in the first quadrant. Now, multiplication by i represents a $\frac{\pi}{2}$ anticlockwise rotation around the origin and so multiplication by $-i$ represents a $\frac{\pi}{2}$ clockwise rotation. Thus, $B$is found by rotating A $\frac{\pi}{2}$ radians around the origin. Multiplying by $2 \text{ cis}\left(\frac{\pi}{3}\right) z$ doubles the distance of the point from the origin and rotates it by $\frac{\pi}{3}$ radians. Multiplying by $\frac{1}{2} \text{ cis}\left(-\frac{\pi}{4}\right)$ halves the distance from the origin and rotates the point $\frac{\pi}{4}$ radians clockwise.



12a From Box 22,

$|wz| = |w||z|$

Replacing z with $z \div w$,

$|w(z \div w)| = |w||z \div w|$

$\left|w\frac{z}{w}\right| = |w|\left|\frac{z}{w}\right|$

$|z| = |w|\left|\frac{z}{w}\right|$

$\frac{|z|}{|w|} = \left|\frac{z}{w}\right|$

12b From Box 22,

$\arg(zw) = \arg(w) + \arg(z)$

Replacing z with $z \div w$,

$\arg\big((z \div w)w\big) = \arg(z \div w) + \arg(w)$

$\arg\left(\frac{\mathrm{z}}{\mathrm{w}}w\right) = \arg\left(\frac{z}{w}\right) + \arg(w)$

$\arg(\mathrm{z}) = \arg\left(\frac{z}{w}\right) + \arg(w)$

Hence $\arg\left(\frac{z}{w}\right) = \arg(\mathrm{z}) - \arg(w)$.

13a $z_1 = \sqrt{\left(\sqrt{3}\right)^2 + (1)^2}\,\mathrm{cis}\,\left(\tan^{-1}\frac{1}{\sqrt{3}}\right)$ (first quadrant)

$= \sqrt{4}\,\mathrm{cis}\left(\tan^{-1}\frac{1}{\sqrt{3}}\right)$

$= 2\,\mathrm{cis}\frac{\pi}{6}$

$z_2 = \sqrt{\left(2\sqrt{2}\right)^2 + \left(2\sqrt{2}\right)^2}\,\mathrm{cis}\left(\tan^{-1}\frac{2\sqrt{2}}{2\sqrt{2}}\right)$ (first quadrant)

$= \sqrt{16}\,\mathrm{cis}\,(\tan^{-1} 1)$

$= 4\,\mathrm{cis}\frac{\pi}{4}$

13b $z_1 z_2 = 2\,\mathrm{cis}\frac{\pi}{6} \times 4\,\mathrm{cis}\frac{\pi}{4}$

$= (2 \times 4)\,\mathrm{cis}\left(\frac{\pi}{6} + \frac{\pi}{4}\right)$

$= 8\,\mathrm{cis}\left(\frac{\pi}{6} + \frac{\pi}{4}\right)$

$= 8\,\mathrm{cis}\frac{5\pi}{12}$

$\frac{z_2}{z_1} = \frac{4\,\mathrm{cis}\frac{\pi}{4}}{2\,\mathrm{cis}\frac{\pi}{6}}$

$= \frac{4}{2}\,\mathrm{cis}\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$

$= 2\,\mathrm{cis}\frac{\pi}{12}$

14 $z_1 = \sqrt{\left(-\sqrt{3}\right) + (1)^2}\ \text{cis}\left(\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)\right)$ (second quadrant)

$$= \sqrt{4}\ \text{cis}\left(\pi - \frac{\pi}{6}\right)$$

$$= 2\ \text{cis}\frac{5\pi}{6}$$

$z_2 = \sqrt{(-1)^2 + (-1)^2}\ \text{cis}\left(\tan^{-1}\left(\frac{-1}{-1}\right)\right)$ (third quadrant)

$$= \sqrt{2}\text{cis}\left(-\frac{3\pi}{4}\right)$$

$$z_1 z_2 = 2\ \text{cis}\frac{5\pi}{6} \times \sqrt{2}\text{cis}\left(-\frac{3\pi}{4}\right)$$

$$= 2\sqrt{2}\ \text{cis}\left(\frac{5\pi}{6} - \frac{3\pi}{4}\right)$$

$$= 2\sqrt{2}\ \text{cis}\frac{\pi}{12}$$

$$\frac{z_2}{z_1} = \frac{\sqrt{2}\ \text{cis}\left(-\frac{3\pi}{4}\right)}{2\ \text{cis}\left(\frac{5\pi}{6}\right)}$$

$$= \frac{\sqrt{2}}{2}\ \text{cis}\ \left(-\frac{3\pi}{4} - \frac{5\pi}{6}\right)$$

$$= \frac{\sqrt{2}}{2}\ \text{cis}\ \left(-\frac{19\pi}{12}\right)$$

$$= \frac{\sqrt{2}}{2}\ \text{cis}\ \left(2\pi - \frac{19\pi}{12}\right)$$

$$= \frac{\sqrt{2}}{2}\ \text{cis}\ \left(\frac{24\pi - 19\pi}{12}\right)$$

$$= \frac{\sqrt{2}}{2}\ \text{cis}\frac{5\pi}{12}$$

15a

$$\frac{\left(1 + i\sqrt{3}\right)}{1 + i}$$

$$= \frac{\left(1 + i\sqrt{3}\right)(1 - i)}{(1 + i)(1 - i)}$$

$$= \frac{1 - i + i\sqrt{3} - i^2\sqrt{3}}{(1+i)(1-i)}$$

$$= \frac{1 - i + i\sqrt{3} + \sqrt{3}}{1^2 - i^2}$$

$$= \frac{\left(\sqrt{3}+1\right) + i\left(\sqrt{3}-1\right)}{2}$$

$$= \frac{1}{2}\left(\left(\sqrt{3}+1\right) + i\left(\sqrt{3}-1\right)\right)$$

15b

$$\frac{1 + i\sqrt{3}}{1+i}$$

$$= \frac{2 \operatorname{cis}\frac{\pi}{3}}{\sqrt{2} \operatorname{cis}\frac{\pi}{4}}$$

$$= \frac{2}{\sqrt{2}} \operatorname{cis}\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$$

$$= \sqrt{2} \operatorname{cis}\frac{\pi}{12}$$

15c $\sqrt{2} \operatorname{cis}\frac{\pi}{12}$

$$= \frac{1 + i\sqrt{3}}{1+i}$$

$$= \frac{1}{2}\left(\left(\sqrt{3}+1\right) + i\left(\sqrt{3}-1\right)\right)$$

Hence $\sqrt{2}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right) = \frac{1}{2}\left(\left(\sqrt{3}+1\right) + i\left(\sqrt{3}-1\right)\right)$

Equating the real parts in the above equation,

$$\sqrt{2}\cos\frac{\pi}{12} = \frac{1}{2}\left(\sqrt{3}+1\right)$$

Hence

$$\cos\frac{\pi}{12} = \frac{1}{2\sqrt{2}}\left(\sqrt{3}+1\right)$$

16a $|z| = \left|\frac{z_1}{z_2}\right|$

$$= \frac{|z_1|}{|z_2|}$$

$$= \frac{\sqrt{1+5^2}}{\sqrt{3^2+2^2}}$$

$$= \frac{\sqrt{1+25}}{\sqrt{9+4}}$$

$$= \frac{\sqrt{26}}{\sqrt{13}}$$

$$= \sqrt{2}$$

16b $\tan\left(\tan^{-1} 5 - \tan^{-1}\frac{2}{3}\right)$

$$= \frac{\tan(\tan^{-1} 5) - \tan\left(\tan^{-1}\frac{2}{3}\right)}{1 + \tan(\tan^{-1} 5)\tan\left(\tan^{-1}\frac{2}{3}\right)}$$

$$= \frac{5 - \frac{2}{3}}{1 + 5\left(\frac{2}{3}\right)}$$

$$= \frac{\frac{13}{3}}{\frac{13}{3}}$$

$$= 1$$

$\arg z = \arg z_1 - \arg z_2$

$$= \tan^{-1} 5 - \tan^{-1}\frac{2}{3}$$

$\tan(\arg z)$

$$= \tan\left(\tan^{-1} 5 - \tan^{-1}\frac{2}{3}\right)$$

$$= 1$$

$\arg z$

$$= \tan^{-1} 1$$

$$= \frac{\pi}{4}$$

16c

$$
\begin{aligned}
z &= \sqrt{2}\operatorname{cis}\frac{\pi}{4} \\
&= \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \\
&= \sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) \\
&= 1 + i
\end{aligned}
$$

17a
$$
\begin{aligned}
z\bar{z} &= (r\operatorname{cis}\theta)\big(r\operatorname{cis}(-\theta)\big) \\
&= r^2\operatorname{cis}(\theta-\theta) \\
&= r^2\operatorname{cis}0 \\
&= r^2 \\
&= |z|^2
\end{aligned}
$$

17b $\arg(z^2)$

$$
\begin{aligned}
&= \arg((r\operatorname{cis}\theta)^2) \\
&= \arg\big(r^2\operatorname{cis}(\theta+\theta)\big) \\
&= \arg(r^2\operatorname{cis}2\theta) \\
&= 2\theta \\
&= 2\arg(r\operatorname{cis}\theta) \\
&= 2\arg(z)
\end{aligned}
$$

17c $|z| = 1$

$z\bar{z} = 1$

$\bar{z} = \dfrac{1}{z}$ (as z is non zero)

$\bar{z} = z^{-1}$

18a Note that $|z|^2$ is a real number and as such its argument must be zero, that is

$\arg(|z|^2) = 0$

$\arg z\bar{z} = 0$

$\arg z + \arg\bar{z} = 0$

$\arg\bar{z} = -\arg z$

19a, b $z^2 = (\cos\theta + i\sin\theta)^2$
$= \cos^2\theta + 2i\sin\theta\cos\theta + i^2\sin^2\theta$
$= \cos^2\theta + 2i\sin\theta\cos\theta - \sin^2\theta$
$= \cos^2\theta - \sin^2\theta + 2i\sin\theta\cos\theta$

$z^2 = (\cos\theta + i\sin\theta)^2$
$= (\operatorname{cis}\theta)^2$
$= \operatorname{cis} 2\theta$
$= \cos 2\theta + i\sin 2\theta$

Thus $\cos^2\theta - \sin^2\theta + 2i\sin\theta\cos\theta = \cos 2\theta + i\sin 2\theta$

Equating the real and imaginary coefficients in the above equation,

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$\sin 2\theta = 2\sin\theta\cos\theta$

20a $|z-1| = 1$

Let $z = x + iy$.

$|x + iy - 1| = 1$

$|x - 1 + iy| = 1$

$|x - 1 + iy|^2 = 1$

$(x-1)^2 + y^2 = 1$

$x^2 - 2x + 1 + y^2 = 1$

$x^2 + y^2 = 2x$

$|z|^2 = 2x$

$|z|^2 = 2\mathrm{Re}(z)$

20b $|z-1| = 1$

$|z-1|^2 = 1$

$(z-1)\overline{z-1} = 1$

$(z-1)(\bar{z} - \bar{1}) = 1$

$(z-1)(\bar{z} - 1) = 1$

$z\bar{z} - z - \bar{z} + 1 = 1$

$z\bar{z} - z - \bar{z} = 0$

$z\bar{z} = z + \bar{z}$

$|z|^2 = z + \bar{z}$

$|z|^2 = x + iy + x - iy$

$|z|^2 = 2x$

$|z|^2 = 2\mathrm{Re}(z)$

21a $|2z - 1| = |z - 2|$

Let $z = x + iy$.

$|2(x + iy) - 1| = |(x + iy) - 2|$

$|(2x - 1) + i2y| = |(x - 2) + iy|$

$|(2x - 1) + i2y|^2 = |(x - 2) + iy|^2$

$(2x - 1)^2 + 4y^2 = (x - 2)^2 + y^2$

$(2x - 1)^2 + 3y^2 = (x - 2)^2$

$4x^2 - 4x + 1 + 3y^2 = x^2 - 4x + 4$

$3x^2 + 3y^2 = 3$

$x^2 + y^2 = 1$

$|z|^2 = 1$

21b $|2z - 1|^2 = |z - 2|^2$

$(2z - 1)\overline{2z - 1} = (z - 2)\overline{z - 2}$

$(2z - 1)(2\bar{z} - 1) = (z - 2)(\bar{z} - 2)$

$4z\bar{z} - 2z - 2\bar{z} + 1 = z\bar{z} - 2z - 2\bar{z} + 4$

$3z\bar{z} = 3$

$z\bar{z} = 1$

$|z|^2 = 1$

22a $z = 1 + \cos\theta + i\sin\theta$

$$\begin{aligned}
|z| &= |1 + \cos\theta + i\sin\theta| \\
&= \sqrt{(1 + \cos\theta)^2 + (\sin\theta)^2} \\
&= \sqrt{1 + 2\cos\theta + \cos^2\theta + \sin^2\theta} \\
&= \sqrt{1 + 2\cos\theta + 1} \\
&= \sqrt{2 + 2\cos\theta} \\
&= \sqrt{2 + 2\cos\left(2\frac{\theta}{2}\right)} \\
&= \sqrt{2 + 2\left(2\cos^2\frac{\theta}{2} - 1\right)} \\
&= \sqrt{4\cos^2\frac{\theta}{2}} \\
&= 2\cos\frac{\theta}{2}
\end{aligned}$$

$\arg z$

$$\begin{aligned}
&= \arg(1 + \cos\theta + i\sin\theta) \\
&= \tan^{-1}\left(\frac{\sin\theta}{1 + \cos\theta}\right) \\
&= \tan^{-1}\left(\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{1 + \left(2\cos^2\frac{\theta}{2} - 1\right)}\right) \\
&= \tan^{-1}\left(\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}\right) \\
&= \tan^{-1}\left(\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\right) \\
&= \tan^{-1}\left(\tan\frac{\theta}{2}\right) \\
&= \frac{\theta}{2}
\end{aligned}$$

22b

$$z = 2\cos\frac{\theta}{2}\operatorname{cis}\frac{\theta}{2}$$

$$\begin{aligned}
z^{-1} &= \frac{1}{z} \\
&= \frac{1}{2\cos\frac{\theta}{2}\operatorname{cis}\frac{\theta}{2}} \\
&= \frac{1}{2\cos\frac{\theta}{2}}\operatorname{cis}\left(-\frac{\theta}{2}\right) \\
&= \frac{1}{2\cos\frac{\theta}{2}}\left(\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}\right) \\
&= \frac{1}{2}\left(1 - i\tan\frac{\theta}{2}\right) \\
&= \frac{1}{2} - \frac{1}{2}i\tan\frac{\theta}{2}
\end{aligned}$$

Solutions to Exercise 1D Enrichment questions

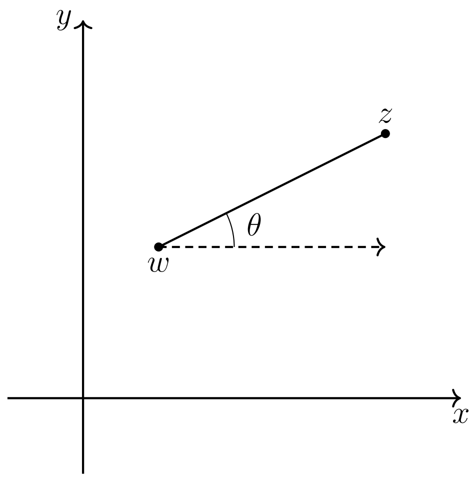
23 **a** $|z-w|^2 = (z-w)\overline{(z-w)}$

$$= ((x_2-x_1)+i(y_2-y_1)) \times ((x_2-x_1)-i(y_2-y_1))$$

$$= (x_2-x_1)^2 + (y_2-y_1)^2$$

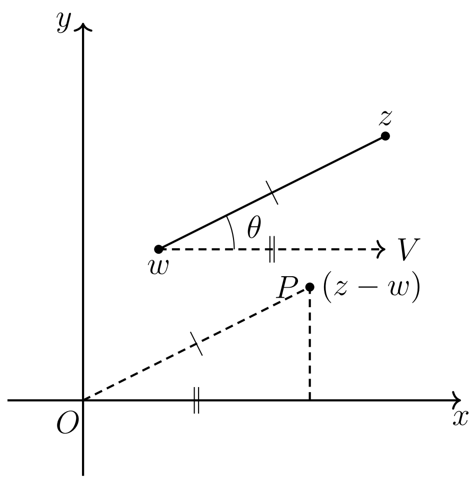
$$= (WZ)^2 \quad \text{from coordinate geometry}$$

So $|z-w| = |WZ|$



b By **a**, $|WZ| = |OP|$ and by shifting $WZ \parallel OP$ and $WV \parallel OX$ so $\angle ZWV = \angle POX$ i.e.

$\theta = \arg(z-w)$



24 $z = \text{cis}\ \theta$

$w = \text{cis}\ \phi$

$$\begin{aligned} z + w &= (\cos\theta + \cos\phi) + i(\sin\theta + \sin\phi) \\ &= 2\cos\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right) + i2\sin\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right) \\ &= 2\cos\left(\frac{\theta-\phi}{2}\right)\left(\cos\left(\frac{\theta+\phi}{2}\right) + i\sin\left(\frac{\theta+\phi}{2}\right)\right) \end{aligned}$$

So, $$\begin{aligned} \arg(z+w) &= \frac{\theta+\phi}{2} \\ &= \frac{1}{2}(\theta+\phi) \\ &= \frac{1}{2}(\arg\ z + \arg\ w) \end{aligned}$$

also notice that $|z+w| = 2\cos\left(\frac{\theta-\phi}{2}\right)$

25 **a** $$\begin{aligned} |z| &= \sqrt{x^2+y^2} \quad \text{where } z = x+iy \\ &\geqslant \sqrt{x^2} \\ &\geqslant |x| \\ &\geqslant x \\ &\geqslant \text{Re}(z) \end{aligned}$$

Hence $|z| \geqslant \text{Re}(z)$ (with equality when $y = 0$)

b $$\begin{aligned} |z+w|^2 &= (z+w)\overline{(z+w)} \\ &= z\bar{z} + z\bar{w} + \bar{z}w + w\bar{w} \\ &= |z|^2 + 2\text{Re}(z\bar{w}) + |w|^2 \\ &\leqslant |z|^2 + 2|z\bar{w}| + |w|^2 \quad (|z\bar{w}| \geqslant \text{Re}(z\bar{w}) \text{ from part a}) \\ &\leqslant |z|^2 + 2|z||\bar{w}| + |w|^2 \\ &\leqslant |z|^2 + 2|z||w| + |w|^2 \\ &\leqslant (|z|+|w|)^2 \end{aligned}$$

Hence $|z+w|^2 \leqslant (|z|+|w|)^2$

and since all quantities are positive or zero

$|z+w| \leqslant |z| + |w|$

Solutions to Exercise 1E Foundation questions

1a $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$

$= \overrightarrow{OA} + \overrightarrow{OC}$

$= (5 + i) + (2 + 3i)$

$= 7 + 4i$

1b $\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC}$

$= -\overrightarrow{OA} + \overrightarrow{OC}$

$= -(5 + i) + (2 + 3i)$

$= -3 + 2i$

1c $\overrightarrow{CA} = -\overrightarrow{AC}$

$= -(-3 + 2i)$

$= 3 - 2i$

2a $\overrightarrow{OR} = \overrightarrow{OP} \times \operatorname{cis} \frac{\pi}{2}$

$= (4 + 3i)\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

$= (4 + 3i)(0 + i \times 1)$

$= (4 + 3i)i$

$= 4i + 3i^2$

$= 4i - 3$

$= -3 + 4i$

2b $\overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{PQ}$

$= \overrightarrow{OP} + \overrightarrow{OR}$

$= (4 + 3i) + (-3 + 4i)$

$= 1 + 7i$

2c $\overrightarrow{QR} = \overrightarrow{PO}$

$= -\overrightarrow{OP}$

$= -(4 + 3i)$

$= -4 - 3i$

2d $\overrightarrow{PR} = \overrightarrow{PO} + \overrightarrow{OR}$

$= -\overrightarrow{OP} + \overrightarrow{OR}$

$= -(4 + 3i) + (-3 + 4i)$

$= -7 + i$

3 $\overrightarrow{OP} = \overrightarrow{AB}$

$= \overrightarrow{AO} + \overrightarrow{OB}$

$= -\overrightarrow{OA} + \overrightarrow{OB}$

$= -(3 + 5i) + (9 + 8i)$

$= 6 + 3i$

$\overrightarrow{OQ} = \overrightarrow{OP} \times \text{cis}\,\dfrac{\pi}{2}$

$= (6 + 3i) \times i$ (using $\text{cis}\,\dfrac{\pi}{2} = i$ from question 2a)

$= 6i + 3i^2$

$= 6i - 3$

$= -3 + 6i$

4a $C = \overrightarrow{OC}$

$= \overrightarrow{OA} \times \text{cis}\,\dfrac{\pi}{2}$

$= (2 + i)i$

$= 2i + i^2$

$= 2i - 1$

$= -1 + 2i$

$B = \overrightarrow{OB}$

$= \overrightarrow{OA} + \overrightarrow{AB}$

$= \overrightarrow{OA} + \overrightarrow{OC}$

$= (2 + i) + (-1 + 2i)$

$= 1 + 3i$

4b $B' = B \times \text{cis } 45°$

$= (1 + 3i) \times \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i\right)$

$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i + \frac{3}{\sqrt{2}} i + \frac{3}{\sqrt{2}} i^2$

$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i + \frac{3}{\sqrt{2}} i - \frac{3}{\sqrt{2}}$

$= \left(\frac{1}{\sqrt{2}} - \frac{3}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}} + \frac{3}{\sqrt{2}}\right) i$

$= \frac{-2}{\sqrt{2}} + \frac{4}{\sqrt{2}} i$

$= -\sqrt{2} + 2\sqrt{2} i$

5a $\overrightarrow{BC} = \overrightarrow{BO} + \overrightarrow{OC}$

$= -\overrightarrow{OB} + \overrightarrow{OC}$

$= -(5 + 3i) + (9 + 6i)$

$= 4 + 3i$

5b $\overrightarrow{BA} = \overrightarrow{BC} \times \text{cis } \frac{\pi}{2}$

$= (4 + 3i) \times i$

$= 4i + 3i^2$

$= 4i - 3$

$= -3 + 4i$

5c $\overrightarrow{OA} = \overrightarrow{OB} + \overrightarrow{BA}$

$= (5 + 3i) + (-3 + 4i)$

$= 2 + 7i$

6a $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$

$= -\overrightarrow{OA} + \overrightarrow{OB}$

$= -(9 + i) + (4 + 13i)$

$= -5 + 12i$

6b $\overrightarrow{AD} = \overrightarrow{AB} \times \text{cis}\ \frac{\pi}{2}$

$= (-5 + 12i) \times i$

$= -5i + 12i^2$

$= -5i - 12$

$= -12 - 5i$

$D = \overrightarrow{OD}$

$= \overrightarrow{OA} + \overrightarrow{AD}$

$= (9 + i) + (-12 - 5i)$

$= -3 - 4i$

Solutions to Exercise 1E Development questions

7 w can be obtained by rotating z by $\frac{\pi}{2}$ radians. Hence it follows that $w = iz$.

Hence

$z^2 + w^2$

$= z^2 + (iz)^2$

$= z^2 - z^2$

$= 0$

8 $E = OE$

$= AB$

$= OB - OA$

$= w_2 - w_1$

F is a $\frac{\pi}{2}$ rotation from E, hence $F = i(w_2 - w_1)$

$C = OC$

$= OB + BC$

$= OB + OF$

$= w_2 + i(w_2 - w_1)$

$D = OD$

$= OA + AD$

$= OA + OF$

$= w_1 + i(w_2 - w_1)$

9a $z_1 - z_2$ represents the vector BA and $z_3 - z_2$ represents the vector BC.

Since BA is $\frac{\pi}{2}$ radians anticlockwise of BC it follows that $(z_1 - z_2) = i(z_3 - z_2)$.

$(z_1 - z_2)^2 + (z_3 - z_2)^2$

$= \left(i(z_3 - z_2)\right)^2 + (z_3 - z_2)^2$

$= -(z_3 - z_2)^2 + (z_3 - z_2)^2$

$= 0$

9b $D = OD$

$= OA + AD$

$= z_1 + BC$

$= z_1 + (z_3 - z_2)$

$= z_1 - z_2 + z_3$

10a OC is formed by taking the vector OA, rotating it by $\frac{\pi}{2}$ (multiplying ω by i) and then doubling the length of the rotated vector. This gives $2i\omega$.

10b The diagonals of a rectangle bisect one another. Hence the point of intersection will be at the midpoint of A and C.

This is at

$$\frac{\omega + 2i\omega}{2} = \frac{1}{2}\omega(1 + 2i)$$

11 The angle subtended by the vertices will be $\frac{2\pi}{3}$ radians (they divide a full rotation, 2π, into 3 equal segments). Hence the other vertices are obtained by rotating by $\frac{2\pi}{3}$ and $-\frac{2\pi}{3}$.

This is equivalent to multiplying $1+\sqrt{3}i$ by

$$\text{cis}\left(\pm\frac{2\pi}{3}\right) = \cos\left(\pm\frac{2\pi}{3}\right) + i\sin\left(\pm\frac{2\pi}{3}\right) = (-\frac{1}{2} \pm i\frac{\sqrt{3}}{2})$$

So the other two vertices are given by

$$(1+\sqrt{3}i)\left(-\frac{1}{2} \pm i\frac{\sqrt{3}}{2}\right)$$

$$= -\frac{1}{2}(1+\sqrt{3}i)(1 \pm \sqrt{3}i)$$

$$= -\tfrac{1}{2}(1-3i^2) \text{ or } -\tfrac{1}{2}(1+2\sqrt{3}i+3i^2)$$

$$= -2 \text{ or } 1 - i\sqrt{3}$$

$$= -2 \text{ and } 1 - \sqrt{3}i$$

12a This is done by simply rotating by $\pm\frac{\pi}{2}$ about the origin. To do this, multiply z by $\pm i$ to give

$$\pm iz$$

$$= \pm i(3+4i)$$

$$= \pm(3i-4)$$

$$= \pm(4-3i)$$

Hence $w = -4+3i$ or $4-3i$.

12b To make the right angle at z itself we simply need to translate the two vectors from part (a) by z. This gives

$$3+4i \pm (4-3i) = 7+i \text{ or } -1+7i$$

Hence $w = -1+7i$ or $7+i$

12c In this case the line OW and WZ must be perpendicular, hence

$w = \pm i(z - w)$

$w = iz - iw$ or $w = -iz + iw$

$(1 + i)w = iz$ or $(1 - i)w = -iz$

$w = \left(\frac{i}{1+i}\right) z$ or $w = \left(\frac{-i}{1-i}\right) z$

$w = \left(\frac{i(1-i)}{2}\right) z$ or $w = \left(-\frac{i(1+i)}{2}\right) z$

$w = \left(\frac{1+i}{2}\right) z$ or $w = \left(\frac{1-i}{2}\right) z$

$w = \left(\frac{1+i}{2}\right)(3 + 4i)$ or $w = \left(\frac{1-i}{2}\right)(3 + 4i)$

$w = \frac{1}{2}(3 + 4i + 3i + 4i^2)$ or $w = \frac{1}{2}(3 + 4i - 3i - 4i^2)$

Hence $w = \frac{1}{2}(-1 + 7i)$ or $\frac{1}{2}(7 + i)$

13 In order to be parallelogram, the opposite sides must be parallel and of equal length. Another way of saying this is that they must be the same vector. Hence, we must solve for the following equations:

$z_4 - z_3 = z_2 - z_1$

$z_4 = z_3 + z_2 - z_1$

$= -1 + 7i + 2 + 6i - (1 + i)$

$= 12i$

$z_4 - z_3 = z_1 - z_2$

$z_4 = z_3 + z_1 - z_2$

$= -1 + 7i + 1 + i - (2 + 6i)$

$= -2 + 2i$

$z_4 - z_1 = z_2 - z_3$

$z_4 = z_2 + z_1 - z_3$

$= 2 + 6i + 1 + i - (-1 + 7i)$

$= 4$

Three possible values of z_4 are $-2 + 2i$, $12i$, 4.

14 Since z has a modulus of 1, the quadrilateral formed by the points O, 1, z and $1+z$ has all sides of equal length. The diagonal of a rhombus bisects the angle at the corner of the rhombus. Thus, the line between 0 and $(1+z)$ bisects the angle between the line from 0 to 1 and the line from 0 to z.

That is, $\operatorname{Arg}(z+1)=\frac{1}{2}\operatorname{Arg} z$ or $2\operatorname{Arg}(z+1)=\operatorname{Arg} z$.

15a $z_1 - z_2$ represents AB and $z_4 - z_3$ represents CD hence $AB = CD$ so the opposite sides are both equal in length and parallel. Thus, $ABCD$ is a parallelogram.

15b $z_1 - z_3$ represents AC whilst $z_4 - z_2$ represents DB. Since $z_1 - z_3 = i(z_4 - z_2)$ it follows that

$\arg(z_1 - z_3)$

$= \arg i(z_4 - z_2)$

$= \arg i + \arg(z_4 - z_2)$

$= \dfrac{\pi}{2} + \arg(z_4 - z_2)$

This means that AC and DB are perpendicular. Since the diagonals of the parallelogram are perpendicular, it must be a square.

16

$$\frac{z_2 - z_1}{z_3 - z_1} = \operatorname{cis}\frac{\pi}{3}$$

$$\left|\frac{z_2 - z_1}{z_3 - z_1}\right| = \left|\operatorname{cis}\frac{\pi}{3}\right|$$

$$\left|\frac{z_2 - z_1}{z_3 - z_1}\right| = 1$$

$$|z_2 - z_1| = |z_3 - z_2|$$

This means that the triangle is isosceles.

$$\arg\left(\frac{z_2 - z_1}{z_3 - z_1}\right) = \arg\left(\operatorname{cis}\frac{\pi}{3}\right)$$

$$\arg\left(\frac{z_2 - z_1}{z_3 - z_1}\right) = \frac{\pi}{3}$$

The angle between the two equal sides is $\frac{\pi}{3}$.

Hence the triangle is equilateral.

17

$$\frac{z_2}{z_1} = \frac{z_3}{z_2}$$

$$\arg\left(\frac{z_2}{z_1}\right) = \arg\left(\frac{z_3}{z_2}\right)$$

$$\arg z_2 - \arg z_1 = \arg z_3 - \arg z_2$$

Hence $\angle P_2OP_1 = \angle P_3OP_2$.

So OP_2 bisects P_1OP_3.

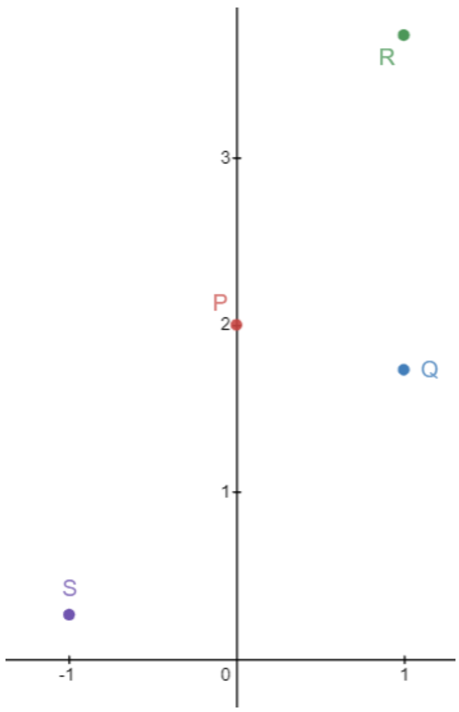
18a $i = \text{cis}\frac{\pi}{2}$ so $z_1 = 2\text{ cis}\frac{\pi}{2}$

$$z_2 = \sqrt{1 + \left(\sqrt{3}\right)^2}\ \text{cis}(\tan^{-1}\sqrt{3})$$

$$= 2\text{ cis }\frac{\pi}{3}$$

$$z_1 = 2\text{ cis}\frac{\pi}{2}, z_2 = 2\text{ cis}\frac{\pi}{3}$$

18b



18c i $\arg(z_1 + z_2)$

$= \arg\left(2i + 1 + \sqrt{3}i\right)$

$= \arg\left(1 + \left(2 + \sqrt{3}\right)i\right)$

$= \tan^{-1}\left(\frac{2 + \sqrt{3}}{1}\right)$

$= \tan^{-1}\left(2 + \sqrt{3}\right)$

$= \frac{5\pi}{12}$

18c ii $\arg(z_1 - z_2)$

$= \arg\left(2i - 1 - \sqrt{3}i\right)$

$= \arg\left(-1 + \left(2 - \sqrt{3}\right)i\right)$

$= \tan^{-1}\left(\frac{2 - \sqrt{3}}{-1}\right)$

$= \tan^{-1}\left(\sqrt{3} - 2\right)$

$= \frac{11\pi}{12}$

19a $|z|^2 = |x + iy|^2$

$= x^2 + y^2$

$= x^2 - i^2y^2$

$= (x + iy)(x - iy)$

$= z\bar{z}$

19b $|z_1 + z_2|^2 + |z_1 - z_2|^2$

$= (z_1 + z_2)\overline{(z_1 + z_2)} + (z_1 - z_2)\overline{(z_1 - z_2)}$

$= (z_1 + z_2)(\overline{z_1} + \overline{z_2}) + (z_1 - z_2)(\overline{z_1} - \overline{z_2})$

$= z_1\overline{z_1} + z_2\overline{z_1} + z_1\overline{z_2} + z_2\overline{z_2} + z_1\overline{z_1} - z_2\overline{z_1} - z_1\overline{z_2} + z_2\overline{z_2}$

$= z_1\overline{z_1} + z_2\overline{z_2} + z_1\overline{z_1} + z_2\overline{z_2}$

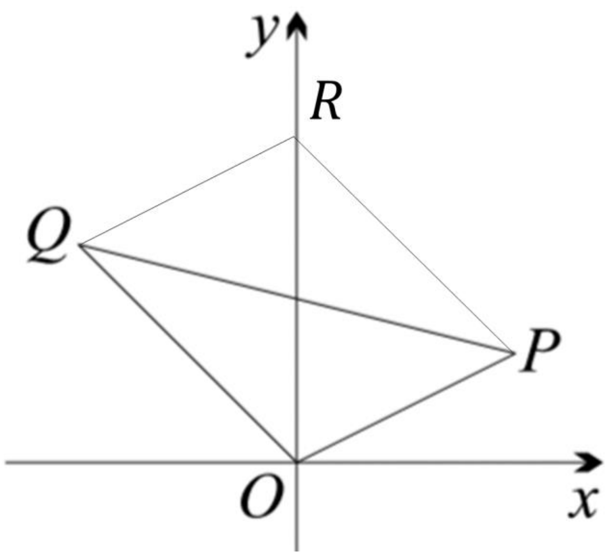
$= 2(z_1\overline{z_1} + z_2\overline{z_2})$

$= 2(|z_1|^2 + |z_2|^2)$

19c The sum of the squares of the diagonals of a parallelogram is equal to the sum of the squares of its sides.

20a $|z - w|$ represents the length PQ, $|z|$ is the length of OP and $|w|$ is the length of OQ. The length of any one side of a triangle must be less than or equal to the sum of the lengths of the other two sides of the triangle (this is the triangle inequality) and hence $|z - w| \leq |z| + |w|$.

20b



b

20c This is a parallelogram as all opposing sides are parallel.

20d This implies that the diagonals are of equal length and thus $OPRQ$ is a square, this means that OP and OQ are perpendicular hence $\frac{w}{z}$ is a multiple of i and purely imaginary.

$\arg\frac{z}{w} = \frac{\pi}{2}$, so $\frac{w}{z}$ is purely imaginary.

21a

If $\frac{z_3-z_1}{z_2-z_1}$ is real, then

$$\arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right) = 0$$

$$\arg(z_3 - z_1) - \arg(z_2 - z_1) = 0$$

So $z_3 - z_1$ and $z_2 - z_1$ are parallel and thus z_1, z_2 and z_3 are collinear.

21b

$$\left(\frac{19 + 29i - (5 + 8i)}{(13 + 20i) - (5 + 8i)}\right)$$

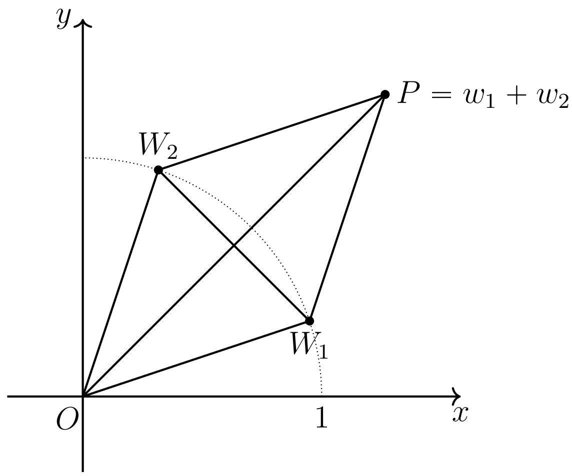
$$= \frac{14 + 21i}{8 + 12i}$$

$$= \frac{7(2 + 3i)}{4(2 + 3i)}$$

$$= \frac{7}{4}$$

Thus $\frac{z_3-z_1}{z_2-z_1}$ is real and hence it follows that the three points are collinear.

Solutions to Exercise 1E Enrichment questions



22

From the given information, w_1 and w_2 lie on the unit circle in the first quadrant.

Let P be the point which corresponds with $w_1 + w_2$.

Clearly OW_1PW_2 is a parallelogram, and since $|w_1| = |w_2|$, it is also a rhombus.

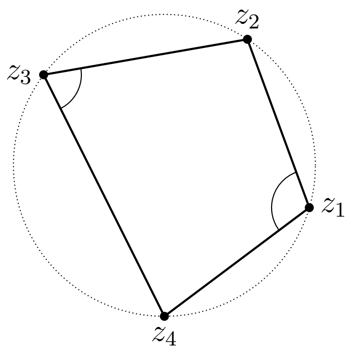
Hence OP bisects $\angle W_1OW_2$, and so

$$\begin{aligned}\arg\,(w_1 + w_2) &= \frac{1}{2}(\arg\,w_1 + \arg\,w_2)\\ &= \frac{1}{2}(\alpha_1 + \alpha_2)\end{aligned}$$

Now $w_1 - w_2$ is represented by $\overrightarrow{w_2w_1}$ which is $\frac{\pi}{2}$ clockwise from $\overrightarrow{OP}$ (diagonals of a rhombus are perpendicular).

$$\begin{aligned}\text{Thus }\arg\,(w_1 - w_2) &= \frac{1}{2}(\alpha_1 + \alpha_2) - \frac{\pi}{2}\\ &= \frac{1}{2}(\alpha_1 + \alpha_2 - \pi) \quad \#\end{aligned}$$

23 For simplicity, let z_1, z_2, z_3 and z_4 be arranged in anticlockwise order, as in the diagram.



Now $\arg\left(\dfrac{z_4 - z_1}{z_2 - z_1}\right)$ represents the angle between vectors $\overrightarrow{z_1 z_4}$ and $\overrightarrow{z_1 z_2}$ i.e. $\angle z_2 z_1 z_4$.

Likewise, $\arg\left(\dfrac{z_2 - z_3}{z_4 - z_3}\right) = \angle z_4 z_3 z_2$

thus $\angle z_2 z_1 z_4 + \angle z_4 z_3 z_2 = \pi$ (as marked above)

i.e. the opposite angles of a quadrilateral are supplementary.

Hence, by the converse of opposite angles in a cyclic quadrilateral, z_1, z_2, z_3 and z_4 are concyclic.

24 $\dfrac{z_1^{-1}-z_2^{-1}}{z_1^{-1}-z_3^{-1}} = \dfrac{z_1^{-1}-z_2^{-1}}{z_1^{-1}-z_3^{-1}} \times \dfrac{z_1z_2z_3}{z_1z_2z_3}$

$$= \frac{z_3(z_2-z_1)}{z_2(z_3-z_1)}$$

$$= \frac{(z_3-0)}{(z_2-0)}\frac{(z_2-z_1)}{(z_3-z_1)}$$

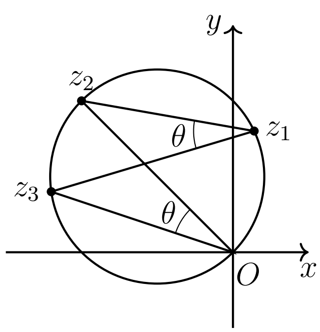
$$=$$

Thus $\arg\left(\dfrac{z_1^{-1}-z_2^{-1}}{z_1^{-1}-z_3^{-1}}\right) = \arg\left(\dfrac{z_3-0}{z_2-0}\right) + \arg\left(\dfrac{z_2-z_1}{z_3-z_1}\right)$

$= \arg\left(\dfrac{z_3-0}{z_2-0}\right) - \arg\left(\dfrac{z_3-z_1}{z_2-z_1}\right)$ (angles in the same segment,

$= \theta - \theta$ see diagram below)

$= 0$



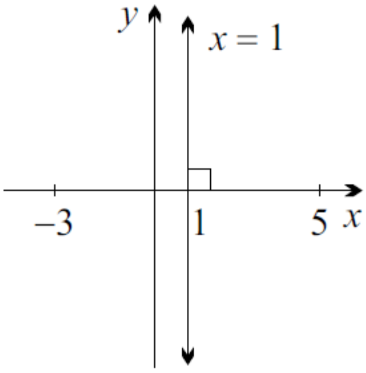
Hence $\dfrac{z_1^{-1}-z_2^{-1}}{z_1^{-1}-z_3^{-1}}$ is real, so $z_1^{-1}-z_2^{-1} = k(z_1^{-1}-z_3^{-1})$ for some real $k>0$, thus the vector from $\dfrac{1}{z_2}$ to $\dfrac{1}{z_1}$ is parallel with the vector from $\dfrac{1}{z_3}$ to $\dfrac{1}{z_1}$ and has common point $\dfrac{1}{z_1}$.

Hence $\dfrac{1}{z_1}$, $\dfrac{1}{z_2}$ and $\dfrac{1}{z_3}$ are collinear.

Solutions to Exercise 1F Foundation questions

1a For $|z+3| = |z-5|$, the midpoint of -3 and 5 is 1. These three points lie on the horizontal x-axis. Hence the perpendicular bisector must be vertical.

Hence the equation of the perpendicular bisector is $x = 1$.



1b For $|z-i| = |z+1|$, the midpoint of i and -1 is $\frac{-1+i}{2}$ which in Cartesian form is $(-\frac{1}{2}, \frac{1}{2})$.

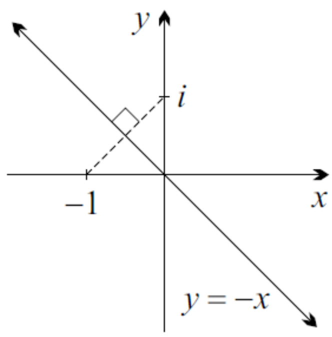
The gradient of the line between the two points is $\frac{0-1}{-1-0} = 1$ and hence the gradient of the perpendicular line is $m = -\frac{1}{1} = -1$.

For a straight line, the equation is $y = mx + b$.

Hence $y = -x + b$.

Substituting $(-\frac{1}{2}, \frac{1}{2})$ gives $\frac{1}{2} = -\left(-\frac{1}{2}\right) + b$ and so $b = 0$.

Thus the equation of the perpendicular bisector is $y = -x$.



1c For $|z+2-2i|=|z|$, the midpoint of $-2+2i$ and 0 is $\frac{-2+2i+0}{2}=-1+i$ which in Cartesian form is $(-1,1)$.

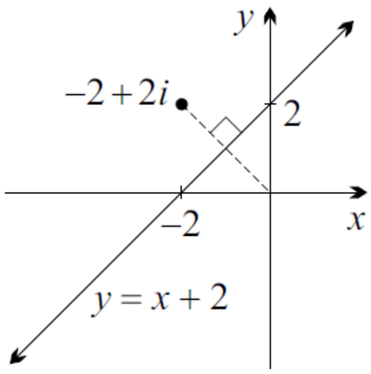
The gradient of the line between the two points is $\frac{2-0}{-2-0}=\frac{2}{-2}=-1$ and hence the gradient of the perpendicular line is $m=-\frac{1}{-1}=1$.

For a straight line, the equation is $y=mx+b$.

Hence $y=x+b$.

Substituting in $(-1,1)$ gives $1=-1+b$ and hence $b=2$.

Thus the equation of the perpendicular bisector is $y=x+2$.



1d For $|z-i|=|z-4+i|$, the midpoint of i and $4-i$ is 2 which in Cartesian form is $(2,0)$.

The gradient of the line between i and $4-i$ is $\frac{-1-1}{4-0}=-\frac{2}{4}=-\frac{1}{2}$ and hence the gradient of the perpendicular line is $m=-\frac{1}{\left(-\frac{1}{2}\right)}=2$.

For a straight line, the equation is $y=mx+b$.

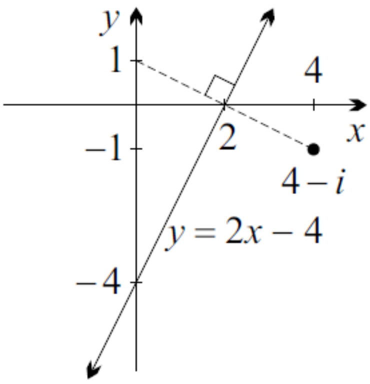
Hence $y=2x+b$.

Substituting in $(2,0)$ gives

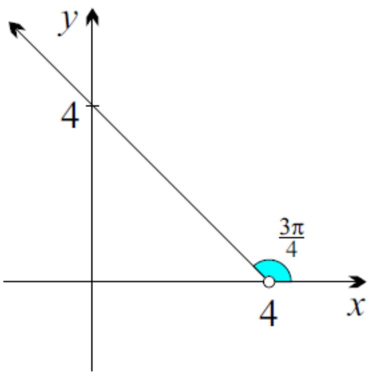
$0=2(2)+b$

$b=-4$

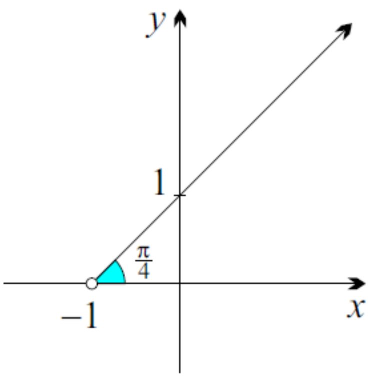
Thus the equation of the perpendicular bisector is $y=2x-4$.



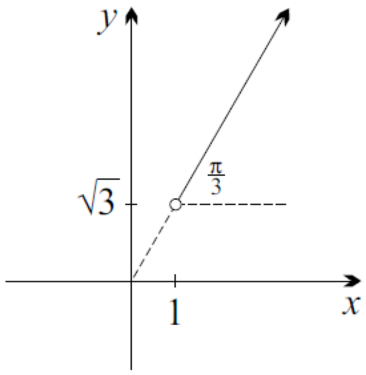
2a From Box 29, $\arg(z-4)=\frac{3\pi}{4}$ is a line originating at the point 4, which in Cartesian form is $(4, 0)$, where the angle made with the horizontal is $\frac{3\pi}{4}$.



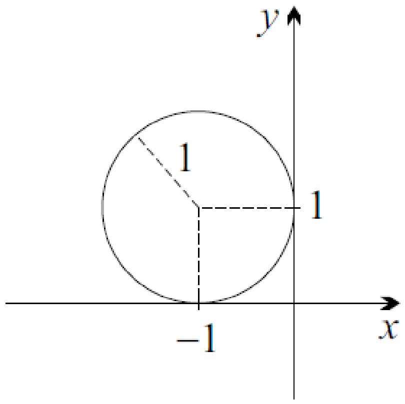
2b From Box 29, $\arg(z+1)=\frac{\pi}{4}$ is a line originating at the point -1, which in Cartesian form is $(-1, 0)$, where the angle made with the horizontal is $\frac{\pi}{4}$.



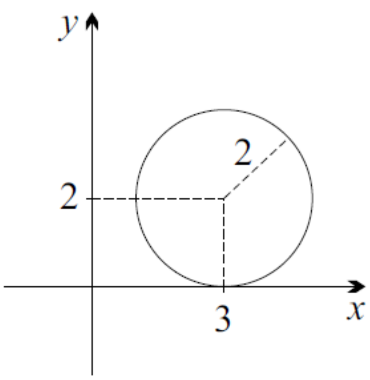
2c From Box 29, $\arg\left(z - 1 - i\sqrt{3}\right) = \frac{\pi}{3}$ is a line originating at the point $1 + i\sqrt{3}$, which in Cartesian form is $(1, \sqrt{3})$, where the angle made with the horizontal is $\frac{\pi}{3}$.



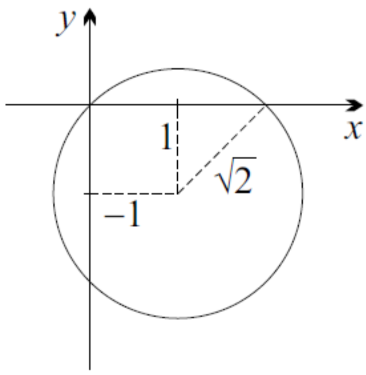
3a From Box 31, $|z + 1 - i| = 1$ is a circle with radius 1 unit and centre $(-1 + i)$, which is $(-1, 1)$ in Cartesian form.



3b From Box 31, $|z - 3 - 2i| = 2$ is a circle with radius 2 units and centre $(3 + 2i)$, which is $(3, 2)$ in Cartesian form.



3c From Box 31, $|z-1+i|=\sqrt{2}$ is a circle with radius $\sqrt{2}$ units and centre $(1-i)$, which is $(1,-1)$ in Cartesian form.



4a For the boundary $|z-8i|=|z-4|$, the midpoint of 4 and $8i$ is $\frac{4+8i}{2}=2+4i$ which in Cartesian form is $(2,4)$.

The gradient of the line between the two points is $\frac{8-0}{0-4}=-2$ and hence the gradient of the perpendicular line is $m=-\frac{1}{-2}=\frac{1}{2}$.

For a straight line, the equation is $y=mx+b$.

Hence $y=\frac{1}{2}x+b$.

Substituting $(2,4)$ gives $4=\frac{1}{2}\times 2+b$ and so $b=3$.

Thus the equation of the boundary is $y=\frac{1}{2}x+3$.

At $(0,0)$, $|-8i|=8$ and $|-4|=4$ so $|-8i|>|-4|$.

Therefore the region $|z-8i|\geq|z-4|$ includes $(0,0)$ so shade to the right of the boundary. The boundary is included.

Alternatively:

$|z-8i|\geq|z-4|$

$|x+iy-8i|\geq|x+iy-4|$

$|x+(y-8)i|\geq|(x-4)+iy|$

$\sqrt{x^2+(y-8)^2}\geq\sqrt{(x-4)^2+y^2}$

$x^2+(y-8)^2\geq(x-4)^2+y^2$

$x^2+y^2-16y+64\geq x^2-8x+16+y^2$

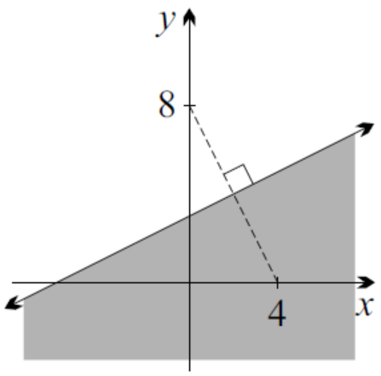
$-16y+64\geq-8x+16$

$-16y + 8x + 48 \geq 0$

$-2y + x + 6 \geq 0$

$y \leq \frac{1}{2}x + 3$

At $(0, 0)$, $0 < 3$ so the region contains the point $(0, 0)$.



4b For the boundary $|z - 2 + i| = |z - 4 + i|$, the midpoint of $2 - i$ and $4 - i$ is $\frac{6-2i}{2} = 3 - i$ which in Cartesian form is $(3, -1)$.

The gradient of the line between the two points is $\frac{-1-(-1)}{4-2} = 0$ and hence the perpendicular line is vertical, passing through $(3, -1)$.

Thus the equation of the boundary is $x = 3$.

At $(0, 0)$, $|-2 + i| = \sqrt{5}$ and $|-4 + i| = \sqrt{17}$ so $|-2 + i| < |-4 + i|$.

Therefore the region $|z - 2 + i| \leq |z - 4 + i|$ includes $(0, 0)$ so shade to the left of the boundary. The boundary is included.

Alternatively:

$|z - 2 + i| \leq |z - 4 + i|$

$|x + iy - 2 + i| \leq |x + iy - 4 + i|$

$|(x - 2) + (y + 1)i| \leq |(x - 4) + (y + 1)i|$

$\sqrt{(x - 2)^2 + (y + 1)^2} \leq \sqrt{(x - 4)^2 + (y + 1)^2}$

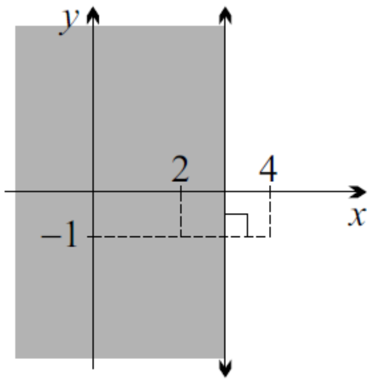
$(x - 2)^2 + (y + 1)^2 \leq (x - 4)^2 + (y + 1)^2$

$x^2 - 4x + 4 + y^2 + 2y + 1 \leq x^2 - 8x + 16 + y^2 + 2y + 1$

$-4x + 4 + 2y + 1 \leq -8x + 16 + 2y + 1$

$4x \leq 12$

$x \le 3$



4c For the boundary $|z + 1 - i| = |z - 3 + i|$, the midpoint of $-1 + i$ and $3 - i$ is $\frac{2+0i}{2} = 1$ which in Cartesian form is $(1, 0)$.

The gradient of the line between the two points is $\frac{-1-1}{3-(-1)} = -\frac{1}{2}$ and hence the gradient of the perpendicular line is $m = -\frac{1}{-\frac{1}{2}} = 2$.

For a straight line, the equation is $y = mx + b$.

Hence $y = 2x + b$.

Substituting $(1, 0)$ gives $0 = 2 \times 1 + b$ and so $b = -2$.

Thus the equation of the boundary is $y = 2x - 2$.

At $(0, 0)$, $|1 - i| = \sqrt{2}$ and $|-3 + i| = \sqrt{10}$ so $|1 - i| < |-3 + i|$.

Therefore the region $|z + 1 - i| \ge |z - 3 + i|$ does not include $(0, 0)$ so shade to the right of the boundary. The boundary is included.

Alternatively:

$|z + 1 - i| \ge |z - 3 + i|$

$|x + iy + 1 - i| \ge |x + iy - 3 + i|$

$|(x + 1) + (y - 1)i| \ge |(x - 3) + (y + 1)i|$

$\sqrt{(x + 1)^2 + (y - 1)^2} \ge \sqrt{(x - 3)^2 + (y + 1)^2}$

$(x + 1)^2 + (y - 1)^2 \ge (x - 3)^2 + (y + 1)^2$

$x^2 + 2x + 1 + y^2 - 2y + 1 \ge x^2 - 6x + 9 + y^2 + 2y + 1$

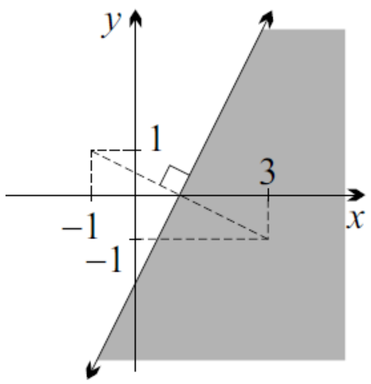
$2x + 1 - 2y + 1 \ge -6x + 9 + 2y + 1$

$8x - 4y \ge 8$

$2x - y \geq 2$

$y \leq 2x - 2$

At (0, 0), $0 > -2$ so the region does not contains the point (0, 0).

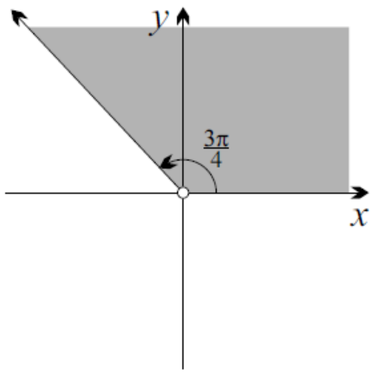


4d The boundaries are $\arg(z) = 0$ and $\arg(z) = \frac{3\pi}{4}$

$\arg(z) = 0$ is a line originating at the origin, where the angle made with the horizontal is 0.

$\arg(z) = \frac{3\pi}{4}$ is a line originating at the origin, where the angle made with the horizontal is $\frac{3\pi}{4}$.

For $0 \leq \arg(z) \leq \frac{3\pi}{4}$, shade the region between the two boundaries. The boundaries are included, except for $z = 0$ where the argument is undefined.

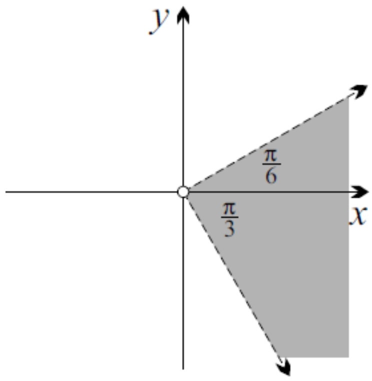


4e The boundaries are $\arg(z) = -\frac{\pi}{3}$ and $\arg(z) = \frac{\pi}{6}$

$\arg(z) = -\frac{\pi}{3}$ is a line originating at the origin, where the angle made with the horizontal is $-\frac{\pi}{3}$. That is, $\frac{\pi}{3}$ in a clockwise direction from the positive x-axis.

$\arg(z) = \frac{\pi}{6}$ is a line originating at the origin, where the angle made with the horizontal is $\frac{\pi}{6}$.

For $-\frac{\pi}{3} < \arg(z) < \frac{\pi}{6}$, shade the region between the two boundaries. The boundaries are not included.



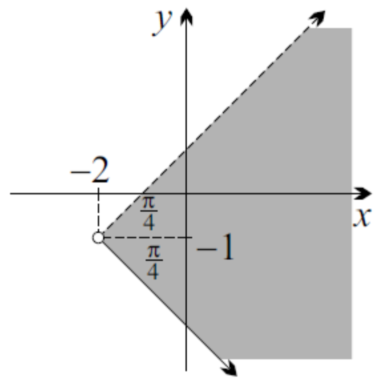
4f The boundaries are $\arg(z + 2 + i) = -\frac{\pi}{4}$ and $\arg(z + 2 + i) = \frac{\pi}{4}$

$\arg(z + 2 + i) = -\frac{\pi}{4}$ is a line originating at the point $-2 - i$, which in Cartesian form is $(-2, -1)$, where the angle made with the horizontal is $-\frac{\pi}{4}$. That is, $\frac{\pi}{4}$ in a clockwise direction from the horizontal.

$\arg(z + 2 + i) = \frac{\pi}{4}$ is a line originating at the point $-2 - i$, which in Cartesian form is $(-2, -1)$, where the angle made with the horizontal is $\frac{\pi}{4}$. That is, $\frac{\pi}{4}$ in an anticlockwise direction from the horizontal.

For $-\frac{\pi}{4} \leq \arg(z + 2 + i) < \frac{\pi}{4}$, shade the region between the two boundaries.

The boundary of $\arg(z + 2 + i) = -\frac{\pi}{4}$ is included but the boundary $\arg(z + 2 + i) = \frac{\pi}{4}$ is not.



4g The boundary is $|z| = 2$ which is a circle with radius 2 and centre at $(0, 0)$.

At $(0, 0)$, $|0| = 0$ and $0 < 2$.

Therefore the region $|z| > 2$ does not include $(0, 0)$ so shade outside the circular boundary. The boundary is not included.

Alternatively:

$|z| > 2$

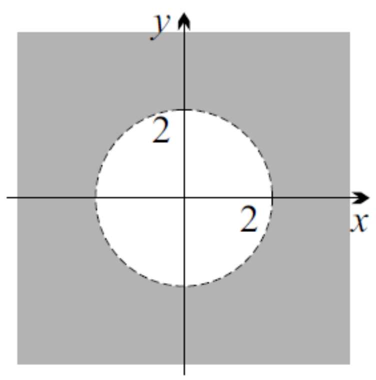
$|x + iy| > 2$

$\sqrt{x^2 + y^2} > 2$

$x^2 + y^2 > 4$

Boundary is $x^2 + y^2 = 4$, which is a circle with centre at $(0, 0)$ and radius of 2 units.

At $(0, 0)$, $x^2 + y^2 = 0 + 0 = 0$ and $0 < 4$ so the region does not contain the point $(0, 0)$. The boundary is not included.



4h The boundary $|z + 2i| = 1$ is a circle with radius 1 and centre at $-2i$, which is $(0, -2)$ in Cartesian form.

At $(0, 0)$, $|2i| = 2$ and $2 > 1$.

Therefore the region $|z + 2i| \leq 1$ does not include $(0, 0)$ so shade inside the circular boundary. The boundary is included.

Alternatively:

$|z + 2i| \leq 1$

$|x + iy + 2i| \leq 1$

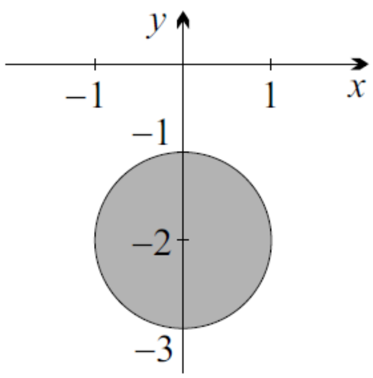
$|x + (y + 2)i| \leq 1$

$\sqrt{x^2 + (y + 2)^2} \leq 1$

$x^2 + (y + 2)^2 \leq 1$

Boundary is $x^2 + (y + 2)^2 = 1$, which is a circle with centre at $(0, -2)$ and radius of 1 unit.

At $(0, 0)$, $x^2 + (y + 2)^2 = 0 + 2^2 = 4$ and $4 > 1$ so the region does not contain the point $(0, 0)$. The boundary is included.



4i The boundary $|z - 2 + i| = 1$ is a circle with radius 1 unit and centre $(2 - i)$, which is $(2, -1)$ in Cartesian form.

The boundary $|z - 2 + i| = 2$ is a circle with radius 2 units and centre $(2 - i)$, which is $(2, -1)$ in Cartesian form.

For the region $1 < |z - 2 + i| \leq 2$, shade between the two boundaries. The boundary of $|z - 2 + i| = 2$ is included but the boundary $|z - 2 + i| = 1$ is not.

Alternatively:

$1 < |z - 2 + i| \leq 2$

$1 < |x + iy - 2 + i| \leq 2$

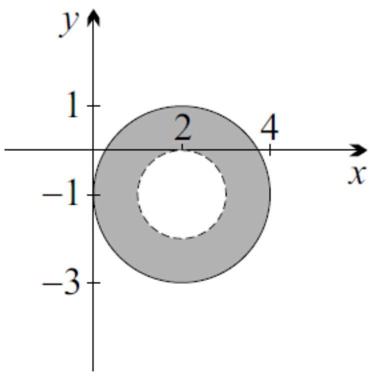
$1 < |(x - 2) + (y + 1)i| \leq 2$

$1 < \sqrt{(x - 2)^2 + (y + 1)^2} \leq 2$

$1 < (x - 2)^2 + (y + 1)^2 \leq 4$

One boundary is $(x - 2)^2 + (y + 1)^2 = 1$, which is a circle with centre at $(2, -1)$ and radius of 1 unit. This boundary is not included in the region.

The other boundary is $(x - 2)^2 + (y + 1)^2 = 4$, which is a circle with centre at $(2, -1)$ and radius of 2 units. This boundary is included in the region.



Solutions to Exercise 1F Development questions

5a $|z-2+i| \le 2$

$|z-2+i| = 2$ is a circle with centre $2-i$ or $(2,-1)$ and radius 2 units.

Alternatively, let $z = x+iy$.

$|(x+iy)-2+i| \le 2$

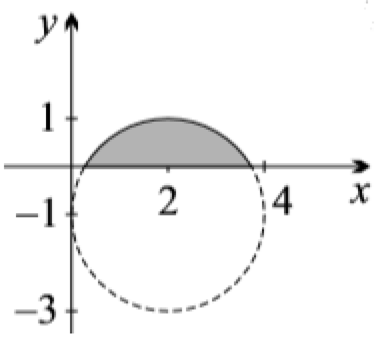
$|(x-2)+(y+1)i| \le 2$

$|(x-2)+(y+1)i|^2 \le 4$

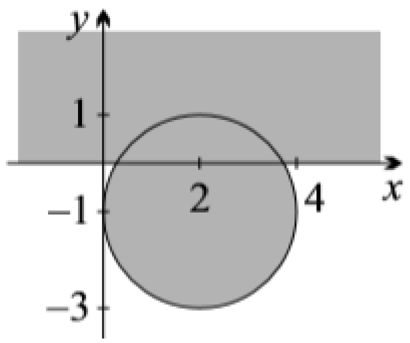
$(x-2)^2+(y+1)^2 \le 4$

$\mathrm{Im}(z) \ge 0$ is the region $y \ge 0$.

5a i



5a ii



5b Let $z = x + iy$.

$0 \le \text{Re}(z) \le 2$

$0 \le x \le 2$

$|z - 1 + i| \le 2$

$|z - 1 + i| = 2$ is a circle with centre $1 - i$ or $(1, -1)$ and radius 2 units.

Alternatively,

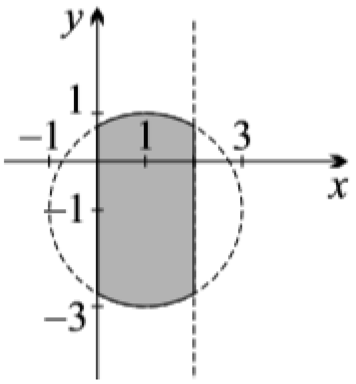
$|x + iy - 1 + i| \le 2$

$|(x - 1) + i(y - 1)| \le 2$

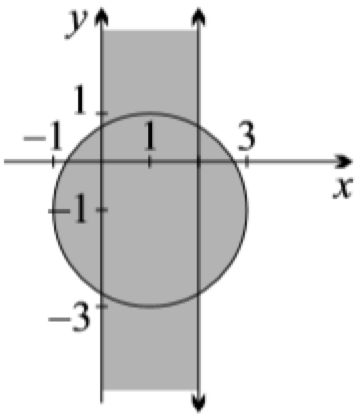
$|(x - 1) + i(y - 1)|^2 \le 4$

$(x - 1)^2 + (y - 1)^2 \le 4$

5b i



5b ii



5c Let $z = x + iy$

$|z - \bar{z}| < 2$

$|(x + iy) - (x - iy)| < 2$

$|2iy| < 2$

$2|y| < 2$

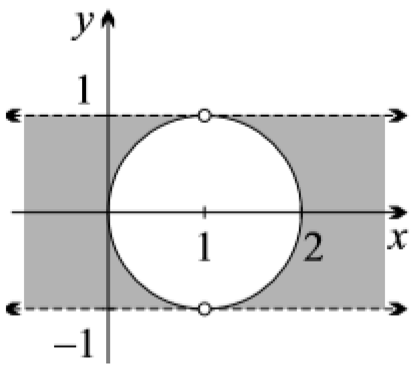
$|y| < 1$

$-1 < y < 1$

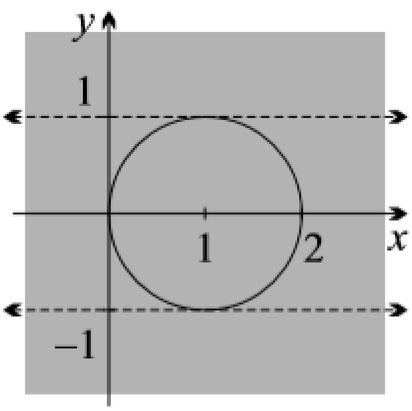
$|z - 1| \geq 1$

$|z - 1| = 1$ is a circle with centre $(1, 0)$ and radius 1 unit.

5c i



5c ii



5d Let $z = x + iy$

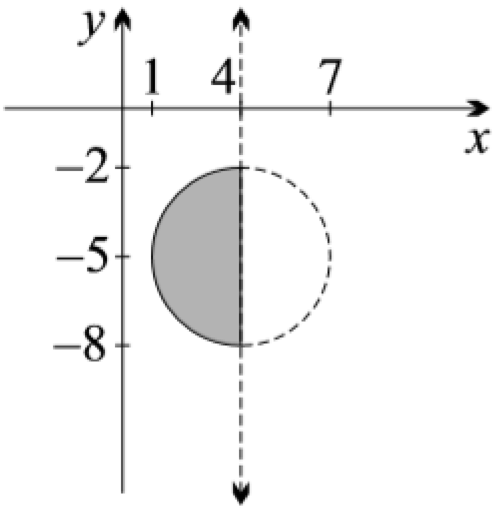
$\text{Re}(z) \leq 4$

$x \leq 4$

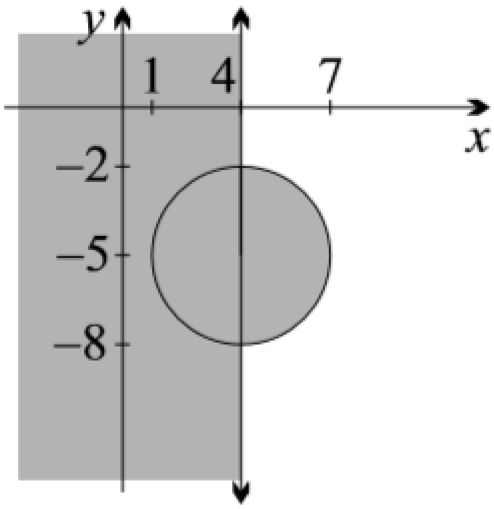
$|z - 4 + 5i| \leq 3$

$|z - 4 + 5i| = 3$ is a circle with centre $4 - 5i$ or $(4, -5)$ and radius 3 units.

5d i



5d ii



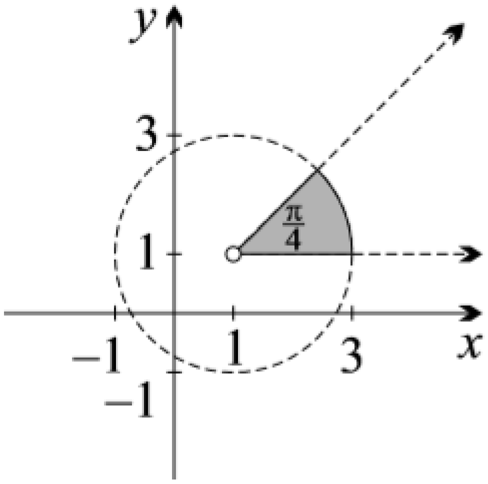
5e $|z - 1 - i| \le 2$

$|z - 1 - i| = 2$ is a circle with centre $1 + i$ or $(1, 1)$ and radius 2 units.

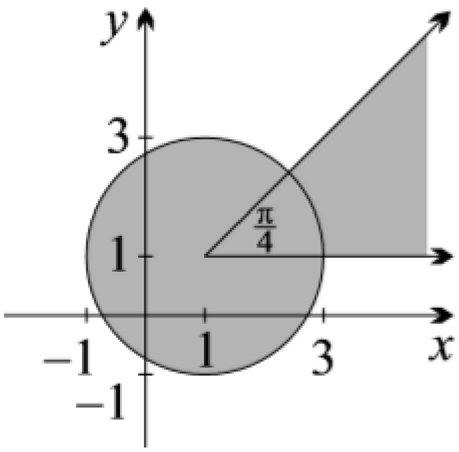
$0 \le \arg(z - 1 - i) \le \frac{\pi}{4}$

Vertex of angle at $1 + i$ or $(1, 1)$.

5e i



5e ii



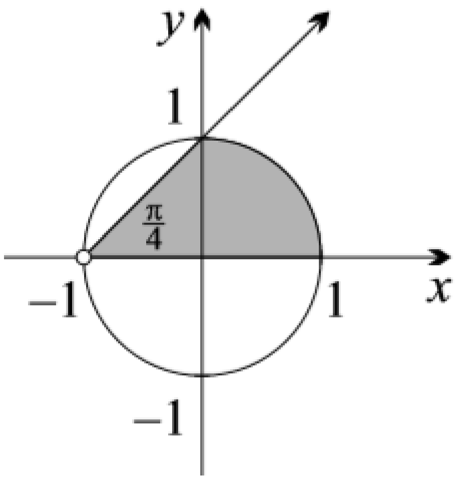
5f $|z| \leq 1$

$|z| = 1$ is a circle with centre $(0, 0)$ and radius 1 unit.

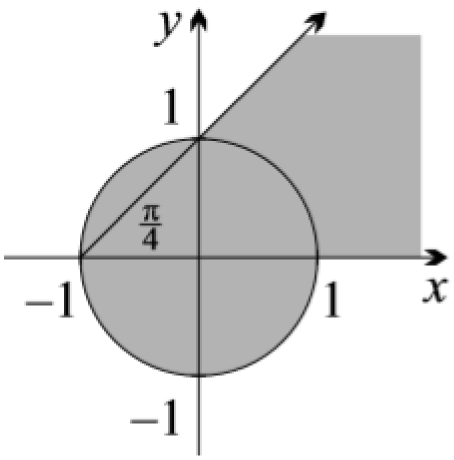
$0 \leq \arg(z + 1) \leq \frac{\pi}{4}$

Vertex of angle at $-1 + 0i$ or $(-1, 0)$.

5f i



5f ii



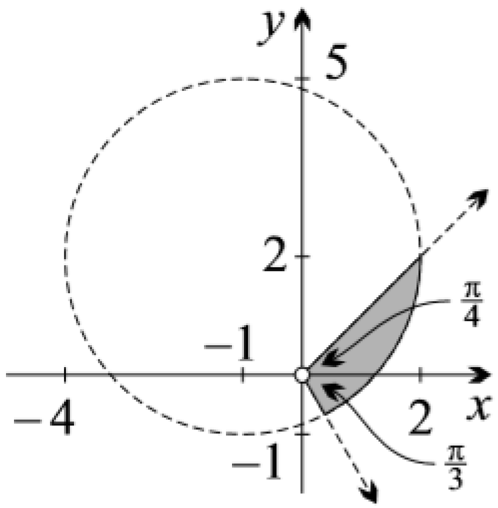
5g $|z + 1 - 2i| \le 3$

$|z + 1 - 2i| = 3$ is a circle with centre $-1 + 2i$ or $(-1, 2)$ and radius 3 units.

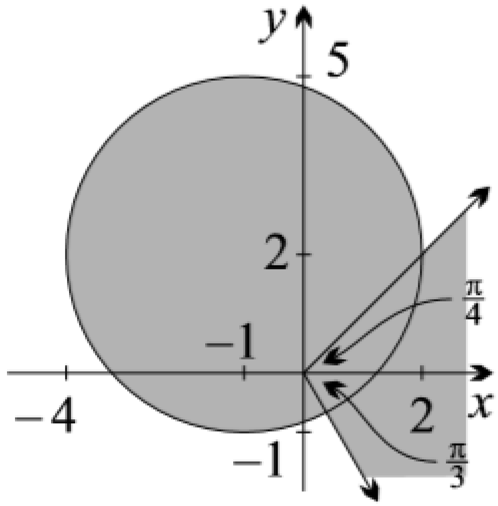
$-\dfrac{\pi}{3} \le \arg z \le \dfrac{\pi}{4}$

Vertex of angle at $(0, 0)$.

5g i



5g ii



5h $|z-3-i| \le 5$

$|z-3-i| = 5$ is a circle with centre $3+i$ or $(3, 1)$ and radius 5 units.

Let $z = x + iy$.

$|z+1| \le |z-1|$

$|(x+iy)+1| \le |(x+iy)-1|$

$|(x+1)+iy| \le |(x-1)+iy|$

$|(x+1)+iy|^2 \le |(x-1)+iy|^2$

$(x+1)^2 + y^2 \le (x-1)^2 + y^2$

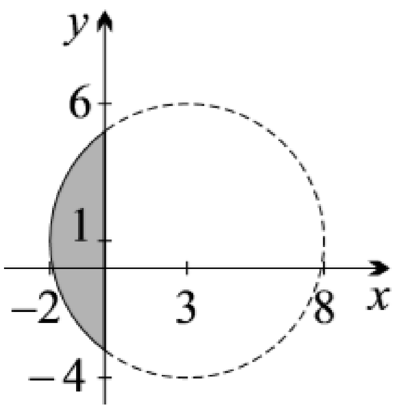
$(x+1)^2 \le (x-1)^2$

$x^2 + 2x + 1 \le x^2 - 2x + 1$

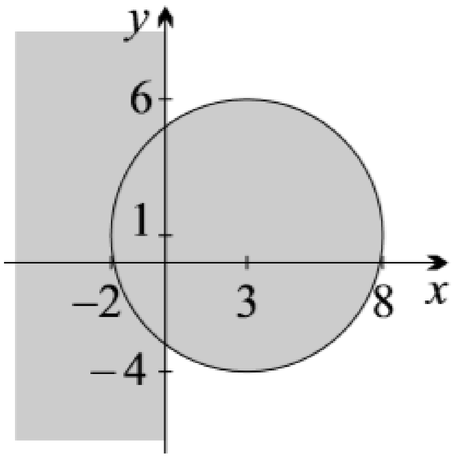
$4x \le 0$

$x \le 0$

5h i



5h ii



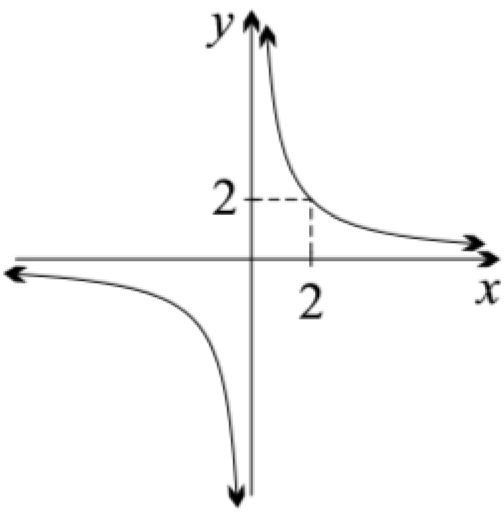
6a $(x + iy)^2 - (x - iy)^2 = 16i$

$x^2 + 2ixy - y^2 - (x^2 - 2ixy - y^2) = 16i$

$4ixy = 16i$

$4xy = 16$

$y = \dfrac{4}{x}$



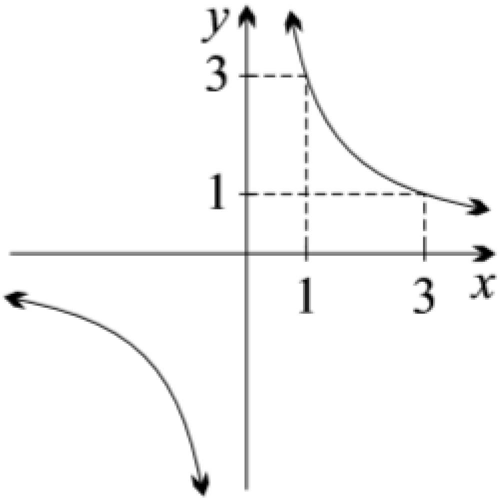
6b $(x + iy)^2 - (x - iy)^2 = 12i$

$x^2 + 2ixy - y^2 - (x^2 - 2ixy - y^2) = 12i$

$4ixy = 12i$

$4xy = 12$

$y = \dfrac{3}{x}$



7a $|z - 3i| = \mathrm{Im}(z)$

$|x + iy - 3i| = y$

$|x + i(y - 3)| = y$

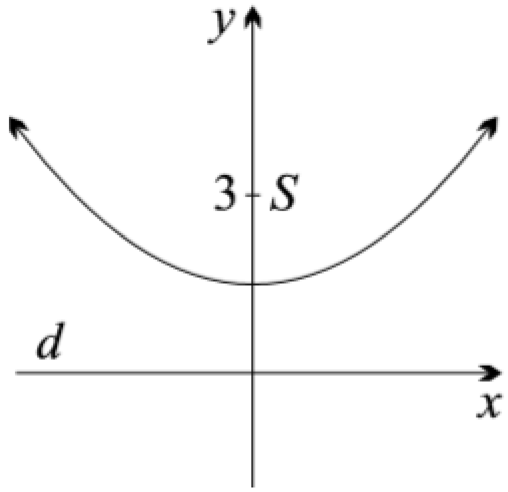
$|x + i(y - 3)|^2 = y^2$

$x^2 + (y - 3)^2 = y^2$

$x^2 + y^2 - 6y + 9 = y^2$

$x^2 - 6y + 9 = 0$

$y = \dfrac{1}{6}(x^2 + 9)$



7b $|z+2| = -\mathrm{Re}(z)$

$|x+iy+2| = -x$

$|(x+2)+iy| = -x$

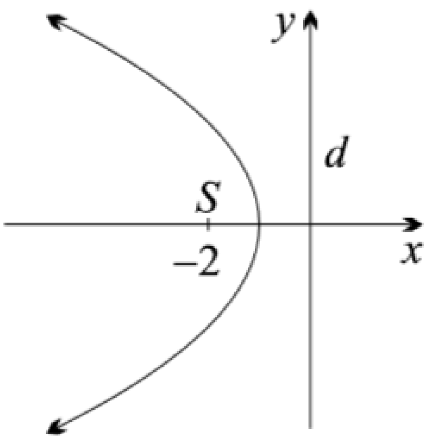
$|(x+2)+iy|^2 = x^2$

$(x+2)^2 + y^2 = x^2$

$x^2 + 4x + 4 + y^2 = x^2$

$4x + 4 + y^2 = 0$

$x = -\left(1 + \frac{1}{4}y^2\right)$



7c $|z| = \mathrm{Re}(z+2)$

$|x+iy| = \mathrm{Re}(x+iy+2)$

$|x+iy| = \mathrm{Re}\big((x+2)+iy\big)$

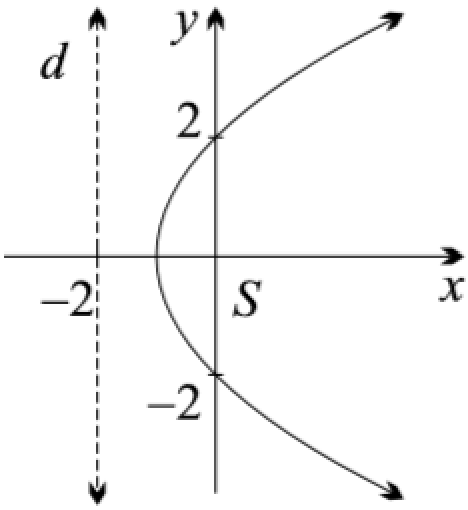
$|x+iy| = x+2$

$|x+iy|^2 = (x+2)^2$

$x^2+y^2 = x^2+4x+4$

$y^2 = 4x+4$

$x = \frac{1}{4}y^2 - 1$



7d $|z-i| = \mathrm{Im}(z+i)$

$|x+iy-i| = \mathrm{Im}(x+iy+i)$

$|x+iy-i| = \mathrm{Im}(x+i(y+1))$

$|x+i(y-1)| = y+1$

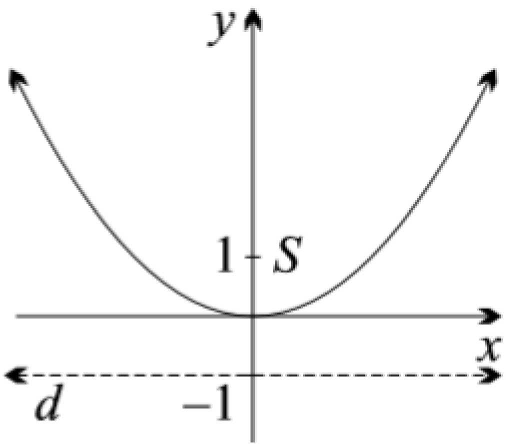
$|x+i(y-1)|^2 = (y+1)^2$

$x^2+(y-1)^2 = y^2+2y+1$

$x^2+y^2-2y+1 = y^2+2y+1$

$x^2 = 4y$

$y = \frac{1}{4}x^2$



8a $\text{Im}(z) = |z|$

$\text{Im}(x + iy) = |x + iy|$

$y = |x + iy|$

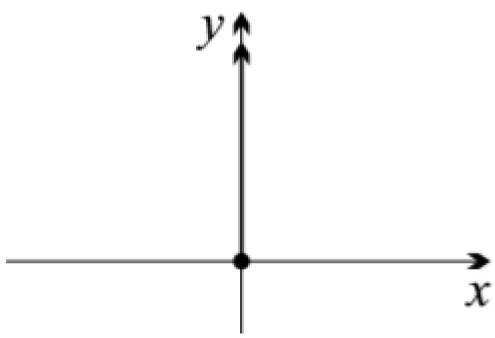
$y^2 = |x + iy|^2$

$y^2 = x^2 + y^2$

$x^2 = 0$

$x = 0$

Since we have taken the square, we must check if this works in the case where $y < 0$. We can show it does not by considering $y = -1$ and $x = 0$. In this case, $\text{Im}(z) = -1$ but $|z| = 1$. Thus, the solution is $x = 0$ for $y \geq 0$.



8b $\mathrm{Re}\left(1-\frac{4}{z}\right)=0$

$$\mathrm{Re}\left(1-\frac{4\bar{z}}{z\bar{z}}\right)=0$$

$$\mathrm{Re}\left(1-\frac{4\bar{z}}{|z|^2}\right)=0$$

$$\mathrm{Re}\left(1-\frac{4(x-iy)}{|z|^2}\right)=0$$

$$\mathrm{Re}\left(1-\frac{4x}{|z|^2}+i\frac{4y}{|z|^2}\right)=0$$

$$1-\frac{4x}{|z|^2}=0$$

$$1=\frac{4x}{|z|^2}$$

$$4x=|z|^2 \qquad (z\neq 0)$$

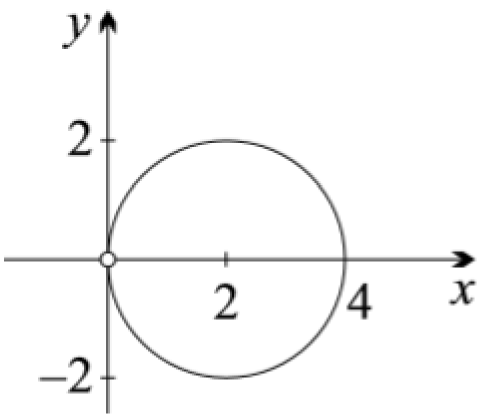
$$4x=x^2+y^2$$

$$x^2-4x+y^2=0$$

$$x^2-4x+4+y^2=4$$

$$(x-2)^2+y^2=4$$

Circle with centre (2, 0) and radius 2 units, omitting the origin.



8c $\mathrm{Re}\left(z-\frac{1}{z}\right)=0$

$$\mathrm{Re}\left(z-\frac{1\bar{z}}{z\bar{z}}\right)=0$$

$$\mathrm{Re}\left(z-\frac{\bar{z}}{|z|^2}\right)=0$$

$$\mathrm{Re}\left(x+iy-\frac{(x-iy)}{|z|^2}\right)=0$$

$$\mathrm{Re}\left(x-\frac{x}{|z|^2}+i\left(y+\frac{y}{|z|^2}\right)\right)=0$$

$$x-\frac{x}{|z|^2}=0$$

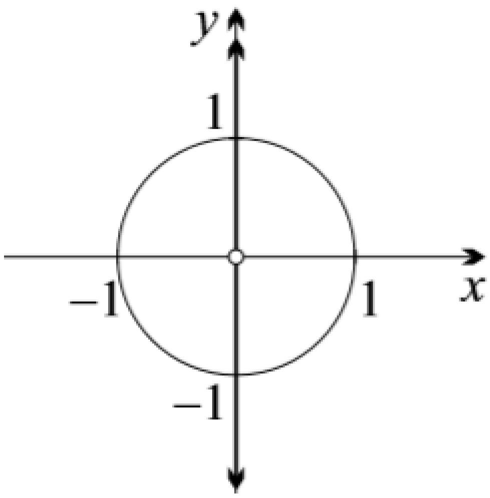
$$x=\frac{x}{|z|^2}$$

$x=x|z|^2$ $\quad$ $(z\neq 0)$

$1=|z|^2$ $\quad$ (since $z\neq 0$)

$1=x^2+y^2$

Circle with centre (0, 0) and radius 1 unit (omitting the origin).



9a

$$\arg\left(\frac{z-2}{z}\right)=\frac{\pi}{2}$$

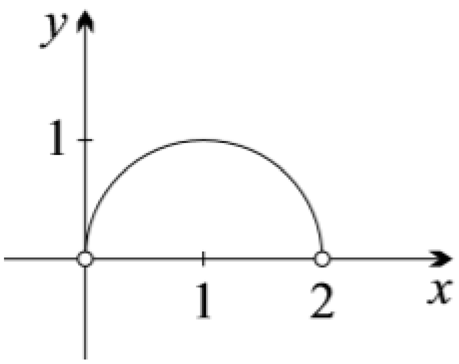
$$\arg(z-2)-\arg z=\frac{\pi}{2}$$

Consider point $A=2+0i=2$ and point $B=0+0i=0$.

The set of points z represents an arc of a circle with centre C and chord AB where the point P moves anticlockwise along the curve from point A to point B, with the endpoints A and B excluded.

$$\angle APB=\frac{\pi}{2}$$

Hence the chord AB forms the diameter of the circle so $C=1+0i$ and radius is 1 unit.



9b

$$\arg\left(\frac{z-1+i}{z-1-i}\right)=\frac{\pi}{2}$$

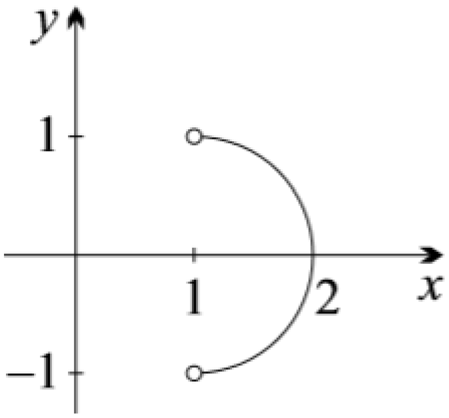
$$\arg(z-1+i)-\arg(z-1-i)=\frac{\pi}{2}$$

Consider point $A=1-i$ and point $B=1+i$.

The set of points z represents an arc of a circle with centre C and chord AB where the point P moves anticlockwise along the curve from point A to point B, with the endpoints A and B excluded.

$$\angle APB=\frac{\pi}{2}$$

Hence the chord AB forms the diameter of the circle so $C=1+0i$ and radius is 1 unit.



9c

$$\arg\left(\frac{z-i}{z+i}\right)=\frac{\pi}{4}$$

$$\arg(z-i)-\arg(z+i)=\frac{\pi}{4}$$

Consider point $A=i$ and point $B=-i$.

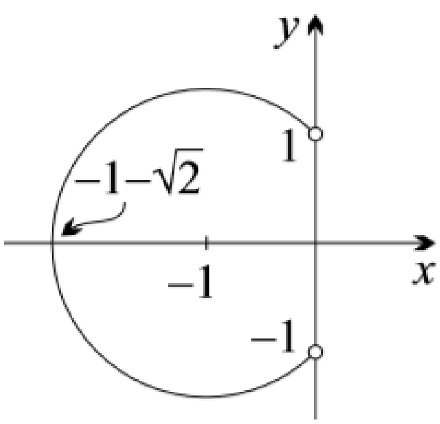
The set of points z represents an arc of a circle with centre C and chord AB where the point P moves anticlockwise along the curve from point A to point B, with the endpoints A and B excluded.

$\angle APB=\frac{\pi}{4}$ and $\angle ACB=\frac{\pi}{2}$ (angles at the centre and circumference)

So $\angle CAB=\frac{\pi}{4}$ (base angles of isosceles triangle)

Using ΔAOC, $\tan\frac{\pi}{4}=\frac{OC}{OA}=\frac{OC}{1}$ so $OC=1$.

Hence $C=-1+0i$ and $AC=\sqrt{1^2+1^2}=\sqrt{2}$ is its radius.



9d

$$\arg\left(\frac{z+1}{z-3}\right) = \frac{\pi}{3}$$

$$\arg(z+1) - \arg(z-3) = \frac{\pi}{3}$$

Consider point $A = -1 + 0i$ and point $B = 3 + 0i$.

The set of points z represents an arc of a circle with centre C and chord AB where the point P moves anticlockwise along the curve from point A to point B, with the endpoints A and B excluded.

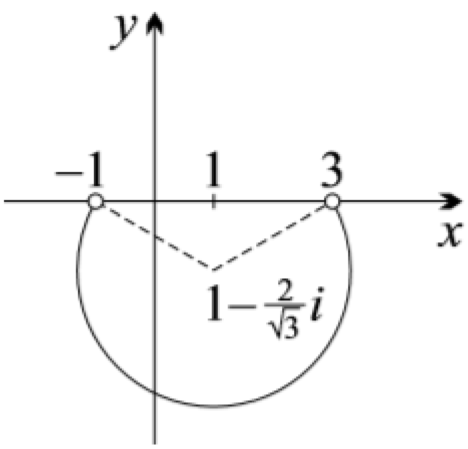
$\angle APB = \dfrac{\pi}{3}$ and $\angle ACB = \dfrac{2\pi}{3}$ (angles at the centre and circumference)

So $\angle CAB = \dfrac{\pi}{6}$ (base angles of isosceles triangle)

Let D be the point at $1 + 0i$ (midpoint of AB).

Using ΔADC, $\tan\dfrac{\pi}{6} = \dfrac{CD}{AD} = \dfrac{CD}{2}$ so $CD = 2 \times \dfrac{1}{\sqrt{3}} = \dfrac{2}{\sqrt{3}}$.

Hence $C = 1 - \frac{2}{\sqrt{3}}i$ and $AC = \sqrt{2^2 + \left(\frac{2}{\sqrt{3}}\right)^2} = \frac{4}{\sqrt{3}}$ is its radius.



9e

$$\arg\left(\frac{z-2i}{z+2i}\right)=\frac{\pi}{6}$$

$$\arg(z-2i)-\arg(z+2i)=\frac{\pi}{6}$$

Consider point $A = 2i$ and point $B = -2i$.

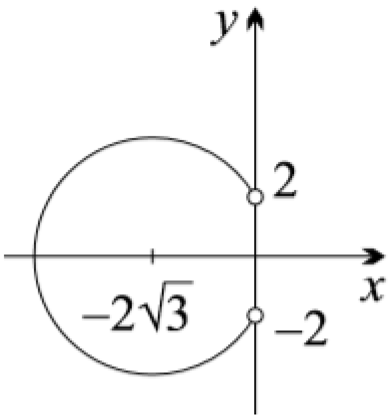
The set of points z represents an arc of a circle with centre C and chord AB where the point P moves anticlockwise along the curve from point A to point B, with the endpoints A and B excluded.

$\angle APB = \frac{\pi}{6}$ and $\angle ACB = \frac{\pi}{3}$ (angles at the centre and circumference)

So $\angle CAB = \frac{\pi}{3}$ (base angles of isosceles triangle)

Using ΔAOC, $\tan\frac{\pi}{3} = \frac{OC}{OA} = \frac{OC}{2}$ so $OC = 2 \times \sqrt{3} = 2\sqrt{3}$.

Hence $C = -2\sqrt{3} + 0i$ and $AC = \sqrt{2^2 + (2\sqrt{3})^2} = 4$ is its radius.



9f

$$\arg\left(\frac{z}{z+4}\right) = \frac{3\pi}{4}$$

$$\arg(z) - \arg(z+4) = \frac{3\pi}{4}$$

Consider point $A = 0 + 0i$ and point $B = -4 + 0i$.

The set of points z represents an arc of a circle with centre C and chord AB where the point P moves anticlockwise along the curve from point A to point B, with the endpoints A and B excluded.

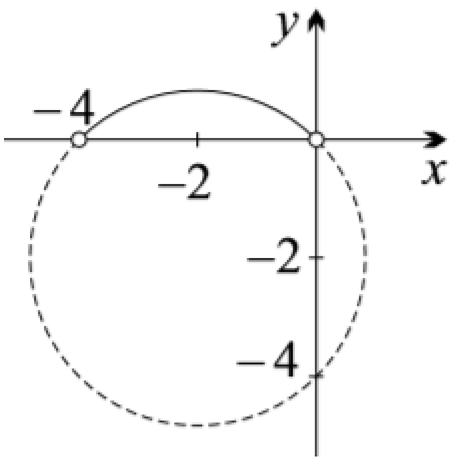
$\angle APB = \dfrac{3\pi}{4}$ and $\angle ACB = \dfrac{\pi}{2}$ (angles at the centre and circumference)

So $\angle CAB = \dfrac{\pi}{4}$ (base angles of isosceles triangle)

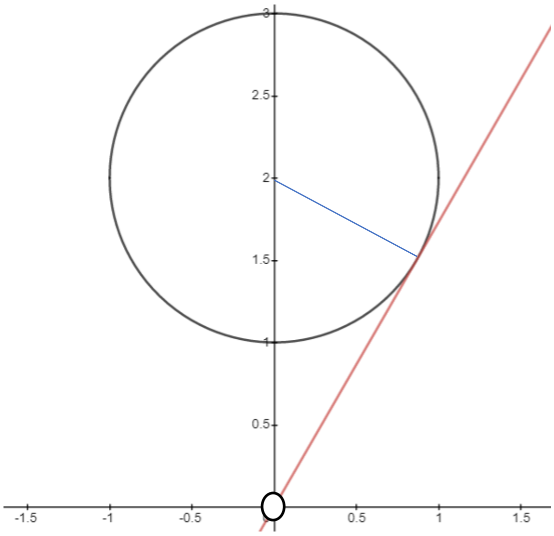
Let D be the point at $-2 + 0i$ (midpoint of AB).

Using ΔADC, $\tan\dfrac{\pi}{4} = \dfrac{CD}{AD} = \dfrac{CD}{2}$ so $CD = 2 \times 1 = 2$.

Hence $C = -2 - 2i$ and $AC = \sqrt{2^2 + 2^2} = \sqrt{8}$ is its radius.



10a



z lies somewhere on the red line (as $\arg z = \frac{\pi}{3}$). Now, $|z - 2i| \geq 1$ on and outside the black circle which has centre $2i$ and radius 1. Hence, we can see that $|z - 2i| \geq 1$.

10b To find the solution we must find the point where the line and the circle in the above diagram touch.

The distance between the origin and the point where the line touches the circle is

$\sqrt{2^2 - 1^2} = \sqrt{3}$ (using the Pythagorean theorem)

Since $\arg z = \frac{\pi}{3}$ and $|z| = \sqrt{3}$, $z = \sqrt{3}\operatorname{cis}\frac{\pi}{3}$.

11a $\arg(z + 3) = \tan^{-1}\frac{3\sqrt{3}}{3}$

$\arg(z + 3) = \dfrac{\pi}{3}$

11b z will take on the minimum value of |z| when it is perpendicular to the line in the argand diagram, hence $\arg z = \frac{5\pi}{6}$. Using trigonometry, we find $\cos\frac{\pi}{6} = \frac{|z|}{3}$, thus $|z| = 3\cos\frac{\pi}{6} = 3\left(\frac{\sqrt{3}}{2}\right)$.

Thus $|z| = \frac{3\sqrt{3}}{2}$, $\arg z = \frac{5\pi}{6}$.

11c

$$z = \frac{3\sqrt{3}}{2}\operatorname{cis}\frac{5\pi}{6}$$

$$= \frac{3\sqrt{3}}{2}\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$$

$$= -\frac{9}{4} + \frac{3\sqrt{3}}{4}i$$

12a $|z - 1| = 2$

Circle with centre at (1, 0) and radius 2 units.

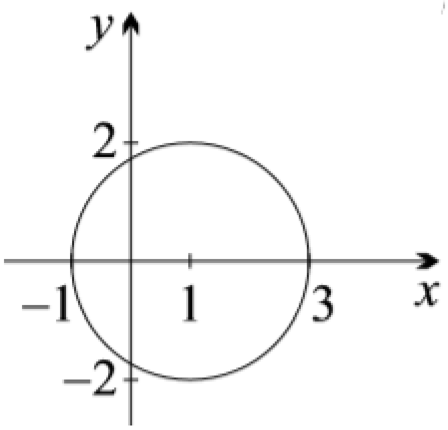
Alternatively,

$|x + iy - 1| = 2$

$|(x - 1) + iy| = 2$

$|(x - 1) + iy|^2 = 4$

$(x - 1)^2 + y^2 = 4$



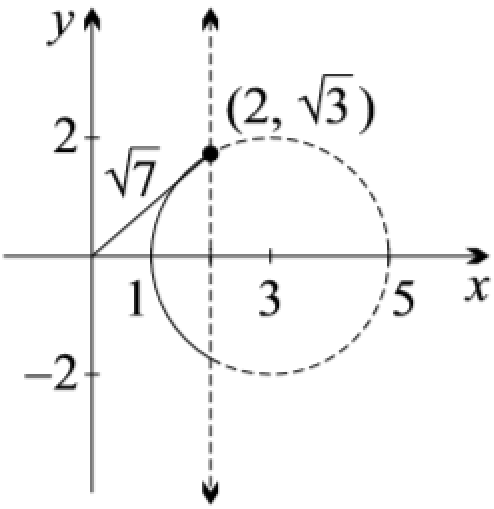
The maximum and minimum values of $|z|$ are given by the maximum and minimum distance of a point on the circle to the origin. From the diagram, the farthest point is at $z = 3$ whilst the closest is at $z = -1$. Thus, the maximum and minimum values for $|z|$ are 3 and 1 respectively.

12b $\text{Re}(z) \le 2$

$x \le 2$

$|z - 3| = 2$

Circle with centre (3, 0) and radius 2 units.



When $x = 2$,

$(2 - 3)^2 + y^2 = 4$

$1 + y^2 = 4$

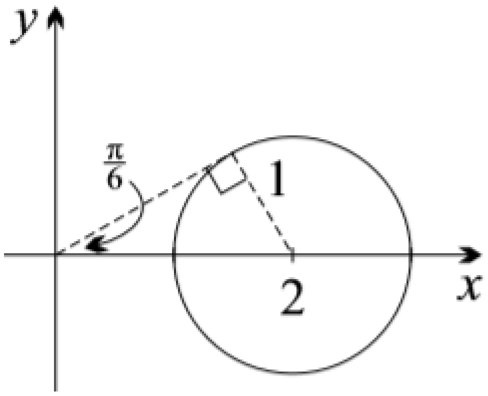
$y^2 = 3$

$y = \pm\sqrt{3}$

So the point with distance on the arc furthest from the origin is $(2, \pm\sqrt{3})$. Hence $|z| = \sqrt{2^2 + \left(\pm\sqrt{3}\right)^2} = \sqrt{4 + 3} = \sqrt{7}$ so the maximum value is $\sqrt{7}$. Observe from the diagram that the minimum value is when the arc intersects the x-axis. This is when $|z| = 1$.

So $1 \le |z| \le \sqrt{7}$.

13a i Circle with centre (2, 0) and radius 1 unit.



13a ii As is shown in the diagram, the maximum value of $\arg z$ occurs when the vector formed by z is a tangent to the circle. This is when $\arg z = \sin^{-1}\frac{1}{2} = \frac{\pi}{6}$.

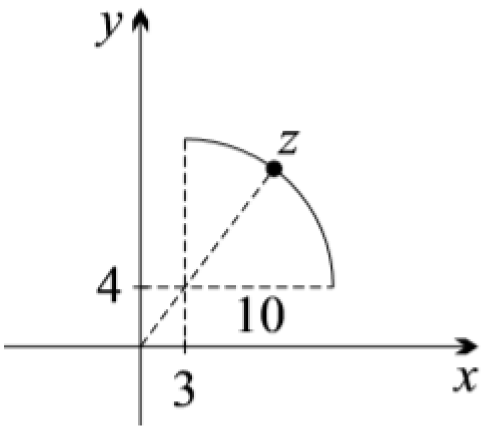
Hence $-\frac{\pi}{6} \leq \arg z \leq \frac{\pi}{6}$.

13b This is simply part (a) shifted left by 2 units.

14a $|w| = 10$

$0 \leq \arg w \leq \dfrac{\pi}{2}$

These two constraints give that w lies on a circle of radius 10, centred at the origin restricted to the first quadrant. z is w translated by $3 + 4i$, hence it is a circle of radius 10, centred $(3, 4)$ restricted to the upper right-hand quarter.



14b The maximum value of z occurs when the vector for z passes through the origin. The distance from the origin to the centre of the circle is $\sqrt{4^2+3^2}=5$. The distance from the centre to the outside of the circle is 10 units. Hence the maximum value of $|z|$ is 15.

14c $\arg z = \tan^{-1}\frac{4}{3}$

Hence the value of z for which this occurs is

$$15 \operatorname{cis}\left(\tan^{-1}\frac{4}{3}\right)$$

$$= 15\cos\left(\tan^{-1}\frac{4}{3}\right) + 15i\sin\left(\tan^{-1}\frac{4}{3}\right)$$

$$= 15 \times \frac{3}{\sqrt{3^2+4^2}} + 15i \times \frac{4}{\sqrt{3^2+4^2}}$$

$$= 9 + 12i$$

15a $|z - z_0| = r$

$$|z - z_0|^2 = r^2$$

$$(z - z_0)\overline{(z - z_0)} = r^2$$

$$(z - z_0)(\bar{z} - \bar{z_0}) = r^2$$

$$z\bar{z} - (z\bar{z_0} + \bar{z}z_0) + z_0\bar{z_0} - r^2 = 0$$

15b i $z\bar{z} - \left(z\overline{(-2)} + \bar{z}(-2)\right) + (-2)\overline{-2} - 2^2 = 0$

Hence using the above result

$$|z + 2| = 2$$

This is a circle with centre $-2 + 0i$ or $(-2, 0)$ and radius 2 units.

15b ii $z\bar{z} - \left(z\overline{(1+i)} + \bar{z}(1+i)\right) + (1+i)(1-i) - 2 + 1 = 0$

$$z\bar{z} - \left(z\overline{(1+i)} + \bar{z}(1+i)\right) + (1+i)(1-i) - 1 = 0$$

$$|z - (1+i)| = 1$$

This is a circle with centre $1 + i$ or $(1, 1)$ and radius 1 unit.

15b iii $\frac{1}{z}+\frac{1}{\bar{z}}=1$

$$\frac{z\bar{z}}{z}+\frac{z\bar{z}}{\bar{z}}=z\bar{z}$$

$$z+\bar{z}=z\bar{z}$$

$$z\bar{z}-(z+\bar{z})+(1)(1)-1=0$$

$$|z-1|=1$$

This is a circle with centre $1+0i$ or $(1, 0)$ and radius 1 unit.

16a The line through 1 and i, omitting i.

$$\frac{z-1}{z-i}$$

$$=\frac{x+iy-1}{x+iy-i}$$

$$=\frac{(x-1)+iy}{x+i(y-1)}$$

$$=\frac{((x-1)+iy)\big(x-i(y-1)\big)}{\big(x+i(y-1)\big)\big(x-i(y-1)\big)}$$

$$=\frac{x(x-1)+i(-(x-1)(y-1)+xy)+y(y-1)}{x^2+(y-1)^2}$$

$$=\frac{x(x-1)+y(y-1)+i(-xy+x+y-1+xy)}{x^2+(y-1)^2}$$

$$=\frac{x(x-1)+y(y-1)+i(x+y-1)}{x^2+(y-1)^2}$$

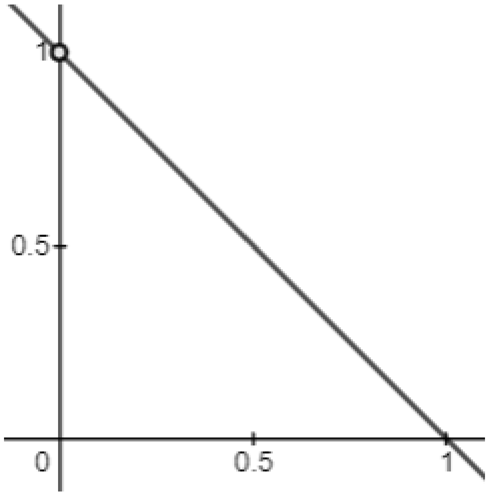
If the fraction is real, then the imaginary part must be equal to zero, hence

$$\frac{(x+y-1)}{x^2+(y-1)^2}=0$$

Now, $x^2+(y-1)^2\neq 0$ if $(x,y)\neq(0,1)$

$x+y-1=0 \qquad (x^2+(y-1)^2\neq 0)$

$x+y=1 \qquad (x,y)\neq(0,1)$



16b The circle with diameter joining 1 and i, omitting these two points.

Using the above equation, if the fraction is to be purely imaginary, it follows that the real component is equal to zero.

$$\frac{x(x-1)+y(y-1)}{x^2+(y-1)^2}=0$$

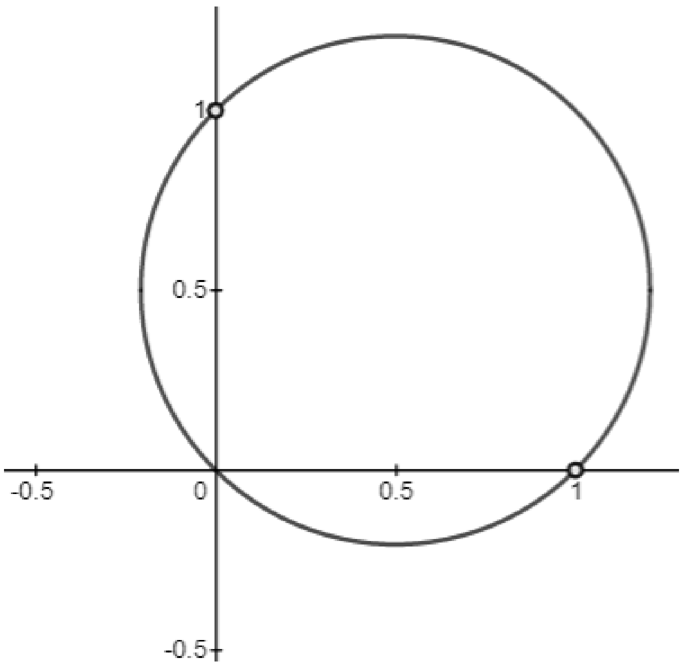
$$\frac{x^2-x+y^2-y}{x^2+(y-1)^2}=0$$

$$x^2-x+y^2-y=0 \qquad (x^2+(y-1)^2\neq 0, (x,y)\neq(0,1))$$

$$\left(x-\frac{1}{2}\right)^2+\left(y-\frac{1}{2}\right)^2=\frac{1}{4}+\frac{1}{4}$$

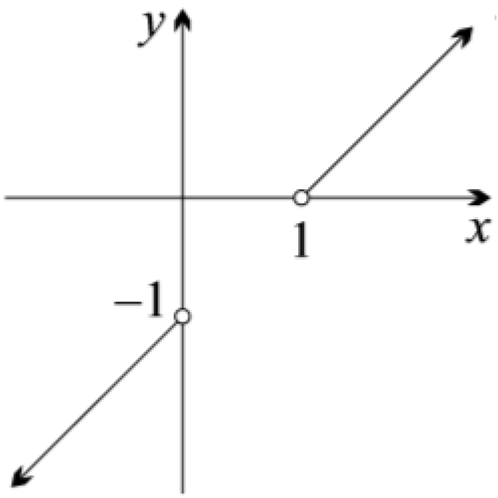
$$\left(x-\tfrac{1}{2}\right)^2+\left(y-\tfrac{1}{2}\right)^2=\tfrac{1}{2}$$

Now this circle goes through the point $(x,y)=(1,0)$, but this would make $z=1$ and so $\frac{z-1}{z-i}=0$, contradicting the requirement for the expression to be imaginary. Hence, we must have $(x,y)\neq(1,0),(0,1)$.



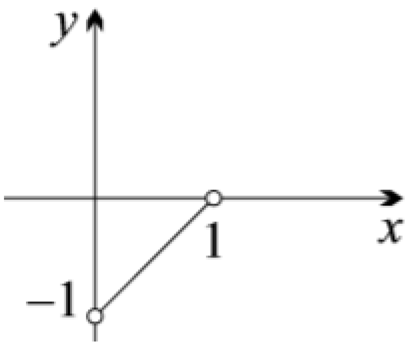
17a $\arg(z+i) = \arg(z-1)$

The expression above implies that that the vector from $-i$ to z and from 1 to z are parallel with common point z and so are collinear. This means z must lie on the line which goes through $-i$ and 1. Further since the arguments are equal z must lie on the same side as $-i$ and 1 (not between them) since otherwise the two vectors would have different signs. Finally, the above expression is not defined at z $= -i$ or 1 since $\arg(1+i) \neq 0$ and $\arg(i-1) \neq 0$, hence we exclude those points giving the final graph to be.

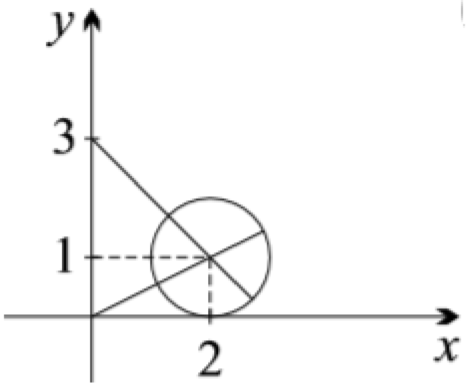


17b $\arg(z+i) = \arg(z-1) + \pi$

The argument is fundamentally the same as part a) but now there is a constant π difference between the two vectors which implies that they point in opposite directions. Thus, z must now like between $-i$ and 1.



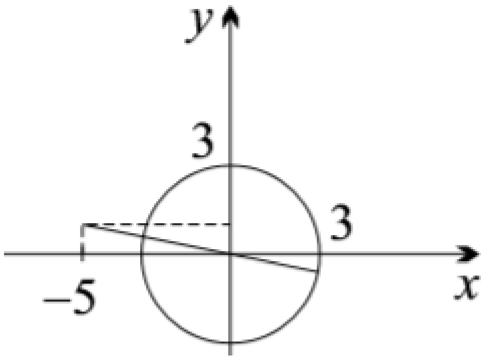
18a i $|z-2-i| = 1$ is a circle with centre $(2, 1)$ and radius 1 unit.



Observe that the vector passing through the origin and the centre of the circle intersects the circle at the maximum and minimum values. The distance to the centre of the circle is $\sqrt{2^2+1^2} = \sqrt{5}$. The distance from the centre of the circle to the circumference is 1 unit. Hence the minimum value of $|z|$ is $\sqrt{5}-1$ and the maximum is $\sqrt{5}+1$.

18 a ii To consider the maximum and minimum values of $|z-3i|$ now consider the vector passing through the point $(0, 3)$ and the centre of the circle (since $|z-3i|$ is the distance from points on the circle to the point $(0, 3)$). The distance between these two points is $\sqrt{(3-1)^2+2^2} = 2\sqrt{2}$. Hence the minimum value of $|z-3i|$ is $2\sqrt{2}-1$ and the maximum is $2\sqrt{2}+1$.

18b The curve representing $|z| = 3$ is a circle with centre at $(0, 0)$ and radius 3 units.



To consider the maximum and minimum values of $|z + 5 - i|$ consider a line passing through the point $(-5, 1)$ and the centre of the circle. The distance between these two points is $\sqrt{(-5)^2 + 1^2} = \sqrt{26}$. Hence the minimum value of $|z + 5 - i|$ is $\sqrt{26} - 3$ and the maximum is $\sqrt{26} + 3$.

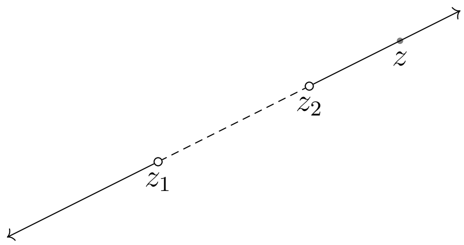
18c i Like before, $|z - z_0| = r$ represents a circle of centre z_0 and radius r. The distance to the centre of the circle from the origin is $|z|$, thus the minimum distance from the origin to the circle circumference is $\big||z_0| - r\big|$ and the maximum distance is $\big||z_0| - r\big|$. Hence $\big||z_0| - r\big| \leq |z| \leq |z_0| + r$.

18c ii Like before, $|z - z_0| = r$ represents a circle of centre z_0 and radius r. The distance to the centre of the circle from the origin is $|z|$, thus the minimum distance from z_1 to the circle circumference is $\big||z_0 - z_1| - r\big|$ and the maximum distance is $\big||z_0 - z_1| + r\big|$. Hence, $\big||z_0 - z_1| - r\big| \leq |z - z_1| \leq |z_0 - z_1| + r$.

18d In each case we let $w = z - z_0$ and z $= z_0$ or $z_0 - z_1$ and observe that the above inequalities are produced.

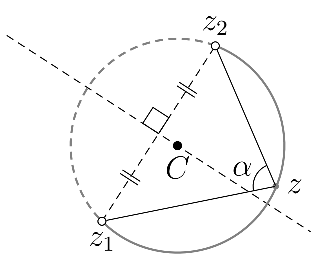
Solutions to Exercise 1F Enrichment questions

19 **a** When $\alpha = 0$, vector $\overrightarrow{z_1 z}$ is parallel with $\overrightarrow{z_2 z}$ with common point z, hence z_1, z_2 and z are collinear, with z on the same side of z_1 and z_2 (not between) so z on either of the opposite rays with end points z_1 and z_2.

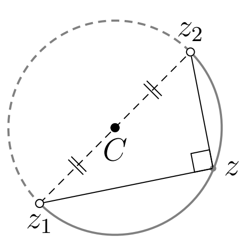


$\arg\left(\dfrac{z - z_1}{z - z_2}\right)$ is undefined at z_1 and z_2 so these points are excluded. The diagram shows one possible location of z on one of these rays.

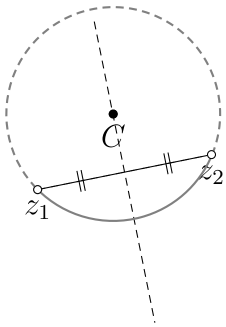
b The angle between vectors $\overrightarrow{z_1 z}$ and $\overrightarrow{z_2 z}$ is constant so by the converse of the angle in the same segment, z lies on an arc of a circle through z_1 and z_2, excluding the endpoints and taken anticlockwise. Since the angle at z is acute, it is the angle in a major segment. The centre will be somewhere on the perpendicular bisector of $z_1 z_2$ and on the same side as the arc.



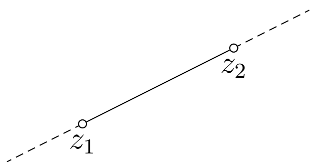
c By similar reasoning to **b**, this is the angle in a semi-circle. The centre is $\dfrac{z_1 + z_2}{2}$



d Likewise this is the angle in a minor segment



e When $\alpha = \pi$, the vector $\overrightarrow{z_1 z}$ and $\overrightarrow{z_2 z}$ are in opposite directions, hence z is in the line segment between z_1 and z_2



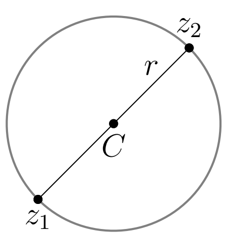
20 $|z - z_1|^2 + |z - z_2|^2 = |z_1 - z_2|^2$

The three distances satisfy Pythagoras theorem, hence $\triangle z z_1 z_2$ is right-angled at z.

Hence z lies on the circle with diameter $z_1 z_2$, including the end points.

The centre is $\dfrac{z_1 + z_2}{2}$

The radius is $r = \dfrac{|z_1 + z_2|}{2}$



21 **a** **i** squaring $k^2|z-z_1|^2 = l^2|z-z_2|^2$

putting $z = x+iy$, $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ and completing squares will give the desired result after a lot of algebra. Alternatively:

Note that $|z-w|^2 = (z-w)(\bar{z}-\bar{w})$

$$= |z|^2 - z\bar{w} - \bar{z}w + |w|^2$$

so $k^2(|z|^2 - z\bar{z}_1 - \bar{z}z_1 + |z_1|^2) = l^2(|z|^2 - z\bar{z}_2 - \bar{z}z_2 + |z_2|^2)$

Rearranging

$(k^2-l^2)|z|^2 - z(k^2\bar{z}_1 - l^2\bar{z}_2) - \bar{z}(k^2z_1 - l^2z_2) = l^2|z_2|^2 - k^2|z_1|^2$

Now complete the square on the LHS

$$|z|^2 - z\frac{k^2\bar{z}_1 - l^2\bar{z}_2}{k^2-l^2} - \bar{z}\frac{k^2z_1 - l^2z_2}{k^2-l^2} = \frac{l^2|z_2|^2 - k^2|z_1|^2}{k^2-l^2}$$

$$|z|^2 - z\left(\overline{\frac{k^2z_1 - l^2z_2}{k^2-l^2}}\right) - \bar{z}\left(\frac{k^2z_1 - l^2z_2}{k^2-l^2}\right) + \left|\frac{k^2z_1 - l^2z_2}{k^2-l^2}\right|^2 = \frac{l^2|z_2|^2 - k^2|z_1|^2}{k^2-l^2} + \left|\frac{k^2z_1 - l^2z_2}{k^2-l^2}\right|^2$$

so LHS$= \left|z - \frac{k^2z_1 - l^2z_2}{k^2-l^2}\right|^2$

and RHS$= \frac{l^2|z_2|^2 - k^2|z_1|^2}{k^2-l^2} + \frac{k^4|z_1|^2 - k^2l^2z_1\bar{z}_2 - k^2l^2\bar{z}_1z_2 + l^4|z_2^2|}{(k^2-l^2)^2}$

$$= \frac{\left[k^2l^2|z_2|^2 - \cancel{l^4|z_2|} - \cancel{k^4|z_1|^2} + k^2l^2|z_1|^2 + \cancel{k^4|z_1|^2} - k^2l^2z_1\bar{z}_2 - k^2l^2\bar{z}_1z_2 + \cancel{l^4|z_2|^2}\right]}{k^2-l^2}$$

$$= \frac{[k^2l^2|z_2|^2 + k^2l^2|z_1|^2 - k^2l^2z_1\bar{z}_2 - k^2l^2\bar{z}_1z_2]}{(k^2-l^2)^2}$$

So RHS$= \frac{k^2l^2|z_1-z_2|^2}{(k^2-l^2)^2}$

Thus $\left|z - \frac{k^2z_1 - l^2z_2}{k^2-l^2}\right|^2 = \frac{k^2l^2|z_1-z_2|^2}{(k^2-l^2)^2}$

Or, taking square roots

Thus $\left|z - \frac{k^2z_1 - l^2z_2}{k^2-l^2}\right| = \frac{kl|z_1-z_2|}{k^2-l^2}$

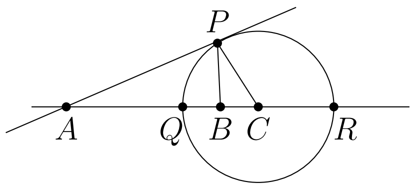
But this is the equation of a circle $|z-c| = r$

where centre $c = \frac{k^2z_1 - l^2z_2}{k^2-l^2}$

and radius $r = \frac{kl|z_1-z_2|}{k^2-l^2}$

as required

ii In the diagram Let $A = z_1$, $B = z_2$. Let Q divide AB internally in the ration $k : l$ and let R divide AB externally in the ratio $k : l$. Let C be the midpoint of QR and let P lie on the circle with diameter QR.



The task is to show that $AP : PB = k : l$

Since ratios are involved, similar $\triangle$s will be used.

First note that $Q = \dfrac{kz_2 + lz_1}{k + l}$ $\quad R = \dfrac{kz_2 - lz_1}{k - l}$

$$\begin{aligned}\text{Then } C &= \frac{1}{2}(Q + R)\\ &= \frac{1}{2}\left(\frac{kz_2 + lz_1}{k + l} + \frac{kz_2 - lz_1}{k - l}\right)\\ &= \frac{1}{2}\left(\frac{k^2z_2 - \cancel{klz_2} + \cancel{klz_1} - l^2z_1 + k^2z_2 + \cancel{klz_2} - \cancel{klz_1} - l^2z_1}{k^2 - l^2}\right)\\ &= \frac{1}{2}\left(\frac{2k^2z_2 - 2l^2z_1}{k^2 - l^2}\right) = \left(\frac{k^2z_2 - l^2z_1}{k^2 - l^2}\right)\end{aligned}$$

$$\begin{aligned}QC &= \left|\frac{k^2z_2 - l^2z_1}{k^2 - l^2} - \frac{kz_2 + lz_1}{k + l}\right|\\ &= \left|\frac{k^2z_2 - l^2z_1 - k(k - l)z_2 - l(k - l)z_1}{k^2 - l^2}\right|\\ &= \left|\frac{klz_2 - klz_1}{k^2 - l^2}\right|\\ &= \frac{kl|z_2 - z_1|}{|k^2 - l^2|}\end{aligned}$$

Next consider $\triangle APC$ and$\triangle PBC$

clearly $\angle C$ is in common

$$\begin{aligned}\text{Next } \frac{AC}{PC} &= \frac{\frac{1}{2}(AR + AQ)}{\frac{1}{2}(AR - AQ)}\\ &= \frac{1 + \left(\dfrac{AQ}{AR}\right)}{1 - \left(\dfrac{AQ}{AR}\right)}\end{aligned}$$

But, by the ratio division, $\dfrac{AQ}{AR} = \dfrac{k - l}{k + l}$ so

$$\begin{aligned}\frac{AC}{PC} &= \frac{(k+l)+(k-l)}{(k+l)-(k-l)}\\ &= \frac{2k}{2l}\\ &= \frac{k}{l}\end{aligned}$$

Lastly $\dfrac{PC}{BC} = \dfrac{\frac{1}{2}(AR-AQ)}{AC-AB}$

$$\begin{aligned}&= \frac{(AR-AQ)}{(AR+AQ)-2AB}\\ &= \frac{1-\frac{AQ}{AR}}{1+\frac{AQ}{AR}-2\frac{AB}{AR}}\end{aligned}$$

but $\dfrac{AB}{AR} = \dfrac{k-l}{k}$ by the ratio division, so

$$\begin{aligned}\frac{PC}{BC} &= \frac{\left(1-\frac{k-l}{k+l}\right)}{\left(1+\frac{k-l}{k+l}-2\frac{k-l}{k}\right)} \times \frac{(k+l)k}{(k+l)k}\\ &= \frac{((k+l)-(k-l))k}{(k+l)k+k(k-l)-2(k^2-l^2)}\\ &= \frac{2kl}{2l^2}\\ &= \frac{k}{l}\end{aligned}$$

Hence $\triangle APC \;|||\; \triangle PBC$ (SAS)

with ratio $\dfrac{k}{l}$

Thus $AP : PB = k : l$ (ratio of matching sides in similar $\triangle$s)

That is P satisfies the given equation, and this is true for any point on the circle. That is, tge circle is the graph of the equation.

b Return to the original equation $k|z-z_1| = l|z-z_2|$

in the limit as $k \to l$ this gives

$k|z-z_1| = k|z-z_2|$

thats is $|z-z_1| = |z-z_2|$

which is the perpendicular bisector of z_1 and z_2 (AB)

Solutions to Exercise 1G Foundation questions

1a $P(x) = x^3 - 6x + 4$

The factors of 4 are $\pm 1, \pm 2, \pm 4$.

Substituting each value for x into the equation to find when $P(x) = 0$ gives $P(2) = 0$ and hence $x - 2$ is a factor of $P(x) = x^3 - 6x + 4$.

Dividing $x^3 - 6x + 4$ by $(x - 2)$ gives $x^2 + 2x - 2$.

Thus

$$\begin{aligned} P(x) &= (x-2)(x^2+2x-2) \\ &= (x-2)((x^2+2x+1)-3) \\ &= (x-2)\left((x+1)^2-(\sqrt{3})^2\right) \qquad \text{(completing the square)} \\ &= (x-2)\left(x+1-\sqrt{3}\right)\left(x+1+\sqrt{3}\right) \qquad \text{(difference of two squares)} \end{aligned}$$

1b $P(x) = x^3 + 3x^2 - 2x - 2$

The factors of 2 are $\pm 1, \pm 2$.

Substituting each value for x into the equation to find when $P(x) = 0$ gives $P(1) = 0$ and hence $x - 1$ is a factor of $P(x) = x^3 + 3x^2 - 2x - 2$.

Dividing $x^3 + 3x^2 - 2x - 2$ by $(x - 1)$ gives $x^2 + 4x + 2$.

Thus

$$\begin{aligned} P(x) &= (x-1)(x^2+4x+2) \\ &= (x-1)((x^2+4x+4)-2) \\ &= (x-1)((x+2)^2-(\sqrt{2})^2) \\ &= (x-1)\left(x+2-\sqrt{2}\right)\left(x+2+\sqrt{2}\right) \end{aligned}$$

1c $P(x) = x^3 - 3x^2 - 2x + 4$

The factors of 4 are ±1, ±2, ±4.

Substituting each value for x into the equation to find when $P(x) = 0$ gives $P(1) = 0$ and hence $x - 1$ is a factor of $P(x) = x^3 - 3x^2 - 2x + 4$.

Dividing $x^3 - 3x^2 - 2x + 4$ by $(x - 1)$ gives $x^2 - 2x - 4$.

Thus

$$\begin{aligned} P(x) &= (x-1)(x^2 - 2x - 4) \\ &= (x-1)((x^2 - 2x + 1) - 5) \\ &= (x-1)((x-1)^2 - (\sqrt{5})^2) \\ &= (x-1)\left(x - 1 - \sqrt{5}\right)\left(x - 1 + \sqrt{5}\right) \end{aligned}$$

2a Since the coefficients of the polynomial are real, all complex roots must also have their conjugate as a root. Complex zeroes occur in conjugate pairs.

2b $1 + i$ and $1 - i$ are zeroes of $P(x)$.

Let the third zero of the polynomial be a, then by the sum of the roots:

$$a + 1 + i + 1 - i = -\frac{(-8)}{1}$$

$$a + 2 = 8$$

$$a = 6$$

Hence the other root is 6.

3a Since the coefficients of the equation are real, the complex conjugate of $1 - 2i$ which is $1 + 2i$ must also be a root.

3b Since $1+2i$ and $1-2i$ are both roots, it follows that $\big(x-(1+2i)\big)$ and $\big(x-(1-2i)\big)$ must be factors of $P(x)$.

$\big(x-(1+2i)\big)\big(x-(1-2i)\big)$

$= x^2-(1+2i+1-2i)x+(1+2i)(1-2i)$

$= x^2-2x+1-4i^2$

$= x^2-2x+5$

Hence x^2-2x+5 must also be a factor of $P(x)$.

3c Let the third root be a. Using the sum of the roots:

$a+1+2i+1-2i=-\dfrac{0}{1}$

$a+2=0$

$a=-2$

So $(x+2)$ is a factor of $P(x)$.

Thus

$P(x)=(x+2)(x^2-2x+5)$

4a Since the coefficients of the equation are real, the complex conjugate of $-3i$ which is $3i$ must also be a root.

4b Since $3i$ and $-3i$ are both roots, it follows that $(z-3i)$ and $(z+3i)$ must be factors of $P(z)$.

$(z-3i)(z+3i)$

$= z^2-9i^2$

$= z^2+9$

Hence z^2+9 is a quadratic factor of $P(z)$.

4c Let the third root be a. Using the sum of the roots:

$$a + 3i + (-3i) = -\frac{3}{2}$$

$$a = -\frac{3}{2}$$

So $\left(z + \frac{3}{2}\right)$ is a factor of $P(z)$ or $(2z + 3)$ is a factor of $P(z)$.

Hence $P(z)$ as a product of factors with real coefficients can be written as:

$$P(z) = (2z + 3)(z^2 + 9)$$

5a $P(3 + i)$

$= 2(3 + i)^3 - 13(3 + i)^2 + 26(3 + i) - 10$

$= 2(9 + 6i + i^2)(3 + i) - 13(9 + 6i + i^2) + 78 + 26i - 10$

$= 2(9 + 6i - 1)(3 + i) - 13(9 + 6i - 1) + 78 + 26i - 10$

$= 2(8 + 6i)(3 + i) - 13(8 + 6i) + 78 + 26i - 10$

$= 2(24 + 8i + 18i + 6i^2) - 13(8 + 6i) + 78 + 26i - 10$

$= 2(24 + 26i - 6) - 13(8 + 6i) + 78 + 26i - 10$

$= 2(18 + 26i) - 13(8 + 6i) + 78 + 26i - 10$

$= 36 + 52i - 104 - 78i + 78 + 26i - 10$

$= 0 + 0i$

$= 0$

5b Since $3 + i$ is a root of $P(z)$, and since $P(z)$ has real coefficients, the complex conjugate of $3 + i$ which is $3 - i$ must also be a root of $P(z)$.

Hence $P(3 - i) = 0$.

5c i Let the third root be a. Using the sum of the roots of $P(z)$:

$$a + 3 + i + 3 - i = -\frac{(-13)}{2}$$

$$a + 6 = \frac{13}{2}$$

$$a = \frac{1}{2}$$

So $\left(z - \frac{1}{2}\right)$ is a factor of $P(z)$ or $(2z - 1)$ is a factor of $P(z)$.

Hence $P(z)$ as a product of linear factors can be written as:

$$P(z) = (2z - 1)\big(z - (3 + i)\big)\big(z - (3 - i)\big)$$

$$= (2z - 1)(z - 3 - i)(z - 3 + i)$$

5c ii $P(z) = (2z - 1)(z - 3 - i)(z - 3 + i)$ (from question 5c i)

$$= (2z - 1)(z^2 - 3z + zi - 3z + 9 - 3i - zi + 3i - i^2)$$

$$= (2z - 1)(z^2 - 6z + 10)$$

So $P(z)$ as a product of a linear factor and a quadratic factor, with real coefficients, can be written as:

$$P(z) = (2z - 1)(z^2 - 6z + 10)$$

Solutions to Exercise 1G Development questions

6a The coefficients of $Q(x)$ are real, so complex zeroes occur in conjugate pairs.

6b $\alpha + \beta + \delta + \gamma = 6$

$2i - 2i + \delta + \gamma = 6$

$\delta + \gamma = 6 \qquad (1)$

$\alpha\beta\delta\gamma = 16$

$(2i)(-2i)\delta\gamma = 16$

$4\delta\gamma = 16$

$\delta\gamma = 4$

$\delta = \frac{4}{\gamma} \qquad (2)$

Subbing (2) into (1) gives,

$\frac{4}{\gamma} + \gamma = 6$

$4 + \gamma^2 = 6\gamma$

$\gamma^2 - 6\gamma + 4 = 0$

$$\begin{aligned}\gamma &= \frac{6 \pm \sqrt{6^2 - 4(1)(4)}}{2(1)}\\ &= \frac{6 \pm \sqrt{20}}{2}\\ &= \frac{6 \pm 2\sqrt{5}}{2}\\ &= 3 \pm \sqrt{5}\end{aligned}$$

So, the two other zeroes are $3 + \sqrt{5}$ and $3 - \sqrt{5}$

6c i $(x - 2i)(x + 2i)(x - 3 - \sqrt{5})(x - 3 + \sqrt{5})$

6c ii $(x-2i)(x+2i)\left(x-3-\sqrt{5}\right)\left(x-3+\sqrt{5}\right)$

$=(x^2-4i^2)\left(x-3-\sqrt{5}\right)\left(x-3+\sqrt{5}\right)$

$=(x^2+4)(x-3-\sqrt{5})(x-3+\sqrt{5})$

6c iii $(x^2+4)(x-3-\sqrt{5})(x-3+\sqrt{5})$

$=(x^2+4)(x^2-3x+x\sqrt{5}-3x+9-3\sqrt{5}-x\sqrt{5}+3\sqrt{5}-5)$

$=(x^2+4)(x^2-6x+4)$

7a Since all the coefficients are real, and since $1+3i$ is a root, its complex conjugate, $1-3i$ must also be a root. The sum of the roots is

$$\alpha+\beta+\gamma+\delta=-\frac{b}{a}=-(-3)$$

$1+3i+1-3i+\gamma+\delta=3$

$2+\gamma+\delta=3$

$\gamma+\delta=1 \qquad (1)$

The product of the roots is

$$\alpha\beta\delta\gamma=\frac{d}{a}=-60$$

$(1-3i)(1+3i)\delta\gamma=-60$

$10\delta\gamma=-60$

$\delta\gamma=-6$

$\delta=-\frac{6}{\gamma} \qquad (2)$

Substituting (2) into (1):

$$\gamma-\frac{6}{\gamma}=1$$

$\gamma^2-6=\gamma$

$\gamma^2-\gamma-6=0$

$(\gamma-3)(\gamma+2)=0$

$\gamma=-2$ or 3

Hence $x = 1 \pm 3i$, 3 or -2.

7b Since all the coefficients are real, and since $1 - i$ is a root, its complex conjugate, $1 + i$ must also be a root. The sum of the roots is

$$\alpha + \beta + \gamma + \delta = -\frac{b}{a} = -(-6)$$

$$1 - i + 1 + i + \gamma + \delta = 6$$

$$2 + \gamma + \delta = 6$$

$$\gamma + \delta = 4 \qquad (1)$$

The product of the roots is

$$\alpha\beta\delta\gamma = \frac{d}{a} = 10$$

$$(1 - i)(1 + i)\delta\gamma = 10$$

$$2\delta\gamma = 10$$

$$\delta\gamma = 5$$

$$\delta = \frac{5}{\gamma} \qquad (2)$$

Substituting (2) into (1):

$$\gamma + \frac{5}{\gamma} = 4$$

$$\gamma^2 + 5 = 4\gamma$$

$$\gamma^2 - 4\gamma + 5 = 0$$

$$\gamma = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(5)}}{2}$$

$$= \frac{4 \pm \sqrt{-4}}{2}$$

$$= \frac{4 \pm 2i}{2}$$

$$= 2 \pm i$$

Hence $x = 2 \pm i$, $1 \pm i$.

8a $P(x) = x^4 - 5x^3 + 4x^2 + 3x + 9$

$P'(x) = 4x^3 - 15x^2 + 8x + 3$

$P'(3) = 4(3)^3 - 15(3)^2 + 8(3) + 3 = 0$

Hence, $x = 3$ is a double root.

8b The sum of the roots of the equation is

$\alpha + \beta + \gamma + \delta = -\dfrac{b}{a} = -\dfrac{(-5)}{1}$

$3 + 3 + \gamma + \delta = 5$

$\gamma + \delta = -1 \qquad (1)$

The product of the roots is

$\alpha\beta\gamma\delta = 9$

$(3)(3)\gamma\delta = 9$

$\gamma\delta = 1$

$\delta = \frac{1}{\gamma} \qquad\qquad (2)$

Substituting (2) into (1):

$\gamma + \dfrac{1}{\gamma} = -1$

$\gamma^2 + 1 = -\gamma$

$\gamma^2 + \gamma + 1 = 0$

$$\gamma = \frac{-1 \pm \sqrt{1^2 - 4(1)(1)}}{2(1)}$$

$$= \frac{-1 \pm \sqrt{-3}}{2}$$

$$= \frac{-1 \pm \sqrt{3}i}{2}$$

Hence $x = 3, -\frac{1}{2} + \frac{\sqrt{3}}{2}i, -\frac{1}{2} - \frac{\sqrt{3}}{2}i.$

9a Since $a + ib$ and $a + 2ib$ are roots, and since the coefficients of the equation are real, their complex conjugates, $a - ib$ and $a - 2ib$ must also be roots.

The sum of the roots is

$$\alpha + \beta + \gamma + \delta = -\frac{b}{a}$$

$$a + ib + a + 2ib + a - ib + a - 2ib = -\frac{-12}{1}$$

$$4a = 12$$

$$a = 3$$

9b The product of the roots is

$$\alpha\beta\gamma\delta = \frac{d}{a}$$

$$\alpha\beta\gamma\delta = 130$$

$$(a + ib)(a + 2ib)(a - ib)(a - 2ib) = 130$$

$$(a + ib)(a - ib)(a + 2ib)(a - 2ib) = 130$$

$$(a^2 + b^2)(a^2 + 4b^2) = 130$$

Since $a = 3$

$$(3^2 + b^2)(3^2 + 4b^2) = 130$$

$$(9 + b^2)(9 + 4b^2) = 130$$

$$81 + 36b^2 + 9b^2 + 4b^4 = 130$$

$$4b^4 + 45b^2 - 49 = 0$$

$$4(b^2)^2 + 45b^2 - 49 = 0$$

$$(4b^2 + 49)(b^2 - 1) = 0$$

So, $b^2 = -\frac{49}{4}$ or $b^2 = 1$

Since b is real, $b^2 \geq 0$, thus $b^2 = 1$.

Hence $b = \pm 1$

Since $b > 0$, $b = 1$.

9c The roots are $3 \pm i$, $3 \pm 2i$, hence as a product of roots, the polynomial is

$(z-3-i)(z-3+i)(z-3-2i)(z-3+2i)$

which is

$(z^2-6z+10)(z^2-6z+13)$

10a $P(2i) = (2i)^3 + k(2i)^2 + 6$

$= -8i - 4k + 6$

$= (6-4k) - 8i$

10b $P(x) = (x^2+4)Q(x) - 4x - 6$

$P(2i) = ((2i)^2+4)Q(2i) - 4(2i) - 6$

$= (-4+4)Q(2i) - 8i - 6$

$= 0 - 8i - 6$

$= -8i - 6$

Equating this to the result in part **a** gives,

$(6-4k) - 8i = -8i - 6$

$4k - 6 = 6$

$4k = 12$

$k = 3$

11a $P(-i) = (-i)^3 - (-i)^2 + m(-i) + n$

$= i + 1 - im + n$

$= (1+n) + i(1-m)$

11b $P(x) = (x^2 + 1)Q(x) + 6x - 3$

$P(-i) = ((-i)^2 + 1)Q(-i) + 6(-i) - 3$

$= (-1 + 1)Q(-i) - 6i - 3$

$= -6i - 3$

Equating this to the result in part **a** gives,

$(1 + n) + i(1 - m) = -6i - 3$

Equating real parts,

$1 + n = -3$

$n = -4$

Equating imaginary parts,

$1 - m = -6$

$m = 7$

Hence $m = 7, n = -4$.

12a By the remainder theorem the remainder is

$P(-2i) = (-2i)^3 + (-2i)^2 + 6(-2i) - 3$

$= 8i - 4 - 12i - 3$

$= -7 - 4i$

12b i By the remainder theorem the remainder is

$P(2i) = (2i)^3 + (2i)^2 + 6(2i) - 3$

$= -8i - 4 + 12i - 3$

$= -7 + 4i$

12b ii Since we are dividing by a degree 2 polynomial, the remainder will be of degree 1, hence

$P(x) = (x^2 + 4)Q(x) + ax + b$

$P(2i) = ((2i)^2 + 4)Q(x) + a(2i) + b$

$-7 + 4i = (-4 + 4)Q(x) + a(2i) + b$

$-7 + 4i = (0)Q(x) + a(2i) + b$

$-7 + 4i = 2ai + b \qquad (1)$

$P(-2i) = ((-2i)^2 + 4)Q(x) + a(-2i) + b$

$-7 - 4i = (-4 + 4)Q(x) + a(-2i) + b$

$-7 - 4i = (0)Q(x) + a(-2i) + b$

$-7 - 4i = -2ai + b \qquad (2)$

Adding (1) and (2):

$-14 = 2b$

$b = -7$

Subtracting (2) from (1):

$8i = 4ai$

$a = 2$

Hence the remainder is $2x - 7$.

13a $P(iw) = (iw)^8 - \frac{5}{2}(iw)^4 + 1$

$= (i^8w^8) - \dfrac{5}{2}(i^4w^4) + 1$

$= w^8 - \dfrac{5}{2}w^4 + 1$

$= P(w)$

$= 0 \qquad$ (since w is a root)

$P\left(\dfrac{1}{w}\right) = \left(\dfrac{1}{w}\right)^8 - \dfrac{5}{2}\left(\dfrac{1}{w}\right)^4 + 1$

$$= \frac{1}{w^8} - \frac{5}{2w^4} + 1$$

$$= \frac{1}{w^8}\left(1 - \frac{5}{2}w^4 + w^8\right)$$

$$= \frac{1}{w^8}\left(w^8 - \frac{5}{2}w^4 + 1\right)$$

$$= \frac{1}{w^8}P(w)$$

$$= \frac{1}{w^8}(0) \qquad \text{(since } w \text{ is a root)}$$

$$= 0$$

Hence w and $\frac{1}{w}$ are both roots of the equation.

13b Factorising the equation (treating it as a quadratic with z^4 as the variable) gives $P(z) = \frac{1}{2}(z^4 - 2)(2z^4 - 1)$ so one root is $z = \sqrt[4]{2}$.

13c The polynomial is of degree 8, so, 8 roots exist.

$z^4 - 2 = 0$

$z = \pm\sqrt[4]{2}$

From parts(a) and (b),

$\pm i\sqrt[4]{2}$; $\pm\frac{1}{\sqrt[4]{2}}$ and $\pm\frac{i}{\sqrt[4]{2}}$ are also the roots of the equation

14a $P(x)$ has minimum value B, when $x = 0$. We know that this is a minimum since $A > 0$, and so for all real x it will be the case that $x^4 + Ax^2 \geq 0$.

Since $B > 0$, it follows that $P(x) > 0$ for all real values of x.

14b $-ic, -id$; the coefficients of $P(x)$ are real, so zeroes occur in conjugate pairs.

14c Since ic is a root,

$P(ic) = 0$

$(ic)^4 + A(ic)^2 + B = 0$

$c^4 - Ac^2 + B = 0$

Similarly, since id is a root,

$d^4 - Ad^2 + B = 0$

Adding these two equations together gives

$c^4 - Ac^2 + B + d^4 - Ad^2 + B = 0$

$c^4 + d^4 = A(c^2 + d^2) - 2B$

Now, the product of the pairs of the roots will be

$$(ic)(-ic) + (ic)(id) + (ic)(-id) + (-ic)(id) + (-ic)(-id) + (id)(-id) = \frac{c}{a}$$

$$c^2 + d^2 = \frac{A}{1}$$

$c^2 + d^2 = A$

Hence

$c^4 + d^4 = A^2 - 2B$

15a They form a conjugate pair, since $P(x)$ has real coefficients.

15b $P'(x) = 3x^2 + c$

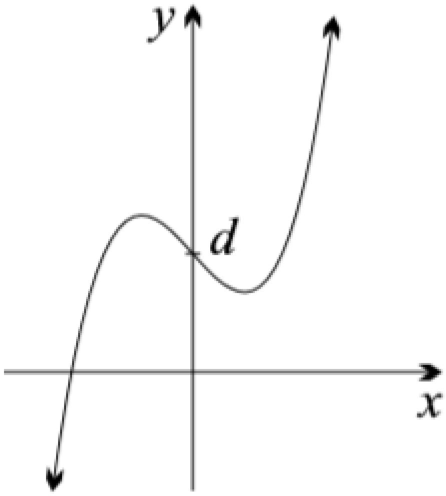
Since the graph has turning points, there must be real values of x such that $P'(x) = 0$. That is for

$3x^2 + c = 0$

$c = -3x^2$

Since x and c are real in this case, it follows that $c < 0$

15c



15d The sum of the products of the roots will be

$(a + ib)(a - ib) + k(a + ib) + k(a - ib) = c$

$a^2 + b^2 + 2ka = c$

$2ka = c - a^2 - b^2$

$k = \dfrac{1}{2a}(c - a^2 - b^2)$

Now since $a^2 + b^2 \geq 0$ it follows that $-a^2 - b^2 \leq 0$ and hence from part (b), $c - a^2 - b^2 \leq 0$. Thus $2ka < 0$. Since $k < 0$ it follows that $a > 0$ as required.

15e Roots of the equation are: $k, a \pm ib$

Sum of the roots:

$0 = k + (a + ib) + (a - ib)$

$k = -2a \qquad (1)$

Sum of products of two roots:

$c = k(a + ib) + k(a - ib) + (a^2 + b^2)$

$\quad = 2ka + a^2 + b^2$

Substituting (1):

$c = 2(-2a)a + a^2 + b^2$

$\quad = b^2 - 3a^2$

$b^2 = c + 3a^2 \qquad (2)$

Product of the roots:

$-d = k(a + ib)(a - ib)$

Substituting (1):

$-d = (-2a)(a^2 + b^2)$

$d = 2a(a^2 + b^2)$

Substituting (2):

$d = 2a(a^2 + 3a^2 + c)$

$\quad = 8a^3 + 2ac$

16a The minimum stationary point is at $x = 1$. $f(1) = k - 2 > 0$. Hence the graph of $f(x)$ has only one x-intercept which lies to the left of the maximum stationary point at $x = -1$.

16b $f(x)$ has real coefficients.

16c Let α be the third root of the equation, the sum of the roots is

$a + ib + a - ib + \alpha = 0$

Hence the third root is

$\alpha = -2a$

The sum of the products of pairs gives:

$-2a(a + ib) - 2a(a - ib) + (a + ib)(a - ib) = -3$

$-4a^2 + a^2 + b^2 = -3$

$b^2 = 3a^2 - 3$

$b^2 = 3(a^2 - 1)$

16d The product of the roots is

$-2a(a+ib)(a-ib) = -2702$

$-2a(a^2+b^2) = -2702$

$-2a\left(a^2+3(a^2-1)\right) = -2702$

$-2a(4a^2-1) = -2702$

$-8a^3+2a = -2702$

$8a^3-2a+2702 = 0$

$4a^3-a+1351 = 0$

Since a is an integer, it must also be a factor of 1351. The smallest factor greater than 1 is 7. Substituting 7 into the equation solves it, hence $a = 7$, $\alpha = -2 \times 7 = -14$ and $b^2 = 3(7^2-1) = 144$ so $b = \pm 12$.

Thus the roots of the equation are $-14, 7 \pm 12i$.

17 $P'(x) = 0 + 1 + \frac{2x}{2!} + \frac{3x^2}{3!} + \cdots + \frac{nx^{n-1}}{n!}$

$$P'(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{n-1}}{(n-1)!}$$

Double roots will occur if any root occurs at maxima or minima. That is,

$P(a) = 0$ for a such that $P'(a) = 0$, this leads to the equation

$P(X) - P'(X) = 0$

For this case we have:

$$P(X) - P'(X) = \frac{x^n}{n!},$$

so the only point where double roots can occur is $x = 0$,

but $P(X = 0) = 1$, hence no double roots.

18a $P(-1+2i)$

$= (-1+2i)^4 + 4(-1+2i)^3 + 14(-1+2i)^2 + 20(-1+2i) + 25$

$= -7 + 24i + 44 - 8i - 42 - 56i - 20 + 40i + 25$

$= 0$

$P'(x) = 4z^3 + 12z^2 + 28z + 20$

$P'(-1+2i)$

$= 4(-1+2i)^3 + 12(-1+2i)^2 + 28(-1+2i) + 20$

$= 44 - 8i - 36 - 48i - 28 + 56i + 20$

$= 0$

18b $-1+2i$ is a double zero of $P(z)$.

18c The coefficients of $P(z)$ are real and $-1+2i$ counts as two of the zeroes of $P(z)$ so its conjugate $-1-2i$ must also count as two zeroes.

18d $P(z) = (z+1-2i)^2(z+1+2i)^2$

$= (z^2 + 2z + 5)^2$

Solutions to Exercise 1G Enrichment questions

19 Let $P(z) = a_n z^n + \cdots + a - 2z^2 + a_1 z + a_0$

where the coefficients are real,

and let $P(\alpha) = \beta$ where α and β are complex (possibly real). Then

$$
\begin{aligned}
P(\overline{\alpha}) &= a_n(\overline{\alpha})^n + \cdots + a_2(\overline{\alpha})^2 + a_1(\bar{\alpha}) + a_0 \\
&= a_n\overline{(\alpha^n)} + \cdots + a_2\overline{(\alpha^2)} + a_1\overline{(\alpha)} + a_0 && \text{since } (\bar{z})^n = \overline{(z^n)} \\
&= \overline{(a_n\alpha^n)} + \cdots + \overline{(a_2\alpha^2)} + \overline{(a_1\alpha)} + \overline{a_0} && \text{since } (a\bar{z}) = \overline{(az)},\ \text{where } a \text{ is real} \\
&= \overline{a_n(\overline{\alpha})^n + \cdots + a_2(\overline{\alpha})^2 + a_1(\bar{\alpha}) + a_0} && \text{since } \bar{w} + \bar{z} = \overline{(w+z)} \\
&= \overline{P(\alpha)} \\
&= \bar{\beta} && \#
\end{aligned}
$$

20 $P\left(\dfrac{p}{q}\right) = 0$ so

$$a_n\frac{p^n}{q^n} + \cdots + a_2\frac{p^2}{q^2} + a_1\frac{p}{q} + a_0 = 0$$

So $a_n p^n + \cdots + a_2 p^2 q^{n-2} + a_1 p q^{n-1} + a_0 q^n = 0$

Thus $a_{n-1}p^{n-1}q + \cdots + a_2 p^2 q^{n-2} + a_1 p q^{n-1} + a_0 q^n = -a_n p^n$

or $q\left(a_{n-1}p^{n-1} + \cdots + a_2 p^2 q^{n-3} + a_1 p q^{n-2} + a_0 q^{n-1}\right) = -a_n p^n$

Hence $a_n p^n$ is a multiple of q and q is not a factor of p^n. Hence q is a factor of a_n.

Likewise

$p\left(a_1 q^{n-1} + a_2 q^{n-2} p + \cdots + a_n p^{n-1}\right) = -a_0 q^n$

Hence $a_0 q^n$ is a multiple of p and p is not a factor of q^n. Hence p is a factor of a_0.

21 By way of contradiction, suppose there exists an odd polynomial with real coefficients that does not have a real zero.

By the fundamental Theorem of Algebra, it has at least one complex zero and by box 38, the number of zeroes counted by multiplicity is odd.

But the coefficients are real and so the zeroes come in conjugate pairs, by which there is an even number of zeroes.

The number of zeroes cannot be both even and odd, so there is a contradiction.

Hence every odd polynomial with real coefficients has at least one real zero.

22 Since $P(z)$ has double zero $z = \alpha$, it has factor $(z-\alpha)^2$.

a Since $P(z)$ has real coefficients, it also has double zero $z = \bar{\alpha}$ and factor $(z-\bar{\alpha})^2$

b Thus it is possible to write

$$\begin{aligned} P(z) &= (z-\alpha)^2(z-\bar{\alpha})^2Q(z) \\ &= \left((z-\alpha)(z-\bar{\alpha})\right)^2 Q(z) \\ &= (z^2 - 2\mathrm{Re}(\alpha)z + |\alpha|^2)^2 Q(z) \end{aligned}$$

Thus $(z^2 - 2\mathrm{Re}(\alpha)z + |\alpha|^2)^2$ is a factor of $P(z)$

c

$$\begin{aligned} P'(z) &= 2(z^2 - 2\mathrm{Re}(\alpha)z + |\alpha|^2).(2z - 2\mathrm{Re}(\alpha)).Q(z) + (z^2 - 2\mathrm{Re}(\alpha)z + |\alpha|^2)^2 Q'(z) \\ &= (z^2 - 2\mathrm{Re}(\alpha)z + |\alpha|^2)\left[4(z - \mathrm{Re}(\alpha))Q(z) + (z^2 - 2\mathrm{Re}(\alpha)z + |\alpha|^2)Q'(z)\right] \\ &= (z-\alpha)(z-\bar{\alpha})\left[4(z - \mathrm{Re}(\alpha))Q(z) + (z-\alpha)(z-\bar{\alpha})Q'(z)\right] \end{aligned}$$

That is $(z-\alpha)$ is a factor of $P'(z)$ and hence $P'(\alpha) = 0$.

(It should also be clear that $P(\bar{\alpha}) = 0$)

d This is essentially the result of box 36 applied to complex zeroes. As noted there, induction may be used. The induction steps will essentially be parts **a**, **b** and **c** above.

23 **a** Let $u = a + b\sqrt{c}$ and $v = p + q\sqrt{c}$ then

i $$\begin{aligned} \text{LHS} &= u^* + v^* \\ &= a - b\sqrt{c} + p - q\sqrt{c} \\ &= (a + p) - (b + q)\sqrt{c} \\ \text{RHS} &= (u + v)^* \\ &= (a + b\sqrt{c} + p + q\sqrt{c})^* \\ &= ((a + p) + (b + q)\sqrt{c})^* \\ &= (a + p) - (b + q)\sqrt{c} \\ &= \text{LHS} \quad \# \end{aligned}$$

ii $$\begin{aligned} \text{LHS} &= \lambda u^* \\ &= \lambda(a - b\sqrt{c}) \\ &= \lambda a - \lambda b\sqrt{c} \\ \text{RHS} &= (\lambda u)^* \\ &= (\lambda a + \lambda b\sqrt{c})^* \\ &= \lambda a - \lambda b\sqrt{c} \\ &= \text{LHS} \quad \# \end{aligned}$$

iii $\underline{\underline{\text{A}}}$ $(u^1)^* = u^*$

$= (u^*)^1$

So the result is true for $n = 1$

$\underline{\underline{\text{B}}}$ Assume the result is true for $n = k$, a positive integer

That is, assume that

$$(u^k)^* = (u^*)^k \qquad (\star)$$

Let $u^k = p + q\sqrt{c}$

Now prove the result is true for $n = k + 1$, that is, prove that $(u^{k+1})^* = (u^*)^{k+1}$

Now LHS $= (u^k u)^*$

$= ((p + q\sqrt{c})(a + b\sqrt{c}))^*$

$= (ap + bqc + (aq + bp)\sqrt{c})^*$

$= ap + bqc - (aq + bp)\sqrt{c}$

RHS $= (u*)^k.u^*$

$= (u^k)^*.u^*$ by $(\star)$, the induction hypothesis

So RHS $= (p - q\sqrt{c})(a - b\sqrt{c})$

$= ap + bqc - (aq + bp)\sqrt{c}$

$=$ LHS #

$\underline{\underline{\text{C}}}$ It follows from parts A and B by mathematical induction that the result is true for all integers $n \geqslant 1$

b Let $P(x) = a_0 + a_1 x + \cdots + a_n x^n$ have rational coefficients and let $P(u) = 0$

then $P(u^*) = a_0 + a_1 u^* + a_2 (u^*)^2 + \cdots + a_n (y^*) x^n$

$= a_0 + a_1 (u)^* + a_2 (u^2)^* + \cdots + a_n (u^n)^*$ by **a iii**

$= a_0^* + (a_1 u)^* + (a_2 u^2)^* + \cdots + (a_n u^n)^*$ by **a ii**

$= (a_0 + a_1 u + a_2 u^2 + \cdots + a_n u^n)^*$ by **a i**

$= (P(u))^*$

$= 0^*$

$= 0$

Hence u^* is also a zero of $P(x)$. #

Solutions to Exercise 1H Chapter Review

1a $6z - \overline{w}$

$= 6(3 - i) - \overline{(17 + i)}$

$= 18 - 6i - (17 - i)$

$= 1 - 5i$

1b z^3

$= (3 - i)^3$

$= (3 - i)(3 - i)^2$

$= (3 - i)(9 - 6i + i^2)$

$= (3 - i)(8 - 6i)$

$= 24 - 18i - 8i + 6i^2$

$= 18 - 26i$

1c

$\frac{w}{z}$

$= \frac{17 + i}{3 - i}$

$= \frac{(17 + i)(3 + i)}{(3 - i)(3 + i)}$

$= \frac{51 + 20i + i^2}{9 - i^2}$

$= \frac{51 + 20i - 1}{9 + 1}$

$= \frac{50 + 20i}{10}$

$= 5 + 2i$

2a $z^2 + 100$

$= z^2 - 100i^2$

$= z^2 - (10i)^2$

$= (z + 10i)(z - 10i)$

2b $z^2 + 10z + 34$

Using the quadratic formula,

$$z = \frac{-10 \pm \sqrt{10^2 - 4(1)(34)}}{2(1)}$$

$$= \frac{-10 \pm \sqrt{-36}}{2}$$

$$= \frac{-10 \pm 6i}{2}$$

$= -5 \pm 3i$

Hence, the equation can be written as

$\big(z - (-5 - 3i)\big)\big(z - (-5 + 3i)\big)$

$= (z + 5 + 3i)(z + 5 - 3i)$

3a $z^2 - 8z + 25 = 0$

Using the quadratic formula,

$$z = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(1)(25)}}{2 \times 1}$$

$$= \frac{8 \pm \sqrt{64 - 100}}{2}$$

$$= \frac{8 \pm \sqrt{-36}}{2}$$

$$= \frac{8 \pm 6i}{2}$$

$= 4 \pm 3i$

3b $16z^2 + 16z + 13 = 0$

Using the quadratic formula,

$$z = \frac{-16 \pm \sqrt{16^2 - 4(16)(13)}}{2(16)}$$

$$= \frac{-16 \pm \sqrt{16^2 - 4(16)(13)}}{2(16)}$$

$$= \frac{-16 \pm \sqrt{256 - 832}}{32}$$

$$= \frac{-16 \pm \sqrt{-576}}{32}$$

$$= \frac{-16 \pm 24i}{32}$$

$$= -\frac{1}{2} \pm \frac{3}{4}i$$

4a Let z be a square root of $5 - 12i$, then

$$\begin{aligned} z^2 &= (a + bi)^2 \\ &= a^2 + 2abi + b^2 i^2 \\ &= a^2 - b^2 + 2abi \end{aligned}$$

Since $z^2 = 5 - 12i$

$5 - 12i = a^2 - b^2 + 2abi$

Equating real parts,

$5 = a^2 - b^2 \quad (1)$

Equating imaginary parts,

$-12 = 2ab$

$b = -\frac{6}{a} \qquad (2)$

Substituting (2) into (1):

$$5 = a^2 - \left(-\frac{6}{a}\right)^2$$

$$5 = a^2 - \frac{36}{a^2}$$

$5a^2 = a^4 - 36$

$a^4 - 5a^2 - 36 = 0$

$(a^2 - 9)(a^2 + 4) = 0$

Since a is real, $a^2 \geq 0$, hence $a^2 - 9 = 0$. Thus $(a - 3)(a + 3) = 0$ so $a = \pm 3$.

When $a = 3$, $b = -2$ and when $a = -3$, $b = 2$ so the roots are $z = \pm(3 - 2i)$.

4b Let z be a square root of $7 + 6\sqrt{2}i$, then

$$\begin{aligned} z^2 &= (a + bi)^2 \\ &= a^2 + 2abi + b^2i^2 \\ &= a^2 - b^2 + 2abi \end{aligned}$$

Since $z^2 = 7 + 6\sqrt{2}i$

$7 + 6\sqrt{2}i = a^2 - b^2 + 2abi$

Equating real parts,

$7 = a^2 - b^2 \qquad (1)$

Equating imaginary parts,

$6\sqrt{2} = 2ab$

$b = \dfrac{3\sqrt{2}}{a} \qquad (2)$

Substituting (2) into (1):

$7 = a^2 - \left(\dfrac{3\sqrt{2}}{a}\right)^2$

$7 = a^2 - \dfrac{18}{a^2}$

$7a^2 = a^4 - 18$

$a^4 - 7a^2 - 18 = 0$

$(a^2 - 9)(a^2 + 2) = 0$

Since a is real, $a^2 \geq 0$, hence $a^2 - 9 = 0$. Thus $(a + 3)(a - 3) = 0$ so $a = \pm 3$.

When $a = 3$, $b = \sqrt{2}$ and when $a = -3$, $b = -\sqrt{2}$

so the roots are $z = \pm(3 + \sqrt{2}i)$.

5a $z^2 - 5z + (7 + i) = 0$

$$\left(z^2 - 5z + \frac{25}{4}\right) - \frac{25}{4} + (7 + i) = 0$$

$$\left(z - \frac{5}{2}\right)^2 + \frac{3}{4} + i = 0$$

$$\left(z - \frac{5}{2}\right)^2 = -\frac{3}{4} - i$$

Let λ^2 be a square root of $-\frac{3}{4} - i$,

$$\begin{aligned}\lambda^2 &= (a + bi)^2 \\ &= a^2 + 2abi + b^2i^2 \\ &= a^2 - b^2 + 2abi\end{aligned}$$

Equating real parts,

$$a^2 - b^2 = -\tfrac{3}{4} \qquad (1)$$

Equating imaginary parts,

$$2ab = -1$$

$$b = -\frac{1}{2a} \qquad (2)$$

Substituting (2) into (1):

$$a^2 - \left(-\frac{1}{2a}\right)^2 = -\frac{3}{4}$$

$$a^2 - \frac{1}{4a^2} = -\frac{3}{4}$$

$$4a^4 - 1 = -3a^2$$

$$4a^4 + 3a^2 - 1 = 0$$

$$\begin{aligned}a^2 &= \frac{-3 \pm \sqrt{(-3)^2 - 4(4)(-1)}}{2(4)} \\ &= \frac{-3 \pm \sqrt{25}}{8} \\ &= \frac{-3 \pm 5}{8}\end{aligned}$$

$= -1 \text{ or } \dfrac{1}{4}$

Since a is real, $a^2 \geq 0$ and hence $a^2 = \frac{1}{4}$ so $a = \pm\frac{1}{2}$.

When $a = \frac{1}{2}$, $b = -1$ and when $a = -\frac{1}{2}$, $b = 1$ so $\lambda = \pm\left(\frac{1}{2} - i\right)$, hence we have

$z - \dfrac{5}{2} = \pm\left(\dfrac{1}{2} - i\right)$

$z = 2 + i \text{ or } 3 - i$

5b $z^2 - (6 + i)z + (14 + 8i) = 0$

$\Delta = b^2 - 4ac$

$= \left(-(6 + i)\right)^2 - 4(1)(14 + 8i)$

$= 36 + 12i - 1 - 56 - 32i$

$= -21 - 20i$

Let λ^2 be a square root of $-21 - 12i$,

$$\begin{aligned}\lambda^2 &= (a + bi)^2 \\ &= a^2 + 2abi + b^2i^2 \\ &= a^2 - b^2 + 2abi\end{aligned}$$

Equating real parts,

$a^2 - b^2 = -21 \qquad (1)$

Equating imaginary parts,

$ab = -10$

$b = \dfrac{-10}{a} \qquad (2)$

Substituting (2) into (1):

$a^2 - \left(-\dfrac{10}{a}\right)^2 = -21$

$a^2 - \dfrac{100}{a^2} = -21$

$a^4 - 100 = -21a^2$

$a^4 + 21a^2 - 100 = 0$

$(a^2 + 25)(a^2 - 4) = 0$

Since $a^2 \geq 0$, $a^2 = 4$ and hence $a = \pm 2$

When $a = 2$, $b = -\frac{10}{2} = -5$

When $a = -2$, $b = -\frac{10}{-2} = 5$

Thus $\lambda = \pm(2 - 5i)$

Hence the roots of the equation are

$$z = \frac{-b \pm \lambda}{2a}$$

$$= \frac{-\big(-(6 + i)\big) \pm (2 - 5i)}{2(1)}$$

$$= \frac{6 + i \pm (2 - 5i)}{2}$$

$z = 2 + 3i$ or $4 - 2i$

6 $\overline{3i} = -3i$ is also a zero as P(z) has real coefficients, so $(z - 3i)(z + 3i) = z^2 + 9$ is a factor.

7a The coefficients of $P(z)$ are real and hence the conjugate is also a root.

7b The sum of the zeroes is

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha + \beta + \gamma = -\frac{-8}{1}$$

$\alpha + \beta + \gamma = 8$

Hence

$2 - 5i + 2 + 5i + \gamma = 8$

$4 + \gamma = 8$

$\gamma = 4$

So the third root is 4.

7c $P(z) = (z-4)\big(z-(2-5i)\big)\big(z-(2+5i)\big)$

$= (z-4)(z^2-4z+29)$

8a $z = \sqrt{1^2+1^2}\,\mathrm{cis}\left(\tan^{-1}\left(-\frac{1}{1}\right)\right)$ (4th quadrant)

$= \sqrt{2}\,\mathrm{cis}\left(-\frac{\pi}{4}\right)$

8b $z = \sqrt{\left(3\sqrt{3}\right)^2+3^2}\,\mathrm{cis}\,\left(\tan^{-1}\left(\frac{3}{-3\sqrt{3}}\right)\right)$ (2nd quadrant)

$= 6\,\mathrm{cis}\left(\frac{5\pi}{6}\right)$

9a $4\,\mathrm{cis}\frac{\pi}{2}$

$= 4\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)$

$= 4(0+i\times 1)$

$= 4i$

9b $\sqrt{6}\,\mathrm{cis}\left(-\frac{3\pi}{4}\right)$

$= \sqrt{6}\left(\cos\frac{3\pi}{4}-i\sin\frac{3\pi}{4}\right)$

$= \sqrt{6}\left(-\frac{1}{\sqrt{2}}-i\frac{1}{\sqrt{2}}\right)$

$= -\sqrt{3}-\sqrt{3}i$

10a $2\,\mathrm{cis}\frac{\pi}{2}\times 3\,\mathrm{cis}\frac{\pi}{3}$

$= (3\times 2)\,\mathrm{cis}\left(\frac{\pi}{2}+\frac{\pi}{3}\right)$

$= 6\,\mathrm{cis}\left(\frac{3\pi}{6}+\frac{2\pi}{6}\right)$

$= 6\,\mathrm{cis}\left(\frac{5\pi}{6}\right)$

10b

$$\frac{10\text{cis}10\theta}{5\text{cis}5\theta}$$

$$= \frac{10}{5}\text{cis}(10\theta - 5\theta)$$

$$= 2\text{cis}(5\theta)$$

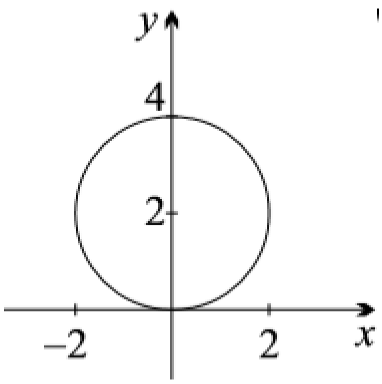
10c $(3\text{cis}3\alpha)^2$

$$= 3^2\text{cis}(3\alpha \times 2)$$

$$= 9\text{cis}(6\alpha)$$

11a $|z - 2i| = 2$

Circle with centre $2i$ or $(0, 2)$ and radius 2 units.



11b $|z| = |z - 2 - 2i|$

Line equidistant from $z = 0$ and $z = 2 + 2i$.

Alternatively,

$$|x + iy| = |x + iy - 2 - 2i|$$

$$|x + iy| = |(x - 2) + (y - 2)i|$$

$$|x + iy|^2 = |(x - 2) + (y - 2)i|^2$$

$$x^2 + y^2 = (x - 2)^2 + (y - 2)^2$$

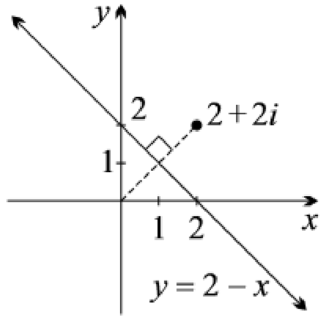
$$x^2 + y^2 = x^2 - 4x + 4 + y^2 - 4y + 4$$

$$0 = -4x + 4 - 4y + 4$$

$4x + 4y = 8$

$x + y = 2$

$y = 2 - x$



11c

$\arg(z + 2) = -\dfrac{\pi}{4}$

Endpoint of line at $-2 + 0i$ (not included) and line extending with angle $-\frac{\pi}{4}$.

Alternatively,

$\arg(x + iy + 2) = -\dfrac{\pi}{4}$

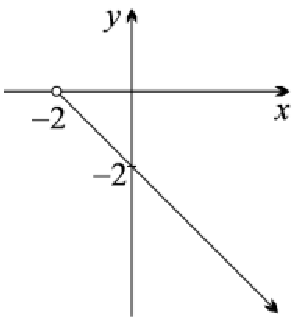
$\arg(x + 2 + iy) = -\dfrac{\pi}{4}$

$\tan^{-1}\left(\dfrac{y}{x + 2}\right) = -\dfrac{\pi}{4}$ $\qquad (x \neq -2)$

$\dfrac{y}{x + 2} = \tan\left(-\dfrac{\pi}{4}\right)$

$\dfrac{y}{x + 2} = -1$

$y = -(x + 2)$ $\qquad$ (For $x < -2, y > 0$ and the slope cannot be $-\dfrac{\pi}{4}$, so $x > -2$)



11d

$$\arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{2}$$

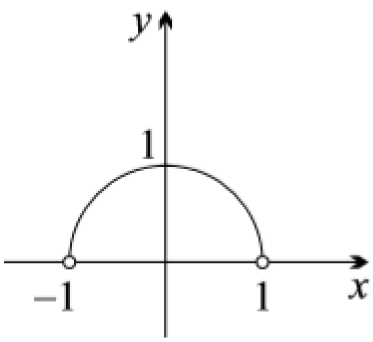
$$\arg(z-1) - \arg(z+1) = \frac{\pi}{2}$$

Consider point $A = 1 + 0i$ and point $B = -1 + 0i$.

The set of points z represents an arc of a circle with centre C and chord AB where the point P moves anticlockwise along the curve from point A to point B, with the endpoints A and B excluded (since the vector 1 to z is perpendicular to the vector from -1 to z, lines from diameter to circumference are perp)

$$\angle APB = \frac{\pi}{2}$$

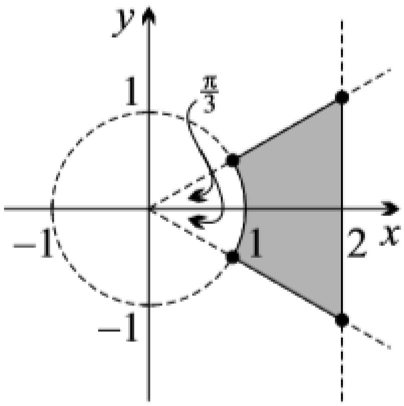
Hence the chord AB forms the diameter of the circle so $C = 0 + 0i$ and radius is 1 unit.



12 From $|z| \geq 1$ we get that $|z|^2 \geq 1^2$ and hence $x^2 + y^2 \geq 1$

From $\text{Re}(z) \leq 2$ we get that $x \leq 2$

From $-\frac{\pi}{3} \leq \arg z \leq \frac{\pi}{3}$ we get that $-\frac{\pi}{3} \leq \tan^{-1}\frac{y}{x} \leq \frac{\pi}{3}$ and hence $-\sqrt{3} \leq \frac{y}{x} \leq \sqrt{3}$ so $-\sqrt{3}x \leq y \leq \sqrt{3}x$.



13a

$$\frac{z}{w}$$

$$= \frac{-1 + \sqrt{3}i}{1 + i}$$

$$= \frac{(-1 + \sqrt{3}i)(1 - i)}{(1 + i)(1 - i)}$$

$$= \frac{-1 + \left(1 + \sqrt{3}\right)i - \sqrt{3}i^2}{1 - i^2}$$

$$= \frac{-1 + \left(1 + \sqrt{3}\right)i + \sqrt{3}}{2}$$

$$= \frac{1}{2}\left(\sqrt{3} - 1\right) + \frac{1}{2}\left(\sqrt{3} + 1\right)i$$

13b $z = \sqrt{(-1)^2 + \left(\sqrt{3}\right)^2} \text{ cis}\left(\tan^{-1}\frac{\sqrt{3}}{-1}\right)$ (2nd quadrant)

$$= \sqrt{4} \text{ cis}\left(\pi - \frac{\pi}{3}\right)$$

$$= 2 \text{ cis}\frac{2\pi}{3}$$

$w = \sqrt{1^2 + 1^2} \text{ cis}\left(\tan^{-1}\frac{1}{1}\right)$ (1st quadrant)

$$= \sqrt{2} \text{ cis}\frac{\pi}{4}$$

13c

$$\frac{z}{w} = \frac{2 \text{ cis}\frac{2\pi}{3}}{\sqrt{2} \text{ cis}\frac{\pi}{4}}$$

$$= \frac{2}{\sqrt{2}} \text{ cis}\left(\frac{2\pi}{3} - \frac{\pi}{4}\right)$$

$$= \sqrt{2} \text{ cis}\left(\frac{8\pi - 3\pi}{12}\right)$$

$$= \sqrt{2} \text{ cis}\frac{5\pi}{12}$$

13d Equating parts (a) and (c),

$$\sqrt{2} \text{ cis}\frac{5\pi}{12} = \frac{1}{2}\left(\sqrt{3} - 1\right) + \frac{1}{2}\left(\sqrt{3} + 1\right)i$$

$$\sqrt{2} \left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right) = \frac{1}{2}\left(\sqrt{3} - 1\right) + \frac{1}{2}\left(\sqrt{3} + 1\right)i$$

Equating the real parts,

$$\sqrt{2}\cos\frac{5\pi}{12} = \frac{1}{2}\left(\sqrt{3} - 1\right)$$

$$\cos\frac{5\pi}{12} = \frac{\sqrt{3} - 1}{2\sqrt{2}}$$

$$= \frac{\left(\sqrt{3} - 1\right)\sqrt{2}}{2(2)}$$

$$= \frac{\sqrt{6} - \sqrt{2}}{4}$$

14a $z\bar{z} = z + \bar{z}$

$(x + iy)(x - iy) = (x + iy) + (x - iy)$

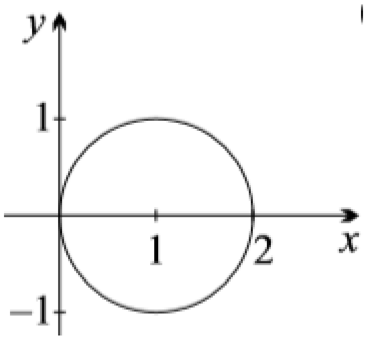
$x^2 - i^2y^2 = 2x$

$x^2 + y^2 = 2x$

$x^2 - 2x + 1 + y^2 = 1$

$(x - 1)^2 + y^2 = 1$

Circle with centre (1, 0) and radius 1 unit.



14b $\bar{z} = iz$

$x - iy = i(x + iy)$

$x - iy = ix + i^2y$

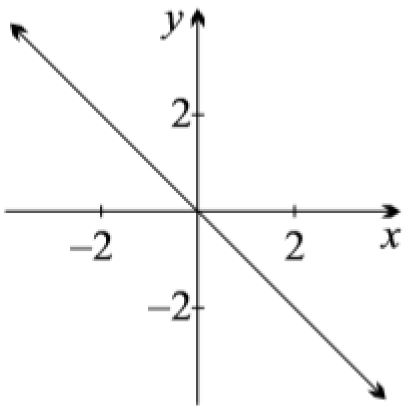
$x - iy = ix - y$

$(x + y) - i(x + y) = 0$

The real and imaginary parts must both be equal to zero, hence

$x + y = 0$

$y = -x$



14c $|z+2| = 2|z-4|$

$|(x+iy)+2| = 2|(x+iy)-4|$

$|(x+2)+iy| = 2|(x-4)+iy|$

$|(x+2)+iy|^2 = 4|(x-4)+iy|^2$

$(x+2)^2 + y^2 = 4((x-4)^2 + y^2)$

$x^2 + 4x + 4 + y^2 = 4(x^2 - 8x + 16 + y^2)$

$x^2 + 4x + 4 + y^2 = 4x^2 - 32x + 64 + 4y^2$

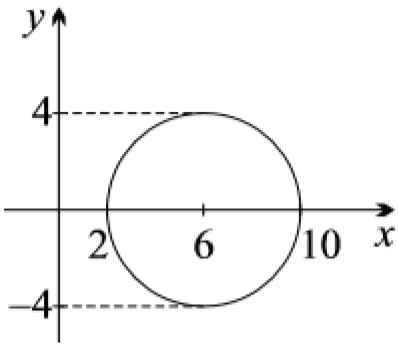
$0 = 3x^2 - 36x + 60 + 3y^2$

$0 = x^2 - 12x + 20 + y^2$

$0 = (x-6)^2 - 36 + 20 + y^2$

$(x-6)^2 + y^2 = 16$

Circle with centre (6, 0) and radius 4 units.



15a $\overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OQ}$

$= -\overrightarrow{OP} + \overrightarrow{OQ}$

$= -(4-2i) + (7+3i)$

$= 3 + 5i$

15b $\overrightarrow{PR} = i\overrightarrow{PQ}$ (since PQR is isosceles $|PR| = |PQ|$ and since $\angle P = 90°$ we have that PR is an anti clock wise rotation by $\frac{\pi}{2}$). Hence

$\overrightarrow{PR} = i(3+5i) = 3i + 5i^2 = -5 + 3i$

15c $\overrightarrow{OR} = \overrightarrow{OP} + \overrightarrow{PR}$

$= 4 - 2i + (-5 + 3i)$

$= -1 + i$

16a Since there is a right angle at z_1 and since two sides of the triangle are of equal length, the sides $z_3 - z_1$ and $z_2 - z_1$ must satisfy the equation.

$(z_3 - z_1) = \pm i(z_2 - z_1)$

$\left(z_3 - (4 - i)\right) = \pm i(2i - (4 - i))$

$z_3 - 4 + i = \pm i(3i - 4)$

$z_3 = (4 - i) \pm i(3i - 4)$

$z_3 = (4 - i) \pm (-3 - 4i)$

$z = 1 - 5i$ or $7 + 3i$

16b Since there is a right angle at z_2 and since two sides of the triangle are of equal length, the sides $z_3 - z_2$ and $z_1 - z_2$ must satisfy the equation.

$(z_3 - z_2) = \pm i(z_1 - z_2)$

$(z_3 - 2i) = \pm i((4 - i) - 2i)$

$z_3 - 2i = \pm i(4 - 3i)$

$z_3 = 2i \pm i(4 - 3i)$

$z_3 = 2i \pm (4i + 3)$

$z = 3 + 6i$ or $-3 - 2i$

17 Without loss of generality label the triangle OPQ in an anticlockwise fashion starting at the origin. Now because it is given that triangle OPQ is equilateral we have that the angle at each vertex must be $\frac{\pi}{3}$ and also that $|z_1| = |z_2| = |z_2 - z_1|$. Now by drawing OPQ if necessary, it becomes apparent that $z_2$must be a $\frac{\pi}{3}$ rotation of z_1, and so we can write $z_2 = \text{cis}\left(\frac{\pi}{3}\right) z_1$. Similarly, but considering the vector $-z_1$ and the triangle OPQ it is apparent that $-z_1$is a $\frac{\pi}{3}$ rotation of the vector $z_2 - z_1$, this is $-z_1 = \text{cis}\left(\frac{\pi}{3}\right)(z_2 - z_1)$. Dividing now the first equation by the second we have,

$$\frac{z_2}{-z_1} = \frac{\text{cis}\left(\frac{\pi}{3}\right) z_1}{\text{cis}\left(\frac{\pi}{3}\right)(z_2 - z_1)}$$

$$-z_1^2 = z_2(z_2 - z_1)$$

$$-z_1^2 = z_2^2 - z_2 z_1$$

$$z_1^2 + z_2^2 = z_1 z_2$$

18a $\arg(z_1 + z_2)$

$$= \arg\left(2\,\text{cis}\frac{\pi}{12} + 2i\right)$$

$$= \arg\left(2\,\text{cis}\frac{\pi}{12} + 2\,\text{cis}\frac{\pi}{2}\right)$$

$$= \arg\left(\text{cis}\frac{\pi}{12} + \text{cis}\frac{\pi}{2}\right)$$

$$= \arg\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12} + \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$

$$= \arg\left(\cos\frac{\pi}{12} + \cos\frac{\pi}{2} + i\left(\sin\frac{\pi}{12} + \sin\frac{\pi}{2}\right)\right)$$

$$= \tan^{-1}\left(\frac{\sin\frac{\pi}{12} + \sin\frac{\pi}{2}}{\cos\frac{\pi}{12} + \cos\frac{\pi}{2}}\right)$$

$$= \tan^{-1} \frac{2\sin\left(\frac{\frac{\pi}{12} + \frac{\pi}{2}}{2}\right)\cos\left(\frac{\frac{\pi}{12} - \frac{\pi}{2}}{2}\right)}{\left(2\cos\left(\frac{\frac{\pi}{12} + \frac{\pi}{2}}{2}\right)\cos\left(\frac{\frac{\pi}{12} - \frac{\pi}{2}}{2}\right)\right)}$$

$$= \tan^{-1} \left(\frac{\sin\left(\frac{\frac{\pi}{12}+\frac{\pi}{2}}{2}\right)}{\cos\left(\frac{\frac{\pi}{12}+\frac{\pi}{2}}{2}\right)} \right)$$

$$= \tan^{-1}\left(\tan\left(\frac{\frac{\pi}{12}+\frac{\pi}{2}}{2}\right)\right)$$

$$= \tan^{-1}\left(\tan\frac{7\pi}{24}\right)$$

$$= \frac{7\pi}{24}$$

18b $\arg(z_2 - z_1)$

$$= \arg\left(2i - 2\operatorname{cis}\frac{\pi}{12}\right)$$

$$= \arg\left(2\operatorname{cis}\frac{\pi}{2} - 2\operatorname{cis}\frac{\pi}{12}\right)$$

$$= \arg\left(\operatorname{cis}\frac{\pi}{2} - \operatorname{cis}\frac{\pi}{12}\right)$$

$$= \arg\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2} - \cos\frac{\pi}{12} - i\sin\frac{\pi}{12}\right)$$

$$= \arg\left(\cos\frac{\pi}{2} - \cos\frac{\pi}{12} + i\left(\sin\frac{\pi}{2} - \sin\frac{\pi}{12}\right)\right)$$

$$= \tan^{-1}\left(\frac{\sin\frac{\pi}{2} - \sin\frac{\pi}{12}}{\cos\frac{\pi}{2} - \cos\frac{\pi}{12}}\right)$$

$$= \tan^{-1}\left(\frac{2\cos\left(\frac{\frac{\pi}{2}+\frac{\pi}{12}}{2}\right)\sin\left(\frac{\frac{\pi}{2}-\frac{\pi}{12}}{2}\right)}{-2\sin\left(\frac{\frac{\pi}{2}+\frac{\pi}{12}}{2}\right)\sin\left(\frac{\frac{\pi}{2}-\frac{\pi}{12}}{2}\right)}\right)$$

$$= \tan^{-1}\left(-\cot\left(\frac{\frac{\pi}{2}+\frac{\pi}{12}}{2}\right)\right)$$

$$= \tan^{-1}\left(-\cot\frac{7\pi}{24}\right)$$

$= \tan^{-1}\left(\cot\left(-\frac{7\pi}{24}\right)\right)$

$= \tan^{-1}\left(\cot\left(\pi - \frac{7\pi}{24}\right)\right)$

$= \tan^{-1}\left(\cot\frac{17\pi}{24}\right)$

$= \tan^{-1}\left(\tan\left(\frac{\pi}{2} - \frac{17\pi}{24}\right)\right)$

$= \tan^{-1}\left(\tan\left(-\frac{5\pi}{24}\right)\right)$

$= \tan^{-1}\left(\tan\left(\pi - \frac{5\pi}{24}\right)\right)$

$= \tan^{-1}\left(\tan\frac{19\pi}{24}\right)$

$= \frac{19\pi}{24}$

19 $|z_1| = |z_2|$

$\arg((z_1 + z_2)^2) = 2\arg(z_1 + z_2)$

Now $z_1 + z_2$ bisects the angle between z_1 and z_2 as z_1 and z_2 represent two sides of a rhombus whilst $z_1 + z_2$ is the diagonal, thus

$\arg(z_1 + z_2)$

$= \frac{1}{2}(\arg(z_1) + \arg(z_2))$

$= \frac{1}{2}\arg(z_1 z_2)$

Hence $\arg((z_1 + z_2)^2)$

$= 2\arg(z_1 + z_2)$

$= 2\left(\frac{1}{2}\arg(z_1 z_2)\right)$

$= \arg(z_1 z_2)$

20

$$\frac{z^2-1}{z^2+1}$$

$$=\frac{(\text{cis}\theta)^2-1}{(\text{cis}\theta)^2+1}$$

$$=\frac{(\cos\theta+i\sin\theta)^2-1}{(\cos\theta+i\sin\theta)^2+1}$$

$$=\frac{\cos^2\theta+2i\sin\theta\cos\theta+i^2\sin^2\theta-1}{\cos^2\theta+2i\sin\theta\cos\theta+i^2\sin^2\theta+1}$$

$$=\frac{\cos^2\theta+2i\sin\theta\cos\theta-\sin^2\theta-1}{\cos^2\theta+2i\sin\theta\cos\theta-\sin^2\theta+1}$$

$$=\frac{-(1-\cos^2\theta)+2i\sin\theta\cos\theta-\sin^2\theta}{\cos^2\theta+2i\sin\theta\cos\theta+1-\sin^2\theta}$$

$$=\frac{-\sin^2\theta+2i\sin\theta\cos\theta-\sin^2\theta}{\cos^2\theta+2i\sin\theta\cos\theta+\cos^2\theta}$$

$$=\frac{2i\sin\theta\cos\theta-2\sin^2\theta}{2\cos^2\theta+2i\sin\theta\cos\theta}$$

$$=\frac{i\sin\theta\cos\theta-\sin^2\theta}{\cos^2\theta+i\sin\theta\cos\theta}$$

$$=\frac{\sin\theta\,(i\cos\theta-\sin\theta)}{\cos\theta\,(\cos\theta+i\sin\theta)}$$

$$=\tan\theta\left(\frac{i\cos\theta-\sin\theta}{\cos\theta+i\sin\theta}\right)$$

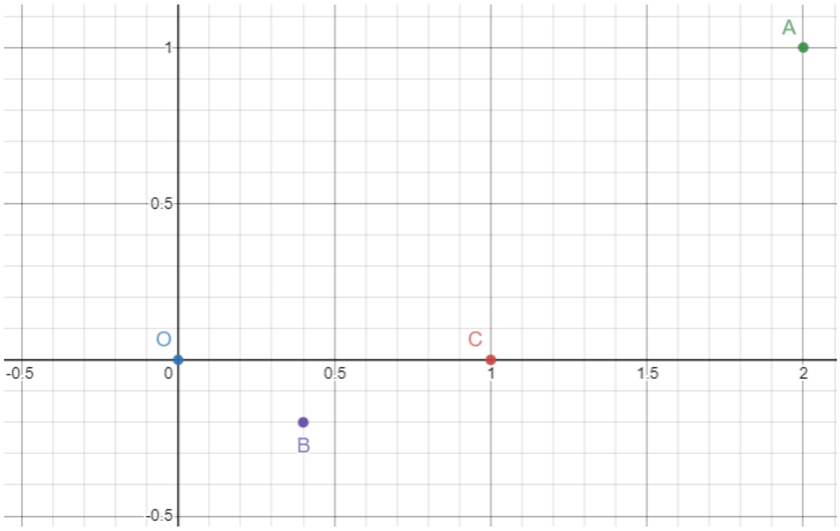
$$=i\tan\theta\left(\frac{i\cos\theta-\sin\theta}{i(\cos\theta+i\sin\theta)}\right)$$

$$=i\tan\theta\left(\frac{i\cos\theta-\sin\theta}{i\cos\theta+i^2\sin\theta}\right)$$

$$=i\tan\theta\left(\frac{i\cos\theta-\sin\theta}{i\cos\theta-\sin\theta}\right)$$

$$=i\tan\theta$$

21



Given that $0 < \arg(z) < \frac{\pi}{2}$, z is in the first quadrant and the 4 points will appear similar to the above diagram. Now by definition we have, $|\overrightarrow{OA}| = |z|$, $|\overrightarrow{OC}| = 1$ and $|\overrightarrow{OB}| = \frac{1}{|z|}$. Further $|\overrightarrow{CA}| = |z-1|$ and also $|\overrightarrow{BC}| = |1 - 1/z|$. Now by comparing sides of triangles OBC and OAC we see that,

$$\frac{|\overrightarrow{OA}|}{|\overrightarrow{OC}|} = |z|, \frac{|\overrightarrow{OC}|}{|\overrightarrow{OB}|} = \frac{1}{\frac{1}{|z|}} = |z| \text{ and } \frac{|\overrightarrow{CA}|}{|\overrightarrow{CB}|} = \frac{|z-1|}{\left|1-\frac{1}{z}\right|} = \left|\frac{z-1}{1-\frac{1}{z}}\right| = \left|\frac{z(z-1)}{z-1}\right| = |z|$$

Thus we have,

$$\frac{|\overrightarrow{OA}|}{|\overrightarrow{OC}|} = \frac{|\overrightarrow{OC}|}{|\overrightarrow{OB}|} = \frac{|\overrightarrow{CA}|}{|\overrightarrow{CB}|} = |z|$$

And so by the SSS test we conclude that triangles OBC and OAC are similar, hence $\angle OAC = \angle OCB$

Alternatively, without using similar triangles

$\angle OAC$

$= \arg(\overrightarrow{AO}) - \arg(\overrightarrow{AC})$

$= \arg(\overrightarrow{AO}) - \arg(\overrightarrow{AO} + \overrightarrow{OC})$

$= \arg(-\overrightarrow{OA}) - \arg(-\overrightarrow{OA} + \overrightarrow{OC})$

$= \arg(-z) - \arg(-z + 1)$

$$= \arg\left(\frac{-z}{-z+1}\right)$$

$$= \arg\left(\frac{z}{z-1}\right)$$

$\angle OCB$

$$= \arg \overrightarrow{CO} - \arg \overrightarrow{CB}$$

$$= \arg\left(-\overrightarrow{OC}\right) - \arg(\overrightarrow{CO} + \overrightarrow{OB})$$

$$= \arg\left(-\overrightarrow{OC}\right) - \arg(-\overrightarrow{OC} + \overrightarrow{OB})$$

$$= \arg 1 - \arg(-1 + \frac{1}{z})$$

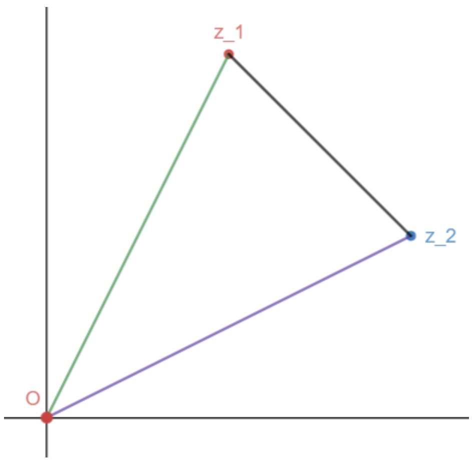
$$= \arg\left(\frac{1}{-1+\frac{1}{z}}\right)$$

$$= \arg\left(\frac{z}{-z+1}\right)$$

$$= \arg\left(\frac{z}{1-z}\right)$$

Hence $\angle OAC = \angle OCB$.

22a



$|z_1 - z_2|$ denotes the length of the black line in the diagram above. On the other hand $|z_1| - |z_2|$ is the difference in length between the green and blue lines. The latter distance must be shorter (unless the blue and green lines overlap in which case the two distances are equal). Thus $|z_1 - z_2| \geq |z_1| - |z_2|$.

22b

$$\left|z - \frac{4}{z}\right| = 2$$

$$\left|\frac{z^2 - 4}{z}\right| = 2$$

$$\frac{|z^2 - 4|}{|z|} = 2$$

$$|z^2 - 4| = 2|z|$$

$$|z^2| - |4| \leq 2|z|$$

$$|z|^2 - 4 \leq 2|z|$$

$$|z|^2 - 2|z| - 4 \leq 0$$

The maximum value of $|z|$ will occur when $|z|^2 - 2|z| - 4 = 0$.

$$|z| = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-4)}}{2(1)}$$

$$= \frac{2 \pm \sqrt{4 + 16}}{2}$$

$$= \frac{2 \pm \sqrt{20}}{2}$$

$$= \frac{2 \pm 2\sqrt{5}}{2}$$

$$= 1 \pm \sqrt{5}$$

So the maximum value is $\sqrt{5} + 1$.