Solutions to Exercise 13A

Let the integration constants be A, B, C or D.

1a $y' - y = x$

first-order differential equation

1b $y'y = 3x$

first-order differential equation

1c $y'' + 4y' - y = \sin x$

second-order differential equation

1d $y' + y\cos x = e^x$

first-order differential equation

1e $y'' - \frac{1}{2}(y')^2 = 0$

second-order differential equation

1f $y' + y^2 = 1$

first-order differential equation

1g $y' + xy = 0$

first-order differential equation

1h $xy'' + y' = x^2$

second-order differential equation

1i $y'' - xy' + e^x y = 0$

second-order differential equation

2a linear

2b non-linear

2d linear

2f linear

2g linear

3a one arbitrary constant

3b one arbitrary constant

3c two arbitrary constants

3d one arbitrary constant

3e two arbitrary constants

3f one arbitrary constant

3g one arbitrary constant

3h two arbitrary constants

3i two arbitrary constants

4a $y = 5x^3$

$y' = 15x^2$

Substituting for y and y' in $xy' - 3y = 0$:

$\text{LHS} = x(15x^2) - 3(5x^3)$

$= 15x^3 - 15x^3$

$= 0$

$= \text{RHS}$

Therefore, $y = 5x^3$ is a solution of $xy' - 3y = 0$.

4b $y = x^2 - 1$

$y' = 2x$

Substituting for y and y' in $xy' - 2y = 2$:

$\text{LHS} = x(2x) - 2(x^2 - 1)$

$= 2x^2 - 2x^2 + 2$

$= 2$

$= \text{RHS}$

Therefore, $y = x^2 - 1$ is a solution of $xy' - 2y = 2$.

4c $y = 3e^{-x}$

$y' = -3e^{-x}$

Substituting for y and y' in $y' + y = 0$:

$\text{LHS} = -3e^{-x} + 3e^{-x}$

$= 0$

$= \text{RHS}$

Therefore, $y = 3e^{-x}$ is a solution of $y' + y = 0$.

4d $y = \sqrt{x^2 + 4}$

$= (x^2 + 4)^{\frac{1}{2}}$

$y' = \frac{1}{2}(x^2 + 4)^{-\frac{1}{2}} \times 2x$

$= x(x^2 + 4)^{-\frac{1}{2}}$

Substituting for y and y' in $y'y = x$:

$\text{LHS} = x(x^2 + 4)^{-\frac{1}{2}} \times (x^2 + 4)^{\frac{1}{2}}$

$= x(x^2 + 4)^0$

$= x \times 1$

$= x$

$= \text{RHS}$

Therefore $y = \sqrt{x^2 + 4}$ is a solution of $y'y = x$.

5a $y = \int (2x - 3)dx$

$= \frac{2x^2}{2} - 3x + C$

$= x^2 - 3x + C$

5b $y = \int (12e^{-2x} + 4)dx$

$= -\frac{12}{2}e^{-2x} + 4x + C$

$= -6e^{-2x} + 4x + C$

5c $y = \int (\sec^2 x)dx$

$= \tan x + C$

5d $y = \int (6\cos 2x + 9\sin 3x)\,dx$

$= \frac{6}{2}\sin 2x - \frac{9}{3}\cos 3x + C$

$= 3\sin 2x - 3\cos 3x + C$

5e $y = \int \sqrt{1-5x}\ dx$

$= \int (1-5x)^{\frac{1}{2}}\ dx$

$= \dfrac{(1-5x)^{\frac{3}{2}}}{(-5)\times\left(\frac{3}{2}\right)} + C$

$= \dfrac{(1-5x)^{\frac{3}{2}}}{\left(-\frac{15}{2}\right)} + C$

$= -\dfrac{2(1-5x)^{\frac{3}{2}}}{15} + C$

5f $y = \int 4x\cos x^2\ dx$

Let $u = x^2$, $\frac{du}{dx} = 2x$ so $\frac{du}{2x} = dx$

$y = \int 4x\cos u\ \frac{du}{2x}$

$= \int 2\cos u\ du$

$= 2\int \cos u\ du$

$= 2\sin u + C$

$= 2\sin x^2 + C$

6a $y = Ce^x - x - 1$

$y' = Ce^x - 1$

Substituting for y and y' in $y' = x + y$:

$\text{LHS} = Ce^x - 1$

$\text{RHS} = x + Ce^x - x - 1$

$= Ce^x - 1$

$= \text{LHS}$

Therefore $y = Ce^x - x - 1$ is a solution of $y' = x + y$.

6b $y = Cxe^{-x}$

$y' = C(e^{-x} \times 1 + x \times -e^{-x})$

$y' = C(e^{-x} - xe^{-x})$

Substituting for y and y' in $xy' = y(1-x)$:

$\text{LHS} = x(C(e^{-x} - xe^{-x}))$

$= Cx(e^{-x} - xe^{-x})$

$= Cxe^{-x}(1-x)$

$= y(1-x)$

$= \text{RHS}$

Therefore $y = Cxe^{-x}$ is a solution of $xy' = y(1-x)$.

6c $y = \sin\ (x + C)$

$y' = \cos\ (x + C)$

Substituting for y and y' in $(y')^2 = 1 - y^2$:

$\text{LHS} = (\cos\ (x + C))^2$

$= \cos^2\ (x + C)$

$\text{RHS} = 1 - \sin^2\ (x + C)$

$= \cos^2\ (x + C)$

$= \text{LHS}$

Therefore $y = \sin\ (x + C)$ is a solution of $(y')^2 = 1 - y^2$.

6d

$$y = \frac{C}{x} + 2$$

$$= Cx^{-1} + 2$$

$$\frac{dy}{dx} = -Cx^{-2}$$

$$= -\frac{C}{x^2}$$

Substituting for y and $\frac{dy}{dx}$ in $\frac{dy}{dx} = \frac{2-y}{x}$:

$$\text{RHS} = \frac{2-\left(\frac{C}{x}+2\right)}{x}$$

$$= \frac{-\frac{C}{x}}{x}$$

$$= -\frac{C}{x^2}$$

$$= \frac{dy}{dx}$$

$$= \text{LHS}$$

Therefore $y = \frac{C}{x} + 2$ is a solution of $\frac{dy}{dx} = \frac{2-y}{x}$.

7a $y = x^2 - 2x + 3$

$y' = 2x - 2$

$y'' = 2$

Substituting for y, y' and y'' in $x^2y'' - 2xy' + 2y = 6$:

$\text{LHS} = x^2(2) - 2x(2x-2) + 2(x^2 - 2x + 3)$

$= 2x^2 - 4x^2 + 4x + 2x^2 - 4x + 6$

$= 6$

$= \text{RHS}$

Therefore $y = x^2 - 2x + 3$ is a solution of $x^2y'' - 2xy' + 2y = 6$.

7b $y = 2e^x + e^{5x}$

$y' = 2e^x + 5e^{5x}$

$y'' = 2e^x + 25e^{5x}$

Substituting for y, y' and y'' in $y'' - 6y' + 5y = 0$:

$\text{LHS} = 2e^x + 25e^{5x} - 6(2e^x + 5e^{5x}) + 5(2e^x + e^{5x})$

$= 2e^x + 25e^{5x} - 12e^x - 30e^{5x} + 10e^x + 5e^{5x}$

$= 0$

$= \text{RHS}$

Therefore $y = 2e^{x} + e^{5x}$ is a solution of $y'' - 6y' + 5y = 0$.

7c $y = \cos \pi x - 3 \sin \pi x$

$y' = -\pi \sin \pi x - 3\pi \cos \pi x$

$y'' = -\pi^2 \cos \pi x + 3\pi^2 \sin \pi x$

Substituting for y and y'' in $y'' + \pi^2 y = 0$:

LHS $= -\pi^2 \cos \pi x + 3\pi^2 \sin \pi x + \pi^2(\cos \pi x - 3 \sin \pi x)$

$= -\pi^2 \cos \pi x + 3\pi^2 \sin \pi x + \pi^2 \cos \pi x - 3\pi^2 \sin \pi x$

$= 0$

$=$ RHS

Therefore $y = \cos \pi x - 3 \sin \pi x$ is a solution of $y'' + \pi^2 y = 0$.

7d $y = e^{-2x} \sin x$

$y' = (\sin x)(-2e^{-2x}) + (e^{-2x})(\cos x)$

$= -2e^{-2x} \sin x + e^{-2x} \cos x$

$y'' = (\sin x)(4e^{-2x}) + (-2e^{-2x})(\cos x) + (\cos x)(-2e^{-2x}) + (e^{-2x})(-\sin x)$

$= 4e^{-2x} \sin x - 2e^{-2x} \cos x - 2e^{-2x} \cos x - e^{-2x} \sin x$

$= 3e^{-2x} \sin x - 4e^{-2x} \cos x$

Substituting for y, y' and y'' in $y'' + 4y' + 5y = 0$:

LHS $= 3e^{-2x} \sin x - 4e^{-2x} \cos x + 4(-2e^{-2x} \sin x + e^{-2x} \cos x) + 5(e^{-2x} \sin x)$

$= 3e^{-2x} \sin x - 4e^{-2x} \cos x - 8e^{-2x} \sin x + 4e^{-2x} \cos x + 5e^{-2x} \sin x$

$= 0$

$=$ RHS

Therefore $y = e^{-2x} \sin x$ is a solution of $y'' + 4y' + 5y = 0$.

7e $y = \cos(\log_e x)$

$$y' = -\sin(\log_e x) \times \frac{1}{x}$$

$$= -\frac{\sin(\log_e x)}{x}$$

$$y'' = -\frac{x \times (\cos(\log_e x) \times \frac{1}{x} - \sin(\log_e x)(1)}{x^2}$$

$$= -\frac{\cos(\log_e x) - \sin(\log_e x)}{x^2}$$

Substituting for y, y' and y'' in $x^2y'' + xy' + y = 0$:

$$\text{LHS} = x^2\left(-\frac{\cos(\log_e x) - \sin(\log_e x)}{x^2}\right) + x\left(-\frac{\sin(\log_e x)}{x}\right) + \cos(\log_e x)$$

$$= -\cos(\log_e x) + \sin(\log_e x) - \sin(\log_e x) + \cos(\log_e x)$$

$$= 0$$

$$= \text{RHS}$$

Therefore $y = \cos(\log_e x)$ is a solution of $x^2y'' + xy' + y = 0$.

8a $y'' = 2$

$$y' = \int 2\,dx = 2x + A$$

$$y = \int (2x + A)dx = \frac{2x^2}{2} + Ax + B = x^2 + Ax + B$$

8b $y'' = \cos 2x$

$$y' = \int \cos 2x\ dx = \frac{1}{2}\sin 2x + A$$

$$y = \int \left(\frac{1}{2}\sin 2x + A\right)dx = -\frac{1}{4}\cos 2x + Ax + B$$

8c $y'' = e^{\frac{1}{2}x}$

$$y' = \int e^{\frac{1}{2}x}\,dx = 2e^{\frac{1}{2}x} + A$$

$$y = \int \left(2e^{\frac{1}{2}x} + A\right)dx = 4e^{\frac{1}{2}x} + Ax + B$$

8d $y'' = \sec^2 x$

$y' = \int \sec^2 x \, dx = \tan x + A$

$y = \int (\tan x + A) \, dx$

$y = \int \left(\frac{\sin x}{\cos x} + A\right) \, dx$

Using substitution:

Let $u = \cos x$

$\frac{du}{dx} = -\sin x$

$dx = -\frac{du}{\sin x}$

$y = \int \left(\frac{\sin x}{\cos x} + A\right) \, dx$

$= \int \frac{\sin x}{\cos x} \, dx + \int A \, dx$

$= \int \frac{\sin x}{u}\left(-\frac{du}{\sin x}\right) + Ax + B$

$= -\int \frac{1}{u} \, du + Ax + B$

$= -\log_e |u| + Ax + B$

$= -\log_e |\cos x| + Ax + B$

9a For $y = e^{-x}$

$y' = -e^{-x}$

$y'' = e^{-x}$

Substituting y, y' and y'' into $y'' - 2y' - 3y = 0$:

$\text{LHS} = e^{-x} - 2(-e^{-x}) - 3e^{-x}$

$= e^{-x} + 2e^{-x} - 3e^{-x}$

$= 0$

$= \text{RHS}$

Therefore $y = e^{-x}$ is a solution of $y'' - 2y' - 3y = 0$.

For $y = e^{3x}$

$y' = 3e^{3x}$

$y'' = 9e^{3x}$

Substituting y, y' and y'' into $y'' - 2y' - 3y = 0$:

$\text{LHS} = 9e^{3x} - 2(3e^{3x}) - 3e^{3x}$

$= 9e^{3x} - 6e^{3x} - 3e^{3x}$

$= 0$

$= \text{RHS}$

Therefore $y = e^{3x}$ is a solution of $y'' - 2y' - 3y = 0$.

9b $y = Ae^{-x} + Be^{3x}$

$y' = -Ae^{-x} + 3Be^{3x}$

$y'' = Ae^{-x} + 9Be^{3x}$

Substituting y, y' and y'' into $y'' - 2y' - 3y = 0$

$\text{LHS} = Ae^{-x} + 9Be^{3x} - 2(-Ae^{-x} + 3Be^{3x}) - 3(Ae^{-x} + Be^{3x})$

$= Ae^{-x} + 9Be^{3x} + 2Ae^{-x} - 6Be^{3x} - 3Ae^{-x} - 3Be^{3x}$

$= 0$

$= \text{RHS}$

Therefore $y = Ae^{-x} + Be^{3x}$ is a solution of $y'' - 2y' - 3y = 0$.

10a $y = x^3 + Ax^2 + Bx + C$

$y' = 3x^2 + 2Ax + B$

$y'' = 6x + 2A$

$y''' = 6$

Substituting y''' into $y''' = 6$:

$\text{LHS} = 6$

$= \text{RHS}$

Therefore $y = x^3 + Ax^2 + Bx + C$ is a solution of $y''' = 6$.

10b $y = Ae^{-x} + Be^{-2x} + 2x - 3$

$y' = -Ae^{-x} - 2Be^{-2x} + 2$

$y'' = Ae^{-x} + 4Be^{-2x}$

Substituting y, y' and y'' into $y'' + 3y' + 2y = 4x$:

$\text{LHS} = Ae^{-x} + 4Be^{-2x} + 3(-Ae^{-x} - 2Be^{-2x} + 2) + 2(Ae^{-x} + Be^{-2x} + 2x - 3)$

$= Ae^{-x} + 4Be^{-2x} - 3Ae^{-x} - 6Be^{-2x} + 6 + 2Ae^{-x} + 2Be^{-2x} + 4x - 6$

$= (Ae^{-x} - 3Ae^{-x} + 2Ae^{-x}) + (4Be^{-2x} - 6Be^{-2x} + 2Be^{-2x}) + 6 + 4x - 6$

$= 4x$

$= \text{RHS}$

Therefore $y = Ae^{-x} + Be^{-2x} + 2x - 3$ is a solution of $y'' + 3y' + 2y = 4x$.

10c $y = A\cos 2x + B\sin 2x$

$y' = -2A\sin 2x + 2B\cos 2x$

$y'' = -4A\cos 2x - 4B\sin 2x$

Substituting y and y'' into $y'' + 4y = 0$:

$\text{LHS} = -4A\cos 2x - 4B\sin 2x + 4(A\cos 2x + B\sin 2x)$

$= -4A\cos 2x - 4B\sin 2x + 4A\cos 2x + 4B\sin 2x$

$= 0$

$= \text{RHS}$

Therefore $y = A\cos 2x + B\sin 2x$ is a solution of $y'' + 4y = 0$.

10d $y = A\,e^{-x}\cos x$

$y' = A(\cos x \times (-e^{-x}) + e^{-x} \times (-\sin x))$

$= A(-e^{-x}\cos x - e^{-x}\sin x)$

$= -Ae^{-x}\cos x - Ae^{-x}\sin x$

$y'' = \text{A}\,((e^{-x}\cos x + e^{-x}\sin x) + e^{-x}\sin x - e^{-x}\cos x)$

$= \text{A}\,e^{-x}\cos x + Ae^{-x}\sin x + Ae^{-x}\sin x - A\,e^{-x}\cos x$

$= 2Ae^{-x}\sin 2x$

Substituting y, y' and y'' into $y'' + 2y' + 2y = 0$:

$\text{LHS} = 2Ae^{-x}\sin 2x + 2(-Ae^{-x}\cos x - Ae^{-x}\sin x) + 2A\,e^{-x}\cos x$

$= 2Ae^{-x}\sin 2x - 2Ae^{-x}\cos x - 2\,Ae^{-x}\sin x + 2A\,e^{-x}\cos x$

$= 0$

$= \text{RHS}$

Therefore $y = A\,e^{-x}\cos x$ is a solution of $y'' + 2y' + 2y = 0$.

10e $y = Ae^{-\frac{1}{2}x^2}$

$y' = A\left(-\frac{1}{2} \times 2xe^{-\frac{1}{2}x^2}\right)$

$= -Axe^{-\frac{1}{2}x^2}$

$y'' = -A\left(e^{-\frac{1}{2}x^2} \times 1 + x \times -\frac{1}{2} \times 2xe^{-\frac{1}{2}x^2}\right)$

$= -Ae^{-\frac{1}{2}x^2} + Ax^2e^{-\frac{1}{2}x^2}$

Substituting y and y'' into $y'' = y(x^2 - 1)$:

$\text{LHS} = -Ae^{-\frac{1}{2}x^2} + Ax^2e^{-\frac{1}{2}x^2}$

$\text{RHS} = Ae^{-\frac{1}{2}x^2}(x^2 - 1)$

$= Ax^2e^{-\frac{1}{2}x^2} - Ae^{-\frac{1}{2}x^2}$

$= -Ae^{-\frac{1}{2}x^2} + Ax^2e^{-\frac{1}{2}x^2}$

$= \text{LHS}$

Therefore $y = A\,e^{-x}\cos x$ is a solution of $y'' = y(x^2 - 1)$.

11a $y' = 1$

$y = x + A$

Substituting $(2, 1)$:

$1 = 2 + \text{A}$

$A = -1$

Therefore $y = x - 1$

11b $y' = 2x - 3$

$y = \frac{2x^2}{2} - 3x + A$

$y = x^2 - 3x + A$

Substituting $(0, 2)$:

$2 = (0)^2 - 3(0) + A$

$A = 2$

Therefore $y = x^2 - 3x + 2$

11c $y' = 3x^2 + 6x - 9$

$y = \frac{3x^3}{3} + \frac{6x^2}{2} - 9x + A$

$y = x^3 + 3x^2 - 9x + A$

Substituting $(1, 2)$:

$2 = (1)^3 + 3(1)^3 - 9(1) + A$

$2 = 1 + 3 - 9 + A$

$2 = -5 + A$

$A = 7$

Therefore $y = x^3 + 3x^2 - 9x + 7$

11d $y' = \sin x$

$y = -\cos x + A$

Substituting $(\pi, 3)$:

$3 = -\cos \pi + A$

$3 = 1 + A$

$A = 2$

Therefore $y = -\cos x + 2$

$y = 2 - \cos x$

11e $y' = 6e^{2x}$

$y = 3e^{2x} + A$

Substituting (0, 0):

$0 = 3e^0 + A$

$0 = 3 + A$

$A = -3$

Therefore $y = 3e^{2x} - 3$

$y = 3(e^{2x} - 1)$

11f $y' = 3\sqrt{x} - 2$

$= 3x^{\frac{1}{2}} - 2$

$y = \frac{3}{\frac{3}{2}} \times x^{\frac{3}{2}} - 2x + A$

$= 2x^{\frac{3}{2}} - 2x + A$

Substituting (4, 7):

$7 = 2(4)^{\frac{3}{2}} - 2(4) + A$

$7 = 2(8) - 8 + A$

$7 = 8 + A$

$A = -1$

Therefore $y = 2x^{\frac{3}{2}} - 2x - 1$

$y = 2x\sqrt{x} - 2x - 1$

12a i

$$\text{RHS} = \frac{1}{x} + \frac{1}{1-x}$$

$$= \frac{1(1-x) + 1(x)}{x(1-x)}$$

$$= \frac{1 - x + x}{x(1-x)}$$

$$= \frac{1}{x(1-x)}$$

$$= \text{LHS}$$

Hence:

$$\frac{1}{x(1-x)} = \frac{1}{x} + \frac{1}{1-x}$$

Alternatively, using a partial fractions approach:

$$\frac{1}{x(1-x)} = \frac{A}{x} + \frac{B}{1-x}$$

$$= \frac{A(1-x) + Bx}{x(1-x)}$$

$$= \frac{A - Ax + Bx}{x(1-x)}$$

$$= \frac{A - x(A-B)}{x(1-x)}$$

Equating coefficients in the numerators:

$1 + 0x = A - (A - B)x$

$A = 1 \qquad (1)$

$A - B = 0 \qquad (2)$

Substituting (1) into (2):

$1 - B = 0$

$B = 1$

Therefore:

$$\frac{1}{x(1-x)} = \frac{1}{x} + \frac{1}{1-x}$$

12a ii

$$y = \int \frac{1}{x(1-x)}\,dx$$

$$= \int \left(\frac{1}{x} + \frac{1}{1-x}\right) dx$$

$$= \log_e|x| + (-\log_e|1-x|) + C$$

$$= \log_e|x| - \log_e|1-x| + C$$

$$= \log_e\left|\frac{x}{1-x}\right| + C \qquad (3)$$

12a iii Substituting $y\left(\frac{1}{2}\right) = 0$ into (3):

$$0 = \log_e\left|\frac{\frac{1}{2}}{1-\frac{1}{2}}\right| + C$$

$$0 = \log_e|1| + C$$

$$0 = 0 + C$$

$$C = 0$$

Therefore:

$$y = \log_e\left|\frac{x}{1-x}\right|$$

12b i

$$\text{RHS} = \frac{1}{2-x} + \frac{1}{2+x}$$

$$= \frac{1(2+x) + 1(2-x)}{(2-x)(2+x)}$$

$$= \frac{2+x+2-x}{(2-x)(2+x)}$$

$$= \frac{4}{(2-x)(2+x)}$$

$$= \text{LHS}$$

Hence:

$$\frac{4}{(2-x)(2+x)} = \frac{1}{2-x} + \frac{1}{2+x}$$

Alternatively, using a partial fractions approach:

$$\begin{aligned}\frac{4}{(2-x)(2+x)} &= \frac{A}{2-x} + \frac{B}{2+x}\\ &= \frac{A(2+x)+B(2-x)}{(2+x)(2-x)}\\ &= \frac{2A+Ax+2B-Bx}{(2+x)(2-x)}\\ &= \frac{x(A-B)+2A+2B}{(2+x)(2-x)}\end{aligned}$$

Equating coefficients in the numerators:

$0x + 4 = x(A-B) + 2A + 2B$

$A - B = 0 \qquad (1)$

$2A + 2B = 4 \qquad (2)$

From (1), $A = B$.

Substituting $A = B$ into (2):

$2B + 2B = 4$

$4B = 4$

$B = 1$

Hence $A = 1$.

Therefore:

$$\frac{4}{(2-x)(2+x)} = \frac{1}{2-x} + \frac{1}{2+x}$$

12b ii

$$y = \int \frac{4}{(2-x)(2+x)}\,dx$$

$$= \int \left(\frac{1}{2-x} + \frac{1}{2+x}\right) dx$$

$$= -\log_e|2-x| + \log_e|2+x| \ + C$$

$$= \log_e|2+x| - \log_e|2-x| + C$$

$$= \log_e\left|\frac{2+x}{2-x}\right| + C \qquad (3)$$

12b iii Substituting $y(0) = 1$ into (3):

$$1 = \log_e\left|\frac{2+0}{2-0}\right| + C$$

$$1 = \log_e|1| + C$$

$$1 = 0 + C$$

$$1 = C$$

Hence:

$$y = \log_e\left|\frac{2+x}{2-x}\right| + 1$$

13a i

$$\frac{d(x^2)}{dx} + \frac{d(y^2)}{dx} = \frac{d(9)}{dx}$$

$$2x + \frac{d(y^2)}{dy} \times \frac{dy}{dx} = 0$$

$$2x + 2y\frac{dy}{dx} = 0$$

(or $2x + 2yy' = 0$)

13a ii

From $2x + 2y\dfrac{dy}{dx} = 0$

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = -\frac{2x}{2y}$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

13a iii y' is undefined when $y = 0$. That is, at $(3, 0)$ and $(-3, 0)$ where the tangent to the circle is vertical.

13b i $y^2 = x + 4$

$$\frac{d(y^2)}{dx} = \frac{d(x)}{dx} + \frac{d(4)}{dx}$$

$$2y\frac{dy}{dx} = 1 + 0$$

$$\frac{dy}{dx} = \frac{1}{2y}$$

13b ii $xy = c^2$

$$y\frac{d(x)}{dx} + x\frac{d(y)}{dx} = \frac{d(c^2)}{dx}$$

$$y(1) + x\frac{dy}{dx} = 0$$

$$x\frac{dy}{dx} = -y$$

$$\frac{dy}{dx} = -\frac{y}{x}$$

13b iii $9x^2 + 16y^2 = 144$

$$\frac{d(9x^2)}{dx} + \frac{d(16y^2)}{dx} = \frac{d(144)}{dx}$$

$$18x + 32y\frac{dy}{dx} = 0$$

$$32y\frac{dy}{dx} = -18x$$

$$\frac{dy}{dx} = -\frac{18x}{32y}$$

$$\frac{dy}{dx} = -\frac{9x}{16y}$$

13b iv $x^2 - 4y^2 = 4$

$$\frac{d(x^2)}{dx} - \frac{d(4y^2)}{dx} = \frac{d(4)}{dx}$$

$$2x - 8y\frac{dy}{dx} = 0$$

$$8y\frac{dy}{dx} = 2x$$

$$\frac{dy}{dx} = \frac{2x}{8y}$$

$$\frac{dy}{dx} = \frac{x}{4y}$$

13b v $xy - y^2 = 1$

$$\frac{d(xy)}{dx} - \frac{d(y^2)}{dx} = \frac{d(1)}{dx}$$

$$y\frac{d(x)}{dx} + x\frac{d(y)}{dx} - 2y\frac{dy}{dx} = 0$$

$$y + x\frac{dy}{dx} - 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(x - 2y) = -y$$

$$\frac{dy}{dx} = -\frac{y}{x - 2y}$$

$$\frac{dy}{dx} = \frac{y}{2y - x}$$

13b vi $x^3 + y^3 = 3xy$

$$\frac{d(x^3)}{dx} + \frac{d(y^3)}{dx} = \frac{d(3xy)}{dx}$$

$$3x^2 + 3y^2\frac{dy}{dx} = 3y\frac{d(x)}{dx} + 3x\frac{d(y)}{dx}$$

$$3x^2 + 3y^2\frac{dy}{dx} = 3y + 3x\frac{dy}{dx}$$

$$x^2 + y^2\frac{dy}{dx} = y + x\frac{dy}{dx}$$

$$y^2\frac{dy}{dx} - x\frac{dy}{dx} = y - x^2$$

$$\frac{dy}{dx}(y^2 - x) = y - x^2$$

$$\frac{dy}{dx} = \frac{y - x^2}{y^2 - x}$$

14a $y = \sin x$

$y' = \cos x$

$y'' = -\sin x$

Substituting y and y'' into $y'' + y = 0$:

LHS $= -\sin x + \sin x$

$= 0$

$=$ RHS

Therefore $y = \sin x$ is a solution of $y'' + y' = 0$.

14b $y'' = -\sin x$

$y'' = -y$

When $y = 12$,

$y'' = -12$

14c $f''(x)$ is the negative of $f(x)$ so it is a reflection in the x-axis.

15 $y'' = \sec^2 x$

$y' = \tan x + A$

$y' = \dfrac{\sin x}{\cos x} + A$

Since $y'(0) = 1$,

$1 = \dfrac{\sin 0}{\cos 0} + A$

$1 = 0 + A$

$A = 1$

Therefore $y' = \dfrac{\sin x}{\cos x} + 1$

Let $u = \cos x$

$\dfrac{du}{dx} = -\sin x$

$dx = -\dfrac{du}{\sin x}$

So:

$$y = \int \left(\frac{\sin x}{\cos x} + 1\right) dx$$

$$= \int \frac{\sin x}{\cos x}\, dx + \int 1\, dx$$

$$= \int \frac{\sin x}{u}\left(-\frac{du}{\sin x}\right) + x + B$$

$= -\int \frac{1}{u}(du) + x + B$

$= -\log_e|u| + x + B$

$= -\log_e|\cos x| + x + B$

Since $y(0) = 1$,

$1 = -\log_e|\cos 0| + 0 + B$

$1 = -\log_e 1 + 0 + B$

$1 = B$

Therefore $y = -\log_e|\cos x| + x + 1$

$y = 1 + x - \log_e(\cos x)$

16 $y = \tan x$

$y' = \sec^2 x$

$= (\cos x)^{-2}$

$y'' = -2(\cos x)^{-3}(-\sin x)$

$= \frac{2\sin x}{\cos^3 x}$

$= \frac{2\sin x}{\cos x} \times \frac{1}{\cos^2 x}$

$= 2\tan x \sec^2 x$

First check

Substituting $y\left(\frac{\pi}{4}\right) = 1$ into $y = \tan x$:

$\text{RHS} = \tan\left(\frac{\pi}{4}\right)$

$= 1$

$= \text{LHS}$

Second check

Substituting $y'\left(\frac{\pi}{4}\right) = 2$ into $y' = \sec^2 x$:

$\text{RHS} = \sec^2\left(\frac{\pi}{4}\right)$

$= \dfrac{1}{\cos^2\left(\frac{\pi}{4}\right)}$

$= \dfrac{1}{\left(\frac{1}{\sqrt{2}}\right)^2}$

$= 2$

$= \text{LHS}$

Substituting y, y' and y'' into $y'' = 2yy'$:

$\text{RHS} = 2 \times \tan x \times \sec^2 x$

$= 2\tan x \sec^2 x$

$= \text{LHS}$

Therefore $y = \tan x$ is a solution.

17 $y = Ae^{\lambda x}$

$y' = A\lambda e^{\lambda x}$

$y'' = A\lambda^2 e^{\lambda x}$

17a Substituting y, y' and y'' into $y'' - 4y' + 3y = 0$:

$A\lambda^2 e^{\lambda x} - 4A\lambda e^{\lambda x} + 3\,Ae^{\lambda x} = 0$

$Ae^{\lambda x}(\lambda^2 - 4\lambda + 3) = 0$

$Ae^{\lambda x}(\lambda - 1)(\lambda - 3) = 0$

$(\lambda - 1)(\lambda - 3) = 0 \quad (\text{as } Ae^{\lambda x} \neq 0)$

$\lambda = 1$ or $\lambda = 3$

17b Substituting y, y' and y'' into $y'' + 2y' + y = 0$:

$A\lambda^2 e^{\lambda x} + 2A\lambda e^{\lambda x} + Ae^{\lambda x} = 0$

$Ae^{\lambda x}(\lambda^2 + 2\lambda + 1) = 0$

$Ae^{\lambda x}(\lambda + 1)^2 = 0$

$(\lambda + 1)^2 = 0 \quad (\text{as } Ae^{\lambda x} \neq 0)$

$\lambda = -1$

17c Substituting y, y' and y'' into $y'' - 3y' + 4y = 0$:

$A\lambda^2 e^{\lambda x} - 3A\lambda e^{\lambda x} + 4Ae^{\lambda x} = 0$

$Ae^{\lambda x}(\lambda^2 - 3\lambda + 4) = 0$

$\lambda^2 - 3\lambda + 4 = 0 \quad$ (as $Ae^{\lambda x} \neq 0$)

Using the discriminant for a quadratic to check for any real solutions:

$\Delta = b^2 - 4ac$

$= (-3)^2 - 4(1)(4)$

$= 9 - 16$

$= -7$

Since $\Delta < 0$, no real solutions exist.

18a $y = \sec x$

$= (\cos x)^{-1}$

$y' = -(\cos x)^{-2} \times -\sin x$

$= \dfrac{\sin x}{\cos x} \times \dfrac{1}{\cos x}$

$= \tan x \sec x$

$y' - y \tan x$

$= \tan x \sec x - \sec x \tan x$

$= 0$

Therefore required IVP is $y' - y\tan x = 0, y(0) = 1$

18b $y^2 = \sec^2 x$

$(y')^2 = (\tan x \sec x)^2$

$= \tan^2 x \sec^2 x$

$= \tan^2 x \times y^2$

Now $\tan^2 x + 1 = \sec^2 x$

$\tan^2 x = \sec^2 x - 1$

$\tan^2 x = y^2 - 1$

So $(y')^2 = \tan^2 x \times y^2$

$= (y^2 - 1) \times y^2$

Therefore required IVP is $(y')^2 = y^2(y^2 - 1), y(0) = 1$

19a $y(0) = 1$

19b $y' = -2xy$

Substituting $y(0) = 1$:

$y' = -2(0)(1)$

$= 0$

19c i $y' = -2xy$

$y'' = -2\left(y \times 1 + x \times 1\dfrac{dy}{dx}\right)$

$= -2y - 2xy'$

$= -2y - 2x(-2xy)$

$= -2y + 4x^2y$

$= y(4x^2 - 2)$

or $y'' = (4x^2 - 2)y$

19c ii $y'' = (4x^2 - 2)y$

At $(0, 1)$:

$y'' = (4 \times 0^2 - 2) \times 1$

$= -2$

Since $y'' < 0$, $y = f(x)$ is concave down.

19d $y'' = (4x^2 - 2)y$

$= 4x^2y - 2y$

$y''' = 4y \times \frac{d(x^2)}{dx} + 4x^2 \times \frac{d(y)}{dx} - 2 \times \frac{d(y)}{dx}$

$= 4y \times 2x + 4x^2y' - 2y'$

$= 8xy + 4x^2y' - 2y'$

$= 8xy + 4x^2(-2xy) - 2(-2xy)$

$= 8xy - 8x^3y + 4xy$

At $(0, 1)$:

$y''' = 8 \times 0 \times 1 - 8 \times 0 \times 1 + 4 \times 0 \times 1$

$= 0$

20a $y = cx - c^2$

$y' = c$

Substituting y and y' into $(y')^2 - xy' + y = 0$:

$\text{LHS} = (c)^2 - x(c) + cx - c^2$

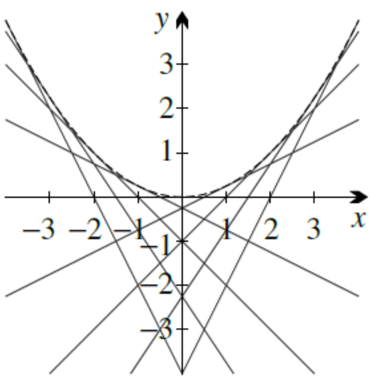
$= c^2 - cx + cx - c^2$

$= 0$

$= \text{RHS}$

Therefore $y = cx - c^2$ is a general solution of $(y')^2 - xy' + y = 0$.

20b



They seem to form the outline of a curve (shown dotted)

20c For $c = p$,

$y = px - p^2$

For $c = p + h$,

$y = (p + h)x - (p + h)^2$

At point of intersection:

$(p + h)x - (p + h)^2 = px - p^2$

$px + hx - p^2 - 2ph - h^2 = px - p^2$

$hx - 2ph - h^2 = 0$

$hx = 2ph + h^2$

$hx = h(2p + h)$

$x = 2p + h \qquad (h \neq 0)$

$y = p(2p + h) - p^2$

$= 2p^2 + ph - p^2$

$y = p^2 + ph$

Therefore point of intersection is $(2p + h, p^2 + ph)$.

20d $\lim_{h \to 0}(x)$

$= \lim_{h \to 0}(2p + h)$

$= 2p$

$\lim_{h \to 0}(y)$

$= \lim_{h \to 0}(p^2 + ph)$

$= p^2$

Therefore, as $h \to 0$, point of intersection is $(2p, p^2)$.

20e $x = 2p$

so $p = \dfrac{x}{2}$

Substituting $p = \frac{x}{2}$ into $y = p^2$:

$$y = \left(\frac{x}{2}\right)^2$$

$$y = \frac{1}{4}x^2$$

$$y' = \frac{1}{2}x$$

Substituting y and y' into $(y')^2 - xy' + y = 0$:

$$\text{LHS} = \left(\frac{1}{2}x\right)^2 - x\left(\frac{1}{2}x\right) + \frac{1}{4}x^2$$

$$\text{LHS} = \frac{1}{4}x^2 - \frac{1}{2}x^2 + \frac{1}{4}x^2$$

$$= 0$$

$$= \text{RHS}$$

Therefore $y = \frac{1}{4}x^2$ is a solution of $(y')^2 - xy' + y = 0$.

This is called a singular solution. This means the solution is not obtained from the general solution but obtained by the usual method of solving the differential equation. In solving this differential equation, a general solution consisting of a family of curves is obtained.

Each line in part b is tangent to the parabola. Thus every point in the parabola is also a point in a general solution. Hence the parabola itself is also a solution.

21 $y^{(n)} = 1$

$$y^{(n-1)} = \int 1\,dx$$

$$= x + A_1$$

$$y^{(n-2)} = \int (x + A_1)\,dx$$

$$= \frac{x^2}{2} + A_1 x + A_2$$

$$y^{(n-3)} = \int \left(\frac{x^2}{2} + \frac{A_1 x}{1} + A_2\right) dx$$

$$= \frac{x^3}{3 \times 2} + \frac{A_1 x^2}{2 \times 1} + \frac{A_2 x}{1} + A_3$$

$$y^{(n-4)} = \int \left(\frac{x^3}{3 \times 2} + \frac{A_1 x^2}{2 \times 1} + \frac{A_2 x}{1} + A_3\right) dx$$

$$= \frac{x^4}{4 \times 3 \times 2} + \frac{A_1 x^3}{3 \times 2} + \frac{A_2 x^2}{2 \times 1} + \frac{A_3 x}{1} + A_4$$

Following this pattern gives:

$$y^{(2)} = \frac{x^{n-2}}{(n-2)!} + \frac{A_1 x^{n-3}}{(n-3)!} + \frac{A_2 x^{n-4}}{(n-4)!} + \cdots + \frac{A_{n-3} x}{1!} + A_{n-2}$$

$$y^{(1)} = \frac{x^{n-1}}{(n-1)!} + \frac{A_1 x^{n-2}}{(n-2)!} + \frac{A_2 x^{n-3}}{(n-3)!} + \cdots + \frac{A_{n-2} x}{1!} + A_{n-1}$$

$$y = y^{(0)} = \frac{x^n}{n!} + \frac{A_1 x^{n-1}}{(n-1)!} + \frac{A_2 x^{n-2}}{(n-2)!} + \cdots + \frac{A_{n-1} x}{1!} + A_n$$

Or, absorbing the factorial terms into the constants:

$$y = \frac{x^n}{n!} + C_1 x^{n-1} + C_2 x^{n-2} + \cdots + C_n$$

There are n arbitrary constants.

Solutions to Exercise 13B

1a $y' = 2(1) - 3 = -1$

1b $y' = 2\cos(0) - 1 = 1$

1c $y' = 4 - (1)^2 = 3$

1d $y' = \frac{1}{1+(1)} = \frac{1}{2}$

1e $y' = \frac{(1)}{(-2)} + 1 = \frac{1}{2}$

1f $y' = (1)(-2) - (1) = -3$

2a

y \ x	-1	0	1	2	3	4	5
3	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$
2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$
1	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$
0	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$
-1	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$
-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$
-3	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$

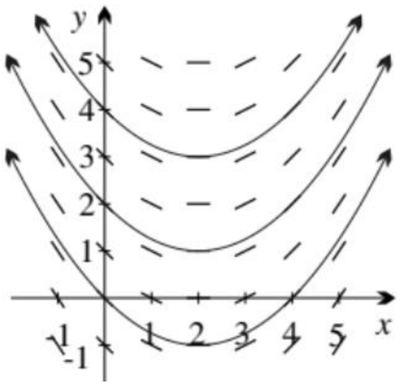
2d Every entry in that column is the same.

2e Every vertical line $x = k$ is an isocline.

2f concave up with a minimum turning point at $x = 2$

2g a parabola

2h



3a $y' = x - y$

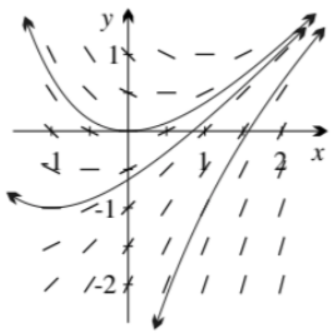
3b

y \ x	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
1	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1
$\frac{1}{2}$	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$
0	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
$-\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$
-1	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3
$-\frac{3}{2}$	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	$\frac{7}{2}$
-2	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	$\frac{7}{2}$	4

3e The lines $y = x + k$ are isoclines.

3f concave up

3g, h



3i $y = x - 1$

3j $y' = x - y$

$\text{LHS} = \frac{d}{dx}(x - 1)$

$= 1$

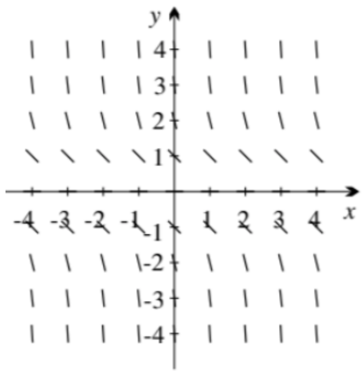
$\text{RHS} = x - y$

$= x - (x - 1)$

$= 1$

Therefore it is a solution.

4a



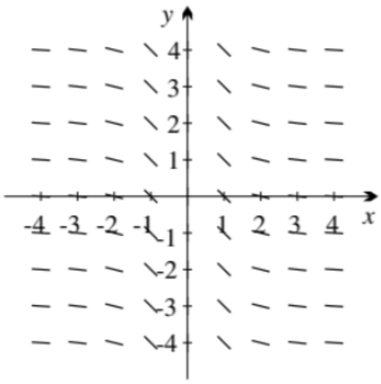
4a i x axis

4a ii any horizontal line

4a iii The gradients decrease to zero then increase.

4a iv The gradients are the same. It is an isocline.

4b



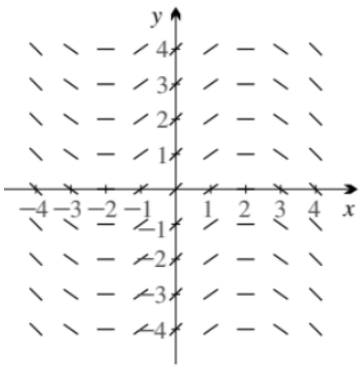
4b i There are no points or isoclines where $y' = 0$.

4b ii any vertical line

4b iii The gradients are the same. It is an isocline.

4b iv The gradients decrease to vertical then increase.

4c



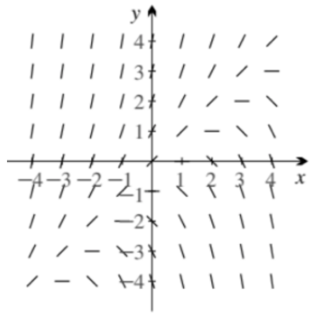
4c i $x = -2, x = 2$

4c ii any vertical line

4c iii The gradients are the same. It is an isocline.

4c iv The gradients increase from -1 to 1 then back again.

4d



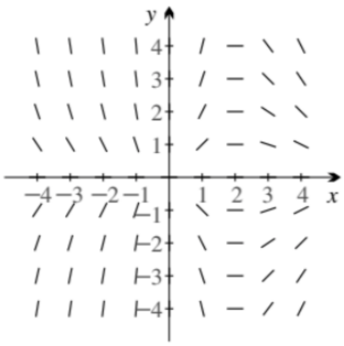
4d i $y = x - 1$

4d ii $y = x + C$

4d iii The gradients increase.

4d iv The gradients decrease.

4e



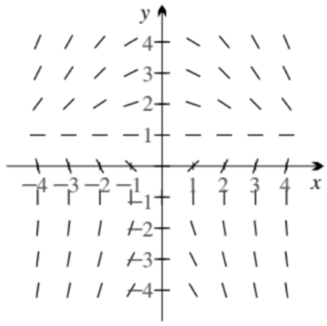
4e i $x = 2$

4e ii $y = 0$ (excepting the point $(0, 0)$ where the gradient is undefined)

4e iii The gradients increase.

4e iv The gradients decrease, but the gradient is undefined at the y-axis.

4f



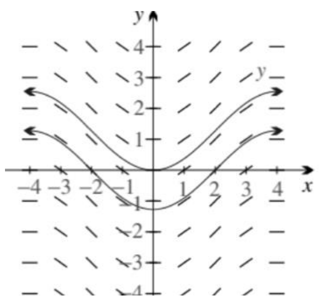
4f i $y = 1$ or $x = 0$

4f ii Undefined at $y = -1$.

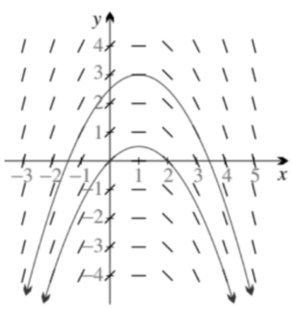
4f iii The gradients decrease, but the gradient is undefined at $y = -1$.

4f iv The gradients decrease.

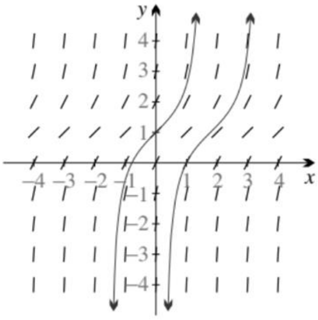
5a



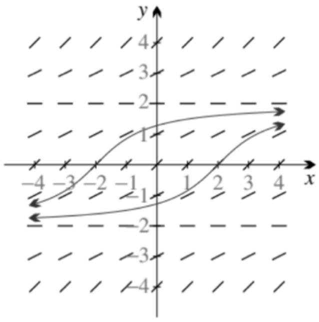
5b



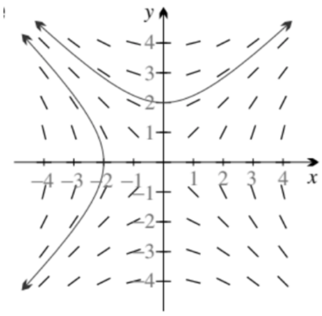
5c



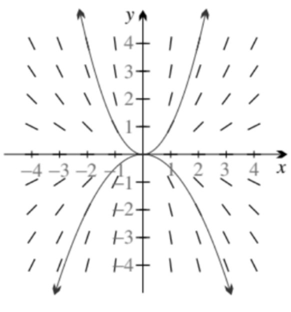
5d



5e



5f



6a All vertical lines are isoclines so $y' = f(x)$.

6b All vertical lines are isoclines so $y' = f(x)$.

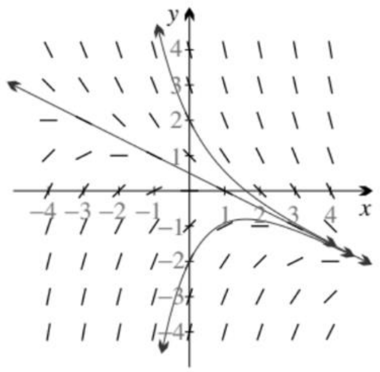
6c All horizontal lines are isoclines so $y' = g(y)$.

6d All horizontal lines are isoclines so $y' = g(y)$.

6e Not all horizontals and not all verticals are isoclines, so y' is a combination.

6f Not all horizontals and not all verticals are isoclines, so y' is a combination.

7a Solution curves through the points $(0, -2)$ and $(0, 2)$ are shown.



7b i The gradients decrease.

7b ii The solution curves converge as they cross $x = 1$ from left to right.

7c $y' = -\frac{1}{2}$ everywhere on that line

7d

$$y = \frac{1}{2} - \frac{1}{2}x$$

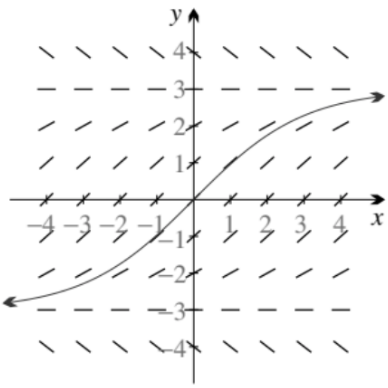
$$y' = -\frac{1}{2}x - y$$

$$\text{LHS} = \frac{d}{dx}\left(\frac{1}{2} - \frac{1}{2}x\right)$$
$$= -\frac{1}{2}$$
$$\text{RHS} = -\frac{1}{2}x - y$$
$$= -\frac{1}{2}x - \left(\frac{1}{2} - \frac{1}{2}x\right)$$
$$= -\frac{1}{2}$$

Therefore it is a solution.

7e The isocline is an asymptote for each one.

8a



8b $y = -3, 3$

8c Yes, these constant solutions are isoclines.

8d i converge

8d ii diverge

8d iii yes

8d iv They are the asymptotes for the solution curves.

8e See part a.

9 $y' = -2 - y$

Since gradient depends only on y, all horizontal lines should be isoclines (eliminating options A and B).

Both the remaining solutions show $y' = 0$ at $y = -2$ which fits with the DE given.

Checking D we can confirm that the slope field seems consistent with the equation given.

10 All vertical lines are isoclines but horizontal lines are not, so y' must be a function of x, eliminating options B and D. At $x = 0$, the slope is positive, which eliminates A. Checking C we can confirm that the slope field seems consistent with the equation given.

11 $y' = 1 - \frac{x}{y}$

For $x = 0, y \neq 0$, slope should equal 1. This eliminates options C and D.

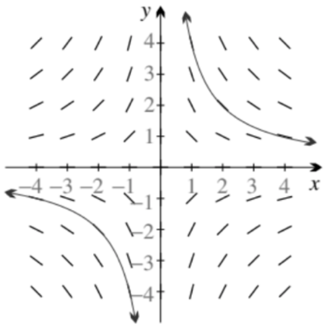
For $x = y \neq 0$, slope should equal 0. This eliminates option A.

Checking B we can confirm that the slope field seems consistent with the equation given.

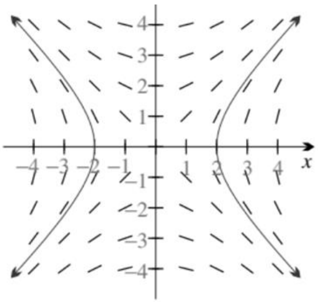
12a Along vertical and horizontal lines, the slope increases with increasing x, and decreases with increasing y. Of the four options given, only B is consistent with this pattern.

12b Along all horizontal lines, the slope is 0 at $x = 0$, and asymptotes to zero again as x becomes large in either direction. However, under options A and C, the slope would tend to $\pm\infty$ as x becomes large, so we can rule these out. At (1, 1), option B predicts a slope of 1 (inconsistent with the graph shown) and D predicts slope of -1 (consistent with the graph shown) so the only valid answer is D.

13a i,ii



13b i,ii



13c i $xy = 4$

Need to show $y' = -\frac{y}{x}$

$\frac{d}{dx}(xy) = \frac{d}{dx}(4)$

$(1)(y) + (x)\frac{dy}{dx} = 0$

$\frac{dy}{dx} = -\frac{y}{x}$

Hyperbola passes through $(2, 2)$ and $(-2, -2)$.

Therefore it is a solution of the IVP in part a.

13c ii $x^2 - y^2 = 4$

Need to show $y' = -\frac{x}{y}$

$\frac{d}{dx}(x^2 - y^2) = \frac{d}{dx}(4)$

$2x - (2y)\frac{dy}{dx} = 0$

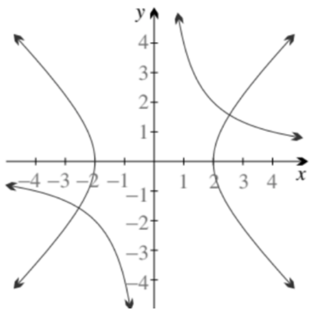
$$\frac{dy}{dx} = \frac{-2x}{-2y}$$

$$\frac{dy}{dx} = \frac{x}{y}$$

Hyperbola passes through $(2, 0)$ and $(-2, 0)$.

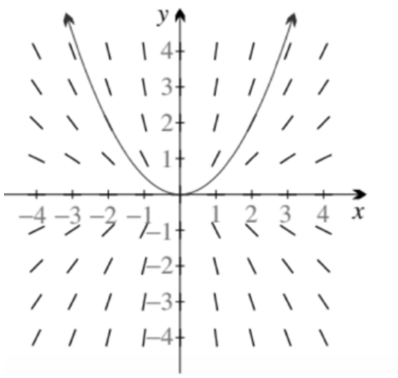
Therefore it is a solution of the IVP in part b.

13c iii

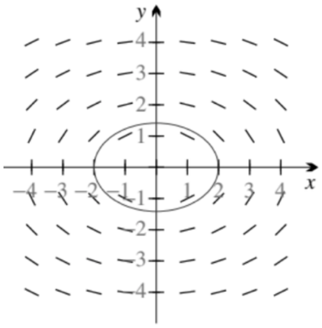


13d At any point where these curves intersect, gradient of the first curve is $-\frac{y}{x}$ and gradient of the second curve is $\frac{x}{y}$. The product of these two is always -1, hence the hyperbolas are perpendicular where they intersect.

14a i, ii



14b i, ii



14c i $y = \frac{1}{2}x^2$

Need to show $y' = \dfrac{2y}{x}$

$\text{LHS} = y'$

$= \dfrac{d}{dx}(y)$

$= \dfrac{d}{dx}\left(\dfrac{1}{2}x^2\right)$

$= x$

$\text{RHS} = \dfrac{2y}{x}$

$= \dfrac{2\left(\frac{1}{2}x^2\right)}{x}$

$= x$

Also the parabola passes through $(2, 2)$.

Therefore it is a solution of the IVP in part a.

14c ii $x^2 + 2y^2 = 4$

Need to show $y' = -\dfrac{x}{2y}$

$\dfrac{d}{dx}(x^2 + 2y^2) = \dfrac{d}{dx}(4)$

$(2x) + (4y)\dfrac{dy}{dx} = 0$

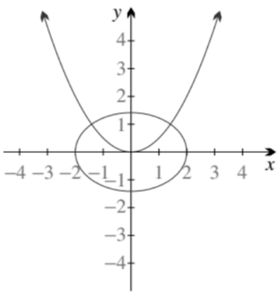
$$\frac{dy}{dx} = -\frac{2x}{4y}$$

$$\frac{dy}{dx} = -\frac{x}{2y}$$

Also the ellipse passes through $(2, 0)$ and $(-2, 0)$.

Therefore it is a solution of the IVP in part b.

14c iii



14d The gradient of the parabola is $\frac{2y}{x}$ and the gradient of the ellipse is $-\frac{x}{27}$. At any given point the product of these gradients is -1 so the parabola and the ellipse are perpendicular where they intersect.

15a $x^2 + y^2 = 16$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(16)$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = -\frac{2x}{2y}$$

$$\frac{dy}{dx} = -\frac{x}{y} \text{ as required}$$

15b $(x-3)^2 + (y-1)^2 = 4$

15c $(x-3)^2+(y-1)^2=4$

$$\frac{d}{dx}(x-3)^2+\frac{d}{dx}(y-1)^2=\frac{d}{dx}(16)$$

$$\frac{d}{dx}(x^2-6x+9)+\frac{d}{dx}(y^2-2y+1)=\frac{d}{dx}(16)$$

$$(2x-6)+(2y-2)\frac{dy}{dx}=0$$

$$(2y-2)\frac{dy}{dx}=-(2x-6)$$

$$\frac{dy}{dx}=-\frac{(2x-6)}{(2y-2)}$$

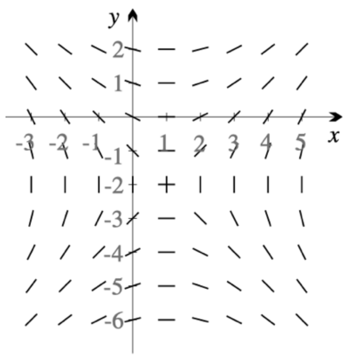
$$\frac{dy}{dx}=-\frac{2(x-3)}{2(y-1)}$$

$$\frac{dy}{dx}=-\frac{x-3}{y-1}$$

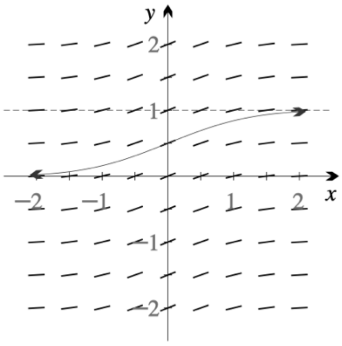
15d i

$$\frac{dy}{dx}=\frac{x-1}{y+2}$$

15d ii



16a, b

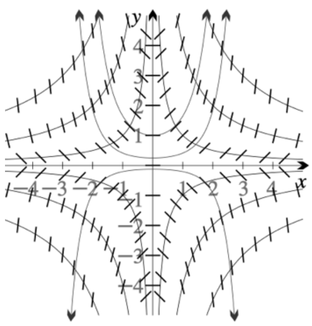


16c $\quad \Phi(1) \doteqdot 0.8$; a better approximation is 0.8413.

17a $\quad x = 0$ and $y = 0$

17b $\quad$ The rectangular hyperbola $xy = C$

17c-i



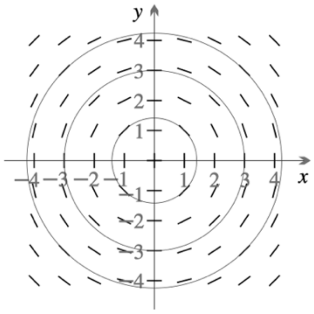
17j i $\quad$ The advantage of this technique is the exact gradient of the integral curve is known as an isocline is crossed.

17j ii $\quad$ The disadvantage of this technique is it takes a lot of time to sketch the isoclines.

18a i $\quad$ The line elements of the slope field for points on the y-axis other than the origin are horizontal.

18a ii $\quad$ The line elements of the slope field for points on the x-axis are vertical.

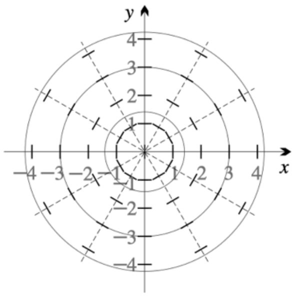
18a iii



18a iv circle

18b i $y = -\frac{1}{C}x$

18b ii



The straight lines at 30° and 60° to the x-axis.

18b iii Each line element is perpendicular to its isocline because the product of the gradients

$$C \times \frac{-1}{C} = -1$$

18b iv See graph in part b ii.

18b v Yes, the shape of part c is clearer now. Notice that the innermost line elements almost join up to give the outline of a circle.

19a Any two points on the same vertical line have the same x-value.

Therefore, if $y' = f(x)$, then any two points on the same vertical line give the same slope, making this line an isocline.

Let D be the domain of $f(x)$. For each a in D, the point (a, y) gives $y' = f(a)$ for all values of y. Hence $x = a$ is an isocline for each a in D.

19b Any two points on the same horizontal line have the same y-value. Therefore, if $y' = g(y)$, then any two points on the same horizontal line give the same slope, making this line an isocline.

19c If $y = cx + b$ is a solution to this DE, then $y' = \frac{d(cx+b)}{dx} = c$ everywhere along the line. Hence all points on the line give the same gradient, so it is an isocline.

20a $x^3 + y^3 = 3xy$

Differentiating:

$$\frac{d(x^3)}{dx} + \frac{d(y^3)}{dx} = \frac{d(3xy)}{dx}$$

$$3x^2 + 3y^2\frac{dy}{dx} = 3y + 3x\frac{dy}{dx}$$

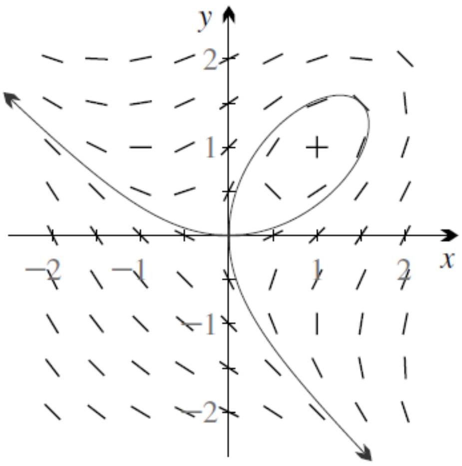
Grouping terms:

$$\frac{dy}{dx}(3y^2 - 3x) = 3y - 3x^2$$

$$y' = \frac{3y - 3x^2}{3y^2 - 3x}$$

$$= \frac{y - x^2}{y^2 - x}$$

20b



20c $y' = 0$ when $y - x^2 = 0$, that is, when $y = x^2$, excluding the points $(0, 0)$ and $(1, 1)$ where the gradient is undefined.

20d $\frac{1}{y'}$ approaches zero as $y^2 - x$ approaches zero, that is, the integral curve is vertical when $x = y^2$, again excluding the points $(0, 0)$ and $(1, 1)$.

20e See part b.

20f Although gradients here are undefined, we can see that the curve crosses itself here.

20g $y = -x - 1$ appears to be an isocline.

Substituting this into the DE from part a, we find:

$$y' = \frac{y - x^2}{y^2 - x}$$

$$= \frac{-x - 1 - x^2}{(-x - 1)^2 - x}$$

$$= \frac{-x^2 - x - 1}{x^2 + 2x + 1 - x}$$

$$= \frac{-x^2 - x - 1}{x^2 + x + 1}$$

$$= -1$$

Hence the slope field equals -1 everywhere along this line, so it is an isocline. The gradient of the line is also equal to -1 so it is a solution of the DE.

20h The curve is symmetrical in the line $y = x$.

20h i The equation is symmetric in x and y, hence we expect the solution to be symmetric when we swap x and y by reflecting in $y = x$.

20h ii Swapping x and y in the RHS of the differential equation has the same effect as taking the reciprocal of the gradient, hence the gradient field is symmetric around $y = x$ and so we expect the curve to have the same symmetry.

Solutions to Exercise 13C

Let the integration constants be A, B, C or D.

1a

$$\frac{dy}{dx} = \frac{x-1}{y+1}$$

$$(y+1)\frac{dy}{dx} = x - 1$$

$$(y+1)\,dy = (x-1)\,dx$$

$$\int (y+1)dy = \int (x-1)\,dx$$

1b

$$\int (y+1)dy = \int (x-1)dx$$

$$\frac{(y+1)^2}{2} = \frac{(x-1)^2}{2} + C$$

$$(y+1)^2 = (x-1)^2 + 2C$$

$$(y+1)^2 = (x-1)^2 + D,\ \text{where } D = 2C$$

2a

$$\frac{dy}{dx} = xe^{-y}$$

$$\left(\frac{1}{e^{-y}}\right)dy = x\,dx$$

$$\int \left(\frac{1}{e^{-y}}\right)dy = \int x\,dx$$

$$\int (e^y)\,dy = \int x\,dx$$

$$e^y = \frac{x^2}{2} + C$$

$$\ln(e^y) = \ln\left|\frac{x^2}{2} + C\right|$$

$$y = \ln\left|\frac{x^2}{2} + C\right|$$

2b

$$\frac{dy}{dx} = 4x^3(1 + y^2)$$

$$\left(\frac{1}{1 + y^2}\right) dy = 4x^3\, dx$$

$$\int \left(\frac{1}{1 + y^2}\right) dy = \int 4x^3\, dx$$

$$\tan^{-1} y = \frac{4x^4}{4} + C$$

$$y = \tan(x^4 + C)$$

3a Substitute $y = 0$, where $x \neq 0$. Then LHS and RHS are both zero.

3b

$$\frac{dy}{dx} = -\frac{y^2}{x}$$

$$\frac{1}{y^2}\, dy = -\frac{1}{x}\, dx$$

$$\int \left(\frac{1}{y^2}\right) dy = \int \left(-\frac{1}{x}\right) dx$$

$$\int (y^{-2})\, dy = -\int \frac{1}{x}\, dx$$

$$\frac{y^{-1}}{-1} = -\ln|x| + C$$

$$-\frac{1}{y} = -\ln|x| + C$$

$$-\frac{y}{1} = -\frac{1}{\ln|x| + C}$$

$$y = \frac{1}{\ln|x| + C}$$

4a

$$\frac{dy}{dx} = -\frac{x}{y}$$

$$y\,dy = -x\,dx$$

$$\int y\,dy = -\int x\,dx$$

4b

$$\int y\,dy = -\int x\,dx$$

$$\frac{y^2}{2} = -\frac{x^2}{2} + C$$

$$y^2 = -x^2 + 2C$$

$y^2 = -x^2 + D$, where $D = 2C$

4c Substituting $(1, \sqrt{3})$:

$$\left(\sqrt{3}\right)^2 = -(1)^2 + D$$

$$3 = -1 + D$$

$$4 = D$$

Therefore $y^2 = -x^2 + 4$ or $y^2 + x^2 = 4$

5a

$$\frac{dy}{dx} = \frac{x}{y}$$

$$y\,dy = x\,dx$$

$$\int y\,dy = \int x\,dx$$

$$\frac{y^2}{2} = \frac{x^2}{2} + C$$

$$y^2 = x^2 + 2C$$

$y^2 = x^2 + D$, where $D = 2C$

Substituting $(0, 1)$:

$(1)^2 = (0)^2 + D$

$1 = D$

Therefore $y^2 = x^2 + 1$

5b

$$\frac{dy}{dx} = (1 + x)\,(1 + y^2)$$

$$\left(\frac{1}{1 + y^2}\right)\,dy = (1 + x)\,dx$$

$$\int \left(\frac{1}{1 + y^2}\right) dy = \int (1 + x) dx$$

$$\tan^{-1} y = \frac{(1 + x)^2}{2} + C$$

$$y = \tan\left(\frac{1}{2}(1 + x)^2 + C\right)$$

Substituting $(-1, 0)$:

$0 = \tan\left(\frac{1}{2}(1 + (-1))^2 + C\right)$

$0 = \tan(C)$

$0 = C$

Therefore, $y = \tan\left(\frac{1}{2}(1 + x)^2\right)$

5c

$$\frac{dy}{dx} = -2y^2x$$

$$\frac{1}{y^2}\,dy = -2x\,dx$$

$$\int \frac{1}{y^2} dy = \int (-2x) dx$$

$$\int (y^{-2})\,dy = \int (-2x) dx$$

$$\frac{y^{-1}}{-1} = \frac{-2x^2}{2} + C$$

$$-y^{-1} = -x^2 + C$$

$$y^{-1} = x^2 - C$$

$$\frac{1}{y} = \frac{x^2 - C}{1}$$

$$\frac{y}{1} = \frac{1}{x^2 - C}$$

$$y = \frac{1}{x^2 - C}$$

Substituting $\left(1, \frac{1}{2}\right)$:

$$\frac{1}{2} = \frac{1}{(1)^2 - C}$$

$$2 = 1 - C$$

$$C = -1$$

Therefore $y = \dfrac{1}{x^2 + 1}$

5d

$$\frac{dy}{dx} = e^{-y}\sec^2 x$$

$$\left(\frac{1}{e^{-y}}\right) dy = \sec^2 x\, dx$$

$$\int \left(\frac{1}{e^{-y}}\right) dy = \int \sec^2 x\, dx$$

$$\int (e^y)\, dy = \int \sec^2 x\, dx$$

$$e^y = \tan x + C$$

$$y = \ln(C + \tan x)$$

Substituting $\left(\frac{\pi}{4}, \ln 2\right)$:

$$\ln 2 = \ln(C + \tan\frac{\pi}{4})$$

$2 = C + \tan\dfrac{\pi}{4}$

$2 = C + 1$

$1 = C$

Therefore $y = \ln\,(1 + \tan x)$

(More strictly, the solution is only valid in the interval $-\frac{\pi}{2} < x < \frac{\pi}{2}$.)

6a

$\dfrac{dy}{dx} = \dfrac{2y+4}{x}$

$y = -2$ is a constant solution

$\text{LHS} = \dfrac{d(-2)}{dx} = 0$

$\text{RHS} = \dfrac{2(-2)+4}{x} = 0$

$\text{LHS} = \text{RHS}$

Therefore $y = -2$ is a solution of the DE.

6b

$\dfrac{dy}{dx} = \dfrac{2y+4}{x}$

$\dfrac{1}{2y+4}\,dy = \dfrac{1}{x}\,dx$

$\displaystyle\int \frac{1}{2y+4}\,dy = \int \frac{1}{x}\,dx$

$\displaystyle\frac{1}{2}\int \frac{1}{y+2}\,dy = \int \frac{1}{x}\,dx$

$\dfrac{1}{2}\ln|y+2| = \ln|x| + B$

$\ln|y+2| = 2\ln|x| + 2B$

$\ln|y+2| = \ln(x^2) + \ln(e^{2B})$

$\ln|y+2| = \ln(e^{2B}x^2)$

$y + 2 = e^{2B}x^2$ or $-e^{2B}x^2$

$y + 2 = Cx^2$ where $C = e^{2B}$ or $-e^{2B}$ and $C \neq 0$

$y = Cx^2 - 2$

6c Allow $C = 0$.

7a $y' = -xy$

$y = 0$ is a constant solution

$\text{LHS} = \dfrac{d(0)}{dx} = 0$

$\text{RHS} = -x \times 0 = 0$

$\text{LHS} = \text{RHS}$

Therefore $y = 0$ is a solution of the DE.

7b

$$\frac{dy}{dx} = -xy$$

$$\frac{1}{y}\,dy = -x\,dx$$

$$\int \frac{1}{y}\,dy = \int (-x)\,dx$$

$$\int \frac{1}{y}\,dy = -\int x\,dx$$

$$\ln|y| = -\frac{x^2}{2} + B$$

$$|y| = e^{\left(-\frac{1}{2}x^2 + B\right)}$$

$$y = e^{-\frac{1}{2}x^2} \times e^B \text{ or } -e^{-\frac{1}{2}x^2} \times e^B$$

$$y = Ce^{-\frac{1}{2}x^2} \text{ where } C = e^B \text{ or } -e^B \text{ and } C \neq 0$$

7c Allow $C = 0$

8a

$$\frac{dy}{dx} = \frac{2-y}{x} = \frac{y-2}{-x}$$

$$\frac{1}{y-2}\,dy = -\frac{1}{x}\,dx$$

$$\int \frac{1}{y-2}\,dy = -\int \frac{1}{x}\,dx$$

$$\ln|y-2| = -\ln|x| + B$$

$$\ln|y-2| = \ln|x^{-1}| + \ln(e^B)$$

$$\ln|y-2| = \ln(e^B|x^{-1}|)$$

$$|y-2| = e^B|x^{-1}|$$

$$y - 2 = e^B x^{-1} \text{ or } -e^B x^{-1}$$

$$y - 2 = Cx^{-1} \text{ where } C = e^B \text{ or } -e^B$$

$$y - 2 = \frac{C}{x}$$

$$y = \frac{C}{x} + 2$$

8b

$$\frac{dy}{dx} = \frac{xy}{1+x^2}$$

$$\frac{1}{y}\,dy = \frac{x}{1+x^2}\,dx$$

$$\int \frac{1}{y}\,dy = \int \frac{x}{1+x^2}\,dx$$

$$\int \frac{1}{y}\,dy = \frac{1}{2}\int \frac{2x}{1+x^2}\,dx$$

$$2\ln|y| = \ln|1+x^2| + B$$

$$\ln(y^2) = \ln(1+x^2) + B$$

$$\ln(y^2) = \ln(1+x^2) + \ln(C) \quad \text{where } B = \ln(C)$$

$$\ln(y^2) = \ln(C(1+x^2))$$

$$y^2 = C(1+x^2)$$

8c

$$\frac{dy}{dx} = \frac{-2y}{x}$$

$$\frac{1}{y}\,dy = \left(-\frac{2}{x}\right)\,dx$$

$$\int \frac{1}{y}\,dy = \int \left(-\frac{2}{x}\right)\,dx$$

$$\ln|y| = -2\ln|x| + B$$

$$\ln|y| = \ln\,(x^{-2}) + B$$

$$\ln|y| = \ln\,(x^{-2}) + \ln(e^B)$$

$$\ln|y| = \ln(e^B x^{-2})$$

$$|y| = e^B x^{-2}$$

$$y = \frac{e^B}{x^2} \text{ or } -\frac{e^B}{x^2}$$

$$y = \frac{C}{x^2} \text{ where } C = e^B \text{ or} - e^B$$

8d

$$\frac{dy}{dx} = y\sin x$$

$$\frac{1}{y}\,dy = \sin x\,\,dx$$

$$\int \frac{1}{y}\,dy = \int \sin x\,\,dx$$

$$\ln|y| = -\cos x + B$$

$$\ln|y| = \ln(e^{-\cos x}) + \ln(e^B)$$

$$\ln|y| = \ln(e^B e^{-\cos x})$$

$$|y| = e^B e^{-\cos x}$$

$$y = e^B e^{-\cos x} \text{ or } -e^B e^{-\cos x}$$

$$y = Ce^{-\cos x} \text{ where } C = e^B \text{ or } -e^B$$

8e

$$\frac{dy}{dx} = \frac{3y}{x^2}$$

$$\frac{1}{3y}\,dy = \frac{1}{x^2}\,dx$$

$$\int \frac{1}{3y}\,dy = \int \frac{1}{x^2}\,dx$$

$$\frac{1}{3}\int \frac{1}{y}\,dy = \int \frac{1}{x^2}\,dx$$

$$\frac{1}{3}\ln|y| = \frac{x^{-1}}{-1} + B$$

$$\frac{1}{3}\ln|y| = -\frac{1}{x} + B$$

$$\ln|y| = -\frac{3}{x} + 3B$$

$$\ln|y| = \ln\left(e^{-\frac{3}{x}}\right) + \ln(e^{3B})$$

$$\ln|y| = \ln\left(e^{3B}e^{-\frac{3}{x}}\right)$$

$$|y| = e^{3B}e^{-\frac{3}{x}}$$

$$y = e^{3B}e^{-\frac{3}{x}} \text{ or } -e^{3B}e^{-\frac{3}{x}}$$

$$y = Ce^{-\frac{3}{x}} \text{ where } C = e^{3B} \text{ or } -e^{3B}$$

8f

$$\frac{dy}{dx} = \frac{y(1-x)}{x}$$

$$\frac{1}{y}\,dy = \frac{1-x}{x}\,dx$$

$$\int \frac{1}{y}\,dy = \int \frac{1-x}{x}\,dx$$

$$\int \frac{1}{y}\,dy = \int \left(\frac{1}{x} - \frac{x}{x}\right)\,dx$$

$$\int \frac{1}{y}\, dy = \int \left(\frac{1}{x} - 1\right) dx$$

$$\ln|y| = \ln|x| - x + B$$

$$\frac{1}{2}\ln|y^2| = \frac{1}{2}\ln|x^2| - x + B$$

$$\ln(y^2) = \ln(x^2) - 2x + 2B$$

$$\ln(y^2) = \ln(x^2) + \ln(e^{-2x}) + \ln(e^{2B})$$

$$\ln(y^2) = \ln(e^{2B}x^2e^{-2x})$$

$$y^2 = e^{2B}x^2e^{-2x}$$

$$y = e^B xe^{-x} \text{ or } -e^B xe^{-x}$$

$$y = Cxe^{-x} \text{ where } C = e^B \text{ or } -e^B$$

9a $\cos y = 0$

$$y = -\frac{3\pi}{2}, -\frac{\pi}{2}, -\frac{3\pi}{2}, \frac{\pi}{2} \text{ and } \frac{3\pi}{2}$$

9b

$$\frac{dy}{dx} = 3x^2\cos^2 y$$

$$\frac{1}{\cos^2 y}\, dy = 3x^2\, dx$$

$$\int \frac{1}{\cos^2 y}\, dy = \int 3x^2\, dx$$

$$\int \sec^2 y\, dy = \int 3x^2\, dx$$

$$\tan y = \frac{3x^3}{3} + C$$

$$\tan y = x^3 + C$$

$$y = \tan^{-1}(x^3 + C)$$

10a Constant solution is $y = 0$ which does not match the initial value.

10b

$$\frac{dy}{dx} = \frac{2y}{x-1}$$

$$\frac{1}{y}\,dy = \frac{2}{x-1}\,dx$$

$$\int \frac{1}{y}\,dy = \int \frac{2}{x-1}\,dx$$

$$\int \frac{1}{y}\,dy = 2\int \frac{1}{x-1}\,dx$$

$$\ln|y| = 2\ln|x-1| + B$$

$$\ln|y| = \ln\left((x-1)^2\right) + \ln(e^B)$$

$$\ln|y| = \ln\left(e^B(x-1)^2\right)$$

$$|y| = e^B(x-1)^2 \text{ or } -e^B(x-1)^2$$

$$y = C(x-1)^2 \text{ where } C = e^B \text{ or } -e^B$$

10c Substituting $y(2) = 1$:

$$1 = C(2-1)^2$$

$$1 = C$$

$$y = (x-1)^2$$

11a Constant solution is $y = 1$ which does not satisfy the IVP.

11b

$$\frac{dy}{dx} = (y-1)\tan x$$

$$\frac{1}{y-1}\,dy = \tan x\,dx$$

$$\int \frac{1}{y-1}\,dy = \int \tan x\,dx$$

$$\int \frac{1}{y-1}\,dy = -\int \left(\frac{-\sin x}{\cos x}\right)dx$$

$$\ln|y-1| = -\ln|\cos x| + B$$

$\ln|y-1| = \ln|\sec x| + \ln(e^B)$

$\ln|y-1| = \ln(e^B|\sec x|)$

$|y-1| = e^B|\sec x|$

$y-1 = e^B \sec x \text{ or } -e^B \sec x$

$y-1 = C\sec x \quad \text{where } C = e^B \text{ or } -e^B$

$y = 1 + C\sec x$

11c Substituting $y\left(\frac{\pi}{4}\right) = 3$:

$3 = 1 + C\sec\left(\frac{\pi}{4}\right)$

$3 = 1 + C \times \frac{\sqrt{2}}{1}$

$3 - 1 = \sqrt{2}C$

$C = \frac{2}{\sqrt{2}} = \sqrt{2}$

Therefore $y = 1 + \sqrt{2}\sec x$

12a

$\frac{dy}{dx} = \frac{y}{x}$

$\frac{1}{y}\,dy = \frac{1}{x}\,dx$

$\int \frac{1}{y}\,dy = \int \frac{1}{x}\,dx$

$\ln|y| = \ln|x| + B$

$\ln|y| = \ln|x| + \ln(e^B)$

$\ln|y| = \ln(e^B|x|)$

$|y| = e^B|x|$

$y = e^B x \text{ or } -e^B x$

$y = Cx \quad \text{where } C = e^B \text{ or } -e^B$

Substituting $y(2) = 1$:

$1 = C \times 2$

$\frac{1}{2} = C$

Therefore $y = \frac{1}{2}x$

12b

$\frac{dy}{dx} = \frac{y}{2x}$

$\frac{1}{y}\,dy = \frac{1}{2x}\,dx$

$\int \frac{1}{y}\,dy = \int \frac{1}{2x}\,dx$

$\int \frac{1}{y}\,dy = \frac{1}{2}\int \frac{1}{x}\,dx$

$\ln|y| = \frac{1}{2}\ln|x| + B$

$\ln|y| = \ln|x|^{\frac{1}{2}} + \ln(e^B)$

$\ln|y| = \ln\left(e^B|x|^{\frac{1}{2}}\right)$

$|y| = e^B|x|^{\frac{1}{2}}$

$y = e^B x^{\frac{1}{2}}$ or $-e^B x^{\frac{1}{2}}$

$y = C\sqrt{x}$ where $C = e^B$ or $-e^B$

Substituting $y(1) = 2$:

$2 = C \times 1$

$2 = C$

Therefore $y = 2\sqrt{x}$

12c

$$\frac{dy}{dx} = \frac{-2xy}{1 + x^2}$$

$$\frac{1}{y}\,dy = \frac{-2x}{1 + x^2}\,dx$$

$$\int \frac{1}{y}\,dy = \int \frac{-2x}{1 + x^2}\,dx$$

$$\int \frac{1}{y}\,dy = -\int \frac{2x}{1 + x^2}\,dx$$

$$\ln|y| = -\ln|1 + x^2| + B$$

$$\ln|y| = \ln((1 + x^2)^{-1}) + \ln(e^B)$$

$$\ln|y| = \ln(e^B(1 + x^2)^{-1})$$

$$|y| = e^B(1 + x^2)^{-1}$$

$$y = \frac{e^B}{1 + x^2} \text{ or } -\frac{e^B}{1 + x^2}$$

$$y = \frac{C}{1 + x^2} \text{ where } C = e^B \text{ or } -e^B$$

Substituting $y(1) = 2$:

$$2 = \frac{C}{1 + 1}$$

$$4 = C$$

Therefore $y = \dfrac{4}{1 + x^2}$

12d

$$\frac{dy}{dx} = -\frac{y}{x}$$

$$\frac{1}{y}\,dy = -\frac{1}{x}\,dx$$

$$\int \frac{1}{y}\,dy = -\int \frac{1}{x}\,dx$$

$$\ln|y| = -\ln|x| + B$$

$$\ln|y| = \ln|x|^{-1} + \ln(e^B)$$

$\ln|y| = \ln(e^B|x|^{-1})$

$|y| = e^B|x|^{-1}$

$y = \dfrac{e^B}{x}$ or $-\dfrac{e^B}{x}$

$y = \dfrac{C}{x}$ where $C = e^B$ or $-e^B$

Substituting $y(2) = 1$:

$1 = \dfrac{C}{2}$

$2 = C$

Therefore $y = \dfrac{2}{x}$

12e

$\dfrac{dy}{dx} = y\cos x$

$\dfrac{1}{y}\,dy = \cos x\ dx$

$\displaystyle\int \frac{1}{y}\,dy = \int \cos x\ dx$

$\ln|y| = \sin x + B$

$|y| = e^{(\sin x + B)}$

$|y| = e^{\sin x}e^B$

$y = e^B e^{\sin x}$ or $-e^B e^{\sin x}$

$y = Ce^{\sin x}$ where $C = e^B$ or $-e^B$

Substituting $y\left(\frac{\pi}{2}\right) = 1$:

$1 = Ce^{\sin\frac{\pi}{2}}$

$1 = Ce^1$

$C = \dfrac{1}{e}$

Therefore $y = \dfrac{1}{e}e^{\sin x}$

$$= e^{\sin x} \times e^{-1}$$

$$= e^{\sin x - 1}$$

12f

$$\frac{dy}{dx} = \frac{y(2-x)}{x^2}$$

$$\frac{1}{y}\,dy = \frac{2-x}{x^2}\,dx$$

$$\int \frac{1}{y}\,dy = \int \frac{2-x}{x^2}\,dx$$

$$\int \frac{1}{y}\,dy = \int \left(\frac{2}{x^2} - \frac{x}{x^2}\right)\,dx$$

$$\int \frac{1}{y}\,dy = \int \left(\frac{2}{x^2} - \frac{1}{x}\right)\,dx$$

$$\ln|y| = \frac{2x^{-1}}{-1} - \ln|x| + C$$

$$\ln|y| = -\frac{2}{x} - \ln|x| + C$$

$$\ln|y| + \ln|x| = -\frac{2}{x} + C$$

$$\ln|xy| = -\frac{2}{x} + C$$

$$xy = e^{-\frac{2}{x}+C}$$

$$y = \frac{e^{-\frac{2}{x}+C}}{x}$$

Substituting $y(2) = \frac{1}{2}$:

$$\frac{1}{2} = \frac{e^{-\frac{2}{2}+C}}{2}$$

$$\frac{1}{2} = \frac{e^{-1+\mathrm{C}}}{2}$$

$$1 = e^{-1-C}$$

$$-1 - C = 0$$

$$C = -1$$

$$y = \frac{e^{-\frac{2}{x}+1}}{x}$$

$$y = \frac{e^{1-\frac{2}{x}}}{x}$$

13a

$$\frac{d(\log_e(\log_e x))}{dx}$$

$$= \frac{1}{x} \times \frac{1}{\log_e x}$$

$$= \frac{1}{x\ \log_e x}$$

13b $(x \log_e x)y' = y$

$$y' = \frac{y}{x \log_e x}$$

$$\frac{dy}{dx} = \frac{y}{x \log_e x}$$

$$\frac{1}{y}\ dy = \frac{1}{x \log_e x}\ dx$$

$$\int \frac{1}{y}\ dy = \int \frac{1}{x \log_e x}\ dx$$

$$\log_e|y| = \log_e(\log_e x) + B$$

$$\log_e|y| = \log_e(\log_e x) + \log_e e^B$$

$$\log_e|y| = \log_e(e^B \log_e x)$$

$$|y| = e^B \log_e x$$

$$y = e^B \log_e x \text{ or } -e^B \log_e x$$

$$y = C\log_e x \quad \text{where } C = e^B \text{ or } -e^B$$

14a

$$\frac{x}{x+2}=\frac{x+2-2}{x+2}$$
$$=\frac{x+2}{x+2}-\frac{2}{x+2}$$
$$=1-\frac{2}{x+2}$$

14b $(x+2)y'-xy=0$

$$\frac{dy}{dx}=\frac{xy}{x+2}$$
$$\frac{1}{y}\,dy=\frac{x}{x+2}\,dx$$
$$\int\frac{1}{y}\,dy=\int\frac{x}{x+2}dx$$
$$\int\frac{1}{y}\,dy=\int\left(1-\frac{2}{x+2}\right)dx$$
$$\ln|y|=x-2\ln|x+2|+B$$
$$\ln|y|=\ln(e^x)-\ln(|x+2|^2)+\ln(e^B)$$
$$\ln|y|=\ln(e^x)-\ln((x+2)^2)+\ln(e^B)$$
$$\ln|y|=\ln\left(\frac{e^Be^x}{(x+2)^2}\right)$$
$$|y|=\frac{e^Be^x}{(x+2)^2}$$
$$y=\frac{e^Be^x}{(x+2)^2}\ \text{ or }\ -\frac{e^Be^x}{(x+2)^2}$$
$$y=\frac{Ce^x}{(x+2)^2}\ \text{ where } C=e^B \text{ or } -e^B$$

Substituting $y(0)=1$:

$$1=\frac{Ce^0}{(0+2)^2}$$
$$1=\frac{C}{4}$$

$C = 4$

Therefore $y = \dfrac{4e^x}{(x+2)^2}$

15a Since $\cos 2x = 2\cos^2 x - 1$

$2\cos^2 x = 1 + \cos 2x$

15b

$\dfrac{dy}{dx} = \dfrac{2\cos^2 x}{y}$

$y\,dy = 2\cos^2 x\;dx$

$\int y\,dy = \int 2\cos^2 x\,dx$

$\int y\,dy = \int (1 + \cos 2x)\,dx$

$\dfrac{y^2}{2} = x + \dfrac{1}{2}\sin 2x + C$

$y^2 = 2x + \sin 2x + 2C$

Substituting $y(0) = \sqrt{2}$:

$(\sqrt{2})^2 = 2(0) + \sin(0) + 2C$

$2C = 2$

Therefore $y^2 = 2x + \sin 2x + 2$

16a $y = x \times u$

$y' = u \times 1 + x \times u'$

$y' = u + xu'$

16b i $xy' = 2x + 2y$

$$y' = \frac{2x + 2y}{x}$$

Substituting for y and y':

$$u + xu' = \frac{2x + 2xu}{x}$$

$$u + xu' = \frac{2x(1 + u)}{x}$$

$$u + xu' = 2(1 + u)$$

$$xu' = 2(1 + u) - u$$

$$xu' = 2 + 2u - u$$

$$xu' = 2 + u$$

$$u' = \frac{2 + u}{x}$$

16b ii

$$\frac{du}{dx} = \frac{2 + u}{x}$$

$$\frac{1}{2 + u}\ du = \frac{1}{x}\ dx$$

$$\int \frac{1}{2 + u}\ du = \int \frac{1}{x}\ dx$$

$$\ln|2 + u| = \ln|x| + B$$

$$\ln|2 + u| = \ln|x| + \ln(e^B)$$

$$\ln|2 + u| = \ln(e^B|x|)$$

$$|2 + u| = e^B|x|$$

$$2 + u = e^B x \text{ or } -e^B x$$

$$2 + u = Cx \text{ where } C = e^B \text{ or } -e^B$$

$$u = Cx - 2$$

16b iii $u = Cx - 2$

From part a, $u = \frac{y}{x}$.

$\frac{y}{x} = Cx - 2$

$y = Cx^2 - 2x$

17a $(x^2 + 1)y' + (y^2 + 1) = 0$

$(x^2 + 1)\frac{dy}{dx} = -(y^2 + 1)$

$\frac{1}{y^2 + 1}\, dy = -\frac{1}{x^2 + 1}\, dx$

$\int \frac{1}{y^2 + 1}\, dy = -\int \frac{1}{x^2 + 1} dx$

$\tan^{-1} y = -\tan^{-1} x + C$

$\tan^{-1} y + \tan^{-1} x = C$

17b $\tan^{-1} y + \tan^{-1} x = C$

$\tan(\tan^{-1} y + \tan^{-1} x) = \tan C$

Using the formula for tangent of the sum of two angles,

$\frac{y + x}{1 - xy} = D$ where $D = \tan C$

17c Substituting $y(0) = 1$:

$\frac{1 + 0}{1 - 0} = D$

$D = 1$

Therefore $\frac{y + x}{1 - xy} = 1$

$y + x = 1 - xy$

$y + xy = 1 - x$

$y(1 + x) = 1 - x$

$$y = \frac{1 - x}{1 + x}$$

18a

$$y_1 = \frac{1}{16}x^4$$

Substituting $x = 2$:

$$y_1(2) = \frac{1}{16} \times 2^4$$

$$= \frac{16}{16}$$

$$= 1$$

$$y_2 = \frac{1}{16}(x^2 - 8)^2$$

Substituting $x = 2$:

$$y_2(2) = \frac{1}{16}(2^2 - 8)^2$$

$$= \frac{1}{16}(-4)^2$$

$$= \frac{16}{16}$$

$$= 1$$

Both y_1 and y_2 satisfy the initial condition $y(2) = 1$.

18b

$$y_1' = \frac{d}{dx}\left(\frac{x^4}{16}\right)$$

$$= \frac{x^3}{4}$$

$$(y_1')^2 = \left(\frac{x^3}{4}\right)^2$$

$$= \frac{x^6}{16}$$

$$= x^2 \times \frac{x^4}{16}$$

$$= x^2 y_1$$

$$y_2' = \frac{d}{dx}\left(\frac{1}{16}(x^2-8)^2\right)$$

$$= \frac{1}{16} \times 2(x^2-8) \times 2x$$

$$= \frac{x}{4}(x^2-8)$$

$$(y_2')^2 = \left(\frac{x}{4}(x^2-8)\right)^2$$

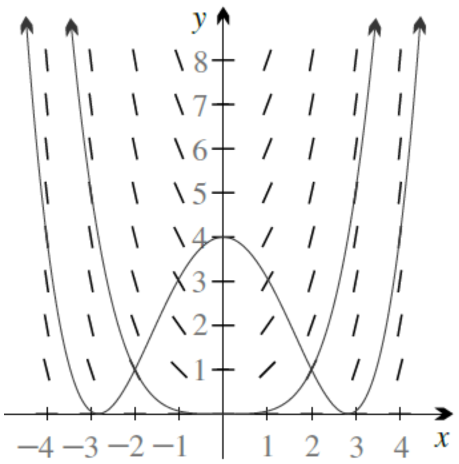
$$= \frac{x^2}{16}(x^2-8)^2$$

$$= x^2 \times \frac{1}{16}(x^2-8)^2$$

$$= x^2 y_2$$

Both y_1 and y_2 satisfy the DE $(y')^2 = x^2 y$.

18c



18d $y_2' = xy_2^{\frac{1}{2}}$

So $y_2' = \frac{x}{4}(x^2 - 8)$ becomes

$$xy_2^{\frac{1}{2}} = \frac{x}{4}\sqrt{(x^2-8)^2}$$

$$= \frac{x}{4}|x^2 - 8|$$

When $x^2 - 8 < 0$ these two expressions are not equal.

To derive the solution correctly:

$$\frac{dy}{dx} = xy^{\frac{1}{2}}$$

$$y^{-\frac{1}{2}}\,dy = x\,dx$$

$$\int y^{-\frac{1}{2}}\,dy = \int x\,dx$$

$$2y^{\frac{1}{2}} = \frac{x^2}{2} + B$$

$$y^{\frac{1}{2}} = \frac{x^2}{4} + C \text{ where } C = \frac{B}{2}$$

Substituting $y(2) = 1$:

$$1^{\frac{1}{2}} = \frac{2^2}{4} + C$$

$$1 = \frac{4}{4} + C$$

$$C = 0$$

Hence the solution is

$$y^{\frac{1}{2}} = \frac{x^2}{4} \text{ or } y = \frac{x^4}{16}$$

19a The result follows from the fundamental theorem of calculus.

19b i

$$\frac{dy}{dx} = -xy$$

$$\frac{1}{y}\,dy = -x\,dx$$

$$\int \frac{1}{y}\,dy = -\int x\,dx$$

$$\log|y| = -\frac{x^2}{2} + B$$

$$|y| = e^{-\frac{1}{2}x^2 + B}$$

$$|y| = e^B e^{-\frac{1}{2}x^2}$$

$$y = e^B e^{-\frac{1}{2}x^2} \text{ or } -e^B e^{-\frac{1}{2}x^2}$$

$$y = Ce^{-\frac{1}{2}x^2} \text{ where } C = e^B \text{ or } -e^B$$

Substituting $y(0) = 1$:

$$1 = Ce^{-\frac{1}{2}\times 0^2}$$

$$1 = Ce^0$$

$$C = 1$$

Therefore $y = e^{-\frac{1}{2}x^2}$

19b ii

$$y_1(x) = y_0 + \int_0^x f(t, y_0)\,dt$$

$$= y_0 + \int_0^x (-y_0 t\,)dt$$

$$= 1 - \int_0^x t\,dt \qquad (\text{as } y_0 = 1)$$

$$= 1 - \left[\frac{t^2}{2}\right]_0^x$$

$$= 1 - \frac{1}{2}x^2$$

$$y_2(x) = y_0 + \int_0^x f\big(t, y_1(t)\big)\, dt$$

$$= y_0 + \int_0^x (-y_1(t)\ t)\, dt$$

$$= 1 - \int_0^x \left(1 - \frac{1}{2}t^2\right) t\, dt$$

$$= 1 - \int_0^x \left(t - \frac{1}{2}t^3\right)\, dt$$

$$= 1 - \left[\frac{1}{2}t^2 - \frac{1}{8}t^4\right]_0^x$$

$$= 1 - \frac{1}{2}x^2 + \frac{1}{8}x^4$$

$$y_3(x) = y_0 + \int_0^x f\big(t, y_2(t)\big)\, dt$$

$$= y_0 + \int_0^x (-y_2(t)t\ )\, dt$$

$$= 1 - \int_0^x \left(1 - \frac{1}{2}t^2 + \frac{1}{8}t^4\right) t\, dt$$

$$= 1 - \int_0^x \left(t - \frac{1}{2}t^3 + \frac{1}{8}t^5\right) dt$$

$$= 1 - \left[\frac{1}{2}t^2 - \frac{1}{8}t^4 + \frac{1}{48}t^6\right]_0^x$$

$$= 1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \frac{1}{48}x^6$$

$$y_4(x) = y_0 + \int_0^x f\big(t, y_3(t)\big)\, dt$$

$$= y_0 + \int_0^x (-y_3(t)t\ )dt$$

$$= 1 - \int_0^x \left(1 - \frac{1}{2}t^2 + \frac{1}{8}t^4 - \frac{1}{48}t^6\right) t\, dt$$

$$= 1 - \int_0^x \left(t - \frac{1}{2}t^3 + \frac{1}{8}t^5 - \frac{1}{48}t^7\right) dt$$

$$= 1 - \left[\frac{1}{2}t^2 - \frac{1}{8}t^4 + \frac{1}{48}t^6 - \frac{1}{384}t^8\right]_0^x$$

$$= 1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \frac{1}{48}x^6 + \frac{1}{384}x^8$$

19b iii

$$y\left(\frac{1}{2}\right) = e^{-\frac{1}{2}\left(\frac{1}{2}\right)^2} = e^{-\frac{1}{8}}$$

Therefore $e = \left(y\left(\frac{1}{2}\right)\right)^{-8}$

Applying the formula above,

$$y_4\left(\frac{1}{2}\right) \doteqdot 0.882\,497\,2$$

and $e \doteqdot \left(y_4\left(\frac{1}{2}\right)\right)^{-8} \doteqdot 2.7183$

Solutions to Exercise 13D

Let the integration constants be A, B, C or D.

1a $y = 0$

1b

$$\frac{dy}{dx} = -y$$

$$\left(\frac{1}{y}\right) dy = (-1)\, dx$$

$$\int \left(\frac{1}{y}\right) dy = \int (-1)\, dx$$

$$\ln|y| = -x + C$$

1c $\ln|y| = \ln e^{-x+C}$

$$|y| = e^{-x+C}$$

$$|y| = e^{C} e^{-x}$$

$$y = e^{C} e^{-x} \text{ or } -e^{C} e^{-x}$$

$$y = Ae^{-x} \text{ where } A = e^{C} \text{ or } -e^{C}$$

1d $y = 0$ when $A = 0$

1e Substituting $y(0) = 2$:

$$2 = Ae^{-0}$$

$$A = 2$$

Therefore $y = 2e^{-x}$

2a $y = 0$

2b

$$\frac{dx}{dy} = \frac{1}{3y}$$

2c

$$\int \frac{dx}{dy}\, dy = \int \frac{1}{3y}\, dy$$

$$\int \frac{dx}{dy}\, dy = \frac{1}{3}\int \frac{1}{y}\, dy$$

$$x = \frac{1}{3}\ln|y| + C$$

2d

$$x - C = \frac{1}{3}\ln|y|$$

$$\ln|y| = 3(x - C)$$

$$|y| = e^{3(x-C)}$$

$$|y| = e^{-3C}e^{3x}$$

$$y = e^{-3C}e^{3x} \text{ or} - e^{-3C}e^{3x}$$

$$y = Ae^{3x} \text{ where } A = e^{-3C} \text{ or } -e^{-3C}$$

2e $y = 0$ when $A = 0$

2f Substituting $y(0) = -1$:

$$-1 = Ae^{3(0)}$$

$$A = -1$$

Therefore $y = -e^{3x}$

3a $y' - y = 0$

$y' = y$

$\frac{dy}{dx} = y$

$\frac{dx}{dy} = \frac{1}{y}$

$\int \frac{dx}{dy}\, dy = \int \frac{1}{y}\, dy$

$x = \ln|y| + C$

$x - C = \ln|y|$

$|y| = e^{x-C}$

$|y| = e^{-C}e^{x}$

$y = e^{-C}e^{x}$ or $-e^{-C}e^{x}$

$y = Ae^{x}$ where $A = e^{-C}$ or $-e^{-C}$

Substituting $y(0) = -3$:

$-3 = Ae^{0}$

$-3 = A$

Therefore $y = -3e^{x}$

3b $y' + 2y = 0$

$y' = -2y$

$\frac{dy}{dx} = -2y$

$\frac{dx}{dy} = -\frac{1}{2y}$

$\int \frac{dx}{dy}\, dy = -\frac{1}{2}\int \frac{1}{y}\, dy$

$x = -\frac{1}{2}\ln|y| + C$

$\ln|y| = 2C - 2x$

$|y| = e^{2C-2x}$

$|y| = e^{2C}e^{-2x}$

$y = e^{2C}e^{-2x}$ or $- e^{2C}e^{-2x}$

$y = Ae^{-2x}$ where $A = e^{2C}$ or $-e^{2C}$

Substituting $y(0) = 1$:

$1 = Ae^0$

$1 = A$

Therefore $y = e^{-2x}$

3c $y' = -3y$

$\frac{dy}{dx} = -3y$

$\int \frac{dx}{dy}\, dy = -\frac{1}{3}\int \frac{1}{y}\, dy$

$x = -\frac{1}{3}\ln|y| + C$

$\ln|y| = 3C - 3x$

$|y| = e^{3C-3x}$

$|y| = e^{3C}e^{-3x}$

$y = e^{3C}e^{-3x}$ or $- e^{3C}e^{-3x}$

$y = Ae^{-3x}$ where $A = e^{3C}$ or $-e^{3C}$

Substituting $y(0) = 2$:

$2 = Ae^0$

$2 = A$

Therefore $y = 2e^{-3x}$

3d $y' = 2y$

$\frac{dy}{dx} = 2y$

$\frac{dx}{dy} = \frac{1}{2y}$

$$\int \frac{dx}{dy}\,dy = \frac{1}{2}\int \frac{1}{y}\,dy$$

$$x = \frac{1}{2}\ln|y| + C$$

$$2x - 2C = \ln|y|$$

$$|y| = e^{2x-2C}$$

$$|y| = e^{-2C}e^{2x}$$

$$y = e^{-2C}e^{2x} \text{ or } -e^{-2C}e^{2x}$$

$$y = Ae^{2x} \text{ where } A = e^{-2C} \text{ or } -e^{-2C}$$

Substituting $y(0) = -1$:

$$-1 = Ae^{0}$$

$$-1 = A$$

Therefore $y = -e^{2x}$

4a $\quad y = 2$

4b

$$\frac{dy}{dx} = 2 - y$$

$$\frac{dy}{dx} = -1(y-2)$$

$$\frac{1}{y-2}\,dy = (-1)\,dx$$

$$\int \frac{1}{y-2}\,dy = \int (-1)\,dx$$

$$\ln|y-2| = -x + C$$

4c $\ln|y-2| = -x + C$

$|y-2| = e^{-x+C}$

$y-2 = e^{C}e^{-x} \text{ or } -e^{C}e^{-x}$

$y-2 = Ae^{-x} \text{ where } A = e^{C} \text{ or } -e^{C}$

$y = 2 + Ae^{-x}$

4d $y = 2$ when $A = 0$

4e Substituting $y(0) = 3$:

$3 = 2 + Ae^{0}$

$1 = A$

$y = 2 + e^{-x}$

5a $y' = 1 - y$

$$\frac{dy}{dx} = 1 - y$$

$$\frac{dy}{dx} = -1(y-1)$$

$$\frac{1}{y-1}\,dy = (-1)\,dx$$

$$\int \frac{1}{y-1}\,dy = \int (-1)\,dx$$

$\ln|y-1| = -x + C$

$|y-1| = e^{-x+C}$

$y-1 = e^{C}e^{-x} \text{ or } -e^{C}e^{-x}$

$y-1 = Ae^{-x} \text{ where } A = e^{C} \text{ or } -e^{C}$

$y = 1 + Ae^{-x}$

Substituting $y(0) = 3$:

$3 = 1 + Ae^{0}$

$3 = 1 + A$

$2 = A$

Therefore $y = 1 + 2e^{-x}$

5b $y' = y - 1$

$\frac{dy}{dx} = y - 1$

$\frac{1}{y-1}\ dy = 1\ dx$

$\int \frac{1}{y-1}\ dy = \int 1\ dx$

$\ln|y - 1| = x + C$

$|y - 1| = e^{x+C}$

$y - 1 = e^C e^x$ or $-e^C e^x$

$y - 1 = Ae^x$ where $A = e^C$ or $-e^C$

$y = 1 + Ae^x$

Substituting $y(0) = 0$:

$0 = 1 + Ae^0$

$0 = 1 + A$

$-1 = A$

Therefore $y = 1 - e^x$

5c $y' = \frac{1}{2}(y + 1)$

$\frac{dy}{dx} = \frac{1}{2}(y + 1)$

$\frac{1}{y+1}\ dy = \frac{1}{2}\ dx$

$\int \frac{1}{y+1}\ dy = \int \frac{1}{2}\ dx$

$\ln|y + 1| = \frac{1}{2}x + C$

$|y + 1| = e^{\frac{1}{2}x+C}$

$y + 1 = e^C e^{\frac{1}{2}x}$ or $-e^C e^{\frac{1}{2}x}$

$y + 1 = Ae^{\frac{1}{2}x}$ where $A = e^C$ or $-e^C$

$y = -1 + Ae^{\frac{1}{2}x}$

Substituting $y(0) = 1$:

$1 = -1 + Ae^0$

$1 = -1 + A$

$2 = A$

Therefore $y = -1 + 2e^{\frac{1}{2}x}$

5d $y' = 2(3 - y)$

$\frac{dy}{dx} = 2(3 - y)$

$\frac{dy}{dx} = -2(y - 3)$

$\frac{1}{y - 3}\,dy = (-2)\,dx$

$\int \frac{1}{y - 3}\,dy = \int (-2)\,dx$

$\ln|y - 3| = -2x + C$

$|y - 3| = e^{-2x+C}$

$y - 3 = e^C e^{-2x}$ or $-e^C e^{-2x}$

$y - 3 = Ae^{-2x}$ where $A = e^C$ or $-e^C$

$y = 3 + Ae^{-2x}$

Substituting $y(0) = 4$:

$4 = 3 + Ae^0$

$4 = 3 + A$

$1 = A$

Therefore $y = 3 + e^{-2x}$

6a $y' = 2y^2$

$$\frac{dy}{dx} = 2y^2$$

$$\int \frac{1}{y^2}\, dy = \int 2\, dx$$

$$-y^{-1} = 2x + C$$

$$-\frac{1}{y} = 2x + C$$

Substituting $y(0) = 3$:

$$-\frac{1}{3} = 0 + C$$

$$C = -\frac{1}{3}$$

Therefore $-\frac{1}{y} = 2x - \frac{1}{3}$

$$-\frac{1}{y} = \frac{6x - 1}{3}$$

$$-\frac{y}{1} = \frac{3}{6x - 1}$$

$$y = -\frac{3}{6x - 1}$$

$$y = \frac{3}{1 - 6x}$$

6b $y' = -y^2$

$$\frac{dy}{dx} = -y^2$$

$$\int \left(-\frac{1}{y^2}\right) dy = \int 1\, dx$$

$$y^{-1} = x + C$$

$$\frac{1}{y} = x + C$$

Substituting $y(0) = 1$:

$$\frac{1}{1} = 0 + C$$

$$C = 1$$

Therefore $\frac{1}{y} = x + 1$

$$\frac{y}{1} = \frac{1}{x+1}$$

$$y = \frac{1}{x+1}$$

6c $y' = 1 + y^2$

$$\frac{dy}{dx} = 1 + y^2$$

$$\int \frac{1}{1+y^2}\,dy = \int 1\,dx$$

$$\tan^{-1} y = x + C$$

Substituting $y\left(\frac{\pi}{4}\right) = 1$:

$$\tan^{-1} 1 = \frac{\pi}{4} + C$$

$$\frac{\pi}{4} = \frac{\pi}{4} + C$$

$$0 = C$$

Therefore $\tan^{-1} y = x$

$$y = \tan x$$

6d $y' = -e^y$

$$\frac{dy}{dx} = -e^y$$

$$\int \left(-\frac{1}{e^y}\right)\,dy = \int 1\,dx$$

$$\int (-e^{-y})\,dy = \int 1\,dx$$

$e^{-y} = x + C$

$\ln(e^{-y}) = \ln(x + C)$

$-y \ln e = \ln(x + C)$

$-y = \ln(x + C)$

$y = -\ln(x + C)$

Substituting $y(0) = 0$:

$0 = -\ln C$

$C = e^0$

$C = 1$

Therefore $y = -\ln(x + 1)$

6e $y' = e^{-y}$

$\frac{dy}{dx} = e^{-y}$

$\int \frac{1}{e^{-y}}\, dy = \int 1\, dx$

$\int e^y\, dy = \int 1\, dx$

$e^y = x + C$

$\ln e^y = \ln(x + C)$

$y = \ln(x + C)$

Substituting $y(3) = 0$:

$0 = \ln(3 + C)$

$e^0 = 3 + C$

$1 = 3 + C$

$C = -2$

Therefore $y = \ln(x - 2)$

6f $y' = y^{\frac{2}{3}}$

$\frac{dy}{dx} = y^{\frac{2}{3}}$

$\int \frac{1}{y^{\frac{2}{3}}}\, dy = \int 1\, dx$

$\int y^{-\frac{2}{3}}\, dy = \int 1\, dx$

$\frac{y^{\frac{1}{3}}}{\frac{1}{3}} = x + C$

$3y^{\frac{1}{3}} = x + C$

Substituting $y(0) = 1$:

$3(1^{\frac{1}{3}}) = 0 + C$

$C = 3$

Therefore $3y^{\frac{1}{3}} = x + 3$

$y^{\frac{1}{3}} = \frac{x+3}{3}$

$y = \left(\frac{x+3}{3}\right)^3$

7a $y' = ky$

$\frac{dy}{dx} = ky$

$\int \frac{1}{y}\, dy = \int k\, dx$

$\ln|y| = kx + C$

$|y| = e^{kx+C}$

$y = e^C e^{kx}\ \ or\ \ -e^C e^{kx}$

$y = Ae^{kx}$ where $A = e^C$ or $-e^C$

7b Substituting $y(0) = 20$:

$20 = Ae^{k(0)}$

$A = 20$

7c $y = 20e^{kx}$

Substituting $y(2) = 5$:

$5 = 20e^{k(2)}$

$\frac{1}{4} = e^{2k}$

$2k = \ln\left(\frac{1}{4}\right)$

$k = \frac{1}{2}\ln\left(\frac{1}{4}\right)$

$k = \frac{1}{2}\ln(2^{-2})$

$k = -2 \times \frac{1}{2}\ln 2$

$k = -\ln 2$

7d $y = 20e^{(-\ln 2)x}$

$y = 20e^{-x\ln 2}$

$y = 20e^{\ln 2^{-x}}$

$y = 20 \times 2^{-x}$

$y(3) = 20 \times 2^{-3}$

$= \frac{20}{8}$

$= 2\frac{1}{2}$

8a $y' = ky$

$\frac{dy}{dx} = ky$

$$\int \frac{1}{y}\,dy = \int k\,dx$$

$$\ln|y| = kx + C$$

$$|y| = e^{kx+C}$$

$$y = e^C e^{kx} \ \ or \ -e^C e^{kx}$$

$$y = Ae^{kx} \text{ where } A = e^C \text{ or } -e^C$$

8b Substituting $y(0) = 8$:

$$8 = Ae^{k(0)}$$

$$A = 8$$

8c $y = 8e^{kx}$

Substituting $y(2) = 18$:

$$18 = 8e^{k(2)}$$

$$\frac{18}{8} = e^{2k}$$

$$2k = \ln\left(\frac{9}{4}\right)$$

$$k = \frac{1}{2}\ln\left(\frac{9}{4}\right)$$

$$k = \frac{1}{2}\ln\left(\frac{3}{2}\right)^2$$

$$k = 2 \times \frac{1}{2}\ln\frac{3}{2}$$

$$k = \ln\frac{3}{2}$$

8d

$$y = 8e^{(\ln\frac{3}{2})x}$$

$$y = 8e^{x\ln\frac{3}{2}}$$

$$y = 8e^{\ln\left(\frac{3}{2}\right)^x}$$

$$y = 8 \times \left(\frac{3}{2}\right)^x$$

$$y(4) = 8 \times \left(\frac{3}{2}\right)^4$$

$$= 8 \times \frac{81}{16}$$

$$= 40\frac{1}{2}$$

9a $y = Ae^{\lambda x} + Be^{-\lambda x} + C\cos\lambda x + D\sin\lambda x$

$$y' = A\lambda e^{\lambda x} - B\lambda e^{-\lambda x} - C\lambda\sin\lambda x + D\lambda\cos\lambda x$$

$$y'' = A\lambda^2 e^{\lambda x} + B\lambda^2 e^{-\lambda x} - C\lambda^2\cos\lambda x - D\lambda^2\sin\lambda x$$

$$y''' = A\lambda^3 e^{\lambda x} - B\lambda^3 e^{-\lambda x} + C\lambda^3\sin\lambda x - D\lambda^3\cos\lambda x$$

$$y'''' = A\lambda^4 e^{\lambda x} + B\lambda^4 e^{-\lambda x} + C\lambda^4\cos\lambda x + D\lambda^4\sin\lambda x$$

$$\text{LHS} = y^{(4)}$$

$$= A\lambda^4 e^{\lambda x} + B\lambda^4 e^{-\lambda x} + C\lambda^4\cos\lambda x + D\lambda^4\sin\lambda x$$

$$\text{RHS} = \lambda^4 y$$

$$= \lambda^4(Ae^{\lambda x} + Be^{-\lambda x} + C\cos\lambda x + D\sin\lambda x)$$

$$= A\lambda^4 e^{\lambda x} + B\lambda^4 e^{-\lambda x} + C\lambda^4\cos\lambda x + D\lambda^4\sin\lambda x$$

$$\text{LHS} = \text{RHS}$$

Therefore y is a solution of the DE.

9b Substituting $y(0) = 0$:

$0 = Ae^{\lambda(0)} + Be^{-\lambda(0)} + C\cos\lambda(0) + D\sin\lambda(0)$

$0 = A + B + C \qquad (1)$

Substituting $y''(0) = 0$:

$0 = A\lambda^2 e^{\lambda(0)} + B\lambda^2 e^{-\lambda(0)} - C\lambda^2\cos\lambda(0) - D\lambda^2\sin\lambda(0)$

$0 = A\lambda^2 + B\lambda^2 - C\lambda^2$

$0 = \lambda^2(A + B - C)$

$0 = A + B - C \quad \text{where } \lambda \neq 0 \qquad (2)$

$(1) - (2)$:

$2C = 0$

$C = 0$

9c i Substituting $y(10) = 0$ and $C = 0$:

$0 = Ae^{\frac{n\pi}{10}(10)} + Be^{-\frac{n\pi}{10}(10)} + D\sin\left(\frac{n\pi}{10}(10)\right)$

$0 = Ae^{n\pi} + Be^{-n\pi} + D\sin n\pi \qquad (1)$

Substituting $y''(10) = 0$ and $C = 0$:

$0 = A\lambda^2 e^{10\lambda} + B\lambda^2 e^{-10\lambda} - C\lambda^2\cos 10\lambda - D\lambda^2\sin 10\lambda$

$0 = \lambda^2(Ae^{n\pi} + Be^{-n\pi} - D\sin n\pi)$

$0 = Ae^{n\pi} + Be^{-n\pi} - D\sin n\pi \quad \text{where } \lambda \neq 0 \qquad (2)$

$(1) + (2)$:

$2Ae^{n\pi} + 2Be^{-n\pi} + D\sin n\pi - D\sin n\pi = 0$

$2Ae^{n\pi} + 2Be^{-n\pi} = 0$

$Ae^{n\pi} + Be^{-n\pi} = 0$

$Ae^{n\pi} = -Be^{-n\pi}$

$A = -Be^{-2n\pi} \qquad (3)$

From part b above, $A + B - C = 0$ and $C = 0$

Therefore $A + B = 0$

$A = -B \qquad (4)$

Combining (3) and (4):

$-B = -Be^{-2n\pi}$

$B = Be^{-2n\pi}$

Either $B = 0$ or $e^{-2n\pi} = 1$, i.e. $n = 0$.

If $n = 0$ then $\lambda = 0$ and the beam equation simplifies to $y = A + B = 0$.

Without loss of generality we can then set $A = B = 0$.

If $n \neq 0$ then $B = 0$ and therefore $A = 0$.

9c ii Since $A = B = C = 0$,

$y = D\sin\left(\frac{n\pi}{10}x\right)$

10a $y' = e^{-y}$

$\frac{dy}{dx} = e^{-y}$

$\int \frac{1}{e^{-y}}\,dy = \int 1\,dx$

$\int e^{y}\,dy = \int 1\,dx$

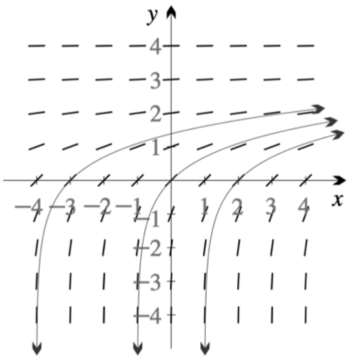
$e^{y} = x + C$

$\ln e^{y} = \ln(x + C)$

$y = \ln(x + C)$

10b Log curves with different x-intercepts

10c i,ii



10c iii Shift left or right

10c iv The isoclines are horizontal lines.

10d $y = \ln(x + C)$

Substituting $(0, 1)$:

$1 = \ln(0 + C)$

$e^1 = 0 + C$

$C = e$

Therefore $y = \ln(x + e)$

11a $y'y = 2$

$\frac{dy}{dx}y = 2$

$\int y\,dy = \int 2\,dx$

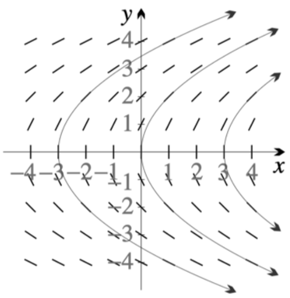
$\frac{y^2}{2} = 2x + B$

$y^2 = 4x + 2B$

$y^2 = 4x + C$ where $C = 2B$

11b Concave right parabolas with vertex on the x-axis.

11c i, ii



11c iii Shift left or right

11c iv The isoclines are horizontal lines.

11d $y^2 = 4x + C$

Substituting $(0, 1)$:

$1 = 4(0) + C$

$C = 1$

Therefore $y^2 = 4x + 1$

12a

$$L(x) = \frac{1}{1 + e^{-x}}$$

For y-intercept, $x = 0$.

$$L(0) = \frac{1}{1 + e^0}$$

$$L(0) = \frac{1}{2}$$

y-intercept at $\left(0, \frac{1}{2}\right)$

12b $e^{-x} > 0$ for all x so $L(x)$ is always positive

12c Since $e^{-x} \to 0$ as $x \to \infty$, $\lim_{x\to\infty} L(x) = 1$

and $e^{-x} \to \infty$ as $x \to -\infty$, so $\lim_{x\to-\infty} L(x) = 0$

12d $L(x) = (1 + e^{-x})^{-1}$

$L'(x) = -(1 + e^{-x})^{-2} \times -e^{-x}$

$= \dfrac{e^{-x}}{(1 + e^{-x})^2}$

For stationary points to exist, $L'(x) = 0$.

$\dfrac{e^{-x}}{(1 + e^{-x})^2} = 0$

$e^{-x} = 0$ which has no solution

Therefore, there are no stationary points.

12e i $L'(x)$

$= \dfrac{e^{-x}}{(1 + e^{-x})^2}$

$= \dfrac{1}{e^{x}(1 + e^{-x})^2}$

$= \dfrac{1}{e^{x}(1 + 2e^{-x} + e^{-2x})}$

$= \dfrac{1}{(e^{x} + 2 + e^{-x})}$

$= \dfrac{1}{\left(e^{\frac{x}{2}} + e^{-\frac{x}{2}}\right)^2}$

12e ii

$$L'(x) = \left(e^{\frac{x}{2}} + e^{-\frac{x}{2}}\right)^{-2}$$

$$L''(x) = -2\left(e^{\frac{x}{2}} + e^{-\frac{x}{2}}\right)^{-3} \times \left(\frac{1}{2}e^{\frac{x}{2}} - \frac{1}{2}e^{-\frac{x}{2}}\right)$$

$$= -2\left(e^{\frac{x}{2}} + e^{-\frac{x}{2}}\right)^{-3} \times \frac{1}{2}\left(e^{\frac{x}{2}} - e^{-\frac{x}{2}}\right)$$

$$= -\left(e^{\frac{x}{2}} + e^{-\frac{x}{2}}\right)^{-3} \times \left(e^{\frac{x}{2}} - e^{-\frac{x}{2}}\right)$$

$$= -\frac{\left(e^{\frac{x}{2}} - e^{-\frac{x}{2}}\right)}{\left(e^{\frac{x}{2}} + e^{-\frac{x}{2}}\right)^{3}}$$

12e iii Point of inflection when $L''(x) = 0$.

$$-\frac{\left(e^{\frac{x}{2}} - e^{-\frac{x}{2}}\right)}{\left(e^{\frac{x}{2}} + e^{-\frac{x}{2}}\right)^{3}} = 0$$

$$e^{\frac{x}{2}} - e^{-\frac{x}{2}} = 0$$

$$e^{-\frac{x}{2}}(e^{-x} - 1) = 0$$

$$e^{-\frac{x}{2}} = 0 \Longrightarrow \text{ No solution}$$

$$e^{-x} - 1 = 0$$

$$e^{-x} = 1$$

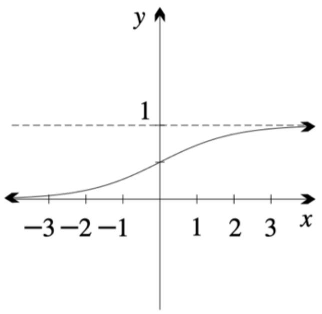
$$-x = \ln 1$$

$$x = 0$$

$$L(0) = (1 + e^{0})^{-1} = (2)^{-1} = \frac{1}{2}$$

$$\text{POI} = \left(0, \frac{1}{2}\right)$$

12f



12g i $y' = y(1-y)$

$$\text{LHS} = y' = L'(x) = \frac{e^{-x}}{(1+e^{-x})^2} \qquad \text{(from part a)}$$

$$\text{RHS} = y(1-y)$$

$$= \left(\frac{1}{1+e^{-x}}\right)\left(1-\frac{1}{1+e^{-x}}\right)$$

$$= \left(\frac{1}{1+e^{-x}}\right)\left(\frac{1+e^{-x}-1}{1+e^{-x}}\right)$$

$$= \left(\frac{1}{1+e^{-x}}\right)\left(\frac{e^{-x}}{1+e^{-x}}\right)$$

$$= \frac{e^{-x}}{(1+e^{-x})^2}$$

Therefore, since LHS = RHS, $L(x)$ is a solution.

12g ii $y' = y(1-y)$

$$= \left(\frac{1}{1+e^{-x}}\right)\left(1-\frac{1}{1+e^{-x}}\right)$$

$$= \left(\frac{1}{1+e^{-x}}\right)\left(\frac{1+e^{-x}-1}{1+e^{-x}}\right)$$

$$= \left(\frac{1}{1+e^{-x}}\right)\left(\frac{e^{-x}}{1+e^{-x}}\right)$$

$$= \frac{e^{-x}}{(1+e^{-x})^2}$$

$$= \frac{1}{e^{x}(1+e^{-x})^2}$$

$$= \frac{1}{e^x(1 + 2e^{-x} + e^{-2x})}$$

$$= \frac{1}{(e^x + 2 + e^{-x})}$$

$$= \frac{1}{\left(e^{\frac{x}{2}} + e^{-\frac{x}{2}}\right)^2}$$

12g iii $y' = y(1 - y) = y - y^2$

$$y'' = \frac{d}{dy}(y - y^2) \times \frac{dy}{dx}$$

$$= (1 - 2y) \times y'$$

$$= \left(1 - 2\left(\frac{1}{1 + e^{-x}}\right)\right) \times \left(e^{\frac{x}{2}} + e^{-\frac{x}{2}}\right)^{-2}$$

$$= \left(\frac{e^{-x} - 1}{1 + e^{-x}}\right) \times \left(e^{\frac{x}{2}} + e^{-\frac{x}{2}}\right)^{-2}$$

$$= \left(\frac{e^{-x} - 1}{1 + e^{-x}}\right) \times \frac{1}{\left(e^{\frac{x}{2}} + e^{-\frac{x}{2}}\right)^2}$$

$$= \left(\frac{1 - e^{-x}}{1 + e^{-x}}\right)\left(\frac{e^{\frac{x}{2}}}{e^{\frac{x}{2}}}\right) \times \frac{-1}{\left(e^{\frac{x}{2}} + e^{-\frac{x}{2}}\right)^2}$$

$$= \frac{\left(e^{\frac{x}{2}} - e^{-\frac{x}{2}}\right)}{\left(e^{\frac{x}{2}} + e^{-\frac{x}{2}}\right)} \times \frac{-1}{\left(e^{\frac{x}{2}} + e^{-\frac{x}{2}}\right)^2}$$

$$= -\frac{\left(e^{\frac{x}{2}} - e^{-\frac{x}{2}}\right)}{\left(e^{\frac{x}{2}} + e^{-\frac{x}{2}}\right)^3}$$

13a

Let $\dfrac{1}{y(1-y)} = \dfrac{A}{y} + \dfrac{B}{1-y}$

$\text{RHS} = \dfrac{A(1-y) + By}{y(1-y)}$

$= \dfrac{A - Ay + By}{y(1-y)}$

$= \dfrac{A + y(-A + B)}{y(1-y)}$

Equating coefficients of numerators in LHS and RHS:

$A = 1$

$-A + B = 0$

$-1 + B = 0$

$B = 1$

Therefore $\dfrac{1}{y(1-y)} = \dfrac{1}{y} + \dfrac{1}{1-y}$

13b i $y = 0$ and $y = 1$

13b ii $y' = y(1-y)$

$\dfrac{dy}{dx} = y(1-y)$

$\displaystyle\int \frac{1}{y(1-y)}\,dy = \int 1\ dx$

$\displaystyle\int \left(\frac{1}{y} + \frac{1}{1-y}\right) dy = \int 1\ dx$

$\ln|y| - \ln|1-y| = x + C$

$\ln\left|\dfrac{y}{1-y}\right| = x + C$

$\left|\dfrac{y}{1-y}\right| = e^{x+C}$

$$\left|\frac{y}{1-y}\right| = e^C \times e^x$$

$$\frac{y}{1-y} = e^C \times e^x \text{ or } -e^C \times e^x$$

$$\frac{y}{1-y} = Ae^x \text{ where } A = e^C \text{ or } -e^C$$

$$y = Ae^x(1-y)$$

$$y = Ae^x - yAe^x$$

$$y + yAe^x = Ae^x$$

$$y(1+Ae^x) = Ae^x$$

$$y = \frac{Ae^x}{1+Ae^x}$$

$$y = \frac{1}{A^{-1}e^{-x}(1+Ae^x)}$$

$$y = \frac{1}{A^{-1}e^{-x}+1}$$

$$y = \frac{1}{Be^{-x}+1} \text{ where } B = A^{-1}$$

$$y = \frac{1}{1+Be^{-x}}$$

13c

$$L(x) = \frac{1}{1+e^{-x}} \longrightarrow y = \frac{1}{1+Be^{-x}}$$

$$y = \frac{1}{1+Be^{-x}}$$

$$= \frac{1}{1+e^{\ln B}e^{-x}}$$

$$= \frac{1}{1+e^{-x+\ln B}}$$

$$= \frac{1}{1+e^{-(x-\ln B)}}$$

Therefore there is a shift of $\ln B$ to the right.

13d i

$$\lim_{B\to\infty}\frac{1}{1+Be^{-x}}=0$$

Yes, this is one of the constant solutions.

13d ii

$$\lim_{B\to 0^+}\frac{1}{1+Be^{-x}}=1$$

Yes, this is one of the constant solutions.

14a $y=0$ and $y=1$

14b $y'=ry(1-y)$

$$\frac{dy}{dx}=ry(1-y)$$

$$\int\frac{1}{y(1-y)}\,dy=\int r\ dx$$

$$\int\left(\frac{1}{y}+\frac{1}{1-y}\right)\,dy=\int r\ dx$$

$$\ln|y|-\ln|1-y|=rx+C$$

$$\ln\left|\frac{y}{1-y}\right|=rx+C$$

$$\left|\frac{y}{1-y}\right|=e^{rx+C}$$

$$\left|\frac{y}{1-y}\right|=e^{C}\times e^{rx}$$

$$\frac{y}{1-y}=e^{C}\times e^{rx}\ \text{ or }\ -e^{C}\times e^{rx}$$

$$\frac{y}{1-y}=Ae^{rx}\text{ where }A=e^{C}\text{ or }-e^{C}$$

$$y=Ae^{rx}(1-y)$$

$$y=Ae^{rx}-yAe^{rx}$$

$$y+yAe^{rx}=Ae^{rx}$$

$$y(1 + Ae^{rx}) = Ae^{rx}$$

$$y = \frac{Ae^{rx}}{1 + Ae^{rx}}$$

$$y = \frac{1}{A^{-1}e^{-rx}(1 + Ae^{rx})}$$

$$y = \frac{1}{A^{-1}e^{-rx} + 1}$$

$$y = \frac{1}{Be^{-rx} + 1} \quad \text{where } B = A^{-1}$$

$$y = \frac{1}{1 + Be^{-rx}}$$

14c Substituting $y(0) = y_0$:

$$y_0 = \frac{1}{1 + Be^{-r(0)}}$$

$$y_0 = \frac{1}{1 + B}$$

$$1 + B = \frac{1}{y_0}$$

$$B = \frac{1}{y_0} - 1$$

14d

$$y = \frac{1}{1 + Be^{-rx}}$$

$$= \frac{1}{1 + e^{\ln B}e^{-rx}}$$

$$= \frac{1}{1 + e^{-rx + \ln B}}$$

$$= \frac{1}{1 + e^{-r\left(x - \frac{1}{r}\ln B\right)}}$$

Therefore, there is a shift of $\frac{1}{r}\ln B$ to the right.

14e i

$$B = \frac{1}{y_0} - 1$$

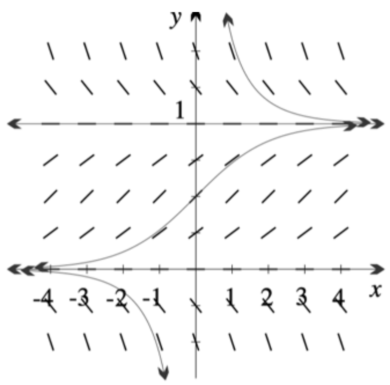
As $y_0 \to 0^+, B \to \infty$ so $y = 0$ in the limit.

14e ii

$$B = \frac{1}{y_0} - 1$$

As $y_0 \to 1^-, B \to 0^+$ so $y = 1$ in the limit.

15a,b,d



15c

$$y = \frac{1}{1 - e^{-x}}$$

$$= (1 - e^{-x})^{-1}$$

$$y' = (1 - e^{-x})^{-2} \times -e^{-x}$$

$$= -e^{-x}(1 - e^{-x})^{-2}$$

$$y' = y(1 - y)$$

$$\text{RHS} = \frac{1}{1 - e^{-x}}\left(1 - \frac{1}{1 - e^{-x}}\right)$$

$$= \frac{1}{1 - e^{-x}}\left(\frac{1 - e^{-x} - 1}{1 - e^{-x}}\right)$$

$$= -e^{-x}(1 - e^{-x})^{-2}$$

$= \text{LHS}$

So $y = \dfrac{1}{1 - e^{-x}}$ is a solution of the DE $y' = y(1 - y)$.

16a

Let $\dfrac{1}{(y-1)(y-3)} = \dfrac{A}{y-1} + \dfrac{B}{y-3}$

$\text{RHS} = \dfrac{A(y-3) + B(y-1)}{(y-1)(y-3)}$

$= \dfrac{Ay - 3A + By - B}{(y-1)(y-3)}$

$= \dfrac{y(A+B) - 3A - B}{(y-1)(y-3)}$

Equating coefficients of numerators in LHS and RHS:

$A + B = 0 \qquad (1)$

$-3A - B = 1 \qquad (2)$

$(1) + (2)$:
$-2A = 1$

$A = -\dfrac{1}{2}$

Substituting $A = -\frac{1}{2}$ into (1):

$-\dfrac{1}{2} + B = 0$

$B = \dfrac{1}{2}$

Therefore

$$\frac{1}{(y-1)(y-3)} = \frac{-\frac{1}{2}}{y-1} + \frac{\frac{1}{2}}{y-3}$$

$$= \frac{1}{2(y-3)} - \frac{1}{2(y-1)}$$

$$= \frac{1}{2}\left(\frac{1}{(y-3)} - \frac{1}{(y-1)}\right)$$

16b i $y = 1$ and $y = 3$

16b ii $y' = -(1-y)(3-y)$

$$\frac{dy}{dx} = -(1-y)(3-y)$$

$$\int \frac{1}{(1-y)(3-y)}\,dy = \int (-1)\,dx$$

$$\int \frac{1}{(y-1)(y-3)}\,dy = \int (-1)\,dx$$

$$\frac{1}{2}\int \left(\frac{1}{(y-3)} - \frac{1}{(y-1)}\right)dy = \int (-1)\,dx$$

$$\int \left(\frac{1}{(y-3)} - \frac{1}{(y-1)}\right)dy = \int (-2)\,dx$$

$$\ln|y-3| - \ln|y-1| = -2x + C$$

$$\ln\left|\frac{y-3}{y-1}\right| = -2x + C$$

$$\left|\frac{y-3}{y-1}\right| = e^{-2x+C}$$

$$\left|\frac{y-3}{y-1}\right| = e^{C} \times e^{-2x}$$

$$\frac{y-3}{y-1} = e^{C} \times e^{-2x} \text{ or } -e^{C} \times e^{-2x}$$

$$\frac{y-3}{y-1} = Ae^{-2x} \text{ where } A = e^{C} \text{ or } -e^{C}$$

$$y - 3 = Ae^{-2x}(y-1)$$

$$y - 3 = yAe^{-2x} - Ae^{-2x}$$

$$y - yAe^{-2x} = 3 - Ae^{-2x}$$

$$y(1 - Ae^{-2x}) = 3 - Ae^{-2x}$$

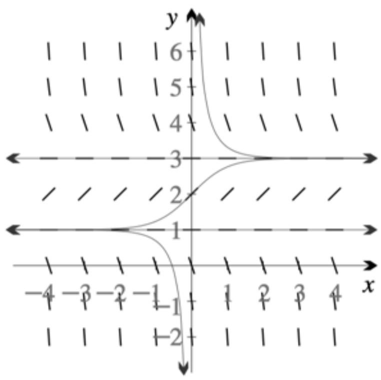
$$y = \frac{3 - Ae^{-2x}}{1 - Ae^{-2x}}$$

$$y = \frac{3 + Be^{-2x}}{1 + Be^{-2x}} \text{ where } B = -A$$

16c i As $B \to 0^+$, $y \to 3$, so $y = 3$ is the constant solution.

16c ii As $B \to \infty$, $y \to 1$, so $y = 1$ is the constant solution.

16d



16e i $y' = -(1-y)(3-y)$

$$y'' = -\frac{d}{dy}(1-y)(3-y) \times \frac{dy}{dx}$$

$$= -\frac{d}{dy}(3 - 4y + y^2) \times y'$$

$$= -(-4 + 2y) \times -(1-y)(3-y)$$

$$= 2(-2+y)(1-y)(3-y)$$

$$= -2(2-y)(1-y)(3-y)$$

$$= -2(1-y)(2-y)(3-y)$$

16e ii For points of inflection, $y'' = 0$

$-2(1-y)(2-y)(3-y) = 0$

$y = 1, 2$ or 3

However, $y = 1$ and $y = 3$ are constant solutions so only point of inflection at $y = 2$.

For $B = 1$

$$y = \frac{3 + e^{-2x}}{1 + e^{-2x}}$$

$$2 = \frac{3 + e^{-2x}}{1 + e^{-2x}}$$

$$2 + 2e^{-2x} = 3 + e^{-2x}$$

$$e^{-2x} = 1$$

$$x = 0$$

Point of inflection at $(0, 2)$.

17a

$$v = \frac{1}{y}$$

$$v' = -\frac{1}{y^2}y'$$

$$= -v^2y' \qquad \left(\text{as } v^2 = \frac{1}{y^2}\right)$$

$$= -v^2ry(1-y) \qquad (\text{as } y' = ry(1-y))$$

$$= -v^2ry^2\left(\frac{1-y}{y}\right)$$

$$= -rv^2y^2\left(\frac{1}{y} - 1\right)$$

$$= -r\left(\frac{1}{y} - 1\right) \qquad (\text{as } v^2y^2 = 1)$$

$$= -r(v-1)$$

$$= r(1-v)$$

17b $v' = r(1-v)$

$$\frac{dv}{dx} = r(1-v)$$

$$\frac{dv}{dx} = -r(v-1)$$

$$\int \frac{1}{v-1}\,dy = \int (-r)\,dx$$

$$\ln|v-1| = -rx + C$$

$$|v-1| = e^{-rx+C}$$

$$|v-1| = e^C e^{-rx}$$

$$v - 1 = e^C e^{-rx} \text{ or } -e^C e^{-rx}$$

$$v - 1 = Be^{-rx} \text{ where } B = e^C \text{ or } -e^C$$

$$v = 1 + Be^{-rx}$$

17c

$$v = \frac{1}{y}$$

$$y = \frac{1}{v}$$

$$y = \frac{1}{1 + Be^{-rx}}$$

18a $y'' = 2(1 - y')$

$$v' = 2(1 - v)$$

18b $y'(0) = 0$ so $v(0) = 0$

18c $v' = 2(1 - v)$

$$\frac{dv}{dx} = 2(1 - v)$$

$$\frac{dv}{dx} = -2(v - 1)$$

$$\int \frac{1}{v-1}\,dy = \int (-2)\,dx$$

$$\ln|v-1| = -2x + C$$

$$|v-1| = e^{-2x+C}$$

$$|v-1| = e^C e^{-2x}$$

$$v - 1 = e^C e^{-2x} \text{ or } -e^C e^{-2x}$$

$v - 1 = Be^{-2x}$ where $B = e^{C}$ or $- e^{C}$

$v = 1 + Be^{-2x}$

(Alternatively, use $r = 2$ in your answer to Q17b.)

Substituting $v(0) = 0$ from part b:

$0 = 1 + Be^{-2(0)}$

$0 = 1 + B$

$-1 = B$

Therefore $v = 1 - e^{-2x}$

18d Since $v = y'$,

$y' = 1 - e^{-2x}$

18e

$$\int y'\,dx = \int (1 - e^{-2x})dx$$

$y = x + \frac{1}{2}e^{-2x} + C$

Substituting $y(0) = 1$:

$$1 = 0 + \frac{1}{2}e^{-2(0)} + C$$

$$1 = \frac{1}{2} + C$$

$$\frac{1}{2} = C$$

Therefore $y = x + \dfrac{1}{2}e^{-2x} + \dfrac{1}{2}$

19a We are given:

$$\frac{df(x)}{dx} = g\big(f(x)\big)$$

Substitute $u = x - C$

Note that $\frac{du}{dx} = 1$

If $y = f(x - C)$

$y = f(u)$

$$y' = \frac{df(u)}{du} \times \frac{du}{dx}$$

$$= \frac{df(u)}{du} \times 1$$

$$= \frac{df(u)}{du}$$

$$= g\big(f(u)\big)$$

$$= g\big(f(x - C)\big)$$

$$= g\big(f(y)\big)$$

So $y = f(x - C)$ is also a solution.

19b Shift right by C

19c Since translating a solution horizontally gives another solution, all horizontal lines are isoclines.

19d If a graph is shifted right, its gradient at a given height is unchanged by the shift.

20a i

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{dx}\right)\\ &= \frac{dv}{dx} \qquad \left(\text{as } v = \frac{dy}{dx}\right)\\ &= \frac{dv}{dy} \times \frac{dy}{dx}\\ &= \frac{dv}{dy} \times v\\ &= v\frac{dv}{dy}\end{aligned}$$

20a ii

$$\frac{d^2y}{dx^2} + y = 0$$

$$v\frac{dv}{dy} + y = 0$$

$$v\frac{dv}{dy} = -y$$

20b

$$v\frac{dv}{dy} = -y$$

$$v\,dv = -y\,dy$$

$$\int v\,dv = -\int y\,dy$$

$$\frac{v^2}{2} = -\frac{y^2}{2} + B$$

$$v^2 = -y^2 + 2B$$

$$v^2 + y^2 = C \text{ where } C = 2B$$

20c If C is negative, there is no solution.

If C is zero, the solution is a single point, the origin.

If C is positive, the solution is a circle with centre the origin and radius $\sqrt{C}$.

20d Standard parametrisation for a circle: $v = r\cos x\,, y = r\sin x$

20e $y = r\sin x$

$$y' = \frac{d}{dx}(r\sin x)$$

$$= r\cos x$$

$$= v$$

20f The original DE is autonomous. From Question 19 we know that if $y = f(x)$ is a solution to the DE, then $y = f(x - D)$ is also a solution.

Hence the general solution is $y = r\sin(x - D)$.

20g i $y = r\sin(x - D)$

$$= r\cos D\sin x - r\sin D\cos x$$

$$= A\cos x + B\sin x \text{ where } A = -r\sin D \text{ and } B = r\cos D$$

20g ii $y = A\cos x + B\sin x$

$$\frac{dy}{dx} = -A\sin x + B\cos x$$

$$\frac{d^2y}{dx^2} = -A\cos x - B\sin x$$

$$= -y$$

$$\frac{d^2y}{dx^2} + y = 0$$

21a

$$\frac{dy}{dx} = -\sqrt{1-y^2}$$

$$\int \frac{-1}{\sqrt{1-y^2}}\, dy = \int 1\, dx$$

$$\cos^{-1} y = x + C$$

21b Substituting $y(0) = 1$:

$$\cos^{-1} 1 = 0 + C$$

$$0 = 0 + C$$

$$C = 0$$

Therefore $\cos^{-1} y = x$

21c

$$\frac{dy}{dx} = -\sqrt{1-y^2}$$

If $y = \cos x$,

$$\text{LHS} = \frac{d}{dx}(\cos x) = -\sin x$$

$$\text{RHS} = -\sqrt{1-\cos^2 x} = -|\sin x|$$

So the two solutions differ when $\sin x$ is negative.

21d The DE holds when $\sin x$ is non-negative. The largest region containing the initial point where this is true is $0 \le x \le \pi$ so the solution is:

$$y = \cos x, 0 \le x \le \pi$$

22a $y = x^3$

$$\frac{dy}{dx} = 3x^2$$

Substituting into the initial value problem $\frac{dy}{dx} = 3y^{\frac{2}{3}}$:

LHS $= 3x^2$

RHS $= 3(x^3)^{\frac{2}{3}}$

$= 3x^2$

$=$ LHS

Hence $y = x^3$ is a solution of this IVP.

22b $y = 0$

$\frac{dy}{dx} = 0$

Substituting into the initial value problem $\frac{dy}{dx} = 3y^{\frac{2}{3}}$:

LHS $= 0$

RHS $= 3(0)^{\frac{2}{3}}$

$= 0$

$=$ LHS

Hence $y = 0$ is a solution of this IVP.

22c $y(0) = 0$

From parts a and b above, we know the DE is satisfied for $x \neq 0$

For $x < 0, y = 0$ and $\frac{dy}{dx} = 0$

As $x \to 0^+, y$ tends to zero and $\frac{dy}{dx} = 3x^2$ also tends to zero.

Therefore at $x = 0, y = 0$ and $\frac{dy}{dx} = 0$ so the DE and initial values are satisfied here too.

22d $y(0) = 0$

From parts a and b above, we know the DE is satisfied for $x \neq 0$

For $x > 0, y = 0$ and $\dfrac{dy}{dx} = 0$

As $x \to 0^-$, y tends to zero and $\frac{dy}{dx} = 3x^2$ also tends to zero.

Therefore at $x = 0$, $y = 0$ and $\frac{dy}{dx} = 0$ so the DE and initial values are satisfied here too.

Solutions to Exercise 13E

Let the integration constants be A, B, C or D.

1a i $y = ax^2 + bx$

$y' = 2ax + b$

Substituting for y' in $y' = 1 - 4x$:

$2ax + b = 1 - 4x$

Equating coefficients:

$2a = -4$ and $b = 1$

$a = -2$ and $b = 1$

Therefore $y = -2x^2 + x$

1a ii $y = e^{-x}(a\cos x + b\sin x)$

$y' = -e^{-x}(a\cos x + b\sin x) + e^{-x}(-a\sin x + b\cos x)$

$y' = e^{-x}\big(-(a\cos x + b\sin x) + (-a\sin x + b\cos x)\big)$

$y' = e^{-x}(-a\cos x - b\sin x - a\sin x + b\cos x)$

Substituting for y' in $y' = 2e^{-x}\sin x$:

$e^{-x}(-a\cos x - b\sin x - a\sin x + b\cos x) = 2e^{-x}\sin x$

Equating coefficients:

$-a - b = 2$ (1)

$-a + b = 0$ (2)

(2) + (3):

$-2a = 2$

$a = -1$

Substituting $a = -1$ into (2):

$1 + b = 0$

$b = -1$

Therefore $y = e^{-x}(-\cos x - \sin x)$ or $y = -e^{-x}(\cos x + \sin x)$

1a iii $y = ax + b + 3e^{-x}$

$y' = a - 3e^{-x}$

Substituting for y' and y in $y' = x - y$:

$a - 3e^{-x} = x - (ax + b + 3e^{-x})$

$a - 3e^{-x} = x - ax - b - 3e^{-x}$

$-x + a = -ax - b$

Equating coefficients:

$-a = -1$ so $a = 1$

$-b = a$ so $b = -1$

Therefore $y = x - 1 + 3e^{-x}$

1b i $y = ax^2 + bx + c + 4e^{-2x}$

$y' = 2ax + b - 8e^{-2x}$

Substituting for y' and y in $y' + 2y = x^2 - 3x - 4$:

$2ax + b - 8e^{-2x} + 2(ax^2 + bx + c + 4e^{-2x}) = x^2 - 3x - 4$

$2ax + b - 8e^{-2x} + 2ax^2 + 2bx + 2c + 8e^{-2x} = x^2 - 3x - 4$

$2ax + b + 2ax^2 + 2bx + 2c = x^2 - 3x - 4$

$2ax^2 + 2ax + 2bx + b + 2c = x^2 - 3x - 4$

$2ax^2 + (2a + 2b)x + (b + 2c) = x^2 - 3x - 4$

Equating coefficients:

$2a = 1$ so $a = \frac{1}{2}$

$2a + 2b = -3$

$2\left(\frac{1}{2}\right) + 2b = -3$

$1 + 2b = -3$

$2b = -4$

$b = -2$

$b + 2c = -4$

$-2 + 2c = -4$

$2c = -2$

$c = -1$

Therefore $y = \frac{1}{2}x^2 - 2x - 1 + 4e^{-2x}$

1b ii $y = ax^2 + bx + c - \sin 2x$

$y' = 2ax + b - 2\cos 2x$

$y'' = 2a + 4\sin 2x$

Substituting for y'', $y'$and y in $y'' + 4y = x^2 + 5x$:

$2a + 4\sin 2x + 4(ax^2 + bx + c - \sin 2x) = x^2 + 5x$

$2a + 4\sin 2x + 4ax^2 + 4bx + 4c - 4\sin 2x = x^2 + 5x$

$2a + 4ax^2 + 4bx + 4c = x^2 + 5x$

$4ax^2 + 4bx + (2a + 4c) = x^2 + 5x$

Equating coefficients:

$4a = 1$

$a = \frac{1}{4}$

$4b = 5$

$b = \frac{5}{4}$

$2a + 4c = 0$

$2\left(\frac{1}{4}\right) + 4c = 0$

$\frac{1}{2} + 4c = 0$

$4c = -\frac{1}{2}$

$c = -\frac{1}{8}$

Therefore $y = \frac{1}{4}x^2 + \frac{5}{4}x - \frac{1}{8} - \sin 2x$

or $y = \frac{1}{8}(2x^2 + 10x - 1) - \sin 2x$

1c $y = 5e^{\lambda x}$

$y' = 5\lambda e^{\lambda x}$

$y'' = 5\lambda^2 e^{\lambda x}$

Substituting for y'', $y'$and y in $y'' + 5y' + 6y = 0$:

$5\lambda^2 e^{\lambda x} + 5(5\lambda e^{\lambda x}) + 6\left(5e^{\lambda x}\right) = 0$

$5\lambda^2 e^{\lambda x} + 25\lambda e^{\lambda x} + 30e^{\lambda x} = 0$

$5e^{\lambda x}(\lambda^2 + 5\lambda + 6) = 0$

$5e^{\lambda x} = 0$ (no solution)

or $\lambda^2 + 5\lambda + 6 = 0$

$(\lambda + 3)(\lambda + 2) = 0$

$\lambda = -3, -2$

Therefore $y = 5e^{-3x}$ or $5e^{-2x}$

2a

$$\frac{dR}{dt} = kR$$

$$\frac{1}{R}\, dR = k\, dt$$

$$\int \frac{1}{R}\, dR = \int k\, dt$$

$\ln|R| = kt + C$

$R = e^{kt+C}$ since R must be positive

$R = e^C e^{kt}$

$R = Ae^{kt}$ where $A = e^C$

2b $R = Ae^{kt}$

When $t = 0, R = 100,$ so:

$100 = Ae^{k\times 0}$

$A = 100$

2c $R = 100e^{kt}$

When $t = 4, R = 20,$ so:

$20 = 100e^{4k}$

$\frac{1}{5} = e^{4k}$

$\ln\left(\frac{1}{5}\right) = 4k$

$k = \frac{1}{4}\ln(5^{-1})$

$k = -\frac{1}{4}\ln 5$

2d $R = 100e^{kt}$ where $k = -\frac{1}{4}\ln 5$

When $t = 12,$

$$\begin{aligned} R &= 100e^{\left(-\frac{1}{4}\ln 5\right)\times 12} \\ &= 100e^{(-3\ln 5)} \\ &= 100e^{\ln 5^{-3}} \\ &= 100 \times 5^{-3} \\ &= \frac{100}{125} \\ &= \frac{4}{5} \end{aligned}$$

Amount present after 12 days is $\frac{4}{5}$ gram.

2e $R = 100e^{\left(-\frac{1}{4}\ln 5\right)t}$

$R = 100e^{\left(-\frac{1}{4}t\ln 5\right)}$

$R = 100e^{\left(\ln 5^{-\frac{1}{4}t}\right)}$

$R = 100 \times 5^{-\frac{1}{4}t}$

$R = 100 \times \left(\frac{1}{5}\right)^{\frac{1}{4}t}$

When $t = 12$,

$R = 100 \times \left(\frac{1}{5}\right)^3 = 100 \times \frac{1}{125} = \frac{4}{5}$

3a

$\frac{dH}{dt} = k(H - 25)$

$\frac{1}{H - 25}\, dH = k\, dt$

$\int \frac{1}{H - 25}\, dR = \int k\, dt$

$\ln|H - 25| = kt + C$

$|H - 25| = e^{kt+C}$

$|H - 25| = e^C e^{kt}$

$H - 25 = e^C e^{kt}\ or - e^C e^{kt}$

$H - 25 = Ae^{kt}$ where $A = e^C$ or $-e^C$

$H = 25 + Ae^{kt}$

3b When $t = 0, H = 5,$ so:

$5 = 25 + Ae^{k(0)}$

$5 = 25 + A$

$-20 = A$

Therefore $H = 25 - 20e^{kt}$

3c When $t = 10, H = 15$, so:

$$15 = 25 - 20e^{10k}$$

$$20e^{10k} = 10$$

$$e^{10k} = \frac{10}{20}$$

$$10k = \ln\frac{1}{2}$$

$$k = \frac{1}{10}\ln(2^{-1})$$

$$k = -\frac{1}{10}\ln 2$$

3d When $H = 24$,

$$24 = 25 - 20e^{\left(-\frac{1}{10}\ln 2\right)t}$$

$$20e^{\left(-\frac{1}{10}\ln 2\right)t} = 1$$

$$2^{-\frac{1}{10}t} = \frac{1}{20}$$

$$t \doteqdot 43 \text{ mins}$$

4a

$$\frac{dV}{dt} = k\pi r^2, \qquad \text{for some constant } k$$

4b

$$\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt}$$

$$\frac{dr}{dt} = \frac{dr}{dV} \times k\pi r^2$$

$h = 3r$ (from initial proportions) so

$$V = \frac{1}{3}\pi r^2 \times 3r = \pi r^3$$

Therefore $\dfrac{dV}{dr} = 3\pi r^2$

$\dfrac{dr}{dt} = \dfrac{1}{3\pi r^2} \times k\pi r^2$

$\dfrac{dr}{dt} = \dfrac{1}{3}k$

4c

$\dfrac{dr}{dt} = \dfrac{1}{3}k$

$\int 3\,dr = \int k\,dt$

$3r = kt + C$

$r = \dfrac{1}{3}kt + D$ where $D = \dfrac{1}{3}C$

Substituting $r(0) = 4$:

$4 = \dfrac{1}{3}k(0) + D$

$D = 4$

Therefore $r = \dfrac{1}{3}kt + 4$

4d Substituting $r(6) = 3.5$:

$3.5 = \dfrac{1}{3}k(6) + 4$

$-0.5 = 2k$

$2k = -\dfrac{1}{2}$

$k = -\dfrac{1}{4}$

Therefore $r = \dfrac{1}{3}\left(-\dfrac{1}{4}\right)t + 4$

$r = 4 - \dfrac{1}{12}t$

4e $V = \pi r^3$

$V = \pi\left(4 - \frac{1}{12}t\right)^3$

Since r and V cannot be negative, $4 - \frac{1}{12}t$ must also be non-negative therefore $0 \le t \le 48$.

5a $h(0) = 400$

The height decreases so $\frac{dh}{dt} < 0$.

Since $\sqrt{h} \ge 0$, the constant k must be negative.

5b

$$\frac{dh}{dt} = k\sqrt{h}$$

$$\int \frac{1}{\sqrt{h}}\,dh = \int k\,dt$$

$$\int h^{-\frac{1}{2}}\,dh = \int k\,dt$$

$$2h^{\frac{1}{2}} = kt + C$$

$$h^{\frac{1}{2}} = \frac{1}{2}(kt + C)$$

$$h = \frac{1}{4}(kt + C)^2$$

Substituting $h(0) = 400$:

$$400 = \frac{1}{4}(k(0) + C)^2$$

$$1600 = C^2$$

$$C = 40$$

Therefore $h = \frac{1}{4}(kt + 40)^2$

5c Substituting $h(20) = 100$:

$100 = \frac{1}{4}(k \times 20 + 40)^2$

$400 = (20k + 40)^2$

$20 = 20k + 40$

$20k = -20$

$k = -1$

5d

$$h = \frac{1}{4}(-t + 40)^2$$

When $h = 0$,

$$0 = \frac{1}{4}(-t + 40)^2$$

$0 = (-t + 40)^2$

$0 = -t + 40$

$t = 40$ mins

5e No, $h(t)$ increases for $t > 40$, which is impossible.

6a

$$\text{Gradient} = \frac{y - 0}{x - 0} = \frac{y}{x}$$

6b

$$\frac{dy}{dx} = \frac{y}{x}$$

6c

$$\int \frac{1}{y}\,dy = \int \frac{1}{x}\,dx$$

$$\ln|y| = \ln|x| + B$$

$$\ln|y| - \ln|x| = B$$

$$\ln\left|\frac{y}{x}\right| = B$$

$$\left|\frac{y}{x}\right| = e^B$$

$$\frac{y}{x} = e^B \text{ or } -e^B$$

$$\frac{y}{x} = C \text{ where } C = e^B \text{ or } -e^B$$

$$y = Cx$$

6d $x = 0$ because then y' undefined.

7a

$$\text{Gradient} = \frac{y-0}{x-0} = \frac{y}{x}$$

7b Normal has gradient of $-\frac{x}{y}$.

$$\frac{dy}{dx} = -\frac{x}{y}$$

7c

$$\int y\ dy = -\int x\ dx$$

$$\frac{y^2}{2} = -\frac{x^2}{2} + C$$

$$\frac{x^2}{2} + \frac{y^2}{2} = C$$

$$x^2 + y^2 = 2C$$

$x^2 + y^2 = r^2$ where $r^2 = 2C$ and r is the radius of the circle

8a

$$\text{Gradient} = \frac{y - 0}{x - (x - 1)} = y$$

8b

$$\frac{dy}{dx} = y$$

8c

$$\int \frac{1}{y} dy = \int 1 \ dx$$

$\ln|y| = x + C$

$|y| = e^{x+C}$

$y = e^C e^x \text{ or} - e^C e^x$

$y = Ae^x$ where $A = e^C$ or $-e^C$

8d Substituting $(0, 1)$:

$1 = Ae^0$

$A = 1$

Therefore $y = e^x$

9a

$$\frac{dP}{dh} = kP$$

$$\int \frac{1}{P} \ dP = \int k \ dh$$

$ln\ |P| = kh + C$

$P = e^{kh+C}$ (from the context of the problem, we can ignore the negative case)

$P = e^C e^{kh}$

$P = Ae^{kh}$ where $A = e^C$

Substituting $(0, P_0)$:

$P_0 = Ae^0$

$A = P_0$

Therefore $P = P_0 e^{kh}$

9b Substituting $(10\,000, 40)$:

$40 = P_0 e^{10\,000k} \qquad (1)$

Substituting $(6000, 80)$:

$80 = P_0 e^{6000k} \qquad (2)$

$(2) \div (1)$:

$\dfrac{80}{40} = \dfrac{e^{6000k}}{e^{10\,000k}}$

$2 = e^{6000k - 10\,000k}$

$2 = e^{-4000k}$

$\ln 2 = -4000k$

$k = -\dfrac{\ln 2}{4000}$

9c Substituting $k = -\frac{\ln 2}{4000}$ into (1):

$40 = P_0\, e^{10\,000\left(-\frac{\ln 2}{4000}\right)}$

$40 = P_0\, e^{\left(-\frac{5}{2}\ln 2\right)}$

$40 = P_0\, 2^{\left(-\frac{5}{2}\right)}$

$\dfrac{40}{2^{\left(-\frac{5}{2}\right)}} = P_0$

$160\sqrt{2} = P_0$

Therefore $P = 160\sqrt{2}\, e^{\left(-\frac{\ln 2}{4000}\right)h}$

When $h = 0$,

$$P = 160\sqrt{2} \times e^{\left(-\frac{\ln 2}{4000}\right)\times 0}$$

$$P = 160\sqrt{2}$$

$$P \doteqdot 226 \text{ kPa}$$

10a $A = (2x, 0), B = (0, 2y)$

10b

$$\frac{dy}{dx} = \frac{2y - 0}{0 - 2x} = -\frac{y}{x}$$

10c

$$\int \frac{1}{y}\,dy = -\int \frac{1}{x}\,dx$$

$$\ln|y| = -\ln|x| + B$$

$$\ln|y| + \ln|x| = B$$

$$\ln|xy| = B$$

$$|xy| = e^B$$

$$xy = e^B \text{ or } -e^B$$

$$xy = C \text{ where } C = e^B \text{ or } -e^B$$

$$\text{or } y = \frac{C}{x}$$

11a

$$\frac{dC}{dt} = k(S - C)$$

$$\int \frac{1}{S - C}\,dC = \int k\ dt$$

$$\int \frac{1}{C - S}\,dC = -\int k\ dt$$

$$ln\,|C - S| = -kt + B$$

$$|C - S| = e^{-kt+B}$$

$|C - S| = e^B e^{-kt}$

$C - S = e^B e^{-kt}$ or $e^B e^{-kt}$

$C - S = Ae^{-kt}$ where $A = e^{-B}$ or $-e^{-B}$

$C = S + Ae^{-kt}$

11b Substituting $(0, C_0)$:

$C_0 = S + Ae^{-k(0)}$

$C_0 = S + A$

$C_0 - S = A$

Therefore $C = S + (C_0 - S)e^{-kt}$

12a

$$\begin{aligned}\text{RHS} &= \frac{1}{N} + \frac{1}{1000 - N}\\ &= \frac{1000 - N + N}{N(P - N)}\\ &= \frac{1000}{N(1000 - N)}\\ &= \text{LHS}\end{aligned}$$

Therefore $\dfrac{1000}{N(1000 - N)} = \dfrac{1}{N} + \dfrac{1}{1000 - N}$

Alternatively, let

$$\begin{aligned}\frac{1000}{N(1000 - N)} &= \frac{A}{N} + \frac{B}{1000 - N}\\ &= \frac{A(1000 - N) + B(N)}{N(1000 - N)}\\ &= \frac{1000A - AN + BN}{(y - 1)(y - 3)}\\ &= \frac{N(-A + B) + 1000A}{(y - 1)(y - 3)}\end{aligned}$$

Equating coefficients:

$1000A = 1000$

$A = 1$

$-A + B = 0$

Since $A = 1$,

$-1 + B = 0$

$B = 1$

Therefore $\dfrac{1000}{N(1000-N)} = \dfrac{1}{N} + \dfrac{1}{1000-N}$

12b

$\dfrac{dN}{dt} = kN(1000-N)$

$\displaystyle\int \frac{1}{N(1000-N)}\, dN = \int k\, dt$

$\displaystyle\frac{1}{1000}\int \frac{1000}{N(1000-N)}\, dN = \int k\, dt$

$\displaystyle\int \left(\frac{1}{N} + \frac{1}{1000-N}\right) dN = \int 1000k\, dt$

$\ln|N| - \ln|1000-N| = 1000kt + C$

$\ln\left|\dfrac{N}{1000-N}\right| = 1000kt + C$

$\left|\dfrac{N}{1000-N}\right| = e^{1000kt+C}$

$\dfrac{N}{1000-N} = e^{C}e^{1000kt}$ or $-e^{C}e^{1000kt}$

$\dfrac{N}{1000-N} = Ae^{1000kt}$ where $A = e^{C}$ or $-e^{C}$

$N = Ae^{1000kt}(1000-N)$

$N = 1000Ae^{1000kt} - NAe^{1000kt}$

$N + NAe^{1000kt} = 1000Ae^{1000kt}$

$N(1 + Ae^{1000kt}) = 1000Ae^{1000kt}$

$N = \dfrac{1000Ae^{1000kt}}{1 + Ae^{1000kt}}$

$$N = \frac{1000Ae^{1000kt}}{1 + Ae^{1000kt}} \times \frac{A^{-1}e^{-1000kt}}{A^{-1}e^{-1000kt}}$$

$$N = \frac{1000}{A^{-1}e^{-1000kt} + 1}$$

$$N = \frac{1000}{Be^{-1000kt} + 1} \text{ where } B = A^{-1}$$

$$N = \frac{1000}{1 + Be^{-1000kt}}$$

12c Substituting $N(0) = 40$:

$$40 = \frac{1000}{1 + Be^{-1000k(0)}}$$

$$40 = \frac{1000}{1 + B}$$

$$40(1 + B) = 1000$$

$$40 + 40B = 1000$$

$$B = \frac{1000 - 40}{40}$$

$$B = 24$$

Therefore $N = \dfrac{1000}{1 + 24e^{-1000kt}}$

12d Substituting $N(1) = 80$:

$$80 = \frac{1000}{1 + 24e^{-1000k(1)}}$$

$$80 + 1920e^{-1000k} = 1000$$

$$1920e^{-1000k} = 920$$

$$e^{-1000k} = \frac{920}{1920}$$

$$-1000k = \ln\left(\frac{23}{48}\right)$$

$$k = -\frac{1}{1000}\ln\left(\frac{23}{48}\right)$$

$$k \doteqdot 7.357 \times 10^{-4}$$

12e When $t = 5$,

$$N = \frac{1000}{1 + 24e^{-1000k(5)}}$$

$$\doteqdot 622.57$$

Population will be 623 after 5 years.

13a

$$\text{RHS} = \frac{1}{N} + \frac{1}{P - N}$$

$$= \frac{P - N + N}{N(P - N)}$$

$$= \frac{P}{N(P - N)}$$

$$= \text{LHS}$$

Therefore $\dfrac{P}{N(P - N)} = \dfrac{1}{N} + \dfrac{1}{P - N}$

Alternatively, let

$$\frac{P}{N(P - N)} = \frac{A}{N} + \frac{B}{P - N}$$

$$= \frac{A(P - N) + B(N)}{N(P - N)}$$

$$= \frac{AP - AN + BN}{N(P - N)}$$

$$= \frac{N(-A + B) + AP}{N(P - N)}$$

Equating coefficients:

$AP = P$

$A = 1$

$-A + B = 0$

Since $A = 1$,

$-1 + B = 0$

$B = 1$

Therefore $\dfrac{P}{N(P-N)} = \dfrac{1}{N} + \dfrac{1}{P-N}$

13b (Same as Q12b with $P = 1000$.)

$\dfrac{dN}{dt} = kN(P-N)$

$\displaystyle\int \frac{1}{N(P-N)}\,dN = \int k\,dt$

$\displaystyle\frac{1}{P}\int \frac{P}{N(P-N)}\,dN = \int k\,dt$

$\displaystyle\int \left(\frac{1}{N} + \frac{1}{P-N}\right) dN = \int kP\,dt$

$\ln|N| - \ln|P-N| = kPt + C$

$\ln\left|\dfrac{N}{P-N}\right| = kPt + C$

$\left|\dfrac{N}{P-N}\right| = e^{kPt+C}$

$\dfrac{N}{P-N} = e^{C}e^{kPt}$ or $- e^{C}e^{kPt}$

$\dfrac{N}{P-N} = Ae^{kPt}$ where $A = e^{C}$ or $-e^{C}$

$N = Ae^{kPt}(P-N)$

$N = PAe^{kPt} - NAe^{kPt}$

$N + NAe^{kPt} = PAe^{kPt}$

$N(1 + Ae^{kPt}) = PAe^{kPt}$

$$N = \frac{PAe^{kPt}}{1 + Ae^{kPt}}$$

$$N = \frac{PAe^{kPt}}{1 + Ae^{kPt}} \times \frac{A^{-1}e^{-kPt}}{A^{-1}e^{-kPt}}$$

$$N = \frac{P}{A^{-1}e^{-kPt} + 1}$$

$$N = \frac{P}{Be^{-kPt} + 1} \text{ where } B = A^{-1}$$

$$N = \frac{P}{1 + Be^{-kPt}}$$

13c Substituting $N(0) = 23.2$ and $P = 187.5$:

$$23.2 = \frac{187.5}{1 + Be^{-187.5k(0)}}$$

$$23.2 = \frac{187.5}{1 + B}$$

$$23.2(1 + B) = 187.5$$

$$23.2 + 23.2B = 187.5$$

$$B = \frac{187.5 - 23.2}{23.2}$$

$$B = \frac{164.3}{23.2}$$

$$\text{Therefore } N = \frac{187.5}{1 + \frac{164.3}{23.2}e^{-187.5kt}}$$

$$N = \frac{187.5}{1 + \frac{164.3}{23.2}e^{-187.5kt}} \times \frac{23.2}{23.2}$$

$$N = \frac{187.5 \times 23.2}{23.21 + 164.3e^{-187.5kt}}$$

13d Substituting $N(40, 63.0)$:

$$63.0 = \frac{187.5 \times 23.2}{23.21 + 164.3e^{-187.5(40)k}}$$

$$63.0(23.21 + 164.3e^{-7500k}) = 187.5 \times 23.2$$

$$23.21 + 164.3e^{-7500k} = \frac{4350}{63.0}$$

$$164.3e^{-7500k} = \frac{4350}{63.0} - 23.21$$

$$e^{-7500k} = \frac{4350 - 23.21 \times 63.0}{63.0 \times 164.3}$$

$$-7500k = \ln\left(\frac{4350 - 1462.23}{10\,350.9}\right)$$

$$k = -\frac{1}{7500}\ln\left(\frac{2887.77}{10\,350.9}\right)$$

$$k \doteqdot 1.702 \times 10^{-4}$$

13e When $t = 80$ and $k = 1.702 \times 10^{-4}$,

$$N = \frac{187.5 \times 23.2}{23.21 + 164.3e^{-187.5(1.702 \times 10^{-4})(80)}}$$

$$N = \frac{4350}{23.21 + 164.3e^{-2.553}}$$

$$N = 120.832\ldots$$

Predicted population is about 120.9 million.

13f The mathematical model needs to be revised. Clearly the carrying capacity is much larger than the figure of 187.5 estimated in 1850.

Using the model, when $t = 168$ and $k = 1.702 \times 10^{-4}$,

$$N = \frac{187.5 \times 23.2}{23.21 + 164.3e^{-187.5(1.702 \times 10^{-4})(168)}}$$

$$N = \frac{4350}{23.21 + 164.3e^{-5.3613}}$$

$$N = 181.401\ldots$$

Predicted population from the model is about 181.4 million.

14a Using the results from Question 13a and 13b,

$$N = \frac{P}{1 + Be^{-kPt}}$$

Alternatively, the full working is shown below.

From Q13a, $\frac{P}{N(P-N)} = \frac{1}{N} + \frac{1}{P-N}$

$$\frac{dN}{dt} = kN(P-N)$$

$$\int \frac{1}{N(P-N)}\,dN = \int k\,dt$$

$$\frac{1}{P}\int \frac{P}{N(P-N)}\,dN = \int k\,dt$$

$$\int \left(\frac{1}{N} + \frac{1}{P-N}\right) dN = \int kP\,dt$$

$$\ln|N| - \ln|P-N| = kPt + C$$

$$\ln\left|\frac{N}{P-N}\right| = kPt + C$$

$$\left|\frac{N}{P-N}\right| = e^{kPt+C}$$

$$\frac{N}{P-N} = e^{C}e^{kPt} \text{ or } -e^{C}e^{kPt}$$

$$\frac{N}{P-N} = Ae^{kPt} \text{ where } A = e^{C} \text{ or } -e^{C}$$

$$N = Ae^{kPt}(P-N)$$

$$N = PAe^{kPt} - NAe^{kPt}$$

$$N + NAe^{kPt} = PAe^{kPt}$$

$$N(1 + Ae^{kPt}) = PAe^{kPt}$$

$$N = \frac{PAe^{kPt}}{1 + Ae^{kPt}}$$

$$N = \frac{PAe^{kPt}}{1 + Ae^{kPt}} \times \frac{A^{-1}e^{-kPt}}{A^{-1}e^{-kPt}}$$

$$N = \frac{P}{A^{-1}e^{-kPt} + 1}$$

$$N = \frac{P}{Be^{-kPt} + 1} \text{ where } B = A^{-1}$$

$$N = \frac{P}{1 + Be^{-kPt}}$$

14b Substituting $N(0) = N_0$:

$$N_0 = \frac{P}{1 + Be^{-kP(0)}}$$

$$N_0 = \frac{P}{1 + B}$$

$$N_0(1 + B) = P$$

$$N_0 + BN_0 = P$$

$$BN_0 = P - N_0$$

$$B = \frac{P - N_0}{N_0}$$

$$\text{Therefore } N = \frac{P}{1 + \left(\frac{P - N_0}{N_0}\right) e^{-kPt}}$$

$$N = \frac{P}{1 + \left(\frac{P - N_0}{N_0}\right) e^{-kPt}} \times \frac{N_0}{N_0}$$

$$N = \frac{N_0 P}{N_0 + (P - N_0)e^{-kPt}}$$

14c Substituting $N(t_1) = N_1$:

$$N_1 = \frac{N_0 P}{N_0 + (P - N_0)e^{-kPt_1}}$$

$$\frac{N_1}{N_0 P} = \frac{1}{N_0 + (P - N_0)e^{-kPt_1}}$$

$$\frac{N_0 P}{N_1} = \frac{N_0 + (P - N_0)e^{-kPt_1}}{1}$$

$$\frac{N_0 P}{N_1} - N_0 = (P - N_0)e^{-kPt_1}$$

$$\frac{N_0P - N_0N_1}{N_1} = (P - N_0)e^{-kPt_1}$$

$$\frac{N_0P - N_0N_1}{N_1(P - N_0)} = e^{-kPt_1}$$

$$\frac{N_0(P - N_1)}{N_1(P - N_0)} = e^{-kPt_1}$$

$$\frac{N_1(P - N_0)}{N_0(P - N_1)} = e^{kPt_1}$$

$$kPt_1 = \ln\left(\frac{N_1(P - N_0)}{N_0(P - N_1)}\right)$$

$$k = \frac{1}{t_1P}\ln\left(\frac{N_1(P - N_0)}{N_0(P - N_1)}\right)$$

14d

$$N = \frac{N_0P}{N_0 + (P - N_0)e^{-Pk(t)}} \text{ and } k = \frac{1}{t_1P}\ln\left(\frac{N_1(P - N_0)}{N_0(P - N_1)}\right)$$

By the same logic (as in part c),

$$k = \frac{1}{t_2P}\ln\left(\frac{N_2(P - N_0)}{N_0(P - N_{2)}}\right)$$

Substituting $t_1 = 1$ and $t_2 = 2$:

$$\frac{1}{P}\ln\left(\frac{N_1(P - N_0)}{N_0(P - N_1)}\right) = k = \frac{1}{2P}\ln\left(\frac{N_2(P - N_0)}{N_0(P - N_2)}\right)$$

$$2\ln\left(\frac{N_1(P - N_0)}{N_0(P - N_1)}\right) = \ln\left(\frac{N_2(P - N_0)}{N_0(P - N_2)}\right)$$

$$\ln\left(\frac{N_1(P - N_0)}{N_0(P - N_1)}\right)^2 = \ln\left(\frac{N_2(P - N_0)}{N_0(P - N_2)}\right)$$

$$\left(\frac{N_1(P - N_0)}{N_0(P - N_1)}\right)^2 = \frac{N_2(P - N_0)}{N_0(P - N_2)}$$

$$N_0(P - N_2)\big(N_1(P - N_0)\big)^2 = N_2(P - N_0)\big(N_0(P - N_1)\big)^2$$

$$(P - N_2)(P - N_0)(N_1)^2 = N_0N_2(P - N_1)^2$$

15a It represents the 200 fish harvested each year.

15b

$$\frac{dy}{dt} = -2 + \frac{1}{24}y(16 - y)$$

$$\frac{dy}{dt} = \frac{1}{24}(-48 + 16y - y^2)$$

$$\frac{dy}{dt} = -\frac{1}{24}(y^2 - 16y + 48)$$

$$\frac{dy}{dt} = -\frac{1}{24}(y - 4)(y - 12)$$

Initial condition: $y(0) = 5 - 2 = 3$ (where y is measured in hundreds)

15c RHS

$$= \frac{3}{y - 12} - \frac{3}{y - 4}$$

$$= \frac{3(y - 4) - 3(y - 12)}{(y - 12)(y - 4)}$$

$$= \frac{3y - 12 - 3y + 36}{(y - 12)(y - 4)}$$

$$= \frac{24}{(y - 12)(y - 4)}$$

$= \text{LHS}$

15d

$$\frac{dy}{dt} = -\frac{1}{24}(y - 4)(y - 12)$$

$$\int \frac{24}{(y - 12)(y - 4)}\,dy = -\int 1\,dt$$

$$\int \left(\frac{3}{y - 12} - \frac{3}{y - 4}\right) dy = -\int 1\,dt$$

$$\int \left(\frac{1}{y - 12} - \frac{1}{y - 4}\right) dy = -\frac{1}{3}\int 1\,dt$$

$$\ln|y-12|-\ln|y-4|=-\frac{1}{3}t+C$$

$$\ln\left|\frac{y-12}{y-4}\right|=-\frac{1}{3}t+C$$

$$\left|\frac{y-12}{y-4}\right|=e^{-\frac{1}{3}t+C}$$

$$\frac{y-12}{y-4}=e^{C}e^{-\frac{1}{3}t}\text{ or }e^{C}e^{-\frac{1}{3}t}$$

$$\frac{y-12}{y-4}=Ae^{-\frac{1}{3}t}\text{ where }A=e^{C}\text{ or }-e^{C}$$

$$y-12=(y-4)Ae^{-\frac{1}{3}t}$$

$$y-12=yAe^{-\frac{1}{3}t}-4Ae^{-\frac{1}{3}t}$$

$$y\left(1-Ae^{-\frac{1}{3}t}\right)=12-4Ae^{-\frac{1}{3}t}$$

$$y=\frac{12-4Ae^{-\frac{1}{3}t}}{1-Ae^{-\frac{1}{3}t}}$$

Substituting $y(0)=3$:

$$3=\frac{12-4Ae^{0}}{1-Ae^{0}}$$

$$3=\frac{12-4A}{1-A}$$

$$3-3A=12-4A$$

$$A=9$$

Therefore $y=\dfrac{12-36e^{-\frac{1}{3}t}}{1-9e^{-\frac{1}{3}t}}$

$$y=\frac{12\left(1-3e^{-\frac{1}{3}t}\right)}{1-9e^{-\frac{1}{3}t}}$$

15e Substituting $y = 0$:

$$0 = \frac{12\left(1 - 3e^{-\frac{1}{3}t}\right)}{1 - 9e^{-\frac{1}{3}t}}$$

$$1 - 3e^{-\frac{1}{3}t} = 0$$

$$e^{-\frac{1}{3}t} = \frac{1}{3}$$

$$t = -3\ln\frac{1}{3}$$

$$= 3\ln 3$$

$$= 3.295\,8\ldots$$

$$\doteqdot 3.3$$

Fish will die out in approximately 3.3 years.

15f i $y(0) = 5$

15f ii

$$y = \frac{4\left(3 + 7e^{-\frac{1}{3}t}\right)}{1 + 7e^{-\frac{1}{3}t}}$$

As $t \to \infty$, $e^{-\frac{1}{3}t} \to 0$

Therefore y approaches:

$$\frac{4(3 + 7(0))}{1 + 7(0)}$$

$$= \frac{12}{1}$$

$$= 12$$

So skipping the harvest for the first year allows the fish population to stabilise at 1200. This suggests the harvest should be stopped for one year to save the species.

16a i Let $u = log_e x$

$$\frac{du}{dx} = \frac{1}{x}$$

$$x\,du = dx$$

$$I = \int \frac{dx}{x \log_e x}$$

$$I = \int \frac{x\,du}{xu}$$

$$I = \int \frac{du}{u}$$

16a ii $I = log_e|u| + C$

$$= log_e|log_e x| + C$$

16b

$$\frac{dN}{dt} = kN \log_e N$$

$$\int \frac{1}{N log_e N}\,dN = \int k\,dt$$

$$log_e|log_e N| = kt + C \qquad \text{(using part a)}$$

$$|log_e N| = e^{kt+C}$$

$$log_e N = e^C e^{kt} \text{ or } -e^C e^{kt}$$

$$log_e N = Ae^{kt} \text{ where } A = e^C \text{ or } -e^C$$

$$N = e^{Ae^{kt}}$$

17 $M_0 = 200$

$$\frac{dM}{dt} = -\text{outflow} - \text{decay}$$

$$\frac{dM}{dt} = -\frac{5}{100}M - \frac{10}{100}M$$

$$= -\frac{15}{100}M$$

$$= -\frac{3}{20}M$$

$$\int \frac{1}{M}\,dM = -\frac{3}{20}\int 1\,dt$$

$\ln M = -\frac{3}{20}t + C$ (we can omit absolute value signs since M cannot be negative)

$M = e^{-\frac{3}{20}t+C}$

$M = e^{-\frac{3}{20}}e^{C}$

$M = Ae^{-\frac{3}{20}t}$ where $A = e^{C}$

Substituting $M(0) = 200$:

$200 = Ae^{0}$

$200 = A$

Therefore $M = 200e^{-\frac{3}{20}t}$

18a

$$\frac{dh}{dt} = k(h-20)$$

$$\int \frac{1}{h-20}\,dh = \int k\,dt$$

$\ln|h-20| = kt + C$

$|h-20| = e^{kt+C}$

$|h-20| = e^{kt}e^{C}$

$h - 20 = Ae^{kt}$ where $A = e^{C}$ or $-e^{C}$

$h = 20 + Ae^{kt}$

Substituting $h(0) = 100$:

$100 = 20 + Ae^{k(0)}$

$A = 80$

Therefore $h = 20 + 80e^{kt}$

$h = 20(1 + 4e^{kt})$

Substituting $h(10) = 80$:

$80 = 20(1 + 4e^{10k})$

$1 + 4e^{10k} = 4$

$4e^{10k} = 3$

$e^{10k} = \frac{3}{4}$

$10k = \ln\frac{3}{4}$

$k = \frac{1}{10}\ln\frac{3}{4}$

18b i

$\frac{dH}{dt} = k(H - 20 + t)$

$H(0) = 80$

18b ii $y = H - 20 + t$

$\frac{dy}{dt} = \frac{dH}{dt} + 1$

$\frac{dy}{dt} = k(H - 20 + t) + 1$

$\frac{dy}{dt} = ky + 1$

$y(0) = 80 - 20 = 60$

18b iii

$\frac{dy}{dt} = ky + 1$

$\int \frac{1}{ky + 1}\, dy = \int 1\, dt$

$\frac{1}{k}\int \frac{1}{y + \frac{1}{k}} dy = \int 1\, dt$

$$\int \frac{1}{y+\frac{1}{k}}\,dy = k\int 1\,dt$$

$$\ln\left|y+\frac{1}{k}\right| = kt + C$$

$$\left|y+\frac{1}{k}\right| = e^{kt+C}$$

$$\left|y+\frac{1}{k}\right| = e^{C}e^{kt}$$

$$y+\frac{1}{k} = Ae^{kt} \text{ where } A = e^{C} \text{ or } -e^{C}$$

$$y = Ae^{kt} - \frac{1}{k}$$

Substituting $y(0) = 60$:

$$60 = Ae^{0t} - \frac{1}{k}$$

$$A = 60 + \frac{1}{k}$$

Therefore $y = \left(60+\frac{1}{k}\right)e^{kt} - \frac{1}{k}$

Since $y = H - 20 + t,\qquad H = y + 20 - t$

$$H = \left(60+\frac{1}{k}\right)e^{kt} - \frac{1}{k} + 20 - t$$

or $H(t) = 20 - \frac{1}{k} - t + \left(60+\frac{1}{k}\right)e^{kt}$

19a

$$\frac{dN}{dt} = kN(P-N)$$

$$\frac{d(Py)}{dt} = kPy(P-Py)$$

$$P\frac{dy}{dt} = kP^2y(1-y)$$

$$\frac{dy}{dt} = kPy(1-y)$$

$\frac{dy}{dt} = ry(1-y) \qquad \text{(since } r = kP\text{)}$

$y(0) = y_0 = \frac{N_0}{P}$

19b $x = rt$

$\frac{dx}{dt} = r$

Using the chain rule:

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$= ry(1-y) \times \frac{1}{r}$

$= y(1-y)$

Initial condition: $y(0) = y_0$

19c

$v = \frac{1}{y}$

$\frac{dv}{dy} = -\frac{1}{y^2}$

Using the chain rule:

$\frac{dv}{dx} = \frac{dv}{dy} \times \frac{dy}{dx}$

$= -\frac{1}{y^2} \times y(1-y)$

$= -\frac{1}{y}(1-y)$

$= -v\left(1 - \frac{1}{v}\right)$

$= 1 - v$

or $v' = 1 - v$

19d $v' = 1 - v$

$$\int \frac{1}{1-v}\,dv = \int 1\,dx$$

$$\int \frac{1}{v-1}\,dv = -\int 1\,dx$$

$$\ln|v-1| = -x + C$$

$$|v-1| = e^{-x+C}$$

$$v - 1 = e^C e^{-x} \text{ or } -e^C e^{-x}$$

$$v - 1 = Ae^{-x} \text{ where } A = e^C \text{ or } -e^C$$

$$v = 1 + Ae^{-x}$$

Since $v = \frac{1}{y}$ or $y = \frac{1}{v}$

$$y = \frac{1}{1 + Ae^{-x}}$$

19e Substituting $y(0) = y_0 = \frac{N_0}{P}$:

$$\frac{N_0}{P} = \frac{1}{1 + Ae^{-0}}$$

$$1 + A = \frac{P}{N_0}$$

$$A = \frac{P}{N_0} - 1$$

Therefore $y = \dfrac{1}{1 + \left(\frac{P}{N_0} - 1\right)e^{-x}}$

$$y = \frac{1}{1 + \left(\frac{P}{N_0} - 1\right)e^{-x}} \times \frac{N_0}{N_0}$$

$$y = \frac{N_0}{N_0 + (P - N_0)e^{-x}}$$

19f Substituting $x = rt = kPt$:

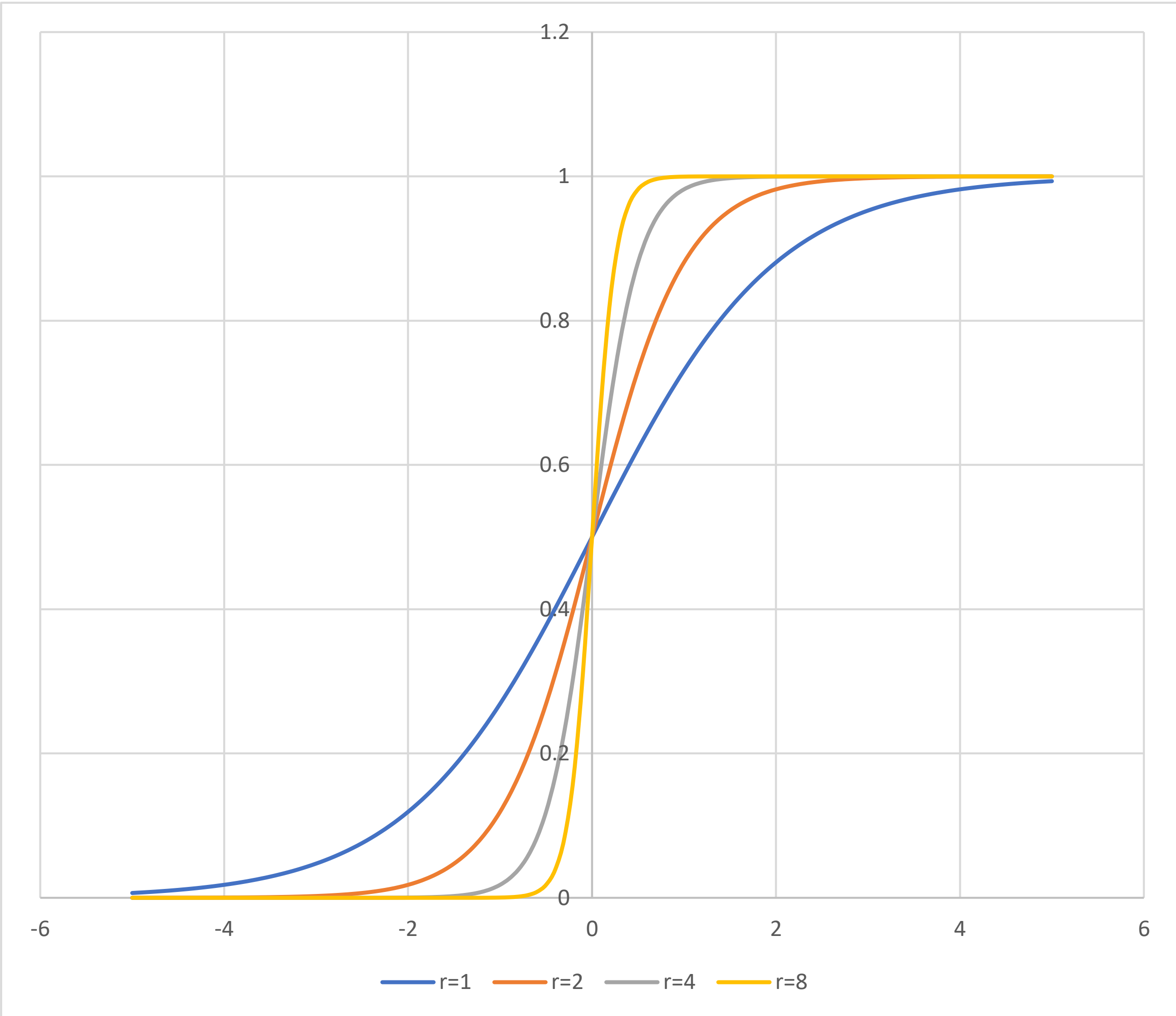
$$y = \frac{N_0}{N_0 + (P - N_0)e^{-kPt}}$$

$$N = Py$$

$$= P \times \frac{N_0}{N_0 + (P - N_0)e^{-kPt}}$$

$$= \frac{N_0 P}{N_0 + (P - N_0)e^{-kPt}}$$

20a The curves all have an asymptote $y = 0$ on the left and an asymptote $y = 1$ on the right, an the curves become steeper at $\left(0, \frac{1}{2}\right)$ as the value of r increases.

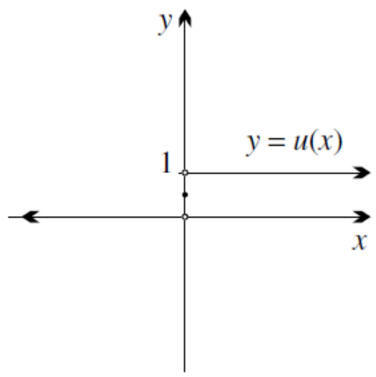


20b i As $r \to \infty, -rx \to -\infty$ so $e^{-rx} \to 0$ and $y \to 1$

20b ii As $r \to \infty, -rx = 0$ so $e^{-rx} = 1$ and $y = \frac{1}{2}$

20b iii As $r \to \infty, -rx \to \infty$ so $e^{-rx} \to \infty$ and $y \to 0$

20c



Solutions to Chapter review

Let the integration constants be A, B, C or D.

1a 1st-order, linear, one arbitrary constant

1b 2nd-order, two arbitrary constants

1c 3rd-order, three arbitrary constants

2a $xy' = y(1 - x^2)$

$y' = \frac{y}{x}(1 - x^2)$

2b

y \ x	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
2	3	$\frac{4}{3}$	0	-3	*	3	0	$-\frac{4}{3}$	-3
$\frac{3}{2}$	$\frac{9}{4}$	$\frac{5}{4}$	0	$-\frac{5}{4}$	*	$\frac{5}{4}$	0	$-\frac{5}{4}$	$-\frac{9}{4}$
1	$\frac{3}{2}$	$\frac{5}{6}$	0	$-\frac{3}{2}$	*	$\frac{3}{2}$	0	$-\frac{5}{6}$	$-\frac{3}{2}$
$\frac{1}{2}$	$\frac{3}{4}$	$\frac{5}{12}$	0	$-\frac{5}{12}$	*	$\frac{5}{12}$	0	$-\frac{5}{12}$	$-\frac{3}{4}$
0	0	0	0	0	*	0	0	0	0
$-\frac{1}{2}$	$-\frac{3}{4}$	$-\frac{5}{12}$	0	$\frac{5}{12}$	*	$-\frac{5}{12}$	0	$\frac{5}{12}$	$\frac{3}{4}$
-1	$-\frac{3}{2}$	$-\frac{5}{6}$	0	$\frac{3}{2}$	*	$-\frac{3}{2}$	0	$\frac{5}{6}$	$\frac{3}{2}$
$-\frac{3}{2}$	$-\frac{9}{4}$	$-\frac{5}{4}$	0	$\frac{5}{4}$	*	$-\frac{5}{4}$	0	$\frac{5}{4}$	$\frac{9}{4}$
-2	-3	$-\frac{4}{3}$	0	3	*	-3	0	$\frac{4}{3}$	3

2c See part h.

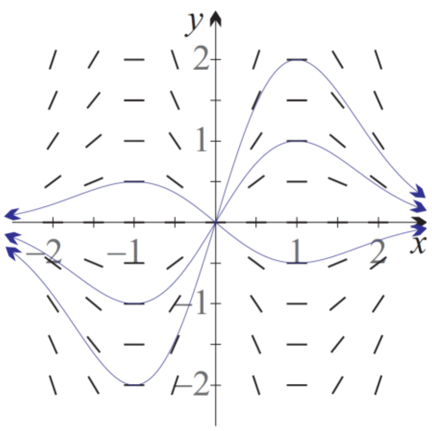
2d See part h.

2e The lines $x = 1$ and $x = -1$ are isoclines. $y = 0$ is also an isocline, ignoring the point $(0, 0)$ where the gradient is undefined.

2f $y = 0$. Yes, $y = 0$ is a constant solution.

2g Odd: y' is unchanged when x is replaced with $-x$ and y is replaced with $-y$.

2c,d,h



3a $y = Cx^2 e^x$

$y' = C(e^x \times 2x + x^2 \times e^x)$

Substituting for y and y' in $xy' = y(2 + x)$:

LHS $= xy'$

$= Cx(2xe^x + x^2e^x)$

$= Cx^2e^x(2 + x)$

$= y(2 + x)$

$=$ RHS

Therefore, $y = Cx^2 e^x$ is a solution of $xy' = y(2 + x)$.

3b $y = \sqrt{x^2+C} = (x^2+C)^{\frac{1}{2}}$

$y' = \frac{1}{2}(x^2+C)^{-\frac{1}{2}} \times 2x$

$y' = x(x^2+C)^{-\frac{1}{2}}$

Substituting for y and y' in $yy' = x$:

$\text{LHS} = yy'$

$= (x^2+C)^{\frac{1}{2}} \times x(x^2+C)^{-\frac{1}{2}}$

$= x(x^2+C)^0$

$= x$

$= \text{RHS}$

Therefore, $y = \sqrt{x^2+C}$ is a solution of $yy' = x$.

3c

$y = \dfrac{1}{x^2+C} = (x^2+C)^{-1}$

$y' = -(x^2+C)^{-2} \times 2x$

$= -2x(x^2+C)^{-2}$

$= -\dfrac{2x}{(x^2+C)^2}$

Substituting for y and y' in $y' = -2xy^2$

$\text{RHS} = -2x \times \left(\dfrac{1}{x^2+C}\right)^2$

$= -\dfrac{2x}{(x^2+C)^2}$

$= \text{LHS}$

Therefore, $y = \dfrac{1}{x^2+C}$ is a solution of $y' = -2xy^2$.

4a $y = 0$

4b Taking the reciprocal of the DE gives

$$\frac{dx}{dy} = -\frac{2}{y}$$

4c

$$\int \frac{dx}{dy}\, dy = -\int \frac{2}{y}\, dy$$

$$x = -2\ln|y| + C$$

4d $x = -2\ln|y| + C$

$$x - C = -2\ln|y|$$

$$\frac{C - x}{2} = \ln|y|$$

$$|y| = e^{\left(\frac{C-x}{2}\right)}$$

$$y = e^{\left(\frac{C}{2}\right)} \times e^{\left(-\frac{x}{2}\right)} \text{ or } -e^{\left(\frac{C}{2}\right)} \times e^{\left(-\frac{x}{2}\right)}$$

$$y = Ae^{-\frac{x}{2}} \text{ where } A = e^{\left(\frac{C}{2}\right)} \text{or } -e^{\left(\frac{C}{2}\right)}$$

4e $y = 0$ when $A = 0$

4f Substituting $y(0) = 3$:

$$3 = Ae^{-\frac{0}{2}}$$

$$A = 3$$

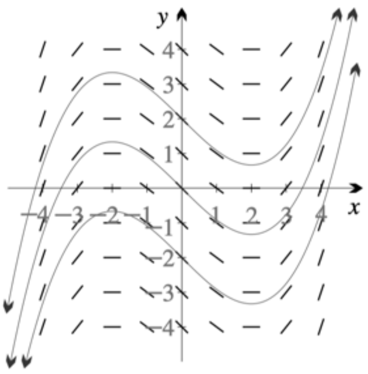
Therefore $y = 3e^{-\frac{x}{2}}$

5a i $y' = 0$ when $x = 2$ or $x = -2$

5a ii It is an isocline.

5a iii Gradients decrease to -1 then increase.

5a iv

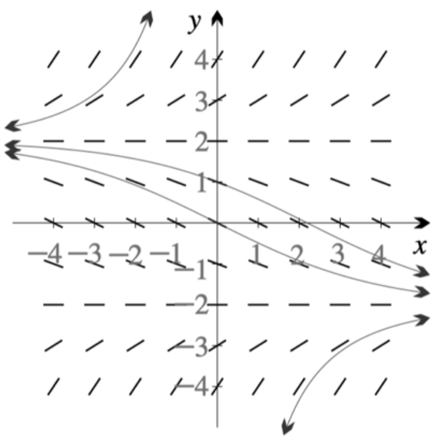


5b i $y' = 0$ when $y = 2$ or $y = -2$

5b ii Gradients decrease to $-\frac{1}{2}$ then increase.

5b iii It is an isocline.

5b iv

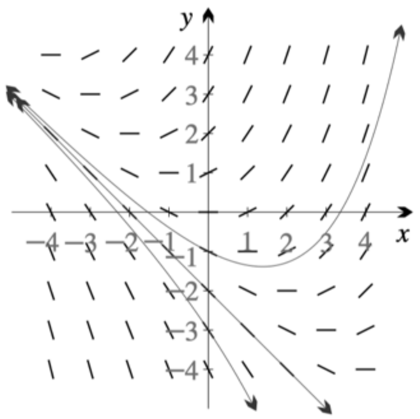


5c i $y' = 0$ when $y = -x$

5c ii Gradients decrease

5c iii Gradients increase

5c iv



6a $y = Ae^{-x} + Be^{-2x}$

$y' = -Ae^{-x} - 2Be^{-2x}$

$y'' = Ae^{-x} + 4Be^{-2x}$

Substituting for y'', y' and y in $y'' + 3y' + 2y = 0$:

$\text{LHS} = Ae^{-x} + 4Be^{-2x} + 3(-Ae^{-x} - 2Be^{-2x}) + 2(Ae^{-x} + Be^{-2x})$

$= Ae^{-x} + 4Be^{-2x} - 3Ae^{-x} - 6Be^{-2x} + 2Ae^{-x} + 2Be^{-2x}$

$= 3Ae^{-x} - 3Ae^{-x} + 6Be^{-2x} - 6Be^{-2x}$

$= 0$

$= \text{RHS}$

Therefore, $y = Ae^{-x} + Be^{-2x}$ is a solution of $y'' + 3y' + 2y = 0$.

6b $y = Ae^{-x}\cos 2x + Be^{-x}\sin 2x$

$y' = A(\cos 2x \times -e^{-x} + e^{-x} \times -2\sin 2x) + B(\sin 2x \times -e^{-x} + e^{-x} \times 2\cos 2x)$

$= A(-e^{-x}\cos 2x - 2e^{-x}\sin 2x) + B(-e^{-x}\sin 2x + 2e^{-x}\cos 2x)$

$= -Ae^{-x}(\cos 2x + 2\sin 2x) - Be^{-x}(\sin 2x - 2\cos 2x)$

$y'' = -A\big(-e^{-x}(\cos 2x + 2\sin 2x) + e^{-x}(-2\sin 2x + 4\cos 2x)\big) - B\big(-e^{-x}(\sin 2x - 2\cos 2x) + e^{-x}(2\cos 2x + 4\sin 2x)\big)$

$= -Ae^{-x}(-\cos 2x - 2\sin 2x - 2\sin 2x + 4\cos 2x) - Be^{-x}(-\sin 2x + 2\cos 2x + 2\cos 2x + 4\sin 2x)$

$= -Ae^{-x}(3\cos 2x - 4\sin 2x) - Be^{-x}(3\sin 2x + 4\cos 2x)$

Substituting y'', y' and y in $y'' + 2y' + 5y = 0$:

LHS

$= -Ae^{-x}(3\cos 2x - 4\sin 2x) - Be^{-x}(3\sin 2x + 4\cos 2x) + 2\big(-Ae^{-x}(\cos 2x + 2\sin 2x) - Be^{-x}(\sin 2x - 2\cos 2x)\big) + 5Ae^{-x}\cos 2x + 5Be^{-x}\sin 2x$

$= e^{-x}\cos 2x\,(-3A - 4B - 2A + 4B + 5A) + e^{-x}\sin 2x\,(4A - 3B - 4A - 2B + 5B)$

$= e^{-x}\cos 2x\,(0) + e^{-x}\sin 2x\,(0)$

$= 0$

$=$ RHS

Therefore, $y = Ae^{-x}\cos 2x + Be^{-x}\sin 2x$ is a solution of $y'' + 2y' + 5y = 0$.

6c $y = A\cos x + B\sin x + Ce^{2x}$

$y' = -A\sin x + B\cos x + 2Ce^{2x}$

$y'' = -A\cos x - B\sin x + 4Ce^{2x}$

$y''' = A\sin x - B\cos x + 8Ce^{2x}$

Substituting y''', y'', y' and y in $y''' - 2y'' + y' - 2y = 0$:

LHS

$= A\sin x - B\cos x + 8Ce^{2x} - 2(-A\cos x - B\sin x + 4Ce^{2x}) + (-A\sin x + B\cos x + 2Ce^{2x}) - 2(A\cos x + B\sin x + Ce^{2x})$

$= A\sin x - B\cos x + 8Ce^{2x} + 2A\cos x + 2B\sin x - 8Ce^{2x} - A\sin x + B\cos x + 2Ce^{2x} - 2A\cos x - 2B\sin x - 2Ce^{2x}$

$= 0$

$= \text{RHS}$

Therefore, $y = A\cos x + B\sin x + Ce^{2x}$ is a solution of $y''' - 2y'' + y' - 2y = 0$.

6d $y = Ae^{2x} + Be^{-2x} + C\cos 2x + D\cos 2x$

$y' = 2Ae^{2x} - 2Be^{-2x} - 2C\sin 2x - 2D\sin 2x$

$y'' = 4Ae^{2x} + 4Be^{-2x} - 4C\cos 2x - 4D\cos 2x$

$y''' = 8Ae^{2x} - 8Be^{-2x} + 8C\sin 2x + 8D\sin 2x$

$y'''' = 16Ae^{2x} + 16Be^{-2x} + 16C\cos 2x + 16D\cos 2x$

Substituting y'''' and y in $y'''' - 16y = 0$:

$\text{LHS} = 16Ae^{2x} + 16Be^{-2x} + 16C\cos 2x + 16D\cos 2x$

$-16(Ae^{2x} + Be^{-2x} + C\cos 2x + D\cos 2x)$

$= 0$

$= \text{RHS}$

Therefore, $y = Ae^{2x} + Be^{-2x} + C\cos 2x + D\cos 2x$ is a solution of $y'''' - 16y = 0$.

7a

$$\frac{dy}{dx} = \frac{-2xy}{1+x^2}$$

$$\int \frac{1}{y}\,dy = -\int \frac{2x}{1+x^2}\,dx$$

$\ln|y| = -\ln|1+x^2| + B$

$\ln|y| + \ln|1+x^2| = B$

$\ln|y(1+x^2)| = B$

$|y(1+x^2)| = e^B$

$y(1+x^2) = e^B$ or $-e^B$

$y(1+x^2) = C$ where $C = e^B$ or $-e^B$

$$y = \frac{C}{1+x^2}$$

7b

$$\frac{dy}{dx} = \frac{1-x}{2+y}$$

$$\int (2+y)\,dy = \int (1-x)\,dx$$

$$2y + \frac{y^2}{2} = x - \frac{x^2}{2} + B$$

$$y^2 + 4y = -x^2 + 2x + 2B$$

$$y^2 + 4y + 4 - 4 = -\big((x^2 - 2x + 1) - 1\big) + 2B$$

$$(y+2)^2 - 4 = -((x-1)^2 - 1) + 2B$$

$$(x-1)^2 + (y+2)^2 = 1 + 4 + 2B$$

$$(x-1)^2 + (y+2)^2 = 5 + 2B$$

$$(x-1)^2 + (y+2)^2 = C \text{ where } C = 5 + 2B$$

7c

$$\frac{dy}{dx} = \frac{y(x-1)}{x}$$

$$\int \frac{1}{y}\,dy = \int \frac{x-1}{x}\,dx$$

$$\int \frac{1}{y}\,dy = \int \left(1 - \frac{1}{x}\right)dx$$

$$\ln|y| = x - \ln|x| + B$$

$$\ln|x| + \ln|y| = x + B$$

$$\ln|xy| = x + B$$

$$|xy| = e^{x+B}$$

$$xy = e^x \times e^B \text{ or} - e^x \times e^B$$

$$xy = e^x \times C \text{ where } C = e^B \text{ or} - e^B$$

$$y = \frac{Ce^x}{x}$$

8a

$$\frac{dy}{dx} = \frac{1}{2}(1 - y) = \frac{1 - y}{2}$$

$$\frac{dx}{dy} = \frac{2}{1 - y}$$

$$\frac{dx}{dy} = \frac{-2}{y - 1}$$

$$\int \frac{dx}{dy} dy = -\int \frac{2}{y - 1} dy$$

$$x = -2\log_e|y - 1| + C$$

$$\log_e|y - 1| = -\frac{1}{2}(x - C)$$

$$|y - 1| = e^{-\frac{1}{2}(x-C)}$$

$$y - 1 = e^{-\frac{1}{2}(x-C)} \text{ or } -e^{-\frac{1}{2}(x-C)}$$

$$y - 1 = e^{-\frac{1}{2}x} \times e^{\frac{1}{2}C} \text{ or} - e^{-\frac{1}{2}x} \times e^{\frac{1}{2}C}$$

$$y - 1 = Ae^{-\frac{1}{2}x} \text{ where } A = e^{\frac{1}{2}C} \text{ or } -e^{\frac{1}{2}C}$$

$$y = 1 + Ae^{-\frac{1}{2}x}$$

Substituting $y(0) = 2$:

$$2 = 1 + Ae^{-\frac{1}{2}\times 0}$$

$$A = 2 - 1$$

$$A = 1$$

Therefore $y = 1 + e^{-\frac{1}{2}x}$

8b

$$\frac{dy}{dx} = \frac{1}{5}(5 - y) = \frac{5 - y}{5}$$

$$\frac{dx}{dy} = \frac{5}{5 - y}$$

$$\frac{dx}{dy} = \frac{-5}{y - 5}$$

$$\int \frac{dx}{dy}\, dy = -\int \frac{5}{y - 5}\, dy$$

$$x = -5 \log_e|y - 5| + C$$

$$\log_e|y - 5| = -\frac{1}{5}(x - C)$$

$$|y - 5| = e^{-\frac{1}{5}(x-C)}$$

$$y - 5 = e^{-\frac{1}{5}(x-C)} \text{ or } -e^{-\frac{1}{5}(x-C)}$$

$$y - 5 = e^{-\frac{1}{5}x} \times e^{\frac{1}{5}C} \text{ or} - e^{-\frac{x}{5}} \times e^{\frac{1}{5}C}$$

$$y - 5 = Ae^{-\frac{1}{5}x} \text{ where } A = e^{\frac{1}{5}C} \text{ or } -e^{\frac{1}{5}C}$$

$$y = 5 + Ae^{-\frac{1}{5}x}$$

Substituting $y(0) = 2$:

$$2 = 5 + Ae^{-\frac{1}{5}\times 0}$$

$$A = 2 - 5$$

$$A = -3$$

Therefore $y = 5 - 3e^{-\frac{1}{5}x}$

9a Let

$$\frac{1}{1 - x^2} = \frac{A}{1 - x} + \frac{B}{1 + x}$$

$$= \frac{A(1 + x) + B(1 - x)}{(1 + x)(1 - x)}$$

$$= \frac{A + Ax + B - Bx}{(1 + x)(1 - x)}$$

$$= \frac{(A-B)x + A + B}{(1+x)(1-x)}$$

Equating coefficients in the numerators:

$A + B = 1 \qquad (1)$

$A - B = 0 \qquad (2)$

$(1) + (2)$:

$2A = 1$

$A = \frac{1}{2}$

Substituting $A = \frac{1}{2}$ into (1):

$\frac{1}{2} + B = 1$

$B = \frac{1}{2}$

So $\frac{1}{1-x^2} = \frac{1}{2(1-x)} + \frac{1}{2(1+x)}$

or $\frac{1}{1-x^2} = \frac{1}{2}\left(\frac{1}{1-x} + \frac{1}{1+x}\right)$

9b

$\frac{dy}{dx} = \frac{y}{1-x^2}$

$\frac{1}{y}\,dy = \frac{1}{1-x^2}\,dx$

$\int \frac{1}{y}\,dy = \int \frac{1}{1-x^2}\,dx$

$\int \frac{1}{y}dy = \int \left(\frac{\frac{1}{2}}{1-x} + \frac{\frac{1}{2}}{1+x}\right)dx$

$\log_e |y| = -\frac{1}{2}\log_e |1-x| + \frac{1}{2}\log_e |1+x| + B$

$\log_e |y| = \frac{1}{2}\log_e \left|\frac{1+x}{1-x}\right| + B$

$$\log_e |y| = \log_e \left|\frac{1+x}{1-x}\right|^{\frac{1}{2}} + \log_e e^B$$

$$\log_e |y| = \log_e \left(e^B \left|\frac{1+x}{1-x}\right|^{\frac{1}{2}}\right)$$

$$|y| = e^B \left|\frac{1+x}{1-x}\right|^{\frac{1}{2}}$$

$$y = e^B \left|\frac{1+x}{1-x}\right|^{\frac{1}{2}} \text{ or } - e^B \left|\frac{1+x}{1-x}\right|^{\frac{1}{2}}$$

$$y = C \left|\frac{1+x}{1-x}\right|^{\frac{1}{2}} \text{ where } C = e^B \text{ or } -e^B$$

$$y = C\sqrt{\left|\frac{1+x}{1-x}\right|}$$

10a For domain: $1 - e^{-x} \neq 0$

$e^{-x} \neq 1$

$-x \neq \log_e 1$

$x \neq 0$

Also $L(x) \neq 0$, so there are no intercepts.

10b

As $x \to \infty, e^{-x} \to 0$ so $L(x) \to \dfrac{1}{1-0}$. Hence $\lim\limits_{x \to \infty} L(x) = 1$

As $x \to -\infty, e^{-x} \to \infty$ so $L(x) \to \dfrac{1}{-\infty}$. Hence $\lim\limits_{x \to -\infty} L(x) = 0$

10c $x \neq 0$ (from part a) so there is a vertical asymptote at $x = 0$.

As $x \to 0^+, e^{-x} \to 1^-$ so $L(x) \to \dfrac{1}{0^+}$. That is $L(x) \longrightarrow \infty$.

As $x \to 0^-, e^{-x} \to 1^+$ so $L(x) \to \dfrac{1}{0^-}$. That is $L(x) \longrightarrow -\infty$.

10d For $x = \log_e 2$:

$$y = L(\log_e 2)$$

$$= \frac{1}{1 - e^{-(\log_e 2)}}$$

$$= \frac{1}{1 - e^{\log_e 2^{\frac{1}{2}}}}$$

$$= \frac{1}{1 - \frac{1}{2}}$$

$$= 2$$

For $x = -\log_e 2$:

$$y = L(-\log_e 2)$$

$$= \frac{1}{1 - e^{\log_e 2}}$$

$$= \frac{1}{1 - 2}$$

$$= -1$$

10e i $L(x) = (1 - e^{-x})^{-1}$

$$L'(x) = -(1 - e^{-x})^{-2} \times e^{-x}$$

$$= -\frac{e^{-x}}{(1 - e^{-x})^2}$$

$$= -\frac{e^{-x}}{1 - 2e^{-x} + e^{-2x}}$$

$$= -\frac{1}{e^x(1 - 2e^{-x} + e^{-2x})}$$

$$= \frac{-1}{e^x - 2 + e^{-x}}$$

$$= \frac{-1}{e^x - 2\left(e^{\frac{x}{2}}\right)\left(e^{-\frac{x}{2}}\right) + e^{-x}}$$

$$= \frac{-1}{\left(e^{\frac{x}{2}} - e^{-\frac{x}{2}}\right)^2}$$

10e ii $L'(x) = -\left(e^{\frac{x}{2}} - e^{-\frac{x}{2}}\right)^{-2}$

$$L''(x) = 2\left(e^{\frac{x}{2}} - e^{-\frac{x}{2}}\right)^{-3} \times \left(\frac{e^{\frac{x}{2}}}{2} + \frac{e^{-\frac{x}{2}}}{2}\right)$$

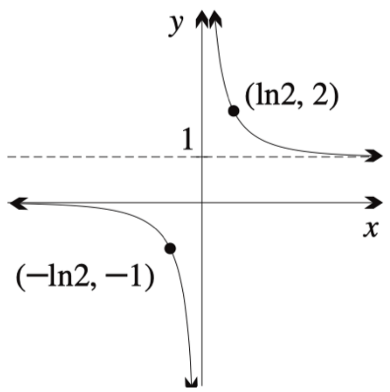
$$= \frac{2}{\left(e^{\frac{x}{2}} - e^{-\frac{x}{2}}\right)^3} \times \left(\frac{e^{\frac{x}{2}} + e^{-\frac{x}{2}}}{2}\right)$$

$$= \frac{e^{\frac{x}{2}} + e^{-\frac{x}{2}}}{\left(e^{\frac{x}{2}} - e^{-\frac{x}{2}}\right)^3}$$

10f When $x > 0$, $L''(x) > 0$ and when $x < 0, L''(x) < 0$.

Hence graph is concave up for $x > 0$ and concave down for $x < 0$.

10g



10h Substituting for y' and y in $y' = y(1 - y)$:

$$\text{LHS} = -\frac{e^{-x}}{(1 - e^{-x})^2}$$

$$\text{RHS} = \frac{1}{1 - e^{-x}}\left(1 - \frac{1}{1 - e^{-x}}\right)$$

$$= \frac{1}{1 - e^{-x}} - \frac{1}{(1 - e^{-x})^2}$$

$$= \frac{(1-e^{-x})-1}{(1-e^{-x})^2}$$

$$= -\frac{e^{-x}}{(1-e^{-x})^2}$$

$$= \text{LHS}$$

Therefore, since LHS = RHS, $y = L(x)$ is a solution of the DE.

11 The graph in option C corresponds to $y' = \frac{1}{4}(x^2 + y^2)$.

Slope must be zero at the origin (eliminates A) and positive everywhere else (eliminates B and D which have zero slope on the x and y-axes respectively).

12 Option B $y' = 1 + \frac{y}{x}$ corresponds to the graph.

The diagram shows slope of zero on the isocline $y = -x$, which eliminates options C and D. For $x = 1$, slope grows more extreme with increasing y, which eliminates A.

13a

$$\frac{dy}{dx} = -\frac{y}{x}$$

$$\int \frac{1}{y}\,dy = -\int \frac{1}{x}\,dx$$

$$\log_e|y| = -\log_e|x| + C$$

$$\log_e|x| + \log_e|y| = C$$

$$\log_e|xy| = C$$

$$|xy| = e^C$$

$$xy = e^C \text{ or } -e^C$$

$$xy = A \text{ where } A = e^C \text{ or } -e^C$$

$$y = \frac{A}{x}$$

Substituting $y(2) = 1$:

$1 = \frac{A}{2}$

$A = 2$

Therefore $y = \frac{2}{x}$

13b

$\frac{dy}{dx} = (1 + 2x)e^{-y}$

$\int \frac{1}{e^{-y}}\, dy = \int (1 + 2x)\, dx$

$\int e^{y}\, dy = \int (1 + 2x)\, dx$

$e^{y} = x + x^2 + C$

$y = \ln(x + x^2 + C)$

Substituting $y(1) = 0$:

$0 = \ln(1 + 1 + C)$

$0 = \ln(2 + C)$

$1 = 2 + C$

$C = -1$

Therefore $y = \ln(x + x^2 - 1)$

13c

$\frac{dy}{dx} = \frac{y^2}{\sqrt{x}}$

$\int \frac{1}{y^2}\, dy = \int \frac{1}{\sqrt{x}}\, dx$

$\int y^{-2}\, dy = \int x^{-\frac{1}{2}}\, dx$

$-y^{-1} = 2x^{\frac{1}{2}} + C$

$$-\frac{1}{y} = 2x^{\frac{1}{2}} + C$$

$$-y = \frac{1}{2x^{\frac{1}{2}} + C}$$

$$y = -\frac{1}{2\sqrt{x} + C}$$

Substituting $y(0) = -1$:

$$-1 = -\frac{1}{2\sqrt{0} + C}$$

$$-1 = -\frac{1}{C}$$

$$C = 1$$

Therefore $y = -\dfrac{1}{2\sqrt{x} + 1}$

14a Let

$$\frac{1}{y(1-y)} = \frac{A}{y} + \frac{B}{1-y}$$

$$= \frac{A(1-y) + By}{y(1-y)}$$

$$= \frac{A - Ay + By}{y(1-y)}$$

$$= \frac{(B-A)y + A}{y(1-y)}$$

Equating coefficients in the numerators:

$A = 1$

$B - A = 0 \qquad (1)$

Substituting $A = 1$ into (1):

$B - 1 = 0$

$B = 1$

Therefore $\dfrac{1}{y(1-y)} = \dfrac{1}{y} + \dfrac{1}{1-y}$

14b i $y = 0$ and $y = 1$

14b ii

$$\frac{dy}{dx} = y(1-y)$$

$$\int \frac{1}{y(1-y)}\,dy = \int 1\,dx$$

$$\int \left(\frac{1}{y} + \frac{1}{1-y}\right)\,dy = \int 1\,dx$$

$$\log_e |y| - \log_e |1-y| = x + C$$

$$\log_e \left|\frac{y}{1-y}\right| = x + C$$

$$\left|\frac{y}{1-y}\right| = e^{x+C}$$

$$\frac{y}{1-y} = e^x e^C \text{ or } -e^x e^C$$

$$\frac{y}{1-y} = Ae^x \text{ where } A = e^C \text{ or } -e^C$$

$$y = Ae^x(1-y)$$

$$y = Ae^x - Ae^x y$$

$$Ae^x y + y = Ae^x$$

$$y(Ae^x + 1) = Ae^x$$

$$y = \frac{Ae^x}{Ae^x + 1}$$

$$y = \frac{Ae^x}{Ae^x + 1} \times \frac{A^{-1}e^{-x}}{A^{-1}e^{-x}}$$

$$y = \frac{1}{A^{-1}e^{-x}(Ae^x + 1)}$$

$$y = \frac{1}{1 + A^{-1}e^{-x}}$$

$$y = \frac{1}{1 + Be^{-x}} \text{ where } B = A^{-1}$$

14b iii Substituting $\left(0,\ \frac{1}{4}\right)$:

$$\frac{1}{4} = \frac{1}{1 + Be^0}$$

$$\frac{1}{4} = \frac{1}{1 + B}$$

$$B = 3$$

Therefore $y = \dfrac{1}{1 + 3e^{-x}}$

15a Let

$$\frac{1}{(2-y)(3-y)} = \frac{A}{2-y} + \frac{B}{3-y}$$

$$= \frac{A(3-y) + B(2-y)}{(2-y)(3-y)}$$

$$= \frac{3A - Ay + 2B - By}{(2-y)(3-y)}$$

$$= \frac{(-A-B)y + 3A + 2B}{(2-y)(1-y)}$$

Equating coefficients in the numerators:

$-A - B = 0$ (1)

$3A + 2B = 1$ (2)

$2 \times (1) + (2)$:

$A = 1$

Substituting $A = 1$ into (1):

$-1 - B = 0$

$B = -1$

Therefore $\dfrac{1}{(2-y)(3-y)} = \dfrac{1}{2-y} - \dfrac{1}{3-y}$

or $\dfrac{1}{(2-y)(3-y)} = \dfrac{1}{y-3} - \dfrac{1}{y-2}$

15b i $y = 2$ and $y = 3$

15b ii $y' = -\frac{1}{5}(3-y)(2-y)$

$$\frac{dy}{dx} = -\frac{1}{5}(3-y)(2-y)$$

$$\frac{1}{(3-y)(2-y)}dy = -\frac{1}{5}\,dx$$

$$\int\left(\frac{1}{y-3} - \frac{1}{y-2}\right)\,dy = \int -\frac{1}{5}\,dx$$

$$\log_e|y-3| - \log_e|y-2| = -\frac{1}{5}x + C$$

$$\log_e\left|\frac{y-3}{y-2}\right| = -\frac{1}{5}x + C$$

$$\left|\frac{y-3}{y-2}\right| = e^{-\frac{1}{5}x+C}$$

$$\frac{y-3}{y-2} = e^{-\frac{1}{5}x}e^C \text{ or } -e^{-\frac{1}{5}x}e^C$$

$$\frac{y-3}{y-2} = Ae^{-\frac{1}{5}x} \text{ where } A = e^C \text{ or } -e^C$$

$$y - 3 = Ae^{-\frac{1}{5}x}(y-2)$$

$$y - 3 = Ae^{-\frac{1}{5}x}y - 2Ae^{-\frac{1}{5}x}$$

$$y - Ae^{-\frac{1}{5}x}y = 3 - 2Ae^{-\frac{1}{5}x}$$

$$y\left(1 - Ae^{-\frac{1}{5}x}\right) = 3 - 2Ae^{-\frac{1}{5}x}$$

$$y = \frac{3 - 2Ae^{-\frac{1}{5}x}}{1 - Ae^{-\frac{1}{5}x}}$$

Substituting $y(0) = 1$:

$$1 = \frac{3 - 2Ae^0}{1 - Ae^0}$$

$$1 = \frac{3 - 2A}{1 - A}$$

$1 - A = 3 - 2A$

$A = 2$

Therefore $y = \dfrac{3 - 4e^{-\frac{1}{5}x}}{1 - 2e^{-\frac{1}{5}x}}$

15b iii $y = 0$ when:

$0 = \dfrac{3 - 4e^{-\frac{1}{5}x}}{1 - 2e^{-\frac{1}{5}x}}$

$3 - 4e^{-\frac{1}{5}x} = 0$

$e^{-\frac{1}{5}x} = \dfrac{3}{4}$

$-\dfrac{1}{5}x = \ln\dfrac{3}{4}$

$x = -5\ln\dfrac{3}{4}$

$x = 5\ln\left(\dfrac{3}{4}\right)^{-1}$

$x = 5\ln\dfrac{4}{3}$

or $x \doteqdot 1.44$

16a Let

$$\frac{5}{N(5-N)} = \frac{A}{N} + \frac{B}{5-N}$$

$$= \frac{A(5-N) + BN}{N(5-N)}$$

$$= \frac{5A - AN + BN}{N(5-N)}$$

$$= \frac{(B-A)N + 5A}{N(5-N)}$$

Equating coefficients in the numerators:

$5A = 5$ so $A = 1$

$B - A = 0 \qquad (1)$

Substituting $A = 1$ into (1):

$B - 1 = 0$

$B = 1$

Therefore $\dfrac{5}{N(5-N)} = \dfrac{1}{N} + \dfrac{1}{5-y}$

16b

$$\frac{dN}{dt} = kN(5-N)$$

$$\frac{1}{N(5-N)}\,dN = k\,dt$$

$$\int \frac{1}{N(5-N)}\,dN = \int k\,dt$$

$$\int \frac{5}{N(5-N)}\,dN = \int 5k\,dt$$

$$\int \left(\frac{1}{N} + \frac{1}{5-N}\right) dN = \int 5k\,dt$$

$$\log_e |N| - \log_e |5-N| = 5kt + C$$

$$\log_e \left|\frac{N}{5-N}\right| = 5kt + C$$

$$\left|\frac{N}{5-N}\right| = e^{5kt+C}$$

$$\frac{N}{5-N} = e^{5kt+C} \text{ or } -e^{5kt+C}$$

$$\frac{N}{5-N} = e^C e^{5kt} \text{ or } -e^C e^{5kt}$$

$$\frac{N}{5-N} = Ae^{5kt} \text{ where } A = e^C \text{ or } -e^C$$

$$N = Ae^{5kt}(5-N)$$

$$N = 5Ae^{5kt} - NAe^{5kt}$$

$NAe^{5kt} + N = 5Ae^{5kt}$

$N(Ae^{5kt} + 1) = 5Ae^{5kt}$

$N = \dfrac{5Ae^{5kt}}{Ae^{5kt} + 1}$

$N = \dfrac{5Ae^{5kt}}{Ae^{5kt} + 1} \times \dfrac{A^{-1}e^{-5kt}}{A^{-1}e^{-5kt}}$

$N = \dfrac{5}{A^{-1}e^{-5kt}(Ae^{5kt} + 1)}$

$N = \dfrac{5}{1 + A^{-1}e^{-5kt}}$

$N = \dfrac{5}{1 + Be^{-5kt}}$ where $B = A^{-1}$

16c Substituting $N(0) = 1$:

$1 = \dfrac{5}{1 + Be^{0}}$

$1 + B = 5$

$B = 4$

Therefore $N = \dfrac{5}{1 + 4e^{-5kt}}$

16d Total sales at $t = 2$ is 1 million phones already sold at $t = 0$, plus 0.4 million phones sold between $t = 0$ and $t = 2$. So $N(2) = 1.4$ (as N is the number of people in millions).

Substituting $N(2) = 1.4$:

$1.4 = \dfrac{5}{1 + 4e^{-5k(2)}}$

$1 + 4e^{-10k} = \dfrac{5}{1.4}$

$1 + 4e^{-10k} = \dfrac{25}{7}$

$4e^{-10k} = \dfrac{25}{7} - 1$

$4e^{-10k} = \frac{18}{7}$

$e^{-10k} = \frac{9}{14}$

$-10k = \ln\frac{9}{14}$

$k = -\frac{1}{10}\ln\frac{9}{14}$

$k \doteqdot 4.418 \times 10^{-2}$

16e When $t = 3$,

$N = \frac{5}{1 + 4e^{-5\left(-\frac{1}{10}\ln\frac{9}{14}\right)(3)}}$

$\doteqdot 1.633$

Sales during year 3 equals total sales in three years (1.633 million) minus sales up to year 2 (1.4 million).

That is, the projected sales in the third year will be 0.233 million or 233,000.

17a $y' = x(1 - 2y)$

$= x - 2xy$

$y'' = 1 - 2y \times 1 - 2x \times y'$

$= 1 - 2y - 2xy'$

17b Turning point when $y' = 0$:

$x(1 - 2y) = 0$

$x = 0$

When $x = 0, y = 1$, so

$y'' = 1 - 2 \times 1 - 2 \times 0 \times 0$

$= 1 - 2$

$= -1$

Since $y'' < 0$, the turning point at $x = 0$ is a maximum.

17c $y' = x(1-2y)$

$$\frac{dy}{dx} = x(1-2y)$$

$$\int \frac{1}{1-2y}\,dy = \int x\,dx$$

$$-\frac{1}{2}\int \frac{2}{2y-1}\,dy = \int x\,dx$$

$$\int \frac{2}{2y-1}\,dy = -2\int x\,dx$$

$$\ln|2y-1| = -x^2 + C$$

$$|2y-1| = e^{-x^2+C}$$

$$|2y-1| = e^C e^{-x^2}$$

$$2y-1 = e^C e^{-x^2} \text{ or } -e^C e^{-x^2}$$

$$2y-1 = Ae^{-x^2} \text{ where } A = e^C \text{ or } -e^C$$

$$2y = 1 + Ae^{-x^2}$$

$$y = \frac{1}{2}\left(1 + Ae^{-x^2}\right)$$

Substituting $y(0) = 1$:

$$1 = \frac{1}{2}(1 + Ae^0)$$

$$2 = 1 + A$$

$$A = 1$$

Therefore $y = \frac{1}{2}\left(1 + e^{-x^2}\right)$

18a The solutions are $y = F(x) + C$ where $F'(x) = f(x)$.

Each is a vertical shift of the other.

18b The solutions are $y = G^{-1}(x + C)$ where $G'(y) = \frac{1}{g(y)}$.

Each is a horizontal shift of the other.

19a $y = 1, y = -1$

19b $y_1 = \cos(x + A)$

$y_1{}' = -\sin(x + A)$

Substituting y' and y in $(y')^2 = 1 - y^2$:

$$\begin{aligned}\text{LHS} &= (-\sin(x + A))^2 \\ &= \sin^2(x + A) \\ &= 1 - \cos^2(x + A) \\ &= 1 - y^2 \\ &= \text{RHS}\end{aligned}$$

Therefore $y_1 = \cos(x + A)$ is a solution of $(y')^2 = 1 - y^2$.

$y_2 = \sin(x + B)$

$y_2{}' = \cos(x + B)$

Substituting y' and y in $(y')^2 = 1 - y^2$:

$$\begin{aligned}\text{LHS} &= (\cos(x + B))^2 \\ &= \cos^2(x + B) \\ &= 1 - \sin^2(x + B) \\ &= 1 - y^2 \\ &= \text{RHS}\end{aligned}$$

Therefore $y_2 = \sin(x + B)$ is a solution of $(y')^2 = 1 - y^2$.

19c $\cos(x + A) = \cos x \cos A - \sin x \sin A$

$\sin(x + B) = \sin x \cos B + \sin B \cos x$

Let $B = A + \frac{\pi}{2} + 2n\pi$ for integer n

Then $\cos B = \cos\left(A + \frac{\pi}{2} + 2n\pi\right)$

$$\begin{aligned}\cos B &= \cos\left(A + \frac{\pi}{2}\right) \\ &= \cos A \cos\frac{\pi}{2} - \sin A \sin\frac{\pi}{2}\end{aligned}$$

$= \cos A \times 0 - \sin A \times 1$

$= -\sin A$

$\sin B = \sin\left(A + \frac{\pi}{2} + 2n\pi\right)$

$\sin B = \sin\left(A + \frac{\pi}{2}\right)$

$= \sin A \cos\frac{\pi}{2} + \cos A \sin\frac{\pi}{2}$

$= \sin A \times 0 + \cos A \times 1$

$= \cos A$

Substituting $\cos B = -\sin A$ and $\sin B = \cos A$ in the expansion for $\sin(x + B)$:

$\sin(x + B) = \sin x \times -\sin A + \cos A \times \cos x$

$= -\sin x \sin A + \cos A \cos x$

$= \cos(x + A)$

So the two solutions are identical.

19d $B = A + \frac{\pi}{2}$

(More precisely, $B = A + \frac{\pi}{2} + 2n\pi$, for some integer n.)

20a Using Pythagoras' theorem:

$AB^2 + AC^2 = BC^2$

$y^2 + AC^2 = 4^2$

$AC^2 = 16 - y^2$

$AC = \sqrt{16 - y^2}$

$\text{Gradient} = -\frac{AB}{AC}$

$\frac{dy}{dx} = -\frac{y}{\sqrt{16 - y^2}}$

20b $-4 \le y \le 4; y(0) = 4$

20c $y = 0$, not a solution of the IVP

20d