Solutions to Exercise 4A Foundation questions

Let C be a constant.

1a

$$\int e^{4x}\,dx$$

$$= \frac{1}{4}e^{4x} + C$$

1b

$$\int \sin 5x\,dx$$

$$= -\frac{1}{5}\cos 5x + C$$

1c

$$\int \sec^2 \frac{1}{2}x\,dx$$

$$= 2\tan\frac{1}{2}x + C$$

1d

$$\int \frac{1}{3x-4}\,dx$$

Let $u = 3x - 4$

$$\frac{du}{dx} = 3$$

Hence,

$$\int \frac{1}{3x-4}\,dx$$

$$= \frac{1}{3}\int \frac{3}{u}\,dx$$

$$= \frac{1}{3}\int \frac{1}{u}\frac{du}{dx}\,dx$$

$$= \frac{1}{3}\int \frac{1}{u}\,du$$

$$= \frac{1}{3}\ln|u| + C$$

$$= \frac{1}{3}\ln|3x - 4| + C$$

1e

$$\int \frac{2}{\sqrt{x}}\,dx$$

$$= 2\int x^{-\frac{1}{2}}\,dx$$

$$= 2 \times \left[\frac{x^{\frac{1}{2}}}{\frac{1}{2}}\right] + C$$

$$= 4\sqrt{x} + C$$

1f

$$\int 3^x\,dx$$

$$= \frac{3^x}{\ln 3} + C$$

2a

$$\int \frac{1}{(2x-1)^2}\,dx$$

$$= \int (2x-1)^{-2}\,dx$$

$$= \frac{(2x-1)^{-1}}{2 \times -1} + C$$

$$= -\frac{1}{2(2x-1)} + C$$

2b

$$\int \frac{1}{\sqrt{25-x^2}}\,dx$$

$$= \int \frac{1}{\sqrt{5^2-x^2}}\,dx$$

Let $f(x) = x$, $f'(x) = 1$

Hence,

$$\int \frac{1}{\sqrt{5^2-x^2}}\,dx$$

$$= \sin^{-1}\left(\frac{x}{5}\right) + C$$

2c

$$\int x^2 e^{x^3}\,dx$$

$f(x) = x^3$, $f'(x) = 3x^2$

Hence,

$$\int x^2 e^{x^3}\,dx$$

$$= \frac{1}{3}\int 3x^2 e^{x^3}\,dx$$

$$= \frac{1}{3}\int f'(x)e^{f(x)}\,dx$$

$$= \frac{1}{3}\frac{e^{f(x)}}{\ln e} + C$$

$$= \frac{1}{3}e^{x^3} + C$$

2d

$$\int \frac{1}{9+x^2}\,dx$$

$$= \int \frac{1}{3^2+x^2}\,dx$$

Let $f(x) = x, f'(x) = 1$

Hence,

$$\int \frac{1}{3^2+x^2}\,dx$$

$$= \frac{1}{3}\tan^{-1}\frac{x}{3} + C$$

2e

$$\int \frac{4x+2}{x^2+x+1}\,dx$$

Let $f(x) = x^2+x+1, f'(x) = 2x+1$

Hence,

$$\int \frac{4x+2}{x^2+x+1}\,dx$$

$$= 2\int \frac{2x+1}{x^2+x+1}\,dx$$

$$= 2\ln|x^2+x+1| + C$$

$$= 2\ln(x^2+x+1) + C \qquad (\text{since } x^2+x+1 > 0)$$

2f

$$\int 2x(x^2+1)^4 dx$$

Let $f(x) = x^2+1, f'(x) = 2x$

Hence,

$$\int 2x(x^2+1)^4 dx$$

$$= \frac{1}{5}(x^2+1)^5 + C$$

3a

$$\int_0^4 e^{\frac{x}{2}}\, dx$$

Let $f(x) = \dfrac{x}{2}, f'(x) = \dfrac{1}{2}$

Hence,

$$\int_0^4 e^{\frac{x}{2}}\, dx$$

$$= 2\int_0^4 \frac{1}{2} e^{\frac{x}{2}}\, dx$$

$$= 2\left[e^{\frac{x}{2}}\right]_0^4$$

$$= 2(e^2 - e^0)$$

$$= 2(e^2 - 1)$$

3b

$$\int_0^{\frac{\pi}{8}} \sec^2 2x\ \ dx$$

Let $f(x) = 2x, f'(x) = 2$

Hence,

$$\int_0^{\frac{\pi}{8}} \sec^2 2x\ \ dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{8}} 2\sec^2 2x\ \ dx$$

$$= \frac{1}{2}[\tan 2x]_0^{\frac{\pi}{8}}$$

$$= \frac{1}{2}\left(\tan\frac{\pi}{4} - \tan 0\right)$$

$$= \frac{1}{2}$$

3c

$$\int_{-4}^{4} \frac{1}{16+x^2}\ dx$$

$$= \int_{-4}^{4} \frac{1}{4^2+x^2}\ dx$$

Let $f(x) = x$, $f'(x) = 1$

Hence,

$$\int_{-4}^{4} \frac{1}{16+x^2} dx$$

$$= \left[\frac{1}{4}\tan^{-1}\frac{x}{4}\right]_{-4}^{4}$$

$$= \frac{1}{4}\tan^{-1} 1 - \frac{1}{4}\tan^{-1}(-1)$$

$$= \frac{1}{4}\left(\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right)$$

$$= \frac{\pi}{8}$$

3d

$$\int_{0}^{1} \frac{1}{\sqrt{2-x^2}} dx$$

$$= \int_{0}^{1} \frac{1}{\sqrt{\left(\sqrt{2}\right)^2 - x^2}} dx$$

Let $f(x) = x$, $f'(x) = 1$

Hence,

$$\int_{0}^{1} \frac{1}{\sqrt{\left(\sqrt{2}\right)^2 - x^2}} dx$$

$$= \left[\sin^{-1}\left(\frac{x}{\sqrt{2}}\right)\right]_{0}^{1}$$

$$= \left(\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) - \sin^{-1} 0\right)$$

$$= \frac{\pi}{4}$$

3e

$$\int_{-2}^{-1} \frac{3}{2-3x}\, dx$$

Let $f(x) = 2 - 3x$, $f'(x) = -3$

Hence,

$$\int_{-2}^{-1} \frac{3}{2-3x}\, dx$$

$$= -\int_{-2}^{-1} \frac{-3}{2-3x}\, dx$$

$$= -[\ln|2-3x|]_{-2}^{-1}$$

$$= -(\ln 5 - \ln 8)$$

$$= \ln 8 - \ln 5$$

$$= \frac{\ln 8}{\ln 5}$$

3f

$$\int_0^{\frac{\pi}{4}} \cos x \sin^3 x\, dx$$

Let $f(x) = \sin x$, $f'(x) = \cos x$

Hence,

$$\int_0^{\frac{\pi}{4}} \cos x \sin^3 x\, dx$$

$$= \left[\frac{1}{4}\sin^4 x\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{4}\left(\sin^4 \frac{\pi}{4} - \sin^4 0\right)$$

$$= \frac{1}{4}\left(\left(\frac{1}{\sqrt{2}}\right)^4 - 0\right)$$

$$= \frac{1}{16}$$

Solutions to Exercise 4A Development questions

4a

$$\int -\frac{1}{x^2}e^{\frac{1}{x}}\,dx$$

Let $u = \frac{1}{x}$

$du = -\frac{1}{x^2}dx$

Hence

$$\int -\frac{1}{x^2}e^{\frac{1}{x}}\,dx$$

$$= \int e^u\,du$$

$$= e^u + C$$

$$= e^{\frac{1}{x}} + C$$

4b

$$\int \frac{\cos 3x}{1 + \sin 3x}dx$$

Let $u = 1 + \sin 3x$

$du = 3\cos 3x\,dx$

Hence

$$\int \frac{\cos 3x}{1 + \sin 3x}dx$$

$$= \frac{1}{3}\int \frac{1}{u}du$$

$$= \frac{1}{3}\ln|u| + C$$

$$= \frac{1}{3}\ln(1 + \sin 3x) + C$$

Note: Modulus function not needed since $1 + \sin 3x$ is never less than 0.

4c

$$\int x\sec^2(x^2)\,dx$$

Let $u = x^2$

$du = 2x\,dx$

Hence

$$\int x\sec^2(x^2)\,dx$$

$$= \frac{1}{2}\int 2x\sec^2(x^2)\,dx$$

$$= \frac{1}{2}\int \sec^2 u\,du$$

$$= \frac{1}{2}\tan u + C$$

$$= \frac{1}{2}\tan(x^2) + C$$

4d

$$\int 5^{2x}\,dx$$

$$= \int \left(e^{\ln 5}\right)^{2x}\,dx$$

$$= \int e^{2\ln(5)x}\,dx$$

$$= \frac{1}{2\ln 5}e^{2\ln(5)x} + C$$

$$= \frac{1}{2\ln 5}5^{2x} + C$$

4e

$$\int \frac{1+\sec^2 x}{x+\tan x}dx$$

Let $u = x + \tan x$

$du = 1 + \sec^2 x\,dx$

Hence

$$\int \frac{1 + \sec^2 x}{x + \tan x} dx$$

$$= \int \frac{1}{u} du$$

$$= \ln|u| + C$$

$$= \ln|x + \tan x| + C$$

4f

$$\int \frac{e^x}{\sqrt{1 - e^{2x}}} dx$$

Let $u = e^x$

$du = e^x dx$

Hence

$$\int \frac{e^x}{\sqrt{1 - (e^x)^2}} dx$$

$$= \int \frac{1}{\sqrt{1 - u^2}} du$$

$$= \sin^{-1} u + C$$

$$= \sin^{-1}(e^x) + C$$

5a

$$\int_0^4 (1 - x)^3 \, dx$$

Let $u = 1 - x$

$du = -1 \, dx$

$x = 4, u = -3$

$x = 0, u = 1$

Hence

$$\int_0^4 (1 - x)^3 \, dx$$

$= -\int_{1}^{-3} u^3\, du$

$= \int_{-3}^{1} u^3\, du$

$= \left[\frac{1}{4}u^4\right]_{-3}^{1}$

$= \frac{1}{4} - \frac{1}{4} \times (-3)^4$

$= \frac{1}{4} - \frac{1}{4} \times 81$

$= -20$

5b

$\int_0^1 \frac{x^2}{1+x^3}\, dx$

Let $u = 1 + x^3$

$du = 3x^2\, dx$

$x = 1, u = 2$

$x = 0, u = 1$

Hence

$\int_0^1 \frac{x^2}{1+x^3}\, dx$

$= \frac{1}{3}\int_0^1 \frac{3x^2}{1+x^3}\, dx$

$= \frac{1}{3}\int_1^2 \frac{1}{u}\, du$

$= \frac{1}{3}[\ln|u|]_1^2$

$= \frac{1}{3}(\ln 2 - \ln 1)$

$= \frac{1}{3}\ln 2$

5c

$$\int_0^1 \frac{dx}{1+3x^2}$$

$$= \frac{1}{3}\int_0^1 \frac{dx}{\frac{1}{3}+x^2}$$

$$= \frac{1}{3}\int_0^1 \frac{1}{\left(\frac{1}{\sqrt{3}}\right)^2+x^2}dx$$

$$= \frac{\sqrt{3}}{3}\left[\tan^{-1}\left(\sqrt{3}x\right)\right]_0^1$$

$$= \frac{\sqrt{3}}{3}\left(\frac{\pi}{3}-0\right)$$

$$= \frac{\sqrt{3}\pi}{9}$$

Note: This is the rationalised answer. $\frac{\pi}{3\sqrt{3}}$ is also acceptable.

5d

$$\int_0^1 \frac{e^{2x}}{e^{2x}+1}dx$$

Let $u = e^{2x}+1$

$du = 2e^{2x}dx$

$x = 1, u = e^2+1$

$x = 0, u = 2$

Hence

$$\int_0^1 \frac{e^{2x}}{e^{2x}+1}dx$$

$$= \frac{1}{2}\int_0^1 \frac{2e^{2x}}{e^{2x}+1}dx$$

$$= \frac{1}{2}\int_2^{e^2+1} \frac{1}{u}du$$

$$= \frac{1}{2}[\ln|u|]_2^{e^2+1}$$

$$= \frac{1}{2}(\ln(e^2+1) - \ln 2)$$

$$= \frac{1}{2}\ln\left(\frac{e^2+1}{2}\right)$$

5e

$$\int_0^{\frac{1}{3}} \frac{dx}{\sqrt{4-9x^2}}$$

$$= \frac{1}{2}\int_0^{\frac{1}{3}} \frac{dx}{\sqrt{1-\frac{9}{4}x^2}}$$

Let $u = \frac{3}{2}x$

$$du = \frac{3}{2}dx$$

$x = 0, u = 0$

$x = \frac{1}{3}, u = \frac{1}{2}$

Hence,

$$\int_0^{\frac{1}{3}} \frac{dx}{\sqrt{4-9x^2}}$$

$$= \frac{1}{3}\int_0^{\frac{1}{2}} \frac{du}{\sqrt{1-u^2}}$$

$$= \frac{1}{3}[\sin^{-1} u]_0^{\frac{1}{2}}$$

$$= \frac{1}{3}\sin^{-1}\frac{1}{2}$$

$$= \frac{\pi}{18}$$

5f

$$\int_0^{\frac{\pi}{4}} \frac{\sec^2 x}{1+\tan x}\, dx$$

Let $u = 1 + \tan x$

$du = \sec^2 x\, dx$

$x = \frac{\pi}{4}, u = 2$

$x = 0, u = 1$

Hence

$$\int_0^{\frac{\pi}{4}} \frac{\sec^2 x}{1+\tan x}\, dx$$

$= \int_1^2 \frac{1}{u}\, du$

$= [\ln|u|]_1^2$

$= \ln|2| - \ln|1|$

$= \ln(2) - 0$

$= \ln 2$

Solutions to Exercise 4A Enrichment questions

6 $\int_e^{e^2} \frac{1}{x \ln x} dx$

$= \int_e^{e^2} \frac{\frac{1}{x}}{\ln x} dx$

$= [\ln(\ln x)]_e^{e^2}$

$= \ln 2 - \ln 1$

$= \ln 2$

7 Let $y = \frac{\ln x}{x} = \frac{1}{x} \cdot \ln x$

Then, $y' = \frac{1}{x^2} - \frac{\ln x}{x^2}$ (product rule)

So, $\frac{\ln x}{x^2} = \frac{1}{x^2} - y'$

Hence,

$\int \frac{\ln x}{x^2} dx$

$= -\frac{1}{x} - y + C$

$= -\frac{1+\ln x}{x} + C$

8 Let $y = x \sin^{-1} x$

Then, $y' = \sin^{-1} x + \frac{x}{\sqrt{1-x^2}}$

So, $\sin^{-1} x = y' - \frac{x}{\sqrt{1-x^2}}$

Hence,

$\int_0^{\frac{1}{2}} \sin^{-1} x \, dx$

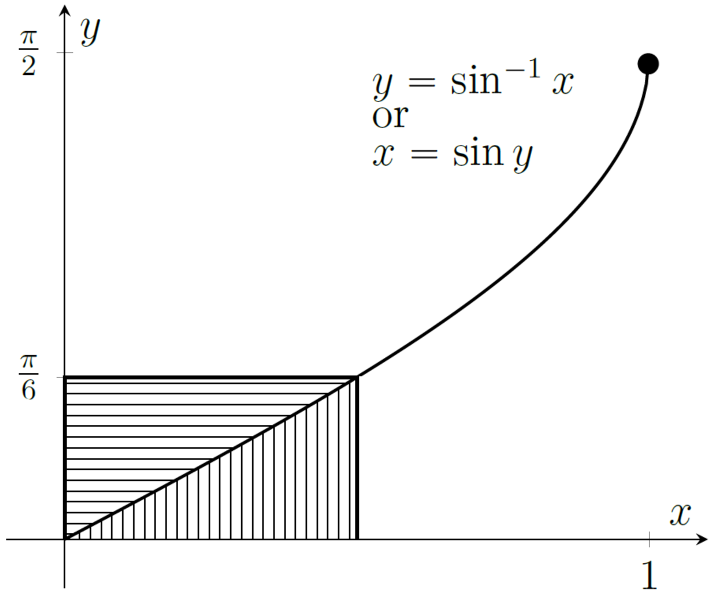
$= \left[y - \sqrt{1-x^2}\right]_0^{\frac{1}{2}}$

$= \left[x\sin^{-1} x - \sqrt{1-x^2}\right]_0^{\frac{1}{2}}$

$= \left(\frac{1}{2} \cdot \frac{\pi}{6} - \sqrt{\frac{3}{4}}\right) - (0 - 1)$

$= \frac{\pi}{12} - \frac{\sqrt{3}}{2} + 1$

Better still, use inverse functions and subtraction of areas, as indicated in the diagram below.



9 Let $y = \tan^3 x$

Then,

$y' = 3\tan^2 x \sec^2 x$

$y' = 3\tan^2 x + 3\tan^4 x$

So, $\tan^4 x = \frac{1}{3}y' - \tan^2 x$

$\int_0^{\frac{\pi}{4}} \tan^4 x \, dx$

$= \int_0^{\frac{\pi}{4}} \frac{1}{3}y' - \tan^2 x \, dx$

$= \int_0^{\frac{\pi}{4}} \frac{1}{3}y' + 1 - \sec^2 x \, dx$ (by Pythagoras; viz $\tan^2 x = \sec^2 x - 1$.)

$= \left[\frac{1}{3}y + x - \tan x\right]_0^{\frac{\pi}{4}}$

$= \left[\frac{1}{3}\tan^3 x + x - \tan x\right]_0^{\frac{\pi}{4}}$

$= \left(\frac{1}{3} + \frac{\pi}{4} - 1\right) - (0)$

$= \frac{\pi}{4} - \frac{2}{3}$

Alternatively,

$\int_0^{\frac{\pi}{4}} \tan^4 x \, dx$

$= \int_0^{\frac{\pi}{4}} \tan^2 x \sec^2 x - \tan^2 x \, dx$ (by Pythagoras)

$= \int_0^{\frac{\pi}{4}} \tan^2 x \sec^2 x + 1 - \sec^2 x \, dx$ (by Pythagoras again)

$= \left[\frac{1}{3}\tan^3 x + x - \tan x\right]_0^{\frac{\pi}{4}}$

$= \left(\frac{1}{3} + \frac{\pi}{4} - 1\right) - (0)$

$= \frac{\pi}{4} - \frac{2}{3}$ (as before)

Solutions to Exercise 4B Foundation questions

1a

$$\int \frac{x}{x-1}\,dx$$

$$= \int \frac{x-1+1}{x-1}\,dx$$

$$= \int \left(\frac{x-1}{x-1} + \frac{1}{x-1}\right) dx$$

$$= \int \left(1 + \frac{1}{x-1}\right) dx$$

$$= x + \ln|x-1| + C$$

1b

$$\int \frac{x-1}{x+1}\,dx$$

$$= \int \frac{x+1-2}{x+1}\,dx$$

$$= \int \left(\frac{x+1}{x+1} - \frac{2}{x+1}\right) dx$$

$$= \int \left(1 - \frac{2}{x+1}\right) dx$$

$$= x - 2\ln|x+1| + C$$

1c

$$\int \frac{x+1}{x-1}\,dx$$

$$= \int \frac{x-1+2}{x-1}\,dx$$

$$= \int \left(\frac{x-1}{x-1} + \frac{2}{x-1}\right) dx$$

$$= \int \left(1 + \frac{2}{x-1}\right) dx$$

$$= x + 2\ln|x-1| + C$$

2a

$$\int_0^1 \frac{x-1}{x+1}\,dx$$

$$= \int_0^1 \frac{x+1-2}{x+1}\,dx$$

$$= \int_0^1 \left(\frac{x+1}{x+1} - \frac{2}{x+1}\right)dx$$

$$= \int_0^1 \left(1 - \frac{2}{x+1}\right)dx$$

$$= [x - 2\ln|x+1|]_0^1$$

$$= (1 - 2\ln 2) - (0 - 2\ln 1)$$

$$= 1 - \ln 2^2 - 0 + 0$$

$$= 1 - \ln 4$$

2b

$$\int_0^2 \frac{x}{2x+1}\,dx$$

$$= \int_0^2 \left(\frac{1}{2} + \frac{-\frac{1}{2}}{(2x+1)}\right)dx \qquad \text{(by long division)}$$

$$= \int_0^2 \left(\frac{1}{2} - \frac{1}{2(2x+1)}\right)dx$$

Let $f(x) = 2x+1, f'(x) = 2$

Hence,

$$\int_0^2 \left(\frac{1}{2} - \frac{1}{2(2x+1)}\right)dx$$

$$= \int_0^2 \left(\frac{1}{2} - \frac{2}{4(2x+1)}\right)dx$$

$$= \int_0^2 \frac{1}{2}\,dx - \frac{1}{4}\int_0^2 \frac{2}{2x+1}\,dx$$

$$= \left[\frac{x}{2} - \frac{1}{4}\ln|2x+1|\right]_0^2$$

$$= \left(1 - \frac{1}{4}\ln 5\right) - \left(0 - \frac{1}{4}\ln 1\right)$$

$$= 1 - \frac{1}{4}\ln 5$$

2c

$$\int_0^1 \frac{3 - x^2}{1 + x^2}\, dx$$

$$= \int_0^1 \left(-1 + \frac{4}{1 + x^2}\right) dx \qquad \text{(by long division)}$$

$$= \int_0^1 -1\, dx + 4\int_0^1 \frac{1}{1 + x^2}\, dx$$

$$= [-x]_0^1 + 4[\tan^{-1} x]_0^1$$

$$= -1 + 0 + 4\tan^{-1} 1 - 0$$

$$= -1 + 4 \times \frac{\pi}{4}$$

$$= -1 + \pi$$

$$= \pi - 1$$

3a

$$\int_0^{\frac{\sqrt{3}}{2}} \frac{1 - x}{\sqrt{1 - x^2}}\, dx$$

$$= \int_0^{\frac{\sqrt{3}}{2}} \left(\frac{1}{\sqrt{1 - x^2}} - \frac{x}{\sqrt{1 - x^2}}\right) dx$$

$$= \int_0^{\frac{\sqrt{3}}{2}} \frac{1}{\sqrt{1 - x^2}}\, dx - \int_0^{\frac{\sqrt{3}}{2}} \frac{x}{\sqrt{1 - x^2}}\, dx$$

If $y = (1 - x^2)^{\frac{1}{2}}$, then

$$\frac{dy}{dx} = \frac{1}{2}(1 - x^2)^{-\frac{1}{2}} \times -2x$$

$$= \frac{-x}{\sqrt{1 - x^2}}$$

Hence,

$$\int_0^{\frac{\sqrt{3}}{2}} \frac{1}{\sqrt{1-x^2}}\,dx + \int_0^{\frac{\sqrt{3}}{2}} \frac{-x}{\sqrt{1-x^2}}\,dx$$

$$= [\sin^{-1} x]_0^{\frac{\sqrt{3}}{2}} + \left[\sqrt{1-x^2}\right]_0^{\frac{\sqrt{3}}{2}}$$

$$= \left(\sin^{-1}\frac{\sqrt{3}}{2} - \sin^{-1} 0\right) + \left(\frac{1}{2} - 1\right)$$

$$= \frac{\pi}{3} - \frac{1}{2}$$

3b

$$\int_0^1 \frac{2x+1}{1+x^2}\,dx$$

$$= \int_0^1 \left(\frac{2x}{1+x^2} + \frac{1}{1+x^2}\right) dx$$

$$= \int_0^1 \frac{2x}{1+x^2}\,dx + \int_0^1 \frac{1}{1+x^2}\,dx$$

If $y = 1 + x^2$, then

$$\frac{dy}{dx} = 2x$$

Hence,

$$\int_0^1 \frac{2x}{1+x^2}\,dx + \int_0^1 \frac{1}{1+x^2}\,dx$$

$$= [\ln(1+x^2)]_0^1 + [\tan^{-1}(x)]_0^1$$

$$= (\ln 2 - \ln 1) + (\tan^{-1} 1 - \tan^{-1} 0)$$

$$= \ln 2 - 0 + \frac{\pi}{4} - 0$$

$$= \frac{\pi}{4} + \ln 2$$

3c

$$\int_0^1 \frac{1-x}{1+x^2}\,dx$$

$$= \int_0^1 \left(\frac{1}{1+x^2} - \frac{x}{1+x^2}\right) dx$$

$$= \int_0^1 \frac{1}{1+x^2}\,dx - \int_0^1 \frac{x}{1+x^2}\,dx$$

If $y = 1 + x^2$, then

$$\frac{dy}{dx} = 2x$$

Hence,

$$\int_0^1 \frac{1}{1+x^2}\,dx - \int_0^1 \frac{x}{1+x^2}\,dx$$

$$= \int_0^1 \frac{1}{1+x^2}\,dx - \frac{1}{2}\int_0^1 \frac{2x}{1+x^2}\,dx$$

$$= [\tan^{-1} x]_0^1 - \frac{1}{2}[\ln(1+x^2)]_0^1 \qquad \text{(since } 1 + x^2 > 0\text{)}$$

$$= (\tan^{-1} 1 - \tan^{-1} 0) - \frac{1}{2}(\ln 2 - \ln 1)$$

$$= \frac{\pi}{4} - 0 - \frac{\ln 2}{2} + 0$$

$$= \frac{\pi}{4} - \frac{2\ln 2}{4}$$

$$= \frac{1}{4}(\pi - \ln 2^2)$$

$$= \frac{1}{4}(\pi - \ln 4)$$

3d

$$\int_0^2 \frac{1+x}{4+x^2}\,dx$$

$$= \int_0^2 \left(\frac{1}{4+x^2} + \frac{x}{4+x^2}\right) dx$$

$$= \int_0^2 \frac{1}{4+x^2}\,dx + \int_0^2 \frac{x}{4+x^2}\,dx$$

If $y = 4 + x^2$, then

$$\frac{dy}{dx} = 2x$$

Hence,

$$\int_0^2 \frac{1}{4+x^2}\,dx + \int_0^2 \frac{x}{4+x^2}\,dx$$

$$= \int_0^2 \frac{1}{4+x^2}\,dx + \frac{1}{2}\int_0^2 \frac{2x}{4+x^2}\,dx$$

$$= \left[\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right]_0^2 + \frac{1}{2}[\ln(4+x^2)]_0^2 \qquad (\text{since } 4 + x^2 > 0)$$

$$= \frac{1}{2}[(\tan^{-1}1 - \tan^{-1}0) + (\ln 8 - \ln 4)]$$

$$= \frac{1}{2}\left[\frac{\pi}{4} - 0 + \ln 8 - \ln 4\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{4} + \ln 2^3 - \ln 2^2\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{4} + 3\ln 2 - 2\ln 2\right]$$

$$= \frac{\pi}{8} + \frac{1}{2}\ln 2$$

4a

$$y = \log\left(x + \sqrt{x^2 + a^2}\right)$$

Let $u = x + (x^2 + a^2)^{\frac{1}{2}}$

$$\frac{du}{dx} = 1 + \frac{1}{2}(x^2 + a^2)^{-\frac{1}{2}} \times 2x$$

$$= 1 + \frac{x}{(x^2 + a^2)^{\frac{1}{2}}}$$

Hence,

$$y = \log u$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{1}{u} \times \left(1 + \frac{x}{\sqrt{x^2 + a^2}}\right)$$

$$\frac{dy}{dx} = \frac{\left(\frac{x + (x^2 + a^2)^{\frac{1}{2}}}{(x^2 + a^2)^{\frac{1}{2}}}\right)}{x + (x^2 + a^2)^{\frac{1}{2}}}$$

$$\frac{dy}{dx} = \frac{1}{(x^2 + a^2)^{\frac{1}{2}}}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{x^2 + a^2}}$$

4b

$$\int \frac{1}{\sqrt{x^2 + a^2}}\, dx$$

$$= \log\left(x + \sqrt{x^2 + a^2}\right) + C$$

(since $\sqrt{x^2 + a^2} \geq |x|$ and so $x + \sqrt{x^2 + a^2}$ is not negative)

4c i

$$\int \frac{1}{\sqrt{x^2+3}}\,dx$$

$$= \int \frac{1}{\sqrt{x^2+\left(\sqrt{3}\right)^2}}\,dx$$

$$= \log\left(x+\sqrt{x^2+3}\right)+C$$

(since $\sqrt{x^2+3} \geq |x|$ and so $x+\sqrt{x^2+3}$ is not negative)

4c ii

$$\int_{-4}^{4} \frac{1}{\sqrt{x^2+9}}$$

$$= \int_{-4}^{4} \frac{1}{\sqrt{x^2+3^2}}$$

$$= \left[\log\left(x+\sqrt{x^2+9}\right)\right]_{-4}^{4}$$

$$= \log 9 - \log 1$$

$$= \log 9 - 0$$

$$= \log 9$$

$$= \log 3^2$$

$$= 2\log 3$$

Solutions to Exercise 4B Development questions

5a

$$\int \frac{x^3}{x^2+1}\,dx$$

$x^3 = x(x^2+1) - x$

Hence

$$\int \frac{x^3}{x^2+1}\,dx$$

$$= \int \frac{x(x^2+1) - x}{x^2+1}\,dx$$

$$= \int \left(x - \frac{x}{x^2+1}\right) dx$$

$$= \frac{1}{2}x^2 - \frac{1}{2}\ln(x^2+1) + C$$

Note: Modulus function not needed since $x^2 + 1$ is always greater than 0.

5b

$$\int \frac{x^3}{x+1}\,dx$$

$x^3 = (x^3+1) - 1$

$x^3 + 1 = (x+1)(x^2 - x + 1)$

Hence

$$\int \frac{x^3}{x+1}\,dx$$

$$= \int \frac{(x^3+1) - 1}{x+1}\,dx$$

$$= \int \frac{(x+1)(x^2-x+1) - 1}{x+1}\,dx$$

$$= \int \left(x^2 - x + 1 - \frac{1}{x+1}\right) dx$$

$$= \frac{1}{3}x^3 - \frac{1}{2}x^2 + x - \ln|x+1| + C$$

5c i

$$\int \frac{x^3}{x-1}\,dx$$

$$x^3 = x^3 - 1 + 1$$

$$x^3 - 1 = (x-1)(x^2+x+1)$$

Hence

$$\int \frac{x^3}{x-1}\,dx$$

$$= \int \frac{(x-1)(x^2+x+1)+1}{x-1}\,dx$$

$$= \int \left(x^2 + x + 1 + \frac{1}{x-1}\right) dx$$

$$= \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + \ln|x-1| + C$$

5c ii

$$\int \frac{x^4}{x^2+1}\,dx$$

$$x^4 = x^4 - 1 + 1$$

$$x^4 - 1 = (x^2-1)(x^2+1)$$

Hence

$$\int \frac{x^4}{x^2+1}\,dx$$

$$= \int \frac{(x^2-1)(x^2+1)+1}{x^2+1}\,dx$$

$$= \int \left(x^2 - 1 + \frac{1}{x^2+1}\right) dx$$

$$= \frac{1}{3}x^3 - x + \tan^{-1} x + C$$

5c iii

$$\int \frac{1}{1+e^x}dx$$

$$1 = 1 + e^x - e^x$$

Hence

$$\int \frac{1}{1+e^x}dx$$

$$= \int \frac{1+e^x-e^x}{1+e^x}dx$$

$$= \int \left(1 - \frac{e^x}{1+e^x}\right)dx$$

$$= \int dx - \int \frac{e^x}{1+e^x}dx$$

Let $u = 1 + e^x$

$du = e^x dx$

$$= \int dx - \int \frac{1}{u}du$$

$$= x - \ln(1+e^x) + C$$

Note: Modulus function not needed since $1 + e^x$ is always greater than 0.

5c iv

$$\int \frac{x}{\sqrt{2+x}}dx$$

$$x = 2 + x - 2$$

Hence

$$\int \frac{x}{\sqrt{2+x}}dx$$

$$= \int \frac{2+x-2}{\sqrt{2+x}}dx$$

$$= \int \sqrt{2+x} - \frac{2}{\sqrt{2+x}}dx$$

$$= \frac{2}{3}(2+x)^{\frac{3}{2}} - 4(2+x)^{\frac{1}{2}} + C$$

Note: This can be further factorised and simplified as below.

$$= \left(\frac{2}{3}(2+x) - 4\right)\sqrt{2+x} + C$$

$$= \frac{2}{3}\left(2 + x - \left(\frac{3}{2} \times 4\right)\right)\sqrt{2+x} + C$$

$$= \frac{2}{3}(x-4)\sqrt{2+x} + C$$

5c v

$$\int \frac{x}{\sqrt{1-x}}\,dx$$

$$x = -(1-x) + 1$$

Hence

$$\int \frac{x}{\sqrt{1-x}}\,dx$$

$$= \int \frac{-(1-x)+1}{\sqrt{1-x}}\,dx$$

$$= \int -\sqrt{1-x} + \frac{1}{\sqrt{1-x}}\,dx$$

$$= \frac{2}{3}(1-x)^{\frac{3}{2}} - 2(1-x)^{\frac{1}{2}} + C$$

Note: This can be further factorised and simplified as below.

$$= \sqrt{1-x}\left(\frac{2}{3}(1-x) - 2\right) + C$$

$$= \frac{2}{3}\sqrt{1-x}\left(1 - x - \left(\frac{3}{2} \times 2\right)\right) + C$$

$$= -\frac{2}{3}(2+x)\sqrt{1-x} + C$$

5c vi

$$\int \frac{x^3}{x^2+4}dx$$

$$= \int \frac{x^3+4x-4x}{x^2+4}dx$$

$$= \int \frac{x(x^2+4)-4x}{x^2+4}dx$$

$$= \int \left(x - \frac{4x}{x^2+4}\right)dx$$

$$= \frac{1}{2}x^2 - 2\ln|x^2+4| + C$$

$$= \frac{1}{2}x^2 - 2\ln(x^2+4) + C$$

Note: Modulus function not needed since $x^2 + 4$ is always greater than 0.

6a

$$\int_1^2 \frac{e^{2x}+1}{e^{2x}-1}dx$$

$$= \int_1^2 \frac{e^{2x}+1}{e^{2x}-1} \times \frac{e^{-x}}{e^{-x}}dx$$

$$= \int_1^2 \frac{e^x+e^{-x}}{e^x-e^{-x}}dx$$

Let $u = e^x - e^{-x}$

$du = e^x + e^{-x}dx$

$x = 2, u = e^2 - e^{-2}$

$x = 1, u = e - e^{-1}$

Hence

$$\int_1^2 \frac{e^x+e^{-x}}{e^x-e^{-x}}dx$$

$$= \int_{e-e^{-1}}^{e^2-e^{-2}} \frac{1}{u}du$$

$$= [\ln|u|]_{e-e^{-1}}^{e^2-e^{-2}}$$

$$= \ln(e^2 - e^{-2}) - \ln(e - e^{-1})$$

$$= \ln\left(\frac{e^2 - e^{-2}}{e - e^{-1}}\right)$$

$$= \ln\left(\frac{(e + e^{-1})(e - e^{-1})}{e - e^{-1}}\right)$$

$$= \ln(e + e^{-1})$$

6b

$$\int_0^1 \frac{e^x}{e^x + e^{-x}}\,dx$$

$$= \int_0^1 \frac{e^x}{e^x + e^{-x}} \times \frac{e^x}{e^x}\,dx$$

$$= \frac{1}{2}\int_0^1 \frac{2e^{2x}}{e^{2x} + 1}\,dx$$

$$= \frac{1}{2}[\ln|e^{2x} + 1|]_0^1$$

$$= \frac{1}{2}(\ln(e^2 + 1) - \ln(2))$$

$$= \frac{1}{2}\ln\left(\frac{e^2 + 1}{2}\right)$$

6c

$$\int_1^{\sqrt{3}} \frac{2 + \frac{1}{x}}{x + \frac{1}{x}}\,dx$$

$$= \int_1^{\sqrt{3}} \frac{2 + \frac{1}{x}}{x + \frac{1}{x}} \times \frac{x}{x}\,dx$$

$$= \int_1^{\sqrt{3}} \frac{2x + 1}{x^2 + 1}\,dx$$

$$= \int_1^{\sqrt{3}} \frac{2x}{x^2 + 1}\,dx + \int_1^{\sqrt{3}} \frac{1}{x^2 + 1}\,dx$$

$$= [\ln(x^2+1)]_1^{\sqrt{3}} + [\tan^{-1} x]_1^{\sqrt{3}}$$

$$= \ln 4 - \ln 2 + \tan^{-1}\sqrt{3} - \tan^{-1} 1$$

$$= \ln 2 + \frac{\pi}{3} - \frac{\pi}{4}$$

$$= \ln 2 + \frac{\pi}{12}$$

7a

$$\int \frac{x^2+x+1}{x+1}\,dx$$

$$= \int \left(\frac{x^2}{x+1} + 1\right)\,dx$$

$$= \int \left(\frac{x^2-1+1}{x+1} + 1\right)\,dx$$

$$= \int \left(\frac{(x-1)(x+1)+1}{x+1} + 1\right)\,dx$$

$$= \int \left(x - 1 + \frac{1}{x+1} + 1\right)\,dx$$

$$= \int \left(x + \frac{1}{x+1}\right)\,dx$$

$$= \frac{1}{2}x^2 + \ln|x+1| + C$$

7b

$$\int \frac{x^3-2x^2+3}{x-2}\,dx$$

$$= \int \frac{x^2(x-2)+3}{x-2}\,dx$$

$$= \int \left(x^2 + \frac{3}{x-2}\right)\,dx$$

$$= \frac{1}{3}x^3 + 3\ln|x-2| + C$$

7c

$$\int \frac{(x+1)^2}{1+x^2}\,dx$$

$$= \int \frac{x^2+2x+1}{1+x^2}\,dx$$

$$= \int \left(1+\frac{2x}{1+x^2}\right)dx$$

$$= x + \ln(1+x^2) + C$$

Note: Modulus function not needed since $1+x^2$ is always greater than 0.

8a

$$y = \ln\left(x+\sqrt{x^2-a^2}\right), x > |a|$$

$$\frac{dy}{dx} = \frac{1}{x+\sqrt{x^2-a^2}} \times \left(1+\frac{x}{\sqrt{x^2-a^2}}\right)$$

$$\frac{dy}{dx} = \frac{1}{x+\sqrt{x^2-a^2}} \times \left(\frac{x+\sqrt{x^2-a^2}}{\sqrt{x^2-a^2}}\right)$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{x^2-a^2}}$$

8b

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right) + C$$

8c i

$$\int \frac{1}{\sqrt{x^2-5}}\,dx$$

$$= \int \frac{1}{\sqrt{x^2-\sqrt{5}^2}}\,dx$$

$$= \ln\left(x+\sqrt{x^2-5}\right) + C$$

8c ii

$$\int_{\sqrt{5}}^{3} \frac{1}{\sqrt{x^2-4}}\,dx$$

$$= \left[\ln\left(x+\sqrt{x^2-4}\right)\right]_{\sqrt{5}}^{3}$$

$$= \ln\left(3+\sqrt{5}\right) - \ln(\sqrt{5}+1)$$

$$= \ln\left(\frac{\sqrt{5}+3}{\sqrt{5}+1}\right)$$

$$= \ln\left(\frac{\sqrt{5}+1+2}{\sqrt{5}+1}\right)$$

$$= \ln\left(1+\frac{2}{\sqrt{5}+1}\times\frac{\sqrt{5}-1}{\sqrt{5}-1}\right)$$

$$= \ln\left(1+\frac{\sqrt{5}-1}{2}\right)$$

$$= \ln\left(\frac{\sqrt{5}+1}{2}\right)$$

Solutions to Exercise 4B Enrichment questions

9 $\int \frac{dx}{x+\sqrt{x}}$

$= 2\int \frac{\frac{1}{2}\cdot\frac{1}{\sqrt{x}}}{\sqrt{x}+1}\,dx$

$= 2\ln\left(\sqrt{x}+1\right) + C$ (note: $x > 0$)

<u>Note</u>: There is no need for any absolute value here since $\sqrt{x}+1 > 0$ for all $x \geq 0$.

In these solutions, absolute values will only be included when there is a need.

Alternatively, this can be also done by substitution (as in 4C).

Put $u = \sqrt{x}$, then $u^2 = x$ and so $2udu = dx$.

Hence,

$\int \frac{dx}{x+\sqrt{x}}$

$= \int \frac{2udu}{u^2+u}$

$= 2\int \frac{du}{u+1}$ (for $u > 0$)

$= 2\ln(u+1) + C$

$= 2\ln\left(\sqrt{x}+1\right) + C$ (as before)

10 $\int \frac{dx}{\sqrt{x^2-a^2}}$

$= \int \frac{x+\sqrt{x^2-a^2}}{\sqrt{x^2-a^2}\left(x+\sqrt{x^2-a^2}\right)}\,dx$

$= \int \frac{1+\frac{x}{\sqrt{x^2-a^2}}}{\left(x+\sqrt{x^2-a^2}\right)}\,dx$

$= \ln\left|\left(x+\sqrt{x^2-a^2}\right)\right| + C$, for $|x| > |a|$

Solutions to Exercise 4C Foundation questions

1a Using reverse chain rule:

$\int 2x(x^2+1)^4\,dx$

Let $f(x) = x^2 + 1$

$f'(x) = 2x$

$$\int 2x(x^2+1)^4\,dx$$

$$= \frac{1}{5}(x^2+1)^5 + C$$

Using substitution:

Let $u = x^2 + 1$

$$\frac{du}{dx} = 2x$$

$du = 2x\,dx$

Thus

$$\int 2x(x^2+1)^4\,dx$$

$= \int u^4\,du$

$$= \frac{1}{5}u^5 + C$$

$$= \frac{1}{5}(x^2+1)^5 + C$$

1b Using reverse chain rule:

$\int 3x^2(1+x^3)^6\,dx$

Let $f(x) = 1 + x^3$

$f'(x) = 3x^2$

$\int 3x^2(1+x^3)^6\,dx$

$$= \frac{1}{7}(1+x^3)^7 + C$$

Using substitution:

Let $u = 1 + x^3$

$\frac{du}{dx} = 3x^2$

$du = 3x^2\ dx$

Thus

$\int 3x^2(1 + x^3)^6\ dx$

$= \int u^6\ du$

$= \frac{1}{7}u^7 + C$

$= \frac{1}{7}(1+x^3)^7 + C$

1c Using reverse chain rule:

$\int \frac{6x^2}{(1 + x^3)^2}\ dx$

Let $f(x) = 1 + x^3$

$f'(x) = 3x^2$

$\int \frac{6x^2}{(1 + x^3)^2}\ dx$

$= 2\int \frac{3x^2}{(1 + x^3)^2}\ dx$

$= \frac{2}{-1}(1 + x^3)^{-1} + C$

$= -\frac{2}{1 + x^3} + C$

Using substitution:

Let $u = 1 + x^3$

$\frac{du}{dx} = 3x^2$

$du = 3x^2 dx$

Thus

$$\int \frac{6x^2}{(1+x^3)^2}\,dx$$

$$= 2\int \frac{3x^2}{(1+x^3)^2}\,dx$$

$$= 2\int u^{-2}\,du$$

$$= -2u^{-1} + C$$

$$= -\frac{2}{1+x^3} + C$$

1d Using reverse chain rule:

$$\int \frac{4x}{(3-x^2)^5}\,dx$$

Let $f(x) = 3 - x^2$

$f'(x) = -2x$

$$\int \frac{4x}{(3-x^2)^5}\,dx$$

$$= -2\int \left(-2x(3-x^2)^{-5}\right)dx$$

$$= \frac{-2}{-4}(3-x^2)^{-4} + C$$

$$= \frac{1}{2(3-x^2)^4} + C$$

Using substitution:

Let $u = 3 - x^2$

$\frac{du}{dx} = -2x$

$du = -2x\,dx$

Thus

$$\int \frac{4x}{(3-x^2)^5}\,dx$$

$$= -2\int \frac{-2x}{(3-x^2)^5}\,dx$$

$$= -2\int u^{-5}\,du$$

$= \dfrac{-2}{-4} u^{-4} + C$

$= \dfrac{1}{2(3 - x^2)^4} + C$

1e Using reverse chain rule:

$\displaystyle\int \frac{x}{\sqrt{x^2 - 2}} \, dx$

Let $f(x) = x^2 - 2$

$f'(x) = 2x$

$\displaystyle\int \frac{x}{\sqrt{x^2 - 2}} \, dx$

$= \dfrac{1}{2} \displaystyle\int 2x(x^2 - 2)^{-\frac{1}{2}} \, dx$

$= \dfrac{1}{2} \times 2(x^2 - 2)^{\frac{1}{2}} + C$

$= \sqrt{x^2 - 2} + C$

Using substitution:

Let $u = x^2 - 2$

$\dfrac{du}{dx} = 2x$

$du = 2x \, dx$

Thus,

$\displaystyle\int \frac{x}{\sqrt{x^2 - 2}} \, dx$

$= \dfrac{1}{2} \displaystyle\int \frac{2x}{\sqrt{x^2 - 2}} \, dx$

$= \dfrac{1}{2} \displaystyle\int u^{-\frac{1}{2}} \, du$

$= \dfrac{1}{2} \times 2u^{\frac{1}{2}} + C$

$= \sqrt{x^2 - 2} + C$

1f Using reverse chain rule:

$$\int \frac{x^3}{\sqrt{1+x^4}}\,dx$$

Let $f(x) = 1 + x^4$

$f'(x) = 4x^3$

$$\int \frac{x^3}{\sqrt{1+x^4}}\,dx$$

$$= \frac{1}{4}\int 4x^3(1+x^4)^{-\frac{1}{2}}\,dx$$

$$= \frac{1}{4} \times 2(1+x^4)^{\frac{1}{2}} + C$$

$$= \frac{1}{2}\sqrt{1+x^4} + C$$

Using substitution:

Let $u = 1 + x^4$

$$\frac{du}{dx} = 4x^3$$

$du = 4x^3\,dx$

Thus,

$$\int \frac{x^3}{\sqrt{1+x^4}}\,dx$$

$$= \frac{1}{4}\int \frac{4x^3}{\sqrt{1+x^4}}\,dx$$

$$= \frac{1}{4}\int u^{-\frac{1}{2}}\,du$$

$$= \frac{1}{4} \times 2u^{\frac{1}{2}} + C$$

$$= \frac{1}{2}\sqrt{1+x^4} + C$$

2a

$$\int \frac{\cos x}{\sin^3 x}\ dx$$

Let $u = \sin x$

$$\frac{du}{dx} = \cos x$$

$du = \cos x\ dx$

Hence,

$$\int \frac{\cos x}{\sin^3 x}\ dx$$

$$= \int \frac{1}{u^3}\ du$$

$$= \int u^{-3}\ du$$

$$= \frac{u^{-2}}{-2} + C$$

$$= \frac{-1}{2\sin^2 x} + C$$

2b

$$\int \frac{\sec^2 x}{(1+\tan x)^2}\ dx$$

Let $u = 1 + \tan x$

$$\frac{du}{dx} = \sec^2 x$$

$du = \sec^2 x\ dx$

Hence,

$$\int \frac{\sec^2 x}{(1+\tan x)^2}\ dx$$

$$= \int \frac{1}{u^2}\ du$$

$$= \int u^{-2}\ du$$

$$= -u^{-1} + C$$

$$= \frac{-1}{1 + \tan x} + C$$

2c

$$\int \frac{(\ln x)^2}{x}\, dx$$

Let $u = \ln x$

$$\frac{du}{dx} = \frac{1}{x}$$

$$du = \frac{dx}{x}$$

Hence,

$$\int \frac{(\ln x)^2}{x}\, dx$$

$$= \int u^2\, du$$

$$= \frac{1}{3}u^3 + C$$

$$= \frac{1}{3}(\ln x)^3 + C$$

2d

$$\int \frac{\cos\sqrt{x}}{\sqrt{x}}\, dx$$

Let $u = \sqrt{x}$

$$\frac{du}{dx} = \frac{1}{2\sqrt{x}}$$

$$du = \frac{dx}{2\sqrt{x}}$$

Hence,

$$\int \frac{\cos\sqrt{x}}{\sqrt{x}}\, dx$$

$$= 2\int \frac{\cos\sqrt{x}}{2\sqrt{x}}\ dx$$

$$= 2\int \cos u\ \ du$$

$$= 2\sin u + C$$

$$= 2\sin\sqrt{x} + C$$

2e

$$\int \frac{x}{1+x^4}dx$$

Let $u = x^2$

$$\frac{du}{dx} = 2x$$

$$du = 2x\ dx$$

$$\int \frac{x}{1+x^4}dx$$

$$= \frac{1}{2}\int \frac{2x}{1^2+(x^2)^2}dx$$

$$= \frac{1}{2}\int \frac{1}{1^2+u^2}du$$

$$= \frac{1}{2}\times\frac{1}{1}\tan^{-1}u + C$$

$$= \frac{1}{2}\tan^{-1}x^2 + C$$

2f

$$\int \frac{x^2}{\sqrt{1-x^6}}dx$$

Let $u = x^3$

$$\frac{du}{dx} = 3x^2$$

$$du = 3x^2\ dx$$

$$\int \frac{x^2}{\sqrt{1-x^6}}\,dx$$

$$= \frac{1}{3}\int \frac{3x^2}{\sqrt{1-(x^3)^2}}\,dx$$

$$= \frac{1}{3}\int \frac{1}{\sqrt{1^2-u^2}}\,du$$

$$= \frac{1}{3}\times\frac{1}{1}\sin^{-1}u + C$$

$$= \frac{1}{3}\sin^{-1}x^3 + C$$

3a

$$\int_0^1 x^3(1+3x^4)^2\,dx$$

Let $u = 1+3x^4$

$$\frac{du}{dx} = 12x^3$$

$$du = 12x^3 dx$$

When $x = 1, u = 4$.

When $x = 0, u = 1$.

Hence

$$\int_0^1 x^3(1+3x^4)^2\,dx$$

$$= \frac{1}{12}\int_0^1 12x^3(1+3x^4)^2\,dx$$

$$= \frac{1}{12}\int_1^4 u^2\,du$$

$$= \left[\frac{1}{36}u^3\right]_1^4$$

$$= \frac{64}{36} - \frac{1}{36}$$

$$= \frac{63}{36}$$

$= \frac{7}{4}$

3b

$\int_0^1 \frac{x}{\sqrt{4-x^2}}\,dx$

Let $u = 4 - x^2$

$\frac{du}{dx} = -2x$

$du = -2x\,dx$

When $x = 1, u = 3.$

When $x = 0, u = 4.$

Hence

$\int_0^1 \frac{x}{\sqrt{4-x^2}}\,dx$

$= -\frac{1}{2}\int_0^1 \frac{-2x}{\sqrt{4-x^2}}\,dx$

$= -\frac{1}{2}\int_4^3 \frac{du}{\sqrt{u}}$

$= \frac{1}{2}\int_3^4 \frac{du}{\sqrt{u}}$

$= \left[\frac{1}{2} \times 2u^{\frac{1}{2}}\right]_3^4$

$= \left[\sqrt{u}\right]_3^4$

$= 2 - \sqrt{3}$

3c

$$\int_3^4 \frac{x+1}{\sqrt{x^2+2x+3}}dx$$

Let $u = x^2 + 2x + 3$

$$\frac{du}{dx} = 2x + 2 = 2(x+1)du = 2(x+1)\,dx$$

When $x = 4, u = 27.$

When $x = 3, u = 18.$

Hence

$$\int_3^4 \frac{x+1}{\sqrt{x^2+2x+3}}dx$$

$$= \frac{1}{2}\int_3^4 \frac{2(x+1)}{\sqrt{x^2+2x+3}}dx$$

$$= \frac{1}{2}\int_{18}^{27} \frac{du}{\sqrt{u}}$$

$$= \left[\frac{1}{2} \times 2u^{\frac{1}{2}}\right]_{18}^{27}$$

$$= \left[\sqrt{u}\right]_{18}^{27}$$

$$= \sqrt{27} - \sqrt{18}$$

$$= 3\sqrt{3} - 3\sqrt{2}$$

$$= 3\left(\sqrt{3} - \sqrt{2}\right)$$

3d

$$\int_0^{\frac{\pi}{2}} \sin^4 x\ \cos x dx$$

Let $u = \sin x$

$$\frac{du}{dx} = \cos x$$

$$du = \cos x\,dx$$

When $x = \frac{\pi}{2}, u = 1$.

When $x = 0, u = 0$.

Hence,

$$\int_0^{\frac{\pi}{2}} \sin^4 x \cos x dx$$

$$= \int_0^1 u^4 \, du$$

$$= \left[\frac{1}{5} u^5\right]_0^1$$

$$= \frac{1}{5} - 0$$

$$= \frac{1}{5}$$

3e

$$\int_0^{\frac{\pi}{4}} \tan^2 x \sec^2 x \, dx$$

Let $u = \tan x$

$$\frac{du}{dx} = \sec^2 x$$

$du = \sec^2 x \, dx$

When $x = \frac{\pi}{4}, u = 1$.

When $x = 0, u = 0$.

Hence,

$$\int_0^{\frac{\pi}{4}} \tan^2 x \sec^2 x \, dx$$

$$= \int_0^1 u^2 \, du$$

$$= \left[\frac{1}{3} u^3\right]_0^1$$

$= \frac{1}{3} - 0$

$= \frac{1}{3}$

3f

$\int_1^{e^2} \frac{\ln x}{x}\ dx$

Let $u = \ln x$

$\frac{du}{dx} = \frac{1}{x}$

$du = \frac{dx}{x}$

When $x = e^2, u = 2$.

When $x = 1, u = 0$.

Hence,

$\int_1^{e^2} \frac{\ln x}{x}\ dx$

$= \int_0^2 u\ \ du$

$= \left[\frac{1}{2}u^2\right]_0^2$

$= \frac{1}{2} \times 4 - 0$

$= 2$

Solutions to Exercise 4C Development questions

4a

$$\int_0^1 x(x-1)^5\,dx$$

Let $u = x - 1$

$du = dx$

$x = u + 1$

$x = 1, u = 0$

$x = 0, u = -1$

Hence

$$\int_0^1 x(x-1)^5\,dx$$

$$= \int_{-1}^0 (u+1)u^5\,du$$

$$= \int_{-1}^0 (u^6 + u^5)\,du$$

$$= \left[\frac{1}{7}u^7 + \frac{1}{6}u^6\right]_{-1}^0$$

$$= \left(\frac{1}{7}(0)^7 + \frac{1}{6}(0)^6\right) - \left(\frac{1}{7}(-1)^7 + \frac{1}{6}(-1)^6\right)$$

$$= \frac{1}{7} - \frac{1}{6}$$

$$= -\frac{1}{42}$$

4b

$$\int_0^1 x(x-1)^5\,dx$$

$x = (x-1) + 1$

Hence

$$\int_0^1 x(x-1)^5\,dx$$

$$= \int_0^1 \big((x-1)+1\big)(x-1)^5\,dx$$

$$= \int_0^1 (x-1)^6 + (x-1)^5\,dx$$

$$= \left[\frac{1}{7}(x-1)^7 + \frac{1}{6}(x-1)^6\right]_0^1$$

$$= \left(\frac{1}{7}(0) + \frac{1}{6}(0)\right) - \left(\frac{1}{7}(-1)^7 + \frac{1}{6}(-1)^6\right)$$

$$= \frac{1}{7} - \frac{1}{6}$$

$$= -\frac{1}{42}$$

5a

$$\int x\sqrt{x+1}\,dx$$

Let $u = \sqrt{x+1}$

$x = u^2 - 1$

$dx = \ 2u\,du$

Hence

$$\int x\sqrt{x+1}\,dx$$

$$= \int (u^2-1)u \times 2u\,du$$

$$= 2\int (u^4 - u^2)\,du$$

$$= 2\left(\frac{1}{5}u^5 - \frac{1}{3}u^3\right) + C$$

$$= \frac{2}{5}\left(\sqrt{x+1}\right)^5 - \frac{2}{3}\left(\sqrt{x+1}\right)^3 + C$$

$$= \frac{2}{15}\big(3(x+1)^2\sqrt{x+1} - 5(x+1)\sqrt{x+1}\big) + C$$

$$= \frac{2}{15}\left((x+1)\sqrt{x+1}(3(x+1)-5)\right) + C$$

$$= \frac{2}{15}(x+1)\sqrt{x+1}(3x-2) + C$$

$$= \frac{2}{15}(3x-2)(x+1)\sqrt{x+1} + C$$

5b

$$\int \frac{1}{1+\sqrt{x}}\,dx$$

Let $u = 1 + \sqrt{x}$

$x = (u-1)^2$

$dx = 2(u-1)\,du$

Hence

$$\int \frac{1}{1+\sqrt{x}}\,dx$$

$$= 2\int \frac{1}{u}(u-1)\,du$$

$$= 2\int \left(1 - \frac{1}{u}\right)\,du$$

$$= 2(u - \ln|u|) + C$$

$$= 2 + 2\sqrt{x} - 2\ln\left(1+\sqrt{x}\right) + C$$

$$= 2\left(1 + \sqrt{x} - \ln\left(1+\sqrt{x}\right)\right) + C$$

5c

$$\int \frac{1}{1+x^{\frac{1}{4}}}\,dx$$

Let $u = x^{\frac{1}{4}}$

$x = u^4$

$dx = 4u^3\,du$

Hence

$$\int \frac{1}{1+x^{\frac{1}{4}}} dx$$

$$= 4\int \frac{u^3}{1+u} du$$

$u^3 = u^3 + 1 - 1$

$u^3 + 1 = (u+1)(u^2 - u + 1)$

$u^3 = (u+1)(u^2 - u + 1) - 1$

Hence

$$4\int \frac{u^3}{1+u} du$$

$$= 4\int \frac{(u+1)(u^2-u+1)-1}{1+u} du$$

$$= 4\int \left(u^2 - u + 1 - \frac{1}{1+u}\right) du$$

$$= 4\left(\frac{1}{3}u^3 - \frac{1}{2}u^2 + u - \ln|1+u|\right) + C$$

$$= 4\left(\frac{1}{3}x^{\frac{3}{4}} - \frac{1}{2}x^{\frac{1}{2}} + x^{\frac{1}{4}} - \ln\left(1 + x^{\frac{1}{4}}\right)\right) + C$$

$$= 4\left(\frac{1}{3}x^{\frac{3}{4}} - \frac{1}{2}\sqrt{x} + x^{\frac{1}{4}} - \ln\left(1 + x^{\frac{1}{4}}\right)\right) + C$$

5d

$$\int \frac{1}{\sqrt{e^{2x}-1}} dx$$

Let $u = \sqrt{e^{2x} - 1}$

$e^{2x} = u^2 + 1$

$$du = \frac{e^{2x}}{\sqrt{e^{2x}-1}} dx$$

$$du = \frac{u^2+1}{u} dx$$

$$dx = \frac{u}{u^2+1} du$$

Hence,

$$\int \frac{1}{\sqrt{e^{2x}-1}}\,dx$$

$$= \int \frac{1}{u} \times \frac{u}{u^2+1}\,du$$

$$= \int \frac{1}{u^2+1}\,du$$

$$= \tan^{-1}(u) + C$$

$$= \tan^{-1}\left(\sqrt{e^{2x}-1}\right) + C$$

6a

$$\int_0^1 \frac{2-x}{(2+x)^3}\,dx$$

Let $u = 2 + x$

$x = u - 2$

$x = 1, u = 3$

$x = 0, u = 2$

$dx = du$

Hence

$$\int_0^1 \frac{2-x}{(2+x)^3}\,dx$$

$$= \int_2^3 \frac{2-u+2}{u^3}\,du$$

$$= \int_2^3 (4u^{-3} - u^{-2})\,du$$

$$= [-2u^{-2} + u^{-1}]_2^3$$

$$= \left(-\frac{2}{9} + \frac{1}{3}\right) - \left(-\frac{1}{2} + \frac{1}{2}\right)$$

$$= \frac{1}{9}$$

6b

$$\int_0^4 x\sqrt{4-x}\,dx$$

Let $u = \sqrt{4-x}$

$x = 4, u = 0$

$x = 0, u = 2$

$x = 4 - u^2$

$dx = -2u\,du$

Hence

$$\int_0^4 x\sqrt{4-x}\,dx$$

$$= \int_2^0 (4-u^2)u \times -2u\,du$$

$$= 2\int_0^2 (4-u^2)u^2\,du$$

$$= 2\int_0^2 (4u^2-u^4)\,du$$

$$= 2\left[\frac{4}{3}u^3 - \frac{1}{5}u^5\right]_0^2$$

$$= 2\left[\frac{4}{3}\times 8 - \frac{1}{5}\times 32\right]$$

$$= \frac{128}{15}$$

6c

$$\int_0^4 \frac{1}{5+\sqrt{x}}\,dx$$

Let $u = \sqrt{x}$

$x = 4, u = 2$

$x = 0, u = 0$

$x = u^2$

$dx = 2u\,du$

Hence

$\int_0^4 \frac{1}{5+\sqrt{x}}\,dx$

$= \int_0^2 \frac{1}{5+u} \times 2u\,du$

$= 2\int_0^2 \frac{u}{5+u}\,du$

$= 2\int_0^2 \frac{u+5-5}{5+u}\,du$

$= 2\int_0^2 \left(1 - \frac{5}{5+u}\right)du$

$= 2[u - 5\ln|5+u|]_0^2$

$= 2((2 - 5\ln 7) - (0 - 5\ln 5))$

$= 4 + 10\ln\left(\frac{5}{7}\right)$

6d

$\int_4^{12} \frac{1}{(4+x)\sqrt{x}}\,dx$

Let $u = \sqrt{x}$

$x = 12, u = 2\sqrt{3}$

$x = 4, u = 2$

$x = u^2$

$dx = 2u\,du$

Hence

$\int_4^{12} \frac{1}{(4+x)\sqrt{x}}\,dx$

$= \int_2^{2\sqrt{3}} \frac{1}{(4+u^2)u} \times 2u\,du$

$$= \int_2^{2\sqrt{3}} \frac{2}{2^2 + u^2} du$$

$$= \left[\tan^{-1}\left(\frac{u}{2}\right)\right]_2^{2\sqrt{3}}$$

$$= \tan^{-1}(\sqrt{3}) - \tan^{-1}(1)$$

$$= \frac{\pi}{3} - \frac{\pi}{4}$$

$$= \frac{\pi}{12}$$

7a

$$\int \frac{1}{(1+x)\sqrt{x}} dx$$

Let $u = \sqrt{x}$

$x = u^2$

$dx = 2u\,du$

Hence

$$\int \frac{1}{(1+x)\sqrt{x}} dx$$

$$= \int \frac{1}{(1+u^2)u} 2u\,du$$

$$= \int \frac{2}{(1+u^2)}\,du$$

$$= 2\tan^{-1}(u) + C$$

$$= 2\tan^{-1}(\sqrt{x}) + C$$

7b

$$\int \frac{x}{\sqrt{x+1}}\,dx$$

Let $u = \sqrt{x+1}$

$x = u^2 - 1$

$dx = 2u\,du$

Hence

$$\int \frac{x}{\sqrt{x+1}}\,dx$$

$$= \int \frac{u^2-1}{u} 2u\,du$$

$$= 2\int (u^2 - 1)\,du$$

$$= 2\left(\frac{1}{3}u^3 - u\right) + C$$

$$= \frac{2}{3}(x+1)\sqrt{x+1} - 2\sqrt{x+1} + C$$

$$= \left(\frac{2}{3}(x+1) - 2\right)\sqrt{x+1} + C$$

$$= \left(\frac{2}{3}x - \frac{4}{3}\right)\sqrt{x+1} + C$$

$$= \frac{2}{3}(x-2)\sqrt{x+1} + C$$

8a

$$\int \frac{1}{(1+x^2)^{\frac{3}{2}}}\,dx$$

Let $x = \tan\theta$

$dx = \sec^2\theta\ d\theta$

Hence

$$\int \frac{1}{(1+x^2)^{\frac{3}{2}}}\,dx$$

$$= \int \frac{1}{(1+\tan^2\theta)^{\frac{3}{2}}} \sec^2\theta \, d\theta$$

$$= \int \frac{\sec^2\theta}{(\sec^2\theta)^{\frac{3}{2}}} d\theta$$

$$= \int \frac{\sec^2\theta}{\sec^3\theta} d\theta$$

$$= \int \cos\theta \;\, d\theta$$

$$= \sin\theta + C$$

$$\theta = \tan^{-1} x$$

$$\sin\theta + C$$

$$= \sin(\tan^{-1} x) + C$$

$$= \frac{x}{\sqrt{x^2+1}} + C$$

8b

$$\int \frac{x^2}{\sqrt{4-x^2}} dx$$

Let $x = 2\sin\theta$

$dx = 2\cos\theta \;\, d\theta$

Hence

$$\int \frac{x^2}{\sqrt{4-x^2}} dx$$

$$= \int \frac{4\sin^2\theta}{\sqrt{4-4\sin^2\theta}} 2\cos\theta \;\, d\theta$$

$$= \int \frac{8\sin^2\theta\cos\theta}{\sqrt{4(1-\sin^2\theta)}} \; d\theta$$

$$= \int \frac{8\sin^2\theta\cos\theta}{2\cos\theta} \; d\theta$$

$$= \int 4\sin^2\theta \;\, d\theta$$

$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$

Hence

$\int 4\sin^2\theta \ d\theta$

$= \int (2-2\cos 2\theta) \ d\theta$

$= 2\theta - \sin 2\theta + C$

$\theta = \sin^{-1}\left(\frac{x}{2}\right)$

$\sin 2\theta = 2\sin\theta\cos\theta = 2\sin\theta\sqrt{1-\sin^2\theta}$

Hence

$2\theta - \sin 2\theta + C$

$= 2\sin^{-1}\left(\frac{x}{2}\right) - x\sqrt{1-\frac{x^2}{4}} + C$

$= 2\sin^{-1}\left(\frac{x}{2}\right) - x\sqrt{\frac{4-x^2}{4}} + C$

$= 2\sin^{-1}\left(\frac{x}{2}\right) - \frac{x}{2}\sqrt{4-x^2} + C$

8c

$\int \frac{1}{x^2\sqrt{25-x^2}} dx$

Let $x = 5\cos\theta$

$dx = -5\sin\theta \ d\theta$

Hence

$\int \frac{1}{x^2\sqrt{25-x^2}} dx$

$= \int \frac{1}{25\cos^2\theta\sqrt{25-25\cos^2\theta}}(-5\sin\theta) \ d\theta$

$= \int \frac{-5\sin\theta}{25\cos^2\theta\sqrt{25\sin^2\theta}} \ d\theta$

$$= \int \frac{-5\sin\theta}{25\cos^2\theta\, 5\sin\theta}\ d\theta$$

$$= \int \frac{-1}{25\cos^2\theta}\ d\theta$$

$$= -\frac{1}{25}\int \sec^2\theta\ d\theta$$

$$= -\frac{1}{25}\tan\theta + C$$

$$\theta = \cos^{-1}\left(\frac{x}{5}\right)$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sqrt{1-\cos^2\theta}}{\cos\theta}$$

Hence

$$-\frac{1}{25}\tan\theta + C$$

$$= -\frac{1}{25}\frac{\sqrt{1-\cos^2\theta}}{\cos\theta} + C$$

$$= -\frac{1}{25}\frac{\sqrt{1-\frac{x^2}{25}}}{\frac{x}{5}} + C$$

$$= \frac{-\sqrt{\frac{25-x^2}{25}}}{5x} + C$$

$$= \frac{-\frac{1}{5}\sqrt{25-x^2}}{5x} + C$$

$$= \frac{-\sqrt{25-x^2}}{25x} + C$$

8d

$$\int \frac{1}{x^2\sqrt{1+x^2}}\,dx$$

Let $x = \tan\theta$

$dx = \sec^2\theta\, d\theta$

Hence

$$\int \frac{1}{x^2\sqrt{1+x^2}}\,dx$$

$$= \int \frac{1}{\tan^2\theta\,\sqrt{1+\tan^2\theta}}\sec^2\theta\, d\theta$$

$$= \int \frac{\sec^2\theta}{\tan^2\theta\sec\theta}\,d\theta$$

$$= \int \frac{\sec\theta}{\tan^2\theta}\,d\theta$$

$$= \int \frac{1}{\cos\theta}\times\frac{\cos^2\theta}{\sin^2\theta}\,d\theta$$

$$= \int \frac{\cos\theta}{\sin^2\theta}\,d\theta$$

Let $u = \sin\theta$

$du = \cos\theta\, d\theta$

Hence

$$\int \frac{\cos\theta}{\sin^2\theta}\,d\theta$$

$$= \int \frac{1}{u^2}\,du$$

$$= -\frac{1}{u} + C$$

$$= -\frac{1}{\sin\theta} + C$$

$$= -\frac{1}{\sin(\tan^{-1}x)} + C$$

$$= -\frac{1}{\sin(\tan^{-1}x)} + C$$

$$= \frac{-\sqrt{x^2+1}}{x} + C$$

9a

$$\int_0^{\sqrt{2}} \frac{x^3}{\sqrt{x^2+1}}\, dx$$

Let $x = \tan\theta$

$dx = \sec^2\theta\, d\theta$

$x = \sqrt{2}, \theta = \tan^{-1}\sqrt{2}$

$x = 0, \theta = 0$

Hence

$$\int_0^{\sqrt{2}} \frac{x^3}{\sqrt{x^2+1}}\, dx$$

$$= \int_0^{\tan^{-1}\sqrt{2}} \frac{\tan^3\theta}{\sqrt{\tan^2\theta+1}} \sec^2\theta\, d\theta$$

$$= \int_0^{\tan^{-1}\sqrt{2}} \frac{\tan^3\theta}{\sec\theta} \sec^2\theta\, d\theta$$

$$= \int_0^{\tan^{-1}\sqrt{2}} \tan^3\theta \sec\theta\, d\theta$$

$$= \int_0^{\tan^{-1}\sqrt{2}} \sec\theta \tan\theta \tan^2\theta\, d\theta$$

$$= \int_0^{\tan^{-1}\sqrt{2}} \sec\theta \tan\theta\, (\sec^2\theta - 1)\, d\theta$$

Let $u = \sec\theta$

$du = \sec\theta \tan\theta\, d\theta$

$\theta = \tan^{-1}\sqrt{2}, u = \sqrt{3}$

$\theta = 0, u = 1$

Hence

$$\int_0^{\tan^{-1}\sqrt{2}} \sec\theta \tan\theta\, (\sec^2\theta - 1)\, d\theta$$

$$= \int_1^{\sqrt{3}} (u^2 - 1)\, du$$

$$= \left[\frac{1}{3}u^3 - u\right]_1^{\sqrt{3}}$$

$$= \left(\sqrt{3} - \sqrt{3}\right) - \left(\frac{1}{3} - 1\right)$$

$$= \frac{2}{3}$$

9b

$$\int_0^{\sqrt{2}} \frac{x^3}{\sqrt{x^2+1}}\, dx$$

$$x^3 = x(x^2+1) - x$$

$$\int_0^{\sqrt{2}} \frac{x^3}{\sqrt{x^2+1}}\, dx$$

$$= \int_0^{\sqrt{2}} \frac{x(x^2+1) - x}{\sqrt{x^2+1}}\, dx$$

$$= \int_0^{\sqrt{2}} \left(x\sqrt{x^2+1} - \frac{x}{\sqrt{x^2+1}}\right) dx$$

From here on, the integral can be evaluated by a substitution $u = x^2$.

10a

$$\int_1^2 \sqrt{4 - x^2}\, dx$$

Let $x = 2\sin\theta$

$dx = 2\cos\theta\, d\theta$

$x = 2, \theta = \frac{\pi}{2}$

$x = 1, \theta = \frac{\pi}{6}$

Hence

$$\int_{1}^{2} \sqrt{4-x^2}\,dx$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sqrt{4-(2\sin\theta)^2}\,2\cos\theta\,d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sqrt{4(1-\sin^2\theta)}\,2\cos\theta\,d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 2\cos\theta\,2\cos\theta\,d\theta$$

$$= 4\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos^2\theta\,d\theta$$

$$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$$

Hence

$$4\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos^2\theta\,d\theta$$

$$= 4\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1}{2}(1+\cos 2\theta)\,d\theta$$

$$= 2\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (1+\cos 2\theta)\,d\theta$$

$$= 2\left[\theta+\frac{1}{2}\sin 2\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= 2\left(\left(\frac{\pi}{2}+\frac{1}{2}\sin\pi\right)-\left(\frac{\pi}{6}+\frac{1}{2}\sin\frac{\pi}{3}\right)\right)$$

$$= (\pi+0)-\left(\frac{\pi}{3}+\frac{\sqrt{3}}{2}\right)$$

$$= \frac{2\pi}{3}-\frac{\sqrt{3}}{2}$$

10b

$$\int_1^2 \sqrt{4-x^2}\,dx$$

The function represents a semi-circle of radius 2 centred at the origin.

The area between $x = 1$ and $x = 2$ represents half of the minor segment created of a circle of radius 2.

The formula for the area of a segment:

$$\text{Area} = \frac{1}{2}r^2(\theta - \sin\theta)$$

$r = 2, \cos\left(\frac{\theta}{2}\right) = \frac{1}{2}$ (draw out the diagram to see the triangle)

$$\theta = \frac{2\pi}{3}$$

Hence

$$\int_1^2 \sqrt{4-x^2}\,dx$$

$$= \frac{1}{2} \times \text{Area of segment}$$

$$= \frac{1}{2} \times \frac{1}{2} 2^2 \left(\frac{2\pi}{3} - \sin\frac{2\pi}{3}\right)$$

$$= \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$$

11a

$$I = \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x}\,dx$$

Let $u = \frac{\pi}{2} - x$

$$x = \frac{\pi}{2} - u$$

$$du = -dx$$

$$x = \frac{\pi}{2}, u = 0$$

$x = 0, u = \frac{\pi}{2}$

Hence

$$I = -\int_{\frac{\pi}{2}}^{0} \frac{\sin\left(\frac{\pi}{2} - u\right)}{\sin\left(\frac{\pi}{2} - u\right) + \cos\left(\frac{\pi}{2} - u\right)} du$$

$$I = \int_{0}^{\frac{\pi}{2}} \frac{\cos u}{\cos u + \sin u} du$$

u is a dummy variable and can be replaced by x.

$$I = \int_{0}^{\frac{\pi}{2}} \frac{\cos x}{\sin x + \cos x} dx$$

11b

$$I = \int_{0}^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x} dx$$

$$I = \int_{0}^{\frac{\pi}{2}} \frac{\cos x}{\sin x + \cos x} dx$$

Hence

$$I + I = \int_{0}^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x} dx + \int_{0}^{\frac{\pi}{2}} \frac{\cos x}{\sin x + \cos x} dx$$

$$2I = \int_{0}^{\frac{\pi}{2}} \frac{\sin x + \cos x}{\sin x + \cos x} dx$$

$$2I = \int_{0}^{\frac{\pi}{2}} 1\, dx$$

$$2I = \frac{\pi}{2}$$

$$I = \frac{\pi}{4}$$

12a

$$\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx$$

Let $u = \cos x$

$du = -\sin x \, dx$

$x = \pi, u = -1$

$x = 0, u = 1$

Hence

$$\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx$$

$$= -\int_1^{-1} \frac{1}{1+u^2}\,du$$

$$= \int_{-1}^{1} \frac{1}{1+u^2}\,du$$

$$= [\tan^{-1} u]_{-1}^{1}$$

$$= \tan^{-1} 1 - \tan^{-1}(-1)$$

$$= \frac{\pi}{4} + \frac{\pi}{4}$$

$$= \frac{\pi}{2}$$

12b i

$$I = \int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\,dx$$

Let $u = \pi - x$

$x = \pi - u$

$du = -dx$

$x = \pi, u = 0$

$x = 0, u = \pi$

Hence

$$I = \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx$$

$$I = -\int_{\pi}^{0} \frac{(\pi - u)\sin(\pi - u)}{1 + \cos^2(\pi - u)} du$$

$$I = \int_0^{\pi} \frac{(\pi - u)\sin(u)}{1 + \cos^2(u)} du$$

$$I = \int_0^{\pi} \frac{\pi \sin(u)}{1 + \cos^2(u)} du - \int_0^{\pi} \frac{u \sin(u)}{1 + \cos^2(u)} du$$

$$I = \pi \int_0^{\pi} \frac{\sin(u)}{1 + \cos^2(u)} du - I$$

$$I = \pi \times \frac{\pi}{2} - I$$

$$I = \frac{\pi^2}{2} - I$$

12b ii

$$I = \frac{\pi^2}{2} - I$$

$$2I = \frac{\pi^2}{2}$$

$$I = \frac{\pi^2}{4}$$

Solutions to Exercise 4C Enrichment questions

13a Let $x = \sin\theta, 0 \le \theta \le \frac{\pi}{2}$ so that $\cos\theta \ge 0$.

$dx = \cos\theta\, d\theta$

At $x = 0,\qquad \theta = 0$

At $x = \frac{1}{2},\qquad \theta = \frac{\pi}{6}$

So,

$\int_0^{\frac{1}{2}} \frac{x^2}{\sqrt{1-x^2}} dx$

$= \int_0^{\frac{\pi}{6}} \frac{\sin^2\theta\cos\theta}{\sqrt{1-\sin^2\theta}} d\theta$

$= \int_0^{\frac{\pi}{6}} \frac{\sin^2\theta\cos\theta}{\sqrt{\cos^2\theta}} d\theta$ (by Pythagoras)

$= \int_0^{\frac{\pi}{6}} \frac{\sin^2\theta\cos\theta}{\cos\theta} d\theta$ (since $\theta \ge 0$)

$= \int_0^{\frac{\pi}{6}} \sin^2\theta\, d\theta$

$= \frac{1}{2}\int_0^{\frac{\pi}{6}} (1 - \cos 2\theta) d\theta$ (double angle)

$= \frac{1}{2}\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{6}}$

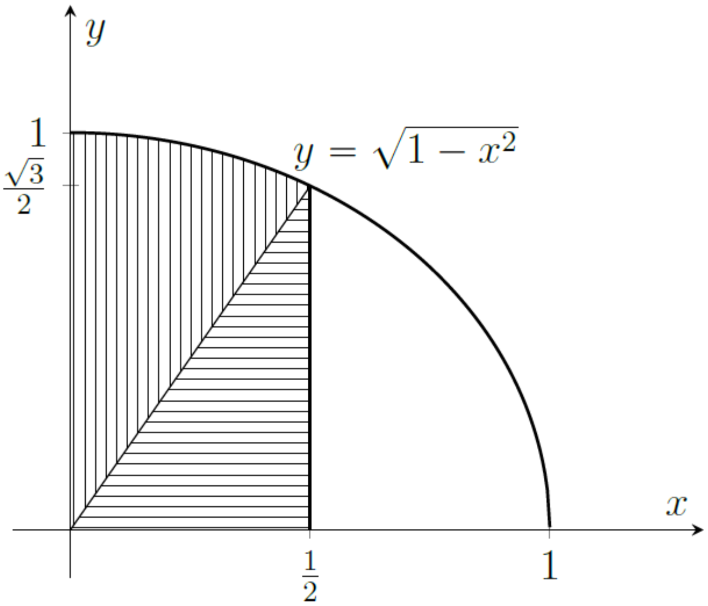
$= \frac{1}{2}\left(\frac{\pi}{6} - \frac{1}{2}\frac{\sqrt{3}}{2}\right) - 0$

$= \frac{\pi}{12} - \frac{\sqrt{3}}{8}$

13b $\int_0^{\frac{1}{2}} \frac{x^2-1}{\sqrt{1-x^2}} dx + \int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^2}}$

$= -\int_0^{\frac{1}{2}} \sqrt{1-x^2}\, dx + \int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^2}}$

$= \text{sector } -\Delta + [\sin^{-1} x]_0^{\frac{1}{2}}$ (see diagram below)



$= \left(\frac{1}{2} \cdot 1^2 \cdot \frac{\pi}{6}\right) - \left(\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{2}\right) + \frac{\pi}{6}$

$= \frac{\pi}{12} - \frac{\sqrt{3}}{8}$

14a Let $u = \sqrt{x^2 - 1}$, so that $x^2 = u^2 + 1$

$du = \frac{x}{\sqrt{x^2-1}} dx$

So,

$I = \int \frac{xdx}{x^2\sqrt{x^2-1}}$

$= \int \frac{du}{u^2+1}$

$= \tan^{-1} u + C_1$

$= \tan^{-1} \sqrt{x^2 - 1} + C_1$

14b Let $u = -\sqrt{x^2 - 1}$, so that $x^2 = u^2 + 1$ again.

$du = \frac{-x}{\sqrt{x^2-1}} dx$

So,

$I = -\int \frac{-xdx}{x^2\sqrt{x^2-1}}$

$= -\int \frac{du}{u^2+1}$

$= -\tan^{-1} u + C_2$

$= -\tan^{-1}\left(-\sqrt{x^2-1}\right) + C_2$

$= \tan^{-1}\sqrt{x^2-1} + C_2$ (since $\tan^{-1}$ is odd)

15a $I = \int_{2+\epsilon}^{4} \frac{dx}{x^2\sqrt{x^2-4}}$

Let $x = 2\sec\theta$, where $0 \le \theta < \frac{\pi}{2}$, so that $x > 0$.

Hence,

$\sqrt{x^2-4}$

$= \sqrt{4\sec^2\theta - 4}$

$= \sqrt{4\tan^2\theta}$

$= 2\tan\theta$, for this domain

$dx = 2\sec\theta\tan\theta\, d\theta$

Hence,

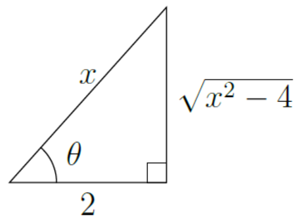
$\int \frac{dx}{x^2\sqrt{x^2-4}}$

$= \int \frac{2\sec\theta\tan\theta\, d\theta}{4\sec^2\theta \cdot 2\tan\theta}$

$= \frac{1}{4}\int \cos\theta\, d\theta$

$= \frac{1}{4}\sin\theta + C$

$= \frac{1}{4}\cdot\frac{\sqrt{x^2-4}}{x} + C$ (see diagram below)



Thus,

$I = \left[\frac{\sqrt{x^2-4}}{4x}\right]_{2+\epsilon}^{4}$

$= \frac{\sqrt{12}}{16} - \frac{\sqrt{4\epsilon+\epsilon^2}}{4(2+\epsilon)}$

$= \frac{\sqrt{3}}{8} - \frac{\sqrt{\epsilon(4+\epsilon)}}{4(2+\epsilon)}$

15a Alternative solution:

$I = \int \frac{dx}{x^2\sqrt{x^2-4}}$, with $x > 2$

$= \int \frac{dx}{x^3\sqrt{1-\frac{4}{x^2}}}$, (since $x > 0$ and $\sqrt{x^2} = x$)

Let $\theta = \sin^{-1}\frac{2}{x}$ so that $\cos\theta > 0$

Then, $\sin\theta = \frac{2}{x}$

$\cos\theta d\theta = \frac{-2}{x^2}dx$

$-\frac{1}{2}\cos\theta d\theta = \frac{1}{x^2}dx$

Hence,

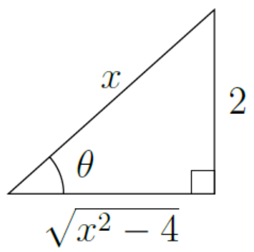
$I = \frac{1}{2}\int \frac{2}{x}.\frac{1}{\sqrt{1-\left(\frac{2}{x}\right)^2}}.\frac{dx}{x^2}$

$= \frac{1}{2}\int \sin\theta.\frac{1}{\cos\theta}.\left(-\frac{1}{2}\right)\cos\theta\, d\theta$

$= -\frac{1}{4}\int \sin\theta\, d\theta$

$= \frac{1}{4}\cos\theta + C$

$= \frac{1}{4}.\frac{\sqrt{x^2-4}}{x} + C$ (see diagram below)



Hence,

$\int_{2+\epsilon}^{4} \frac{dx}{x^2\sqrt{x^2-4}}$

$= \left[\frac{\sqrt{x^2-4}}{4x}\right]_{2+\epsilon}^{4}$

$= \frac{\sqrt{12}}{16} - \frac{\sqrt{\epsilon(4+\epsilon)}}{4(2+\epsilon)}$ (difference of two squares in the numerator)

$= \frac{\sqrt{3}}{8} - \frac{\sqrt{\epsilon(4+\epsilon)}}{4(2+\epsilon)}$ (as before)

15b $\lim_{\epsilon \to 0^+} I = \lim_{\epsilon \to 0^+} \frac{\sqrt{3}}{8} - \frac{\sqrt{4\epsilon+\epsilon^2}}{4(2+\epsilon)}$

$= \frac{\sqrt{3}}{8} - 0$

$= \frac{\sqrt{3}}{8}$

Solutions to Exercise 4D Foundation questions

1a

Let $\dfrac{2}{(x-1)(x+1)} = \dfrac{A}{x-1} + \dfrac{B}{x+1}$

$$\frac{2}{(x-1)(x+1)} = \frac{A(x+1)+B(x-1)}{(x-1)(x+1)}$$

$2 = A(x+1) + B(x-1)$

When $x = 1$,

$2 = 2A$

$A = 1$

When $x = -1$,

$2 = -2B$

$B = -1$

Thus,

$$\frac{2}{(x-1)(x+1)} = \frac{1}{x-1} - \frac{1}{x+1}$$

1b

Let $\dfrac{1}{(x-4)(x-1)} = \dfrac{A}{x-4} + \dfrac{B}{x-1}$

$$\frac{1}{(x-4)(x-1)} = \frac{A(x-1)+B(x-4)}{(x-4)(x-1)}$$

$1 = A(x-1) + B(x-4)$

When $x = 4$,

$1 = 3A$

$A = \dfrac{1}{3}$

When $x = 1$,

$1 = -3B$

$B = -\dfrac{1}{3}$

Thus,

$$\frac{1}{(x-4)(x-1)} = \frac{1}{3(x-4)} - \frac{1}{3(x-1)}$$

1c

$$\frac{4x}{x^2-9} = \frac{4x}{(x-3)(x+3)}$$

Let $\frac{4x}{(x-3)(x+3)} = \frac{A}{x-3} + \frac{B}{x+3}$

$$\frac{4x}{(x-3)(x+3)} = \frac{A(x+3)+B(x-3)}{(x-3)(x+3)}$$

$4x = A(x+3) + B(x-3)$

When $x = 3$,

$12 = 6A$

$A = 2$

When $x = -3$,

$-12 = -6B$

$B = 2$

Thus,

$$\frac{4x}{x^2-9} = \frac{2}{x-3} + \frac{2}{x+3}$$

1d

$$\frac{x}{x^2-3x+2} = \frac{x}{(x-2)(x-1)}$$

Let $\frac{x}{(x-2)(x-1)} = \frac{A}{x-2} + \frac{B}{x-1}$

$$\frac{x}{(x-2)(x-1)} = \frac{A(x-1)+B(x-2)}{(x-2)(x-1)}$$

$x = A(x-1) + B(x-2)$

When $x = 2$,

$A = 2$

When $x = 1$,

$1 = -B$

$B = -1$

Thus,

$$\frac{x}{x^2 - 3x + 2} = \frac{2}{x - 2} - \frac{1}{x - 1}$$

1e

$$\frac{x - 1}{x^2 + x - 6} = \frac{x - 1}{(x - 2)(x + 3)}$$

Let $\dfrac{x - 1}{(x - 2)(x + 3)} = \dfrac{A}{x - 2} + \dfrac{B}{x + 3}$

$$\frac{x - 1}{(x - 2)(x + 3)} = \frac{A(x + 3) + B(x - 2)}{(x - 2)(x + 3)}$$

$x - 1 = A(x + 3) + B(x - 2)$

When $x = 2$,

$1 = 5A$

$A = \dfrac{1}{5}$

When $x = -3$,

$-4 = -5B$

$B = \dfrac{4}{5}$

Thus,

$$\frac{x - 1}{x^2 + x - 6} = \frac{1}{5(x - 2)} + \frac{4}{5(x + 3)}$$

1f

Let $\dfrac{3x+1}{(x-1)(x^2+3)} = \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+3}$

$\dfrac{3x+1}{(x-1)(x^2+3)} = \dfrac{A(x^2+3)+(Bx+C)(x-1)}{(x-1)(x^2+3)}$

$3x+1 = A(x^2+3)+(Bx+C)(x-1)$

When $x = 1$,

$4 = 4A$

$A = 1$

Equating coefficients of x^2 yields:

$0 = A + B$

$0 = 1 + B$

$B = -1$

When $x = 0$,

$1 = 3A - C$

$1 = 3 - C$

$C = 2$

Thus,

$\dfrac{3x+1}{(x-1)(x^2+3)} = \dfrac{1}{x-1} + \dfrac{2-x}{x^2+3}$

2a

Let $\dfrac{2}{(x-4)(x-2)} = \dfrac{A}{x-4} + \dfrac{B}{x-2}$

$\dfrac{2}{(x-4)(x-2)} = \dfrac{A(x-2)+B(x-4)}{(x-4)(x-2)}$

$2 = A(x-2) + B(x-4)$

When $x = 4$,

$2 = 2A$

$A = 1$

When $x = 2$,

$2 = -2B$

$B = -1$

Thus,

$$\int \frac{2}{(x-4)(x-2)}\,dx$$

$$= \int \left(\frac{1}{x-4} - \frac{1}{x-2}\right) dx$$

$$= \int \frac{1}{x-4}\,dx - \int \frac{1}{x-2}\,dx$$

$$= \ln|x-4| - \ln|x-2| + C$$

2b

$$\frac{4}{x^2+4x+3} = \frac{4}{(x+1)(x+3)}$$

Let $\frac{4}{(x+1)(x+3)} = \frac{A}{x+1} + \frac{B}{x+3}$

$$\frac{4}{(x+1)(x+3)} = \frac{A(x+3)+B(x+1)}{(x+1)(x+3)}$$

$4 = A(x+3) + B(x+1)$

When $x = -1$,

$4 = 2A$

$A = 2$

When $x = -3$,

$4 = -2B$

$B = -2$

Thus,

$$\int \frac{4}{x^2+4x+3}\,dx$$

$$= \int \left(\frac{2}{x+1} - \frac{2}{x+3}\right) dx$$

$$= 2\int \frac{1}{x+1}\,dx - 2\int \frac{1}{x+3}\,dx$$

$$= 2\ln|x+1| - 2\ln|x+3| + C$$

2c

Let $\dfrac{3x-2}{(x-1)(x-2)} = \dfrac{A}{x-1} + \dfrac{B}{x-2}$

$$\frac{3x-2}{(x-1)(x-2)} = \frac{A(x-2)+B(x-1)}{(x-1)(x-2)}$$

$$3x-2 = A(x-2)+B(x-1)$$

When $x = 1$,

$1 = -1A$

$A = -1$

When $x = 2$,

$4 = B$

$B = 4$

Thus,

$$\int \frac{3x-2}{(x-1)(x-2)}\,dx$$

$$= \int \left(\frac{-1}{x-1} + \frac{4}{x-2}\right)dx$$

$$= -\int \frac{1}{x-1}\,dx + 4\int \frac{1}{x-2}\,dx$$

$$= -\ln|x-1| + 4\ln|x-2| + C$$

$$= 4\ln|x-2| - \ln|x-1| + C$$

2d

$$\frac{2x+10}{x^2+2x-3} = \frac{2x+10}{(x-1)(x+3)}$$

Let $\dfrac{2x+10}{(x-1)(x+3)} = \dfrac{A}{x-1} + \dfrac{B}{x+3}$

$$\frac{2x+10}{(x-1)(x+3)}=\frac{A(x+3)+B(x-1)}{(x-1)(x+3)}$$

$2x+10=A(x+3)+B(x-1)$

When $x=1$,

$12=4A$

$A=3$

When $x=-3$,

$4=-4B$

$B=-1$

Thus,

$$\int\frac{2x+10}{x^2+2x-3}dx$$

$$=\int\left(\frac{3}{x-1}-\frac{1}{x+3}\right)dx$$

$$=3\int\frac{1}{x-1}dx-\int\frac{1}{x+3}dx$$

$$=3\ln|x-1|-\ln|x+3|+C$$

2e

Let $\dfrac{4x+5}{(2x+3)(x+1)}=\dfrac{A}{2x+3}+\dfrac{B}{x+1}$

$$\frac{4x+5}{(2x+3)(x+1)}=\frac{A(x+1)+B(2x+3)}{(2x+3)(x+1)}$$

$4x+5=A(x+1)+B(2x+3)$

When $x=-\dfrac{3}{2}$,

$-1=-\dfrac{A}{2}$

$A=2$

When $x=-1$,

$B=1$

Thus,

$$\int \frac{4x+5}{(2x+3)(x+1)} dx$$

$$= \int \left(\frac{2}{2x+3} + \frac{1}{x+1}\right) dx$$

$$= \int \frac{2}{2x+3} dx + \int \frac{1}{x+1} dx$$

$$= \ln|2x+3| + \ln|x+1| + C$$

$$= \ln|x+1| + \ln|2x+3| + C$$

2f

$$\frac{10x}{2x^2 - x - 3} = \frac{10x}{(x+1)(2x-3)}$$

Let $\dfrac{10x}{(x+1)(2x-3)} = \dfrac{A}{x+1} + \dfrac{B}{2x-3}$

$$\frac{10x}{(x+1)(2x-3)} = \frac{A(2x-3) + B(x+1)}{(x+1)(2x-3)}$$

$$10x = A(2x-3) + B(x+1)$$

When $x = -1$,

$$-10 = -5A$$

$$A = 2$$

When $x = \dfrac{3}{2}$,

$$15 = \frac{5B}{2}$$

$$B = 6$$

Thus,

$$\int \frac{10x}{2x^2 - x - 3} dx$$

$$= \int \left(\frac{2}{x+1} + \frac{6}{2x-3}\right) dx$$

$$= 2\int \frac{1}{x+1} dx + 3\int \frac{2}{2x-3} dx$$

$$= 2\ln|x+1| + 3\ln|2x-3| + C$$

3a

$$\frac{1}{x^2-4}=\frac{1}{(x-2)(x+2)}$$

Let $\dfrac{1}{(x-2)(x+2)}=\dfrac{A}{x-2}+\dfrac{B}{x+2}$

$$\frac{1}{(x-2)(x+2)}=\frac{A(x+2)+B(x-2)}{(x-2)(x+2)}$$

$1=A(x+2)+B(x-2)$

When $x=2$,

$1=4A$

$A=\dfrac{1}{4}$

When $x=-2$,

$1=-4B$

$B=-\dfrac{1}{4}$

Thus,

$$\int_4^6 \frac{1}{x^2-4}\,dx$$

$$=\int_4^6\left(\frac{1}{4(x-2)}-\frac{1}{4(x+2)}\right)dx$$

$$=\frac{1}{4}[\ln|x-2|-\ln|x+2|]_4^6$$

$$=\frac{1}{4}\left[\ln\left|\frac{x-2}{x+2}\right|\right]_4^6$$

$$=\frac{1}{4}\left[\ln\frac{4}{8}-\ln\frac{2}{6}\right]$$

$$=\frac{1}{4}\ln\left(\frac{1}{2}\div\frac{1}{3}\right)$$

$$=\frac{1}{4}\ln\frac{3}{2}$$

3b

$$\frac{3}{x^2+x-2} = \frac{3}{(x-1)(x+2)}$$

Let $\dfrac{3}{(x-1)(x+2)} = \dfrac{A}{x-1} + \dfrac{B}{x+2}$

$$\frac{3}{(x-1)(x+2)} = \frac{A(x+2)+B(x-1)}{(x-1)(x+2)}$$

$3 = A(x+2) + B(x-1)$

When $x = 1$,

$3 = 3A$

$A = 1$

When $x = -2$,

$3 = -3B$

$B = -1$

Thus,

$$\int_2^4 \frac{3}{x^2+x-2}\,dx$$

$$= \int_2^4 \left(\frac{1}{x-1} - \frac{1}{x+2}\right) dx$$

$$= [\ln|x-1| - \ln|x+2|]_2^4$$

$$= \left[\ln\left|\frac{x-1}{x+2}\right|\right]_2^4$$

$$= \ln\left(\frac{3}{6}\right) - \ln\left(\frac{1}{4}\right)$$

$$= \ln\left(\frac{1}{2} \div \frac{1}{4}\right)$$

$$= \ln 2$$

3c

$$\frac{11}{2x^2+5x-12}=\frac{11}{(2x-3)(x+4)}$$

Let $\dfrac{11}{(2x-3)(x+4)}=\dfrac{A}{2x-3}+\dfrac{B}{x+4}$

$$\frac{11}{(2x-3)(x+4)}=\frac{A(x+4)+B(2x-3)}{(2x-3)(x+4)}$$

$$11=A(x+4)+B(2x-3)$$

When $x=\dfrac{3}{2}$,

$$11=\frac{11A}{2}$$

$$A=2$$

When $x=-4$,

$$11=-11B$$

$$B=-1$$

Thus,

$$\int_2^5 \frac{11}{2x^2+5x-12}\,dx$$

$$=\int_2^5\left(\frac{2}{2x-3}-\frac{1}{x+4}\right)dx$$

$$=[\ln|2x-3|-\ln|x+4|]_2^5$$

$$=\left[\ln\left|\frac{2x-3}{x+4}\right|\right]_2^5$$

$$=\ln\left(\frac{7}{9}\right)-\ln\left(\frac{1}{6}\right)$$

$$=\ln\left(\frac{7}{9}\div\frac{1}{6}\right)$$

$$=\ln\frac{14}{3}$$

3d

$$\frac{1}{3x^2-4x+1}=\frac{1}{(x-1)(3x-1)}$$

Let $\dfrac{1}{(x-1)(3x-1)}=\dfrac{A}{x-1}+\dfrac{B}{3x-1}$

$$\frac{1}{(x-1)(3x-1)}=\frac{A(3x-1)+B(x-1)}{(x-1)(3x-1)}$$

$1=A(3x-1)+B(x-1)$

When $x=1$,

$1=2A$

$A=\dfrac{1}{2}$

When $x=\dfrac{1}{3}$,

$1=-\dfrac{2B}{3}$

$B=-\dfrac{3}{2}$

Thus,

$$\int_{-1}^{0}\frac{1}{3x^2-4x+1}\,dx$$

$$=\int_{-1}^{0}\left(\frac{1}{2(x-1)}-\frac{3}{2(3x-1)}\right)dx$$

$$=\frac{1}{2}\int_{-1}^{0}\left(\frac{1}{x-1}-\frac{3}{3x-1}\right)dx$$

$$=\frac{1}{2}[\ln|x-1|-\ln|3x-1|]_{-1}^{0}$$

$$=\frac{1}{2}\left[\ln\left|\frac{x-1}{3x-1}\right|\right]_{-1}^{0}$$

$$=\frac{1}{2}\left(\ln\left(\frac{-1}{-1}\right)-\ln\left(\frac{-2}{-4}\right)\right)$$

$$=\frac{1}{2}\left(\ln 1-\ln\frac{1}{2}\right)$$

$= \frac{1}{2}\ln\left(1 \div \frac{1}{2}\right)$

$= \frac{1}{2}\ln 2$

4a

Let $\frac{x^2 - 2x + 5}{(x-2)(x^2+1)} = \frac{A}{x-2} + \frac{Bx + C}{x^2+1}$

$\frac{x^2 - 2x + 5}{(x-2)(x^2+1)} = \frac{A(x^2+1) + (Bx+C)(x-2)}{(x-2)(x^2+1)}$

$x^2 - 2x + 5 = A(x^2+1) + (Bx+C)(x-2)$

When $x = 2$,

$5 = 5A$

$A = 1$

Equating coefficients of x^2 yields:

$1 = A + B$

$1 = 1 + B \qquad (\text{since } A = 1)$

$B = 0$

When $x = 0$,

$5 = A - 2C$

$5 = 1 - 2C$

$C = -2$

Thus,

$\int \frac{x^2 - 2x + 5}{(x-2)(x^2+1)}\, dx$

$= \int \left(\frac{1}{x-2} - \frac{2}{x^2+1}\right) dx$

$= \int \frac{1}{x-2}\, dx - 2\int \frac{1}{x^2+1}\, dx$

$= \ln|x-2| - 2\tan^{-1} x + C$

4b

Let $\dfrac{6-x}{(2x+1)(x^2+3)} = \dfrac{A}{2x+1} + \dfrac{Bx+C}{x^2+3}$

$\dfrac{6-x}{(2x+1)(x^2+3)} = \dfrac{A(x^2+3)+(Bx+C)(2x+1)}{(2x+1)(x^2+3)}$

$6-x = A(x^2+3)+(Bx+C)(2x+1)$

When $x = -\dfrac{1}{2}$,

$\dfrac{13}{2} = \dfrac{13A}{4}$

$A = 2$

Equating coefficients of x^2 yields:

$0 = A + 2B$

$0 = 2 + 2B$ (since $A = 2$)

$B = -1$

When $x = 0$,

$6 = 3A + C$

$6 = 6 + C$

$C = 0$

Thus,

$\displaystyle\int \frac{6-x}{(2x+1)(x^2+3)}\,dx$

$\displaystyle= \int \left(\frac{2}{2x+1} - \frac{x}{x^2+3}\right)dx$

$\displaystyle= \int \frac{2}{2x+1}\,dx - \frac{1}{2}\int \frac{2x}{x^2+3}\,dx$

$= \ln|2x+1| - \dfrac{1}{2}\ln(x^2+3) + C$ (since x^2+3 is positive)

4c

$$\frac{x^2+x+3}{x^3+x} = \frac{x^2+x+3}{x(x^2+1)}$$

Let $\dfrac{x^2+x+3}{x(x^2+1)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+1}$

$$\frac{x^2+x+3}{x(x^2+1)} = \frac{A(x^2+1)+(Bx+C)(x)}{x(x^2+1)}$$

$$x^2+x+3 = A(x^2+1)+(Bx+C)(x)$$

When $x = 0$,

$A = 3$

Equating coefficients of x^2 yields:

$1 = A + B$

$1 = 3 + B \qquad \text{(since } A = 3\text{)}$

$B = -2$

Equating coefficients of x yields:

$C = 1$

Thus,

$$\int \frac{x^2+x+3}{x^3+x}\,dx$$

$$= \int \left(\frac{3}{x} + \frac{-2x+1}{x^2+1}\right) dx$$

$$= \int \frac{3}{x}\,dx - \int \frac{2x}{x^2+1}\,dx + \int \frac{1}{x^2+1}\,dx$$

$$= 3\ln|x| - \ln(x^2+1) + \tan^{-1} x + C \qquad \text{(since } x^2+1 \text{ is positive)}$$

$$= \tan^{-1} x + 3\ln|x| - \ln(x^2+1) + C$$

5a

Let $\dfrac{1+2x-4x^2}{(x+1)(4x^2+1)} = \dfrac{A}{x+1} + \dfrac{Bx+C}{4x^2+1}$

$$\frac{1+2x-4x^2}{(x+1)(4x^2+1)} = \frac{A(4x^2+1)+(Bx+C)(x+1)}{(x+1)(4x^2+1)}$$

$1 + 2x - 4x^2 = A(4x^2 + 1) + (Bx + C)(x + 1)$

When $x = -1$,

$-5 = 5A$

$A = -1$

Equating coefficients of x^2 yields:

$-4 = 4A + B$

$-4 = -4 + B \qquad \text{(since } A = -1\text{)}$

$B = 0$

When $x = 0$,

$1 = A + C$

$1 = -1 + C$

$C = 2$

Thus,

$$\int_0^{\frac{1}{2}} \frac{1 + 2x - 4x^2}{(x + 1)(4x^2 + 1)}\,dx$$

$$= \int_0^{\frac{1}{2}} \left(\frac{-1}{x + 1} + \frac{2}{4x^2 + 1}\right) dx$$

$$= \int_0^{\frac{1}{2}} \left(\frac{2}{4x^2 + 1} - \frac{1}{x + 1}\right) dx$$

$$= \int_0^{\frac{1}{2}} \frac{2}{(2x)^2 + 1}\,dx - \int_0^{\frac{1}{2}} \frac{1}{x + 1}\,dx$$

$$= [\tan^{-1} 2x]_0^{\frac{1}{2}} - [\ln|x + 1|]_0^{\frac{1}{2}}$$

$$= (\tan^{-1} 1 - \tan^{-1} 0) - \left(\ln\frac{3}{2} - \ln 1\right)$$

$$= \frac{\pi}{4} - 0 - \ln\frac{3}{2} - 0$$

$$= \frac{\pi}{4} - \ln\frac{3}{2}$$

5b

Let $\dfrac{7-x}{(x+3)(x^2+1)} = \dfrac{A}{x+3} + \dfrac{Bx+C}{x^2+1}$

$\dfrac{7-x}{(x+3)(x^2+1)} = \dfrac{A(x^2+1)+(Bx+C)(x+3)}{(x+3)(x^2+1)}$

$7-x = A(x^2+1)+(Bx+C)(x+3)$

When $x=-3$,

$10 = 10A$

$A = 1$

Equating coefficients of x^2 yields:

$0 = A + B$

$0 = 1 + B \qquad \text{(since } A = 1\text{)}$

$B = -1$

When $x = 0$,

$7 = A + 3C$

$7 = 1 + 3C$

$C = 2$

Thus,

$\displaystyle\int_{-1}^{1} \frac{7-x}{(x+3)(x^2+1)}\,dx$

$\displaystyle= \int_{-1}^{1} \left(\frac{1}{x+3} + \frac{2-x}{x^2+1}\right) dx$

$\displaystyle= \int_{-1}^{1} \left(\frac{1}{x+3} + \frac{2}{x^2+1} - \frac{x}{x^2+1}\right) dx$

$\displaystyle= \int_{-1}^{1} \frac{1}{x+3}\,dx + 2\int_{-1}^{1} \frac{1}{x^2+1}\,dx - \frac{1}{2}\int_{-1}^{1} \frac{2x}{x^2+1}\,dx$

$= [\ln|x+3|]_{-1}^{1} + 2[\tan^{-1} x]_{-1}^{1} - \dfrac{1}{2}[\ln(x^2+1)]_{-1}^{1} \qquad \text{(since } x^2+1 \text{ is positive)}$

$= (\ln 4 - \ln 2) + 2(\tan^{-1} 1 - \tan^{-1}(-1)) - \dfrac{1}{2}(\ln 2 - \ln 2)$

$$= \ln 2 + 2\left(\frac{\pi}{4} - \left(\frac{\pi}{4}\right)\right) - 0$$

$$= \ln 2 + \pi$$

$$= \pi + \ln 2$$

5c

$$\frac{x^2 - 4}{x^3 + 2x} = \frac{x^2 - 4}{x(x^2 + 2)}$$

Let $\dfrac{x^2 - 4}{x(x^2 + 2)} = \dfrac{A}{x} + \dfrac{Bx + C}{x^2 + 2}$

$$\frac{x^2 - 4}{x(x^2 + 2)} = \frac{A(x^2 + 2) + (Bx + C)(x)}{x(x^2 + 2)}$$

$$x^2 - 4 = A(x^2 + 2) + (Bx + C)(x)$$

When $x = 0$,

$$-4 = 2A$$

$$A = -2$$

Equating coefficients of x^2 yields:

$$1 = A + B$$

$$1 = -2 + B \qquad \text{(since } A = -2\text{)}$$

$$B = 3$$

Equating coefficients of x yields:

$$C = 0$$

Thus,

$$\frac{x^2 - 4}{x^3 + 2x} = \frac{3x}{x^2 + 2} - \frac{2}{x}$$

Hence,

$$\int_1^{\sqrt{2}} \frac{x^2 - 4}{x^3 + 2x}\, dx$$

$$= \int_1^{\sqrt{2}} \left(\frac{3x}{x^2 + 2} - \frac{2}{x}\right) dx$$

$$= \frac{3}{2}\int_1^{\sqrt{2}} \frac{2x}{x^2+2}\,dx - 2\int_1^{\sqrt{2}} \frac{1}{x}\,dx$$

$$= \frac{3}{2}\left[\ln(x^2+2)\right]_1^{\sqrt{2}} - 2\left[\ln|x|\right]_1^{\sqrt{2}} \qquad \text{(since } x^2+2 \text{ is positive)}$$

$$= \frac{3}{2}(\ln 4 - \ln 3) - 2\left(\ln\sqrt{2} - \ln 1\right)$$

$$= \frac{3}{2}\ln 4 - \frac{3}{2}\ln 3 - 2\ln\sqrt{2} + 0$$

$$= \frac{3}{2}\ln 4 - \frac{3}{2}\ln 3 - \ln\left(\sqrt{2}\right)^2$$

$$= 3\ln(4)^{\frac{1}{2}} - \frac{3}{2}\ln 3 - \ln 2$$

$$= 3\ln 2 - \ln 2 - \frac{3}{2}\ln 3$$

$$= 2\ln 2 - \frac{3}{2}\ln 3$$

$$= \ln 2^2 - \frac{3}{2}\ln 3$$

$$= \ln 4 - \frac{3}{2}\ln 3$$

Solutions to Exercise 4D Development questions

6a

$$\int \frac{2x+3}{(x-1)(x-2)(2x-3)}dx$$

$$\frac{2x+3}{(x-1)(x-2)(2x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{2x-3}$$

Using cover-up rule:

$$A = \frac{2(1)+3}{(1-2)(2(1)-3)} = 5$$

$$B = \frac{2(2)+3}{(2-1)(2(2)-3)} = 7$$

$$C = \frac{2\left(\frac{3}{2}\right)+3}{\left(\frac{3}{2}-1\right)\left(\frac{3}{2}-2\right)} = -24$$

Hence

$$\frac{2x+3}{(x-1)(x-2)(2x-3)} = \frac{5}{x-1} + \frac{7}{x-2} - \frac{24}{2x-3}$$

$$\int \frac{2x+3}{(x-1)(x-2)(2x-3)}dx$$

$$= \int \left(\frac{5}{x-1} + \frac{7}{x-2} - \frac{24}{2x-3}\right)dx$$

$$= 5\ln|x-1| + 7\ln|x-2| - 12\ln|2x-3| + C$$

6b

$$\int \frac{4x+12}{x^3-6x^2+8}dx$$

$$= \int \frac{4x+12}{x(x^2-6x+8)}dx$$

$$= \int \frac{4x+12}{x(x-4)(x-2)}dx$$

$$\frac{4x+12}{x(x-4)(x-2)} = \frac{A}{x} + \frac{B}{x-4} + \frac{C}{x-2}$$

Using cover-up rule:

$$A = \frac{4(0) + 12}{(0 - 4)(0 - 2)} = \frac{3}{2}$$

$$B = \frac{4(4) + 12}{(4)(4 - 2)} = \frac{7}{2}$$

$$C = \frac{4(2) + 12}{(2)(2 - 4)} = -5$$

Hence

$$\frac{4x + 12}{x(x - 4)(x - 2)} = \frac{\frac{3}{2}}{x} + \frac{\frac{7}{2}}{(x - 4)} - \frac{5}{x - 2}$$

$$\int \frac{4x + 12}{x(x - 4)(x - 2)} dx$$

$$= \int \left(\frac{3}{2x} + \frac{7}{2(x - 4)} - \frac{5}{x - 2}\right) dx$$

$$= \frac{3}{2}\ln|x| + \frac{7}{2}\ln|x - 4| - 5\ln|x - 2| + C$$

7a

$$\int_2^7 \frac{3x + 5}{(x - 1)(x + 2)(x + 1)} dx$$

$$\frac{3x + 5}{(x - 1)(x + 2)(x + 1)} = \frac{A}{x - 1} + \frac{B}{x + 2} + \frac{C}{x + 1}$$

Using cover-up rule:

$$A = \frac{3(1) + 5}{(1 + 2)(1 + 1)} = \frac{4}{3}$$

$$B = \frac{3(-2) + 5}{(-2 - 1)(-2 + 1)} = -\frac{1}{3}$$

$$C = \frac{3(-1) + 5}{(-1 - 1)(-1 + 2)} = -1$$

Hence

$$\frac{3x + 5}{(x - 1)(x + 2)(x + 1)} = \frac{\frac{4}{3}}{x - 1} - \frac{\frac{1}{3}}{x + 2} - \frac{1}{x + 1}$$

$$\int_2^7 \frac{3x+5}{(x-1)(x+2)(x+1)}\,dx$$

$$= \int_2^7 \left(\frac{\frac{4}{3}}{x-1} - \frac{\frac{1}{3}}{x+2} - \frac{1}{x+1}\right) dx$$

$$= \left[\frac{4}{3}\ln|x-1| - \frac{1}{3}\ln|x+2| - \ln|x+1|\right]_2^7$$

$$= \left(\frac{4}{3}\ln 6 - \frac{1}{3}\ln 9 - \ln 8\right) - \left(\frac{4}{3}\ln 1 - \frac{1}{3}\ln 4 - \ln 3\right)$$

$$= \left(\frac{4}{3}\ln(3\times 2) - \frac{1}{3}\ln 3^2 - \ln 2^3\right) - \left(-\frac{1}{3}\ln 2^2 - \ln 3\right)$$

$$= \frac{4}{3}\ln 3 + \frac{4}{3}\ln 2 - \frac{2}{3}\ln 3 - 3\ln 2 + \frac{2}{3}\ln 2 + \ln 3$$

$$= \frac{5}{3}\ln 3 - \ln 2$$

7b

$$\int_1^2 \frac{13x+6}{x^3-x^2-6x}\,dx$$

$$= \int_1^2 \frac{13x+6}{x(x^2-x-6)}\,dx$$

$$= \int_1^2 \frac{13x+6}{x(x-3)(x+2)}\,dx$$

$$\frac{13x+6}{x(x-3)(x+2)} = \frac{A}{x} + \frac{B}{x-3} + \frac{C}{x+2}$$

Using cover-up rule:

$$A = \frac{13(0)+6}{(0-3)(0+2)} = -1$$

$$B = \frac{13(3)+6}{3(3+2)} = 3$$

$$C = \frac{13(-2)+6}{(-2)(-2-3)} = -2$$

Hence

$$\frac{13x+6}{x(x-3)(x+2)} = -\frac{1}{x} + \frac{3}{x-3} - \frac{2}{x+2}$$

$$\int_1^2 \frac{13x+6}{x(x-3)(x+2)}\,dx$$

$$= \int_1^2 \left(-\frac{1}{x} + \frac{3}{x-3} - \frac{2}{x+2}\right) dx$$

$$= [-\ln|x| + 3\ln|x-3| - 2\ln|x+2|]_1^2$$

$$= (-\ln 2 + 3\ln 1 - 2\ln 4) - (-\ln 1 + 3\ln 2 - 2\ln 3)$$

$$= (-\ln 2 - 2\ln 2^2) - (3\ln 2 - 2\ln 3)$$

$$= -\ln 2 - 4\ln 2 - 3\ln 2 + 2\ln 3$$

$$= -8\ln 2 + 2\ln 3$$

8a i

$$\frac{2x^2+1}{(x-1)(x+2)} = A + \frac{B}{x-1} + \frac{C}{x+2}$$

$$2x^2 + 1 = A(x-1)(x+2) + B(x+2) + C(x-1)$$

Let $x = -2$,

$$9 = -3C$$

$$C = -3$$

Let $x = 1$,

$$3 = 3B$$

$$B = 1$$

Let $x = 0$,

$$1 = -2A + 2 + 3$$

$$A = 2$$

Hence

$$\frac{2x^2+1}{(x-1)(x+2)} = 2 + \frac{1}{x-1} - \frac{3}{x+2}$$

8a ii

$$\int \frac{2x^2+1}{(x-1)(x+2)}\,dx$$

$$= \int \left(2 + \frac{1}{x-1} - \frac{3}{x+2}\right) dx$$

$$= 2x + \ln|x-1| - 3\ln|x+2| + C$$

8b i

$$\int \frac{x^2-2x+3}{(x+1)(x-2)}\,dx$$

$$\frac{x^2-2x+3}{(x+1)(x-2)} = A + \frac{B}{x+1} + \frac{C}{x-2}$$

$$x^2-2x+3 = A(x+1)(x-2) + B(x-2) + C(x+1)$$

Let $x = 2$,

$3 = 3C$

$C = 1$

Let $x = -1$,

$6 = -3B$

$B = -2$

Let $x = 0$,

$3 = -2A + 4 + 1$

$A = 1$

Hence

$$\frac{x^2-2x+3}{(x+1)(x-2)} = 1 - \frac{2}{x+1} + \frac{1}{x-2}$$

$$\int \frac{x^2-2x+3}{(x+1)(x-2)}\,dx$$

$$= \int \left(1 - \frac{2}{x+1} + \frac{1}{x-2}\right) dx$$

$$= x - 2\ln|x+1| + \ln|x-2| + C$$

8b ii

$$\int \frac{3x^2-66}{(x+4)(x-5)}\,dx$$

$$\frac{3x^2-66}{(x+4)(x-5)} = A + \frac{B}{x+4} + \frac{C}{x-5}$$

$3x^2-66 = A(x+4)(x-5) + B(x-5) + C(x+4)$

Let $x = 5$,

$9 = 9C$

$C = 1$

Let $x = -4$,

$-18 = -9B$

$B = 2$

Let $x = 0$,

$-66 = -20A - 10 + 4$

$A = 3$

Hence

$$\frac{3x^2-66}{(x+4)(x-5)} = 3 + \frac{2}{x+4} + \frac{1}{x-5}$$

$$\int \frac{3x^2-66}{(x+4)(x-5)}\,dx$$

$$= \int \left(3 + \frac{2}{x+4} + \frac{1}{x-5}\right) dx$$

$= 3x + 2\ln|x+4| + \ln|x-5| + C$

9a i

$$\frac{x^3-3x^2-4}{(x+1)(x-3)} = Ax + B + \frac{C}{x+1} + \frac{D}{x-3}$$

$x^3-3x^2-4 = (Ax+B)(x+1)(x-3) + C(x-3) + D(x+1)$

Equating x^3 coefficients gives $A = 1$.

Let $x = 3$,

$-4 = 4D$

$D = -1$

Let $x = -1$,

$-8 = -4C$

$C = 2$

Let $x = 0$,

$-4 = -3B - 6 - 1$

$B = -1$

9a ii

$$\int_0^1 \frac{x^3 - 3x^2 - 4}{(x+1)(x-3)}\,dx$$

$$= \int_0^1 \left(x - 1 + \frac{2}{x+1} - \frac{1}{x-3}\right) dx$$

$$= \left[\frac{1}{2}x^2 - x + 2\ln|x+1| - \ln|x-3|\right]_0^1$$

$$= \left(\frac{1}{2} - 1 + 2\ln 2 - \ln 2\right) - (-\ln 3)$$

$$= \ln 2 + \ln 3 - \frac{1}{2}$$

9b

$$\int_2^4 \frac{x^3 + 4x^2 + x - 3}{(x+2)(x-1)}\,dx$$

$$\frac{x^3 + 4x^2 + x - 3}{(x+2)(x-1)} = Ax + B + \frac{C}{x+2} + \frac{D}{x-1}$$

$x^3 + 4x^2 + x - 3 = (Ax + B)(x+2)(x-1) + C(x-1) + D(x+2)$

Equating x^3 coefficients gives $A = 1$.

Let $x = 1$,

$3 = 3D$

$D = 1$

Let $x = -2$,

$3 = -3C$

$C = -1$

Let $x = 0$,

$-3 = -2B + 1 + 2$

$B = 3$

Hence

$$\int_2^4 \frac{x^3 + 4x^2 + x - 3}{(x+2)(x-1)}\,dx$$

$$= \int_2^4 \left(x + 3 - \frac{1}{x+2} + \frac{1}{x-1}\right) dx$$

$$= \left[\frac{1}{2}x^2 + 3x - \ln|x+2| + \ln|x-1|\right]_2^4$$

$$= (8 + 12 - \ln 6 + \ln 3) - (2 + 6 - \ln 4 + \ln 1)$$

$$= (20 - \ln 3 - \ln 2 + \ln 3) - (8 - 2\ln 2)$$

$$= 12 + \ln 2$$

10ai

$$\frac{3x^2 - 10}{x^2 - 4x + 4} = A + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$

$$\frac{3x^2 - 10}{(x-2)^2} = A + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$

$3x^2 - 10 = A(x-2)^2 + B(x-2) + C$

Let $x = 2$,

$12 - 10 = C$

$C = 2$

$3x^2 - 10 = A(x^2 - 4x + 4) + Bx - 2B + 2$

$3x^2 - 10 = Ax^2 - 4Ax + 4A + Bx - 2B + 2$

$3x^2 - 10 = Ax^2 + (B - 4A)x + 4A - 2B + 2$

Equating x^2 coefficients gives $A = 3$.

Equating x coefficients gives:

$B - 12 = 0$

$B = 12$

10a ii

$$\int \frac{3x^2 - 10}{x^2 - 4x + 4} dx$$

$$= \int \left(3 + \frac{12}{x-2} + \frac{2}{(x-2)^2}\right) dx$$

$$= 3x + 12\ln|x-2| - \frac{2}{x-2} + C$$

10b i

$$\frac{3x+7}{(x-1)^2(x-2)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x-2} + \frac{D}{(x-2)^2}$$

$3x + 7 = A(x-1)(x-2)^2 + B(x-2)^2 + C(x-1)^2(x-2) + D(x-1)^2$

Let $x = 1$,

$3 + 7 = B$

$B = 10$

Let $x = 2$,

$6 + 7 = D$

$D = 13$

$3x + 7 = A(x-1)(x-2)^2 + 10(x-2)^2 + C(x-1)^2(x-2) + 13(x-1)^2$

Equating x^3 coefficients gives $A + C = 0$.

Let $x = 0$,

$7 = -4A + 40 - 2C + 13$

$-46 = -4A - 2C$

$23 = 2A + C$

$A = -C$ from previous calculation

$23 = -2C + C = -C$

$C = -23$

$A = 23$

10b ii

$$\int \frac{3x+7}{(x-1)^2(x-2)^2}\,dx$$

$$= \int \left(\frac{23}{x-1} + \frac{10}{(x-1)^2} - \frac{23}{x-2} + \frac{13}{(x-2)^2}\right) dx$$

$$= 23\ln|x-1| - \frac{10}{x-1} - 23\ln|x-2| - \frac{13}{x-2} + C$$

$$= 23\ln\left|\frac{x-1}{x-2}\right| - \frac{10}{x-1} - \frac{13}{x-2} + C$$

11a

$$\int_4^6 \frac{x^2-8}{x^3+4x}\,dx$$

$$= \int_4^6 \frac{x^2-8}{x(x^2+4)}\,dx$$

$$\frac{x^2-8}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

$$x^2 - 8 = A(x^2+4) + (Bx+C)x$$

$$x^2 - 8 = Ax^2 + 4A + Bx^2 + Cx$$

$$x^2 - 8 = (A+B)x^2 + Cx + 4A$$

Equating constant coefficients gives:

$4A = -8$

$A = -2$

Equating x^2 coefficients gives:

$1 = -2 + B$

$B = 3$

Equating x coefficients gives $C = 0$.

Hence

$$\int_4^6 \frac{x^2-8}{x(x^2+4)}\,dx$$

$$= \int_4^6 \left(\frac{-2}{x} + \frac{3x}{x^2+4}\right) dx$$

$$= \left[-2\ln|x| + \frac{3}{2}\ln|x^2+4|\right]_4^6$$

$$= \left(-2\ln 6 + \frac{3}{2}\ln 40\right) - \left(-2\ln 4 + \frac{3}{2}\ln 20\right)$$

$$= 2\ln\frac{2}{3} + \frac{3}{2}\ln 2$$

$$= \frac{3}{2}\ln 2 - 2\ln\frac{3}{2}$$

11b

$$\int_0^2 \frac{1+4x}{(4-x)(x^2+1)}\,dx$$

$$\frac{1+4x}{(4-x)(x^2+1)} = \frac{A}{4-x} + \frac{Bx+C}{x^2+1}$$

$$1+4x = A(x^2+1) + (Bx+C)(4-x)$$

Let $x = 4$,

$$17 = 17A$$

$$A = 1$$

$$1+4x = x^2+1-Bx^2+4Bx-Cx+4C$$

$$1+4x = (1-B)x^2 + (4B-C)x + 4C + 1$$

Equating x^2 coefficients gives:

$$0 = 1-B$$

$$B = 1$$

Equating x coefficients gives:

$$4 = 4-C$$

$$C = 0$$

Hence

$$\int_0^2 \frac{1+4x}{(4-x)(x^2+1)}\,dx$$

$$= \int_0^2 \left(\frac{1}{4-x} + \frac{x}{x^2+1}\right) dx$$

$$= \left[-\ln|4-x| + \frac{1}{2}\ln|x^2+1|\right]_0^2$$

$$= \left(-\ln 2 + \frac{1}{2}\ln 5\right) - \left(-\ln 4 + \frac{1}{2}\ln 1\right)$$

$$= -\ln 2 + \frac{1}{2}\ln 5 + 2\ln 2$$

$$= \frac{1}{2}\ln 4 + \frac{1}{2}\ln 5$$

$$= \frac{1}{2}\ln 20$$

12a

$$\frac{x^2-1}{x^4+x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1}$$

$$\frac{x^2-1}{x^2(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1}$$

$$x^2 - 1 = Ax(x^2+1) + B(x^2+1) + (Cx+D)x^2$$

$$x^2 - 1 = Ax^3 + Ax + Bx^2 + B + Cx^3 + Dx^2$$

$$x^2 - 1 = (A+C)x^3 + (B+D)x^2 + Ax + B$$

Equating constant coefficients gives $B = -1$.

Equating x coefficients gives $A = 0$.

Equating x^2 coefficients gives:

$1 = B + D$

$D = 2$

Equating x^3 coefficients gives:

$0 = A + C$

$C = 0$

12b

$$\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{x^2-1}{x^4+x^2}\,dx$$

$$= \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \left(\frac{2}{x^2+1} - \frac{1}{x^2}\right) dx$$

$$= \left[2\tan^{-1}(x) + \frac{1}{x}\right]_{\frac{1}{\sqrt{3}}}^{\sqrt{3}}$$

$$= \left(2\tan^{-1}(\sqrt{3}) + \frac{1}{\sqrt{3}}\right) - \left(2\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) + \sqrt{3}\right)$$

$$= \frac{2\pi}{3} + \frac{1}{\sqrt{3}} - \frac{\pi}{3} - \sqrt{3}$$

$$= \frac{\pi}{3} + \frac{\sqrt{3}}{3} - \frac{3\sqrt{3}}{3}$$

$$= \frac{\pi}{3} - \frac{2\sqrt{3}}{3}$$

$$= \frac{1}{3}\left(\pi - 2\sqrt{3}\right)$$

13a

$$\int \frac{x^2+1}{x^2-1}\,dx$$

$$= \int \frac{x^2+1}{(x-1)(x+1)}\,dx$$

$$\frac{x^2+1}{(x-1)(x+1)} = A + \frac{B}{x-1} + \frac{C}{x+1}$$

$$x^2+1 = A(x-1)(x+1) + B(x+1) + C(x-1)$$

$$x^2+1 = Ax^2 - A + Bx + B + Cx - C$$

$$x^2+1 = Ax^2 + (B+C)x + B - A - C$$

Equating x^2 coefficients gives $A = 1$.

Equating x coefficients gives $B + C = 0$.

Equating constant coefficients gives:

$B - 1 - C = 1$

$B - C = 2$

$\therefore B = 1$

$\therefore C = -1$

Hence

$\int \frac{x^2+1}{(x-1)(x+1)} dx$

$= \int \left(1 + \frac{1}{x-1} - \frac{1}{x+1}\right) dx$

$= x + \ln|x-1| - \ln|x+1| + C$

13b

$\int \frac{x^2+1}{x^2-x} dx$

$= \int \frac{x^2+1}{x(x-1)} dx$

$\frac{x^2+1}{x(x-1)} = A + \frac{B}{x} + \frac{C}{x-1}$

$x^2 + 1 = Ax(x-1) + B(x-1) + Cx$

Let $x = 1$,

$C = 2$

Let $x = 0$,

$1 = -B$

$B = -1$

Equating x^3 coefficients gives $A = 1$.

Hence

$\int \frac{x^2+1}{x(x-1)} dx$

$= \int \left(1 - \frac{1}{x} + \frac{2}{x-1}\right) dx$

$= x - \ln|x| + 2\ln|x-1| + C$

13c

$\int \frac{x^3+1}{x^3+x}\,dx$

$= \int \frac{x^3+1}{x(x^2+1)}\,dx$

$\frac{x^3+1}{x(x^2+1)} = A + \frac{B}{x} + \frac{Cx+D}{x^2+1}$

$x^3+1 = Ax(x^2+1) + B(x^2+1) + (Cx+D)x$

$x^3+1 = Ax^3 + Ax + Bx^2 + B + Cx^2 + Dx$

$x^3+1 = Ax^3 + (B+C)x^2 + (A+D)x + B$

Equating x^3 coefficients gives $A = 1$.

Equating constant coefficients gives $B = 1$.

Equating x^2 coefficients gives:

$0 = 1 + C$

$C = -1$

Equating x coefficients gives:

$0 = 1 + D$

$D = -1$

Hence

$\int \frac{x^3+1}{x(x^2+1)}\,dx$

$= \int \left(1 + \frac{1}{x} + \frac{-x-1}{x^2+1}\right) dx$

$= \int \left(1 + \frac{1}{x} - \frac{x}{x^2+1} - \frac{1}{x^2+1}\right) dx$

$= x + \ln|x| - \frac{1}{2}\ln(x^2+1) - \tan^{-1}(x) + C$

13d

$$\int \frac{x^2}{x^2 - 5x + 6} dx$$

$$= \int \frac{x^2}{(x-2)(x-3)} dx$$

$$\frac{x^2}{(x-2)(x-3)} = A + \frac{B}{x-2} + \frac{C}{x-3}$$

$$x^2 = A(x-2)(x-3) + B(x-3) + C(x-2)$$

Let $x = 3$,

$C = 9$

Let $x = 2$

$4 = -B$

$B = -4$

Equating x^2 coefficients gives $A = 1$.

Hence

$$\int \frac{x^2}{(x-2)(x-3)} dx$$

$$= \int \left(1 - \frac{4}{x-2} + \frac{9}{x-3}\right) dx$$

$$= x - 4\ln|x-2| + 9\ln|x-3| + C$$

13e

$$\int \frac{x^3 + 5}{x^2 + x} dx$$

$$= \int \frac{x^3 + 5}{x(x+1)} dx$$

$$\frac{x^3 + 5}{x(x+1)} = Ax + B + \frac{C}{x} + \frac{D}{x+1}$$

$$x^3 + 5 = Ax^2(x+1) + Bx(x+1) + C(x+1) + Dx$$

$$x^3 + 5 = Ax^3 + (A+B)x^2 + (B+C+D)x + C$$

Equating constant coefficients gives $C = 5$.

Equating x^3 coefficients gives $A = 1$.

Equating x^2coefficients gives:

$0 = 1 + B$

$B = -1$

Equating x coefficients gives:

$0 = -1 + 5 + D$

$D = -4$

Hence

$$\int \frac{x^3 + 5}{x(x+1)}\,dx$$

$$= \int \left(x - 1 + \frac{5}{x} - \frac{4}{x+1}\right) dx$$

$$= \frac{1}{2}x^2 - x + 5\ln|x| - 4\ln|x+1| + C$$

13f

$$\int \frac{x^4}{x^2 - 3x + 2}\,dx$$

$$= \int \frac{x^4}{(x-2)(x-1)}\,dx$$

$$\frac{x^4}{(x-2)(x-1)} = Ax^2 + Bx + C + \frac{D}{x-2} + \frac{E}{x-1}$$

$x^4 = (Ax^2 + Bx + C)(x-2)(x-1) + D(x-1) + E(x-2)$

Let $x = 1$,

$1 = -E$

$E = -1$

Let $x = 2$,

$D = 16$

$x^4 = (Ax^2 + Bx + C)(x-2)(x-1) + 16(x-1) - (x-2)$

Equating x^4 coefficients gives $A = 1$.

Equating x^3 coefficients gives:

$B - 3A = 0$

$B = 3$

Equating constant coefficients gives:

$0 = 2C - 16 + 2$

$C = 7$

Hence

$$\int \frac{x^4}{(x-2)(x-1)}\,dx$$

$$= \int \left(x^2 + 3x + 7 + \frac{16}{x-2} - \frac{1}{x-1}\right) dx$$

$$= \frac{1}{3}x^3 + \frac{3}{2}x^2 + 7x + 16\ln|x-2| - \ln|x-1| + C$$

Solutions to Exercise 4D Enrichment questions

14 Let $\frac{5x-x^2}{(x+1)^2(x-1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$

Then, $5x - x^2 = A(x+1)^2 + B(x+1)(x-1) + C(x-1)$

When $x = 1$,

$4 = 4A$

$A = 1$

When $x = -1$,

$-6 = -2C$

$C = 3$

When $x = 0$,

$0 = A - B - C$

$B = A - C$

$B = -2$

$\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{5x-x^2}{(x+1)^2(x-1)} dx$

$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{x-1} - \frac{2}{x+1} + \frac{3}{(x+1)^2} dx$

$= \left[\ln|x-1| - 2\ln(x+1) - \frac{3}{x+1}\right]_{-\frac{1}{2}}^{\frac{1}{2}}$

$= \left(\ln\frac{1}{2} - 2\ln\frac{3}{2} - 2\right) - \left(\ln\frac{3}{2} - 2\ln\frac{1}{2} - 6\right)$

$= 4 + 3\ln\frac{1}{2} - 3\ln\frac{3}{2}$

$= 4 - 3\ln 3$

15a $C_k = \lim_{x \to a_k} P(x) \cdot \frac{x-a_k}{Q(x)}$

$= \lim_{x \to a_k} P(x) \cdot \frac{x-a_k}{Q(x)-Q(a_k)}$ (since $Q(a_k) = 0$)

$= P(a_k) \cdot \lim_{x \to a_k} \frac{1}{\left(\frac{Q(x)-Q(a_k)}{x-a_k}\right)}$

$= P(a_k) \cdot \frac{1}{Q'(a_k)}$

15b From question 6b,

$P(x) = 4x + 12$

$Q(x) = x^3 - 6x^2 + 8x = x(x-4)(x-2)$

$Q'(x) = 3x^2 - 12x + 8$

So, $C_0 = \frac{12}{8} = \frac{3}{2}$ $(x = 0)$

$C_2 = \frac{20}{-4} = -5$ $(x = 2)$

$C_4 = \frac{28}{8} = \frac{7}{2}$ $(x = 4)$

$\int \frac{4x+12}{x^3-6x^2+8x} dx$

$= \int \frac{\frac{3}{2}}{x} - \frac{5}{x-2} + \frac{\frac{7}{2}}{x-4} dx$

$= \frac{3}{2}\ln|x| - 5\ln|x-2| + \frac{7}{2}\ln|x-4| + C$

From question 7b,

$P(x) = 13x + 6$

$Q(x) = x^3 - x^2 - 6x = x(x-3)(x+2)$

$Q'(x) = 3x^2 - 2x - 6$

So, $C_0 = \frac{6}{-6} = -1$ $(x = 0)$

$C_3 = \frac{45}{15} = 3$ $(x = 3)$

$C_{-2} = \frac{-20}{10} = -2$ $(x = -2)$

$\int_1^2 \frac{13x+6}{x^3-x^2-6x} dx$

$= \int_1^2 \frac{3}{x-3} - \frac{1}{x} - \frac{2}{x+2} dx$

$= [3\ln|x-3| - \ln|x| - 2\ln|x+2|]_1^2$

$= (\cancel{3\ln 1} - \ln 2 - 2\ln 4) - (3\ln 2 - \cancel{\ln 1} - 2\ln 3)$ $(\ln 1 = 0)$

$= 2\ln 3 - \ln 2 - 4\ln 2 - 3\ln 2$

$= 2\ln 3 - 8\ln 2$

Solutions to Exercise 4E Foundation questions

1a

$$\int \frac{1}{9+x^2}\,dx$$

$$= \int \frac{1}{3^2+x^2}\,dx$$

$$= \frac{1}{3}\int \frac{3}{3^2+x^2}\,dx$$

$$= \frac{1}{3}\tan^{-1}\frac{x}{3} + C$$

1b

$$\int \frac{1}{\sqrt{9-x^2}}\,dx$$

$$= \int \frac{1}{\sqrt{3^2-x^2}}\,dx$$

$$= \sin^{-1}\frac{x}{3} + C$$

1c

$$\frac{1}{x^2-9} = \frac{1}{(x-3)(x+3)}$$

Let $\frac{1}{(x-3)(x+3)} = \frac{A}{x-3} + \frac{B}{x+3}$

$$\frac{1}{(x-3)(x+3)} = \frac{A(x+3)+B(x-3)}{(x-3)(x+3)}$$

$1 = A(x+3) + B(x-3)$

When $x = 3$,

$1 = 6A$

$A = \frac{1}{6}$

When $x = -3$,

$1 = -6B$

$B = -\frac{1}{6}$

Hence,

$\int \frac{1}{x^2 - 9}\,dx$

$= \int \left(\frac{1}{6(x-3)} - \frac{1}{6(x+3)}\right) dx$

$= \frac{1}{6}\int \left(\frac{1}{x-3} - \frac{1}{x+3}\right) dx$

$= \frac{1}{6}(\ln|x-3| - \ln|x+3|) + C$

$= \frac{1}{6}\ln\left|\frac{x-3}{x+3}\right| + C$

1d

$\frac{1}{9 - x^2} = \frac{1}{(3-x)(3+x)}$

Let $\frac{1}{(3-x)(3+x)} = \frac{A}{3-x} + \frac{B}{3+x}$

$\frac{1}{(3-x)(3+x)} = \frac{A(3+x) + B(3-x)}{(3-x)(3+x)}$

$1 = A(3+x) + B(3-x)$

When $x = 3$,

$1 = 6A$

$A = \frac{1}{6}$

When $x = -3$,

$1 = 6B$

$B = \frac{1}{6}$

Hence,

$\int \frac{1}{9 - x^2}\,dx$

$$= \int \left(\frac{1}{6(3-x)} + \frac{1}{6(3+x)}\right) dx$$

$$= \frac{1}{6}\int \left(-\frac{1}{x-3} + \frac{1}{x+3}\right) dx$$

$$= \frac{1}{6}(-\ln|3-x| + \ln|3+x|) + C$$

$$= \frac{1}{6}\ln\left|\frac{3+x}{3-x}\right| + C$$

Alternatively:

$$\int \frac{1}{9-x^2}\,dx$$

$$= -\int \frac{1}{x^2-9}\,dx$$

$$= -\frac{1}{6}\ln\left|\frac{x-3}{x+3}\right| + C \qquad \text{(using answer from part c)}$$

$$= \frac{1}{6}\ln\left|\frac{x-3}{x+3}\right|^{-1} + C$$

$$= \frac{1}{6}\ln\left|\frac{x+3}{x-3}\right| + C$$

$$= \frac{1}{6}\ln\left|\frac{3+x}{3-x}\right| + C$$

1e

$$\int \frac{1}{\sqrt{9+x^2}}\,dx$$

$$= \int \frac{1}{\sqrt{3^2+x^2}}\,dx$$

$$= \ln\left(x + \sqrt{9+x^2}\right) + C \qquad \text{(using standard integral)}$$

1f

$$\int \frac{1}{\sqrt{x^2-9}}\,dx$$

$$= \int \frac{1}{\sqrt{x^2-3^2}}\,dx$$

$$= \ln\left|x+\sqrt{x^2-9}\right| + C \qquad \text{(using standard integral)}$$

2a

$$\int \frac{1}{x^2+4x+5}\,dx$$

$$= \int \frac{1}{(x^2+4x+4)+1}\,dx$$

$$= \int \frac{1}{(x+2)^2+1^2}\,dx$$

$$= \tan^{-1}(x+2) + C$$

2b

$$\int \frac{1}{x^2-4x+20}\,dx$$

$$= \int \frac{1}{(x^2-4x+4)+16}\,dx$$

$$= \int \frac{1}{(x-2)^2+4^2}\,dx$$

$$= \frac{1}{4}\int \frac{4}{(x-2)^2+4^2}\,dx$$

$$= \frac{1}{4}\tan^{-1}\left(\frac{x-2}{4}\right) + C$$

2c

$$\int \frac{1}{\sqrt{9+8x-x^2}}\,dx$$

$$= \int \frac{1}{\sqrt{-(x^2-8x-9)}}\,dx$$

$$= \int \frac{1}{\sqrt{-\left((x^2-8x+16)-25\right)}}\,dx$$

$$= \int \frac{1}{\sqrt{-((x-4)^2-25)}}\,dx$$

$$= \int \frac{1}{\sqrt{25-(x-4)^2}}\,dx$$

$$= \int \frac{1}{\sqrt{5^2-(x-4)^2}}\,dx$$

$$= \sin^{-1}\frac{x-4}{5} + C$$

2d

$$\int \frac{1}{\sqrt{20-8x-x^2}}\,dx$$

$$= \int \frac{1}{\sqrt{-(x^2+8x-20)}}\,dx$$

$$= \int \frac{1}{\sqrt{-\left((x^2+8x+16)-36\right)}}\,dx$$

$$= \int \frac{1}{\sqrt{-((x+4)^2-36)}}\,dx$$

$$= \int \frac{1}{\sqrt{36-(x+4)^2}}\,dx$$

$$= \int \frac{1}{\sqrt{6^2-(x+4)^2}}\,dx$$

$$= \sin^{-1}\frac{x+4}{6} + C$$

2e

$$\int \frac{1}{\sqrt{x^2 - 6x + 13}} dx$$

$$= \int \frac{1}{\sqrt{(x^2 - 6x + 9) + 4}} dx$$

$$= \int \frac{1}{\sqrt{(x-3)^2 + 4}} dx$$

$$= \int \frac{1}{\sqrt{(x-3)^2 + 2^2}} dx$$

$$= \ln\left((x-3) + \sqrt{(x-3)^2 + 2^2}\right) + C \qquad \text{(using standard integral)}$$

$$= \ln\left(x - 3 + \sqrt{x^2 - 6x + 13}\right) + C$$

Note: we don't need to take absolute values here, because the log expression is always positive:

$$(x-3)^2 + 2^2 > (x-3)^2 \geq 0$$

$$\sqrt{(x-3)^2 + 2^2} > \sqrt{(x-3)^2}$$

$$\sqrt{(x-3)^2 + 2^2} > |(x-3)|$$

$$\sqrt{(x-3)^2 + 2^2} + (x-3) > 0$$

2f

$$\int \frac{1}{\sqrt{4x^2 + 8x + 6}} dx$$

$$= \int \frac{1}{\sqrt{4\left(x^2 + 2x + \frac{3}{2}\right)}} dx$$

$$= \int \frac{1}{\sqrt{4\left((x^2 + 2x + 1) + \frac{1}{2}\right)}} dx$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{(x+1)^2 + \frac{1}{2}}} dx$$

$$= \frac{1}{2}\ln\left|(x+1)+\sqrt{(x+1)^2+\frac{1}{2}}\right|+C \qquad \text{(using standard integral)}$$

$$= \frac{1}{2}\ln\left|x+1+\sqrt{x^2+2x+\frac{3}{2}}\right|+C$$

3a

$$\int_1^3 \frac{1}{x^2-2x+5}\,dx$$

$$= \int_1^3 \frac{1}{(x^2-2x+1)+4}\,dx$$

$$= \int_1^3 \frac{1}{(x-1)^2+4}\,dx$$

$$= \int_1^3 \frac{1}{(x-1)^2+2^2}\,dx$$

$$= \frac{1}{2}\int_1^3 \frac{2}{(x-1)^2+2^2}\,dx$$

$$= \frac{1}{2}\left[\tan^{-1}\frac{x-1}{2}\right]_1^3$$

$$= \frac{1}{2}(\tan^{-1}1-\tan^{-1}0)$$

$$= \frac{1}{2}\left(\frac{\pi}{4}-0\right)$$

$$= \frac{\pi}{8}$$

3b

$$\int_1^5 \frac{4}{x^2 - 6x + 13} dx$$

$$= \int_1^5 \frac{4}{(x^2 - 6x + 9) + 4} dx$$

$$= \int_1^5 \frac{4}{(x-3)^2 + 4} dx$$

$$= 2\int_1^5 \frac{2}{(x-3)^2 + 2^2} dx$$

$$= 2\left[\tan^{-1}\frac{x-3}{2}\right]_1^5$$

$$= 2(\tan^{-1} 1 - \tan^{-1}(-1))$$

$$= 2\left(\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right)$$

$$= 2 \times \frac{\pi}{2}$$

$$= \pi$$

3c

$$\int_{-1}^0 \frac{1}{\sqrt{3 - 2x - x^2}} dx$$

$$= \int_{-1}^0 \frac{1}{\sqrt{-(x^2 + 2x - 3)}} dx$$

$$= \int_{-1}^0 \frac{1}{\sqrt{-\left((x^2 + 2x + 1) - 4\right)}} dx$$

$$= \int_{-1}^0 \frac{1}{\sqrt{-((x-1)^2 - 4)}} dx$$

$$= \int_{-1}^0 \frac{1}{\sqrt{4 - (x-1)^2}} dx$$

$$= \frac{1}{2}\int_{-1}^0 \frac{2}{\sqrt{2^2 - (x-1)^2}} dx$$

$$= \frac{1}{2}\left[\sin^{-1}\frac{x-1}{2}\right]_{-1}^0$$

$$= \frac{1}{2}\left(\sin^{-1}\left(-\frac{1}{2}\right) - \sin^{-1}(-1)\right)$$

$$= \frac{1}{2}\left(-\frac{\pi}{6} - \left(-\frac{\pi}{2}\right)\right)$$

$$= \frac{1}{2}\left(-\frac{\pi}{6} + \frac{\pi}{2}\right)$$

$$= \frac{\pi}{6}$$

3d

$$\int_0^1 \frac{3}{\sqrt{3 + 4x - 4x^2}}\,dx$$

$$= \int_0^1 \frac{3}{\sqrt{-(4x^2 - 4x - 3)}}\,dx$$

$$= \int_0^1 \frac{3}{\sqrt{-\left((4x^2 - 4x + 1) - 4\right)}}\,dx$$

$$= \int_0^1 \frac{3}{\sqrt{-((2x - 1)^2 - 4)}}\,dx$$

$$= \int_0^1 \frac{3}{\sqrt{4 - (2x - 1)^2}}\,dx$$

$$= \frac{3}{2}\int_0^1 \frac{2}{\sqrt{2^2 - (2x - 1)^2}}\,dx$$

$$= \frac{3}{2}\left[\sin^{-1}\frac{2x - 1}{2}\right]_0^1$$

$$= \frac{3}{2}\left(\sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}\left(-\frac{1}{2}\right)\right)$$

$$= \frac{3}{2}\left(\frac{\pi}{6} - \left(-\frac{\pi}{6}\right)\right)$$

$$= \frac{3}{2}\left(\frac{\pi}{6} + \frac{\pi}{6}\right)$$

$$= \frac{3}{2} \times \frac{\pi}{3}$$

$$= \frac{\pi}{2}$$

3e

$$\int_{-1}^{3} \frac{1}{\sqrt{x^2 + 2x + 10}} dx$$
$$= \int_{-1}^{3} \frac{1}{\sqrt{(x^2 + 2x + 1) + 9}} dx$$
$$= \int_{-1}^{3} \frac{1}{\sqrt{(x + 1)^2 + 3^2}} dx$$
$$= \left[\ln\left((x + 1) + \sqrt{(x + 1)^2 + 9}\right)\right]_{-1}^{3} \qquad \text{(using standard integral)}$$
$$= \ln 9 - \ln 3$$
$$= \ln\left(\frac{9}{3}\right)$$
$$= \ln 3$$

3f

$$\int_{\frac{1}{2}}^{1} \frac{2}{\sqrt{x^2 - x + 1}} dx$$
$$= \int_{\frac{1}{2}}^{1} \frac{2}{\sqrt{\left(x^2 - x + \frac{1}{4}\right) + \frac{3}{4}}} dx$$
$$= 2\int_{\frac{1}{2}}^{1} \frac{1}{\sqrt{\left(x - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}} dx$$
$$= 2\left[\ln\left(\left(x - \frac{1}{2}\right) + \sqrt{\left(x - \frac{1}{2}\right)^2 + \frac{3}{4}}\right)\right]_{\frac{1}{2}}^{1}$$
$$= 2\left(\ln\left(\frac{3}{2}\right) - \ln\left(\frac{\sqrt{3}}{2}\right)\right)$$
$$= 2\ln\left(\frac{3}{2}\right) - 2\ln\left(\frac{\sqrt{3}}{2}\right)$$
$$= \ln\left(\frac{3}{2}\right)^2 - \ln\left(\frac{\sqrt{3}}{2}\right)^2$$

$$= \ln\left(\frac{9}{4}\right) - \ln\left(\frac{3}{4}\right)$$

$$= \ln\left(\frac{9}{4} \div \frac{3}{4}\right)$$

$$= \ln 3$$

Solutions to Exercise 4E Development questions

4a

$$\int \frac{2x+1}{x^2+2x+2}\,dx$$

$$= \int \frac{2x+1}{x^2+2x+1+1}\,dx$$

$$= \int \frac{2x+1}{(x+1)^2+1}\,dx$$

Let $u = x + 1$

$du = dx$

$x = u - 1$

Hence

$$\int \frac{2x+1}{(x+1)^2+1}\,dx$$

$$= \int \frac{2(u-1)+1}{u^2+1}\,du$$

$$= \int \frac{2u-1}{u^2+1}\,du$$

$$= \int \frac{2u}{u^2+1}\,du - \int \frac{1}{u^2+1}\,du$$

$$= \ln|u^2+1| - \tan^{-1} u + C$$

$$= \ln((x+1)^2+1) - \tan^{-1}(x+1) + C$$

$$= \ln(x^2+2x+2) - \tan^{-1}(x+1) + C$$

4b

$$\int \frac{x}{x^2+2x+10}\,dx$$

$$= \int \frac{x}{x^2+2x+1+9}\,dx$$

$$= \int \frac{x}{(x+1)^2+9}\,dx$$

Let $u = x + 1$

$du = dx$

$x = u - 1$

Hence

$\int \frac{x}{(x+1)^2 + 9} dx$

$= \int \frac{u-1}{u^2+9} du$

$= \int \frac{u}{u^2+9} du - \int \frac{1}{u^2+9} du$

$= \frac{1}{2} \int \frac{2u}{u^2+9} du - \frac{1}{3} \int \frac{3}{u^2+9} du$

$= \frac{1}{2} \ln(u^2+9) - \frac{1}{3} \tan^{-1}\left(\frac{u}{3}\right) + C$

$= \frac{1}{2} \ln((x+1)^2+9) - \frac{1}{3} \tan^{-1}\left(\frac{x+1}{3}\right) + C$

$= \frac{1}{2} \ln(x^2+2x+10) - \frac{1}{3} \tan^{-1}\left(\frac{x+1}{3}\right) + C$

4c

$\int \frac{x}{\sqrt{6x - x^2}} dx$

$= \int \frac{x}{\sqrt{-(x^2-6x)}} dx$

$= \int \frac{x}{\sqrt{-((x-3)^2-9)}} dx$

$= \int \frac{x}{\sqrt{9-(x-3)^2}} dx$

Let $u = x - 3$

$du = dx$

$x = u + 3$

Hence

$\int \frac{x}{\sqrt{9-(x-3)^2}} dx$

$$= \int \frac{u+3}{\sqrt{9-u^2}}\,du$$

$$= \int \frac{u}{\sqrt{9-u^2}}\,du + \int \frac{3}{\sqrt{9-u^2}}\,du$$

$$= -\frac{1}{2}\int \frac{-2}{\sqrt{9-u^2}}\,du + 3\int \frac{1}{\sqrt{9-u^2}}\,du$$

$$= -\sqrt{9-u^2} + 3\sin^{-1}\left(\frac{u}{3}\right) + C$$

$$= -\sqrt{9-(x-3)^2} + 3\sin^{-1}\left(\frac{x-3}{3}\right) + C$$

$$= -\sqrt{6x-x^2} + 3\sin^{-1}\left(\frac{x-3}{3}\right) + C$$

4d

$$\int \frac{x+3}{\sqrt{4-2x-x^2}}\,dx$$

$$= \int \frac{x+3}{\sqrt{-(x^2+2x-4)}}\,dx$$

$$= \int \frac{x+3}{\sqrt{-((x+1)^2-5)}}\,dx$$

$$= \int \frac{x+3}{\sqrt{5-(x+1)^2}}\,dx$$

Let $u = x + 1$

$du = dx$

$x = u - 1$

Hence

$$\int \frac{x+3}{\sqrt{5-(x+1)^2}}\,dx$$

$$= \int \frac{u+2}{\sqrt{5-u^2}}\,du$$

$$= -\frac{1}{2}\int \frac{-2u}{\sqrt{5-u^2}}\,du + \int \frac{2}{\sqrt{5-u^2}}\,du$$

$$= -\sqrt{5-u^2} + 2\sin^{-1}\left(\frac{u}{\sqrt{5}}\right) + C$$

$$= -\sqrt{5-(x+1)^2} + 2\sin^{-1}\left(\frac{x+1}{\sqrt{5}}\right) + C$$

$$= -\sqrt{4-2x-x^2} + 2\sin^{-1}\left(\frac{x+1}{\sqrt{5}}\right) + C$$

4e

$$\int \frac{x}{\sqrt{x^2+2x+10}}\,dx$$

$$= \int \frac{x}{\sqrt{(x+1)^2+9}}\,dx$$

Let $u = x+1$

$du = dx$

$x = u-1$

Hence

$$\int \frac{x}{\sqrt{(x+1)^2+9}}\,dx$$

$$= \int \frac{u-1}{\sqrt{u^2+9}}\,du$$

$$= \frac{1}{2}\int \frac{2u}{\sqrt{u^2+9}}\,du - \int \frac{1}{\sqrt{u^2+9}}\,du$$

$$= \sqrt{u^2+9} - \ln\left(u+\sqrt{u^2+9}\right) + C$$

$$= \sqrt{(x+1)^2+9} - \ln\left(x+1+\sqrt{(x+1)^2+9}\right) + C$$

$$= \sqrt{x^2+2x+10} - \ln\left(x+1+\sqrt{x^2+2x+10}\right) + C$$

4f

$$\int \frac{x+3}{\sqrt{x^2-2x-4}}dx$$

$$= \int \frac{x+3}{\sqrt{(x-1)^2-5}}dx$$

Let $u = x - 1$

$du = dx$

$x = u + 1$

Hence

$$\int \frac{x+3}{\sqrt{(x-1)^2-5}}dx$$

$$= \int \frac{u+4}{\sqrt{u^2-5}}du$$

$$= \frac{1}{2}\int \frac{2u}{\sqrt{u^2-5}}du + \int \frac{4}{\sqrt{u^2-5}}du$$

$$= \sqrt{u^2-5} + 4\ln\left|u + \sqrt{u^2-5}\right| + C$$

$$= \sqrt{(x-1)^2-5} + 4\ln\left|x - 1 + \sqrt{(x-1)^2-5}\right| + C$$

$$= \sqrt{x^2-2x-4} + 4\ln\left|x - 1 + \sqrt{x^2-2x-4}\right| + C$$

5a

$$\int_0^2 \frac{x+1}{x^2+4}dx$$

$$= \int_0^2 \frac{x}{x^2+4} + \frac{1}{x^2+4}dx$$

$$= \int_0^2 \frac{1}{2}\frac{2x}{x^2+4} + \frac{1}{2}\frac{2}{x^2+4}dx$$

$$= \left[\frac{1}{2}\ln|x^2+4| + \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right]_0^2$$

$$= \left(\frac{1}{2}\ln 8 + \frac{\pi}{8}\right) - \left(\frac{1}{2}\ln 4 + 0\right)$$

$$= \frac{1}{2}\ln 2 + \frac{\pi}{8}$$

5b

$$\int_1^2 \frac{x+1}{x^2-4x+5}\,dx$$

$$= \int_1^2 \frac{x+1}{(x-2)^2+1}\,dx$$

Let $u = x - 2$

$du = dx$

$x = 2, u = 0$

$x = 1, u = -1$

$x = u + 2$

Hence

$$\int_1^2 \frac{x+1}{(x-2)^2+1}\,dx$$

$$= \int_{-1}^0 \frac{u+3}{u^2+1}\,du$$

$$= \int_{-1}^0 \left(\frac{u}{u^2+1} + \frac{3}{u^2+1}\right)du$$

$$= \left[\frac{1}{2}\ln(u^2+1) + 3\tan^{-1} u\right]_{-1}^0$$

$$= (0+0) - \left(\frac{1}{2}\ln 2 - 3\tan^{-1}(-1)\right)$$

$$= -\frac{1}{2}\ln 2 + \frac{3\pi}{4}$$

$$= -\frac{1}{4}\times 2\ln 2 + \frac{3\pi}{4}$$

$$= -\frac{1}{4}\ln 4 + \frac{3\pi}{4}$$

$$= \frac{1}{4}(3\pi - \ln 4)$$

5c

$$\int_1^2 \frac{2x-3}{x^2-2x+2}\,dx$$

$$= \int_1^2 \frac{2x-3}{(x-1)^2+1}\,dx$$

Let $u = x - 1$

$du = dx$

$x = 2, u = 1$

$x = 1, u = 0$

$x = u + 1$

Hence

$$\int_1^2 \frac{2x-3}{(x-1)^2+1}\,dx$$

$$= \int_0^1 \frac{2(u+1)-3}{u^2+1}\,du$$

$$= \int_0^1 \frac{2u-1}{u^2+1}\,du$$

$$= \int_0^1 \left(\frac{2u}{u^2+1} - \frac{1}{u^2+1}\right) du$$

$$= [\ln(u^2+1) - \tan^{-1} u]_0^1$$

$$= \ln 2 - \frac{\pi}{4}$$

5d

$$\int_{-1}^0 \frac{x}{\sqrt{3-2x-x^2}}\,dx$$

$$= \int_{-1}^0 \frac{x}{\sqrt{-(x^2+2x-3)}}\,dx$$

$$= \int_{-1}^0 \frac{x}{\sqrt{-((x+1)^2-4)}}\,dx$$

$$= \int_{-1}^{0} \frac{x}{\sqrt{4-(x+1)^2}} dx$$

Let $u = x + 1$

$du = dx$

$x = 0, u = 1$

$x = -1, u = 0$

$x = u - 1$

Hence

$$\int_{-1}^{0} \frac{x}{\sqrt{4-(x+1)^2}} dx$$

$$= \int_{0}^{1} \frac{u-1}{\sqrt{4-u^2}} du$$

$$= \int_{0}^{1} \left(-\frac{1}{2}\frac{-2u}{\sqrt{4-u^2}} - \frac{1}{\sqrt{4-u^2}}\right) du$$

$$= \left[-\sqrt{4-u^2} - \sin^{-1}\left(\frac{u}{2}\right)\right]_0^1$$

$$= -\sqrt{3} - \frac{\pi}{6} + 2$$

5e

$$\int_{-1}^{3} \frac{1-2x}{\sqrt{x^2+2x+3}} dx$$

$$= \int_{-1}^{3} \frac{1-2x}{\sqrt{(x+1)^2+2}} dx$$

Let $u = x + 1$

$du = dx$

$x = 3, u = 4$

$x = -1, u = 0$

$x = u - 1$

Hence

$$\int_{-1}^{3} \frac{1-2x}{\sqrt{(x+1)^2+2}}\,dx$$

$$= \int_{0}^{4} \frac{1-2(u-1)}{\sqrt{u^2+2}}\,du$$

$$= \int_{0}^{4} \frac{3-2u}{\sqrt{u^2+2}}\,du$$

$$= \int_{0}^{4} \left(\frac{3}{\sqrt{u^2+2}} - \frac{2u}{\sqrt{u^2+2}}\right) du$$

$$= \left[3\ln\left|u+\sqrt{u^2+2}\right| - 2\sqrt{u^2+2}\right]_0^4$$

$$= 3\ln\left(4+3\sqrt{2}\right) - 6\sqrt{2} - 3\ln(\sqrt{2}) + 2\sqrt{2}$$

$$= 3\ln\left(\frac{4+3\sqrt{2}}{\sqrt{2}}\right) - 4\sqrt{2}$$

$$= 3\ln\left(\frac{4}{\sqrt{2}} + 3\right) - 4\sqrt{2}$$

$$= 3\ln\left(2\sqrt{2}+3\right) - 4\sqrt{2}$$

5f

$$\int_{0}^{1} \frac{x+3}{\sqrt{x^2+4x+1}}\,dx$$

$$= \int_{0}^{1} \frac{x+3}{\sqrt{(x+2)^2-3}}\,dx$$

Let $u = x+2$

$du = dx$

$x = 1, u = 3$

$x = 0, u = 2$

$x = u-2$

Hence

$$\int_{0}^{1} \frac{x+3}{\sqrt{(x+2)^2-3}}\,dx$$

$$= \int_2^3 \frac{u+1}{\sqrt{u^2-3}}\,du$$

$$= \int_2^3 \left(\frac{u}{\sqrt{u^2-3}} + \frac{1}{\sqrt{u^2-3}}\right) du$$

$$= \left[\sqrt{u^2-3} + \ln\left|u + \sqrt{u^2-3}\right|\right]_2^3$$

$$= \sqrt{6} + \ln\left(3+\sqrt{6}\right) - 1 - \ln 3$$

$$= \ln\left(\frac{3+\sqrt{6}}{3}\right) + \sqrt{6} - 1$$

$$= \ln\left(1 + \sqrt{\frac{2}{3}}\right) + \sqrt{6} - 1$$

6a

$$\int \sqrt{\frac{1+x}{1-x}}\,dx$$

$$= \int \sqrt{\frac{1+x}{1-x}} \times \sqrt{\frac{1+x}{1+x}}\,dx$$

$$= \int \frac{1+x}{\sqrt{1-x^2}}\,dx$$

$$= \int \frac{1}{\sqrt{1-x^2}}\,dx + \int \frac{x}{\sqrt{1-x^2}}\,dx$$

$$= \sin^{-1} x - \sqrt{1-x^2} + C$$

6b

$$\int \sqrt{\frac{3-x}{2+x}}\,dx$$

$$= \int \sqrt{\frac{3-x}{2+x}} \times \sqrt{\frac{3-x}{3-x}}\,dx$$

$$= \int \frac{3-x}{\sqrt{6+x-x^2}}\,dx$$

$$= \int \frac{3-x}{\sqrt{-(x^2-x-6)}}\,dx$$

$$= \int \frac{3-x}{\sqrt{\frac{25}{4}-\left(x-\frac{1}{2}\right)^2}}\,dx$$

Let $u = x - \frac{1}{2}$

$du = dx$

$x = u + \frac{1}{2}$

Hence

$$\int \frac{3-x}{\sqrt{\frac{25}{4}-\left(x-\frac{1}{2}\right)^2}}\,dx$$

$$= \int \frac{3-\left(u+\frac{1}{2}\right)}{\sqrt{\frac{25}{4}-u^2}}\,du$$

$$= \int \frac{\frac{5}{2}-u}{\sqrt{\frac{25}{4}-u^2}}\,du$$

$$= \int \left(\frac{\frac{5}{2}}{\sqrt{\frac{25}{4}-u^2}} - \frac{u}{\sqrt{\frac{25}{4}-u^2}}\right)du$$

$$= \frac{5}{2}\sin^{-1}\left(\frac{2u}{5}\right) + \sqrt{\frac{25}{4} - u^2} + C$$

$$= \frac{5}{2}\sin^{-1}\left(\frac{2x-1}{5}\right) + \sqrt{6 + x - x^2} + C$$

6c

$$\int \sqrt{\frac{x-1}{x+1}}\,dx$$

$$= \int \sqrt{\frac{x-1}{x+1}} \times \sqrt{\frac{x-1}{x-1}}\,dx$$

$$= \int \frac{x-1}{\sqrt{x^2-1}}\,dx$$

$$= \int \left(\frac{x}{\sqrt{x^2-1}} - \frac{1}{\sqrt{x^2-1}}\right) dx$$

$$= \sqrt{x^2-1} - \ln\left|x + \sqrt{x^2-1}\right| + C$$

7a

$$\int_{-1}^{0} \sqrt{\frac{1-x}{x+3}}\,dx$$

$$= \int_{-1}^{0} \sqrt{\frac{1-x}{x+3}} \times \sqrt{\frac{1-x}{1-x}}\,dx$$

$$= \int_{-1}^{0} \frac{1-x}{\sqrt{3-2x-x^2}}\,dx$$

$$= \int_{-1}^{0} \frac{1-x}{\sqrt{-(x^2+2x-3)}}\,dx$$

$$= \int_{-1}^{0} \frac{1-x}{\sqrt{4-(x+1)^2}}\,dx$$

Let $u = x + 1$

$du = dx$

$x = 0, u = 1$

$x = -1, u = 0$

$x = u - 1$

$= \int_0^1 \frac{1 - u + 1}{\sqrt{4 - u^2}} du$

$= \int_0^1 \frac{2 - u}{\sqrt{4 - u^2}} du$

$= \int_0^1 \left(\frac{2}{\sqrt{4 - u^2}} - \frac{u}{\sqrt{4 - u^2}}\right) du$

$= \left[2 \sin^{-1}\left(\frac{u}{2}\right) + \sqrt{4 - u^2}\right]_0^1$

$= \frac{\pi}{3} + \sqrt{3} - 2$

7b

$\int_{-1}^0 \sqrt{\frac{x + 2}{1 - x}} dx$

$= \int_{-1}^0 \sqrt{\frac{x + 2}{1 - x}} \times \sqrt{\frac{x + 2}{x + 2}} dx$

$= \int_{-1}^0 \frac{x + 2}{\sqrt{2 - x - x^2}} dx$

$= \int_{-1}^0 \frac{x + 2}{\sqrt{-(x^2 + x - 2)}} dx$

$= \int_{-1}^0 \frac{x + 2}{\sqrt{\frac{9}{4} - \left(x + \frac{1}{2}\right)^2}} dx$

Let $u = x + \frac{1}{2}$

$du = dx$

$x = 0, u = \frac{1}{2}$

$x = -1, u = -\frac{1}{2}$

$x = u - \frac{1}{2}$

Hence

$$\int_{-1}^{0} \frac{x+2}{\sqrt{\frac{9}{4} - \left(x + \frac{1}{2}\right)^2}} dx$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{u - \frac{1}{2} + 2}{\sqrt{\frac{9}{4} - u^2}} du$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{u + \frac{3}{2}}{\sqrt{\frac{9}{4} - u^2}} du$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{u}{\sqrt{\frac{9}{4} - u^2}} + \frac{\frac{3}{2}}{\sqrt{\frac{9}{4} - u^2}} \right) du$$

$$= \left[\sqrt{\frac{9}{4} - u^2} + \frac{3}{2}\sin^{-1}\left(\frac{2u}{3}\right) \right]_{-\frac{1}{2}}^{\frac{1}{2}}$$

$$= \left(\sqrt{2} + \frac{3}{2}\sin^{-1}\left(\frac{1}{3}\right)\right) - \left(\sqrt{2} - \frac{3}{2}\sin^{-1}\left(\frac{1}{3}\right)\right)$$

$$= 3\sin^{-1}\left(\frac{1}{3}\right)$$

7c

$$\int_{0}^{1} \sqrt{\frac{x+1}{x+3}} dx$$

$$= \int_{0}^{1} \sqrt{\frac{x+1}{x+3}} \times \sqrt{\frac{x+1}{x+1}} dx$$

$$= \int_{0}^{1} \frac{x+1}{\sqrt{x^2 + 4x + 3}} dx$$

$$= \int_0^1 \frac{x+1}{\sqrt{(x+2)^2 - 1}}\,dx$$

Let $u = x + 2$

$du = dx$

$x = 1, u = 3$

$x = 0, u = 2$

$x = u - 2$

Hence

$$\int_0^1 \frac{x+1}{\sqrt{(x+2)^2 - 1}}\,dx$$

$$= \int_2^3 \frac{u - 2 + 1}{\sqrt{u^2 - 1}}\,du$$

$$= \int_2^3 \frac{u - 1}{\sqrt{u^2 - 1}}\,du$$

$$= \int_2^3 \left(\frac{u}{\sqrt{u^2 - 1}} - \frac{1}{\sqrt{u^2 - 1}}\right) du$$

$$= \left[\sqrt{u^2 - 1} - \ln\left|u + \sqrt{u^2 - 1}\right|\right]_2^3$$

$$= 2\sqrt{2} - \ln\left(3 + 2\sqrt{2}\right) - \sqrt{3} + \ln\left(2 + \sqrt{3}\right)$$

$$= 2\sqrt{2} - \sqrt{3} + \ln\left(\frac{2 + \sqrt{3}}{3 + 2\sqrt{2}}\right)$$

Solutions to Exercise 4E Enrichment questions

8a Rationalising the numerator:

$\sqrt{\frac{x}{4-x}} = \frac{x}{\sqrt{4x-x^2}}$, for $0 < x < 4$

This is undefined at $x = 0$, the lower limit of the integral.

8b $\int_{\epsilon}^{2} \sqrt{\frac{x}{4-x}}\, dx$

$= \int_{\epsilon}^{2} \frac{x}{\sqrt{4x-x^2}}\, dx$

$= \int_{\epsilon}^{2} \frac{2}{\sqrt{4-(x-2)^2}}\, dx - \int_{\epsilon}^{2} \frac{2-x}{\sqrt{4x-x^2}}\, dx$

$= \left[2\sin^{-1}\left(\frac{x-2}{2}\right) - \sqrt{4x-x^2}\right]_{\epsilon}^{2}$

$= \left(2\sin^{-1} 0 - \sqrt{4}\right) - \left(2\sin^{-1}\left(\frac{\epsilon-2}{2}\right) - \sqrt{4\epsilon-\epsilon^2}\right)$

So taking the limit as $\epsilon \to 0^+$,

$\int_{0}^{2} \sqrt{\frac{x}{4-x}}\, dx$

$= \lim_{\epsilon\to 0^+} \int_{\epsilon}^{2} \sqrt{\frac{x}{4-x}}\, dx$

$= \lim_{\epsilon\to 0^+} \left(-2 - 2\sin^{-1}\left(\frac{\epsilon-2}{2}\right) + \sqrt{4\epsilon-\epsilon^2}\right)$

$= -2 - 2\sin^{-1}(-1) + 0$

$= \pi - 2$

9a RHS

$= (x+1)(x^2+2x+2) + (x-1)$

$= x^3 + 2x^2 + 2x + x^2 + 2x + 2 + x - 1$

$= x^3 + 3x^2 + 5x + 1$

$=$ LHS

9b $\int_{-1}^{0} \frac{x^3+3x^2+5x+1}{\sqrt{x^2+2x+2}}\,dx$

$= \int_{-1}^{0} \left(\frac{(x+1)(x^2+2x+2)}{\sqrt{x^2+2x+2}} + \frac{(x-1)}{\sqrt{x^2+2x+2}}\right) dx$

$= \int_{-1}^{0} (x+1)\sqrt{x^2+2x+2}\,dx + \int_{-1}^{0} \frac{(x+1)}{\sqrt{x^2+2x+2}}\,dx - 2\int_{-1}^{0} \frac{dx}{\sqrt{(x+1)^2+1}}\,dx$

$= \frac{1}{3}\left[(x^2+2x+2)^{\frac{3}{2}}\right]_{-1}^{0} + \left[(x^2+2x+2)^{\frac{1}{2}}\right]_{-1}^{0} - 2\left[\ln\left(x+1+\sqrt{x^2+2x+2}\right)\right]_{-1}^{0}$

$= \frac{1}{3}\left(2^{\frac{3}{2}}-1\right) + \left(2^{\frac{1}{2}}-1\right) - \left(2\ln\left(1+\sqrt{2}\right) - 2\ln 1\right)$

$= \frac{1}{3}\left(5\sqrt{2}-4\right) - 2\ln\left(1+\sqrt{2}\right)$

Solutions to Exercise 4F Foundation questions

1a

$$\int xe^x\,dx$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Therefore

$u = x, v' = e^x$

$u' = 1, v = e^x$

Hence

$$\int xe^x\,dx$$

$$= xe^x - \int e^x\,dx$$

$= xe^x - e^x + C$

$= e^x(x-1) + C$

1b

$$\int xe^{-x}\,dx$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Therefore

$u = x, v' = e^{-x}$

$u' = 1, v = -e^{-x}$

Hence

$$\int xe^{-x}\,dx$$

$$= -xe^{-x} + \int e^{-x}\,dx$$

$= -xe^{-x} - e^{-x} + C$

$= -e^{-x}(x+1) + C$

1c

$$\int (x+1)e^{3x}\,dx$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Therefore

$u = x+1, v' = e^{3x}$

$u' = 1, v = \frac{1}{3}e^{3x}$

Hence

$$\int (x+1)e^{3x}\,dx$$

$$= \frac{1}{3}(x+1)e^{3x} - \frac{1}{3}\int e^{3x}\,dx$$

$$= \frac{1}{3}(x+1)e^{3x} - \frac{1}{3}\left(\frac{1}{3}e^{3x}\right) + C$$

$$= \frac{1}{3}(x+1)e^{3x} - \frac{1}{9}e^{3x} + C$$

$$= \frac{1}{9}e^{3x}(3x+3-1) + C$$

$$= \frac{1}{9}e^{3x}(3x+2) + C$$

1d

$$\int x\cos x\,\,dx$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Therefore

$u = x, v' = \cos x$

$u' = 1, v = \sin x$

Hence

$$\int x\cos x\,\,dx$$

$$= x\sin x - \int \sin x \ dx$$

$$= x\sin x - (-\cos x) + C$$

$$= x\sin x + \cos x + C$$

1e

$$\int (x-1)\sin 2x\, dx$$

$$\int uv'\, dx = uv - \int u'v\, dx$$

Therefore

$u = x - 1, v' = \sin 2x$

$u' = 1, v = -\frac{1}{2}\cos 2x$

Hence

$$\int (x-1)\sin 2x\, dx$$

$$= -\frac{1}{2}(x-1)\cos 2x + \frac{1}{2}\int \cos 2x\, dx$$

$$= -\frac{1}{2}(x-1)\cos 2x + \frac{1}{4}\sin 2x + C$$

1f

$$\int (2x-3)\ \sec^2 x\, dx$$

$$\int uv'\, dx = uv - \int u'v\, dx$$

Therefore

$u = 2x - 3, v' = \sec^2 x$

$u' = 2, v = \tan x$

Hence

$$\int (2x-3)\ \sec^2 x\, dx$$

$$= (2x - 3)\tan x - \int 2\tan x \, dx$$

$$= (2x - 3)\tan x + 2\int \frac{-\sin x}{\cos x} dx$$

$$= (2x - 3)\tan x + 2\ln|\cos x| + C$$

2a

$$\int_0^{\pi} x \sin x \, dx$$

$$\int uv' \, dx = uv - \int u'v \, dx$$

Therefore

$u = x, v' = \sin x$

$u' = 1, v = -\cos x$

Hence

$$\int_0^{\pi} x \sin x \, dx$$

$$= [-x\cos x]_0^{\pi} + \int_0^{\pi} \cos x \, dx$$

$$= (-\pi\cos\pi - 0) + [\sin x]_0^{\pi}$$

$$= \pi + [\sin x]_0^{\pi}$$

$$= \pi + \sin\pi - \sin 0$$

$$= \pi$$

2b

$$\int_0^{\frac{\pi}{2}} x \cos x \, dx$$

$$\int uv' \, dx = uv - \int u'v \, dx$$

Therefore

$u = x, v' = \cos x$

$u' = 1, v = \sin x$

Hence

$$\int_0^{\frac{\pi}{2}} x \cos x \, dx$$

$$= [x \sin x]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin x \, dx$$

$$= \frac{\pi}{2} \sin \frac{\pi}{2} - 0 - [-\cos x]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2} - \left(-\cos \frac{\pi}{2} + \cos 0\right)$$

$$= \frac{\pi}{2} - 1$$

2c

$$\int_0^{\frac{\pi}{4}} x \sec^2 x \, dx$$

$$\int uv' \, dx = uv - \int u'v \, dx$$

Therefore

$u = x, v' = \sec^2 x$

$u' = 1, v = \tan x$

Hence

$$\int_0^{\frac{\pi}{4}} x \sec^2 x \, dx$$

$$= [x \tan x]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \tan x \, dx$$

$$= \frac{\pi}{4} \tan \frac{\pi}{4} - 0 + \int_0^{\frac{\pi}{4}} \frac{-\sin x}{\cos x} dx$$

$$= \frac{\pi}{4} + [\ln|\cos x|]_0^{\frac{\pi}{4}}$$

$$= \frac{\pi}{4} + \left(\ln\left(\cos \frac{\pi}{4}\right) - \ln(\cos 0)\right)$$

$$= \frac{\pi}{4} + \left(\ln\left(\frac{1}{\sqrt{2}}\right) - \ln 1\right)$$

$$= \frac{\pi}{4} + \left(-\frac{1}{2}\ln 2 - 0\right)$$

$$= \frac{\pi}{4} - \frac{1}{2}\ln 2$$

2d

$$\int_0^1 x\, e^{2x}\, dx$$

$$\int uv'\, dx = uv - \int u'v\, dx$$

Therefore

$u = x, v' = e^{2x}$

$u' = 1, v = \frac{1}{2}e^{2x}$

Hence

$$\int_0^1 x\, e^{2x}\, dx$$

$$= \left[\frac{1}{2}xe^{2x}\right]_0^1 - \frac{1}{2}\int_0^1 e^{2x}\, dx$$

$$= \frac{1}{2}e^2 - \frac{1}{2}\left[\frac{1}{2}e^{2x}\right]_0^1$$

$$= \frac{1}{2}e^2 - \frac{1}{2}\left(\frac{1}{2}e^2 - \frac{1}{2}\right)$$

$$= \frac{1}{2}e^2 - \frac{1}{4}e^2 + \frac{1}{4}$$

$$= \frac{1}{4}e^2 + \frac{1}{4}$$

$$= \frac{1}{4}(e^2 + 1)$$

2e

$$\int_0^1 (1-x)e^{-x}\,dx$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Therefore

$u = 1 - x, v' = e^{-x}$

$u' = -1, v = -e^{-x}$

Hence

$$\int_0^1 (1-x)e^{-x}\,dx$$

$$= [(1-x)(-e^{-x})]_0^1 - \int_0^1 e^{-x}\,dx$$

$$= [(x-1)e^{-x}]_0^1 - [-e^{-x}]_0^1$$

$$= 0 - (-1) + [e^{-x}]_0^1$$

$$= 1 + e^{-1} - 1$$

$$= \frac{1}{e}$$

2f

$$\int_{-2}^0 (x+2)e^{x}\,dx$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Therefore

$u = x + 2, v' = e^{x}$

$u' = 1, v = e^{x}$

Hence

$$\int_{-2}^0 (x+2)e^{x}\,dx$$

$$= [(x+2)e^x]_{-2}^{0} - \int_{-2}^{0} e^x \, dx$$

$$= 2 - 0 - [e^x]_{-2}^{0}$$

$$= 2 - (1 - e^{-2})$$

$$= 1 + e^{-2}$$

3a

$$\int \ln x \, dx$$

$$\int uv' \, dx = uv - \int u'v \, dx$$

Therefore

$u = \ln x \, , v' = 1$

$u' = \dfrac{1}{x}, v = x$

Hence

$$\int \ln x \, dx$$

$$= x \ln x - \int 1 \, dx$$

$$= x \ln x - x + C$$

$$= x(\ln x - 1) + C$$

3b

$$\int \ln(x^2) \, dx$$

$$\int uv' \, dx = uv - \int u'v \, dx$$

Therefore

$u = \ln(x^2) = 2 \ln x \, , v' = 1$

$u' = \dfrac{2}{x}, v = x$

Hence

$$\int \ln(x^2)\,dx$$

$$= x\ln(x^2) - \int 2\,dx$$

$$= x\ln(x^2) - 2x + C$$

$$= 2x\ln x - 2x + C$$

$$= 2x(\ln x - 1) + C$$

3c

$$\int \cos^{-1} x\,dx$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Therefore

$$u = \cos^{-1} x\,, v' = 1$$

$$u' = \frac{-1}{\sqrt{1-x^2}}, v = x$$

Hence

$$\int \cos^{-1} x\,dx$$

$$= x\cos^{-1} x - \int \frac{-x}{\sqrt{1-x^2}}\,dx$$

Let $u = 1 - x^2$

$$du = -2x\,dx$$

Therefore

$$\int \cos^{-1} x\,dx$$

$$= x\cos^{-1} x - \frac{1}{2}\int \frac{-2x}{\sqrt{1-x^2}}\,dx$$

$$= x\cos^{-1} x - \frac{1}{2}\int \frac{1}{\sqrt{u}}\,du$$

$$= x\cos^{-1}x - \frac{1}{2}\left(2\sqrt{u}\right) + C$$

$$= x\cos^{-1}x - \sqrt{u} + C$$

$$= x\cos^{-1}x - \sqrt{1-x^2} + C$$

4a

$$\int_0^1 \tan^{-1}x\,dx$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Therefore

$u = \tan^{-1}x\,, v' = 1$

$u' = \dfrac{1}{1+x^2}, v = x$

Hence

$$\int_0^1 \tan^{-1}x\,dx$$

$$= [x\tan^{-1}x]_0^1 - \int_0^1 \frac{x}{1+x^2}\,dx$$

$$= \tan^{-1}(1) - 0 - \frac{1}{2}\int_0^1 \frac{2x}{1+x^2}\,dx$$

$$= \frac{\pi}{4} - \frac{1}{2}[\ln|1+x^2|]_0^1$$

$$= \frac{\pi}{4} - \frac{1}{2}(\ln 2 - \ln 1)$$

$$= \frac{\pi}{4} - \frac{1}{2}\ln 2$$

4b

$$\int_1^e \ln x \, dx$$

$$= [x(\ln x - 1)]_1^e \qquad \text{(using the result from question 3a)}$$

$$= e(\ln e - 1) - (\ln 1 - 1)$$

$$= e(1 - 1) - (0 - 1)$$

$$= 1$$

4c

$$\int_1^e \ln \sqrt{x} \, dx$$

$$\int uv' \, dx = uv - \int u'v \, dx$$

Therefore

$$u = \ln \sqrt{x} = \frac{1}{2} \ln x, v' = 1$$

$$u' = \frac{1}{2x}, v = x$$

Hence

$$\int_1^e \ln \sqrt{x} \, dx$$

$$= [x \ln \sqrt{x}]_1^e - \int_1^e \frac{1}{2} dx$$

$$= \left[\frac{1}{2} x \ln x\right]_1^e - \left[\frac{1}{2} x\right]_1^e$$

$$= \left(\frac{1}{2} e - 0\right) - \left(\frac{1}{2} e - \frac{1}{2}\right)$$

$$= \frac{1}{2}$$

Alternatively:

Since $\ln \sqrt{x} = \dfrac{1}{2} \ln x$

$$\int_1^e \ln\sqrt{x}\,dx$$

$$= \frac{1}{2}\int_1^e \ln x\,dx$$

$$= \frac{1}{2}\times 1 \qquad \text{(using the result from question 4b)}$$

$$= \frac{1}{2}$$

5a

$$\int x\ln x\,dx$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Therefore

$$u = \ln x\,, v' = x$$

$$u' = \frac{1}{x}, v = \frac{1}{2}x^2$$

Hence

$$\int x\ln x\,dx$$

$$= \frac{1}{2}x^2\ln x - \int \frac{1}{2}x\,dx$$

$$= \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C$$

$$= \frac{1}{4}x^2(2\ln x - 1) + C$$

5b

$$\int x^2\ln x\,dx$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Therefore

$u = \ln x \,, v' = x^2$

$u' = \dfrac{1}{x}, v = \dfrac{1}{3}x^3$

Hence

$\displaystyle\int x^2 \ln x \, dx$

$= \dfrac{1}{3}x^3 \ln x - \displaystyle\int \dfrac{1}{3}x^2 \, dx$

$= \dfrac{1}{3}x^3 \ln x - \dfrac{1}{9}x^3 + C$

$= \dfrac{1}{9}x^3(3 \ln x - 1) + C$

5c

$\displaystyle\int \frac{\ln x}{x^2} dx$

$\displaystyle\int uv' \, dx = uv - \int u'v \, dx$

Therefore

$u = \ln x \,, v' = \dfrac{1}{x^2}$

$u' = \dfrac{1}{x}, v = -\dfrac{1}{x}$

Hence

$\displaystyle\int \frac{\ln x}{x^2} dx$

$= -\dfrac{1}{x}\ln x - \displaystyle\int \left(-\frac{1}{x^2}\right) dx$

$= -\dfrac{1}{x}\ln x - \dfrac{1}{x} + C$

$= -\dfrac{1}{x}(\ln x + 1) + C$

Solutions to Exercise 4F Development questions

6a

$$\int x^2 e^x \, dx$$

$$\int uv' \, dx = uv - \int u'v \, dx$$

Therefore

$u = x^2, v' = e^x$

$u' = 2x, v = e^x$

Hence

$$\int x^2 e^x \, dx$$

$$= x^2 e^x - \int 2xe^x \, dx$$

Consider $\int 2xe^x \, dx,$

$u = 2x, v' = e^x$

$u' = 2, v = e^x$

Hence

$$\int 2xe^x \, dx = 2xe^x - \int 2e^x \, dx$$

Therefore

$$\int x^2 e^x \, dx$$

$$= x^2 e^x - 2xe^x + \int 2e^x \, dx$$

$$= x^2 e^x - 2xe^x + 2e^x + C$$

$$= (2 - 2x + x^2)e^x + C$$

6b

$$\int x^2 \cos x \, dx$$

$$\int uv' \, dx = uv - \int u'v \, dx$$

Therefore

$u = x^2, v' = \cos x$

$u' = 2x, v = \sin x$

Hence

$$\int x^2 \cos x \, dx$$

$$= x^2 \sin x - \int 2x \sin x \, dx$$

Consider $\int 2x \sin x \, dx,$

$u = 2x, v' = \sin x$

$u' = 2, v = -\cos x$

Hence

$$\int 2x \sin x \, dx = -2x \cos x + \int 2 \cos x \, dx$$

Therefore

$$\int x^2 \cos x \, dx$$

$$= x^2 \sin x + 2x \cos x - \int 2 \cos x \, dx$$

$$= x^2 \sin x + 2x \cos x - 2 \sin x + C$$

6c

$$\int (\ln x)^2 \, dx$$

$$\int uv' \, dx = uv - \int u'v \, dx$$

Therefore

$u = (\ln x)^2, v' = 1$

$u' = \dfrac{2\ln x}{x}, v = x$

Hence

$\int (\ln x)^2\,dx$

$= x(\ln x)^2 - \int 2\ln x\,dx$

$= x(\ln x)^2 - 2\int \ln x\,dx$

Consider $\int \ln x\,dx$,

$u = \ln x\,, v' = 1$

$u' = \dfrac{1}{x}, v = x$

Hence

$\int \ln x\,dx = x\ln x - \int 1\,dx$

Therefore

$\int (\ln x)^2\,dx$

$= x(\ln x)^2 - 2x\ln x + 2\int 1\,dx$

$= x(\ln x)^2 - 2x\ln x + 2x + C$

7a Using integration by parts:

$\int_0^1 x(x-1)^5\,dx$

$\int uv'\,dx = uv - \int u'v\,dx$

Therefore

$u = x, v' = (x-1)^5$

$u' = 1, v = \dfrac{1}{6}(x-1)^6$

Hence

$$\int_0^1 x(x-1)^5\,dx$$

$$= \left[\frac{x}{6}(x-1)^6\right]_0^1 - \int_0^1 \frac{1}{6}(x-1)^6\,dx$$

$$= 0 - \left[\frac{1}{42}(x-1)^7\right]_0^1$$

$$= -\frac{1}{42}$$

By substitution:

$$\int_0^1 x(x-1)^5\,dx$$

Let $u = x - 1$

$du = dx$

$x = u + 1$

$x = 1, u = 0$

$x = 0, u = -1$

Hence

$$\int_0^1 x(x-1)^5\,dx$$

$$= \int_{-1}^0 (u+1)u^5\,du$$

$$= \int_{-1}^0 (u^6 + u^5)\,du$$

$$= \left[\frac{1}{7}u^7 + \frac{1}{6}u^6\right]_{-1}^0$$

$$= \left(\frac{1}{7}(0)^7 + \frac{1}{6}(0)^6\right) - \left(\frac{1}{7}(-1)^7 + \frac{1}{6}(-1)^6\right)$$

$$= \frac{1}{7} - \frac{1}{6}$$

$$= -\frac{1}{42}$$

7b Using integration by parts:

$$\int_0^1 x\sqrt{x+1}\,dx$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Therefore

$u = x, v' = \sqrt{x+1}$

$u' = 1, v = \frac{2}{3}(x+1)^{\frac{3}{2}}$

Hence

$$\int_0^1 x\sqrt{x+1}\,dx$$

$$= \left[\frac{2}{3}x(x+1)^{\frac{3}{2}}\right]_0^1 - \int_0^1 \frac{2}{3}(x+1)^{\frac{3}{2}}\,dx$$

$$= \frac{2}{3}(2)^{\frac{3}{2}} - \left[\frac{4}{15}(x+1)^{\frac{5}{2}}\right]_0^1$$

$$= \frac{4\sqrt{2}}{3} - \left(\frac{4}{15}(2)^{\frac{5}{2}} - \frac{4}{15}\right)$$

$$= \frac{4\sqrt{2}}{3} - \left(\frac{16\sqrt{2}}{15} - \frac{4}{15}\right)$$

$$= \frac{20\sqrt{2}}{15} - \frac{16\sqrt{2}-4}{15}$$

$$= \frac{4\sqrt{2}+4}{15}$$

$$= \frac{4}{15}\left(\sqrt{2}+1\right)$$

By substitution:

$$\int_0^1 x\sqrt{x+1}\,dx$$

Let $u = \sqrt{x+1}$

$x = 1, u = \sqrt{2}$

$x = 0, u = 1$

$x = u^2 - 1$

$dx = 2u\,du$

Hence

$$\int_0^1 x\sqrt{x+1}\,dx$$

$$= \int_1^{\sqrt{2}} (u^2 - 1)u \times 2u\,du$$

$$= 2\int_1^{\sqrt{2}} (u^4 - u^2)\,du$$

$$= 2\left[\frac{1}{5}u^5 - \frac{1}{3}u^3\right]_1^{\sqrt{2}}$$

$$= 2\left(\frac{4}{5}\sqrt{2} - \frac{2}{3}\sqrt{2} - \frac{1}{5} + \frac{1}{3}\right)$$

$$= \frac{8\sqrt{2} - 2}{5} - \frac{4\sqrt{2} - 2}{3}$$

$$= \frac{24\sqrt{2} - 6 - 20\sqrt{2} + 10}{15}$$

$$= \frac{4\sqrt{2} + 4}{15}$$

$$= \frac{4}{15}\left(\sqrt{2} + 1\right)$$

7c Using integration by parts:

$$\int_0^4 x\sqrt{4-x}\,dx$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Therefore

$u = x, v' = \sqrt{4-x}$

$u' = 1, v = -\frac{2}{3}(4-x)^{\frac{3}{2}}$

Hence

$$\int_0^4 x\sqrt{4-x}\,dx$$

$$= \left[-\frac{2}{3}x(4-x)^{\frac{3}{2}}\right]_0^4 + \int_0^4 \frac{2}{3}(4-x)^{\frac{3}{2}}\,dx$$

$$= 0 + \left[-\frac{4}{15}(4-x)^{\frac{5}{2}}\right]_0^4$$

$$= (0) - \left(-\frac{4}{15}(4-0)^{\frac{5}{2}}\right)$$

$$= \frac{4 \times 32}{15}$$

$$= \frac{128}{15}$$

By substitution:

$$\int_0^4 x\sqrt{4-x}\,dx$$

Let $u = \sqrt{4-x}$

$x = 4, u = 0$

$x = 0, u = 2$

$x = 4 - u^2$

$dx = -2u\,du$

Hence

$$\int_0^4 x\sqrt{4-x}\,dx$$

$$= \int_2^0 (4-u^2)u \times -2u\,du$$

$$= 2\int_0^2 (4-u^2)u^2\,du$$

$$= 2\int_0^2 (4u^2-u^4)\,du$$

$$= 2\left[\frac{4}{3}u^3 - \frac{1}{5}u^5\right]_0^2$$

$$= 2\left[\frac{4}{3} \times 8 - \frac{1}{5} \times 32\right]$$

$$= \frac{128}{15}$$

8a

$$\int e^x \cos x \, dx$$

$$\int uv' \, dx = uv - \int u'v \, dx$$

Therefore

$u = e^x, v' = \cos x$

$u' = e^x, v = \sin x$

Hence

$$\int e^x \cos x \, dx$$

$$= e^x \sin x - \int e^x \sin x \, dx$$

Consider $\int e^x \sin x \, dx$,

$u = e^x, v' = \sin x$

$u' = e^x, v = -\cos x$

Hence

$$\int e^x \sin x \, dx = -e^x \cos x + \int e^x \cos x \, dx$$

Therefore

$$\int e^x \cos x \, dx = e^x \sin x + e^x \cos x - \int e^x \cos x \, dx$$

$$2\int e^x \cos x \, dx = e^x \sin x + e^x \cos x$$

Therefore

$$\int e^x \cos x \, dx$$

$$= \frac{e^x \sin x + e^x \cos x}{2} + C$$

8b

$$\int e^{-x} \sin x \, dx$$

$$\int uv' \, dx = uv - \int u'v \, dx$$

Therefore

$u = e^{-x}, v' = \sin x$

$u' = -e^{-x}, v = -\cos x$

Hence

$$\int e^{-x} \sin x \, dx$$

$$= -e^{-x} \cos x - \int e^{-x} \cos x \, dx$$

Consider $\int e^{-x} \cos x \, dx$,

$u = e^{-x}, v' = \cos x$

$u' = -e^{-x}, v = \sin x$

Hence

$$\int e^{-x} \cos x \, dx = e^{-x} \sin x + \int e^{-x} \sin x \, dx$$

Therefore

$$\int e^{-x} \sin x \, dx = -e^{-x} \cos x - e^{-x} \sin x - \int e^{-x} \sin x \, dx$$

$$2 \int e^{-x} \sin x \, dx = -e^{-x} \cos x - e^{-x} \sin x$$

Therefore

$$\int e^{-x} \sin x \, dx$$

$$= -\frac{e^{-x} \cos x + e^{-x} \sin x}{2} + C$$

9a

$$\int_0^{\frac{\pi}{2}} e^{2x} \cos x \, dx$$

$$\int uv' \, dx = uv - \int u'v \, dx$$

Therefore

$u = e^{2x}, v' = \cos x$

$u' = 2e^{2x}, v = \sin x$

Hence

$$\int_0^{\frac{\pi}{2}} e^{2x} \cos x \, dx$$

$$= [e^{2x} \sin x]_0^{\frac{\pi}{2}} - 2\int_0^{\frac{\pi}{2}} e^{2x} \sin x \, dx$$

Consider $\int_0^{\frac{\pi}{2}} e^{2x} \sin x \, dx$,

$u = e^{2x}, v' = \sin x$

$u' = 2e^{2x}, v = -\cos x$

Hence

$$\int_0^{\frac{\pi}{2}} e^{2x} \sin x \, dx = [-e^{2x} \cos x]_0^{\frac{\pi}{2}} + 2\int_0^{\frac{\pi}{2}} e^{2x} \cos x \, dx$$

Therefore

$$\int_0^{\frac{\pi}{2}} e^{2x} \cos x \, dx = [e^{2x} \sin x]_0^{\frac{\pi}{2}} - 2[-e^{2x} \cos x]_0^{\frac{\pi}{2}} - 4\int_0^{\frac{\pi}{2}} e^{2x} \cos x \, dx$$

$$5\int_0^{\frac{\pi}{2}} e^{2x} \cos x \, dx = [e^{2x} \sin x]_0^{\frac{\pi}{2}} - 2[-e^{2x} \cos x]_0^{\frac{\pi}{2}}$$

$$\int_0^{\frac{\pi}{2}} e^{2x} \cos x \, dx = \frac{1}{5}\left(e^{\pi} + 2(0-1)\right)$$

$$= \frac{1}{5}(e^{\pi} - 2)$$

9b

$$\int_0^{\frac{\pi}{4}} e^x \sin 2x\, dx$$

$$\int uv'\, dx = uv - \int u'v\, dx$$

Therefore

$$u = e^x, v' = \sin 2x$$

$$u' = e^x, v = -\frac{1}{2}\cos 2x$$

Hence

$$\int_0^{\frac{\pi}{4}} e^x \sin 2x\, dx$$

$$= \frac{1}{2}[-e^x \cos 2x]_0^{\frac{\pi}{4}} + \frac{1}{2}\int_0^{\frac{\pi}{4}} e^x \cos 2x\, dx$$

Consider $\int_0^{\frac{\pi}{4}} e^x \cos 2x\, dx$,

$$u = e^x, v' = \cos 2x$$

$$u' = e^x, v = \frac{1}{2}\sin 2x$$

Hence

$$\int_0^{\frac{\pi}{4}} e^x \cos 2x\, dx = \frac{1}{2}[e^x \sin 2x]_0^{\frac{\pi}{4}} - \frac{1}{2}\int_0^{\frac{\pi}{4}} e^x \sin 2x\, dx$$

Therefore

$$\int_0^{\frac{\pi}{4}} e^x \sin 2x\, dx = \frac{1}{2}[-e^x \cos 2x]_0^{\frac{\pi}{4}} + \frac{1}{4}[e^x \sin 2x]_0^{\frac{\pi}{4}} - \frac{1}{4}\int_0^{\frac{\pi}{4}} e^x \sin 2x\, dx$$

$$\frac{5}{4}\int_0^{\frac{\pi}{4}} e^x \sin 2x\, dx = \frac{1}{2}[-e^x \cos 2x]_0^{\frac{\pi}{4}} + \frac{1}{4}[e^x \sin 2x]_0^{\frac{\pi}{4}}$$

$$\int_0^{\frac{\pi}{4}} e^x \sin 2x\, dx = \frac{4}{5}\left(\frac{1}{2}\big((0) - (-1)\big) + \frac{1}{4}\left(\left(e^{\frac{\pi}{4}}\right) - (0)\right)\right)$$

$$= \frac{4}{5}\left(\frac{1}{2} + \frac{1}{4}e^{\frac{\pi}{4}}\right)$$

$$= \frac{1}{5}\left(e^{\frac{\pi}{4}} + 2\right)$$

10a

$$\int_0^{\frac{\sqrt{3}}{2}} \sin^{-1} x \, dx$$

$$\int uv' \, dx = uv - \int u'v \, dx$$

Therefore

$u = \sin^{-1} x \,, v' = 1$

$u' = \dfrac{1}{\sqrt{1 - x^2}}, v = x$

Hence

$$\int_0^{\frac{\sqrt{3}}{2}} \sin^{-1} x \, dx$$

$$= [x \sin^{-1} x]_0^{\frac{\sqrt{3}}{2}} - \int_0^{\frac{\sqrt{3}}{2}} \frac{x}{\sqrt{1 - x^2}} dx$$

$$= \frac{\sqrt{3}}{2} \times \frac{\pi}{3} - \left[-\sqrt{1 - x^2}\right]_0^{\frac{\sqrt{3}}{2}}$$

$$= \frac{\pi\sqrt{3}}{6} + \frac{1}{2} - 1$$

$$= \frac{\pi\sqrt{3}}{6} - \frac{1}{2}$$

$$= \frac{3\pi}{6\sqrt{3}} - \frac{3\sqrt{3}}{6\sqrt{3}}$$

$$= \frac{3\pi - 3\sqrt{3}}{6\sqrt{3}}$$

$$= \frac{1}{2\sqrt{3}}\left(\pi - \sqrt{3}\right)$$

10b

$$\int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} \cos^{-1} x \, dx$$

$$\int uv' \, dx = uv - \int u'v \, dx$$

Therefore

$u = \cos^{-1} x \,, v' = 1$

$u' = \dfrac{-1}{\sqrt{1-x^2}}, v = x$

Hence

$$\int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} \cos^{-1} x \, dx$$

$$= [x \cos^{-1} x]_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} - \int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} \frac{-x}{\sqrt{1-x^2}} dx$$

$$= \left(\frac{\sqrt{3}}{2} \times \frac{\pi}{6}\right) - \left(-\frac{\sqrt{3}}{2} \times \frac{5\pi}{6}\right) - \left[\sqrt{1-x^2}\right]_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}}$$

$$= \frac{\pi\sqrt{3}}{12} + \frac{5\pi\sqrt{3}}{12} - \frac{1}{2} + \frac{1}{2}$$

$$= \frac{6\pi\sqrt{3}}{12}$$

$$= \frac{\pi\sqrt{3}}{2}$$

10c

$$\int_0^1 4x\tan^{-1} x \, dx$$

$$\int uv' \, dx = uv - \int u'v \, dx$$

Therefore

$u = \tan^{-1} x \,, v' = 4x$

$u' = \dfrac{1}{x^2+1}, v = 2x^2$

Hence

$\int_0^1 4x\tan^{-1} x\, dx$

$= [2x^2 \tan^{-1} x]_0^1 - 2\int_0^1 \dfrac{x^2}{x^2+1}\, dx$

$= \dfrac{\pi}{2} - 2\int_0^1 \dfrac{x^2+1-1}{x^2+1}\, dx$

$= \dfrac{\pi}{2} - 2\int_0^1 1 - \dfrac{1}{x^2+1}\, dx$

$= \dfrac{\pi}{2} - 2[x - \tan^{-1} x]_0^1$

$= \dfrac{\pi}{2} - 2\left(1 - \dfrac{\pi}{4}\right)$

$= \dfrac{\pi}{2} - 2 + \dfrac{\pi}{2}$

$= \pi - 2$

11a

$\int_0^\pi x^2 \cos 2x\, dx$

$\int uv'\, dx = uv - \int u'v\, dx$

Therefore

$u = x^2, v' = \cos 2x$

$u' = 2x, v = \dfrac{1}{2}\sin 2x$

Hence

$\int_0^\pi x^2 \cos 2x\, dx$

$= \left[x^2 \dfrac{1}{2}\sin 2x\right]_0^\pi - \int_0^\pi x \sin 2x\, dx$

Consider $\int_0^{\pi} x \sin 2x \, dx$,

$u = x, v' = \sin 2x$

$u' = 1, v = -\frac{1}{2}\cos 2x$

Hence

$$\int_0^{\pi} x \sin 2x \, dx = \left[-x\frac{1}{2}\cos 2x\right]_0^{\pi} + \frac{1}{2}\int_0^{\pi} \cos 2x \, dx$$

Therefore

$$\int_0^{\pi} x^2 \cos 2x \, dx$$

$$= \left[x^2\frac{1}{2}\sin 2x\right]_0^{\pi} - \left[-x\frac{1}{2}\cos 2x\right]_0^{\pi} - \frac{1}{2}\int_0^{\pi} \cos 2x \, dx$$

$$= \left[x^2\frac{1}{2}\sin 2x\right]_0^{\pi} + \left[x\frac{1}{2}\cos 2x\right]_0^{\pi} - \frac{1}{2}\left[\frac{1}{2}\sin 2x\right]_0^{\pi}$$

$$= 0 + \frac{\pi}{2} - \frac{1}{2}(0)$$

$$= \frac{\pi}{2}$$

11b

$$\int_0^{\pi} x^2 \sin\frac{1}{2}x \, dx$$

$$\int uv' \, dx = uv - \int u'v \, dx$$

Therefore

$u = x^2, v' = \sin\frac{1}{2}x$

$u' = 2x, v = -2\cos\frac{1}{2}x$

Hence

$$\int_0^{\pi} x^2 \sin\frac{1}{2}x \, dx$$

$$= \left[-2x^2 \cos\frac{1}{2}x\right]_0^\pi + 4\int_0^\pi x\cos\frac{1}{2}x\,dx$$

Consider $\int_0^\pi x\cos\frac{1}{2}x\,dx$,

$u = x, v' = \cos\frac{1}{2}x$

$u' = 1, v = 2\sin\frac{1}{2}x$

Hence

$$\int_0^\pi x\cos\frac{1}{2}x\,dx = \left[2x\sin\frac{1}{2}x\right]_0^\pi - 2\int_0^\pi \sin\frac{1}{2}x\,dx$$

Therefore

$$\int_0^\pi x^2\sin\frac{1}{2}x\,dx$$

$$= \left[-2x^2\cos\frac{1}{2}x\right]_0^\pi + 4\left(\left[2x\sin\frac{1}{2}x\right]_0^\pi - 2\int_0^\pi \sin\frac{1}{2}x\,dx\right)$$

$$= 0 + 8\pi - 8\left[-2\cos\frac{1}{2}x\right]_0^\pi$$

$$= 8\pi + 16\left[\cos\frac{1}{2}x\right]_0^\pi$$

$$= 8\pi + 16(-1)$$

$$= 8\pi - 16$$

11c

$$\int_1^e \sin(\ln x)\,dx$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Therefore

$u = \sin(\ln x), v' = 1$

$u' = \dfrac{\cos(\ln x)}{x}, v = x$

Hence

$$\int_1^e \sin(\ln x)\,dx$$

$$= [x\sin(\ln x)]_1^e - \int_1^e \cos(\ln x)\,dx$$

Consider $\int_1^e \cos(\ln x)\,dx$,

$u = \cos(\ln x)\,, v' = 1$

$u' = \dfrac{-\sin(\ln x)}{x}, v = x$

Hence

$$\int_1^e \cos(\ln x)\,dx = [x\cos(\ln x)]_1^e + \int_1^e \sin(\ln x)\,dx$$

Therefore

$$\int_1^e \sin(\ln x)\,dx = [x\sin(\ln x)]_1^e - [x\cos(\ln x)]_1^e - \int_1^e \sin(\ln x)\,dx$$

$$2\int_1^e \sin(\ln x)\,dx = [x\sin(\ln x)]_1^e - [x\cos(\ln x)]_1^e$$

$$\int_1^e \sin(\ln x)\,dx$$

$$= \frac{1}{2}\left([x\sin(\ln x)]_1^e - [x\cos(\ln x)]_1^e\right)$$

$$= \frac{1}{2}(e\sin 1 - 0 - e\cos 1 + 1)$$

$$= \frac{1}{2}e(\sin 1 - \cos 1) + \frac{1}{2}$$

11d

$$\int_1^e \cos(\ln x)\,dx$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Therefore

$u = \cos(\ln x)\,, v' = 1$

$u' = \dfrac{-\sin(\ln x)}{x}, v = x$

Hence

$\displaystyle\int_1^e \cos(\ln x)\,dx$

$= [x\cos(\ln x)]_1^e + \displaystyle\int_1^e \sin(\ln x)\,dx$

Consider $\displaystyle\int_1^e \sin(\ln x)\,dx$,

$u = \sin(\ln x)\,, v' = 1$

$u' = \dfrac{\cos(\ln x)}{x}, v = x$

Hence

$\displaystyle\int_1^e \sin(\ln x)\,dx = [x\sin(\ln x)]_1^e - \int_1^e \cos(\ln x)\,dx$

Therefore

$\displaystyle\int_1^e \cos(\ln x)\,dx = [x\cos(\ln x)]_1^e + [x\sin(\ln x)]_1^e - \int_1^e \cos(\ln x)\,dx$

$2\displaystyle\int_1^e \cos(\ln x)\,dx = [x\cos(\ln x)]_1^e + [x\sin(\ln x)]_1^e$

$\displaystyle\int_1^e \cos(\ln x)\,dx$

$= \dfrac{1}{2}\left([x\cos(\ln x)]_1^e + [x\sin(\ln x)]_1^e\right)$

$= \dfrac{1}{2}(e\cos 1 - 1 + e\sin 1)$

$= \dfrac{1}{2}e(\cos 1 + \sin 1) - \dfrac{1}{2}$

12a

$$\int x \ln x \, dx$$

$$\int uv' \, dx = uv - \int u'v \, dx$$

Therefore

$u = \ln x \,, v' = x$

$u' = \dfrac{1}{x}, v = \dfrac{1}{2}x^2$

Hence

$$\int x \ln x \, dx$$

$$= \frac{1}{2}x^2 \ln x - \frac{1}{2}\int x \, dx$$

$$= \frac{1}{2}x^2 \ln x - \frac{1}{2}\left(\frac{1}{2}x^2\right) + C$$

$$= \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C$$

$$= \frac{1}{4}x^2(\ln x - 1) + C$$

12b

$$\int x(\ln x)^2 \, dx$$

$$\int uv' \, dx = uv - \int u'v \, dx$$

Therefore

$u = (\ln x)^2, v' = x$

$u' = \dfrac{2\ln x}{x}, v = \dfrac{1}{2}x^2$

Hence

$$\int x(\ln x)^2 \, dx$$

$$= \frac{1}{2}x^2(\ln x)^2 - \int x \ln x \, dx$$

$$= \frac{1}{2}x^2(\ln x)^2 - \frac{1}{4}x^2(\ln x - 1) + C$$

$$= \frac{1}{4}x^2(2(\ln x)^2 + \ln x + 1) + C$$

13a

$$\int_0^{\frac{\pi}{2}} x \sin x \cos x \, dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} x \sin 2x \, dx$$

$$\int uv' \, dx = uv - \int u'v \, dx$$

Therefore

$u = x, v' = \sin 2x$

$u' = 1, v = -\frac{1}{2}\cos 2x$

Hence

$$\frac{1}{2}\int_0^{\frac{\pi}{2}} x \sin 2x \, dx$$

$$= \frac{1}{2}\left(\left[-\frac{1}{2}x\cos 2x\right]_0^{\frac{\pi}{2}} + \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos 2x \, dx\right)$$

$$= \frac{1}{2}\left(\frac{\pi}{4} + \frac{1}{2}\left[\frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{2}}\right)$$

$$= \frac{\pi}{8} - 0$$

$$= \frac{\pi}{8}$$

13b

$$\int_0^{\frac{\pi}{2}} x\sin^2 x\,dx$$

$$= \int_0^{\frac{\pi}{2}} x\left(\frac{1}{2}(1-\cos 2x)\right)dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}}(x - x\cos 2x)\,dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} x\,dx - \frac{1}{2}\int_0^{\frac{\pi}{2}} x\cos 2x\,dx$$

$$= \frac{1}{4}[x^2]_0^{\frac{\pi}{2}} - \frac{1}{2}\int_0^{\frac{\pi}{2}} x\cos 2x\,dx$$

$$= \frac{\pi^2}{16} - \frac{1}{2}\int_0^{\frac{\pi}{2}} x\cos 2x\,dx$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Therefore

$u = x, v' = \cos 2x$

$u' = 1, v = \frac{1}{2}\sin 2x$

Hence

$$\int_0^{\frac{\pi}{2}} x\cos 2x\,dx$$

$$= \left[\frac{1}{2}x\sin 2x\right]_0^{\frac{\pi}{2}} - \frac{1}{2}\int_0^{\frac{\pi}{2}}\sin 2x\,dx$$

$$= 0 - \frac{1}{2}\left[-\frac{1}{2}\cos 2x\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{4}[\cos 2x]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{4}(-1-1)$$

$= -\frac{1}{2}$

Therefore

$\int_0^{\frac{\pi}{2}} x \sin^2 x \, dx$

$= \frac{\pi^2}{16} - \frac{1}{2}\int_0^{\frac{\pi}{2}} x \cos 2x \, dx$

$= \frac{\pi^2}{16} - \frac{1}{2}\left(-\frac{1}{2}\right)$

$= \frac{\pi^2}{16} + \frac{1}{4}$

$= \frac{1}{16}(\pi^2 + 4)$

13c

$\int_0^{\frac{\pi}{4}} x \tan^2 x \, dx$

$= \int_0^{\frac{\pi}{4}} x(\sec^2 x - 1) \, dx$

$= \int_0^{\frac{\pi}{4}} (x \sec^2 x - x) \, dx$

$= \int_0^{\frac{\pi}{4}} x \sec^2 x \, dx - \left[\frac{1}{2}x^2\right]_0^{\frac{\pi}{4}}$

$= \int_0^{\frac{\pi}{4}} x \sec^2 x \, dx - \frac{\pi^2}{32}$

$\int uv' \, dx = uv - \int u'v \, dx$

Therefore

$u = x, v' = \sec^2 x$

$u' = 1, v = \tan x$

Hence

$$\int_0^{\frac{\pi}{4}} x \sec^2 x \, dx$$

$$= [x \tan x]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \tan x \, dx$$

$$= [x \tan x]_0^{\frac{\pi}{4}} + [\ln|\cos x|]_0^{\frac{\pi}{4}}$$

$$= \frac{\pi}{4} - 0 + \ln\left(\frac{1}{\sqrt{2}}\right) - 0$$

$$= \frac{\pi}{4} - \frac{1}{2}\ln 2$$

Therefore

$$\int_0^{\frac{\pi}{4}} x \tan^2 x \, dx$$

$$= \frac{\pi}{4} - \frac{\pi^2}{32} - \frac{1}{2}\ln 2$$

13d

$$\int_0^{\pi} x^2(\cos^2 x - \sin^2 x) \, dx$$

$$= \int_0^{\pi} x^2 \cos 2x \, dx$$

$$\int uv' \, dx = uv - \int u'v \, dx$$

Therefore

$$u = x^2, v' = \cos 2x$$

$$u' = 2x, v = \frac{1}{2}\sin 2x$$

Hence

$$\int_0^{\pi} x^2 \cos 2x \, dx$$

$$= \left[\frac{1}{2}x^2 \sin 2x\right]_0^{\pi} - \int_0^{\pi} x \sin 2x \, dx$$

Consider $\int_0^{\pi} x \sin 2x \, dx$,

$u = x, v' = \sin 2x$

$u' = 1, v = -\frac{1}{2}\cos 2x$

Hence

$$\int_0^{\pi} x \sin 2x \, dx = \left[-\frac{1}{2}x\cos 2x\right]_0^{\pi} + \frac{1}{2}\int_0^{\pi} \cos 2x \, dx$$

Therefore

$$\int_0^{\pi} x^2 \cos 2x \, dx$$

$$= \left[\frac{1}{2}x^2 \sin 2x\right]_0^{\pi} - \int_0^{\pi} x \sin 2x \, dx$$

$$= 0 - \left[-\frac{1}{2}x\cos 2x\right]_0^{\pi} - \frac{1}{2}\int_0^{\pi} \cos 2x \, dx$$

$$= \frac{1}{2}[x\cos 2x]_0^{\pi} - \frac{1}{2}\left[\frac{1}{2}\sin 2x\right]_0^{\pi}$$

$$= \frac{1}{2}(\pi - 0) - 0$$

$$= \frac{\pi}{2}$$

Solutions to Exercise 4F Enrichment questions

14a $I = \int \sqrt{a^2 - x^2}dx$

$= x\sqrt{a^2 - x^2} + \int \frac{x^2}{\sqrt{a^2-x^2}}dx$ (by parts with $u = \sqrt{a^2 - x^2}$, $v' = 1$)

$= x\sqrt{a^2 - x^2} - \int \frac{a^2-x^2}{\sqrt{a^2-x^2}}dx + \int \frac{a^2}{\sqrt{a^2-x^2}}dx$

$= x\sqrt{a^2 - x^2} - \int \sqrt{a^2 - x^2}\, dx + a^2 \sin^{-1}\left(\frac{x}{a}\right)$

But $\int \sqrt{a^2 - x^2}dx = I$

Hence,

$2I = x\sqrt{a^2 - x^2} + a^2 \sin^{-1}\left(\frac{x}{a}\right) + 2C$ (for some constant C.)

Thus $I = \frac{1}{2}\left[x\sqrt{a^2 - x^2} + a^2 \sin^{-1}\left(\frac{x}{a}\right)\right] + C$

14b $I = \int \ln\left(x + \sqrt{x^2 + a^2}\right)dx$

Integrating by parts with $u = \ln\left(x + \sqrt{x^2 + a^2}\right)$, $v' = 1$,

$I = x \ln\left(x + \sqrt{x^2 + a^2}\right) - \int \frac{x}{\sqrt{x^2+a^2}}dx$

$= x \ln\left(x + \sqrt{x^2 + a^2}\right) - \sqrt{x^2 + a^2} + C$

14c $I = \int \ln\left(x + \sqrt{x^2 - a^2}\right)dx$

Integrating by parts with $u = \ln\left(x + \sqrt{x^2 - a^2}\right)$, $v' = 1$,

$= x \ln\left(x + \sqrt{x^2 - a^2}\right) - \int \frac{x}{\sqrt{x^2-a^2}}dx$

$= x \ln\left(x + \sqrt{x^2 - a^2}\right) - \sqrt{x^2 - a^2} + C$

15a First note that $\sin(A + B) + \sin(A - B) = 2 \sin A \cos B$

$\int x \sin x \cos 3x\, dx$

$= \frac{1}{2}\int x(\sin 4x + \sin(-2x))\, dx$

$= \frac{1}{2}\int x \sin 4x - x \sin 2x\, dx$

$\int x \sin ax\, dx$

$= -\frac{x}{a}\cos ax + \frac{1}{a}\int \cos ax\, dx$ (by parts)

$= -\frac{x}{a}\cos ax + \frac{1}{a^2}\sin ax + C$

Hence,

$\int x \sin x \cos 3x\, dx$

$= -\frac{x}{8}\cos 4x + \frac{1}{32}\sin 4x + \frac{1}{4}x\cos 2x - \frac{1}{8}\sin 2x + C$

$= \frac{1}{32}(\sin 4x - 4x\cos 4x + 8x\cos 2x - 4\sin 2x) + C$

15b First note that $\cos(A + B) - \cos(A - B) = \cos A \cos B$

$\int x\cos 2x \cos x\, dx = \frac{1}{2}\int x(\cos 3x + \cos x)\, dx$

$\int x \cos ax\, dx$

$= \frac{x}{a}\sin ax - \frac{1}{a}\int \sin ax\, dx$ (by parts)

$= \frac{x}{a}\sin ax + \frac{1}{a^2}\cos ax + C$

Hence,

$\int x\cos 2x\cos x\, dx$

$= \frac{x}{6}\sin 3x + \frac{1}{18}\cos 3x + \frac{x}{2}\sin x + \frac{1}{2}\cos x + C$

$= \frac{1}{18}(3x\sin 3x + \cos 3x + 9x\sin x + 9\cos x) + C$

15c Using the identity in part a, above:

$\int e^x \sin 2x \cos x\, dx = \frac{1}{2}\int e^x(\sin 3x + \sin x)\, dx$

$I = \int e^x \sin ax\, dx$

$= e^x \sin ax - a\int e^x \cos ax\, dx$ (by parts)

$= e^x \sin ax - a(e^x\cos ax + a\int e^x \sin ax\, dx)$ (by parts again)

Hence,

$I = e^x(\sin ax - a\cos ax) - a^2 I$

$(1 + a^2)I = e^x(\sin ax - a\cos ax)$

$I = \frac{1}{1+a^2}e^x(\sin ax - a\cos ax)$

Hence,

$\int e^x \sin 2x \cos x \, dx$

$= \frac{1}{2}\int e^x(\sin 3x + \sin x)\, dx$

$= \frac{1}{2}\left[\frac{1}{10}e^x(\sin 3x - 3\cos 3x) + \frac{1}{2}e^x(\sin x - \cos x)\right] + C$

$= \frac{1}{20}e^x(\sin 3x - 3\cos 3x + 5\sin x - 5\cos x) + C$

16a $I = \int_0^{\frac{1}{2}} x \sin^{-1} x \, dx$

$= \left[\frac{x^2}{2}\sin^{-1} x\right]_0^{\frac{1}{2}} - \frac{1}{2}\int_0^{\frac{1}{2}} \frac{x^2}{\sqrt{1-x^2}}\, dx$ (by parts)

In the integral put $x = \sin\theta$ with $0 \le \theta \le \frac{\pi}{6}$.

So, $dx = \cos\theta d\theta$

And $\sqrt{1-x^2} = |\cos\theta| = \cos\theta$ (for this domain)

Hence,

$I = \left(\frac{1}{8}.\frac{\pi}{6} - 0\right) - \frac{1}{2}\int_0^{\frac{\pi}{6}} \frac{\sin^2\theta}{\cos\theta}\cdot\cos\theta \, d\theta$

$= \frac{\pi}{48} - \frac{1}{2}\int_0^{\frac{\pi}{6}} \sin^2\theta \, d\theta$

$= \frac{\pi}{48} - \frac{1}{4}\int_0^{\frac{\pi}{6}} (1 - \cos 2\theta) d\theta$

$= \frac{\pi}{48} - \frac{1}{4}\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{6}}$

$= \frac{\pi}{48} - \frac{1}{4}\left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right) + 0$

$= \frac{\sqrt{3}}{16} - \frac{\pi}{48}$

$= \frac{1}{48}\left(3\sqrt{3} - \pi\right)$

16b $I = \int_0^1 x^2 \tan^{-1} x \, dx$

$= \left[\frac{x^3}{3}\tan^{-1} x\right]_0^1 - \frac{1}{3}\int_0^1 \frac{x^3}{1+x^2} dx$ (by parts)

$= \left(\frac{\pi}{12} - 0\right) - \frac{1}{3}\int_0^1 \frac{x^3+x}{x^2+1} dx + \frac{1}{3}\int_0^1 \frac{x}{x^2+1} dx$

$= \frac{\pi}{12} - \frac{1}{3}\int_0^1 x dx + \frac{1}{6}\int_0^1 \frac{2x}{x^2+1} dx$

$= \frac{\pi}{12} - \frac{1}{6}[x^2]_0^1 + \frac{1}{6}[\ln(x^2+1)]_0^1$

$= \frac{\pi}{12} - \frac{1}{6} + \frac{1}{6}\ln 2$

$= \frac{1}{12}(\pi + 2\ln 2 - 2)$

17 $\lim_{N\to\infty} \int_0^N te^{-st} dt$

$= \lim_{N\to\infty} \left[\frac{te^{-st}}{-s}\right]_0^N + \frac{1}{s}\int_0^N e^{-st} dt$ (by parts)

$= \lim_{N\to\infty} \frac{Ne^{-sN}}{-s} - \frac{1}{s^2}[e^{-st}]_0^N$

$= \lim_{N\to\infty} \frac{Ne^{-sN}}{-s} - \frac{e^{-sN}}{s^2} + \frac{1}{s^2}$

$= 0 - 0 + \frac{1}{s^2}$

$= \frac{1}{s^2}$

Solutions to Exercise 4G Foundation questions

1a

$$\int \cos x \, dx$$

$$= \sin x + C$$

1b

$$\int \sin x \, dx$$

$$= -\cos x + C$$

1c

$$\int \tan x \, dx$$

$$= \int \frac{\sin x}{\cos x} dx$$

Let $u = \cos x$

$du = -\sin x \, dx$

Hence

$$\int \frac{\sin x}{\cos x} dx$$

$$= \int -\frac{1}{u} du$$

$$= -\ln|u| + C$$

$$= -\ln|\cos x| + C$$

1d

$$\int \cot x \, dx$$

$$= \int \frac{\cos x}{\sin x} dx$$

Let $u = \sin x$

$du = \cos x \, dx$

Hence

$\int \frac{\cos x}{\sin x} dx$

$= \int \frac{1}{u} du$

$= \ln|u| + C$

$= \ln|\sin x| + C$

2a

$\int \cos x \sin^2 x \, dx$

Let $u = \sin x$

$du = \cos x \, dx$

Hence

$\int \cos x \sin^2 x \, dx$

$= \int u^2 \, du$

$= \frac{1}{3} u^3 + C$

$= \frac{1}{3} \sin^3 x + C$

2b

$\int \cos^2 x \sin x \, dx$

Let $u = \cos x$

$du = -\sin x \, dx$

Hence

$\int \cos^2 x \sin x \, dx$

$$= -\int u^2 \, du$$

$$= -\frac{1}{3}u^3 + C$$

$$= -\frac{1}{3}\cos^3 x + C$$

2c

$$\int \sin^3 x \, dx$$

$$= \int \sin^2 x \sin x \, dx$$

$$= \int (1 - \cos^2 x) \sin x \, dx$$

Let $u = \cos x$

$du = -\sin x \, dx$

Hence

$$\int (1 - \cos^2 x) \sin x \, dx$$

$$= \int (u^2 - 1) \, du$$

$$= \frac{1}{3}u^3 - u + C$$

$$= \frac{1}{3}\cos^3 x - \cos x + C$$

2d

$$\int \cos^3 x \, dx$$

$$= \int \cos^2 x \cos x \, dx$$

$$= \int (1 - \sin^2 x) \cos x \, dx$$

Let $u = \sin x$

$du = \cos x\, dx$

Hence

$\int (1 - \sin^2 x) \cos x\, dx$

$= \int (1 - u^2)\, du$

$= u - \frac{1}{3}u^3 + C$

$= \sin x - \frac{1}{3}\sin^3 x + C$

2e

$\int \cos^5 x\, dx$

$= \int \cos^4 x \cos x\, dx$

$= \int (\cos^2 x)^2 \cos x\, dx$

$= \int (1 - \sin^2 x)^2 \cos x\, dx$

Let $u = \sin x$

$du = \cos x\, dx$

Hence

$\int (1 - \sin^2 x)^2 \cos x\, dx$

$= \int (1 - u^2)^2\, du$

$= \int (1 - 2u^2 + u^4)\, du$

$= u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + C$

$= \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C$

2f

$$\int \sin^3 x \cos^3 x \, dx$$

$$= \int \sin^3 x \,(1 - \sin^2 x) \cos x \, dx$$

Let $u = \sin x$

$du = \cos x \, dx$

Hence

$$\int \sin^3 x \,(1 - \sin^2 x) \cos x \, dx$$

$$= \int u^3 (1 - u^2) \, du$$

$$= \int (u^3 - u^5) \, du$$

$$= \frac{1}{4} u^4 - \frac{1}{6} u^6 + C$$

$$= \frac{1}{4} \sin^4 x - \frac{1}{6} \sin^6 x + C$$

3a

$$\int_0^{\frac{\pi}{2}} \sin^2 x \, dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{2} (1 - \cos 2x) \, dx$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{2}} (1 - \cos 2x) \, dx$$

$$= \frac{1}{2} \left[x - \frac{1}{2} \sin 2x \right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2} \left(\left(\frac{\pi}{2} - 0 \right) - (0 - 0) \right)$$

$$= \frac{\pi}{4}$$

3b

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos^2 x\, dx$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{2}(1 + \cos 2x)\, dx$$

$$= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (1 + \cos 2x)\, dx$$

$$= \frac{1}{2}\left[x + \frac{1}{2}\sin 2x\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$= \frac{1}{2}\left(\left(\frac{\pi}{3} + \frac{\sqrt{3}}{4}\right) - \left(\frac{\pi}{6} + \frac{\sqrt{3}}{4}\right)\right)$$

$$= \frac{1}{2}\left(\frac{\pi}{6}\right)$$

$$= \frac{\pi}{12}$$

3c

$$\int_0^{\pi} \sin^2 x \cos^2 x\, dx$$

$$= \int_0^{\pi} \left(\frac{1}{2}\sin 2x\right)^2 dx$$

$$= \frac{1}{4}\int_0^{\pi} \sin^2 2x\, dx$$

$$= \frac{1}{4}\int_0^{\pi} \frac{1}{2}(1 - \cos 4x)\, dx$$

$$= \frac{1}{8}\int_0^{\pi} (1 - \cos 4x)\, dx$$

$$= \frac{1}{8}\left[x - \frac{1}{4}\sin 4x\right]_0^{\pi}$$

$$= \frac{1}{8}\big((\pi - 0) - (0 - 0)\big)$$

$$= \frac{\pi}{8}$$

4a

$$\int \sec^2 x \, dx$$

$$= \tan x + C$$

4b

$$\int \tan^2 x \, dx$$

$$= \int (\sec^2 x - 1) \, dx$$

$$= \tan x - x + C$$

4c

$$\int \sec^4 x \, dx$$

$$= \int \sec^2 x \sec^2 x \, dx$$

$$= \int (1 + \tan^2 x) \sec^2 x \, dx$$

Let $u = \tan x$

$du = \sec^2 x \, dx$

Hence

$$\int (1 + \tan^2 x) \sec^2 x \, dx$$

$$= \int (1 + u^2) \, du$$

$$= u + \frac{1}{3}u^3 + C$$

$$= \tan x + \frac{1}{3}\tan^3 x + C$$

4d

$$\int \tan^4 x \, dx$$

$$= \int \tan^2 x \tan^2 x \, dx$$

$$= \int \tan^2 x \, (\sec^2 x - 1) \, dx$$

$$= \int (\tan^2 x \sec^2 x - \tan^2 x) \, dx$$

$$= \int (\tan^2 x \sec^2 x - (\sec^2 x - 1)) \, dx$$

$$= \int \tan^2 x \sec^2 x \, dx - \int (\sec^2 x - 1) \, dx$$

Let $u = \tan x$

$du = \sec^2 x \, dx$

Hence

$$\int \tan^2 x \sec^2 x \, dx - \int (\sec^2 x - 1) \, dx$$

$$= \int u^2 \, du - \int (\sec^2 x - 1) \, dx$$

$$= \frac{1}{3} u^3 - (\tan x - x) + C$$

$$= \frac{1}{3} \tan^3 x - \tan x + x + C$$

Solutions to Exercise 4G Development questions

5a

$$\int_0^{\frac{\pi}{2}} \cos^3 x \sin x \, dx$$

Let $u = \cos x$

$du = -\sin x \; dx$

$x = \frac{\pi}{2}, u = 0$

$x = 0, u = 1$

Hence

$$\int_0^{\frac{\pi}{2}} \cos^3 x \sin x \, dx$$

$$= -\int_1^0 u^3 \, du$$

$$= \int_0^1 u^3 \, du$$

$$= \left[\frac{1}{4}u^4\right]_0^1$$

$$= \frac{1}{4}$$

5b

$$\int_0^{\frac{\pi}{6}} \cos^3 x \, dx$$

$$= \int_0^{\frac{\pi}{6}} \cos x \cos^2 x \, dx$$

$$= \int_0^{\frac{\pi}{6}} \cos x \,(1 - \sin^2 x) \, dx$$

Let $u = \sin x$

$du = \cos x \; dx$

$x=\frac{\pi}{6}, u=\frac{1}{2}$

$x=0, u=0$

Hence

$\int_0^{\frac{\pi}{6}} \cos x\,(1-\sin^2 x)\,dx$

$=\int_0^{\frac{1}{2}} (1-u^2)\,du$

$=\left[u-\frac{1}{3}u^3\right]_0^{\frac{1}{2}}$

$=\frac{1}{2}-\frac{1}{24}$

$=\frac{11}{24}$

5c

$\int_0^{\frac{\pi}{3}} \sin^3 x\cos x\,dx$

Let $u=\sin x$

$du=\cos x\ dx$

$x=\frac{\pi}{3}, u=\frac{\sqrt{3}}{2}$

$x=0, u=0$

Hence

$\int_0^{\frac{\pi}{3}} \sin^3 x\cos x\,dx$

$=\int_0^{\frac{\sqrt{3}}{2}} u^3\,du$

$=\left[\frac{1}{4}u^4\right]_0^{\frac{\sqrt{3}}{2}}$

$$= \frac{1}{4}\left(\frac{9}{16}\right)$$

$$= \frac{9}{64}$$

5d

$$\int_0^{\frac{\pi}{3}} \sin^5 x \, dx$$

$$= \int_0^{\frac{\pi}{3}} \sin x \sin^4 x \, dx$$

$$= \int_0^{\frac{\pi}{3}} \sin x \,(1 - \cos^2 x)^2 \, dx$$

Let $u = \cos x$

$du = -\sin x \; dx$

$x = \frac{\pi}{3}, u = \frac{1}{2}$

$x = 0, u = 1$

Hence

$$\int_0^{\frac{\pi}{3}} \sin x \,(1 - \cos^2 x)^2 \, dx$$

$$= -\int_1^{\frac{1}{2}} (1 - u^2)^2 \, du$$

$$= \int_{\frac{1}{2}}^{1} (1 - 2u^2 + u^4) \, du$$

$$= \left[u - \frac{2}{3}u^3 + \frac{1}{5}u^5\right]_{\frac{1}{2}}^{1}$$

$$= \left(1 - \frac{2}{3} + \frac{1}{5}\right) - \left(\frac{1}{2} - \frac{2}{3}\left(\frac{1}{8}\right) + \frac{1}{5}\left(\frac{1}{32}\right)\right)$$

$$= \frac{53}{480}$$

5e

$$\int_0^{\pi} \sin^3 x \cos^2 x \, dx$$

$$= \int_0^{\pi} \sin x \sin^2 x \cos^2 x \, dx$$

$$= \int_0^{\pi} \sin x \, (1 - \cos^2 x) \cos^2 x \, dx$$

Let $u = \cos x$

$du = -\sin x \; dx$

$x = \pi, u = -1$

$x = 0, u = 1$

Hence

$$\int_0^{\pi} \sin x \, (1 - \cos^2 x) \cos^2 x \, dx$$

$$= -\int_1^{-1} (1 - u^2) u^2 \, du$$

$$= \int_{-1}^{1} (u^2 - u^4) \, du$$

$$= \left[\frac{1}{3}u^3 - \frac{1}{5}u^5\right]_{-1}^{1}$$

$$= \frac{1}{3} - \frac{1}{5} + \frac{1}{3} - \frac{1}{5}$$

$$= \frac{4}{15}$$

5f

$$\int_0^{\frac{\pi}{4}} \sin^2 x \cos^3 x \, dx$$

$$= \int_0^{\frac{\pi}{4}} \cos x \sin^2 x \cos^2 x \, dx$$

$$= \int_0^{\frac{\pi}{4}} \cos x \sin^2 x \, (1 - \sin^2 x) \, dx$$

Let $u = \sin x$

$du = \cos x \ dx$

$x = \frac{\pi}{4}, u = \frac{1}{\sqrt{2}}$

$x = 0, u = 0$

Hence

$$\int_0^{\frac{\pi}{4}} \cos x \sin^2 x\,(1 - \sin^2 x)\,dx$$

$$= \int_0^{\frac{1}{\sqrt{2}}} u^2(1 - u^2)\,du$$

$$= \int_0^{\frac{1}{\sqrt{2}}} (u^2 - u^4)\,du$$

$$= \left[\frac{1}{3}u^3 - \frac{1}{5}u^5\right]_0^{\frac{1}{\sqrt{2}}}$$

$$= \frac{1}{3}\left(\frac{1}{\sqrt{2}}\right)^3 - \frac{1}{5}\left(\frac{1}{\sqrt{2}}\right)^5$$

$$= \frac{1}{6\sqrt{2}} - \frac{1}{20\sqrt{2}}$$

$$= \frac{10}{60\sqrt{2}} - \frac{3}{60\sqrt{2}}$$

$$= \frac{7}{60\sqrt{2}}$$

6a

$$\int \cos^4 x\,dx$$

$$= \int \cos^2 x \cos^2 x\,dx$$

$$= \int \left(\frac{1}{2}(1 + \cos 2x)\right)\left(\frac{1}{2}(1 + \cos 2x)\right)dx$$

$$= \frac{1}{4}\int (1 + \cos 2x)(1 + \cos 2x)\,dx$$

$$= \frac{1}{4}\int (1 + 2\cos 2x + \cos^2 2x)\,dx$$

$$= \frac{1}{4}\int \left(1 + 2\cos 2x + \frac{1}{2}(1 + \cos 4x)\right) dx$$

$$= \frac{1}{4}\int \left(\frac{3}{2} + 2\cos 2x + \frac{1}{2}\cos 4x\right) dx$$

$$= \frac{1}{4}\left(\frac{3}{2}x + \sin 2x + \frac{1}{8}\sin 4x\right) + C$$

$$= \frac{1}{32}(12x + 8\sin 2x + \sin 4x) + C$$

6b

$$\int \sin^4 x\,dx$$

$$= \int \sin^2 x \sin^2 x\,dx$$

$$= \int \left(\frac{1}{2}(1 - \cos 2x)\right)\left(\frac{1}{2}(1 - \cos 2x)\right) dx$$

$$= \frac{1}{4}\int (1 - \cos 2x)(1 - \cos 2x)\,dx$$

$$= \frac{1}{4}\int (1 - 2\cos 2x + \cos^2 2x)\,dx$$

$$= \frac{1}{4}\int \left(1 - 2\cos 2x + \frac{1}{2}(1 + \cos 4x)\right) dx$$

$$= \frac{1}{4}\int \left(\frac{3}{2} - 2\cos 2x + \frac{1}{2}\cos 4x\right) dx$$

$$= \frac{1}{4}\left(\frac{3}{2}x - \sin 2x + \frac{1}{8}\sin 4x\right) + C$$

$$= \frac{1}{32}(12x - 8\sin 2x + \sin 4x) + C$$

6c

$$\int \sin^4 x \cos^4 x \, dx$$

$$= \int (\sin x \cos x)^4 \, dx$$

$$= \int \left(\frac{1}{2}\sin 2x\right)^4 \, dx$$

$$= \frac{1}{16}\int \sin^4 2x \, dx$$

$$= \frac{1}{16}\int \sin^2 2x \sin^2 2x \, dx$$

$$= \frac{1}{16}\int \left(\frac{1}{2}(1 - \cos 4x)\right)\left(\frac{1}{2}(1 - \cos 4x)\right) dx$$

$$= \frac{1}{64}\int (1 - \cos 4x)(1 - \cos 4x) \, dx$$

$$= \frac{1}{64}\int (1 - 2\cos 4x + \cos^2 4x) \, dx$$

$$= \frac{1}{64}\int \left(1 - 2\cos 4x + \frac{1}{2}(1 + \cos 8x)\right) dx$$

$$= \frac{1}{64}\int \left(\frac{3}{2} - 2\cos 4x + \frac{1}{2}\cos 8x\right) dx$$

$$= \frac{1}{64}\left(\frac{3}{2}x - \frac{1}{2}\sin 4x + \frac{1}{16}\sin 8x\right) + C$$

$$= \frac{1}{1024}(24x - 8\sin 4x + \sin 8x) + C$$

7a

$$\int_0^{\frac{\pi}{3}} \sec^2 x \tan^2 x \, dx$$

Let $u = \tan x$

$du = \sec^2 x \ \ dx$

$x = \frac{\pi}{3}, u = \sqrt{3}$

$x = 0, u = 0$

Hence

$$\int_0^{\frac{\pi}{3}} \sec^2 x \tan^2 x \, dx$$

$$= \int_0^{\sqrt{3}} u^2 \, du$$

$$= \left[\frac{1}{3}u^3\right]_0^{\sqrt{3}}$$

$$= \sqrt{3}$$

7b

$$\int_{-\frac{\pi}{6}}^{\frac{\pi}{3}} \sec^2 x \tan^3 x \, dx$$

Let $u = \tan x$

$du = \sec^2 x \ dx$

$x = \frac{\pi}{3}, u = \sqrt{3}$

$x = -\frac{\pi}{6}, u = -\frac{1}{\sqrt{3}}$

Hence

$$\int_{-\frac{\pi}{6}}^{\frac{\pi}{3}} \sec^2 x \tan^3 x \, dx$$

$$= \int_{-\frac{1}{\sqrt{3}}}^{\sqrt{3}} u^3 \, du$$

$$= \left[\frac{1}{4}u^4\right]_{-\frac{1}{\sqrt{3}}}^{\sqrt{3}}$$

$$= \frac{1}{4}\left(9 - \frac{1}{9}\right)$$

$$= \frac{1}{4}\left(\frac{80}{9}\right)$$

$= \frac{20}{9}$

$= 2\frac{2}{9}$

7c

$\int_0^{\frac{\pi}{4}} \sec^4 x \tan x \, dx$

$= \int_0^{\frac{\pi}{4}} \sec^3 x \sec x \tan x \, dx$

Let $u = \sec x$

$du = \sec x \tan x \; dx$

$x = \frac{\pi}{4}, u = \sqrt{2}$

$x = 0, u = 1$

Hence

$\int_0^{\frac{\pi}{4}} \sec^3 x \sec x \tan x \, dx$

$= \int_1^{\sqrt{2}} u^3 \, du$

$= \left[\frac{1}{4}u^4\right]_1^{\sqrt{2}}$

$= \frac{1}{4}(4) - \frac{1}{4}$

$= \frac{3}{4}$

7d

$\int_0^{\frac{\pi}{4}} \tan^5 x \, dx$

$= \int_0^{\frac{\pi}{4}} (\sec^2 x - 1) \tan^3 x \, dx$

$$= \int_0^{\frac{\pi}{4}} (\sec^2 x \tan^3 x - \tan^3 x)\, dx$$

$$= \int_0^{\frac{\pi}{4}} (\sec x \tan x \sec x\,(\sec^2 x - 1) - (\sec^2 x - 1) \tan x)\, dx$$

$$= \int_0^{\frac{\pi}{4}} \sec x \tan x \sec x\,(\sec^2 x - 1)\, dx - \int_0^{\frac{\pi}{4}} (\sec^2 x - 1) \tan x\, dx$$

$$= \int_0^{\frac{\pi}{4}} \sec x \tan x \sec x\,(\sec^2 x - 1)\, dx - \int_0^{\frac{\pi}{4}} \sec x \sec x \tan x\, dx + \int_0^{\frac{\pi}{4}} \tan x\, dx$$

Let $u = \sec x$

$du = \sec x \tan x \;\; dx$

$x = \frac{\pi}{4}, u = \sqrt{2}$

$x = 0, u = 1$

Hence

$$\int_0^{\frac{\pi}{4}} \sec x \tan x \sec x\,(\sec^2 x - 1)\, dx - \int_0^{\frac{\pi}{4}} \sec x \sec x \tan x\, dx + \int_0^{\frac{\pi}{4}} \tan x\, dx$$

$$= \int_1^{\sqrt{2}} u(u^2 - 1)\, du - \int_1^{\sqrt{2}} u\, du + \int_0^{\frac{\pi}{4}} \tan x\, dx$$

$$= \int_1^{\sqrt{2}} (u^3 - u)\, du - \int_1^{\sqrt{2}} u\, du + \int_0^{\frac{\pi}{4}} \tan x\, dx$$

$$= \int_1^{\sqrt{2}} (u^3 - 2u)\, du + \int_0^{\frac{\pi}{4}} \tan x\, dx$$

$$= \left[\frac{1}{4}u^4 - u^2\right]_1^{\sqrt{2}} + [-\ln|\cos x|]_0^{\frac{\pi}{4}}$$

$$= \left(\frac{1}{4}(4) - 2 - \frac{1}{4} + 1\right) - \ln\left(\frac{1}{\sqrt{2}}\right)$$

$$= -\frac{1}{4} - \frac{1}{2}\ln\left(\frac{1}{2}\right)$$

$$= \frac{1}{2}\ln 2 - \frac{1}{4}$$

$$= \frac{1}{4}(2\ln 2 - 1)$$

8a

$$\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin x}\,dx$$

Let $t = \tan\left(\frac{x}{2}\right)$

$$dx = \frac{2}{1+t^2}\,dt$$

$$x = \frac{\pi}{2}, t = 1$$

$$x = 0, t = 0$$

$$\sin x = \frac{2t}{1+t^2}$$

Hence

$$\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin x}\,dx$$

$$= \int_0^1 \frac{1}{1+\frac{2t}{1+t^2}} \times \frac{2}{1+t^2}\,dt$$

$$= 2\int_0^1 \frac{1}{1+t^2+2t}\,dt$$

$$= 2\int_0^1 \frac{1}{(t+1)^2}\,dt$$

$$= 2\left[-\frac{1}{t+1}\right]_0^1$$

$$= 2\left(-\frac{1}{2}+1\right)$$

$$= 1$$

8b

$$\int_0^{\frac{\pi}{2}} \frac{1}{4+5\cos x}\,dx$$

Let $t = \tan\left(\frac{x}{2}\right)$

$$dx = \frac{2}{1+t^2}\,dt$$

$x = \frac{\pi}{2}, t = 1$

$x = 0, t = 0$

$$\cos x = \frac{1-t^2}{1+t^2}$$

Hence

$$\int_0^{\frac{\pi}{2}} \frac{1}{4+5\cos x}\,dx$$

$$= \int_0^1 \frac{1}{4+5\left(\frac{1-t^2}{1+t^2}\right)} \frac{2}{1+t^2}\,dt$$

$$= 2\int_0^1 \frac{1}{4(1+t^2)+5(1-t^2)}\,dt$$

$$= 2\int_0^1 \frac{1}{4+4t^2+5-5t^2}\,dt$$

$$= 2\int_0^1 \frac{1}{9-t^2}\,dt$$

$$= 2\int_0^1 \frac{1}{(3-t)(3+t)}\,dt$$

$$\frac{1}{(3-t)(3+t)} = \frac{\frac{1}{6}}{3-t} + \frac{\frac{1}{6}}{3+t} \qquad \text{(using cover} - \text{up method)}$$

$$2\int_0^1 \frac{1}{(3-t)(3+t)}\,dt$$

$$= 2\int_0^1 \left(\frac{\frac{1}{6}}{3-t} + \frac{\frac{1}{6}}{3+t}\right)dt$$

$$= \frac{1}{3}[-\ln|3-t| + \ln|3+t|]_0^1$$

$$= \frac{1}{3}\left[\ln\left|\frac{3+t}{3-t}\right|\right]_0^1$$

$$= \frac{1}{3}\ln 2$$

8c

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{5+3\sin x}\,dx$$

Let $t = \tan\left(\frac{x}{2}\right)$

$$dx = \frac{2}{1+t^2}\,dt$$

$$x = \frac{\pi}{2}, t = 1$$

$$x = -\frac{\pi}{2}, t = -1$$

$$\sin x = \frac{2t}{1+t^2}$$

Hence

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{5+3\sin x}\,dx$$

$$= \int_{-1}^{1} \frac{1}{5+3\frac{2t}{1+t^2}} \times \frac{2}{1+t^2}\,dt$$

$$= 2\int_{-1}^{1} \frac{1}{5(1+t^2)+6t}\,dt$$

$$= 2\int_{-1}^{1} \frac{1}{5+5t^2+6t}\,dt$$

$$= \frac{2}{5}\int_{-1}^{1} \frac{1}{t^2+\frac{6}{5}t+1}\,dt$$

$$= \frac{2}{5}\int_{-1}^{1} \frac{1}{\left(t+\frac{3}{5}\right)^2+\frac{16}{25}}\,dt$$

$$= \frac{1}{2}\int_{-1}^{1} \frac{\frac{4}{5}}{\left(t+\frac{3}{5}\right)^2+\left(\frac{4}{5}\right)^2}\,dt$$

$$= \frac{1}{2}\left[\tan^{-1}\left(\frac{5\left(t+\frac{3}{5}\right)}{4}\right)\right]_{-1}^{1}$$

$$= \frac{1}{2}\left[\tan^{-1}\left(\frac{5t+3}{4}\right)\right]_{-1}^{1}$$

$$= \frac{1}{2}\left(\tan^{-1}(2) - \tan^{-1}\left(-\frac{1}{2}\right)\right)$$

$$= \frac{1}{2}\left(\tan^{-1}(2) + \tan^{-1}\left(\frac{1}{2}\right)\right)$$

This can be further simplified by utilising the tangent addition formula to $\frac{\pi}{4}$, however the above expression is acceptable.

9a

$$\int_0^1 \sqrt{1-x^2}\,dx$$

Let $x = \sin\theta$

$dx = \cos\theta\, d\theta$

$x = 1, \theta = \frac{\pi}{2}$

$x = 0, \theta = 0$

Hence

$$\int_0^1 \sqrt{1-x^2}\,dx$$

$$= \int_0^{\frac{\pi}{2}} \sqrt{1-\sin^2\theta}\cos\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \cos\theta\cos\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \cos^2 \theta \, d\theta$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{2}} (1 + \cos 2\theta) \, d\theta$$

$$= \frac{1}{2} \left[\theta + \frac{1}{2} \sin 2\theta\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left(\frac{\pi}{2}\right)$$

$$= \frac{\pi}{4}$$

9b

$$\int_0^1 x^3 \sqrt{1 + x^2} \, dx$$

Let $x = \tan \theta$

$dx = \sec^2 \theta \, d\theta$

$x = 1, \theta = \frac{\pi}{4}$

$x = 0, \theta = 0$

Hence

$$\int_0^1 x^3 \sqrt{1 + x^2} \, dx$$

$$= \int_0^{\frac{\pi}{4}} \tan^3 \theta \sqrt{1 + \tan^2 \theta} \sec^2 \theta \, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \tan^3 \theta \sec^3 \theta \, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \sec \theta \tan \theta \, (\sec^2 \theta - 1) \sec^2 \theta \, d\theta$$

Let $u = \sec \theta$

$du = \sec \theta \tan \theta \, d\theta$

$\theta = \frac{\pi}{4}, u = \sqrt{2}$

$\theta = 0, u = 1$

Hence

$$\int_0^{\frac{\pi}{4}} \sec\theta \tan\theta \,(\sec^2\theta - 1)\sec^2\theta \, d\theta$$

$$= \int_1^{\sqrt{2}} (u^2 - 1)u^2 \, du$$

$$= \int_1^{\sqrt{2}} (u^4 - u^2) \, du$$

$$= \left[\frac{1}{5}u^5 - \frac{1}{3}u^3\right]_1^{\sqrt{2}}$$

$$= \frac{4\sqrt{2}}{5} - \frac{2\sqrt{2}}{3} - \frac{1}{5} + \frac{1}{3}$$

$$= \frac{4\sqrt{2} - 1}{5} + \frac{1 - 2\sqrt{2}}{3}$$

$$= \frac{12\sqrt{2} - 3 + 5 - 10\sqrt{2}}{15}$$

$$= \frac{2\sqrt{2} + 2}{15}$$

$$= \frac{2}{15}(1 + \sqrt{2})$$

9c

$$\int_0^1 x^2\sqrt{1 - x^2} \, dx$$

Let $x = \sin\theta$

$dx = \cos\theta \, d\theta$

$x = 1, \theta = \frac{\pi}{2}$

$x = 0, \theta = 0$

Hence

$$\int_0^1 x^2\sqrt{1 - x^2} \, dx$$

$$= \int_0^{\frac{\pi}{2}} \sin^2\theta \sqrt{1-\sin^2\theta}\cos\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin^2\theta \cos^2\theta\, d\theta$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{2}} \sin^2 2\theta\, d\theta$$

$$= \frac{1}{8}\int_0^{\frac{\pi}{2}} (1-\cos 4\theta)\, d\theta$$

$$= \frac{1}{8}\left[\theta - \frac{1}{4}\sin 4\theta\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{8}\left(\frac{\pi}{2}\right)$$

$$= \frac{\pi}{16}$$

10a

$$\int \sin x \cos x\, dx$$

Let $u = \sin x$

$du = \cos x\, dx$

Hence

$$\int \sin x \cos x\, dx$$

$$= \int u\, du$$

$$= \frac{1}{2}u^2 + C$$

$$= \frac{1}{2}\sin^2 x + C$$

10b

$$\int \sin x \cos x \, dx$$

$$= \int \frac{1}{2} \sin 2x \, dx$$

$$= -\frac{1}{4} \cos 2x + C$$

10c

$$\frac{1}{2} \sin^2 x + C$$

$$= \frac{1}{2}\left(\frac{1}{2}(1 - \cos 2x)\right) + C$$

$$= \frac{1}{4}(1 - \cos 2x) + C$$

$$= \frac{1}{4} - \frac{1}{4} \cos 2x + C$$

$\frac{1}{4} + C$ is still a constant, C

Hence

$$\frac{1}{4} - \frac{1}{4} \cos 2x + C$$

$$= -\frac{1}{4} \cos 2x + C$$

11a

$$\int_0^{\frac{\pi}{4}} (\tan^3 x + \tan x) \, dx$$

$$= \int_0^{\frac{\pi}{4}} \tan x \, (\tan^2 x + 1) \, dx$$

$$= \int_0^{\frac{\pi}{4}} \tan x \sec^2 x \, dx$$

Let $u = \tan x$

$du = \sec^2 x\, dx$

$x = \frac{\pi}{4}, u = 1$

$x = 0, u = 0$

Hence

$\int_0^{\frac{\pi}{4}} \tan x \sec^2 x\, dx$

$= \int_0^1 u\, du$

$= \left[\frac{1}{2}u^2\right]_0^1$

$= \frac{1}{2}$

11b

$\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} (\cos x - \cos^3 x)\, dx$

$= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \cos x\,(1 - \cos^2 x)\, dx$

$= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \cos x \sin^2 x\, dx$

Let $u = \sin x$

$du = \cos x\, dx$

$x = \frac{\pi}{3}, u = \frac{\sqrt{3}}{2}$

$x = -\frac{\pi}{3}, u = -\frac{\sqrt{3}}{2}$

Hence

$\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \cos x \sin^2 x\, dx$

$$= \int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} u^2 \, du$$

$$= \left[\frac{1}{3}u^3\right]_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}}$$

$$= \frac{1}{3}\left(\frac{3\sqrt{3}}{8} + \frac{3\sqrt{3}}{8}\right)$$

$$= \frac{2\sqrt{3}}{8}$$

$$= \frac{\sqrt{3}}{4}$$

12a

$$\int_0^{\frac{\pi}{3}} \sin^3 x \sec^2 x \, dx$$

$$= \int_0^{\frac{\pi}{3}} \frac{\sin^3 x}{\cos^2 x} dx$$

$$= \int_0^{\frac{\pi}{3}} \frac{\sin x \sin^2 x}{\cos^2 x} dx$$

$$= \int_0^{\frac{\pi}{3}} \frac{\sin x \,(1 - \cos^2 x)}{\cos^2 x} dx$$

Let $u = \cos x$

$du = -\sin x \, dx$

$x = \frac{\pi}{3}, u = \frac{1}{2}$

$x = 0, u = 1$

Hence

$$\int_0^{\frac{\pi}{3}} \frac{\sin x \,(1 - \cos^2 x)}{\cos^2 x} dx$$

$$= -\int_1^{\frac{1}{2}} \frac{1 - u^2}{u^2} du$$

$$= \int_{\frac{1}{2}}^{1} \left(\frac{1}{u^2} - 1\right) du$$

$$= \left[-\frac{1}{u} - u\right]_{\frac{1}{2}}^{1}$$

$$= -2 - \left(-2 - \frac{1}{2}\right)$$

$$= -2 + \frac{5}{2}$$

$$= \frac{1}{2}$$

12b

$$\int_0^{\frac{\pi}{3}} \sin^3 x \sec^4 x \, dx$$

$$= \int_0^{\frac{\pi}{3}} \frac{\sin^3 x}{\cos^4 x} dx$$

$$= \int_0^{\frac{\pi}{3}} \frac{\sin x \sin^2 x}{\cos^4 x} dx$$

$$= \int_0^{\frac{\pi}{3}} \frac{\sin x\,(1 - \cos^2 x)}{\cos^4 x} dx$$

Let $u = \cos x$

$du = -\sin x \, dx$

$x = \frac{\pi}{3}, u = \frac{1}{2}$

$x = 0, u = 1$

Hence

$$\int_0^{\frac{\pi}{3}} \frac{\sin x\,(1 - \cos^2 x)}{\cos^4 x} dx$$

$$= -\int_1^{\frac{1}{2}} \frac{1 - u^2}{u^4} du$$

$$= \int_{\frac{1}{2}}^{1} \left(\frac{1}{u^4} - \frac{1}{u^2}\right) du$$

$$= \left[-\frac{1}{3u^3} + \frac{1}{u}\right]_{\frac{1}{2}}^{1}$$

$$= -\frac{1}{3} + 1 + \frac{1}{3}(8) - 2$$

$$= \frac{7}{3} - 1$$

$$= \frac{4}{3}$$

13a

$$\int \sin 3x \cos x \, dx$$

$$= \frac{1}{2}\int (\sin(3x - x) + \sin(3x + x))\, dx$$

$$= \frac{1}{2}\int (\sin 2x + \sin 4x)\, dx$$

$$= \frac{1}{2}\left(-\frac{1}{2}\cos 2x - \frac{1}{4}\cos 4x\right) + C$$

$$= -\frac{1}{8}\cos 4x - \frac{1}{4}\cos 2x + C$$

13b

$$\int \cos 3x \sin x \, dx$$

$$= \frac{1}{2}\int (\sin(x - 3x) + \sin(x + 3x))\, dx$$

$$= \frac{1}{2}\int (-\sin 2x + \sin 4x)\, dx$$

$$= \frac{1}{2}\left(\frac{1}{2}\cos 2x - \frac{1}{4}\cos 4x\right) + C$$

$$= -\frac{1}{8}\cos 4x + \frac{1}{4}\cos 2x + C$$

13c

$$\int \cos 6x \cos 2x \, dx$$

$$= \frac{1}{2}\int (\cos(6x - 2x) + \cos(6x + 2x)) \, dx$$

$$= \frac{1}{2}\int (\cos 4x + \cos 8x) \, dx$$

$$= \frac{1}{2}\left(\frac{1}{4}\sin 4x + \frac{1}{8}\sin 8x\right) + C$$

$$= \frac{1}{16}\sin 8x + \frac{1}{8}\sin 4x + C$$

14a

$$\int_0^{\frac{\pi}{4}} \sin 3x \sin x \, dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}} (\cos(3x - x) - \cos(3x + x)) \, dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}} (\cos 2x - \cos 4x) \, dx$$

$$= \frac{1}{2}\left[\frac{1}{2}\sin 2x - \frac{1}{4}\sin 4x\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{2}\left(\frac{1}{2} - 0 - 0 + 0\right)$$

$$= \frac{1}{4}$$

14b

$$\int_0^{\frac{\pi}{4}} \cos 4x \cos 2x \, dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}} (\cos(4x - 2x) + \cos(4x + 2x)) \, dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}}(\cos 2x + \cos 6x)\,dx$$

$$= \frac{1}{2}\left[\frac{1}{2}\sin 2x + \frac{1}{6}\sin 6x\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{2}\left(\frac{1}{2} - \frac{1}{6} - 0 - 0\right)$$

$$= \frac{1}{2}\left(\frac{1}{3}\right)$$

$$= \frac{1}{6}$$

14c

$$\int_0^{\frac{\pi}{3}}\sin 4x \cos 2x\,dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{3}}(\sin(4x - 2x) + \sin(4x + 2x))\,dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{3}}(\sin 2x + \sin 6x)\,dx$$

$$= \frac{1}{2}\left[-\frac{1}{2}\cos 2x - \frac{1}{6}\cos 6x\right]_0^{\frac{\pi}{3}}$$

$$= \frac{1}{2}\left(\frac{1}{4} - \frac{1}{6} + \frac{1}{2} + \frac{1}{6}\right)$$

$$= \frac{1}{2}\left(\frac{3}{4}\right)$$

$$= \frac{3}{8}$$

15a

$$\int \frac{1}{1+\cos x}\,dx$$

Let $t = \tan\left(\frac{x}{2}\right)$

$$dx = \frac{2}{1+t^2}\,dt$$

$$\cos x = \frac{1-t^2}{1+t^2}$$

Hence

$$\int \frac{1}{1+\cos x}\,dx$$

$$= \int \frac{1}{1+\frac{1-t^2}{1+t^2}} \times \frac{2}{1+t^2}\,dt$$

$$= \int \frac{2}{1+t^2+1-t^2}\,dt$$

$$= \int \frac{2}{2}\,dt$$

$$= \int 1\,dt$$

$$= t + C$$

$$= \tan\frac{x}{2} + C$$

15b

$$\int \frac{1}{1+\sin x-\cos x}\,dx$$

Let $t = \tan\left(\frac{x}{2}\right)$

$$dx = \frac{2}{1+t^2}\,dt$$

$$\sin x = \frac{2t}{1+t^2}$$

$$\cos x = \frac{1-t^2}{1+t^2}$$

Hence

$$\int \frac{1}{1+\sin x-\cos x}\,dx$$

$$= \int \frac{1}{1+\frac{2t}{1+t^2}-\frac{1-t^2}{1+t^2}} \times \frac{2}{1+t^2}\,dt$$

$$= \int \frac{2}{1+t^2+2t-1+t^2}\,dt$$

$$= \int \frac{2}{2t^2+2t}\,dt$$

$$= \int \frac{1}{t(t+1)}\,dt$$

$$= \int \left(\frac{1}{t}-\frac{1}{t+1}\right)dt \quad \text{(using cover − up method)}$$

$$= \ln|t| - \ln|t+1| + C$$

$$= \ln\left|\frac{t}{t+1}\right| + C$$

$$= \ln\left|\frac{\tan\left(\frac{x}{2}\right)}{\tan\left(\frac{x}{2}\right)+1}\right| + C$$

15c

$$\int \frac{1}{3\sin x+4\cos x}\,dx$$

Let $t = \tan\left(\frac{x}{2}\right)$

$$dx = \frac{2}{1+t^2}\,dt$$

$$\sin x = \frac{2t}{1+t^2}$$

$$\cos x = \frac{1-t^2}{1+t^2}$$

Hence

$$\int \frac{1}{3\sin x+4\cos x}\,dx$$

$$= \int \frac{1}{3\left(\frac{2t}{1+t^2}\right) + 4\left(\frac{1-t^2}{1+t^2}\right)} \times \frac{2}{1+t^2}\, dt$$

$$= \int \frac{2}{6t + 4 - 4t^2}\, dt$$

$$= \int \frac{1}{3t + 2 - 2t^2}\, dt$$

$$= \frac{1}{2}\int \frac{1}{\frac{3}{2}t + 1 - t^2}\, dt$$

$$= \frac{1}{2}\int \frac{1}{\frac{25}{16} - \left(t - \frac{3}{4}\right)^2}\, dt$$

$$= \frac{1}{2}\int \frac{1}{\left(\frac{5}{4}\right)^2 - \left(t - \frac{3}{4}\right)^2}\, dt$$

$$= \frac{1}{2}\int \frac{1}{\left(\frac{5}{4} - t + \frac{3}{4}\right)\left(\frac{5}{4} + t - \frac{3}{4}\right)}\, dt$$

$$= \frac{1}{2}\int \frac{1}{(2-t)\left(\frac{1}{2} + t\right)}\, dt$$

$$= \frac{1}{2}\int \left(\frac{\frac{2}{5}}{\frac{1}{2} + t} + \frac{\frac{2}{5}}{2-t}\right) dt \qquad \text{(using cover} - \text{up method)}$$

$$= \frac{1}{2}\int \left(\frac{\frac{4}{5}}{1 + 2t} + \frac{\frac{2}{5}}{2-t}\right) dt$$

$$= \frac{1}{5}\int \left(\frac{2}{1+2t} + \frac{1}{2-t}\right) dt$$

$$= \frac{1}{5}(\ln|1 + 2t| - \ln|2 - t|) + C$$

$$= \frac{1}{5}\ln\left|\frac{1+2t}{2-t}\right| + C$$

$$= \frac{1}{5}\ln\left|\frac{1 + 2\tan\left(\frac{x}{2}\right)}{2 - \tan\left(\frac{x}{2}\right)}\right| + C$$

16a

$$\int \sec x \, dx$$

$$= \int \frac{1}{\cos x} dx$$

Let $t = \tan\left(\frac{x}{2}\right)$

$$dx = \frac{2}{1+t^2} dt$$

$$\cos x = \frac{1-t^2}{1+t^2}$$

Hence

$$\int \frac{1}{\cos x} dx$$

$$= \int \frac{1+t^2}{1-t^2} \times \frac{2}{1+t^2} dt$$

$$= \int \frac{2}{1-t^2} dt$$

$$= \int \frac{2}{(1-t)(1+t)} dt$$

$$= \int \left(\frac{1}{1-t} + \frac{1}{1+t}\right) dt \quad \text{(using cover} - \text{up method)}$$

$$= -\ln|1-t| + \ln|1+t| + C$$

$$= \ln\left|\frac{1+t}{1-t}\right| + C$$

$$= \ln\left|\frac{1+\tan\left(\frac{x}{2}\right)}{1-\tan\left(\frac{x}{2}\right)}\right| + C$$

16b

$$\ln\left|\frac{1+\tan\left(\frac{x}{2}\right)}{1-\tan\left(\frac{x}{2}\right)}\right|+C$$

$$=\ln\left|\frac{1+\tan\left(\frac{x}{2}\right)}{1-\tan\left(\frac{x}{2}\right)}\times\frac{1+\tan\left(\frac{x}{2}\right)}{1+\tan\left(\frac{x}{2}\right)}\right|+C$$

$$=\ln\left|\frac{1+2\tan\left(\frac{x}{2}\right)+\tan^2\left(\frac{x}{2}\right)}{1-\tan^2\left(\frac{x}{2}\right)}\right|+C$$

$$=\ln\left|\frac{1+\tan^2\left(\frac{x}{2}\right)}{1-\tan^2\left(\frac{x}{2}\right)}+\frac{2\tan\left(\frac{x}{2}\right)}{1-\tan^2\left(\frac{x}{2}\right)}\right|+C$$

$$\tan x=\frac{2\tan\left(\frac{x}{2}\right)}{1-\tan^2\left(\frac{x}{2}\right)}$$

$$\cos x=\frac{1-\tan^2\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)}$$

Hence

$$\ln\left|\frac{1+\tan^2\left(\frac{x}{2}\right)}{1-\tan^2\left(\frac{x}{2}\right)}+\frac{2\tan\left(\frac{x}{2}\right)}{1-\tan^2\left(\frac{x}{2}\right)}\right|+C$$

$$=\ln\left|\frac{1}{\cos x}+\tan x\right|+C$$

$$=\ln|\sec x+\tan x|+C$$

Solutions to Exercise 4G Enrichment questions

17 $(\operatorname{cis}\theta)^3 = \operatorname{cis} 3\theta$ (by de Moivre)

$\cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta = \cos 3\theta + i\sin 3\theta$

Equate real parts to get:

$\cos^3\theta - 3\cos\theta\sin^2\theta = \cos 3\theta$

$\cos^3\theta = 3\cos\theta\sin^2\theta + \cos 3\theta$

Hence,

$\int \cos^3\theta\, d\theta$

$= \int (3\cos\theta\sin^2\theta + \cos 3\theta)\, d\theta$

$= \sin^3\theta + \frac{1}{3}\sin 3\theta + C$

18 $I = \int \sec^3 x\, dx$

$= \int \sec^2 x \cdot \sec x\, dx$

$= \tan x\sec x - \int \tan x \cdot \sec x\tan x\, dx$ (by parts with $u = \sec x$, $v' = \sec^2 x$.)

$= \tan x\sec x - \int \tan^2 x \cdot \sec x\, dx$

$= \tan x\sec x - \int \sec^3 x\, dx + \int \sec x\, dx$ (by Pythagoras)

$= \tan x\sec x + \ln|\sec x + \tan x| - I$

Hence,

$2I = \tan x\sec x + \ln|\sec x + \tan x| + 2C$, or

$I = \frac{1}{2}\tan x\sec x + \frac{1}{2}\ln|\sec x + \tan x| + C$, for some constant C.

Thus,

$\int_0^{\frac{\pi}{4}} \sec^3 x\, dx$

$= \left[\frac{1}{2}\tan x\sec x + \frac{1}{2}\ln|\sec x + \tan x|\right]_0^{\frac{\pi}{4}}$

$= \left(\frac{1}{2}\cdot 1 \cdot \sqrt{2} + \frac{1}{2}\ln\left(\sqrt{2}+1\right)\right) - \left(0 + \frac{1}{2}\ln(1+0)\right)$

$= \frac{1}{\sqrt{2}} + \frac{1}{2}\ln\left(1+\sqrt{2}\right)$

Solutions to Exercise 4H Foundation questions

1a

$$I_n = \int \tan^n x\,dx$$

$$= \int \tan^{n-2} x \tan^2 x\,dx$$

$$= \int \tan^{n-2} x\,(\sec^2 x - 1)dx$$

$$= \int \tan^{n-2} x \sec^2 x\,dx - \int \tan^{n-2} x\,dx$$

$$= \int \tan^{n-2} x \sec^2 x\,dx - I_{n-2}$$

Let $u = \tan x$

$du = \sec^2 x\,dx$

Hence

$$I_n = \int u^{n-2} du - I_{n-2}$$

$$= \frac{u^{n-1}}{n-1} - I_{n-2}$$

$$= \frac{\tan^{n-1} x}{n-1} - I_{n-2}$$

1b

$$I_n = \frac{\tan^{n-1} x}{n-1} - I_{n-2}$$

$$I_0 = \int \tan^0 x\,dx$$

$$= \int 1\,dx$$

$$= x$$

$$I_2 = \frac{\tan^{2-1} x}{2-1} - I_0$$

$$= \tan x - x$$

$$I_4 = \frac{\tan^{4-1} x}{4-1} - I_2$$

$$= \frac{1}{3}\tan^3 x - \tan x + x$$

$$I_6 = \frac{\tan^{6-1} x}{6-1} - I_4$$

$$= \frac{1}{5}\tan^5 x - \frac{1}{3}\tan^3 x + \tan x - x + C$$

2a

$$I_n = \int x^n e^x \, dx$$

$$\int uv' \, dx = uv - \int u'v \, dx$$

Therefore

$u = x^n, v' = e^x$

$u' = nx^{n-1}, v = e^x$

Hence

$$I_n = x^n e^x - n\int x^{n-1} e^x \, dx$$

$$= x^n e^x - nI_{n-1}$$

2b $I_n = x^n e^x - nI_{n-1}$

$$I_0 = \int x^0 e^x \, dx$$

$$= \int e^x \, dx$$

$$= e^x$$

$I_1 = xe^x - I_0$

$= xe^x - e^x$

$I_2 = x^2 e^x - 2I_1$

$= x^2 e^x - 2xe^x + 2e^x$

$$I_3 = x^3e^x - 3I_2$$

$$= x^3e^x - 3x^2e^x + 6xe^x - 6e^x + C$$

$$= (x^3 - 3x^2 + 6x - 6)e^x + C$$

3a

$$I_n = \int_1^e x(\ln x)^n\,dx$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Therefore

$$u = (\ln x)^n, v' = x$$

$$u' = \frac{n(\ln x)^{n-1}}{x}, v = \frac{1}{2}x^2$$

Hence

$$I_n = \left[\frac{1}{2}x^2(\ln x)^n\right]_1^e - \frac{1}{2}n\int_1^e x(\ln x)^{n-1}\,dx$$

$$= \left(\frac{1}{2}e^2 - 0\right) - \frac{1}{2}nI_{n-1}$$

$$= \frac{1}{2}e^2 - \frac{1}{2}nI_{n-1}$$

3b

$$I_n = \int_1^e x(\ln x)^n\,dx$$

$$I_0 = \int_1^e x(\ln x)^0\,dx$$

$$= \int_1^e x\,dx$$

$$= \left[\frac{1}{2}x^2\right]_1^e$$

$$= \frac{1}{2}e^2 - \frac{1}{2}$$

$$I_n = \frac{1}{2}e^2 - \frac{1}{2}nI_{n-1}$$

$$I_1 = \frac{1}{2}e^2 - \frac{1}{2}I_0$$

$$= \frac{1}{2}e^2 - \frac{1}{4}e^2 + \frac{1}{4}$$

$$= \frac{1}{4}e^2 + \frac{1}{4}$$

$$I_2 = \frac{1}{2}e^2 - I_1$$

$$= \frac{1}{2}e^2 - \frac{1}{4}e^2 - \frac{1}{4}$$

$$= \frac{1}{4}e^2 - \frac{1}{4}$$

$$I_3 = \frac{1}{2}e^2 - \frac{3}{2}I_2$$

$$= \frac{1}{2}e^2 - \frac{3}{2}\left(\frac{1}{4}e^2 - \frac{1}{4}\right)$$

$$= \frac{4}{8}e^2 - \frac{3}{8}e^2 + \frac{3}{8}$$

$$= \frac{1}{8}e^2 + \frac{3}{8}$$

$$I_4 = \frac{1}{2}e^2 - 2I_3$$

$$= \frac{1}{2}e^2 - 2\left(\frac{1}{8}e^2 + \frac{3}{8}\right)$$

$$= \frac{1}{2}e^2 - \frac{1}{4}e^2 - \frac{3}{4}$$

$$= \frac{1}{4}e^2 - \frac{3}{4}$$

$$= \frac{1}{4}(e^2 - 3)$$

4a

$$u_n = \int_0^{\frac{\pi}{2}} \cos^n x \, dx$$

$$= \int_0^{\frac{\pi}{2}} \cos^{n-1} x \cos x \, dx$$

$$\int uv' \, dx = uv - \int u'v \, dx$$

Therefore

$u = \cos^{n-1} x \, , v' = \cos x$

$u' = -(n-1)\cos^{n-2} x \sin x \, , v = \sin x$

Hence

$$u_n = [\sin x \cos^{n-1} x]_0^{\frac{\pi}{2}} + (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2} x \sin^2 x \, dx$$

$$= (0-0) + (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2} x \sin^2 x \, dx$$

$$= (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2} x \sin^2 x \, dx$$

4b

$$u_n = (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2} x \sin^2 x \, dx$$

$$= (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2} x \, (1-\cos^2 x) \, dx$$

$$= (n-1)\left(\int_0^{\frac{\pi}{2}} \cos^{n-2} x \, dx - \int_0^{\frac{\pi}{2}} \cos^n x \, dx\right)$$

$$= (n-1)(u_{n-2} - u_n)$$

4c

$$u_n = (n-1)(u_{n-2} - u_n)$$

$$= (n-1)u_{n-2} - (n-1)u_n$$

$$u_n + (n-1)u_n = (n-1)u_{n-2}$$

$$nu_n = (n-1)u_{n-2}$$

$$u_n = \frac{n-1}{n}u_{n-2}$$

$$u_1 = \int_0^{\frac{\pi}{2}} \cos x \, dx$$

$$= [\sin x]_0^{\frac{\pi}{2}}$$

$$= 1$$

$$u_3 = \frac{3-1}{3}u_0$$

$$= \frac{2}{3}$$

$$u_5 = \frac{5-1}{5}u_3$$

$$= \frac{8}{15}$$

Solutions to Exercise 4H Development questions

5a

$$T_n = \int_0^{\frac{\pi}{4}} \sec^n x \, dx$$

$$T_n = \int_0^{\frac{\pi}{4}} \sec^{n-2} x \times \sec^2 x \, dx$$

$$\int uv' \, dx = uv - \int u'v \, dx$$

$$u = \sec^{n-2} x\,, v' = \sec^2 x$$

$$u' = (n-2)\sec^{n-3} x \sec x \tan x\,, v = \tan x$$

Hence

$$T_n = [\sec^{n-2} x \tan x]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} (n-2)\sec^{n-3} x \sec x \tan x \times \tan x \, dx$$

$$T_n = \sqrt{2}^{n-2} - (n-2)\int_0^{\frac{\pi}{4}} \sec^{n-2} x \tan^2 x \, dx$$

$$T_n = \sqrt{2}^{n-2} - (n-2)\int_0^{\frac{\pi}{4}} \sec^{n-2} x\,(\sec^2 x - 1) \, dx$$

$$T_n = \sqrt{2}^{n-2} - (n-2)\int_0^{\frac{\pi}{4}} (\sec^n x - \sec^{n-2} x) \, dx$$

$$T_n = \sqrt{2}^{n-2} - (n-2)(T_n - T_{n-2})$$

$$T_n = \sqrt{2}^{n-2} - \big((n-2)T_n - (n-2)T_{n-2}\big)$$

$$T_n = \sqrt{2}^{n-2} - (n-2)T_n + (n-2)T_{n-2}$$

$$T_n + (n-2)T_n = \sqrt{2}^{n-2} + (n-2)T_{n-2}$$

$$T_n(1 + n - 2) = \sqrt{2}^{n-2} + (n-2)T_{n-2}$$

$$T_n(n-1) = \sqrt{2}^{n-2} + (n-2)T_{n-2}$$

$$T_n = \frac{\sqrt{2}^{n-2}}{n-1} + \frac{n-2}{n-1}T_{n-2}$$

5b

$$T_n = \frac{\sqrt{2}^{n-2}}{n-1} + \frac{n-2}{n-1} T_{n-2}$$

$$T_0 = \int_0^{\frac{\pi}{4}} 1\, dx$$

$$= [x]_0^{\frac{\pi}{4}}$$

$$= \frac{\pi}{4}$$

$$T_2 = \frac{\sqrt{2}^{2-2}}{2-1} + \frac{2-2}{2-1} T_0$$

$$= 1 + 0$$

$$= 1$$

$$T_4 = \frac{\sqrt{2}^{4-2}}{4-1} + \frac{4-2}{4-1} T_2$$

$$= \frac{2}{3} + \frac{2}{3}(1)$$

$$= \frac{4}{3}$$

$$T_6 = \frac{\sqrt{2}^{6-2}}{6-1} + \frac{6-2}{6-1} T_4$$

$$= \frac{4}{5} + \frac{4}{5}\left(\frac{4}{3}\right)$$

$$= \frac{28}{15}$$

6a

$$C_n = \int_0^{\frac{\pi}{2}} x^n \cos x\, dx$$

$$\int uv'\, dx = uv - \int u'v\, dx$$

$u = x^n, v' = \cos x$

$u' = nx^{n-1}, v = \sin x$

Hence

$$C_n = [x^n \sin x]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} nx^{n-1} \sin x \, dx$$

$$C_n = \left(\frac{\pi}{2}\right)^n - n\int_0^{\frac{\pi}{2}} x^{n-1} \sin x \, dx$$

Consider $\int_0^{\frac{\pi}{2}} x^{n-1} \sin x \, dx$

$u = x^{n-1}, v' = \sin x$

$u' = (n-1)x^{n-2}, v = -\cos x$

Hence

$$\int_0^{\frac{\pi}{2}} x^{n-1} \sin x \, dx = [-x^{n-1} \cos x]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} (n-1)x^{n-2} \cos x \, dx$$

$$\int_0^{\frac{\pi}{2}} x^{n-1} \sin x \, dx = \int_0^{\frac{\pi}{2}} (n-1)x^{n-2} \cos x \, dx$$

Therefore

$$C_n = \left(\frac{\pi}{2}\right)^n - n\int_0^{\frac{\pi}{2}} (n-1)x^{n-2} \cos x \, dx$$

$$C_n = \left(\frac{\pi}{2}\right)^n - n(n-1)\int_0^{\frac{\pi}{2}} x^{n-2} \cos x \, dx$$

$$C_n = \left(\frac{\pi}{2}\right)^n - n(n-1)C_{n-2}$$

6b

$$C_0 = \int_0^{\frac{\pi}{2}} \cos x \, dx$$

$$= [\sin x]_0^{\frac{\pi}{2}}$$

$$= 1$$

$$C_2 = \left(\frac{\pi}{2}\right)^2 - 2(2-1)C_0$$

$$= \left(\frac{\pi}{2}\right)^2 - 2$$

$$C_4 = \left(\frac{\pi}{2}\right)^4 - 4(4-1)C_2$$

$$= \left(\frac{\pi}{2}\right)^4 - 12\left(\left(\frac{\pi}{2}\right)^2 - 2\right)$$

$$= \left(\frac{\pi}{2}\right)^4 - 12\left(\frac{\pi}{2}\right)^2 + 24$$

$$C_6 = \left(\frac{\pi}{2}\right)^6 - 6(6-1)C_4$$

$$= \left(\frac{\pi}{2}\right)^6 - 30\left(\left(\frac{\pi}{2}\right)^4 - 12\left(\frac{\pi}{2}\right)^2 + 24\right)$$

$$= \left(\frac{\pi}{2}\right)^6 - 30\left(\frac{\pi}{2}\right)^4 + 360\left(\frac{\pi}{2}\right)^2 - 720$$

7a

$$I_n = \int \frac{x^n}{1+x^2}\,dx$$

$$I_n = \int \frac{x^{n-2}x^2}{1+x^2}\,dx$$

$$I_n = \int \frac{x^{n-2}(1+x^2-1)}{1+x^2}\,dx$$

$$I_n = \int \frac{x^{n-2}(1+x^2)}{1+x^2}\,dx - \int \frac{x^{n-2}}{1+x^2}\,dx$$

$$I_n = \int x^{n-2}\,dx - I_{n-2}$$

$$I_n = \frac{x^{n-1}}{n-1} - I_{n-2}$$

7b

$$I_1 = \int \frac{x}{1+x^2}\,dx$$

$$= \frac{1}{2}\ln(1+x^2)$$

$$I_3 = \frac{x^2}{3-1} - I_1$$

$$= \frac{x^2}{2} - \frac{1}{2}\ln(1+x^2)$$

$$I_5 = \frac{x^4}{5-1} - I_3$$

$$= \frac{x^4}{4} - \frac{x^2}{2} + \frac{1}{2}\ln(1+x^2)$$

Hence

$$\int \frac{x^5}{1+x^2}\,dx = \frac{x^4}{4} - \frac{x^2}{2} + \frac{1}{2}\ln(1+x^2) + C$$

8a

$$I_n = \int_0^1 (1-x^2)^n\,dx$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

$$u = (1-x^2)^n, v' = 1$$

$$u' = -2nx(1-x^2)^{n-1}, v = x$$

Hence

$$I_n = \int_0^1 (1-x^2)^n\,dx$$

$$I_n = [x(1-x^2)^n]_0^1 + 2n\int_0^1 x^2(1-x^2)^{n-1}\,dx$$

$$I_n = 2n\int_0^1 \big(1-(1-x^2)\big)(1-x^2)^{n-1}\,dx$$

$$I_n = 2n\int_0^1 ((1-x^2)^{n-1} - (1-x^2)^n)\,dx$$

$$I_n = 2n(I_{n-1} - I_n)$$

$$I_n = 2nI_{n-1} - 2nI_n$$

$$I_n(1+2n) = 2nI_{n-1}$$

$$I_n = \frac{2nI_{n-1}}{1+2n}$$

$$I_n = \frac{2n}{2n+1}I_{n-1}$$

8b

$$I_0 = \int_0^1 (1 - x^2)^0 \, dx$$

$$= \int_0^1 1 \, dx$$

$$= 1$$

$$I_1 = \frac{2(1)}{2(1) + 1} I_0$$

$$= \frac{2}{3}(1)$$

$$= \frac{2}{3}$$

$$I_2 = \frac{2(2)}{2(2) + 1} I_1$$

$$= \frac{4}{5}\left(\frac{2}{3}\right)$$

$$= \frac{8}{15}$$

$$I_3 = \frac{2(3)}{2(3) + 1} I_2$$

$$= \frac{6}{7}\left(\frac{8}{15}\right)$$

$$= \frac{16}{35}$$

$$I_4 = \frac{2(4)}{2(4) + 1} I_3$$

$$= \frac{8}{9}\left(\frac{16}{35}\right)$$

$$= \frac{128}{315}$$

9a

$$u_n = \int_0^1 x(1-x^3)^n\,dx$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

$$u = (1-x^3)^n, v' = x$$

$$u' = -3nx^2(1-x^3)^{n-1}, v = \frac{1}{2}x^2$$

Hence

$$u_n = \int_0^1 x(1-x^3)^n\,dx$$

$$u_n = \left[\frac{1}{2}x^2(1-x^3)^n\right]_0^1 + \frac{3}{2}n\int_0^1 x^4(1-x^3)^{n-1}\,dx$$

$$u_n = 0 + \frac{3n}{2}\int_0^1 x^4(1-x^3)^{n-1}\,dx$$

Since $x^4 = x - x + x^4 = x - x(1-x^3)$,

$$u_n = \frac{3n}{2}\int_0^1 \left(x - x(1-x^3)\right)(1-x^3)^{n-1}\,dx$$

$$u_n = \frac{3n}{2}\int_0^1 x(1-x^3)^{n-1}dx - \frac{3n}{2}\int_0^1 x(1-x^3)(1-x^3)^{n-1}\,dx$$

$$u_n = \frac{3n}{2}\int_0^1 x(1-x^3)^{n-1}dx - \frac{3n}{2}\int_0^1 x(1-x^3)^n\,dx$$

$$u_n = \frac{3n}{2}u_{n-1} - \frac{3n}{2}u_n$$

$$u_n\left(1+\frac{3n}{2}\right) = \frac{3n}{2}u_{n-1}$$

$$u_n\left(\frac{3n+2}{2}\right) = \frac{3n}{2}u_{n-1}$$

$$u_n = \frac{3n}{3n+2}u_{n-1}$$

9b

$$u_0 = \int_0^1 x\,dx$$

$$= \left[\frac{1}{2}x^2\right]_0^1$$

$$= \frac{1}{2}$$

$$u_4 = \frac{3(4)}{3(4)+2}u_3 \qquad \text{(by the formula in part a)}$$

$$= \frac{12}{14} \times \frac{3(3)}{3(3)+2}u_2$$

$$= \frac{12}{14} \times \frac{9}{11} \times \frac{3(2)}{3(2)+2}u_1$$

$$= \frac{12}{14} \times \frac{9}{11} \times \frac{6}{8} \times \frac{3(1)}{3(1)+2}u_0$$

$$= \frac{12}{14} \times \frac{9}{11} \times \frac{6}{8} \times \frac{3}{5} \times \frac{1}{2}$$

$$= \frac{243}{1540}$$

10a

$$J_n = \int \frac{x^n}{\sqrt{1-x^2}}\,dx$$

$$J_n = \int \frac{x^{n-1}x}{\sqrt{1-x^2}}\,dx$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

$$u = x^{n-1}, v' = \frac{x}{\sqrt{1-x^2}}$$

$$u' = (n-1)x^{n-2}, v = -\sqrt{1-x^2}$$

Hence

$$J_n = -x^{n-1}\sqrt{1-x^2} + (n-1)\int x^{n-2}\sqrt{1-x^2}\,dx$$

$$J_n = -x^{n-1}\sqrt{1-x^2} + (n-1)\int x^{n-2}\sqrt{1-x^2} \times \frac{\sqrt{1-x^2}}{\sqrt{1-x^2}}\,dx$$

$$J_n = -x^{n-1}\sqrt{1-x^2} + (n-1)\int \frac{x^{n-2}(1-x^2)}{\sqrt{1-x^2}}\,dx$$

$$J_n = -x^{n-1}\sqrt{1-x^2} + (n-1)\int \left(\frac{x^{n-2}}{\sqrt{1-x^2}} - \frac{x^n}{\sqrt{1-x^2}}\right)dx$$

$$J_n = -x^{n-1}\sqrt{1-x^2} + (n-1)(J_{n-2} - J_n)$$

$$J_n = -x^{n-1}\sqrt{1-x^2} + (n-1)J_{n-2} - (n-1)J_n$$

$$J_n + (n-1)J_n = -x^{n-1}\sqrt{1-x^2} + (n-1)J_{n-2}$$

$$nJ_n = -x^{n-1}\sqrt{1-x^2} + (n-1)J_{n-2}$$

$$J_n = \frac{1}{n}\left((n-1)J_{n-2} - x^{n-1}\sqrt{1-x^2}\right)$$

10b

$$J_0 = \int \frac{x^0}{\sqrt{1-x^2}}\,dx$$
$$= \int \frac{1}{\sqrt{1-x^2}}\,dx$$
$$= \sin^{-1} x$$
$$J_2 = \frac{1}{2}\left((2-1)J_0 - x^{2-1}\sqrt{1-x^2}\right)$$
$$= \frac{1}{2}\left(\sin^{-1} x - x\sqrt{1-x^2}\right)$$

Hence

$$\int \frac{x^2}{\sqrt{1-x^2}}\,dx = J_2 + C$$
$$= \frac{1}{2}\left(\sin^{-1} x - x\sqrt{1-x^2}\right) + C$$

11a

$$u_n = \int_0^{\frac{\pi}{2}} \sin^n x \cos^2 x\,dx$$
$$u_n = \int_0^{\frac{\pi}{2}} \sin^{n-1} x \sin x \cos^2 x\,dx$$
$$\int uv'\,dx = uv - \int u'v\,dx$$
$$u = \sin^{n-1} x\,,\ v' = \sin x \cos^2 x$$
$$u' = (n-1)\sin^{n-2} x \cos x\,,\ v = -\frac{1}{3}\cos^3 x$$

Hence

$$u_n = \left[-\frac{1}{3}\sin^{\mathrm{n}-1} x \cos^3 x\right]_0^{\frac{\pi}{2}} + \frac{1}{3}\int_0^{\frac{\pi}{2}} (n-1)\sin^{n-2} x \cos x \cos^3 x\,dx$$
$$u_n = \frac{n-1}{3}\int_0^{\frac{\pi}{2}} \sin^{n-2} x \cos^4 x\,dx$$
$$u_n = \frac{n-1}{3}\int_0^{\frac{\pi}{2}} \sin^{n-2} x\,(1-\sin^2 x)\cos^2 x\,dx$$

$$u_n = \frac{n-1}{3}\int_0^{\frac{\pi}{2}} (\sin^{n-2} x \cos^2 x - \sin^n x \cos^2 x)\, dx$$

$$u_n = \frac{n-1}{3}(u_{n-2} - u_n)$$

$$u_n = \frac{n-1}{3}u_{n-2} - \frac{n-1}{3}u_n$$

$$u_n + \frac{n-1}{3}u_n = \frac{n-1}{3}u_{n-2}$$

$$u_n\left(1 + \frac{n-1}{3}\right) = \frac{n-1}{3}u_{n-2}$$

$$u_n\left(\frac{n+2}{3}\right) = \frac{n-1}{3}u_{n-2}$$

$$u_n = \frac{n-1}{n+2}u_{n-2}$$

11b

$$u_0 = \int_0^{\frac{\pi}{2}} \sin^0 x \cos^2 x\, dx$$

$$= \int_0^{\frac{\pi}{2}} \cos^2 x\, dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} (1 + \cos 2x)\, dx$$

$$= \frac{1}{2}\left[x + \frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left(\frac{\pi}{2}\right)$$

$$= \frac{\pi}{4}$$

$$u_2 = \frac{2-1}{2+2}u_0$$

$$= \frac{1}{4}\left(\frac{\pi}{4}\right)$$

$$= \frac{\pi}{16}$$

$$u_4 = \frac{4-1}{4+2}u_2$$

$$= \frac{3}{6}\left(\frac{\pi}{16}\right)$$
$$= \frac{\pi}{32}$$

12a

$$I_n = \int_0^1 \frac{x^n}{\sqrt{1+x}}\,dx$$
$$I_0 = \int_0^1 \frac{1}{\sqrt{1+x}}\,dx$$
$$= \int_0^1 (1+x)^{-\frac{1}{2}}\,dx$$
$$= \left[2\sqrt{1+x}\right]_0^1$$
$$= 2\sqrt{2} - 2$$

12b

$$I_n = \int_0^1 \frac{x^n}{\sqrt{1+x}}\,dx$$
$$I_n = \int_0^1 \frac{x^{n-1}x}{\sqrt{1+x}}\,dx$$
$$I_n = \int_0^1 \frac{x^{n-1}(1+x-1)}{\sqrt{1+x}}\,dx$$
$$I_n = \int_0^1 \frac{x^{n-1}(1+x) - x^{n-1}}{\sqrt{1+x}}\,dx$$
$$I_n = \int_0^1 \left(x^{n-1}\sqrt{1+x} - \frac{x^{n-1}}{\sqrt{1+x}}\right)dx$$
$$I_n = \int_0^1 x^{n-1}\sqrt{1+x}\,dx - I_{n-1}$$
$$I_{n-1} + I_n = \int_0^1 x^{n-1}\sqrt{1+x}\,dx$$

12c

$$I_{n-1} + I_n = \int_0^1 x^{n-1}\sqrt{1+x}\,dx$$

Consider $\int_0^1 x^{n-1}\sqrt{1+x}\,dx$

$$\int uv'\,dx = uv - \int u'v\,dx$$

$$u = \sqrt{1+x}, v' = x^{n-1}$$

$$u' = \frac{1}{2\sqrt{1+x}}, v = \frac{1}{n}x^n$$

Hence

$$\int_0^1 x^{n-1}\sqrt{1+x}\,dx$$

$$= \left[\frac{1}{n}x^n\sqrt{1+x}\right]_0^1 - \frac{1}{2n}\int_0^1 \frac{x^n}{\sqrt{1+x}}\,dx$$

$$= \frac{1}{n}\sqrt{2} - \frac{1}{2n}I_n$$

Hence

$$I_{n-1} + I_n = \frac{1}{n}\sqrt{2} - \frac{1}{2n}I_n$$

$$I_n + \frac{1}{2n}I_n = \frac{1}{n}\sqrt{2} - I_{n-1}$$

$$I_n\left(1 + \frac{1}{2n}\right) = \frac{1}{n}\sqrt{2} - I_{n-1}$$

$$I_n = \frac{\frac{1}{n}\sqrt{2} - I_{n-1}}{1 + \frac{1}{2n}}$$

$$I_n = \frac{2\sqrt{2} - 2nI_{n-1}}{2n+1}$$

12d

$$I_1 = \frac{2\sqrt{2} - 2(1)I_0}{2(1) + 1}$$

$$= \frac{2\sqrt{2} - 2\left(2\sqrt{2} - 2\right)}{3}$$

$$= \frac{2\sqrt{2} - 4\sqrt{2} + 4}{3}$$

$$= \frac{4 - 2\sqrt{2}}{3}$$

$$I_2 = \frac{2\sqrt{2} - 2(2)I_1}{2(2) + 1}$$

$$= \frac{2\sqrt{2} - 4\left(\frac{4 - 2\sqrt{2}}{3}\right)}{5}$$

$$= \frac{2\sqrt{2} - \frac{16 - 8\sqrt{2}}{3}}{5}$$

$$= \frac{6\sqrt{2} - 16 + 8\sqrt{2}}{15}$$

$$= \frac{14\sqrt{2} - 16}{15}$$

13a

$$(1 + t^2)^{n-1} + t^2(1 + t^2)^{n-1}$$

$$= (1 + t^2)(1 + t^2)^{n-1}$$

$$= (1 + t^2)^n$$

13b

$$P_n = \int_0^x (1 + t^2)^n \, dt$$

$$P_n = \int_0^x ((1 + t^2)^{n-1} + t^2(1 + t^2)^{n-1}) \, dt$$

$$P_n = \int_0^x (1 + t^2)^{n-1} \, dt + \int_0^x t^2(1 + t^2)^{n-1} \, dt$$

$$P_n = P_{n-1} + \int_0^x t^2(1+t^2)^{n-1}\,dt$$

Consider $\int_0^x t^2(1+t^2)^{n-1}\,dt$

$$\int uv'\,dx = uv - \int u'v\,dx$$

$u = t, v' = t(1+t^2)^{n-1}$

$u' = 1, v = \dfrac{(1+t^2)^n}{2n}$

Hence

$$\int_0^x t^2(1+t^2)^{n-1}\,dt$$

$$= \left[\frac{t(1+t^2)^n}{2n}\right]_0^x - \frac{1}{2n}\int_0^x (1+t^2)^n\,dt$$

Therefore

$$P_n = P_{n-1} + \left[\frac{t(1+t^2)^n}{2n}\right]_0^x - \frac{1}{2n}\int_0^x (1+t^2)^n\,dt$$

$$P_n = P_{n-1} + \frac{x(1+x^2)^n}{2n} - \frac{1}{2n}P_n$$

$$P_n + \frac{1}{2n}P_n = P_{n-1} + \frac{x(1+x^2)^n}{2n}$$

$$P_n\left(1 + \frac{1}{2n}\right) = P_{n-1} + \frac{x(1+x^2)^n}{2n}$$

$$P_n\left(\frac{2n+1}{2n}\right) = P_{n-1} + \frac{x(1+x^2)^n}{2n}$$

$$P_n = \frac{2n}{2n+1}\left(P_{n-1} + \frac{x(1+x^2)^n}{2n}\right)$$

$$P_n = \frac{1}{2n+1}\left((1+x^2)^n x + 2nP_{n-1}\right)$$

13c i

$$P_0=\int_0^x (1+t^2)^0\,dt$$
$$=\int_0^x 1\,dt$$
$$=x$$
$$P_1=\frac{1}{2(1)+1}((1+x^2)^1x+2(1)P_0)$$
$$=\frac{1}{3}\big((1+x^2)x+2x\big)$$
$$=\frac{1}{3}(1+x^2)x+\frac{2}{3}x$$
$$P_2=\frac{1}{2(2)+1}((1+x^2)^2x+2(2)P_1)$$
$$=\frac{1}{5}\left((1+x^2)^2x+4\left(\frac{1}{3}(1+x^2)x+\frac{2}{3}x\right)\right)$$
$$=\frac{1}{5}\left((1+x^2)^2x+\frac{4}{3}(1+x^2)x+\frac{8}{3}x\right)$$
$$=\frac{1}{5}(1+x^2)^2x+\frac{4}{15}(1+x^2)x+\frac{8}{15}x$$
$$P_3=\frac{1}{2(3)+1}((1+x^2)^3x+2(3)P_2)$$
$$=\frac{1}{7}\left((1+x^2)^3x+6\left(\frac{1}{5}(1+x^2)^2x+\frac{4}{15}(1+x^2)x+\frac{8}{15}x\right)\right)$$
$$=\frac{1}{7}\left((1+x^2)^3x+\frac{6}{5}(1+x^2)^2x+\frac{8}{5}(1+x^2)x+\frac{16}{5}x\right)$$
$$=\frac{1}{7}(1+x^2)^3x+\frac{6}{35}(1+x^2)^2x+\frac{8}{35}(1+x^2)x+\frac{16}{35}x$$
$$P_4=\frac{1}{2(4)+1}((1+x^2)^4x+2(4)P_3)$$
$$=\frac{1}{9}\left((1+x^2)^4x+8\left(\frac{1}{7}(1+x^2)^3x+\frac{6}{35}(1+x^2)^2x+\frac{8}{35}(1+x^2)x+\frac{16}{35}x\right)\right)$$
$$=\frac{1}{9}\left((1+x^2)^4x+\frac{8}{7}(1+x^2)^3x+\frac{48}{35}(1+x^2)^2x+\frac{64}{35}(1+x^2)x+\frac{128}{35}x\right)$$
$$=\frac{1}{9}x\left((1+x^2)^4+\frac{8}{7}(1+x^2)^3+\frac{48}{35}(1+x^2)^2+\frac{64}{35}(1+x^2)x+\frac{128}{35}\right)$$

13c ii

$$
\begin{aligned}
P_4 &= \int_0^x (1+t^2)^4\,dt \\
&= \int_0^x (1+4t^2+6t^4+4t^6+t^8)\,dt \\
&= \left[t+\frac{4}{3}t^3+\frac{6}{5}t^5+\frac{4}{7}t^7+\frac{1}{9}t^9\right]_0^x \\
&= x+\frac{4}{3}x^3+\frac{6}{5}x^5+\frac{4}{7}x^7+\frac{1}{9}x^9 \\
&= x\left(1+\frac{4}{3}x^2+\frac{6}{5}x^4+\frac{4}{7}x^6+\frac{1}{9}x^8\right)
\end{aligned}
$$

13d

$$
\begin{aligned}
P_4 &= \frac{1}{9}x\left((1+x^2)^4+\frac{8}{7}(1+x^2)^3+\frac{48}{35}(1+x^2)^2+\frac{64}{35}(1+x^2)x+\frac{128}{35}\right) \\
&= x\left(1+\frac{4}{3}x^2+\frac{6}{5}x^4+\frac{4}{7}x^6+\frac{1}{9}x^8\right)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&1+\frac{4}{3}x^2+\frac{6}{5}x^4+\frac{4}{7}x^6+\frac{1}{9}x^8 \\
&= \frac{1}{9}\left((1+x^2)^4+\frac{8}{7}(1+x^2)^3+\frac{48}{35}(1+x^2)^2+\frac{64}{35}(1+x^2)+\frac{128}{35}\right)
\end{aligned}
$$

14a

$$T_n = \int_0^1 x^n\sqrt{1-x}\,dx$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

$$u = x^n, v' = \sqrt{1-x}$$

$$u' = nx^{n-1}, v = -\frac{2}{3}(1-x)\sqrt{1-x}$$

Hence

$$T_n = \left[-\frac{2}{3}x^n(1-x)\sqrt{1-x}\right]_0^1 + \frac{2n}{3}\int_0^1 x^{n-1}(1-x)\sqrt{1-x}\,dx$$

$$T_n = \frac{2n}{3}\int_0^1 x^{n-1}\sqrt{1-x} - x^n\sqrt{1-x}\,dx$$

$$T_n = \frac{2n}{3}(T_{n-1} - T_n)$$

$$T_n = \frac{2n}{3}T_{n-1} - \frac{2n}{3}T_n$$

$$T_n + \frac{2n}{3}T_n = \frac{2n}{3}T_{n-1}$$

$$T_n\left(1 + \frac{2n}{3}\right) = \frac{2n}{3}T_{n-1}$$

$$T_n\left(\frac{2n+3}{3}\right) = \frac{2n}{3}T_{n-1}$$

$$T_n = \frac{2n}{2n+3}T_{n-1}$$

14b

$$T_0 = \int_0^1 x^0\sqrt{1-x}\,dx$$

$$= \int_0^1 \sqrt{1-x}\,dx$$

$$= \left[-\frac{2}{3}(1-x)^{\frac{3}{2}}\right]_0^1$$

$$= \frac{2}{3}$$

$$T_1 = \frac{2(1)}{2(1)+3}T_0$$

$$= \frac{2}{5}\left(\frac{2}{3}\right)$$

$$= \frac{4}{15}$$

$$T_2 = \frac{2(2)}{2(2)+3}T_1$$

$$= \frac{4}{7}\left(\frac{4}{15}\right)$$

$$= \frac{16}{105}$$

$$T_3 = \frac{2(3)}{2(3)+3}T_2$$
$$= \frac{6}{9}\left(\frac{16}{105}\right)$$
$$= \frac{32}{315}$$

14c

$$T_n = \frac{n!\,(n+1)!}{(2n+3)!}4^{n+1}$$

When $n = 0$:

$$T_0 = \frac{0!\,(0+1)!}{(2(0)+3)!}4^{0+1}$$
$$= \frac{2}{3}$$

As calculated in question 14b. So, the result is true for $n = 0$.

Now assume the statement is true for the positive integer $n = k - 1$

$$T_{k-1} = \frac{(k-1)!\,k!}{(2(k-1)+3)!}4^{k}$$
$$= \frac{(k-1)!\,k!}{(2k+1)!}4^{k}$$

Now using the reduction formula, consider the original proposition:

$$LHS = T_k$$
$$= \frac{2k}{2k+3}T_{k-1}$$
$$= \frac{2k}{2k+3}\frac{(k-1)!\,k!}{(2k+1)!}4^{k}$$
$$= \frac{2k}{2k+3}\frac{(k-1)!\,k!}{(2k+1)!}4^{k} \times \frac{2k+2}{2k+2}$$
$$= \frac{4k(k-1)!\,(k+1)k!}{(2k+3)(2k+2)(2k+1)!}4^{k}$$
$$= \frac{k!\,(k+1)!}{(2k+3)!}4^{k+1}$$
$$= RHS$$

By mathematical induction, this result is true for all integers $n \geq 0$.

Solutions to Exercise 4H Enrichment questions

15a $I_n = \int_0^1 (1-x^2)^n dx$

Integrating by parts with $u = (1-x^2)^n$ and $v' = 1$ gives:

$I_n = [(1-x^2)^n \cdot x]_0^1 - \int_0^1 (-2x)\, n(1-x^2)^{n-1} \cdot x\, dx$

$= 0 - 0 + 2n \int_0^1 x^2(1-x^2)^{n-1}\, dx$

$= 2n\, J_{n-1}$

15b $J_{n-1} = \int_0^1 x^2(1-x^2)^{n-1}\, dx$

$= -\int_0^1 (1-x^2)(1-x^2)^{n-1}\, dx + \int_0^1 (1-x^2)^{n-1}\, dx$

$= -I_n + I_{n-1}$

Hence,

$I_n = -2nI_n + 2nI_{n-1}$

$(2n+1)I_n = 2nI_{n-1}$

Thus,

$I_n = \frac{2n}{2n+1} I_{n-1}$

15c From part b,

$J_{n-1} = I_{n-1} - I_n$

$J_n = I_n - I_{n+1}$

$= I_n - \frac{2(n+1)}{2(n+1)+1} I_n$

$= \frac{2n+3-2(n+1)}{2n+3} I_n$

$= \frac{1}{2n+3} I_n$

15d $J_n = \frac{1}{2n+3} I_n$ (from part c)

$= \frac{1}{2n+3} . 2n\, J_{n-1}$ (from part a)

Hence,

$J_n = \frac{2n}{2n+3} \cdot J_{n-1}$

16a $I_1 = \int_0^{\frac{\pi}{4}} \tan\theta\, d\theta$

$= [-\ln(\cos\theta)]_0^{\frac{\pi}{4}}$ (see Box 9)

$= -\ln\frac{1}{\sqrt{2}} + \ln 1$ (but $\ln 1 = 0$)

$= \frac{1}{2}\ln 2$

16b $I_n + I_{n-2} = \int_0^{\frac{\pi}{4}} (\tan^n\theta + \tan^{n-2}\theta)\, d\theta$

$= \int_0^{\frac{\pi}{4}} \tan^{n-2}\theta(\tan^2\theta + 1) d\theta$

$= \int_0^{\frac{\pi}{4}} \tan^{n-2}\theta \sec^2\theta\, d\theta$

$= \left[\frac{\tan^{n-1}\theta}{n-1}\right]_0^{\frac{\pi}{4}}$

$= \frac{1}{n-1} - 0$

$= \frac{1}{n-1}$

16c $\tan^n\theta \le \tan^{n-2}\theta$ for $0 \le \theta \le \frac{\pi}{4}$, with equality at the end points only.

Hence,

$\int_0^{\frac{\pi}{4}} \tan^n\theta\, d\theta < \int_0^{\frac{\pi}{4}} \tan^{n-2}\theta\, d\theta$

That is, $I_n < I_{n-2}$

From part b,

$I_n + I_n < \frac{1}{n-1}$,

$I_n < \frac{1}{2(n-1)}$.

Also, from part b:

$I_{n-2} + I_{n-2} > \frac{1}{n-1}$

$I_{n-2} > \frac{1}{2(n-1)}$

$I_n > \frac{1}{2(n+1)}$

Combining these:

$\frac{1}{2(n+1)} < I_n < \frac{1}{2(n-1)}$

16d From part b:

$I_5 = \frac{1}{4} - I_3$

$= \frac{1}{4} - \left(\frac{1}{2} - I_1\right)$

$= I_1 - \frac{1}{4}$

$= \frac{1}{2}\ln 2 - \frac{1}{4}$

From part c:

$\frac{1}{12} < I_5 < \frac{1}{8}$

Hence,

$\frac{1}{12} < \frac{1}{2}\ln 2 - \frac{1}{4} < \frac{1}{8}$

$\frac{1}{3} < \frac{1}{2}\ln 2 < \frac{3}{8}$

$\frac{2}{3} < \ln 2 < \frac{3}{4}$

17a $I_n = \int_0^{\frac{\pi}{3}} \cos nx \sec x \, dx$

$= \int_0^{\frac{\pi}{3}} (\cos(n-2)x + 2x) \sec x \, dx$

$= \int_0^{\frac{\pi}{3}} \cos(n-2)x \cos 2x \sec x \, dx - \int_0^{\frac{\pi}{3}} \sin(n-2)x \sin 2x \sec x \, dx$

But, $\cos 2x = 2\cos x^2 - 1$ and $\sin 2x = 2 \sin x$, so,

I_n

$= 2\int_0^{\frac{\pi}{3}} \cos(n-2)x \cos x \, dx - \int_0^{\frac{\pi}{3}} \cos(n-2)x \sec x \, dx - 2\int_0^{\frac{\pi}{3}} \sin(n-2)x \sin x \, dx$

$= 2\int_0^{\frac{\pi}{3}} \cos(n-1)x \, dx - I_{n-2}$

$= \frac{2}{n-1}[\sin(n-1)x]_0^{\frac{\pi}{3}} - I_{n-2}$

$= \frac{2}{n-1} \sin \frac{(n-1)\pi}{3} - I_{n-2}$

17b $I_5 = \frac{2}{4} \sin \frac{4\pi}{3} - I_3$

$I_3 = \frac{2}{2} \sin \frac{2\pi}{3} - I_1$

$I_1 = \int_0^{\frac{\pi}{3}} \cos x \sec x \, dx$

$= \frac{\pi}{3}$

Hence,

$I_3 = \frac{\sqrt{3}}{2} - \frac{\pi}{3}$

$I_5 = \frac{1}{2}\left(\frac{-\sqrt{3}}{2}\right) - \frac{\sqrt{3}}{2} + \frac{\pi}{3}$

$= \frac{\pi}{3} - \frac{3\sqrt{3}}{4}$

Solutions to Exercise 4I Foundation questions

1a

$$\int_{-1}^{1} \frac{x^2}{(5+x^3)^2}\,dx$$

Let $u = 5 + x^3$

$du = 3x^2\,dx$

$x = 1, u = 6$

$x = -1, u = 4$

Hence

$$\int_{-1}^{1} \frac{x^2}{(5+x^3)^2}\,dx$$

$$= \frac{1}{3}\int_{4}^{6} \frac{1}{u^2}\,du$$

$$= \frac{1}{3}\left[-\frac{1}{u}\right]_4^6$$

$$= \frac{1}{3}\left(-\frac{1}{6} + \frac{1}{4}\right)$$

$$= \frac{1}{36}$$

1b

$$\int_{0}^{\pi} x \sin x\,dx$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Therefore

$u = x, v' = \sin x$

$u' = 1, v = -\cos x$

Hence

$$\int_{0}^{\pi} x \sin x\,dx$$

$= [-x\cos x]_0^\pi + \int_0^\pi \cos x\, dx$

$= (\pi - 0) + [\sin x]_0^\pi$

$= \pi + (0 - 0)$

$= \pi$

1c

Let $\dfrac{2x+2}{(x+3)(x-1)} = \dfrac{A}{x+3} + \dfrac{B}{x-1}$

$2x + 2 = A(x-1) + B(x+3)$

When $x = -3$,

$-4 = -4A$

$A = 1$

When $x = 1$,

$4 = 4B$

$B = 1$

Hence

$\dfrac{2x+2}{(x+3)(x-1)} = \dfrac{1}{x+3} + \dfrac{1}{x-1}$

$\int_2^3 \dfrac{2x+2}{(x+3)(x-1)}\, dx$

$= \int_2^3 \left(\dfrac{1}{x+3} + \dfrac{1}{x-1}\right) dx$

$= [\ln|x+3| + \ln|x-1|]_2^3$

$= (\ln 6 + \ln 2) - (\ln 5 + \ln 1)$

$= \ln\left(\dfrac{6\times 2}{5\times 1}\right)$

$= \ln\dfrac{12}{5}$

1d

$$\int_0^2 \frac{x-1}{x+1}\,dx$$

$$= \int_0^2 \frac{x+1-2}{x+1}\,dx$$

$$= \int_0^2 \left(1 - \frac{2}{x+1}\right) dx$$

$$= [x - 2\ln|x+1|]_0^2$$

$$= (2 - 2\ln 3) - (0 - 0)$$

$$= 2 - 2\ln 3$$

1e

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{3\cos x}{\sin^4 x}\,dx$$

Let $u = \sin x$

$du = \cos x\,dx$

When $x = \frac{\pi}{2}, u = 1$

When $x = \frac{\pi}{4}, u = \frac{\sqrt{2}}{2}$

Hence

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{3\cos x}{\sin^4 x}\,dx$$

$$= \int_{\frac{\sqrt{2}}{2}}^{1} \frac{3}{u^4}\,du$$

$$= \left[-\frac{3}{3u^3}\right]_{\frac{\sqrt{2}}{2}}^{1}$$

$$= \left[-\frac{1}{u^3}\right]_{\frac{\sqrt{2}}{2}}^{1}$$

$$= -1 + \frac{8}{2\sqrt{2}}$$

$$= -1 + \frac{4}{\sqrt{2}}$$

$$= -1 + \frac{4}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$

$$= -1 + 2\sqrt{2}$$

$$= 2\sqrt{2} - 1$$

1f

$$\int_0^{\frac{1}{3}} \frac{1}{\sqrt{4 - 9x^2}}\, dx$$

$$= \frac{1}{3}\int_0^{\frac{1}{3}} \frac{3}{\sqrt{2^2 - (3x)^2}}\, dx$$

$$= \frac{1}{3}\left[\sin^{-1}\left(\frac{3x}{2}\right)\right]_0^{\frac{1}{3}}$$

$$= \frac{1}{3}\left(\frac{\pi}{6} - 0\right)$$

$$= \frac{\pi}{18}$$

2a

$$\int \frac{x}{\sqrt{1 + x^2}}\, dx$$

Let $u = 1 + x^2$

$du = 2x\, dx$

Hence

$$\int \frac{x}{\sqrt{1 + x^2}}\, dx$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{u}}\, du$$

$$= \frac{1}{2} \times 2\sqrt{u} + C$$

$$= \sqrt{u} + C$$

$$= \sqrt{1 + x^2} + C$$

2b

$$\int \frac{1 + x}{1 + x^2}\,dx$$

$$= \int \left(\frac{1}{1 + x^2} + \frac{x}{1 + x^2}\right) dx$$

$$= \int \frac{1}{1 + x^2}\,dx + \int \frac{x}{1 + x^2}\,dx$$

$$= \int \frac{1}{1 + x^2}\,dx + \frac{1}{2}\int \frac{2x}{1 + x^2}\,dx$$

$$= \tan^{-1} x + \frac{1}{2}\ln(1 + x^2) + C \qquad \text{(since } 1 + x^2 \text{ is positive)}$$

2c

$$\int \sin x \cos^4 x\,dx$$

Let $u = \cos x$

$du = -\sin x\,dx$

Hence

$$\int \sin x \cos^4 x\,dx$$

$$= -\int u^4\,du$$

$$= -\frac{1}{5}u^5 + C$$

$$= -\frac{1}{5}\cos^5 x + C$$

2d

$$\frac{1}{2x^2+3x+1} = \frac{1}{(x+1)(2x+1)}$$

Let $\dfrac{1}{(x+1)(2x+1)} = \dfrac{A}{x+1} + \dfrac{B}{2x+1}$

$1 = A(2x+1) + B(x+1)$

When $x = -1$,

$1 = -A$

$A = -1$

When $x = -\dfrac{1}{2}$

$1 = \dfrac{1}{2}B$

$B = 2$

Hence

$$\int \frac{1}{(x+1)(2x+1)}\,dx$$

$$= \int \left(\frac{-1}{x+1} + \frac{2}{2x+1}\right) dx$$

$$= -\ln|x+1| + \ln|2x+1| + C$$

$$= \ln\left|\frac{2x+1}{x+1}\right| + C$$

2e

$$\int x^3 \ln x\,dx$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Therefore

$u = \ln x\,, v' = x^3$

$u' = \dfrac{1}{x}, v = \dfrac{1}{4}x^4$

Hence

$$\int x^3 \ln x \, dx$$

$$= \frac{1}{4}x^4 \ln x - \frac{1}{4}\int x^3 \, dx$$

$$= \frac{1}{4}x^4 \ln x - \frac{1}{4}\left(\frac{1}{4}x^4\right) + C$$

$$= \frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 + C$$

2f

$$\int \sin^3 2x \, dx$$

$$= \int \sin^2 2x \sin 2x \, dx$$

$$= \int (1 - \cos^2 2x) \sin 2x \, dx$$

Let $u = \cos 2x$

$du = -2\sin 2x \, dx$

Hence

$$\int (1 - \cos^2 2x) \sin 2x \, dx$$

$$= -\frac{1}{2}\int (1 - u^2) \, du$$

$$= -\frac{1}{2}\left(u - \frac{1}{3}u^3\right) + C$$

$$= -\frac{1}{2}\cos 2x + \frac{1}{6}\cos^3 2x + C$$

$$= \frac{1}{6}\cos^3 2x - \frac{1}{2}\cos 2x + C$$

2g

$$\int \frac{1}{x^2+6x+25}\,dx$$

$$= \int \frac{1}{x^2+6x+9+16}\,dx$$

$$= \int \frac{1}{(x+3)^2+4^2}\,dx$$

$$= \tan^{-1}\left(\frac{x+3}{4}\right) + C$$

2h

$$\int 3x\cos 3x\,dx$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Therefore

$u = 3x, v' = \cos 3x$

$u' = 3, v = \frac{1}{3}\sin 3x$

Hence

$$\int 3x\cos 3x\,dx$$

$$= x\sin 3x - \int \sin 3x\,dx$$

$$= x\sin 3x + \frac{1}{3}\cos 3x + C$$

2i

$$\int \frac{x}{\sqrt{4+x}}\,dx$$

$$= \int \frac{4+x-4}{\sqrt{4+x}}\,dx$$

$$= \int \left(\frac{4+x}{\sqrt{4+x}} - \frac{4}{\sqrt{4+x}}\right)dx$$

$$= \int \left(\sqrt{4+x} - \frac{4}{\sqrt{4+x}}\right) dx$$

Let $u = 4 + x$

$du = dx$

Hence

$$\int \left(\sqrt{4+x} - \frac{4}{\sqrt{4+x}}\right) dx$$

$$= \int \left(\sqrt{u} - \frac{4}{\sqrt{u}}\right) du$$

$$= \int \left(u^{\frac{1}{2}} - 4u^{-\frac{1}{2}}\right) du$$

$$= \frac{2}{3}u^{\frac{3}{2}} - 8u^{\frac{1}{2}} + C$$

$$= \frac{2}{3}\left(\sqrt{4+x}\right)^3 - 8\sqrt{4+x} + C$$

$$= \sqrt{4+x}\left(\frac{2}{3}(4+x) - 8\right) + C$$

$$= \sqrt{4+x}\left(\frac{2}{3}(4+x) - \frac{2}{3}(12)\right) + C$$

$$= \frac{2}{3}(x-8)\sqrt{4+x} + C$$

3a

$$\int_0^1 x^2 e^{-x}\, dx$$

$$\int uv'\, dx = uv - \int u'v\, dx$$

Therefore

$u = x^2, v' = e^{-x}$

$u' = 2x, v = -e^{-x}$

Hence

$$\int_0^1 x^2 e^{-x}\, dx$$

$$= [-x^2e^{-x}]_0^1 + \int_0^1 2xe^{-x}\,dx$$

$$= -\frac{1}{e} + \int_0^1 2xe^{-x}\,dx$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Therefore

$u = 2x, v' = e^{-x}$

$u' = 2, v = -e^{-x}$

Hence

$$-\frac{1}{e} + \int_0^1 2xe^{-x}\,dx$$

$$= -\frac{1}{e} - [2xe^{-x}]_0^1 + \int_0^1 2e^{-x}\,dx$$

$$= -\frac{1}{e} - \frac{2}{e} + [-2e^{-x}]_0^1$$

$$= -\frac{3}{e} - \frac{2}{e} + 2$$

$$= 2 - \frac{5}{e}$$

3b

$$\int_0^{\frac{\pi}{2}} \sin^3 x \cos^5 x\,dx$$

$$= \int_0^{\frac{\pi}{2}} \sin^2 x \cos^5 x \sin x\,dx$$

$$= \int_0^{\frac{\pi}{2}} (1 - \cos^2 x) \cos^5 x \sin x\,dx$$

Let $u = \cos x$

$du = -\sin x\,dx$

When $x = \frac{\pi}{2}, u = 0$

When $x = 0, u = 1$

Hence

$$\int_0^{\frac{\pi}{2}} (1 - \cos^2 x) \cos^5 x \sin x \, dx$$

$$= -\int_1^0 (1 - u^2) u^5 \, du$$

$$= \int_0^1 (u^5 - u^7) \, du$$

$$= \left[\frac{1}{6}u^6 - \frac{1}{8}u^8\right]_0^1$$

$$= \frac{1}{6} - \frac{1}{8}$$

$$= \frac{1}{24}$$

3c

Let $\dfrac{x}{(x+1)(x^2+1)} = \dfrac{A}{x+1} + \dfrac{Bx + C}{x^2+1}$

$x = A(x^2 + 1) + (Bx + C)(x + 1)$

When $x = -1$,

$-1 = 2A$

$A = -\dfrac{1}{2}$

When $x = 0$,

$0 = A + C$

$C = -A$

$C = \dfrac{1}{2}$

When $x = 1$,

$1 = 2A + 2B + 2C$

$1 = -1 + 2B + 1$

$B = \frac{1}{2}$

Hence

$\int_0^1 \frac{x}{(x+1)(x^2+1)}\,dx$

$= \int_0^1 \left(\frac{-\frac{1}{2}}{x+1} + \frac{\frac{1}{2}x + \frac{1}{2}}{x^2+1} \right) dx$

$= \frac{1}{2}\int_0^1 \left(-\frac{1}{x+1} + \frac{x+1}{x^2+1} \right) dx$

$= \frac{1}{2}\int_0^1 \left(-\frac{1}{x+1} + \frac{x}{x^2+1} + \frac{1}{x^2+1} \right) dx$

$= \frac{1}{2}\left[-\ln|x+1| + \frac{1}{2}\ln(x^2+1) + \tan^{-1} x \right]_0^1$ (since $x^2 + 1$ is positive)

$= \frac{1}{2}\left(-\ln 2 + \frac{1}{2}\ln 2 + \frac{\pi}{4} \right) - \frac{1}{2}(0 + 0 + 0)$

$= \frac{1}{8}(-4\ln 2 + 2\ln 2 + \pi)$

$= \frac{1}{8}(\pi - 2\ln 2)$

3d

$\int_0^{\frac{1}{2}} (1 - x^2)^{-\frac{3}{2}}\,dx$

$= \int_0^{\frac{1}{2}} \frac{1}{\left(\sqrt{1-x^2}\right)^3}\,dx$

Let $x = \sin u$

$dx = \cos u\,du$

When $x = \frac{1}{2}, u = \frac{\pi}{6}$

When $x = 0, u = 0$

Therefore

$$\int_0^{\frac{1}{2}} \frac{1}{\left(\sqrt{1-x^2}\right)^3}\,dx$$

$$= \int_0^{\frac{\pi}{6}} \frac{\cos u}{\left(\sqrt{1-\sin^2 u}\right)^3}\,du$$

$$= \int_0^{\frac{\pi}{6}} \frac{\cos u}{\sqrt{\cos^6 u}}\,du \quad \text{(noting that in this interval } \cos u \text{ is the positive square root)}$$

$$= \int_0^{\frac{\pi}{6}} \frac{\cos u}{\cos^3 u}\,du$$

$$= \int_0^{\frac{\pi}{6}} \frac{1}{\cos^2 u}\,du$$

$$= \int_0^{\frac{\pi}{6}} \sec^2 u\,du$$

$$= [\tan u]_0^{\frac{\pi}{6}}$$

$$= \frac{1}{\sqrt{3}}$$

3e

$$\int_0^1 \frac{1-x^2}{1+x^2}\,dx$$

$$= \int_0^1 \left(\frac{1}{1+x^2} - \frac{x^2}{1+x^2}\right)dx$$

$$= \int_0^1 \left(\frac{1}{1+x^2} - \frac{1+x^2-1}{1+x^2}\right)dx$$

$$= \left(\int_0^1 \frac{1}{1+x^2} - \frac{1+x^2}{1+x^2} + \frac{1}{1+x^2}\right)dx$$

$$= \int_0^1 \left(\frac{2}{1+x^2} - 1\right)dx$$

$$= [2\tan^{-1} x - x]_0^1$$

$$= \frac{\pi}{2} - 1$$

3f

$$\int_2^4 \frac{x}{\sqrt{6x-8-x^2}}\,dx$$

$$= \int_2^4 \frac{x}{\sqrt{1-(x-3)^2}}\,dx$$

Let $u = x - 3$

$x = u + 3$

$du = dx$

When $x = 4, u = 1$

When $x = 2, u = -1$

Hence

$$\int_2^4 \frac{x}{\sqrt{1-(x-3)^2}}\,dx$$

$$= \int_{-1}^1 \frac{u+3}{\sqrt{1-u^2}}\,du$$

$$= \int_{-1}^1 \left(\frac{u}{\sqrt{1-u^2}} + \frac{3}{\sqrt{1-u^2}}\right) du$$

$$= \left[-\frac{1}{2}\sqrt{1-u^2} + 3\sin^{-1} u\right]_{-1}^1$$

$$= \left(0 + 3 \times \frac{\pi}{2}\right) - \left(0 + 3 \times -\frac{\pi}{2}\right)$$

$$= \frac{3\pi}{2} + \frac{3\pi}{2}$$

$$= 3\pi$$

3g

$$\int_0^1 \frac{\sqrt{x}}{1+x}\,dx$$

Let $u = \sqrt{x}$

$x = u^2$

$dx = 2u\,du$

When $x = 1, u = 1$

When $x = 0, u = 0$

Hence

$$\int_0^1 \frac{\sqrt{x}}{1+x}\,dx$$

$$= \int_0^1 \frac{u}{1+u^2}\,2u\,du$$

$$= 2\int_0^1 \frac{u^2}{1+u^2}\,du$$

$$= 2\int_0^1 \frac{1+u^2-1}{1+u^2}\,du$$

$$= 2\int_0^1 \left(\frac{1+u^2}{1+u^2} - \frac{1}{1+u^2}\right)du$$

$$= 2\int_0^1 \left(1 - \frac{1}{1+u^2}\right)du$$

$$= 2[u - \tan^{-1} u]_0^1$$

$$= 2\left(1 - \frac{\pi}{4}\right) - 2(0-0)$$

$$= \frac{1}{2}(4-\pi)$$

3h

$$\int_0^{\sqrt{3}} \tan^{-1} x\,dx$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Therefore

$u = \tan^{-1} x\,, v' = 1$

$u' = \dfrac{1}{1+x^2}, v = x$

Hence

$$\int_0^{\sqrt{3}} \tan^{-1} x \, dx$$

$$= [x \tan^{-1} x]_0^{\sqrt{3}} - \int_0^{\sqrt{3}} \frac{x}{1+x^2} \, dx$$

$$= \left(\frac{\sqrt{3}\pi}{3} - 0\right) - \frac{1}{2}\int_0^{\sqrt{3}} \frac{2x}{1+x^2} \, dx$$

$$= \frac{\sqrt{3}\pi}{3} - \frac{1}{2}[\ln|1+x^2|]_0^{\sqrt{3}}$$

$$= \frac{\pi}{\sqrt{3}} - \left(\frac{1}{2}\ln 4 - 0\right)$$

$$= \frac{\pi}{\sqrt{3}} - \ln 2$$

3i

$$\int_0^{\frac{\pi}{4}} \sin 2x \cos 3x \, dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{1}{2}(\sin(2x - 3x) + \sin(2x + 3x)) \, dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}} (-\sin x + \sin 5x) \, dx$$

$$= \frac{1}{2}\left[\cos x - \frac{1}{5}\cos 5x\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{2}\left(\left(\frac{\sqrt{2}}{2} - \frac{1}{5}\left(-\frac{\sqrt{2}}{2}\right)\right) - \left(1 - \frac{1}{5}\right)\right)$$

$$= \frac{1}{2}\left(\frac{3\sqrt{2}}{5} - \frac{4}{5}\right)$$

$$= \frac{1}{10}(3\sqrt{2} - 4)$$

3j

$$\int_0^{\pi} e^{-x}\cos x\,dx$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Therefore

$u = \cos x\,, v' = e^{-x}$

$u' = -\sin x\,, v = -e^{-x}$

Hence

$$\int_0^{\pi} e^{-x}\cos x\,dx$$

$$= [-e^{-x}\cos x]_0^{\pi} - \int_0^{\pi} e^{-x}\sin x\,dx$$

Consider $\int_0^{\pi} e^{-x}\sin x\,dx$

$$\int uv'\,dx = uv - \int u'v\,dx$$

Therefore

$u = \sin x\,, v' = e^{-x}$

$u' = \cos x\,, v = -e^{-x}$

Hence

$$\int_0^{\pi} e^{-x}\sin x\,dx$$

$$= [-e^{-x}\sin x]_0^{\pi} + \int_0^{\pi} e^{-x}\cos x\,dx$$

Therefore

$$\int_0^{\pi} e^{-x}\cos x\,dx = [-e^{-x}\cos x]_0^{\pi} - \int_0^{\pi} e^{-x}\sin x\,dx$$

$$\int_0^{\pi} e^{-x}\cos x\,dx = [-e^{-x}\cos x]_0^{\pi} - [-e^{-x}\sin x]_0^{\pi} - \int_0^{\pi} e^{-x}\cos x\,dx$$

$$2\int_0^{\pi} e^{-x}\cos x\,dx = [-e^{-x}\cos x]_0^{\pi} + [e^{-x}\sin x]_0^{\pi}$$

$$2\int_0^\pi e^{-x}\cos x\,dx = (e^{-\pi}+1)+(0+0)$$

$$2\int_0^\pi e^{-x}\cos x\,dx = e^{-\pi}+1$$

$$\int_0^\pi e^{-x}\cos x\,dx = \frac{1}{2}(1+e^{-\pi})$$

Solutions to Exercise 4I Development questions

4a

$$\frac{x-1}{x^3+1}=\frac{A}{x+1}+\frac{Bx+C}{x^2-x+1}$$

$$x-1=A(x^2-x+1)+(Bx+C)(x+1)$$

$$x-1=Ax^2-Ax+A+Bx^2+Bx+Cx+C$$

$$x-1=(A+B)x^2+(B+C-A)x+A+C$$

Equating coefficients gives:

$$A+B=0$$

$$B+C-A=1$$

$$A+C=-1$$

Therefore

$$B=-A$$

$$C=-1-A$$

Hence

$$B+C-A=1$$

$$(-A)+(-1-A)-A=1$$

$$-3A=2$$

$$\therefore A=-\frac{2}{3}$$

$$\therefore B=\frac{2}{3}$$

$$\therefore C=-\frac{1}{3}$$

4b

$$\int_0^1 \frac{x^3+x}{x^3+1}dx$$

$$=\int_0^1 \frac{x^3+1+x-1}{x^3+1}dx$$

$$= \int_0^1 \left(1 + \frac{x-1}{x^3+1}\right) dx$$

$$= \int_0^1 \left(1 - \frac{\frac{2}{3}}{x+1} + \frac{\frac{2}{3}x - \frac{1}{3}}{x^2-x+1}\right) dx$$

$$\int_0^1 \left(1 - \frac{2}{3} \times \frac{1}{x+1} + \frac{1}{3} \times \frac{2x-1}{x^2-x+1}\right) dx$$

$$= \left[x - \frac{2}{3}\ln|x+1| + \frac{1}{3}\ln(x^2-x+1)\right]_0^1$$

$$= 1 - \frac{2}{3}\ln 2$$

5

$$\int x^3 e^{-x^2}\, dx$$

$$\int uv'\, dx = uv - \int u'v\, dx$$

$$u = x^2, v' = xe^{-x^2}$$

$$u' = 2x, v = -\frac{1}{2}e^{-x^2}$$

Hence

$$\int x^3 e^{-x^2}\, dx$$

$$= -\frac{1}{2}x^2 e^{-x^2} + \int xe^{-x^2}\, dx$$

$$= -\frac{1}{2}x^2 e^{-x^2} - \frac{1}{2}e^{-x^2} + C$$

$$= -\frac{1}{2}e^{-x^2}(1 + x^2) + C$$

6a

$$\int_0^{\frac{\pi}{3}} \sec^4 x\, dx$$

$$= \int_0^{\frac{\pi}{3}} \sec^2 x \sec^2 x\, dx$$

$$= \int_0^{\frac{\pi}{3}} (1 + \tan^2 x) \sec^2 x\, dx$$

Let $u = \tan x$

$du = \sec^2 x\, dx$

$x = \frac{\pi}{3}, u = \sqrt{3}$

$x = 0, u = 0$

Hence

$$\int_0^{\frac{\pi}{3}} (1 + \tan^2 x) \sec^2 x\, dx$$

$$= \int_0^{\sqrt{3}} (1 + u^2)\, du$$

$$= \left[u + \frac{1}{3}u^3\right]_0^{\sqrt{3}}$$

$$= \sqrt{3} + \sqrt{3}$$

$$= 2\sqrt{3}$$

6b

$$\int_0^{\frac{\pi}{3}} \sec^6 x\, dx$$

$$= \int_0^{\frac{\pi}{3}} \sec^4 x \sec^2 x\, dx$$

$$= \int_0^{\frac{\pi}{3}} \sec^4 x\, (1 + \tan^2 x)\, dx$$

$$= \int_0^{\frac{\pi}{3}} \sec^4 x\, dx + \int_0^{\frac{\pi}{3}} \sec^4 x \tan^2 x\, dx$$

$$= 2\sqrt{3} + \int_0^{\frac{\pi}{3}} \sec^2 x \sec^2 x \tan^2 x\, dx$$

$$= 2\sqrt{3} + \int_0^{\frac{\pi}{3}} \sec^2 x\,(1 + \tan^2 x) \tan^2 x\, dx$$

Let $u = \tan x$

$du = \sec^2 x\, dx$

$x = \frac{\pi}{3}, u = \sqrt{3}$

$x = 0, u = 0$

Hence

$$2\sqrt{3} + \int_0^{\frac{\pi}{3}} \sec^2 x\,(1 + \tan^2 x) \tan^2 x\, dx$$

$$= 2\sqrt{3} + \int_0^{\sqrt{3}} (1 + u^2)u^2\, du$$

$$= 2\sqrt{3} + \int_0^{\sqrt{3}} (u^2 + u^4)\, du$$

$$= 2\sqrt{3} + \left[\frac{1}{3}u^3 + \frac{1}{5}u^5\right]_0^{\sqrt{3}}$$

$$= 2\sqrt{3} + \sqrt{3} + \frac{9}{5}\sqrt{3}$$

$$= \frac{10}{5}\sqrt{3} + \frac{5}{5}\sqrt{3} + \frac{9}{5}\sqrt{3}$$

$$= \frac{24\sqrt{3}}{5}$$

7a

$$\int_0^{\frac{\pi}{2}} \frac{1}{3+5\cos x}\,dx$$

Let $t = \tan\left(\frac{x}{2}\right)$

$$dx = \frac{2}{1+t^2}\,dt$$

$$x = \frac{\pi}{2}, t = 1$$

$$x = 0, t = 0$$

$$\cos x = \frac{1-t^2}{1+t^2}$$

Hence

$$\int_0^{\frac{\pi}{2}} \frac{1}{3+5\cos x}\,dx$$

$$= \int_0^1 \frac{1}{3+5\frac{1-t^2}{1+t^2}} \times \frac{2}{1+t^2}\,dt$$

$$= \int_0^1 \frac{2}{3+3t^2+5-5t^2}\,dt$$

$$= \int_0^1 \frac{2}{8-2t^2}\,dt$$

$$= \int_0^1 \frac{1}{4-t^2}\,dt$$

$$= \int_0^1 \frac{1}{(2-t)(2+t)}\,dt$$

$$= \int_0^1 \left(\frac{\frac{1}{4}}{2-t} + \frac{\frac{1}{4}}{2+t}\right) dt \quad \text{(using cover} - \text{up rule)}$$

$$= \frac{1}{4}\int_0^1 \left(\frac{1}{2-t} + \frac{1}{2+t}\right) dt$$

$$= \frac{1}{4}\left[-\ln|2-t| + \ln|2+t|\right]_0^1$$

$$= \frac{1}{4}(\ln 3 + \ln 2 - \ln 2)$$

$$= \frac{1}{4}\ln 3$$

7b

$$\int_0^{\frac{\pi}{2}} \frac{1}{\cos x - 2\sin x + 3}\,dx$$

Let $t = \tan\left(\frac{x}{2}\right)$

$$dx = \frac{2}{1+t^2}\,dt$$

$$x = \frac{\pi}{2}, t = 1$$

$$x = 0, t = 0$$

$$\cos x = \frac{1-t^2}{1+t^2}$$

$$\sin x = \frac{2t}{1+t^2}$$

Hence

$$\int_0^{\frac{\pi}{2}} \frac{1}{\cos x - 2\sin x + 3}\,dx$$

$$= \int_0^1 \frac{1}{\frac{1-t^2}{1+t^2} - 2 \times \frac{2t}{1+t^2} + 3} \times \frac{2}{1+t^2}\,dt$$

$$= \int_0^1 \frac{2}{1 - t^2 - 4t + 3 + 3t^2}\,dt$$

$$= \int_0^1 \frac{2}{2t^2 - 4t + 4}\,dt$$

$$= \int_0^1 \frac{1}{t^2 - 2t + 2}\,dt$$

$$= \int_0^1 \frac{1}{(t-1)^2 + 1}\,dt$$

$$= [\tan^{-1}(t-1)]_0^1$$

$= \tan^{-1} 0 - \tan^{-1}(-1)$

$= \dfrac{\pi}{4}$

8a

$$\frac{4t}{(1+t)^2(1+t^2)} = \frac{A}{1+t} + \frac{B}{(1+t)^2} + \frac{Ct+D}{1+t^2}$$

$4t = A(1+t)(1+t^2) + B(1+t^2) + (Ct+D)(1+t)^2$

$4t = A(t^3+t^2+t+1) + B + Bt^2 + (Ct+D)(1+2t+t^2)$

$4t = At^3 + At^2 + At + A + B + Bt^2 + Ct + 2Ct^2 + Ct^3 + D + 2Dt + Dt^2$

$4t = (A+C)t^3 + (A+B+2C+D)t^2 + (A+C+2D)t + A + B + D$

Equating coefficients gives:

$A + C = 0$

$A + B + 2C + D = 0$

$A + C + 2D = 4$

$A + B + D = 0$

Hence

$A + C + 2D = 4$

$(0) + 2D = 4$

$\therefore D = 2$

$A + B + 2C + D = 0$

$(0) + 2C = 0$

$\therefore C = 0$

$A + C = 0$

$\therefore A = 0$

$A + B + D = 0$

$(0) + B + (2) = 0$

$\therefore B = -2$

8b

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\sin x}\,dx$$

Let $t = \tan\left(\frac{x}{2}\right)$

$$dx = \frac{2}{1+t^2}\,dt$$

$x = \frac{\pi}{2}, t = 1$

$x = 0, t = 0$

$$\sin x = \frac{2t}{1+t^2}$$

Hence

$$= \int_0^1 \frac{\sin x}{1+\sin x}\,dx$$

$$= \int_0^1 \frac{\frac{2t}{1+t^2}}{1+\frac{2t}{1+t^2}} \times \frac{2}{1+t^2}\,dt$$

$$= \int_0^1 \frac{\frac{4t}{1+t^2}}{1+t^2+2t}\,dt$$

$$= \int_0^1 \frac{\frac{4t}{1+t^2}}{(1+t)^2}\,dt$$

$$= \int_0^1 \frac{4t}{(1+t)^2(1+t^2)}\,dt$$

$$= \int_0^1 \left(\frac{-2}{(1+t)^2} + \frac{2}{1+t^2}\right)dt$$

$$= \left[\frac{2}{1+t} + 2\tan^{-1}(t)\right]_0^1$$

$$= \frac{2}{1+1} + 2\times\frac{\pi}{4} - 2$$

$$= 1 + \frac{\pi}{2} - 2$$

$$= \frac{\pi}{2} - 1$$

9

$$\int_1^{64} \frac{1}{\sqrt{x} + \sqrt[3]{x}}\, dx$$

Let $u = \sqrt[6]{x}$

$x = u^6$

$dx = 6u^5\, du$

$x = 64, u = 2$

$x = 1, u = 1$

Hence

$$\int_1^{64} \frac{1}{\sqrt{x} + \sqrt[3]{x}}\, dx$$

$$= \int_1^{2} \frac{1}{\sqrt{u^6} + \sqrt[3]{u^6}} \times 6u^5\, du$$

$$= \int_1^{2} \frac{1}{u^3 + u^2} \times 6u^5\, du$$

$$= 6\int_1^{2} \frac{u^3}{u + 1}\, du$$

$$= 6\int_1^{2} \frac{u^3 + 1 - 1}{u + 1}\, du$$

$$= 6\int_1^{2} \frac{(u + 1)(u^2 - u + 1) - 1}{u + 1}\, du$$

$$= 6\int_1^{2} \left(u^2 - u + 1 - \frac{1}{u + 1}\right) du$$

$$= 6\left[\frac{1}{3}u^3 - \frac{1}{2}u^2 + u - \ln|u + 1|\right]_1^2$$

$$= 6\left(\frac{8}{3} - 2 + 2 - \ln 3 - \frac{1}{3} + \frac{1}{2} - 1 + \ln 2\right)$$

$$= 6\left(\frac{7}{3} - \frac{1}{2} + \ln\frac{2}{3}\right)$$

$$= 14 - 3 + 6\ln\frac{2}{3}$$

$$= 11 - 6\ln\frac{3}{2}$$

10a

$$\int \sqrt{a^2 - x^2}\, dx$$

Let $\theta = \sin^{-1}\left(\frac{x}{a}\right)$

$x = a\sin\theta$

$dx = a\cos\theta\, d\theta$

Hence

$$\int \sqrt{a^2 - x^2}\, dx$$

$$= \int \sqrt{a^2 - (a\sin\theta)^2}\, a\cos\theta\, d\theta$$

$$= \int \sqrt{a^2(1 - \sin^2\theta)}\, a\cos\theta\, d\theta$$

$$= \int \sqrt{a^2\cos^2\theta}\, a\cos\theta\, d\theta$$

$$= \int a\cos\theta\, a\cos\theta\, d\theta$$

$$= a^2 \int \cos^2\theta\, d\theta$$

$$= a^2 \int \frac{1}{2}(1 + \cos 2\theta)\, d\theta$$

$$= \frac{a^2}{2} \int (1 + \cos 2\theta)\, d\theta$$

$$= \frac{a^2}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C$$

$$= \frac{a^2}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C$$

$$= \frac{a^2}{2}\left(\sin^{-1}\left(\frac{x}{a}\right) + \frac{1}{2}\sin\left(2\sin^{-1}\left(\frac{x}{a}\right)\right)\right) + C$$

$$= \frac{a^2}{2}\left(\sin^{-1}\left(\frac{x}{a}\right) + \sin\left(\sin^{-1}\left(\frac{x}{a}\right)\right)\cos\left(\sin^{-1}\left(\frac{x}{a}\right)\right)\right) + C$$

$$= \frac{a^2}{2}\left(\sin^{-1}\left(\frac{x}{a}\right) + \frac{x}{a} \times \frac{\sqrt{a^2 - x^2}}{a}\right) + C$$

$$= \frac{a^2}{2}\left(\sin^{-1}\left(\frac{x}{a}\right) + \frac{x\sqrt{a^2 - x^2}}{a^2}\right) + C$$

$$= \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + \frac{x}{2}\sqrt{a^2 - x^2} + C$$

10b

$$\int \sqrt{a^2 - x^2}\, dx$$

$$\int uv'\, dx = uv - \int u'v\, dx$$

$$u = \sqrt{a^2 - x^2}, v' = 1$$

$$u' = \frac{-x}{\sqrt{a^2 - x^2}}, v = x$$

Hence

$$\int \sqrt{a^2 - x^2}\, dx$$

$$= x\sqrt{a^2 - x^2} + \int \frac{x^2}{\sqrt{a^2 - x^2}}\, dx$$

$$= x\sqrt{a^2 - x^2} - \int \frac{a^2 - x^2 - a^2}{\sqrt{a^2 - x^2}}\, dx$$

$$= x\sqrt{a^2 - x^2} - \int \frac{a^2 - x^2}{\sqrt{a^2 - x^2}}\, dx + \int \frac{a^2}{\sqrt{a^2 - x^2}}\, dx$$

$$= x\sqrt{a^2 - x^2} - \int \sqrt{a^2 - x^2}\, dx + a^2 \sin^{-1}\left(\frac{x}{a}\right)$$

$$2\int \sqrt{a^2 - x^2}\, dx = x\sqrt{a^2 - x^2} + a^2 \sin^{-1}\left(\frac{x}{a}\right)$$

$$\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C$$

11a

$$\int_0^1 \frac{5-5x^2}{(1+2x)(1+x^2)}dx$$

$$\frac{5-5x^2}{(1+2x)(1+x^2)} = \frac{A}{1+2x} + \frac{Bx+C}{1+x^2}$$

$$5-5x^2 = A(1+x^2) + (Bx+C)(1+2x)$$

$$5-5x^2 = A + Ax^2 + Bx + C + 2Bx^2 + 2Cx$$

$$5-5x^2 = (A+2B)x^2 + (B+2C)x + A + C$$

Equating coefficients gives:

$$A + 2B = -5$$

$$B + 2C = 0$$

$$A + C = 5$$

Therefore

$$B = -2C$$

$$A = 5 - C$$

Hence

$$A + 2B = -5$$

$$(5 - C) + 2(-2C) = -5$$

$$5 - C - 4C = -5$$

$$-5C = -10$$

$$\therefore C = 2$$

$$\therefore B = -4$$

$$\therefore A = 3$$

Hence

$$\int_0^1 \frac{5-5x^2}{(1+2x)(1+x^2)}dx$$

$$= \int_0^1 \left(\frac{3}{1+2x} + \frac{-4x+2}{1+x^2}\right)dx$$

$$= \int_0^1 \left(\frac{3}{2} \times \frac{2}{1+2x} - 2 \times \frac{2x}{1+x^2} + \frac{2}{1+x^2}\right)dx$$

$$= \left[\frac{3}{2}\ln|1+2x| - 2\ln|1+x^2| + 2\tan^{-1}x\right]_0^1$$

$$= \frac{3}{2}\ln 3 - 2\ln 2 + \frac{\pi}{2}$$

$$= \frac{1}{2}(3\ln 3 - 4\ln 2 + \pi)$$

$$= \frac{1}{2}(\ln 27 - \ln 16 + \pi)$$

$$= \frac{1}{2}\left(\pi + \ln\frac{27}{16}\right)$$

11b

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{1+\cos x + 2\sin x}\,dx$$

Let $t = \tan\left(\frac{x}{2}\right)$

$$dx = \frac{2}{1+t^2}\,dt$$

$$x = \frac{\pi}{2}, t = 1$$

$$x = 0, t = 0$$

$$\cos x = \frac{1-t^2}{1+t^2}$$

$$\sin x = \frac{2t}{1+t^2}$$

Hence

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{1+\cos x + 2\sin x}\,dx$$

$$= \int_0^1 \frac{\frac{1-t^2}{1+t^2}}{1 + \frac{1-t^2}{1+t^2} + 2\times\frac{2t}{1+t^2}} \times \frac{2}{1+t^2}\,dt$$

$$= \int_0^1 \frac{2\times\frac{1-t^2}{1+t^2}}{1+t^2+1-t^2+4t}\,dt$$

$$= \int_0^1 \frac{\frac{1-t^2}{1+t^2}}{1+2t}\,dt$$

$$= \int_0^1 \frac{1-t^2}{(1+2t)(1+t^2)}\,dt$$

$$= \frac{1}{5}\int_0^1 \frac{5-5t^2}{(1+2t)(1+t^2)}\,dt$$

$$= \frac{1}{5}\left(\frac{1}{2}\left(\pi + \ln\frac{27}{16}\right)\right)$$

$$= \frac{1}{10}\left(\pi + \ln\frac{27}{16}\right)$$

12a

$8\sin x + \cos x - 2 = P(3\sin x + 2\cos x - 1) + Q(3\cos x - 2\sin x)$

$8\sin x + \cos x - 2 = 3P\sin x + 2P\cos x - P + 3Q\cos x - 2Q\sin x$

$8\sin x + \cos x - 2 = (3P - 2Q)\sin x + (2P + 3Q)\cos x - P$

Equating coefficients gives:

$-P = -2$

$3P - 2Q = 8$

$2P + 3Q = 1$

Hence

$\therefore P = 2$

$3P - 2Q = 8$

$3(2) - 2Q = 8$

$-2Q = 2$

$\therefore Q = -1$

12b

$$\int \frac{8\sin x + \cos x - 2}{3\sin x + 2\cos x - 1}\,dx$$

$$= \int \frac{2(3\sin x + 2\cos x - 1) - (3\cos x - 2\sin x)}{3\sin x + 2\cos x - 1}\,dx$$

$$= \int \left(2 - \frac{3\cos x - 2\sin x}{3\sin x + 2\cos x - 1}\right)dx$$

Let $u = 3\sin x + 2\cos x - 1$

$du = 3\cos x - 2\sin x\,dx$

Hence

$$\int \left(2 - \frac{3\cos x - 2\sin x}{3\sin x + 2\cos x - 1}\right)dx$$

$$= \int 2\,dx - \int \frac{3\cos x - 2\sin x}{3\sin x + 2\cos x - 1}\,dx$$

$$= \int 2\,dx - \int \frac{1}{u}\,du$$

$$= 2x - \ln|u| + C$$

$$= 2x - \ln|3\sin x + 2\cos x - 1| + C$$

13a

$$T_n = \int_0^{\pi} \sin^n x\,dx$$

$$T_n = \int_0^{\pi} \sin^{n-1} x \sin x\,dx$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

$u = \sin^{n-1} x\,,\, v' = \sin x$

$u' = (n-1)\sin^{n-2} x \cos x\,,\, v = -\cos x$

Hence

$$T_n = [-\sin^{n-1} x \cos x]_0^{\pi} + \int_0^{\pi} (n-1)\sin^{n-2} x \cos x \cos x\,dx$$

$$T_n = (n-1)\int_0^{\pi} \sin^{n-2} x \cos^2 x\,dx$$

$$T_n = (n-1)\int_0^{\pi} \sin^{n-2} x\,(1-\sin^2 x)\,dx$$

$$T_n = (n-1)\int_0^{\pi} (\sin^{n-2} x - \sin^n x)\,dx$$

$$T_n = (n-1)(T_{n-2} - T_n)$$

$$T_n = (n-1)T_{n-2} - (n-1)T_n$$

$$T_n + (n-1)T_n = (n-1)T_{n-2}$$

$$nT_n = (n-1)T_{n-2}$$

$$T_n = \frac{n-1}{n}T_{n-2}$$

13b

$$\begin{aligned} T_0 &= \int_0^{\pi} \sin^0 x\,dx \\ &= \int_0^{\pi} 1\,dx \\ &= \pi \end{aligned}$$

$$\begin{aligned} T_1 &= \int_0^{\pi} \sin x\,dx \\ &= [-\cos x]_0^{\pi} \\ &= 2 \end{aligned}$$

$$\begin{aligned} T_2 &= \frac{2-1}{2}T_0 \\ &= \frac{1}{2}(\pi) \\ &= \frac{\pi}{2} \end{aligned}$$

$$\begin{aligned} T_4 &= \frac{4-1}{4}T_2 \\ &= \frac{3}{4}\left(\frac{\pi}{2}\right) \\ &= \frac{3\pi}{8} \end{aligned}$$

$$\begin{aligned} T_6 &= \frac{6-1}{6}T_4 \\ &= \frac{5}{6}\left(\frac{3\pi}{8}\right) \end{aligned}$$

$$= \frac{5\pi}{16}$$

$$\begin{aligned} T_3 &= \frac{3-1}{3} T_1 \\ &= \frac{2}{3}(2) \\ &= \frac{4}{3} \end{aligned}$$

$$\begin{aligned} T_5 &= \frac{5-1}{5} T_3 \\ &= \frac{4}{5}\left(\frac{4}{3}\right) \\ &= \frac{16}{15} \end{aligned}$$

Therefore

$$\begin{aligned} T_5 T_6 &= \frac{5\pi}{16} \times \frac{16}{15} \\ &= \frac{\pi}{3} \end{aligned}$$

14a

$$I_n = \int_1^e (\ln x)^n \, dx$$

$$\int uv' \, dx = uv - \int u'v \, dx$$

$$u = (\ln x)^n, v' = 1$$

$$u' = \frac{n(\ln x)^{n-1}}{x}, v = x$$

Hence

$$I_n = [x(\ln x)^n]_1^e - \int_1^e n(\ln x)^{n-1} \, dx$$

$$I_n = e - n\int_1^e (\ln x)^{n-1} \, dx$$

$$I_n = e - nI_{n-1}$$

14b

$$\begin{aligned} I_0 &= \int_1^e (\ln x)^0 \, dx \\ &= \int_1^e 1 \, dx \\ &= e - 1 \end{aligned}$$

$$\begin{aligned} I_1 &= e - (1)I_0 \\ &= 1 \end{aligned}$$

$$\begin{aligned} I_2 &= e - (2)I_1 \\ &= e - 2 \end{aligned}$$

$$\begin{aligned} I_3 &= e - (3)I_2 \\ &= e - 3e + 6 \\ &= 6 - 2e \end{aligned}$$

Solutions to Exercise 4I Enrichment questions

15a $I_n = \int_0^1 \frac{x^{n-1}}{(x+1)^n} dx, \qquad n = 1, 2, 3, \ldots$

$I_1 = \int_0^1 \frac{1}{x+1} dx$

$= [\ln(x+1)]_0^1$

$= \ln 2 - \ln 1$

$= \ln 2$

15b $I_{n+1} = \int_0^1 \frac{x^n}{(x+1)^{n+1}} dx$

Integrating by parts with $u = x^n$ and $v' = (x+1)^{-(n+1)}$ gives:

$I_{n+1} = \left[\frac{x^n(x+1)^{-n}}{-n}\right]_0^1 + \int_0^1 \frac{nx^{n-1}(x+1)^{-n}}{n} dx$

$= -\frac{1}{n \cdot 2^n} - 0 + \int_0^1 \frac{x^{n-1}}{(x+1)^n} dx$

$= I_n - \frac{1}{n \cdot 2^n}$

15c $I_{n+1} = \int_0^1 \frac{x^n}{(x+1)^{n+1}} dx$

$= \int_0^1 \frac{x}{x+1} \cdot \frac{x^{n-1}}{(x+1)^n} dx$

Hence

$I_{n+1} < \int_0^1 \frac{1}{2} \cdot \frac{x^{n-1}}{(x+1)^n} dx$

$< \frac{1}{2} I_n$

15d From parts b and c:

$I_n - \frac{1}{n \cdot 2^n} < \frac{1}{2} I_n$

$\frac{1}{2} I_n < \frac{1}{n \cdot 2^n}$

Hence,

$I_n < \frac{1}{n \cdot 2^{n-1}}$

15e $I_3 = I_2 - \frac{1}{8}$ (by part b)

$= I_1 - \frac{1}{2} - \frac{1}{8}$

$= I_1 - \frac{5}{8}$

$< \frac{1}{12}$ (by part d)

Hence

$\ln 2 < \frac{5}{8} + \frac{1}{12}$

$< \frac{7}{24}$

Also,

$I_4 = I_3 - \frac{1}{24}$ (by part b)

$= \ln 2 - \frac{5}{8} - \frac{1}{24}$ (from above)

$= \ln 2 - \frac{2}{3}$

But, $I_n > 0$ for all n, so:

$\ln 2 > \frac{2}{3}$

Hence, $\frac{2}{3} < \ln 2 < \frac{17}{24}$

16 $\int_0^{\pi} \frac{\cos x + 2\sin x}{5 + 3\cos x}\, dx$

$= \frac{1}{3}\int_0^{\pi} \frac{3\cos x}{5+3\cos x}\, dx + \frac{2}{3}\int_0^{\pi} \frac{3\sin x}{5+3\cos x}\, dx$

$= \frac{1}{3}\int_0^{\pi} \frac{5+3\cos x}{5+3\cos x}\, dx - \frac{5}{3}\int_0^{\pi} \frac{1}{5+3\cos x}\, dx - \frac{2}{3}\int_0^{\pi} \frac{-3\sin x}{5+3\cos x}\, dx$

$= \frac{1}{3}[x]_0^{\pi} - \frac{5}{3}\cdot\frac{\pi}{4} - \frac{2}{3}[\ln(5 + 3\cos x)]_0^{\pi}$

$= \frac{\pi}{3} - \frac{5\pi}{12} - \frac{2}{3}(\ln 2 - \ln 8)$

$= -\frac{\pi}{12} + \frac{2}{3}(3\ln 2 - \ln 2)$

$= -\frac{\pi}{12} + \frac{4}{3}\ln 2$

$= \frac{1}{12}(16\ln 2 - \pi)$

17a $u = t - t^{-1}$

$u^2 = t^2 + t^{-2} - 2$

$du = (1 + t^{-2})dt$

Hence,

$\int \frac{1+t^2}{1+t^4} dt$

$= \int \frac{1+t^{-2}}{t^2+t^{-2}} dt$

$= \int \frac{du}{u^2+2}$

$= \frac{1}{\sqrt{2}} \tan^{-1} \frac{u}{\sqrt{2}} + C$

$= \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{t-t^{-1}}{\sqrt{2}}\right) + C$

$= \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{\sqrt{2}(t^2-1)}{2t}\right) + C$

17b Let $\frac{1+t^2}{1+t^4} = \frac{At+B}{1+\sqrt{2}t+t^2} + \frac{Ct+D}{1-\sqrt{2}t+t^2}$

But this function is even, so:

$\frac{At+B}{1+\sqrt{2}t+t^2} + \frac{Ct+D}{1-\sqrt{2}t+t^2} = \frac{-At+B}{1-\sqrt{2}t+t^2} + \frac{-Ct+D}{1+\sqrt{2}t+t^2}$

Thus, $A = -C$ and $B = D$

Also,

$1 + t^2 = (At + B)\left(1 - \sqrt{2}t + t^2\right) + (Ct + D)\left(1 + \sqrt{2}t + t^2\right)$

At $t = 0$, $\quad 1 = B + D = 2B$

$\quad B = D = \frac{1}{2}$

At $t = 1$, $2 = (A + B)\left(2 - \sqrt{2}\right) + (C + D)\left(2 + \sqrt{2}\right)$

$2 = A\left(2 - \sqrt{2}\right) + \frac{1}{2}\left(2 - \sqrt{2}\right) - A\left(2 + \sqrt{2}\right) + \frac{1}{2}\left(2 + \sqrt{2}\right)$, (since $A = -C$)

$2 = -2A\sqrt{2} + 2$

Hence, $A = -C = 0$

Thus,

$\int \frac{1+t^2}{1+t^4} dt$

$= \int \left(\frac{1}{1+\sqrt{2}t+t^2} + \frac{1}{1-\sqrt{2}t+t^2}\right) dt$

$= \frac{1}{2}\int \left(\frac{1}{\frac{1}{2}+\left(t+\frac{1}{\sqrt{2}}\right)^2} + \frac{1}{\frac{1}{2}+\left(t-\frac{1}{\sqrt{2}}\right)^2}\right) dt$

$= \frac{1}{2}\left[\sqrt{2}\tan^{-1}\left(\sqrt{2}\left(t+\frac{1}{\sqrt{2}}\right)\right) + \sqrt{2}\tan^{-1}\left(\sqrt{2}\left(t-\frac{1}{\sqrt{2}}\right)\right)\right] + C$

$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\sqrt{2}t+1\right) + \frac{1}{\sqrt{2}}\tan^{-1}\left(\sqrt{2}t-1\right) + C$

17c If $pq \leq 0$ then $0 \in [p, q]$, and the substitution

$u = t - t^{-1}$ is undefined when $t = 0$.

18 $I = \int_0^{\ln 2} \frac{1}{5\cos hx - 3\sin hx} dx$

$= \int_0^{\ln 2} \frac{2e^x}{5(e^{2x}+1)-3(e^{2x}-1)} dx$

$= \int_0^{\ln 2} \frac{2e^x}{2e^x+8} dx$

$= \int_0^{\ln 2} \frac{e^x}{(e^x)^2+2^2} dx$

$= \frac{1}{2}\left[\tan^{-1}\frac{e^x}{2}\right]_0^{\ln 2}$

$= \frac{1}{2}\left(\tan^{-1} 1 - \tan^{-1}\frac{1}{2}\right)$

$= \frac{1}{2}(\alpha - \beta)$

Now,

$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}$

$= \frac{1-\frac{1}{2}}{1+1\times\frac{1}{2}}$

$= \frac{2-1}{2+1}$

$= \frac{1}{3}$

So, $\alpha - \beta = \tan^{-1}\frac{1}{3}$

Hence, $I = \frac{1}{2}\tan^{-1}\frac{1}{3}$

19a $I = \int_0^1 \frac{\ln(1+x)}{1+x^2} dx$

Let $x = \tan\theta$ with $0 \le \theta \le \frac{\pi}{4}$, then

$dx = \sec^2\theta \, d\theta$

$= (1 + \tan^2\theta) d\theta$

Or, $\frac{dx}{1+x^2} = d\theta$

So, $I = \int_0^{\frac{\pi}{4}} \ln(1 + \tan\theta) \, d\theta$

19b Let $u = \frac{\pi}{4} - \theta$

Then $du = -d\theta$

When $\theta = 0, u = \frac{\pi}{4}$

When $\theta = \frac{\pi}{4}, u = 0$

Hence,

$I = \int_{\frac{\pi}{4}}^{0} \ln\left(1 + \tan\left(\frac{\pi}{4} - u\right)\right)(-du)$

$= \int_0^{\frac{\pi}{4}} \ln\left(1 + \frac{1-\tan u}{1+\tan u}\right) du$

$= \int_0^{\frac{\pi}{4}} \ln\left(\frac{1+\tan u+1-\tan u}{1+\tan u}\right) du$

$= \int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1+\tan u}\right) du$

$= \int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1+\tan\theta}\right) d\theta$ (since θ is a dummy variable.)

19c $I = \int_0^{\frac{\pi}{4}} (\ln 2 - \ln(1 + \tan\theta)) d\theta$

$= \int_0^{\frac{\pi}{4}} \ln 2 \, d\theta - I$

$2I = \frac{\pi}{4}\ln 2$

Hence, $I = \frac{\pi}{8}\ln 2$

Further,

$I = \int_0^1 \ln(1+x) \cdot \frac{1}{1+x^2}\,dx$

$= [\ln(1+x) \cdot \tan^{-1} x]_0^1 - \int_0^1 \frac{\tan^{-1} x}{1+x}\,dx$

Thus, $\frac{\pi}{8}\ln 2 = \frac{\pi}{4}\ln 2 - \int_0^1 \frac{\tan^{-1} x}{1+x}\,dx$

Or, $\int_0^1 \frac{\tan^{-1} x}{1+x}\,dx = \frac{\pi}{8}\ln 2$

So, $\int_0^1 \frac{\ln(1+x)}{1+x^2}\,dx = \int_0^1 \frac{\tan^{-1} x}{1+x}\,dx$ (!)

Here is a more obscure, but elegant and equivalent solution:

$I = \int_0^1 \frac{\ln(1+x)}{1+x^2}\,dx$

Let $x = \frac{1-u}{1+u}$ (essentially the $\tan\left(\theta - \frac{\pi}{4}\right)$ step above.)

At $u = 0, x = 1$ and at $u = 1, x = 0$ and,

$\frac{dx}{du} = \frac{(1+u)(-1)-(1-u)}{(1+u)^2}$, and so,

$dx = \frac{-2du}{(1+u)^2}$

Hence,

$I = \int_1^0 \frac{\ln\left(1+\frac{1-u}{1+u}\right)}{1+\left(\frac{1-u}{1+u}\right)^2} \cdot \frac{(-2)du}{(1+u)^2}$

$= 2\int_0^1 \frac{\ln\left(\frac{2}{1+u}\right)du}{(1+u)^2+(1-u)^2}$

$= 2\int_0^1 \frac{\ln 2 - \ln(1+u)}{2(1+u^2)}\,du$

$= \int_0^1 \frac{\ln 2}{1+u^2}\,du - \int_0^1 \frac{\ln(1+u)}{1+u^2}\,du$

$= [\ln 2 \cdot \tan^{-1} u]_0^1 - I$

Hence,

$2I = \frac{\pi}{4}\ln 2$ or $I = \frac{\pi}{8}\ln 2$

Solutions to Exercise 4J Chapter review

1a

$$\int xe^{x^2}\,dx$$

Let $u = x^2$

$du = 2x\,dx$

Hence

$$\int xe^{x^2}\,dx$$

$$= \frac{1}{2}\int 2xe^{x^2}\,dx$$

$$= \frac{1}{2}\int e^u\,du$$

$$= \frac{1}{2}e^u + C$$

$$= \frac{1}{2}e^{x^2} + C$$

1b

$$\int \frac{3x}{x^2+1}\,dx$$

Let $u = x^2 + 1$

$du = 2x\,dx$

Hence

$$\int \frac{3x}{x^2+1}\,dx$$

$$= \frac{3}{2}\int \frac{2x}{x^2+1}\,dx$$

$$= \frac{3}{2}\int \frac{1}{u}\,du$$

$$= \frac{3}{2}\ln|u| + C$$

$= \frac{3}{2}\ln(x^2 + 1) + C$

1c

$\int x(1 + x^2)^5 \, dx$

Let $u = 1 + x^2$

$du = 2x \, dx$

Hence

$\int x(1 + x^2)^5 \, dx$

$= \frac{1}{2}\int 2x(1 + x^2)^5 \, dx$

$= \frac{1}{2}\int u^5 \, du$

$= \frac{1}{2}\left(\frac{1}{6}u^6\right) + C$

$= \frac{1}{12}u^6 + C$

$= \frac{1}{12}(1 + x^2)^6 + C$

1d

$\int \cos^3 x \sin x \, dx$

Let $u = \cos x$

$du = -\sin x \, dx$

Hence

$\int \cos^3 x \sin x \, dx$

$= -\int u^3 \, du$

$= -\frac{1}{4}u^4 + C$

$$= -\frac{1}{4}\cos^4 x + C$$

1e

$$\int \frac{4x}{x^2 - 2x - 3}\,dx$$

$$= \int \frac{4x}{(x-3)(x+1)}\,dx$$

$$\frac{4x}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1}$$

Using cover-up method:

$$A = \frac{4(3)}{3+1}$$

$$= 3$$

$$B = \frac{4(-1)}{-1-3}$$

$$= 1$$

Hence

$$\int \frac{4x}{(x-3)(x+1)}\,dx$$

$$= \int \left(\frac{3}{x-3} + \frac{1}{x+1}\right)dx$$

$$= 3\ln|x-3| + \ln|x+1| + C$$

1f

$$\int xe^{-2x}\,dx$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

$$u = x, v' = e^{-2x}$$

$$u' = 1, v = -\frac{1}{2}e^{-2x}$$

Hence

$$\int xe^{-2x}\,dx$$

$$= -\frac{1}{2}xe^{-2x} + \frac{1}{2}\int e^{-2x}\,dx$$

$$= -\frac{1}{2}xe^{-2x} + \frac{1}{2}\left(-\frac{1}{2}e^{-2x}\right) + C$$

$$= -\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + C$$

2a

$$\int \tan^2 x\,dx$$

$$= \int (\sec^2 x - 1)\,dx$$

$$= \tan x - x + C$$

2b

$$\int \frac{x}{\sqrt{3+x}}\,dx$$

Let $u = 3 + x$

$x = u - 3$

$dx = du$

Hence

$$\int \frac{x}{\sqrt{3+x}}\,dx$$

$$= \int \frac{u-3}{\sqrt{u}}\,du$$

$$= \int \left(u^{\frac{1}{2}} - 3u^{-\frac{1}{2}}\right) du$$

$$= \frac{2}{3}u^{\frac{3}{2}} - 6u^{\frac{1}{2}} + C$$

$$= \frac{2}{3}(3+x)^{\frac{3}{2}} - 6(3+x)^{\frac{1}{2}} + C$$

2c

$$\int \frac{1}{x^2+2x+5}\,dx$$

$$= \int \frac{1}{(x+1)^2+4}\,dx$$

$$= \frac{1}{2}\int \frac{2}{(x+1)^2+2^2}\,dx$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right)+C$$

2d

$$\int x\cos\left(\frac{1}{3}x\right)dx$$

$$\int uv'\,dx = uv - \int u'v\,dx$$

$$u = x, v' = \cos\left(\frac{1}{3}x\right)$$

$$u' = 1, v = 3\sin\left(\frac{1}{3}x\right)$$

Hence

$$\int x\cos\left(\frac{1}{3}x\right)dx$$

$$= 3x\sin\left(\frac{1}{3}x\right) - 3\int \sin\left(\frac{1}{3}x\right)dx$$

$$= 3x\sin\left(\frac{1}{3}x\right) + 9\cos\left(\frac{1}{3}x\right) + C$$

2e

$$\int \frac{x+2}{x+1}\,dx$$

$$= \int \frac{x+1+1}{x+1}\,dx$$

$$= \int \left(1+\frac{1}{x+1}\right)dx$$

$= x + \ln|x+1| + C$

2f

$\int \frac{3x^2+2}{x^3+x}dx$

$\int \frac{3x^2+2}{x(x^2+1)}dx$

$\frac{3x^2+2}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$

$A = \frac{3(0)^2+2}{(0)^2+1} = 2 \quad \text{(using cover} - \text{up method)}$

$3x^2+2 = 2(x^2+1) + (Bx+C)x$

$3x^2+2 = 2x^2+2+Bx^2+Cx$

$3x^2+2 = (B+2)x^2+Cx+2$

Equating coefficients gives:

$C = 0$

$B + 2 = 3$

$\therefore B = 1$

Hence

$\int \frac{3x^2+2}{x(x^2+1)}dx$

$= \int \left(\frac{2}{x} + \frac{x}{x^2+1}\right)dx$

$= \int \left(\frac{2}{x} + \frac{1}{2} \times \frac{2x}{x^2+1}\right)dx$

$= 2\ln|x| + \frac{1}{2}\ln(x^2+1) + C$

3a

$$\int \frac{1}{(4-x^2)^{\frac{3}{2}}}\,dx$$

Let $x = 2\sin\theta$

$dx = 2\cos\theta\, d\theta$

Hence

$$\int \frac{1}{(4-x^2)^{\frac{3}{2}}}\,dx$$

$$= \int \frac{1}{(4-4\sin^2\theta)^{\frac{3}{2}}} \times 2\cos\theta\, d\theta$$

$$= \int \frac{1}{(4\cos^2\theta)^{\frac{3}{2}}} \times 2\cos\theta\, d\theta$$

$$= \int \frac{1}{(2\cos\theta)^3} \times 2\cos\theta\, d\theta$$

$$= \int \frac{1}{8\cos^3\theta} \times 2\cos\theta\, d\theta$$

$$= \int \frac{1}{4\cos^2\theta}\, d\theta$$

$$= \frac{1}{4}\int \sec^2\theta\, d\theta$$

$$= \frac{1}{4}\tan\theta + C$$

$$\theta = \sin^{-1}\left(\frac{x}{2}\right)$$

Hence

$$\frac{1}{4}\tan\theta + C$$

$$= \frac{1}{4}\tan\left(\sin^{-1}\left(\frac{x}{2}\right)\right) + C$$

$$= \frac{1}{4}\frac{x}{\sqrt{4-x^2}} + C$$

$$= \frac{x}{4\sqrt{4-x^2}} + C$$

3b

$$\int \frac{e^x}{e^{2x}-1}dx$$

Let $u = e^x$

$du = e^x dx$

Hence

$$\int \frac{e^x}{e^{2x}-1}dx$$

$$= \int \frac{e^x}{(e^x)^2-1}dx$$

$$= \int \frac{1}{u^2-1}du$$

$$= \int \frac{1}{(u+1)(u-1)}du$$

$$= \int \left(\frac{-\frac{1}{2}}{u+1}+\frac{\frac{1}{2}}{u-1}\right)du \quad \text{(using cover} - \text{up method)}$$

$$= \frac{1}{2}\int \left(\frac{1}{u-1}-\frac{1}{u+1}\right)du$$

$$= \frac{1}{2}(\ln|u-1|-\ln|u+1|)+C$$

$$= \frac{1}{2}(\ln|e^x-1|-\ln|e^x+1|)+C$$

$$= \frac{1}{2}\ln\left(\frac{e^x-1}{e^x+1}\right)+C$$

3c

$$\int \frac{1}{2+\sqrt{x}}dx$$

Let $u = \sqrt{x}$

$x = u^2$

$dx = 2u\,du$

Hence

$$\int \frac{1}{2+\sqrt{x}}\,dx$$

$$= \int \frac{1}{2+u} \times 2u\,du$$

$$= 2\int \frac{u}{2+u}\,du$$

$$= 2\int \frac{u+2-2}{2+u}\,du$$

$$= 2\int \left(1 - \frac{2}{2+u}\right) du$$

$$= 2u - 4\ln|2+u| + C$$

$$= 2\sqrt{x} - 4\ln\left(2+\sqrt{x}\right) + C$$

3d

$$\int \frac{1}{5+4\cos x}\,dx$$

Let $t = \tan\left(\frac{x}{2}\right)$

$$dx = \frac{2}{1+t^2}\,dt$$

$$\cos x = \frac{1-t^2}{1+t^2}$$

Hence

$$\int \frac{1}{5+4\cos x}\,dx$$

$$= \int \frac{1}{5+4\frac{1-t^2}{1+t^2}} \times \frac{2}{1+t^2}\,dt$$

$$= \int \frac{2}{5+5t^2+4-4t^2}\,dt$$

$$= \int \frac{2}{t^2+9}\,dt$$

$$= \frac{2}{3}\int \frac{3}{t^2+3^2}\,dt$$

$$= \frac{2}{3}\tan^{-1}\left(\frac{t}{3}\right) + C$$

$$= \frac{2}{3}\tan^{-1}\left(\frac{\tan\left(\frac{x}{2}\right)}{3}\right) + C$$

$$= \frac{2}{3}\tan^{-1}\left(\frac{1}{3}\tan\frac{1}{2}x\right) + C$$

4a

$$\int_{-1}^{2} x^2\sqrt{x^3+1}\,dx$$

Let $u = x^3$

$du = 3x^2$

$x = 2, u = 8$

$x = -1, u = -1$

Hence

$$\int_{-1}^{2} x^2\sqrt{x^3+1}\,dx$$

$$= \frac{1}{3}\int_{-1}^{8}\sqrt{u+1}\,du$$

$$= \frac{1}{3}\int_{-1}^{8}(u+1)^{\frac{1}{2}}\,du$$

$$= \frac{1}{3}\left[\frac{2}{3}(u+1)^{\frac{3}{2}}\right]_{-1}^{8}$$

$$= \frac{2}{9}\left[(u+1)^{\frac{3}{2}}\right]_{-1}^{8}$$

$$= \frac{2}{9}(27-0)$$

$$= 6$$

4b

$$\int_4^5 \frac{2x}{x^2 - 4x + 3}\,dx$$

$$= \int_4^5 \frac{2x}{(x-3)(x-1)}\,dx$$

$$\frac{2x}{(x-3)(x-1)} = \frac{A}{x-3} + \frac{B}{x-1}$$

Using cover-up method:

$$A = \frac{2(3)}{3-1} = 3$$

$$B = \frac{2(1)}{1-3} = -1$$

Hence

$$\int_4^5 \frac{2x}{(x-3)(x-1)}\,dx$$

$$= \int_4^5 \left(\frac{3}{x-3} - \frac{1}{x-1}\right) dx$$

$$= [3\ln|x-3| - \ln|x-1|]_4^5$$

$$= 3\ln 2 - \ln 4 - 3\ln 1 + \ln 3$$

$$= 3\ln 2 - 2\ln 2 + \ln 3$$

$$= \ln 2 + \ln 3$$

$$= \ln 6$$

4c

$$\int_0^{\frac{\pi}{3}} \sin^3 x\,dx$$

$$= \int_0^{\frac{\pi}{3}} \sin x \sin^2 x\,dx$$

$$= \int_0^{\frac{\pi}{3}} \sin x\,(1 - \cos^2 x)\,dx$$

Let $u = \cos x$

$du = -\sin x \, dx$

$x = \frac{\pi}{3}, u = \frac{1}{2}$

$x = 0, u = 1$

Hence

$\int_0^{\frac{\pi}{3}} \sin x \,(1 - \cos^2 x) \, dx$

$= -\int_1^{\frac{1}{2}} (1 - u^2) \, du$

$= \int_{\frac{1}{2}}^1 (1 - u^2) \, du$

$= \left[u - \frac{1}{3}u^3\right]_{\frac{1}{2}}^1$

$= 1 - \frac{1}{3} - \frac{1}{2} + \frac{1}{24}$

$= \frac{24}{24} - \frac{8}{24} - \frac{12}{24} + \frac{1}{24}$

$= \frac{5}{24}$

4d

$\int_0^1 \frac{8x}{3 + 4x} \, dx$

$= \int_0^1 \frac{6 + 8x - 6}{3 + 4x} \, dx$

$= \int_0^1 \left(2 - \frac{6}{3 + 4x}\right) dx$

$= \int_0^1 \left(2 - \frac{6}{4} \times \frac{4}{3 + 4x}\right) dx$

$= \left[2x - \frac{3}{2}\ln|3 + 4x|\right]_0^1$

$$= 2 - \frac{3}{2}\ln 7 + \frac{3}{2}\ln 3$$

$$= 2 + \frac{3}{2}\ln\left(\frac{3}{7}\right)$$

$$= 2 - \frac{3}{2}\ln\left(\frac{7}{3}\right)$$

4e

$$\int_0^1 x^2\sqrt{1-x}\,dx$$

Let $u = 1 - x$

$x = 1 - u$

$du = -dx$

$x = 1, u = 0$

$x = 0, u = 1$

Hence

$$\int_0^1 x^2\sqrt{1-x}\,dx$$

$$= -\int_1^0 (1-u)^2 u^{\frac{1}{2}}\,du$$

$$= \int_0^1 (1 - 2u + u^2)u^{\frac{1}{2}}\,du$$

$$= \int_0^1 \left(u^{\frac{1}{2}} - 2u^{\frac{3}{2}} + u^{\frac{5}{2}}\right)du$$

$$= \left[\frac{2}{3}u^{\frac{3}{2}} - \frac{4}{5}u^{\frac{5}{2}} + \frac{2}{7}u^{\frac{7}{2}}\right]_0^1$$

$$= \frac{2}{3} - \frac{4}{5} + \frac{2}{7}$$

$$= \frac{70}{105} - \frac{84}{105} + \frac{30}{105}$$

$$= \frac{16}{105}$$

4f

$$\int_0^{\frac{\pi}{4}} \sin 5x \cos 3x \, dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}} (\sin(5x - 3x) + \sin(5x + 3x)) \, dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}} (\sin 2x + \sin 8x) \, dx$$

$$= \frac{1}{2}\left[-\frac{1}{2}\cos 2x - \frac{1}{8}\cos 8x\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{2}\left(-\frac{1}{8} + \frac{1}{2} + \frac{1}{8}\right)$$

$$= \frac{1}{4}$$

5a

$$\int_0^{\frac{\pi}{3}} x \sin 3x \, dx$$

$$\int uv' \, dx = uv - \int u'v \, dx$$

$$u = x, v' = \sin 3x$$

$$u' = 1, v = -\frac{1}{3}\cos 3x$$

Hence

$$\int_0^{\frac{\pi}{3}} x \sin 3x \, dx$$

$$= \left[-\frac{1}{3}x\cos 3x\right]_0^{\frac{\pi}{3}} + \int_0^{\frac{\pi}{3}} \frac{1}{3}\cos 3x \, dx$$

$$= \frac{\pi}{9} + \frac{1}{3}\left[\frac{1}{3}\sin 3x\right]_0^{\frac{\pi}{3}}$$

$$= \frac{\pi}{9} + \frac{1}{9}[\sin 3x]_0^{\frac{\pi}{3}}$$

$$= \frac{\pi}{9} + \frac{1}{9}(0)$$

$$= \frac{\pi}{9}$$

5b

$$\int_0^2 \frac{3-7x}{\sqrt{4x-x^2}}\,dx$$

$$= \int_0^2 \frac{3-7x}{\sqrt{4-(x-2)^2}}\,dx$$

Let $u = x - 2$

$x = u + 2$

$du = dx$

$x = 2, u = 0$

$x = 0, u = -2$

Hence

$$\int_0^2 \frac{3-7x}{\sqrt{4-(x-2)^2}}\,dx$$

$$= \int_{-2}^0 \frac{3-7(u+2)}{\sqrt{4-u^2}}\,du$$

$$= \int_{-2}^0 \frac{3-7u-14}{\sqrt{4-u^2}}\,du$$

$$= \int_{-2}^0 \frac{-11-7u}{\sqrt{4-u^2}}\,du$$

$$= \int_{-2}^0 \left(-\frac{11}{\sqrt{4-u^2}} - \frac{7u}{\sqrt{4-u^2}}\right)du$$

$$= \int_{-2}^0 \left(-\frac{11}{\sqrt{4-u^2}} + 7 \times \frac{-2u}{2\sqrt{4-u^2}}\right)du$$

$$= \left[-11\sin^{-1}\left(\frac{u}{2}\right) + 7\sqrt{4-u^2}\right]_{-2}^0$$

$$= 14 + 11\left(-\frac{\pi}{2}\right)$$

$$= 14 - \frac{11\pi}{2}$$

5c

$$\int_0^{\frac{\pi}{2}} \sin^2 x \cos^3 x \, dx$$

$$= \int_0^{\frac{\pi}{2}} \sin^2 x \cos^2 x \cos x \, dx$$

$$= \int_0^{\frac{\pi}{2}} \sin^2 x \,(1 - \sin^2 x) \cos x \, dx$$

Let $u = \sin x$

$du = \cos x \, dx$

$x = \frac{\pi}{2}, u = 1$

$x = 0, u = 0$

Hence

$$\int_0^{\frac{\pi}{2}} \sin^2 x \,(1 - \sin^2 x) \cos x \, dx$$

$$= \int_0^1 u^2(1 - u^2) \, du$$

$$= \int_0^1 (u^2 - u^4) \, du$$

$$= \left[\frac{1}{3}u^3 - \frac{1}{5}u^5\right]_0^1$$

$$= \frac{1}{3} - \frac{1}{5}$$

$$= \frac{2}{15}$$

5d

$$\int_0^3 \frac{x^2+x+18}{x^3+9x^2+9x+81}\,dx$$

$$=\int_0^3 \frac{x^2+x+18}{(x+9)(x^2+9)}\,dx$$

$$\frac{x^2+x+18}{(x+9)(x^2+9)}=\frac{A}{x+9}+\frac{Bx+C}{x^2+9}$$

Using cover-up method:

$$A=\frac{(-9)^2+(-9)+18}{(-9)^2+9}=1$$

$$x^2+x+18=x^2+9+(Bx+C)(x+9)$$

$$x^2+x+18=x^2+9+Bx^2+Cx+9Bx+9C$$

$$x^2+x+18=(B+1)x^2+(9B+C)x+9C+9$$

Equating coefficients gives:

$$B+1=1$$

$$\therefore B=0$$

$$9C+9=18$$

$$\therefore C=1$$

Hence

$$\int_0^3 \frac{x^2+x+18}{(x+9)(x^2+9)}\,dx$$

$$=\int_0^3 \left(\frac{1}{x+9}+\frac{1}{x^2+9}\right)dx$$

$$=\left[\ln|x+9|+\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)\right]_0^3$$

$$=\ln 12+\frac{\pi}{12}-\ln 9$$

$$=\frac{\pi}{12}+\ln\frac{4}{3}$$

5e

$$\int_2^4 \sqrt{16-x^2}\,dx$$

Let $x = 4\sin u$

$$u = \sin^{-1}\left(\frac{x}{4}\right)$$

$$dx = 4\cos u\,du$$

$$x = 4, u = \frac{\pi}{2}$$

$$x = 2, u = \frac{\pi}{6}$$

Hence

$$\int_2^4 \sqrt{16-x^2}\,dx$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sqrt{16-16\sin^2 u} \times 4\cos u\,du$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 4\cos u\, 4\cos u\,du$$

$$= 16\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos^2 u\,du$$

$$= 8\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (1+\cos 2u)\,du$$

$$= 8\left[u + \frac{1}{2}\sin 2u\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= 8\left(\frac{\pi}{2} + 0 - \frac{\pi}{6} - \frac{\sqrt{3}}{4}\right)$$

$$= 8\left(\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right)$$

$$= \frac{8\pi}{3} - 2\sqrt{3}$$

5f

$$\int_0^{\frac{1}{2}} e^{2x} \sin \pi x \, dx$$

$$\int uv' \, dx = uv - \int u'v \, dx$$

$$u = e^{2x}, v' = \sin \pi x$$

$$u' = 2e^{2x}, v = -\frac{1}{\pi} \cos \pi x$$

Hence

$$\int_0^{\frac{1}{2}} e^{2x} \sin \pi x \, dx$$

$$= \left[-\frac{1}{\pi} e^{2x} \cos \pi x\right]_0^{\frac{1}{2}} + \frac{2}{\pi} \int_0^{\frac{1}{2}} e^{2x} \cos \pi x \, dx$$

$$= \frac{1}{\pi} + \frac{2}{\pi} \int_0^{\frac{1}{2}} e^{2x} \cos \pi x \, dx$$

Consider $\int_0^{\frac{1}{2}} e^{2x} \cos \pi x \, dx$

$$\int uv' \, dx = uv - \int u'v \, dx$$

$$u = e^{2x}, v' = \cos \pi x$$

$$u' = 2e^{2x}, v = \frac{1}{\pi} \sin \pi x$$

Hence

$$\int_0^{\frac{1}{2}} e^{2x} \cos \pi x \, dx$$

$$= \left[\frac{1}{\pi} e^{2x} \sin \pi x\right]_0^{\frac{1}{2}} - \frac{2}{\pi} \int_0^{\frac{1}{2}} e^{2x} \sin \pi x \, dx$$

$$= \frac{e}{\pi} - \frac{2}{\pi} \int_0^{\frac{1}{2}} e^{2x} \sin \pi x \, dx$$

Therefore

$$\int_0^{\frac{1}{2}} e^{2x} \sin \pi x \, dx$$

$$= \frac{1}{\pi} + \frac{2}{\pi} \int_0^{\frac{1}{2}} e^{2x} \cos \pi x \, dx$$

$$= \frac{1}{\pi} + \frac{2}{\pi} \left(\frac{e}{\pi} - \frac{2}{\pi} \int_0^{\frac{1}{2}} e^{2x} \sin \pi x \, dx \right)$$

$$\int_0^{\frac{1}{2}} e^{2x} \sin \pi x \, dx = \frac{1}{\pi} + \frac{2e}{\pi^2} - \frac{4}{\pi^2} \int_0^{\frac{1}{2}} e^{2x} \sin \pi x \, dx$$

$$\left(1 + \frac{4}{\pi^2}\right) \int_0^{\frac{1}{2}} e^{2x} \sin \pi x \, dx = \frac{1}{\pi} + \frac{2e}{\pi^2}$$

$$\frac{\pi^2 + 4}{\pi^2} \int_0^{\frac{1}{2}} e^{2x} \sin \pi x \, dx = \frac{1}{\pi} + \frac{2e}{\pi^2}$$

$$\int_0^{\frac{1}{2}} e^{2x} \sin \pi x \, dx = \left(\frac{1}{\pi} + \frac{2e}{\pi^2}\right) \left(\frac{\pi^2}{\pi^2 + 4}\right)$$

$$\int_0^{\frac{1}{2}} e^{2x} \sin \pi x \, dx = (\pi + 2e) \frac{1}{\pi^2 + 4}$$

$$\int_0^{\frac{1}{2}} e^{2x} \sin \pi x \, dx = \frac{2e + \pi}{4 + \pi^2}$$

6a

$$\int_8^{15} \frac{1}{(x-3)\sqrt{x+1}} dx$$

Let $u = \sqrt{x+1}$

$x = u^2 - 1$

$dx = 2u \, du$

$x = 15, u = 4$

$x = 8, u = 3$

Hence

$$\int_8^{15} \frac{1}{(x-3)\sqrt{x+1}}\,dx$$

$$= \int_3^4 \frac{1}{(u^2-1-3)u} 2u\,du$$

$$= \int_3^4 \frac{2}{u^2-4}\,du$$

$$= \int_3^4 \frac{2}{(u-2)(u+2)}\,du$$

$$\frac{2}{(u-2)(u+2)} = \frac{A}{u-2} + \frac{B}{u+2}$$

Using cover-up method:

$$A = \frac{2}{2+2} = \frac{1}{2}$$

$$B = \frac{2}{-2-2} = -\frac{1}{2}$$

Hence

$$\int_3^4 \frac{2}{(u-2)(u+2)}\,du$$

$$= \int_3^4 \left(\frac{\frac{1}{2}}{u-2} - \frac{\frac{1}{2}}{u+2} \right) du$$

$$= \left[\frac{1}{2}\ln|u-2| - \frac{1}{2}\ln|u+2| \right]_3^4$$

$$= \frac{1}{2}\ln 2 - \frac{1}{2}\ln 6 - \frac{1}{2}\ln 1 + \frac{1}{2}\ln 5$$

$$= \frac{1}{2}(\ln 2 - \ln 3 - \ln 2 + \ln 5)$$

$$= \frac{1}{2}(-\ln 3 + \ln 5)$$

$$= \frac{1}{2}\ln\frac{5}{3}$$

6b

$$\int_0^{\frac{\pi}{3}} \frac{1}{9 - 8\sin^2 x}\, dx$$

$$t = \tan x$$

$$dt = \sec^2 x\, dx$$

$$dt = (1 + \tan^2 x)\, dx$$

$$dt = (1 + t^2)\, dx$$

$$dx = \frac{1}{1 + t^2}\, dt$$

$$x = \frac{\pi}{3}, t = \sqrt{3}$$

$$x = 0, t = 0$$

$$\sin^2 x = \frac{\tan^2 x}{1 + \tan^2 x}$$

$$\sin^2 x = \frac{t^2}{1 + t^2}$$

Hence

$$\int_0^{\frac{\pi}{3}} \frac{1}{9 - 8\sin^2 x}\, dx$$

$$= \int_0^{\sqrt{3}} \frac{1}{9 - 8\frac{t^2}{1 + t^2}} \times \frac{1}{1 + t^2}\, dt$$

$$= \int_0^{\sqrt{3}} \frac{1}{9 + 9t^2 - 8t^2}\, dt$$

$$= \int_0^{\sqrt{3}} \frac{1}{t^2 + 9}\, dt$$

$$= \left[\frac{1}{3}\tan^{-1}\left(\frac{t}{3}\right)\right]_0^{\sqrt{3}}$$

$$= \frac{\pi}{18}$$

6c

$$\int_0^2 \sqrt{x(4-x)}\,dx$$

Let $x = 4\sin^2\theta$

$$\theta = \sin^{-1}\left(\frac{\sqrt{x}}{2}\right)$$

$$dx = 8\sin\theta\cos\theta\,d\theta$$

$$x = 2, \theta = \frac{\pi}{4}$$

$$x = 0, \theta = 0$$

Hence

$$\int_0^2 \sqrt{x(4-x)}\,dx$$

$$= \int_0^{\frac{\pi}{4}} \sqrt{4\sin^2\theta\,(4-4\sin^2\theta)} \times 8\sin\theta\cos\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \sqrt{4\sin^2\theta\,4\cos^2\theta} \times 8\sin\theta\cos\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} 4\sin\theta\cos\theta\,8\sin\theta\cos\theta\,d\theta$$

$$= 32\int_0^{\frac{\pi}{4}} \sin^2\theta\cos^2\theta\,d\theta$$

$$= 32\int_0^{\frac{\pi}{4}} (\sin\theta\cos\theta)^2\,d\theta$$

$$= 32\int_0^{\frac{\pi}{4}} \left(\frac{1}{2}\sin 2\theta\right)^2 d\theta$$

$$= 8\int_0^{\frac{\pi}{4}} \sin^2 2\theta\,d\theta$$

$$= 8\int_0^{\frac{\pi}{4}} \frac{1}{2}(1-\cos 4\theta)\,d\theta$$

$$= 4\int_0^{\frac{\pi}{4}} (1 - \cos 4\theta)\, d\theta$$

$$= 4\left[\theta - \frac{1}{4}\sin 4\theta\right]_0^{\frac{\pi}{4}}$$

$$= 4\left(\frac{\pi}{4}\right)$$

$$= \pi$$

6d

$$\int_0^{\frac{\pi}{3}} \frac{1}{\cos x}\, dx$$

Let $t = \tan\left(\frac{x}{2}\right)$

$$dx = \frac{2}{1+t^2}\, dt$$

$$x = \frac{\pi}{3}, t = \frac{1}{\sqrt{3}}$$

$$x = 0, t = 0$$

$$\cos x = \frac{1-t^2}{1+t^2}$$

Hence

$$\int_0^{\frac{\pi}{3}} \frac{1}{\cos x}\, dx$$

$$= \int_0^{\frac{1}{\sqrt{3}}} \frac{1}{\frac{1-t^2}{1+t^2}} \times \frac{2}{1+t^2}\, dt$$

$$= \int_0^{\frac{1}{\sqrt{3}}} \frac{1+t^2}{1-t^2} \times \frac{2}{1+t^2}\, dt$$

$$= \int_0^{\frac{1}{\sqrt{3}}} \frac{2}{1-t^2}\, dt$$

$$= \int_0^{\frac{1}{\sqrt{3}}} \frac{2}{(1-t)(1+t)}\, dt$$

$$= \int_0^{\frac{1}{\sqrt{3}}} \left(\frac{1}{1-t} + \frac{1}{1+t}\right) dt \qquad \text{(using cover up method)}$$

$$= [-\ln|1-t| + \ln|1+t|]_0^{\frac{1}{\sqrt{3}}}$$

$$= -\ln\left(1 - \frac{1}{\sqrt{3}}\right) + \ln\left(1 + \frac{1}{\sqrt{3}}\right)$$

$$= \ln\left(\frac{1 + \frac{1}{\sqrt{3}}}{1 - \frac{1}{\sqrt{3}}}\right)$$

$$= \ln\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)$$

$$= \ln\left(\frac{\sqrt{3}+1}{\sqrt{3}-1} \times \frac{\sqrt{3}+1}{\sqrt{3}+1}\right)$$

$$= \ln\left(\frac{3 + 2\sqrt{3} + 1}{2}\right)$$

$$= \ln(2+\sqrt{3})$$

7a

$$I_n = \int_0^1 x^n e^x \, dx$$

$$\int uv' \, dx = uv - \int u'v \, dx$$

$u = x^n, v' = e^x$

$u' = nx^{n-1}, v = e^x$

Hence

$$I_n = [x^n e^x]_0^1 - n \int_0^1 x^{n-1} e^x \, dx$$

$$I_n = e - nI_{n-1}$$

7b

$I_0 = \int_0^1 x^0 e^x \, dx$

$= \int_0^1 e^x \, dx$

$= e - 1$

$I_1 = e - (1)I_0$

$= e - (e - 1)$

$= 1$

$I_2 = e - (2)I_1$

$= e - 2$

$I_3 = e - (3)I_2$

$= e - 3e + 6$

$= -2e + 6$

$I_4 = e - (4)I_3$

$= e + 8e - 24$

$= 9e - 24$

$I_5 = e - (5)I_4$

$= e - 45e + 120$

$= 120 - 44e$

8a

$I_n = \int x^3 (\ln x)^n \, dx$

$\int uv' \, dx = uv - \int u'v \, dx$

$u = (\ln x)^n, v' = x^3$

$u' = \dfrac{n(\ln x)^{n-1}}{x}, v = \dfrac{1}{4}x^4$

Hence

$$I_n = \frac{1}{4}x^4(\ln x)^n - \frac{1}{4}n\int x^3(\ln x)^{n-1}\,dx$$

$$I_n = \frac{1}{4}x^4(\ln x)^n - \frac{1}{4}nI_{n-1}$$

8b

$$I_0 = \int x^3(\ln x)^0\,dx$$

$$= \int x^3\,dx$$

$$= \frac{1}{4}x^4$$

$$I_1 = \frac{1}{4}x^4\ln x - \frac{1}{4}(1)I_0$$

$$= \frac{1}{4}x^4\ln x - \frac{1}{16}x^4$$

$$I_2 = \frac{1}{4}x^4(\ln x)^2 - \frac{1}{4}(2)I_1$$

$$= \frac{1}{4}x^4(\ln x)^2 - \frac{1}{2}\left(\frac{1}{4}x^4\ln x - \frac{1}{16}x^4\right)$$

$$= \frac{1}{4}x^4(\ln x)^2 - \frac{1}{8}x^4\ln x + \frac{1}{32}x^4$$

$$I_3 = \frac{1}{4}x^4(\ln x)^3 - \frac{1}{4}(3)I_2$$

$$= \frac{1}{4}x^4(\ln x)^3 - \frac{3}{4}\left(\frac{1}{4}x^4(\ln x)^2 - \frac{1}{8}x^4\ln x + \frac{1}{32}x^4\right)$$

$$= \frac{1}{4}x^4(\ln x)^3 - \frac{3}{16}x^4(\ln x)^2 + \frac{3}{32}x^4\ln x - \frac{3}{128}x^4 + C$$

$$= \frac{1}{128}x^4(32(\ln x)^3 - 24(\ln x)^2 + 12\ln x - 3) + C$$

9a

$$I_{2n} = \int_0^{\frac{\pi}{2}} \sin^{2n} x \, dx$$

$$I_{2n} = \int_0^{\frac{\pi}{2}} \sin^{2n-1} x \sin x \, dx$$

$$\int uv' \, dx = uv - \int u'v \, dx$$

$$u = \sin^{2n-1} x \, , v' = \sin x$$

$$u' = (2n-1) \sin^{2\mathrm{n}-2} x \cos x \, , v = -\cos x$$

Hence

$$I_{2n} = [-\sin^{2n-1} x \cos x]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} (2n-1) \sin^{2n-2} x \cos x \cos x \, dx$$

$$= \int_0^{\frac{\pi}{2}} (2n-1) \sin^{2n-2} x \cos^2 x \, dx$$

$$= (2n-1) \int_0^{\frac{\pi}{2}} \sin^{2n-2} x \, (1 - \sin^2 x) \, dx$$

$$= (2n-1) \int_0^{\frac{\pi}{2}} \sin^{2n-2} x - \sin^{2n} x \, dx$$

$$= (2n-1)(I_{2n-2} - I_{2n})$$

$$= (2n-1)I_{2n-2} - (2n-1)I_{2n}$$

$$I_{2n} + (2n-1)I_{2n} = (2n-1)I_{2n-2}$$

$$2nI_{2n} = (2n-1)I_{2n-2}$$

$$I_{2n} = \frac{2n-1}{2n} I_{2n-2}$$

9b

$$I_0 = \int_0^{\frac{\pi}{2}} \sin^0 x \, dx$$

$$= \int_0^{\frac{\pi}{2}} 1 \, dx$$

$$= \frac{\pi}{2}$$

$$I_2 = \frac{2(1) - 1}{2(1)} I_0$$

$$= \frac{\pi}{4}$$

$$I_4 = \frac{2(2) - 1}{2(2)} I_2$$

$$= \frac{3\pi}{16}$$

$$I_6 = \frac{2(3) - 1}{2(3)} I_4$$

$$= \frac{5}{6} \times \frac{3\pi}{16}$$

$$= \frac{5\pi}{32}$$

10a

$$I_n = \int_0^1 (1 + x^2)^n \, dx$$

$$\int uv' \, dx = uv - \int u'v \, dx$$

$$u = (1 + x^2)^n, v' = 1$$

$$u' = 2nx(1 + x^2)^{n-1}, v = x$$

Hence

$$I_n = [x(1 + x^2)^n]_0^1 - 2n \int_0^1 x^2(1 + x^2)^{n-1} \, dx$$

$$= 2^n - 2n\int_0^1 (1 + x^2 - 1)(1 + x^2)^{n-1}\,dx$$

$$= 2^n - 2n\int_0^1 ((1 + x^2)^n - (1 + x^2)^{n-1})\,dx$$

$$= 2^n - 2n(I_n - I_{n-1})$$

$$= 2^n - 2nI_n + 2nI_{n-1}$$

$$I_n + 2nI_n = 2^n + 2nI_{n-1}$$

$$(2n + 1)I_n = 2^n + 2nI_{n-1}$$

10b

$$J_n = \int_0^{\frac{\pi}{4}} \sec^{2n}\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \sec^2\theta \sec^{2n-2}\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \sec^2\theta\,(\sec^2\theta)^{n-1}\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \sec^2\theta\,(1 + \tan^2\theta)^{n-1}\,d\theta$$

Let $u = \tan\theta$

$du = \sec^2\theta\,d\theta$

$\theta = \frac{\pi}{4}, u = 1$

$\theta = 0, u = 0$

$$J_n = \int_0^1 (1 + u^2)^{n-1}\,du$$

$\therefore J_n = I_{n-1}$

$(2n + 1)J_{n+1} = 2^n + 2nJ_n$

Substitute $n = n - 1$

$(2(n - 1) + 1)J_n = 2^{n-1} + 2(n - 1)J_{n-1}$

$(2n - 1)J_n = 2^{n-1} + 2(n - 1)J_{n-1}$

10c

$$\int_0^{\frac{\pi}{4}} \sec^6\theta\, d\theta = J_3$$

$$J_0 = \int_0^{\frac{\pi}{4}} \sec^0\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} 1\, d\theta$$

$$= \frac{\pi}{4}$$

$$J_n = \frac{1}{2n-1}\left(2^{n-1} + 2(n-1)J_{n-1}\right)$$

$$J_1 = \frac{1}{2(1)-1}\left(2^{(1)-1} + 2\big((1)-1\big)J_0\right)$$

$$= 1$$

$$J_2 = \frac{1}{2(2)-1}\left(2^{(2)-1} + 2\big((2)-1\big)J_1\right)$$

$$= \frac{1}{3}\big(2 + 2(1)\big)$$

$$= \frac{4}{3}$$

$$J_3 = \frac{1}{2(3)-1}\left(2^{(3)-1} + 2\big((3)-1\big)J_2\right)$$

$$= \frac{1}{5}\left(4 + \frac{16}{3}\right)$$

$$= \frac{1}{5}\left(\frac{28}{3}\right)$$

$$= \frac{28}{15}$$

$$\therefore \int_0^{\frac{\pi}{4}} \sec^6\theta\, d\theta = \frac{28}{15}$$

11a

$$I_n = \int \frac{\sin 2nx}{\sin x}\,dx$$

$$= \int \frac{\sin 2nx\,(2\sin^2 x + \cos 2x)}{\sin x}\,dx$$

$$= \int \frac{2\sin^2 x \sin 2nx + \sin 2nx \cos 2x}{\sin x}\,dx$$

$$= \int \frac{2\sin^2 x \sin 2nx + \sin 2nx \cos 2x + (\sin 2x \cos 2nx - \sin 2x \cos 2nx)}{\sin x}\,dx$$

$$= \int \frac{(2\sin x \cos x)\cos 2nx + 2\sin^2 x \sin 2nx + \sin 2nx \cos 2x - \sin 2x \cos 2nx}{\sin x}\,dx$$

$$= \int \frac{2\sin x\,(\cos x \cos 2nx + \sin x \sin 2nx) + \sin 2nx \cos 2x - \sin 2x \cos 2nx}{\sin x}\,dx$$

$$= \int \frac{2\sin x\,(\cos(2nx - x)) + \sin(2nx - 2x)}{\sin x}\,dx$$

$$= \int \left(2\cos(2n-1)x + \frac{\sin(2(n-1)x)}{\sin x}\right)dx$$

$$\therefore I_n = \frac{2}{2n-1}\sin(2n-1)x + I_{n-1}$$

11b

$$\int_0^{\frac{\pi}{2}} \frac{\sin 6x}{\sin x}\,dx = I_3$$

$$I_1 = \int_0^{\frac{\pi}{2}} \frac{\sin 2x}{\sin x}\,dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{2\sin x \cos x}{\sin x}\,dx$$

$$= \int_0^{\frac{\pi}{2}} 2\cos x\,dx$$

$$= 2[\sin x]_0^{\frac{\pi}{2}}$$

$$= 2$$

$$I_n = \left[\frac{2}{2n-1}\sin\big((2n-1)x\big)\right]_0^{\frac{\pi}{2}} + I_{n-1}$$

$$= \frac{2}{2n-1}\sin\left(\frac{(2n-1)\pi}{2}\right) + I_{n-1}$$

$$I_2 = \frac{2}{2(2)-1}\sin\left(\frac{(2(2)-1)\pi}{2}\right) + I_1$$

$$= -\frac{2}{3} + 2$$

$$= \frac{4}{3}$$

$$I_3 = \frac{2}{2(3)-1}\sin\left(\frac{(2(3)-1)\pi}{2}\right) + I_2$$

$$= \frac{2}{5} + \frac{4}{3}$$

$$= \frac{26}{15}$$

$$\int_0^{\frac{\pi}{2}} \frac{\sin 6x}{\sin x}\,dx = \frac{26}{15}$$

12a

$$\int_0^a f(x)\,dx$$

Let $u = a - x$

$x = a - u$

$du = -dx$

$x = a, u = 0$

$x = 0, u = a$

Hence

$$\int_0^a f(x)\,dx$$

$$= -\int_1^0 f(a-u)\,du$$

$$= \int_0^1 f(a-u)\,du$$

$$= \int_0^1 f(a-x)\,dx$$

12b

$$I = \int_0^{\pi} \frac{x \sin x}{3+\sin^2 x}\,dx$$

$$I = \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{3+\sin^2(\pi - x)}\,dx$$

$$I = \int_0^{\pi} \frac{(\pi - x)\sin x}{3+\sin^2 x}\,dx$$

$$2I = \int_0^{\pi} \frac{x \sin x}{3+\sin^2 x}\,dx + \int_0^{\pi} \frac{(\pi - x)\sin x}{3+\sin^2 x}\,dx$$

$$2I = \int_0^{\pi} \frac{x \sin x + \pi \sin x - x \sin x}{3+\sin^2 x}\,dx$$

$$2I = \int_0^{\pi} \frac{\pi \sin x}{3+\sin^2 x}\,dx$$

$$2I = \int_0^{\pi} \frac{\pi \sin x}{4-\cos^2 x}\,dx$$

Let $u = \cos x$

$du = -\sin x\,dx$

$x = \pi, u = -1$

$x = 0, u = 1$

Hence

$$2I = -\int_1^{-1} \frac{\pi}{4-u^2}\,du$$

$$2I = \pi \int_{-1}^{1} \frac{1}{4-u^2}\,du$$

$$2I = \pi \int_{-1}^{1} \left(\frac{\frac{1}{4}}{2-u} + \frac{\frac{1}{4}}{2+u} \right) du$$

$$2I = \frac{\pi}{4} \int_{-1}^{1} \left(\frac{1}{2-u} + \frac{1}{2+u} \right) du$$

$$I = \frac{\pi}{8}[-\ln|2-u| + \ln|2+u|]_{-1}^{1}$$

$$I = \frac{\pi}{8}(-\ln 1 + \ln 3 + \ln 3 - \ln 1)$$

$$I = \frac{\pi}{8} \times 2\ln 3$$

$$I = \frac{\pi \ln 3}{4}$$