Solutions to Exercise 12A

1a

x	-1	-0.7	-0.5	-0.2	0	0.3	0.6	0.8	1
$\frac{dy}{dx}$	undefined	1.40	1.16	1.02	1	1.05	1.25	1.67	undefined

1b

x	-1	-0.7	-0.5	-0.2	0	0.3	0.6	0.8	1
$\frac{dy}{dx}$	undefined	1.40	1.16	1.02	1	1.05	1.25	1.67	undefined

2a

$$\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$$

2b

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

2c

$$\frac{d}{dx}(\sin^{-1} 2x)$$

$$= \frac{1}{\sqrt{1-(2x)^2}} \times \frac{d}{dx}(2x)$$

$$= \frac{2}{\sqrt{1-4x^2}}$$

2d

$$\frac{d}{dx}(\tan^{-1} 3x)$$

$$= \frac{1}{1+(3x)^2} \times \frac{d}{dx}(3x)$$

$$= \frac{3}{1+9x^2}$$

2e

$$\frac{d}{dx}(\cos^{-1} 5x)$$

$$= \frac{-1}{\sqrt{1-(5x)^2}} \times \frac{d}{dx}(5x)$$

$$= -\frac{5}{\sqrt{1-25x^2}}$$

2f

$$\frac{d}{dx}(\sin^{-1} -x)$$

$$= \frac{1}{\sqrt{1-(-x)^2}} \times \frac{d}{dx}(-x)$$

$$= \frac{-1}{\sqrt{1-x^2}}$$

2g

$$\frac{d}{dx}(\sin^{-1} x^2)$$

$$= \frac{1}{\sqrt{1-(x^2)^2}} \times \frac{d}{dx}(x^2)$$

$$= \frac{2x}{\sqrt{1-x^4}}$$

2h

$$\frac{d}{dx}(\tan^{-1} x^3)$$

$$= \frac{1}{1+(x^3)^2} \times \frac{d}{dx}(x^3)$$

$$= \frac{3x^2}{1 + x^6}$$

2i

$$\frac{d}{dx}(\tan^{-1}(x+2))$$

$$= \frac{1}{1 + (x+2)^2} \times \frac{d}{dx}(x+2)$$

$$= \frac{1}{x^2 + 4x + 5}$$

2j

$$\frac{d}{dx}(\cos^{-1}(1-x))$$

$$= \frac{-1}{\sqrt{1-(1-x)^2}} \times \frac{d}{dx}(1-x)$$

$$= \frac{-1}{\sqrt{1-(1-2x+x^2)}} \times -1$$

$$= \frac{1}{\sqrt{2x - x^2}}$$

2k

$$\frac{d}{dx}(x\sin^{-1}x)$$

$$= \sin^{-1}x \times \frac{d}{dx}(x) + x \times \frac{d}{dx}(\sin^{-1}x)$$

$$= \sin^{-1}x + \frac{x}{\sqrt{1-x^2}}$$

2l

$$\frac{d}{dx}((1+x^2)\tan^{-1}x)$$

$$= \tan^{-1}x \times \frac{d}{dx}(1+x^2) + (1+x^2) \times \frac{d}{dx}(\tan^{-1}x)$$

$$= \tan^{-1} x \times 2x + (1 + x^2) \times \frac{1}{1 + x^2}$$
$$= 2x \tan^{-1} x + 1$$

2m

$$\frac{d}{dx}\left(\sin^{-1}\frac{1}{5}x\right)$$
$$= \frac{1}{\sqrt{1 - \left(\frac{1}{5}x\right)^2}} \times \frac{d}{dx}\left(\frac{1}{5}x\right)$$
$$= \frac{1}{\sqrt{1 - \frac{x^2}{25}}} \times \frac{1}{5}$$
$$= \frac{1}{\sqrt{25 - x^2}}$$

2n

$$\frac{d}{dx}\left(\tan^{-1}\frac{1}{4}x\right)$$
$$= \frac{1}{1 + \left(\frac{1}{4}x\right)^2} \times \frac{d}{dx}\left(\frac{1}{4}x\right)$$
$$= \frac{1}{1 + \frac{x^2}{16}} \times \frac{1}{4}$$
$$= \frac{4}{16 + x^2}$$

2o

$$\frac{d}{dx}\left(\cos^{-1}\sqrt{x}\right)$$
$$= \frac{-1}{\sqrt{1 - \sqrt{x}^2}} \times \frac{d}{dx}\left(\sqrt{x}\right)$$
$$= \frac{-1}{\sqrt{1 - \sqrt{x}^2}} \times \frac{d}{dx}\left(x^{\frac{1}{2}}\right)$$

$$= \frac{-1}{\sqrt{1 - \sqrt{x}^2}} \times \frac{1}{2} x^{-\frac{1}{2}}$$
$$= \frac{-1}{2\sqrt{x}\sqrt{1 - x}}$$
$$= -\frac{1}{2\sqrt{x - x^2}}$$

2p

$$\frac{d}{dx}\left(\tan^{-1} \sqrt{x}\right)$$
$$= \frac{1}{1 + \left(\sqrt{x}\right)^2} \times \frac{d}{dx}\left(\sqrt{x}\right)$$
$$= \frac{1}{1 + \left(\sqrt{x}\right)^2} \times \frac{d}{dx}\left(x^{\frac{1}{2}}\right)$$
$$= \frac{1}{1 + \left(\sqrt{x}\right)^2} \times -\frac{1}{2} x^{-\frac{1}{2}}$$
$$= \frac{1}{2\sqrt{x}(1 + x)}$$

2q

$$\frac{d}{dx}\left(\tan^{-1} \frac{1}{x}\right)$$
$$= \frac{1}{1 + \left(\frac{1}{x}\right)^2} \times \frac{d}{dx}\left(\frac{1}{x}\right)$$
$$= \frac{1}{1 + \frac{1}{x^2}} \times -\frac{1}{x^2}$$
$$= -\frac{1}{x^2 + 1}$$

3 For this question you only need to provide the gradient. We also provide the equation of the tangent for your benefit.

3a

$$\frac{dy}{dx} = \frac{2}{1+x^2}$$

At $x = 0$, $y = 2\tan^{-1} 0 = 0$ and $\frac{dy}{dx} = \frac{2}{1+0^2} = 2$. Hence, the tangent will be given by

$$y - y_1 = m(x - x_1)$$

$$y - 0 = 2(x - 0)$$

$$y = 2x$$

3b

$$\frac{dy}{dx} = \frac{\sqrt{3}}{\sqrt{1-x^2}}$$

At $x = \frac{1}{2}$, $y = \sqrt{3}\sin^{-1}\frac{1}{2} = \frac{\sqrt{3}}{6}$ and $\frac{dy}{dx} = \frac{\sqrt{3}}{\sqrt{1-\frac{1}{2}^2}} = \frac{\sqrt{3}}{\sqrt{\frac{3}{4}}} = 2$. Hence, the tangent will be given by

$$y - y_1 = m(x - x_1)$$

$$y - \frac{\sqrt{3}}{6} = 2(x - 0)$$

$$y = 2x + \frac{\sqrt{3}}{6}$$

3c

$$\frac{dy}{dx} = \frac{2}{1+4x^2}$$

At $x = -\frac{1}{2}$, $y = \tan^{-1}\left(2\left(-\frac{1}{2}\right)\right) = -\frac{\pi}{4}$ and $\frac{dy}{dx} = \frac{2}{1+4\left(-\frac{1}{2}\right)^2} = \frac{2}{2} = 1$. Hence, the tangent will be given by

$$y - y_1 = m(x - x_1)$$

$$y - (-\frac{\pi}{4}) = 1(x - 0)$$

$$y = x - \frac{\pi}{4}$$

3d

$$\frac{dy}{dx}=\frac{-\frac{1}{2}}{\sqrt{1-\left(\frac{x}{2}\right)^2}}=-\frac{1}{\sqrt{4-x^2}}$$

At $x=\sqrt{3}$, $y=\cos^{-1}\frac{\sqrt{3}}{2}=\frac{\pi}{6}$ and $\frac{dy}{dx}=-\frac{1}{\sqrt{4-\sqrt{3}^2}}=-1$. Hence, the tangent will be given by

$y-y_1=m(x-x_1)$

$y-\frac{\pi}{6}=-1(x-0)$

$y=-x+\frac{\pi}{6}$

4a

$$\frac{dy}{dx}=-\frac{1}{\sqrt{1-(3x)^2}}\times 2\times 3=-\frac{6}{\sqrt{1-9x^2}}$$

When $x=0$

$y=2\cos^{-1}3(0)=2\left(\frac{\pi}{2}\right)=\pi,$

$\frac{dy}{dx}=-\frac{6}{\sqrt{1-9(0)^2}}=-6$ and $m_{norm}=-\frac{1}{\left(\frac{dy}{dx}\right)}=\frac{1}{6}$

For the tangent:

$y-y_1=m(x-x_1)$

$y-\pi=-6(x-0)$

$y=-6x+\pi$

For the normal:

$y-y_1=m(x-x_1)$

$y-\pi=\frac{1}{6}(x-0)$

$y=\frac{1}{6}x+\pi$

4b

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-\left(\frac{x}{2}\right)^2}} \times \frac{1}{2} = \frac{1}{\sqrt{4-x^2}}$$

When $x = \sqrt{2}$

$y = \sin^{-1}\frac{\sqrt{2}}{2} = \frac{\pi}{4}$,

$\frac{dy}{dx} = \frac{1}{\sqrt{4-(\sqrt{2})^2}} = \frac{1}{\sqrt{2}}$ and $m_{norm} = -\frac{1}{\left(\frac{dy}{dx}\right)} = -\sqrt{2}$

For the tangent:

$$y - y_1 = m(x - x_1)$$

$$y - \frac{\pi}{4} = \frac{1}{\sqrt{2}}(x - \sqrt{2})$$

$$y = \frac{x}{\sqrt{2}} - 1 + \frac{\pi}{4}$$

For the normal:

$$y - y_1 = m(x - x_1)$$

$$y - \frac{\pi}{4} = -\sqrt{2}(x - \sqrt{2})$$

$$y = -x\sqrt{2} + 2 + \frac{\pi}{4}$$

5a

$$\frac{d}{dx}(\sin^{-1}x + \cos^{-1}x)$$
$$= \frac{d}{dx}(\sin^{-1}x) + \frac{d}{dx}(\cos^{-1}x)$$
$$= \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}}$$
$$= 0$$

5b As the gradient of the function is zero, it must be flat along its entire domain and hence constant. Letting $x = 0$ gives $\sin^{-1}0 + \cos^{-1}0 = 0 + \frac{\pi}{2} = \frac{\pi}{2}$, hence the function has a constant value of $\frac{\pi}{2}$.

6a

$$\frac{d}{dx}(\cos^{-1}x + \cos^{-1}(-x))$$
$$= -\frac{1}{\sqrt{1-x^2}} - \frac{\frac{d}{dx}(-x)}{\sqrt{1-(-x)^2}}$$
$$= -\frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}}$$
$$= 0$$

As the gradient of the function is zero, it must be flat along its entire domain and hence constant. Letting $x = 0$ gives $\cos^{-1} 0 + \cos^{-1} 0 = \frac{\pi}{2} + \frac{\pi}{2} = \pi$, hence the function has a constant value of π.

6b

$$\frac{d}{dx}\left(2\sin^{-1}\sqrt{x} - \sin^{-1}(2x-1)\right)$$
$$= \frac{1}{\sqrt{1-\sqrt{x}^2}} \times 2 \times \frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{\sqrt{1-(2x-1)^2}} \times (-2)$$
$$= \frac{1}{\sqrt{x}\sqrt{1-x}} - \frac{2}{\sqrt{1-(4x^2-4x+1)}}$$
$$= \frac{1}{\sqrt{x-x^2}} - \frac{2}{\sqrt{4x-4x^2}}$$
$$= \frac{1}{\sqrt{x-x^2}} - \frac{2}{2\sqrt{x-x^2}}$$
$$= \frac{1}{\sqrt{x-x^2}} - \frac{1}{\sqrt{x-x^2}}$$
$$= 0$$

As the gradient of the function is zero, it must be flat along its entire domain and hence constant. Letting $x = 0$ gives $2\sin^{-1}\sqrt{0} - \sin^{-1}(2(0)-1) = 2\sin^{-1} 0 - \sin^{-1}(-1) = 0 - \frac{\pi}{2} = \frac{\pi}{2}$, hence the function has a constant value of $\frac{\pi}{2}$.

7a

$$f'(x) = 1 \times \tan^{-1}x + x \times \frac{1}{1+x^2} - \frac{1}{2} \times \frac{2x}{1+x^2}$$
$$= \tan^{-1}x + \frac{x}{1+x^2} - \frac{x}{1+x^2}$$
$$= \tan^{-1}x$$

Hence

$$f''(x) = \frac{1}{1+x^2}$$

7b $f''(-1) = \frac{1}{2} > 0$, hence the curve is concave up

8 By the quotient rule

$$\frac{dy}{dx} = \frac{x \times \frac{d}{dx}(\sin^{-1} x) - \sin^{-1} x \times \frac{d}{dx}(x)}{x^2}$$

$$= \frac{x \times \frac{1}{\sqrt{1-x^2}} - \sin^{-1} x}{x^2}$$

When $x = \frac{1}{2}$

$$\frac{dy}{dx} = \frac{\frac{1}{2} \times \frac{1}{\sqrt{1-\left(\frac{1}{2}\right)^2}} - \sin^{-1}\left(\frac{1}{2}\right)}{\left(\frac{1}{2}\right)^2}$$

$$= \frac{\frac{1}{2} \times \frac{1}{\sqrt{\frac{3}{4}}} - \frac{\pi}{6}}{\frac{1}{4}}$$

$$= \frac{\frac{1}{2} \times \frac{2}{\sqrt{3}} - \frac{\pi}{6}}{\frac{1}{4}}$$

$$= \frac{\frac{1}{\sqrt{3}} - \frac{\pi}{6}}{\frac{1}{4}}$$

$$= 4\left(\frac{\sqrt{3}}{3} - \frac{\pi}{6}\right)$$

$$= \frac{2}{3}(2\sqrt{3} - \pi)$$

9a

$$\frac{d}{dx}\left(x\cos^{-1}x-\sqrt{1-x^2}\right)$$
$$=x\times-\frac{1}{\sqrt{1-x^2}}+\cos^{-1}x-\frac{1}{2}\times(-2x)\times(1-x^2)^{-\frac{1}{2}}$$
$$=-\frac{x}{\sqrt{1-x^2}}+\cos^{-1}x+\frac{x}{\sqrt{1-x^2}}$$
$$=\cos^{-1}x$$

9b

$$\frac{d}{dx}(\sin^{-1}e^{3x})$$
$$=\frac{1}{\sqrt{1-(e^{3x})^2}}\times\frac{d}{dx}(e^{3x})$$
$$=\frac{3e^{3x}}{\sqrt{1-e^{6x}}}$$

9c

$$\frac{d}{dx}\left(\sin^{-1}\frac{1}{4}(2x-3)\right)$$
$$=\frac{1}{\sqrt{1-\left(\frac{1}{4}(2x-3)\right)^2}}\times\frac{d}{dx}\left(\frac{1}{4}(2x-3)\right)$$
$$=\frac{1}{\sqrt{1-\frac{1}{16}(4x^2-12x+9)}}\times\frac{2}{4}$$
$$=\frac{2}{4}\times\frac{1}{\frac{1}{4}\sqrt{16-(4x^2-12x+9)}}$$
$$=\frac{2}{\sqrt{7-4x^2+12x}}$$

9d

$$\frac{d}{dx}\left(\tan^{-1}\frac{1}{1-x}\right)$$

$$= \frac{1}{1+\left(\frac{1}{1-x}\right)^2} \times \frac{d}{dx}\left(\frac{1}{1-x}\right)$$

$$= \frac{(1-x)^2}{(1-x)^2+1} \times -1 \times -1 \times (1-x)^{-2}$$

$$= \frac{1}{(1-x)^2+1}$$

$$= \frac{1}{1-2x+x^2+1}$$

$$= \frac{1}{x^2-2x+2}$$

9e

$$\frac{d}{dx}(\sin^{-1} e^x)$$

$$= \frac{1}{\sqrt{1-(e^x)^2}} \times \frac{d}{dx}(e^x)$$

$$= \frac{e^x}{\sqrt{1-e^{2x}}}$$

9f

$$\frac{d}{dx}\left(\log_e \sqrt{\sin^{-1} x}\right)$$

$$= \frac{\frac{d}{dx}\left(\sqrt{\sin^{-1} x}\right)}{\sqrt{\sin^{-1} x}}$$

$$= \frac{\frac{1}{2} \times (\sin^{-1} x)^{-\frac{1}{2}} \times \frac{d}{dx}(\sin^{-1} x)}{\sqrt{\sin^{-1} x}}$$

$$= \frac{1}{2\sin^{-1} x} \times \frac{d}{dx}(\sin^{-1} x)$$

$$= \frac{1}{2\sqrt{1-x^2}\sin^{-1} x}$$

9g

$$\frac{d}{dx}\left(\sin^{-1}\sqrt{\log_e x}\right)$$

$$= \frac{1}{\sqrt{1-\left((\log_e x)^{\frac{1}{2}}\right)^2}} \times \frac{d}{dx}\left((\log_e x)^{\frac{1}{2}}\right)$$

$$= \frac{1}{\sqrt{1-\log_e x}} \times \frac{1}{2} \times (\log_e x)^{-\frac{1}{2}} \times \frac{d}{dx}(\log_e x)$$

$$= \frac{1}{\sqrt{\log_e x\,(1-\log_e x)}} \times \frac{1}{2}\left(\frac{1}{x}\right)$$

$$= \frac{1}{2x\sqrt{\log_e x\,(1-\log_e x)}}$$

9h

$$\frac{d}{dx}\left(\sqrt{x}\sin^{-1}\sqrt{1-x}\right)$$

$$= \sin^{-1}\sqrt{1-x} \times \frac{d}{dx}(\sqrt{x}) + \sqrt{x} \times \frac{d}{dx}\left(\sin^{-1}\sqrt{1-x}\right)$$

$$= \sin^{-1}\sqrt{1-x} \times \frac{1}{2\sqrt{x}} + \sqrt{x} \times \frac{1}{\sqrt{1-\left(\sqrt{1-x}\right)^2}} \times \frac{d}{dx}\left(\sqrt{1-x}\right)$$

$$= \frac{1}{2\sqrt{x}} \times \sin^{-1}\sqrt{1-x} + \sqrt{x} \times \frac{1}{\sqrt{1-(1-x)}} \times \frac{1}{2} \times -1 \times \frac{1}{\sqrt{1-x}}$$

$$= \frac{1}{2\sqrt{x}} \times \sin^{-1}\sqrt{1-x} + \sqrt{x} \times \frac{1}{\sqrt{x}} \times \frac{1}{2} \times -1 \times \frac{1}{\sqrt{1-x}}$$

$$= \frac{1}{2\sqrt{x}} \times \sin^{-1}\sqrt{1-x} - \frac{1}{2\sqrt{1-x}}$$

9i

$$\frac{d}{dx}\left(\tan^{-1}\frac{x+2}{1-2x}\right)$$

$$= \frac{1}{1+\left(\frac{x+2}{1-2x}\right)^2} \times \frac{d}{dx}\left(\frac{x+2}{1-2x}\right)$$

$$= \frac{1}{1+\left(\frac{x+2}{1-2x}\right)^2} \times \frac{(1-2x)-(-2)(x+2)}{(1-2x)^2}$$

$$= \frac{1}{1 + \left(\frac{x+2}{1-2x}\right)^2} \times \frac{1 - 2x + 2x + 4}{(1-2x)^2}$$

$$= \frac{5}{(1-2x)^2 + (x+2)^2}$$

$$= \frac{5}{1 - 4x + 4x^2 + x^2 + 4x + 4}$$

$$= \frac{5}{5x^2 + 5}$$

$$= \frac{1}{1 + x^2}$$

10a i

$$y' = 2 \times \frac{1}{\sqrt{1-x^2}} \times \sin^{-1} x = \frac{2\sin^{-1} x}{\sqrt{1-x^2}}$$

$$y'' = \frac{2 \times \frac{1}{\sqrt{1-x^2}} \times \sqrt{1-x^2} - \frac{1}{2} \times (-2x) \times (1-x^2)^{-\frac{1}{2}} \times 2\sin^{-1} x}{\left(\sqrt{1-x^2}\right)^2}$$

$$= \frac{2 + \frac{2x\sin^{-1} x}{\sqrt{1-x^2}}}{1-x^2}$$

$$= \frac{2(1 + x\sin^{-1} x)}{(1-x^2)^{\frac{3}{2}}}$$

10a ii

$$(1-x^2)y'' - xy' - 2$$

$$= (1-x^2)\left(\frac{2 + \frac{2x\sin^{-1} x}{\sqrt{1-x^2}}}{1-x^2}\right) - x\left(\frac{2\sin^{-1} x}{\sqrt{1-x^2}}\right) - 2$$

$$= 2 + \frac{2x\sin^{-1} x}{\sqrt{1-x^2}} - \frac{2x\sin^{-1} x}{\sqrt{1-x^2}} - 2$$

$$= 0$$

10b

$$y' = \frac{1}{\sqrt{1-x^2}}e^{\sin^{-1}x}$$

$$y'' = -\frac{1}{2}(-2x)(1-x^2)^{-\frac{3}{2}}e^{\sin^{-1}x} + \frac{1}{\sqrt{1-x^2}}\frac{1}{\sqrt{1-x^2}}e^{\sin^{-1}x}$$

$$= \frac{xe^{\sin^{-1}x}}{(1-x^2)^{\frac{3}{2}}} + \frac{e^{\sin^{-1}x}}{1-x^2}$$

$$(1-x^2)y'' - xy' - y$$

$$= (1-x^2)\left(\frac{xe^{\sin^{-1}x}}{(1-x^2)^{\frac{3}{2}}} + \frac{e^{\sin^{-1}x}}{1-x^2}\right) - x\left(\frac{1}{\sqrt{1-x^2}}e^{\sin^{-1}x}\right) - e^{\sin^{-1}x}$$

$$= \frac{xe^{\sin^{-1}x}}{\sqrt{1-x^2}} + e^{\sin^{-1}x} - \frac{x}{\sqrt{1-x^2}}e^{\sin^{-1}x} - e^{\sin^{-1}x}$$

$$= 0$$

11a The range is the same as that of $\sin^{-1}x$ which is $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

11b

$$y = \sin^{-1}2x$$

$$2x = \sin y$$

$$x = \frac{1}{2}\sin y$$

11c

$\frac{dx}{dy} = \frac{1}{2}\cos y$, as the square root function always returns positive numbers, and as the division of one number by another positive is always positive, this derivative function will always be positive.

11d

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

$$= \frac{1}{\frac{1}{2}\cos y}$$

$$= \frac{2}{\cos y}$$

$$= \frac{2}{\sqrt{1 - \sin^2 y}}$$

$$= \frac{2}{1 - (2x)^2}$$

$$= \frac{2}{\sqrt{1 - 4x^2}}$$

Note that taking the derivative normally yields

$$\frac{dy}{dx} = \frac{2}{\sqrt{1 - (2x)^2}} = \frac{2}{\sqrt{1 - 4x^2}}$$

12a

$$y = \sin^{-1}\frac{1}{2}x$$

$$\frac{x}{2} = \sin y$$

$$x = 2\sin y$$

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

$$= \frac{1}{2\cos y}$$

$$= \frac{1}{2\sqrt{1 - \left(\frac{x}{2}\right)^2}}$$

$$= \frac{1}{\sqrt{4 - x^2}}$$

12b

$$y = \cos^{-1}(x-1)$$

$$x - 1 = \cos y$$

$$x = \cos y + 1$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{\frac{dx}{dy}}\\ &= \frac{1}{-\sin y}\\ &= \frac{1}{-\sqrt{1-\cos^2 y}}\\ &= \frac{1}{-\sqrt{1-(x-1)^2}}\\ &= \frac{-1}{\sqrt{1-(x^2-2x+1)}}\\ &= \frac{-1}{\sqrt{2x-x^2}}\end{aligned}$$

12c

$$y = \tan^{-1}\sqrt{x}$$

$$\sqrt{x} = \tan y$$

$$x = \tan^2 y$$

$$\frac{dx}{dy} = 2 \times \frac{1}{1+y^2} \times \tan y$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{\frac{dx}{dy}}\\ &= \frac{1}{2 \times \frac{1}{1+y^2} \times \tan y}\\ &= \frac{1+y^2}{2\tan y}\\ &= \frac{1+\left(\tan^{-1}\sqrt{x}\right)^2}{2\tan\left(\tan^{-1}\sqrt{x}\right)}\\ &= \frac{1+\left(\tan^{-1}\sqrt{x}\right)^2}{2\sqrt{x}}\end{aligned}$$

13a From the domain of the inverse cosine function we know that $-1 \le x^2 \le 1$. Furthermore, $x^2 \ge 0$ for all real x. Hence $0 \le x^2 \le 1$.

Hence the domain is $-1 \le x \le 1$

13b $f(-x) = \cos^{-1}(-x)^2 = \cos^{-1} x^2 = f(x)$, hence the function is even and symmetric about the line $x = 0$

13c

$$f'(x) = \frac{-1}{\sqrt{1-(x^2)^2}} \times \frac{d}{dx}(x^2)$$
$$= \frac{-1}{\sqrt{1-(x^2)^2}} \times 2x$$
$$= -\frac{2x}{\sqrt{1-x^4}}$$

13d $f''(x) = 2\cos^{-1} x^2 + 4x^2 \cos^{-1} x^2$

$f'(0) = 0$

$f''(0) = -\pi < 0$

Hence at $x = 0$ the derivative is zero and the curve is concave down, so it is a maximum turning point.

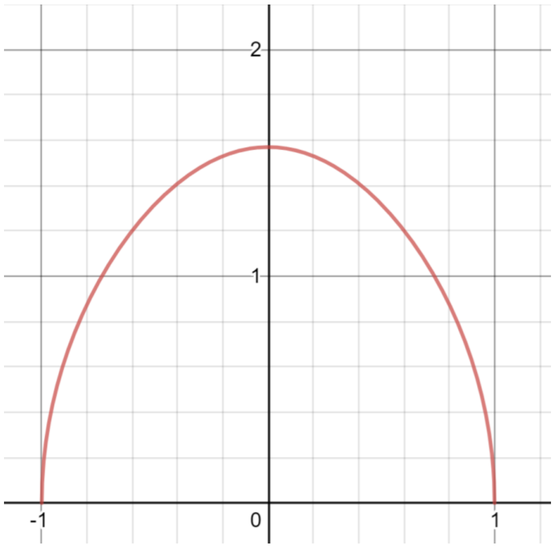
13e

When $x = -1$, $f'(-1) = -\frac{2(-1)}{\sqrt{1-(-1)^4}} = \frac{2}{\sqrt{0}}$ which is undefined.

When $x = 1$, $f'(1) = -\frac{2(1)}{\sqrt{1-(1)^4}} = \frac{2}{\sqrt{0}}$ which is undefined.

This means that the gradient is vertical at that point.

13f



14a

$$\tan \angle TEP = \frac{4}{x}$$

$$\angle TEP = \tan^{-1}\frac{4}{x}$$

$$\tan \angle BEP = \frac{3}{x}$$

$$\angle BEP = \tan^{-1}\frac{3}{x}$$

Hence

$$\theta = \angle TEP - \angle BEP = \tan^{-1}\frac{4}{x} - \tan^{-1}\frac{3}{x}$$

14b

$$\frac{d\theta}{dx} = \frac{1}{\sqrt{1-\left(\frac{4}{x}\right)^2}} \times \frac{d}{dx}\left(\frac{4}{x}\right) - \frac{1}{\sqrt{1-\left(\frac{3}{x}\right)^2}} \times \frac{d}{dx}\left(\frac{3}{x}\right)$$

$$= \frac{1}{\sqrt{1-\frac{16}{x^2}}} \times \left(-\frac{4}{x^2}\right) - \frac{1}{\sqrt{1-\frac{9}{x^2}}} \times \left(-\frac{3}{x^2}\right)$$

$$= \frac{-4}{x\sqrt{x^2-16}} + \frac{3}{x\sqrt{x^2-9}}$$

When $x = 2\sqrt{3}$

$$\frac{d\theta}{dx} = \frac{-4}{(2\sqrt{3})\sqrt{(2\sqrt{3})^2 - 16}} + \frac{3}{(2\sqrt{3})\sqrt{(2\sqrt{3})^2 - 9}}$$
$$= 0$$

Hence there is a stationary point

When $x = \sqrt{3}, \frac{d\theta}{dx} > 0$ and when $x = 3\sqrt{3}, \frac{d\theta}{dx} < 0$. Hence this is a maximum turning point.

14c When $x = 2\sqrt{3}$

$$\theta = \tan^{-1}\frac{4}{2\sqrt{3}} - \tan^{-1}\frac{3}{2\sqrt{3}}$$

$\tan\theta$

$$= \tan\left(\tan^{-1}\frac{4}{2\sqrt{3}} - \tan^{-1}\frac{3}{2\sqrt{3}}\right)$$
$$= \frac{\tan\left(\tan^{-1}\frac{4}{2\sqrt{3}}\right) - \tan\left(\tan^{-1}\frac{3}{2\sqrt{3}}\right)}{1 + \tan\left(\tan^{-1}\frac{4}{2\sqrt{3}}\right)\tan\left(\tan^{-1}\frac{3}{2\sqrt{3}}\right)}$$
$$= \frac{\frac{4}{2\sqrt{3}} - \frac{3}{2\sqrt{3}}}{1 + \frac{4}{2\sqrt{3}}\frac{3}{2\sqrt{3}}}$$
$$= \frac{\frac{1}{2\sqrt{3}}}{1 + 1}$$
$$= \frac{1}{4\sqrt{3}}$$
$$= \frac{\sqrt{3}}{12}$$

Hence

$$\theta = \tan^{-1}\frac{\sqrt{3}}{12}$$

15a

$\theta = \angle OPA$ (equal alternate angles on parallel lines)

$$\tan\theta = \tan\angle OPA = \frac{OA}{AP} = \frac{6}{x}$$

Hence

$$\theta = \tan^{-1}\frac{6}{x}$$

15b

$$\begin{aligned}\frac{d\theta}{dx} &= \frac{1}{1+\left(\frac{6}{x}\right)^2} \times \frac{d}{dx}\left(\frac{6}{x}\right)\\ &= \frac{1}{1+\frac{36}{x^2}} \times \left(-\frac{6}{x^2}\right)\\ &= \frac{-6}{x^2+36}\end{aligned}$$

Now $\frac{dx}{dt} = 600$

Hence

$$\begin{aligned}\frac{d\theta}{dt} &= \frac{d\theta}{dx} \times \frac{dx}{dt}\\ &= \frac{-6}{x^2+36} \times 600\\ &= \frac{-3600}{x^2+36}\end{aligned}$$

15c When $x = 3$,

$$\frac{d\theta}{dt} = \frac{-3600}{3^2+36} = -80$$

Hence it is travelling at 80 radians per hour which is $\frac{1}{45}$ rad/s.

16a Note that $\tan^{-1} x$ is defined for all values of x, the only value for which the function is undefined is when $x = 0$ as $\frac{1}{x}$ is undefined when $x = 0$. Hence the domain is all x such that $x \neq 0$.

$$
\begin{aligned}
f(-x) &= \tan^{-1}(-x) + \tan^{-1}\left(\frac{1}{-x}\right) \\
&= -\tan^{-1}(x) - \tan^{-1}\left(\frac{1}{x}\right) \\
&= -\left(\tan^{-1}(x) + \tan^{-1}\left(\frac{1}{x}\right)\right) \\
&= -f(x)
\end{aligned}
$$

Hence the function is odd so it has rotational symmetry about the origin.

16b

$$
\begin{aligned}
f'(x) &= \frac{1}{1+x^2} + \frac{1}{1+\left(\frac{1}{x}\right)^2} \times \frac{d}{dx}\left(\frac{1}{x}\right) \\
&= \frac{1}{1+x^2} + \frac{1}{1+\frac{1}{x^2}} \times \left(-\frac{1}{x^2}\right) \\
&= \frac{1}{1+x^2} - \frac{1}{1+x^2} \\
&= 0
\end{aligned}
$$

16c

$$
\begin{aligned}
f(x) &= \int f'(x)\, dx \\
&= \int 0\, dx \\
&= C
\end{aligned}
$$

Now as this function has a point of discontinuity at $x = 0$, this constant may be different on either side of the discontinuity.

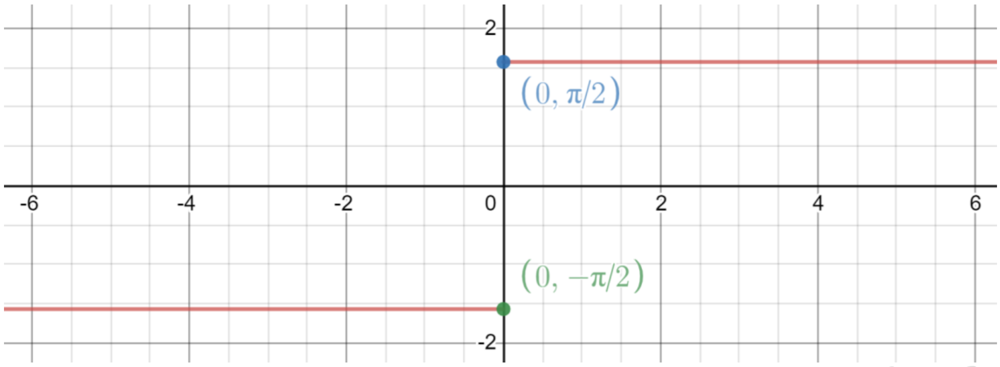
For $x > 0$, substitute in $x = 1$.

$$C = f(1) = \tan^{-1} 1 + \tan^{-1} 1 = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$$

For $x < 0$, substitute in $x = -1$.

$c = f(-1) = -f(1) = -\frac{\pi}{2}$ (as the function is odd)

Hence for $x > 0$, $f(x) = \frac{\pi}{2}$ and for $x < 0$, $f(x) = -\frac{\pi}{2}$



17a

$$\left|\frac{1}{x}\right| \leq 1$$

$$|x| > 1$$

Hence the domain is $x > 1$ or $x < -1$.

17b

$f'(x)$

$$= \frac{-1}{\sqrt{1-\left(\frac{1}{x}\right)^2}} \times \frac{d}{dx}\left(\frac{1}{x}\right)$$

$$= \frac{-1}{\sqrt{1-\frac{1}{x^2}}} \times -\frac{1}{x^2}$$

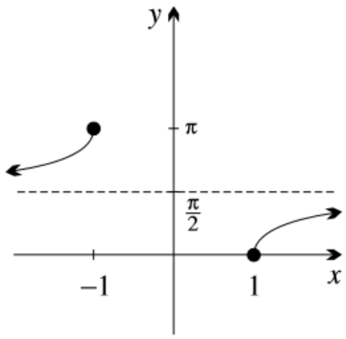
$$= \frac{1}{x^2\sqrt{1-\frac{1}{x^2}}}$$

$$= \frac{1}{\sqrt{x^4}\sqrt{1-\frac{1}{x^2}}}$$

$$= \frac{1}{\sqrt{x^2}\sqrt{x^2-1}}$$

$$= \frac{1}{|x|\sqrt{x^2-1}}$$

17c



17d When $x > 1$,

$$f'(x) = \frac{1}{x\sqrt{x^2 - 1}}$$

When $x < 1$,

$$f'(x) = -\frac{1}{x\sqrt{x^2 - 1}}$$

17e For $x > 1$ we use the equation

$$f'(x) = \frac{1}{x\sqrt{x^2 - 1}}$$

Hence $x > 0$ and $\sqrt{x^2 - 1} > 0$ thus $f'(x) > 0$

For $x < -1$ we use the equation

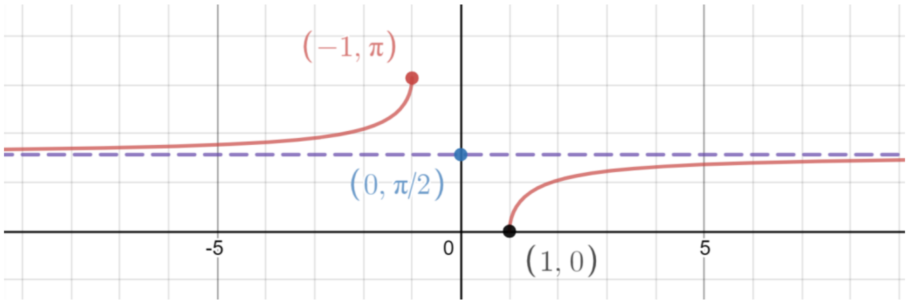
$$f'(x) = -\frac{1}{x\sqrt{x^2 - 1}} = \frac{1}{-x\sqrt{x^2 - 1}}$$

Hence $-x > 0$ and $\sqrt{x^2 - 1} > 0$ thus $f'(x) > 0$

17f i $\lim_{x\to\infty} \cos^{-1}\frac{1}{x} = \lim_{x\to\infty} \cos^{-1} 0^+ = \frac{\pi^+}{2}$

17f ii $\lim_{x\to\infty} \cos^{-1}\frac{1}{x} = \lim_{x\to\infty} \cos^{-1} 0^- = \frac{\pi^-}{2}$

17g



18a $\frac{d}{dx}(\tan^{-1} e^{3x})$

$$= \frac{1}{1 + (e^{3x})^2} \times \frac{d}{dx}(e^{3x})$$

$$= \frac{1}{1 + (e^{3x})^2} \times 3e^{3x}$$

$$= \frac{3e^{3x}}{1 + e^{6x}}$$

18b $\frac{d}{dx}(\sin^{-1} x^3)$

$$= \frac{1}{\sqrt{1 - (x^3)^2}} \times \frac{d}{dx}(x^3)$$

$$= \frac{1}{\sqrt{1 - (x^3)^2}} \times 3x^2$$

$$= \frac{3x^2}{\sqrt{1 - x^6}}$$

18c $\frac{d}{dx}(\cos^{-1} \log_e x)$

$$= \frac{-1}{\sqrt{1 - (\log_e x)^2}} \times \frac{d}{dx}(\log_e x)$$

$$= \frac{-1}{\sqrt{1 - (\log_e x)^2}} \times \frac{1}{x}$$

$$= -\frac{1}{x\sqrt{1 - (\log_e x)^2}}$$

19a The restrictions on x are

$-1 \leq x \leq 1$

and

$-1 \leq \sqrt{1-x^2} \leq 1$

But $\sqrt{1-x^2} \geq 0$, so

$0 \leq \sqrt{1-x^2} \leq 1$

$0 \leq 1-x^2 \leq 1$

$-1 \leq -x^2 \leq 0$

$0 \leq x^2 \leq 1$

Hence the domain is $-1 \leq x \leq 1$.

19b

$$g'(x) = \frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-\left(\sqrt{1-x^2}\right)^2}} \times \frac{d}{dx}\left(\sqrt{1-x^2}\right)$$

$$= \frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-(1-x^2)}} \times \left(\frac{1}{2} \times -2x \times (1-x^2)^{-\frac{1}{2}}\right)$$

$$= \frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{x^2}} \times \left(-\frac{x}{\sqrt{1-x^2}}\right)$$

$$= \frac{1}{\sqrt{1-x^2}} - \frac{1}{|x|} \times \frac{x}{\sqrt{1-x^2}}$$

$$= \frac{1}{\sqrt{1-x^2}} - \frac{x}{|x|\sqrt{1-x^2}}$$

19c $g(x)$ is constant when $g'(x) = 0$, hence

$g'(x) = 0$

$$\frac{1}{\sqrt{1-x^2}} - \frac{x}{|x|\sqrt{1-x^2}} = 0$$

$$\frac{1}{\sqrt{1-x^2}} = \frac{x}{|x|\sqrt{1-x^2}}$$

For $x \neq 1$ and $x \neq 1$

$|x|\sqrt{1-x^2} = x\sqrt{1-x^2}$

$|x| = x$

$|x| - x = 0$

This is true for $x \geq 0$. But $x \neq 0$ so the function is constant for $x > 0$.

20

$\tan^{-1}\frac{x+2}{1-2x}$ is simply $\tan^{-1}x + \tan^{-1}2$ for $x < \frac{1}{2}$, and is $\tan^{-1}x + \tan^{-1}2 - \pi$ for $x > \frac{1}{2}$

(this can be proven by applying tan to both equations and then using double angle formulae to show they have the same result)

21a As $\sin x$ is defined for all real x, and because $-1 \leq \sin x \leq 1$, this function will be defined for all real x.

Now since $-1 \leq \sin x \leq 1$

$\sin^{-1}(-1) \leq \sin^{-1}(\sin x) \leq \sin^{-1}(1)$

$-\frac{\pi}{2} \leq \sin^{-1}(\sin x) \leq \frac{\pi}{2}$

So the range is

$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

21b

$f'(x)$

$= \frac{d}{dx}(\sin^{-1}(\sin(x)))$

$= \frac{1}{\sqrt{1-(\sin x)^2}} \times \frac{d}{dx}(\sin x)$

$= \frac{1}{\sqrt{1-\sin^2 x}} \times \cos x$

$= \frac{\cos x}{\sqrt{\cos^2 x}}$

$= \frac{\cos x}{|\cos x|}$

21c When $\cos x = 0$, $f'(x) = \frac{0}{|0|}$ which is undefined. Hence $f'(x)$ is not defined when $\cos x = 0$.

21d Since $|\cos x| = \cos x$ or $|\cos x| = -\cos x$. Hence

$$f'(x) = \frac{\cos x}{\cos x} = 1$$

or

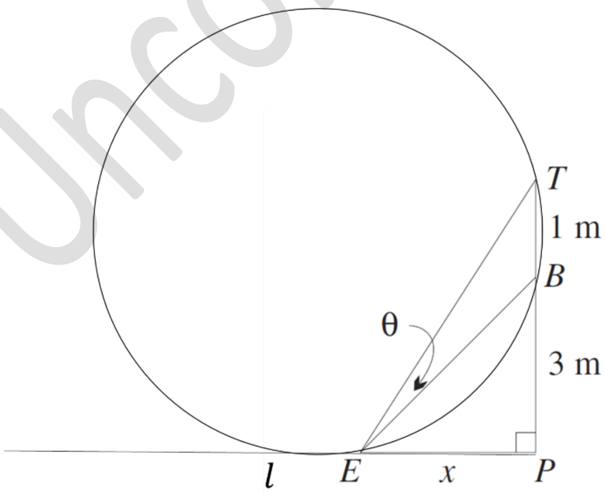
$$f'(x) = \frac{\cos x}{-\cos x} = -1$$

Now $f'(x) = 1$ when $\cos x > 0$ which is when $-\frac{\pi}{2} + 2\pi n < x < -\frac{\pi}{2} + 2\pi n$ for all integers n.

Now $f'(x) = -1$ when $\cos x < 0$ which is when $\frac{\pi}{2} + 2\pi n < x < \frac{3\pi}{2} + 2\pi n$ for all integers n.

21e

22a



Solutions to Exercise 12B

1 Answers will vary

2a

$$\int -\frac{1}{\sqrt{1-x^2}}\,dx = \cos^{-1} x + C$$

2b

$$\int \frac{1}{\sqrt{4-x^2}}\,dx$$

$$= \int \frac{1}{2\sqrt{1-\frac{x^2}{4}}}\,dx$$

$$= \int \frac{1}{2} \times \frac{1}{\sqrt{1-\left(\frac{x}{2}\right)^2}}\,dx$$

$$= \sin^{-1}\left(\frac{x}{2}\right) + C$$

2c

$$\int \frac{1}{9+x^2}\,dx$$

$$= \int \frac{1}{9}\frac{1}{1+\frac{x^2}{9}}\,dx$$

$$= \frac{1}{3}\int \frac{1}{3}\frac{1}{1+\left(\frac{x}{3}\right)^2}\,dx$$

$$= \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + C$$

2d

$$\int \frac{1}{\sqrt{\frac{4}{9}-x^2}}\,dx$$

$$= \int \frac{1}{\frac{2}{3}\sqrt{1-\left(\frac{3}{2}x\right)^2}}\,dx$$

$$= \int \frac{3}{2}\frac{1}{\sqrt{1-\left(\frac{3}{2}x\right)^2}}\,dx$$

$$= \sin^{-1}(\frac{3}{2}x) + C$$

2e

$$\int \frac{1}{2+x^2}\,dx$$

$$= \int \frac{1}{2}\frac{1}{1+\left(\frac{x}{\sqrt{2}}\right)^2}\,dx$$

$$= \frac{\sqrt{2}}{2}\tan^{-1}\frac{x}{\sqrt{2}} + C$$

2f

$$\int -\frac{1}{\sqrt{5-x^2}}\,dx$$

$$= \int \frac{1}{\sqrt{5}}\frac{-1}{\sqrt{1-\left(\frac{x}{\sqrt{5}}\right)^2}}\,dx$$

$$= \cos^{-1}\left(\frac{x}{\sqrt{5}}\right) + C$$

3a

$$\int_0^3 \frac{1}{\sqrt{9-x^2}}\,dx$$

$$= \int_0^3 \frac{1}{3\sqrt{1-\left(\frac{x}{3}\right)^2}}\,dx$$

$$= \left[\sin^{-1}\frac{x}{3}\right]_0^3$$

$$= \sin^{-1} 1 - \sin^{-1} 0$$

$$= \frac{\pi}{2}$$

3b

$$\int_0^2 \frac{1}{4+x^2}\,dx$$

$$= \int_0^2 \frac{1}{4}\frac{1}{1+\left(\frac{x}{2}\right)^2}\,dx$$

$$= \frac{1}{2}\left[\tan^{-1}\left(\frac{x}{2}\right)\right]_0^2$$

$$= \frac{1}{2}(\tan^{-1} 1 - \tan^{-1} 0)$$

$$= \frac{\pi}{8}$$

3c

$$\int_0^1 \frac{1}{\sqrt{2-x^2}}\,dx$$

$$= \int_0^1 \frac{1}{\sqrt{2}\sqrt{1-\left(\frac{x}{\sqrt{2}}\right)^2}}\,dx$$

$$= \left[\sin^{-1}\left(\frac{x}{\sqrt{2}}\right)\right]_0^1$$

$$= \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) - \sin^{-1}\left(\frac{0}{\sqrt{2}}\right)$$

$$= \frac{\pi}{4}$$

3d

$$\int_{\frac{1}{2}}^{\frac{1}{2}\sqrt{3}} \frac{\frac{1}{2}}{\frac{1}{4} + x^2}\,dx$$

$$= \int_{\frac{1}{2}}^{\frac{1}{2}\sqrt{3}} \frac{2}{1 + 4x^2}\,dx$$

$$= \int_{\frac{1}{2}}^{\frac{1}{2}\sqrt{3}} \frac{2}{1 + (2x)^2}\,dx$$

$$= \left[\tan^{-1}(2x)\right]_{\frac{1}{2}}^{\frac{1}{2}\sqrt{3}}$$

$$= \tan^{-1}\left(2\sqrt{3}\right) - \tan^{-1}(1)$$

$$= \frac{\pi}{12}$$

3e

$$\int_{\frac{\sqrt{3}}{6}}^{\frac{1}{6}} \frac{-1}{\sqrt{\frac{1}{9} - x^2}}\,dx$$

$$= \int_{\frac{\sqrt{3}}{6}}^{\frac{1}{6}} \frac{-1}{\frac{1}{3}\sqrt{1 - (3x)^2}}\,dx$$

$$= \int_{\frac{\sqrt{3}}{6}}^{\frac{1}{6}} \frac{-3}{\sqrt{1 - (3x)^2}}\,dx$$

$$= \left[\cos^{-1}(3x)\right]_{\frac{\sqrt{3}}{6}}^{\frac{1}{6}}$$

$$= \cos^{-1}\frac{1}{2} - \cos^{-1}\frac{\sqrt{3}}{2}$$

$$= \frac{\pi}{6}$$

3f

$$\int_{-\frac{3}{4}\sqrt{2}}^{\frac{3}{4}} \frac{1}{\sqrt{\frac{9}{4} - x^2}}\, dx$$

$$= \int_{-\frac{3}{4}\sqrt{2}}^{\frac{3}{4}} \frac{1}{\frac{3}{2}\sqrt{1 - \left(\frac{2x}{3}\right)^2}}$$

$$= \int_{-\frac{3}{4}\sqrt{2}}^{\frac{3}{4}} \frac{2}{3\sqrt{1 - \left(\frac{2x}{3}\right)^2}}$$

$$= \left[\sin^{-1}\left(\frac{2x}{3}\right)\right]_{-\frac{3}{4}\sqrt{2}}^{\frac{3}{4}}$$

$$= \sin^{-1}\frac{1}{2} - \sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$$

$$= \frac{5\pi}{12}$$

4a

$$y' = (1 - x^2)^{-\frac{1}{2}} = \frac{1}{\sqrt{1 - x^2}}$$

$$y = \sin^{-1} x + C$$

Substituting in $(0, \pi)$

$$\pi = \sin^{-1} 0 + C$$

$$\pi = 0 + C$$

$$C = \pi$$

Hence

$$y = \sin^{-1} x + \pi$$

4b

$$y' = 4(16 + x^2)^{-1}$$

$$= \frac{1}{4(16 + x^2)}$$

$$= \frac{4}{4 \times 16} \times \frac{1}{1 + \left(\frac{x}{4}\right)^2}$$

$$= \frac{1}{4} \times \frac{1}{1 + \left(\frac{x}{4}\right)^2}$$

$$y = \tan^{-1}\frac{x}{4} + C$$

Substituting in $(-4, 0)$

$$0 = \tan^{-1} -1 + C$$

$$0 = -\frac{\pi}{4} + C$$

$$C = \frac{\pi}{4}$$

Hence

$$y = \tan^{-1}\frac{x}{4} + \frac{\pi}{4}$$

5a

$$y' = \frac{1}{\sqrt{36 - x^2}} = \frac{1}{6\sqrt{1 - \left(\frac{x}{6}\right)^2}}$$

$$y = \sin^{-1}\left(\frac{x}{6}\right) + C$$

When $x = 3, y = \frac{\pi}{6}$

$$\frac{\pi}{6} = \sin^{-1}\frac{1}{2} + C$$

$$C = 0$$

$$y = \sin^{-1}\left(\frac{x}{6}\right)$$

When $x = 3\sqrt{3}$

$$y = \sin^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{3}$$

5b

$$y' = \frac{2}{4 + x^2} = \frac{1}{2}\frac{1}{1 + \left(\frac{x}{2}\right)^2}$$

$$y = \tan^{-1}\frac{x}{2} + C$$

When $x = 2, y = \frac{\pi}{3}$

$$\frac{\pi}{3} = \tan^{-1} 1 + C$$

$$\frac{\pi}{3} = \frac{\pi}{4} + C$$

$$C = \frac{\pi}{12}$$

$$y = \tan^{-1}\frac{x}{2} + \frac{\pi}{12}$$

When $x = \frac{2}{\sqrt{3}}$

$$y = \tan^{-1}\frac{1}{\sqrt{3}} + \frac{\pi}{12} = \frac{\pi}{4}$$

6a

$$\int \frac{1}{\sqrt{1 - 4x^2}} dx$$

$$= \int \frac{1}{\sqrt{1 - (2x)^2}} dx$$

$$= \frac{1}{2}\sin^{-1} 2x + C$$

6b

$$\int \frac{1}{1 + 16x^2} dx$$

$$= \int \frac{1}{1 + (4x)^2} dx$$

$$= \frac{1}{4}\tan^{-1} 4x + C$$

6c

$$\int -\frac{1}{\sqrt{1-2x^2}}dx$$

$$= \int -\frac{1}{\sqrt{1-\left(\sqrt{2}x\right)^2}}dx$$

$$= \frac{1}{\sqrt{2}}\cos^{-1}\sqrt{2}x + C$$

6d

$$\int \frac{1}{\sqrt{4-9x^2}}dx$$

$$= \int \frac{1}{2\sqrt{1-\left(\frac{3x}{2}\right)^2}}dx$$

$$= \frac{1}{3}\sin^{-1}\frac{3x}{2} + C$$

6e

$$\int \frac{1}{25+9x^2}dx$$

$$= \int \frac{1}{25}\frac{1}{1+\left(\frac{3x}{5}\right)^2}dx$$

$$= \frac{1}{15}\tan^{-1}\frac{3x}{5} + C$$

6f

$$\int -\frac{1}{\sqrt{3-4x^2}}dx$$

$$= \int -\frac{1}{\sqrt{3}\sqrt{1-\left(\frac{2x}{\sqrt{3}}\right)^2}}dx$$

$$= \frac{1}{2}\cos^{-1}\frac{2x}{\sqrt{3}} + C$$

7a

$$\int_0^{\frac{1}{6}} \frac{1}{\sqrt{1-9x^2}}\,dx$$

$$= \int_0^{\frac{1}{6}} \frac{1}{\sqrt{1-(3x)^2}}\,dx$$

$$= \frac{1}{3}[\sin^{-1} 3x]_0^{\frac{1}{6}}$$

$$= \frac{\pi}{18}$$

7b

$$\int_{\frac{1}{2}}^{\frac{1}{2}\sqrt{3}} \frac{2}{1+4x^2}\,dx$$

$$= \int_{\frac{1}{2}}^{\frac{1}{2}\sqrt{3}} \frac{2}{1+(2x)^2}\,dx$$

$$= [\tan^{-1}(2x)]_{\frac{1}{2}}^{\frac{1}{2}\sqrt{3}}$$

$$= \frac{\pi}{12}$$

7c

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\sqrt{1-3x^2}}\,dx$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\sqrt{1-\left(\sqrt{3}x\right)^2}}\,dx$$

$$= \frac{1}{\sqrt{3}}[\sin^{-1}\sqrt{3}x]_{-\frac{1}{2}}^{\frac{1}{2}}$$

$$= \frac{2\pi}{9}\sqrt{3}$$

7d

$$\int_{-\frac{3}{4}}^{\frac{3}{2\sqrt{2}}} \frac{1}{\sqrt{9-4x^2}}\,dx$$

$$= \int_{-\frac{3}{4}}^{\frac{3}{2\sqrt{2}}} \frac{1}{3\sqrt{1-\left(\frac{2x}{3}\right)^2}}\,dx$$

$$= \frac{1}{2}\left[\sin^{-1}\left(\frac{2x}{3}\right)\right]_{-\frac{3}{4}}^{\frac{3}{2\sqrt{2}}}$$

$$= \frac{5\pi}{24}$$

7e

$$\int_{-\frac{1}{2}}^{\frac{3}{2}} \frac{1}{3+4x^2}\,dx$$

$$= \int_{-\frac{1}{2}}^{\frac{3}{2}} \frac{1}{3}\frac{1}{1+\left(\frac{2x}{\sqrt{3}}\right)^2}\,dx$$

$$= \frac{\sqrt{3}}{6}\left[\tan^{-1}\frac{2x}{\sqrt{3}}\right]_{-\frac{1}{2}}^{\frac{3}{2}}$$

$$= \frac{\pi}{12}\sqrt{3}$$

7f

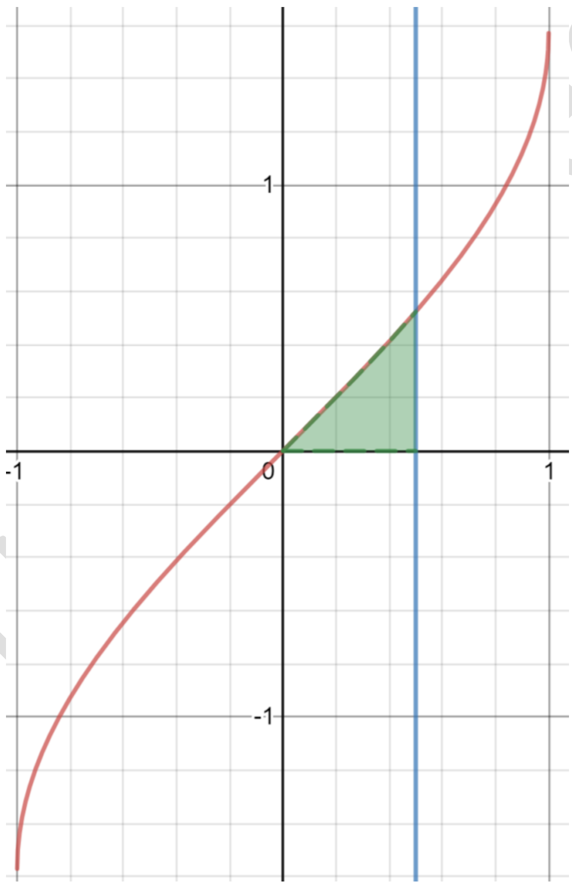
$$\int_{\frac{1}{2}\sqrt{10}}^{\frac{1}{2}\sqrt{30}} \frac{1}{5+2x^2}\,dx$$

$$= \int_{\frac{1}{2}\sqrt{10}}^{\frac{1}{2}\sqrt{30}} \frac{1}{5}\frac{1}{1+\left(\frac{\sqrt{2}x}{\sqrt{5}}\right)^2}\,dx$$

$$= \frac{\sqrt{5}}{5\sqrt{2}}\left[\tan^{-1}\frac{\sqrt{2}x}{\sqrt{5}}\right]_{\frac{1}{2}\sqrt{10}}^{\frac{1}{2}\sqrt{30}}$$

$$= \frac{\pi}{120}\sqrt{10}$$

8a



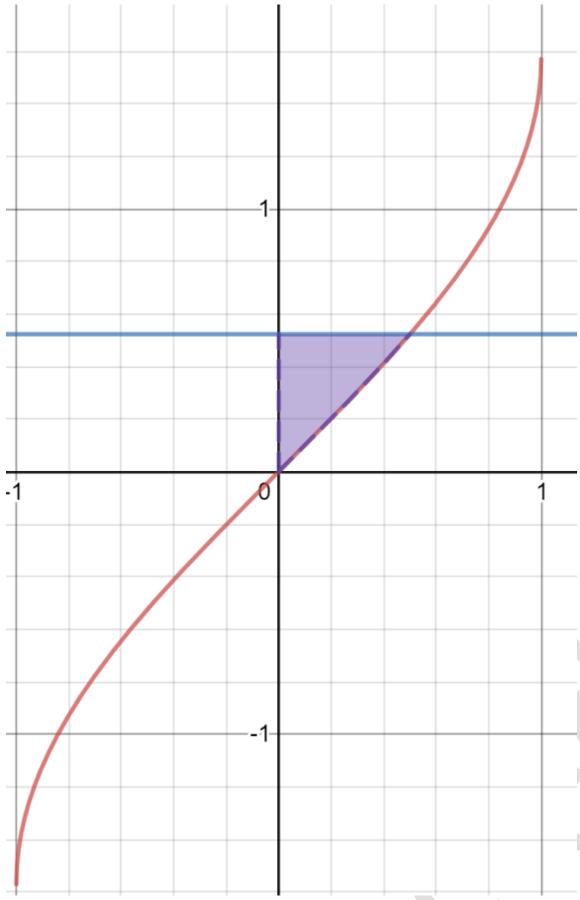
8b

$$
\begin{aligned}
&\frac{d}{dx}\left(x\sin^{-1}x+\sqrt{1-x^2}\right)\\
&=\frac{d}{dx}(x\sin^{-1}x)+\frac{d}{dx}\left(\sqrt{1-x^2}\right)\\
&=\frac{d}{dx}(x)\sin^{-1}x+x\frac{d}{dx}(\sin^{-1}x)+\frac{d}{dx}\left((1-x^2)^{\frac{1}{2}}\right)\\
&=\sin^{-1}x+\frac{x}{\sqrt{1-x^2}}+\frac{1}{2}\times -2x\times\frac{1}{\sqrt{1-x^2}}\\
&=\sin^{-1}x+\frac{x}{\sqrt{1-x^2}}-\frac{x}{\sqrt{1-x^2}}\\
&=\sin^{-1}x
\end{aligned}
$$

8c

$$
\begin{aligned}
&\int_0^{\frac{1}{2}}\sin^{-1}x\,dx\\
&=\int_0^{\frac{1}{2}}\frac{d}{dx}\left(x\sin^{-1}x+\sqrt{1-x^2}\right)dx\\
&=\left[x\sin^{-1}x+\sqrt{1-x^2}\right]_0^{\frac{1}{2}}\\
&=\left(\frac{\pi}{12}+\frac{1}{2}\sqrt{3}-1\right)\text{ square units}
\end{aligned}
$$

9a



9b

$$A=\int_0^{\frac{\pi}{6}}\sin y\,dy$$

$$=-[\cos y]_0^{\frac{\pi}{6}}$$

$$=\left(1-\tfrac{1}{2}\sqrt{3}\right)\text{ square units}$$

9c Note that the area formed by combined regions of the previous two questions is that of a square with area $\frac{\pi}{6}\times\frac{1}{2}=\frac{\pi}{12}$ units².

Thus, the area in this question will be $\frac{\pi}{12}-(\frac{\pi}{12}+\frac{1}{2}\sqrt{3}-1)=1-\frac{1}{2}\sqrt{3}$ units²

10a

$$\frac{d}{dx}(\cos^{-1}(2-x))$$
$$=\frac{-1}{\sqrt{1-(2-x)^2}}\times\frac{d}{dx}(2-x)$$
$$=\frac{-1}{\sqrt{1-(2-x)^2}}\times -1$$
$$=\frac{1}{\sqrt{1-(4-2x+x^2)}}$$
$$=\frac{1}{\sqrt{4x-x^2-3}}$$

10b

$$\int_1^2 \frac{1}{\sqrt{4x-x^2-3}}\,dx$$
$$=[\cos^{-1}(2-x)]_1^2$$
$$=\cos^{-1}0-\cos^{-1}1$$
$$=\frac{\pi}{2}$$

11a

$$\frac{d}{dx}\left(\tan^{-1}\frac{1}{2}x^3\right)$$
$$=\frac{1}{1+\left(\frac{1}{2}x^3\right)^2}\times\frac{d}{dx}\left(\frac{1}{2}x^3\right)$$
$$=\frac{1}{1+\frac{x^6}{4}}\times\frac{3}{2}x^2$$
$$=\frac{3x^2}{2+\frac{x^6}{2}}$$
$$=\frac{6x^2}{4+x^6}$$

11b

$$\int \frac{x^2}{4+x^6}\,dx$$

$$= \frac{1}{6}\int \frac{6x^2}{4+x^6}\,dx$$

$$= \frac{1}{6}\int \frac{d}{dx}\left(\tan^{-1}\frac{1}{2}x^3\right)dx$$

$$= \frac{1}{6}\tan^{-1}\frac{1}{2}x^3 + C$$

12a

$$\frac{d}{dx}(x\tan^{-1}x)$$

$$= \tan^{-1}x \times \frac{d}{dx}(x) + x \times \frac{d}{dx}(\tan^{-1}x)$$

$$= \tan^{-1}x + \frac{x}{1+x^2}$$

12b

$$\frac{d}{dx}(x\tan^{-1}x) = \tan^{-1}x + \frac{x}{1+x^2}$$

$$\tan^{-1}x = \frac{d}{dx}(x\tan^{-1}x) - \frac{x}{1+x^2}$$

$$\int_0^1 \tan^{-1}x\,dx$$

$$= \int_0^1 \left(\frac{d}{dx}(x\tan^{-1}x) - \frac{x}{1+x^2}\right)dx$$

$$= \int_0^1 \frac{d}{dx}(x\tan^{-1}x)\,dx - \int_0^1 \frac{x}{1+x^2}\,dx$$

$$= [x\tan^{-1}x]_0^1 - \left[\frac{1}{2}\ln(1+x^2)\right]_0^1$$

$$= \frac{\pi}{4} - \frac{1}{2}(\ln 2 - 0)$$

$$= \frac{\pi}{4} - \frac{1}{2}\ln 2$$

13a 0 as this is an odd function

13b 0 as this is an odd function

13c

$$\int_{-\frac{3}{4}}^{\frac{3}{4}} \cos^{-1} x \, dx$$

$$= -\left[\frac{1}{\sqrt{1-x^2}}\right]_{-\frac{3}{4}}^{\frac{3}{4}}$$

$$= \frac{3\pi}{4}$$

13d 0 as this is an odd function

13e 0 as this is an odd function

13f

This is a semicircle of radius 6. Hence

$$\int_{-6}^{6} \sqrt{36-x^2} \, dx$$

$$= \frac{1}{2} \times (\pi \times 6^2)$$

$$= 18\pi \text{ square units}$$

14a i $f(0) = \frac{0}{1+0^2} - \tan^{-1} 0 = 0 - 0 = 0$

14a ii

$$f'(x) = \frac{d}{dx}\left(\frac{x}{1+x^2}\right) - \frac{d}{dx}(\tan^{-1} x)$$

$$= \frac{(1+x^2) \times \frac{d}{dx}(x) - x \times \frac{d}{dx}(1+x^2)}{(1+x^2)^2} - \frac{1}{1+x^2}$$
$$= \frac{1+x^2-x(2x)}{(1+x^2)^2} - \frac{1+x^2}{(1+x^2)^2}$$
$$= \frac{-2x^2}{(1+x^2)^2}$$

14b i Firstly consider $f'(x)$

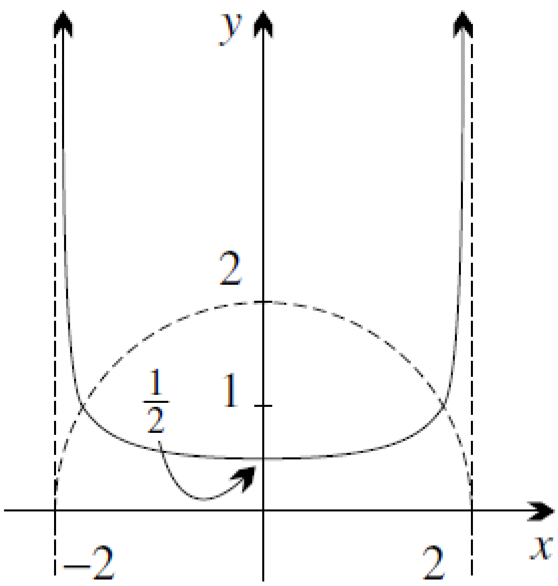
As the numerator $-2x^2 < 0$ for all $x > 0$ and the denominator $(1+x^2)^2 < 0$ for all $x \geq 0$. It follows that $f'(x) < 0$ for all $x < 0$.

Since $f(0) = 0$, and as the function is strictly decreasing for all $x > 0$, it follows that $f(x) < 0$ for all $x > 0$.

14b ii

$$\int_0^1 \frac{x^2}{(1+x^2)^2}\,dx$$
$$= -\frac{1}{2}\int_0^1 \frac{-2x^2}{(1+x^2)^2}\,dx$$
$$= -\frac{1}{2}\int_0^1 f'(x)\,dx$$
$$= -\frac{1}{2}[f(x)]_0^1$$
$$= -\frac{1}{2}[f(1) - f(0)]$$
$$= -\frac{1}{2}f(1)$$
$$= -\frac{1}{2}\left[\frac{1}{2} - \frac{\pi}{4}\right]$$
$$= \frac{1}{8}[\pi - 2]$$

15a,b



15c From the graph the domain is $-2 < x < 2$ (note that this is strictly less than) and the range is $x \geq \frac{1}{2}$. Its symmetry is even as $f(-x) = \frac{1}{\sqrt{4-(-x)^2}} = \frac{1}{\sqrt{4-x^2}} = f(x)$

15d

$$
\begin{aligned}
A &= \int_{-1}^{1} \frac{1}{\sqrt{4-x^2}}\,dx \\
&= \int_{-1}^{1} \frac{1}{2\sqrt{1-\left(\frac{x}{2}\right)^2}}\,dx \\
&= \left[\sin^{-1}\left(\frac{x}{2}\right)\right]_{-1}^{1} \\
&= \sin^{-1}\frac{1}{2} - \sin^{-1}\left(-\frac{1}{2}\right) \\
&= \frac{\pi}{6} - \left(-\frac{\pi}{6}\right) \\
&= \frac{\pi}{3} \text{ square units}
\end{aligned}
$$

15e

$$\begin{aligned}
A &= \int_{-2}^{2} \frac{1}{\sqrt{4-x^2}}\,dx \\
&= \int_{-2}^{2} \frac{1}{2\sqrt{1-\left(\frac{x}{2}\right)^2}}\,dx \\
&= \left[\sin^{-1}\left(\frac{x}{2}\right)\right]_{-2}^{2} \\
&= \sin^{-1} 1 - \sin^{-1}(-1) \\
&= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) \\
&= \pi \ \text{ square units}
\end{aligned}$$

16a

$$f(-x) = \frac{4}{(-x)^2+4} = \frac{4}{x^2+4} = f(x)$$

Hence the function is even and thus has an axis of symmetry about the y-axis.

16b The function is defined for all real x so the domain is all real x.

As $x^2 + 4 \geq 4$ it follows that $0 < \frac{1}{x^2+4} \leq \frac{1}{4}$ and hence $0 < \frac{4}{x^2+4} \leq 1$ thus the range is $0 < f(x) \leq 1$.

16c

$$\begin{aligned}
&f'(x) \\
&= \frac{d}{dx}\left(4(x^2+4)^{-1}\right) \\
&= -1 \times 4(x^2+4)^{-2} \times \frac{d}{dx}(x^2) \\
&= -1 \times 4(x^2+4)^{-2} \times 2x \\
&= -\frac{8x}{(x^2+4)^2}
\end{aligned}$$

Hence

$f'(0) = 0$

and

$f(0) = 1$

So there is a turning point at $(0, 1)$.

Testing for maximum

x	-1	0	1
$f'(x)$	$\frac{8}{25}$	0	$-\frac{8}{25}$

Thus the curve is increasing before the turning point decreasing afterwards.
Thus it is a maximum

16d

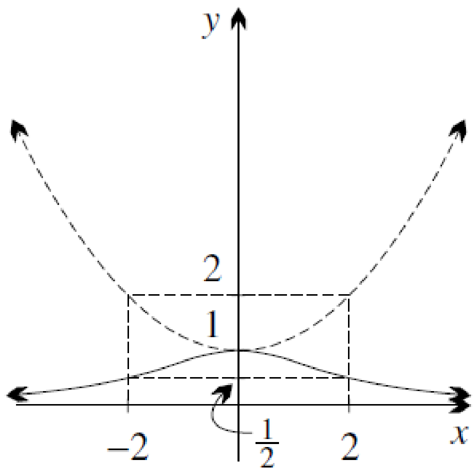
$$\lim_{x\to\infty} f(x)$$

$$= \lim_{x\to\infty} \frac{4}{x^2+4}$$

$$= \lim_{x\to\infty} \frac{\frac{4}{x^2}}{1+\frac{4}{x^2}}$$

$$= \frac{0}{1+0}$$

$$= 0$$



16e

$$\begin{aligned}
A &= \int_{-2\sqrt{3}}^{2\sqrt{3}} \frac{4}{x^2+4}\,dx \\
&= \int_{-2\sqrt{3}}^{2\sqrt{3}} \frac{1}{\left(\frac{x}{2}\right)^2+1}\,dx \\
&= 2\left[\tan^{-1}\frac{x}{2}\right]_{-2\sqrt{3}}^{2\sqrt{3}} \\
&= 2\left(\tan^{-1}\sqrt{3}-\tan^{-1}(-\sqrt{3})\right) \\
&= \pi \text{ square units}
\end{aligned}$$

16f

$$\begin{aligned}
A &= \int_{-a}^{a} \frac{4}{x^2+4}\,dx \\
&= \int_{-a}^{a} \frac{1}{\left(\frac{x}{2}\right)^2+1}\,dx \\
&= 2\left[\tan^{-1}\frac{x}{2}\right]_{-a}^{a} \\
&= 2\left(\tan^{-1}\frac{a}{2}-\tan^{-1}\left(-\frac{a}{2}\right)\right) \\
&= 4\tan^{-1}\frac{a}{2} \text{ square units}
\end{aligned}$$

16g

$$\lim_{a\to\infty} 4\tan^{-1} a$$

$$= 4\lim_{a\to\infty}\tan^{-1} a$$

$$= 4\left(\frac{\pi}{2}\right)$$

$$= 2\pi \text{ square units}$$

17

Since $y = \sin^{-1} x, \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$.

When $x = \frac{\sqrt{3}}{2}, y = \frac{\pi}{3}, \frac{dy}{dx} = 2$. Hence the tangent is

$$y - \frac{\pi}{3} = 2\left(x - \frac{\sqrt{3}}{2}\right)$$

$$y = 2x - \sqrt{3} + \frac{\pi}{3}$$

Rearranging this gives

$$x = \frac{1}{2}\left(y + \sqrt{3} - \frac{\pi}{3}\right)$$

Also note that for the tangent, when $x = 0, y = -\sqrt{3} + \frac{\pi}{3}$

Hence the area can be calculated by

$$\begin{aligned}A &= \int_{\sqrt{3}+\frac{\pi}{3}}^{\frac{\pi}{3}} y_{tangent}\, dx - \int_0^{\frac{\pi}{3}} y_{curve}\, dx \\ &= \int_{\sqrt{3}+\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{1}{2}\left(y + \sqrt{3} - \frac{\pi}{3}\right) dx - \int_0^{\frac{\pi}{3}} \sin y\, dx \\ &= \frac{1}{2}\left[\frac{y^2}{2} + y\left(\sqrt{3} - \frac{\pi}{3}\right)\right]_{\sqrt{3}+\frac{\pi}{3}}^{\frac{\pi}{3}} + [\cos y]_0^{\frac{\pi}{3}} \\ &= \frac{1}{4} \text{ square units}\end{aligned}$$

18a

$$\begin{aligned}I &= \int_0^1 f(x)\, dx \\ &\doteqdot \frac{b-a}{2n}(f(a) + f(b) + 2(f(x_1) + f(x_2) + \cdots)) \\ &= \frac{1-0}{2(4)}(f(0) + f(1) + 2(f(0.25) + f(0.5) + f(0.75)) \\ &= \frac{1}{8}\left(1 + \frac{1}{2} + 2\left(\frac{16}{17} + 0.8 + 0.64\right)\right) \\ &= \frac{5323}{6800} \text{ square units}\end{aligned}$$

18b

$$I = \int_0^1 \frac{1}{1+x^2}\,dx = [\tan^{-1} x]_0^1 = \frac{\pi}{4}$$

Hence

$$\frac{\pi}{4} \doteqdot \frac{5323}{6800}$$

so

$$\pi \doteqdot \frac{5323}{1700}$$

19

$$\int_{-\frac{1}{4}}^{\frac{3}{5}} \frac{1}{1+x^2}\,dx$$

$$= [\tan^{-1} x]_{-\frac{1}{4}}^{\frac{3}{5}}$$

$$= \tan^{-1}\frac{3}{5} + \tan^{-1}\left(-\frac{1}{4}\right)$$

Hence

$$\tan \int_{-\frac{1}{4}}^{\frac{3}{5}} \frac{1}{1+x^2}\,dx$$

$$= \tan\left(\tan^{-1}\frac{3}{5} + \tan^{-1}\left(-\frac{1}{4}\right)\right)$$

$$= \frac{\tan\left(\tan^{-1}\frac{3}{5}\right) + \tan\left(\tan^{-1}\left(-\frac{1}{4}\right)\right)}{1 - \tan\left(\tan^{-1}\frac{3}{5}\right)\tan\left(\tan^{-1}\left(-\frac{1}{4}\right)\right)}$$

$$= \frac{\frac{3}{5} - \frac{1}{4}}{1 - \left(\frac{3}{5}\right)\left(-\frac{1}{4}\right)}$$

$$= 1$$

Hence

$$\int_{-\frac{1}{4}}^{\frac{3}{5}} \frac{1}{1+x^2}\,dx = \frac{\pi}{4}$$

20a Let $x = u^2$, $dx = 2u\,du$

$$\int \frac{1}{\sqrt{x}(1-x)}\,dx = \int \frac{1}{\sqrt{u^2}(1+u^2)} \times 2u\,du$$

For $u > 0$

$$\int \frac{1}{\sqrt{x}(1-x)}\,dx$$

$$= \int \frac{1}{u(1+u^2)} \times 2u\,du$$
$$= 2\int \frac{1}{(1+u^2)}\,du$$
$$= 2\tan^{-1} u + c$$
$$= 2\tan^{-1} \sqrt{x} + \mathrm{c}$$

For $u < 0$

$$\int \frac{1}{\sqrt{x}(1-x)}\,dx$$

$$= \int \frac{1}{-u(1+u^2)} \times 2u\,du$$
$$= -2\int \frac{1}{(1+u^2)}\,du$$
$$= -2\tan^{-1} u + c$$
$$= -2\tan^{-1} \sqrt{x} + \mathrm{c}$$

20b Let $u = e^x$, $du = e^x\,dx$ hence $du = u\,dx$ and so $\frac{1}{u}\,du = dx$

$$\int_0^1 \frac{1}{e^{-x} + e^x}\,dx$$

$$= \int_1^e \frac{1}{u^{-1} + u} \times \frac{1}{u}du$$
$$= \int_1^e \frac{1}{1+u^2}du$$
$$= [\tan^{-1} u]_1^e$$
$$= \tan^{-1} e - \frac{\pi}{4}$$

21a

$$\begin{aligned}
&\frac{d}{dx}\left(\tan^{-1}\left(\frac{3}{2}\tan x\right)\right)\\
&= \frac{1}{1+\left(\frac{3}{2}\tan x\right)^2} \times \frac{d}{dx}\left(\frac{3}{2}\tan x\right)\\
&= \frac{1}{1+\frac{9}{4}\tan^2 x} \times \frac{3}{2}\sec^2 x\\
&= \frac{1}{1+\frac{9}{4}\frac{\sin^2 x}{\cos^2 x}} \times \frac{3}{2}\frac{1}{\cos^2 x}\\
&= \frac{3}{2} \times \frac{1}{\cos^2 x + \frac{9}{4}\sin^2 x}\\
&= \frac{6}{4\cos^2 x + 9\sin^2 x}\\
&= \frac{6}{4\sin^2 x + 5(\sin^2 x + \cos^2 x)}\\
&= \frac{6}{4\sin^2 x + 5}
\end{aligned}$$

21b

$$\begin{aligned}
A &= \int_0^7 \frac{6}{4\sin^2 x + 5}\,dx\\
&= \int_0^7 \frac{d}{dx}\left(\tan^{-1}\left(\frac{3}{2}\tan x\right)\right)dx\\
&= \left[\tan^{-1}\left(\frac{3}{2}\tan x\right)\right]_0^7\\
&= \tan^{-1}\left(\frac{3}{2}\tan 7\right) - \tan^{-1}\left(\frac{3}{2}\tan 0\right)\\
&\doteqdot 0.153 \text{ square unit}
\end{aligned}$$

21c The integrand is well-defined in the interval $[0, 7]$, and lies between 14 and 19, so the area lies between 74 and 79, which is much larger than the answer of 0.153 that was calculated in part (b). The primitive, however, is undefined at two values within the interval, at $x = \frac{\pi}{2}$ and at $x = \frac{3\pi}{2}$, which renders the argument completely invalid.

22a This is a GP with $a = 1$ and $r = -t^2$. Hence $T_k = (-t)^{k-1}$, solving for $T_k = t^{4n}$

$t^{4n} = (-t^2)^{k-1}$

$t^{4n} = (-1)^{k-1}(t^2)^{k-1}$

As $4n$ is even, $k - 1$ must also be even and hence $(-1)^{k-1} = 1$. Thus

$t^{4n} = (t^2)^{k-1}$

$4n = 2k - 2$

$k = 2n + 1$

Hence there are $2n + 1$ terms so the sum must be

$$S_{2n+1} = \frac{a(r^n - 1)}{r - 1} = \frac{(-t^2)^{2n+1} - 1}{-t^2 - 1}$$

Since $2n + 1$ is odd

$$S_{2n+1} = \frac{-t^{4n+2} - 1}{-t^2 - 1} = \frac{t^{4n+2} + 1}{t^2 + 1}$$

Hence it follows that for positive values of t

$$\frac{1}{t^2 + 1} < \frac{t^{4n+2} + 1}{t^2 + 1}$$

And thus

$$\frac{1}{t^2 + 1} < 1 - t^2 + t^4 - \cdots + t^{4n}$$

22b As we are adding one extra term to the previous sum, this will be S_{2n+2}

Hence there are $2n + 2$ terms so the sum must be

$$S_{2n+2} = \frac{a(r^n - 1)}{r - 1} = \frac{(-t^2)^{2n+2} - 1}{-t^2 - 1}$$

Since $2n + 2$ is even

$$S_{2n+2} = \frac{t^{4n+4} - 1}{-t^2 - 1} = \frac{-t^{4n+4} + 1}{t^2 + 1}$$

Hence it follows that for positive values of t

From part **a**

$$\frac{1}{t^2+1} < 1 - t^2 + t^4 - t^6 + \cdots + t^{4n}$$

$$\frac{1}{t^2+1} - t^{4n+2} < 1 - t^2 + t^4 - t^6 + \cdots + t^{4n} - t^{4n+2}$$

$$\frac{1}{t^2+1} - t^{4n+2} < \frac{-t^{4n+4}+1}{t^2+1}$$

$$\frac{1}{t^2+1} < \frac{-t^{4n+4}+1}{t^2+1} + t^{4n+2}$$

$$\frac{1+t^{4n+4}}{t^2+1} < \frac{1}{t^2+1} + t^{4n+2}$$

$$1 - t^2 + t^4 - t^6 + \cdots + t^{4n} < \frac{1}{t^2+1} + t^{4n+2}$$

$$\frac{1}{t^2+1} < \frac{-t^{4n+4}+1}{t^2+1}$$

$$\frac{1}{t^2+1} < \frac{-t^{4n+4}+1}{t^2+1}$$

$$\frac{1}{t^2+1} < \frac{1+t^{4n+2}+t^{4n+4}}{1+t^2}$$

$$\frac{1}{t^2+1} < \frac{1}{1+t^2} + \frac{t^{4n+2}+t^{4n+4}}{1+t^2}$$

$$\frac{1}{t^2+1} < \frac{1}{1+t^2} + \frac{t^{4n+2}(1+t^2)}{1+t^2}$$

$$\frac{1}{t^2+1} < \frac{1}{1+t^2} + t^{4n+2}$$

22c The inequality

$$\frac{1}{1+t^2} < 1 - t^2 + t^4 - t^6 + \cdots t^{4n} < \frac{1}{1+t^2} + t^{4n+2}$$

Is true for all $0 < t < x$, hence

$$\int_0^x \frac{1}{1+t^2}\,dt < \int_0^x (1 - t^2 + t^4 - t^6 + \cdots t^{4n})\,dt < \int_0^x (\frac{1}{1+t^2} + t^{4n+2})\,dt$$

$$[\tan^{-1} x]_0^x < \left[t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \cdots + \frac{t^{4n+1}}{4n+1}\right]_0^x < \left[\tan^{-1} x + \frac{t^{4n+3}}{4n+3}\right]_0^x$$

Thus

$$\tan^{-1} x < x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + \frac{x^{4n+1}}{4n+1} < \tan^{-1} x + \frac{x^{4n+3}}{4n+3}$$

22d

For $0 \le x \le 1$ we know that $0 \le x^{4n+3} \le 1$ and as $n \to \infty, \frac{1}{4n+3} \to 0$, hence it follows that as $n \to \infty, \frac{x^{4n+3}}{4n+3} \to 0$.

Thus as $n \to \infty$

$$\tan^{-1} x < x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots < \tan^{-1} x + 0$$

$$\tan^{-1} x < x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots < \tan^{-1} x$$

And so

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

22e As $\tan x$ is odd, $\tan^{-1} -x = -\tan^{-1} x$

and hence for $-1 \le x < 0$, (where $0 < -x \le 1$)

$\tan^{-1} x$

$$= \tan^{-1}\big(-(-x)\big)$$
$$= -\tan^{-1}(-x)$$
$$= -\left((-1) - \frac{(-1)^3}{3} + \frac{(-1)^5}{5} - \frac{(-1)^7}{7} + \cdots\right)$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

22f Substituting $x = 1$ into the above equation

$$\tan^{-1} 1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

22g

$$\frac{\pi}{4} = \frac{3-1}{3 \times 5} + \frac{7-5}{5 \times 7} + \frac{11-9}{9 \times 11} \ldots$$

$$\frac{\pi}{4} = \frac{2}{3 \times 5} + \frac{2}{5 \times 7} + \frac{2}{9 \times 11} \ldots$$

$$\frac{\pi}{8} = \frac{1}{3 \times 5} + \frac{1}{5 \times 7} + \frac{1}{9 \times 11} \ldots$$

Using 10 terms gives

$\pi \doteqdot 3.092$, error $\doteqdot 0.050$

23a All rectangles have the same width of 1 unit and their height is given by $\tan^{-1} n$ for the nth rectangle . This means that

$$S_n = \tan^{-1} 1 + \tan^{-1} 2 + \cdots + \tan^{-1} n$$

23b

$$\frac{d}{dx}(x \tan x) = \tan x + x \times \frac{1}{1 + x^2}$$

Hence

$$\int \frac{d}{dx}(x \tan x)\, dx = \int \tan x + x \times \frac{1}{1 + x^2} dx$$

$$x \tan x = \int \tan x\, dx + \int \frac{x}{1 + x^2} dx$$

$$x \tan x = \int \tan x\, dx + \frac{1}{2}\ln(1 + x^2) + C$$

$$\int \tan x\, dx = x \tan x - \frac{1}{2}\ln(1 + x^2) + C$$

23c Firstly note that since

$$\int \tan x \, dx = x \tan x - \frac{1}{2}\ln(1 + x^2) + C$$

Substituting $x = u - 1$ into the equation gives

$$\int \tan(u - 1) \, du$$

$$= (u - 1)\tan(u - 1) - \frac{1}{2}\ln(1 + (u - 1)^2) + C$$
$$= (u - 1)\tan^{-1}(u - 1) - \frac{1}{2}\ln(1 + u^2 - 2u + 1) + C$$
$$= (u - 1)\tan^{-1}(u - 1) - \frac{1}{2}\ln(u^2 - 2u + 2) + C$$

The area in the boxes is overestimated by the area under $\tan^{-1} x$ but underestimated by the area under $\tan^{-1}(x - 1)$, hence

$$\int_1^{n+1} \tan(x - 1) \, dx < S_n < \int_0^{n+1} \tan(x) \, dx$$

$$\int_1^{n+1} \tan(x - 1) \, dx < S_n < \int_0^{n+1} \tan(x) \, dx$$

$$\left[(x - 1)\tan^{-1}(x - 1) - \frac{1}{2}\ln(x^2 - 2x + 2)\right]_1^{n+1} < S_n$$
$$< \left[(x)\tan^{-1}(x) - \frac{1}{2}\ln(1 + x^2)\right]_1^{n+1}$$

$$(n)\tan^{-1}(n) - \frac{1}{2}\ln(1 + n^2) - (0)\tan^{-1}(0) - \frac{1}{2}\ln(1) < S_n$$
$$< (n + 1)\tan^{-1}(n + 1) - \frac{1}{2}\ln(1 + (n + 1)^2) - (1)\tan^{-1}(1) - \frac{1}{2}\ln(2)$$

$$(n)\tan^{-1}(n) - \frac{1}{2}\ln(1 + n^2) < S_n$$
$$< (n + 1)\tan^{-1}(n + 1) - \frac{1}{2}\ln(1 + (n + 1)^2) - \frac{\pi}{4} - \frac{1}{2}\ln(2)$$

$$(n)\tan^{-1}(n) - \frac{1}{2}\ln(1 + n^2) < S_n$$
$$< (n + 1)\tan^{-1}(n + 1) - \frac{1}{2}\ln\left(\frac{n^2 + 2n + 2}{2}\right) - \frac{\pi}{4}$$

$$n\tan^{-1} n - \frac{1}{2}\ln(n^2 + 1) < S_n < (n + 1)\tan^{-1}(n + 1) - \frac{1}{2}\ln\left(\frac{n^2}{2} + n + 1\right) - \frac{\pi}{4}$$

23d Substituting $n = 1000$ into the above equation gives

$$1000\tan^{-1}1000 - \frac{1}{2}\ln(1000^2+1) < S_{1000}$$
$$< (1000+1)\tan^{-1}(1000+1) - \frac{1}{2}\ln\left(\frac{1000^2}{2} + 1000 + 1\right) - \frac{\pi}{4}$$

$1562.89 < S_{100} < 1564.02$

$1563 < S_{1000} < 1565$

Hence

$1563 < \tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3 + \cdots + \tan^{-1}1000 < 1565$

Solutions to Exercise 12C

1a

$$\cos 2x = \cos^2 x - \sin^2 x$$

$$\cos 2x = 1 - \sin^2 x - \sin^2 x$$

$$\cos 2x = 1 - 2\sin^2 x$$

$$2\sin^2 x = 1 - \cos 2x$$

$$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$$

1b

$$\cos 4x = \cos 2(2x)$$

$$\cos 4x = \cos^2 2x - \sin^2 2x$$

$$\cos 4x = \cos^2 2x - (1 - \cos^2 2x)$$

$$\cos 4x = 2\cos^2 2x - 1$$

$$2\cos^2 2x = 1 + \cos 4x$$

$$\cos^2 2x = \frac{1}{2} + \frac{1}{2}\cos 4x$$

1c

$$\frac{1}{2}\sin 6x = \frac{1}{2}\sin 2(3x)$$

$$= \frac{1}{2}(2\sin 3x \cos 3x)$$

$$= \sin 3x \cos 3x$$

1d

$$\cos x = \cos 2\left(\frac{x}{2}\right) = \cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right)$$

$$\cos x = 1 - \sin^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right)$$

$$\cos x = 1 - 2\sin^2\left(\frac{x}{2}\right)$$

$$2\sin^2\left(\frac{x}{2}\right) = 1 - \cos x$$

2a $\cos^2 15°$

$$= \frac{1}{2} + \frac{1}{2}\cos 30°$$

$$= \frac{1}{2} + \frac{1}{2} \times \frac{\sqrt{3}}{2}$$

$$= \frac{1}{4}(2 + \sqrt{3})$$

2b

$$\sin^2 \frac{5\pi}{12}$$

$$= \frac{1}{2} - \frac{1}{2}\cos\frac{5\pi}{6}$$

$$= \frac{1}{2} + \frac{1}{2} \times \frac{\sqrt{3}}{2}$$

$$= \frac{1}{4}(2 + \sqrt{3})$$

2c $\sin 105° \cos 105°$

$$= \frac{1}{2}\sin 210°$$

$$= \frac{1}{2} \times \left(-\frac{1}{2}\right)$$

$$= -\frac{1}{4}$$

2d

$$\sin^2 \frac{7\pi}{8}$$

$$= \frac{1}{2} - \frac{1}{2}\cos\frac{7\pi}{4}$$

$$= \frac{1}{2} - \frac{1}{2} \times \left(\frac{1}{\sqrt{2}}\right)$$

$$= \frac{2 - \sqrt{2}}{4}$$

3 $\quad \sin^2 \theta = \frac{1}{2} - \frac{1}{2} \cos 2\theta \text{ (see 1a)}$

3a

$$\int \sin^2 x \, dx$$

$$= \int \left(\frac{1}{2} - \frac{1}{2} \cos 2x\right) dx$$

$$= \frac{x}{2} - \frac{\sin 2x}{4} + C$$

3b

$$\int \sin^2 2x \, dx$$

$$= \int \left(\frac{1}{2} - \frac{1}{2} \cos 4x\right) dx$$

$$= \frac{x}{2} - \frac{\sin 4x}{8} + C$$

3c

$$\int \sin^2 \frac{1}{4} x \, dx$$

$$= \int \left(\frac{1}{2} - \frac{1}{2} \cos \frac{1}{2} x\right) dx$$

$$= \frac{x}{2} - \frac{\sin \frac{1}{2} x}{2\left(\frac{1}{2}\right)} + C$$

$$= \frac{x}{2} - \sin \frac{1}{2} x + C$$

3d

$$\int \sin^2 3x\, dx$$

$$= \int \left(\frac{1}{2} - \frac{1}{2}\cos 6x\right) dx$$

$$= \frac{x}{2} - \frac{\sin 6x}{12} + C$$

4 $\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$ (see 1b)

4a

$$\int \cos^2 x\, dx = \int \left(\frac{1}{2} + \frac{1}{2}\cos 2x\right) dx = \frac{x}{2} + \frac{\sin 2x}{4} + C$$

4b

$$\int \cos^2 6x\, dx = \int \left(\frac{1}{2} + \frac{1}{2}\cos 12x\right) dx = \frac{x}{2} + \frac{\sin 12x}{24} + C$$

4c

$$\int \cos^2 \frac{1}{2}x\, dx = \int \left(\frac{1}{2} + \frac{1}{2}\cos x\right) dx = \frac{x}{2} + \frac{\sin x}{2} + C$$

4d

$$\int \cos^2 10x\, dx = \int \left(\frac{1}{2} + \frac{1}{2}\cos 20x\right) dx = \frac{x}{2} + \frac{\sin 20x}{40} + C$$

5a

$$\int_0^{\pi} \sin^2 x\, dx$$

$$= \int_0^{\pi} \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right) dx$$

$$= \left[\frac{x}{2} - \frac{\sin 2x}{4}\right]_0^{\pi}$$

$$= \frac{\pi}{2}$$

5b

$$\int_0^{\frac{\pi}{4}} \cos^2 x \, dx$$

$$= \int_0^{\frac{\pi}{4}} \left(\frac{1}{2} + \frac{1}{2}\cos 2x\right) dx$$

$$= \left[\frac{x}{2} + \frac{\sin 2x}{4}\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{8}(\pi + 2)$$

5c

$$\int_0^{\frac{\pi}{6}} \sin^2 \frac{1}{2}x \, dx$$

$$= \int_0^{\frac{\pi}{6}} \left(\frac{1}{2} - \frac{1}{2}\cos x\right) dx$$

$$= \left[\frac{x}{2} - \frac{\sin x}{2}\right]_0^{\frac{\pi}{6}}$$

$$= \frac{1}{12}(\pi - 3)$$

5d

$$\int_0^{\frac{\pi}{16}} \cos^2 2x \, dx$$

$$= \int_0^{\frac{\pi}{16}} \left(\frac{1}{2} + \frac{1}{2}\cos 4x\right) dx$$

$$= \left[\frac{x}{2} + \frac{\sin 4x}{8}\right]_0^{\frac{\pi}{6}}$$

$$= \frac{1}{32}(\pi + 2\sqrt{2})$$

5e

$$\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \cos^2\left(x + \frac{\pi}{12}\right) dx$$

$$= \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{1}{2} + \frac{1}{2}\cos 2\left(x + \frac{\pi}{12}\right) dx$$

$$= \left[\frac{x}{2} + \frac{\sin 2\left(x + \frac{\pi}{12}\right)}{4}\right]_{-\frac{\pi}{6}}^{\frac{\pi}{6}}$$

$$= \frac{1}{24}(4\pi + 9)$$

5f

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin^2\left(x - \frac{\pi}{6}\right) dx$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \left(\frac{1}{2} - \frac{1}{2}\cos 2\left(x - \frac{\pi}{6}\right)\right) dx$$

$$= \left[\frac{x}{2} - \frac{\sin 2\left(x - \frac{\pi}{6}\right)}{4}\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$= \frac{1}{24}(2\pi - 3\sqrt{3})$$

6a

$$\int \sin 3x \cos 2x \, dx$$
$$= \int \left(\frac{1}{2}\sin(2x + 3x) + \frac{1}{2}\sin(3x - 2x)\right) dx$$
$$= \int \left(\frac{1}{2}\sin 5x + \frac{1}{2}\sin x\right) dx$$
$$= -\frac{1}{10}\cos 5x - \frac{1}{2}\cos x + C$$

6b

$$\int \cos 3x \sin x \, dx$$
$$= \int \left(\frac{1}{2}\sin 2x - \frac{1}{2}\sin 4x\right) dx$$
$$= \frac{1}{4}\cos 2x - \frac{1}{8}\cos 4x + C$$

6c

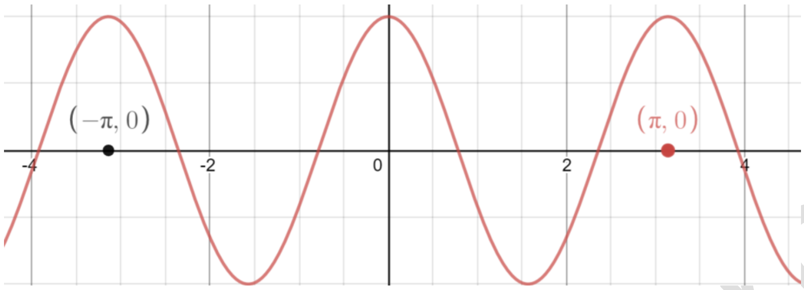
$$\int_0^{\frac{\pi}{4}} 2\cos 2x \cos x \, dx$$
$$= \int_0^{\frac{\pi}{4}} 2 \times \frac{1}{2}(\cos x + \cos 3x) \, dx$$
$$= \int_0^{\frac{\pi}{4}} (\cos x + \cos 3x) \, dx$$
$$= \left[\sin x + \frac{1}{3}\sin x\right]_0^{\frac{\pi}{4}}$$
$$= \frac{2\sqrt{2}}{3}$$

6d

$$\int_0^{\frac{\pi}{3}} \sin 5x \sin 2x \, dx$$
$$= \int_0^{\frac{\pi}{3}} \frac{1}{2}(\sin 3x - \sin 7x) \, dx$$

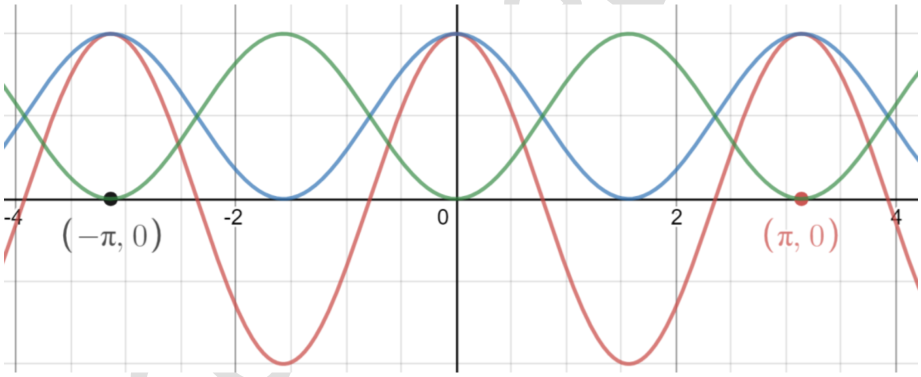
$$= \frac{1}{2}\left[-\frac{1}{3}\cos 3x + \frac{1}{7}\cos 7x\right]_0^{\frac{\pi}{3}}$$
$$= \frac{\sqrt{3}}{28}$$

7a



7b

$y = \frac{1}{2}(1 + \cos 2x)$ is shown in blue and $y = \frac{1}{2}(1 - \cos 2x)$ is shown in green.



7c

By observation firstly note that $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ and that $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$. By observation of the graph, we see that these two graphs always add to 1 and hence $\sin^2 x + \cos^2 x = 1$.

8a Let $u = \sin x$, $du = \cos x\, dx$

$$\int \sin^3 x \cos x\, dx$$

$$= \int u^3\, du$$

$$= \frac{u^4}{4} + C$$

$$= \frac{\sin^4 x}{4} + C$$

8b Let $u = \sin x$, $du = \cos x\, dx$

$$\int \sin^6 x \cos x\, dx$$

$$= \int u^6\, du$$

$$= \frac{u^7}{7} + C$$

$$= \frac{\sin^7 x}{7} + C$$

8c Let $u = \cos x$, $du = -\sin x\, dx$

$$\int \cos^5 x \sin x\, dx$$

$$= -\int u^5\, du$$

$$= -\frac{u^6}{6} + C$$

$$= -\frac{\cos^6 x}{6} + C$$

8d Let $u = \cos x$, $du = -\sin x$

$$\int \cos^8 x \sin x\, dx$$

$$= \int -u^8\, du$$

$= -\dfrac{u^9}{9} + C$

$= -\dfrac{\cos^9 x}{9} + C$

8e Let $u = e^x$, $du = e^x dx$

$\int e^x \sin e^x \, dx$

$= \int \sin u \, du$

$= -\cos u + C$

$= -\cos e^x + C$

8f Let $u = e^x$, $du = e^x dx$

$\int e^x \cos 5e^x \, dx$

$= \int \cos 5u \, du$

$= \dfrac{1}{5} \sin 5u + C$

$= \dfrac{1}{5} \sin 5e^x + C$

8g Let $u = \cos x$, $du = -\sin x$

$\int \tan x \, dx$

$= \int \dfrac{\sin x}{\cos x} dx$

$= \int -\dfrac{1}{u} du$

$= -\ln|u| + C$

$= -\ln|\cos x| + c$

8h Let $u = \sin 7x$, $du = 7\cos x$

$$\int \cot 7x \, dx$$

$$= \int \frac{\cos 7x}{\sin 7x} dx$$

$$= \int \frac{1}{7u} du$$

$$= \frac{1}{7}\ln|u| + C$$

$$= \frac{1}{7}\ln|\sin 7x| + C$$

9a

$$y = \cos x \sin x = \frac{1}{2}\sin 2x$$

Hence the range is

$$-\frac{1}{2} \le y \le \frac{1}{2}$$

9b i

$$\int \cos x \ \sin x \, dx$$

$$= \int \frac{1}{2}\sin 2x \, dx$$

$$= -\frac{1}{4}\cos 2x + C$$

9b ii

Method 1:

Let $u = \sin x$, $du = \cos x \, dx$

$$\int \sin x \cos x \, dx$$

$$= \int u \, du$$

$= \dfrac{u^2}{2} + C$

$= \dfrac{\sin^2 x}{2} + C$

Method 2:

Let $u = \cos x$, $du = -\sin x\, dx$

$\int \sin x \cos x\, dx$

$= \int -u\, du$

$= -\dfrac{u^2}{2} + C$

$= -\dfrac{\cos^2 x}{2} + C$

9c Note that $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$. Thus

$-\dfrac{1}{4}\cos 2x + c$

$= -\dfrac{1}{4}(2\cos^2 x - 1)$

$= -\dfrac{\cos^2 x}{2} + c_2$

$= \dfrac{\sin^2 x}{2} + c_3$

Note at c, c_1 and c_2 are not the same constant.

10a $\sin^4 x = (\sin^2 x)^2$

$= \left(\dfrac{1}{2} - \dfrac{1}{2}\cos 2x\right)^2$

$= \dfrac{1}{4} - \dfrac{1}{2}\cos 2x + \dfrac{1}{4}\cos^2 2x$

$= \dfrac{1}{4} - \dfrac{1}{2}\cos 2x + \dfrac{1}{4}\left(\dfrac{1}{2} + \cos 4x\right)$

$$= \frac{3}{8} - \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x$$

10b $\cos^4 x = (\cos^2 x)^2$

$$= \left(\frac{1}{2} + \frac{1}{2}\cos 2x\right)^2$$

$$= \frac{1}{4} + \frac{1}{2}\cos 2x + \frac{1}{4}\cos^2 2x$$

$$= \frac{1}{4} + \frac{1}{2}\cos 2x + \frac{1}{4}\left(\frac{1}{2} + \cos 4x\right)$$

$$= \frac{3}{8} + \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x$$

10c i

$$\int_0^{\pi} \sin^4 x \, dx$$

$$= \int_0^{\pi} \left(\frac{3}{8} - \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x\right) dx$$

$$= \left[\frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{2}\sin 4x\right]_0^{\pi}$$

$$= \frac{3\pi}{8}$$

10c ii

$$\int_0^{\frac{\pi}{4}} \cos^4 x \, dx$$

$$= \int_0^{\frac{\pi}{4}} \left(\frac{3}{8} + \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x\right) dx$$

$$= \left[\frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{2}\sin 4x\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{32}(3\pi + 8)$$

10d

$$\int_0^{\frac{\pi}{3}} \sin^3 x\, dx$$

$$= \int_0^{\frac{\pi}{3}} \sin x\,(\sin^2 x) dx$$

$$= \int_0^{\frac{\pi}{3}} \sin x\,(1 - \cos^2 x) dx$$

$$= \int_0^{\frac{\pi}{3}} (\sin x - \sin x \cos^2 x) dx$$

$$= \int_0^{\frac{\pi}{3}} \sin x\, dx - \int_0^{\frac{\pi}{3}} \sin x \cos^2 x\, dx$$

Now for $\int_0^{\frac{\pi}{3}} \sin x \cos^2 x\, dx$, let $u = \cos x$, $du = -\sin x$

$$\int_0^{\frac{\pi}{3}} \sin x \cos^2 x\, dx = \int_1^{\frac{1}{2}} u^2\, du = \left[\frac{u^3}{3}\right]_1^{\frac{1}{2}} = \frac{1}{24}$$

Hence

$$\int_0^{\frac{\pi}{3}} \sin^3 x\, dx = [-\cos x]_0^{\frac{\pi}{3}} - \left[\frac{u^3}{3}\right]_1^{\frac{1}{2}} = \frac{5}{24}$$

11a

$$\int \tan^2 2x\, dx$$

$$= \int (\sec^2 2x - 1)\, dx$$

$$= \frac{1}{2}\tan 2x - x + C$$

11b

$$\int \cot^2 \frac{1}{2}x\, dx = \int (\operatorname{cosec}^2(\frac{1}{2}x) - 1)\, dx$$

$$= -2\cot\frac{1}{2}x - x + C$$

11c

$$\int_{\frac{\pi}{12}}^{\frac{\pi}{9}} 3\tan^2 3x\, dx$$

$$= \int_{\frac{\pi}{12}}^{\frac{\pi}{9}} 3(\sec^2 3x + 1)\, dx$$

$$= \int_{\frac{\pi}{12}}^{\frac{\pi}{9}} (3\sec^2 3x + 3\,)dx$$

$$= [\tan 3x + 3x]_{\frac{\pi}{12}}^{\frac{\pi}{9}}$$

$$= \sqrt{3} - 1 - \frac{\pi}{12}$$

11d

$$\int_{\frac{\pi}{24}}^{\frac{\pi}{8}} \cot^2 4x\, dx$$

$$= \int_{\frac{\pi}{24}}^{\frac{\pi}{8}} (\operatorname{cosec}^2 4x + 1)\, dx$$

$$= \left[-\frac{1}{4}\cot\frac{1}{2}x + x\right]_{\frac{\pi}{24}}^{\frac{\pi}{8}}$$

$$= \frac{1}{4}\sqrt{3} - \frac{\pi}{12}$$

12a Let $u = \tan x$, $du = \sec^2 x\, dx$

$$\int \tan x \sec^2 x\, dx$$

$$= \int u\, du$$

$$= \frac{u^2}{2} + C$$

$$= \frac{\tan^2 x}{2} + C$$

12b

$$\int \frac{\sin^2 x}{1 + \cos x} dx$$

$$= \int \frac{1 - \cos^2 x}{1 + \cos x} dx$$

$$= \int \frac{(1 - \cos x)(1 + \cos x)}{1 + \cos x} dx$$

$$= \int (1 - \cos x) dx$$

$$= x - \sin x + C$$

12c

$$\int \frac{1 + \cos^3 x}{\cos^2 x} dx$$

$$= \int \left(\frac{1}{\cos^2 x} + \cos x\right) dx$$

$$= \int (\sec^2 x + \cos x) dx$$

$$= \tan x + \sin x + C$$

13a

$$F(x) = \int_0^x \sin^2 t\, dt$$

$$= \int_0^x \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right) dx$$

$$= \left[\frac{1}{2}x - \frac{1}{4}\sin 2x\right]_0^x$$

$$= \frac{1}{2}x - \frac{1}{4}\sin 2x - (0 - 0)$$

$$= \frac{1}{2}x - \frac{1}{4}\sin 2x$$

13b

$$F'(x) = \frac{1}{2} - \frac{1}{4} \times 2\cos 2x$$
$$= \frac{1}{2} - \frac{1}{2}\cos 2x$$
$$= \sin^2 x$$

13b i $F(x)$ is stationary when $F'(x) = 0$ this is when $\sin^2 x = 0$, $\sin x = 0$ and hence when $x = 0, \pi, 2\pi$.

13b ii $F(x)$ is increasing when $F'(x) > 0$ this is when $\sin^2 x > 0$. This is all x such that $\sin x \neq 0$ and hence when $x \neq 0, \pi, 2\pi$.

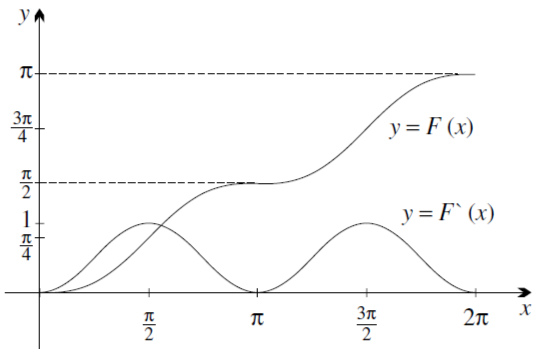
13b iii The curve is decreasing when $F'(x) < 0$. However, $\sin^2 x \geq 0$ for all x. Hence there are no values of x for which the curve is decreasing.

13c

Since $-1 \leq \sin 2x \leq 1$, $-\frac{1}{4} \leq \frac{1}{4}\sin 2x \leq \frac{1}{4}$ and thus $\frac{1}{2}x - \frac{1}{4} \leq \frac{1}{2}x + \frac{1}{4}\sin 2x \leq \frac{1}{2}x + \frac{1}{4}$, thus $\frac{1}{2}x - \frac{1}{4} \leq F(x) \leq \frac{1}{2}x + \frac{1}{4}$. Thus $F(x)$ never differs from $\frac{1}{2}x$ by more than $\frac{1}{4}$.

13d We know that the stationary points are $x = 0, \pi, 2\pi$. As the curve is always increasing, these will all be points of inflection (if there is a maximum or minimum the slope would have to be decreasing on one side).

13e



13f i

$$\int_0^k \sin^2 x\, dx = \frac{3\pi}{2}$$

$$\int_0^k \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right) dx = \frac{3\pi}{2}$$

$$\left[\frac{x}{2} - \frac{\sin 2x}{4}\right]_0^k = \frac{3\pi}{2}$$

$$\frac{k}{2} - \frac{\sin 2k}{4} = \frac{3\pi}{2}$$

$$\frac{k}{2} = \frac{3\pi}{2}$$

$$k = 3\pi$$

13f ii

$$\int_0^k \sin^2 x\, dx = \frac{n\pi}{2}$$

$$\int_0^k \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right) dx = \frac{n\pi}{2}$$

$$\left[\frac{x}{2} - \frac{\sin 2x}{4}\right]_0^k = \frac{n\pi}{2}$$

$$\frac{k}{2} - \frac{\sin 2k}{4} = \frac{n\pi}{2}$$

$$\frac{k}{2} = \frac{n\pi}{2}$$

$k = n\pi$

14

$$\lim_{R\to\infty}\left(\frac{1}{R}\int_0^R \sin^2 t\, dt\right)$$

$$= \lim_{R\to\infty}\left(\frac{1}{R}\left(\frac{R}{2} - \frac{\sin 2R}{4}\right)\right)$$

$$= \lim_{R\to\infty}\left(\frac{1}{2} - \frac{1}{4}\left(\frac{\sin 2R}{R}\right)\right)$$

Now as $-1 \le \sin 2R \le 0$, then as R gets large $\frac{\sin 2R}{R} \to 0$, hence

$$\lim_{R\to\infty}\left(\frac{1}{R}\int_0^R \sin^2 t\, dt\right) = \frac{1}{2} - \frac{1}{4}(0) = \frac{1}{2}$$

Solutions to Exercise 12D

1a

$$
\begin{aligned}
du &= \frac{d}{dx}(1+x^2)\\
&= \frac{d}{dx}(1) + \frac{d}{dx}(x^2)\\
&= 0 + 2x\\
&= 2x
\end{aligned}
$$

1b

$$\int 2x(1+x^2)^3\,dx$$

$$= \int (1+x^2)^3\,(2x\,dx)$$

$$= \int u^3\,du \qquad \text{where } u = 1+x^2$$

$$= \frac{u^4}{4} + c$$

$$= \frac{(1+x^2)^4}{4} + C$$

1c

$$\frac{(1+x^2)^4}{4} + C$$

1d

$$\frac{d}{dx}\left(\frac{(1+x^2)^4}{4}\right)$$

$$= \frac{4(2x)(1+x^2)^3}{4}$$

$$= 2x(1+x^2)^3$$

2a Let $u = 2x + 3$, $du = 2\,dx$

$$\int 2(2x+3)^3\,dx$$

$$= \int u^3\,du$$

$$= \frac{u^4}{4} + C$$

$$= \frac{(2x+3)^4}{4} + C$$

2b Let $u = 1 + x^3$, $du = 3x^2\,dx$

$$\int 3x^2(1+x^3)^4\,dx$$

$$= \int u^4\,du$$

$$= \frac{u^5}{5} + C$$

$$= \frac{(1+x^3)^5}{5} + C$$

2c Let $u = 1 + x^2$, $du = 2x\,dx$

$$\int \frac{2x}{(1+x^2)^2}\,dx$$

$$= \int \frac{1}{u^2}\,du$$

$$= \int u^{-2}\,du$$

$$= -u^{-1} + C$$

$$= -\frac{1}{1+x^2} + C$$

2d Let $u = 3x - 5$, $du = 3\,dx$

$$\int \frac{3}{\sqrt{3x-5}}\,dx$$

$$= \int \frac{1}{\sqrt{u}}\,du$$

$$= \int u^{-\frac{1}{2}}\,du$$

$$= 2u^{\frac{1}{2}} + C$$

$$= 2\sqrt{3x-5} + C$$

2e Let $u = \sin x$, $du = \cos x\,dx$

$$\int \sin^3 x \cos x\,dx$$

$$= \int u^3\,du$$

$$= \frac{u^4}{4} + C$$

$$= \frac{\sin^4 x}{4} + C$$

2f Let $u = 1 + x^4$, $du = 4x^3$

$$\int \frac{4x^3}{1+x^4}\,dx$$

$$= \int \frac{1}{u}\,du$$

$$= \ln|u| + C$$

$$= \ln(1 + x^4) + C \qquad \text{(as } 1 + x^4 \text{ is positive)}$$

3a Let $u = 1 - x^2$, $du = -2x\,dx$ hence $x\,dx = -\frac{1}{2}du$

3b

$$\int \frac{x}{\sqrt{1-x^2}}\,dx$$

$$= \int \frac{1}{\sqrt{u}} \times -\frac{1}{2}\,du$$

$$= -\frac{1}{2}\int u^{-\frac{1}{2}}\,du$$

3c

$$\int \frac{x}{\sqrt{1-x^2}}\,dx$$

$$= \int \frac{1}{\sqrt{u}} \times -\frac{1}{2}\,du$$

$$= -\frac{1}{2}\int u^{-\frac{1}{2}}\,du$$

$$= -u^{\frac{1}{2}} + C$$

$$= -\sqrt{1-x^2} + C$$

4a Let $u = x^4 + 1$, $du = 4x^3\,dx$

$$\int x^3(x^4+1)^5\,dx$$

$$= \int \frac{1}{4}u^5\,du$$

$$= \frac{u^6}{24} + C$$

$$= \frac{(x^4+1)^6}{24} + C$$

4b Let $u = x^3 - 1$, $du = 3x^2\,dx$

$$\int x^2\sqrt{x^3 - 1}\,dx$$

$$= \int \frac{1}{3}\sqrt{u}\,du$$

$$= \frac{1}{3} \times \frac{2}{3}u^{\frac{3}{2}} + C$$

$$= \frac{2}{9}u^{\frac{3}{2}} + C$$

$$= \frac{2}{9}(x^3 - 1)^{\frac{3}{2}} + C$$

4c Let $u = x^3$, $du = 3x^2\,dx$

$$\int x^2 e^{x^3}\,dx$$

$$= \int \frac{1}{3}e^u\,du$$

$$= \frac{e^u}{3} + C$$

$$= \frac{e^{x^3}}{3} + C$$

4d Let $u = 1 + \sqrt{x}$, hence $du = \frac{1}{2}x^{-\frac{1}{2}}dx$, hence $2\,du = \frac{1}{\sqrt{x}}\,dx$

$$\int \frac{1}{\sqrt{x}\left(1 + \sqrt{x}\right)^3}\,dx$$

$$= \int \frac{2}{u^3}\,du$$

$$= -\frac{1}{u^2} + C$$

$$= -\frac{1}{\left(1 + \sqrt{x}\right)^2} + C$$

4e Let $u = \tan 2x$, $du = 2\sec^2 2x\, dx$

$$\int \tan^2 2x \sec^2 2x\, dx$$

$$= \int \frac{u^2}{2}\, du$$

$$= \frac{u^3}{6} + C$$

$$= \frac{\tan^3 2x}{6} + C$$

4f Let $u = \frac{1}{x}$, $du = -\frac{1}{x^2}\, dx$

$$\int \frac{e^{\frac{1}{x}}}{x^2}\, dx$$

$$= \int -e^u\, du$$

$$= -e^u + C$$

$$= -e^{\frac{1}{x}} + C$$

5a Let $u = 2 + x^3$, $du = 3x^2\, dx$

When $x = 0, u = 2$

When $x = 1, u = 3$

$$\int_0^1 x^2(2 + x^3)^3\, dx$$

$$= \int_2^3 \frac{1}{3}u^3\, du$$

$$= \left[\frac{u^9}{9}\right]_2^3$$

$$= \frac{1}{9}(3^9 - 2^9)$$

$= \dfrac{65}{12}$

5b When $u = 1 + x^4, du = 4x^3$

When $x = 0, u = 1$

When $x = 1, u = 2$

$\int_0^1 \dfrac{2x^3}{\sqrt{1+x^4}} dx$

$= \dfrac{1}{2}\int_0^1 \dfrac{4x^3}{\sqrt{1+x^4}}$

$= \dfrac{1}{2}\int_1^2 \dfrac{du}{\sqrt{u}}$

$= \dfrac{1}{2}\left[2u^{\frac{1}{2}}\right]_1^2$

$= \left[\sqrt{u}\right]_1^2$

$= \sqrt{2} - 1$

5c Let $u = \cos x\,, du = -\sin x$

When $x = 0, u = 1$

When $x = \frac{\pi}{2}, u = 0$

$\int_0^{\frac{\pi}{2}} \cos^2 x \sin x\, dx$

$= \int_1^0 -u^2\, du$

$= \int_0^1 u^2\, du$

$= \left[\dfrac{u^3}{3}\right]_0^1$

$= \dfrac{1}{3}$

5d Let $u = 1 - x^2$, $du = 2x\,dx$

When $x = \frac{1}{2}\sqrt{3}$, $u = \frac{1}{4}$

When $x = 1$, $u = 0$

$$\int_{\frac{1}{2}\sqrt{3}}^{1} x\sqrt{1-x^2}\,dx$$

$$= \int_0^{\frac{1}{4}} \frac{1}{2}\sqrt{u}\,du$$

$$= \frac{1}{2}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_0^{\frac{1}{4}}$$

$$= \frac{1}{3}\left[u^{\frac{3}{2}}\right]_0^{\frac{1}{4}}$$

$$= \frac{1}{3}\left(\frac{1}{4}\right)^{\frac{3}{2}}$$

$$= \frac{1}{24}$$

5e Let $u = \ln x$, $du = \frac{1}{x}dx$

When $x = 1$, $u = \ln 1 = 0$

When $x = e^2$, $u = \ln e^2 = 2\ln e = 2$

$$\int_1^{e^2} \frac{\ln x}{x}\,dx$$

$$= \int_0^2 u\,du$$

$$= \left[\frac{u^2}{2}\right]_0^2$$

$$= 2$$

5f Let $u = \sqrt{x}, du = \frac{1}{2}x^{-\frac{1}{2}}dx = \frac{1}{2\sqrt{x}}\ dx$

$$\int_0^4 \frac{e^{\sqrt{x}}}{4\sqrt{x}}dx$$

$$= \int_0^2 \frac{e^u}{2}du$$

$$= \frac{1}{2}[e^u]_0^2$$

$$= \frac{1}{2}(e^2 - 1)$$

5g Let $u = \sin 2x, du = 2\cos 2x\ \ dx$

$$\int_0^{\frac{\pi}{4}} \sin^4 2x \cos 2x\, dx$$

$$= \int_0^1 \frac{u^4}{2}du$$

$$= \frac{1}{2}\left[\frac{u^5}{5}\right]_0^1$$

$$= \frac{1}{10}$$

5h Let $u = \sin^{-1} x, du = \frac{1}{\sqrt{1-x^2}}\ dx$

$$\int_0^1 \frac{(\sin^{-1} x)^3}{\sqrt{1-x^2}}\ dx$$

$$= \int_0^{\frac{\pi}{2}} u^3\, du$$

$$= \left[\frac{u^4}{4}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{\left(\frac{\pi}{2}\right)^4}{4}$$

$$= \frac{\pi^4}{64}$$

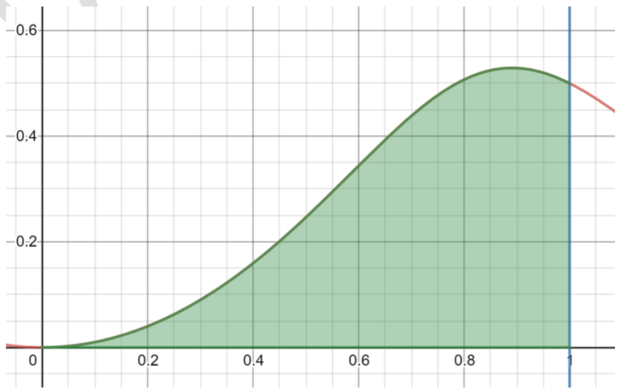
5i Let $u = x^2 + 2x$, $du = 2x + 2\ dx$

$$\int_0^2 \frac{x+1}{\sqrt[3]{x^2+2x}}\, dx$$
$$= \frac{1}{2}\int_0^2 \frac{2x+2}{\sqrt[3]{x^2+2x}}\, dx$$
$$= \frac{1}{2}\int_0^8 \frac{1}{\sqrt[3]{u}}\, du$$
$$= \frac{1}{2}\int_0^8 u^{-\frac{1}{3}}\, du$$
$$= \frac{1}{2}\left[\frac{3}{2}u^{\frac{2}{3}}\right]_0^8$$
$$= \frac{3}{4}\left(8^{\frac{2}{3}}\right)$$
$$= 3$$

5j Let $u = \tan x$, $du = \sec^2 x$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec^2 x}{\tan x}\, dx$$
$$= \int_1^{\sqrt{3}} \frac{1}{u} du$$
$$= [\ln|u|]_1^{\sqrt{3}}$$
$$= \ln\sqrt{3} - \ln 1$$
$$= \frac{1}{2}\ln 3$$

6 Note that the area to be found is shown in green in the diagram below



Let $u = x^3$, $du = 3x^2\, dx$. The area bounded will be given by

$$A = \int_0^1 \frac{x^2}{1 + x^6} dx$$

$$= \frac{1}{3}\int_0^1 \frac{3x^2}{1 + x^6} dx$$

$$= \frac{1}{3}\int_0^1 \frac{1}{1 + u^2} du$$

$$= \frac{1}{3}[\tan^{-1} u]_0^1$$

$$= \frac{\pi}{12} \text{ square units}$$

7 For all following subsections, let $u = \sin x$, $du = \cos x\, dx$

7a

$$\int_0^{\frac{\pi}{6}} \frac{\cos x}{1 + \sin x} dx$$

$$= \int_0^{\frac{1}{2}} \frac{1}{1 + u} du$$

$$= [\ln|1 + u|]_0^{\frac{1}{2}}$$

$$= \ln\frac{3}{2}$$

7b

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{1 + \sin^2 x} dx$$

$$= \int_0^1 \frac{1}{1 + u^2} du$$

$$= [\tan|u|]_0^1$$

$$= \frac{\pi}{4}$$

7c

$$\int_0^{\frac{\pi}{2}} \cos^3 x\, dx$$

$$= \int_0^{\frac{\pi}{2}} \cos x \cos^2 x\, dx$$

$$= \int_0^{\frac{\pi}{2}} \cos x\,(1 - \sin^2 x)\, dx$$

$$= \int_0^{\frac{\pi}{2}} \cos x\ \, dx - \int_0^{\frac{\pi}{2}} \cos x \sin^2 x\, dx$$

$$= [\sin x]_0^{\frac{\pi}{2}} - \int_0^1 u^2\, du$$

$$= 1 - \left[\frac{u^3}{3}\right]_0^1$$

$$= 1 - \frac{1}{3}$$

$$= \frac{2}{3}$$

7d

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos^3 x}{\sin^4 x}\, dx$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos x\,(1 - \sin^2 x)}{\sin^4 x}\, dx$$

$$= \left(\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos x}{\sin^4 x} - \frac{\cos x}{\sin^2 x}\right) dx$$

$$= \int_{\frac{1}{2}}^{1} \left(\frac{1}{u^4} - \frac{1}{u^2}\right) du$$

$$= \int_{\frac{1}{2}}^{1} (u^{-4} - u^{-2})\, du$$

$$= \left[\frac{u^{-3}}{3} - \frac{u^{-1}}{1}\right]_{\frac{1}{2}}^{1}$$

$$= \frac{4}{3}$$

8a Let $u = e^{2x}$, $du = 2e^{2x}\,dx$

$$\int \frac{e^{2x}}{\sqrt{1+e^{2x}}}\,dx$$

$$= \int \frac{1}{2\sqrt{1+u}}\,du$$

$$= \int \frac{1}{2}(1+u)^{-\frac{1}{2}}\,du$$

$$= (1+u)^{\frac{1}{2}} + C$$

$$= \sqrt{1+u} + C$$

$$= \sqrt{1+e^{2x}} + C$$

8b Let $u = \ln x$, $du = \frac{1}{x}\,dx$

$$\int \frac{1}{x\ln x}\,dx$$

$$= \int \frac{1}{u}\,du$$

$$= \ln|u| + C$$

$$= \ln|\ln x| + C$$

8c Let $u = \ln\cos x$, $du = -\frac{\sin x}{\cos x}\,dx = -\tan x\,dx$

$$\int \frac{\tan x}{\ln\cos x}\,dx$$

$$= \int -\frac{1}{u}\,du$$

$$= -\ln|u| + C$$

$$= -\ln|\ln\cos x| + C$$

8d Let $u = \tan x$, $du = \sec^2 x \ dx$

$$\int \tan^3 x \sec^4 x \, dx$$
$$= \int u^3 \sec^2 x \, du$$
$$= \int u^3 (1 + \tan^2 x) \, du$$
$$= \int u^3 (1 + u^2) \, du$$
$$= \int (u^3 + u^5) \, du$$
$$= \frac{u^4}{4} + \frac{u^6}{6} + C$$
$$= \frac{\tan^4 x}{4} + \frac{\tan^6 x}{6} + C$$

9a

$$\frac{dy}{dx} = \frac{e^{2x}}{1 + e^{4x}}$$
$$y = \int \frac{e^{2x}}{1 + e^{4x}} \, dx$$

Let $u = e^{2x}$, $du = 2e^{2x} \ dx$

$$y = \int \frac{e^{2x}}{1 + e^{4x}} \, dx$$
$$= \frac{1}{2} \int \frac{1}{1 + u^2} \, du$$
$$= \frac{1}{2} \tan^{-1} u + C$$
$$= \frac{1}{2} \tan^{-1} e^{2x} + C$$

When $x = 0$, $y = \frac{\pi}{8}$

$$\frac{\pi}{8} = \frac{1}{2} \tan^{-1} e^0 + C$$
$$\frac{\pi}{8} = \frac{1}{2} \tan^{-1} 1 + C$$
$$\frac{\pi}{8} = \frac{\pi}{8} + C$$

Hence $C = 0$ so the equation of the line is

$$y = \frac{1}{2}\tan^{-1} e^{2x}$$

9b

$$y'' = \frac{x}{(4 - x^2)^{\frac{3}{2}}}$$

Let $u = 4 - x^2$, $du = -2x\,dx$ hence $-\frac{1}{2}du = dx$

$$y' = \int \frac{x}{(4 - x^2)^{\frac{3}{2}}}dx$$

$$= \int \frac{1}{u^{\frac{3}{2}}}\left(-\frac{1}{2}\right)du$$

$$= -\frac{1}{2}\int u^{-\frac{3}{2}}$$

$$= u^{-\frac{1}{2}} + C$$

$$= (4 - x^2)^{-\frac{1}{2}} + C$$

When $x = 0, y' = 1$

$$1 = 4^{-\frac{1}{2}} + C$$

$$1 = \frac{1}{2} + C$$

$$C = \frac{1}{2}$$

So

$$y' = (4 - x^2)^{-\frac{1}{2}} + \frac{1}{2} = \frac{1}{\sqrt{4 - x^2}} + \frac{1}{2}$$

$$y = \int \left(\frac{1}{\sqrt{4 - x^2}} + \frac{1}{2}\right) dx$$

$$= \int \left(\frac{1}{2\sqrt{1 - \left(\frac{x}{2}\right)^2}} + \frac{1}{2}\right) dx$$

$= \sin^{-1}\left(\frac{x}{2}\right) + \frac{x}{2} + D$

When $x = 0, y = \frac{1}{2}$

$\frac{1}{2} = 0 + 0 + D$

$D = \frac{1}{2}$

$y = \sin^{-1}\left(\frac{x}{2}\right) + \frac{x}{2} + \frac{1}{2}$

10a

$\frac{d}{dx}(\sec x)$

$= \frac{d}{dx}(\cos^{-1} x)$

$= -1 \times -\sin x \times (\cos x)^{-2}$

$= \frac{1}{\cos x} \times \frac{\sin x}{\cos x}$

$= \sec x \tan x$

10b i Let $u = \sec x$, $du = \sec x \tan x \, dx$

$\int_0^{\frac{\pi}{3}} 2^{\sec x} \sec x \tan x \, dx$

$= \int_1^2 2^u \, du$

$= \left[\frac{2^u}{\ln 2}\right]_1^2$

$= \frac{4}{\ln 2} - \frac{2}{\ln 2}$

$= \frac{2}{\ln 2}$

10b ii

$$\int_0^{\frac{\pi}{4}} \sec^5 x \tan x \, dx$$

$$= \int_1^{\sqrt{2}} u^4 \, du$$

$$= \frac{1}{5}[u^5]_1^{\sqrt{2}}$$

$$= \frac{1}{5}(4\sqrt{2} - 1)$$

11a Let $u = \sin^2 x$, $du = 2\cos x \sin x \, dx = \sin 2x \, dx$

$$\int_0^{\frac{\pi}{2}} \frac{\sin 2x}{1 + \sin^2 x} \, dx$$

$$= \int_0^1 \frac{1}{1 + u^2} \, du$$

$$= [\tan^{-1} u]_0^1$$

$$= \frac{\pi}{4}$$

11b Let $u = x \ln x$, $du = 1 \times \ln x + x \times \frac{1}{x} \, dx = \ln x + 1 \, dx$

$$\int_1^e \frac{\ln x + 1}{(x \ln x + 1)^2} \, dx$$

$$= \int_0^e \frac{1}{(u + 1)^2} \, du$$

$$= \left[-\frac{1}{u + 1}\right]_0^e$$

$$= -\frac{1}{e + 1} - (-1)$$

$$= 1 - \frac{1}{e + 1}$$

$$= \frac{e + 1}{e + 1} - \frac{1}{e + 1}$$

$$= \frac{e}{e + 1}$$

12 Let $u = \sqrt{x-1}$, $du = \frac{1}{2\sqrt{x-1}}\,dx$

If $u = \sqrt{x-1}$, then $u^2 = x - 1$ and $x = 1 + u^2$

$$\int \frac{1}{2x\sqrt{x-1}}\,dx$$

$$= \int \frac{1}{x}\,du$$

$$= \int \frac{1}{1+u^2}\,du$$

$$= \tan^{-1} u + C$$

$$= \tan^{-1} \sqrt{x-1} + C$$

13a Let $u = \sqrt{x}$, $du = \frac{1}{2\sqrt{x}}\,dx$ and $u^2 = x$

$$\int \frac{1}{\sqrt{x(1-x)}}\,dx$$

$$= \int \frac{2}{\sqrt{(1-x)}}\frac{dx}{2\sqrt{x}}$$

$$= \int \frac{2}{\sqrt{1-u^2}}\,du$$

$$= 2\sin^{-1} u + C_1$$

$$= 2\sin^{-1}\sqrt{x} + C_1$$

13b Let $u = x - \frac{1}{2}$, $du = dx$, $x = \frac{1}{2} + u$

$$\int \frac{1}{\sqrt{x(1-x)}}\,dx$$

$$= \int \frac{1}{\sqrt{\left(\frac{1}{2}+u\right)\left(\frac{1}{2}-u\right)}}\,du$$

$$= \int \frac{1}{\sqrt{\frac{1}{4}-u^2}}\,du$$

$$= \int \frac{2}{\sqrt{1-(2u)^2}}\,du$$

$$= \sin^{-1}(2u) + C_2$$

$$= \sin^{-1}(2x-1) + C_2$$

13c Since

$$\int \frac{1}{\sqrt{x(1-x)}}\,dx$$

$$= 2\sin^{-1}\sqrt{x} + C_1$$

$$= \sin^{-1}(2x-1) + C_2$$

It follows that

$$\sin^{-1}(2x-1) = 2\sin^{-1}\sqrt{x} + C_3$$

Substituting $x = 0$

$$\sin^{-1}(-1) = 2\sin^{-1}\sqrt{0} + C_3$$

$$C_3 = -\frac{\pi}{2}$$

Hence we can conclude that

$$\sin^{-1}(2x-1) = 2\sin^{-1}\sqrt{x} - \frac{\pi}{2}$$

14 Let $u = x - \frac{1}{x}, du = 1 + \frac{1}{x^2}\,dx,$

Hence $x^2\,du = x^2 + 1\,dx$

When $x = 1, u = 0$

When $x = \frac{1}{2}(\sqrt{6}+\sqrt{2}),$

$$u = \frac{1}{2}(\sqrt{6}+\sqrt{2}) - \frac{1}{\frac{1}{2}(\sqrt{6}+\sqrt{2})}$$
$$= \frac{1}{2}(\sqrt{6}+\sqrt{2}) - \frac{\sqrt{6}-\sqrt{2}}{\frac{1}{2}(6-2)}$$
$$= \frac{1}{2}(\sqrt{6}+\sqrt{2}) - \frac{1}{2}(\sqrt{6}-\sqrt{2})$$
$$= \sqrt{2}$$

$$u^2 = x^2 - 2 + \frac{1}{x^2}$$

$$u^2 = \frac{x^4+1}{x^2} - 2$$

$$u^2 + 2 = \frac{x^4+1}{x^2}$$

$$\frac{1}{u^2+2}=\frac{x^2}{1+x^4}$$

$$\int_1^{\frac{1}{2}(\sqrt{6}+\sqrt{2})}\frac{1+x^2}{1+x^4}\,dx$$

$$=\int_0^{\sqrt{2}}\frac{x^2}{1+x^4}\,du$$

$$=\int_0^{\sqrt{2}}\frac{1}{u^2+2}\,du$$

$$=\frac{1}{2}\int_0^{\sqrt{2}}\frac{1}{1+\left(\frac{u}{\sqrt{2}}\right)^2}\,du$$

$$=\frac{\sqrt{2}}{2}\left[\tan^{-1}\left(\frac{u}{\sqrt{2}}\right)\right]_0^{\sqrt{2}}$$

$$=\frac{\sqrt{2}}{2}(\tan^{-1}1-0)$$

$$=\frac{1}{\sqrt{2}}\left(\frac{\pi}{4}\right)$$

$$=\frac{\pi}{4\sqrt{2}}$$

Solutions to Exercise 12E

1a $\quad dx = \frac{d}{du}(u) + \frac{d}{du}(1)du = 1du = du$

1b $\quad I = \int x(x-1)^5\,dx$

$$= \int (u+1)(u)^5\,du$$

1c $\quad I = \int (u+1)(u)^5\,du$

$$= \int (u^6 + u^5)\,du$$

$$= \frac{u^7}{7} + \frac{u^6}{6} + C$$

$$= \frac{(x-1)^7}{7} + \frac{(x-1)^6}{6} + C$$

1d

$$\frac{d}{dx}\left(\frac{(x-1)^7}{7} + \frac{(x-1)^6}{6}\right)$$

$$= (x-1)^6 + (x-1)^5$$

$$= (x-1+1)(x-1)^5$$

$$= x(x-1)^5 + C$$

2a

$$\int \frac{x}{\sqrt{x-1}}\,dx$$

$$= \int \frac{u+1}{\sqrt{u}}\,du$$

$$= \int \left(u^{\frac{1}{2}} + u^{-\frac{1}{2}}\right)du$$

$$= \frac{2}{3}u^{\frac{3}{2}} + 2u^{\frac{1}{2}} + C$$

$$= \frac{2}{3}(x-1)^{\frac{3}{2}} + 2\sqrt{x-1} + C$$

2b

$$\int \frac{x}{(x-1)^2}\,dx$$
$$= \int \frac{u+1}{u^2}\,du$$
$$= \int \left(\frac{1}{u} + \frac{1}{u^2}\right)\,du$$
$$= \ln|u| - \frac{1}{u} + C$$
$$= \ln|x-1| - \frac{1}{x-1} + C$$

3a $dx = 2u\,du$

3b

$$J = \int x\sqrt{x+1}\,dx$$
$$= \int (u^2-1)\sqrt{u^2-1+1}\,2u\,du$$
$$= 2\int (u^2-1)\sqrt{u^2}\,u\,du$$
$$= 2\int (u^2-1)u^2\,du$$
$$= 2\int (u^4-u^2)\,du$$

3c

$$J = 2\int (u^4-u^2)\,du$$
$$= 2\left(\frac{u^5}{5} - \frac{u^3}{3}\right) + C$$
$$= 2\left(\frac{\left(\sqrt{x+1}\right)^5}{5} - \frac{\left(\sqrt{x+1}\right)^3}{3}\right) + C$$

3d

$$\frac{d}{dx}\left(2\left(\frac{\left(\sqrt{x+1}\right)^5}{5}-\frac{\left(\sqrt{x+1}\right)^3}{3}\right)\right)$$

$$=2\left(\frac{\frac{5}{2}\left(\sqrt{x+1}\right)^3}{5}-\frac{\frac{3}{2}\left(\sqrt{x+1}\right)}{3}\right)$$

$$=\left(\sqrt{x+1}\right)^3-\left(\sqrt{x+1}\right)$$

$$=(x+1)\sqrt{x+1}-\sqrt{x+1}$$

$$=(x+1-1)\sqrt{x+1}$$

$$=x\sqrt{x+1}$$

4a

$$\int x^2\sqrt{x+1}\,dx$$

$$=\int (u^2-1)^2\sqrt{u^2}\,2u\,du$$

$$=2\int (u^2-1)^2u\,u\,du$$

$$=2\int (u^2-1)^2u^2\,du$$

$$=2\int (u^4-2u^2+1)u^2\,du$$

$$=2\int (u^6-2u^4+u^2)\,du$$

$$=2\left(\frac{u^7}{7}-\frac{2u^5}{5}+\frac{u^3}{3}\right)+C$$

$$=\frac{2(x+1)^{\frac{7}{2}}}{7}-\frac{4(x+1)^{\frac{5}{2}}}{5}+\frac{2(x+1)^{\frac{3}{2}}}{3}+C$$

4b

$$\int \frac{2x+3}{\sqrt{x+1}}dx$$

$$=\int \left(\frac{2x+2}{\sqrt{x+1}}+\frac{1}{\sqrt{x+1}}\right)dx$$

$$=\int \left(\frac{2(x+1)}{\sqrt{x+1}}+\frac{1}{\sqrt{x+1}}\right)dx$$

$$= \int (\frac{2(u^2)}{\sqrt{u^2}} + \frac{1}{\sqrt{u^2}} 2u)\, du$$
$$= \int (4u^2 + 2u)\, du$$
$$= \frac{4}{3}u^3 + u^2 + C$$
$$= \frac{4}{3}(x+1)^{\frac{3}{2}} + 2(x+1)^{\frac{1}{2}} + C$$

5a Let $x = u - 2$, $dx = du$

$$\int \frac{x-2}{x+2} dx$$
$$= \int \frac{u-2-2}{u-2+2} du$$
$$= \int \frac{u-4}{u} du$$
$$= \int \left(1 - \frac{4}{u}\right) du$$
$$= u - 4\ln|u| + C$$
$$= x + 2 - 4\ln|x+2| + C$$

5b Let $x = \frac{1}{2}(u+1)$, $dx = \frac{1}{2}du$, $u = 2x - 1$

$$\int \frac{2x+1}{\sqrt{2x-1}} dx$$
$$= \int \frac{2\left(\frac{1}{2}(u+1)\right) + 1}{\sqrt{2\left(\frac{1}{2}(u+1)\right) - 1}} \frac{1}{2}\, du$$
$$= \int \frac{u+1+1}{\sqrt{u+1-1}} \frac{1}{2}\, du$$
$$= \frac{1}{2}\int \frac{u+2}{\sqrt{u}}\, du$$
$$= \frac{1}{2}\int \left(\sqrt{u} + \frac{2}{\sqrt{u}}\right) du$$
$$= \frac{1}{2}\left(\frac{2}{3}u^{\frac{3}{2}} + 4\sqrt{u}\right) + C$$

$$= \frac{1}{3}u^{\frac{3}{2}} + 2\sqrt{u} + C$$
$$= \frac{1}{3}(2x-1)^{\frac{3}{2}} + 2\sqrt{2x-1} + C$$

5c $\quad x = \frac{1}{4}(u^2+5), dx = \frac{1}{2}u\,du, u^2 = 4x-5, u = \sqrt{4x-5}$

$$\int 3x\sqrt{4x-5}\,dx$$
$$= 3\int \frac{1}{4}(u^2+5)\sqrt{4\left(\frac{1}{4}(u^2+5)\right)-5} \times \frac{1}{2}u\,du$$
$$= 3\int \frac{1}{4}(u^2+5)\sqrt{u^2+5-5} \times \frac{1}{2}u\,du$$
$$= 3\int \frac{1}{4}(u^2+5)\sqrt{u^2} \times \frac{1}{2}u\,du$$
$$= 3\int \frac{1}{8}(u^2+5)u^2\,du$$
$$= \frac{3}{8}\int (u^4+5u^2)\,du$$
$$= \frac{3}{8}\left(\frac{u^5}{5}+\frac{5u^3}{3}\right)+C$$
$$= \frac{3}{8}\left(\frac{u^5}{5}+\frac{5u^3}{3}\right)+C$$
$$= \frac{3}{8}\left(\frac{(\sqrt{4x-5})^5}{5}+\frac{5(\sqrt{4x-5})^3}{3}\right)+C$$

5d $\quad x = (u-1)^2, dx = 2(u-1)\,du$

Hence $\sqrt{x} = u-1, u = 1+\sqrt{x}$

$$\int \frac{1}{1+\sqrt{x}}dx$$
$$= \int \frac{1}{u} \times 2(u-1)\,du$$
$$= 2\int \left(1-\frac{1}{u}\right)du$$
$$= 2(u - \ln|u|)$$
$$= 2\left((1+\sqrt{x}) - \ln\left(1+\sqrt{x}\right)\right) + C \qquad \text{(as } \sqrt{x} \text{ positive)}$$
$$= 2(1+\sqrt{x}) - 2\ln\left(1+\sqrt{x}\right) + C$$

6a $x = u - 1, dx = du$

$$\int_0^1 x(x+1)^3\,dx$$

$$= \int_1^2 (u-1)u^3\,du$$

$$= \int_1^2 (u^4 - u^3)\,du$$

$$= \left[\frac{u^5}{5} - \frac{u^4}{4}\right]_1^2$$

$$= \frac{49}{20}$$

6b $x = 1 - u, dx = -du$

$$\int_0^{\frac{1}{2}} \frac{1+x}{1-x}\,dx$$

$$= \int_1^{\frac{1}{2}} \frac{2-u}{u}\,(-du)$$

$$= \int_{\frac{1}{2}}^1 \left(\frac{2}{u} - 1\right) du$$

$$= [2\ln|u| - u]_{\frac{1}{2}}^1$$

$$= 2\ln 2 - \frac{1}{2}$$

6c Let $x = \frac{1}{3}(u-1), dx = \frac{1}{3}du, u = 3x + 1$

$$\int_0^1 \frac{3x}{\sqrt{3x+1}}\,dx$$

$$= \int_1^4 \frac{u-1}{\sqrt{u}}\,du$$

$$= \int_1^4 \left(\sqrt{u} - u^{-\frac{1}{2}}\right) du$$

$$= \left[\frac{2}{3}u^{\frac{3}{2}} - 2u^{\frac{1}{2}}\right]_1^4$$

$$= \frac{8}{9}$$

6d Let $x = u - 2, dx = du, u = x + 2$

$$\int_0^1 \frac{2 - x}{(2 + x)^3}\, dx$$

$$= \int_2^3 \frac{4 - u}{(u)^3}\, dx$$

$$= \int_2^3 \left(\frac{4}{u^3} - \frac{1}{u^2}\right) dx$$

$$= \left[-\frac{2}{u^2} + \frac{1}{u}\right]_2^3$$

$$= \frac{1}{9}$$

6e Let $x = 4 - u^2, dx = -2u\, du, u = \sqrt{4 - x}$

$$\int_0^4 x\sqrt{4 - x}\, dx$$

$$= \int_2^0 (4 - u^2)u\ (-2u\, du)$$

$$= \int_0^2 (4 - u^2)u\,(2u\, du)$$

$$= \int_0^2 (8u^2 - 2u^4)\, du$$

$$= \left[\frac{8u^3}{3} - \frac{2u^5}{5}\right]_0^2$$

$$= \frac{128}{15}$$

6f $x = \frac{1}{2}(u^2 + 1), dx = u\, du, u^2 = 2x - 1, u = \sqrt{2x - 1}$

$$\int_1^5 \frac{x}{(2x - 1)^{\frac{3}{2}}}\, dx$$

$$= \int_1^5 \frac{x}{(2x-1)^{\frac{3}{2}}}\, dx$$

$$= \int_1^3 \frac{\frac{1}{2}(u^2+1)}{(u^2)^{\frac{3}{2}}}\, u\, du$$

$$= \frac{1}{2}\int_1^3 \frac{(u^2+1)}{u^3}\, u\, du$$

$$= \frac{1}{2}\int_1^3 \frac{(u^2+1)}{u^2}\, du$$

$$= \frac{1}{2}\int_1^3 \left(1+\frac{1}{u^2}\right) du$$

$$= \frac{1}{2}\left[u-\frac{1}{u}\right]_1^3$$

$$= \frac{4}{3}$$

6g $x=(u-3)^2, dx = 2(u-3)\, du, u = \sqrt{x}+3$

$$\int_0^4 \frac{1}{3+\sqrt{x}}\, dx$$

$$= \int_3^5 \frac{1}{u}\, 2(u-3)\, du$$

$$= \int_3^5 2-\frac{6}{u}\, du$$

$$= [2u - 6\ln|u|]_3^5$$

$$= 4 - 6\ln\frac{5}{3}$$

6h $x = u^3 - 1, dx = 3u^2\, du, u = \sqrt[3]{x+1}$

$$\int_0^7 \frac{x^2}{\sqrt[3]{x+1}}\, dx$$

$$= \int_1^2 \frac{(u^3-1)^2}{u}\, 3u^2\, du$$

$$= \int_1^2 3u(u^3-1)^2\, du$$

$$= \int_1^2 3u(u^6 - 2u^3 + 1)du$$
$$= \int_1^2 (3u^7 - 6u^4 + 3u)\,du$$
$$= \left[\frac{3u^8}{8} - \frac{6u^5}{5} + \frac{3u^2}{2}\right]_1^2$$
$$= \frac{2517}{40}$$

7a Let $x = u - 2$, $dx = du$

$$I = \int \frac{1}{\sqrt{5 - 4x - x^2}}dx$$
$$= \int \frac{1}{\sqrt{5 - 4(u-2) - (u-2)^2}}du$$
$$= \int \frac{1}{\sqrt{5 - 4u + 8 - (u^2 - 4u + 4)}}$$
$$= \int \frac{1}{\sqrt{9 - u^2}}$$
$$= \int \frac{1}{3\sqrt{1 - \left(\frac{u}{3}\right)^2}}du$$
$$= \sin^{-1}\left(\frac{u}{3}\right) + C$$
$$= \sin^{-1}(\frac{x+2}{3}) + C$$

7b i Let $x = u - 1$, $dx = du$

$$\int \frac{1}{x^2 + 2x + 4}dx$$
$$= \int \frac{1}{(u-1)^2 + 2(u-1) + 4}du$$
$$= \int \frac{1}{u^2 - 2u + 1 + 2u - 2 + 4}du$$
$$= \int \frac{1}{u^2 + 3}du$$
$$= \frac{1}{3}\int \frac{1}{1 + \left(\frac{u}{\sqrt{3}}\right)^2}\,du$$

$$= \frac{\sqrt{3}}{3}\tan^{-1}\left(\frac{u}{\sqrt{3}}\right) + C$$
$$= \frac{1}{\sqrt{3}}\tan^{-1}\frac{x+1}{\sqrt{3}} + C$$

7b ii Let $x = u - 1$, $dx = du$

$$\int \frac{1}{\sqrt{4 - 2x - x^2}}\,dx$$
$$= \int \frac{1}{\sqrt{4 - 2(u-1) - (u-1)^2}}\,du$$
$$= \int \frac{1}{\sqrt{4 - 2u + 2 - (u^2 - 2u + 1)}}\,du$$
$$= \int \frac{1}{\sqrt{5 - u^2}}\,du$$
$$= \int \frac{1}{\sqrt{5}\sqrt{1 - \left(\frac{u}{\sqrt{5}}\right)^2}}\,du$$
$$= \sin^{-1}\left(\frac{u}{\sqrt{5}}\right) + C$$
$$= \sin^{-1}\frac{x+1}{\sqrt{5}} + C$$

7b iii Let $x = u + 1$, $dx = du$

$$\int_1^2 \frac{1}{\sqrt{3 + 2x - x^2}}\,dx$$
$$= \int_0^1 \frac{1}{\sqrt{3 + 2(u+1) - (u+1)^2}}\,du$$
$$= \int_0^1 \frac{1}{\sqrt{3 + 2u + 2 - (u^2 + 2u + 1)}}\,du$$
$$= \int_0^1 \frac{1}{\sqrt{4 - u^2}}\,du$$
$$= \left[\sin^{-1}\frac{u}{2}\right]_0^1$$
$$= \frac{\pi}{6}$$

7b iv Let $x = u + 3, dx = du$

$$\int_3^7 \frac{1}{x^2 - 6x + 25} dx$$
$$= \int_0^4 \frac{1}{(u+3)^2 - 6(u+3) + 25} du$$
$$= \int_0^4 \frac{1}{(u+3)^2 - 6(u+3) + 25} du$$
$$= \int_0^4 \frac{1}{u^2 + 6u + 9 - 6u - 18 + 25} du$$
$$= \int_0^4 \frac{1}{u^2 + 16} du$$
$$= \frac{1}{4}\left[\tan^{-1}\frac{u}{4}\right]_0^4$$
$$= \frac{\pi}{16}$$

8a Let $x = 2\sin\theta, dx = 2\cos\theta\, d\theta$

Hence

$$J = \int \frac{1}{\sqrt{4 - x^2}} dx$$
$$= \int \frac{1}{\sqrt{4 - (2\sin\theta)^2}} \times 2\cos\theta\, d\theta$$
$$= \int \frac{1}{\sqrt{4 - 4\sin^2\theta}} \times 2\cos\theta\, d\theta$$
$$= \int \frac{1}{2\sqrt{1 - \sin^2\theta}} \times 2\cos\theta\, d\theta$$
$$= \int \frac{1}{2\sqrt{\cos^2\theta}} \times 2\cos\theta\, d\theta$$
$$= \int \frac{1}{2\cos\theta} \times 2\cos\theta\, d\theta$$
$$= \int 1\, d\theta$$
$$= \theta + C$$

Since

$x = 2\sin\theta$

$\sin\theta = \frac{x}{2}$

Hence

$\theta = \sin^{-1}\dfrac{x}{2}$

Thus

$J = \sin^{-1}\dfrac{x}{2} + C$

8b i Let $x = 3\tan\theta$, $dx = 3\sec^2\theta\, d\theta$ and $\theta = \tan^{-1}\frac{x}{3}$

$$\int \frac{1}{9+x^2}\,dx$$

$$= \int \frac{1}{9+(3\tan\theta)^2} 3\sec^2\theta\, d\theta$$

$$= \int \frac{1}{9+9\tan^2\theta} 3\sec^2\theta\, d\theta$$

$$= \frac{1}{3}\int \frac{\sec^2\theta}{1+\tan^2\theta}\,d\theta$$

$$= \frac{1}{3}\int \frac{\sec^2\theta}{\sec^2\theta}\,d\theta$$

$$= \frac{1}{3}\int 1\,d\theta$$

$$= \frac{1}{3}\theta + C$$

$$= \tfrac{1}{3}\tan^{-1}\tfrac{x}{3} + C$$

8b ii Let $x = \sqrt{3}\cos\theta$, $dx = -\sqrt{3}\sin\theta\ \ d\theta$, $\theta = \cos^{-1}\frac{x}{\sqrt{3}}$

$$\int -\frac{1}{\sqrt{3-x^2}}\,dx$$

$$= \int -\frac{1}{\sqrt{3-\left(\sqrt{3}\cos\theta\right)^2}} \times -\sqrt{3}\sin\theta\, d\theta$$

$$= \int \frac{\sqrt{3}\sin\theta}{\sqrt{3}\sqrt{1-\cos^2\theta}}\,d\theta$$

$$= \int \frac{\sqrt{3}\sin\theta}{\sqrt{3}\sqrt{\sin^2\theta}}\,d\theta$$

$$= \int \frac{\sqrt{3}\sin\theta}{\sqrt{3}\sin\theta}\,d\theta$$

$$= \int 1\,d\theta$$

$$= \theta + C$$
$$= \cos^{-1}\frac{x}{\sqrt{3}} + C$$

8b iii Let $x = \frac{1}{2}\sin\theta$, $dx = \frac{1}{2}\cos\theta\ d\theta$ and $\theta = \sin^{-1} 2x$

$$\int \frac{1}{\sqrt{1-4x^2}}dx$$
$$= \int \frac{1}{\sqrt{1-4\left(\frac{1}{2}\sin\theta\right)^2}}\frac{1}{2}\cos\theta\, d\theta$$
$$= \frac{1}{2}\int \frac{\cos\theta}{\sqrt{1-\sin^2\theta}}d\theta$$
$$= \frac{1}{2}\int \frac{\cos\theta}{\sqrt{\cos^2\theta}}d\theta$$
$$= \frac{1}{2}\int \frac{\cos\theta}{\cos\theta}d\theta$$
$$= \frac{1}{2}\int 1\, d\theta$$
$$= \frac{1}{2}\theta + C$$
$$= \frac{1}{2}\sin^{-1} 2x + C$$

8b iv Let $x = \frac{1}{4}\tan\theta$, $dx = \frac{1}{4}\sec^2\theta\, d\theta$ and $\theta = \tan^{-1} 4x$

$$\int \frac{1}{1+16x^2}dx$$
$$= \int \frac{1}{1+16\left(\frac{1}{4}\tan\theta\right)^2}\frac{1}{4}\sec^2\theta\, d\theta$$
$$= \frac{1}{4}\int \frac{\sec^2\theta}{1+\tan^2\theta}d\theta$$
$$= \frac{1}{4}\int \frac{\sec^2\theta}{\sec^2\theta}d\theta$$
$$= \frac{1}{4}\int 1\, d\theta$$
$$= \frac{1}{4}\theta + C$$
$$= \frac{1}{4}\tan^{-1} 4x + C$$

8bv Let $x = 6\sin\theta$, $dx = 6\cos\theta\, d\theta$ and $\theta = \sin^{-1}\frac{x}{6}$.

When $x = 0, \theta = 0$

When $x = 3, \theta = \sin^{-1}\frac{1}{2} = \frac{\pi}{6}$

$$\int_0^3 \frac{1}{\sqrt{36 - x^2}}\,dx$$
$$= \int_0^{\frac{\pi}{6}} \frac{1}{\sqrt{36 - (6\sin\theta)^2}}\,6\cos\theta\, d\theta$$
$$= \int_0^{\frac{\pi}{6}} \frac{1}{6\sqrt{1 - \sin^2\theta}}\,6\cos\theta\, d\theta$$
$$= \int_0^{\frac{\pi}{6}} \frac{6\cos\theta}{6\sqrt{\cos^2\theta}}\,d\theta$$
$$= \int_0^{\frac{\pi}{6}} \frac{6\cos\theta}{6\cos\theta}\,d\theta$$
$$= \int_0^{\frac{\pi}{6}} 1\, d\theta$$
$$= [\theta]_0^{\frac{\pi}{6}}$$
$$= \frac{\pi}{6}$$

8b vi

Let $x = \frac{2}{3}\tan\theta$, $dx = \frac{2}{3}\sec^2\theta\, d\theta$ and $\theta = \tan^{-1}\frac{3}{2}x$

When $x = 0, \theta = 0$

When $x = \frac{2}{3}, \theta = \tan^{-1} 1 = \frac{\pi}{4}$

$$\int_0^{\frac{2}{3}} \frac{1}{4 + 9x^2}\,dx$$
$$= \int_0^{\frac{\pi}{4}} \frac{1}{4 + 9\left(\frac{2}{3}\tan\theta\right)^2}\,\frac{2}{3}\sec^2\theta\, d\theta$$
$$= \int_0^{\frac{\pi}{4}} \frac{1}{4 + 4\tan^2\theta}\,\frac{2}{3}\sec^2\theta\, d\theta$$
$$= \frac{1}{6}\int_0^{\frac{\pi}{4}} \frac{\sec^2\theta}{\sec^2\theta}\,d\theta$$

$$= \frac{1}{6}\int_0^{\frac{\pi}{4}} 1\, d\theta$$
$$= \frac{1}{6}[\theta]_0^{\frac{\pi}{4}}$$
$$= \frac{\pi}{24}$$

9a Let $x = \sin\theta$, $dx = \cos\theta\, d\theta$ and $\theta = \sin^{-1} x$

$$I = \int \frac{1}{(1-x^2)^{\frac{3}{2}}}\, dx$$
$$= \int \frac{1}{(1-\sin^2\theta)^{\frac{3}{2}}}\cos\theta\, d\theta$$
$$= \int \frac{1}{(\cos^2\theta)^{\frac{3}{2}}}\cos\theta\, d\theta$$
$$= \int \frac{1}{\cos^3\theta}\cos\theta\, d\theta$$
$$= \int \frac{1}{\cos^2\theta}\, d\theta$$
$$= \int \sec^2\theta\, d\theta$$
$$= \tan\theta + C$$

Now since $\sin\theta = x = \frac{x}{1}$, it follows, from the Pythagorean theorem that

$$\tan\theta = \frac{x}{\sqrt{1-x^2}}$$

Hence

$$I = \frac{x}{\sqrt{1-x^2}} + C$$

9b i Let $x = 2\tan\theta$, $dx = 2\sec^2\theta\, d\theta$

$$\int \frac{1}{(4+x^2)^{\frac{3}{2}}}\, dx$$
$$= \int \frac{1}{(4+4\tan^2\theta)^{\frac{3}{2}}} 2\sec^2\theta\, d\theta$$
$$= \int \frac{1}{(4\sec^2\theta)^{\frac{3}{2}}} 2\sec^2\theta\, d\theta$$

$$= \int \frac{1}{8\sec^3\theta} 2\sec^2\theta \, d\theta$$
$$= \frac{1}{4}\int \frac{1}{\sec\theta} d\theta$$
$$= \frac{1}{4}\int \cos\theta \, d\theta$$
$$= \frac{1}{4}\sin\theta + C$$
$$= \frac{1}{4}\frac{x}{\sqrt{4+x^2}} + C$$

9b ii Let $x = \sin\theta$, $dx = \cos\theta \, d\theta$

$$\int_0^{\frac{1}{2}} \frac{x^2}{\sqrt{1-x^2}} dx$$
$$= \int_0^{\frac{\pi}{6}} \frac{\sin^2\theta}{\sqrt{1-\sin^2\theta}} \cos\theta \, d\theta$$
$$= \int_0^{\frac{\pi}{6}} \frac{\sin^2\theta}{\sqrt{\cos^2\theta}} \cos\theta \, d\theta$$
$$= \int_0^{\frac{\pi}{6}} \frac{\sin^2\theta}{\cos\theta} \cos\theta \, d\theta$$
$$= \int_0^{\frac{\pi}{6}} \sin^2\theta \, d\theta$$
$$= \frac{1}{2}\int_0^{\frac{\pi}{6}} (1-\cos 2\theta) \, d\theta$$
$$= \frac{1}{2}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{6}}$$
$$= \frac{\pi}{12} - \frac{\sqrt{3}}{8}$$

9b iii Let $x = 2\sin\theta$, $dx = 2\cos\theta \, d\theta$

$$\int_0^2 \sqrt{4-x^2} \, dx$$
$$= \int_0^{\frac{\pi}{2}} \sqrt{4-(2\sin\theta)^2} \, 2\cos\theta \, d\theta$$
$$= \int_0^{\frac{\pi}{2}} 2\sqrt{1-\sin^2\theta} \, 2\cos\theta \, d\theta$$

$$= 4\int_0^{\frac{\pi}{2}} \cos^2\theta \, d\theta$$
$$= 2\int_0^{\frac{\pi}{2}} (1+\cos 2\theta)\, d\theta$$
$$= 2\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}}$$
$$= \pi$$

9b iv Let $x = 5\cos\theta$, $dx = -5\sin\theta \, d\theta$

$$\int \frac{1}{x^2\sqrt{25-x^2}}\,dx$$
$$= \int \frac{1}{(5\cos\theta)^2\sqrt{25-(5\cos\theta)^2}} \times -5\sin\theta\, d\theta$$
$$= \int \frac{1}{25\cos^2\theta \, 5\sqrt{1-\cos^2\theta}} \times -5\sin\theta\, d\theta$$
$$= \int \frac{1}{25\cos^2\theta \, 5\sin\theta} \times -5\sin\theta\, d\theta$$
$$= -\frac{1}{25}\int \frac{1}{\cos^2\theta}\, d\theta$$
$$= -\frac{1}{25}\int \sec^2\theta\, d\theta$$
$$= -\frac{1}{25}\tan\theta + C$$
$$= -\frac{1}{25}\times\frac{\sqrt{25-x^2}}{x} + C$$
$$= -\frac{\sqrt{25-x^2}}{25x} + C$$

9b v Let $x = 3\tan\theta$, $dx = 3\sec^2\theta\, d\theta$

$$\int \frac{1}{x^2\sqrt{9+x^2}}\,dx$$
$$= \int \frac{1}{(3\tan\theta)^2\sqrt{9+(3\tan\theta)^2}} 3\sec^2\theta\, d\theta$$
$$= \int \frac{1}{(3\tan\theta)^2 3\sqrt{1+\tan^2\theta}} 3\sec^2\theta\, d\theta$$
$$= \int \frac{1}{(3\tan\theta)^2 3\sec\theta} 3\sec^2\theta\, d\theta$$

$$= \int \frac{\sec\theta}{(3\tan\theta)^2}\,d\theta$$
$$= \frac{1}{9}\int \frac{\cos^2\theta}{\sin^2\theta} \times \frac{1}{\cos\theta}\,d\theta$$
$$= \frac{1}{9}\int \frac{\cos\theta}{\sin\theta} \times \frac{1}{\sin\theta}\,d\theta$$
$$= \frac{1}{9}\int \cot\theta \operatorname{cosec}\theta\,d\theta$$
$$= -\frac{1}{9}\operatorname{cosec}\theta + C$$
$$= -\frac{1}{9} \times \frac{\sqrt{9+x^2}}{x} + C$$
$$= -\frac{\sqrt{9+x^2}}{9x} + C$$

9b vi Let $x = 2\sec\theta$, $dx = 2\sec\theta\tan\theta\,d\theta$

$$\int_2^4 \frac{1}{x^2\sqrt{x^2-4}}\,dx$$
$$= \int_0^{\frac{\pi}{3}} \frac{1}{(2\sec\theta)^2\sqrt{(2\sec\theta)^2-4}}2\sec\theta\tan\theta\,d\theta$$
$$= \int_0^{\frac{\pi}{3}} \frac{1}{(2\sec\theta)^2 2\sqrt{\sec^2\theta-1}}2\sec\theta\tan\theta\,d\theta$$
$$= \int_0^{\frac{\pi}{3}} \frac{1}{(2\sec\theta)^2 2\sqrt{\tan^2\theta}}2\sec\theta\tan\theta\,d\theta$$
$$= \int_0^{\frac{\pi}{3}} \frac{1}{(2\sec\theta)^2 2\tan\theta}2\sec\theta\tan\theta\,d\theta$$
$$= \int_0^{\frac{\pi}{3}} \frac{1}{4\sec\theta}\,d\theta$$
$$= \frac{1}{4}\int_0^{\frac{\pi}{3}} \cos\theta\,d\theta$$
$$= \frac{1}{4}[\sin\theta]_0^{\frac{\pi}{3}}$$
$$= \frac{\sqrt{3}}{8}$$

10 Let $x = 3\sec\theta$, $dx = 3\sec\theta\tan\theta\,d\theta$ and $\theta = \sec^{-1}\frac{x}{3}$

$f(x)$

$$= \int f'(x)\,dx$$
$$= \int \frac{\sqrt{x^2-9}}{x}\,dx$$
$$= \int \frac{\sqrt{9\sec^2\theta - 9}}{3\sec\theta} 3\sec\theta\tan\theta\,d\theta$$
$$= \int \frac{3\tan\theta}{3\sec\theta} 3\sec\theta\tan\theta\,d\theta$$
$$= 3\int \tan^2\theta\,d\theta$$
$$= 3\int (\sec^2\theta - 1)\,d\theta$$
$$= 3(\tan\theta - \theta) + C$$
$$= \sqrt{x^2-9} - 3\sec^{-1}\frac{x}{3} + C$$

Now note that

$$\sqrt{x^2-9} = \sqrt{9\sec^2\theta - 9} = 3\tan\theta$$

$$\therefore\ \theta = \tan^{-1}\frac{\sqrt{x^2-9}}{3}$$

$$f(3) = 0$$

$$\sqrt{3^2-9} - 3\sec^{-1}\frac{3}{3} + C = 0$$
$$C = 3$$

Hence,

$$f(x) = 3[\tan\theta - \theta] + C$$
$$= \sqrt{x^2-9} - 3\tan^{-1}\frac{\sqrt{x^2-9}}{3}$$

11 Let $x = \sqrt{3}\sin\theta$, $dx = \sqrt{3}\cos\theta\,d\theta$

$$A = \int_0^1 \frac{x^3}{\sqrt{3-x^2}}\,dx$$
$$= \int_0^{\frac{\pi}{6}} \frac{3\sqrt{3}\sin^3\theta}{\sqrt{3-3\sin^2\theta}}\sqrt{3}\cos\theta\,d\theta$$
$$= \int_0^{\frac{\pi}{6}} \frac{3\sqrt{3}\sin^3\theta}{\sqrt{3}\cos\theta}\sqrt{3}\cos\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{6}} 3\sqrt{3}\sin^3\theta\, d\theta$$

Let $u = \cos\theta$, $du = -\sin\theta\, d\theta$

$$A = \int_1^{\frac{\sqrt{3}}{2}} -3\sqrt{3}\sin^2\theta\, du$$

$$= 3\sqrt{3}\int_{\frac{\sqrt{3}}{2}}^{1} 1 - \cos^2\theta\, du$$

$$= 3\sqrt{3}\int_{\frac{\sqrt{3}}{2}}^{1} 1 - u^2\, du$$

$$= 3\sqrt{3}\left[u - \frac{u^3}{3}\right]_{\frac{\sqrt{3}}{2}}^{1}$$

$$= \frac{1}{3}\left(6\sqrt{3} - 7\sqrt{2}\right) \text{ square units}$$

12a Let $x = r\sin\theta$, $dx = r\cos\theta\, d\theta$

$$A = 2\int_{-r}^{r} \sqrt{r^2 - x^2}\, dx$$

$$= 2\int_{-r}^{r} \sqrt{r^2 - x^2}\, dx$$

$$= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{r^2 - (r\sin\theta)^2}\, r\cos\theta\, d\theta$$

$$= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} r\sqrt{1 - \sin^2\theta}\, r\cos\theta\, d\theta$$

$$= 2r^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta\, d\theta$$

$$= r^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 - \cos 2\theta)\, d\theta$$

$$= r^2\left[\theta - \frac{\sin 2\theta}{2}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \pi r^2$$

12b i Let P be the x-position of chord AB

$$\sin\frac{\alpha}{2} = \frac{P}{OA} = \frac{P}{r}$$

Hence

$$P = r\sin\frac{\alpha}{2}$$

The area is twice the area from P to r under the semicircle

$$A = 2\int_P^r \sqrt{r^2 - x^2}\, dx = 2\int_{r\sin\frac{\alpha}{2}}^r \sqrt{r^2 - x^2}\, dx$$

12b ii Let $x = r\cos\theta$, $dx = -r\sin\theta\, d\theta$

$$\begin{aligned} A &= 2\int_{r\sin\frac{\alpha}{2}}^r \sqrt{r^2 - x^2}\, dx \\ &= 2\int_{r\sin\frac{\alpha}{2}}^r \sqrt{r^2 - x^2}\, dx \\ &= 2\int_{r\sin\frac{\alpha}{2}}^{\frac{\pi}{2}} -\sqrt{r^2 - (r\cos\theta)^2}\, r\sin\theta\, d\theta \\ &= 2\int_{\frac{1}{2}\alpha}^{0} -r\sqrt{1 - \cos^2\theta}\, r\sin\theta\, d\theta \\ &= 2\int_{\frac{1}{2}\alpha}^{0} -r\sin\theta\, r\sin\theta\, d\theta \\ &= -2r^2\int_{\frac{1}{2}\alpha}^{0} \sin^2\theta\, d\theta \end{aligned}$$

12b iii

$$\begin{aligned} A &= 2\int_{r\sin\frac{\alpha}{2}}^r \sqrt{r^2 - x^2}\, dx \\ &= -2r^2\int_{\frac{1}{2}\alpha}^{0} \sin^2\theta\, d\theta \\ &= 2r^2\int_0^{\frac{1}{2}\alpha} \sin^2\theta\, d\theta \\ &= 2r^2\int_0^{\frac{1}{2}\alpha} (1 - \cos 2\theta)\, d\theta \end{aligned}$$

$$= 2r^2\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\frac{1}{2}\alpha}$$
$$= \frac{1}{2}r^2(\alpha - \sin\alpha)$$

12c For an ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$
$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$
$$y = b\sqrt{1 - \frac{x^2}{a^2}}$$

Let $x = a\sin\theta, dx = a\cos\theta\, d\theta$

The area will be given by

$$A = 2\int_{-a}^{a} b\sqrt{1 - \frac{x^2}{a^2}}\,dx$$
$$= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} b\sqrt{1 - \frac{a^2\sin^2\theta}{a^2}}\,a\cos\theta\, d\theta$$
$$= 2ab\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1 - \sin^2\theta}\cos\theta\, d\theta$$
$$= 2ab\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta\, d\theta$$
$$= 2ab\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta\, d\theta$$
$$= ab\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1 + \cos 2\theta\, d\theta$$
$$= ab\left[\theta + \frac{1}{2}\sin 2\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$
$$= \pi ab$$

The unit circle has area π, dilating it a units horizontally causes it to become $a\pi$ and then dilating it a further b units vertically causes the area to become πab.

13a Let $x = -u, dx = -du$

$$\int_{-2}^{2} \frac{x^2}{e^x + 1} dx$$

$$= \int_{2}^{-2} \frac{(-u)^2}{e^{-u} + 1} (-du) \qquad \text{(using when } x = -2, u = 2 \text{ and when } x = 2, u = -2\text{)}$$

$$= \int_{-2}^{2} \frac{u^2}{e^{-u} + 1} \, du$$

$$= \int_{-2}^{2} \frac{u^2 e^u}{1 + e^u} \, du$$

$$= \int_{-2}^{2} \frac{u^2 e^u}{e^u + 1} \, du$$

Now let $x = u$

$$\int_{-2}^{2} \frac{u^2 e^u}{e^u + 1} du = \int_{-2}^{2} \frac{x^2 e^x}{e^x + 1} dx$$

Hence

$$\int_{-2}^{2} \frac{x^2}{e^x + 1} dx = \int_{-2}^{2} \frac{x^2 e^x}{e^x + 1} dx$$

13b

$$\int_{-2}^{2} \frac{x^2}{e^x + 1} dx$$

$$= \frac{1}{2}\left(\int_{-2}^{2} \frac{x^2}{e^x + 1} dx + \int_{-2}^{2} \frac{x^2}{e^x + 1} dx\right)$$

$$= \frac{1}{2}\left(\int_{-2}^{2} \frac{x^2}{e^x + 1} dx + \int_{-2}^{2} \frac{x^2 e^x}{e^x + 1} dx\right)$$

$$= \frac{1}{2}\left(\int_{-2}^{2} \frac{x^2 + x^2 e^x}{e^x + 1} dx\right)$$

$$= \frac{1}{2}\left(\int_{-2}^{2} \frac{x^2(1 + e^x)}{e^x + 1} dx\right)$$

$$= \frac{1}{2}\left(\int_{-2}^{2} x^2 \, dx\right)$$

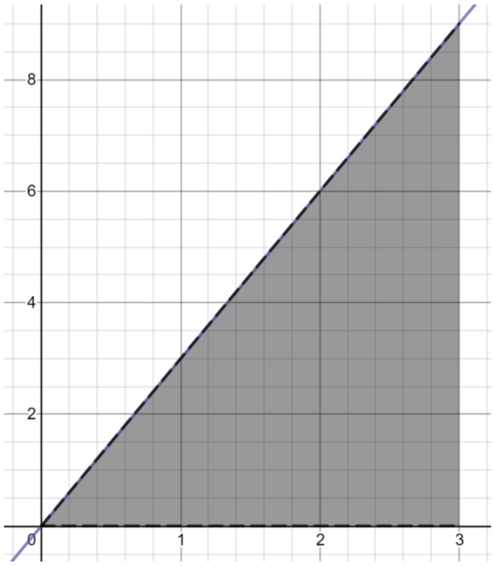
$$= \frac{1}{2}\left[\frac{x^3}{3}\right]_{-2}^{2}$$

$$= \frac{1}{2}\left(\frac{8}{3} - \left(-\frac{8}{3}\right)\right)$$

$$= \frac{8}{3}$$

Solutions to Exercise 12F

1a



1b The radius will be $r = 3(3) = 9$ units and the height will be 3 units.

Hence

$$V = \frac{1}{3}\pi r^2$$

$$= \frac{1}{3}\pi(9)^2(3)$$

$$= 81\pi \text{ cubic units}$$

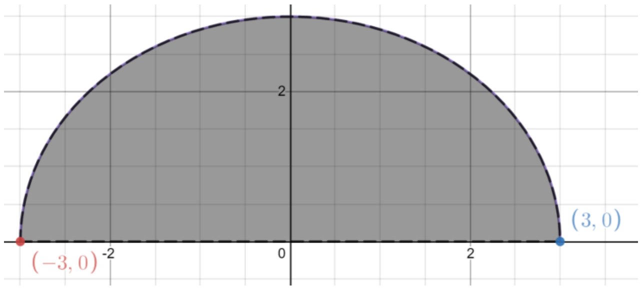
1c

$$\pi\int_0^3 9x^2\,dx$$

$$= 3\pi[x^3]_0^3$$

$$= 81\pi \text{ cubic units}$$

2a



2b This is a sphere of radius 3, hence the volume is

$$V = \frac{4}{3}\pi r^3$$

$$= \frac{4}{3}\pi(3)^3$$

$$= 36\pi \text{ cubic units}$$

2c

$$\pi\int_{-3}^{3}(9 - x^2)\,dx$$

$$= \pi\left[9x - \frac{x^3}{3}\right]_{-3}^{3}$$

$$= 36\pi \text{ cubic units}$$

3a This is a cylinder of radius 2, height 4. Hence

$$V = \pi r^2 h$$

$$= \pi(2)^2 \times 4$$

$$= 16\pi \text{ cubic units}$$

3b This is a cone of radius 3 and height 3. Hence

$$V = \frac{1}{3}\pi r^2 h$$

$$= \frac{1}{3}\pi(3)^2 \times 3$$

$$= 9\pi \text{ cubic units}$$

3c

$$V = \pi \int_0^2 y^2\, dx$$

$$= \pi \int_0^2 x^4\, dx$$

$$= \pi \left[\frac{x^5}{5}\right]_0^2$$

$$= \frac{32\pi}{5} \text{ cubic units}$$

3d

$$V = \pi \int_2^4 y^2\, dx$$

$$= \pi \int_2^4 x\, dx$$

$$= \frac{\pi}{2}[x^2]_2^4$$

$$= 6\pi \text{ cubic units}$$

3e

$$V = \pi \int_0^2 y^2\, dx$$

$$= \int_0^2 (4 - x^2)\, dx$$

$$= \left[4x - \frac{x^3}{3}\right]_0^2$$

$$= \frac{16\pi}{3} \text{ cubic units}$$

3f

$$\begin{aligned} V &= \pi \int_{-1}^{0} y^2 \, dx \\ &= \pi \int_{-1}^{0} x^6 \, dx \\ &= \pi \left[\frac{x^7}{7} \right]_{-1}^{0} \\ &= \frac{\pi}{7} \text{ cubic units} \end{aligned}$$

3g

$$\begin{aligned} V &= \pi \int_{-5}^{-2} y^2 \, dx \\ &= \pi \int_{-5}^{-2} (x+2)^2 \, dx \\ &= \pi \int_{-5}^{-2} (x^2 + 4x + 4) \, dx \\ &= \pi \left[\frac{x^3}{3} + 2x^2 + 4x \right]_{-5}^{-2} \\ &= 9\pi \text{ cubic units} \end{aligned}$$

3h

$$\begin{aligned} V &= \pi \int_{-3}^{3} y^2 \, dx \\ &= \pi \int_{-3}^{3} \left(4 - \frac{4}{9} x^2 \right) dx \\ &= \pi \left[4x - \frac{4}{27} x^3 \right]_{-3}^{3} \\ &= \frac{16\pi}{3} \text{ cubic units} \end{aligned}$$

4a This is a cylinder of radius 1, height 3, hence

$$V = \pi r^2 h$$
$$= \pi(1)^2(3)$$
$$= 3\pi \text{ cubic units}$$

4b This is a cone with radius 2, height 4, hence

$$V = \frac{1}{3}\pi r^2 h$$
$$= \frac{1}{3}\pi(4)^2(2)$$
$$= \frac{256}{3}\pi \text{ cubic units}$$

4c

$$V = \pi \int_2^5 x^2\, dy$$
$$= \pi \int_2^5 y^4\, dy$$
$$= \left[\frac{\pi y^5}{5}\right]_2^5$$
$$= \frac{3093\pi}{5} \text{ cubic units}$$

4d

$$V = \pi \int_0^1 x^2\, dy$$
$$= \pi \int_0^1 y\, dy$$
$$= \pi \left[\frac{y^2}{2}\right]_0^1$$
$$= \frac{\pi}{2} \text{ cubic units}$$

4e This is a sphere of radius 4, hence

$$V = \frac{4}{3}\pi r^3$$

$$= \frac{4}{3}\pi(4)^3$$

$$= \frac{256}{3}\pi \text{ cubic units}$$

4f

$$V = \pi\int_{-3}^{0} x^2\,dy$$

$$= \pi\int_{-3}^{0} (-y^2)^2\,dy$$

$$= \pi\int_{-3}^{0} y^4\,dy$$

$$= \pi\left[\frac{y^5}{5}\right]_{-3}^{0}$$

$$= \frac{243\pi}{5} \text{ cubic units}$$

4g

$$V = \pi\int_{0}^{2} x^2\,dy$$

$$= \pi\int_{0}^{2} (2y - y^2)^2\,dy$$

$$= \pi\int_{0}^{2} (4y^2 - 4y^3 + y^4)\,dy$$

$$= \pi\left[\frac{4}{3}y^3 - y^4 + \frac{y^5}{5}\right]_{0}^{2}$$

$$= \frac{16\pi}{15} \text{ cubic units}$$

4h

$$
\begin{aligned}
V &= \int_{-1}^{1} x^2\, dy \\
&= \int_{-1}^{1} (4 - 4y^2)\, dy \\
&= \left[4y - \frac{4y^3}{3}\right]_{-1}^{1} \\
&= \frac{16\pi}{3} \ \text{ cubic units}
\end{aligned}
$$

5

$$
\begin{aligned}
V &= \pi \int_{0}^{1} y^2\, dx \\
&= \pi \int_{0}^{1} e^{2x}\, dx \\
&= \frac{\pi}{2}\left[e^{2x}\right]_0^1 \\
&= \frac{\pi}{2}(e^2 - 1) \ \text{ cubic units}
\end{aligned}
$$

6

$$
\begin{aligned}
V &= \pi \int_{2}^{4} y^2\, dx \\
&= \pi \int_{2}^{4} \frac{1}{x} dx \\
&= \pi[\ln x]_2^4 \\
&= \pi[\ln 4 - \ln 2] \\
&= \pi \ln 2 \ \text{ cubic units}
\end{aligned}
$$

7

$$V = \pi \int_1^6 x^2 \, dy$$

$$= \pi \int_1^6 \frac{1}{y} dy$$

$$= \pi [\ln y]_1^6$$

$$= \pi \ln 6 \text{ cubic units}$$

8a $\tan^2 x = \sec^2 x - 1$

8b

$$V = \pi \int_1^{\frac{\pi}{3}} y^2 \, dx$$

$$= \pi \int_1^{\frac{\pi}{3}} \tan^2 x \, dx$$

$$= \pi \int_1^{\frac{\pi}{3}} (\sec^2 x - 1) \, dx$$

$$= \pi [\tan x - x]_1^{\frac{\pi}{3}}$$

$$= \pi \left(\sqrt{3} - \frac{\pi}{3}\right) \text{ cubic units}$$

9a $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$

9b

$$V = \pi \int_0^{\frac{\pi}{2}} y^2 \, dx$$

$$= \pi \int_0^{\frac{\pi}{2}} \sin^2 x \, dx$$

$$= \pi \int_0^{\frac{\pi}{2}} \sin^2 x \, dx$$

$$= \pi \int_0^{\frac{\pi}{2}} \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right) dx$$

$$= \pi \left[\frac{1}{2}x - \frac{1}{4}\sin 2x\right]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi^2}{4} \ \text{ cubic units}$$

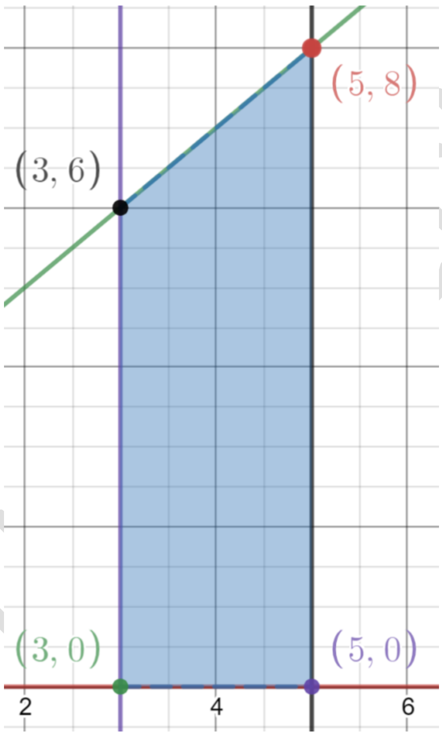
10a

$$V = \pi \int_3^5 y^2 \, dx$$

$$= \int_3^5 (x+3)^2 \, dx$$

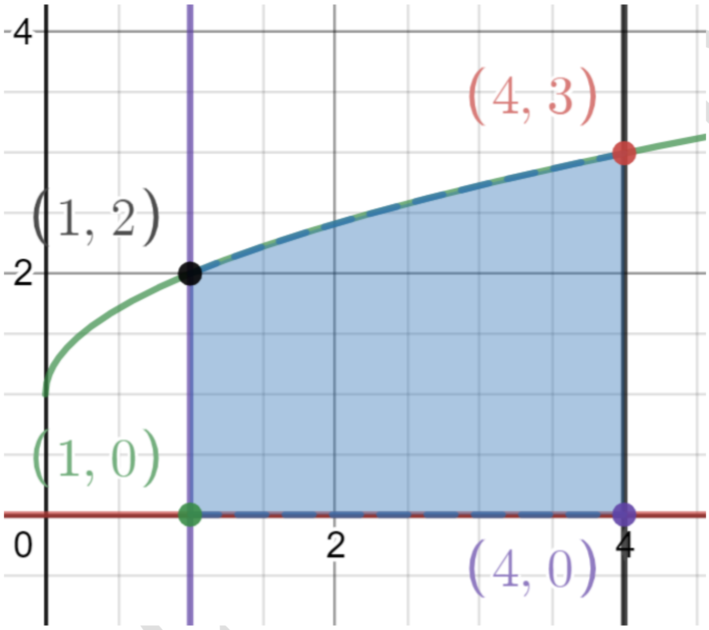
$$= \pi \left[\frac{(x+3)^3}{3}\right]_3^5$$

$$= \frac{296\pi}{3} \ \text{ cubic units}$$



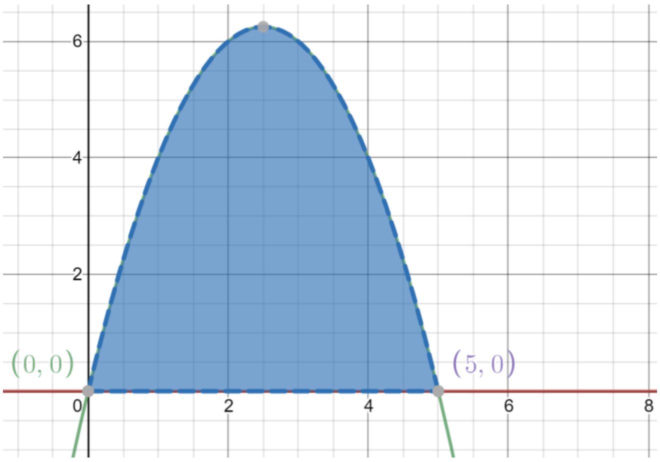
10b

$$\begin{aligned}
V &= \pi \int_1^4 y^2\,dx \\
&= \pi \int_1^4 \left(1+\sqrt{x}\right)^2 dx \\
&= \pi \int_1^4 1 + 2\sqrt{x} + x\,dx \\
&= \pi \int_1^4 1 + 2x^{\frac{1}{2}} + x\,dx \\
&= \pi \left[x + \frac{4}{3}x^{\frac{3}{2}} + \frac{x^2}{2}\right]_1^4 \\
&= \frac{119\pi}{6} \text{ cubic units}
\end{aligned}$$



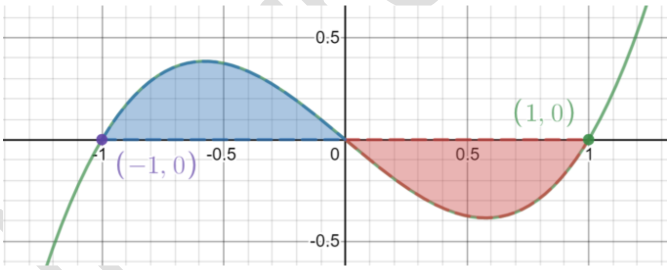
10c

$$\begin{aligned}
V &= \pi \int_0^5 y^2\,dx \\
&= \pi \int_0^5 (5x - x^2)^2\,dx \\
&= \pi \int_0^5 (25x^2 - 10x^3 + x^4)\,dx \\
&= \pi \left[\frac{25x^3}{3} - \frac{10x^4}{4} + \frac{x^5}{5}\right]_0^5 \\
&= \frac{625\pi}{6} \text{ cubic units}
\end{aligned}$$



10d

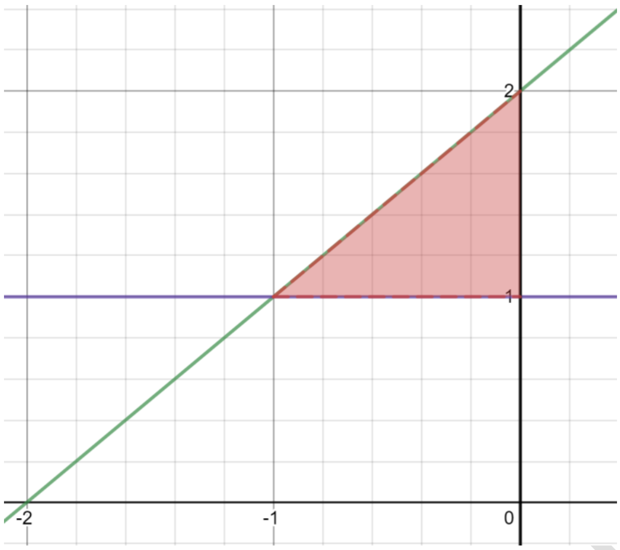
$$
\begin{aligned}
V &= \pi \int_{-1}^{1} y^2 \, dx \\
&= \pi \int_{-1}^{1} (1 - x^2)^2 \, dx \\
&= \pi \int_{-1}^{1} (1 - 2x^2 + x^4) \, dx \\
&= \pi \left[x - \frac{2x^3}{3} + \frac{x^5}{5} \right]_{-1}^{1} \\
&= \frac{16\pi}{15} \text{ cubic units}
\end{aligned}
$$



11a

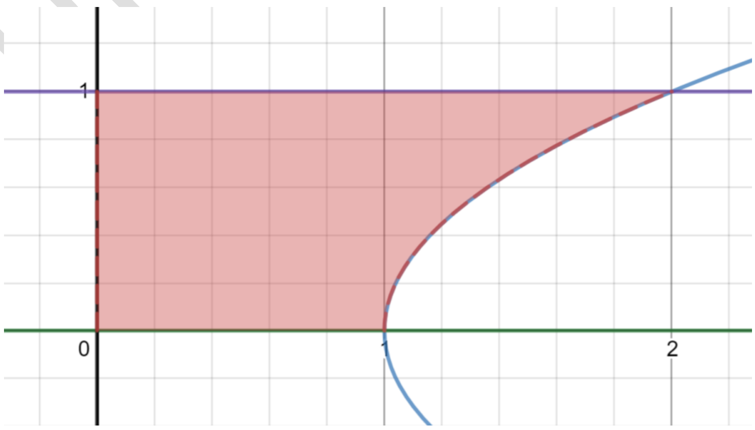
$$
\begin{aligned}
V &= \pi \int_{1}^{2} x^2 \, dy \\
&= \pi \int_{1}^{2} (y - 2)^2 \, dy \\
&= \pi \int_{1}^{2} (y^2 - 4y + 4) \, dy
\end{aligned}
$$

$$= \pi \left[\frac{y^3}{3} - 2y^2 + 4y\right]_1^2$$
$$= \frac{\pi}{3} \text{ cubic units}$$



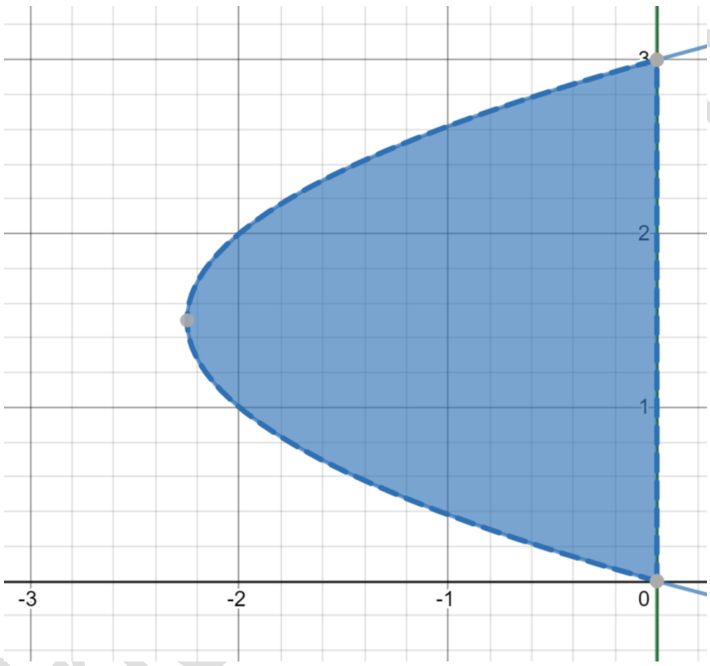
11b

$$V = \pi \int_0^1 x^2 \, dy$$
$$= \pi \int_0^1 (y^2 + 1)^2 \, dy$$
$$= \pi \int_0^1 (y^4 + 2y^2 + 1) \, dy$$
$$= \pi \left[\frac{y^5}{5} + \frac{2y^3}{3} + y\right]_0^1$$
$$= \frac{28\pi}{15} \text{ cubic units}$$



11c

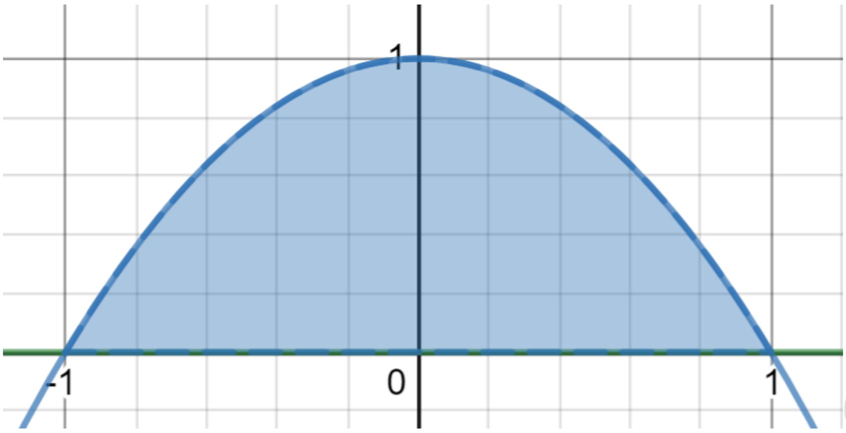
$$\begin{aligned}V &= \pi\int_0^2 x^2\,dy\\ &= \pi\int_0^2 y^2(y-3)^2\,dy\\ &= \pi\int_0^2 y^2(y^2-6y+9)\,dy\\ &= \pi\int_0^2 (y^4-6y^3+9y^2)\,dy\\ &= \pi\left[\frac{y^5}{5}-\frac{6y^4}{4}+3y^3\right]_0^2\\ &= \frac{81\pi}{10}\text{ cubic units}\end{aligned}$$



11d

$$\begin{aligned}V &= \pi\int_0^1 x^2\,dy\\ &= \pi\int_0^1 (1-y)\,dy\\ &= \pi\left[y-\frac{y^2}{2}\right]_0^1\end{aligned}$$

$$= \frac{\pi}{2} \text{ cubic units}$$



12

$$V = \pi \int_{\frac{1}{2}}^{3} y^2 \, dx$$
$$= \pi \int_{\frac{1}{2}}^{3} \left(1 + \frac{1}{x}\right)^2 dx$$
$$= \pi \int_{\frac{1}{2}}^{3} \left(1 + \frac{2}{x} + \frac{1}{x^2}\right) dx$$
$$= \pi \left[x + 2\ln|x| - \frac{1}{x}\right]_{\frac{1}{2}}^{3}$$
$$= \frac{\pi}{6}(25 + 12\ln 6) \text{ cubic units}$$

13

$$V = \pi \int_{0}^{\frac{1}{2}} y^2 \, dx$$
$$= \pi \int_{0}^{\frac{1}{2}} (e^x - e^{-x})^2 \, dx$$
$$= \pi \int_{0}^{\frac{1}{2}} (e^{2x} - 2 + e^{-2x}) \, dx$$
$$= \pi \left[\frac{e^{2x}}{2} - 2x - \frac{e^{-2x}}{2}\right]_{0}^{\frac{1}{2}}$$
$$= \frac{\pi}{2}(e - 2 - e^{-1}) \text{ cubic units}$$

14

$$
\begin{aligned}
V &= \pi \int_{4\pi}^{6\pi} y^2 \, dx \\
&= \pi \int_{4\pi}^{6\pi} \left(4 + 4\sin\frac{x}{4}\right)^2 dx \\
&= 16\pi \int_{4\pi}^{6\pi} \left(1 + 2\sin\frac{x}{4} + \sin^2\frac{x}{4}\right) dx \\
&= 16\pi \int_{4\pi}^{6\pi} \left(1 + 2\sin\frac{x}{4} + \frac{1}{2}\left(1 - \cos\frac{x}{2}\right)\right) dx \\
&= 16\pi \int_{4\pi}^{6\pi} \left(\frac{3}{2} + 2\sin\frac{x}{4} - \cos\frac{x}{2}\right) dx \\
&= 16\pi \left[x - 8\cos\frac{x}{4} - 2\sin\frac{x}{2}\right]_{4\pi}^{6\pi} \\
&= 71.62 \text{ mL}
\end{aligned}
$$

15a

$$
\begin{aligned}
V &= \pi \int_{0}^{16} y^2 \, dx \\
&= \pi \int_{0}^{16} 4^2 \, dx \\
&= \pi [4^2 x]_0^{16} \\
&= 256\pi \text{ cubic units}
\end{aligned}
$$

15b

$$
\begin{aligned}
V &= \pi \int_{0}^{16} x^2 \, dy \\
&= \pi \int_{0}^{16} y \, dy \\
&= \pi \left[\frac{y^2}{2}\right]_0^{16} \\
&= 128\pi \text{ cubic units}
\end{aligned}
$$

15c $V = 256\pi - 128\pi = 128\pi$ cubic units

16a i

$$V = \pi \int_0^2 y^2 \, dx$$

$$= \pi \int_0^2 x^4 \, dx$$

$$= \pi \left[\frac{x^5}{5} \right]_0^2$$

$$= \frac{32\pi}{5} \text{ cubic units}$$

16a ii

$$V = \pi \int_0^4 x^2 \, dy$$

$$= \pi \int_0^4 y \, dy$$

$$= \pi \left[\frac{y^2}{2} \right]$$

$$= 8\pi \text{ cubic units}$$

16b i

$$V = \pi \times 5^2 \times 1 - \pi \int_0^1 y^2 \, dx$$

$$= \pi \times 5^2 \times 1 - \pi \int_0^1 25x^2 \, dx$$

$$= \pi \times 5^2 \times 1 - \pi \left[\frac{25x^3}{3} \right]_0^1$$

$$= \frac{50\pi}{3} \text{ cubic units}$$

16b ii

$$V = \pi \int_0^5 x^2 \, dy$$

$$= \pi \int_0^5 \left(\frac{y}{5}\right)^2 dy$$

$$= \frac{\pi}{25} \int_0^5 y^2 \, dy$$

$$= \frac{\pi}{25} \left[\frac{y^3}{3}\right]_0^5$$

$$= \frac{5\pi}{3} \text{ cubic units}$$

16c i

$$V = \pi \int_0^4 y^2 \, dx$$

$$= \pi \int_0^4 x \, dx$$

$$= \pi \left[\frac{x^2}{2}\right]_0^4$$

$$= 8\pi \text{ cubic units}$$

16c ii

$$V = \pi \times 4^2 \times 2 - \pi \int_0^2 x^2 \, dy$$

$$= \pi \times 4^2 \times 2 - \pi \int_0^2 y^4 \, dy$$

$$= \pi \times 4^2 \times 2 - \pi \left[\frac{y^5}{5}\right]_0^2$$

$$= \frac{128\pi}{5} \text{ cubic units}$$

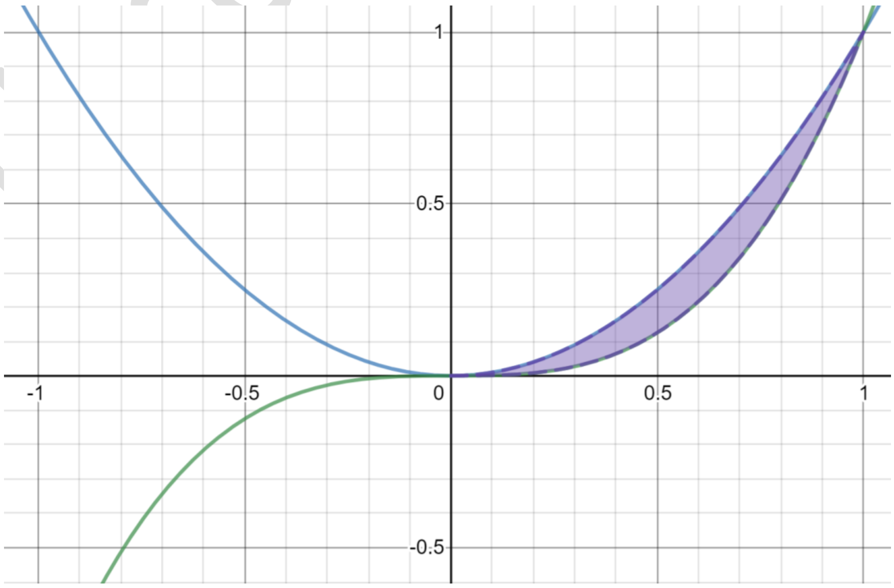
16d i

$$\begin{aligned} V &= \pi \times 4^2 \times 1 - \int_0^1 y^2\,dx \\ &= 16\pi - \pi \int_0^1 (x^2+3)^2\,dx \\ &= 16\pi - \pi \int_0^1 (x^4+6x^2+9)\,dx \\ &= 16\pi - \pi \left[\frac{x^5}{5} + 2x^3 + 9x\right]_0^1 \\ &= \frac{24\pi}{5} \text{ cubic units} \end{aligned}$$

16d ii

$$\begin{aligned} V &= \pi \int_3^4 x^2\,dy \\ &= \pi \int_3^4 (y-3)\,dy \\ &= \pi \left[\frac{y^2}{2} - 3y\right]_3^4 \\ &= \frac{\pi}{2} \text{ cubic units} \end{aligned}$$

17a



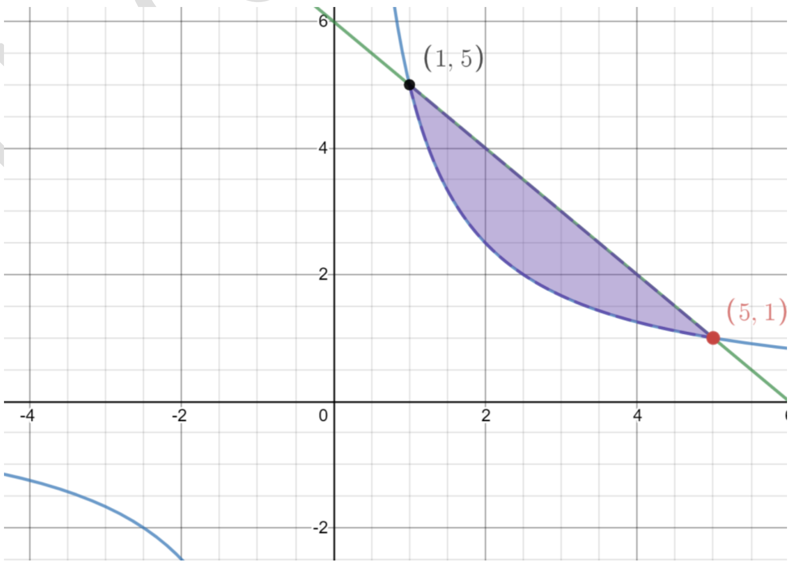
17b i

$$\begin{aligned} V &= \pi \int_0^1 y_1^2\,dx - \pi \int_0^1 y_2^2\,dx \\ &= \pi \int_0^1 x^4\,dx - \pi \int_0^1 x^6\,dx \\ &= \pi \int_0^1 (x^4 - x^6)\,dx \\ &= \pi \left[\frac{x^5}{5} - \frac{x^7}{7}\right]_0^1 \\ &= \frac{2\pi}{35} \text{ cubic units} \end{aligned}$$

17b ii

$$\begin{aligned} V &= \pi \int_0^1 x_1^2\,dy - \pi \int_0^1 x_2^2\,dy \\ &= \pi \int_0^1 y^1\,dy - \pi \int_0^1 y^{\frac{3}{2}}\,dy \\ &= \pi \int_0^1 \left(y^1 - y^{\frac{3}{2}}\right) dy \\ &= \pi \left[\frac{y^2}{2} - \frac{2}{5}y^{\frac{5}{2}}\right]_0^1 \\ &= \frac{\pi}{10} \text{ cubic units} \end{aligned}$$

18a



18b

$$\begin{aligned}V &= \pi\int_1^5 (y_1^2 - y^2)\,dx\\ &= \pi\int_1^5 \left((6-x)^2 - \left(\frac{5}{x}\right)^2\right)dx\\ &= \pi\int_1^5 \left(36 - 12x + x^2 - \frac{25}{x^2}\right)dx\\ &= \pi\left[36x - 6x^2 + \frac{x^3}{3} + \frac{25}{x}\right]_1^5\\ &= \frac{64\pi}{3}\text{ cubic units}\end{aligned}$$

19

$$\begin{aligned}V &= \pi\int_1^2 (y_1^2 - y_2^2)\,dx\\ &= \pi\int_1^2 \left(1^2 - \left(2 - \frac{2}{x}\right)^2\right)dx\\ &= \pi\int_1^2 (1 - \frac{4}{x^2} - 4 + \frac{8}{x})\,dx\\ &= \pi\int_1^2 (-3 - \frac{4}{x^2} + \frac{8}{x})\,dx\\ &= \pi\left[-3x + \frac{4}{x} + 8\ln x\right]_1^2\\ &= \pi(8\ln 2 - 5)\text{ cubic units}\end{aligned}$$

20a $y = x^3 + 2$

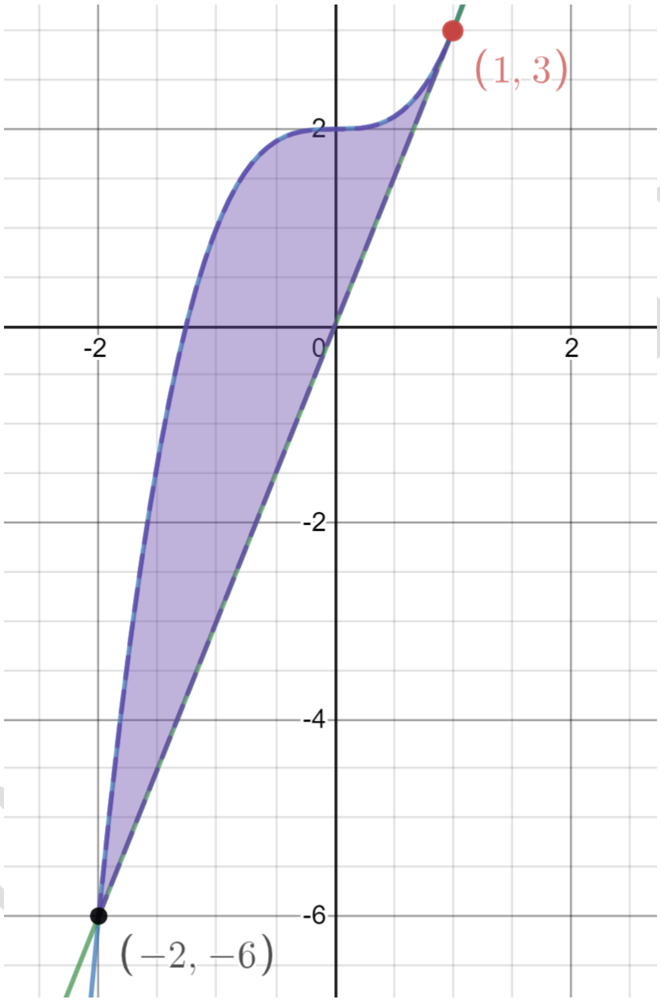
$\dfrac{dy}{dx} = 3x^2$

When $x = 1, y = 3$ and $\frac{dy}{dx} = 3$. Hence the equation of the tangent is

$y - 3 = 3(x - 1)$

$y = 3x$

20b



20c i

$$V = \pi \int_{-2}^{1} (y_1^2 - y_2^2)\, dx$$
$$= \pi \int_{-2}^{1} (x^6 - 9x^2)\, dx$$
$$= \pi \left[\frac{x^7}{7} - 3x^3\right]_{-2}^{1}$$
$$= \frac{15\pi}{7} \text{ cubic units}$$

20c ii

$$V = \pi \int_{-6}^{3} (x_1^2 - x_2^2)\, dy$$
$$= \pi \int_{-6}^{3} \left(y^{\frac{2}{3}} - \frac{y^2}{9}\right) dy$$
$$= \pi \left[\frac{3}{2} y^{\frac{2}{3}} - \frac{y^3}{27}\right]_{-6}^{3}$$
$$= \frac{2\pi}{5} \text{ cubic units}$$

21

$$V = \pi \times 2^2 \times \ln 2 - \pi \int_0^{\ln 2} x^2\, dy$$
$$= 4\pi \ln 2 - \pi \int_0^{\ln 2} (e^y)^2\, dy$$
$$= 4\pi \ln 2 - \pi \int_0^{\ln 2} e^{2y}\, dy$$
$$= 4\pi \ln 2 - \pi \left[\frac{e^{2y}}{2}\right]_0^{\ln 2}$$
$$= 4\pi \ln 2 - \pi \left(\frac{e^{2\ln 2}}{2} - \frac{e^0}{2}\right)$$
$$= 4\pi \ln 2 - \pi \left(\frac{e^{\ln 4}}{2} - \frac{e^0}{2}\right)$$

$$= 4\pi \ln 2 - \pi\left(\frac{4}{2} - \frac{1}{2}\right)$$
$$= 4\pi \ln 2 - \pi\left(\frac{3}{2}\right)$$
$$= \frac{\pi}{2}(8\ln 2 - 3)\ \text{ cubic units}$$

22a

$$V = \pi\int_0^h y^2\,dx$$
$$= \pi\int_0^h \left(\frac{rx}{h}\right)^2 dx$$
$$= \pi\int_0^h \frac{r^2x^2}{h^2}\,dx$$
$$= \pi\left[\frac{r^2x^3}{3h^2}\right]_0^h$$
$$= \pi\left[\frac{r^2h^3}{3h^2}\right]$$
$$= \frac{1}{3}\pi r^2 h$$

22b

$$V = \pi\int_0^h y^2\,dx$$
$$= \pi\int_0^h r^2\,dx$$
$$= \pi[r^2x]_0^h$$
$$= \pi r^2 h$$

22c i

$$\begin{aligned}V &= \pi \int_{-r}^{r} y^2\,dx \\ &= \pi \int_{-r}^{r} (r^2 - x^2)\,dx \\ &= \pi \left[r^2 x - \frac{x^3}{3}\right]_{-r}^{r} \\ &= \pi \left[\left(r^2 r - \frac{r^3}{3}\right) - \left(r^2(-r) - \frac{(-r)^3}{3}\right)\right] \\ &= \frac{4}{3}\pi r^3\end{aligned}$$

22c ii

$$\begin{aligned}V &= \pi \int_{r-h}^{r} y^2\,dx \\ &= \pi \int_{r-h}^{r} r^2 - x^2\,dx \\ &= \pi \left[r^2 x - \frac{x^3}{3}\right]_{r-h}^{r} \\ &= \pi \left[\left(r^2 r - \frac{r^3}{3}\right) - \left(r^2(r-h) - \frac{(r-h)^3}{3}\right)\right] \\ &= \pi \left[\left(r^2 r - \frac{r^3}{3}\right) - \left(r^3 - r^2 h - \frac{r^3 - 3r^2 h + 3rh^2 - h^3}{3}\right)\right] \\ &= \frac{1}{3}\pi h^2(3r - h)\end{aligned}$$

23a $\sec\theta$

$$\begin{aligned}&= \sec\theta \times \frac{\sec\theta + \tan\theta}{\sec\theta + \tan\theta} \\ &= \frac{\sec^2\theta + \sec\theta\tan\theta}{\sec\theta + \tan\theta} \\ &= \frac{\sec\theta\tan\theta + \sec^2\theta}{\sec\theta + \tan\theta}\end{aligned}$$

$$\int \sec\theta\,d\theta$$

$$= \int \frac{\sec\theta\tan\theta + \sec^2\theta}{\sec\theta + \tan\theta}\,d\theta$$

$$= \ln(\sec\theta + \tan\theta) + C$$

23b

$$y = \frac{x}{\sqrt{x^2+16}}$$

$$y^2 = \frac{x^2}{x^2+16}$$

$$x^2 + 16 = \frac{x^2}{y^2}$$

$$16 = \frac{x^2}{y^2} - x^2$$

$$16 = x^2\left(\frac{1}{y^2} - 1\right)$$

$$x^2 = \frac{16}{\left(\frac{1}{y^2} - 1\right)}$$

$$= \frac{16}{\left(\frac{1}{y^2} - \frac{y^2}{y^2}\right)}$$

$$= \frac{16}{\left(\frac{1-y^2}{y^2}\right)}$$

$$= \frac{16y^2}{1-y^2}$$

$$V = \pi\int_0^{\frac{1}{\sqrt{2}}} x^2\,dy$$

$$= \pi\int_0^{\frac{1}{\sqrt{2}}} \frac{16y^2}{1-y^2}\,dy$$

Let $y = \sin\theta, dy = \cos\theta\,d\theta$

$$V = \pi\int_0^{\frac{1}{\sqrt{2}}} x^2\,dy$$

$$= \pi\int_0^{\frac{\pi}{4}} \frac{16\sin^2\theta}{1-\sin^2\theta}\cos\theta\,d\theta$$

$$= \pi \int_0^{\frac{\pi}{4}} \frac{16\sin^2\theta}{\cos^2\theta} \cos\theta \, d\theta$$
$$= \pi \int_0^{\frac{\pi}{4}} \frac{16\sin^2\theta}{\cos\theta} \, d\theta$$
$$= \pi \int_0^{\frac{\pi}{4}} \frac{16(1-\cos^2\theta)}{\cos\theta} \, d\theta$$
$$= 16\pi \int_0^{\frac{\pi}{4}} (\sec\theta - \cos\theta) \, d\theta$$
$$= 16\pi[\ln(\sec\theta + \tan\theta) + \sin\theta]_0^{\frac{\pi}{4}}$$
$$= 16\pi\left(\sqrt{2} - \ln\left(\sqrt{2}+1\right)\right) \text{ cubic units}$$

24a

$$x^n = x^{n+1}$$
$$x^n - x^{n+1} = 0$$
$$x^n(1-x) = 0$$

Hence $x = 0$ or 1. Substituting back into the equations gives the points of intersection as $(0, 0)$ and $(1, 1)$.

24b

$$V = \pi \int_0^1 f(x)^2 \, dx - \pi \int_0^1 f(x)^2 \, dx$$
$$= \pi \int_0^1 (x^n)^2 \, dx - \pi \int_0^1 (x^{n+1})^2 \, dx$$
$$= \pi \int_0^1 x^{2n} \, dx - \pi \int_0^1 x^{2n+2} \, dx$$
$$= \pi \left[\frac{x^{2n+1}}{2n+1}\right]_0^1 - \pi \left[\frac{x^{2n+3}}{2n+3}\right]_0^1$$
$$= \pi \left(\frac{1}{2n+1} - \frac{1}{2n+3}\right)$$

24c It is the cone formed by rotating the line $y = x$ from $x = 0$ to $x = 1$ about the x-axis.

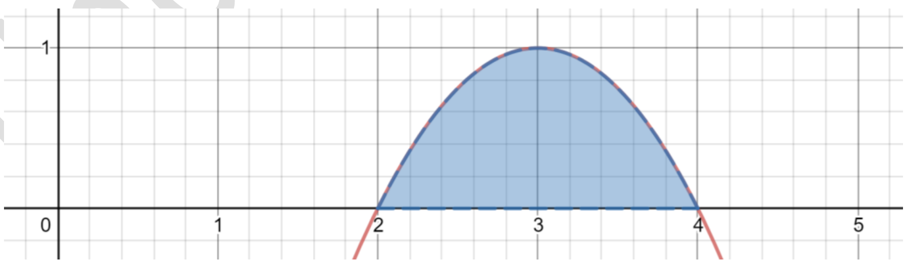
24d

$$\begin{aligned}
&\lim_{n\to\infty}(V_1+V_2+V_3+\cdots+V_n)\\
&=\lim_{n\to\infty}\left(\pi\left(\frac{1}{2+1}-\frac{1}{2+3}\right)+\pi\left(\frac{1}{2(2)+1}-\frac{1}{2(2)+3}\right)+\pi\left(\frac{1}{2(3)+1}-\frac{1}{2(3)+3}\right)\right.\\
&\qquad\qquad\left.+\cdots+\pi\left(\frac{1}{2(n)+1}-\frac{1}{2(n)+3}\right)\right)\\
&=\pi\lim_{n\to\infty}\left(\frac{1}{3}-\frac{1}{5}+\frac{1}{5}-\frac{1}{7}+\frac{1}{7}-\frac{1}{9}+\cdots+\frac{1}{2n+1}-\frac{1}{2n+3}\right)\\
&=\pi\lim_{n\to\infty}\left(\frac{1}{3}-\frac{1}{2n+3}\right)\\
&=\pi\left(\frac{1}{3}-0\right)\\
&=\frac{\pi}{3}\text{ cubic units}
\end{aligned}$$

24e

$$\begin{aligned}
&\frac{1}{3\times 5}+\frac{1}{5\times 7}+\frac{1}{7\times 9}+\cdots\\
&=\frac{1}{2}\left(\frac{1}{3}-\frac{1}{5}\right)+\frac{1}{2}\left(\frac{1}{5}-\frac{1}{7}\right)+\frac{1}{2}\left(\frac{1}{7}-\frac{1}{9}\right)+\cdots\\
&=\frac{1}{2}\left(\frac{1}{3}-\frac{1}{5}+\frac{1}{5}-\frac{1}{7}+\frac{1}{7}-\frac{1}{9}+\cdots\right)\\
&=\frac{1}{2}\left(\frac{1}{3}\right)\\
&=\frac{1}{6}
\end{aligned}$$

25a



25b $y=-x^2+6x-8$

$y=-x^2+6x-9+1$

$y=-(x^2-6x+9)+1$

$y = -(x-3)^2 + 1$

$(x-3)^2 = 1 - y$

$x - 3 = \sqrt{1-y}$ for $3 \le x \le 4$ and $x - 3 = -\sqrt{1-y}$ for $2 \le x \le 3$

Hence

$x = 3 + \sqrt{1-y}$ for $3 \le x \le 4$ and $x = 3 - \sqrt{1-y}$ for $2 \le x \le 3$

25c

$$\begin{aligned}
V &= \pi \int_0^1 \left(3 + \sqrt{1-y}\right)^2 dy - \pi \int_0^1 \left(3 - \sqrt{1-y}\right)^2 dy \\
&= \pi \int_0^1 \left(\left(3 + \sqrt{1-y}\right)^2 - \left(3 - \sqrt{1-y}\right)^2\right) dy \\
&= \pi \int_0^1 \left(\left(9 + 6\sqrt{1-y} + (1-y)\right) - \left(9 - 6\sqrt{1-y} + (1-y)\right)\right) dy \\
&= \pi \int_0^1 \left(12\sqrt{1-y}\,\right) dy \\
&= \int_0^1 \left(12\pi\sqrt{1-y}\,\right) dy \\
&= \int_0^1 \left(12\pi(1-y)^{\frac{1}{2}}\right) dy \\
&= 12\pi \left[-\frac{2}{3}(1-y)^{\frac{3}{2}}\right]_0^1 \\
&= 12\pi \left(-\frac{2}{3} \times 0 + \frac{2}{3} \times 1\right) \\
&= 8\pi \text{ cubic units}
\end{aligned}$$

Solutions to Chapter review

1a

$$\frac{d}{dx}(\sin^{-1} 3x)$$

$$= \frac{1}{\sqrt{1-(3x)^2}} \times \frac{d}{dx}(3x)$$

$$= \frac{3}{\sqrt{1-9x^2}}$$

1b

$$\frac{d}{dx}\left(\tan^{-1}\left(\frac{x}{3}\right)\right)$$

$$= \frac{1}{1+\left(\frac{x}{3}\right)^2} \times \frac{d}{dx}\left(\frac{x}{3}\right)$$

$$= \frac{1}{1+\frac{x^2}{9}} \times \frac{1}{3}$$

$$= \frac{3}{9+x^2}$$

1c

$$\frac{d}{dx}(\cos^{-1}(1-x))$$

$$= \frac{-1}{\sqrt{1-(1-x)^2}} \times \frac{d}{dx}(1-x)$$

$$= \frac{-1}{\sqrt{1-(1-2x+x^2)}} \times -1$$

$$= \frac{1}{\sqrt{2x-x^2}}$$

1d

$$\frac{d}{dx}(x^2\tan^{-1}x)$$
$$= (\tan^{-1}x)\times\frac{d}{dx}(x^2) + x^2\times\frac{d}{dx}(\tan^{-1}x)$$
$$= 2x\tan^{-1}x + x^2\times\frac{1}{1+x^2}$$
$$= \frac{x^2}{1+x^2} + 2x\tan^{-1}x$$

1e

$$\frac{d}{dx}\left(\tan^{-1}\left(\frac{1}{2}x+1\right)\right)$$
$$= \frac{1}{1+\left(\frac{1}{2}x+1\right)^2}\times\frac{d}{dx}\left(\frac{1}{2}x+1\right)$$
$$= \frac{1}{1+\frac{x^2}{4}+x+1}\times\frac{1}{2}$$
$$= \frac{1}{\frac{x^2}{4}+x+2}\times\frac{1}{2}$$
$$= \frac{4}{x^2+4x+8}\times\frac{1}{2}$$
$$= \frac{2}{x^2+4x+8}$$

1f

$$\frac{d}{dx}\left(\sin^{-1}\frac{1}{x}\right)$$
$$= \frac{1}{\sqrt{1-\left(\frac{1}{x}\right)^2}}\times\frac{d}{dx}\left(\frac{1}{x}\right)$$
$$= \frac{1}{\sqrt{1-\frac{1}{x^2}}}\times\left(-\frac{1}{x^2}\right)$$
$$= -\frac{1}{\sqrt{x^4}\sqrt{1-\frac{1}{x^2}}}$$
$$= -\frac{1}{\sqrt{x^4-x^2}}$$

2a $y = \tan^{-1} x$

$$y' = \frac{1}{1+x^2} = (1+x^2)^{-1}$$

$$y'' = -1 \times (1+x^2)^{-2} \times 2x = -\frac{2x}{(1+x^2)^2}$$

Substituting for y' and y'':

$$(1+x^2)y'' + 2xy'$$
$$= (1+x^2) \times -\frac{2x}{(1+x^2)^2} + 2x \times \frac{1}{(1+x^2)}$$
$$= -\frac{2x}{(1+x^2)} + \frac{2x}{(1+x^2)}$$
$$= 0$$

3a

$$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(\sin^{-1}\sqrt{1-x^2}\right)$$

$$= \frac{1}{\sqrt{1-\left(\sqrt{1-x^2}\right)^2}} \times \frac{d}{dx}\left(\sqrt{1-x^2}\right)$$

$$= \frac{1}{\sqrt{1-(1-x^2)}} \times \frac{1}{2} \times (1-x^2)^{-\frac{1}{2}} \times -2x$$

$$= \frac{1}{\sqrt{x^2}} \times -\frac{x}{\sqrt{1-x^2}}$$

$$= -\frac{1}{\sqrt{x^2}} \times \frac{x}{\sqrt{1-x^2}}$$

For $0 \le x \le 1$, $\sqrt{x^2} = x$

Hence

$$\frac{d}{dx}\left(\sin^{-1}\sqrt{1-x^2}\right)$$

$$= -\frac{1}{x} \times \frac{x}{\sqrt{1-x^2}}$$

$$= -\frac{1}{\sqrt{1-x^2}}$$

$$= \frac{d}{dx}(\cos^{-1} x)$$

3b The functions differ by at most a constant.

This constant is zero, so $\cos^{-1} x = \sin^{-1} \sqrt{1-x^2}$ for $0 \le x \le 1$.

4a

$$\int \frac{1}{3+x^2}\,dx$$

$$= \frac{1}{3}\int \frac{1}{1+\left(\frac{x}{\sqrt{3}}\right)^2}\,dx$$

$$= \frac{\sqrt{3}}{3}\tan^{-1}\frac{x}{\sqrt{3}} + C$$

4b

$$\int \frac{1}{\sqrt{3-x^2}}\,dx$$

$$= \int \frac{1}{\sqrt{3}\sqrt{1-\left(\frac{x}{\sqrt{3}}\right)^2}}\,dx$$

$$= \sin^{-1}\frac{x}{\sqrt{3}} + C$$

4c

$$\int \frac{1}{9+4x^2}\,dx$$

$$= \frac{1}{4}\int \frac{1}{\frac{9}{4}+\left(\frac{x}{2}\right)^2}\,dx$$

$$= \frac{1}{9}\int \frac{1}{1+\left(\frac{2x}{3}\right)^2}\,dx$$

$$= \frac{1}{6}\tan^{-1}\frac{2x}{3} + C$$

4d

$$\int -\frac{1}{\sqrt{16 - 9x^2}}\,dx$$

$$= \int -\frac{1}{4\sqrt{1 - \left(\frac{3x}{4}\right)^2}}\,dx$$

$$= \frac{1}{3}\cos^{-1}\frac{3x}{4} + C$$

5a

$$\int_{\frac{1}{3}}^{\frac{1}{\sqrt{3}}} \frac{1}{1 + 9x^2}\,dx$$

$$= \int_{\frac{1}{3}}^{\frac{1}{\sqrt{3}}} \frac{1}{1 + (3x)^2}\,dx$$

$$= \frac{1}{3}\left[\tan^{-1} 3x\right]_{\frac{1}{3}}^{\frac{1}{\sqrt{3}}}$$

$$= \frac{\pi}{36}$$

5b

$$\int_{-\frac{3}{4}}^{\frac{3}{4}} \frac{1}{\sqrt{3 - 4x^2}}\,dx$$

$$= \int_{-\frac{3}{4}}^{\frac{3}{4}} \frac{1}{\sqrt{3}\sqrt{1 - \left(\frac{2x}{\sqrt{3}}\right)^2}}\,dx$$

$$= \frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{2}\left[\sin^{-1}\frac{2x}{\sqrt{3}}\right]_{-\frac{3}{4}}^{\frac{3}{4}}$$

$$= \frac{1}{2}\left[\sin^{-1}\frac{2x}{\sqrt{3}}\right]_{-\frac{3}{4}}^{\frac{3}{4}}$$

$$= \frac{\pi}{3}$$

6a

$$\int \cos^2 x \, dx$$

$$= \int \left(\frac{1}{2} + \frac{1}{2}\cos 2x\right) dx$$

$$= \frac{1}{2}x + \frac{1}{4}\sin 2x + C$$

6b

$$\int \sin^2 x \, dx$$

$$= \int \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right) dx$$

$$= \frac{1}{2}x - \frac{1}{4}\sin 2x + C$$

6c

$$\int \cos^2 2x \, dx$$

$$= \int \left(\frac{1}{2} + \frac{1}{2}\cos 4x\right) dx$$

$$= \frac{1}{2}x + \frac{1}{8}\sin 4x + C$$

6d

$$\int \sin^2 4x \, dx$$

$$= \int \left(\frac{1}{2} - \frac{1}{2}\cos 8x\right) dx$$

$$= \frac{1}{2}x - \frac{1}{16}\sin 4x + C$$

7a

$$\int_0^{\frac{\pi}{3}} \sin^2 3x\, dx$$

$$= \int_0^{\frac{\pi}{3}} \left(\frac{1}{2} - \frac{1}{2}\cos 6x\right) dx$$

$$= \left[\frac{1}{2}x - \frac{1}{16}\sin 4x\right]_0^{\frac{\pi}{3}}$$

$$= \frac{\pi}{6}$$

7b

$$\int_0^{\frac{\pi}{6}} \cos^2 \frac{1}{2}x\, dx$$

$$= \int_0^{\frac{\pi}{6}} \left(\frac{1}{2} + \frac{1}{2}\sin x\right) dx$$

$$= \left[\frac{1}{2}x - \frac{1}{2}\cos x\right]_0^{\frac{\pi}{6}}$$

$$= \frac{1}{12}(\pi + 3)$$

8

$$\int_0^{\frac{\pi}{2}} \cos^2 x\, dx = \int_0^{\frac{\pi}{2}} \left(\frac{1}{2} + \frac{1}{2}\cos 2x\right) dx = \left[\frac{1}{2}x + \frac{1}{4}\sin 2x\right]_0^{\frac{\pi}{2}} = \frac{\pi}{4}$$

$$\int_0^{\frac{\pi}{2}} \cos^2 2x\, dx = \int_0^{\frac{\pi}{2}} \left(\frac{1}{2} + \frac{1}{2}\cos 4x\right) dx = \left[\frac{1}{2}x + \frac{1}{8}\sin 4x\right]_0^{\frac{\pi}{2}} = \frac{\pi}{4}$$

$$\int_0^{\frac{\pi}{2}} \cos^2 4x\, dx = \int_0^{\frac{\pi}{2}} \left(\frac{1}{2} + \frac{1}{2}\cos 8x\right) dx = \left[\frac{1}{2}x + \frac{1}{16}\sin 8x\right]_0^{\frac{\pi}{2}} = \frac{\pi}{4}$$

9a Let $u = 5x - 1$, $du = 5\,dx$

$$\int 5(5x-1)^5$$

$$= \int u^5\,du$$

$$= \frac{u^6}{6} + C$$

$$= \frac{(5x-1)^6}{6} + C$$

9b Let $u = x^2 + 2$

$du = 2x\,dx$

$$\int 2x(x^2+2)^2\,dx$$

$$= \int u^2\,du$$

$$= \frac{u^3}{3} + C$$

$$= \frac{(x^2+2)^3}{3} + C$$

9c Let $u = x^4 + 1$, $du = 4x^3$

$$\int \frac{4x^3}{(x^4+1)^2}dx$$

$$= \int \frac{1}{u^2}\,du$$

$$= \int u^{-2}\,du$$

$$= -u^{-1} + C$$

$$= \frac{1}{x^4+1} + C$$

9d Let $u = 4x + 3, du = 4\,dx$

$$\int \frac{1}{\sqrt{4x+3}}\,dx$$

$$= \int \frac{1}{\sqrt{u}} \times \frac{1}{4}\,du$$

$$= \frac{1}{4}\int u^{-\frac{1}{2}}\,du$$

$$= \frac{1}{4} \times 2u^{\frac{1}{2}} + C$$

$$= \frac{1}{2}\sqrt{4x+3} + C$$

9e Let $u = \sin x, du = \cos x\,dx$

$$\int \sin^2 x \cos x\,dx$$

$$= \int u^2\,du$$

$$= \frac{u^3}{3} + C$$

$$= \frac{\sin^3 x}{3} + C$$

9f Let $u = \tan x, du = \sec^2 x\,dx$

$$\int \tan^3 x \sec^2 x\,dx$$

$$= \int u^3\,du$$

$$= \frac{u^4}{4} + C$$

$$= \frac{\tan^4 x}{4} + C$$

10a Let $u = 1 + x^3$, $du = 3x^2\,dx$ and $\frac{1}{3}du = x^2dx$

$$\int_{-1}^{0} x^2(1 + x^3)^4\,dx$$

$$= \frac{1}{3}\int_{0}^{1} u^4\,du$$

$$= \frac{1}{3}\left[\frac{u^5}{5}\right]_0^1$$

$$= \frac{1}{15}$$

10b Let $u = \cos x$, $du = -\sin x$

$$\int_{0}^{\frac{\pi}{2}} \cos^3 x \sin x\,dx$$

$$= \frac{1}{4}[-u^4]_1^0$$

$$= -\frac{1}{4}(0^3 - 1^2)$$

$$= \frac{1}{4}$$

10c Let $u = x^2 - 1$, $du = 2x\,dx$ and $\frac{1}{2}du = x\,dx$

$$\int_{1}^{\sqrt{2}} x\sqrt{x^2 - 1}\,dx$$

$$= \frac{1}{2}\int_{0}^{1} \sqrt{u}\,du$$

$$= \frac{1}{2}\int_{0}^{1} u^{\frac{1}{2}}\,du$$

$$= \frac{1}{2}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_0^1$$

$$= \frac{1}{2}\left(\frac{2}{3} - 0\right)$$

$$= \frac{1}{3}$$

10d Let $u = \ln x$, $du = \frac{1}{x}\ dx$

$$\int_1^e \frac{(\ln x)^2}{x} dx$$

$$= \int_0^1 u^2\, du$$

$$= \left[\frac{1}{3}u^3\right]_0^1$$

$$= \frac{1}{3}$$

10e Let $u = \frac{1}{x}$, $du = -\frac{1}{x^2}\ dx$

$$\int_{\frac{1}{2}}^1 \frac{e^{\frac{1}{x}}}{x^2} dx$$

$$= \int_2^1 -e^u\, du$$

$$= \int_1^2 e^u\, du$$

$$= [e^u]_1^2$$

$$= e^2 - e$$

10f Let $u = \tan 2x$, $du = 2\sec^2 2x\ \ dx$

$$\int_0^{\frac{\pi}{8}} \frac{\sec^2 2x}{1 + \tan 2x} dx$$

$$= \frac{1}{2}\int_0^1 \frac{1}{1+u} du$$

$$= \frac{1}{2}[\ln|1+u|]_0^1$$

$= \frac{1}{2}(\ln 2 - \ln 1)$

$= \frac{1}{2}\ln 2$

11a Let $x = u + 1$, $du = dx$

$\int \frac{x}{x-1}\,dx$

$= \int \frac{u+1}{u}\,du$

$= \int \left(1 + \frac{1}{u}\right) du$

$= u + \ln|u| + C$

$= x - 1 + \ln|x-1| + C$

11b Let $x = u - 2$, $du = dx$

$\int \frac{x-1}{\sqrt{x+2}}\,dx$

$= \int \frac{u-3}{\sqrt{u}}\,du$

$= \int (u^{\frac{1}{2}} - 3u^{-\frac{1}{2}})\,du$

$= \frac{2}{3}u^{\frac{3}{2}} - 6u^{\frac{1}{2}} + C$

$= \frac{2}{3}(x+2)^{\frac{3}{2}} - 6\sqrt{x+2} + C$

11c Let $x = \frac{1}{2}u^2 - \frac{1}{2}$, $dx = u\,du$

Note that $2x + 1 = u^2$

$\int x\sqrt{2x+1}\,dx$

$= \int \left(\frac{1}{2}u^2 - \frac{1}{2}\right)\left(\sqrt{u^2}\right) u\,du$

$= \frac{1}{2}\int (u^4 - u^2)\,du$

$$= \frac{1}{2}\left(\frac{u^5}{5} - \frac{u^3}{3}\right) + C$$

$$= \frac{1}{2}\left(\frac{(2x+1)^{\frac{5}{2}}}{5} - \frac{(2x+1)^{\frac{3}{2}}}{3}\right) + C$$

$$= \frac{1}{10}(2x+1)^{\frac{5}{2}} - \frac{1}{6}(2x+1)^{\frac{3}{2}} + C$$

11d Let $x = (u-4)^2$, $dx = 2(u-4)\,du$

$$\int \frac{1}{4+\sqrt{x}}\,dx$$

$$= \int \frac{2(u-4)}{4+u-4}\,du$$

$$= \int \left(2 - \frac{8}{u}\right) du$$

$$= 2u - 8\ln|u| + C$$

$$= 2(\sqrt{x}+4) - 8\ln(\sqrt{x}+4) + C$$

12a Let $x = u + 1$, $dx = du$

$$\int_1^2 x(x-1)^4\,dx$$

$$= \int_0^1 (u+1)u^4\,du$$

$$= \int_0^1 u^5 + u^4\,du$$

$$= \left[\frac{u^6}{6} + \frac{u^5}{5}\right]_0^1$$

$$= \frac{1}{6} + \frac{1}{5}$$

$$= \frac{11}{30}$$

12b Let $x = u - 3$, $dx = du$

$$\int_1^5 \frac{x}{x+3}\,dx$$

$$= \int_4^8 \frac{u-3}{u}\,du$$

$$= \int_4^8 \left(1 - \frac{3}{u}\right) du$$

$$= [u - 3\ln|u|]_4^8$$

$$= (8 - 3\ln 8) - (4 - 3\ln 4)$$

$$= 4 - 3\ln 2$$

12c Let $x = u^2 - 1$, $dx = 2u\,du$

$$\int_0^{15} \frac{x}{\sqrt{x+1}}\,dx$$

$$= \int_1^4 \frac{u^2 - 1}{\sqrt{u^2}} \times 2u\,du$$

$$= \int_1^4 (2u^2 - 2)\,du$$

$$= \left[\frac{2}{3}u^3 - 2u\right]_1^4$$

$$= 36$$

12d Let $x = u^2 + 2$, $dx = 2u\,du$

$$\int_2^3 \frac{1}{2}x\sqrt{x-2}\,dx$$

$$= \int_0^1 \frac{1}{2}(u^2+2)\sqrt{u^2} \times 2u\,du$$

$$= \int_0^1 (u^4 + 2u^2)\,du$$

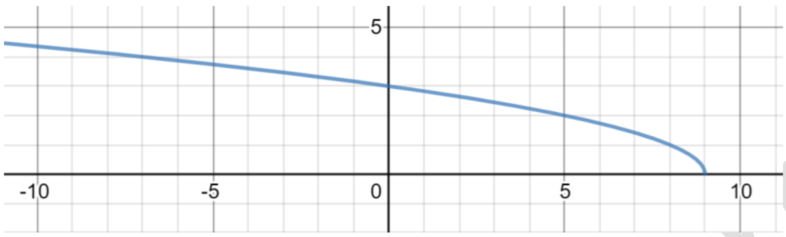
$$= \left[\frac{u^5}{5} + \frac{2}{3}u^3\right]_0^1$$

$$= \frac{1}{5} + \frac{2}{3}$$

$$= \frac{13}{15}$$

13a The domain is all x such that $9 - x \geq 0$ and hence the domain is $x \leq 9$. As the square root function always returns a positive number, the range is $y \geq 0$.

13b



13c

$$A = \int_0^9 \sqrt{9-x}\,dx$$

$$= \int_0^9 (9-x)^{\frac{1}{2}}\,dx$$

$$= -\frac{4}{3}\left[(9-x)^{\frac{3}{2}}\right]_0^9$$

$$= 18 \text{ square units}$$

13d i

$$V = \pi\int_0^9 y^2\,dx$$

$$= \pi\int_0^9 (9-x)\,dx$$

$$= \pi\left[9x - \frac{x^2}{2}\right]_0^9$$

$$= \frac{81\pi}{2} \text{ cubic units}$$

13d ii

$$V = \pi \int_0^3 x^2 \, dy$$

$$= \pi \int_0^3 (y^2 - 9)^2 \, dy$$

$$= \pi \int_0^3 (y^4 - 18y^2 + 81) \, dy$$

$$= \pi \left[\frac{y^5}{5} - 6y^3 + 81y\right]_0^3$$

$$= \frac{648\pi}{5} \text{ cubic units}$$

14

$$V = \int_0^{3\frac{3}{4}} \pi y^2 \, dy$$

$$= \int_0^{3\frac{3}{4}} \pi \left(\frac{1}{\sqrt{4-x}}\right)^2 dx$$

$$= \int_0^{3\frac{3}{4}} \frac{\pi}{4-x} dx$$

$$= -\pi[\ln|4-x|]_0^{3\frac{3}{4}}$$

$$= 4\pi \ln 2 \text{ cubic units}$$

15 $y^2 = 18(x-6)$

$$x = \frac{y^2}{18} + 6$$

$$V = \pi \int_{-6}^6 x^2 \, dy$$

$$= \pi \int_{-6}^6 \left(\frac{y^2}{18} + 6\right)^2 dy$$

$$= \pi \int_{-6}^6 \left(\frac{y^4}{18^2} + \frac{12}{18}y^2 + 36\right) dy$$

$$= \pi \left[\frac{y^5}{5 \times 18^2} + \frac{12}{18 \times 3} y^3 + 36y\right]_{-6}^{6}$$
$$= \frac{2688\pi}{5} \text{ cubic units}$$

16 $\cos^2 2x = \frac{1}{2} + \frac{1}{2}\cos 4x$

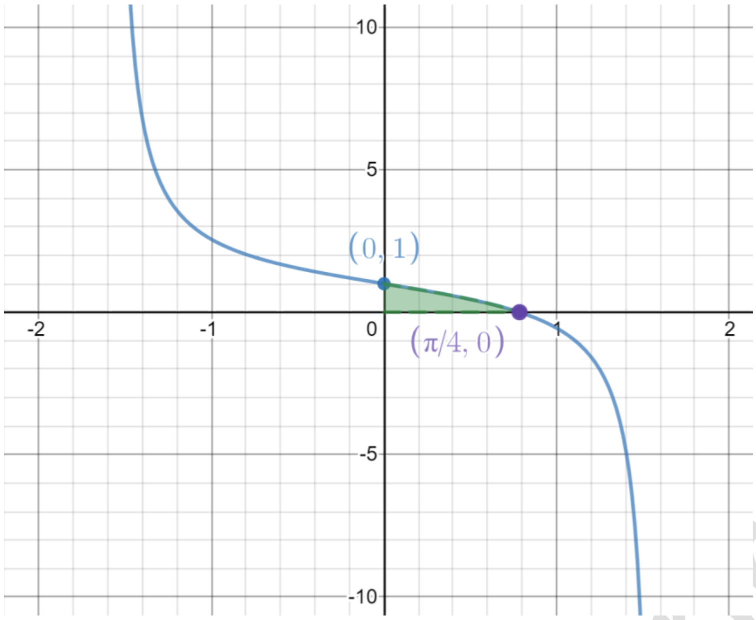
16b

$$V = \pi \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} y^2 \, dx$$
$$= \pi \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \cos^2 2x \, dx$$
$$= \pi \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \left(\frac{1}{2} + \frac{1}{2}\cos 4x\right) dx$$
$$= \pi \left[\frac{1}{2}x + \frac{1}{8}\sin 4x\right]_{-\frac{\pi}{6}}^{\frac{\pi}{6}}$$
$$= \frac{\pi}{24}\left(4\pi + 3\sqrt{3}\right) \text{ cubic units}$$

17

$$V = \pi \int_1^3 y^2 \, dx$$
$$= \pi \int_1^3 (1 + e^{-x})^2 \, dx$$
$$= \pi \int_1^3 (1 + 2e^{-x} + e^{-2x}) \, dx$$
$$= \pi \left[x - 2e^{-x} - \frac{1}{2}e^{-2x}\right]_1^3$$
$$= \pi\left(\left(3 - 2e^{-3} - \frac{1}{2}e^{-6}\right) - \left(1 - 2e^{-1} - \frac{1}{2}e^{-2}\right)\right)$$
$$\doteqdot 8.49 \text{ cubic units}$$

18a



18b

$$V = \pi\int_0^{\frac{\pi}{4}} y^2\,dx$$

$$= \pi\int_0^{\frac{\pi}{4}} (1 - \tan x)^2\,dx$$

$$= \pi\int_0^{\frac{\pi}{4}} (1 - 2\tan x + \tan^2 x)\,dx$$

$$= \pi\int_0^{\frac{\pi}{4}} (1 - 2\tan x + \sec^2 x - 1)\,dx$$

$$= \pi\int_0^{\frac{\pi}{4}} \left(-\frac{2\sin x}{\cos x} + \sec^2 x\right)dx$$

$$= \pi[2\ln|\cos x| + \tan x]_0^{\frac{\pi}{4}}$$

$$= \pi\left(\left(2\ln\left(2^{-\frac{1}{2}}\right) + 1\right) - (-2\ln 1 + 0)\right)$$

$$= \pi\big((-\ln 2 + 1) - (0 + 0)\big)$$

$$= \pi(1 - \ln 2) \text{ cubic units}$$

19a

$$\frac{dy}{dx} = 1 - \frac{1}{x^2}$$

$$\frac{d^2y}{dx^2} = \frac{2}{x^3}$$

Stationary points occur when $\frac{dy}{dx} = 0$

$$1 - \frac{1}{x^2} = 0$$

$$1 = \frac{1}{x^2}$$

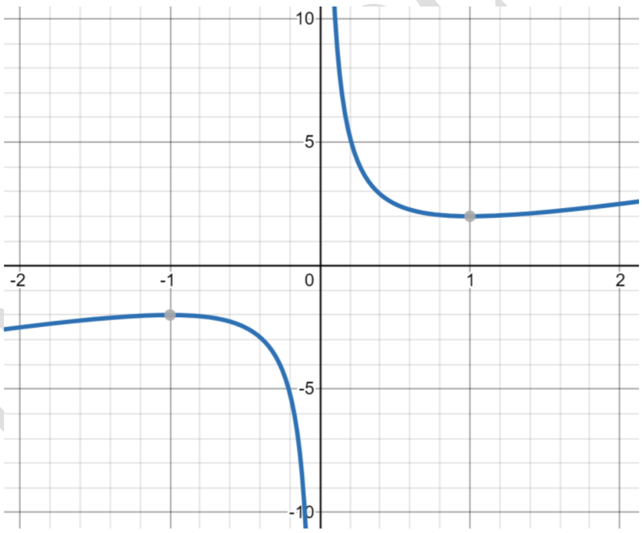
$$x^2 = 1$$

$$x = \pm 1$$

When $x = -1$, $y = -2$, $\frac{d^2y}{dx^2} = -2 < 0$, hence the curve is concave down and this is a maximum.

When $x = 1$, $y = 2$, $\frac{d^2y}{dx^2} = 2 > 0$, hence the curve is concave up and this is a minimum.

Hence there is a minimum at $(1, 2)$ and a maximum at $(-1, -2)$.



19b

$$\frac{5}{2} = x + \frac{1}{x}$$

$$5 = 2x + \frac{2}{x}$$

$$5x = 2x^2 + 2$$

$$2x^2 - 5x + 2 = 0$$

$$(2x - 1)(x - 2) = 0$$

Hence

$x = \frac{1}{2}$ or 2

19c

$$\begin{aligned} V &= \pi \int_{\frac{1}{2}}^{2} \left(\frac{5}{2}\right)^2 dx - \pi \int_{\frac{1}{2}}^{2} \left(x + \frac{1}{x}\right)^2 dx \\ &= \pi \int_{\frac{1}{2}}^{2} \left(\frac{25}{4} - x^2 - 2 - \frac{1}{x^2}\right) dx \\ &= \pi \int_{\frac{1}{2}}^{2} \left(\frac{17}{4} - x^2 - \frac{1}{x^2}\right) dx \\ &= \pi \left[\frac{17}{4}x - \frac{x^3}{3} + \frac{1}{x}\right]_{\frac{1}{2}}^{2} \\ &= \pi \left(\left(\frac{17}{2} - \frac{8}{3} + \frac{1}{2}\right) - \left(\frac{17}{8} - \frac{1}{24} + 2\right)\right) \\ &= \frac{9\pi}{4} \text{ cubic units} \end{aligned}$$

20a <To come>

21a

$$V = \pi \int_1^3 y^2 \, dx$$

$$= \pi \int_1^3 (2^{x+1})^2 \, dx$$

$$= \pi \int_1^3 2^{2x+2} \, dx$$

21b

$$V \doteqdot \pi \times \frac{3-1}{2(4)} \big(f(1) + f(3) + 2\big(f(1.5) + f(2) + f(2.5)\big)\big)$$
$$= \frac{\pi}{4}(2^2 + 2^8 + 2(2^5 + 2^6 + 2^7)$$
$$= 180\pi \text{ cubic units}$$

21c

$$\begin{aligned}
V &= \pi \int_1^3 y^2 \, dx \\
&= \pi \int_1^3 (2^{x+1})^2 \, dx \\
&= \pi \int_1^3 2^{2x+2} \, dx \\
&= \pi \int_1^3 e^{\ln 2^{2x+2}} \, dx \\
&= \pi \int_1^3 e^{(2x+2)\ln 2} \, dx \\
&= \frac{\pi}{2\ln 2}\left[e^{(2x+2)\ln 2}\right]_1^3 \\
&= \frac{120\pi}{\ln 2} \\
&\doteqdot 173\pi \text{ cubic units}
\end{aligned}$$

The exact answer is smaller than the trapezoidal-rule approximation because the curve is concave up.