Solutions to Exercise 6A

1a $2^3 \times 2^7 = 2^{3+7} = 2^{10}$

1b $e^4 \times e^3 = e^{4+3} = e^7$

1c $2^6 \div 2^2 = 2^{6-2} = 2^4$

1d $e^8 \div e^5 = e^{8-5} = e^3$

1e $(2^3)^4 = 2^{3 \times 4} = 2^{12}$

1f $(e^5)^6 = e^{5 \times 6} = e^{30}$

2a $e^{2x} \times e^{5x} = e^{2x+5x} = e^{7x}$

2b $e^{10x} \div e^{8x} = e^{10x-8x} = e^{2x}$

2c $(e^{2x})^5 = e^{2x \times 5} = e^{10x}$

2d $e^{2x} \times e^{-7x} = e^{2x+(-7x)} = e^{-5x}$

2e $e^x \div e^{-4x} = e^{x-(-4x)} = e^{5x}$

2f $(e^{-3x})^4 = e^{-3x \times 4} = e^{-12x}$

3a $e^2 \doteqdot 7.389$

3b $e^{-3} \doteqdot 0.04979$

3c $e = e^1 \doteqdot 2.718$

3d $\frac{1}{e} = e^{-1} \doteqdot 0.3679$

3e $\sqrt{e} = e^{\frac{1}{2}} \doteqdot 1.649$

3f $\frac{1}{\sqrt{e}} = e^{-\frac{1}{2}} \doteqdot 0.6065$

4a $y = e^x$
$y' = e^x$
$y'' = e^x$

4b The curve $y = e^x$ is always concave up, and is always increasing at an increasing rate.

5a Gradient at $P(1, e) = \frac{d}{dx}e^x$, where $x = 1$
$= e^x$, where $x = 1$
$= e$
Tangent at $P(1, e)$:
$y - e = e(x - 1)$
$y = ex$

x-intercept when $y = 0$
$0 = ex$
$x = 0$

5b Gradient at $Q(0, 1) = \frac{d}{dx}e^x$, where $x = 0$
$= e^x$, where $x = 0$
$= 1$
Tangent at $Q(0, 1)$:
$y - 1 = 1(x - 0)$
$y = x + 1$

x-intercept when $y = 0$
$0 = x + 1$
$x = -1$

5c Gradient at $Q\left(-1, \frac{1}{e}\right) = \frac{d}{dx}e^x$, where $x = -1$
$= e^x$, where $x = -1$
$= \frac{1}{e}$

Tangent at $Q\left(-1, \frac{1}{e}\right)$:
$y - \frac{1}{e} = \frac{1}{e}(x - (-1))$
$y - \frac{1}{e} = \frac{1}{e}x + \frac{1}{e}$
$y = \frac{1}{e}x + \frac{2}{e}$
$y = \frac{1}{e}(x + 2)$

$x -$ intercept when $y = 0$
$0 = \frac{1}{e}(x + 2)$
$x = -2$

6a $x = 1, y = e^1 - 1$
$P = (1, e - 1)$

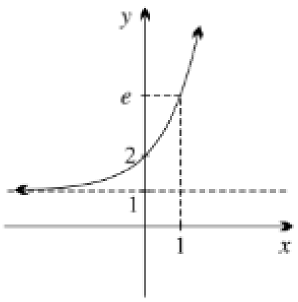
6b $\frac{dy}{dx} = e^x$
When $x = 1, \frac{dy}{dx} = e$

6c Tangent at $P(1, e - 1)$:
$y - (e - 1) = e(x - 1)$
$y = ex - e + e - 1$
$y = ex - 1$
$ex - y - 1 = 0$

Normal at $P(1, e - 1)$ has gradient $= -\frac{1}{\mathrm{e}}$
$y - (e - 1) = -\frac{1}{e}(x - 1)$

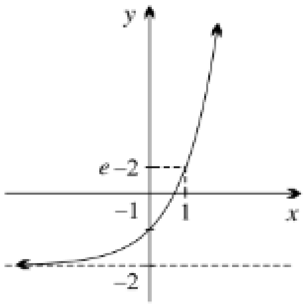
$ey - e^2 + e = -x + 1$
$x + ey - e^2 + e - 1 = 0$

7a



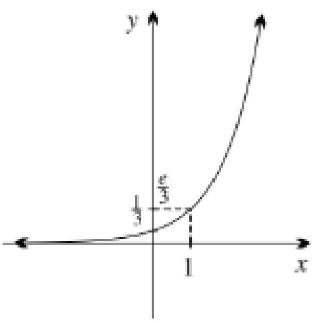
Shift e^x up 1

7b



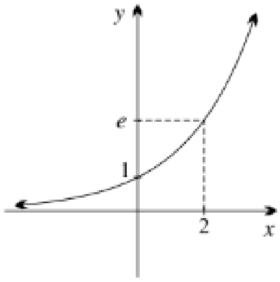
Shift e^x down 2

7c



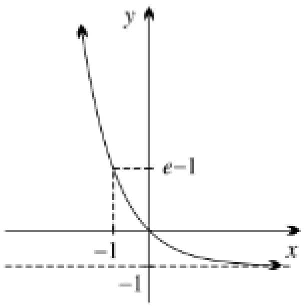
Stretch e^x vertically with factor $\frac{1}{3}$

7d



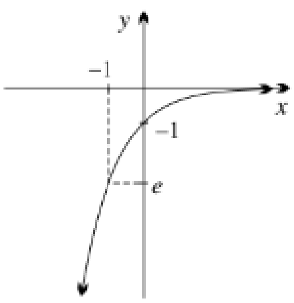
Stretch e^x horizontally with factor 2

8a



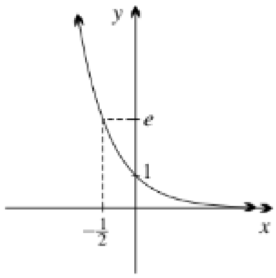
Shift e^{-x} down 1

8b



Reflect e^{-x} in x-axis

8c



Stretch e^{-x} horizontally with factor $\frac{1}{2}$

9 It is a vertical dilation of $y = e^x$ with factor $-\frac{1}{3}$. Its equation is $y = -\frac{1}{3}e^x$.

10a $(e^x + 1)(e^x - 1) = e^{x+x} + e^x - e^x - 1$

$= e^{2x} - 1$

10b $(e^{4x} + 3)(e^{2x} + 3) = e^{4x+2x} + 3e^{4x} + 3e^{2x} + 9$

$= e^{6x} + 3e^{4x} + 3e^{2x} + 9$

10c $(e^{-3x} - 2)e^{3x} = e^{-3x+3x} - 2e^{3x}$

$= 1 - 2e^{3x}$

10d $(e^{-2x} + e^{2x})^2 = (e^{-2x} + e^{2x})(e^{-2x} + e^{2x})$

$= e^{-2x-2x} + e^{-2x+2x} + e^{2x-2x} + e^{2x+2x}$

$= e^{-4x} + 1 + 1 + e^{4x}$

$= e^{-4x} + 2 + e^{4x}$

11a $\frac{e^{4x}+e^{3x}}{e^{2x}} = \frac{e^{4x}}{e^{2x}} + \frac{e^{3x}}{e^{2x}}$

$= e^{4x-2x} + e^{3x-2x}$

$= e^{2x} + e^x$

11b $\frac{e^{2x}-e^{3x}}{e^{4x}} = \frac{e^{2x}}{e^{4x}} - \frac{e^{3x}}{e^{4x}}$
$= e^{2x\,-\,4x} - e^{3x\,-\,4x}$
$= e^{-2x} - e^{-x}$

11c $\frac{e^{10x}+5e^{20x}}{e^{-10x}} = \frac{e^{10x}}{e^{-10x}} + \frac{5e^{20x}}{e^{-10x}}$
$= e^{10x\,-(-10x)} + 5e^{20x\,-(-10x)}$
$= e^{20x} + 5e^{30x}$

11d $\frac{6e^{-x}+9e^{-2x}}{3e^{3x}} = \frac{6e^{-x}}{3e^{3x}} + \frac{9e^{-2x}}{3e^{3x}}$
$= 2e^{-x\,-\,3x} + 3e^{-2x\,-\,3x}$
$= 2e^{-4x} + 3e^{-5x}$

12a y-intercept when $x = 0$
$y = e^0$
$= 1$

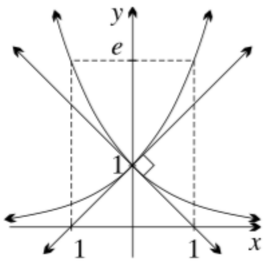
Tangent at $(0{,}1) = \frac{d}{dx}e^x$, where $x = 0$
$= e^0$
$= 1$

12b Reflection in y-axis

12c y-intercept when $x = 0$
$y = e^{-0}$
$= 1$

Gradient at $(0, 1) = \frac{d}{dx}e^{-x}$, where $x = 0$
$= -e^0$
$= -1$

12d



12e Horizontal dilation with factor -1

13a $y = e^x + 5$

$y' = e^x$

$y'' = e^x$

$y''' = e^x$

$y'''' = e^x$

13b $y = e^x + x^3$

$y' = e^x + 3x^2$

$y'' = e^x + 6x$

$y''' = e^x + 6$

$y'''' = e^x$

13c $y = 4e^x$

$y' = 4e^x$

$y'' = 4e^x$

$y''' = 4e^x$

$y'''' = 4e^x$

Here, the gradient equals the height.

13d $y = 5e^x + 5x^2$

$y' = 5e^x + 10x$

$y'' = 5e^x + 10$

$y''' = 5e^x$

$y'''' = 5e^x$

14a $\frac{dy}{dx} = e^x$

At $x = 0$, gradient $= e^0 = 1$

Angle of inclination $= \tan^{-1} 1 = 45^\circ$

14b At $x = 1$, gradient $= e^1 = e$

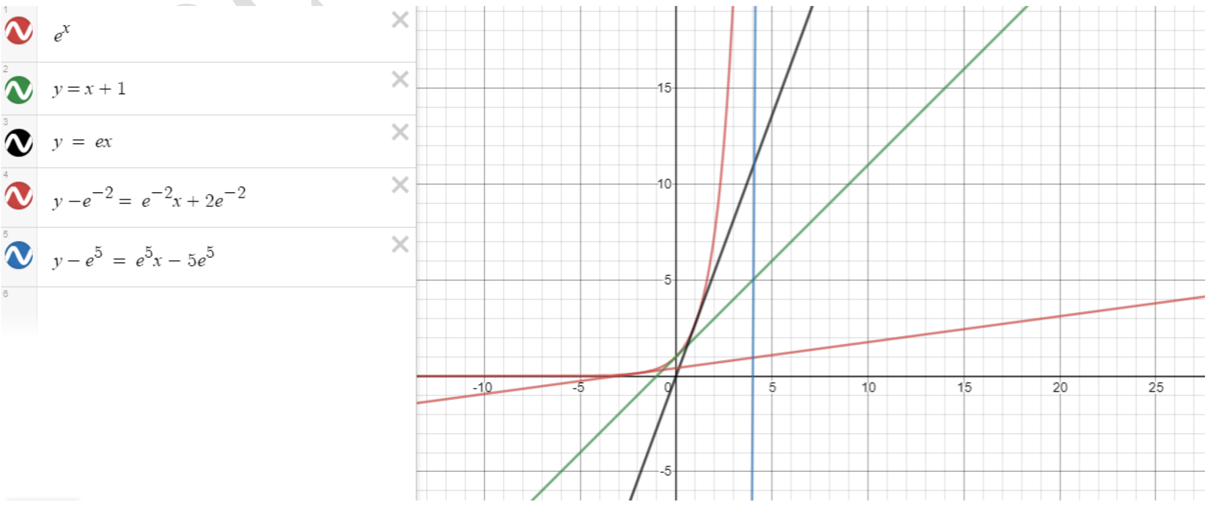
Angle of inclination $= \tan^{-1} e \doteqdot 69^\circ\, 48'$

14c At $x = -2$, gradient $= e^{-2}$

Angle of inclination $= \tan^{-1} e^{-2} \doteqdot 7^\circ\, 42'$

14d At $x = 5$, gradient $= e^5$

Angle of inclination $= \tan^{-1} e^5 \doteqdot 89^\circ\, 37'$



15a $x = 1,\ y = e^1 - 1 = e - 1$

15b $\frac{dy}{dx} = e^x$

$x = 1, \frac{dy}{dx} = e^1 = e$

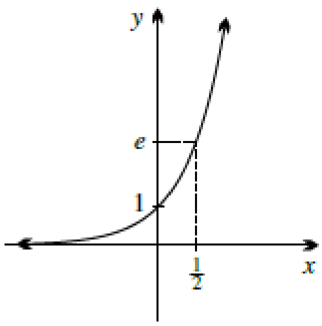
15c Tangent at $P(1, e - 1)$:

$y - (e - 1) = e(x - 1)$

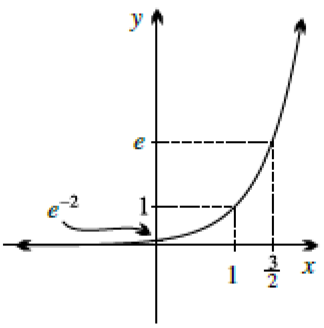
$y = ex - e + e - 1$

$y = ex - 1$

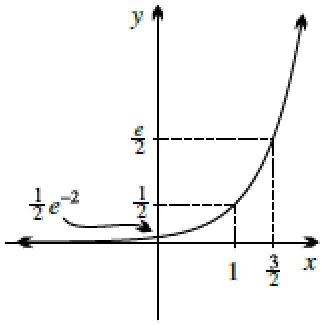
16a Stretch horizontally with factor of $\frac{1}{2}$.



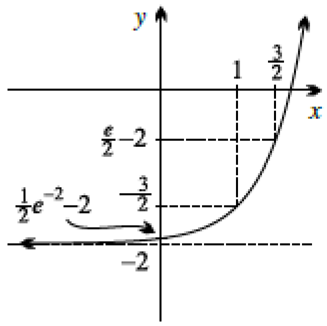
16b Shift right 1 unit.



16c Stretch vertically with factor of $\frac{1}{2}$.



16d Shift down 2 units.



17a As a translation, the transformation is shift left 2 units.

Alternatively, $y = e^2 e^x$ so it is a vertical dilation with factor of e^2.

17b As a dilation, the transformation is dilate vertically with factor of 2.

Alternatively, $y = e^{\log_e 2} e^x = e^{x + \log_e 2}$ so it is a shift left of $\log_e 2$ units.

Solutions to Exercise 6B

1a $\frac{dy}{dx} = 7e^{7x}$

1b $\frac{dy}{dx} = 3 \times 4e^{3x} = 12e^{3x}$

1c $\frac{dy}{dx} = \frac{1}{3} \times 6e^{\frac{1}{3}x} = 2e^{\frac{1}{3}x}$

1d $\frac{dy}{dx} = -2 \times -\frac{1}{2}e^{-2x} = e^{-2x}$

1e $\frac{dy}{dx} = 3e^{3x+4}$

1f $\frac{dy}{dx} = 4e^{4x-3}$

1g $\frac{dy}{dx} = -3e^{-3x+4}$

1h $\frac{dy}{dx} = -2e^{-2x-7}$

2a $\frac{dy}{dx} = e^{x} - e^{-x}$

2b $\frac{dy}{dx} = 2e^{2x} + 3e^{-3x}$

2c $\frac{dy}{dx} = \frac{e^{x}+e^{-x}}{2}$

2d $\frac{dy}{dx} = \frac{e^{x}-e^{-x}}{3}$

2e $\frac{dy}{dx} = \frac{2e^{2x}}{2} + \frac{3e^{3x}}{3} = e^{2x} + e^{3x}$

2f $\frac{dy}{dx} = \frac{4e^{4x}}{4} + \frac{5e^{5x}}{5} = e^{4x} + e^{5x}$

3a $y = e^{x+2x} = e^{3x}$

$\frac{dy}{dx} = 3e^{3x}$

3b $y = e^{3x-x} = e^{2x}$

$\frac{dy}{dx} = 2e^{2x}$

3c $y = e^{x\times 2} = e^{2x}$

$\frac{dy}{dx} = 2e^{2x}$

3d $y = e^{2x\times 3} = e^{6x}$

$\frac{dy}{dx} = 6e^{6x}$

3e $y = e^{4x-x} = e^{3x}$

$\frac{dy}{dx} = 3e^{3x}$

3f $y = e^{x-2x} = e^{-x}$

$\frac{dy}{dx} = -e^{-x}$

3g $y = e^{0-3x} = e^{-3x}$

$\frac{dy}{dx} = -3e^{-3x}$

3h $y = e^{0-5x} = e^{-5x}$

$\frac{dy}{dx} = -5e^{-5x}$

4a i $f'(x) = -e^{-x}$

$f''(x) = e^{-x}$

$f'''(x) = -e^{-x}$

$f^{(4)}(x) = e^{-x}$

4a ii Successive derivatives alternate in signs. More precisely,

$f^{(n)}(x) = \begin{cases} e^{-x}, & \text{if } n \text{ is even,} \\ -e^{-x}, & \text{if } n \text{ is odd} \end{cases}$

4b i $f'(x) = 2e^{2x}$

$f''(x) = 4e^{2x}$

$f'''(x) = 8e^{2x}$

$f^{(4)}(x) = 16e^{2x}$

4b ii Each derivative is twice the previous one. More precisely

$f^{(n)}(x) = 2^n e^{2x}$

5a $\frac{d}{dx}\left(e^x(e^x + 1)\right) = \frac{d}{dx}(e^{x+x} + e^x)$

$= \frac{d}{dx}(e^{2x} + e^x)$

$= 2e^{2x} + e^x$

5b $\frac{d}{dx}\left(e^{-x}(2e^{-x} - 1)\right) = \frac{d}{dx}(2e^{-x-x} - e^{-x})$

$= \frac{d}{dx}(2e^{-2x} - e^{-x})$

$= -4e^{-2x} + e^{-x}$

5c $\frac{d}{dx}(e^x+1)^2 = \frac{d}{dx}((e^x)^2 + e^x + e^x + 1)$

$$= \frac{d}{dx}(e^{2x} + 2e^x + 1)$$

$$= 2e^{2x} + 2e^x$$

5d $\frac{d}{dx}(e^x+3)^2 = \frac{d}{dx}((e^x)^2 + 3e^x + 3e^x + 9)$

$$= \frac{d}{dx}(e^{2x} + 6e^x + 9)$$

$$= 2e^{2x} + 6e^x$$

5e $\frac{d}{dx}(e^x-1)^2 = \frac{d}{dx}((e^x)^2 - e^x - e^x + 1)$

$$= \frac{d}{dx}(e^{2x} - 2e^x + 1)$$

$$= 2e^{2x} - 2e^x$$

5f $\frac{d}{dx}(e^x-2)^2 = \frac{d}{dx}((e^x)^2 - 2e^x - 2e^x + 4)$

$$= \frac{d}{dx}(e^{2x} - 4e^x + 4)$$

$$= 2e^{2x} - 4e^x$$

5g $\frac{d}{dx}\big((e^x+e^{-x})(e^x-e^{-x})\big) = \frac{d}{dx}((e^x)^2 - e^{x-x} + e^{-x+x} - (e^{-x})^2)$

$$= \frac{d}{dx}(e^{2x} - e^{-2x})$$

$$= 2e^{2x} + 2e^{-2x}$$

5h $\frac{d}{dx}\big((e^{5x}+e^{-5x})(e^{5x}-e^{-5x})\big) = \frac{d}{dx}((e^{5x})^2 - e^{5x-5x} + e^{-5x+5x} - (e^{-5x})^2)$

$$= \frac{d}{dx}(e^{10x} - e^{-10x})$$

$= 10e^{10x} + 10e^{-10x}$

6a Let $u = ax + b$

Then $y = e^u$

Hence $\frac{du}{dx} = a$ and $\frac{dy}{du} = e^u$

$\frac{dy}{dx} = a \times e^{ax+b} = ae^{ax+b}$

6b Let $u = x^2$

Then $y = e^u$

Hence $\frac{du}{dx} = 2x$ and $\frac{dy}{du} = e^u$

$\frac{dy}{dx} = 2x \times e^{x^2} = 2xe^{x^2}$

6c Let $u = -\frac{1}{2}x^2$

Then $y = e^u$

Hence $\frac{du}{dx} = -x$ and $\frac{dy}{du} = e^u$

$\frac{dy}{dx} = -x \times e^{-\frac{1}{2}x^2} = -xe^{-\frac{1}{2}x^2}$

6d Let $u = x^2 + 1$

Then $y = e^u$

Hence $\frac{du}{dx} = 2x$ and $\frac{dy}{du} = e^u$

$\frac{dy}{dx} = 2x \times e^{x^2+1} = 2xe^{x^2+1}$

6e Let $u = 1 - x^2$

Then $y = e^u$

Hence $\frac{du}{dx} = -2x$ and $\frac{dy}{du} = e^u$

$\frac{dy}{dx} = -2x \times e^{1-x^2} = -2xe^{1-x^2}$

6f Let $u = x^2 + 2x$

Then $y = e^u$

Hence $\frac{du}{dx} = 2x + 2$ and $\frac{dy}{du} = e^u$

$\frac{dy}{dx} = (2x + 2) \times e^{x^2+2x} = 2(x + 1)e^{x^2+2x}$

6g Let $u = 6 + x - x^2$

Then $y = e^u$

Hence $\frac{du}{dx} = 1 - 2x$ and $\frac{dy}{du} = e^u$

$\frac{dy}{dx} = (1 - 2x) \times e^{6+x-x^2} = (1 - 2x)e^{6+x-x^2}$

6h Let $u = 3x^2 - 2x + 1$

Then $y = \frac{1}{2}e^u$

Hence $\frac{du}{dx} = 6x - 2$ and $\frac{dy}{du} = \frac{1}{2}e^u$

$\frac{dy}{dx} = (6x - 2) \times \frac{1}{2}e^{3x^2-2x+1} = (3x - 1)e^{3x^2-2x+1}$

7a Let $u = x$ and $v = e^x$

Then $\frac{du}{dx} = 1$ and $\frac{dv}{dx} = e^x$

$\frac{dy}{dx} = (e^x \times 1) + (x \times e^x)$

$= e^x(1 + x)$

7b Let $u = x$ and $v = e^{-x}$

Then $\frac{du}{dx} = 1$ and $\frac{dv}{dx} = -e^{-x}$

$\frac{dy}{dx} = (e^{-x} \times 1) + (x \times -e^{-x})$

$= e^{-x} - xe^{-x}$

$= e^{-x}(1 - x)$

7c Let $u = x - 1$ and $v = e^x$

Then $\frac{du}{dx} = 1$ and $\frac{dv}{dx} = e^x$

$\frac{dy}{dx} = (e^x \times 1) + \big((x - 1) \times e^x\big)$

$= e^x(1 + x - 1)$

$= xe^x$

7d Let $u = x + 1$ and $v = e^{3x - 4}$

Then $\frac{du}{dx} = 1$ and $\frac{dv}{dx} = 3e^{3x-4}$

$\frac{dy}{dx} = (e^{3x-4} \times 1) + \big((x + 1) \times 3e^{3x-4}\big)$

$= e^{3x-4}\big(1 + 3(x + 1)\big)$

$= e^{3x-4}(3x + 4)$

7e Let $u = x^2$ and $v = e^{-x}$

Then $\frac{du}{dx} = 2x$ and $\frac{dv}{dx} = -e^{-x}$

$\frac{dy}{dx} = (e^{-x} \times 2x) + (x^2 \times -e^{-x})$

$= e^{-x}(2x - x^2)$

7f Let $u = 2x - 1$ and $v = e^{2x}$

Then $\frac{du}{dx} = 2$ and $\frac{dv}{dx} = 2e^{2x}$

$\frac{dy}{dx} = (e^{2x} \times 2) + \big((2x - 1) \times 2e^{2x}\big)$

$= 2e^{2x}(1 + 2x - 1)$

$= 4xe^{2x}$

7g Let $u = x^2 - 5$ and $v = e^x$

Then $\frac{du}{dx} = 2x$ and $\frac{dv}{dx} = e^x$

$\frac{dy}{dx} = (e^x \times 2x) + \left((x^2 - 5) \times e^x\right)$

$= e^x(2x + x^2 - 5)$

7h Let $u = x^3$ and $v = e^{2x}$

Then $\frac{du}{dx} = 3x^2$ and $\frac{dv}{dx} = 2e^{2x}$

$\frac{dy}{dx} = (e^{2x} \times 3x^2) + (x^3 \times 2e^{2x})$

$= x^2e^{2x}(3 + 2x)$

8a Let $u = e^x$ and $v = x$

Then $\frac{du}{dx} = e^x$ and $\frac{dv}{dx} = 1$

$\frac{dy}{dx} = \frac{(x \times e^x) - (e^x \times 1)}{x^2}$

$= \frac{xe^x - e^x}{x^2}$

$= \frac{e^x(x - 1)}{x^2}$

8b Let $u = x$ and $v = e^x$

Then $\frac{du}{dx} = 1$ and $\frac{dv}{dx} = e^x$

$\frac{dy}{dx} = \frac{(e^x \times 1) - (x \times e^x)}{(e^x)^2}$

$= \frac{e^x - xe^x}{e^{2x}}$

$= \frac{e^x(1 - x)}{e^{2x}}$

$= \frac{1 - x}{e^x}$

8c Let $u = e^x$ and $v = x^2$

Then $\frac{du}{dx} = e^x$ and $\frac{dv}{dx} = 2x$

$$\frac{dy}{dx} = \frac{(x^2 \times e^x) - (e^x \times 2x)}{(x^2)^2}$$

$$= \frac{x^2 e^x - 2xe^x}{x^4}$$

$$= \frac{xe^x(x-2)}{x^4}$$

$$= \frac{e^x(x-2)}{x^3}$$

8d Let $u = x^2$ and $v = e^x$

Then $\frac{du}{dx} = 2x$ and $\frac{dv}{dx} = e^x$

$$\frac{dy}{dx} = \frac{(e^x \times 2x) - (x^2 \times e^x)}{(e^x)^2}$$

$$= \frac{2xe^x - x^2 e^x}{e^{2x}}$$

$$= \frac{e^x(2x - x^2)}{e^{2x}}$$

$$= \frac{2x - x^2}{e^x}$$

8e Let $u = e^x$ and $v = x + 1$

Then $\frac{du}{dx} = e^x$ and $\frac{dv}{dx} = 1$

$$\frac{dy}{dx} = \frac{((x+1) \times e^x) - (e^x \times 1)}{(x+1)^2}$$

$$= \frac{e^x(x + 1 - 1)}{(x+1)^2}$$

$$= \frac{xe^x}{(x+1)^2}$$

8f Let $u = x + 1$ and $v = e^x$

Then $\frac{du}{dx} = 1$ and $\frac{dv}{dx} = e^x$

$$\frac{dy}{dx} = \frac{(e^x \times 1) - ((x+1) \times e^x)}{(e^x)^2}$$

$$= \frac{e^x\big(1-(x+1)\big)}{e^{2x}}$$

$$= \frac{-x}{e^x}$$

8g Let $u = x - 3$ and $v = e^{2x}$

Then $\frac{du}{dx} = 1$ and $\frac{dv}{dx} = 2e^{2x}$

$$\frac{dy}{dx} = \frac{(e^{2x} \times 1) - \big((x-3) \times 2e^{2x}\big)}{(e^{2x})^2}$$

$$= \frac{e^{2x}\big(1-(2x-6)\big)}{e^{4x}}$$

$$= \frac{7-2x}{e^{2x}}$$

8h Let $u = 1 - x^2$ and $v = e^x$

Then $\frac{du}{dx} = -2x$ and $\frac{dv}{dx} = e^x$

$$\frac{dy}{dx} = \frac{(e^x \times -2x) - \big((1-x^2) \times e^x\big)}{(e^x)^2}$$

$$= \frac{e^x\big(-2x-(1-x^2)\big)}{e^{2x}}$$

$$= \frac{x^2-2x-1}{e^x}$$

9a $\frac{d}{dx}\big((e^x+1)(e^x+2)\big) = \frac{d}{dx}((e^x)^2 + 2e^x + e^x + 2)$

$$= \frac{d}{dx}(e^{2x} + 3e^x + 2)$$

$$= 2e^{2x} + 3e^x$$

9b $\frac{d}{dx}\big((e^{2x}+3)(e^{2x}-2)\big) = \frac{d}{dx}((e^{2x})^2 - 2e^{2x} + 3e^{2x} - 6)$

$$= \frac{d}{dx}(e^{4x} + e^{2x} - 6)$$

$$= 4e^{4x} + 2e^{2x}$$

9c $\frac{d}{dx}\left((e^{-x}+2)(e^{-x}+4)\right)=\frac{d}{dx}((e^{-x})^2+4e^{-x}+2e^{-x}+8)$

$=\frac{d}{dx}(e^{-2x}+6e^{-x}+8)$

$=-2e^{-2x}-6e^{-x}$

9d $\frac{d}{dx}\left((e^{-3x}-1)(e^{-3x}-5)\right)=\frac{d}{dx}((e^{-3x})^2-5e^{-3x}-e^{-3x}+5)$

$=\frac{d}{dx}(e^{-6x}-6e^{-3x}+5)$

$=-6e^{-6x}+18e^{3x}$

9e $\frac{d}{dx}\left((e^{2x}+1)(e^{x}+1)\right)=\frac{d}{dx}(e^{2x+x}+e^{2x}+e^{x}+1)$

$=\frac{d}{dx}(e^{3x}+e^{2x}+e^{x}+1)$

$=3e^{3x}+2e^{2x}+e^{x}$

9f $\frac{d}{dx}\left((e^{3x}-1)(e^{-x}+4)\right)=\frac{d}{dx}(e^{3x-x}+4e^{3x}-e^{-x}-4)$

$=\frac{d}{dx}(e^{2x}+4e^{3x}-e^{-x}-4)$

$=2e^{2x}+12e^{3x}+e^{-x}$

10a Let $u=1-e^{x}$

Then $y=u^5$

Hence $\frac{du}{dx}=-e^{x}$ and $\frac{dy}{du}=5u^4$

$\frac{dy}{dx}=-e^{x}\times 5(1-e^{x})^4=-5e^{x}(1-e^{x})^4$

10b Let $u=e^{4x}-9$

Then $y=u^4$

Hence $\frac{du}{dx}=4e^{4x}$ and $\frac{dy}{du}=4u^3$

$\frac{dy}{dx}=4e^{4x}\times 4(e^{4x}-9)^3=16e^{4x}(e^{4x}-9)^3$

10c Let $u = e^x - 1$

Then $y = \frac{1}{u}$

Hence $\frac{du}{dx} = e^x$ and $\frac{dy}{du} = -\frac{1}{u^2}$

$\frac{dy}{dx} = e^x \times -\frac{1}{(e^x - 1)^2} = -\frac{e^x}{(e^x - 1)^2}$

10d Let $u = e^{3x} + 4$

Then $y = \frac{1}{u^2} = u^{-2}$

Hence $\frac{du}{dx} = 3e^{3x}$ and $\frac{dy}{du} = -\frac{2}{u^3}$

$\frac{dy}{dx} = 3e^{3x} \times -\frac{2}{(e^{3x} + 4)^3} = -\frac{6e^{3x}}{(e^{3x} + 4)^3}$

11a Let $u = 2x$

Then $y = 3e^u$

Hence $\frac{du}{dx} = 2$ and $\frac{dy}{du} = 3e^u$

$\frac{dy}{dx} = 2 \times 3e^{2x} = 2y$

11b Let $u = -4x$

Then $y = 5e^u$

Hence $\frac{du}{dx} = -4$ and $\frac{dy}{du} = 5e^u$

$\frac{dy}{dx} = -4 \times 5e^{-4x} = -4y$

12a $f'(x) = 2e^{2x+1}$

$f'(0) = 2e$

$f''(x) = 2 \times 2e^{2x+1} = 4e^{2x+1}$

$f''(0) = 4e$

12b $f'(x) = -3e^{-3x}$

$f'(1) = -3e^{-3}$

$f''(x) = -3 \times -3e^{-3x} = 9e^{-3x}$

$f''(1) = 9e^{-3}$

12c Let $u = x$ and $v = e^{-x}$

Then $\frac{du}{dx} = 1$ and $\frac{dv}{dx} = -e^{-x}$

$f'(x) = (e^{-x} \times 1) + (x \times -e^{-x}) = (1 - x)e^{-x}$

$f'(2) = -e^{-2}$

Let $u = 1 - x$ and $v = e^{-x}$

Then $\frac{du}{dx} = -1$ and $\frac{dv}{dx} = -e^{-x}$

$f''(x) = (e^{-x} \times -1) + \big((1 - x) \times -e^{-x}\big)$

$= \big(-1 - (1 - x)\big)e^{-x}$

$= (x - 2)e^{-x}$

$f''(2) = 0$

12d Let $u = -x^2$

Then $y = e^u$

Hence $\frac{du}{dx} = -2x$ and $\frac{dy}{du} = e^u$

$f'(x) = -2x \times e^{-x^2} = -2xe^{-x^2}$

$f'(0) = 0$

Let $u = -2x$ and $v = e^{-x^2}$

Then $\frac{du}{dx} = -2$ and $\frac{dv}{dx} = -2xe^{-x^2}$

$f''(x) = \left(e^{-x^2} \times -2\right) + \left(-2x \times -2xe^{-x^2}\right)$

$= -2e^{-x^2} + \left(4x^2 \times e^{-x^2}\right)$

$= e^{-x^2}(4x^2 - 2)$

$f''(0) = -2e^0(1 - 0) = -2$

13a $\frac{dy}{dx} = ae^{ax}$

13b $\frac{dy}{dx} = -ke^{-kx}$

13c $\frac{dy}{dx} = Ake^{kx}$

13d $\frac{dy}{dx} = -Ble^{-lx}$

13e $\frac{dy}{dx} = pe^{px+q}$

13f $\frac{dy}{dx} = Cpe^{px+q}$

13g $\frac{dy}{dx} = \frac{pe^{px} - qe^{-qx}}{r}$

13h $\frac{dy}{dx} = e^{ax} - e^{-px}$

14a Let $u = e^x + 1$

Then $y = u^3$

Hence $\frac{du}{dx} = e^x$ and $\frac{dy}{du} = 3u^2$

$\frac{dy}{dx} = e^x \times 3(e^x + 1)^2 = 3e^x(e^x + 1)^2$

14b Let $u = e^x + e^{-x}$

Then $y = u^4$

Hence $\frac{du}{dx} = e^x - e^{-x}$ and $\frac{dy}{du} = 4u^3$

$\frac{dy}{dx} = (e^x - e^{-x}) \times 4(e^x + e^{-x})^3 = 4(e^x - e^{-x})(e^x + e^{-x})^3$

14c Let $u = 1 + x^2$ and $v = e^{1+x}$

Then $\frac{du}{dx} = 2x$ and $\frac{dv}{dx} = e^{1+x}$

$\frac{dy}{dx} = (e^{1+x} \times 2x) + \left((1 + x^2) \times e^{1+x}\right)$

$= e^{1+x}(2x + 1 + x^2)$

$= e^{1+x}(1 + x)^2$

14d Let $u = x^2 - x$ and $v = e^{2x-1}$

Then $\frac{du}{dx} = 2x - 1$ and $\frac{dv}{dx} = 2e^{2x-1}$

$\frac{dy}{dx} = \left(e^{2x-1} \times (2x - 1)\right) + \left((x^2 - x) \times 2e^{2x-1}\right)$

$= e^{2x-1}(2x - 1 + 2x^2 - 2x)$

$= e^{2x-1}(2x^2 - 1)$

14e Let $u = e^x$ and $v = e^x + 1$

Then $\frac{du}{dx} = e^x$ and $\frac{dv}{dx} = e^x$

$\frac{dy}{dx} = \frac{\left((e^x + 1) \times e^x\right) - (e^x \times e^x)}{(e^x)^2}$

$= \frac{e^{2x} + e^x - e^{2x}}{(e^x + 1)^2}$

$= \frac{e^x}{(e^x + 1)^2}$

14f Let $u = e^x + 1$ and $v = e^x - 1$

Then $\frac{du}{dx} = e^x$ and $\frac{dv}{dx} = e^x$

$\frac{dy}{dx} = \frac{\left((e^x - 1) \times e^x\right) - \left((e^x + 1) \times e^x\right)}{(e^x - 1)^2}$

$= \frac{e^x\left(e^x - 1 - (e^x + 1)\right)}{(e^x - 1)^2}$

$= \frac{-2e^x}{(e^x - 1)^2}$

15a $y' = \frac{d}{dx}\left(e^{x-x} + \frac{1}{e^x}\right)$

$= \frac{d}{dx}(e^0 + e^{-x})$

$= -e^{-x}$

15b $y' = \frac{d}{dx}(e^{2x-x} + e^{x-x})$

$= \frac{d}{dx}(e^x + e^0)$

$= e^x$

15c $y' = \frac{d}{dx}(2e^{-2x} - e^{x-2x})$

$= \frac{d}{dx}(2e^{-2x} - e^{-x})$

$= -4e^{-2x} + e^{-x}$

15d $y' = \frac{d}{dx}(3e^{-4x} + e^{x-4x})$

$= \frac{d}{dx}(3e^{-4x} + e^{-3x})$

$= -12e^{-4x} - 3e^{-3x}$

15e $y' = \frac{d}{dx}(e^{x-x} + e^{2x-x} - 3e^{4x-x})$

$= \frac{d}{dx}(e^0 + e^x - 3e^{3x})$

$= e^x - 9e^{3x}$

15f $y' = \frac{d}{dx}(e^{2x-2x} + 2e^{x-2x} + e^{0-2x})$

$= \frac{d}{dx}(e^0 + 2e^{-x} + e^{-2x})$

$= -2e^{-x} - 2e^{-2x}$

16a i Given $y = 2e^{3x}$

LHS: $y' = 6e^{3x}$

RHS: $3y = 3 \times 2e^{3x} = 6e^{3x}$

LHS = RHS and so $y = 2e^{3x}$ is a solution of $y' = 3y$

16a ii Given $y = 2e^{3x}$

$y' = 6e^{3x}$ and $y'' = 18e^{3x}$

LHS:

$$\begin{aligned} y'' - 9y &= 18e^{3x} - 9 \times 2e^{3x} \\ &= 18e^{3x} - 18e^{3x} \\ &= 0 \\ &= \text{RHS} \end{aligned}$$

LHS = RHS and so $y = 2e^{3x}$ is a solution of $y'' - 9y = 0$

16b $y = \frac{1}{2}e^{-3x} + 4$

LHS: $\frac{dy}{dx} = -\frac{3}{2}e^{-3x}$

RHS: $-3(y-4) = -3\left(\frac{1}{2}e^{-3x} + 4 - 4\right) = -\frac{3}{2}e^{-3x}$

LHS = RHS and so $y = \frac{1}{2}e^{-3x} + 4$ is a solution of $\frac{dy}{dx} = -3(y-4)$

16c i $y = e^{-3x} + x - 1$

Substituting $y = e^{-3x} + x - 1$, $y' = -3e^{-3x} + 1$ and $y'' = 9e^{-3x}$ into $y'' + 2y' - 3y = 5 - 3x$ we obtain:

$$\begin{aligned} \text{LHS} &= 9e^{-3x} + 2(-3e^{-3x} + 1) - 3(e^{-3x} + x - 1) \\ &= 9e^{-3x} - 6e^{-3x} + 2 - 3e^{-3x} - 3x + 3 \\ &= 5 - 3x \\ &= \text{RHS} \end{aligned}$$

LHS = RHS and so $y = e^{-3x} + x - 1$ is a solution of $y'' + 2y' - 3y = 5 - 3x$

16d i $y = e^{-x}$

Substituting $y = e^{-x}$, $y' = -e^{-x}$ and $y'' = e^{-x}$ into $y'' + 2y' + y = 0$ we obtain:

$$\begin{aligned} \text{LHS} &= e^{-x} - 2e^{-x} + e^{-x} \\ &= 0 \\ &= \text{RHS} \end{aligned}$$

LHS = RHS and so $y = e^{-x}$ is a solution of $y'' + 2y' + y = 0$

16d ii $y = xe^{-x}$

Applying the product rule on $y = xe^{-x}$:

Let $u = x$ and $v = e^{-x}$.

Then $u' = 1$ and $v' = -e^{-x}$.

$$\begin{aligned} y' &= vu' + uv' \\ &= e^{-x} \times 1 + x \times -e^{-x} \\ &= e^{-x} - xe^{-x} \end{aligned}$$

Differentiating again (including use of the product rule) we obtain:

$$\begin{aligned} y'' &= -e^{-x} - \left(e^{-x} - xe^{-x}\right) \\ &= -2e^{-x} + xe^{-x} \end{aligned}$$

Substituting $y = xe^{-x}$, $y' = e^{-x} - xe^{-x}$ and $y'' = -2e^{-x} + xe^{-x}$ into $y'' + 2y' + y = 0$ we obtain:

$$\begin{aligned} \text{LHS} &= -2e^{-x} + xe^{-x} + 2\left(e^{-x} - xe^{-x}\right) + xe^{-x} \\ &= (-2 + 2)e^{-x} + (x - 2x + x)e^{-x} \\ &= 0 \\ &= \text{RHS} \end{aligned}$$

LHS = RHS and so $y = xe^{-x}$ is a solution of $y'' + 2y' + y = 0$

17a $y = \sqrt{e^x}$

$y = e^{\frac{1}{2}x}$

$y' = \frac{1}{2}e^{\frac{1}{2}x}$

$y' = \frac{1}{2}\sqrt{e^x}$

17b $y = \sqrt[3]{e^x}$

$y = e^{\frac{1}{3}x}$

$y' = \frac{1}{3}e^{\frac{1}{3}x}$

$y' = \frac{1}{3}\sqrt[3]{e^x}$

17c $y = \frac{1}{\sqrt{e^x}}$

$y = e^{-\frac{1}{2}x}$

$y¢= -\frac{1}{2}e^{-\frac{1}{2}x}$

$y' = -\frac{1}{2\sqrt{e^x}}$

17d $y = \frac{1}{\sqrt[3]{e^x}}$

$y = e^{-\frac{1}{3}x}$

$y¢= -\frac{1}{3}e^{-\frac{1}{3}x}$

$y' = -\frac{1}{3\sqrt[3]{e^x}}$

17e $y = e^{\sqrt{x}}$

Using $\frac{d}{dx}e^{f(x)} = e^{f(x)}f'(x)$ with $f(x) = \sqrt{x}$ and $f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$ we obtain:

$y' = \frac{1}{2\sqrt{x}}e^{\sqrt{x}}$

17f $y = e^{-\sqrt{x}}$

Using $\frac{d}{dx}e^{f(x)} = e^{f(x)}f'(x)$ with $f(x) = -\sqrt{x}$ and $f'(x) = -\frac{1}{2}x^{-\frac{1}{2}} = -\frac{1}{2\sqrt{x}}$ we obtain:

$y' = -\frac{1}{2\sqrt{x}}e^{-\sqrt{x}}$

17g $y = e^{\frac{1}{x}}$

Using $\frac{d}{dx}e^{f(x)} = e^{f(x)}f'(x)$ with $f(x) = \frac{1}{x}$ and $f'(x) = -\frac{1}{x^2}$ we obtain:

$y' = -\frac{1}{x^2}e^{\frac{1}{x}}$

17h $y = e^{-\frac{1}{x}}$

Using $\frac{d}{dx}e^{f(x)} = e^{f(x)}f'(x)$ with $f(x) = -\frac{1}{x}$ and $f'(x) = \frac{1}{x^2}$ we obtain:

$y' = \frac{1}{x^2}e^{-\frac{1}{x}}$

17i $y = e^{x-\frac{1}{x}}$

Using $\frac{d}{dx}e^{f(x)} = e^{f(x)}f'(x)$ with $f(x) = x - \frac{1}{x}$ and $f'(x) = 1 + \frac{1}{x^2}$ we obtain:

$$y' = \left(1 + \frac{1}{x^2}\right)e^{x - \frac{1}{x}}$$

17j $y = e^{e^x}$

Using $\frac{d}{dx}e^{f(x)} = e^{f(x)}f'(x)$ with $f(x) = e^x$ and $f'(x) = e^x$ we obtain:

$$y' = e^x e^{e^x} = e^{x + e^x}$$

18a $\cosh x = \frac{e^x + e^{-x}}{2}$ and $\sinh x = \frac{e^x - e^{-x}}{2}$

$$\text{LHS}: \frac{d}{dx}\cosh x = \frac{e^x - e^{-x}}{2} = \sinh x$$

$$\text{RHS} = \sinh x$$

$$\text{LHS} = \text{RHS and so } \frac{d}{dx}\cosh x = \sinh x$$

$$\text{LHS}: \frac{d}{dx}\sinh x = \frac{e^x + e^{-x}}{2} = \cosh x$$

$$\text{RHS} = \cosh x$$

$$\text{LHS} = \text{RHS and so } \frac{d}{dx}\sinh x = \cosh x$$

18b Let $y = \cosh x = \frac{e^x + e^{-x}}{2}$.

$$y' = \frac{e^x - e^{-x}}{2} \text{ and } y'' = \frac{e^x + e^{-x}}{2}$$

$$\text{LHS}: y'' = \frac{e^x + e^{-x}}{2} \text{ and RHS}: y = \frac{e^x + e^{-x}}{2}$$

$$\text{LHS} = \text{RHS and so } y = \cosh x = \frac{e^x + e^{-x}}{2} \text{ satisfies } y'' = y$$

Let $y = \sinh x = \dfrac{e^x - e^{-x}}{2}$.

$y' = \dfrac{e^x + e^{-x}}{2}$ and $y'' = \dfrac{e^x - e^{-x}}{2}$

LHS: $y'' = \dfrac{e^x - e^{-x}}{2}$ and RHS: $y = \dfrac{e^x - e^{-x}}{2}$

LHS = RHS and so $y = \sinh x = \dfrac{e^x - e^{-x}}{2}$ satisfies $y'' = y$

18c Given $\cosh^2 x - \sinh^2 x = 1$.

$$\begin{aligned}\text{LHS} &= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2\\ &= \frac{1}{4}\left(\left(e^{2x} + 2 + e^{-2x}\right) - \left(e^{2x} - 2 + e^{-2x}\right)\right)\\ &= \frac{1}{4}(4)\\ &= 1\\ &= \text{RHS}\end{aligned}$$

LHS = RHS and so $\cosh^2 x - \sinh^2 x = 1$

19a i $y = Ae^{kx}$

$$\begin{aligned}\text{LHS} &= y'\\ &= Ake^{kx}\\ &= k(Ae^{kx})\\ &= ky\\ &= \text{RHS}\end{aligned}$$

19a ii $y = Ae^{kx}$

Substituting $y = Ae^{kx}$, $y' = kAe^{kx}$ and $y'' = k^2Ae^{kx}$ into $y'' - k^2y = 0$ we obtain:

$\text{LHS} = k^2Ae^{kx} - k^2(Ae^{kx})$

$= k^2Ae^{kx} - k^2Ae^{kx}$

$= 0$

$= \text{RHS}$

19b $y = Ae^{kx} + C$

Substituting $y = Ae^{kx} + C$, and $\frac{dy}{dx} = kAe^{kx}$ into $\frac{dy}{dx} = k(y - C)$ we obtain:

$\text{LHS} = kAe^{kx}$

$= k(Ae^{kx} + C - C)$

$= k(y - C)$

$= \text{RHS}$

19c $y = (Ax - B)e^{3x}$

Applying the product rule on $y = (Ax - B)e^{3x}$:

Let $u = Ax - B$ and $v = e^{3x}$.

Then $u' = A$ and $v' = 3e^{3x}$.

$y' = e^{3x} \times A + (Ax - B) \times 3e^{3x}$

$= Ae^{3x} + (Ax - B)3e^{3x}$

Differentiating again (including use of the product rule) we obtain:

Let $u = Ax - B$ and $v = 3e^{3x}$.

Then $u' = A$ and $v' = 9e^{3x}$.

$y'' = 3Ae^{3x} + (3e^{3x} \times A + (Ax - B) \times 9e^{3x})$

$= 3Ae^{3x} + 3Ae^{3x} + (Ax - B)9e^{3x}$

$= 6Ae^{3x} + (Ax - B)9e^{3x}$

Substituting $y = (Ax - B)e^{3x}$, $y' = Ae^{3x} + (Ax - B)3e^{3x}$and $y'' = 6Ae^{3x} + (Ax - B)9e^{3x}$ into $y'' + y' - y = 0$ we obtain:

$\text{LHS} = 6Ae^{3x} + (Ax - B)9e^{3x} - Ae^{3x} + (Ax - B)3e^{3x} - (Ax - B)e^{3x}$

$= \big(6A + 9(Ax - B) - A + 3(Ax - B) - (Ax - B)\big)e^{3x}$

$= \big(5A + 11(Ax - B)\big)e^{3x}$

20a Given $y = e^{\lambda x}$ is a solution of $y'' + 3y' - 10y = 0$.

Substituting $y' = \lambda e^{\lambda x}$ and $y'' = \lambda^2 e^{\lambda x}$ into $y'' + 3y' - 10y = 0$ we obtain:

$\lambda^2 e^{\lambda x} + 3\lambda e^{\lambda x} - 10e^{\lambda x} = 0$

Taking out $e^{\lambda x}$ as a common factor we obtain:

$e^{\lambda x}\left(\lambda^2 + 3\lambda - 10\right) = 0$

$e^{\lambda x} \neq 0$ and so $\lambda^2 + 3\lambda - 10 = 0$

$(\lambda + 5)(\lambda - 2) = 0$

So $\lambda = -5$ or $\lambda = 2$.

20b Given $y = e^{\lambda x}$ is a solution of $y'' + y' - y = 0$.

Substituting $y' = \lambda e^{\lambda x}$ and $y'' = \lambda^2 e^{\lambda x}$ into $y'' + y' - y = 0$ we obtain:

$\lambda^2 e^{\lambda x} + \lambda e^{\lambda x} - e^{\lambda x} = 0$

Taking out $e^{\lambda x}$ as a common factor we obtain:

$e^{\lambda x}\left(\lambda^2 + \lambda - 1\right) = 0$

$e^{\lambda x} \neq 0$ and so $\lambda^2 + \lambda - 1 = 0$

$$\lambda = \frac{-1 \pm \sqrt{1^2 - 4 \times 1 \times (-1)}}{2} = \frac{-1 \pm \sqrt{5}}{2}$$

So $\lambda = -\frac{1}{2}\left(1 + \sqrt{5}\right)$ or $\lambda = -\frac{1}{2}\left(1 - \sqrt{5}\right)$.

Solutions to Exercise 6C

1a When $x = \frac{1}{2}$, $y = e^{2\left(\frac{1}{2}\right) - 1}$

$= e^0$

$= 1$

1b $\frac{dy}{dx} = 2e^{2x-1}$

Gradient of tangent at $A\left(\frac{1}{2}, 1\right) = 2e^{2\left(\frac{1}{2}\right)-1} = 2$

1c $y - 1 = 2\left(x - \frac{1}{2}\right)$

$y = 2x - 1 + 1$

$y = 2x$

When $x = 0, y = 2 \times 0 = 0$

Hence, it passes through O.

2a When $x = -\frac{1}{3}$, $y = e^{3\left(-\frac{1}{3}\right) + 1}$

$= e^0$

$= 1$

$R\left(-\frac{1}{3}, 1\right)$

2b $\frac{dy}{dx} = 3e^{3x+1}$

Gradient of tangent at $R\left(-\frac{1}{3}, 1\right) = 3e^{3\left(-\frac{1}{3}\right)-1} = 3$

2c Gradient of normal at $R\left(-\frac{1}{3}, 1\right) = -\frac{1}{3}$

2d $y - 1 = -\frac{1}{3}\left(x - \left(-\frac{1}{3}\right)\right)$

$y - 1 = -\frac{1}{3}x - \frac{1}{9}$

$9y - 9 = -3x - 1$

$3x + 9x - 8 = 0$

3a $\frac{dy}{dx} = -e^{-x}$

Gradient of tangent at $P(-1, e) = -e^{-(-1)} = -e$

Gradient of normal at $P(-1, e) = \frac{1}{e}$

$y - e = \frac{1}{e}(x - (-1))$

$ey - e^2 = x + 1$

$x - ey + e^2 + 1 = 0$

3b When $y = 0$,

$x - 0 + e^2 + 1 = 0$

$x = -e^2 - 1$

When $x = 0$,

$0 - ey + e^2 + 1 = 0$

$ey = e^2 + 1$

$y = e + \frac{1}{e}$

3c Area $= \frac{1}{2}\left((0 - (-e^2 - 1)) \times \left(e + \frac{1}{e}\right)\right)$

$= \frac{1}{2}\left((e^2 + 1) \times \left(e + \frac{1}{e}\right)\right)$

$= \frac{1}{2}\left(e^3 + e + e + \frac{1}{e}\right)$

$= \frac{1}{2}\left(e^3 + 2e + \frac{1}{e}\right)$

4a $\frac{dy}{dx} = e^x$

Gradient of tangent at $B(0, 1) = e^0 = 1$

$y - 1 = 1(x - 0)$

$y = x + 1$

4b $\frac{dy}{dx} = -e^{-x}$

Gradient of tangent at $B(0, 1) = -e^0 = -1$

$y - 1 = -1(x - 0)$

$y = -x + 1$

4c For part a,

At x-axis, $y = 0$

$0 = x + 1$

$x = -1$

Hence, tangent meets x-axis at $F(-1, 0)$

For part b,

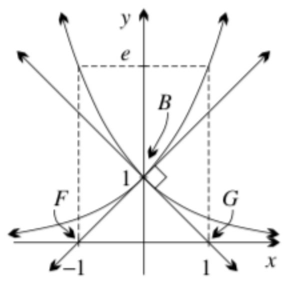
At x-axis, $y = 0$

$0 = -x + 1$

$x = 1$

Hence, tangent meets x-axis at $G(1, 0)$

4d



4e Isosceles right triangle

$\text{Area} = \frac{1}{2}\big((1-(-1))\times 1\big)$

$= \frac{1}{2}(2\times 1)$

$= 1$

5a $\frac{dy}{dx} = e^{-x}$

Gradient of tangent at origin $(0,0) = e^{-0} = 1$

$y - 0 = 1(x - 0)$

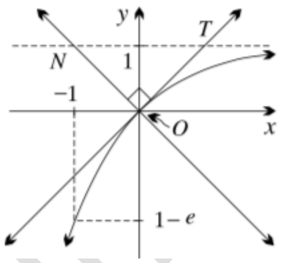
$y = x$

5b $y = -x$

5c Asymptote of $y = -e^{-x}$ is $y = 0$

Hence, asymptote of $y = 1 - e^{-x}$ is $y = 1$

5d



5e $\text{Area} = \frac{1}{2}\big((1-(-1))\times 1\big)$

$= \frac{1}{2}(2\times 1)$

$= 1$

6a $y' = 1 - e^x$

$y'' = -e^x$

6b e^x will always be positive for all x

So $y'' = -e^x$ will always be negative for all x

6c Stationary points when:

$y' = 0$

$1 - e^x = 0$

$e^x = 1$

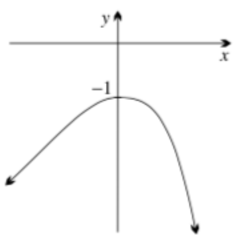
$x = 0$

When $x = 0, y = 0 - e^0 = -1$

When $x = 0, y'' = -e^0 = -1$

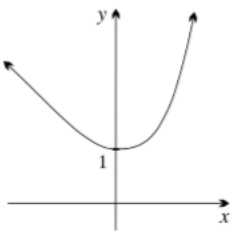
Maximum turning point at $(0, -1)$

6d



Range: $y \leq -1$

6e



7a $\frac{dy}{dx} = e^x$

Gradient of tangent at $T(t, e^t) = e^t$

7b $y - e^t = e^t(x - t)$

$y = e^t(x - t + 1)$

At x-intercept, $y = 0$,

$0 = e^t(x - t + 1)$

$x - t + 1 = 0$

$x = t - 1$

7c The x-intercept of each tangent to $y = e^x$ is 1 unit left of the x-value of the point of contact.

8a

x	-1	0	1
y	$-e^{-1}$	0	e
sign	$-$	0	$+$

There is a zero at $x = 0$, it is positive for $x > 0$ and negative for $x < 0$.

$f(x) = xe^x$

$f(-x) = -xe^{-x} \neq f(x) \neq -f(x)$

Therefore, it is neither even nor odd.

8b Let $u = x$ and $v = e^x$

Then $\frac{du}{dx} = 1$ and $\frac{dv}{dx} = e^x$

$y' = (e^x \times 1) + (x \times e^x)$

$\quad = e^x(1 + x)$

Let $u = e^x$ and $v = 1 + x$

Then $\frac{du}{dx} = e^x$ and $\frac{dv}{dx} = 1$

$y'' = \left((1 + x) \times e^x\right) + (e^x \times 1)$

$= (1 + x + 1)e^x$

$= (2 + x)e^x$

8c Stationary point is when $y' = 0$

$0 = e^x(1 + x)$

$1 + x = 0$

$x = -1$

Therefore, only one stationary point.

When $x = -1$

$y'' = \left(2 + (-1)\right)e^{-1}$

$= e^{-1} > 0$

Minimum turning point at $x = -1$

8d Point of inflection is when $y'' = 0$

$0 = (2 + x)e^x$

$2 + x = 0$

$x = -2$

When $x = -2, y = -2e^{-2}$.

8e

x	2	5	10	20	40
e^x	7.3891	148.4132	22026.4658	485165195.4	2.3539×10^{17}

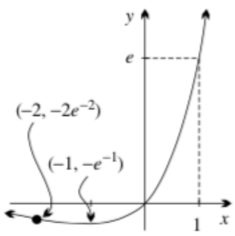
As $x \to \infty$, $e^x \to \infty$

Hence, their product $xe^x \to \infty$.

Since $(1 + x)e^x > xe^x$ and $(2 + x)e^x > xe^x$, these must also tend towards ∞.

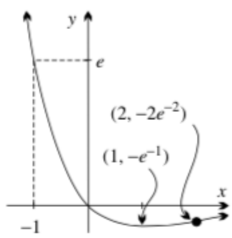
Hence y, y' and y'' must all tend towards ∞.

8f



Range: $y \geq -e^{-1}$

8g



9a When $y = 0$,

$0 = (1 - x)e^x$

$1 - x = 0$

$x = 1$

x	0	1	2
y	1	0	$-e^2$
sign	$+$	0	$-$

9b Let $u = 1 - x$ and $v = e^x$

Then $\frac{du}{dx} = -1$ and $\frac{dv}{dx} = e^x$

$\frac{dy}{dx} = (e^x \times -1) + \left((1 - x) \times e^x\right)$

$= e^x(-1 + 1 - x)$

$= -xe^x$

Let $u = -x$ and $v = e^x$

Then $\frac{du}{dx} = -1$ and $\frac{dv}{dx} = e^x$

$y' = (e^x \times -1) + (-x \times e^x)$

$= e^x(-1 - x)$

$= -(x + 1)e^x$

9c When $y' = 0$,

$-xe^x = 0$

$x = 0$, which is the y-intercept.

When $x = 0$,

$y'' = -(0 + 1)e^0$

$= -1 < 0$

Therefore, there is a maximum turning point at the y-intercept.

Inflection point is when $y'' = 0$,

$0 = -(x + 1)e^x$

$x + 1 = 0$

$x = -1$

When $x = -1, y = \big(1 - (-1)\big)e^{-1} = 2e^{-1}$

Therefore, point of inflection at $(-1, 2e^{-1})$.

9d

x	2	5	10	20	40
$-xe^x$	-14.7781	-742.0658	-220264.6579	-9703303908	-9.4154×10^{18}

As $x \to \infty$, $-xe^x \to -\infty$

Hence y' tends towards $-\infty$. To show that y tends towards $-\infty$:

Let $u = x - 1$.

By substitution, $y = -ue^{u+1}$

$= e \times -ue^u$

As x tends to ∞, so does u.

Hence $-ue^{u} \to -\infty$.

Since y is a constant positive multiple of $-ue^{u}$, it must also tend towards $-\infty$.

To show that y'' also tends towards $-\infty$:

Let $v = x + 1$.

By substitution, $y'' = -ve^{v-1}$

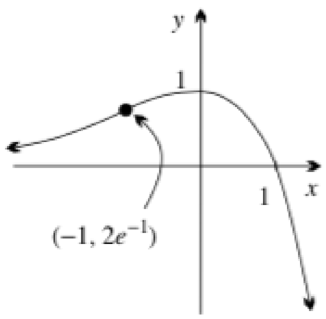
$= -ve^{v}e^{-1}$

As x tends to ∞, so does v.

Hence $-ve^{v} \to -\infty$.

Since y is a constant positive multiple of $-ve^{v}$, it must also tend towards $-\infty$.

9e



Range: $y \le 1$

10a $y = x^2e^{-x}$

Applying the product rule on $y = x^2e^{-x}$:

Let $u = x^2$ and $v = e^{-x}$.

Then $u' = 2x$ and $v' = -e^{-x}$.

$$\begin{aligned} y' &= vu' + uv' \\ &= 2xe^{-x} - x^2e^{-x} \\ &= xe^{-x}(2-x) \end{aligned}$$

So $y' = x(2-x)e^{-x}$.

Differentiating again (including use of the product rule) we obtain:

Let $u = xe^{-x}$ and $v = 2-x$.

Then $u' = e^{-x} - xe^{-x}$ and $v' = -1$.

$$\begin{aligned} y'' &= vu' + uv' \\ &= \left(e^{-x} - xe^{-x}\right)(2-x) - xe^{-x} \\ &= 2e^{-x} - xe^{-x} - 2xe^{-x} + x^2e^{-x} - xe^{-x} \\ &= e^{-x}\left(x^2 - 4x + 2\right) \end{aligned}$$

So $y'' = \left(2 - 4x + x^2\right)e^{-x}$.

10b There are stationary points where $y' = 0$.

$x(2-x)e^{-x} = 0 \Rightarrow x = 0, 2$ noting that $e^{-x} > 0$ for all real values of x

So there are stationary points at $x = 0$ and $x = 2$.

x	-1	0	1	2	3
y'	$-3e$	0	e^{-1}	0	$-3e^{-3}$
slope	\	–	/	–	\

When $x = 0$, $y = 0$ and when $x = 2$, $y = 4e^{-2}$.

Hence $(0,0)$ is a minimum turning point and $\left(2, 4e^{-2}\right)$ is a maximum turning point.

10c i So $y'' = \left(2 - 4x + x^2\right)e^{-x}$.

$y'' = 0 \Rightarrow x^2 - 4x + 2 = 0$ noting that $e^{-x} > 0$ for all real values of x.

$$x^2 - 4x + 2 = 0$$

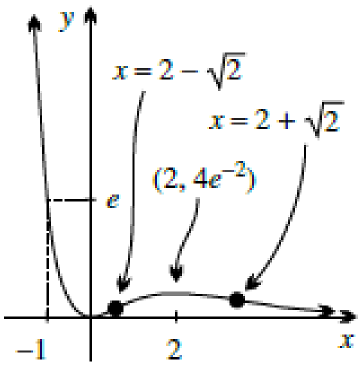
$$\begin{aligned} x &= \frac{4 \pm \sqrt{(-4)^2 - 4\times1\times2}}{2} \\ &= \frac{4 \pm 2\sqrt{2}}{2} \\ &= 2 \pm \sqrt{2} \end{aligned}$$

So there are inflection points at $x = 2 - \sqrt{2}$ and $x = 2 + \sqrt{2}$.

10c ii

x	0	$2-\sqrt{2}$	1	$2+\sqrt{2}$	4
$f''(x)$	2	0	$-e^{-1}$	0	$2e^{-4}$
concavity	up		down		up

10d The range is $y \geq 0$.



11a y-intercept is when $x = 0$,

$y = (1+0)^2 e^{-0}$

$= 1$

x-intercept is when $y = 0$,

$0 = (1+x)^2 e^{-x}$

Since e^{-x} can never be 0,

$0 = (1+x)^2$

$x = -1$

11b Applying the product rule:

Let $u = (1+x)^2$ and $v = e^{-x}$.

Then $u' = 2(1+x)$ and $v' = -e^{-x}$.

$y' = e^{-x} \times 2(1+x) + (1+x)^2 \times -e^{-x}$

$= 2(1+x)e^{-x} - (1+x)^2 e^{-x}$

$= (2(1+x) - (1+x)^2)e^{-x}$

$= \left(2 + 2x - (x^2 + 2x + 1)\right)e^{-x}$

$= (1 - x^2)e^{-x}$

Turning point when $y' = 0$:

$0 = (1 - x^2)e^{-x}$

Since e^{-x} can never be 0,

$0 = 1 - x^2$

$x^2 = 1$

$x = \pm 1$

Applying the product rule again:

Let $u = 1 - x^2$ and $v = e^{-x}$.

Then $u' = -2x$ and $v' = -e^{-x}$.

$y'' = e^{-x} \times -2x + (1 - x^2) \times -e^{-x}$

$= -2xe^{-x} - (1 - x^2)e^{-x}$

$= (x^2 - 2x - 1)e^{-x}$

When $x = -1$,

$y'' = ((-1)^2 - 2(-1) - 1)e^{-(-1)}$

$= (1 + 2 - 1)e$

$= 2 > 0$

Hence, it is a minimum turning point

When $x = 1$,

$y'' = (1^2 - 2(1) - 1)e^{-1}$

$= \dfrac{(1 - 2 - 1)}{e}$

$= -\frac{2}{e} < 0$

Hence, it is a maximum turning point

11c

x	-2	-1	1	5	10	20
y	7.389056	0	1.471518	0.242566	0.005493	9.089687×10^{-7}

Hence $y \to 0$ as $x \to \infty$.

So $y'' = (x^2 - 2x - 1)e^{-x}$.

$y'' = 0 \Longrightarrow x^2 - 2x - 1 = 0$ noting that $e^{-x} > 0$ for all real values of x.

$x^2 - 2x - 1 = 0$

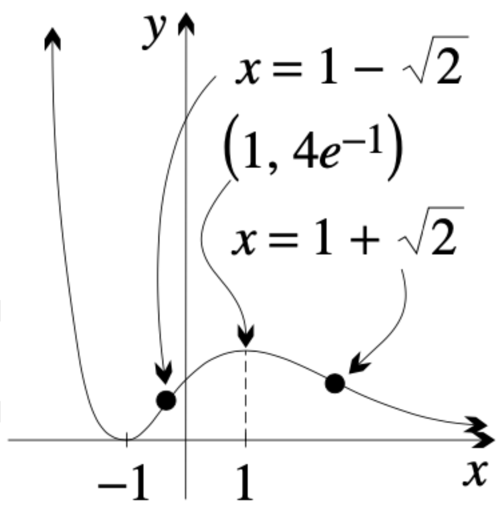
$x = \frac{2 \pm \sqrt{(-2)^2 - 4 \times 1 \times -1}}{2}$

$= \frac{2 \pm 2\sqrt{2}}{2}$

$= 1 \pm \sqrt{2}$

So there are inflection points at $x = 1 + \sqrt{2}$ and $x = 1 - \sqrt{2}$.

11d Range $=$ $y \geq 0$



12 y-intercept is when $x = 0$,

$y = (0 + 0 + 2)e^0$

$= 2$

x-intercept is when $y = 0$,

$0 = (x^2 + 3x + 2)e^x$

Since e^x can never be 0,

$0 = x^2 + 3x + 2$

$x = \frac{-3 \pm \sqrt{3^2 - 4 \times 1 \times 2}}{2}$

$= \frac{-3 \pm 1}{2}$

Hence, x-intercept at $x = -2$ and $x = -1$.

Applying the product rule:

Let $u = x^2 + 3x + 2$ and $v = e^x$.

Then $u' = 2x + 3$ and $v' = e^x$.

$y' = e^x \times (2x + 3) + (x^2 + 3x + 2) \times e^x$

$= (2x + 3 + x^2 + 3x + 2)e^x$

$= (x^2 + 5x + 5)e^x$

Turning point when $y' = 0$:

$0 = (x^2 + 5x + 5)e^x$

Since e^x can never be 0,

$0 = x^2 + 5x + 5$

$x = \frac{-5 \pm \sqrt{5^2 - 4 \times 1 \times 5}}{2}$

$= \frac{-5 \pm \sqrt{5}}{2}$

Hence, turning points at $x = \frac{-5+\sqrt{5}}{2}$ and $x = \frac{-5-\sqrt{5}}{2}$.

Applying the product rule again:

Let $u = x^2 + 5x + 5$ and $v = e^x$.

Then $u' = 2x + 5$ and $v' = e^x$.

$y'' = e^x \times (2x + 5) + (x^2 + 5x + 5) \times e^x$

$= (2x + 5 + x^2 + 5x + 5)e^x$

$= (x^2 + 7x + 10)e^x$

When $x = \frac{-5+\sqrt{5}}{2}$,

$$y'' = \left(\left(\frac{-5+\sqrt{5}}{2}\right)^2 + 7\left(\frac{-5+\sqrt{5}}{2}\right) + 10\right)e^{\frac{-5+\sqrt{5}}{2}}$$

$= \sqrt{5}e^{\frac{-5+\sqrt{5}}{2}} > 0$

Hence, it is a minimum turning point.

When $x = \frac{-5-\sqrt{5}}{2}$,

$$y'' = \left(\left(\frac{-5-\sqrt{5}}{2}\right)^2 + 7\left(\frac{-5-\sqrt{5}}{2}\right) + 10\right)e^{\frac{-5-\sqrt{5}}{2}}$$

$= -\sqrt{5}e^{\frac{-5-\sqrt{5}}{2}} < 0$

Hence, it is a maximum turning point.

Inflection points when $y'' = 0$,

$0 = (x^2 + 7x + 10)e^x$

Since e^x can never be 0,

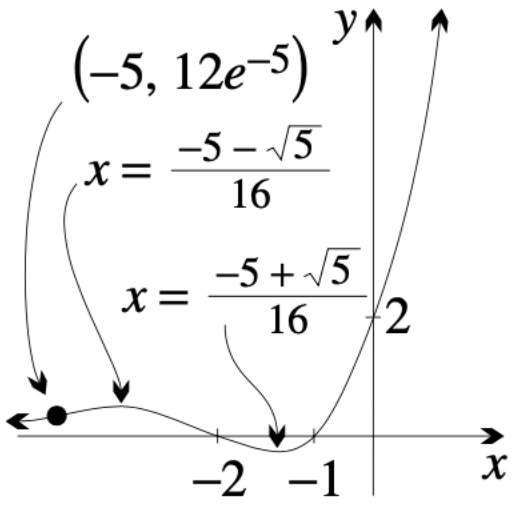
$0 = x^2 + 7x + 10$

$x = \frac{-7\pm\sqrt{7^2 - 4\times1\times10}}{2}$

$= \frac{-7\pm3}{2}$

Hence, turning points at $x = -5$ and $x = -2$.

From before, we know that $x = -2$ is the x-intercept.



13a $x \neq 0$

13b By the quotient rule,

$$y' = \frac{x\frac{d}{dx}(e^x) - e^x\frac{d}{dx}(x)}{x^2}$$

$$= \frac{(xe^x - e^x)}{x^2}$$

$$= \frac{(x-1)e^x}{x^2}$$

Turning point when $y' = 0$:

$$0 = \frac{(x-1)e^x}{x^2}$$

Since x^2 and e^x can never be 0,

$0 = x - 1$

$x = 1$

When $x = 1$,

$y = \frac{e}{1} = e$

Applying the quotient rule again:

$$y'' = \frac{x^2 \frac{d}{dx}\left((x-1)e^x\right) - (x-1)e^x \frac{d}{dx}(x^2)}{(x^2)^2}$$

Applying the product rule:

Let $u = x - 1$ and $v = e^x$.

Then $u' = 1$ and $v' = e^x$.

$$\frac{d}{dx}\left((x-1)e^x\right) = e^x \times 1 + (x-1) \times e^x$$

$$= (1 + x - 1)e^x$$

$$= xe^x$$

$$y'' = \frac{\left(x^2(xe^x) - (x-1)e^x(2x)\right)}{x^4}$$

$$= \frac{(x^3 - 2x^2 + 2x)e^x}{x^4}$$

Since x^4 and e^x can never be 0,

$0 = x^3 - 2x^2 + 2x$

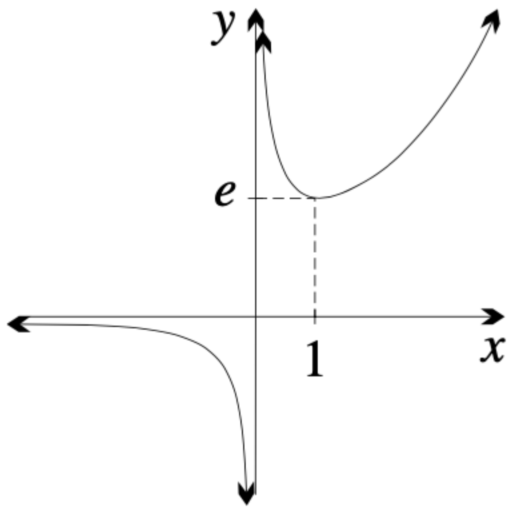
$= x(x^2 - 2x + 2)$

$\sqrt{b^2 - 4ac} = (-2)^2 - 4 \times 1 \times 2$

$= -4 < 0$

Therefore, no solutions. Hence, no inflection points

13c Range $= y < 0$ or $y \geq e$



14a Let $u = -\frac{1}{2}x^2$

Then $y = e^u$

Hence $\frac{du}{dx} = -x$ and $\frac{dy}{du} = e^u$

$y' = -x \times e^{-\frac{1}{2}x^2} = -xe^{-\frac{1}{2}x^2}$

Let $u = -x$ and $v = e^{-\frac{1}{2}x^2}$

Then $\frac{du}{dx} = -1$ and $\frac{dv}{dx} = -xe^{-\frac{1}{2}x^2}$

$y'' = \left(e^{-\frac{1}{2}x^2} \times -1\right) + \left(-x \times -xe^{-\frac{1}{2}x^2}\right)$

$= -e^{-\frac{1}{2}x^2} + x^2e^{-\frac{1}{2}x^2}$

$= (x^2 - 1)e^{-\frac{1}{2}x^2}$

14b Turning point is when

$y' = 0$

$-xe^{-\frac{1}{2}x^2} = 0$

Since $e^{-\frac{1}{2}x^2} > 0$ for all x,

$x = 0$

Hence, turning point at the y-intercept.

When $x = 0$,

$y'' = (0 - 1)e^0$

$\quad = -1 < 0$

Hence, it is a maximum turning point

Inflection point is when

$y'' = 0$

$(x^2 - 1)e^{-\frac{1}{2}x^2}$

Since $e^{-\frac{1}{2}x^2} > 0$ for all x,

$x^2 - 1 = 0$

$x = \pm 1$

When $x = 1, y = e^{-\frac{1}{2}(1^2)} = e^{-\frac{1}{2}}$

When $x = -1, y = e^{-\frac{1}{2}((-1)^2)} = e^{-\frac{1}{2}}$

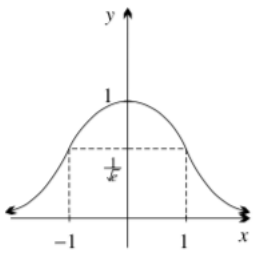
14c

x	2	5	10	20	40
y	0.1353	0.00000373	-1.929×10^{-22}	1.3839×10^{-87}	0

Hence $y \to 0$ as $x \to \infty$.

Since y is an even function, that is $f(x) = f(-x)$, $y \to 0$ as $x \to -\infty$.

14d



Range $= 0 < y \le 1$

15a By product rule,

$$y' = \frac{d}{dx}(x)e^{-x^2} + x\frac{d}{dx}\left(e^{-x^2}\right)$$

Consider $\frac{d}{dx}\left(e^{-x^2}\right)$, let $u = -x^2, \frac{du}{dx} = -2x$

By chain rule,

$$\frac{d}{dx}\left(e^{-x^2}\right) = \frac{d}{du}(e^u)\frac{du}{dx} = e^u(-2x) = -2xe^{-x^2}$$

$$y' = \frac{d}{dx}(x)e^{-x^2} + x\frac{d}{dx}\left(e^{-x^2}\right)$$

$$= e^{-x^2} + x\left(-2xe^{-x^2}\right)$$

$$= e^{-x^2}(1 - 2x^2)$$

There are stationary points where $y' = 0$.

$e^{-x^2}(1 - 2x^2) = 0$

Since $e^{-x^2} > 0$ for all x,

$0 = 1 - 2x^2$

$2x^2 = 1$

$x^2 = \frac{1}{2}$

$x = \pm\frac{1}{\sqrt{2}}$

So there are stationary points at $x = -\frac{1}{\sqrt{2}}$ and $x = \frac{1}{\sqrt{2}}$.

x	-1	$-\frac{1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{2}}$	1
y'	$-e^{-1}$	0	1	0	$-e^{-1}$
slope	\	–	/	–	\

When $x = -\frac{1}{\sqrt{2}}, y = -\frac{1}{\sqrt{2e}}$ and when $x = \frac{1}{\sqrt{2}}, y = \frac{1}{\sqrt{2e}}$.

Hence $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2e}}\right)$ is a minimum turning point and $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2e}}\right)$ is a maximum turning point.

15b By product rule,

$$y'' = \frac{d}{dx}(1-2x^2)e^{-x^2} + (1-2x^2)\frac{d}{dx}\left(e^{-x^2}\right)$$

$$= -4xe^{-x^2} + (1-2x^2)\left(-2xe^{-x^2}\right)$$

$$= -2xe^{-x^2}\left(2+(1-2x^2)\right)$$

$$= -2xe^{-x^2}(3-2x^2)$$

Inflection points when $y'' = 0$,

$0 = -2xe^{-x^2}(3-2x^2)$

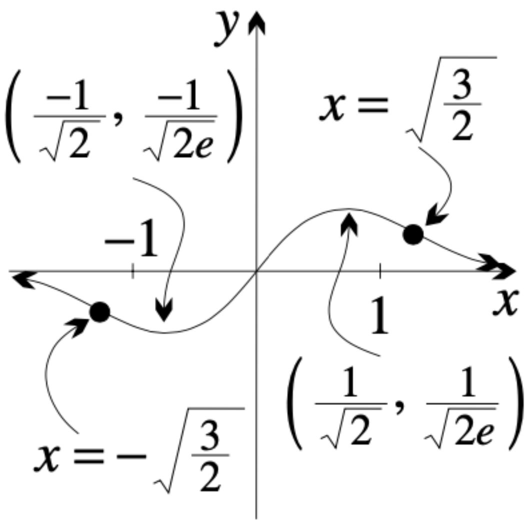
Since e^x can never be 0,

$0 = (-2x)(3-2x^2)$

$0 = -2x, 0 = 3-2x^2$

$x = 0, x = \pm\sqrt{\frac{3}{2}}$

Hence, inflection points at $x = 0, x = -\sqrt{\frac{3}{2}}$ and $x = \sqrt{\frac{3}{2}}$.



Range $= -\frac{1}{\sqrt{2e}} \le y \le \frac{1}{\sqrt{2e}}$

16a i

x	-2	-5	-10	-20	-40
y	-0.0677	-0.0013	-4.540×10^{-6}	-1.0306×10^{-10}	-1.0621×10^{-19}

Hence $y \to 0$ as $x \to -\infty$.

16a ii

x	2	5	10	20	40
y	3.6945	29.6826	2202.6466	24258259.77	5.8846×10^{15}

Hence $y \to \infty$ as $x \to \infty$.

16b i

x	-2	-5	-10	-20	-40
y	-3.6945	-29.6826	-2202.6466	-24258259.77	-5.8846×10^{15}

Hence $y \to -\infty$ as $x \to -\infty$.

16b ii

x	2	5	10	20	40
y	0.0677	0.0013	4.540×10^{-6}	1.0306×10^{-10}	1.0621×10^{-19}

Hence $y \to 0$ as $x \to \infty$.

16c Let $y = x^k e^k$

x \ k	-20	-15	-10	-5	-2	-1	1	2	5	10	20	40
-40	1.87461E-41	-2.849E-31	4.32967E-21	-6.58E-11	8.4585E-05	-0.009197	-108.73127	11822.4898	-1.52E+10	2.3096E+20	5.3344E+40	2.8456E+81
-20	1.96567E-35	-9.335E-27	4.43359E-18	-2.106E-09	0.00033834	-0.018394	-54.365637	2955.62244	-474922109	2.2555E+17	5.0873E+34	2.5881E+69
-10	2.06115E-29	-3.059E-22	4.53999E-15	-6.738E-08	0.00135335	-0.0367879	-27.182818	738.90561	-14841316	2.2026E+14	4.8517E+28	2.3539E+57
-5	2.16128E-23	-1.002E-17	4.64895E-12	-2.156E-06	0.00541341	-0.0735759	-13.591409	184.726402	-463791.12	2.151E+11	4.6269E+22	2.1408E+45
-2	1.96567E-15	-9.335E-12	4.43359E-08	-0.0002106	0.03383382	-0.1839397	-5.4365637	29.5562244	-4749.2211	22555101	5.0873E+14	2.5881E+29
2	1.96567E-15	9.3354E-12	4.43359E-08	0.00021056	0.03383382	0.18393972	5.43656366	29.5562244	4749.22109	22555101	5.0873E+14	2.5881E+29
5	2.16128E-23	1.0024E-17	4.64895E-12	2.1561E-06	0.00541341	0.07357589	13.5914091	184.726402	463791.122	2.151E+11	4.6269E+22	2.1408E+45
10	2.06115E-29	3.059E-22	4.53999E-15	6.7379E-08	0.00135335	0.03678794	27.1828183	738.90561	14841315.9	2.2026E+14	4.8517E+28	2.3539E+57
20	1.96567E-35	9.3354E-27	4.43359E-18	2.1056E-09	0.00033834	0.01839397	54.3656366	2955.62244	474922109	2.2555E+17	5.0873E+34	2.5881E+69
40	1.87461E-41	2.8489E-31	4.32967E-21	6.58E-11	8.4585E-05	0.00919699	108.731273	11822.4898	1.5198E+10	2.3096E+20	5.3344E+40	2.8456E+81

As $k \to -\infty, e^k \to 0$ and $x^k \to 0$. Hence, $y \to 0$

As $k \to \infty, e^k \to \infty$ and $x^k \to \infty$. Hence, $y \to \infty$

17 $|x| = \pm x$

Case when x is positive,

Applying the product rule:

Let $u = x$ and $v = e^{-x}$.

Then $u' = 1$ and $v' = -e^{-x}$.

$y' = e^{-x} \times 1 + x \times -e^{-x}$

$= (1 - x)e^{-x}$

Stationary point when $y' = 0$,

$0 = (1 - x)e^{-x}$

Since $e^{-x} > 0$ for all real x,

$0 = 1 - x$

$x = 1$

Case when x is negative,

Applying the product rule:

Let $u = x$ and $v = e^{-(-x)} = e^x$.

Then $u' = 1$ and $v' = e^x$.

$y' = e^x \times 1 + x \times e^x$

$= (1 + x)e^x$

Stationary point when $y' = 0$,

$0 = (1 + x)e^x$

Since $e^{-x} > 0$ for all real x,

$0 = 1 + x$

$x = -1$

18a $x \neq 0$

18b Let $u = \frac{1}{x}$

Then $y = e^u$

Hence $\frac{du}{dx} = -\frac{1}{x^2}$ and $\frac{dy}{du} = e^u$

$y' = -\frac{1}{x^2} \times e^{\frac{1}{x}} = -\frac{1}{x^2} e^{\frac{1}{x}}$

x	-2	-5	-10	-20	-40
y	0.6065	0.8187	0.9048	0.9512	0.9753
y'	-0.1516	-0.0327	-0.0090	-0.0024	-0.0006

Hence $y \to 1$ and $y' \to 0$ as $x \to -\infty$.

x	-1	$-\frac{1}{2}$	$-\frac{1}{5}$	$-\frac{1}{10}$	$-\frac{1}{20}$
y	0.3679	0.1353	0.0067	0.0000454	2.0612×10^{-9}
y'	-0.3679	-0.5413	-0.1684	-0.0045	-8.2446×10^{-7}

x	1	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{20}$
y	2.7183	7.3891	148.4132	22026.4658	4.8517×10^{8}
y'	-2.7183	-29.5562	-3710.3290	-2.2026×10^{6}	-1.9407×10^{11}

Hence $y \to 0$ and $y' \to 0$ as $x \to 0$ from the left, and $y \to \infty$ and $y' \to -\infty$ as $x \to 0$ from the right.

x	2	5	10	20	40
y	1.6487	1.2214	1.1052	1.0513	1.0253
y'	-0.4122	-0.04886	-0.0111	-0.0026	-0.0006

Hence $y \to 1$ and $y' \to 0$ as $x \to \infty$.

18c y' decreased from when x went from 1 to $\frac{1}{2}$ and continued increasing after, so there must be an inflection point.

Applying the product rule:

Let $u = -\frac{1}{x^2}$ and $v = e^{\frac{1}{x}}$.

Then $u' = \frac{2}{x^3}$ and $v' = -\frac{1}{x^2}e^{\frac{1}{x}}$.

$y'' = e^{\frac{1}{x}} \times \frac{2}{x^3} + -\frac{1}{x^2} \times -\frac{1}{x^2}e^{\frac{1}{x}}$

$= e^{\frac{1}{x}}\left(\frac{2}{x^3} + \frac{1}{x^4}\right)$

Inflection points when $y'' = 0$,

$0 = e^{\frac{1}{x}}\left(\frac{2}{x^3} + \frac{1}{x^4}\right)$

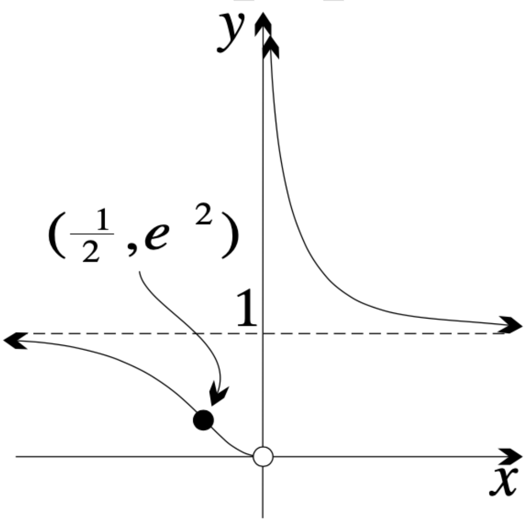
Since $e^{\frac{1}{x}}$ can never be 0,

$0 = \frac{2}{x^3} + \frac{1}{x^4}$

$x = -\frac{1}{2}$

Hence, inflection points at $x = -\frac{1}{2}$

18d



Range $= y > 0, y \neq 1$

18e $x \neq 0$

Applying the product rule:

Let $u = x$ and $v = e^{\frac{1}{x}}$.

Then $u' = 1$ and $v' = -\frac{1}{x^2}e^{\frac{1}{x}}$.

$y' = e^{\frac{1}{x}} \times 1 + x \times -\frac{1}{x^2}e^{\frac{1}{x}}$

$= e^{\frac{1}{x}}\left(1 - \frac{1}{x}\right)$

x	-2	-5	-10	-20	-40
y	-1.2131	-4.0937	-9.0484	-19.0246	-39.0124
y'	0.9098	0.9825	0.9953	0.9988	0.9997

Hence $y \to -\infty$ and $y' \to 1$ as $x \to -\infty$.

x	-1	$-\frac{1}{2}$	$-\frac{1}{5}$	$-\frac{1}{10}$	$-\frac{1}{20}$
y	-0.3679	-0.0677	-0.0013	-4.5400×10^{-6}	-1.0306×10^{-10}
y'	0.7358	0.4060	0.0404	0.0005	4.3284×10^{-8}

x	1	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{20}$
y	2.7183	3.6945	29.6826	2202.6466	2.4258×10^{7}
y'	0	-7.3891	-593.6526	-198238.1922	-9.2181×10^{9}

Hence $y \to 0$ and $y' \to 0$ as $x \to 0$ from the left, and $y \to \infty$ and $y' \to -\infty$ as $x \to 0$ from the right.

x	2	5	10	20	40
y	3.2974	6.1070	11.0517	21.0254	41.0126
y'	0.8244	0.9771	0.9947	0.9988	0.9997

Hence $y \to \infty$ and $y' \to 1$ as $x \to \infty$.

Applying the product rule:

Let $u = 1 - \frac{1}{x}$ and $v = e^{\frac{1}{x}}$.

Then $u' = \frac{1}{x^2}$ and $v' = -\frac{1}{x^2}e^{\frac{1}{x}}$.

$y'' = e^{\frac{1}{x}} \times \frac{1}{x^2} + \left(1 - \frac{1}{x}\right) \times -\frac{1}{x^2}e^{\frac{1}{x}}$

$= e^{\frac{1}{x}}\left(\frac{1}{x^2} - \frac{1}{x^2} + \frac{1}{x^3}\right)$

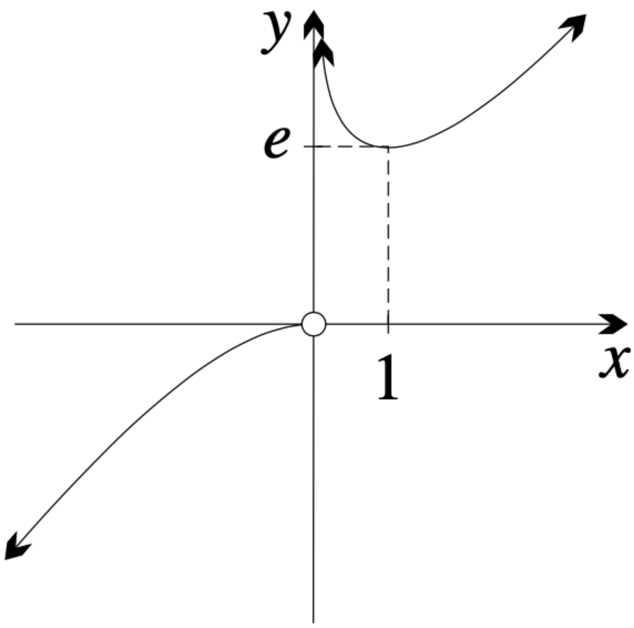
$= \frac{1}{x^3}e^{\frac{1}{x}}$

Inflection points when $y'' = 0$,

$0 = \frac{1}{x^3}e^{\frac{1}{x}}$

Since both $\frac{1}{x^3}$ and $e^{\frac{1}{x}}$ can never be 0,

There are no inflection points.



Range $= y < 0, y \geq e$

Solutions to Exercise 6D

Let C be a constant.

1a $\int e^{2x}\,dx = \frac{1}{2}e^{2x} + C$

1b $\int e^{3x}\,dx = \frac{1}{3}e^{3x} + C$

1c $\int e^{\frac{1}{3}x}\,dx = \frac{1}{\frac{1}{3}}e^{\frac{1}{3}x} + C = 3e^{\frac{1}{3}x} + C$

1d $\int e^{\frac{1}{2}x}\,dx = \frac{1}{\frac{1}{2}}e^{\frac{1}{2}x} + C = 2e^{\frac{1}{2}x} + C$

1e $\int 10e^{2x}\,dx = \frac{10}{2}e^{2x} + C = 5e^{2x} + C$

1f $\int 12e^{3x}\,dx = \frac{12}{3}e^{3x} + C = 4e^{3x} + C$

1g $\int e^{4x+5}\,dx = \frac{1}{4}e^{4x+5} + C$

1h $\int e^{4x-2}\,dx = \frac{1}{4}e^{4x-2} + C$

1i $\int 6e^{3x+2}\,dx = \frac{6}{3}e^{3x+2} + C = 2e^{3x+2} + C$

1j $\int 4e^{4x+3}\,dx = \frac{4}{4}e(4x+3) + C = e^{4x+3} + C$

1k $\int e^{7-2x}\,dx = -\frac{1}{2}e^{7-2x} + C$

1l $\int \frac{1}{2}e^{1-3x}\,dx = -\frac{\frac{1}{2}}{3}e^{1-3x} + C = -\frac{1}{6}e^{1-3x} + C$

2a $\int_0^1 e^x\ dx = [e^x]_0^1$

$= e^1 - e^0$

$= e - 1$

2b $\int_1^2 e^x\ dx = [e^x]_1^2$

$= e^2 - e$

2c $\int_{-1}^3 e^{-x}\ dx = [-e^{-x}]_{-1}^3$

$= -e^{-(3)} - \left(-e^{-(-1)}\right)$

$= e - e^{-3}$

2d $\int_{-2}^0 e^{-x}\ dx = [-e^{-x}]_{-2}^0$

$= -e^0 - \left(-e^{-(-2)}\right)$

$= e^2 - 1$

2e $\int_0^2 e^{2x}\ dx = \left[\frac{1}{2}e^{2x}\right]_0^2$

$= \frac{1}{2}e^{2(2)} - \frac{1}{2}e^0$

$= \frac{1}{2}(e^4 - 1)$

2f $\int_{-1}^2 20e^{-5x}\ dx = \left[-\frac{20}{5}e^{-5x}\right]_{-1}^2$

$= -4e^{-5(2)} - \left(-4e^{-5(-1)}\right)$

$= 4(e^5 - e^{-10})$

2g $\int_{-3}^{1} 8e^{-4x}\ dx = \left[-\frac{8}{4}e^{-4x}\right]_{-3}^{1}$

$= -2e^{-4(1)} - \left(-2e^{-4(-3)}\right)$

$= -2(e^{-4} - e^{12})$

$= 2(e^{12} - e^{-4})$

2h $\int_{-1}^{3} 9e^{6x}\ dx = \left[\frac{9}{6}e^{6x}\right]_{-1}^{3}$

$= \frac{3}{2}e^{6(3)} - \frac{3}{2}e^{6(-1)}$

$= \frac{3}{2}(e^{18} - e^{-6})$

2i $\int_{-1}^{1} e^{2x+1}\ dx = \left[\frac{1}{2}e^{2x+1}\right]_{-1}^{1}$

$= \frac{1}{2}e^{2(1)+1} - \frac{1}{2}e^{2(-1)+1}$

$= \frac{1}{2}(e^{3} - e^{-1})$

2j $\int_{-2}^{0} e^{4x-3}\ dx = \left[\frac{1}{4}e^{4x-3}\right]_{-2}^{0}$

$= \frac{1}{4}e^{4(0)-3} - \frac{1}{4}e^{4(-2)-3}$

$= \frac{1}{4}(e^{-3} - e^{-11})$

2k $\int_{-2}^{-1} e^{3x+2}\ dx = \left[\frac{1}{3}e^{3x+2}\right]_{-2}^{-1}$

$= \frac{1}{3}e^{3(-1)+2} - \frac{1}{3}e^{3(-2)+2}$

$= \frac{1}{3}(e^{-1} - e^{-4})$

2l $\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{3-2x}\ dx = \left[-\frac{1}{2}e^{3-2x}\right]_{-\frac{1}{2}}^{\frac{1}{2}}$

$= -\frac{1}{2}e^{3-2\left(\frac{1}{2}\right)} - \left(-\frac{1}{2}e^{3-2\left(-\frac{1}{2}\right)}\right)$

$$= -\tfrac{1}{2}(e^2 - e^4)$$

$$= -\tfrac{e^2}{2}(1 - e^2)$$

$$= \tfrac{e^2}{2}(e^2 - 1)$$

2m $\quad \int_{-\frac{1}{3}}^{\frac{1}{3}} e^{2+3x}\ dx = \left[\tfrac{1}{3}e^{2+3x}\right]_{-\frac{1}{3}}^{\frac{1}{3}}$

$$= \tfrac{1}{3}e^{2+3\left(\frac{1}{3}\right)} - \tfrac{1}{3}e^{2+3\left(-\frac{1}{3}\right)}$$

$$= \tfrac{1}{3}(e^3 - e)$$

$$= \tfrac{e}{3}(e^2 - 1)$$

2n $\quad \int_1^2 6e^{3x+1}\ dx = \left[\tfrac{6}{3}e^{3x+1}\right]_1^2$

$$= 2e^{3(2)+1} - 2e^{3(1)+1}$$

$$= 2(e^7 - e^4)$$

$$= 2e^4(e^3 - 1)$$

2o $\quad \int_2^3 12e^{4x-5}\ dx = \left[\tfrac{12}{4}e^{4x-5}\right]_2^3$

$$= 3e^{4(3)-5} - 3e^{4(2)-5}$$

$$= 3(e^7 - e^3)$$

$$= 3e^3(e^4 - 1)$$

2p $\quad \int_1^2 12e^{8-3x}\ dx = \left[-\tfrac{12}{3}e^{8-3x}\right]_1^2$

$$= -4e^{8-3(2)} - \left(-4e^{8-3(1)}\right)$$

$$= -4(e^2 - e^5)$$

$$= -4e^2(1 - e^3)$$

$$= 4e^2(e^3 - 1)$$

3a $\frac{1}{e^x} = e^{-x}$

$\int e^{-x} dx = -e^{-x} + C$

3b $\frac{1}{e^{2x}} = e^{-2x}$

$\int e^{-2x} dx = -\frac{1}{2}e^{-2x} + C$

3c $\frac{1}{e^{3x}} = e^{-3x}$

$\int e^{-3x} dx = -\frac{1}{3}e^{-3x} + C$

3d $-\frac{3}{e^{3x}} = -3e^{-3x}$

$\int -3e^{-3x}\, dx = -\left(-\frac{3}{3}\right)e^{-3x} = e^{-3x} + C$

3e $\frac{6}{e^{2x}} = 6e^{-2x}$

$\int 6e^{-2x} dx = -\frac{6}{2}e^{-2x} = -3e^{-2x} + C$

3f $\frac{8}{e^{-2x}} = 8e^{2x}$

$\int 8e^{2x} dx = \frac{8}{2}e^{2x} = 4e^{2x} + C$

4a $f'(x) = e^{2x}$

$f(x) = \int f'(x)\, dx$

$\quad = \int e^{2x}\, dx$

$\quad = \frac{1}{2}e^{2x} + C$

4b $f(0) = -2$

$\frac{1}{2}e^{2(0)} + C = -2$

$\frac{1}{2} + C = -2$

$C = -2\frac{1}{2}$

$f(x) = \frac{1}{2}e^{2x} - 2\frac{1}{2}$

4c $f(1) = \frac{1}{2}e^{2(1)} - 2\frac{1}{2} = \frac{1}{2}e^{2} - 2\frac{1}{2}$

$f(2) = \frac{1}{2}e^{2(2)} - 2\frac{1}{2} = \frac{1}{2}e^{4} - 2\frac{1}{2}$

5a $f(x) = \int 1 + 2e^{x}\,dx$

$= x + 2e^{x} + C$

$f(0) = 1$

$0 + 2e^{0} + C = 1$

$2 + C = 1$

$C = -1$

$f(x) = x + 2e^{x} - 1$

$f(1) = 1 + 2e - 1 = 2e$

5b $f(x) = \int 1 - 3e^{x}\,dx$

$= x - 3e^{x} + C$

$f(0) = -1$

$0 - 3e^{0} + C = -1$

$C - 3 = -1$

$C = 2$

$f(x) = x - 3e^{x} + 2$

$f(1) = 1 - 3e + 2 = 3 - 3e$

5c $f(x) = \int 2 + e^{-x}\,dx$

$= 2x - e^{-x} + C$

$f(0) = 0$

$2(0) - e^0 + C = 0$

$C - 1 = 0$

$C = 1$

$f(x) = 2x - e^{-x} + 1$

$f(1) = 2 - e^{-1} + 1 = 3 - e^{-1}$

5d $f(x) = \int 4 - e^{-x}\, dx$

$= 4x + e^{-x} + C$

$f(0) = 2$

$4(0) + e^0 + C = 2$

$1 + C = 2$

$C = 1$

$f(x) = 4x + e^{-x} + 1$

$f(1) = 4 + e^{-1} + 1 = 5 + e^{-1}$

5e $f(x) = \int e^{2x-1}\, dx$

$= \frac{1}{2}e^{2x-1} + C$

$f\left(\frac{1}{2}\right) = 3$

$\frac{1}{2}e^{2\left(\frac{1}{2}\right)-1} + C = 3$

$\frac{1}{2}e^0 + C = 3$

$C = \frac{5}{2}$

$f(x) = \frac{1}{2}e^{2x-1} + \frac{5}{2}$

$f(1) = \frac{1}{2}e^{2(1)-1} + \frac{5}{2} = \frac{1}{2}e + \frac{5}{2}$

5f $f(x) = \int e^{1-3x}\, dx$

$= -\frac{1}{3}e^{1-3x} + C$

$f\left(\frac{1}{3}\right) = \frac{2}{3}$

$-\frac{1}{3}e^{1-3\left(\frac{1}{3}\right)} + C = \frac{2}{3}$

$-\frac{1}{3}e^{0} + C = \frac{2}{3}$

$C = 1$

$f(x) = 1 - \frac{1}{3}e^{1-3x}$

$f(1) = 1 - \frac{1}{3}e^{1-3(1)} = 1 - \frac{1}{3}e^{-2}$

5g $f(x) = \int e^{\frac{1}{2}x+1}\, dx$

$= \frac{1}{\frac{1}{2}}e^{\frac{1}{2}x+1} + C$

$= 2e^{\frac{1}{2}x+1} + C$

$f(-2) = -4$

$2e^{\frac{1}{2}(-2)+1} + C = -4$

$2e^{0} + C = -4$

$C = -6$

$f(x) = 2e^{\frac{1}{2}x+1} - 6$

$f(1) = 2e^{\frac{1}{2}(1)+1} - 6 = 2e^{\frac{3}{2}} - 6$

5h $f(x) = \int e^{\frac{1}{3}x+2}\, dx$

$= \frac{1}{\frac{1}{3}}e^{\frac{1}{3}x+2} + C$

$= 3e^{\frac{1}{3}x+2} + C$

$f(-6) = 2$

$3e^{\frac{1}{3}(-6)+2} + C = 2$

$3e^0 + C = 2$

$C = -1$

$f(x) = 3e^{\frac{1}{3}x+2} - 1$

$f(1) = 3e^{\frac{1}{3}(1)+2} - 1 = 3e^{\frac{7}{3}} - 1$

6a $\int e^x(e^x + 1)\,dx = \int e^{2x} + e^x\,dx$

$= \frac{1}{2}e^{2x} + e^x + C$

6b $\int e^x(e^x - 1)\,dx = \int e^{2x} - e^x\,dx$

$= \frac{1}{2}e^{2x} - e^x + C$

6c $\int e^{-x}(2e^{-x} - 1)\,dx = \int 2e^{-2x} - e^{-x}dx$

$= -\frac{2}{2}e^{-2x} - (-e^{-x})$

$= e^{-x} - e^{-2x} + C$

6d $\int (e^x + 1)^2\,dx = \int (e^x)^2 + e^x + e^x + 1\,dx$

$= \int e^{2x} + 2e^x + 1\,dx$

$= \frac{1}{2}e^{2x} + 2e^x + x + C$

6e $\int (e^x + 3)^2\,dx = \int (e^x)^2 + 3e^x + 3e^x + 9\,dx$

$= \int e^{2x} + 6e^x + 9\,dx$

$= \frac{1}{2}e^{2x} + 6e^x + 9x + C$

6f $\int (e^x - 1)^2\,dx = \int (e^x)^2 - e^x - e^x + 1\,dx$

$= \int e^{2x} - 2e^x + 1\,dx$

$= \frac{1}{2}e^{2x} - 2e^x + x + C$

6g $\int (e^x + e^{-x})(e^x - e^{-x})\,dx = \int (e^x)^2 - e^{x-x} + e^{-x+x} - (e^{-x})^2\,dx$

$$= \int e^{2x} - e^{-2x}\,dx$$

$$= \tfrac{1}{2}e^{2x} - \left(-\tfrac{1}{2}e^{-2x}\right)$$

$$= \tfrac{1}{2}(e^{2x} + e^{-2x}) + C$$

6h $\int (e^{5x} + e^{-5x})(e^{5x} - e^{-5x})\,dx = \int (e^{5x})^2 - e^{5x-5x} + e^{-5x+5x} - (e^{-5x})^2\,dx$

$$= \int e^{10x} - e^{-10x}\,dx$$

$$= \tfrac{1}{10}e^{10x} - \left(-\tfrac{1}{10}e^{-10x}\right)$$

$$= \tfrac{1}{10}(e^{10x} + e^{-10x}) + C$$

7a $\int e^{7x+q}\,dx = \tfrac{1}{7}e^{7x+q} + C$

7b $\int e^{3x-k}\,dx = \tfrac{1}{3}e^{3x-k} + C$

7c $\int e^{sx+1}\,dx = \tfrac{1}{s}e^{sx+1} + C$

7d $\int e^{kx-1}\,dx = \tfrac{1}{k}e^{kx-1} + C$

7e $\int pe^{px+q}\,dx = \tfrac{p}{p}e^{px+q} + C = e^{px+q} + C$

7f $\int me^{mx+k}\,dx = \tfrac{m}{m}e^{mx+k} + C = e^{mx+k} + C$

7g $\int Ae^{sx-t}\,dx = \tfrac{A}{s}e^{sx-t} + C$

7h $\int Be^{kx-\ell}\,dx = \tfrac{B}{k}e^{kx-\ell} + C$

8a $\frac{1}{e^{x-1}} = e^{-(x-1)} = e^{1-x}$

$\int e^{1-x}\,dx = -e^{1-x} + C$

8b $\frac{1}{e^{3x-1}} = e^{-(3x-1)} = e^{1-3x}$

$\int e^{1-3x}\,dx = -\frac{1}{3}e^{1-3x} + C$

8c $\frac{1}{e^{2x+5}} = e^{-(2x+5)} = e^{-2x-5}$

$\int e^{-2x-5}\,dx = -\frac{1}{2}e^{-2x-5} + C$

8d $\frac{4}{e^{2x-1}} = 4e^{-(2x-1)} = 4e^{1-2x}$

$\int 4e^{1-2x}\,dx = -\frac{4}{2}e^{1-2x} = -2e^{1-2x} + C$

8e $\frac{10}{e^{2-5x}} = 10e^{-(2-5x)} = 10e^{5x-2}$

$\int 10e^{5x-2}\,dx = \frac{10}{5}e^{5x-2} = 2e^{5x-2} + C$

8f $\frac{12}{e^{3x-5}} = 12e^{-(3x-5)} = 12e^{5-3x}$

$\int 12e^{5-3x}\,dx = -\frac{12}{3}e^{5-3x} = -4e^{5-3x} + C$

9a $\int \frac{e^x+1}{e^x}\,dx = \int 1 + \frac{1}{e^x}\,dx$

$= \int 1 + e^{-x}\,dx$

$= x - e^{-x} + C$

9b $\int \frac{e^{2x}+1}{e^x}\,dx = \int e^{2x-x} + \frac{1}{e^x}\,dx$

$= \int e^x + e^{-x}\,dx$

$= e^x - e^{-x} + C$

9c $\int \frac{e^x-1}{e^{2x}}\,dx = \int e^{x-2x} - \frac{1}{e^{2x}}\,dx$

$= \int e^{-x} - e^{-2x}\,dx$

$= -e^{-x} - \left(-\frac{1}{2}e^{-2x}\right) + C$

$= \frac{1}{2}e^{-2x} - e^{-x} + C$

9d $\int \frac{e^x-3}{e^{3x}}\,dx = \int e^{x-3x} - \frac{3}{e^{3x}}\,dx$

$= \int e^{-2x} - 3e^{-3x}\,dx$

$= -\frac{1}{2}e^{-2x} - \left(-\frac{3}{3}e^{-3x}\right) + C$

$= e^{-3x} - \frac{1}{2}e^{-2x} + C$

9e $\int \frac{2e^{2x}-3e^x}{e^{4x}}\,dx = \int 2e^{2x-4x} - 3e^{x-4x}\,dx$

$= \int 2e^{-2x} - 3e^{-3x}\,dx$

$= -\frac{2}{2}e^{-2x} - \left(-\frac{3}{3}e^{-3x}\right) + C$

$= e^{-3x} - e^{-2x} + C$

9f $\int \frac{2e^x-e^{2x}}{e^{3x}}\,dx = \int 2e^{x-3x} - e^{2x-3x}\,dx$

$= \int 2e^{-2x} - e^{-x}\,dx$

$= -\frac{2}{2}e^{-2x} - (-e^{-x}) + C$

$= e^{-x} - e^{-2x} + C$

10a $y = \int e^{x-1}\,dx = e^{x-1} + C$

When $x = 1$,

$y = 1$

$e^{1-1} + C = 1$

$e^0 + C = 1$

$C = 0$

$y = e^{x-1}$

At y-intercept, $x = 0$

$y = e^{0-1} = e^{-1}$

10b $y = \int e^{2-x}\,dx = -e^{2-x} + C$

When $x = 0$,

$y = 1$

$-e^{2-0} + C = 1$

$-e^{2} + C = 1$

$C = 1 + e^{2}$

$y = e^{2} + 1 - e^{2-x}$

Horizontal asymptote: $y = e^{2} + 1$

10c $f(x) = \int e^{x} + \frac{1}{e}\,dx$

$= e^{x} + \frac{x}{e} + C$

$f(-1) = -1$

$e^{-1} - \frac{1}{e} + C = -1$

$e^{-1} - e^{-1} + C = -1$

$C = -1$

$f(x) = e^{x} + \frac{x}{e} - 1$

$f(0) = e^{0} + \frac{0}{e} - 1$

$= 0$

10d $f(x) = \int e^{x} - e^{-x}\,dx$

$= e^{x} - (-e^{-x}) + C$

$= e^{x} + e^{-x} + C$

$f(0) = 0$

$e^0 + e^0 + C = 0$

$C = -2$

$f(x) = e^x + e^{-x} - 2$

11a $\int_0^1 e^x(2e^x - 1)\ dx = \int_0^1 2e^{2x} - e^x\ dx$

$= \left[\frac{2}{2}e^{2x} - e^x\right]_0^1$

$= \left(e^{2(1)} - e\right) - \left(e^0 - e^0\right)$

$= e^2 - e$

11b $\int_{-1}^1 (e^x + 2)^2\ dx = \int_{-1}^1 (e^x)^2 + 2e^x + 2e^x + 4\ dx$

$= \int_{-1}^1 e^{2x} + 4e^x + 4\ dx$

$= \left[\frac{1}{2}e^{2x} + 4e^x + 4x\right]_{-1}^1$

$= \left(\frac{1}{2}e^{2(1)} + 4e + 4\right) - \left(\frac{1}{2}e^{2(-1)} + 4e^{-1} - 4\right)$

$= \frac{1}{2}e^2 + 4e - \frac{1}{2}e^{-2} - 4e^{-1} + 8$

11c $\int_0^1 (e^x - 1)(e^{-x} + 1)\ dx = \int_0^1 e^{x-x} + e^x - e^{-x} - 1\ dx$

$= \int_0^1 e^0 + e^x - e^{-x} - 1\ dx$

$= \int_0^1 e^x - e^{-x}\ dx$

$= [e^x - (-e^{-x})]_0^1$

$= (e + e^{-1}) - (e^0 + e^0)$

$= e + e^{-1} - 2$

11d $\int_{-1}^{1}(e^{2x}+e^{-x})(e^{2x}-e^{-x})\ dx=\int_{-1}^{1}(e^{2x})^2-e^{2x-x}+e^{-x+2x}-(e^{-x})^2\ dx$

$$=\int_{-1}^{1}e^{4x}-e^{-2x}\ dx$$

$$=\left[\tfrac{1}{4}e^{4x}-\left(-\tfrac{1}{2}e^{-2x}\right)\right]_{-1}^{1}$$

$$=\left(\tfrac{1}{4}e^{4}+\tfrac{1}{2}e^{-2}\right)-\left(\tfrac{1}{4}e^{4(-1)}+\tfrac{1}{2}e^{-2(-1)}\right)$$

$$=\tfrac{1}{4}e^{4}+\tfrac{1}{2}e^{-2}-\tfrac{1}{4}e^{-4}-\tfrac{1}{2}e^{2}$$

11e $\int_{0}^{1}\frac{e^{3x}+e^{x}}{e^{2x}}\ dx=\int_{0}^{1}e^{3x-2x}+e^{x-2x}\ dx$

$$=\int_{0}^{1}e^{x}+e^{-x}\ dx$$

$$=[e^{x}-e^{-x})]_{0}^{1}$$

$$=(e-e^{-1})-(e^{0}-e^{0})$$

$$=e-e^{-1}$$

11f $\int_{-1}^{1}\frac{e^{x}-1}{e^{2x}}\ dx=\int_{-1}^{1}e^{x-2x}-\frac{1}{e^{2x}}\ dx$

$$=\int_{-1}^{1}e^{-x}-e^{-2x}\ dx$$

$$=\left[-e^{-x}-\left(-\tfrac{1}{2}e^{-2x}\right)\right]_{-1}^{1}$$

$$=\left(\tfrac{1}{2}e^{-2(1)}-e^{-1}\right)-\left(\tfrac{1}{2}e^{-2(-1)}-e^{-(-1)}\right)$$

$$=\tfrac{1}{2}e^{-2}-e^{-1}-\tfrac{1}{2}e^{2}+e$$

12a i Let $y=e^{x^2+3}$.

Applying the chain rule:

Let $u=x^2+3$ and so $y=e^{u}$.

Hence $\dfrac{du}{dx}=2x$ and $\dfrac{dy}{du}=e^{u}$.

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= 2xe^{x^2+3}$$

So $\frac{dy}{dx} = 2xe^{x^2+3}$.

12a ii From part (a) (i), $\frac{d}{dx}\left(e^{x^2+3}\right) = 2xe^{x^2+3}$.

Reversing this to give a primitive we obtain:

$$\int 2xe^{x^2+3}\ dx = e^{x^2+3} + C$$

12b i Let $y = e^{x^2-2x+3}$.

Applying the chain rule:

Let $u = x^2 - 2x + 3$ and so $y = e^u$.

Hence $\frac{du}{dx} = 2(x-1)$ and $\frac{dy}{du} = e^u$.

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= 2(x-1)e^{x^2-2x+3}$$

So $\frac{dy}{dx} = 2(x-1)e^{x^2-2x+3}$.

12b ii From part b i, $\frac{d}{dx}\left(e^{x^2-2x+3}\right) = 2(x-1)e^{x^2-2x+3}$.

Reversing this to give a primitive we obtain:

$$\frac{1}{2}\int 2(x-1)e^{x^2-2x+3}\ dx = \frac{1}{2}e^{x^2-2x+3} + C$$

12c i Let $y = e^{3x^2+4x+1}$.

Applying the chain rule:

Let $u = 3x^2 + 4x + 1$ and so $y = e^u$.

Hence $\dfrac{du}{dx} = 2(3x+2)$ and $\dfrac{dy}{du} = e^u$.

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= 2(3x+2)e^{3x^2+4x+1}\end{aligned}$$

So $\dfrac{dy}{dx} = 2(3x+2)e^{3x^2+4x+1}$.

12c ii From part c i, $\dfrac{d}{dx}\left(e^{3x^2+4x+1}\right) = 2(3x+2)e^{3x^2+4x+1}$.

Reversing this to give a primitive we obtain:

$$\frac{1}{2}\int 2(3x+2)e^{3x^2+4x+1}\ dx = \frac{1}{2}e^{3x^2+4x+1} + C$$

12d i Let $y = e^{x^3}$.

Applying the chain rule:

Let $u = x^3$ and so $y = e^u$.

Hence $\dfrac{du}{dx} = 3x^2$ and $\dfrac{dy}{du} = e^u$.

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= 3x^2 e^{x^3}\end{aligned}$$

So $\dfrac{dy}{dx} = 3x^2 e^{x^3}$.

12d ii From part d i, $\frac{d}{dx}\left(e^{x^3}\right)=3x^2e^{x^3}$.

Reversing this to give a primitive we obtain:

$$\frac{1}{3}\int_{-1}^{0}3x^2e^{x^3}\,dx=\frac{1}{3}\left[e^{x^3}\right]_{-1}^{0}$$
$$=\frac{1}{3}\left(e^0-e^{-1}\right)$$
$$=\frac{1}{3}\left(1-e^{-1}\right)$$

13a Given $\int\frac{1}{\left(e^x\right)^2}\,dx$.

$$\int\frac{1}{\left(e^x\right)^2}\,dx=\int e^{-2x}dx$$

Using $\int e^{ax+b}dx=\frac{1}{a}e^{ax+b}+C$ for some constant C we obtain:

$$\int e^{-2x}dx=-\frac{1}{2}e^{-2x}+C$$

13b Given $\int\frac{1}{\left(e^x\right)^3}\,dx$.

$$\int\frac{1}{\left(e^x\right)^3}\,dx=\int e^{-3x}dx$$

Using $\int e^{ax+b}dx=\frac{1}{a}e^{ax+b}+C$ for some constant C we obtain:

$$\int e^{-3x}dx=-\frac{1}{3}e^{-3x}+C$$

13c Given $\int \sqrt{e^x}\ dx$.

$\int \sqrt{e^x}\ dx = \int e^{\frac{1}{2}x} dx$

Using $\int e^{ax+b} dx = \frac{1}{a}e^{ax+b} + C$ for some constant C we obtain:

$\int e^{\frac{1}{2}x} dx = 2e^{\frac{1}{2}x} + C$

13d Given $\int \sqrt[3]{e^x}\ dx$.

$\int \sqrt[3]{e^x}\ dx = \int e^{\frac{1}{3}x} dx$

Using $\int e^{ax+b} dx = \frac{1}{a}e^{ax+b} + C$ for some constant C we obtain:

$\int e^{\frac{1}{3}x} dx = 3e^{\frac{1}{3}x} + C$

13e Given $\int \frac{1}{\sqrt{e^x}}\ dx$.

$\int \frac{1}{\sqrt{e^x}}\ dx = \int e^{-\frac{1}{2}x} dx$

Using $\int e^{ax+b} dx = \frac{1}{a}e^{ax+b} + C$ for some constant C we obtain:

$\int e^{-\frac{1}{2}x} dx = -2e^{-\frac{1}{2}x} + C$

13f Given $\int \frac{1}{\sqrt[3]{e^x}}\ dx$.

$\int \frac{1}{\sqrt[3]{e^x}}\ dx = \int e^{-\frac{1}{3}x} dx$

Using $\int e^{ax+b} dx = \frac{1}{a}e^{ax+b} + C$ for some constant C we obtain:

$$\int e^{-\frac{1}{3}x}dx = -3e^{-\frac{1}{3}x} + C$$

14a Given $y = xe^x$

Applying the product rule on $\frac{d}{dx}\left(xe^x\right)$:

Let $u = x$ and $v = e^x$.

Then $u' = 1$ and $v' = e^x$.

$y' = e^x \times 1 + x \times e^x$

$\quad = xe^x + e^x$

Reversing this to give a primitive we obtain:

$$\begin{aligned}\int_0^2 xe^x \ dx &= \int_0^2 (xe^x + e^x - e^x) \ dx \\ &= \int_0^2 (xe^x + e^x)dx - \int_0^2 e^x \, dx \\ &= [xe^x - e^x]_0^2 \\ &= 2e^2 - e^2 - (0 - 1) \\ &= e^2 + 1\end{aligned}$$

So $\int_0^2 xe^x dx = e^2 + 1$.

14b Given $y = xe^{-x}$.

Applying the product rule on $\frac{d}{dx}\left(xe^{-x}\right)$:

Let $u = x$ and $v = e^{-x}$.

Then $u' = 1$ and $v' = -e^x$.

$y' = e^{-x} \times 1 + x \times -e^{-x}$

$\quad = e^{-x} - xe^{-x}$

Reversing this to give a primitive we obtain:

$$\int_{-2}^0 xe^{-x} \ dx = -\int_{-2}^0 e^{-x} - xe^{-x} + e^{-x} \ dx$$

$$= \left[-xe^{-x} - e^{-x}\right]_{-2}^{0}$$

$$= 0 - 1 - (2e^2 - e^2)$$

$$= -1 - e^2$$

So $\int_{-2}^{0} xe^{-x}dx = -1 - e^2$.

15a $\int \frac{e^x - e^{-x}}{\sqrt{e^x}}\, dx = \int e^{\frac{1}{2}x} - e^{-\frac{3}{2}x} dx$

Using $\int e^{ax+b}dx = \frac{1}{a}e^{ax+b} + C$ for some constant C we obtain:

$$\int e^{\frac{1}{2}x} - e^{-\frac{3}{2}x} dx = 2e^{\frac{1}{2}x} + \frac{2}{3}e^{-\frac{3}{2}x} + C$$

So $\int \frac{e^x - e^{-x}}{\sqrt{e^x}}\, dx = 2e^{\frac{1}{2}x} + \frac{2}{3}e^{-\frac{3}{2}x} + C$.

15b $\int \frac{e^x + e^{-x}}{\sqrt[3]{e^x}}\, dx = \int e^{\frac{2}{3}x} + e^{-\frac{4}{3}x} dx$

Using $\int e^{ax+b}dx = \frac{1}{a}e^{ax+b} + C$ for some constant C we obtain:

$$\int e^{\frac{2}{3}x} + e^{-\frac{4}{3}x} dx = \frac{3}{2}e^{\frac{2}{3}x} - \frac{3}{4}e^{-\frac{4}{3}x} + C$$

So $\int \frac{e^x + e^{-x}}{\sqrt[3]{e^x}}\, dx = \frac{3}{2}e^{\frac{2}{3}x} - \frac{3}{4}e^{-\frac{4}{3}x} + C$.

16a Given $f(x) = xe^{-x^2}$.

If the function is odd, then $f(-x) = -f(x)$.

$$f(-x) = (-x)e^{-(-x)^2}$$
$$= -xe^{-x^2}$$
$$= -\left(xe^{-x^2}\right)$$
$$= -f(x)$$

Hence the function is odd.

16b The graph has point symmetry in the origin.

So $\int_0^{\sqrt{2}} xe^{-x^2}\,dx + \int_{-\sqrt{2}}^{0} xe^{-x^2}\,dx = 0$.

Hence $\int_{-\sqrt{2}}^{\sqrt{2}} xe^{-x^2}\,dx = 0$.

17a Let $u = x^2, \frac{du}{dx} = 2x$

$$\int xe^{x^2}dx = \int \frac{1}{2}e^u \frac{du}{dx}\,dx$$
$$= \frac{1}{2}\int e^u \frac{du}{dx}\,dx$$
$$= \frac{1}{2}e^{x^2} + C$$

17b Let $u = x^2 - 7, \frac{du}{dx} = 2x$

$$\int 4xe^{x^2-7}dx = \int 2e^u \frac{du}{dx}\,dx$$
$$= 2\int e^u \frac{du}{dx}\,dx$$
$$= 2e^{x^2-7} + C$$

17c Let $u = 3x^2 + 4x + 1, \frac{du}{dx} = 6x + 4$

$$\int (3x+2)e^{x^2}dx = \int \frac{1}{2}e^u \frac{du}{dx}\,dx$$
$$= \frac{1}{2}\int e^u \frac{du}{dx}\,dx$$

$= \frac{1}{2}e^{3x^2+4x+1} + C$

17d Let $u = x^3 - 3x^2, \frac{du}{dx} = 3x^2 - 6x$

$\int (x^2 - 2x)e^{x^2}dx = \int \frac{1}{3}e^u \frac{du}{dx}\, dx$

$= \frac{1}{3}\int e^u \frac{du}{dx}\, dx$

$= \frac{1}{3}e^{x^3-3x^2} + C$

17e Let $u = x^{-1}, \frac{du}{dx} = -x^{-2}$

$\int x^{-2}e^{x^2}dx = \int -e^u \frac{du}{dx}\, dx$

$= -\int e^u \frac{du}{dx}\, dx$

$= -e^{x^{-1}} + C$

17f Let $u = -x\sqrt{x} = -x^{\frac{3}{2}},\ \frac{du}{dx} = -\frac{3}{2}x^{\frac{1}{2}} = -\frac{3}{2}\sqrt{x}$

$\int -\sqrt{x}e^{-x\sqrt{x}}dx = \int \frac{2}{3}e^u \frac{du}{dx}\, dx$

$= \frac{2}{3}\int e^u \frac{du}{dx}\, dx$

$= \frac{2}{3}e^{x\sqrt{x}} + C$

18 LHS $= \int \frac{e^x+1}{e^{\frac{1}{2}x}+e^{-\frac{1}{2}x}}dx$

Let $u = e^{x/2}, \frac{du}{dx} = \frac{(e^{x/2})}{2} \Rightarrow du = \frac{(e^{x/2})}{2}\, dx$

LHS $= \int \left[\frac{u^2+1}{u+\frac{1}{u}}\right] \times \frac{2}{u}du$

$= \int 2du$

$= 2u + C$

$= 2e^{\frac{x}{2}} + C$

$=$ RHS

19a We have been given the series of e^x and the inequality of $1 < e^t < e^R$

Integrating this inequality, we get:

⇨ $\int_0^x 1dt < \int_0^x e^t\, dt < \int_0^x e^R dt$

⇨ $x < e^x - 1 < e^R x$

19b Integrating the inequality in 19a again, we get:

⇨ $\int_0^x tdt < \int_0^x (e^t - 1)\, dt < \int_0^x e^R tdt$

⇨ $\frac{x^2}{2} < e^x - 1 - x < \frac{e^R x^2}{2}$

Hence proved

19c i Integrating the inequality in 19b again, we get:

⇨ $\int_0^x \frac{t^2}{2} dt < \int_0^x (e^t - 1 - t)\, dt < \int_0^x \frac{e^R t^2}{2} dt$

⇨ $\frac{x^3}{3!} < e^x - 1 - x - \frac{x^2}{2} < \frac{e^R x^3}{3!}$

19c ii Integrating the inequality in 19c i again, we get the final inequality as:

⇨ $\int_0^x \frac{t^3}{3!} dt < \int_0^x (e^t - 1 - t - \frac{t^2}{2})\, dt < \int_0^x \frac{e^R t^3}{3!} dt$

⇨ $\frac{x^4}{4!} < e^x - 1 - x - \frac{x^2}{2} < \frac{e^R x^4}{4!}$

19d The induction given as $\frac{x^{n+1}}{(n+1)!} < e^x - 1 - x - \frac{x^2}{2} - \cdots . - \frac{x^n}{n!} < \frac{e^R x^{n+1}}{(n+1)!}$

Lets assume the value of n = 1. After substituting the value of n, we get the same inequality as we had got in 19b, which is true.

Now, lets assume that for n = k, the above inequality holds true. Now to prove that for n = k+1, the above inequality also holds true. Lets only consider $\frac{x^{n+1}}{(n+1)!} <$ $e^x - 1 - x - \frac{x^2}{2} - \cdots . - \frac{x^n}{n!}$ For now. We substitute n with k+1

⇨ RHS – LHS = $e^x - 1 - x - \frac{x^2}{2} - \cdots . - \frac{x^k}{k!} - \frac{x^{k+1}}{(k+1)!} - \frac{x^{k+2}}{(k+2)!}$

⇨ RHS – LHS $= e^x - \left(1 + x + \frac{x^2}{2} + \cdots.. + \frac{x^{k+2}}{2}\right)$

⇨ RHS – LHS > 0, since the power series term of e^x goes till infinity terms, but we are only subtracting the first k+2 terms.

⇨ $\frac{x^{k+2}}{(k+2)!} < e^x - 1 - x - \frac{x^2}{2} - \cdots. - \frac{x^{k+1}}{k+1!}$ Holds true.

Hence, our inequality holds true as the induction method works on the inequality.

19e For the left expression, assume that we increase n by 1 unit. We get the ratio of the new term and old term as:

⇨ $\frac{x^{n+2}}{(n+2)!} \div \frac{x^{n+1}}{(n+1)!}$

⇨ $\frac{x}{n+1}$

As n increases and becomes more than x, the ratio above will tend to 0 as n tends to infinity. Hence, the left (same holds for right) expression will converge to 0 as n approaches infinity.

Hence, $e^x - 1 - x - \frac{x^2}{2} - \cdots. - \frac{x^n}{n!} \to 0$

⇨ $e^x = 1 + x + \frac{x^2}{2} + \cdots. + \frac{x^n}{n!}$

19f In 19e, the left and right expression will converge to 0 regardless of the value of x.

⇨ For $x < 0$, we will have to substitute the inequalities in 19e with '$-x$' instead of 'x'

⇨ $e^{-x} - 1 - (-x) - \frac{(-x)^2}{2} - \cdots. - \frac{(-x)^n}{n!} \to 0$

⇨ $e^{-x} = 1 - x + \frac{x^2}{2} - \frac{x^3}{3!} + \cdots. + \frac{(-1)^n x^n}{n!}$

⇨ Which is the same as substituting with '$-x$' in the power series provided in the question.

20a $e^x = 1 + x + \frac{x^2}{2} + \cdots . + \frac{x^n}{n!}$

And $e^{-x} = 1 - x + \frac{x^2}{2} - \frac{x^3}{3!} + \cdots . + \frac{(-1)^n x^n}{n!}$

$e^x + e^{-x} = 2 \times (1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots . + \frac{x^{2n}}{2n!} + \cdots .)$

Hence, $\frac{(e^x + e^{-x})}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots . + \frac{x^{2n}}{2n!} + \cdots .$

20b After substituting x with 0.5, we get the answer as 1.1276.

20c $u = e^{0.5}$

⇨ $u^2 - 2 \propto u + 1 = e^1 - 2 \times 1.1276 \times e^{0.5} + 1$

⇨ $e + 1 - 2 \times 1.1276 \times e^{0.5} = 0$

20d $u^2 - 2 \propto u + 1 = 0$ is a quadratic equation. Hence its roots are:

$u = \frac{[2\propto \mp \sqrt{4\propto^2 - 4}]}{2} = \propto \mp \sqrt{\propto^2 - 1}$

Hence, $u = 1.6486\ or\ u = 0.6065$

Hence, $e^{0.5} = 1.64$ and $e^{-0.5} = 0.60$. These values match with the calculator obtained values exactly.

Solutions to Exercise 6E

1a i $\int_0^1 e^x \, dx = [e^x]_0^1$

$= e^1 - e^0$

$= e - 1$

$\doteqdot 1.72$

1a ii $\int_{-1}^0 e^x \, dx = [e^x]_{-1}^0$

$= e^0 - e^{-1}$

$= 1 - e^{-1}$

$\doteqdot 0.63$

1a iii $\int_{-2}^0 e^x \, dx = [e^x]_{-2}^0$

$= e^0 - e^{-2}$

$= 1 - e^{-2}$

$\doteqdot 0.86$

1a iv $\int_{-3}^0 e^x \, dx = [e^x]_{-3}^0$

$= e^0 - e^{-3}$

$= 1 - e^{-3}$

$\doteqdot 0.95$

1b $\int_0^1 e^x \, dx \doteqdot 1.72$

1c i $\int_{-1}^0 e^x \, dx \doteqdot 0.63$

1c ii $\int_{-2}^0 e^x \, dx \doteqdot 0.86$

1c iii $\int_{-3}^{0} e^{x}\ dx \doteqdot 0.95$

1d The total area is exactly 1.

2a $\int_{0}^{3} e^{2x}\ dx = \left[\frac{1}{2}e^{2x}\right]_{0}^{3}$

$= \frac{1}{2}e^{2(3)} - \frac{1}{2}e^{0}$

$= \frac{1}{2}e^{6} - \frac{1}{2}$

$\doteqdot 201.2$

2b $\int_{0}^{1} e^{-x}\ dx = [-e^{-x}]_{0}^{1}$

$= -e^{-1} - (-e^{0})$

$= 1 - e^{-1}$

$\doteqdot 0.6321$

2c $\int_{-3}^{0} e^{\frac{1}{3}x}\ dx = \left[\frac{1}{\frac{1}{3}}e^{\frac{1}{3}x}\right]_{-3}^{0}$

$= \left[3e^{\frac{1}{3}x}\right]_{-3}^{0}$

$= 3e^{0} - 3e^{\frac{1}{3}(-3)}$

$= 3 - 3e^{-1}$

$\doteqdot 1.896$

3a $\int_{-2}^{0} e^{x+3}\ dx = [e^{x+3}]_{-2}^{0}$

$= e^{0+3} - e^{-2+3}$

$= e^{3} - e$

$= e(e^{2} - 1)$

3b $\int_0^1 e^{2x-1}\ dx = \left[\frac{1}{2}e^{2x-1}\right]_0^1$

$= \frac{1}{2}e^{2(1)-1} - \frac{1}{2}e^{0-1}$

$= \frac{1}{2}e - \frac{1}{2}e^{-1}$

$= \frac{1}{2}(e - e^{-1})$

3c $\int_{-2}^{-1} e^{-2x-1}\ dx = \left[-\frac{1}{2}e^{-2x-1}\right]_{-2}^{-1}$

$= -\frac{1}{2}e^{-2(-1)-1} - \left(-\frac{1}{2}e^{-2(-2)-1}\right)$

$= \frac{1}{2}e^3 - \frac{1}{2}e$

$= \frac{1}{2}e(e^2 - 1)$

3d $\int_0^3 e^{\frac{1}{3}x+2}\ dx = \left[\frac{1}{\frac{1}{3}}e^{\frac{1}{3}x+2}\right]_0^3$

$= \left[3e^{\frac{1}{3}x+2}\right]_0^3$

$= 3e^{\frac{1}{3}(3)+2} - 3e^{0+2}$

$= 3e^3 - 3e^2$

$= 3e^2(e - 1)$

4a $\int_{-1}^2 e^{\frac{1}{2}x}\ dx = \left[\frac{1}{\frac{1}{2}}e^{\frac{1}{2}x}\right]_{-1}^2$

$= \left[2e^{\frac{1}{2}x}\right]_{-1}^2$

$= 2e^{\frac{1}{2}(2)} - 2e^{\frac{1}{2}(-1)}$

$= 2e - 2e^{-\frac{1}{2}}$

$= 2\left(e - e^{-\frac{1}{2}}\right)$

4b $\int_0^1 e^{-x}\ dx = [-e^{-x}]_0^1$

$= -e^{-1} - (-e^0)$

$= 1 - e^{-1}$

5a $\int_{-2}^2 e^x + e^{-x}\ dx = [e^x - e^{-x}]_{-2}^2$

$= (e^2 - e^{-2}) - \left(e^{-2} - e^{-(-2)}\right)$

$= e^2 - e^{-2} - e^{-2} + e^2$

$= 2e^2 - 2e^{-2}$

$= 2(e^2 - e^{-2})$

$\doteqdot 14.51$

5b $\int_{-3}^3 x^2 + e^x\ dx = \left[\frac{1}{3}x^3 + e^x\right]_{-3}^3$

$= \left(\frac{1}{3}(3)^3 + e^3\right) - \left(\frac{1}{3}(-3)^3 + e^{-3}\right)$

$= 9 + e^3 + 9 - e^{-3}$

$= 18 + e^3 - e^{-3}$

$\doteqdot 38.04$

6a $\int_0^2 1 - e^{-x}\ dx = [x + e^{-x}]_0^2$

$= (2 + e^{-2}) - (0 + e^0)$

$= 2 + e^{-2} - 0 - 1$

$= 1 + e^{-2}$

6b $\int_0^1 e - e^x\ dx = [xe - e^x]_0^1$

$= (e - e) - (0 - e^0)$

$= 1$

6c $\int_{-1}^{0} e^x - 1 \ dx = [e^x - x]_{-1}^{0}$

$= (e^0 - 0) - \left(e^{-1} - (-1)\right)$

$= 1 - e^{-1} - 1$

$= -e^{-1}$

Area $= e^{-1}$

6d $\int_{0}^{2} e^{-x} - 2 \ dx = [-e^{-x} - 2x]_{0}^{2}$

$= \left(-e^{-2} - 2(2)\right) - (-e^0 - 0)$

$= -e^{-2} - 4 + 1$

$= -e^{-2} - 3$

Area $= 3 + e^{-2}$

6e $\int_{-1}^{0} e^{-x} - e \ dx = [-e^{-x} - xe]_{-1}^{0}$

$= (-e^0 - 0) - \left(-e^{-(-1)} - (-1)e\right)$

$= -1 + e - e$

$= -1$

Area $= 1$

6f $\int_{-1}^{2} 3 - e^{-x} \ dx = [3x + e^{-x}]_{-1}^{2}$

$= (3(2) + e^{-2}) - (3(-1) + e^{-(-1)})$

$= 6 + e^{-2} + 3 - e$

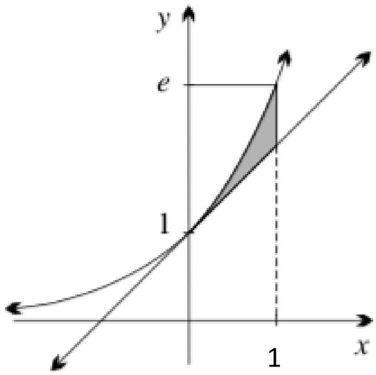
$= 9 + e^{-2} - e$

7a $\int_{0}^{1} e^x - 1 - x \ dx = \left[e^x - x - \frac{1}{2}x^2\right]_{0}^{1}$

$= \left(e - 1 - \frac{1}{2}\right) - (e^0 - 0 - 0)$

$= e - 1 - \frac{1}{2} - 1$

$$= e - 2\frac{1}{2}$$

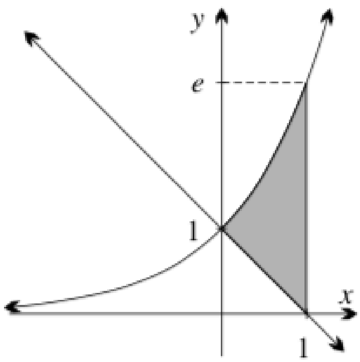


7b $\int_0^1 e^x - 1 + x \ dx = \left[e^x - x + \frac{1}{2}x^2\right]_0^1$

$$= \left(e - 1 + \tfrac{1}{2}\right) - (e^0 - 0 + 0)$$

$$= e - 1 + \tfrac{1}{2} - 1$$

$$= e - 1\frac{1}{2}$$



8a The region is symmetric, so the area is twice the area in the first quadrant.

8b $2\int_0^1 e^{-x} \ dx = 2[-e^{-x}]_0^1$

$$= 2\left(-e^{-1} - (-e^0)\right)$$

$$= 2(1 - e^{-1})$$

$= 2 - \frac{2}{e}$

9a The region is symmetric, so the area is twice the area in the first quadrant.

9b $2\int_0^1 e - e^x \, dx = 2[ex - e^x]_0^1$

$= 2\big((e - e) - (0 - e^0)\big)$

$= 2(1)$

$= 2$

10a To show that y is an odd function, we must show that $f(-x) = -f(x)$

$f(-x) = e^{-x} - e^{-(-x)}$

$= e^{-x} - e^x$

$= -(e^x - e^{-x})$

$= -f(x)$

10b 0

10c The region is symmetric, so the area is twice the area in the first quadrant.

10d $2\int_0^3 e^x - e^{-x} \, dx = 2[e^x + e^{-x}]_0^3$

$= 2\big((e^3 + e^{-3}) - (e^0 + e^0)\big)$

$= 2(e^3 + e^{-3} - 1 - 1)$

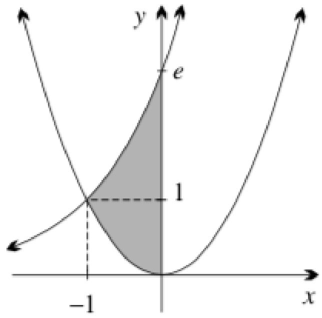
$= 2(e^3 + e^{-3} - 2)$

11a For $y = x^2$: when $x = -1, y = (-1)^2 = 1$

For $y = e^{x+1}$: when $x = -1, y = e^{-1+1} = e^0 = 1$

So the two curves intersect at $(-1, 1)$.

11b



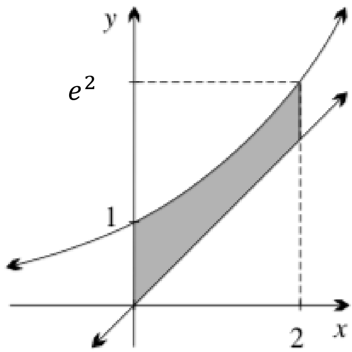
11c $\int_{-1}^{0} e^{x+1} - x^2 \ dx = \left[e^{x+1} - \frac{1}{3}x^3\right]_{-1}^{0}$

$= (e - 0) - \left(e^{-1+1} - \frac{1}{3}(-1)^3\right)$

$= e - e^0 - \frac{1}{3}$

$= e - 1\frac{1}{3}$

12a



$\int_{0}^{2} e^x - x \ dx = \left[e^x - \frac{1}{2}x^2\right]_{0}^{2}$

$= \left(e^2 - \frac{1}{2}(2)^2\right) - (e^0 - 0)$

$= e^2 - 2 - 1$

$= e^2 - 3$

12b y-intercept when $x = 0$,

$y = 8 - 2^0 = 7$

x-intercept when $y = 0$

$0 = 8 - 2^x$

$x = 3$

$\text{Area} = \int_0^3 8 - 2^x \; dx$

$= \left[8x - \dfrac{2^x}{\log_e 2}\right]_0^3$

$= \left(24 - \frac{8}{\log_e 2}\right) - \left(0 - \frac{1}{\log_e 2}\right)$

$= 24 - \frac{7}{\log_e 2}$

13a $\int_0^1 e^x \; dx = [e^x]_0^1$

$= e^1 - e^0$

$= e - 1$

$\doteqdot 1.7183$

13b $\text{Area} = \frac{1}{2}\left(\frac{e^0 + e^{\frac{1}{2}}}{2}\right) + \frac{1}{2}\left(\frac{e^{\frac{1}{2}} + e^1}{2}\right)$

$= \frac{1}{4}\left(1 + 2e^{\frac{1}{2}} + e\right)$

$\doteqdot 1.7539$

13c The trapezoidal rule approximation is greater. The curve is concave up, so all the chords are above the curve.

14a Let $y = e^{-x^2}$.

Applying the chain rule:

Let $u = -x^2$ and so $y = e^u$.

Hence $\frac{du}{dx} = -2x$ and $\frac{dy}{du} = e^u$.

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= -2xe^{-x^2}$$

So $\frac{dy}{dx} = -2xe^{-x^2}$.

From above, $\frac{d}{dx}\left(e^{-x^2}\right) = -2xe^{-x^2}$.

Reversing this to give a primitive we obtain:

$$-\frac{1}{2}\int -2xe^{-x^2}\,dx = -\frac{1}{2}e^{-x^2}$$

14b $\displaystyle\int_0^2 xe^{-x^2}\,dx = \left[-\frac{1}{2}e^{-x^2}\right]_0^2 = -\frac{1}{2}e^{-4} - \left(-\frac{1}{2}\right) = \frac{1}{2} - \frac{1}{2}e^{-4}$

So from $x = 0$ to $x = 2$, the area is $\frac{1}{2} - \frac{1}{2}e^{-4}$ square units.

The function is odd, so the area (not signed) from $x = -2$ to $x = 2$ is $2\left(\frac{1}{2} - \frac{1}{2}e^{-4}\right)$ square units i.e. $1 - e^{-4}$ square units.

15a i $\displaystyle\int_N^0 e^x\,dx = \left[e^x\right]_N^0 = 1 - e^N$

15a ii As $N \to -\infty, e^N \to 0$ and so $\displaystyle\lim_{n\to-\infty}\left(\int_N^0 e^x\,dx\right) = 1$.

15b i $\displaystyle\int_0^N e^{-x}\,dx = \left[-e^{-x}\right]_0^N = -e^{-N} - (-1) = 1 - e^{-N}$

15b ii As $N \to \infty, e^{-N} \to 0$ and so $\displaystyle\lim_{n\to\infty}\left(\int_0^N e^{-x}\,dx\right) = 1$.

15c Let $u = -x^2, \frac{du}{dx} = -2x$

$$\int_0^N 2xe^{-x^2}dx = \int_0^N -e^u \frac{du}{dx}\, dx$$
$$= -\int_0^N e^u \frac{du}{dx}\, dx$$
$$= \left[-e^{-x^2}\right]_0^N$$
$$= -e^{-N^2} - -e^{-0}$$
$$= 1 - e^{-N^2}$$

As $N \to \infty, e^{-N^2} \to 0$ and so $\lim_{N\to\infty}\left(\int_0^N 2xe^{-x^2}dx\right) = 1$.

16a Let $u = \sqrt{x}, \frac{du}{dx} = \frac{1}{2\sqrt{x}}$

$$\int_\delta^1 \frac{e^{\sqrt{x}}}{\sqrt{x}}dx = \int_\delta^1 2e^u \frac{du}{dx}\, dx$$
$$= 2\int_\delta^1 e^u \frac{du}{dx}\, dx$$
$$= 2\left[e^{\sqrt{x}}\right]_\delta^1$$
$$= 2\left(e^1 - e^{\sqrt{\delta}}\right)$$

16b As $\delta \to 0^+, e^{\sqrt{\delta}} \to 1$ and so $\int_\delta^1 \frac{e^{\sqrt{x}}}{\sqrt{x}}dx \to 2(e-1)$

17a Given $y = xe^{-x}$.

Applying the product rule on $\frac{d}{dx}\left(xe^{-x}\right)$:

Let $u = x$ and $v = e^{-x}$.

Then $u' = 1$ and $v' = -e^{x}$.

$$y' = e^{-x} \times 1 + x \times -e^{-x}$$
$$= e^{-x} - xe^{-x}$$

Reversing this to give a primitive we obtain:

$$\int_0^N xe^{-x}\ dx = -\int_0^N e^{-x} - xe^{-x} + e^{-x}\ dx$$

$$= [-xe^{-x} - e^{-x}]_0^N$$

$$= -Ne^{-N} - e^{-N} - (0 - 1)$$

$$= 1 - (1 + N)e^{-N}$$

17b As $N \to \infty, e^{-N} \to 1$ and so $(1 + N)e^{-N} \to 1$. Hence $\lim_{N\to\infty}\left(\int_0^N xe^{-x}dx\right) = 1$

17c Given $y = x^2e^{-x}$.

Applying the product rule:

Let $u = x^2$ and $v = e^{-x}$.

Then $u' = 2x$ and $v' = -e^{x}$.

$y' = e^{-x} \times 2x + x^2 \times -e^{-x}$

$\quad = 2xe^{-x} - x^2e^{-x}$

Reversing this to give a primitive we obtain:

$$\int_0^\infty x^2e^{-x}\ dx = -\int_0^\infty 2xe^{-x} - x^2e^{-x} - 2xe^{-x}\ dx$$

$$= [-x^2e^{-x} + 2(-xe^{-x} - e^{-x})]_0^\infty$$

$$= \left[-\frac{(x^2+2x+2)}{e^x}\right]_0^\infty$$

As $e^x \to \infty, \frac{k}{e^x} \to 0$ given that k grows slower than e^x, Hence, $\lim_{x\to\infty}\left(\frac{(x^2+2x+2)}{e^x}\right) = 0$

Therefore, $\left[-\frac{(x^2+2x+2)}{e^x}\right]_0^\infty = 0 - \left(-\frac{0+0+2}{1}\right) = 2$

Hence, $\int_0^\infty x^2e^{-x}\ dx = 2$

Solutions to Exercise 6F

1a 2.303

1b -2.303

1c 11.72

1d -12.02

1e 3.912

1f -3.912

2a $\ln 5 + \ln 4 = \ln(5 \times 4) = \ln 20$

2b $\ln 30 - \log_e 6 = \ln(30 \div 6) = \ln 5$

2c $\ln 12 - \ln 15 + \ln 10^2 = \ln(12 \div 15 \times 100) = \ln 80$

3a $\log_e e^3 = 3$

3b $\log_e e^{-1} = -1$

3c $\log_e \frac{1}{e^2} = \log_e e^{-2} = -2$

3d $\log_e \sqrt{e} = \log_e e^{\frac{1}{2}} = \frac{1}{2}$

3e $e^{\ln 5} = e^{\log_e 5} = 5$

3f $e^{\ln 0.05} = e^{\log_e 0.05} = 0.05$

3g $e^{\ln 1} = e^{\log_e 1} = 1$

3h $e^{\ln e} = e^{\log_e e} = e$

4a $\log_e 1 = 0$

4b $1 = e^0$

$\log_e 1 = \log_e e^0 = 0$

4c $\log_e e = 1$

4d $e = e^1$

$\log_e e = \log_e e^1 = 1$

5a $\log_e x = 6$

5b $x = e^{-2}$

5c $e^x = 24$

5d $x = \log_e \frac{1}{3}$

6a $\log_2 7 = \frac{\log_e 7}{\log_e 2} \doteqdot 2.807$

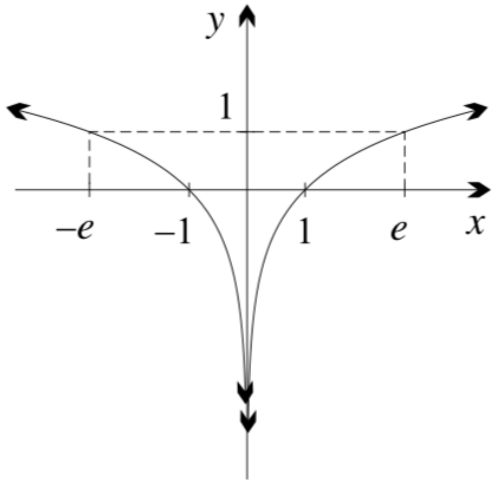
6b $\log_{10} 25 = \frac{\log_e 25}{\log_e 10} \doteqdot 1.398$

6c $\log_3 0.04 = \frac{\log_e 0.04}{\log_e 3} \doteqdot -2.930$

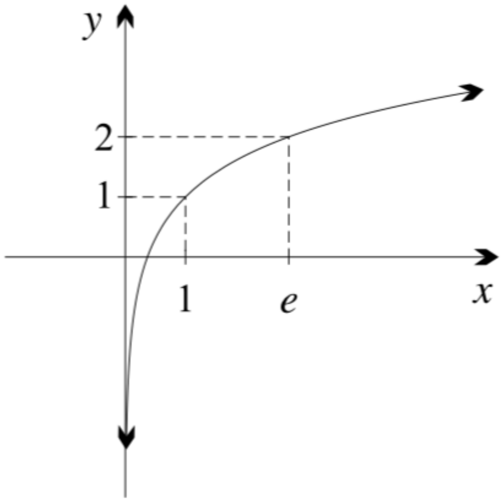
7a Reflection in $y = x$, which reflects lines with gradient 1 to lines of gradient 1. The tangent to $y = e^x$ at its y-intercept has gradient 1, so its reflection also has gradient 1.

7b Reflection in the y-axis, which is also horizontal dilation with factor -1.

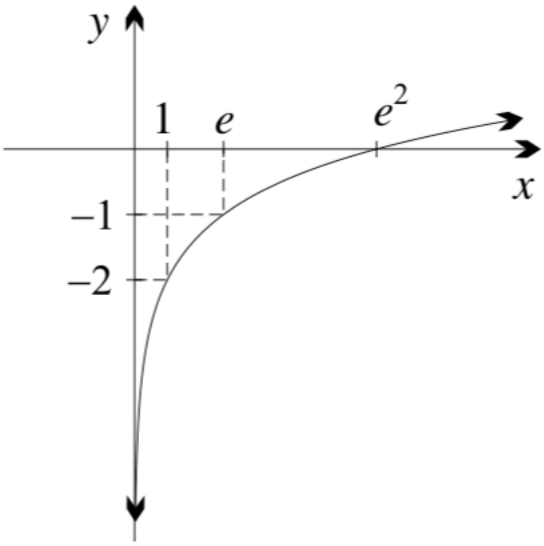
7c



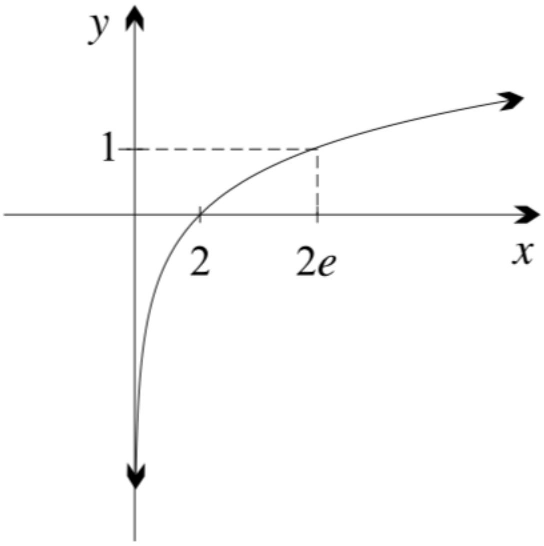
8a Shift $y = \log_e x$ up 1



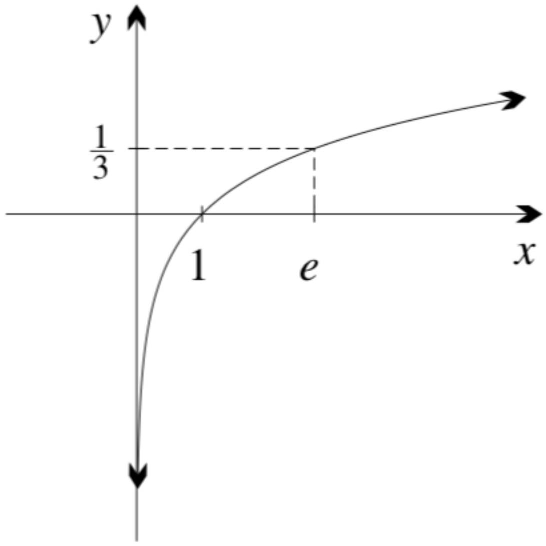
8b Shift $y = \log_e x$ down 2



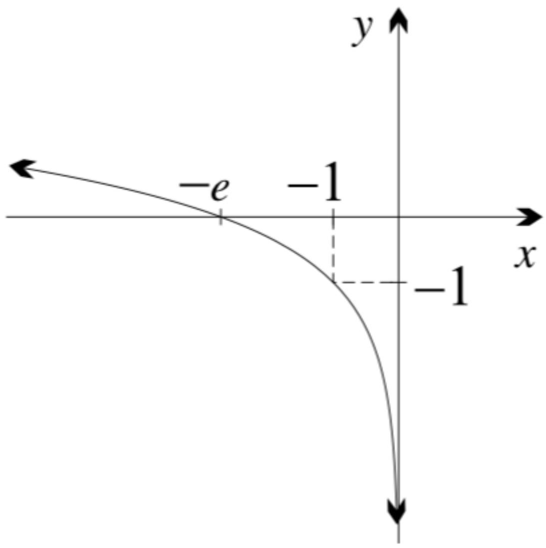
8c Stretch $y = \log_e x$ horizontally with factor 2



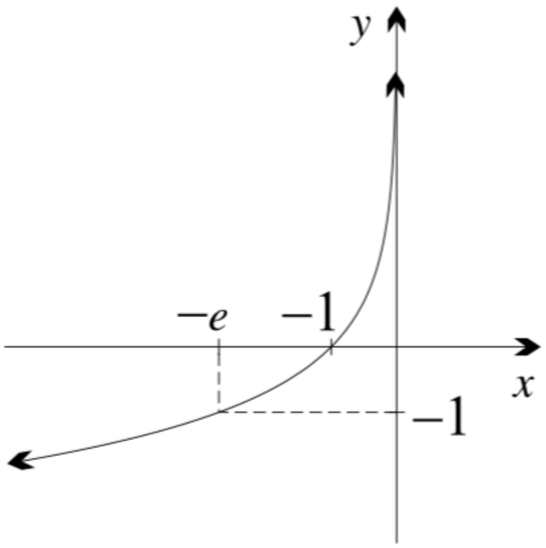
8d Stretch $y = \log_e x$ vertically with factor $\frac{1}{3}$



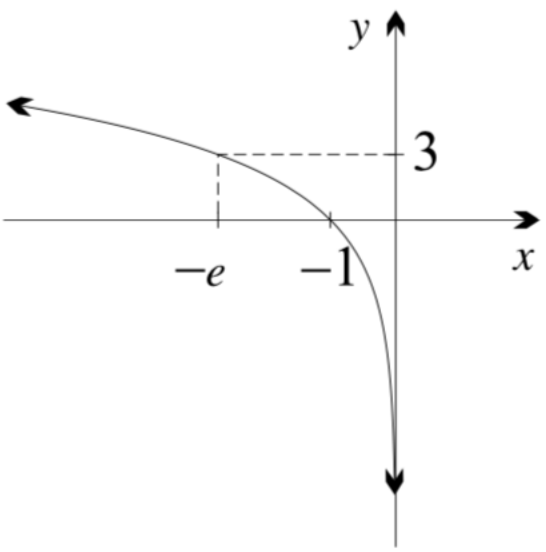
9a Shift $y = \log_e(-x)$ down 1.



9b Reflect $y = \log_e(-x)$ in the x-axis.



9c Stretch $y = \log_e(-x)$ vertically with factor 3.



10a $e \log_e e = e$

10b $\frac{1}{e}\ln\frac{1}{e} = \frac{1}{e}(-1) = -\frac{1}{e}$

10c $3\log_e e^2 = 3 \times 2 = 6$

10d $\ln\sqrt{e} = \ln e^{\frac{1}{2}} = \frac{1}{2}$

10e $e\log_e e^3 - e\log_e e = 3e - e = 2e$

10f $\log_e e + \log_e \frac{1}{e} = 1 + (-1) = 0$

10g $\log_e e^e = e$

10h $\log_e(\log_e e^e) = \log_e e = 1$

10i $\log_e(\log_e(\log_e e^e)) = \log_e 1 = 0$

11 It is a horizontal dilation of $y = \log_e(-x)$ with factor $\frac{1}{2}$.

Its equation is $y = \log_e(-2x)$.

12a $4^x - 9 \times 2^x + 14 = 0$

$(2^2)^x - 9 \times 2^x + 14 = 0$

$(2^x)^2 - 9 \times 2^x + 14 = 0$

Let $u = 2^x$.

$u^2 - 9u + 14 = 0$

$(u-2)(u-7) = 0$

$u = 2, 7$

$2^x = 2,7$

For $2^x = 2$

$x = 1$

For $2^x = 7$

$x = \log_2 7$

12b $3^{2x} - 8 \times 3^x - 9 = 0$

$(3^x)^2 - 8 \times 3^x - 9 = 0$

Let $u = 3^x$.

$u^2 - 8u - 9 = 0$

$(u + 1)(u - 9) = 0$

$u = -1, 9$

$3^x = -1, 9$

For $3^x = 9$

$x = 2$

For $3^x = -1$ there are no solutions.

12c i $25^x - 26 \times 5^x + 25 = 0$

$(5^2)^x - 26 \times 5^x + 25 = 0$

$(5^x)^2 - 26 \times 2^x + 25 = 0$

Let $u = 5^x$.

$u^2 - 26u + 25 = 0$

$(u - 1)(u - 25) = 0$

$u = 1, 25$

$5^x = 1, 25$

For $5^x = 1$

$x = 0$

For $5^x = 25$

$x = 2$

12c ii $9^x - 5 \times 3^x + 4 = 0$

$(3^2)^x - 5 \times 3^x + 4 = 0$

$(3^x)^2 - 5 \times 3^x + 4 = 0$

Let $u = 3^x$.

$u^2 - 5u + 4 = 0$

$(u-1)(u-4)=0$

$u=1,4$

$3^x=1,4$

For $3^x=1$

$x=0$

For $3^x=4$

$x=\log_3 4$

12c iii $3^{2x}-3^x-20=0$

$(3^x)^2-3^x-20=0$

Let $u=3^x$.

$u^2-u-20=0$

$(u+4)(u-5)=0$

$u=-4,5$

$3^x=-4,5$

For $3^x=-4$ there are no solutions.

For $3^x=5$

$x=\log_3 5$

12c iv $7^{2x}+7^x+1=0$

$(7^x)^2+7^x+1=0$

Let $u=7^x$,

$u^2+u+1=0$

The quadratic has no solutions because $\triangle=1-4(1)(1)=-3<0$

12c v $3^{5x}=9^{x+3}$

$3^{5x}=(3^2)^{x+3}$

$3^{5x}=3^{2x+6}$

$5x=2x+6$

$3x = 6$

$x = 2$

12c vi $4^x - 3 \times 2^{x+1} + 2^3 = 0$

$(2^2)^x - 3 \times 2^x \times 2 + 8 = 0$

$(2^x)^2 - 6 \times 2^x + 8 = 0$

Let $u = 2^x$.

$u^2 - 6u + 8 = 0$

$(u - 2)(u - 4) = 0$

$u = 2, 4$

$2^x = 2, 4$

For $2^x = 2$

$x = 1$

For $2^x = 4$

$x = 2$

13a $e^{2x} - 2e^x + 1 = 0$

$(e^2)^x - 2e^x + 1 = 0$

$(e^x)^2 - 2e^x + 1 = 0$

Let $u = e^x$.

$u^2 - 2u + 1 = 0$

$(u - 1)(u - 1) = 0$

$u = 1$

$e^x = 1$

$x = 0$

13b $e^{2x} + e^x - 6 = 0$

$(e^2)^x + e^x - 6 = 0$

$(e^x)^2 + e^x - 6 = 0$

Let $u = e^x$.

$u^2 + u - 6 = 0$

$(u + 3)(u - 2) = 0$

$u = -3, 2$

$e^x = -3, 2$

For $e^x = -3$ there are no solutions

For $e^x = 2$

$x = \log_e 2$

13c $e^{4x} - 10e^{2x} + 9 = 0$

$(e^2)^{2x} - 10e^{2x} + 9 = 0$

$(e^{2x})^2 - 10e^{2x} + 9 = 0$

Let $u = e^{2x}$.

$u^2 - 10u + 9 = 0$

$(u - 1)(u - 9) = 0$

$u = 1, 9$

$e^{2x} = 1, 9$

For $e^{2x} = 1$

$x = 0$

For $e^{2x} = 9$

$(e^x)^2 = 3^2$

$e^x = \pm 3$

$x = \log_e 3$ as $e^x = -3$ has no solutions.

13d $e^{4x} - e^{2x} = 0$

$(e^2)^{2x} - e^{2x} = 0$

$(e^{2x})^2 - e^{2x} = 0$

Let $u = e^{2x}$.

$u^2 - u = 0$

$u(u-1)=0$

$u=0,1$

$e^{2x}=0,1$

For $e^{2x}=0$ there are no solutions

For $e^{2x}=1$

$x=0$

14a $2^{4x}-7\times 2^{2x}+12=0$

Let $u=2^{2x}$ and so $u=4^x$.

$\left(2^{2x}\right)^2-7\times 2^{2x}+12=0$

$$\begin{aligned} u^2-7u+12&=0\\ (u-3)(u-4)&=0\\ u&=3,4 \end{aligned}$$

So $4^x=3$ or $4^x=4$.

Hence $x=\log_4 3\approx 0.792$ or $x=1$.

14b $100^x-10^x-1=0$

Let $u=10^x$.

$10^{2x}-10^x-1=0\Rightarrow\left(10^x\right)^2-10^x-1=0$

$$\begin{aligned} u^2-u-1&=0\\ u&=\frac{1\pm\sqrt{1-4\times1\times-1}}{2}\\ &=\frac{1\pm\sqrt{5}}{2} \end{aligned}$$

So $10^x=\frac{1\pm\sqrt{5}}{2}$.

$\log_{10}\frac{1-\sqrt{5}}{2}$ does not exist because $\frac{1-\sqrt{5}}{2}$ is negative.

Hence $x=\log_{10}\frac{1+\sqrt{5}}{2}\approx 0.209$.

14c $\left(\frac{1}{5}\right)^{2x}-7\times\left(\frac{1}{5}\right)^{x}+10=0$

Let $u=\left(\frac{1}{5}\right)^{x}$.

$\left(\left(\frac{1}{5}\right)^{x}\right)^{2}-7\times\left(\frac{1}{5}\right)^{x}+10=0$

$$\begin{aligned}u^2-7u+10&=0\\(u-2)(u-5)&=0\\u&=2,5\end{aligned}$$

So $\left(\frac{1}{5}\right)^{x}=2$ or $\left(\frac{1}{5}\right)^{x}=5$.

Hence $x=-1$ or $x=\log_{\frac{1}{5}}2\approx-0.431$.

15a $(\log_e x)^2-5\log_e x+4=0$

Let $u=\log_e x$,

$u^2-5u+4=0$

$(u-1)(u-4)=0$

$u=1,4$

$\log_e x=1,4$

For $\log_e x=1$

$x=e$

For $\log_e x=4$

$x=e^4$

15b $(\log_e x)^2 = 3\log_e x$

Let $u = \log_e x$,

$u^2 = 3u$

$u^2 - 3u = 0$

$u(u-3) = 0$

$u = 0, 3$

$\log_e x = 0, 3$

For $\log_e x = 0$

$x = 1$

For $\log_e x = 3$

$x = e^3$

16a $\ln(x^2 + 5x) = 2\ln(x+1)$

$= \ln((x+1)^2)$

$x^2 + 5x = (x+1)^2$

$= x^2 + 2x + 1$

$3x = 1$

$x = \frac{1}{3}$

16b $\log_e(7x - 12) = 2\log_e x$

$= \log_e x^2$

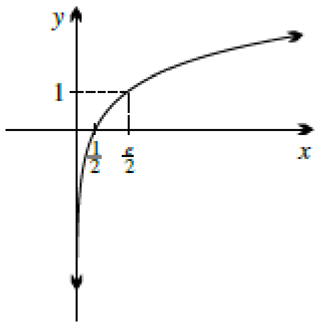
$7x - 12 = x^2$

$x^2 - 7x + 12 = 0$

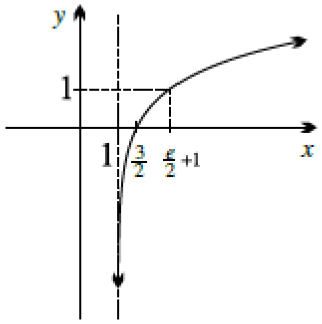
$(x-3)(x-4) = 0$

$x = 3 \text{ or } 4$

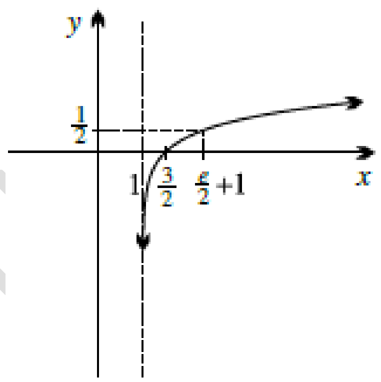
17a Stretch horizontally with factor $\frac{1}{2}$.



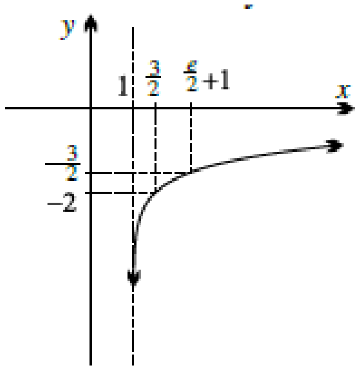
17b Shift right 1.



17c Stretch vertically with factor $\frac{1}{2}$.



17d Shift down 2.



18 First, the base must be positive because powers of negative numbers are not well defined when the index is a real number, so a negative number can't be used as a base for logarithms. Secondly, the base cannot be 1 because all powers of 1 are 1, and in any case, $\log_e 1 = 0$ and you can't divide by zero.

19a As a dilation, the transformation is stretch horizontally with factor $\dfrac{1}{5}$.

Alternatively, $y = \log_e x + \log_e 5$, so it is a shift up $\log_e 5$.

19b As a translation, the transformation is shift up 2.

Alternatively, $y = \log_e x + \log_e e^2 = \log_e e^2 x$ so it is a horizontal dilation with factor e^{-2}.

20 The continued fraction gives an approximation of $e - 1 \doteqdot 1\frac{28}{39}$

Hence $e \doteqdot 2\frac{28}{39}$

Solutions to Exercise 6G

1a $\frac{dy}{dx} = \frac{1}{x+2}$

1b $\frac{dy}{dx} = \frac{1}{x-3}$

1c $\frac{dy}{dx} = \frac{3}{3x+4}$

1d $\frac{dy}{dx} = \frac{2}{2x-1}$

1e $\frac{dy}{dx} = -\frac{4}{-4x+1}$

1f $\frac{dy}{dx} = -\frac{3}{-3x+4}$

1g $\frac{dy}{dx} = -\frac{2}{-2x-7} = \frac{2}{2x+7}$

1h $\frac{dy}{dx} = 3\left(\frac{2}{2x+4}\right)$

$= \frac{6}{2x+4}$

$= \frac{3}{x+2}$

1i $\frac{dy}{dx} = 5\left(\frac{3}{3x-2}\right) = \frac{15}{3x-2}$

2a $\frac{dy}{dx} = \frac{2}{2x} = \frac{1}{x}$

2b $\frac{dy}{dx} = \frac{5}{5x} = \frac{1}{x}$

2c $\frac{dy}{dx} = \frac{3}{3x} = \frac{1}{x}$

2d $\frac{dy}{dx} = \frac{7}{7x} = \frac{1}{x}$

2e $\frac{dy}{dx} = 4\left(\frac{7}{7x}\right) = \frac{4}{x}$

2f $\frac{dy}{dx} = 3\left(\frac{5}{5x}\right) = \frac{3}{x}$

2g $\frac{dy}{dx} = 4\left(\frac{6}{6x}\right) = \frac{4}{x}$

2h $\frac{dy}{dx} = 3\left(\frac{9}{9x}\right) = \frac{3}{x}$

3a $\frac{dy}{dx} = \frac{1}{x+1}$

At $x = 3, \frac{dy}{dx} = \frac{1}{3+1} = \frac{1}{4}$

3b $\frac{dy}{dx} = \frac{2}{2x-1}$

At $x = 3, \frac{dy}{dx} = \frac{2}{2(3)-1} = \frac{2}{5}$

3c $\frac{dy}{dx} = \frac{2}{2x-5}$

At $x = 3, \frac{dy}{dx} = \frac{2}{2(3)-5} = 2$

3d $\frac{dy}{dx} = \frac{4}{4x+3}$

At $x = 3, \frac{dy}{dx} = \frac{4}{4(3)+3} = \frac{4}{15}$

3e $\frac{dy}{dx} = 5\left(\frac{1}{x+1}\right) = \frac{5}{x+1}$

At $x = 3, \frac{dy}{dx} = \frac{5}{3+1} = \frac{5}{4}$

3f $\frac{dy}{dx} = 6\left(\frac{2}{2x+9}\right) = \frac{12}{2x+9}$

At $x = 3, \frac{dy}{dx} = \frac{12}{2(3)+9}$

$= \frac{12}{15}$

$= \frac{4}{5}$

4a $\frac{d}{dx}(2 + \log_e x) = \frac{1}{x}$

4b $\frac{d}{dx}(5 - \log_e(x+1)) = -\frac{1}{x+1}$

4c $\frac{d}{dx}(x + 4\log_e x) = 1 + 4\left(\frac{1}{x}\right) = 1 + \frac{4}{x}$

4d $\frac{d}{dx}(2x^4 + 1 + 3\log_e x) = 2(4x^3) + 3\left(\frac{1}{x}\right) = 8x^3 + \frac{3}{x}$

4e $\frac{d}{dx}(\ln(2x-1) + 3x^2) = \frac{2}{2x-1} + 3(2x) = \frac{2}{2x-1} + 6x$

4f $\frac{d}{dx}(x^3 - 3x + 4 + \ln(5x-7)) = 3x^2 - 3 + \frac{5}{5x-7}$

5a $y = \ln x^3 = 3\ln x$

$\frac{dy}{dx} = 3\left(\frac{1}{x}\right) = \frac{3}{x}$

5b $y = \ln x^2 = 2\ln x$

$\frac{dy}{dx} = 2\left(\frac{1}{x}\right) = \frac{2}{x}$

5c $y = \ln x^{-3} = -3\ln x$

$\frac{dy}{dx} = -3\left(\frac{1}{x}\right) = -\frac{3}{x}$

5d $y = \ln x^{-2} = -2\ln x$

$\frac{dy}{dx} = -2\left(\frac{1}{x}\right) = \frac{-2}{x}$

5e $y = \ln\sqrt{x}$

$= \ln x^{\frac{1}{2}}$

$= \frac{1}{2}\ln x$

$\frac{dy}{dx} = \frac{1}{2}\left(\frac{1}{x}\right) = \frac{1}{2x}$

5f $y = \ln\sqrt{x+1}$

$= \ln(x+1)^{\frac{1}{2}}$

$= \frac{1}{2}\ln(x+1)$

$\frac{dy}{dx} = \frac{1}{2}\left(\frac{1}{x+1}\right) = \frac{1}{2(x+1)}$

6a $\frac{dy}{dx} = \frac{\frac{1}{2}}{\frac{1}{2}x} = \frac{1}{x}$

6b $\frac{dy}{dx} = \frac{\frac{1}{3}}{\frac{1}{3}x} = \frac{1}{x}$

6c $\frac{dy}{dx} = 3\left(\frac{\frac{1}{5}}{\frac{1}{5}x}\right) = \frac{3}{x}$

6d $\frac{dy}{dx} = -6\left(\frac{\frac{1}{2}}{\frac{1}{2}x}\right) = -\frac{6}{x}$

6e $\frac{dy}{dx} = 1 + \frac{\frac{1}{7}}{\frac{1}{7}x} = 1 + \frac{1}{x}$

6f $\frac{dy}{dx} = 4(3x^2) - \frac{\frac{1}{5}}{\frac{1}{5}x} = 12x^2 - \frac{1}{x}$

7a Let $u = x^2 + 1$

Then $y = \ln u$

Hence $\frac{du}{dx} = 2x$ and $\frac{dy}{du} = \frac{1}{u}$

$\frac{dy}{dx} = 2x \times \frac{1}{x^2+1} = \frac{2x}{x^2+1}$

7b Let $u = 2 - x^2$

Then $y = \ln u$

Hence $\frac{du}{dx} = -2x$ and $\frac{dy}{du} = \frac{1}{u}$

$\frac{dy}{dx} = -2x \times \frac{1}{2-x^2} = -\frac{2x}{2-x^2}$

7c Let $u = 1 + e^x$

Then $y = \ln u$

Hence $\frac{du}{dx} = e^x$ and $\frac{dy}{du} = \frac{1}{u}$

$\frac{dy}{dx} = e^x \times \frac{1}{1+e^x} = \frac{e^x}{1+e^x}$

8a $\frac{d}{dx}\log_e(x^2 + 3x + 2) = \frac{2x+3}{x^2+3x+2}$

8b $\frac{d}{dx}\log_e(1 + 2x^3) = \frac{2(3x^2)}{1+2x^3} = \frac{6x^2}{1+2x^3}$

8c $\frac{d}{dx}\ln(e^x - 2) = \frac{e^x}{e^x-2}$

8d $\frac{d}{dx}(x + 3 - \ln(x^2 + x)) = 1 - \frac{2x+1}{x^2+x}$

8e $\frac{d}{dx}(x^2 + \ln(x^3 - x)) = 2x + \frac{3x^2-1}{x^3-x}$

8f $\frac{d}{dx}(4x^3 - 5x^2 + \ln(2x^2 - 3x + 1)) = 4(3x^2) - 5(2x) + \frac{2(2x)-3}{2x^2-3x+1}$

$$= 12x^2 - 10x + \frac{4x - 3}{2x^2 - 3x + 1}$$

9a $\frac{dy}{dx} = \frac{1}{x}$

At $x = 1$, gradient $= \frac{1}{1} = 1$

Angle of inclination $= \tan^{-1} 1 = 45^\circ$

9b At $x = 3$, gradient $= \frac{1}{3}$

Angle of inclination $= \tan^{-1}\frac{1}{3} \doteqdot 18^\circ\, 26'$

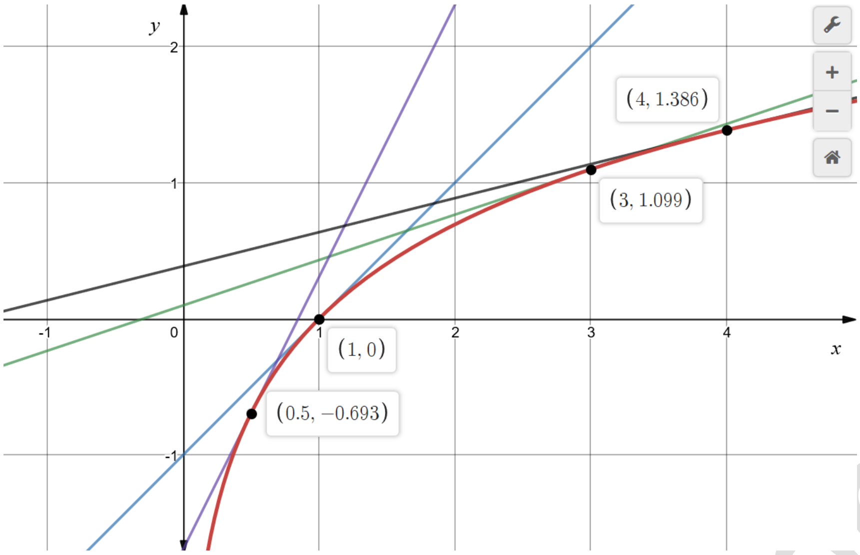
9c At $x = \frac{1}{2}$, gradient $= \frac{1}{\frac{1}{2}} = 2$

Angle of inclination $= \tan^{-1} 2 \doteqdot 63^\circ\, 26'$

9d At $x = 4$, gradient $= \frac{1}{4}$

Angle of inclination $= \tan^{-1}\frac{1}{4} \doteqdot 14^\circ\, 2'$

The graph of $y = \ln x$ with the four tangents at $x = 1, 3, \frac{1}{2}$ and 4 is shown below.



10a Let $u = x$ and $v = \log_e x$

Then $\frac{du}{dx} = 1$ and $\frac{dv}{dx} = \frac{1}{x}$

$\frac{dy}{dx} = (\log_e x \times 1) + \left(x \times \frac{1}{x}\right) = \log_e x + 1$

10b Let $u = x$ and $v = \log_e(2x + 1)$

Then $\frac{du}{dx} = 1$ and $\frac{dv}{dx} = \frac{2}{2x+1}$

$\frac{dy}{dx} = (\log_e(2x + 1) \times 1) + \left(x \times \frac{2}{2x+1}\right) = \log_e(2x + 1) + \frac{2x}{2x+1}$

10c Let $u = 2x + 1$ and $v = \log_e x$

Then $\frac{du}{dx} = 2$ and $\frac{dv}{dx} = \frac{1}{x}$

$\frac{dy}{dx} = (\log_e x \times 2) + \left((2x + 1) \times \frac{1}{x}\right) = 2\log_e x + 2 + \frac{1}{x}$

10d Let $u = x^4$ and $v = \log_e x$

Then $\frac{du}{dx} = 4x^3$ and $\frac{dv}{dx} = \frac{1}{x}$

$\frac{dy}{dx} = (\log_e x \times 4x^3) + \left(x^4 \times \frac{1}{x}\right)$

$= 4x^3 \log_e x + x^3$

$= x^3(4\log_e x + 1)$

10e Let $u = x + 3$ and $v = \log_e(x+3)$

Then $\frac{du}{dx} = 1$ and $\frac{dv}{dx} = \frac{1}{x+3}$

$\frac{dy}{dx} = (\log_e(x+3) \times 1) + \left((x+3) \times \frac{1}{x+3}\right)$

$= \log_e(x+3) + 1$

10f Let $u = x - 1$ and $v = \log_e(2x+7)$

Then $\frac{du}{dx} = 1$ and $\frac{dv}{dx} = \frac{2}{2x+7}$

$\frac{dy}{dx} = (\log_e(2x+7) \times 1) + \left((x-1) \times \frac{2}{2x+7}\right)$

$= \log_e(2x+7) + \frac{2(x-1)}{2x+7}$

10g Let $u = e^x$ and $v = \log_e x$

Then $\frac{du}{dx} = e^x$ and $\frac{dv}{dx} = \frac{1}{x}$

$\frac{dy}{dx} = (\log_e x \times e^x) + \left(e^x \times \frac{1}{x}\right)$

$= e^x\left(\log_e x + \frac{1}{x}\right)$

10h Let $u = e^{-x}$ and $v = \log_e x$

Then $\frac{du}{dx} = -e^{-x}$ and $\frac{dv}{dx} = \frac{1}{x}$

$\frac{dy}{dx} = (\log_e x \times -e^{-x}) + \left(e^{-x} \times \frac{1}{x}\right)$

$= e^{-x}\left(\frac{1}{x} - \log_e x\right)$

11a Let $u = \log_e x$ and $v = x$

Then $\frac{du}{dx} = \frac{1}{x}$ and $\frac{dv}{dx} = 1$

$$\frac{dy}{dx} = \frac{\left(x \times \frac{1}{x}\right) - (\log_e x \times 1)}{x^2}$$

$$= \frac{1 - \log_e x}{x^2}$$

11b Let $u = \log_e x$ and $v = x^2$

Then $\frac{du}{dx} = \frac{1}{x}$ and $\frac{dv}{dx} = 2x$

$$\frac{dy}{dx} = \frac{\left(x^2 \times \frac{1}{x}\right) - (\log_e x \times 2x)}{(x^2)^2}$$

$$= \frac{x - 2x \log_e x}{x^4}$$

$$= \frac{1 - 2\log_e x}{x^3}$$

11c Let $u = x$ and $v = \log_e x$

Then $\frac{du}{dx} = 1$ and $\frac{dv}{dx} = \frac{1}{x}$

$$\frac{dy}{dx} = \frac{(\log_e x \times 1) - \left(x \times \frac{1}{x}\right)}{(\log_e x)^2}$$

$$= \frac{\log_e x - 1}{(\log_e x)^2}$$

11d Let $u = x^2$ and $v = \log_e x$

Then $\frac{du}{dx} = 2x$ and $\frac{dv}{dx} = \frac{1}{x}$

$$\frac{dy}{dx} = \frac{(\log_e x \times 2x) - \left(x^2 \times \frac{1}{x}\right)}{(\log_e x)^2}$$

$$= \frac{2x \log_e x - x}{(\log_e x)^2}$$

$$= \frac{x(2\log_e x - 1)}{(\log_e x)^2}$$

11e Let $u = \log_e x$ and $v = e^x$

Then $\frac{du}{dx} = \frac{1}{x}$ and $\frac{dv}{dx} = e^x$

$$\frac{dy}{dx} = \frac{\left(e^x \times \frac{1}{x}\right) - (\log_e x \times e^x)}{(e^x)^2}$$

$$= \frac{e^x\left(\frac{1}{x} - \log_e x\right)}{(e^x)^2}$$

$$= \frac{\left(\frac{1}{x} - \log_e x\right)}{e^x}$$

$$= \frac{(1 - x\log_e x)}{xe^x}$$

11f Let $u = e^x$ and $v = \log_e x$

Then $\frac{du}{dx} = e^x$ and $\frac{dv}{dx} = \frac{1}{x}$

$$\frac{dy}{dx} = \frac{(\log_e x \times e^x) - \left(e^x \times \frac{1}{x}\right)}{(\log_e x)^2}$$

$$= \frac{e^x\left(\log_e x - \frac{1}{x}\right)}{(\log_e x)^2}$$

$$= \frac{e^x(x\log_e x - 1)}{x(\log_e x)^2}$$

12a $\frac{dy}{dx} = \frac{d}{dx}(\log_e 5 + \log_e x^3)$

$$= \frac{d}{dx}(\log_e 5 + 3\log_e x)$$

$$= \frac{3}{x}$$

12b $\frac{dy}{dx} = \frac{d}{dx}\left(\log_e x^{\frac{1}{3}}\right)$

$$= \frac{d}{dx}\left(\frac{1}{3}\log_e x\right)$$

$$= \frac{1}{3x}$$

12c $\frac{dy}{dx} = \frac{d}{dx}(\log_e 3 - \log_e x)$

$= -\frac{1}{x}$

12d $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{2}\ln(2-x)\right)$

$= \frac{1}{2}\left(-\frac{1}{2-x}\right)$

$= \frac{1}{2x-4}$

12e $\frac{dy}{dx} = \frac{d}{dx}(\log_e 3 - \log_e x)$

$= -\frac{1}{x}$

12f $y = \ln\frac{1+x}{1-x}$

$\ln\frac{1+x}{1-x} = \ln(1+x) - \ln(1-x)$

So $\frac{dy}{dx} = \frac{1}{1+x} + \frac{1}{1-x}$.

12g $y = \log_e 2^x$

$y = x\log_e 2$

So $y' = \log_e 2$.

12h $y = \log_e e^x$

$y = x$

So $y' = 1$.

12i $y = \log_e x^x$

$y = x \log_e x$

Applying the product rule on $y = x \log_e x$:

Let $u = x$ and $v = \log_e x$.

Then $u' = 1$ and $v' = \frac{1}{x}$.

$$\begin{aligned} y' &= vu' + uv' \\ &= (\log_e x)(1) + (x)\left(\frac{1}{x}\right) \\ &= 1 + \log_e x \end{aligned}$$

So $y' = 1 + \log_e x$.

13a $f'(x) = \frac{1}{x-1}$

$f''(x) = -\frac{1}{(x-1)^2}$

$f'(3) = \frac{1}{3-1} = \frac{1}{2}$

$f''(3) = -\frac{1}{(3-1)^2} = -\frac{1}{4}$

13b $f'(x) = \frac{2}{2x+1}$

Let $u = 2$ and $v = 2x + 1$

Then $\frac{du}{dx} = 0$ and $\frac{dv}{dx} = 2$

$$\begin{aligned} f''(x) &= \frac{((2x+1) \times 0) - (2 \times 2)}{(2x+1)^2} \\ &= -\frac{4}{(2x+1)^2} \end{aligned}$$

$f'(0) = \frac{2}{1} = 2$

$f''(3) = -\frac{4}{(0+1)^2} = -4$

13c $f'(x) = \frac{d}{dx} 2\log x$

$= \frac{2}{x}$

$f''(x) = -\frac{2}{x^2}$

$f'(2) = \frac{2}{2} = 1$

$f''(2) = -\frac{2}{2^2} = -\frac{1}{2}$

13d Let $u = x$ and $v = \log x$

Then $\frac{du}{dx} = 1$ and $\frac{dv}{dx} = \frac{1}{x}$

$f'(x) = (\log x \times 1) + \left(x \times \frac{1}{x}\right) = \log x + 1$

$f''(x) = \frac{1}{x}$

$f'(e) = \log e + 1$

$= 1 + 1$

$= 2$

$f''(e) = \frac{1}{e}$

14a Let $u = x$ and $v = \log_e x$

Then $\frac{du}{dx} = 1$ and $\frac{dv}{dx} = \frac{1}{x}$

$\frac{dy}{dx} = \left((\log_e x \times 1) + \left(x \times \frac{1}{x}\right)\right) - 1$

$= \log_e x + 1 - 1$

$= \log_e x$

$\log_e x = 0$

$x = 1$

14b Let $u = x^2$ and $v = \log_e x$

Then $\frac{du}{dx} = 2x$ and $\frac{dv}{dx} = \frac{1}{x}$

$\frac{dy}{dx} = (\log_e x \times 2x) + \left(x^2 \times \frac{1}{x}\right)$

$= 2x \log_e x + x$

$= x(2 \log_e x + 1)$

$x(2 \log_e x + 1) = 0$

$x = 0$ which is not a valid solution as $\log_e 0$ is undefined.

$2 \log_e x + 1 = 0$

$\log_e x = -\frac{1}{2}$

$x = e^{-\frac{1}{2}}$

14c Let $u = \log_e x$ and $v = x$

Then $\frac{du}{dx} = \frac{1}{x}$ and $\frac{dv}{dx} = 1$

$\frac{dy}{dx} = \frac{\left(x \times \frac{1}{x}\right) - (\log_e x \times 1)}{x^2}$

$= \frac{1 - \log_e x}{x^2}$

$\frac{1 - \log_e x}{x^2} = 0$

$1 - \log_e x = 0$

$\log_e x = 1$

$x = e$

14d Let $u = \log_e x$

Then $y = u^4$

Hence $\frac{du}{dx} = \frac{1}{x}$ and $\frac{dy}{du} = 4u^3$

$\frac{dy}{dx} = \frac{1}{x} \times 4(\log_e x)^3 = \frac{4(\log_e x)^3}{x}$

$\frac{4(\log_e x)^3}{x} = 0$

$4(\log_e x)^3 = 0$

$\log_e x = 0$

$x = 1$

14e Let $u = 2\log_e x - 3$

Then $y = u^4$

Hence $\frac{du}{dx} = \frac{2}{x}$ and $\frac{dy}{du} = 4u^3$

$\frac{dy}{dx} = \frac{2}{x} \times 4(2\log_e x - 3)^3 = \frac{8(2\log_e x - 3)^3}{x}$

$\frac{8(2\log_e x - 3)^3}{x} = 0$

$8(2\log_e x - 3)^3 = 0$

$2\log_e x - 3 = 0$

$\log_e x = \frac{3}{2}$

$x = e^{\frac{3}{2}}$

14f Let $u = \log_e x$

Then $y = \frac{1}{u}$

Hence $\frac{du}{dx} = \frac{1}{x}$ and $\frac{dy}{du} = -\frac{1}{u^2}$

$\frac{dy}{dx} = \frac{1}{x} \times -\frac{1}{(\log_e x)^2} = -\frac{1}{x(\log_e x)^2}$, which is never zero.

14g Let $u = \log_e x$

Then $y = \log_e u$

Hence $\frac{du}{dx} = \frac{1}{x}$ and $\frac{dy}{du} = \frac{1}{u}$

$\frac{dy}{dx} = \frac{1}{x} \times \frac{1}{\log_e x} = \frac{1}{x\log_e x}$, which is never zero.

14h Let $u = x$ and $v = \ln x$

Then $\frac{du}{dx} = 1$ and $\frac{dv}{dx} = \frac{1}{x}$

$\frac{dy}{dx} = (\ln x \times 1) + \left(x \times \frac{1}{x}\right) = \ln x + 1$

$\ln x + 1 = 0$

$\ln x = -1$

$x = e^{-1}$

14i $\frac{dy}{dx} = -\frac{1}{x^2} + \frac{1}{x}$

$\frac{1}{x} - \frac{1}{x^2} = 0$

$\frac{1}{x} = \frac{1}{x^2}$

$1 = \frac{1}{x}$

$x = 1$

15a Need to show that $y = \frac{x}{\ln x}$ is a solution of $\frac{dy}{dx} = \left(\frac{y}{x}\right) - \left(\frac{y}{x}\right)^2$

$y = \frac{x}{\ln x}$

Applying the quotient rule on $y = \frac{x}{\ln x}$:

Let $u = x$ and $v = \ln x$.

Then $u' = 1$ and $v' = \frac{1}{x}$.

$$y' = \frac{vu' - uv'}{v^2}$$

$$= \frac{\ln x - (x)\left(\frac{1}{x}\right)}{(\ln x)^2}$$

$$= \frac{\ln x - 1}{(\ln x)^2}$$

So $y' = \dfrac{\ln x - 1}{(\ln x)^2}$.

LHS: $y' = \dfrac{\ln x - 1}{(\ln x)^2}$

RHS: $\left(\dfrac{y}{x}\right) - \left(\dfrac{y}{x}\right)^2 = \dfrac{\frac{x}{\ln x}}{x} - \left(\dfrac{\frac{x}{\ln x}}{x}\right)^2$

$= \dfrac{1}{\ln x} - \dfrac{1}{(\ln x)^2}$

$= \dfrac{\ln x - 1}{(\ln x)^2}$

$= \text{LHS}$

LHS = RHS and so $y = \dfrac{x}{\ln x}$ is a solution of $\dfrac{dy}{dx} = \left(\dfrac{y}{x}\right) - \left(\dfrac{y}{x}\right)^2$

15b Let $u = \log_e x$

Then $y = \log_e u$

Hence $\frac{du}{dx} = \frac{1}{x}$ and $\frac{dy}{du} = \frac{1}{u}$

$\frac{dy}{dx} = \frac{1}{x} \times \frac{1}{\log_e x} = \frac{1}{x \log_e x}$

Let $u = \frac{1}{\log_e x}$ and $v = x$

Then $\frac{du}{dx} = -\frac{1}{x(\log_e x)^2}$ and $\frac{dv}{dx} = 1$

$\dfrac{d^2y}{dx^2} = \dfrac{\left(x \times -\frac{1}{x(\log_e x)^2}\right) - \left(\frac{1}{\log_e x} \times 1\right)}{x^2}$

$= \dfrac{-\frac{1}{(\log_e x)^2} - \frac{1}{\log_e x}}{x^2}$

$= \dfrac{-\log_e x - 1}{x^2(\log_e x)^2}$

$\text{LHS} = x\left(\dfrac{-\log_e x - 1}{x^2(\log_e x)^2}\right) + x\left(\dfrac{1}{x \log_e x}\right)^2 + \dfrac{1}{x \log_e x}$

$= \dfrac{-\log_e x - 1}{x(\log_e x)^2} + x\left(\dfrac{1}{x^2(\log_e x)\text{^}2}\right) + \left(\dfrac{1}{x \log_e x} \times \dfrac{\log_e x}{\log_e x}\right)$

$= \frac{-\log_e x - 1}{x(\log_e x)^2} + \frac{1}{x(\log_e x)\text{^}2} + \frac{\log_e x}{x(\log_e x)^2}$

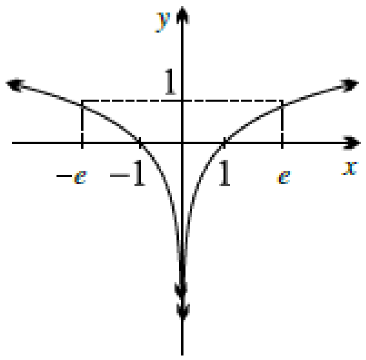
$= \frac{-\log_e x - 1 + 1 + \log_e x}{x(\log_e x)^2}$

$= 0$

$= \text{RHS}$

16a $\log_e |x| = \begin{cases} \log_e x, & \text{for } x > 0 \\ \log_e(-x), & \text{for } x < 0 \end{cases}$

16b



16c For $x > 0$, $\log_e |x| = \log_e x$.

So $\frac{d}{dx}\log_e x = \frac{1}{x}$.

For $x < 0$, $\log_e |x| = \log_e(-x)$.

Using the standard form, $\frac{d}{dx}\log_e(-x) = -\frac{1}{-x} = \frac{1}{x}$.

16d $x = 0$ was excluded in this discussion because $\log_e 0$ is undefined.

In fact, $\log_e x \to -\infty$ as $x \to 0$, so $x = 0$ is an asymptote.

17a By differentiation with first principles, we have:

⇨ $y' = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

⇨ $y' = \lim_{h \to 0} \frac{\log(x+h) - logx}{h}$

⇨ $y' = \lim_{h \to 0} \frac{\log\frac{(x+h)}{x}}{h}$

⇨ $y' = \lim_{h \to 0} \frac{\log\left(1+\frac{h}{x}\right)}{h}$

⇨ $y' = \lim_{h \to 0} \log\left(1 + \frac{h}{x}\right)^{\frac{1}{h}}$

Hence proved.

17b Solving the limits equation obtained in 17a,

⇨ $y' = \lim_{h \to 0} \frac{\log\left(1+\frac{h}{x}\right)}{x \times \frac{h}{x}}$

⇨ $y' = \frac{1}{x} \times \lim_{h \to 0} \frac{\log\left(1+\frac{h}{x}\right)}{\frac{h}{x}}$

⇨ Now, we have a property of limits which states that $\lim_{k \to 0} \frac{\log(1+k)}{k} = 1$

⇨ Hence, $y' = \frac{1}{x}$

17c i $y' = \log \lim_{h \to 0} (1 + \frac{h}{x})^{\frac{1}{h}} = \frac{1}{x}$

Now we are substituting h with $\frac{1}{n}$ and $\frac{1}{x}$ with u. Hence, as h tends to 0, n will tend to infinity.

Hence,

⇨ $u = \log \lim_{n \to \infty} (1 + \frac{u}{n})^n$

⇨ $e^u = \lim_{n \to \infty} (1 + \frac{u}{n})^n$

17c ii Substituting 1 for u in the equation obtained in 17c i, we get the desired result.

17d i When $n = 1$, $\left(1 + \frac{1}{n}\right)^n = 2$

17d ii When $n = 10, \left(1 + \frac{1}{n}\right)^n \doteqdot 2.5937$

17d iii When $n = 100, \left(1 + \frac{1}{n}\right)^n \doteqdot 2.7048$

17d iv When $n = 1000, \left(1 + \frac{1}{n}\right)^n \doteqdot 2.7169$

17d v When $n = 10\,000, \left(1 + \frac{1}{n}\right)^n \doteqdot 2.7181$

Solutions to Exercise 6H

1a

$$\frac{dy}{dx} = \frac{d}{dx}(\log_e x)$$
$$= \frac{1}{x}$$

The gradient of the tangent at point $P(e, 1)$ is the derivative of the function $y = \log_e x$

Since $\frac{dy}{dx} = \frac{d}{dx}(\log_e x) = \frac{1}{x}$, the gradient of the tangent at point $P(e, 1)$ is $\frac{1}{e}$

As the gradient of the tangent is constant, it is a straight line. By employing the formula for a straight line: $y = mx + c, m = \frac{1}{e}$

$$y = \frac{1}{e}x + c$$

Since this line passes through point $P(e, 1)$,

$$1 = \frac{1}{e}(e) + c$$

$$c = 1 - 1 = 0$$

The equation for the gradient is $y = \frac{1}{e}x$

At $= 0, y = \frac{1}{e}(0) = 0$. Therefore the tangent passes through the origin (0,0)

1b The gradient of the tangent at point $Q(1,0)$ is the derivative of the function $y = \log_e x$

Since $\frac{dy}{dx} = \frac{d}{dx}(\log_e x) = \frac{1}{x}$, the gradient of the tangent at point $Q(1,0)$ is $\frac{1}{1} = 1$

As the gradient of the tangent is constant, it is a straight line. By employing the formula for a straight line: $y = mx + c, m = 1$

$$y = x + c$$

Since this line passes through point $Q(1,0)$,

$$0 = 1(1) + c$$

$$c = 0 - 1 = -1$$

The equation for the gradient is $y = x - 1$

At $x = 0, y = (0) - 1 = -1$. Therefore the tangent passes through the point $A(0, -1)$

1c The gradient of the tangent at point $R(\frac{1}{e}, -1)$ is the derivative of the function $y = \log_e x$

Since $\frac{dy}{dx} = \frac{d}{dx}(\log_e x) = \frac{1}{x}$, the gradient of the tangent at point $R(\frac{1}{e}, -1)$ is $\frac{1}{\frac{1}{e}} = e$

As the gradient of the tangent is constant, it is a straight line. By employing the formula for a straight line: $y = mx + c, m = e$

$y = ex + c$

Since this line passes through point $R(\frac{1}{e}, -1)$,

$-1 = e\left(\frac{1}{e}\right) + c$

$c = -1 - 1 = -2$

The equation for the gradient is $y = ex - 2$

At $x = 0, y = e(0) - 2 = -2$. Therefore the tangent passes through the point $B(0, -2)$

1d The gradient of the tangent at point $A(1,0)$ is the derivative of the function $y = \log_e x$

Since $\frac{dy}{dx} = \frac{d}{dx}(\log_e x) = \frac{1}{x}$, the gradient of the tangent at point $A(1,0)$ is $\frac{1}{1} = 1$

As the gradient of the tangent is 1, the gradient of the normal is $-\frac{1}{m_{tangent}} = -1$

The equation of the normal is

$y = mx + c, m = -1$

and passes through the point $A(1, 0)$.

$\therefore 0 = -1(1) + c$

$c = 1$

The equation of the normal is, therefore, $y = -x + 1$

When $x = 0, y = -(0) + 1 = 1$

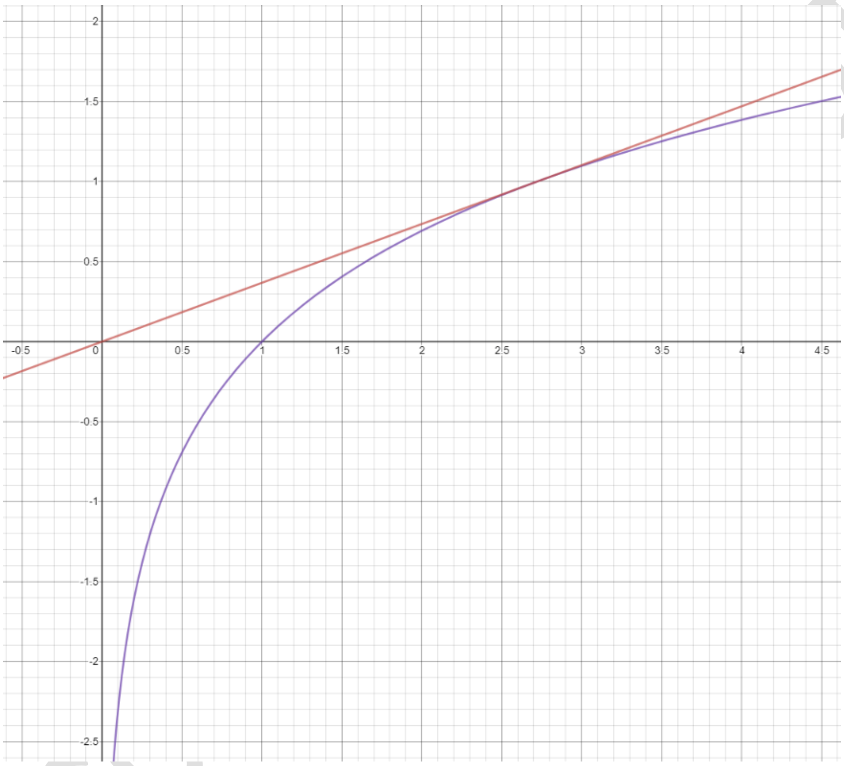
The y-intercept is 1.

2a The diagram shows the tangent $y = \frac{1}{e}x$ and the graph $y = \log_e x$

Let $x = a$ for some $0 < a < e$. As a traverses from 0 to e, the tangent becomes less steep, and as the tangents are straight lines, they will intersect the y-axis at various points where $y < 0$.

Let $x = b$ for some $e < b < \infty$. As b increases from e, the tangent becomes less steep, and as the tangents are straight lines, they will intersect the y-axis at various points where $y > 0$.

Therefore, only the tangent at $(e, 1)$ passes through the origin.



2b

Observe the points below the curve. Because the curve is convex everywhere, there are no tangents that can possibly intersect any point that lies below the curve.

Observe the points above the curve, located within the domain $x > 0$. Any point above the curve in this domain can be intersected by two tangents.

Any point located outside of the domain, that is, $x \leq 0$, is intersected by one tangent.

Any point located on the curve is intersected by only one tangent.

3a $y = 4\log_e x$

The gradient of the tangent at any point is $\frac{d}{dx}(4\log_e x) = \frac{4}{x}$

The gradient of the tangent at $Q(1,0)$ is $\frac{4}{1} = 4$

The equation of the tangent is $y = mx + c, m = 4$

At $Q(1,0)$, $0 = 4(1) + c$

$c = -4$

The equation of the tangent is therefore $y = 4x - 4$

The gradient of the normal to the tangent is $-\frac{1}{m_{tangent}} = -\frac{1}{4}$

The equation of the normal is $y = mx + c, m = -\frac{1}{4}$

At $Q(1,0)$, $0 = -\frac{1}{4}(1) + c$

$$c = \frac{1}{4}$$

The equation of the normal is therefore $y = -\frac{1}{4}x + \frac{1}{4}$.

3b $y = \log_e x + 3$

The gradient of the tangent at any point is $\frac{d}{dx}(\log_e x + 3) = \frac{1}{x}$

The gradient of the tangent at $R(1,3)$ is $\frac{1}{1} = 1$

The equation of the tangent is $y = mx + c, m = 1$

At $R(1,3)$, $3 = 1(1) + c$

$c = 2$

The equation of the tangent is therefore $y = x + 2$.

The gradient of the normal to the tangent is $-\frac{1}{m_{tangent}} = -1$

The equation of the normal is $y = mx + c, m = -1$

At $R(1,3)$, $3 = -1(1) + c$

$c = 4$

The equation of the normal is therefore $y = -x + 4$.

3c $y = 2\log_e x - 2$

The gradient of the tangent at any point is $\frac{d}{dx}(2\log_e x - 2) = \frac{2}{x}$

The gradient of the tangent at $S(1, -2)$ is $\frac{2}{1} = 2$

The equation of the tangent is $y = mx + c, m = 2$

At $S(1, -2)$, $-2 = 2(1) + c$

$c = -4$

The equation of the tangent is therefore $y = 2x - 4$

The gradient of the normal to the tangent is $-\frac{1}{m_{tangent}} = -\frac{1}{2}$

The equation of the normal is $y = mx + c, m = -\frac{1}{2}$

At $S(1, -2)$, $-2 = -\frac{1}{2}(1) + c$

$c = -1\frac{1}{2}$

The equation of the normal is therefore $y = -\frac{1}{2}x - 1\frac{1}{2}$.

3d $y = 1 - 3\log_e x$

The gradient of the tangent at any point is $\frac{d}{dx}(1 - 3\log_e x) = -\frac{3}{x}$

The gradient of the tangent at $T(1,1)$ is $-\frac{3}{1} = -3$

The equation of the tangent is $y = mx + c, m = -3$

At $T(1,1)$, $1 = -3(1) + c$

$c = 4$

The equation of the tangent is therefore $y = -3x + 4$

The gradient of the normal to the tangent is $-\frac{1}{m_{\text{tangent}}} = \frac{1}{3}$

The equation of the normal is $y = mx + c, m = \frac{1}{3}$

At $T(1,1)$, $1 = \frac{1}{3}(1) + c$

$c = \frac{2}{3}$

The equation of the normal is therefore $y = \frac{1}{3}x + \frac{2}{3}$.

4a Let $x = 1$

$y = \log_e(3(1) - 2) = 0$

Therefore the point $P(1,0)$ lies on the curve.

4b The gradient of the tangent at any point is $\frac{d}{dx}(\log_e(3x - 2))$

By chain rule, $\frac{d}{dx}(\log_e(3x - 2)) = \frac{1}{3x-2}\frac{d}{dx}(3x - 2) = \frac{3}{3x-2}$

At $P(1,0)$, $x = 1$

Therefore the gradient of the tangent at $P(1,0)$ is $\frac{3}{3(1)-2} = 3$

The gradient of the normal at $P(1,0)$ is $-\frac{1}{m_{\text{tangent}}} = -\frac{1}{3}$.

Let the equation of the tangent be $y = mx + c, m = 3$

At $P(1,0)$, $0 = 3(1) + c$

$c = -3$

The equation of the tangent is therefore $y = 3x - 3$

When $x = 0$, $y = 3(0) - 3 = -3$

The y-intercept is -3

Let the equation of the normal be $y = mx + c, m = -\frac{1}{3}$

At $P(1,0)$, $0 = -\frac{1}{3}(1) + c$

$c = \frac{1}{3}$

The equation of the normal is therefore $y = -\frac{1}{3}x + \frac{1}{3}$

When $x = 0$, $y = -\frac{1}{3}x + \frac{1}{3} = \frac{1}{3}$

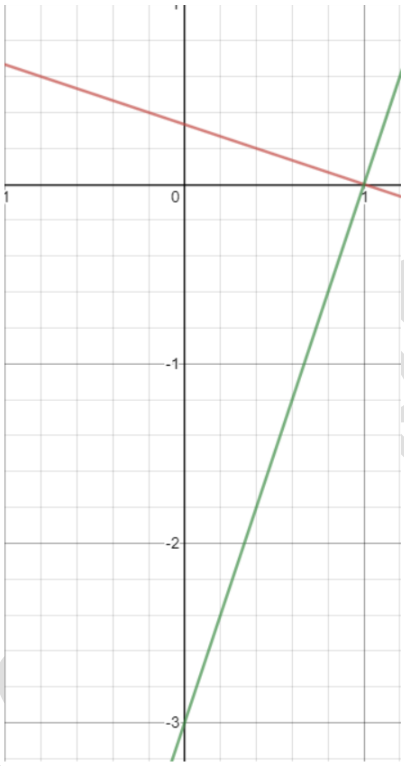
The y-intercept is $\frac{1}{3}$.

4c A triangle is created with the points $(0,-3)$, $\left(0,\frac{1}{3}\right)$, and $(1,0)$

The base of the triangle is therefore $\left(\frac{1}{3}-(-3)\right)=3\frac{1}{3}$

The altitude of the triangle is therefore 1

The area of the triangle is therefore $\frac{1}{2}\left(3\frac{1}{3}\right)(1)=\frac{5}{3}$ square units



5a $y=\log_e x, y'=\frac{1}{x}$. y' describes the gradient of any tangent to y.

$\therefore$ if $y'=\frac{1}{2}, x=2$, and $y=\log_e 2$. The tangent of gradient $\frac{1}{2}$ occurs at point $(2,\log_e 2)$.

Let $y=mx+c$ be the equation of the tangent, $m=\frac{1}{2}$

At $(2,\log_e 2)$, $\log_e 2=\frac{1}{2}(2)+c$

$c=\log_e 2-1$

The equation of the tangent is therefore $y=\frac{1}{2}x+\log_e 2-1$

The gradient of the normal to the tangent at $(2,\log_e 2)$ is $-\frac{1}{m_{\text{tangent}}}=-2$

Let $y = mx + c$ be the equation of the normal, $m = -2$

At $(2, \log_e 2)$, $\log_e 2 = -2(2) + c$

$c = \log_e 2 + 4$

The equation of the normal is therefore $y = -2x + \log_e 2 + 4$.

5b $y = \log_e x$, $y' = \frac{1}{x}$. y' describes the gradient of any tangent to y.

$\therefore$ if $y' = 2$, $x = \frac{1}{2}$, and $y = \log_e \frac{1}{2} = -\log_e 2$. The tangent of gradient 2 occurs at point $\left(\frac{1}{2}, -\log_e 2\right)$.

Let $y = mx + c$ be the equation of the tangent, $m = 2$

At $\left(\frac{1}{2}, -\log_e 2\right)$, $-\log_e 2 = 2\left(\frac{1}{2}\right) + c$

$c = -\log_e 2 - 1$

The equation of the tangent is therefore $y = 2x - \log_e 2 - 1$

The gradient of the normal to the tangent at $\left(\frac{1}{2}, -\log_e 2\right)$ is $-\frac{1}{m_{tangent}} = -\frac{1}{2}$

Let $y = mx + c$ be the equation of the normal, $m = -\frac{1}{2}$

At $\left(\frac{1}{2}, -\log_e 2\right)$, $-\log_e 2 = -\frac{1}{2}\left(\frac{1}{2}\right) + c$

$$c = -\log_e 2 + \frac{1}{4}$$

The equation of the normal is therefore $y = -\frac{1}{2}x - \log_e 2 + \frac{1}{4}$.

6a The domain is $x > 0$. As the domain is not symmetric about the y-axis, the function is neither odd or even.

6b $y' = \frac{d}{dx}(x - \log_e x) = 1 - \frac{1}{x}$

$$y'' = \frac{d}{dx}(y') = \frac{1}{x^2}$$

6c As the domain is $x > 0$, $\frac{1}{x^2}$ is necessarily positive. As y'' is always positive, the function is concave up for all values of x in its domain.

6d The minimum turning point is when $y' = 0$

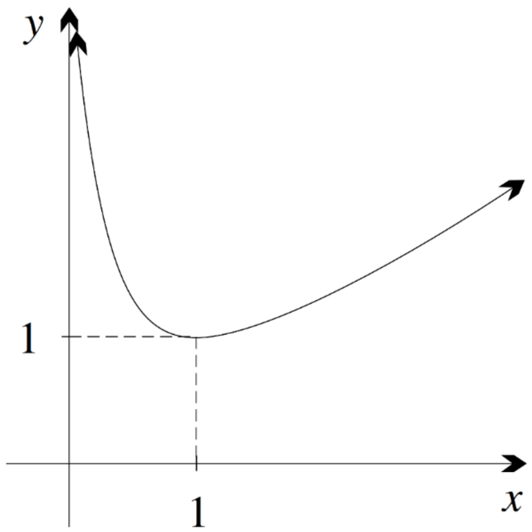
$$1 - \frac{1}{x} = 0$$

$$x = 1$$

$$y = (1) - \log_e(1) = 1$$

The minimum turning point is therefore at $(1, 1)$.

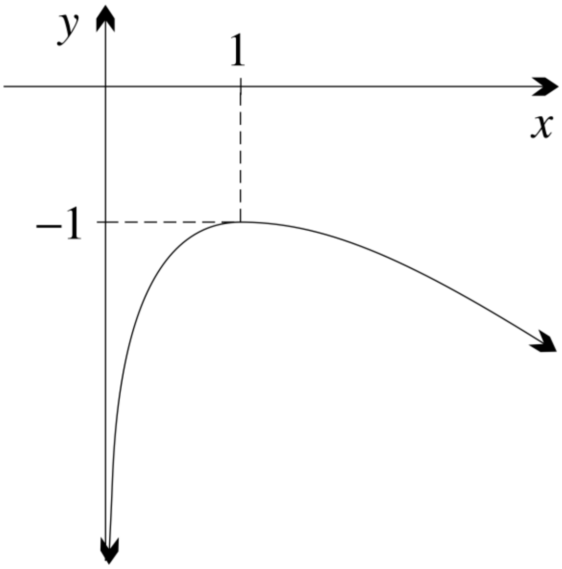
6e



The range for $y = x - \log_e x$ is $y \geq 1$

6f Let $f(y) = x - \log_e x$ and $g(y) = \log_e x - x$

Since $f(y) = -g(y)$, the transformation is a reflection about the x-axis.



7a The domain of $y = \frac{1}{x} + \ln x$ is $x > 0$

7b $y' = \frac{d}{dx}\left(\frac{1}{x} + \ln x\right) = -\frac{1}{x^2} + \frac{1}{x} = \frac{-1+x}{x^2} = \frac{x-1}{x^2}$

$y'' = \frac{d}{dx}(y') = \frac{d}{dx}\left(\frac{x-1}{x^2}\right) = \frac{d}{dx}(x-1) \times \frac{1}{x^2} + \frac{d}{dx}\left(\frac{1}{x^2}\right) \times (x-1)$ by the product rule.

$$y'' = \frac{1}{x^2} + \left(-\frac{2}{x^3}\right)(x-1) = \frac{x+2-2x}{x^3} = \frac{2-x}{x^3}$$

7c Minimum is located at $y' = 0$,

$$\frac{x-1}{x^2} = 0$$

$$x = 1$$

$$y = \frac{1}{1} + \ln 1 = 1$$

Therefore the minimum is at $(1,1)$

Inflection is located at $y'' = 0$,

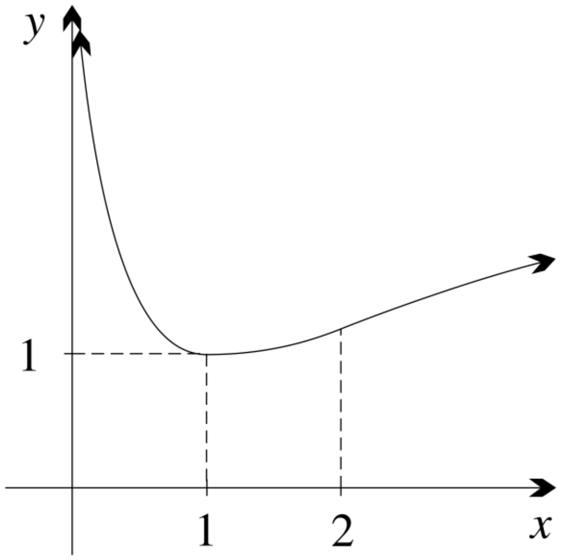
$$\frac{2-x}{x^3} = 0$$

$$x = 2$$

$$y = \frac{1}{2} + \ln 2$$

Therefore the inflection point is located at $\left(2, \frac{1}{2} + \ln 2\right)$.

7d The range is $y \geq 1$.



8a $y = x\log_e x - x$

The domain is $x > 0$.

The x-intercept occurs when $y = 0$.

$x(\log_e x - 1) = 0$

$\log_e x - 1 = 0 \Rightarrow x = e$

So the x-intercept is $(e, 0)$.

8b

x	1	e	e^2
y	-1	0	e^2
sign	–	0	+

8c $y = x\log_e x - x$

Applying the product rule on $y = x\log_e x$:

Let $u = x$ and $v = \log_e x$.

Then $u' = 1$ and $v' = \dfrac{1}{x}$.

$$\begin{aligned}\frac{d}{dx}(uv) &= vu' + uv'\\ &= (\log_e x)(1) + (x)\left(\frac{1}{x}\right)\\ &= 1 + \log_e x\end{aligned}$$

$$\begin{aligned}y' &= 1 + \log_e x - \frac{d}{dx}(x)\\ &= 1 + \log_e x - 1\end{aligned}$$

So $y' = \log_e x$.

And so $y'' = \dfrac{1}{x}$.

8d $y' = \log_e x$

There are stationary points where $y' = 0$.

$\log_e x = 0 \Rightarrow x = 1$

So there is a stationary point at $x = 1$.

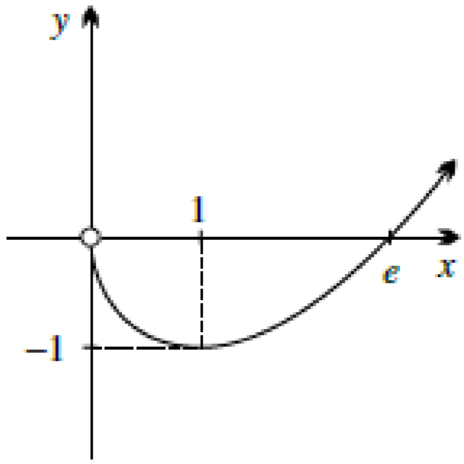
x	$\frac{1}{2}$	1	e
y'	–0.693...	0	1
slope	\	–	/

When $x=1$, $y=-1$.

Hence $(1,-1)$ is a minimum turning point.

8e $y''=\dfrac{1}{x}$ for $x>0$ tells us that the curve is concave up throughout its domain.

8f Given that $y\to 0^-$ as $x\to 0^+$ and the tangent approaches vertical as $x\to 0^+$.



The range is $y\geq -1$.

9a $y=\log_e\left(1+x^2\right)$

$1+x^2>0$ for all real values of x

So the domain is all real values of x.

9b Algebraically, a function $f(x)$ is even if $f(-x)=f(x)$ for all x in the domain.

Replacing x with $-x$ we obtain:

$y=\log_e\left(1+(-x)^2\right)=\log_e\left(1+x^2\right)$

So $y=\log_e\left(1+x^2\right)$ is an even function.

9c $\log_e\left(1+x^2\right)=0 \Rightarrow 1+x^2=1$

Hence the function is zero at $x=0$, and is positive otherwise because the logs of numbers greater than 1 are positive.

9d $y=\log_e\left(1+x^2\right)$

Applying the chain rule:

Let $u=1+x^2$ and so $y=\log_e u$.

Hence $\frac{du}{dx}=2x$ and $\frac{dy}{du}=\frac{1}{u}$.

$$\begin{aligned}\frac{dy}{dx}&=\frac{dy}{du}\times\frac{du}{dx}\\&=\frac{2x}{1+x^2}\end{aligned}$$

So $\frac{dy}{dx}=\frac{2x}{1+x^2}$.

Applying the quotient rule on $\frac{dy}{dx}=\frac{2x}{1+x^2}$:

Let $u=2x$ and $v=1+x^2$.

Then $u'=2$ and $v'=2x$.

$$\begin{aligned}y'' &= \frac{vu' - uv'}{v^2}\\ &= \frac{2\left(1+x^2\right)-(2x)(2x)}{\left(1+x^2\right)^2}\\ &= \frac{2+2x^2-4x^2}{\left(1+x^2\right)^2}\\ &= \frac{2-2x^2}{\left(1+x^2\right)^2}\end{aligned}$$

So $y'' = \dfrac{2\left(1-x^2\right)}{\left(1+x^2\right)^2}$.

9e There are stationary points where $y' = 0$.

$2x = 0 \Rightarrow x = 0$

So there is a stationary point at $x = 0$.

x	-1	0	1
y'	-1	0	1
slope	\	–	/

When $x = 0$, $y = 0$.

So $(0,0)$ is a minimum turning point.

9f $y'' = \dfrac{2\left(1-x^2\right)}{\left(1+x^2\right)^2}$

There are points of inflection where $y'' = 0$.

$y'' = 0 \Rightarrow 2\left(1-x^2\right) = 0$

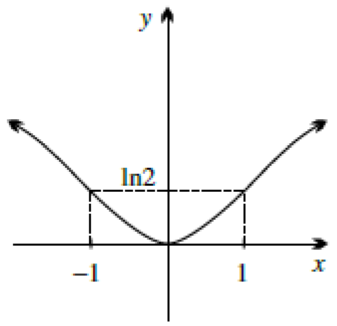
Solving $2\left(1-x^2\right) = 0$ for x we obtain $x = \pm 1$.

So there are points of inflection at $x = \pm 1$.

x	-2	-1	0	1	2
y''	$-\frac{6}{25}$	0	2	0	$-\frac{6}{25}$
concavity	down		up		down

So the points of inflection are $(\pm 1, \log_e 2)$.

9g



The range is $y \geq 0$.

10a $y = (\ln x)^2$

The domain is $x > 0$.

10b Solving $(\ln x)^2 = 0$ for x we obtain $x = 1$.

So the function is zero at $x = 1$, and is positive otherwise because squares cannot be negative.

10c Applying the chain rule:

Let $u = \ln x$ and so $y = u^2$.

Hence $\frac{du}{dx} = \frac{1}{x}$ and $\frac{dy}{du} = 2u$.

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$$
$$=\frac{2}{x}\ln x$$

So $y'=\frac{2}{x}\ln x$.

Applying the product rule on $y'=\frac{2}{x}\ln x$:

Let $u=\frac{2}{x}$ and $v=\ln x$.

Then $u'=-\frac{2}{x^2}$ and $v'=\frac{1}{x}$.

$$y''=vu'+uv'$$
$$=-\frac{2}{x^2}\times\ln x+\frac{2}{x}\times\frac{1}{x}$$
$$=\frac{2}{x^2}(1-\ln x)$$

So $y''=\frac{2(1-\ln x)}{x^2}$.

10d $y''=\frac{2(1-\ln x)}{x^2}$

There are points of inflection where $y''=0$.

$y''=0\Rightarrow 2(1-\ln x)=0$

Solving $2(1-\ln x)=0$ for x we obtain $x=e$.

So there is a point of inflection at $x=e$.

x	2	e	3
y''	0.153...	0	–0.021...
concavity	up		down

So the point of inflection is $(e,1)$.

10e $y' = \frac{2}{x}\ln x$

There are stationary points where $y' = 0$.

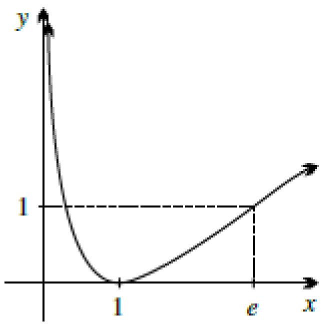
$2\ln x = 0 \Rightarrow x = 1$

So there is a stationary point at $x = 1$.

x	$\frac{1}{e}$	1	e
y'	$-2e$	0	$\frac{2}{e}$
slope	\	–	/

When $x = 1$, $y = 0$.

So $(1, 0)$ is a minimum turning point.



The range is $y \geq 0$.

11a Let $u = x^2$ and $v = \log_e x$

Then $\frac{du}{dx} = 2x$ and $\frac{dv}{dx} = \frac{1}{x}$

$\frac{dy}{dx} = (\log_e x \times 2x) + \left(x^2 \times \frac{1}{x}\right)$

$= 2x\log_e x + x$

$= x(2\log_e x + 1)$

$x(2\log_e x + 1) = 0$

$x = 0$ which is not a valid solution as $\log_e 0$ is undefined.

$2\log_e x + 1 = 0$

$\log_e x = -\frac{1}{2}$

$x = e^{-\frac{1}{2}}$

When $x = e^{-\frac{1}{2}}$,

$y = \left(e^{-\frac{1}{2}}\right)^2 \log_e e^{-\frac{1}{2}} = -\frac{1}{2e}$

Let $u = x$ and $v = 2\log_e x + 1$

Then $\frac{du}{dx} = 1$ and $\frac{dv}{dx} = \frac{2}{x}$

$\frac{d^2y}{dx^2} = \left((2\log_e x + 1) \times 1\right) + \left(x \times \frac{2}{x}\right)$

$= 2\log_e x + 1 + 2$

$= 2\log_e x + 3$

When $= e^{-\frac{1}{2}}$,

$\frac{d^2y}{dx^2} = 2\log_e e^{-\frac{1}{2}} + 3$

$= 2 > 0$

Hence the point $\left(\frac{1}{\sqrt{e}}, -\frac{1}{2e}\right)$ is a minimum point.

11b Inflection point when $\frac{d^2y}{dx^2} = 0$,

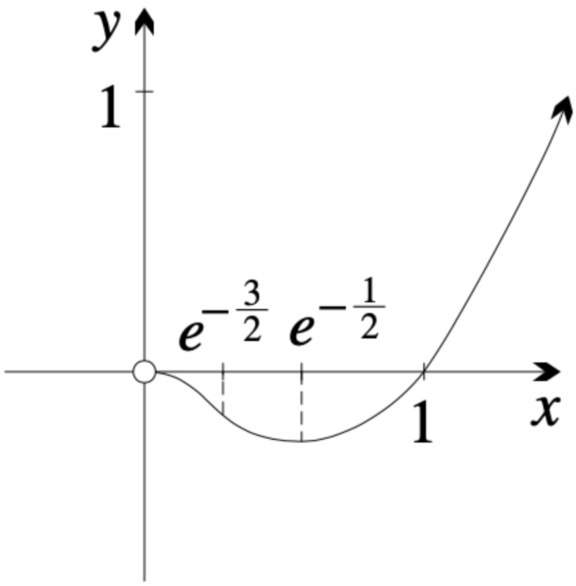
$0 = 2\log_e x + 3$

$\log_e x = -\frac{3}{2}$

$x = e^{-\frac{3}{2}}$

11c As $x \to 0^+, y \to 0$ and $y' \to 0$

11d



Range $= y \geq -\frac{1}{2e}$

12a $y = \dfrac{\log_e x}{x}$

The domain is $x > 0$.

12b Applying the quotient rule on $y = \dfrac{\log_e x}{x}$:

Let $u = \log_e x$ and $v = x$.

Then $u' = \dfrac{1}{x}$ and $v' = 1$.

$$\begin{aligned} y' &= \frac{vu' - uv'}{v^2} \\ &= \frac{x \times \frac{1}{x} - 1 \times \log_e x}{x^2} \\ &= \frac{1 - \log_e x}{x^2} \end{aligned}$$

So $y' = \dfrac{1 - \log_e x}{x^2}$.

Applying the quotient rule on $y' = \dfrac{1-\log_e x}{x^2}$:

Let $u = 1-\log_e x$ and $v = x^2$.

Then $u' = -\dfrac{1}{x}$ and $v' = 2x$.

$$\begin{aligned}
y'' &= \frac{vu' - uv'}{v^2} \\
&= \frac{\left(x^2\right)\left(-\frac{1}{x}\right) - 2x\left(1-\log_e x\right)}{\left(x^2\right)^2} \\
&= \frac{-x - 2x + 2x\log_e x}{x^4} \\
&= \frac{x\left(2\log_e x - 3\right)}{x^4} \\
&= \frac{2\log_e x - 3}{x^3} \quad (x > 0)
\end{aligned}$$

So $y'' = \dfrac{2\log_e x - 3}{x^3}$.

12c $\quad y' = \dfrac{1-\log_e x}{x^2}$

There are stationary points where $y' = 0$.

$1 - \log_e x = 0 \Rightarrow x = e$

So there is a stationary point at $x = e$.

x	2	e	3
y'	0.076…	0	–0.010…
slope	/	–	\

When $x = e$, $y = e^{-1}$.

So $\left(e, e^{-1}\right)$ is a maximum turning point.

12d $y'' = \dfrac{2\log_e x - 3}{x^3}$

There are points of inflection where $y'' = 0$.

$y'' = 0 \Rightarrow 2\log_e x - 3 = 0$

Solving $2\log_e x - 3 = 0$ for x we obtain $x = e^{\frac{3}{2}}$.

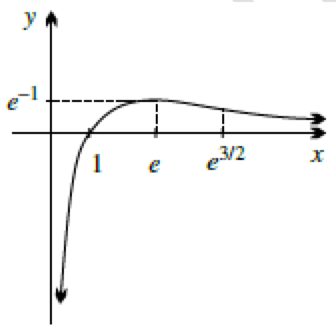
So there is a point of inflection at $x = e^{\frac{3}{2}}$.

x	e	$e^{\frac{3}{2}}$	e^2
y''	$-e^{-3}$	0	e^{-6}
concavity	down		up

When $x = e^{\frac{3}{2}}$, $y = \dfrac{\log_e e^{\frac{3}{2}}}{e^{\frac{3}{2}}} = \dfrac{3}{2}e^{-\frac{3}{2}}$.

So the point of inflection is $\left(e^{\frac{3}{2}}, \frac{3}{2}e^{-\frac{3}{2}}\right)$.

12e Given $y \to 0$ as $x \to \infty$ and that $y \to -\infty$ as $x \to 0^+$.



The range is $y \le e^{-1}$.

13 Let $u = x$ and $v = \log_e x$

Then $\frac{du}{dx} = 1$ and $\frac{dv}{dx} = \frac{1}{x}$

$\frac{dy}{dx} = \frac{(\log_e x \times 1) - \left(x \times \frac{1}{x}\right)}{(\log_e x)^2}$

$= \frac{\log_e x - 1}{(\log_e x)^2}$

Turning points when $\frac{dy}{dx} = 0$,

$0 = \frac{\log_e x - 1}{(\log_e x)^2}$

$x \neq 1$ as the denominator cannot be zero,

Hence,

$0 = \log_e x - 1$

$x = e$

Let $u = \log_e x - 1$ and $v = (\log_e x)^2$

Then $\frac{du}{dx} = \frac{1}{x}$ and $\frac{dv}{dx} = 2\log_e x \times \frac{1}{x} = \frac{2}{x}\log_e x$

$\frac{d^2y}{dx^2} = \frac{\left((\log_e x)^2 \times \frac{1}{x}\right) - \left((\log_e x - 1) \times \frac{2}{x}\log_e x\right)}{((\log_e x)^2)^2}$

$= \frac{(\log_e x - 2\log_e x + 2)\frac{1}{x}\log_e x}{(\log_e x)^4}$

$= \frac{2 - \log_e x}{x(\log_e x)^3}$

Inflection points when $\frac{d^2y}{dx^2} = 0$,

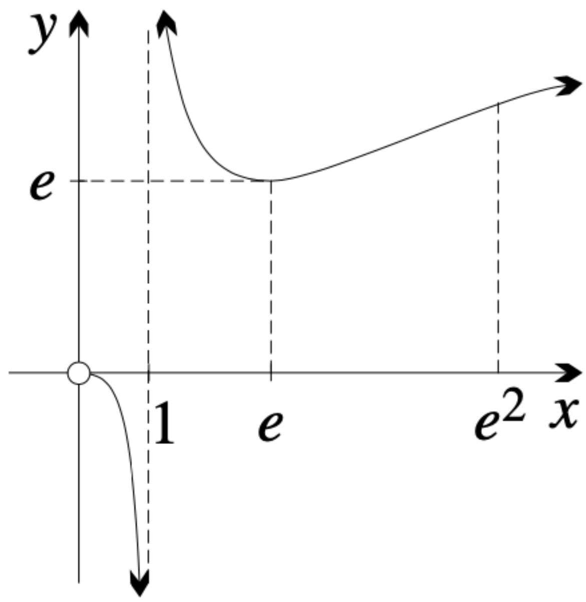
$0 = \frac{2 - \log_e x}{x(\log_e x)^3}$

$x \neq 0$ and $x \neq 1$ as the denominator cannot be zero,

Hence,

$0 = 2 - \log_e x$

$x = e^2$



14a $x > -1$

14b $y' = \frac{d\left(\log_e\left(\frac{x^2}{x+1}\right)\right)}{dx}$

⇨ $y' = \frac{x+1}{x^2} \times \frac{d\left(\frac{x^2}{x+1}\right)}{dx}$

Let $u = x^2$ and $v = x + 1$

⇨ $\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{vdu - udv}{v^2}$

⇨ $y' = \frac{2x(x+1) - x^2}{(x+1)^2} \times \frac{x+1}{x^2}$

⇨ $y' = \frac{x+2}{x(x+1)}$

14c As $y' = \frac{x+2}{x(x+1)}$, substituting x = -2, will yield $y' = \frac{x+2}{x(x+1)} = 0$

But, $x = -2$ is out of the domain, hence, $x = -2$ is not a stationary point.

14d Inflection points occur when $y'' = 0$

First, we have to calculate y''

$\Rightarrow \quad y'' = \frac{d\left(\frac{x+2}{x(x+1)}\right)}{dx}$

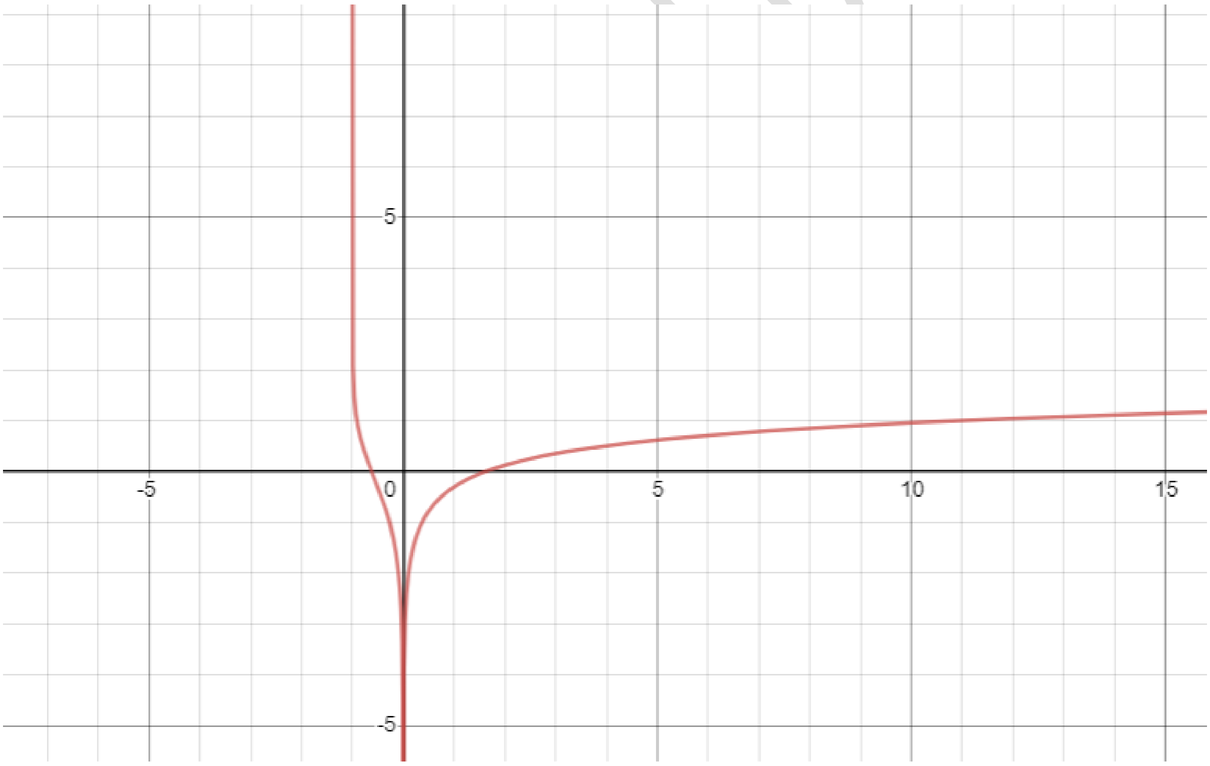
Let $u = x + 2, v = \ x^2 + x$

Hence, $y'' = \frac{vdu-udv}{v^2}$

$y'' = \frac{(x^2+x)-(x+2)(2x+1)}{x^2(x+1)^2} = \frac{x^2+4x+2}{x^2(x+1)^2}$

As we can see, the numeration of the above equation has no real roots. Hence, this curve has no inflection points

14e



15a The domain is $x > 1$.

15b x-intercept means $y \ = \ 0$.

$\Rightarrow \quad \ln \ln x = 0$

⇨ $\ln x = 1$
⇨ $x = e$

15c $y' = \frac{d(\ln \ln x)}{dx}$

⇨ $y' = \frac{1}{\ln x} \times \frac{1}{x}$

$y'' = \frac{d\left(\frac{1}{x \ln x}\right)}{dx}$

Let $u = \frac{1}{x}$ and $v = \frac{1}{\ln x}$

$y'' = vdu + udv$

$y'' = -\frac{1}{x^2 \ln x} - \frac{1}{x\,(\ln x)^2}$

As we can see from equation of y', there are no solutions to that equation. Hence, there are no stationary points for the curve

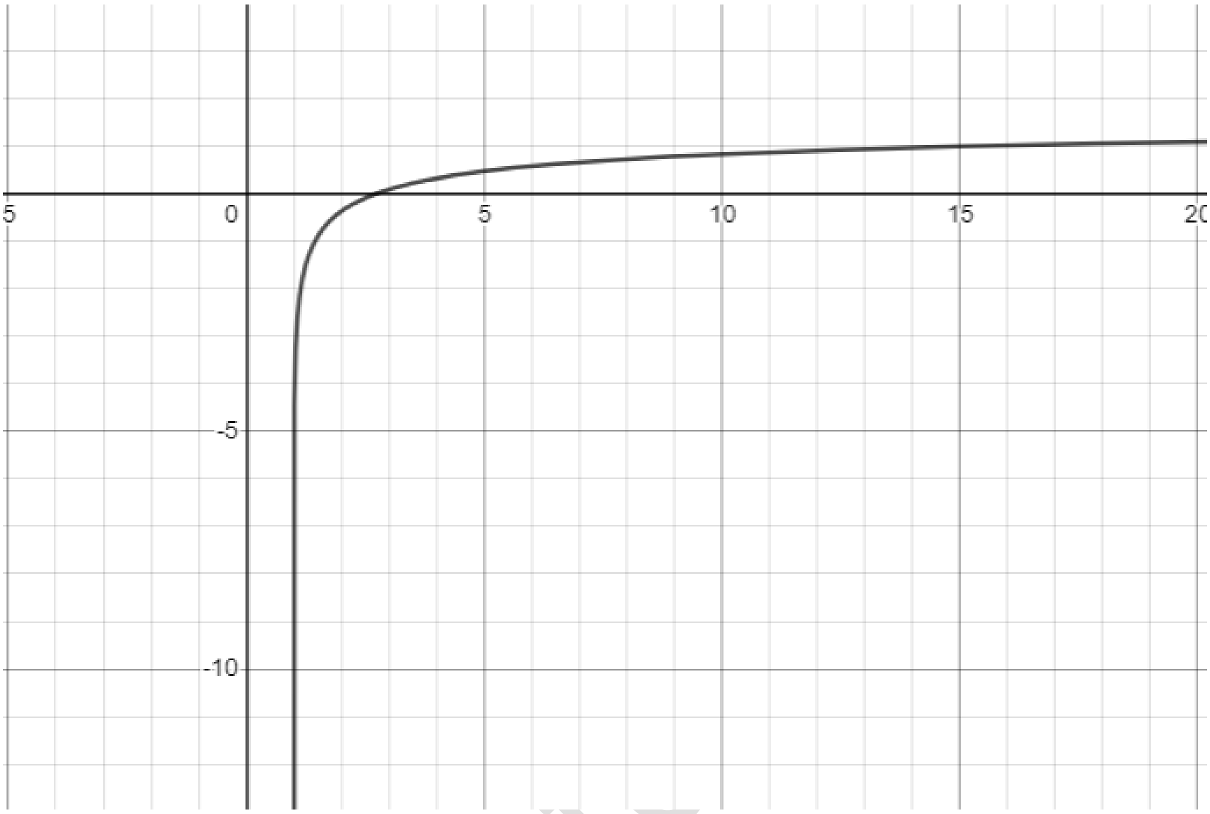
15d Inflection points are those where $y'' = 0$.

⇨ $y'' = -\frac{1}{x^2 \ln x} - \frac{1}{x \ln x^2} = 0$
⇨ $\frac{1}{x^2 \ln x} + \frac{1}{x\,(\ln x)^2} = 0$
⇨ $(x + \ln x) = 0$

Hence, $x \doteqdot 0.567$

But since that point lies outside the domain, there are no inflection points for this curve.

15e



16

x	2	5	10	20	40	4000
$\dfrac{\log_e x}{x}$	0.347	0.322	0.230	0.150	0.092	0.002

Based on this trend, $\lim\limits_{x\to\infty} \dfrac{\log_e x}{x} = 0$.

x	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{20}$	$\frac{1}{40}$	$\frac{1}{4000}$
$x\log_e x$	-0.347	-0.322	-0.230	-0.150	-0.092	-0.002

Based on this trend, $\lim\limits_{x\to 0^+} x\log_e x = 0$.

17 $y = x^{\frac{1}{\ln x}}$

To show that this is a constant function, its derivative must be 0. Hence we must calculate y'.

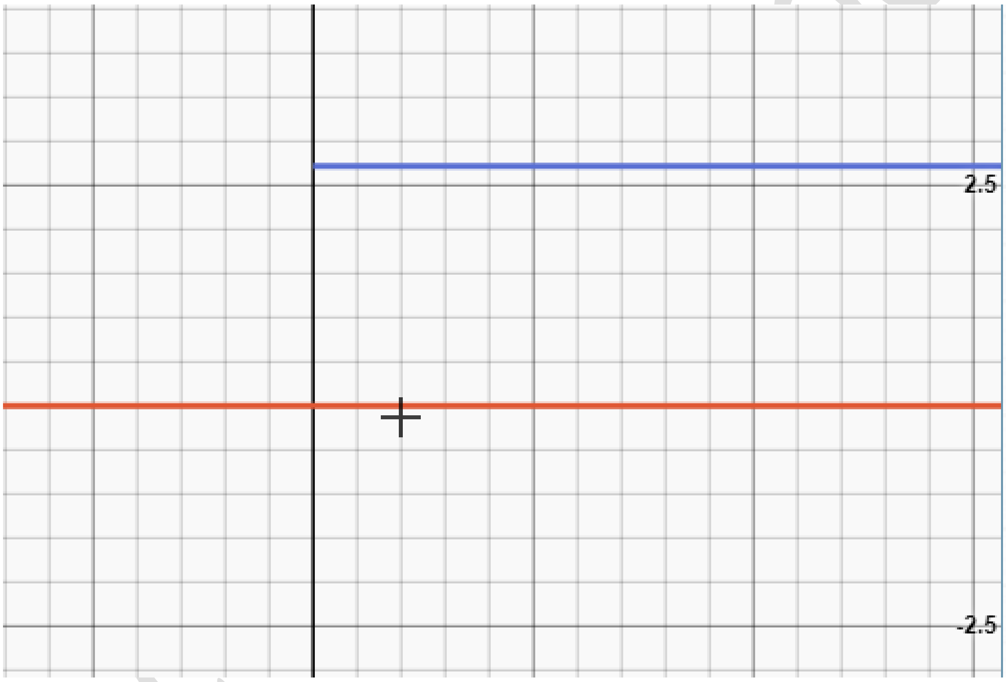
$\Rightarrow \quad y' = \dfrac{d\left(x^{\frac{1}{\ln x}}\right)}{dx}$

$\Rightarrow \quad y' = x^{\frac{1}{\ln x}} \times \dfrac{d\left(\ln x \times \frac{1}{\ln x}\right)}{dx}$

$\Rightarrow \quad y' = x^{\frac{1}{\ln x}} \times \dfrac{d(1)}{dx} = 0$

Hence, it is a constant function.

Its value is e and the domain is $x > 0$.



18a $y = x^x$

Taking log of both sides and differentiating, we get:

$\Rightarrow \quad d(\log y) = d(x \log x)$

$\Rightarrow \quad \frac{1}{y} dy = 1 + \log x$

$\Rightarrow \quad \frac{dy}{dx} = x^x(1 + \log x)$

As the value of x comes near 0, the value of the curve tends to 1.

18b Stationary points occur when $y' = 0$

⇨ $x^x(1 + \log x) = 0$
⇨ $\log x = -1$
⇨ $x = \frac{1}{e}$

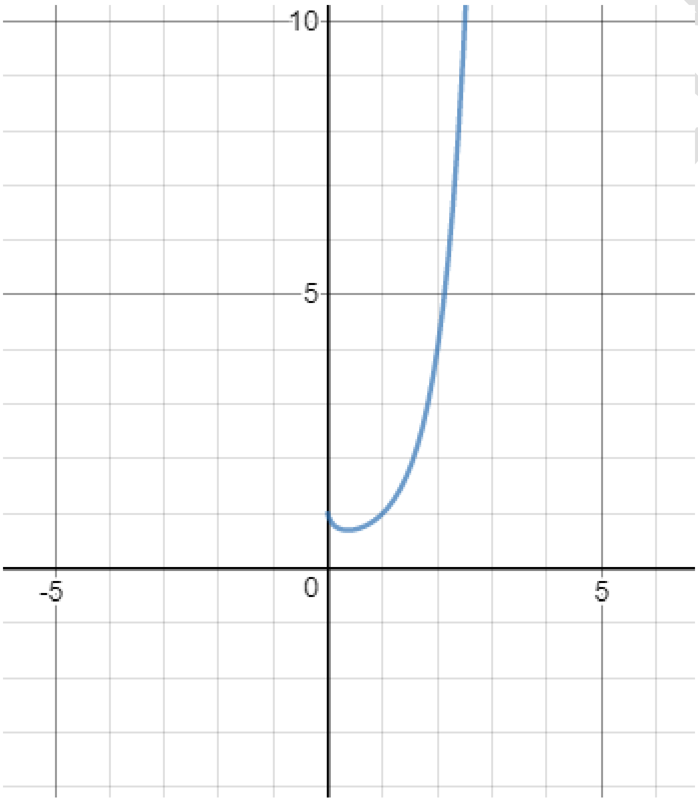
Hence, stationary point is at $(\frac{1}{e}, e^{-\frac{1}{e}})$

$y' = 1$

⇨ $x^x(1 + \log x) = 1$
⇨ $x = 1$

Hence, the gradient of 1 occurs at $x = 1$.

18c Domain is $x > 0$ and range is $y \geq e^{-\frac{1}{e}}$



19a $\lim_{x \to 0+} x^{\frac{1}{x}}$

$\Rightarrow \lim_{x \to 0+} e^{\ln x^{\frac{1}{x}}}$

$\Rightarrow \lim_{x \to 0+} e^{\frac{\ln x}{x}}$

$\Rightarrow e^{\lim_{x \to 0+} \frac{\ln x}{x}}$

$\Rightarrow e^{-\frac{\infty}{0+}}$

$\Rightarrow e^{-\infty} = 0$

As we change the limit to $x \to 0+$, the function $x^{\frac{1}{x}} \to 0$

Now, we calculate $\lim_{x \to \infty} x^{\frac{1}{x}}$

$\Rightarrow \lim_{x \to \infty} e^{\ln x^{\frac{1}{x}}}$

$\Rightarrow e^{\lim_{x \to \infty} \frac{\ln x}{x}}$

$\Rightarrow e^{\lim_{x \to \infty} \frac{1}{x}}$

$\Rightarrow e^{0} = 1$

19b To find the turning point, we have to equate $y' = 0$.

$$y' = \frac{d\left(x^{\frac{1}{x}}\right)}{dx}$$

$$y' = x^{\frac{1}{x}} \times \frac{d\left(\ln x \times \frac{1}{x}\right)}{dx}$$

Let $u = \ln x$ and $v = x$

Hence,

$$y' = x^{\frac{1}{x}} \times \frac{[vu' - uv']}{v^2}$$

$$= \frac{x^{\frac{1}{x}}(1 - \ln x)}{x^2}$$

Hence, the stationary point is at $y' = 0$.

$\frac{x^{\frac{1}{x}}(1-\ln x)}{x^2} = 0$

$1 - \ln x = 0$

$\ln x = 1$

$x = e^1$

$x = e$

19c As we had calculated the $y_1{}'$for $y_1 = x^x$ in question 18a and $y_2{}'$for $y_2 = x^{\frac{1}{x}}$ in question 19b, we have to show that they have a common tangent at $y_1{}'$for $x = 1$

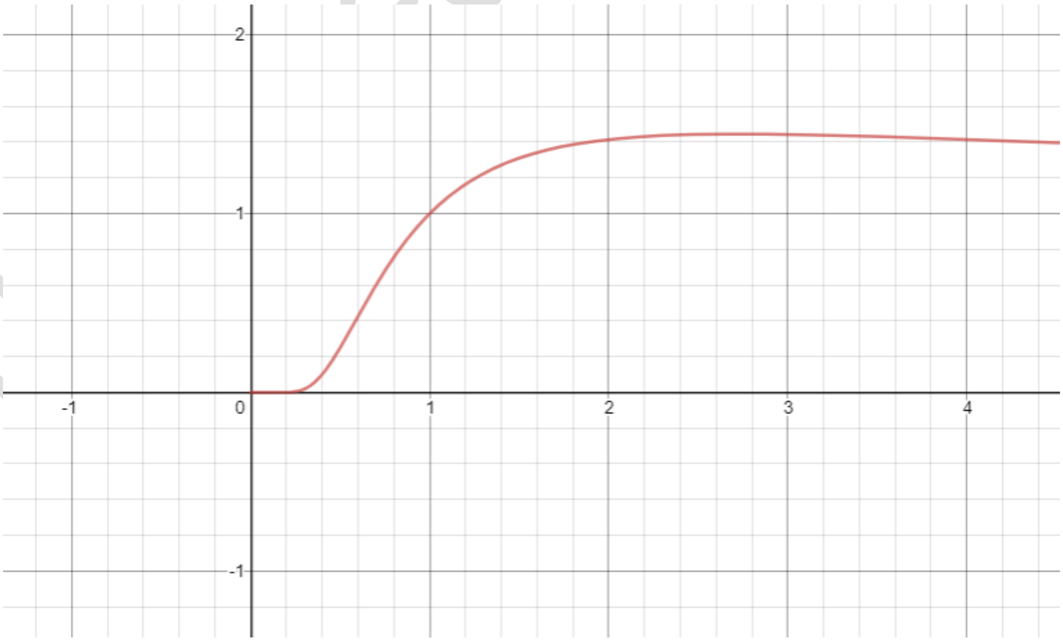
This means that $y_1' = y_2{}'$

$x^x(1 + \ln x) = \frac{x^{\frac{1}{x}}(1-\ln x)}{x^2}$

$x^x(1 + \ln x) \times x^2 \times \frac{x^{-\frac{1}{x}}}{(1 - \ln x)} = 1$

The solution of the above equation is $x = 1$.

19d The sketch is as follows:



Solutions to Exercise 6I

Let C be a constant.

1a

$$\int \frac{2}{x}\,dx$$

$$= 2\int \frac{1}{x}\,dx$$

$$= 2\log_e|x| + C$$

1b

$$\int \frac{1}{3x}\,dx$$

$$= \frac{1}{3}\int \frac{1}{x}\,dx$$

$$= \frac{1}{3}\log_e|x| + C$$

1c

$$\int \frac{4}{5x}\,dx$$

$$= \frac{4}{5}\int \frac{1}{x}\,dx$$

$$= \frac{4}{5}\log_e|x| + C$$

1d

$$\int \frac{3}{2x}\,dx$$

$$= \frac{3}{2}\int \frac{1}{x}\,dx$$

$$= \frac{3}{2}\log_e|x| + C$$

2a

$$\int \frac{1}{4x+1}\,dx$$

$$= \frac{1}{4}\log_e|4x+1| + C$$

2b

$$\int \frac{1}{5x-3}\,dx$$

$$= \frac{1}{5}\log_e|5x-3| + C$$

2c

$$\int \frac{6}{3x+2}\,dx$$

$$= \frac{6}{3}\log_e|3x+2| + C$$

$$= 2\log_e|3x+2| + C$$

2d

$$\int \frac{15}{5x+1}\,dx$$

$$= \frac{15}{5}\log_e|5x+1| + C$$

$$= 3\log_e|5x+1| + C$$

2e

$$\int \frac{4}{4x+3}\,dx$$

$$= \frac{4}{4}\log_e|4x+3| + C$$

$$= \log_e|4x+3| + C$$

2f

$$\int \frac{dx}{3-x}$$

$$= \frac{1}{-1}\log_e|3-x| + C$$

$$= -\log_e|3-x| + C$$

2g

$$\int \frac{dx}{7-2x}$$

$$= \frac{1}{-2}\log_e|7-2x| + C$$

$$= -\frac{1}{2}\log_e|7-2x| + C$$

2h

$$\int \frac{4}{5x-1}dx$$

$$= \frac{4}{5}\log_e|5x-1| + C$$

2i

$$\int \frac{12}{1-3x}dx$$

$$= \frac{12}{-3}\log_e|1-3x| + C$$

$$= -4\log_e|1-3x| + C$$

3a

$$\int_1^5 \frac{1}{x}dx$$

$$= [\log_e|x|]_1^5$$

$$= \log_e 5 - \log_e 1$$

$= \log_e 5$

3b

$\int_1^3 \frac{1}{x} dx$

$= [\log_e |x|]_1^3$

$= \log_e 3 - \log_e 1$

$= \log_e 3$

3c

$\int_{-8}^{-2} \frac{1}{x} dx$

$= [\log_e |x|]_{-8}^{-2}$

$= \log_e |-2| - \log_e |-8|$

$= \log_e 2 - \log_e 8$

$= \log_e \frac{2}{8}$

$= \log_e \frac{1}{4}$

$= -\log_e 4$

$= -\log_e 2^2$

$= -2\log_e 2$

3d

$\int_{-3}^{9} \frac{1}{x} dx = [\log_e |x|]_{-3}^{9}$

This definite integral is meaningless, as it passes over an asymptote at $x = 0$, over which the function $\frac{1}{x}$ is undefined.

3e

$$\int_1^4 \frac{1}{2x}dx$$

$$= \left[\frac{1}{2}\log_e|2x|\right]_1^4$$

$$= \frac{1}{2}\log_e 8 - \frac{1}{2}\log_e 2$$

$$= \frac{1}{2}(\log_e 4)$$

$$= \log_e 4^{\frac{1}{2}}$$

$$= \log_e 2$$

3f

$$\int_{-15}^{-5} \frac{1}{5x}dx$$

$$= \left[\frac{1}{5}\log_e|5x|\right]_{-15}^{-5}$$

$$= \frac{1}{5}\log_e|-25| - \frac{1}{5}\log_e|-75|$$

$$= \frac{1}{5}(\log_e 25 - \log_e 75)$$

$$= \frac{1}{5}\left(\log_e \frac{25}{75}\right)$$

$$= \frac{1}{5}\log_e \frac{1}{3}$$

$$= \frac{1}{5}\log_e 3^{-1}$$

$$= -\frac{1}{5}\log_e 3$$

4a

$$\int_0^1 \frac{dx}{x+1}$$

$= [\log_e|x+1|]_0^1$

Check for asymptote.

$x + 1 = 0$

$x = -1$, outside of range of limits.

$[\log_e|x+1|]_0^1$

$= \log_e|1+1| - \log_e|0+1|$

$= \log_e 2 - \log_e 1$

$= \log_e 2$

$\doteqdot 0.6931$

4c

$$\int_{-5}^{-2} \frac{dx}{2x+3}$$

$= \left[\frac{1}{2}\log_e|2x+3|\right]_{-5}^{-2}$

Check for asymptote.

$2x + 3 = 0$

$x = -\frac{3}{2} = -1\frac{1}{2}$, outside of range of limits.

$\left[\frac{1}{2}\log_e|2x+3|\right]_{-5}^{-2}$

$= \frac{1}{2}\log_e|2(-2)+3| - \frac{1}{2}\log_e|2(-5)+3|$

$= \frac{1}{2}(\log_e 1 - \log_e 7)$

$= \frac{1}{2}\log_e \frac{1}{7}$

$= \frac{1}{2}\log_e 7^{-1}$

$= -\frac{1}{2}\log_e 7$

$\doteqdot -0.9730$

4d

$\int_1^2 \frac{3}{5-2x}\,dx$

$= \left[-\frac{3}{2}\log_e|5-2x|\right]_1^2$

Check for asymptote.

$5 - 2x = 0$

$x = \frac{5}{2} = 2\frac{1}{2}$, outside of range of limits.

$\left[-\frac{3}{2}\log_e|5-2x|\right]_1^2$

$= -\frac{3}{2}\log_e|5-2(2)| - \left(-\frac{3}{2}\log_e|5-2(1)|\right)$

$= -\frac{3}{2}(\log_e 1 - \log_e 3)$

$= \frac{3}{2}\log_e 3$

$\doteqdot 1.648$

4e

$\int_{-1}^1 \frac{3}{7-3x}\,dx$

$= \left[-\frac{3}{3}\log_e|7-3x|\right]_{-1}^1$

Check for asymptote.

$7 - 3x = 0$

$x = \frac{7}{3} = 2\frac{1}{3}$, outside of range of limits.

$$\left[-\frac{3}{3}\log_e|7-3x|\right]_{-1}^{1}$$

$$= -\log_e|7-3(1)| - (-\log_e|7-3(-1)|)$$

$$= -(\log_e 4 - \log_e 10)$$

$$= -\log_e \frac{4}{10}$$

$$= -\log_e \left(\frac{5}{2}\right)^{-1}$$

$$= \log_e \frac{5}{2}$$

$$\doteqdot 0.9163$$

4f

$$\int_0^{11} \frac{5}{2x-11}\,dx$$

$$= \left[\frac{5}{2}\log_e|2x-11|\right]_0^{11}$$

Check for asymptote.

$2x - 11 = 0$

$x = \frac{11}{2} = 5\frac{1}{2}$, the integral is meaningless because it runs across an asymptote at $x = 5\frac{1}{2}$

5a

$$\int_1^{e} \frac{dx}{x}$$

$$= [\log_e|x|]_1^e$$

$$= \log_e|e| - \log_e|1|$$

$$= \log_e e$$

$$= 1$$

5b

$$\int_1^{e^2} \frac{dx}{x}$$

$$= [\log_e|x|]_1^{e^2}$$

$$= \log_e|e^2| - \log_e|1|$$

$$= 2\log_e e$$

$$= 2$$

5c

$$\int_1^{e^4} \frac{dx}{x}$$

$$= [\log_e|x|]_1^{e^4}$$

$$= \log_e|e^4| - \log_e|1|$$

$$= 4\log_e e$$

$$= 4$$

5d

$$\int_{\sqrt{e}}^{e} \frac{dx}{x} = [\log_e|x|]_{\sqrt{e}}^{e}$$

$$= \log_e|e| - \log_e\left|e^{\frac{1}{2}}\right|$$

$$= 1\log_e e - \frac{1}{2}\log_e e$$

$$= \frac{1}{2}$$

6a

$$\frac{x+1}{x} = \frac{x}{x} + \frac{1}{x} = 1 + \frac{1}{x}$$

The primitive of the function above is

$$\int\left(1+\frac{1}{x}\right)dx = x+\log_e|x|+C$$

6b

$$\frac{x+3}{5x}=\frac{x}{5x}+\frac{3}{5x}=\frac{1}{5}+\frac{3}{5x}$$

The primitive of the function above is

$$\int\left(\frac{1}{5}+\frac{3}{5x}\right)dx=\frac{x}{5}+\frac{3}{5}\log_e|x|+C$$

6c

$$\frac{1-8x}{9x}=\frac{1}{9x}-\frac{8x}{9x}=\frac{1}{9x}-\frac{8}{9}$$

The primitive of the function above is

$$\int\left(\frac{1}{9x}-\frac{8}{9}\right)dx=\frac{1}{9}\log_e|x|-\frac{8}{9}x+C$$

6d

$$\frac{3x^2-2x}{x^2}=\frac{3x^2}{x^2}-\frac{2x}{x^2}=3-\frac{2}{x}$$

The primitive of the function above is

$$\int 3-\frac{2}{x}\,dx=3x-2\log_e|x|+C$$

6e

$$\frac{2x^2+x-4}{x}=\frac{2x^2}{x}+\frac{x}{x}-\frac{4}{x}=2x+1-\frac{4}{x}$$

The primitive of the function above is

$$\int\left(2x+1-\frac{4}{x}\right)dx=x^2+x-4\log_e|x|+C$$

6f

$$\frac{x^4 - x + 2}{x^2} = \frac{x^4}{x^2} - \frac{x}{x^2} + \frac{2}{x^2} = x^2 - \frac{1}{x} + \frac{2}{x^2}$$

The primitive of the function above is

$$\int \left(x^2 - \frac{1}{x} + \frac{2}{x^2}\right) dx = \frac{x^3}{3} - \log_e|x| - \frac{2}{x} + C$$

7a

$$\frac{d}{dx}(x^2 - 9) = 2x$$

The numerator is a derivative of the denominator.

Let $f(x) = x^2 - 9$

$$\int \frac{f'(x)}{f(x)} dx = \log_e|f(x)| + C$$

$$\int \frac{2x}{x^2 - 9} dx = \log_e|x^2 - 9| + C$$

7b

$$\frac{d}{dx}(3x^2 + x) = 6x + 1$$

The numerator is a derivative of the denominator.

Let $f(x) = 3x^2 + x$

$$\int \frac{f'(x)}{f(x)} dx = \log_e|f(x)| + C$$

$$\int \frac{6x + 1}{3x^2 + x} dx = \log_e|3x^2 + x| + C$$

7c

$$\frac{d}{dx}(x^2 + x - 3) = 2x + 1$$

The numerator is a derivative of the denominator.

Let $f(x) = x^2 + x - 3$

$$\int \frac{f'(x)}{f(x)}\,dx = \log_e|f(x)| + C$$

$$\int \frac{2x+1}{x^2+x-3}\,dx = \log_e|x^2+x-3| + C$$

7d

$$\frac{d}{dx}(2+5x-3x^2) = 5-6x$$

The numerator is a derivative of the denominator.

Let $f(x) = 2+5x-3x^2$

$$\int \frac{f'(x)}{f(x)}\,dx = \log_e|f(x)| + C$$

$$\int \frac{5-6x}{2+5x-3x^2}\,dx = \log_e|2+5x-3x^2| + C$$

7e

$$\frac{d}{dx}(x^2+6x-1) = 2x+6 = 2(x+3)$$

The numerator is a derivative of the denominator, subjected to a factor of $\frac{1}{2}$

Let $f(x) = x^2+6x-1$

$$\int \frac{f'(x)}{f(x)}\,dx = \log_e|f(x)| + C$$

$$\int \frac{x+3}{x^2+6x-1}\,dx$$

$$= \frac{1}{2}\int \frac{2x+6}{x^2+6x-1}\,dx$$

$$= \frac{1}{2}\log_e|x^2+6x-1| + C$$

7f

$$\frac{d}{dx}(12x - 3 - 2x^2) = 12 - 4x = 4(3 - x)$$

The numerator is a derivative of the denominator, subjected to a factor of $\frac{1}{4}$

Let $f(x) = x^2 + 6x - 1$

$$\int \frac{f'(x)}{f(x)} dx = \log_e|f(x)| + C$$

$$\int \frac{3 - x}{12x - 3 - 2x^2} dx$$

$$= \frac{1}{4}\int \frac{4(3 - x)}{12x - 3 - 2x^2} dx$$

$$= \frac{1}{4}\log_e|12x - 3 - 2x^2| + C$$

7g

$$\frac{d}{dx}(1 + e^x) = e^x$$

The numerator is a derivative of the denominator

Let $f(x) = 1 + e^x$

$$\int \frac{f'(x)}{f(x)} dx = \log_e|f(x)| + C$$

$$\int \frac{e^x}{1 + e^x} dx = \log_e|1 + e^x| + C$$

It is unnecessary to use absolute sign here as $1 + e^x$ is always positive.

7h

$$\frac{d}{dx}(1 + e^{-x}) = -e^{-x}$$

The numerator is a derivative of the denominator, subjected to a factor of -1

Let $f(x) = 1 + e^{-x}$

$$\int \frac{f'(x)}{f(x)} dx = \log_e|f(x)| + C$$

$$\int \frac{e^{-x}}{1+e^{-x}}dx$$

$$= -\int \frac{-e^{-x}}{1+e^{-x}}dx$$

$$= -\log_e|1+e^{-x}| + C$$

7i

$$\frac{d}{dx}(e^x + e^{-x}) = e^x - e^{-x}$$

The numerator is a derivative of the denominator.

Let $f(x) = e^x + e^{-x}$

$$\int \frac{f'(x)}{f(x)}dx = \log_e|f(x)| + C$$

$$\int \frac{e^x - e^{-x}}{e^x + e^{-x}}dx = \log_e|e^x + e^{-x}| + C$$

The absolute value signs are not required because the answers to part g-i are always positive, therefore are not affected by the absolute value signs.

8a

$$\int \frac{1}{3x-k}dx = \frac{1}{3}\log_e|3x-k| + C$$

8b

$$\int \frac{1}{mx-2}dx = \frac{1}{m}\log_e|mx-2| + C$$

8c

$$\int \frac{p}{px+q}dx = \frac{p}{p}\log_e|px+q| + C = \log_e|px+q| + C$$

8d

$$\int \frac{A}{sx - t}\,dx = \frac{A}{s}\log_e|sx - t| + C$$

9a

$$f'(x) = 1 + \frac{2}{x}$$

$$f(x) = x + 2\log_e|x| + C$$

Since $f(1) = 1$,

$$1 + 2\log_e|1| + C = 1$$

$$C = 0$$

$$f(x) = x + 2\log_e|x|$$

$$f(2) = 2 + 2\log_e|2|$$

9b

$$f'(x) = 2x + \frac{1}{3x}$$

$$f(x) = x^2 + \frac{1}{3}\log_e|x| + C$$

Since $f(1) = 2$,

$$(1)^2 + \frac{1}{3}\log_e|1| + C = 2$$

$$C = 1$$

$$f(x) = x^2 + \frac{1}{3}\log_e|x| + 1$$

$$f(2) = (2)^2 + \frac{1}{3}\log_e|2| + 1 = 5 + \frac{1}{3}\log_e 2$$

9c

$$f'(x) = 3 + \frac{5}{2x-1}$$

$$f(x) = 3x + \frac{5}{2}\log_e|2x-1| + C$$

Since $f(1) = 0$,

$$3(1) + \frac{5}{2}\log_e|2(1)-1| + C = 0$$

$$C = -3$$

$$f(x) = 3x + \frac{5}{2}\log_e|2x-1| - 3$$

$$f(2) = 3(2) + \frac{5}{2}\log_e|2(2)-1| - 3 = 3 + \frac{5}{2}\log_e 3$$

9d

$$f'(x) = 6x^2 + \frac{15}{3x+2}$$

$$f(x) = 2x^3 + \frac{15}{3}\log_e|3x+2| + C$$

Since $f(1) = 5\log_e 5$,

$$2(1)^3 + 5\log_e|3(1)+2| + C = 5\log_e 5$$

$$C = -2$$

$$f(x) = 2x^3 + 5\log_e|3x+2| - 2$$

$$f(2) = 2(2)^3 + 5\log_e|3(2)+2| - 2 = 14 + 5\log_e 8$$

10a

$$f'(x) = \frac{x^2+x+1}{x} = \frac{x^2}{x} + \frac{x}{x} + \frac{1}{x} = x + 1 + \frac{1}{x}$$

$$f(x) = \int x + 1 + \frac{1}{x}\,dx = \frac{x^2}{2} + x + \log_e|x| + C$$

$$f(1) = \frac{(1)^2}{2} + (1) + \log_e(1) + C = 1\frac{1}{2}$$

$$C = 1\frac{1}{2} - 1 - \frac{1}{2} = 0$$

$$f(x) = \frac{x^2}{2} + x + \log_e|x|$$

10b

$$g'(x) = \frac{2x^3 - 3x - 4}{x^2}$$

$$= \frac{2x^3}{x^2} - \frac{3x}{x^2} - \frac{4}{x^2}$$

$$= 2x - \frac{3}{x} - \frac{4}{x^2}$$

$$g(x) = \int \left(2x - \frac{3}{x} - \frac{4}{x^2}\right) dx$$

$$= x^2 - 3\log_e|x| + \frac{4}{x} + C$$

$$g(2) = (2)^2 - 3\log_e|2| + \frac{4}{2} + C = -3\log_e 2$$

$$C = -2 - 4 = -6$$

$$g(x) = x^2 - 3\log_e|x| + \frac{4}{x} - 6$$

11a $y' = \frac{1}{4x}$

$$y = \frac{1}{4}\log_e|x| + C$$

Since $x = e^2, y = 1$

$$1 = \frac{1}{4}\log_e e^2 + C$$

$$C = 1 - \frac{1}{4}(2)\log_e e = \frac{1}{2}$$

$$y = \frac{1}{4}\log_e|x| + \frac{1}{2}$$

When $y = 0$,

$\frac{1}{4}\log_e|x| + \frac{1}{2} = 0$

$\frac{1}{4}\log_e|x| = -\frac{1}{2}$

$\log_e|x| = -2$

$|x| = e^{-2}$

The curve meets the x-axis on the right-hand side of the origin at $x = e^{-2}$

11b $y' = \frac{2}{x+1}$

$y = 2\log_e|x+1| + C$

When $x = 0, y = 1$

$1 = 2\log_e(0+1) + C$

$C = 1 - 2\log_e 1 = 1$

$y = 2\log_e|x+1| + 1$

11c $y' = \frac{2x+5}{x^2+5x+4}$

Let $u = x^2 + 5x + 4$

$u' = 2x + 5$

The numerator for y' is the derivative of its denominator.

$y = \int \frac{u'}{u}\,dx = \log_e|u| + C$

$y = \log_e|x^2 + 5x + 4| + C$

At $x = 1, y = 1$

$1 = \log_e|(1)^2 + 5(1) + 4| + C$

$C = 1 - \log_e 10$

$y = \log_e|x^2 + 5x + 4| + 1 - \log_e 10$

$\quad = \log_e\left|\frac{x^2 + 5x + 4}{10}\right| + 1$

$y(0)$

$= \log_e \left|\frac{(0)^2 + 5(0) + 4}{10}\right| + 1$

$= \log_e \frac{4}{10} + 1$

or $\log_e \frac{2}{5} + 1$

11d $y' = \frac{(2+x)}{x} = \frac{2}{x} + 1$

$y = 2\log_e|x| + x + C$

At $x = 1, y = 1$

$1 = 2\log_e|1| + 1 + C$

$C = 0$

$y = 2\log_e|x| + x$

Let $x = 2,$

$y = 2\log_e|2| + 2 = \log_e 4 + 2$

11e $f''(x) = \frac{1}{x^2}$

$f'(x) = -\frac{1}{x} + C_1$

At $x = 1, f'(x) = 0$

$0 = -\frac{1}{1} + C_1$

$C_1 = 1$

$f'(x) = -\frac{1}{x} + 1$

$f(x) = -\log_e|x| + x + C_2$

At $x = 1, f(x) = 3$

$3 = -\log_e 1 + 1 + C_2$

$C_2 = 2$

$f(x) = -\log_e|x| + x + 2$

$f(e) = -\log_e e + e + 2 = -1 + e + 2 = e + 1$

12a Let $u = x^3 - 5$

$u' = 3x^2$

$\int \frac{u'}{u} dx = \log_e|u| + C$

$\int \frac{3x^2}{x^3 - 5} dx = \log_e|x^3 - 5| + C$

12b Let $u = x^4 + x - 5$

$u' = 4x^3 + 1$

$\int \frac{u'}{u} dx = \log_e|u| + C$

$\int \frac{4x^3 + 1}{x^4 + x - 5} dx = \log_e|x^4 + x - 5| + C$

12c Let $u = x^4 - 6x^2$

$u' = 4x^3 - 12x$

$\int \frac{u'}{u} dx = \log_e|u| + C$

$\int \frac{x^3 - 3x}{x^4 - 6x^2} dx$

$= \frac{1}{4}\int \frac{4x^3 - 12x}{x^4 - 6x^2} dx$

$= \frac{1}{4}\log_e|x^4 - 6x^2| + C$

12d Let $u = 5x^4 - 7x^2 + 8$

$u' = 20x^3 - 14x$

$\int \frac{u'}{u} dx = \log_e|u| + C$

$\int \frac{10x^3 - 7x}{5x^4 - 7x^2 + 8} dx$

$$= \frac{1}{2}\int \frac{20x^3 - 14x}{5x^4 - 7x^2 + 8}dx$$

$$= \frac{1}{2}\log_e|5x^4 - 7x^2 + 8| + C$$

12e Let $u = x^3 - x$

$u' = 3x^2 - 1$

$$\int \frac{u'}{u}dx = \log_e|u| + C$$

$$\int_2^3 \frac{3x^2 - 1}{x^3 - x}dx$$

$$= [\log_e|x^3 - x|]_2^3$$

$$= \log_e|3^3 - 3| - \log_e|2^3 - 2|$$

$$= \log_e|24| - \log_e|6|$$

$$= \log_e 4$$

$$= 2\log_e 2$$

12f Let $u = x^2 + 2x$

$u' = 2x + 2$

$$\int \frac{u'}{u}dx = \log_e|u| + C$$

$$\int_e^{2e} \frac{2x + 2}{x^2 + 2x}dx$$

$$= [\log_e|x^2 + 2x|]_e^{2e}$$

$$= \log_e|4e^2 + 4e| - \log_e|e^2 + 2e|$$

$$= \log_e\left(\frac{4e^2 + 4e}{e^2 + 2e}\right)$$

$$= \log_e\frac{4(e + 1)}{e + 2}$$

13a i $y = x\log_e x - x$

Applying the product rule on $\frac{d}{dx}(x\log_e x)$:

Let $u = x$ and $v = \log_e x$.

Then $u' = 1$ and $v' = \frac{1}{x}$.

$$\begin{aligned}\frac{d}{dx}(uv) &= vu' + uv' \\ &= (\log_e x)(1) + (x)\left(\frac{1}{x}\right) \\ &= \log_e x + 1\end{aligned}$$

$$\begin{aligned}y' &= \log_e x + 1 - \frac{d}{dx}(x) \\ &= \log_e x + 1 - 1 \\ &= \log_e x\end{aligned}$$

So $y' = \log_e x$.

13a ii From part (i), $\frac{d}{dx}(x\log_e x - x) = \log_e x$.

Reversing this to give a primitive we obtain:

$\int \log_e x \, dx = x\log_e x - x + C$ for some constant C

Reversing this to give a primitive we obtain:

$$\begin{aligned}\int_{\sqrt{e}}^{e} \log_e x \, dx &= \left[x\log_e x - x\right]_{\sqrt{e}}^{e} \\ &= e - e - \left(\sqrt{e} \times \log_e e^{\frac{1}{2}} - \sqrt{e}\right) \\ &= \sqrt{e} - \frac{\sqrt{e}}{2} \\ &= \frac{\sqrt{e}}{2}\end{aligned}$$

So $\int_{\sqrt{e}}^{e} \log_e x \, dx = \frac{\sqrt{e}}{2}$.

13b i $y = 2x^2 \log_e x - x^2$

Applying the product rule on $\frac{d}{dx}\left(2x^2 \log_e x\right)$:

Let $u = 2x^2$ and $v = \log_e x$.

Then $u' = 4x$ and $v' = \frac{1}{x}$.

$$\begin{aligned}\frac{d}{dx}(uv) &= vu' + uv' \\ &= (\log_e x)(4x) + (2x^2)\left(\frac{1}{x}\right) \\ &= 4x\log_e x + 2x\end{aligned}$$

$$\begin{aligned}y' &= 4x\log_e x + 2x - \frac{d}{dx}\left(x^2\right) \\ &= 4x\log_e x + 2x - 2x \\ &= 4x\log_e x\end{aligned}$$

So $y' = 4x\log_e x$.

13b ii From part (i), $\frac{d}{dx}\left(2x^2 \log_e x - x^2\right) = 4x\log_e x$.

Reversing this to give a primitive we obtain:

$$\frac{1}{4}\int 4x\log_e x\, dx = \frac{1}{4}\left(2x^2 \log_e x - x^2\right)$$

So $\int x\log_e x\, dx = \frac{1}{2}x^2 \log_e x - \frac{1}{4}x^2 + C$

$$\begin{aligned}\int_e^2 x\log_e x\, dx &= \left[\frac{1}{2}x^2 \log_e x - \frac{1}{4}x^2\right]_e^2 \\ &= (2\log_e 2 - 1) - \left(\frac{1}{2}e^2 - \frac{1}{4}e^2\right) \\ &= 2\log_e 2 - 1 - \frac{1}{4}e^2\end{aligned}$$

So $\int_e^2 x\log_e x\, dx = 2\log_e 2 - 1 - \frac{e^2}{4}$.

13c Let $y=(\log_e x)^2$.

Applying the chain rule:

Let $u=\log_e x$ and so $y=u^2$.

Hence $\dfrac{du}{dx}=\dfrac{1}{x}$ and $\dfrac{dy}{du}=2u$.

$$\begin{aligned}\frac{dy}{dx}&=\frac{dy}{du}\times\frac{du}{dx}\\&=\frac{2}{x}\log_e x\end{aligned}$$

So $\dfrac{dy}{dx}=\dfrac{2\log_e x}{x}$.

Reversing this to give a primitive we obtain:

$$\begin{aligned}\frac{1}{2}\int_{\sqrt{e}}^{\sqrt{e}}\frac{2\log_e x}{x}\,dx&=\left[\frac{1}{2}(\log_e x)^2\right]_{\sqrt{e}}^{e}\\&=\frac{1}{2}-\frac{1}{2}\times\left(\frac{1}{2}\right)^2\\&=\frac{1}{2}-\frac{1}{8}\\&=\frac{3}{8}\end{aligned}$$

So $\displaystyle\int_{\sqrt{e}}^{\sqrt{e}}\frac{\log_e x}{x}\,dx=\frac{3}{8}$.

13d Let $y=\ln(\ln x)$.

Applying the chain rule:

Let $u=\ln x$ and so $y=\ln u$.

Hence $\dfrac{du}{dx}=\dfrac{1}{x}$ and $\dfrac{dy}{du}=\dfrac{1}{u}$.

$$\begin{aligned}\frac{dy}{dx}&=\frac{dy}{du}\times\frac{du}{dx}\\&=\frac{1}{x\ln x}\end{aligned}$$

So $\dfrac{dy}{dx} = \dfrac{1}{x \ln x}$.

From above, $\dfrac{d}{dx}\left(\ln\left(\ln x\right)\right) = \dfrac{1}{x \ln x}$.

Reversing this to give a primitive we obtain:

$\int \dfrac{1}{x \ln x}\ dx = \ln\left(\ln x\right) + C$ for some constant C

14a Given $\int_1^a \dfrac{1}{x} dx = 5$ and $a > 0$.

$$\int_1^a \frac{1}{x} dx = \left[\ln|x|\right]_1^a$$
$$= \ln a$$

$\ln a = 5 \Rightarrow a = e^5$

So $a = e^5$.

14b $\int_a^e \dfrac{1}{x} dx = 5$ and $a > 0$

$$\int_a^e \frac{1}{x} dx = \left[\ln|x|\right]_a^e$$
$$= 1 - \ln a$$

$1 - \ln a = 5$
$\ln a = -4$
$a = e^{-4}$

So $a = e^{-4}$.

14c $\int_{a}^{-1} \frac{1}{x}\,dx = -2$ and $a < 0$

$$\int_{a}^{-1} \frac{1}{x}\,dx = \left[\ln|x|\right]_{a}^{-1}$$
$$= -\ln|a|$$

$$-\ln|a| = -2$$
$$\ln|a| = 2$$
$$|a| = e^{2}$$
$$a = \pm e^{2}$$

So $a = -e^{2}$ as $a < 0$.

14d $\int_{-e}^{a} \frac{1}{x}\,dx = -2$ and $a < 0$

$$\int_{-e}^{a} \frac{1}{x}\,dx = \left[\ln|x|\right]_{-e}^{a}$$
$$= \ln|a| - \ln|-e|$$
$$= \ln|a| - 1$$

$$\ln|a| - 1 = -2$$
$$\ln|a| = -1$$
$$|a| = e^{-1}$$
$$a = \pm e^{-1}$$

So $a = -e^{-1}$ as $a < 0$.

15a We have to find $\int \frac{e^x}{e^x+1}\,dx$

If we observe the numeration and denominator of the above integral, we can say that the integral is of the form $\int \frac{u'}{u}\,du$

Hence, the value of the integral will be $\log|e^x + 1| + C$

15b

$$\int_1^e \left(x+\frac{1}{x^2}\right)^2 dx = \int_1^e \left(x^2+\frac{2}{x}+\frac{1}{x^4}\right)dx$$

$$\begin{aligned}\int_1^e \left(x^2+\frac{2}{x}+\frac{1}{x^4}\right)dx &= \left[\frac{1}{3}x^3+2\ln x-\frac{1}{3x^3}\right]_1^e\\ &= \left(\frac{1}{3}e^3+2-\frac{1}{3}e^{-3}\right)-\left(\frac{1}{3}+0-\frac{1}{3}\right)\\ &= \frac{1}{3}e^3+2-\frac{1}{3}e^{-3}\end{aligned}$$

So $\int_1^e \left(x+\frac{1}{x}\right)^2 dx = \frac{1}{3}\left(e^3-e^{-3}\right)+2$.

15c $xe^x = e^{x+\log_e x}$

$$\begin{aligned}\text{LHS} &= xe^x\\ &= e^{\log_e x}\times e^x\\ &= e^{x+\log_e x}\\ &= \text{RHS}\end{aligned}$$

Hence, $\int xe^x\ dx = \int e^{x+\log_e x}\ dx$

$$=$$

$$= (x-1)e^x + C$$

16 The key to all this is that $\log_e|5x| = \log_e 5 + \log_e|x|$ and so $\log_e|x|$ and $\log_e|5x|$ differ only by a constant $\log_e 5$.

Thus $C_2 = C_1 - \frac{1}{5}\log_e 5$.

As C_1 and C_2 are arbitrary constants, it does not matter at all.

In particular, in a definite integral, adding a constant does not change the answer because it cancels out when we take $F(b)-F(a)$.

17 $y' = \frac{1}{x}$

$\int y'\,dx = \log_e x + C$

Since $x \neq 0$, we have two cases where $x < 0$ and $x > 0$

When $x > 0$,

Let $x = 1, y = 1$,

$1 = \log_e 1 + C$

$1 = 0 + C$

$C = 1$

When $x < 0$,

Let $x = -1, y = 2$,

Since $\log_e x$ cannot have negative values, let $y = \log_e(-x) + C$

$2 = \log_e 1 + C$

$2 = 0 + C$

$C = 2$

Hence,

$$y = \begin{cases} \log_e(-x) + 2, & \text{for } x < 0, \\ \log_e x + 1, & \text{for } x > 0 \end{cases}$$

18a The sum of the sequence on the LHS is a geometric progression with first term = 1 and the total number of terms as $2n + 1$. The geometric ratio is $-t$

Hence, the sum is:

$\Rightarrow\ 1 \times \frac{\left(1-\left(-t^{2n+1}\right)\right)}{1+t}$

$\Rightarrow\ \frac{\left(1+\left(t^{2n+1}\right)\right)}{1+t} = \frac{1}{1+t} + \frac{t^{2n+1}}{1+t} = \text{RHS}$

18b Integrating both the sides of the equation in question 18a, we get:

$\Rightarrow \int_0^x (1 - t + t^2 - \cdots .. + t^{2n})dt = \int_0^x \frac{1}{1+t}dt + \int_0^x \frac{t^{2n+1}}{1+t}dt$

$\Rightarrow (x - 0) - \left(\frac{x^2}{2} - 0\right) + \left(\frac{x^3}{3} - 0\right) - \cdots .. + \left(\frac{x^{2n+1}}{2n+1} - 0\right) = (\log|1 + x| - \log 1) + \int_0^x \frac{t^{2n+1}}{1+t}dt$

As $x > -1, 1 + x > 0$. Hence, the absolute sign is unnecessary.

$\Rightarrow \log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots . + \frac{x^{2n+1}}{2n+1} - \int_0^x \frac{t^{2n+1}}{1+t}dt$

18c It is given that $0 \le t \le 1$. Hence, $1 \le 1 + t \le 2$

Therefore, $\frac{t^{2n+1}}{1+t} \le t^{2n+1}$ as the denominator is more than 1

As the above inequality holds true, we integrate both the sides.

$\Rightarrow \int_0^x \frac{t^{2n+1}}{1+t}dt \le \int_0^x t^{2n+1}dt$

$\Rightarrow \int_0^x \frac{t^{2n+1}}{1+t}dt \le \frac{x^{2n+2}}{2n+2}$

As $0 \le x \le 1$ and $n \to \infty$, the RHS of the inequality tends to 0

Hence, $\int_0^x \frac{t^{2n+1}}{1+t}dt \le 0$

But, the LHS integral can never be negative as $x \ge 0$ *and* $t \ge 0$

Hence, $\int_0^x \frac{t^{2n+1}}{1+t}dt$ converges to 0

Now, lets consider the equation we had got in 18b,

$\Rightarrow \log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots . + \frac{x^{2n+1}}{2n+1} - \int_0^x \frac{t^{2n+1}}{1+t}dt$

As the last term converges to 0 as n tends to infinity, we can say that,

$\Rightarrow \log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots .$

18d i Substituting $x = \frac{1}{2}$ in the equation, we get $\log\frac{3}{2} \doteqdot 0.41$.

18d ii $\log_e 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots ...$

18e The series will be:

$$\Rightarrow \log(1-x) = (-x) - \frac{(-x)^2}{2} + \frac{(-x)^3}{3} - \frac{(-x)^4}{4} + \cdots \ldots. = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots..\right)$$

Hence, $\log_e \frac{1}{2} \doteqdot -0.69$

18f LHS $= \log\frac{1+x}{1-x} = \log(1+x) - \log(1-x)$

LHS $= \left[x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots..\right] - \left[-\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots..\right)\right]$

LHS $= 2\left[x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots \ldots\right] =$ RHS

Putting $x = \frac{1}{2}$, we get $\log_e 3 \doteqdot 1.0986$

Solutions to Exercise 6J

1a

$$\int_1^e \frac{1}{x}\,dx$$

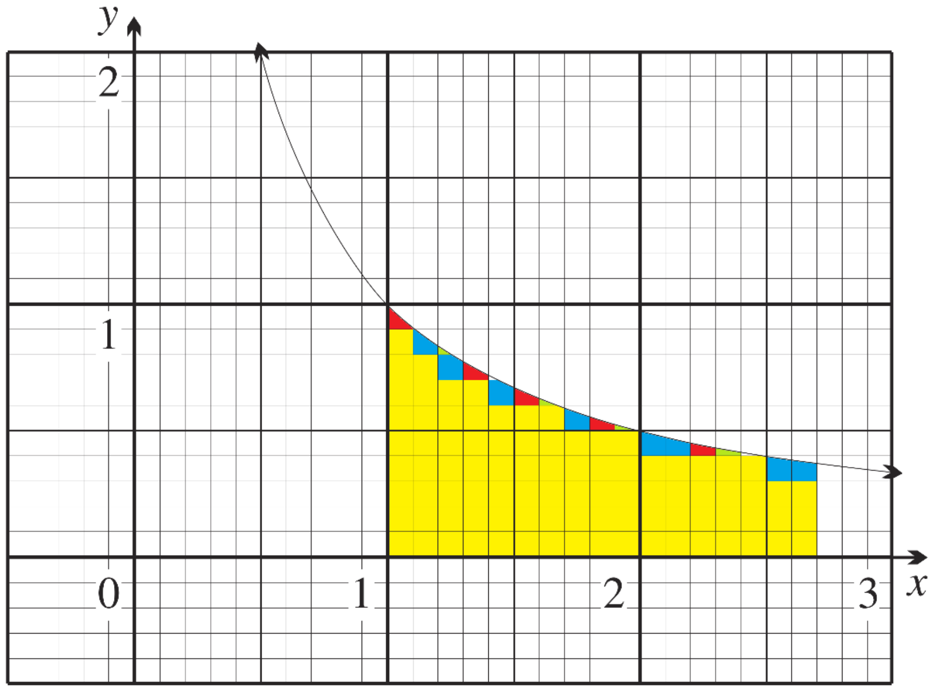
$= [\ln|x|]_1^e$

$= \ln e - \ln 1$

$= 1 - 0$

$= 1$ square unit

1b



Based on the diagram above, the 100 square mark occurs at approximately 2.7.

2i

$$\int_1^5 \frac{1}{x}\,dx$$

$= [\ln|x|]_1^5$

$= \ln 5 - \ln 1$

$= \ln 5$ square units

$\doteqdot 1.609$ square units

2ii The area between the curve is calculated as the definite integral.

$$\int_e^{e^2} \frac{1}{x}\,dx$$

$= [\ln|x|]_e^{e^2}$

$= \ln e^2 - \ln e$

$= 2 - 1$

$= 1$ square unit

2iii

$$\int_2^8 \frac{1}{x}\,dx$$

$= [\ln|x|]_2^8$

$= \ln 8 - \ln 2$

$= \ln 4$

$= 2\ln 2$ square units

$\doteqdot 1.386$ square units

3a

$$\int_2^3 \frac{1}{x}\,dx$$

$= [\ln|x|]_2^3$

$= (\ln 3 - \ln 2)$ square units

3b

$$\int_{\frac{1}{2}}^{2} \frac{1}{x} dx$$

$$= [\ln|x|]_{\frac{1}{2}}^{2}$$

$$= \ln 2 - \ln \frac{1}{2}$$

$$= \ln 2 - (-\ln 2)$$

$$= 2 \ln 2 \text{ square units}$$

4a

$$\int_{0}^{1} \frac{1}{3x+2} dx$$

$$= \left[\frac{1}{3} \ln|3x+2|\right]_{0}^{1}$$

$$= \frac{1}{3} \ln|3(1)+2| - \frac{1}{3} \ln|3(0)+2|$$

$$= \frac{1}{3}(\ln 5 - \ln 2) \text{ square units}$$

$$\doteqdot 0.3054 \text{ square units}$$

4b

$$\int_{2}^{e^3+1} \frac{3}{x-1} dx$$

$$= [3 \ln|x-1|]_{2}^{e^3+1}$$

$$= 3 \ln|(e^3+1)-1| - 3 \ln|(2)-1|$$

$$= 3(\ln e^3 - \ln 1)$$

$$= 9 \ln e$$

$$= 9 \text{ square units}$$

4c

$$\int_{\frac{1}{2}}^{2} \frac{1}{x} + x\,dx$$

$$= \left[\ln|x| + \frac{x^2}{2}\right]_{\frac{1}{2}}^{2}$$

$$= \left(\ln|2| + \frac{2^2}{2}\right) - \left(\ln\left|\frac{1}{2}\right| + \frac{\left(\frac{1}{2}\right)^2}{2}\right)$$

$$= \ln 2 + 2 + \ln 2 - \frac{1}{8}$$

$$= \left(2\ln 2 + \tfrac{15}{8}\right) \text{ square units}$$

4d

$$\int_{1}^{3} \frac{1}{x} + x^2\,dx$$

$$= \left[\ln|x| + \frac{x^3}{3}\right]_{1}^{3}$$

$$= \left(\ln|3| + \frac{3^3}{3}\right) - \left(\ln|1| + \frac{1^3}{3}\right)$$

$$= \ln 3 + \frac{27}{3} - \frac{1}{3}$$

$$= \left(\ln 3 + 8\tfrac{2}{3}\right) \text{ square units}$$

5a

$$\int_{1}^{3} 3 - \frac{3}{x}\,dx$$

$$= [3x - 3\ln|x|]_1^3$$

$$= (3(3) - 3\ln|3|) - (3(1) - 3\ln|1|)$$

$$= 9 - 3\ln 3 - 3 + 0$$

$$= (6 - 3\ln 3) \text{ square units}$$

5b

$$\int_1^3 2 - \frac{1}{x}\,dx$$

$$= [2x - \ln|x|]_1^3$$

$$= (2(3) - \ln|3|) - (2(1) - \ln|1|)$$

$$= 6 - \ln 3 - 2 + 0$$

$$= (4 - \ln 3) \text{ square units}$$

6a

$$\int_1^4 \left(2 - \frac{2}{x}\right) dx - \int_1^4 \frac{1}{2}(x-1)\,dx$$

$$= \int_1^4 \left(2 - \frac{2}{x} - \frac{x}{2} + \frac{1}{2}\right) dx$$

$$= \int_1^4 \left(\frac{5}{2} - \frac{2}{x} - \frac{x}{2}\right) dx$$

$$= \left[\frac{5x}{2} - 2\ln|x| - \frac{x^2}{4}\right]_1^4$$

$$= \left(\frac{5(4)}{2} - 2\ln|4| - \frac{(4)^2}{4}\right) - \left(\frac{5(1)}{2} - 2\ln|1| - \frac{(1)^2}{4}\right)$$

$$= 10 - 2\ln 4 - 4 - \frac{5}{2} + 0 + \frac{1}{4}$$

$$= \left(\frac{15}{4} - 2\ln 4\right) \text{ square units}$$

6b

Rearrange $x + 2y - 5 = 0$ to become $y = \frac{5-x}{2}$

$$\int_1^4 \frac{5-x}{2}\,dx - \int_1^4 \frac{2}{x}\,dx$$

$$= \int_1^4 \left(\frac{5}{2} - \frac{x}{2} - \frac{2}{x}\right) dx$$

$$= \left[\frac{5x}{2} - \frac{x^2}{4} - 2\ln|x|\right]_1^4$$

$$= \left(\frac{5(4)}{2} - \frac{(4)^2}{4} - 2\ln|4|\right) - \left(\frac{5(1)}{2} - \frac{(1)^2}{4} - 2\ln|1|\right)$$

$$= 10 - 4 - 2\ln 4 - \frac{5}{2} + \frac{1}{4} + 0$$

$$= \left(\frac{15}{4} - 2\ln 4\right) \text{ square units}$$

7a

$$\int_1^2 \left(2 - \left(2 - \frac{2}{x}\right)\right) dx$$

$$= \int_1^2 \frac{2}{x}\, dx$$

$$= [2\ln|x|]_1^2$$

$$= (2\ln|2|) - (2\ln|1|)$$

$$= 2\ln 2 \text{ square units}$$

7b

$$\int_{-1}^0 \left(1 - \frac{1}{x+2}\right) dx$$

$$= [x - \ln|x+2|]_{-1}^0$$

$$= (0 - \ln|0+2|) - ((-1) - \ln|(-1)+2|)$$

$$= 0 - \ln 2 + 1 + \ln 1$$

$$= (1 - \ln 2) \text{ square units}$$

8a

$$\int_1^4 \left(0 - \left(-\frac{1}{x}\right)\right) dx$$

$$= [\ln|x|]_1^4$$

$$= (\ln|4|) - (\ln|1|)$$

$= \ln 4 - \ln 1$

$= \ln 4$ square units

8b

$$\int_1^3 \left(0 - \left(\frac{3}{x} - 3\right)\right) dx$$

$= [3x - 3\ln|x|]_1^3$

$= (3(3) - 3\ln|3|) - (3(1) - 3\ln|1|)$

$= 9 - 3\ln 3 - 3 - 3\ln 1$

$= (6 - 3\ln 3)$ square units

9a

$$\int_{\frac{1}{2}}^1 \left(\frac{1}{x} - 1\right) dx + \int_1^2 \left(0 - \left(\frac{1}{x} - 1\right)\right) dx$$

$= [\ln|x| - x]_{\frac{1}{2}}^1 + [x - \ln|x|]_1^2$

$= \left[(\ln|1| - 1) - \left(\ln\left|\frac{1}{2}\right| - \frac{1}{2}\right)\right] + [(2 - \ln|2|) - (1 - \ln|1|)]$

$= 0 - 1 - \ln\frac{1}{2} + \frac{1}{2} + 2 - \ln 2 - 1 + \ln 1$

$= 0 - 1 + \frac{1}{2} + 2 - 1 + \ln 2 - \ln 2 + \ln 1$

$= \frac{1}{2}$ square units

9b

$$\int_1^2 0 - 1 + \frac{2}{x} dx + \int_2^3 1 - \frac{2}{x} dx$$

$= [2\ln|x| - x]_1^2 + [x - 2\ln|x|]_2^3$

$= [(2\ln|2| - 2) - (2\ln|1| - 1)] + [(3 - 2\ln|3|) - (2 - 2\ln|2|)]$

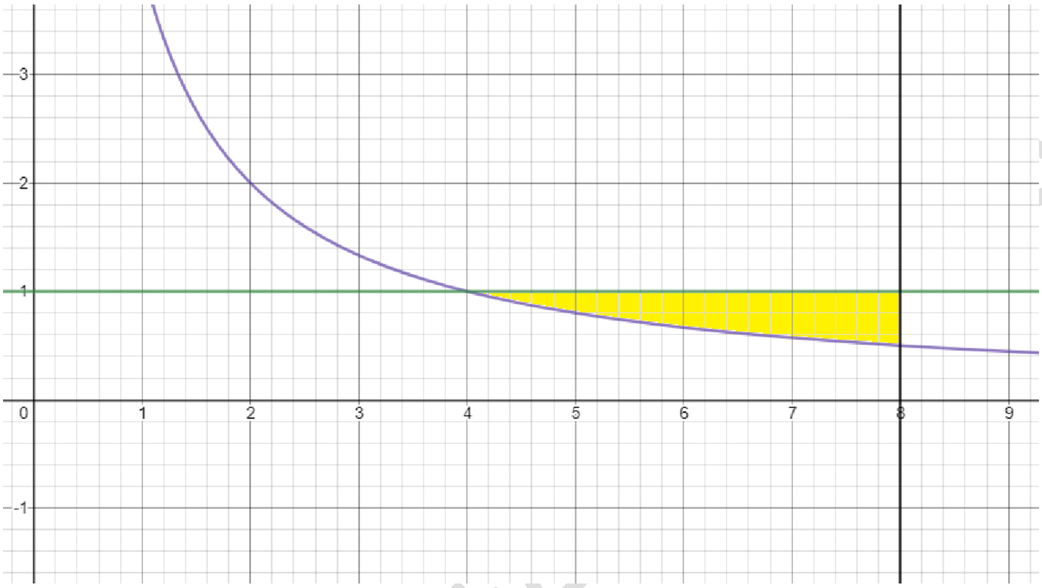
$= 2\ln 2 - 2 - 2\ln 1 + 1 + 3 - 2\ln 3 - 2 + 2\ln 2$

$= 1 - 2 + 3 - 2 + 2\ln 2 - 2\ln 1 - 2\ln 3 + 2\ln 2$

$= 2\ln 4 - 2\ln 3$

$= 2\ln\frac{4}{3}$ square units

10a



10b

$$\int_4^8 1 - \frac{4}{x}\, dx$$

$= [x - 4\ln|x|]_4^8$

$= (8 - 4\ln|8|) - (4 - 4\ln|4|)$

$= 4 - 4 \times 3\ln 2 + 4 \times 2\ln 2$

$= 4 - 12\ln 2 + 8\ln 2$

$= 4 - 4\ln 2$

$= 4(1 - \ln 2)$ square units

11a First, solve the equations simultaneously to determine the intersection point.

$y = \frac{1}{x}, y = 4 - 3x$

$$\frac{1}{x} = 4 - 3x$$

$1 = 4x - 3x^2$

$3x^2 - 4x + 1 = 0$

$(3x-1)(x-1) = 0$

$x = \frac{1}{3}, x = 1$

When $x = \frac{1}{3}, y = \frac{1}{\frac{1}{3}} = 3$

When $x = 1, y = \frac{1}{1} = 1$

The intersection points are $\left(\frac{1}{3}, 3\right), (1,1)$

11b Given that there are only two intersection points, we only need to determine which curve lies above the other, pick a value of $x \in \left(\frac{1}{3}, 1\right)$ between the points of intersection.

Try $x = \frac{2}{3}$, let $f(x) = \frac{1}{x}$, $g(x) = 4 - 3x$

$f\left(\frac{2}{3}\right) = \frac{3}{2}, g\left(\frac{2}{3}\right) = 4 - 3\left(\frac{2}{3}\right) = 2$

As $g\left(\frac{2}{3}\right) > f\left(\frac{2}{3}\right)$ the line $g(x)$ lies above $f(x)$

The integral required to determine the area between the curves is

$$\int_{\frac{1}{3}}^{1} 4 - 3x - \frac{1}{x}\,dx$$

$$= \left[4x - \frac{3}{2}x^2 - \ln|x|\right]_{\frac{1}{3}}^{1}$$

$$= \left(4(1) - \frac{3}{2}(1)^2 - \ln|1|\right) - \left(4\left(\frac{1}{3}\right) - \frac{3}{2}\left(\frac{1}{3}\right)^2 - \ln\left|\frac{1}{3}\right|\right)$$

$$= 4 - \frac{3}{2} - 0 - \frac{4}{3} + \frac{1}{6} + \ln\frac{1}{3}$$

$$= \left(\frac{4}{3} - \ln 3\right) \text{ square units}$$

12a Let $u = x^2 + 1$

$u' = 2x$

Given the standard form $\int \frac{u'}{u} dx = \ln |u| + C$

$$\int_0^2 \frac{x}{x^2 + 1} dx$$

$$= \frac{1}{2} \int_0^2 \frac{2x}{x^2 + 1}$$

$$= \left[\frac{1}{2} \ln|x^2 + 1|\right]_0^2$$

$$= \left(\frac{1}{2} \ln|(2)^2 + 1|\right) - \left(\frac{1}{2} \ln|(0)^2 + 1|\right)$$

$$= \frac{1}{2} \ln 5 - \frac{1}{2} \ln 1$$

$= \frac{1}{2} \ln 5$ square units

12b Let $u = x^2 + 2x + 3$

$u' = 2x + 2$

Given the standard form $\int \frac{u'}{u} dx = \ln |u| + C$

$$\int_0^1 \frac{x + 1}{x^2 + 2x + 3} dx$$

$$= \frac{1}{2} \int_0^1 \frac{2x + 2}{x^2 + 2x + 3}$$

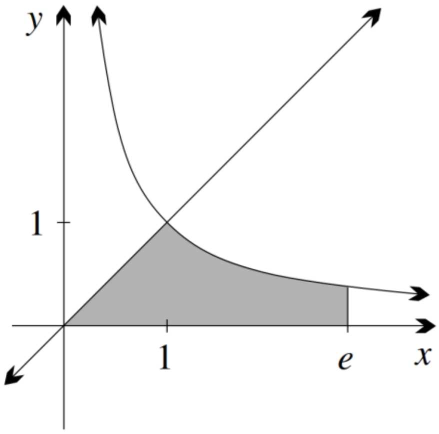
$$= \left[\frac{1}{2} \ln|x^2 + 2x + 3|\right]_0^1$$

$$= \left(\frac{1}{2} \ln|(1)^2 + 2(1) + 3|\right) - \left(\frac{1}{2} \ln|(0)^2 + 2(0) + 3|\right)$$

$$= \frac{1}{2} \ln 6 - \frac{1}{2} \ln 3$$

$= \frac{1}{2} \ln 2$ square units

13a



13b The appropriate integral is a sum of integrals from $x = 0$ to $x = 1$ and from $x = 1$ to $x = e$.

$$\int_0^1 x\,dx + \int_1^e \frac{1}{x}\,dx$$

$$= \left[\frac{x^2}{2}\right]_0^1 + [\ln|x|]_1^e$$

$$= \left(\frac{1^2}{2} - \frac{0^2}{2}\right) + (\ln|e| - \ln|1|)$$

$$= \frac{1}{2} + 1$$

$= 1\frac{1}{2}$ square units

14a Four subintervals require 5 function values between $1 \le x \le 5$

$At\ x = 1, y = \ln 1 = 0$

$At\ x = 2, y = \ln 2$

$At\ x = 3, y = \ln 3$

$At\ x = 4, y = \ln 4 = 2\ln 2$

$At\ x = 5, y = \ln 5$

Dimensions of trapezoid 1:

$a_1 = y(1) = 0$

$b_1 = y(2) = \ln 2$

$h_1 = 2 - 1 = 1$

Area of trapezoid 1, $A_1 = \frac{1}{2}(a_1 + b_1)(h_1) = \frac{1}{2}(0 + \ln 2)(1) = \frac{1}{2}\ln 2$

Dimensions of trapezoid 2:

$a_2 = y(2) = \ln 2$

$b_2 = y(3) = \ln 3$

$h_2 = 3 - 2 = 1$

Area of trapezoid 1, $A_2 = \frac{1}{2}(a_2 + b_2)(h_2) = \frac{1}{2}(\ln 2 + \ln 3)(1) = \frac{1}{2}\ln 6$

Dimensions of trapezoid 3:

$a_3 = y(3) = \ln 3$

$b_3 = y(4) = \ln 4$

$h_3 = 4 - 3 = 1$

Area of trapezoid 1, $A_3 = \frac{1}{2}(a_3 + b_3)(h_3) = \frac{1}{2}(\ln 3 + \ln 4)(1) = \frac{1}{2}\ln 12$

Dimensions of trapezoid 4:

$a_4 = y(4) = \ln 4$

$b_4 = y(5) = \ln 5$

$h_4 = 5 - 4 = 1$

Area of trapezoid 1, $A_4 = \frac{1}{2}(a_4 + b_4)(h_4) = \frac{1}{2}(\ln 4 + \ln 5)(1) = \frac{1}{2}\ln 20$

Total area of trapezoids, $A_1 + A_2 + A_3 + A_4$

$= \frac{1}{2}\ln 2 + \frac{1}{2}\ln 6 + \frac{1}{2}\ln 12 + \frac{1}{2}\ln 20$

$= \frac{1}{2}(\ln(2 \times 6 \times 12 \times 20))$

$= \frac{1}{2}\ln 2880$

$\doteqdot 3.928$ square units

14b Let $u = x$ and $v = \log_e x$

Then $\frac{du}{dx} = 1$ and $\frac{dv}{dx} = \frac{1}{x}$

$\frac{dy}{dx} = (\log_e x \times 1) + \left(x \times \frac{1}{x}\right)$

$= \log_e x + 1$

$= \ln x + 1$

$\int_1^5 \ln x \, dx = \int_1^5 \ln x + 1 - 1 \, dx$

$= \int_1^5 \ln x + 1 \;\, dx - \int_1^5 1 \;\, dx$

$= [x \log_e x - x]_1^5$

$= 5 \log_e 5 - 5 - (0 - 1)$

$= 5 \log_e 5 - 4$

$\doteqdot 4.0472$ square units

14c The estimate is less. The curve is concave down, so the chords are below the curve.

15a $4x = 2(2x + 1) - 2$

LHS $= 4x + 2 - 2$

$= 2(2x) + 2 - 2$

$= 2(2x + 1) - 2$

$=$ RHS

15b $y = \frac{4x}{2x+1}$

$\int_0^1 \frac{4x}{2x+1} dx = \int_0^1 \frac{2(2x+1)-2}{2x+1} dx$

$= \int_0^1 \frac{2(2x+1)}{2x+1} - \frac{2}{2x+1} dx$

$= \int_0^1 2 - \frac{2}{2x+1} dx$

$= \left[2x - \frac{2}{2}\log_e(2x + 1)\right]_0^1$

$$= [2x - \log_e(2x+1)]_0^1$$

$$= 2 - \log_e 3 - (0 - 0)$$

$$= 2 - \log_e 3 \text{ u}^2$$

16a $\frac{6}{x} = x^2 - 6x + 11$

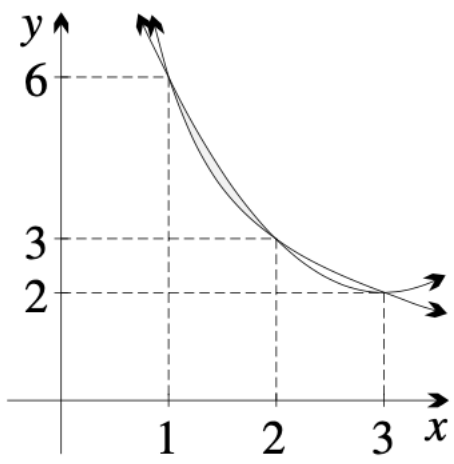
$6 = x^3 - 6x^2 + 11x$

$x^3 - 6x^2 + 11x - 6 = 0$

$(x-1)(x^2 - 5x + 6) = 0$

$(x-1)(x-2)(x-3) = 0$

16b



16c Area $= \int_1^2 x^2 - 6x + 11 - \frac{6}{x}\,dx + \int_2^3 \frac{6}{x} - x^2 + 6x - 11\,dx$

$$= \left[\frac{1}{3}x^3 - 3x^2 + 11x - 6\log_e x\right]_1^2 + \left[6\log_e x - \frac{1}{3}x^3 + 3x^2 - 11x\right]_2^3$$

$$= \frac{8}{3} - 3(4) + 11(2) - 6\log_e 2 - \left(\frac{1}{3} - 3 + 11 - 0\right) + 6\log_e 3 - \frac{27}{3} + 3(9)$$

$$-11(3) - \left(6\log_e 2 - \frac{8}{3} + 3(4) - 11(2)\right)$$

$$= 2 - 6\log_e 2 + 6\log_e 3 + 6\log_e 2$$

$$= \left(2 - 6\log_e \frac{4}{3}\right) \text{ square units}$$

17a The upper rectangle will have a height of $\frac{1}{2^n}$ and the lower rectangle will have a height of $\frac{1}{2^{n+1}}$. The width of both the rectangles will be $2^{n+1} - 2^n$

Hence, the actual area will lie between the area of the lower rectangle and the upper rectangle.

$$\frac{2^{n+1} - 2^n}{2^{n+1}} < \int_{2^n}^{2^{n+1}} \frac{1}{x} dx < \frac{2^{n+1} - 2^n}{2^n}$$

$$\frac{1}{2} < \int_{2^n}^{2^{n+1}} \frac{1}{x} dx < 1$$

17b $\int_1^{2^n} \frac{1}{x} dx = \ln 2^n - \ln 1 = n \ln 2$

Therefore, the integral will tend to ∞ as $n \to \infty$

18a Consider the curves $y = 6e^{-x}$ and $y = e^x - 1$.

The x- coordinate of the intersection point of these two curves satisfies the equation $e^x - 1 = 6e^{-x}$.

Multiplying both sides by e^x we obtain $e^{2x} - e^x = 6$.

Let $u = e^x$ and given that $e^{2x} = (e^x)^2$:

$$(e^x)^2 - e^x - 6 = 0$$

So $u^2 - u - 6 = 0$.

18b $u^2 - u - 6 = 0$

$$(u-3)(u+2) = 0$$

$$u = -2, 3$$

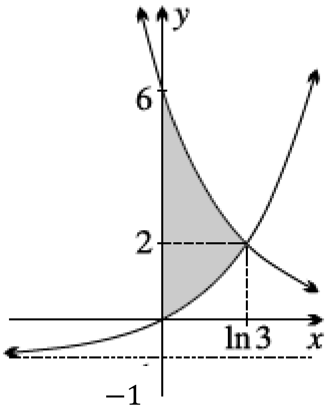
So $e^x = -2$ or $e^x = 3$.

$\log_e(-2)$ does not exist

Hence $x = \ln 3$.

Substituting $x = \ln 3$ into $y = e^x - 1$ we obtain $y = 2$.

18c



18d The area of the shaded region is given $\text{Area} = \int_0^{\ln 3} (\text{top curve} - \text{bottom curve})\, dx$.

$$\int_0^{\ln 3} \left(6e^{-x} - \left(e^x - 1\right)\right) dx = \left[-6e^{-x} - e^x + x\right]_0^{\ln 3}$$
$$= (-2 - 3 + \ln 3) - (-6 - 1 + 0)$$
$$= 2 + \ln 3$$

So the area of the shaded region is $(2 + \ln 3)$ square units.

19a The required area is given by $\int_{-e}^{-1} \left(\frac{1}{x} + 1\right) dx$.

$$\int_{-e}^{-1} \left(\frac{1}{x} + 1\right) dx = \left[\ln|x| + x\right]_{-e}^{-1}$$
$$= (0 - 1) - (1 - e)$$
$$= e - 2$$

So the required area is $(e - 2)$ square units.

19b In this region the sign of y is negative so

the required area is given by $-\int_{-1}^{-e^{-1}} \left(\frac{1}{x} + 1\right)\, dx$.

$\int_{-1}^{-e^{-1}} \left(\frac{1}{x} + 1\right)\, dx$

$= [\ln|x| + x]_{-1}^{-e^{-1}}$

$= (\ln e^{-1} - e^{-1}) - (\ln 1 - 1)$

$= -1 - e^{-1} - 0 + 1$

$= -e^{-1}$

So the required area is e^{-1} square units.

19c The required area is given by $(e-2) + e^{-1}$ or $(e-2+e^{-1})$ square units.

20 $\int \frac{1}{x+\sqrt{x}}\, dx$

Let $u = \sqrt{x}$ and $du = \frac{1}{2\sqrt{x}} dx$, $dx = 2u\, du$

Hence, $\int \frac{1}{x+\sqrt{x}}\, dx = \int \frac{2u}{u^2+u}\, du$

$= 2\int \frac{1}{u+1} du$

$= 2\log_e(u+1) + C$

$= 2\log_e\left(\sqrt{x}+1\right) + C$

21a The derivatives of both the equations have been solved below:

$\frac{d\left(\log(x+\sqrt{x^2+a^2})\right)}{dx} = \frac{1}{x+\sqrt{x^2+a^2}} \times \frac{d\left((x+\sqrt{x^2+a^2})\right)}{dx}$

$\frac{d\left(\log(x+\sqrt{x^2+a^2})\right)}{dx} = \frac{1}{x+\sqrt{x^2+a^2}} \times \left[1 + \frac{2x}{2\sqrt{x^2+a^2}}\right]$

$\frac{d\left(\log(x+\sqrt{x^2+a^2})\right)}{dx} = \frac{1}{\sqrt{x^2+a^2}}$

Now we calculate the other derivative:

$\frac{d\left(\log(x+\sqrt{x^2-a^2})\right)}{dx} = \frac{1}{x+\sqrt{x^2-a^2}} \times \frac{d\left((x+\sqrt{x^2-a^2})\right)}{dx}$

$$\frac{d\left(\log(x+\sqrt{x^2-a^2})\right)}{dx} = \frac{1}{x+\sqrt{x^2-a^2}} \times \left[1+\frac{2x}{2\sqrt{x^2-a^2}}\right]$$

$$\frac{d\left(\log(x+\sqrt{x^2-a^2})\right)}{dx} = \frac{1}{\sqrt{x^2-a^2}}$$

21b i Substituting the value of $a = 1$ in the equation above, we take the definite integral of the derivative obtained in 21a to find the value:

$$\int_0^1 \frac{1}{\sqrt{1+x^2}} = \log\left(1+\sqrt{1+1}\right) - \log 1 = \log(1+\sqrt{2})$$

21b ii Substituting the value of $a = 4$ in the equation in 21a, we take the definite integral of the derivative obtained in 21a to find the value:

$$\int_4^8 \frac{1}{\sqrt{x^2-16}}$$

$$= \log\left(8+\sqrt{64-16}\right) - \log\left(4+\sqrt{16-16}\right)$$

$$= \log\left(2+\sqrt{3}\right) + \log 4 - \log 4$$

$$= \log\left(2+\sqrt{3}\right)$$

22a The height of the upper rectangle is $\frac{1}{n}$ and the height of the lower rectangle is $\frac{1}{n+1}$. The width of both the rectangles is 1 unit. Hence, the actual area obtained from the integral of the function given will lie between the area of the lower and upper rectangle. Hence,

$$\frac{1}{n+1} < \int_n^{n+1} \frac{1}{x}\,dx < \frac{1}{n}$$

22b Expanding the above inequality, we get:

$$\frac{1}{n+1} < \ln(n+1) - \ln n < \frac{1}{n}$$

$$\frac{1}{n+1} < \ln\frac{(n+1)}{n} < \frac{1}{n}$$

Multiplying the inequality by ‘n’, we get:

$$\frac{n}{n+1} < \mathrm{n}\ln(1+\frac{1}{n}) < 1$$

$$\frac{n}{n+1} < \ln(1+\frac{1}{n})^n < 1$$

22c $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$

$$= \lim_{n\to\infty} e^{\ln\left(1+\frac{1}{n}\right)^n}$$

$$= \lim_{n\to\infty} e^{n\ln(1+\frac{1}{n})}$$

$$= \lim_{n\to\infty} e^{\frac{\ln(1+\frac{1}{n})}{\frac{1}{n}}}$$

Applying L'Hopital rule, we get:

$$\Rightarrow \; e^{\frac{\lim_{n\to\infty}\left(\left(-\frac{1}{n^2}\right)(-n^2)\right)}{1+\frac{1}{n}}}$$

$$\Rightarrow \; e^{\frac{\lim_{n\to\infty}(1)}{1+\frac{1}{n}}}$$

$$\Rightarrow \; e^1 = e$$

22d $\lim_{n\to\infty}\left(1+\frac{t}{n}\right)^n$

$$= \lim_{n\to\infty} e^{\ln\left(1+\frac{t}{n}\right)^n}$$

$$= \lim_{n\to\infty} e^{n\ln(1+\frac{t}{n})}$$

$$= \lim_{n\to\infty} e^{\frac{\ln(1+\frac{t}{n})}{\frac{1}{n}}}$$

Applying L'Hopital rule, we get:

$$e^{\frac{\lim_{n\to\infty}\left(\left(-\frac{t}{n^2}\right)(-n^2)\right)}{1+\frac{t}{n}}}$$

$$= e^t$$

23a i $ABCO$ can be considered as the upper rectangle. The area of the curve will always be greater than 0 as the graph lies above the x axis.

Hence, the inequality will go as:

$$0 < \int_1^{\sqrt{x}} \frac{1}{t} dt < \text{area of } ABCO$$

23a ii The width of ABCO is $\sqrt{x}$ and height is 1.

Hence, expanding the inequality above, we get:

$$0 < \ln \sqrt{x} < \sqrt{x}$$

Multiplying the inequality by $\frac{2}{x}$, we get:

$$0 < \frac{\ln x}{x} < \frac{2}{\sqrt{x}}$$

Note that the sign of the inequality did not change as $x > 1$.

23a iii L'Hopital rule can be directly used on $\lim\limits_{x\to\infty} \frac{\ln x}{x}$

$$\frac{\lim\limits_{x\to\infty} 1}{x} = 0$$

23b $\lim\limits_{u\to 0+} \frac{\ln\frac{1}{u}}{\frac{1}{u}} = \lim\limits_{u\to 0+} (-u \ln u)$

L'Hopital rule can be applied directly. We get:

$$\frac{\lim\limits_{u\to 0+} \left(-\frac{1}{u}\right)/}{-\frac{1}{u^2}}$$

$$\lim_{u\to 0+} u = 0$$

23c We know that $\lim\limits_{x\to 0+} x \ln x = 0$. Substituting $x = e^u$, we get $\lim\limits_{u\to -\infty} ue^u$

We have to note that the limit changes from $x \to 0+$ to $u \to -\infty$ to keep the original limit valid.

Hence, $\lim\limits_{u \to -\infty} ue^{u} = 0$

23d We know that $\lim\limits_{x \to \infty} \frac{\ln x}{x} = 0$. Substituting $x = e^{u}$, we get $\lim\limits_{u \to \infty} ue^{-u}$ after simple transformation.

Hence, $\lim\limits_{u \to \infty} ue^{-u} = 0$

Solutions to Exercise 6K

Let C be a constant.

1a

$$\log_2 3 = \frac{\log_e 3}{\log_e 2} \doteqdot 1.58$$

Since $\log_2 3 \doteqdot 1.58$, $2^{1.58} \doteqdot 2.99$, which approximates to 3

1b

$$\log_2 10 = \frac{\log_e 10}{\log_e 2} \doteqdot 3.32$$

Since $\log_2 10 \doteqdot 3.32$, $2^{3.32} \doteqdot 9.99$, which approximates to 10

1c

$$\log_5 26 = \frac{\log_e 26}{\log_e 5} \doteqdot 2.02$$

Since $\log_5 26 \doteqdot 2.02$, $5^{2.02} \doteqdot 25.82$, which approximates to 26

1d

$$\log_3 0.0047 = \frac{\log_e 0.0047}{\log_e 3} \doteqdot -4.88$$

Since $\log_3 0.0047 \doteqdot -4.88$, $3^{-4.88} \doteqdot 0.004\,695$, which approximates to 0.0047

2a

$$y = \log_2 x = \frac{\log_e x}{\log_e 2}$$

Since $\frac{1}{\log_e 2}$ is a constant,

$$y' = \frac{d}{dx}(\log_e x) \times \frac{1}{\log_e 2}$$

$$= \frac{1}{x} \times \frac{1}{\log_e 2}$$

$$= \frac{1}{x \log_e 2}$$

2b

$$y = \log_{10} x = \frac{\log_e x}{\log_e 10}$$

Since $\frac{1}{\log_e 10}$ is a constant,

$$y' = \frac{d}{dx}(\log_e x) \times \frac{1}{\log_e 10}$$

$$= \frac{1}{x} \times \frac{1}{\log_e 10}$$

$$= \frac{1}{x \log_e 10}$$

2c

$$y = 3\log_5 x = \frac{3\log_e x}{\log_e 5}$$

Since $\frac{3}{\log_e 5}$ is a constant,

$$y' = \frac{d}{dx}(\log_e x) \times \frac{3}{\log_e 5}$$

$$= \frac{1}{x} \times \frac{3}{\log_e 5}$$

$$= \frac{3}{x \log_e 5}$$

3a Standard form $\frac{d}{dx}\log_a x = \frac{1}{x \log_e a}$

$$y = \log_3 x$$

$$y' = \frac{1}{x \log_e 3}$$

3b Standard form $\frac{d}{dx}\log_a x = \frac{1}{x\log_e a}$

$y = \log_7 x$

$y' = \dfrac{1}{x\log_e 7}$

3c Standard form $\frac{d}{dx}\log_a x = \frac{1}{x\log_e a}$

$y = 5\log_6 x$

$y' = 5\dfrac{d}{dx}\log_6 x$

$= \dfrac{5}{x\log_e 6}$

4a $y = 3^x$

$= \left(e^{\log_e 3}\right)^x$

$= e^{x\log_e 3}$

$y' = e^{x\log_e 3}\,\frac{d}{dx}(x\log_e 3)$ by the chain rule

$= e^{x\log_e 3}\,\log_e 3$

$= 3^x\log_e 3$

4b $y = 4^x$

$= \left(e^{\log_e 4}\right)^x$

$= e^{x\log_e 4}$

$y' = e^{x\log_e 4}\,\frac{d}{dx}(x\log_e 4)$ by the chain rule

$= e^{x\log_e 4}\,\log_e 4$

$= 4^x\log_e 4$

4c $y = 2^x$

$= \left(e^{\log_e 2}\right)^x$

$= e^{x \log_e 2}$

$y' = e^{x \log_e 2} \frac{d}{dx}(x \log_e 2)$ by the chain rule

$= e^{x \log_e 2} \log_e 2$

$= 2^x \log_e 2$

5a Standard form:

$\frac{d}{dx} a^x = a^x \log_e a$

$y = 10^x$

$y' = 10^x \log_e 10$

5b Standard form:

$\frac{d}{dx} a^x = a^x \log_e a$

$y = 8^x$

$y' = 8^x \log_e 8$

5c Standard form:

$\frac{d}{dx} a^x = a^x \log_e a$

$y = 3 \times 5^x$

$y' = 3(5^x \log_e 5)$

6a

$$\int 2^x \, dx = \int e^{x \log_e 2} \, dx$$

Integrate by substitution.

Let $u = x \log_e 2$, $\frac{du}{dx} = \log_e 2$ so $du = \log_e 2 \;\, dx$

$$\int e^{x\log_e 2}\,dx$$

$$= \frac{1}{\log_e 2}\int e^{x\log_e 2}\log_e 2\,dx$$

$$= \frac{1}{\log_e 2}\int e^u\,du$$

$$= \frac{1}{\log_e 2}\times e^u + C$$

$$= \frac{e^{x\log_e 2}}{\log_e 2} + C$$

$$= \frac{2^x}{\log_e 2} + C$$

6b

$$\int 6^x\,dx = \int e^{x\log_e 6}\,dx$$

Integrate by substitution.

Let $u = x\log_e 6\,, \frac{du}{dx} = \log_e 6$ so $du = \log_e 6\ \ dx$

$$\int e^{x\log_e 6}\,dx$$

$$= \frac{1}{\log_e 6}\int e^{x\log_e 6}\log_e 6\,dx$$

$$= \frac{1}{\log_e 6}\int e^u\,du$$

$$= \frac{1}{\log_e 6}\times e^u + C$$

$$= \frac{e^{x\log_e 6}}{\log_e 6} + C$$

$$= \frac{6^x}{\log_e 6} + C$$

6c

$$\int 7^x \, dx = \int e^{x \log_e 7} \, dx$$

Integrate by substitution.

Let $u = x \log_e 7, \frac{du}{dx} = \log_e 7$ so $du = \log_e 7 \; dx$

$$\int e^{x \log_e 7} \, dx$$

$$= \frac{1}{\log_e 7} \int e^{x \log_e 7} \log_e 7 \, dx$$

$$= \frac{1}{\log_e 7} \int e^u \, du$$

$$= \frac{1}{\log_e 7} \times e^u + C$$

$$= \frac{e^{x \log_e 7}}{\log_e 7} + C$$

$$= \frac{7^x}{\log_e 7} + C$$

6d

$$\int 3^x \, dx = \int e^{x \log_e 3} \, dx$$

Integrate by substitution.

Let $u = x \log_e 3, \frac{du}{dx} = \log_e 3$ so $du = \log_e 3 \; dx$

$$\int e^{x \log_e 3} \, dx$$

$$= \frac{1}{\log_e 3} \int e^{x \log_e 3} \log_e 3 \, dx$$

$$= \frac{1}{\log_e 3} \int e^u \, du$$

$$= \frac{1}{\log_e 3} \times e^u + C$$

$$= \frac{e^{x \log_e 3}}{\log_e 3} + C$$

$$= \frac{3^x}{\log_e 3} + C$$

7a Standard form:

$$\int a^x \, dx = \frac{a^x}{\log_e a} + C$$

$$\int_0^1 2^x \, dx$$

$$= \left[\frac{2^x}{\log_e 2}\right]_0^1$$

$$= \frac{2^1}{\log_e 2} - \frac{2^0}{\log_e 2}$$

$$= \frac{2-1}{\log_e 2}$$

$$= \frac{1}{\log_e 2}$$

$$\doteqdot 1.443$$

7b Standard form:

$$\int a^x \, dx = \frac{a^x}{\log_e a} + C$$

$$\int_0^1 3^x \, dx$$

$$= \left[\frac{3^x}{\log_e 3}\right]_0^1$$

$$= \frac{3^1}{\log_e 3} - \frac{3^0}{\log_e 3}$$

$$= \frac{3-1}{\log_e 3}$$

$$= \frac{2}{\log_e 3}$$

$$\doteqdot 1.820$$

7c Standard form:

$$\int a^x \, dx = \frac{a^x}{\log_e a} + C$$

$$\int_{-1}^{1} 5^x \, dx$$

$$= \left[\frac{5^x}{\log_e 5}\right]_{-1}^{1}$$

$$= \frac{5^1}{\log_e 5} - \frac{5^{-1}}{\log_e 5}$$

$$= \frac{5 - \frac{1}{5}}{\log_e 5}$$

$$= \frac{\frac{24}{5}}{\log_e 5}$$

$$= \frac{24}{5 \log_e 5}$$

$$\doteqdot 2.982$$

7d Standard form:

$$\int a^x \, dx = \frac{a^x}{\log_e a} + C$$

$$\int_{0}^{2} 4^x \, dx$$

$$= \left[\frac{4^x}{\log_e 4}\right]_{0}^{2}$$

$$= \frac{4^2}{\log_e 4} - \frac{4^0}{\log_e 4}$$

$$= \frac{16-1}{\log_e 4}$$

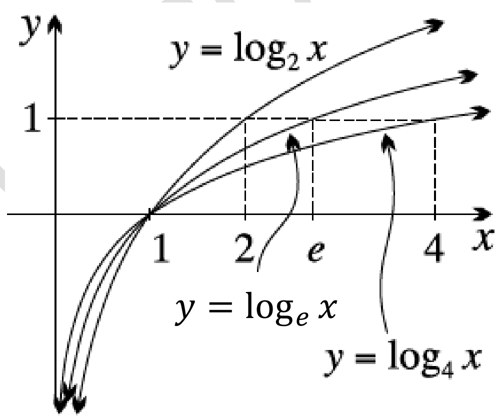
$$= \frac{15}{\log_e 4}$$

$$\doteqdot 10.82$$

8a

x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
$\log_2 x$	$\log_2 2^{-2}$ $= -2$	$\log_2 2^{-1}$ $= -1$	$\log_2 1 = 0$	$\log_2 2 = 1$	$\log_2 2^2 = 2$
$\log_e x$	$\log_e 2^{-2}$ $= -2\log_e 2$ $= -1.39$	$\log_e 2^{-1}$ $= -1\log_e 2$ $= -0.69$	$\log_e 1 = 0$	$\log_e 2$ $= 0.69$	$\log_e 2^2$ $= 2\log_e 2$ $= 1.39$
$\log_4 x$	$\log_4 4^{-1}$ $= -\log_4 4$ $= -1$	$\log_4 4^{-\frac{1}{2}}$ $= -\frac{1}{2}\log_4 4$ $= -\frac{1}{2}$	$\log_4 1 = 0$	$\log_4 4^{\frac{1}{2}}$ $= \frac{1}{2}\log_4 4$ $= \frac{1}{2}$	$\log_4 4$ $= 1$

8b



9a $y = \log_2 x$

$y' = \dfrac{1}{x \log_e 2}$

The tangent of the gradient to the curve at $x = 1$ is $y' = \frac{1}{\log_e 2}$.

9b Let the equation of the tangent be $f(x) = mx + b, m = \frac{1}{\log_e 2}$

At $x = 1, y = \log_2 1 = 0$

The tangent passes through the point $(1, 0)$.

$f(1) = \dfrac{1}{\log_e 2} + b = 0$

$b = -\dfrac{1}{\log_e 2}$

Equation of tangent, $f(x) = \frac{x}{\log_e 2} - \frac{1}{\log_e 2} = \frac{1}{\log_e 2}(x - 1)$

9c i $y = \log_3 x$

$y' = \dfrac{1}{x \log_e 3}$

The tangent of the gradient to the curve at $x = 1$ is $y' = \frac{1}{\log_e 3}$.

Let the equation of the tangent be $f(x) = mx + b, m = \frac{1}{\log_e 3}$.

At $x = 1, y = \log_3 1 = 0$

The tangent passes through the point $(1, 0)$.

$f(1) = \dfrac{1}{\log_e 3} + b = 0$

$b = -\dfrac{1}{\log_e 3}$

Equation of tangent, $f(x) = \frac{x}{\log_e 3} - \frac{1}{\log_e 3} = \frac{1}{\log_e 3}(x - 1)$

9c ii $y = \log_5 x$

$$y' = \frac{1}{x \log_e 5}$$

The tangent of the gradient to the curve at $x = 1$ is $y' = \frac{1}{\log_e 5}$.

Let the equation of the tangent be $f(x) = mx + b, m = \frac{1}{\log_e 5}$.

At $x = 1, y = \log_5 1 = 0$

The tangent passes through the point $(1, 0)$.

$$f(1) = \frac{1}{\log_e 5} + b = 0$$

$$b = -\frac{1}{\log_e 5}$$

Equation of tangent, $f(x) = \frac{x}{\log_e 5} - \frac{1}{\log_e 5} = \frac{1}{\log_e 5}(x - 1)$

10a

$$\int_1^3 2^x \, dx$$

$$= \left[\frac{2^x}{\log_e 2}\right]_1^3$$

$$= \frac{2^3}{\log_e 2} - \frac{2^1}{\log_e 2}$$

$$= \frac{8 - 2}{\log_e 2}$$

$$= \frac{6}{\log_e 2}$$

$$\doteqdot 8.6562$$

10b

$$\int_{-1}^{1} 3^x + 1\,dx$$

$$= \left[\frac{3^x}{\log_e 3} + x\right]_{-1}^{1}$$

$$= \left(\frac{3^1}{\log_e 3} + 1\right) - \left(\frac{3^{-1}}{\log_e 3} - 1\right)$$

$$= \frac{3 - \frac{1}{3}}{\log_e 3} + 2$$

$$= \frac{8}{3\log_e 3} + 2$$

$$\doteqdot 4.4273$$

10c

$$\int_{0}^{2} (10^x - 10x)\,dx$$

$$= \left[\frac{10^x}{\log_e 10} - 5x^2\right]_{0}^{2}$$

$$= \left(\frac{10^2}{\log_e 10} - 5(2)^2\right) - \left(\frac{10^0}{\log_e 10} - 5(0)^2\right)$$

$$= \frac{100 - 1}{\log_e 10} - 20$$

$$= \frac{99}{\log_e 10} - 20$$

$$\doteqdot 22.9952$$

11a

$$y = \log_{10} x = \frac{\log_e x}{\log_e 10}$$

$$y' = \frac{1}{\log_e 10}\frac{d}{dx}\log_e x$$

$$y' = \frac{1}{x \log_e 10}$$

At $(10, 1)$,

$$y' = \frac{1}{10 \log_e 10}$$

11b Let the equation of the tangent be $y = mx + b, m = \frac{1}{10 \log_e 10}$

The tangent passes through the point $(10, 1)$.

$$y = \frac{10}{10 \log_e 10} + b = 1$$

$$b = 1 - \frac{10}{10 \log_e 10} = 1 - \frac{1}{\log_e 10}$$

Equation of tangent, $y = \frac{x}{10 \log_e 10} + 1 - \frac{1}{\log_e 10}$

Rearranging gives:

$$\frac{x}{10 \log_e 10} + 1 - \frac{1}{\log_e 10} - y = 0$$

$$x - 10y \log_e 10 + 10 \log_e 10 - 10 = 0$$

11c The tangent has a gradient of 1 when $y' = 1$

$$\frac{1}{x \log_e 10} = 1$$

$$x = \frac{1}{\log_e 10}$$

12a For $y = \log_2 x$

$$y' = \frac{1}{x \log_e 2}$$

At $x = 3, y = \log_2 3\,, y' = \frac{1}{3 \log_e 2}$

Let the equation of the tangent be $y = mx + b, m = \frac{1}{3 \log_e 2}$

$\log_2 3 = \dfrac{3}{3\log_e 2} + b$

$b = \log_2 3 - \dfrac{1}{\log_e 2} = \dfrac{\log_e 3 - 1}{\log_e 2}$

Equation of tangent to $y = \log_2 x$ at $x = 3$ is

$y = \dfrac{x}{3\log_e 2} + \dfrac{\log_e 3 - 1}{\log_e 2}$

$y = \dfrac{1}{\log_e 2}\left(\dfrac{x}{3} + \log_e 3 - 1\right)$

For $y = \log_e x$

$y' = \dfrac{1}{x}$

At $x = 3, y = \log_e 3\,, y' = \frac{1}{3}$

Let the equation of the tangent be $y = mx + b, m = \frac{1}{3}$

$\log_e 3 = \dfrac{3}{3} + b$

$b = \log_e 3 - 1$

Equation of tangent to $y = \log_e x$ at $x = 3$ is

$y = \dfrac{x}{3} + \log_e 3 - 1$

For $y = \log_4 x$

$y' = \dfrac{1}{x\log_e 4}$

At $x = 3, y = \log_4 3\,, y' = \frac{1}{3\log_e 4}$

Let the equation of the tangent be $y = mx + b, m = \frac{1}{3\log_e 4}$

$\log_4 3 = \dfrac{3}{3\log_e 4} + b$

$b = \log_4 3 - \dfrac{1}{\log_e 4} = \dfrac{\log_e 3 - 1}{\log_e 4}$

Equation of tangent to $y = \log_4 x$ at $x = 3$ is

$y = \dfrac{x}{3\log_e 4} + \dfrac{\log_e 3 - 1}{\log_e 4}$

$y = \dfrac{1}{\log_e 4}\left(\dfrac{x}{3} + \log_e 3 - 1\right)$

12b Check when $y = 0$ for each curve.

For $y = \frac{1}{\log_e 2}\left(\frac{x}{3} + \log_e 3 - 1\right)$,

$0 = \dfrac{1}{\log_e 2}\left(\dfrac{x}{3} + \log_e 3 - 1\right)$

$x = 3 - 3\log_e 3$

For $y = \frac{x}{3} + \log_e 3 - 1$,

$0 = \dfrac{x}{3} + \log_e 3 - 1$

$x = 3 - 3\log_e 3$

For $y = \frac{1}{\log_e 4}\left(\frac{x}{3} + \log_e 3 - 1\right)$,

$0 = \dfrac{1}{\log_e 4}\left(\dfrac{x}{3} + \log_e 3 - 1\right)$

$x = 3 - 3\log_e 3$

All derived tangents above meet at $(3 - 3\log_e 3\,, 0)$.

13a At $x = 0$,

$y = 2^0 = 1$

$y = 1 + 2(0) - (0)^2 = 1$

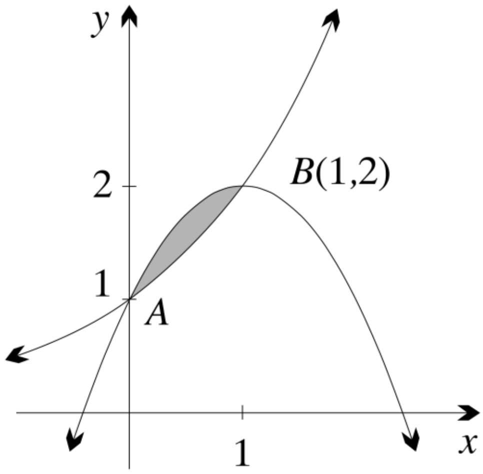
At $x = 1$,

$y = 2^1 = 2$

$y = 1 + 2(1) - 1^2 = 2$

Therefore, both $y = 2^x$ and $y = 1 + 2x - x^2$ intersect at points $A(0, 1)$ and $B(1, 2)$.

13b



The area shaded is defined by the definite integral:

$$\int_0^1 (1 + 2x - x^2 - 2^x)\,dx$$

$$= \left[x + x^2 - \frac{x^3}{3} - \frac{2^x}{\log_e 2}\right]_0^1$$

$$= \left(1 + 1^2 - \frac{1^3}{3} - \frac{2^1}{\log_e 2}\right) - \left(0 + 0^2 - \frac{0^3}{3} - \frac{2^0}{\log_e 2}\right)$$

$$= \left(1 + 1 - \frac{1}{3} - \frac{2}{\log_e 2}\right) - \left(-\frac{1}{\log_e 2}\right)$$

$$= \left(1\frac{2}{3} - \frac{1}{\log_e 2}\right) \text{ square units}$$

14 x-intercept, $y = 0 = 8 - 2^x$

$2^x = 8$

$x = \log_2 8$

$x = \log_2 2^3 = 3$

y-intercept, $x = 0$,

$y = 8 - 2^0 = 7$

Intercepts are: $(0, 7)$, $(3, 0)$

The area under the curve bounded by the coordinate axes is

$$\int_0^3 (8 - 2^x)\,dx$$

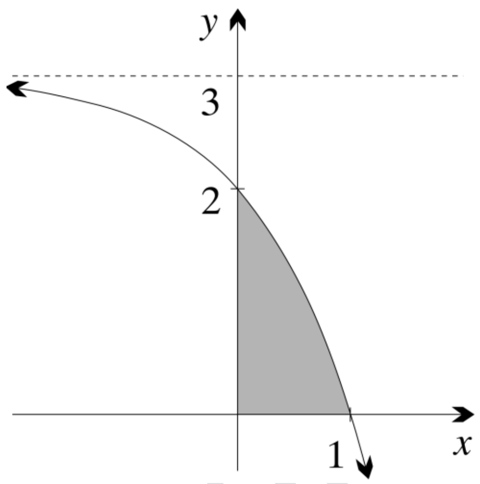
$$= \left[8x - \frac{2^x}{\log_e 2}\right]_0^3$$

$$= \left(8(3) - \frac{2^3}{\log_e 2}\right) - \left(8(0) - \frac{2^0}{\log_e 2}\right)$$

$$= 24 - \frac{8}{\log_e 2} + \frac{1}{\log_e 2}$$

$$= \left(24 - \frac{7}{\log_e 2}\right) \text{ square units}$$

15a



15b

$$\int_0^1 (3 - 3^x)\,dx$$

$$= \left[3x - \frac{3^x}{\log_e 3}\right]_0^1$$

$$= \left(3(1) - \frac{3^1}{\log_e 3}\right) - \left(3(0) - \frac{3^0}{\log_e 3}\right)$$

$$= 3 - \frac{3}{\log_e 3} + \frac{1}{\log_e 3}$$

$= \left(3 - \frac{2}{\log_e 3}\right)$ square units

16a First find the y-intercept.

At $x = 0, y = 4^0 = 1, y = 0 + 1 = 1$

At $x = -\frac{1}{2}, y = 4^{-\frac{1}{2}} = \frac{1}{4^{\frac{1}{2}}} = \frac{1}{2}, y = -\frac{1}{2} + 1 = \frac{1}{2}$

Both $y = 4^x$ and $y = x + 1$ intersect at y-intercept $(0, 1)$ and the point $\left(-\frac{1}{2}, \frac{1}{2}\right)$.

16b Determine which of the curves lies above the other, between the interval $\left[-\frac{1}{2}, 0\right]$

Choose $x = -\frac{1}{4}$,

$$y = 4^{-\frac{1}{4}} = \frac{1}{4^{\frac{1}{4}}} = \frac{1}{\sqrt{2}} \doteqdot 0.7071$$

$$y = -\frac{1}{4} + 1 = \frac{3}{4} = 0.75 > 0.7071$$

$y = x + 1$ is located above $y = 4^x$ over the interval of $\left(-\frac{1}{2}, 0\right)$.

The integral which defines the area of the enclosed region is:

$$\int_{-\frac{1}{2}}^{0} (x + 1 - 4^x)\, dx$$

16c $\int_{-\frac{1}{2}}^{0} (x + 1 - 4^x)\, dx$

$$= \left[\frac{x^2}{2} + x - \frac{4^x}{\log_e 4}\right]_{-\frac{1}{2}}^{0}$$

$$= \left(\frac{0^2}{2} + 0 - \frac{4^0}{\log_e 4}\right) - \left(\frac{\left(-\frac{1}{2}\right)^2}{2} + \left(-\frac{1}{2}\right) - \frac{4^{-\frac{1}{2}}}{\log_e 4}\right)$$

$$= -\frac{1}{\log_e 4} - \frac{1}{8} + \frac{1}{2} + \frac{1}{2\log_e 4}$$

$$= \frac{3}{8} - \frac{1}{2\log_e 4}$$

17a $y = \log_3 x$

$$\frac{dy}{dx} = \frac{1}{x\log_e 3}$$

So the tangent at $A(e, \log_3 e)$ has gradient $\frac{1}{e\log_e 3}$.

Using change of base, $\log_3 e = \frac{\log_e e}{\log_e 3} = \frac{1}{\log_e 3}$.

The tangent is $y - \frac{1}{\log_e 3} = \frac{1}{e\log_e 3}(x - e)$.

$$y = \frac{x}{e\log_e 3} - \frac{1}{\log_e 3} + \frac{1}{\log_e 3}$$
$$= \frac{x}{e\log_e 3}$$

This tangent has gradient $\frac{1}{e\log_e 3}$ and passes through the origin.

17b $y = \log_5 x$

$$\frac{dy}{dx} = \frac{1}{x\log_e 5}$$

So the tangent at $A(e, \log_5 e)$ has gradient $\frac{1}{e\log_e 5}$.

Using change of base, $\log_5 e = \frac{\log_e e}{\log_e 5} = \frac{1}{\log_e 5}$.

The tangent is $y - \frac{1}{\log_e 5} = \frac{1}{e\log_e 5}(x - e)$.

$$y = \frac{x}{e\log_e 5} - \frac{1}{\log_e 5} + \frac{1}{\log_e 5}$$
$$= \frac{x}{e\log_e 5}$$

This tangent has gradient $\frac{1}{e\log_e 5}$ and passes through the origin.

17c $y = \log_a x$

$$\frac{dy}{dx} = \frac{1}{x\log_e a}$$

So the tangent at $A(e, \log_a e)$ has gradient $\frac{1}{e\log_e a}$.

Using change of base, $\log_a e = \frac{\log_e e}{\log_e a} = \frac{1}{\log_e a}$.

The tangent is $y - \frac{1}{\log_e a} = \frac{1}{e\log_e a}(x - e)$.

$$y = \frac{x}{e\log_e a} - \frac{1}{\log_e a} + \frac{1}{\log_e a}$$
$$= \frac{x}{e\log_e a}$$

This tangent has gradient $\frac{1}{e\log_e a}$ and passes through the origin.

18a Let $y = x\log_e x - x$.

Applying the product rule on $\frac{d}{dx}(x\log_e x)$:

Let $u = x$ and $v = \log_e x$.

Then $u' = 1$ and $v' = \frac{1}{x}$.

$$\begin{aligned}\frac{d}{dx}(uv) &= vu' + uv' \\ &= (\log_e x)(1) + (x)\left(\frac{1}{x}\right) \\ &= \log_e x + 1\end{aligned}$$

$$\begin{aligned}y' &= \log_e x + 1 - \frac{d}{dx}(x) \\ &= \log_e x + 1 - 1 \\ &= \log_e x\end{aligned}$$

So $y' = \log_e x$.

From above, $\frac{d}{dx}(x\log_e x - x) = \log_e x$.

Reversing this to give a primitive we obtain:

$\int \log_e x \, dx = x\log_e x - x + C$ for some constant C

18b $\int \log_e x \, dx = x\log_e x - x + C$ for some constant C.

Using $\log_a x = \frac{\log_e x}{\log_e a}$, we obtain $\log_{10} x = \frac{\log_e x}{\log_e 10}$.

$$\begin{aligned}\int_1^{10} \log_{10} x \, dx &= \int_1^{10} \frac{\log_e x}{\log_e 10} dx \\ &= \frac{1}{\log_e 10}\left[x\log_e x - x\right]_1^{10} \\ &= \frac{1}{\log_e 10}\left(10\log_e 10 - 10 - (0-1)\right) \\ &= 10 - \frac{9}{\log_e 10}\end{aligned}$$

So $\int_1^{10} \log_{10} x \, dx = 10 - \frac{9}{\log_e 10}$.

19a i $y = \log_3 x$

Using $\dfrac{d}{dx}\log_a(mx+b) = \dfrac{m}{(mx+b)\log_e a}$ with $a = 3, m = 1$ and $b = 0$ we

obtain $\dfrac{d}{dx}\log_3 x = \dfrac{1}{x\log_e 3}$.

So $y' = \dfrac{1}{x\log_e 3}$.

19a ii $y = \log_7(2x+3)$

Using $\dfrac{d}{dx}\log_a(mx+b) = \dfrac{m}{(mx+b)\log_e a}$ with $a = 7, m = 2$ and $b = 3$ we

obtain $\dfrac{d}{dx}\log_7(2x+3) = \dfrac{2}{(2x+3)\log_e 7}$.

So $y' = \dfrac{2}{(2x+3)\log_e 7}$.

19a iii $y = 5\log_6(4-9x)$

Using $\dfrac{d}{dx}\log_a(mx+b) = \dfrac{m}{(mx+b)\log_e a}$ with $a = 6, m = -9$ and $b = 4$ we

obtain $5\times\dfrac{d}{dx}\log_6(4-9x) = -\dfrac{45}{(4-9x)\log_e 6}$.

So $y' = -\dfrac{45}{(4-9x)\log_e 6}$.

19b i $y = 10^x$

Using $\dfrac{d}{dx}a^{mx+b} = ma^{mx+b}\log_e a$ with $a = 10, m = 1$ and $b = 0$ we

obtain $\dfrac{d}{dx}10^x = 10^x\log_e 10$.

So $y' = 10^x \log_e 10$.

19b ii $y = 8^{4x-3}$

Using $\dfrac{d}{dx} a^{mx+b} = ma^{mx+b} \log_e a$ with $a = 8, m = 4$ and $b = -3$ we

obtain $\dfrac{d}{dx} 8^{4x-3} = 4 \times 8^{4x-3} \log_e 8$.

So $y' = 4 \times 8^{4x-3} \log_e 8$.

19b iii $y = 3 \times 5^{2-7x}$

Using $\dfrac{d}{dx} a^{mx+b} = ma^{mx+b} \log_e a$ with $a = 5, m = -7$ and $b = 2$ we

obtain $3 \times \dfrac{d}{dx} 5^{2-7x} = 3 \times -7 \times 5^{2-7x} \log_e 5$.

So $y' = -21 \times 5^{2-7x} \log_e 5$.

19c i Using $\int a^{mx+b} dx = \dfrac{a^{mx+b}}{m \log_e a} + C$ with $a = 3, m = 5$ and $b = 0$ we obtain:

$\int 3^{5x} dx = \dfrac{3^{5x}}{5 \log_e 3} + C$ for some constant C.

19c ii Using $\int a^{mx+b} dx = \dfrac{a^{mx+b}}{m \log_e a} + C$ with $a = 6, m = 2$ and $b = 7$ we obtain:

$\int 6^{2x+7} dx = \dfrac{6^{2x+7}}{2 \log_e 6} + C$ for some constant C.

19c iii Using $\int a^{mx+b}dx = \dfrac{a^{mx+b}}{m\log_e a} + C$ with $a = 7, m = -9$ and $b = 4$ we obtain:

$$\int 5\times 7^{9-4x}\,dx = -\frac{5\times 7^{9-4x}}{9\log_e 7} + C$$ for some constant C.

20a Let $y = a^x$,

$y = e^{x\log_e a} = e^{kx}$

Let $y = \log_a x$,

$y = \frac{\log_e x}{\log_e a} = \frac{1}{k}\log_e x$

20b The functions $y = a^x$ and $y = \log_e x$ are inverse, so they are symmetric in the line $y = x$. The common tangent is therefore the line $y = x$, which has gradient 1. (This argument would be invalid if there were more than one intersection point.)

20c Let $y = e^{kx}$,

$y' = ke^{kx} = 1$

Let $y = \frac{1}{k}\log_e x$,

$y' = \frac{1}{kx} = 1$

20d $ke^{kx} = \frac{1}{kx}$

$k^2 e^{kx} = \frac{1}{x}$

$e^{\log_e k^2} \times e^{kx} = \frac{1}{x}$

$e^{kx+2\log_e k} = \frac{1}{x}$

$kx + 2\log_e k = \log_e \frac{1}{x}$

$\frac{x}{e} - 2 = \log_e \frac{1}{x}$

$e^{kx-2} = \frac{1}{x}$

$e^{\frac{x}{e}-2} = \frac{1}{x}$

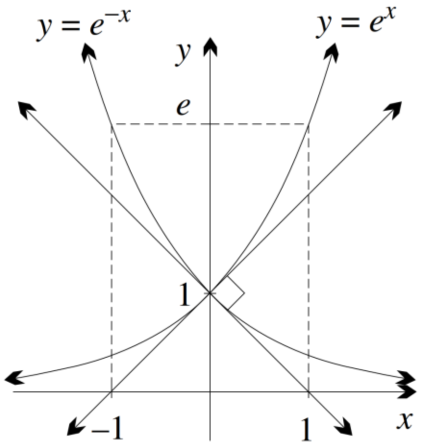
$\frac{1}{e^2} e^{\frac{x}{e}} = \frac{1}{x}$

$\frac{1}{e} e^{\frac{x}{e}} = \frac{e}{x}$

$k = \frac{1}{e}$

Solutions to Chapter review

1a



The graph of $y = e^{-x}$ is the reflection of $y = e^{x}$ about the y-axis.

The y-intercepts are identical, when $x = 0, y = e^{0} = e^{-0} = 1$.

Observe $y = e^{-x}, y' = -e^{-x}$. At $x = 0, y' = -1$ (gradient of the tangent).

The equation of the tangent of $y = e^{-x}$ is $y = mx + b, m = -1$

At $x = 0, y = 1 \Rightarrow\ 1 = -0 + b \Rightarrow b = 1$

$\therefore y = -x + 1$

Observe $y = e^{x}, y' = e^{x}$. At $x = 0, y' = 1$ (gradient of the tangent).

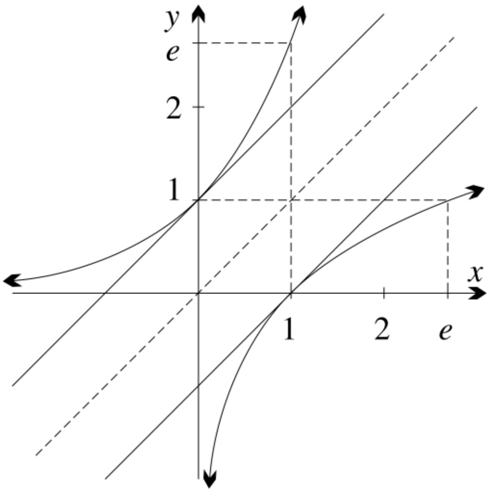
The equation of the tangent of $y = e^{x}$ is $y = mx + b, m = 1$

At $x = 0, y = 1 \Rightarrow\ 1 = x + b \Rightarrow b = 1$

$\therefore y = x + 1$

The tangents meet at right angles.

1b



The graph of $y = e^x$ is the reflection of $y = \log_e x$ about the line $y = x$, as they are inverse functions.

The y-intercept occurs on the curve $y = e^x$, as $x = 0$ is only within the domain of $y = e^x$. At $x = 0, y = e^0 = 1$.

The x-intercept occurs on the curve $y = \log_e x$, as $y = 0$ is outside of the range of $y = e^x$. At $y = 0, \log_e x = 0 \Rightarrow x = 1$.

Observe $y = e^x, y' = e^x$. At $x = 0, y' = 1$ (gradient of the tangent).

The equation of the tangent of $y = e^{-x}$ is $y = mx + b, m = 1$

At $x = 0, y = 1 \Rightarrow \; 1 = 0 + b \Rightarrow b = 1$

$\therefore y = x + 1$

Observe $y = \log_e x , y' = \frac{1}{x}$. At $x = 1, y' = 1$ (gradient of the tangent).

The equation of the tangent of $y = \log_e x$ is $y = mx + b, m = 1$

At $x = 1, y = 0 \Rightarrow \; 0 = 1 + b \Rightarrow b = -1$

$\therefore y = x - 1$

The tangents have the same gradients but different intercepts, indicating they are parallel in the same plane.

2a $\quad e^4 \doteqdot 54.60$ (use your calculator)

2b $\quad e \doteqdot 2.718$ (use your calculator)

2c $e^{-\frac{3}{2}} \doteqdot 0.2231$ (use your calculator)

2d $\log_e 2 \doteqdot 0.6931$ (use your calculator with the "ln" function)

2e $\log_{10} \frac{1}{2} = \frac{\log_e \frac{1}{2}}{\log_e 10} \doteqdot -0.3010$ (use your calculator with the "ln" function)

2f $\log_2 0.03 = \frac{\log_e 0.03}{\log_e 2} \doteqdot -5.059$ (use your calculator with the "ln" function)

2g $\log_{1.05} 586 = \frac{\log_e 586}{\log_e 1.05} \doteqdot 130.6$ (use your calculator with the "ln" function)

2h $\log_8 3\frac{3}{7} = \frac{\log_e \frac{24}{7}}{\log_e 8} = \frac{(\log_e 24 - \log_e 7)}{\log_e 8} \doteqdot 0.5925$ (use your calculator with the "ln" function)

3a $3^x = 14$

$x = \log_3 14$

$= \frac{\log_e 14}{\log_e 3}$

$\doteqdot 2.402$

3b $2^x = 51$

$x = \log_2 51$

$= \frac{\log_e 51}{\log_e 2}$

$\doteqdot 5.672$

3c $4^x = 1345$

$x = \log_4 1345$

$= \dfrac{\log_e 1345}{\log_e 4}$

$\doteqdot 5.197$

3d $5^x = 132$

$x = \log_5 132$

$= \dfrac{\log_e 132}{\log_e 5}$

$\doteqdot 3.034$

4a $e^{2x} \times e^{3x} = e^{2x+3x} = e^{5x}$

4b $e^{7x} \div e^x = e^{7x-x} = e^{6x}$

4c $\frac{e^{2x}}{e^{6x}} = e^{2x-6x} = e^{-4x} = \frac{1}{e^{4x}}$

4d $(e^{3x})^3 = e^{9x}$

5a $9^x - 7 \times 3^x - 18 = 0$

$e^{x\log_e 9} - 7 \times e^{x\log_e 3} - 18 = 0$

$e^{2x\log_e 3} - 7e^{x\log_e 3} - 18 = 0$

Let $u = e^{x\log_e 3}$

$u^2 - 7u - 18 = 0$

$(u-9)(u+2) = 0$

$\therefore u = 9, or\ u = -2$

$e^{x\log_e 3} = 9$ or $e^{x\log_e 3} = -2$

$3^x = 9$ or $3^x = -2$

$3^x = 3^2$ or $x = \log_3 -2$

As $\log_3 -2$ is undefined, $x = 2$.

5b $e^{2x} - 11e^x + 28 = 0$

Let $u = e^x$

$u^2 - 11u + 28 = 0$

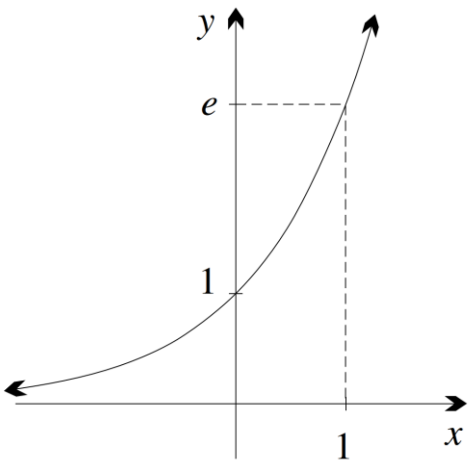
$(u - 7)(u - 4) = 0$

$\therefore u = 7$, or $u = 4$

$e^x = 7$, or $e^x = 4$

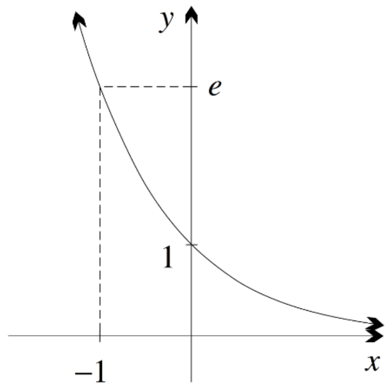
$x = \log_e 7$ or $x = \log_e 4$

6a



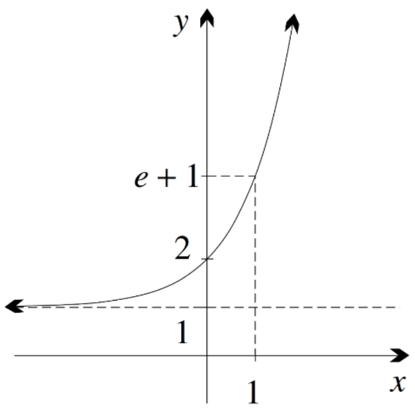
The range of the function $y = e^x$ is $y > 0$.

6b



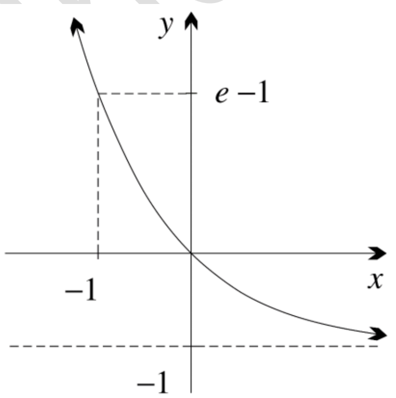
The range of the function $y = e^{-x}$ is $y > 0$.

6c



The range of the function $y = e^{x} + 1$ is $y > 1$.

6d



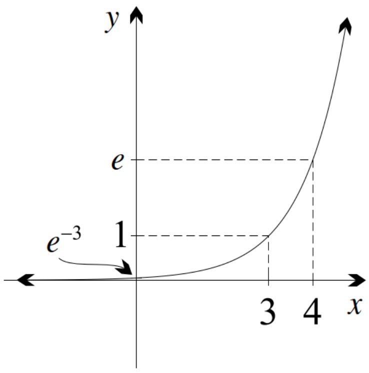
The range of the function $y = e^{-x} - 1$ is $y > -1$.

7a i $y = e^{x-3}$ is a translation of $y = e^x$ 3 units to the right.

7a ii $y = e^{x-3} = e^{x+(-3)} = e^{-3}e^x$

$\frac{y}{e^{-3}} = e^x$

This statement implies that $y = e^{x-3}$ is dilated by a factor of e^{-3}. As the y-value is transformed, the dilation is vertical.

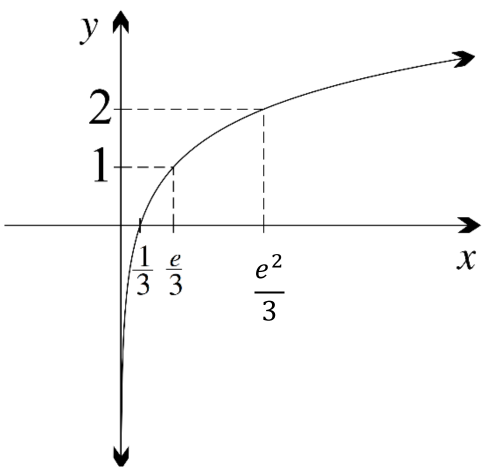


7b i $y = \log_e 3x = \log_e \frac{x}{\frac{1}{3}}$

This statement implies that $y = \log_e x$ is dilated by a factor of $\frac{1}{3}$. As the x-value is transformed, the dilation is horizontal.

7b ii $y = \log_e 3x = \log_e x + \log_e 3$

This statement implies that $y = \log_e 3x$ is a translation of $y = \log_e x$ by $\log_e 3$ units upwards (y- intercept increases by $\log_e 3$ units).



8a $y = e^x$

$$y' = \frac{d}{dx}(e^x) = e^x$$

8b $y = e^{3x}$

$$y' = \frac{d}{dx}(e^u), u = 3x, \frac{du}{dx} = 3$$

By the chain rule,

$$y' = \frac{d}{du}(e^u)\frac{du}{dx} = e^u(3) = 3e^{3x}$$

8c $y = e^{2x+3}$

$$y' = \frac{d}{dx}(e^u), u = 2x + 3, \frac{du}{dx} = 2$$

By the chain rule,

$$y' = \frac{d}{du}(e^u)\frac{du}{dx} = e^u(2) = 2e^{2x+3}$$

8d $y = e^{-x}$

$$y' = \frac{d}{dx}(e^u), u = -x, \frac{du}{dx} = -1$$

By the chain rule,

$$y' = \frac{d}{du}(e^u)\frac{du}{dx} = e^u(-1) = -e^{-x}$$

8e $y = e^{-3x}$

$$y' = \frac{d}{dx}(e^u), u = -3x, \frac{du}{dx} = -3$$

By the chain rule,

$$y' = \frac{d}{du}(e^u)\frac{du}{dx} = e^u(-3) = -3e^{-3x}$$

8f $y = 3e^{2x+5}$

$y' = \frac{d}{dx}(3e^{u}), u = 2x + 5, \frac{du}{dx} = 2$

By the chain rule,

$y' = \frac{d}{du}(3e^{u})\frac{du}{dx} = 3e^{u}(2) = 6e^{2x+5}$

8g $y = 4e^{\frac{1}{2}x}$

$y' = \frac{d}{dx}(4e^{u}), u = \frac{1}{2}x, \frac{du}{dx} = \frac{1}{2}$

By the chain rule,

$y' = \frac{d}{du}(4e^{u})\frac{du}{dx} = 4e^{u}\left(\frac{1}{2}\right) = 2e^{\frac{1}{2}x}$

8h $y = \frac{2}{3}e^{6x-5}$

$y' = \frac{d}{dx}\left(\frac{2}{3}e^{u}\right), u = 6x - 5, \frac{du}{dx} = 6$

By the chain rule,

$y' = \frac{d}{du}\left(\frac{2}{3}e^{u}\right)\frac{du}{dx} = \frac{2}{3}e^{u}(6) = 4e^{6x-5}$

9a $y = e^{3x} \times e^{2x}$

$y = e^{3x+2x} = e^{5x}$

$y' = 5e^{5x}$

9b $y = \frac{e^{7x}}{e^{3x}}$

$y = e^{7x-3x} = e^{4x}$

$y' = 4e^{4x}$

9c $y = \frac{e^x}{e^{4x}}$

$y = e^{x-4x} = e^{-3x}$

$y' = -3e^{-3x}$

9d $y = (e^{-2x})^3$

$y = e^{-6x}$

$y' = -6e^{-6x}$

10a $y = e^{x^3}$

Let $u = x^3, \frac{du}{dx} = 3x^2$

By chain rule,

$$y' = \frac{d}{du}(e^u)\frac{du}{dx}$$

$$= e^u(3x^2)$$

$$= 3x^2e^{x^3}$$

10b $y = e^{x^2-3x}$

Let $u = x^2 - 3x, \frac{du}{dx} = 2x - 3$

By chain rule,

$$y' = \frac{d}{du}(e^u)\frac{du}{dx}$$

$$= e^u(2x - 3)$$

$$= (2x - 3)e^{x^2-3x}$$

10c $y = xe^{2x}$

By product rule,

$$y' = \frac{d}{dx}(x)e^{2x} + x\frac{d}{dx}(e^{2x})$$

Consider $\frac{d}{dx}(e^{2x})$, let $u = 2x, \frac{du}{dx} = 2$

By chain rule,

$$\frac{d}{dx}(e^{2x}) = \frac{d}{du}(e^u)\frac{du}{dx} = e^u(2) = 2e^{2x}$$

$$y' = e^{2x} + x(2e^{2x})$$

$$= e^{2x} + 2xe^{2x}$$

$$= e^{2x}(1 + 2x)$$

10d $y = (e^{2x} + 1)^3$

Let $v = e^{2x} + 1, \frac{dv}{dx} = \frac{d}{dx}(e^{2x} + 1) = \frac{d}{dx}(e^{2x})$

Consider $\frac{d}{dx}(e^{2x})$, let $u = 2x, \frac{du}{dx} = 2$

By chain rule,

$$\frac{dv}{dx} = \frac{d}{dx}(e^{2x}) = \frac{d}{du}(e^u)\frac{du}{dx} = e^u(2) = 2e^{2x}$$

By chain rule,

$$y' = \frac{d}{dv}(v^3)\frac{dv}{dx}$$

$$= 3v^2(2e^{2x})$$

$$= 6e^{2x}(e^{2x} + 1)^2$$

10e $y = \frac{e^{3x}}{x}$

Let $f(x) = e^{3x}, f'(x) = 3e^{3x}$

Let $g(x) = x, g'(x) = 1$

By quotient rule,

$$y = \frac{f(x)}{g(x)}$$

$$y' = \frac{g(x)f'(x) - f(x)g'(x)}{\left(g(x)\right)^2}$$

$$y' = \frac{(x)(3e^{3x}) - e^{3x}(1)}{(x)^2}$$

$$= \frac{3xe^{3x} - e^{3x}}{x^2}$$

$$= \frac{e^{3x}}{x^2}(3x - 1)$$

10f $y = x^2 e^{x^2}$

By product rule,

$$y' = \frac{d}{dx}(x^2)e^{x^2} + x^2\frac{d}{dx}(e^{x^2})$$

Consider $\frac{d}{dx}(e^{x^2})$, let $u = x^2, \frac{du}{dx} = 2x$

By chain rule,

$$\frac{d}{dx}(e^{x^2}) = \frac{d}{du}(e^u)\frac{du}{dx} = e^u(2x) = 2xe^{x^2}$$

$$y' = \frac{d}{dx}(x^2)e^{x^2} + x^2\frac{d}{dx}(e^{x^2})$$

$$= 2xe^{x^2} + x^2 2xe^{x^2}$$

$$= 2xe^{x^2}(1 + x^2)$$

10g $y = (e^x - e^{-x})^5$

Let $u = e^x - e^{-x}$,

$$\frac{du}{dx} = \frac{d}{dx}(e^x) - \frac{d}{dx}(e^{-x}) = e^x + e^{-x}$$

By chain rule,

$$y' = \frac{d}{du}(u^5)\frac{du}{dx}$$

$$= 5u^4(e^x + e^{-x})$$

$$= 5(e^x - e^{-x})^4(e^x + e^{-x})$$

10h $y = \frac{e^{2x}}{2x+1}$

Let $f(x) = e^{2x}, f'(x) = 2e^{2x}$

Let $g(x) = 2x + 1, g'(x) = 2$

By quotient rule,

$$y = \frac{f(x)}{g(x)}$$

$$y' = \frac{g(x)f'(x) - f(x)g'(x)}{\left(g(x)\right)^2}$$

$$= \frac{(2x+1)(2e^{2x}) - (e^{2x})(2)}{(2x+1)^2}$$

$$= \frac{e^{2x}(4x + 2 - 2)}{(2x+1)^2}$$

$$= \frac{4xe^{2x}}{(2x+1)^2}$$

11a $y = e^{2x+1}$

Let $u = 2x + 1, \frac{du}{dx} = 2$

By chain rule,

$$y' = \frac{d}{du}(e^u)\frac{du}{dx}$$

$$= (e^u)(2)$$

$$= 2e^{2x+1}$$

$$y'' = 2\frac{d}{dx}(e^{2x+1})$$

By chain rule,

$$y'' = 2\frac{d}{du}(e^u)\frac{du}{dx}$$

$$= 2(e^u)(2)$$

$$= 4e^{2x+1}$$

11b $y = e^{x^2+1}$

Let $u = x^2 + 1, \frac{du}{dx} = 2x$

By chain rule,

$$y' = \frac{d}{du}(e^u)\frac{du}{dx}$$

$$= (e^u)(2x)$$

$$= 2xe^{x^2+1}$$

$$y'' = 2\frac{d}{dx}\left(xe^{x^2+1}\right)$$

By product rule,

$$y'' = 2\left[\frac{d}{dx}(x)e^{x^2+1} + x\frac{d}{dx}\left(e^{x^2+1}\right)\right]$$

Given that $\frac{d}{dx}\left(e^{x^2+1}\right) = 2xe^{x^2+1}$ from above by the chain rule,

$$y'' = 2\left[(1)e^{x^2+1} + x\left(2xe^{x^2+1}\right)\right]$$

$$= 2e^{x^2+1} + 4x^2e^{x^2+1}$$

$$= 2e^{x^2+1}(1 + 2x^2)$$

12 Gradient of the tangent of the curve $y = e^x$ at $x = 2$

$y' = e^x$

Let $x = 2, y' = e^2$

Let the equation of the tangent be of the form $y = mx + b, m = e^2$

$at\ x = 2, y = e^2$

$e^2 = 2e^2 + b$

$b = e^2 - 2e^2 = -e^2$

The equation of the tangent of the curve $y = e^x$ at $x = 2$ is

$y = e^2(x - 1)$

The x-intercept is when $y = 0$,

$0 = e^2(x - 1) \Rightarrow x = 1$

The y-intercept is when $x = 0$,

$y = e^2(0 - 1) = -e^2$

13a First, find the gradient of the tangent of the curve $y = e^{-3x}$ at $x = 0$.

$y' = -3e^{-3x}$

At $x = 0$,

$y' = -3$

The gradient of the normal is $-\frac{1}{y'} = \frac{1}{3}$

13b $y'' = (-3)(-3e^{-3x})$

$= 9e^{-3x}$

When $x = 0$, $y'' = 9$. As this is a positive value, the curve is concave up.

14a $y = e^x - x$

$$\begin{aligned} y' &= \frac{d}{dx}(e^x) - \frac{d}{dx}(x) \\ &= e^x - 1 \end{aligned}$$

$$\begin{aligned} y'' &= \frac{d}{dx}(e^x) - \frac{d}{dx}(1) \\ &= e^x \end{aligned}$$

14b A stationary point is determined when $y' = 0$

$$\begin{aligned} e^x - 1 &= 0 \\ e^x &= 1 \\ x &= \ln 1 \\ x &= 0 \end{aligned}$$

At $x = 0, y = e^0 - 0 = 1$

The stationary point is at $(0,1)$

14c The concavity of the curve is determined by the second derivative, y''

First, we have to determine the domain of the original equation, $y = e^x - x$

The equation of y is defined for all real values of x

Therefore, $y'' = e^x$ is defined for all real values of x.

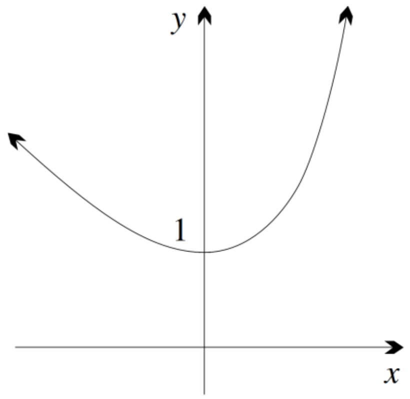
For all values of x, the range of y'' is in the interval $(0, \infty)$

As such, $y'' > 0 \; \forall \; x$

The curve is always concave up.

14d Since we have determined that a stationary point exists at (0,1) and the curve is concave up for all values of x, we can infer that the stationary point (0,1) is a local minimum.

We can therefore, infer the range of $y \geq 1$



15 The stationary point $y = xe^{-2x}$ is when $y' = 0$

By the chain rule,

$$\begin{aligned} y' &= \frac{d}{dx}(x)e^{-2x} + x\frac{d}{dx}(e^{-2x}) \\ &= e^{-2x} - 2xe^{-2x} \\ &= e^{-2x}(1 - 2x) \end{aligned}$$

When $y' = 0$,

$e^{-2x}(1 - 2x) = 0$

$\Rightarrow e^{-2x} = 0$ or $1 - 2x = 0$

As the range of e^u does not include 0 for any value of u,

$1 - 2x = 0$

$x = \frac{1}{2}$

At $x = \frac{1}{2}, y = \left(\frac{1}{2}\right) e^{-2\left(\frac{1}{2}\right)} = \frac{1}{2} e^{-1} = \frac{1}{2e}$

The stationary point is therefore at $\left(\frac{1}{2}, \frac{1}{2e}\right)$.

To determine the nature of the stationary point, the second derivative y'' should be determined.

By the chain rule,

$$\begin{aligned} y'' &= \frac{d}{dx}(e^{-2x})(1-2x) + e^{-2x}\frac{d}{dx}(1-2x) \\ &= -2e^{-2x}(1-2x) + e^{-2x}(-2) \\ &= -2e^{-2x} + 4xe^{-2x} - 2e^{-2x} \\ &= 4xe^{-2x} - 4e^{-2x} \\ &= 4e^{-2x}(x-1) \end{aligned}$$

At $x = \frac{1}{2}$,

$$y'' = 4e^{-2\left(\frac{1}{2}\right)}\left(\frac{1}{2} - 1\right) = -2e^{-1} = -\frac{2}{e} < 0$$

As this is a negative value, the curve is concave *down* at the stationary point.

The stationary point is therefore a maximum turning point.

16a

$$\int e^{5x}\,dx$$

Let $u = 5x, u' = 5$

$$\begin{aligned} \int e^{5x}\,dx &= \frac{1}{5}\int 5e^{5x}\,dx \\ &= \frac{1}{5}\int e^u\,du \\ &= \frac{1}{5}e^u + C \\ &= \frac{1}{5}e^{5x} + C \end{aligned}$$

16b

$$\int 10e^{2-5x}\,dx$$

Let $u = 2 - 5x, u' = -5$

$$\begin{aligned}\int 10e^{2-5x}\,dx &= -\int 2(-5)e^{2-5x}\,dx\\ &= -\int 2e^u\,du\\ &= -2e^u + C\\ &= -2e^{2-5x} + C\end{aligned}$$

16c

$$\int e^{\frac{1}{5}x}\,dx$$

Let $u = \frac{1}{5}x, u' = \frac{1}{5}$

$$\begin{aligned}\int e^{\frac{1}{5}x}\,dx &= 5\int \frac{1}{5}e^{\frac{1}{5}x}\,dx\\ &= 5\int e^u\,du\\ &= 5e^u + C\\ &= 5e^{\frac{1}{5}x} + C\end{aligned}$$

16d

$$\int 3e^{5x-4}\,dx$$

Let $u = 5x - 4, u' = 5$

$$\begin{aligned}\int 3e^{5x-4}\,dx &= \frac{1}{5}\int 3(5)e^{5x-4}\,dx\\ &= \frac{3}{5}\int e^u\,du\\ &= \frac{3}{5}e^u + C\\ &= \frac{3}{5}e^{5x-4} + C\end{aligned}$$

17a

$$\begin{aligned}\int_0^2 e^x\,dx &= [e^x]_0^2\\ &= e^2 - e^0\\ &= e^2 - 1\end{aligned}$$

17b

$$\int_0^1 e^{2x}\,dx$$

Let $u = 2x, u' = 2$

$$\begin{aligned}\int_0^1 e^{2x}\,dx &= \frac{1}{2}\int_0^1 2e^{2x}\,dx\\ &= \frac{1}{2}\int_0^2 e^u\,du\\ &= \frac{1}{2}[e^u]_0^2\\ &= \frac{1}{2}(e^2 - e^0)\\ &= \frac{1}{2}(e^2 - 1)\end{aligned}$$

17c

$$\int_{-1}^0 e^{-x}\,dx$$

Applying the standard form:

$$\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + C,$$

$$\begin{aligned}\int_{-1}^0 e^{-x}\,dx &= [-e^{-x}]_{-1}^0\\ &= -e^0 - (-e^1)\\ &= e^1 - e^0\\ &= e - 1\end{aligned}$$

17d

$$\int_{-\frac{2}{3}}^{0} e^{3x+2}\,dx$$

Applying the standard form:

$$\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + C,$$

$$\begin{aligned}\int_{-\frac{2}{3}}^{0} e^{3x+2}\,dx &= \left[\frac{1}{3}e^{3x+2}\right]_{-\frac{2}{3}}^{0}\\ &= \frac{1}{3}e^{2} - \frac{1}{3}e^{0}\\ &= \frac{1}{3}(e^{2}-1)\end{aligned}$$

17e

$$\int_{0}^{\frac{1}{2}} e^{3-2x}\,dx$$

Applying the standard form:

$$\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + C,$$

$$\begin{aligned}\int_{0}^{\frac{1}{2}} e^{3-2x}\,dx &= \left[-\frac{1}{2}e^{3-2x}\right]_{0}^{\frac{1}{2}}\\ &= \left(-\frac{1}{2}e^{3-2\left(\frac{1}{2}\right)}\right) - \left(-\frac{1}{2}e^{3-2(0)}\right)\\ &= \left(-\frac{1}{2}e^{2}\right) - \left(-\frac{1}{2}e^{3}\right)\\ &= \frac{1}{2}e^{3} - \frac{1}{2}e^{2}\\ &= \frac{1}{2}e^{2}(e-1)\end{aligned}$$

17f

$$\int_0^2 2e^{\frac{1}{2}x}\,dx$$

Applying the standard form:

$$\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + C,$$

$$\begin{aligned}\int_0^2 2e^{\frac{1}{2}x}\,dx &= \left[4e^{\frac{1}{2}x}\right]_0^2\\ &= \left(4e^{\frac{1}{2}(2)}\right) - \left(4e^{\frac{1}{2}(0)}\right)\\ &= 4e - 4\\ &= 4(e-1)\end{aligned}$$

18a Let $y = \frac{1}{e^{5x}} = e^{-5x}$

Applying the standard form:

$$\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + C,$$

$$\int e^{-5x}\,dx = -\frac{1}{5}e^{-5x} + C$$

18b Let $y = e^{3x} \times e^{x} = e^{4x}$

Applying the standard form:

$$\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + C,$$

$$\int e^{4x}\,dx = \frac{1}{4}e^{4x} + C$$

18c Let $y = \frac{6}{e^{3x}} = 6e^{-3x}$

Applying the standard form:

$$\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + C,$$

$$\int 6e^{-3x}\,dx = -2e^{-3x} + C$$

18d Let $y = (e^{3x})^2 = e^{6x}$

Applying the standard form:

$$\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + C,$$
$$\int e^{6x}\,dx = \frac{1}{6}e^{6x} + C$$

18e Let $y = \frac{e^{3x}}{e^{5x}} = e^{-2x}$

Applying the standard form:

$$\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + C,$$
$$\int e^{-2x}\,dx = -\frac{1}{2}e^{-2x} + C$$

18f Let $y = \frac{e^{3x}+1}{e^{2x}} = e^{x} + e^{-2x}$

Applying the standard form:

$$\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + C,$$
$$\int (e^{x} + e^{-2x})\,dx = e^{x} - \frac{1}{2}e^{-2x} + C$$

18g Let $y = e^{2x}(e^{x} + e^{-x}) = e^{3x} + e^{x}$

Applying the standard form:

$$\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + C,$$
$$\int (e^{3x} + e^{x})\,dx = \frac{1}{3}e^{3x} + e^{x} + C$$

18h Let $y = (1 + e^{-x})^2 = 1 + 2e^{-x} + e^{-2x}$

Applying the standard form:

$$\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + C,$$

$$= \int (1 + 2e^{-x} + e^{-2x})\, dx = x - 2e^{-x} - \frac{1}{2}e^{-2x} + C$$

19a By applying the standard form:

$$\int e^{ax+b}\, dx = \frac{1}{a}e^{ax+b} + C,$$

$$\begin{aligned}\int_0^1 (1 + e^{-x})\, dx &= [x - e^{-x}]_0^1 \\ &= (1 - e^{-1}) - (0 - e^{-0}) \\ &= \left(1 - \frac{1}{e}\right) + 1 \\ &= 2 - \frac{1}{e}\end{aligned}$$

19b By applying the standard form:

$$\int e^{ax+b}\, dx = \frac{1}{a}e^{ax+b} + C,$$

$$\begin{aligned}\int_0^2 (e^{2x} + x)\, dx &= \left[\frac{1}{2}e^{2x} + \frac{x^2}{2}\right]_0^2 \\ &= \left(\frac{1}{2}e^{2(2)} + \frac{(2)^2}{2}\right) - \left(\frac{1}{2}e^{2(0)} + \frac{(0)^2}{2}\right) \\ &= \left(\frac{1}{2}e^4 + 2\right) - \left(\frac{1}{2}\right) \\ &= \frac{1}{2}(e^4 + 3)\end{aligned}$$

19c By applying the standard form:

$$\int e^{ax+b}\, dx = \frac{1}{a}e^{ax+b} + C,$$

$$\begin{aligned}\int_0^1 \frac{2}{e^x}\, dx &= \int_0^1 2e^{-x}\, dx \\ &= [-2e^{-x}]_0^1 \\ &= (-2e^{-1}) - (-2e^0) \\ &= (-2e^{-1}) - (-2) \\ &= 2 - 2e^{-1}\end{aligned}$$

19d By applying the standard form:

$$\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + C,$$

$$\begin{aligned}\int_0^{\frac{1}{3}} e^{3x}(1-e^{-3x})\,dx &= \int_0^{\frac{1}{3}}(e^{3x}-e^{0})\,dx\\ &= \int_0^{\frac{1}{3}}(e^{3x}-1)\,dx\\ &= \left[\frac{1}{3}e^{3x}-x\right]_0^{\frac{1}{3}}\\ &= \left(\frac{1}{3}e^{3\left(\frac{1}{3}\right)}-\left(\frac{1}{3}\right)\right)-\left(\frac{1}{3}e^{3(0)}-(0)\right)\\ &= \left(\frac{1}{3}e-\frac{1}{3}\right)-\left(\frac{1}{3}\right)\\ &= \frac{1}{3}(e-2)\end{aligned}$$

19e By applying the standard form:

$$\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + C,$$

$$\begin{aligned}\int_0^1 \frac{e^{2x}+1}{e^x}\,dx &= \int_0^1 (e^x+e^{-x})\,dx\\ &= [e^x-e^{-x}]_0^1\\ &= (e^1-e^{-1})-(e^0-e^{-0})\\ &= \left(e-\frac{1}{e}\right)-(1-1)\\ &= e-\frac{1}{e}\end{aligned}$$

19f By applying the standard form:

$$\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + C,$$

$$\begin{aligned}\int_0^1 (e^x+1)^2\,dx &= \int_0^1 (e^{2x}+2e^x+1)\,dx\\ &= \left[\frac{1}{2}e^{2x}+2e^x+x\right]_0^1\\ &= \left(\frac{1}{2}e^{2(1)}+2e^{(1)}+1\right)-\left(\frac{1}{2}e^{2(0)}+2e^{(0)}+0\right)\end{aligned}$$

$$= \left(\frac{1}{2}e^2 + 2e + 1\right) - \frac{1}{2} - 2$$
$$= \frac{1}{2}e^2 + 2e - \frac{3}{2}$$
$$= \frac{1}{2}(e^2 + 4e - 3)$$

20 $f'(x) = e^x - e^{-x} - 1$

$$f(x) = \int (e^x - e^{-x} - 1)\,dx$$
$$= e^x + e^{-x} - x + C$$

Given $f(0) = 3$,

$$f(0) = e^0 + e^{-0} - 0 + C$$
$$3 = 1 + 1 + C$$
$$C = 1$$

$$f(x) = e^x + e^{-x} - x + 1$$

$$f(1) = e^1 + e^{-1} - 1 + 1$$
$$= e + e^{-1}$$

21a Let $y = e^{x^3}$

Let $u = x^3, \frac{du}{dx} = 3x^2$

$$y' = \frac{d}{du}(e^u)\frac{du}{dx}$$
$$= e^u(3x^2)$$
$$= 3x^2e^{x^3}$$

21b

$$\int_0^1 x^2e^{x^3}\,dx = \frac{1}{3}\int_0^1 3x^2e^{x^3}\,dx$$
$$= \frac{1}{3}\left[e^{x^3}\right]_0^1$$
$$= \frac{1}{3}\left[e^{1^3} - e^{0^3}\right]$$
$$= \frac{1}{3}(e - 1)$$

22a The area under the curve can be expressed by the following integral:

$$\int_0^1 e^{2x}\,dx$$

By applying the standard form:

$$\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + C,$$

$$\int_0^1 e^{2x}\,dx$$

$$= \left[\frac{1}{2}e^{2x}\right]_0^1$$

$$= \frac{1}{2}e^{2(1)} - \frac{1}{2}e^{2(0)}$$

$$= \frac{1}{2}(e^2 - 1)$$

$\doteqdot$ 3.19 square units

22b The area under the curve can be expressed by the following integral:

$$\int_0^1 1 - e^{-x}\,dx$$

By applying the standard form:

$$\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + C,$$

$$\int_0^1 1 - e^{-x}\,dx$$

$$= [x + e^{-x}]_0^1$$

$$= (1 + e^{-1}) - (0 + e^{-0})$$

$$= 1 + \frac{1}{e} - 1$$

$$= \frac{1}{e}$$

$\doteqdot$ 0.368 square units

23a The area of the shaded region can be expressed by the following integral:

$$\int_{-1}^{0} -(y)\,dx = \int_{-1}^{0} 1 - e^{2x}\,dx$$

By applying the standard form:

$$\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + C,$$

$$\int_{-1}^{0} 1 - e^{2x}\,dx$$

$$= \left[x - \frac{1}{2}e^{2x}\right]_{-1}^{0}$$

$$= \left(0 - \frac{1}{2}e^{2(0)}\right) - \left(-1 - \frac{1}{2}e^{2(-1)}\right)$$

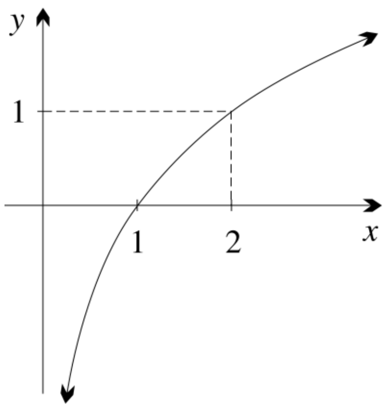
$$= -\frac{1}{2} + 1 + \frac{1}{2}e^{-2}$$

$$= \frac{1}{2}(1 + e^{-2}) \text{ square units}$$

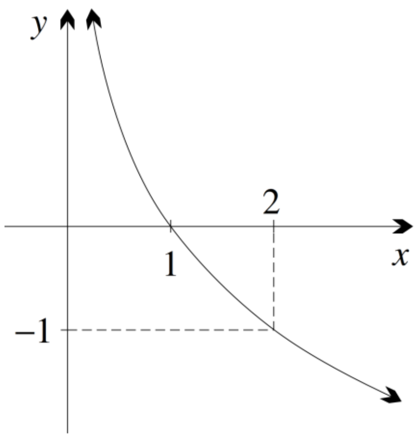
23b The area of the shaded region can be expressed by the following integral:

$$\begin{aligned}
\int_{0}^{1} (e-1)x - (e^x - 1)\,dx &= \left[\frac{e-1}{2}x^2 - e^x + x\right]_0^1 \\
&= \left(\frac{e-1}{2}(1)^2 - e^1 + 1\right) - \left(\frac{e-1}{2}(0)^2 - e^0 + 0\right) \\
&= \frac{e-1}{2} - e + 1 + 1 \\
&= \frac{e}{2} - \frac{1}{2} - e + 2 \\
&= \frac{3}{2} - \frac{e}{2} \\
&= \frac{1}{2}(3 - e) \text{ square units}
\end{aligned}$$

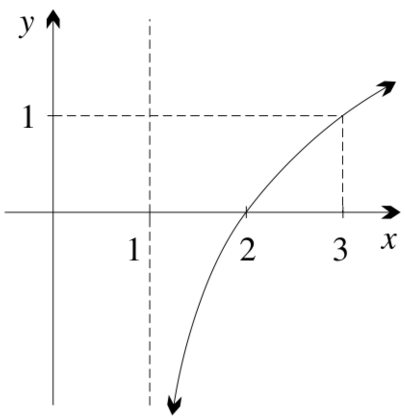
24a The vertical asymptote is the y-axis, as the domain of the function is $x > 0$.



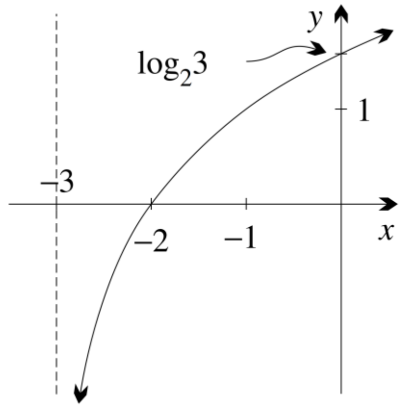
24b The vertical asymptote is the y-axis, as the domain of the function is $x > 0$.



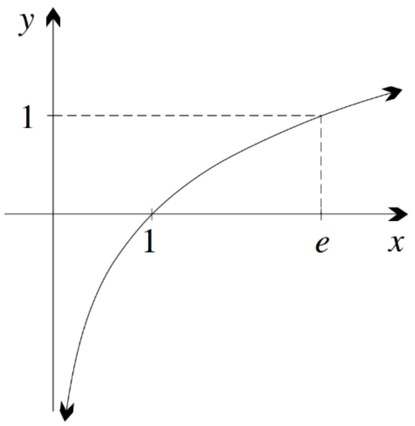
24c The vertical asymptote is the line $x = 1$, as the domain of the function is $x > 1$.



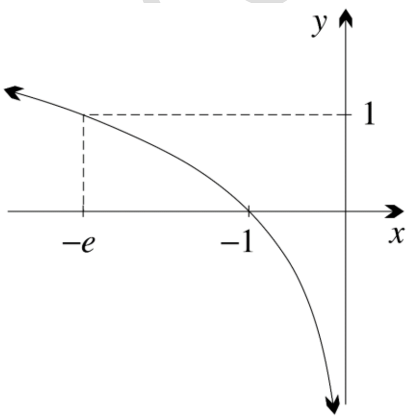
24d The vertical asymptote is the line $x = -3$, as the domain of the function is $x > -3$.



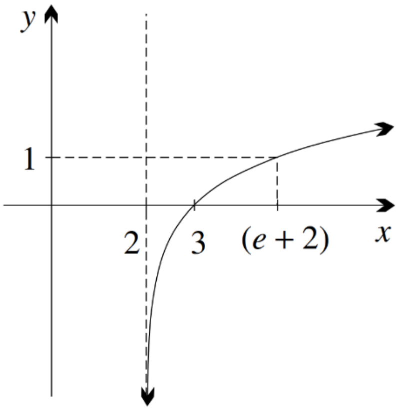
25a The vertical asymptote is the y-axis, as the domain of the function is $x > 0$.



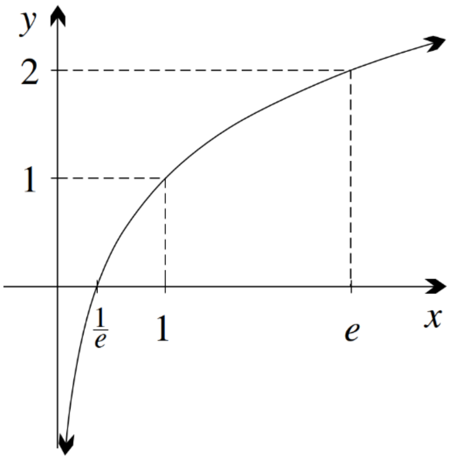
25b The vertical asymptote is the y-axis, as the domain of the function is $x < 0$.



25c The vertical asymptote is the line $x = 2$, as the domain of the function is $x > 2$.



24d The vertical asymptote is the line y-axis, as the domain of the function is $x > 0$.



26a $e \log_e e = e \times 1$
$= e$

26b $\log_e e^3 = 3 \log_e e$
$= 3 \times 1$
$= 3$

26c

$$\begin{aligned}\ln\left(\frac{1}{e}\right) &= \ln e^{-1}\\ &= -\ln e\\ &= -1\end{aligned}$$

26d

$$\begin{aligned}2e\ln\sqrt{e} &= 2e\ln e^{\frac{1}{2}}\\ &= 2e\left(\frac{1}{2}\right)\ln e\\ &= e\times 1\\ &= e\end{aligned}$$

27a Let $y = \log_e x$

$$y' = \frac{1}{x}$$

27b Let $y = \log_e 2x$

$$y' = \frac{2}{2x} == \frac{1}{x}$$

27c Let $y = \log_e(x+4)$

Let $u = x + 4, \frac{du}{dx} = 1$

By the chain rule,

$$\begin{aligned}y' &= \frac{d}{du}\log_e u\frac{du}{dx}\\ &= \frac{1}{u}\times 1\\ &= \frac{1}{x+4}\end{aligned}$$

27d Let $y = \log_e(2x - 5)$

Let $u = 2x - 5, \frac{du}{dx} = 2$

By the chain rule,

$$\begin{aligned} y' &= \frac{d}{du}\log_e u \frac{du}{dx} \\ &= \frac{1}{u} \times 2 \\ &= \frac{2}{2x - 5} \end{aligned}$$

27e Let $y = 2\log_e(5x - 1)$

Let $u = 5x - 1, \frac{du}{dx} = 5$

By the chain rule,

$$\begin{aligned} y' &= \frac{d}{du}2\log_e u \frac{du}{dx} \\ &= \frac{2}{u} \times 5 \\ &= \frac{10}{5x - 1} \end{aligned}$$

27f Let $y = x + \log_e x$

$$\begin{aligned} y' &= \frac{d}{dx}(x) + \frac{d}{dx}(\log_e x) \\ &= 1 + \frac{1}{x} \end{aligned}$$

27g Let $y = \ln(x^2 - 5x + 2)$

Let $u = x^2 - 5x + 2, \frac{du}{dx} = 2x - 5$

By the chain rule,

$$\begin{aligned} y' &= \frac{d}{du}(\ln u)\frac{du}{dx} \\ &= \frac{1}{u}(2x - 5) \\ &= \frac{2x - 5}{x^2 - 5x + 2} \end{aligned}$$

27h Let $y = \ln(1 + 3x^5)$

Let $u = 1 + 3x^5, \frac{du}{dx} = 15x^4$

By the chain rule,

$$\begin{aligned} y' &= \frac{d}{du}(\ln u)\frac{du}{dx} \\ &= \frac{1}{u}(15x^4) \\ &= \frac{15x^4}{1 + 3x^5} \end{aligned}$$

27i Let $y = 4x^2 - 8x^3 + \ln(x^2 - 2)$

Let $u = x^2 - 2, \frac{du}{dx} = 2x$

By the chain rule,

$$\begin{aligned} y' &= \frac{d}{dx}(4x^2) - \frac{d}{dx}(8x^3) + \frac{d}{du}(\ln u)\frac{du}{dx} \\ &= 8x - 24x^2 + \frac{1}{u}(2x) \\ &= 8x - 24x^2 + \frac{2x}{x^2 - 2} \end{aligned}$$

28a Let $y = \log_e x^3$

$y = 3\log_e x$

$y' = \frac{3}{x}$

28b Let $y = \log_e \sqrt{x}$

$$\begin{aligned} y &= \log_e x^{\frac{1}{2}} \\ &= \frac{1}{2}\log_e x \end{aligned}$$

$y' = \frac{1}{2x}$

28c Let $y = \ln x(x+2)$

$y = \ln x + \ln(x+2)$

$y' = \dfrac{1}{x} + \dfrac{1}{x+2}$

28d Let $y = \ln \frac{x}{x-1}$

$y = \ln x - \ln(x-1)$

$y' = \dfrac{1}{x} - \dfrac{1}{x-1}$

29a Let $y = x \log x$

By the product rule,

$$y' = \frac{d}{dx}(x)\log x + x\frac{d}{dx}(\log x)$$
$$= \log x + 1$$

29b Let $y = e^x \log x$

By the product rule,

$$y' = \frac{d}{dx}(e^x)\log x + e^x\frac{d}{dx}(\log x)$$
$$= e^x \log x + \frac{e^x}{x}$$
$$= e^x\left(\log x + \frac{1}{x}\right)$$

29c Let $y = \frac{x}{\ln x}$

By the quotient rule,

$$y' = \frac{\ln x \frac{d}{dx}(x) - x\frac{d}{dx}(\ln x)}{(\ln x)^2}$$
$$= \frac{\left(\ln x - \frac{x}{x}\right)}{(\ln x)^2}$$

$= \frac{\ln x - 1}{(\ln x)^2}$

29d Let $y = \frac{\ln x}{x^2}$

By the quotient rule,

$$\begin{aligned} y' &= \frac{x^2 \frac{d}{dx}(\ln x) - \ln x \frac{d}{dx}(x^2)}{(x^2)^2} \\ &= \frac{\frac{x^2}{x} - \ln x \times 2x}{x^4} \\ &= \frac{x - 2x \ln x}{x^4} \\ &= \frac{1 - 2\ln x}{x^3} \end{aligned}$$

30 $y = 3\log_e x + 4$

The gradient of the tangent at any point is $y' = \frac{3}{x}$

At $(1, 4)$, $x = 1$

$$y' = \frac{3}{1} = 3$$

Let the equation of the tangent at point $(1, 4)$ be $y = mx + b, m = 3$

At $(1, 4)$, $x = 1, y = 4$

$4 = 3(1) + b$

$b = 1$

$y = 3x + 1$

31a $y = x - \log_e x$

$$\begin{aligned} y' &= \frac{d}{dx}(x) - \frac{d}{dx}(\log_e x) \\ &= 1 - \frac{1}{x} \\ &= \frac{x}{x} - \frac{1}{x} \\ &= \frac{x - 1}{x} \end{aligned}$$

31b The turning point is located when the gradient of the graph reaches 0, which is

$y' = 0$

$\frac{x-1}{x} = 0 \Rightarrow x = 1$

At $x = 1$, $y = 1 - \log_e 1$

$y = 1$

To determine if the turning point is a local minimum or maximum, the concavity of the graph must be determined by finding y''

$$y'' = \frac{d}{dx}\left(\frac{x-1}{x}\right)$$

By the quotient rule,

$$\begin{aligned} y'' &= \frac{x\frac{d}{dx}(x-1) - \frac{d}{dx}(x)(x-1)}{x^2} \\ &= \frac{x(1) - 1 \times (x-1)}{x^2} \\ &= \frac{x - x + 1}{x^2} \\ &= \frac{1}{x^2} \end{aligned}$$

At $x = 1, y'' = 1 > 0$

The second derivative of the function is positive, indicating that the curve is concave up at the point $(1, 1)$.

Therefore, it can be concluded that the graph of $y = x - \log_e x$ has a minimum turning point at point $(1, 1)$.

32a

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

32b

$$\int \frac{3}{x}\,dx = 3\ln|x| + C$$

32c

$$\int \frac{1}{5x}dx$$

$$= \frac{1}{5}\int \frac{1}{x}dx$$

$$= \frac{1}{5}\ln|x| + C$$

32d

$$\int \frac{1}{x+7}dx$$

Let $u = x + 7, \frac{du}{dx} = 1$

$$\begin{aligned}\int \frac{1}{x+7}dx &= \int \frac{1}{u}du \\ &= \ln|u| + C \\ &= \ln|x+7| + C\end{aligned}$$

32e

$$\int \frac{1}{2x-1}dx$$

Let $u = 2x - 1, \frac{du}{dx} = 2$

$$\int \frac{1}{2x-1}dx$$

$$= \frac{1}{2}\int \frac{2}{u}dx$$

$$= \frac{1}{2}\int \frac{1}{u}du$$

$$= \frac{1}{2}\ln|u| + C$$

$$= \frac{1}{2}\ln|2x-1| + C$$

32f

$$\int \frac{1}{2-3x}\,dx$$

Let $u = 2 - 3x, \frac{du}{dx} = -3$

$$\begin{aligned}\int \frac{1}{2-3x}\,dx &= -\frac{1}{3}\int -\frac{3}{2-3x}\,dx \\ &= -\frac{1}{3}\int \frac{1}{u}\,du \\ &= -\frac{1}{3}\ln|u| + C \\ &= -\frac{1}{3}\ln|2-3x| + C\end{aligned}$$

32g

$$\int \frac{2}{2x+9}\,dx$$

Let $u = 2x + 9, \frac{du}{dx} = 2$

$$\begin{aligned}\int \frac{2}{2x+9}\,dx &= \int \frac{1}{u}\,du \\ &= \ln|u| + C \\ &= \ln|2x+9| + C\end{aligned}$$

32h

$$\int \frac{8}{1-4x}\,dx$$

Let $u = 1 - 4x, \frac{du}{dx} = -4$

$$\begin{aligned}\int \frac{8}{1-4x}\,dx &= -\int \frac{2(-4)}{1-4x}\,du \\ &= -\int \frac{2}{u}\,du \\ &= -2\ln|u| + C \\ &= -2\ln|1-4x| + C\end{aligned}$$

33a

$$\int_0^1 \frac{1}{x+2}\,dx = [\ln|x+2|]_0^1$$
$$= \ln(1+2) - \ln(0+2)$$
$$= \ln 3 - \ln 2$$
$$= \ln\frac{3}{2}$$

33b

$$\int_1^4 \frac{1}{4x-3}\,dx = \left[\frac{1}{4}\ln|4x-3|\right]_1^4$$
$$= \frac{1}{4}\ln|4(4)-3| - \frac{1}{4}\ln|4(1)-3|$$
$$= \frac{1}{4}(\ln 13 - \ln 1)$$
$$= \frac{1}{4}\ln 13$$

33c

$$\int_1^e \frac{1}{x}\,dx = [\ln|x|]_1^e$$
$$= \ln e - \ln 1$$
$$= 1 - 0$$
$$= 1$$

33d

$$\int_{e^2}^{e^3} \frac{1}{x}\,dx = [\ln|x|]_{e^2}^{e^3}$$
$$= \ln e^3 - \ln e^2$$
$$= 3\ln e - 2\ln e$$
$$= 3 - 2$$
$$= 1$$

34a By applying the standard form:

$$\int \frac{u'}{u} dx = \log_e|u| + C,$$

Let $u = x^2 + 4, u' = 2x$

$$\int \frac{2x}{x^2 + 4} dx = \log_e|x^2 + 4| + C$$

Since $x^2 + 4$ is always greater than zero, the absolute value can be ignored.

$$\int \frac{2x}{x^2 + 4} dx = \log_e(x^2 + 4) + C$$

34b By applying the standard form:

$$\int \frac{u'}{u} dx = \log_e|u| + C,$$

Let $u = x^3 - 5x + 7, u' = 3x^2 - 5$

$$\int \frac{3x^2 - 5}{x^3 - 5x + 7} dx = \log_e|x^3 - 5x + 7| + C$$

34c By applying the standard form:

$$\int \frac{u'}{u} dx = \log_e|u| + C,$$

Let $u = x^2 - 3, u' = 2x$

$$\begin{aligned}\int \frac{x}{x^2 - 3} dx &= \frac{1}{2}\int \frac{2x}{x^2 - 3} dx \\ &= \frac{1}{2}\log_e|x^2 - 3| + C\end{aligned}$$

34d By applying the standard form:

$$\int \frac{u'}{u} dx = \log_e|u| + C,$$

Let $u = x^4 - 4x, u' = 4x^3 - 4$

$$\int \frac{x^3 - 1}{x^4 - 4x} dx = \frac{1}{4}\int \frac{4x^3 - 4}{x^4 - 4x} dx$$

$$= \frac{1}{4}\log_e|x^4 - 4x| + C$$

35 The integral that describes the area of the bounded region is as follows

$$\begin{aligned}\int_2^4 \frac{1}{x}\,dx &= [\ln|x|]_2^4 \\ &= \ln|4| - \ln|2| \\ &= \ln\frac{4}{2} \\ &= \ln 2 \text{ square units}\end{aligned}$$

36a $y_1 = \frac{5}{x}$

$y_2 = 6 - x$

Let $y_1 = y_2$

$\frac{5}{x} = 6 - x$

$5 = 6x - x^2$

$x^2 - 6x + 5 = 0$

$(x-5)(x-1) = 0$

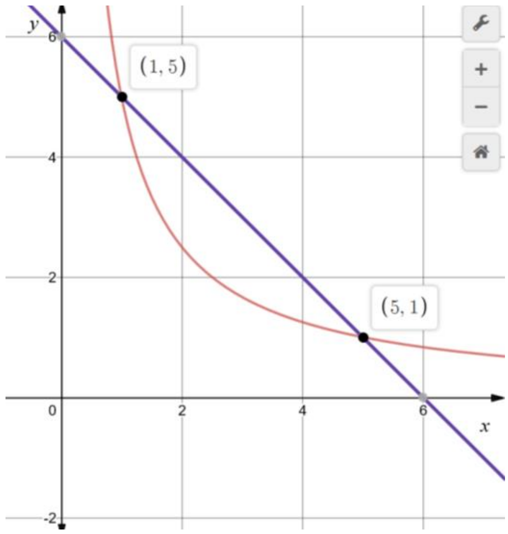
$\therefore x = 5$ or $x = 1$

Let $x = 1, y = \frac{5}{1} = 5$

Let $x = 5, y = \frac{5}{5} = 1$

The points of intersection are (1 ,5) and (5 ,1).

36b The graphs of $y = \frac{5}{x}$ (red curve) and $y = 6 - x$ (purple line) are shown below.



The area of the enclosed region is evaluated by the following integral.

$$\begin{aligned}\int_1^5 6 - x - \frac{5}{x}\,dx &= \left[6x - \frac{x^2}{2} - 5\ln|x|\right]_1^5 \\ &= \left(6(5) - \frac{5^2}{2} - 5\ln|5|\right) - \left(6(1) - \frac{1^2}{2} - 5\ln|1|\right) \\ &= 30 - \frac{25}{2} - 5\ln 5 - 6 + \frac{1}{2} \\ &= (12 - 5\ln 5) \text{ square units}\end{aligned}$$

37a Let $y = e^x$

$$y' = \frac{d}{dx}(e^x) = e^x$$

37b Let $y = 2^x$

$$y = e^{\log_e 2^x} = e^{x\log_e 2}$$

Let $u = x\log_e 2$, $u' = \log_e 2$

By the chain rule,

$$\begin{aligned}y' &= \frac{d}{du}(e^u)\frac{du}{dx} \\ &= e^u(\log_e 2) \\ &= e^{x\log_e 2}\log_e 2 \\ &= e^{\log_e 2^x}\log_e 2\end{aligned}$$

$= 2^x \log_e 2$

37c Let $y = 3^x$

$y = e^{\log_e 3^x} = e^{x \log_e 3}$

Let $u = x \log_e 3, u' = \log_e 3$

By the chain rule,

$$\begin{aligned} y' &= \frac{d}{du}(e^u)\frac{du}{dx} \\ &= e^u(\log_e 3) \\ &= e^{x \log_e 3} \log_e 3 \\ &= e^{\log_e 3^x} \log_e 3 \\ &= 3^x \log_e 3 \end{aligned}$$

37d Let $y = 5^x$

$y = e^{\log_e 5^x} = e^{x \log_e 5}$

Let $u = x \log_e 5, u' = \log_e 5$

By the chain rule,

$$\begin{aligned} y' &= \frac{d}{du}(e^u)\frac{du}{dx} \\ &= e^u(\log_e 5) \\ &= e^{x \log_e 5} \log_e 5 \\ &= e^{\log_e 5^x} \log_e 5 \\ &= 5^x \log_e 5 \end{aligned}$$

38a

$$\int e^x \, dx = e^x + C$$

38b

$$\int 2^x \, dx = \int e^{\log_e 2^x} \, dx = \int e^{x \log_e 2} \, dx$$

Let $u = x \log_e 2 \,, u' = \log_e 2$

$$\begin{aligned}\int e^{x\log_e 2}\,dx &= \frac{1}{\log_e 2}\int e^u \log_e 2\,dx\\ &= \frac{1}{\log_e 2}\int e^u\,du\\ &= \frac{1}{\log_e 2}e^u + C\\ &= \frac{1}{\log_e 2}e^{x\log_e 2} + C\\ &= \frac{2^x}{\log_e 2} + C\end{aligned}$$

38c

$$\int 3^x\,dx = \int e^{\log_e 3^x}\,dx = \int e^{x\log_e 3}\,dx$$

Let $u = x\log_e 3\,, u' = \log_e 3$

$$\begin{aligned}\int e^{x\log_e 3}\,dx &= \frac{1}{\log_e 3}\int e^u \log_e 3\,dx\\ &= \frac{1}{\log_e 3}\int e^u\,du\\ &= \frac{1}{\log_e 3}e^u + C\\ &= \frac{1}{\log_e 3}e^{x\log_e 3} + C\\ &= \frac{3^x}{\log_e 3} + C\end{aligned}$$

38d

$$\int 5^x\,dx = \int e^{\log_e 5^x}\,dx = \int e^{x\log_e 5}\,dx$$

Let $u = x\log_e 5\,, u' = \log_e 5$

$$\begin{aligned}\int e^{x\log_e 5}\,dx &= \frac{1}{\log_e 5}\int e^u \log_e 5\,dx\\ &= \frac{1}{\log_e 5}\int e^u\,du\\ &= \frac{1}{\log_e 5}e^u + C\\ &= \frac{1}{\log_e 5}e^{x\log_e 5} + C\end{aligned}$$

$$= \frac{5^x}{\log_e 5} + C$$

39a Let $y = x \log_e x$

By the product rule,

$$y' = \frac{d}{dx}(x)\log_e x + x\frac{d}{dx}(\log_e x)$$
$$= \log_e x + x\left(\frac{1}{x}\right)$$
$$= \log_e x + 1$$

$$\int \log_e x \, dx = \int (\log_e x + 1 - 1)\, dx$$
$$= \int (\log_e x + 1)\, dx - \int 1\, dx$$
$$= \int y'\, dx - \int 1\, dx$$
$$= y - x + C$$
$$= x\log_e x - x + C$$

39b Let $y = x\, e^x$

By the product rule,

$$y' = \frac{d}{dx}(x)e^x + x\frac{d}{dx}(e^x)$$
$$= e^x + x\, e^x$$

$$\int x\, e^x\, dx = \int (x\, e^x + e^x - e^x)\, dx$$
$$= \int (x\, e^x + e^x)\, dx - \int e^x\, dx$$
$$= \int y'\, dx - \int e^x\, dx$$
$$= x\, e^x - e^x + C$$

39c Consider each term individually:

$$\int_1^e \frac{1}{x}\,dx = [\ln|x|]_1^e$$
$$= \ln e - \ln 1$$
$$= 1 - 0$$
$$= 1$$

From 39a,

$$\int_1^e \log_e x\,dx$$
$$= [x\log_e x - x]_1^e$$
$$= \big((e)\log_e e - e\big) - (\log_e 1 - 1)$$
$$= e - e - 0 + 1$$
$$= 1$$

From 39b,

$$\int_0^1 xe^x\,dx$$
$$= [x\,e^x - e^x]_0^1$$
$$= (\,e^1 - e^1) - (\,0 - e^0)$$
$$= e^0$$
$$= 1$$

40a $y = 2^x$

$y = e^{\log_e 2^x} = e^{x\log_e 2}$

The gradient of the graph at any point is the first derivative, y'

Let $u = x\log_e 2$, $u' = \log_e 2$

By the chain rule,

$$y' = \frac{d}{du}(e^u)\frac{du}{dx}$$
$$= e^u(\log_e 2)$$
$$= e^{x\log_e 2}\log_e 2$$
$$= e^{\log_e 2^x}\log_e 2$$
$$= 2^x\log_e 2$$

At $A(3, 8)$, $y' = 2^3\log_e 2$

$y' = 8\log_e 2$

40b $y = \log_2 x$

$y = \dfrac{\log_e x}{\log_e 2}$

The gradient of the graph at any point is the first derivative, y'

$y' = \dfrac{1}{\log_e 2}\dfrac{d}{dx}(\log_e x)$

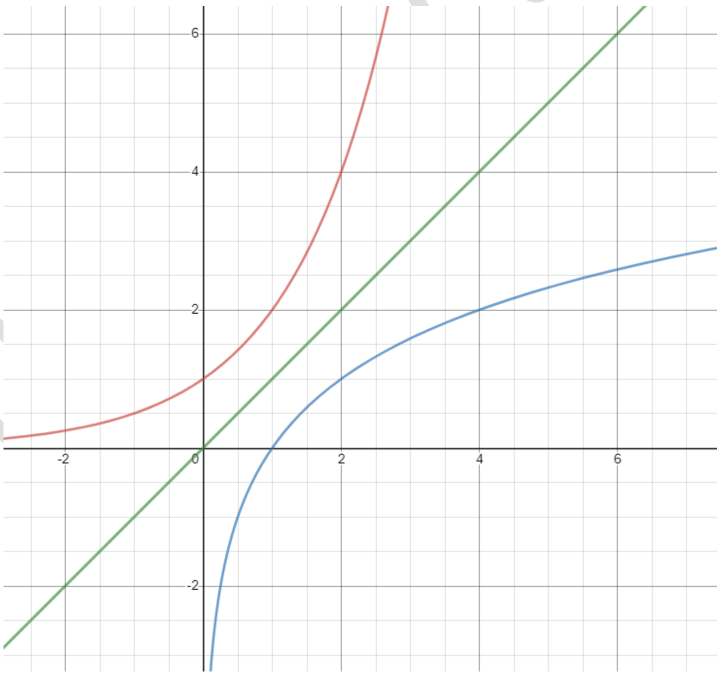
$= \dfrac{1}{\log_e 2}\left(\dfrac{1}{x}\right)$

$= \dfrac{1}{x\log_e 2}$

At $B(8, 3)$,

$y' = \dfrac{1}{8\log_e 2}$

40c



The graph $y = 2^x$ (red curve) is a reflection of $y = \log_2 x$ (blue curve) about $y = x$ (green line). Given that $A(3, 8)$ is a reflection of $B(8, 3)$ about $y = x$, the gradients of those points are by definition reciprocal (the rise and runs are switched).

41a

$$\int_0^3 2^x\,dx = \int_0^3 e^{\log_e 2^x}\,dx = \int_0^3 e^{x\log_e 2}\,dx$$

Let $u = x\log_e 2\,, u' = \log_e 2$

$$\int_0^3 e^{x\log_e 2}\,dx$$

$$= \frac{1}{\log_e 2}\int_0^3 e^u \log_e 2\,dx$$
$$= \frac{1}{\log_e 2}\int_0^{3\log_e 2} e^u\,du$$
$$= \frac{1}{\log_e 2}[e^u]_0^{3\log_e 2}$$
$$= \frac{1}{\log_e 2}\left[e^{3\log_e 2} - e^{0\log_e 2}\right]$$
$$= \frac{1}{\log_e 2}[2^3 - 2^0]$$
$$= \frac{(8-1)}{\log_e 2}$$
$$= \frac{7}{\log_e 2}$$

$$\int_{-3}^0 2^x\,dx = \int_{-3}^0 e^{\log_e 2^x}\,dx = \int_{-3}^0 e^{x\log_e 2}\,dx$$

Let $u = x\log_e 2\,, u' = \log_e 2$

$$\int_{-3}^0 e^{x\log_e 2}\,dx$$

$$= \frac{1}{\log_e 2}\int_{-3}^0 e^u \log_e 2\,dx$$
$$= \frac{1}{\log_e 2}\int_{-3\log_e 2}^0 e^u\,du$$
$$= \frac{1}{\log_e 2}[e^u]_{-3\,log_e 2}^0$$
$$= \frac{1}{\log_e 2}\left[e^{0\log_e 2} - e^{-3\log_e 2}\right]$$

$$= \frac{1}{\log_e 2}[2^0 - 2^{-3}]$$
$$= \frac{1 - \frac{1}{8}}{\log_e 2}$$
$$= \frac{7}{8\log_e 2}$$

41b The region in the first integral is 8 times larger than the first because the area is equivalent to a vertical dilation by a factor of 8. If the graph is inspected such that $y = 2^x$ is dilated vertically by a factor of 8, (draw the graph $\frac{y}{8} = 2^x$), it is equivalent to a translation to the left of the same graph ($y = 2^{x+3}$). Therefore, the region of the first integral is simply transformed by a vertical dilation of 8 and translated 3 units to the right to be the region of the second integral.