Solutions to Exercise 3A

1a i $\quad -1 \le x \le 2$

1a ii $\quad [-1, 2]$

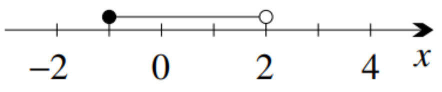
1b i $\quad -1 < x \le 2$

1b ii $\quad (-1, 2]$

1c i $\quad x > -1$

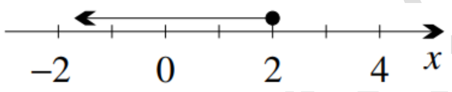
1c ii $\quad (-1, \infty)$

2a i



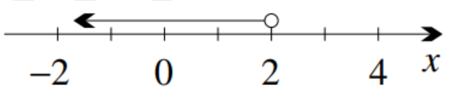
2a ii $\quad [-1, 2)$

2b i



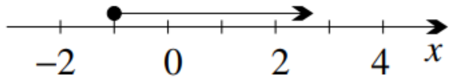
2b ii $\quad (-\infty, 2]$

2c i



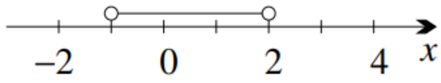
2c ii $\quad (-\infty, 2)$

3a i



3a ii $x \geq -1$

3b i



3b ii $-1 < x < 2$

3c i

3c ii R (all real numbers). Note that there is no way of writing this in terms of inequalities.

4a i $g \circ f(3) = g\big(f(3)\big) = g(3+1) = g(4) = 2^4 = 16$

4a ii $f \circ g(3) = f\big(g(3)\big) = f(2^3) = f(8) = 8+1 = 9$

4a iii $g \circ g(3) = g\big(g(3)\big) = g(2^3) = g(8) = 2^8 = 256$

4a iv $f \circ f(3) = f\big(f(3)\big) = f(3+1) = f(4) = 4+1 = 5$

4b i $g \circ f(x) = g\big(f(x)\big) = g(x+1) = 2^{x+1}$

4b ii $f \circ g(x) = f(g(x)) = f(2^x) = 2^x + 1$

4b iii $g \circ g(x) = g(g(x)) = g(2^x) = 2^{2^x} = (2^2)^x = 4^x$

4b iv $f \circ f(x) = f(f(x)) = f(x+1) = x + 1 + 1 = x + 2$

5a The function is negative for all x values where the function is below the x-axis.

$(-\infty, 1)$

5b The function is negative for all x values where the function is below the x-axis.

$(0, 2)$

5c The function is negative for all x values where the function is below the x-axis.

$(0, 1)$

5d The function is negative for all x values where the function is below the x-axis.

$(4, \infty)$

6a The inequation is true when the curve is above, or on, the x-axis. This is when

$-1 \le x \le 0$ or $x \ge 1$

6b The inequation is true when the curve is below, or on, the x-axis. This is when

$-5 \le x \le -2$ or $x \ge 1$

6c The inequation is true when the curve is strictly above the x-axis. This is when

$x < -2$ or $x > 4$

6d The inequation is true when the curve is above, or on, the x-axis. This is when

$-2 \leq x \leq 2$

6e The inequation is true when the curve is strictly below the x-axis. This is when

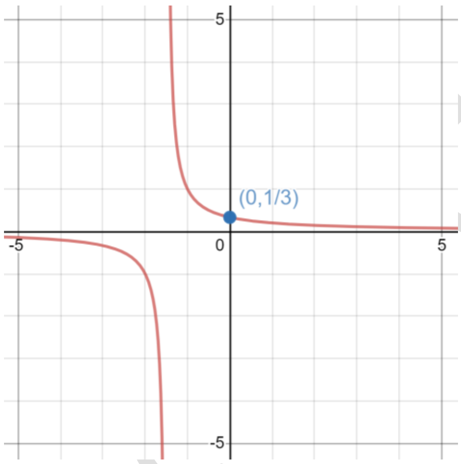
$0 < x < 2$ or $x < -2$

6f The inequation is true when the curve is below, or on, the x-axis. This is when

$-1 \leq x < 0$ or $2 < x \leq 3$

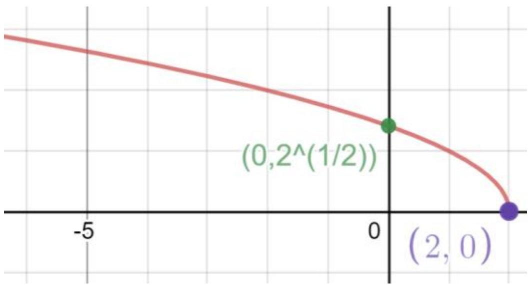
7a The curve is defined for all values of x such that the denominator of $f(x) = \frac{1}{2x+3}$ is non-zero. This is when $2x + 3 \neq 0$ or $x \neq -\frac{3}{2}$.

Hence the domain is all x where $x \neq -\frac{3}{2}$.



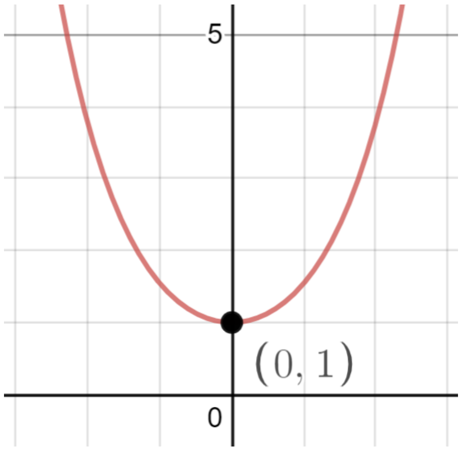
7b The curve is defined for all values of x such that the values under the square root sign $f(x) = \sqrt{2 - x}$ is greater than or equal to zero. This is when $2 - x \geq 0$.

Hence the domain is $2 \geq x$ or $x \leq 2$.



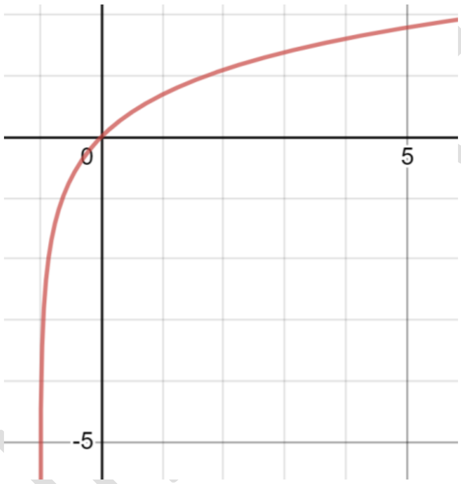
7c The function is defined for all values of x.

Hence the domain is all x. We can see this on the graph below.



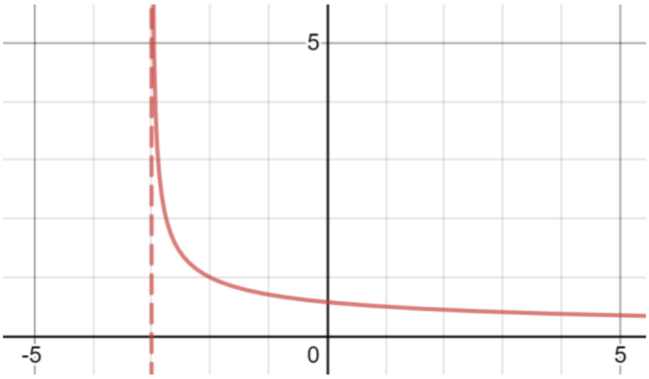
7d The function $f(x) = \log_e(x + 1)$ is defined for all values of x such that $x + 1 > 0$.

Hence the domain is all values of x such that $x > -1$.



7e Note, in order for this function to be defined, the values inside the square root function must be greater than or equal to zero. The denominator of the function must also be non-zero. Hence we must have $x + 3 > 0$.

Thus the domain is all x such that $x > -3$.

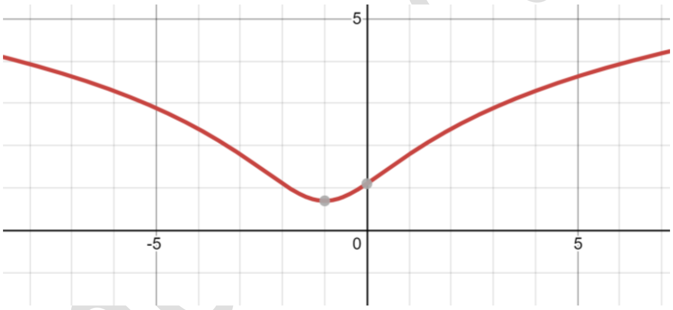


7f The function is defined for all values such that the expression inside the logarithm is greater than zero. Hence this is when:

$x^2 + 2x + 3 > 0$

$(x + 1)^2 + 2 > 0$

This is true for all values of x. Hence the domain is all values of x.



8a i Note that an open (white) circle, indicates an open interval so we use $<$ and $>$ for this section of the interval.

$-1 < x < 1$ or $2 \le x \le 3$

8a ii Note that an open (white) circle, indicates an open interval so we use rounded brackets for this section of the interval.

$(-1, 1) \cup [2, 3]$

8b i Note that an open (white) circle, indicates an open interval so we use $<$ and $>$ for this section of the interval.

$x < 1$ or $x \geq 2$

8b ii Note that an open (white) circle, indicates an open interval so we use rounded brackets for this section of the interval.

$(-\infty, 1) \cup [2, \infty)$

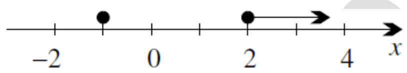
8c i Note that an open (white) circle, indicates an open interval so we use $<$ and $>$ for this section of the interval.

$x < 1$ or $2 \leq x < 3$

8c ii Note that an open (white) circle, indicates an open interval so we use rounded brackets for this section of the interval.

$(-\infty, 1) \cup [2, 3)$

9a i

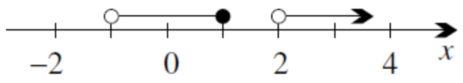


9a ii $[-1, -1] \cup [2, \infty)$

9b i

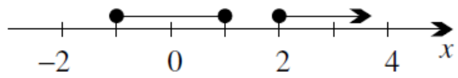
9b ii $(-\infty, -1] \cup (2, 3]$

9c i



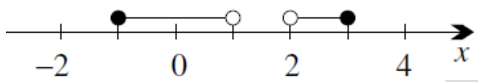
9c ii $(-1, 1] \cup (2, \infty)$

10a i Note that square brackets denote closed intervals and are denoted by a black circle when drawn on a number line.



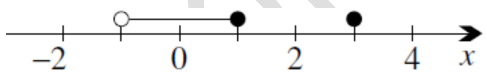
10a ii $-1 \leq x \leq 1$ or $x \geq 2$

10b i



10b ii $-1 \leq x < 1$ or $x \geq 2$

10c i



10c ii $-1 < x \leq 1$ or $x = 3$

11a $[-1, 0] \cup [1, \infty)$

11b $[-5, -2] \cup [1, \infty)$

11c $(-\infty, -2) \cup (4, \infty)$

11d $[-2, 2]$

11e $(-\infty, -2) \cup (0, 2)$

11f $[-1, 0) \cup (2, 3]$

12a The function is undefined when the denominator is zero, that is when $x = -1$.
The zeroes occur when $\frac{x-2}{x+1} = 0$, that is when $x = 2$.

Drawing a table of values gives

x	-2	-1	0	1	2	3
$f(x)$	4	$*$	-2	$-\frac{1}{2}$	0	$\frac{1}{4}$
sign	$+$	$*$	$-$	$-$	$*$	$+$

Thus, the inequation is true when $x < -1$ and when $x \geq 2$.

12b The function is undefined when the denominator is zero, that is when

$(x-3)(x+1) = 0$, or when $x = -1$ or $x = 3$.

The zeroes occur when $\frac{x-1}{x^2-2x-3} = 0$, that is when $x = 1$.

Drawing a table of values gives

x	-2	-1	0	1	2	3	4
$f(x)$	$-\frac{3}{5}$	$*$	$\frac{1}{3}$	0	$-\frac{1}{3}$	$*$	$\frac{3}{5}$
sign	$-$	$*$	$+$	$*$	$-$	$*$	$+$

Thus, the inequation is true when $-1 < x \leq 1$ and when $x > 3$.

12c The function is undefined when the denominator is zero, that is when

$x - 2 = 0$, ie when $x = 2$. The zeroes occur when $\frac{x^2+2x+1}{x-2} = \frac{(x+1)^2}{x-2} = 0$, that is when $x = -1$.

Drawing a table of values gives

x	-2	-1	0	1	2	3
$f(x)$	$-\frac{1}{4}$	0	$-\frac{1}{2}$	-4	$*$	16
sign	$-$	$*$	$-$	$-$	$*$	$+$

Thus, the inequation is true when $x < -1$ and when $-1 < x < 2$.

13a The square root function is only defined for $x \geq 0$, this is our first restriction. We also cannot divide by zero, thus $x^2 - 1 \neq 0$, $x^2 \neq 1$ and thus $x \neq \pm 1$. Hence the domain is $[0, 1) \cup (1, \infty)$.

13b The domain is all x such that the denominator is non-zero, and such that the expression inside of the square root is non-negative.

That is when $x^2 - 5x - 6 > 0$ which is when $(x - 6)(x + 1) > 0$.

As $y = x^2 - 5x - 6$ is a concave up parabola, it is greater than zero when $x < -1$ and when $x > 6$. In interval notation this is $(-\infty, -1) \cup (6, \infty)$.

13c The domain is all x such that the denominator is non-zero, and such that the expression inside the square root is non-negative. That is when $3 + 2x - x^2 > 0$ which is when $(1 + x)(3 - x) > 0$.

As $y = 3 + 2x - x^2$ is a concave down parabola, it is greater than zero when $-1 < x < 3$. In interval notation this is $(-1, 3)$.

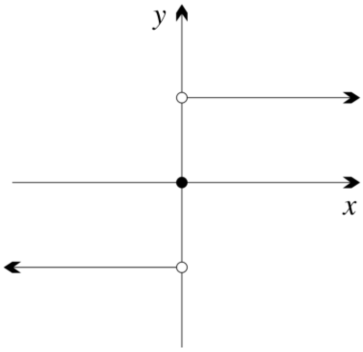
13d The domain is all x such that the denominator is non-zero, and such that the expression inside the square root is non-negative. That is when $x^2 - 2x + 3 > 0$ which is when $x^2 - 2x + 1 + 2 > 0$ and thus when $(x - 1)^2 + 2 > 0$. This is true for all values of x and hence in interval notation the domain is $(-\infty, \infty)$.

14a When $x < 0$, $f(x) = \frac{-x}{x} = -1$.

When $x = 0$, $f(x) = 0$.

When $x > 0$, $f(x) = \frac{x}{x} = 1$.

Drawing this gives:



14b By observation of the graph, $f(x) \geq 0$ when $x \geq 0$.

15a The function is defined for all values such that the denominator is non-zero. Hence this is when $e^x - e^{-x} = 0$. Solving this gives:

$$e^x = e^{-x}$$

$$\frac{e^x}{e^{-x}} = 1$$

$$e^{2x} = 1$$

$$2x = 0$$

$$x = 0$$

Thus, the only point at which the function is undefined is when $x = 0$. Thus the domain is all values of x such that $x \neq 0$. That is $x \in (-\infty, 0) \cup (0, \infty)$.

15b Using the quotient rule:

$$\begin{aligned} h'(x) &= \frac{(e^x - e^{-x})\frac{d}{dx}(e^x + e^{-x}) - (e^x + e^{-x})\frac{d}{dx}(e^x - e^{-x})}{(e^x - e^{-x})^2} \\ &= \frac{(e^x - e^{-x})(e^x - e^{-x}) - (e^x + e^{-x})(e^x + e^{-x})}{(e^x - e^{-x})^2} \\ &= \frac{e^{2x} - e^0 - e^0 + e^{-2x} - (e^{2x} + e^0 + e^0 + e^{-2x})}{(e^x - e^{-x})^2} \\ &= \frac{e^{2x} - 1 - 1 + e^{-2x} - (e^{2x} + 1 + 1 + e^{-2x})}{(e^x - e^{-x})^2} \\ &= -\frac{4}{(e^x - e^{-x})^2} \end{aligned}$$

Since $(e^x - e^{-x})^2 > 0$ for all values of x such that $x \neq 0$, it follows that $-\frac{4}{(e^x-e^{-x})^2} < 0$ and hence we have that $f'(x) < 0$ for all values of x in the domain.

16a i The function is defined for all values such that the denominator is non-zero, hence, it is defined for all $x - 1 \neq 0$ which in turn is all $x \neq 1$.

16a ii The intercepts with the x-axis occur when $y = 0$. This is when:

$$0 = \frac{|x|}{x-1}$$

$$0 = |x|$$

$$x = 0$$

Hence there is a x-intercept at $(0, 0)$.

The intercepts with the y-axis occur when $x = 0$. This is when:

$$y = \frac{|0|}{0-1}$$

$$y = 0$$

Hence there is a y-intercept at $(0, 0)$.

16a iii There are three regions we need to test based on the above information: when $x < 0$, when $0 < x < 1$ and when $x > 1$. This is because there is an intercept at $x = 0$ and a discontinuity at $x = 1$.

For $x < 0$, choose $x = -1$. $y = -\frac{1}{2} < 0$ and hence the function is negative in this region.

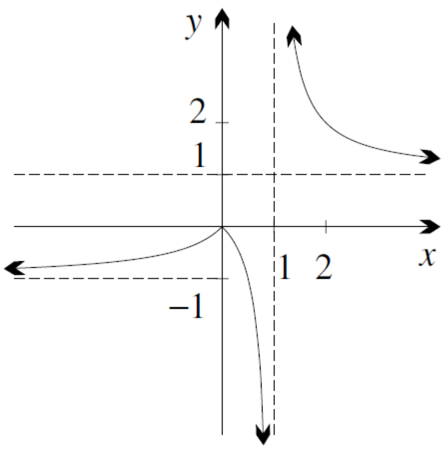
For $0 < x < 1$, choose $x = \frac{1}{2}$, $y = -1 < 0$ and hence the function is negative in this region.

For $x > 1$, choose $x = 2$, $y = 2$. Hence the function is positive in this region.

For $x = 0, y = 0$. Hence the function is positive in this region.

Combining these results, the function is positive in the region $[0, 0] \cup (1, \infty)$.

16a iv



16b i The function is defined for all values such that the denominator is non-zero. Hence, it is defined for:

$x^2 - 1 \neq 0$

$(x-1)(x+1) \neq 0$

$x \neq \pm 1$

16b ii The intercepts with the x-axis occur when $y = 0$. This is when:

$0 = \dfrac{|x|}{x^2 - 1}$

$0 = |x|$

$x = 0$

Hence there is a x-intercept at $(0, 0)$.

The intercepts with the y-axis occur when $x = 0$. This is when:

$y = \dfrac{|0|}{0^2 - 1}$

$y = 0$

Hence there is a y-intercept at $(0, 0)$.

16b iii There are four regions we need to test based on the above information: when $x < -1$, when $-1 < x < 0$, when $0 < x < 1$ and when $x > 1$.

This is because there is an intercept at $x = 0$ and a discontinuity at $x = \pm 1$.

For $-1 < x < 0$, choose $x = -\frac{1}{2}$.

$y = -\frac{2}{3} < 0$ and hence the function is negative in this region.

For $0 < x < 1$, choose $x = \frac{1}{2}$.

$y = -\frac{2}{3} < 0$ and hence the function is negative in this region.

For $x > 1$, choose $x = 2$.

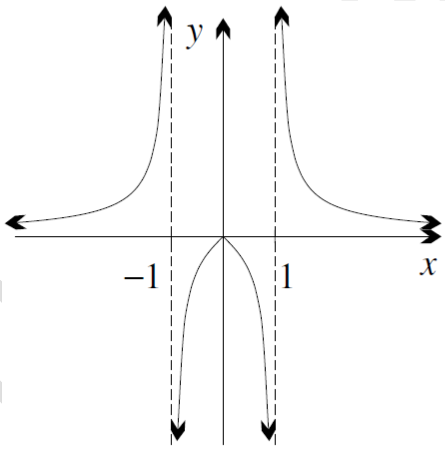
$y = \frac{2}{3}$ and hence the function is positive in this region.

For $x \leq -1$, choose $x = -2$.

$y = \frac{2}{3}$, and hence the function is positive in this region.

Thus, in interval notation the domain in which the function is positive we have $(-\infty, -1) \cup [0, 0] \cup (1, \infty)$.

16b iv



17a $(f \circ g)(x) = f(g(x)) = \sin\left(g(x) + \frac{\pi}{3}\right) = \sin\left(e^x + \frac{\pi}{3}\right)$

$((f \circ g) \circ h)(x) = (f \circ g)(h(x)) = \sin\left(e^{h(x)} + \frac{\pi}{3}\right) = \sin\left(e^{1-x^2} + \frac{\pi}{3}\right)$

$(g \circ h)(x) = g(h(x)) = e^{h(x)} = e^{1-x^2}$

$$(f\circ(g\circ h))(x) = f\big((g\circ h)(x)\big) = \sin\left((g\circ h)(x) + \frac{\pi}{3}\right) = \sin\left(e^{1-x^2} + \frac{\pi}{3}\right)$$

Hence $\big((f\circ g)\circ h\big)(x) = \big(f\circ(g\circ h)\big)(x) = \sin\left(e^{1-x^2} + \frac{\pi}{3}\right)$.

17b $\big((f\circ g)\circ h\big)(x)$

$= (f\circ g)\big(h(x)\big)$
$= f\left(g\big(h(x)\big)\right)$
$= f\big((g\circ h)(x)\big)$
$= \big(f\circ(g\circ h)\big)(x)$

18a It has one endpoint 5, which it contains. Thus it must be closed.

18b It does not contain any endpoints.

18c It contains all its endpoints (there are none).

19a

We factor $f(x)$ by grouping the terms in pairs, noting that $2n+1$ is odd,

$$\begin{aligned} f(x) &= (1+x) + (x^2+x^3) + (x^4+x^5) + \cdots + \left(x^{2n}+x^{2n+1}\right) \\ &= (1+x)\left(1+x^2+x^4+\cdots+x^{2n}\right). \end{aligned}$$

The second factor is always at least 1 because squares are never negative.

Hence the sign of $f(x)$ is the sign of the first factor $1+x$,

that is, $f(x) < 0$, when $x < -1$,

$f(x) = 0$, when $x = -1$,

$f(x) > 0$, when $x > -1$.

19b

When $f(x)$ is differentiated, the first term disappears because it is a constant, and
$f'(x) = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots + (2n-1)x^{2n-2} + (2n)x^{2n-1} + (2n+1)x^{2n}$.
This time we group in triplets, splitting each term of even index (apart from the first),

$$\begin{aligned} f'(x) &= (1 + 2x + x^2) + (2x^2 + 4x^3 + 2x^4) + (3x^4 + 6x^5 + 3x^6) + \cdots \\ &\qquad + \left(nx^{2n-2} + 2nx^{2n-1} + nx^{2n}\right) + (n+1)x^{2n} \\ &= (1+x)^2(1 + 2x^2 + 3x^4 + \cdots + nx^{2n-2}) + (n+1)x^{2n}, \end{aligned}$$

which gives the result in part (b).

19c

In the final expression for $f'(x)$ above, the first term is the product of two factors,
of which the second factor is always at least 1,
and the first factor is positive except at $x = -1$ where it is zero.
Hence the first term is positive, except that it is zero at $x = -1$.
The second term is positive except that it is zero at $x = 0$.
Hence $f'(x)$ is always positive, that is, every tangent to $y = f(x)$ slopes upwards.

Solutions to Exercise 3B

1a Dividing through by the highest power of x in the denominator gives:

$$f(x) = \frac{\frac{1}{x}}{1 - \frac{2}{x}}$$

Hence as $x \to \pm\infty, f(x) \to \dfrac{0}{1-0} = 0$

1b Dividing through by the highest power of x in the denominator gives:

$$f(x) = \frac{1 - \frac{3}{x}}{1 + \frac{4}{x}}$$

Hence as $x \to \pm\infty, f(x) \to \dfrac{1-0}{1+0} = 1$

1c Dividing through by the highest power of x in the denominator gives:

$$f(x) = \frac{2 + \frac{1}{x}}{\frac{3}{x} - 1}$$

Hence as $x \to \pm\infty, f(x) \to \dfrac{2+0}{0-1} = -2$

1d Dividing through by the highest power of x in the denominator gives:

$$f(x) = \frac{\frac{5}{x} - 1}{\frac{4}{x} - 2}$$

Hence as $x \to \pm\infty, f(x) \to \dfrac{0-1}{0-2} = \dfrac{1}{2}$

1e Dividing through by the highest power of x in the denominator gives:

$$f(x) = \frac{\frac{1}{x^2}}{1 + \frac{1}{x^2}}$$

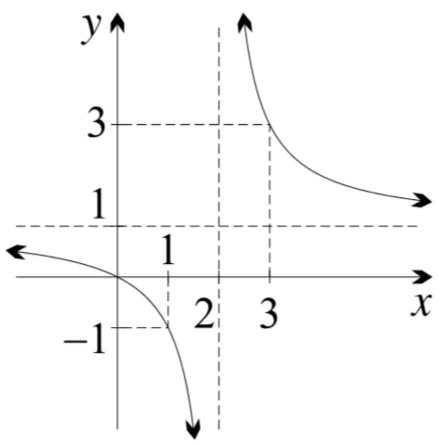
Hence as $x \to \pm\infty, f(x) \to \dfrac{0}{1+0} = 0$

1f Dividing through by the highest power of x in the denominator gives

$$f(x) = \frac{\frac{x}{x^2}}{1+\frac{4}{x^2}} = \frac{\frac{1}{x}}{1+\frac{4}{x^2}}$$

Hence as $x \to \pm\infty, f(x) \to \dfrac{0}{1+0} = 0$

2



2a The function is defined for all x where $x - 2 \neq 0$ (so as to avoid dividing by zero). Hence the natural domain is all real numbers except for $x = 2$.

2b The y-intercept occurs at the point where $x = 0$. Hence the y-intercept is when:

$$y = \frac{0}{0-2} = 0$$

Thus the y-intercept is at $(0, 0)$.

The x-intercept occurs at the point where $y = 0$. Hence the x-intercept is when:

$$\frac{x}{x-2} = 0$$

$$x = 0$$

Thus the x-intercept is at $(0, 0)$.

2c Dividing the denominator and numerator by x gives:

$$y=\frac{\frac{x}{x}}{\frac{x}{x}-\frac{1}{x}}=\frac{1}{1-\frac{1}{x}}$$

Hence as $x\to\infty, y\to\frac{1}{1-0}=\frac{1}{1}=1$

Furthermore as $x\to-\infty, y\to\frac{1}{1-0}=\frac{1}{1}=1$

and thus $y=1$ is a horizontal asymptote.

2d Vertical asymptotes occur when the function on the denominator is zero. This is when $x-2=0$ and hence is $x=2$. $y\to\infty$ as $x\to2^+$ and $y\to-\infty$ as $x\to2^-$. This confirms that $x=2$ is a vertical asymptote.

3a The function is undefined when the denominator of the function is zero. This is when $x+3=0$ and hence is when $x=-3$.

3b The y-intercept occurs at the point where $x=0$. Hence the y-intercept is when:

$$y=\frac{0-1}{0+3}=-\frac{1}{3}$$

Thus the y-intercept is at $(0,-\frac{1}{3})$.

The x-intercept occurs at the point where $y=0$. Hence the x-intercept is when:

$$\frac{x-1}{x+3}=0$$

$$x-1=0$$

$$x=1$$

Thus the x-intercept is at $(1,0)$.

The sign of the function can only change at intercepts or discontinuities, that is when $x=-3$ or $x=1$.

When $x=-4, y=\frac{-5}{-1}=5$

When $x=0, y=-\frac{1}{3}$ as above

When $x = 2, y = \frac{1}{5}$

Thus the function is positive in the interval $(-\infty, -3) \cup [1, \infty)$ and negative in the interval $(-3, 1)$.

3c Vertical asymptotes occur when the function in the denominator is zero. This is when $x + 3 = 0$ and hence is $x = -3$.

$y \to \infty$ as $x \to -3^+$ and $y \to -\infty$ as $x \to -3^-$. This confirms that $x = -3$ is a vertical asymptote.

Dividing the denominator and numerator by x gives:

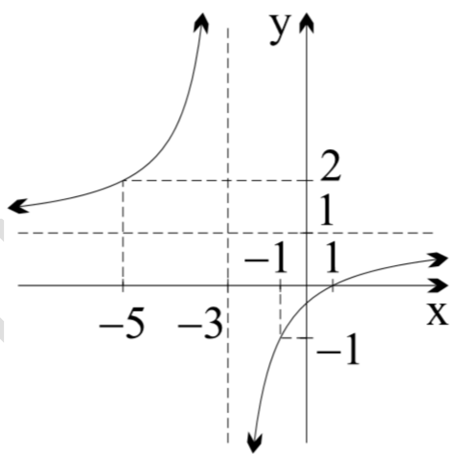
$$y = \frac{\frac{x}{x} - \frac{1}{x}}{\frac{x}{x} + \frac{3}{x}} = \frac{1 - \frac{1}{x}}{1 + \frac{3}{x}}$$

Hence as $x \to \infty, y \to \frac{1-0}{1+0} = \frac{1}{1} = 1$

Furthermore as $x \to -\infty, y \to \frac{1-0}{1+0} = \frac{1}{1} = 1$

and thus $y = 1$ is a horizontal asymptote.

3d



3e This curve is one-to-one as it passes the horizontal line test.

4 The function is defined for all x where $(x-2)^2 \neq 0$ (so as to avoid dividing by zero). Taking the square root of both sides of the equation, gives $x - 2 \neq 0$. Hence the natural domain is all real numbers except for $x = 2$. This means there is a vertical asymptote at $x = 2$.

As $x \to 2^+, y < 0$, and hence $y \to -\infty$

As $x \to 2^-, y < 0$, and hence $y \to -\infty$

As $x \to \infty$, we find that $y \to 0^-$. Similarly, as $x \to -\infty$, we find that $y \to 0^-$. So there is a horizontal asymptote at $y = 0$.

The y-intercept occurs at the point where $x = 0$. Hence the y-intercept is when:

$$y = -\frac{1}{(x-1)^2} = -\frac{1}{(0-1)^2} = -1$$

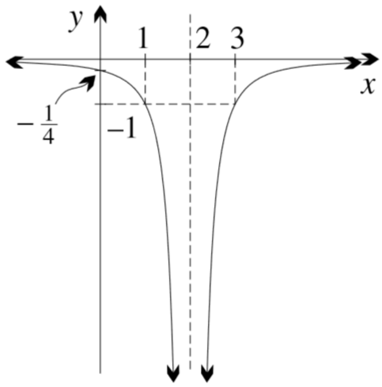
Thus the y-intercept is at $(0, -1)$.

The x-intercepts occur when $y = 0$, which is when:

$$0 = -\frac{1}{(x-1)^2}$$

But there are no solutions to this equation and hence there are no x-intercepts.

The sketch graph is:



5a Dividing through by the highest power of x in the denominator gives:

$$f(x) = \frac{\frac{2}{x^2}}{1 + \frac{1}{x^2}}$$

As $x \to \pm\infty, f(x) \to \frac{0}{1-0} = 0$ so $y = 0$ is a horizontal asymptote.

5b As $x^2 + 1 > 0$ for all values of x. The denominator of the function is never zero and hence the function is defined for all values of x.

5c

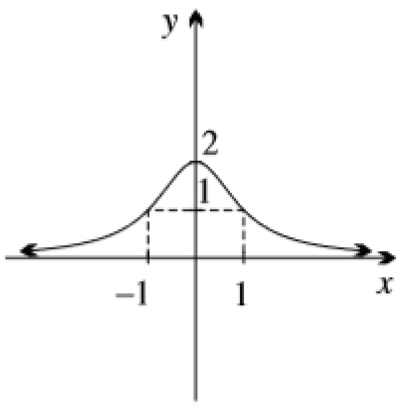
$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{2}{x^2+1}\right)$$
$$= \frac{d}{dx}\left(2(x^2+1)^{-1}\right)$$
$$= -2(x^2+1)^{-2} \times \frac{d}{dx}(x^2+1)$$
$$= -\frac{4x}{(x^2+1)^2}$$

When $x = 0$ (at the y-intercept),

$$\frac{dy}{dx} = -\frac{4(0)}{(0^2+1)} = 0$$

And hence the tangent is horizontal at the y-intercept.

5d



5e As can be seen from the graph, the range of the function is $0 < y \le 2$ (note it is strictly greater than zero as the curve never touches the x-axis).

5f As can be seen from the graph, the function is many-to-one as it fails the horizontal line test.

6a The function is defined for all x where $(x+1)(x-3) \neq 0$ (so as to avoid dividing by zero). Hence the natural domain is all real numbers except for $x = -1, 3$.

6b The y-intercept occurs at the point where $x = 0$. Hence the y-intercept is when:

$$y = \frac{3}{(x+1)(x-3)} = \frac{3}{(0+1)(0-3)} = \frac{3}{-3} = -1$$

Thus the y-intercept is at $(0, -1)$.

6c Dividing the denominator and numerator by x^2 gives:

$$y = \frac{\frac{3}{x^2}}{(1+\frac{1}{x})(1-\frac{3}{x})}$$

Hence as $x \to \infty, y \to \dfrac{0}{(1+0)(1-0)} = 0$

Furthermore as $x \to -\infty, y \to \dfrac{0}{(1+0)(1-0)} = 0$

and thus $y = 0$ is a horizontal asymptote.

6d

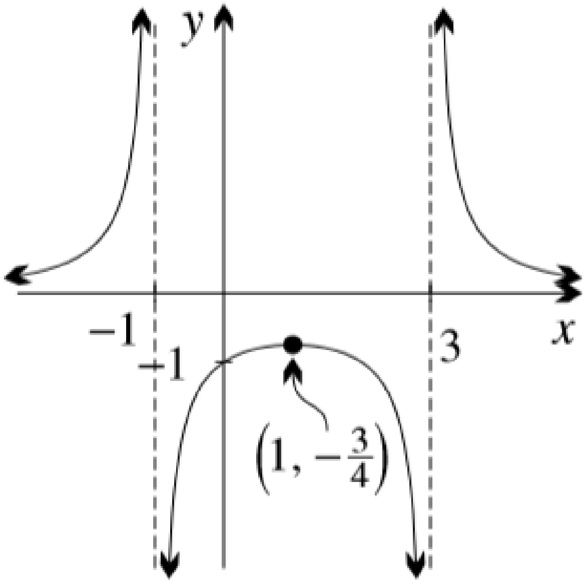
x	-2	-1	0	1	2	3	4
y	$\frac{3}{5}$	*	-1	$-\frac{3}{4}$	-1	*	$\frac{3}{5}$
sign	$+$	*	$-$	$-$	$-$	*	$+$

6e The vertical asymptotes occur at the two values of x where the function is undefined. Hence the vertical asymptotes are at $x = -1$ and at $x = 3$.

As $x \to 3^+, y > 0$ so $y \to \infty$, and as $x \to 3^-, y < 0$ so $y \to -\infty$.

As $x \to -1^+, y < 0$ so $y \to -\infty$, and as $x \to -1^-, y > 0$ so $y \to \infty$.

6f



As can be seen from the graph, the range is $y > 0$ and $y \le -\frac{3}{4}$.

7a

$$y = \frac{4}{4 - x^2} = \frac{4}{(2 - x)(2 + x)}$$

This function is defined for all $(2 - x)(2 + x) \ne 0$ and hence it is defined for all $x \ne \pm 2$. Furthermore, this means there will be vertical asymptotes at $x = \pm 2$.

The y-intercept occurs when $x = 0$.

$$y = \frac{4}{(2 - 0)(2 + 0)} = \frac{4}{4} = 1$$

Thus the curve passes through $(0, 1)$.

The x-intercept occurs when $y = 0$.

$$0 = \frac{4}{(2 - x)(2 + x)}$$ which has no solution.

Thus there are no x-intercepts.

Now $\lim\limits_{x\to\pm\infty} \dfrac{4}{4-x^2}$

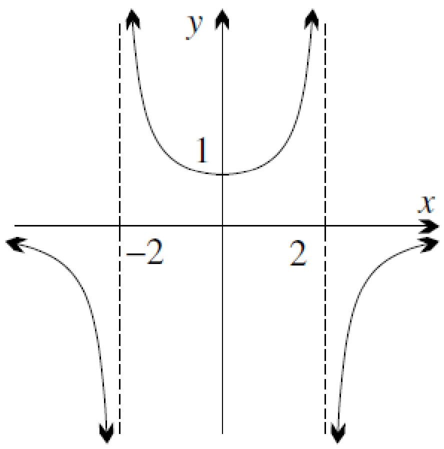
$= \lim\limits_{x\to\pm\infty} \dfrac{\frac{4}{x^2}}{\frac{4}{x^2}-1}$

$= \dfrac{0}{0-1}$

$= 0$

Hence the x-axis acts as an asymptote as values of x tend towards positive and negative infinity.

The sketch graph is:



7b From the graph, the range is $y < 0, y \geq 1$.

8a Let $y = f(x)$

$f(-x) = \dfrac{-3x}{(-x)^2+1} = -\left(\dfrac{3x}{x^2+1}\right) = -f(x)$

Hence by definition the function is odd.

8b x-intercepts occur when $y = 0$. This is when:

$\dfrac{3x}{x^2+1} = 0$

$3x = 0$

$x = 0$

y-intercepts occur when $x = 0$. This is when $y = \frac{3(0)}{0^2+1} = 0$.

Hence there is only one intercept which is at the origin.

8c $\lim\limits_{x \to \infty} f(x)$

$= \lim\limits_{x \to \infty} \dfrac{3x}{x^2 + 1}$

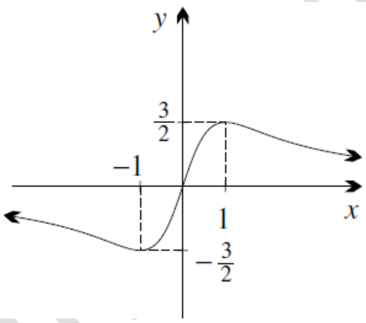
$= \lim\limits_{x \to \infty} \dfrac{\frac{3}{x}}{1 + \frac{1}{x^2}}$

$= \dfrac{0}{1 + 0}$

$= 0$

Hence the line $y = 0$ (the x-axis) acts as a horizontal asymptote.

8d



9a Let $y = f(x)$

$f(-x) = \dfrac{4 - (-x)^2}{4 + (-x)^2} = \dfrac{4 - x^2}{4 + x^2} = f(x)$

Hence by definition the function is even.

9b x-intercepts occur when $y = 0$. This is when:

$$\frac{4 - x^2}{4 + x^2} = 0$$

$$4 - x^2 = 0$$

$$x^2 = 4$$

$$x = \pm 2$$

y-intercepts occur when $x = 0$.

$$y = \frac{4 - 0^2}{4 + 0^2} = 1$$

Hence there are three intercepts at $(-2, 0)$, $(2, 0)$ and $(0, 1)$.

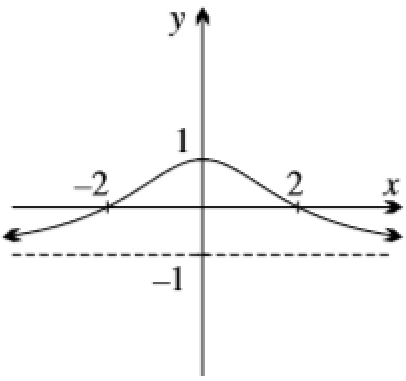
9c Finding the horizontal asymptotes:

Dividing through by the highest power of x in the denominator gives:

$$f(x) = \frac{\frac{4}{x^2} - 1}{\frac{4}{x^2} + 1}$$

Hence as $x \to \infty$, $f(x) \to \frac{0-1}{0+1} = -1$ so $y = -1$ is a horizontal asymptote.

9d



10a

$$\lim_{x\to\pm\infty} f(x)$$

$$= \lim_{x\to\pm\infty} \frac{x^2+5x+6}{x^2-4x+3}$$

$$= \lim_{x\to\pm\infty} \frac{1+\frac{5}{x}+\frac{6}{x^2}}{1-\frac{4}{x}+\frac{3}{x^2}}$$

$$= \frac{1+0+0}{1-0+0}$$

$$= 1$$

Hence there is a horizontal asymptote at $y = 1$.

$$y = \frac{(x+2)(x+3)}{(x-1)(x-3)}$$

Vertical asymptotes occur when the function is undefined. This is when $(x-1)(x-3) = 0$ and hence $x = 1$ and $x = 3$ are vertical asymptotes.

10b Note that

$$y = \frac{(x-1)^2}{(x+1)(x+4)}$$

$$\lim_{x\to\pm\infty} f(x)$$

$$= \lim_{x\to\pm\infty} \frac{x^2-2x+1}{x^2+5x+4}$$

$$= \lim_{x\to\pm\infty} \frac{1-\frac{2}{x}+\frac{1}{x^2}}{1+\frac{5}{x}+\frac{4}{x^2}}$$

$$= \frac{1-0+0}{1+0+0}$$

$$= 1$$

Hence there is a horizontal asymptote at $y = 1$.

Vertical asymptotes occur when the function is undefined. This is when $(x+1)(x+4) = 0$ and hence $x = -1$, and $x = -4$ are vertical asymptotes.

So the asymptotes are $x = -1$, $x = -4$ and $y = 1$.

10c Note that

$$y = \frac{x-5}{(x-2)(x+5)}$$

$$\lim_{x\to\pm\infty} f(x)$$

$$= \lim_{x\to\pm\infty} \frac{x-5}{x^2+3x-10}$$

$$= \lim_{x\to\pm\infty} \frac{\frac{1}{x}-\frac{5}{x^2}}{1+\frac{3}{x}-\frac{10}{x^2}}$$

$$= \frac{0-0}{1+0+0}$$

$$= 0$$

Hence there is a horizontal asymptote at $y = 0$.

Vertical asymptotes occur when the function is undefined. This is when $(x-2)(x+5) = 0$ and hence $x = 2$ and $x = -5$ are vertical asymptotes.

So the asymptotes are $x = -5$, $x = 2$ and $y = 0$.

10d Note that

$$y = \frac{(1-2x)(1+2x)}{(1-3x)(1+3x)}$$

$$\lim_{x\to\pm\infty} f(x)$$

$$= \lim_{x\to\pm\infty} \frac{1-4x^2}{1-9x^2}$$

$$= \lim_{x\to\pm\infty} \frac{\frac{1}{x^2}-4}{\frac{1}{x^2}-9}$$

$$= \frac{0-4}{0-9}$$

$$= \frac{4}{9}$$

Hence there is a horizontal asymptote at $y = \frac{4}{9}$.

Vertical asymptotes occur when the function is undefined, this is when $(1-3x)(1+3x)=0$ and hence $x=\frac{1}{3}$ and $x=-\frac{1}{3}$ are vertical asymptotes.

So the asymptotes are $x=\frac{1}{3}$, $x=-\frac{1}{3}$ and $y=\frac{4}{9}$

11a

$$f(-x)=\frac{-x}{(-x)^2-4}=-\left(\frac{x}{x^2-4}\right)=-f(x)$$

Hence by definition this function is odd and must have rotational (point) symmetry around the origin.

11b The function is defined for all values for which its denominator is non-zero, hence it is defined for all x such that $x^2-4\neq 0$ which is when $(x-2)(x+2)\neq 0$ and hence $x\neq \pm 2$. That is, the domain is all x such that $x\neq 2$ and $x\neq -2$.

Thus the domain is all x such that $x\neq \pm 2$. The asymptotes will be at these points where the function is discontinuous, hence they will be at $x=2$ and $x=-2$.

11c

x	-3	-2	-1	0	1	2	3
y	$-\frac{3}{5}$	*	$\frac{1}{3}$	0	$-\frac{1}{3}$	*	$\frac{3}{5}$
sign	$-$	*	$+$		$-$	*	$+$

11d

$$\lim_{x\to\pm\infty} f(x)$$

$$=\lim_{x\to\pm\infty}\frac{x}{x^2-4}$$

$$=\lim_{x\to\pm\infty}\frac{\frac{1}{x}}{1-\frac{4}{x^2}}$$

$$=\frac{0}{1-0}$$

$= 0$

So $y = 0$ is the horizontal asymptote.

11e

$$f'(x) = \frac{1(x^2-4) - 2x(x)}{(x^2-4)^2}$$

$$= \frac{x^2 - 4 - 2x^2}{(x^2-4)^2}$$

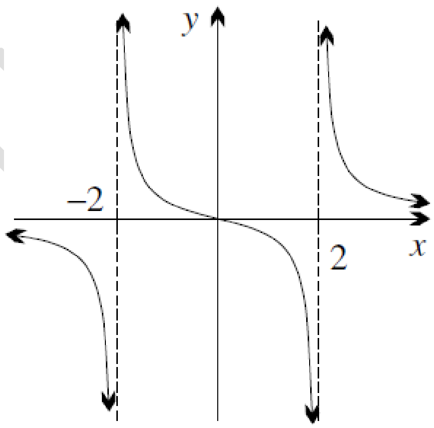
$$= \frac{-4 - x^2}{(x^2-4)^2}$$

$$= -\frac{x^2+4}{(x^2-4)^2}$$

Now $x^2 + 4 > 0$ and $(x^2-4)^2 > 0$ for all x, hence at all points for which the curve is defined, $-\frac{x^2+4}{(x^2-4)^2} < 0$ so $f'(x) < 0$ and thus the curve is decreasing.

11f As $x^2 \geq 0$, $x^2 + 4 > 0$ and hence as $-\frac{1}{(x^2-4)^2}$ is strictly negative, it follows that $-\frac{x^2+4}{(x^2-4)^2} < 0$ for all values of x. It also follows that $f'(x) < 0$ for all defined values of x.

Hence the curve always has a negative gradient (and is thus always decreasing). Furthermore, this means that it will have no stationary points and hence no tangent that is horizontal.

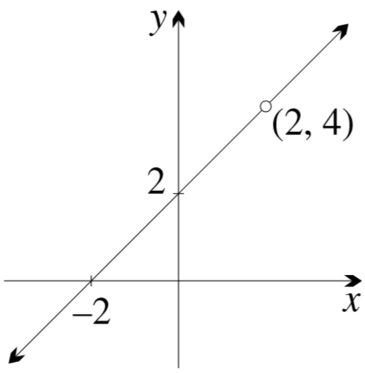


11g The range of the function is all real y or R.

12a

$$\frac{x^2-4}{x-2} = \frac{(x-2)(x+2)}{x-2}$$

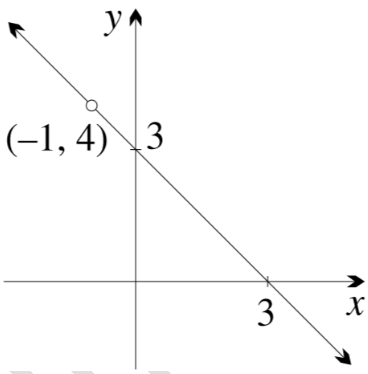
$$= x+2 \quad \text{provided } x-2 \neq 0 \text{ or } x \neq 2.$$



12b i

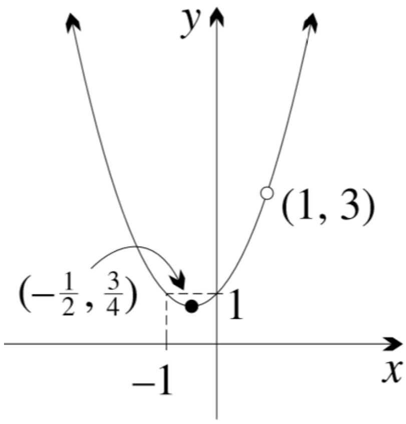
$$y = \frac{(x+1)(3-x)}{(x+1)}$$

$$= 3-x \text{ provided } x+1 \neq 0 \text{ or } x \neq -1$$



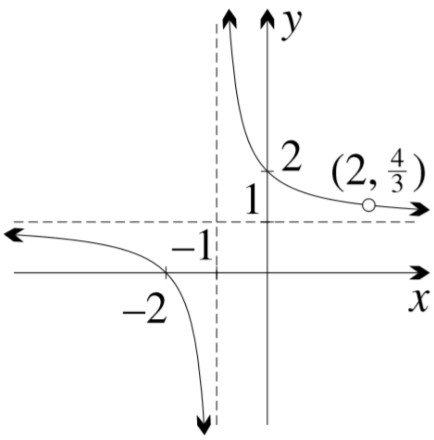
12b ii

$$y = \frac{x^3 - 1}{x - 1}$$
$$= \frac{(x-1)(x^2 + x + 1)}{x - 1}$$
$$= x^2 + x + 1 \text{ provided } x - 1 \neq 0 \text{ or } x \neq 1.$$



12b iii

$$y = \frac{(x+2)(x-2)}{(x-2)(x+1)}$$
$$= \frac{x+2}{x+1} \text{ provided } x - 2 \neq 0 \text{ or } x \neq 2$$



13a Let $f(x) = x + \frac{1}{x}$

$$f(-x) = (-x) + \frac{1}{-x} = -\left(x + \frac{1}{x}\right) = -f(x)$$

Hence the function is odd. A property of odd functions is that they have rotational (point) symmetry around the origin.

13b The function is defined for all real values of x except for $x = 0$ so as to avoid dividing by 0. This means the domain is $x \neq 0$. The vertical asymptote occurs along the vertical line where x is undefined and is thus $x = 0$.

13c

x	-1	0	1
y	-2	$*$	2
sign	$-$	$*$	$+$

13d $y = x + x^{-1}$

$y' = 1 - x^{-2}$

$$= \frac{x^2}{x^2} - \frac{1}{x^2}$$

$$= \frac{x^2 - 1}{x^2}$$

13e The tangent is horizontal when $y' = 0$. This is when:

$$\frac{x^2 - 1}{x^2} = 0$$

$x^2 - 1 = 0$

$x^2 = 1$

$x = \pm 1$

When $x = 1, y = 2$ and when $x = -1, y = -2$.

So the points where the tangent is horizontal are $(1, 2)$ and $(-1, -2)$.

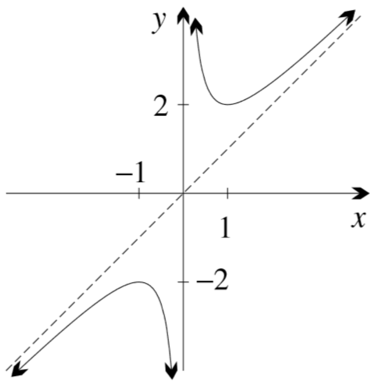
13f

$$\lim_{x\to\infty}(y-x)=\lim_{x\to\infty}\left(x+\frac{1}{x}-x\right)=\lim_{x\to\infty}\left(\frac{1}{x}\right)=0$$

$$\lim_{x\to-\infty}(y-x)=\lim_{x\to-\infty}\left(x+\frac{1}{x}-x\right)=\lim_{x\to-\infty}\left(\frac{1}{x}\right)=0$$

Thus $\lim_{|x|\to\infty}\left(x+\frac{1}{x}\right)=0.$

13g



13h By observation of the graph, and noting that our local minima and maxima are $(1, 2)$ and $(-1, -2)$ respectively, we can conclude that the range is $y \geq 2$ and $y \leq -2$.

14a

$$\lim_{x\to-\infty} y$$

$$= \lim_{x\to-\infty}\frac{1-e^x}{1+e^x}$$

$$= \frac{1-\lim_{x\to-\infty}e^x}{1+\lim_{x\to-\infty}e^x}$$

$$= \frac{1-0}{1+0}$$

$$= 1$$

14b

$$y = \frac{1 - e^x}{1 + e^x} \times \frac{e^{-x}}{e^{-x}}$$

$$= \frac{e^{-x} - e^0}{e^{-x} + e^0}$$

$$= \frac{e^{-x} - 1}{e^{-x} + 1}$$

Hence:

$$\lim_{x \to -\infty} y$$

$$= \lim_{x \to \infty} \frac{e^{-x} - 1}{e^{-x} + 1}$$

$$= \frac{0 - 1}{0 + 1}$$

$$= -1$$

14c x-intercepts occur when $y = 0$

$$0 = \frac{1 - e^x}{1 + e^x}$$

$$0 = 1 - e^x$$

$$e^x = 1$$

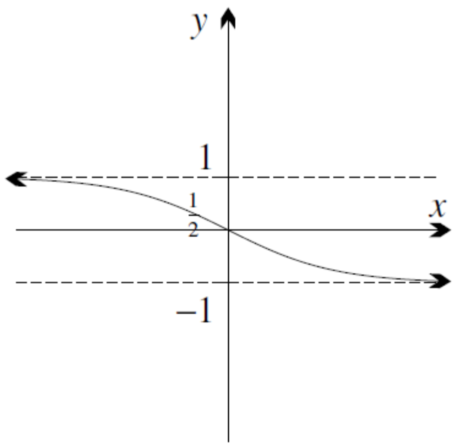
$$x = 0$$

y-intercepts occur when $x = 0$

$$y = \frac{1 - e^0}{1 + e^0} = \frac{1 - 1}{1 + 1} = 0$$

Thus the intercept is at $(0, 0)$.

14d



14e Let $f(x) = y = \frac{1-e^x}{1+e^x}$

$$f(-x) = \frac{1-e^{-x}}{1+e^{-x}}$$

$$= \frac{1-e^{-x}}{1+e^{-x}} \times \frac{e^x}{e^x}$$

$$= \frac{e^x - e^0}{e^x + e^0}$$

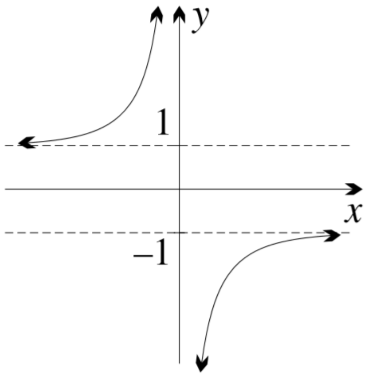
$$= \frac{e^x - 1}{e^x + 1}$$

$$= -\frac{1-e^x}{1+e^x}$$

$$= -f(x)$$

Hence the function is odd.

14f



15a

$$\frac{x^2}{x-1} = \frac{x^2-1+1}{x-1}$$

$$= \frac{(x-1)(x+1)+1}{x-1}$$

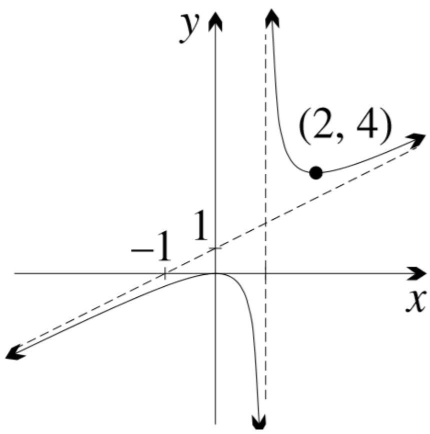
$$= x+1+\frac{1}{x-1}$$

As $x \to \infty, \dfrac{1}{x-1} \to 0^+$ and $y \to x+1$ from above

As $x \to -\infty, \dfrac{1}{x-1} \to 0^-$ and $y \to x+1$ from below

Thus the function approaches an oblique asymptote $y = x + 1$.

The curve also has a vertical asymptote at $x - 1 = 0$ or $x = 1$.



15b

$$\frac{x^2-4}{x+1}$$

$$= \frac{x^2-1-3}{x+1}$$

$$= \frac{(x-1)(x+1)-3}{x+1}$$

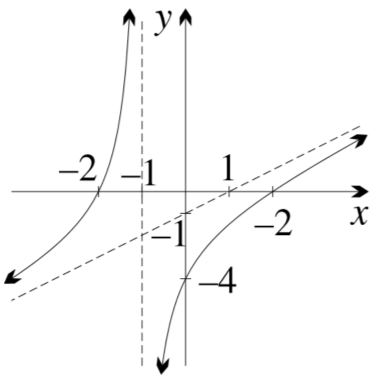
$$= x-1-\frac{3}{x+1}$$

As $x \to \infty$, $\frac{3}{x+1} \to 0^+$ and $y \to x-1$ from below

As $x \to -\infty$, $\frac{3}{x+1} \to 0^-$ and $y \to x-1$ from above

Thus the function approaches an oblique asymptote $y = x - 1$.

The curve also has a vertical asymptote at $x + 1 = 0$ or $x = -1$.



16a

$$y = \frac{x^3-1}{x} = x^2 - \frac{1}{x}$$

As $x \to \infty$, $\frac{1}{x} \to 0^+$ and $y \to x^2$ from below

As $x \to -\infty$, $\frac{1}{x} \to 0^-$ and $y \to x^2$ from above

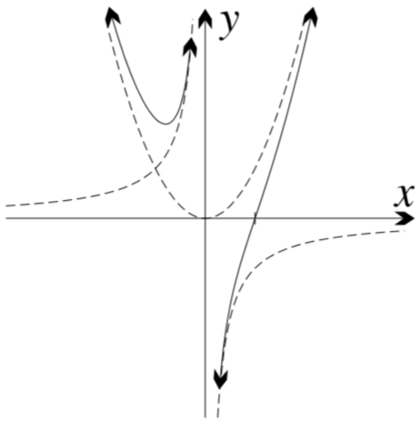
Thus the function approaches an oblique asymptote $y = x^2$.

As $x \to 0^+$, $x^2 \to 0^+$ and $y \to -\frac{1}{x}$ from above

As $x \to 0^-$, $x^2 \to 0^+$ and $y \to -\dfrac{1}{x}$ from above

Thus the function approaches an oblique asymptote $y = -\frac{1}{x}$.

Hence there are two oblique asymptotes, $y = x^2$ and $y = -\frac{1}{x}$.



16b

$y = \dfrac{1}{x} + \sqrt{x}$ (note that $x > 0$)

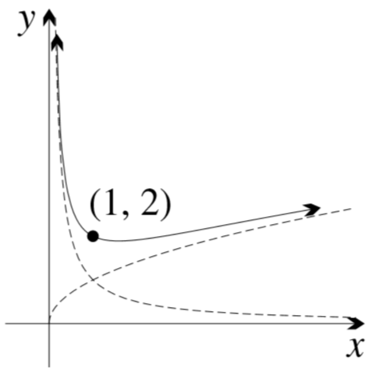
As $x \to \infty$, $\dfrac{1}{x} \to 0^+$ and $y \to \sqrt{x}$ from above

Thus the function approaches an oblique asymptote $y = \sqrt{x}$.

As $x \to 0^+$, $\sqrt{x} \to 0^+$ and $y \to \dfrac{1}{x}$ from above

Thus the function approaches an oblique asymptote $y = \frac{1}{x}$.

Hence there are two oblique asymptotes, $y = \sqrt{x}$ and $y = \frac{1}{x}$.



16c

$$y = |x| + \frac{1}{x}$$

As $x \to \infty$, $\frac{1}{x} \to 0^+$ and $y \to |x|$ from above

As $x \to -\infty$, $\frac{1}{x} \to 0^-$ and $y \to |x|$ from below

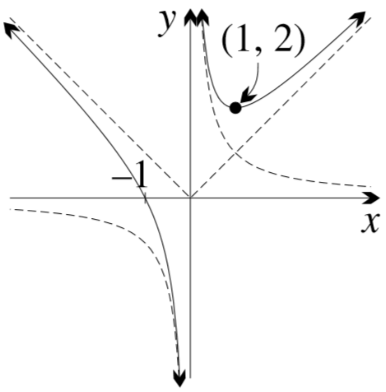
Thus the function approaches an oblique asymptote $y = |x|$.

As $x \to 0^+$, $|x| \to 0^+$ and $y \to \frac{1}{x}$ from above

As $x \to 0^-$, $|x| \to 0^+$ and $y \to \frac{1}{x}$ from above

Thus the function approaches an oblique asymptote $y = \frac{1}{x}$.

Hence there are two oblique asymptotes, $y = |x|$ and $y = \frac{1}{x}$.



Solutions to Exercise 3C

1a Using the difference of two squares:

$$y = \frac{9}{x^2 - 9} = \frac{9}{(x-3)(x+3)}$$

1b The curve is undefined when $(x-3)(x+3) = 0$. This is when $x = \pm 3$.

x is defined for all other values and hence the domain is:

$(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$ or $R\backslash\{-3, 3\}$.

1c

$$f(-x) = \frac{9}{(-x)^2 - 9} = \frac{9}{x^2 - 9} = f(x)$$

Hence the function is even and thus has reflective symmetry about the y-axis.

1d The y-intercepts occur when $x = 0$. This is when:

$$y = \frac{9}{0^2 - 9} = \frac{9}{-9} = -1$$

Hence there is a y-intercept at $(0, -1)$.

The x-intercepts occur when $y = 0$. This is when:

$$0 = \frac{9}{x^2 - 9}$$

This has no solutions. Hence there are no intercepts with the x-axis.

1e

x	-4	-3	-2	0	2	3	4
y	$\frac{9}{7}$	*	$-\frac{9}{5}$	-1	$-\frac{9}{5}$	*	$\frac{9}{7}$
sign	$+$	*	$-$	$-$	$-$	*	$+$

Hence $y \le 0$ for $-3 < x < 3$.

1f The equations of the vertical asymptotes occur when y is undefined. Thus the equations are $x = -3$ and $x = 3$.

As $x \to -3^+, y \to \infty$ and as $x \to -3^-, y \to -\infty$.

As $x \to 3^+, y \to \infty$ and as $x \to 3^-, y \to -\infty$.

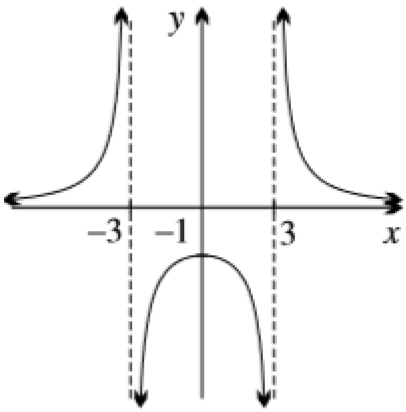
1g Dividing through by the highest power of x in the denominator gives:

$$y = \frac{\frac{9}{x^2}}{1 - \frac{9}{x^2}}$$

Hence as $x \to \infty, y \to \frac{0^+}{1-0} = 0^+$ and as $x \to -\infty, y \to \frac{0^+}{1-0} = 0^+$.

So the horizontal asymptote is $y = 0$.

1h



1i $y = 9(x^2 - 9)^{-1}$

$y' = -9 \times (x^2 - 9)^{-2} \times 2x$

$= -\frac{18x}{(x^2 - 9)^2}$

At $(0, 0), y' = 0$

This means that the graph is horizontal at that point.

2a

$$y = \frac{x}{4 - x^2} = \frac{x}{(2 + x)(2 - x)}$$

2b The curve is undefined when $(2 + x)(2 - x) = 0$. This is when $x = \pm 2$.

x is defined for all other values and hence the domain is:

$(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$ or $R\backslash\{-2, 2\}$.

2c $f(-x)$

$= \dfrac{(-x)}{4 - (-x)^2}$

$= \dfrac{-x}{4 - x^2}$

$= -f(x)$

Hence by definition the function is odd. All odd functions have point symmetry at the origin.

2d Finding the x-intercepts:

The x-intercepts occur when $y = 0$. This is when;

$0 = \dfrac{x}{4 - x^2}$

$x = 0$

x-intercept at $(0, 0)$.

Finding the y-intercepts:

The y-intercepts occur when $x = 0$. This is when:

$y = \dfrac{0}{4 - 0^2}$

$y = 0$

y-intercept at $(0, 0)$.

2e

x	-3	-2	-1	0	1	2	3
y	$\frac{3}{5}$	*	$-\frac{1}{3}$	0	$\frac{1}{3}$	*	$-\frac{3}{5}$
sign	$+$	*	$-$	0	$+$	*	$-$

Thus we see that $y \geq 0$ when $x < -2$ or $0 \leq x < 2$.

2f The equations of the vertical asymptotes occur when y is undefined. Thus the equations are $x = -2$ and $x = 2$.

As $x \to -2^+, y \to -\infty$ and as $x \to -2^-, y \to \infty$.

As $x \to 2^+, y \to -\infty$ and as $x \to 2^-, y \to \infty$.

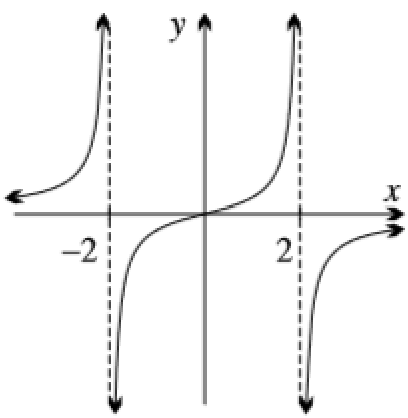
2g Dividing through by the highest power of x in the denominator gives:

$$y = \frac{\frac{1}{x}}{\frac{4}{x^2} - 1}$$

Hence as $x \to \infty, y \to \frac{0^+}{0-1} = 0^-$ and as $x \to -\infty, y \to \frac{0^-}{0-1} = 0^+$.

So the horizontal asymptote is $y = 0$.

2h



2i

$$y = \frac{x}{4 - x^2}$$

Hence

$$y' = \frac{1(4 - x^2) - (-2x)(x)}{(4 - x^2)^2}$$

$$= \frac{4 - x^2 + 2x^2}{(4 - x^2)^2}$$

$$= \frac{x^2 + 4}{(4 - x^2)^2}$$

So since $x^2 \geq 0$, $x^2 + 4 > 0$ and furthermore $(4 - x^2)^2 \geq 0$ so the fraction must be greater than zero for all values for which it is defined.

3a

$$f(x) = \frac{1}{x - 1} + \frac{1}{x - 4}$$

$$= \frac{x - 4}{(x - 1)(x - 4)} + \frac{x - 1}{(x - 1)(x - 4)}$$

$$= \frac{2x - 5}{(x - 1)(x - 4)}$$

3b The function is defined for all values when the denominator is not equal to zero. The denominator is equal to zero when

$$(x - 1)(x - 4) = 0$$

$$x = 1, 4$$

Hence the function is defined for all x except $x = 1, 4$.

3c Consider

$$f(-x) = \frac{2(-x) - 5}{\big((-x) - 1\big)\big((-x) - 4\big)}$$

$$= \frac{-2x - 5}{(x + 1)(x + 4)}$$

It is clear that this is not equal to $f(x)$ or $-f(x)$, and hence, by definition is neither even nor odd. Thus it is not symmetric about $x = 0$.

3d Finding the x-intercepts:

The x-intercepts occur when $y = 0$. This is when:

$$0 = \frac{2x - 5}{(x-1)(x-4)}$$

$$2x - 5 = 0$$

$$x = \frac{5}{2}$$

x-intercept at $\left(2\frac{1}{2}, 0\right)$.

Finding the y-intercept:

The y-intercepts occur when $x = 0$. This is when:

$$y = \frac{2(0) - 5}{(0-1)(0-4)}$$

$$y = -\frac{5}{4}$$

y-intercept at $(0, -\frac{5}{4})$.

3e

x	0	1	2	$2\frac{1}{2}$	3	4	5
y	$-\frac{5}{4}$	*	$\frac{1}{2}$	0	$-\frac{1}{2}$	*	$\frac{5}{4}$
sign	$-$	*	$+$	0	$-$	*	$+$

3f Vertical asymptotes will occur at the values of x where the function is undefined. This is at $x = 1$ and $x = 4$.

As $x \to 1^+, y \to \infty$ and as $x \to 1^-, y \to -\infty$.

As $x \to 4^+, y \to \infty$ and as $x \to 4^-, y \to -\infty$.

3g Consider the limiting points of the function to find horizontal asymptotes.

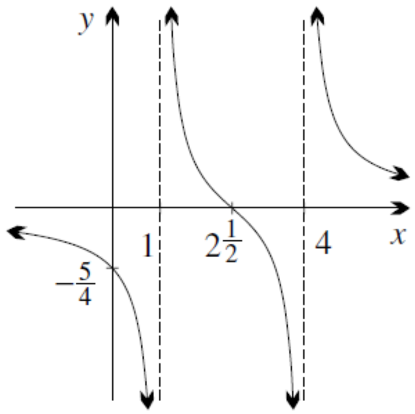
Dividing through by the highest power of x in the denominator gives:

$$y=\frac{\frac{2}{x}-\frac{5}{x^2}}{1-\frac{5}{x}+\frac{4}{x^2}}$$

Hence as $x\to\infty$, $y\to\frac{0-0}{1-0+0}=0^+$ and as $x\to-\infty$, $y\to\frac{-0-0}{1+0+0}=0^-$.

Hence there is a horizontal asymptote at $y=0$.

3h



4a $y=x(x-2)(x+2)$

4b As the function is defined for all values of x the domain is $(-\infty,\infty)$

4c Finding the x-intercepts:

The x-intercepts occur when $y=0$. This is when:

$0=x(x-2)(x+2)$

$x=0,\pm 2$

x-intercepts at $(-2,0),(0,0),(2,0)$.

Finding the y-intercepts:

The y-intercepts occur when $x=0$. This is when:

$y = (0)(0 - 2)(0 + 2)$

$y = 0$

y-intercept at $(0, 0)$.

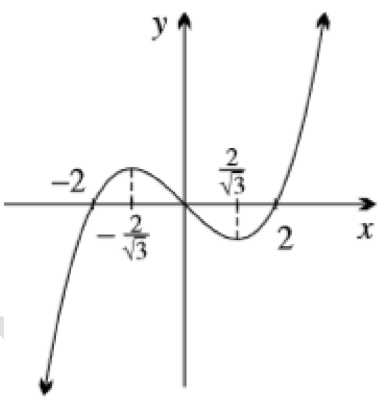
4d $f(-x) = (-x)^3 - 4(-x) = -x^3 + 4x = -(x^3 - 4x) = -f(x)$

Hence by definition the function is odd. All odd functions have point symmetry at the origin.

4e The function is defined for all values of x and tends towards infinity as $x \to \pm\infty$. Hence, we cannot find any asymptotes.

4f, g

x	-3	-2	-1	0	1	2	3
$f(x)$	-15	*	3	*	-3	*	15
sign	$-$	*	$+$	*	$-$	*	$+$



$y = x^3 - 4x$

$y' = 3x^2 - 4$

Stationary points occur when $y' = 0$

$3x^2 - 4 = 0$

$3x^2 = 4$

$x^2 = \frac{4}{3}$

$$x = \pm\frac{2}{\sqrt{3}}$$

Thus the x-coordinates of the stationary points are at $x = \pm\frac{2}{\sqrt{3}}$

5a The function can be rewritten as

$$y = \frac{3(x-1)}{(x-3)(x+1)}$$

The function is defined for all values when the denominator is not equal to zero.

The denominator is equal to zero when:

$$(x-3)(x+1) = 0$$

$$x = 3, -1$$

So the domain is $x \neq -1$ and $x \neq 3$ or all x values except $x = -1, 3$.

Finding the x-intercepts:

The x-intercepts occur when $y = 0$. This is when:

$$0 = \frac{3(x-1)}{(x-3)(x+1)}$$

$$x - 1 = 0$$

$$x = 1$$

x-intercept at $(1, 0)$.

Finding the y-intercepts:

The y-intercepts occur when $x = 0$. This is when:

$$y = \frac{3(0-1)}{(0-3)(0+1)}$$

$$y = \frac{-3}{-3}$$

$$y = 1$$

y-intercept at $(0, 1)$.

So we have intercepts at $(1, 0)$ and $(0, 1)$.

5b The domain is not symmetric about $x = 0$.

5c Vertical asymptotes occur at values of x where the function is undefined, this is when the denominator is equal to zero.

The denominator is equal to zero when:

$(x-3)(x+1)=0$

$x=3,-1$

As $x \to -1^{+}, y \to \infty$ and as $x \to -1^{-}, y \to -\infty$.

As $x \to 3^{+}, y \to \infty$ and as $x \to 3^{-}, y \to -\infty$.

Now we consider the limiting points of the function to find horizontal asymptotes.

Dividing through by the highest power of x in the denominator gives:

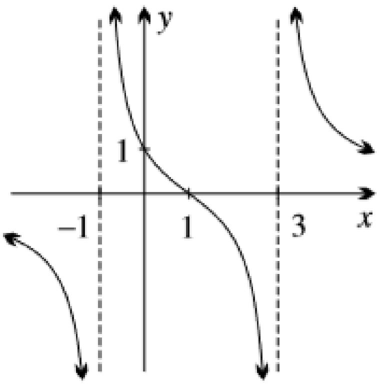
$$y=\frac{\frac{3}{x}-\frac{3}{x^2}}{1-\frac{2}{x}-\frac{3}{x^2}}$$

Hence as $x \to \infty, y \to \frac{0-0}{1-0-0}=0^{+}$ and as $x \to -\infty, y \to \frac{0-0}{1+0-0}=0^{-}$.

Hence the horizontal asymptote is $y=0$.

So the asymptotes are $x=-1$, $x=3$, and $y=0$.

5d



6 $y = -x^3 + 6x^2 - 8x = -x(x-2)(x-4)$

6a It is defined for all values of x so the domain is $-\infty < x < \infty$.

Intercepts with the y-axis occur when $x = 0$. This when:

$y = -0^3 + 6(0)^2 - 8(0) = 0 + 0 - 0 = 0.$

Hence there is an intercept at $(0, 0)$.

Intercepts with the x-axis occur when $x = 0$. This is when:

$0 = -x^3 + 6x^2 - 8x$

$0 = -x(x^2 - 6x + 8)$

$0 = -x(x-2)(x-4)$

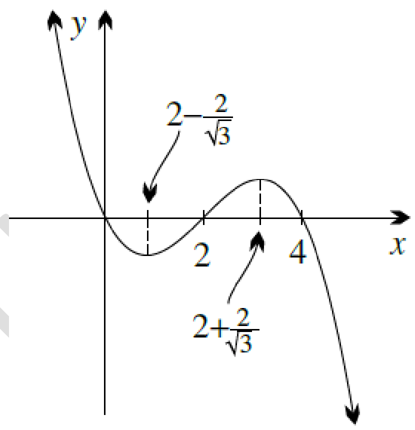
$x = 0, 2, 4$

Hence there are intercepts at $(0, 0), (2, 0), (4, 0)$.

6b Table of signs:

x	-1	0	1	2	3	4	5
y	15	0	-3	0	3	0	-15
sign	$+$	0	$-$	0	$+$	0	$-$

This information gives us the graph:



6c $y' = -3x^2 + 12x - 8$

$y' = 0$ for horizontal/stationary points. This is when:

$-3x^2 + 12x - 8 = 0$

$$3x^2 - 12x + 8 = 0$$

$$x = \frac{-(-12) \pm \sqrt{(-12)^2 - 4 \times 3 \times 8}}{2 \times 3}$$

$$= \frac{12 \pm \sqrt{48}}{6}$$

$$= \frac{12 \pm 4\sqrt{3}}{6}$$

$$= 2 \pm \frac{2\sqrt{3}}{3}$$

$$= 2 \pm \frac{2}{\sqrt{3}}$$

7a

$$y = \frac{x^2 + 2x + 1}{x^2 + 2x - 3}$$

$$= \frac{(x+1)^2}{(x+3)(x-1)}$$

The function is defined for all values when the denominator is not equal to zero.

The denominator is equal to zero when:

$(x+3)(x-1) = 0$

$x = -3, 1$

So the domain is $x \neq 1$ and $x \neq -3$ or all x values.

Finding the x-intercepts:

The x-intercepts occur when $y = 0$. This is when:

$$0 = \frac{(x+1)^2}{(x+3)(x-1)}$$

$x + 1 = 0$

$x = -1$

x-intercept at $(-1, 0)$.

Finding the y-intercepts:

The y-intercepts occur when $x = 0$. This is when:

$$y = \frac{(0+1)^2}{(0-1)(0+3)}$$

$$y = \frac{1}{-3}$$

$$y = -\frac{1}{3}$$

y-intercept at $(0, -\frac{1}{3})$.

So we have intercepts at $(-1, 0)$ and $(0, -\frac{1}{3})$.

7b The domain is not symmetric about $x = 0$ (algebraically $f(x) \neq f(-x)$ and $f(x) \neq -f(x)$) so the function is neither even nor odd).

7c Vertical asymptotes occur at values of x where the function is undefined, this is when the denominator is equal to zero.

The denominator is equal to zero when:

$(x+3)(x-1) = 0$

$x = -3, 1$

As $x \to -3^+, y \to -\infty$ and as $x \to -3^-, y \to \infty$.

As $x \to 1^+, y \to \infty$ and as $x \to 1^-, y \to -\infty$.

Now we consider the limiting points of the function to find horizontal asymptotes.

Dividing through by the highest power of x in the denominator gives:

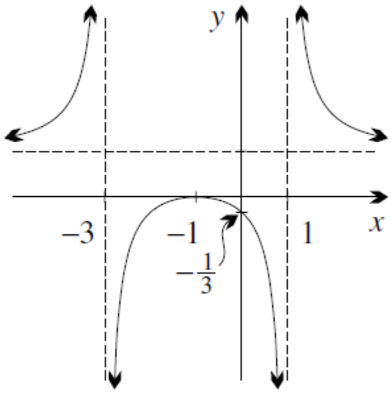
$$y = \frac{1 + \frac{2}{x} + \frac{1}{x^2}}{1 + \frac{2}{x} - \frac{3}{x^2}}$$

Hence as $x \to \infty, y \to \frac{1+0+0}{1+0-0} = 1^+$ and as $x \to -\infty, y \to \frac{1-0+0}{1-0-0} = 1^+$.

Hence the horizontal asymptote is $y = 1$.

So the asymptotes are $x = -3$, $x = 1$ and $y = 1$.

7d



7e From the graph, the range is $y \le 0$ and $y > 1$.

8a

$$f(x) = \frac{x^2 - 4}{x^2 - 4x}$$
$$= \frac{(x-2)(x+2)}{x(x-4)}$$

The function is defined for all values when the denominator is not equal to zero.

The denominator is equal to zero when:

$x(x-4) = 0$

$x = 0, 4$

So the domain is $x \neq 0$ and $x \neq 4$ or all x values.

Finding the x-intercepts:

The x-intercepts occur when $y = 0$. This is when:

$$0 = \frac{(x-2)(x+2)}{x(x-4)}$$

$(x-2)(x+2) = 0$

$x = \pm 2$

x-intercepts at $(2, 0)$ and $(-2, 0)$.

Finding the y-intercepts:

The y-intercepts occur when $x = 0$. This is when:

$$y = \frac{(0-2)(0+2)}{0(0-4)}$$

This is undefined so there are no y-intercepts.

8b Vertical asymptotes occur at values of x where the function is undefined, this is when the denominator is equal to zero.

The denominator is equal to zero when:

$x(x-4) = 0$

$x = 0, 4$

As $x \to 0^+, y \to \infty$ and as $x \to 0^-, y \to -\infty$.

As $x \to 4^+, y \to \infty$ and as $x \to 4^-, y \to -\infty$.

Now we consider the limiting points of the function to find horizontal asymptotes.

Dividing through by the highest power of x in the denominator gives:

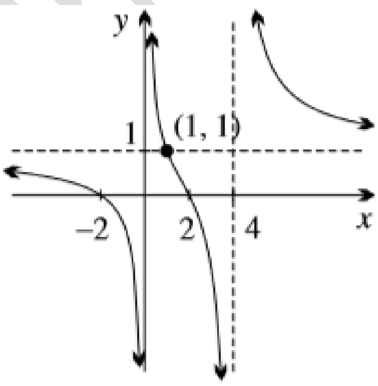
$$y = \frac{1 - \frac{4}{x^2}}{1 - \frac{4}{x}}$$

Hence as $x \to \infty, y \to \frac{1-0}{1-0} = 1^+$ and as $x \to -\infty, y \to \frac{1-0}{1+0} = 1^-$.

Hence the horizontal asymptote is $y = 1$.

So we have the asymptotes $x = 0$, $x = 4$, and $y = 1$.

8c



8d From the graph, we can see that it has a range of all real y.

8e The graph crosses the horizontal asymptote when $y = 1$.

$$\frac{x^2 - 4}{x^2 - 4x} = 1$$

$$x^2 - 4 = x^2 - 4x$$

$$-4 = -4x$$

$$x = 1$$

So the graph crossed the horizontal asymptote at $(1, 1)$.

9a

$$y = \frac{1}{x+1} - \frac{1}{x}$$

$$= \frac{x - (x+1)}{x(x+1)}$$

$$= -\frac{1}{x(x+1)}$$

The domain is all x such that the denominator is non-zero, this is all x such that $x(x+1) \neq 0$ and is hence all x such that $x \neq 0$ and $x \neq -1$. There are no zeroes.

Vertical asymptotes occur at values of x where the function is undefined, this is when the denominator is equal to zero.

The denominator is equal to zero when:

$$x(x+1) = 0$$

$$x = 0, -1$$

As $x \to -1^+, y \to \infty$ and as $x \to -1^-, y \to -\infty$.

As $x \to 0^+, y \to -\infty$ and as $x \to 0^-, y \to \infty$.

Now we consider the limiting points of the function to find horizontal asymptotes.

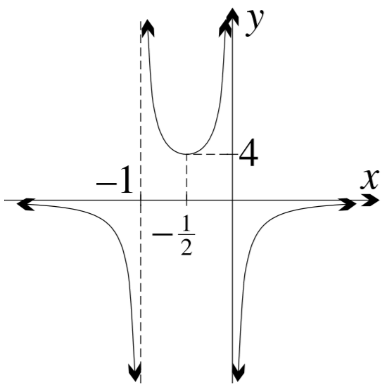
Dividing through by the highest power of x in the denominator gives:

$$y = \frac{-\frac{1}{x^2}}{1 + \frac{1}{x}}$$

Hence as $x \to \infty$, $y \to \frac{-0}{1+0} = 0^-$ and as $x \to -\infty$, $y \to \frac{-0}{1-0} = 0^-$.

Hence the horizontal asymptote is $y = 0$.

So we have the asymptotes $x = -1$, $x = 0$, and $y = 0$.



9b

$$y = \frac{1}{x+3} + \frac{1}{x-3}$$
$$= \frac{x-3+x+3}{(x+3)(x-3)}$$
$$= \frac{2x}{(x+3)(x-3)}$$

The x-intercepts occur when $y = 0$. This is when:

$$0 = \frac{2x}{(x+3)(x-3)}$$

$2x = 0$

$x = 0$

x-intercept at $(0, 0)$.

Finding the y-intercepts:

The y-intercepts occur when $x = 0$. This is when:

$$y = \frac{2(0)}{(0+3)(0-3)} = 0$$

y-intercept at $(0, 0)$.

Vertical asymptotes occur at values of x where the function is undefined, this is when the denominator is equal to zero.

The denominator is equal to zero when:

$(x+3)(x-3)=0$

$x=-3,3$

As $x \to -3^+, y \to \infty$ and as $x \to -3^-, y \to -\infty$.

As $x \to 3^+, y \to \infty$ and as $x \to 3^-, y \to -\infty$.

Now we consider the limiting points of the function to find horizontal asymptotes.

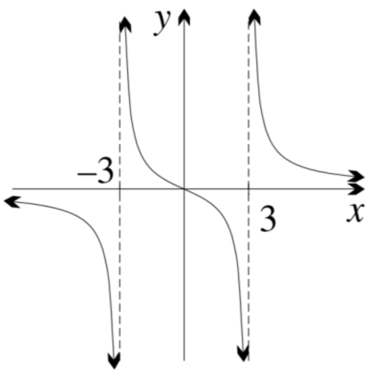
Dividing through by the highest power of x in the denominator gives:

$$y=\frac{\frac{2}{x}}{1-\frac{9}{x^2}}$$

Hence as $x \to \infty, y \to \frac{0}{1-0}=0^+$ and as $x \to -\infty, y \to \frac{-0}{1-0}=0^-$.

Hence the horizontal asymptote is $y=0$.

So we have the asymptotes $x=-3, x=3$, and $y=0$.



10a There are vertical asymptotes when the function is undefined. This is when $x(x-2)=0$ and hence is when $x=0$ or $x=2$.

As $x \to 0^+, y \to -\infty$ and as $x \to 0^-, y \to \infty$.

As $x \to 2^+, y \to \infty$ and as $x \to 2^-, y \to -\infty$.

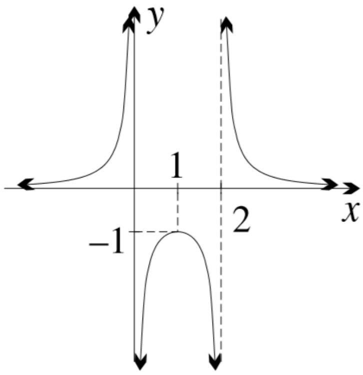
Now we consider the limiting points of the function to find horizontal asymptotes.

Dividing through by the highest power of x in the denominator gives:

$$y = \frac{\frac{1}{x^2}}{(1 - \frac{2}{x})}$$

Hence as $x \to \infty$, $y \to \frac{0^+}{1-0} = 0^+$ and as $x \to -\infty$, $y \to \frac{0^+}{1-0} = 0^+$.

Hence there is a horizontal asymptote at $y = 0$.



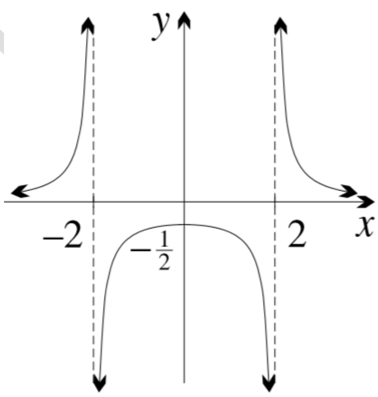
10b There are vertical asymptotes when the function is undefined.

This is when $x^2 - 4 = 0$ and hence is when $(x-2)(x+2) = 0$ and hence $x = \pm 2$.

$$\lim_{x\to\infty} \frac{2}{x^2-4} = \lim_{x\to\infty} \frac{\frac{2}{x^2}}{1-\frac{4}{x^2}} = \frac{0}{1-0} = 0$$

$$\lim_{x\to-\infty} \frac{2}{x^2-4} = \lim_{x\to-\infty} \frac{\frac{2}{x^2}}{1-\frac{4}{x^2}} = \frac{0}{1-0} = 0$$

Hence there is a horizontal asymptote at $y = 0$.



11a

Preparation:

$$y = \frac{1+x^2}{1-x^2} = \frac{1+x^2}{(1-x)(1+x)}$$

Domain:

All x such that the denominator is nonzero. This is all x such that $(1-x)(1+x) \neq 0$. Which is all x such that $x \neq \pm 1$.

Symmetry:

$f(-x) = \frac{1+(-x)^2}{1-(-x)^2} = \frac{1+x^2}{1-x^2} = f(x)$ so the function is even.

Intercepts:

The y-intercepts occur when $x = 0$.

$$y = \frac{1+0^2}{1-0^2} = 1$$

Hence the y-intercept is at $(0, 1)$.

The x-intercepts occur when $y = 0$.

$$0 = \frac{1+x^2}{1-x^2}$$

$$0 = 1 + x^2$$

$$x^2 = -1$$

Which has no integer solutions. Thus, there are no x-intercepts.

Sign:

x	-2	-1	0	1	2
$f(x)$	$-\frac{5}{3}$	*	1	*	$-\frac{5}{3}$
sign	$-$	*	$+$	*	$-$

Vertical asymptotes:

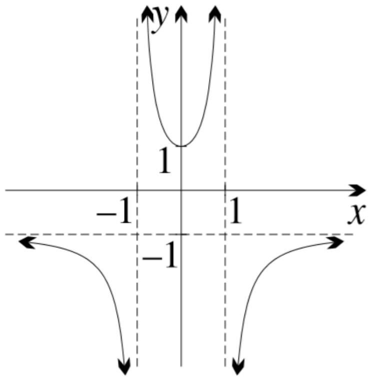
At $x = -1$ and $x = 1$, the denominator vanishes, but the numerator does not, so $x = -1$ and $x = 1$ are vertical asymptotes.

Horizontal Asymptotes:

$\lim_{x\to\infty}\frac{1+x^2}{1-x^2} = \lim_{x\to\infty}\frac{\frac{1}{x^2}+1}{\frac{1}{x^2}-1} = \frac{0+1}{0-1} = -1$

$\lim_{x\to-\infty}\frac{1+x^2}{1-x^2} = \lim_{x\to-\infty}\frac{\frac{1}{x^2}+1}{\frac{1}{x^2}-1} = \frac{0+1}{0-1} = -1$

Hence there is a horizontal asymptote at $y = -1$.



11b

Preparation:

$$y = \frac{x+1}{x(x-3)}$$

Domain:

All x such that the denominator is nonzero. This is all x such that $x(x-3) \neq 0$.
Which is all x such that $x \neq 0$ and $x \neq 3$.

Symmetry:

There are no symmetries.

Intercepts:

The y-intercepts occur when $x = 0$.

$$y = \frac{0+1}{0(0-3)}$$

But this is undefined so there are no y-intercepts.

The x-intercepts occur when $y = 0$.

$$0 = \frac{x+1}{x(x-3)}$$

$x = -1$

Thus the x-intercepts is $(-1, 0)$

Sign:

x	-2	$-\frac{1}{2}$	0	2	3	4
$f(x)$	$-\frac{1}{10}$	$\frac{2}{7}$	$*$	$-\frac{3}{2}$	$*$	$\frac{5}{4}$
sign	$-$	$+$	$*$	$-$	$*$	$+$

Vertical asymptotes:

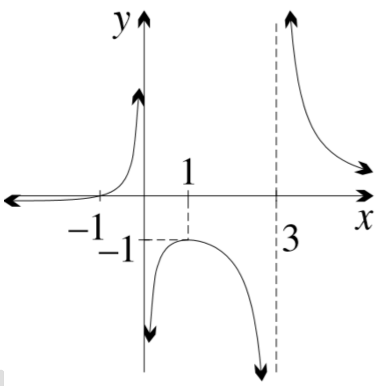
At $x = 0$ and $x = 3$, the denominator vanishes, but the numerator does not, so $x = 0$ and $x = 3$ are vertical asymptotes.

Horizontal Asymptotes:

$$\lim_{x\to\infty} \frac{x+1}{x(x-3)} = \lim_{x\to\infty} \frac{\frac{1}{x}+\frac{1}{x^2}}{\left(1-\frac{3}{x}\right)} = \frac{0+0}{1-0} = 0$$

$$\lim_{x\to-\infty} \frac{x+1}{x(x-3)} = \lim_{x\to-\infty} \frac{\frac{1}{x}+\frac{1}{x^2}}{\left(1-\frac{3}{x}\right)} = \frac{0+0}{1-0} = 0$$

Hence there is a horizontal asymptote at $y = 0$.



11c

Preparation:

$$y = \frac{x-1}{(x+1)(x-2)}$$

Domain:

All x such that the denominator is nonzero. This is all x such that $(x+1)(x-2) \neq 0$. Which is all x such that $x \neq -1$ and $x \neq 2$.

Symmetry:

There are no symmetries.

Intercepts:

The y-intercepts occur when $x = 0$.

$$y = \frac{0-1}{(0+1)(0-2)} = -\frac{1}{(1)(-2)} = \frac{1}{2}$$

Hence the y-intercept is $\left(0, \frac{1}{2}\right)$

The x-intercepts occur when $y = 0$.

$$0 = \frac{x-1}{(x+1)(x-2)}$$

$$x - 1 = 0$$
$$x = 1$$

Thus the x-intercepts is $(1, 0)$

Sign:

x	-2	-1	0	1	$1\frac{1}{2}$	2	3
$f(x)$	$-\frac{3}{4}$	$*$	$\frac{1}{2}$	0	-0.4	$*$	$\frac{1}{2}$
sign	$-$	$*$	$+$	$*$	$-$	$*$	$+$

Vertical asymptotes:

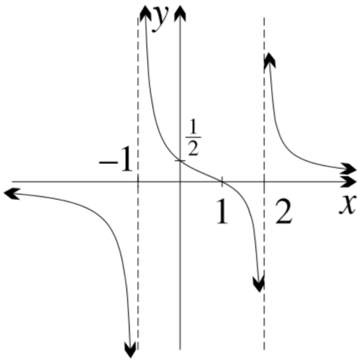
At $x = -1$ and $x = 2$, the denominator vanishes, but the numerator does not, so $x = -1$ and $x = 2$ are vertical asymptotes.

Horizontal Asymptotes:

$$\lim_{x\to\infty} \frac{x-1}{(x+1)(x-2)} = \lim_{x\to\infty} \frac{\frac{1}{x}-\frac{1}{x^2}}{\left(1-\frac{1}{x}\right)\left(1-\frac{2}{x}\right)} = \frac{0+0}{(1-0)(1-0)} = 0$$

$$\lim_{x\to-\infty} \frac{x-1}{(x+1)(x-2)} = \lim_{x\to-\infty} \frac{\frac{1}{x}-\frac{1}{x^2}}{\left(1-\frac{1}{x}\right)\left(1-\frac{2}{x}\right)} = \frac{0+0}{(1-0)(1-0)} = 0$$

Hence there is a horizontal asymptote at $y = 0$.



11d

Preparation:

$$y = \frac{x^2 - 2x}{x^2 - 2x + 2}$$

Domain:

$x^2 - 2x + 2 = (x-1)^2 + 1 > 0$ for all x so the function is defined for the entire domain.

Symmetry:

There are no symmetries.

Intercepts:

The y-intercepts occur when $x = 0$.

$$y = \frac{0^2 - 2(0)}{0^2 - 2(0) + 2} = 0$$

Hence the y-intercept is $(0, 0)$

The x-intercepts occur when $y = 0$.

$$0 = \frac{x^2 - 2x}{x^2 - 2x + 2}$$

$$x^2 - 2x = 0$$
$$x(x-2) = 0$$

$$x = 0, 2$$

Thus the x-intercepts are $(0, 0)$ and $(2, 0)$

Sign:

x	-1	0	1	2	3

$f(x)$	$\frac{3}{5}$	$*$	-1	$*$	$\frac{3}{5}$
sign	$+$	$*$	$-$	$*$	$+$

Vertical asymptotes:

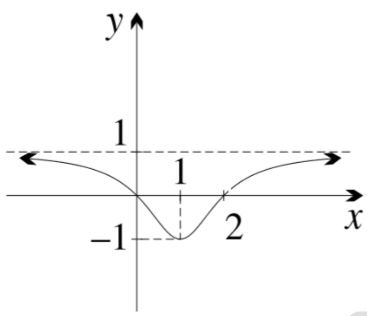
There are no vertical asymptotes.

Horizontal asymptotes:

$$\lim_{x\to\infty}\frac{x^2-2x}{x^2-2x+2}=\lim_{x\to\infty}\frac{1-\frac{2}{x}}{1-\frac{2}{x}-\frac{2}{x^2}}=\frac{1-0}{1-0-0}=1$$

$$\lim_{x\to-\infty}\frac{x^2-2x}{x^2-2x+2}=\lim_{x\to-\infty}\frac{1-\frac{2}{x}}{1-\frac{2}{x}-\frac{2}{x^2}}=\frac{1+0}{1+0-0}=1$$

Hence there is a horizontal asymptote at $y=1$.



11e

Preparation:

$$y=\frac{x^2-4}{(x+2)(x-1)}=\frac{(x-2)(x+2)}{(x+2)(x-1)}=\frac{x-2}{x-1}\text{ when }x\neq-2.$$

Domain:

The function is undefined when the denominator is zero, that is when $(x+2)(x-1)=0$ and hence is when $x=1$ or $x=-2$. So the function is defined for all x such that $x\neq 1$ and $x\neq-2$.

Symmetry:

There are no symmetries.

Intercepts:

The y-intercepts occur when $x=0$.

$$y=\frac{0-2}{0-1}=2$$

Hence the y-intercept is $(0, 2)$

The x-intercepts occur when $y = 0$.

$0 = \dfrac{x-2}{x-1}$

$x - 2 = 0$
$x = 2$

Thus the x-intercept is $(2, 0)$

Sign:

x	-3	-2	-1	0	1	1.5	2	3
$f(x)$	$\frac{5}{4}$	$*$	$\frac{3}{2}$	2	$*$	-1	0	$\frac{1}{2}$
sign	$+$	$*$	$+$	$+$	$*$	$-$	$*$	$-$

Vertical asymptotes:

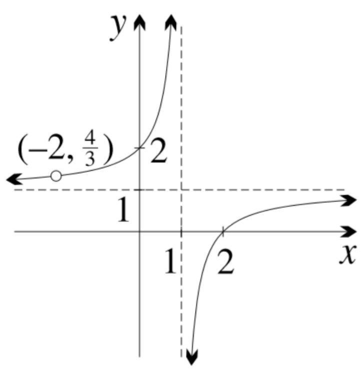
The function is undefined at $x = 1$ so there is a vertical asymptote at this point.

Horizontal Asymptotes:

$\lim\limits_{x\to\infty} \frac{x-2}{x-1} = \lim\limits_{x\to\infty} \frac{1-\frac{2}{x}}{1-\frac{2}{x}} = \frac{1-0}{1-0} = 1$

$\lim\limits_{x\to-\infty} \frac{x-2}{x-1} = \lim\limits_{x\to-\infty} \frac{1-\frac{2}{x}}{1-\frac{2}{x}} = \frac{1+0}{1+0} = 1$

Hence there is a horizontal asymptote at $y = 1$.



11f

Preparation:

$y = \frac{x^2-2}{x} = x - \frac{2}{x}$.

Domain:

The function is undefined when the denominator is zero, that is when $x = 0$. So the function is defined for all x such that $x \neq 0$.

Symmetry:

$f(-x) = \frac{(-x)^2-2}{(-x)} = \frac{x^2-2}{-x} = -\frac{x^2-2}{x} = -f(x)$ so the function is odd.

Intercepts:

The y-intercepts occur when $x = 0$. The function is undefined at this point.

Hence there are no y-intercepts

The x-intercepts occur when $y = 0$.

$$0 = \frac{x^2 - 2}{x}$$

$x^2 - 2 = 0$

$x^2 = 2$

$x = \pm\sqrt{2}$

Thus the x-intercepts are at $(\sqrt{2}, 0)$ and $(-\sqrt{2}, 0)$

Sign:

x	-2	$-\sqrt{2}$	-1	0	1	$\sqrt{2}$	2
$f(x)$	-1	0	1	$*$	-1	0	1
sign	$-$	$*$	$+$	$*$	$-$	$*$	$+$

Vertical asymptotes:

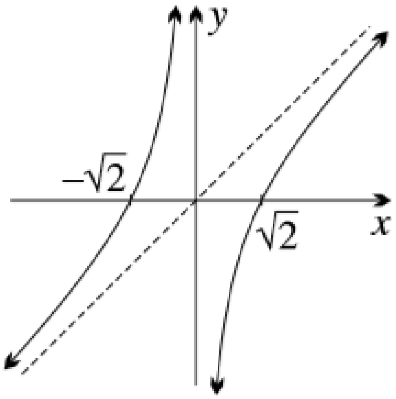
The function is undefined at $x = 0$ so there is a vertical asymptote at this point

Other Asymptotes:

$\lim_{x\to\infty} x - \frac{2}{x} = x - 0 = x$

$\lim_{x\to-\infty} x - \frac{2}{x} = x - 0 = x$

Hence there is an asymptote at $y = x$.



12a Exponentials are defined for all values of x so this function has a domain of all real x.

Finding the x-intercepts:

The x-intercepts occur when $y = 0$. This is when:

$e^{-\frac{1}{2}x^2} = 0$

This has no solutions and hence there are no x-intercepts.

Finding the y-intercepts:

The y-intercepts occur when $x = 0$. This is when:

$y = e^{-\frac{1}{2}(0)^2} = e^0 = 1$

Hence there is a y-intercept at $(0, 1)$.

12b $f(-x) = e^{-\frac{1}{2}(-x)^2} = e^{-\frac{1}{2}x^2} = f(x)$, hence the function is even.

This function is defined for all real x (and is continuous), hence this means that it will have no vertical asymptotes.

Horizontal asymptotes are given by finding the limit as $x \to \pm\infty$.

$\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} e^{-\frac{1}{2}x^2} = 0$

So there is an asymptote at $y = 0$.

12c For all values of $x, -x^2 \leq 0$.

Hence the maximum value of $-x^2$ is 0 when $x = 0$.

This means that the highest point on the curve is (0, 1).

Alternatively:

$$y' = -\frac{1}{2} \times 2x \times e^{-\frac{1}{2}x^2}$$

$y' = 0$ for stationary points, so:

$$-\frac{1}{2} \times 2x \times e^{-\frac{1}{2}x^2} = 0$$

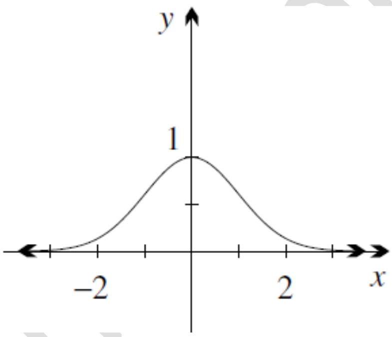
$$x = 0$$

When $x = 0, y = e^0 = 1$, hence there is a stationary point at $(0, 1)$

12d $\frac{dy}{dx} = \frac{d}{dx}\left(-\frac{1}{2}x^2\right)e^{-\frac{1}{2}x^2} = -xe^{-\frac{1}{2}x^2}$

When $x = 0, \frac{dy}{dx} = 0$, hence there is a horizontal tangent there.

12e



As can be seen from the graph, the range is $0 < y \leq 1$.

12f As $e > 2$ it follows that $e^{-1} < 2^{-1}$ and in turn $e^{-\frac{1}{2}} < 2^{-\frac{1}{2}}$.

As $x^2 > 0$ for all x except $x = 0$, $e^{-\frac{1}{2}x^2} < 2^{-\frac{1}{2}x^2}$ for all x except $x = 0$ at which point they are equal. So $y = 2^{-\frac{1}{2}x^2}$ is higher than $y = e^{-\frac{1}{2}x^2}$ except at $x = 0$ where they are equal.

13a i Note that for all $x > 0$, $e^x > 1$ and hence $\frac{d}{dx}(e^x) > \frac{d}{dx}(x)$, this means that the function $y = e^x$ grows at a faster rate than the function $y = x$. Now, note that when $x = 0$, $e^0 = 1 > 0$ and thus the function $y = e^x$ is above the function $y = x$. Hence it follows that $y = e^x$ is greater than $y = x$ at $x = 0$ and then increases at a faster rate than $y = x$ for all $x > 0$. This means that $e^x > x$ for $x \geq 0$ so the function $y = e^x$ is greater.

13a ii $\frac{x^2}{2} \geq 0$ for all x which means if we substitute $x = \frac{x^2}{2}$ into the inequality $e^x > x$ (which is true for all $x \geq 0$), we obtain that $e^{\frac{x^2}{2}} > \frac{x^2}{2}$ for all x. Taking the reciprocal of this then gives that $\frac{1}{\left(e^{\frac{x^2}{2}}\right)} < \frac{1}{\frac{x^2}{2}}$ (for $x \neq 0$) and thus we have that

$e^{-\frac{1}{2}x^2} < \frac{2}{x^2}$ as required.

13a iii Noting that $0 \leq e^{-\frac{1}{2}x^2}$ we have that $0 \leq e^{-\frac{1}{2}x^2} \leq \frac{2}{x^2}$ and thus

$$\lim_{x\to\infty} 0 \leq \lim_{x\to\infty} e^{-\frac{1}{2}x^2} \leq \lim_{x\to\infty} \frac{2}{x}$$

$$0 \leq \lim_{x\to\infty} e^{-\frac{1}{2}x^2} \leq 0$$

And so we conclude that $\lim_{x\to\infty} e^{-\frac{1}{2}x^2} = 0$.

13b i The function $f(x) = -xe^{-\frac{1}{2}x^2}$ is defined for all x and hence has a domain of all real x.

x-intercepts occur when $y = 0$. This is when:

$0 = -xe^{-\frac{1}{2}x^2}$

$x = 0$ (as the exponential function is non-zero for all x)

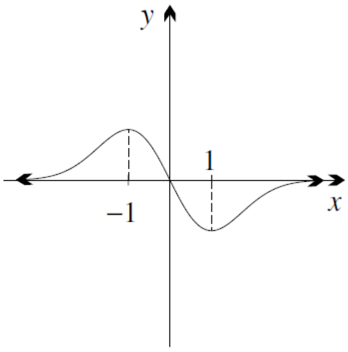
y-intercepts occur when $x = 0$. This is when:

$y = -(0)e^{-\frac{1}{2}0^2} = 0$

Hence the only intercept is at $(0, 0)$.

As $x \to \pm\infty$, $x^2 \to \infty$ and hence $y = xe^{-\frac{1}{2}x^2} \to 0$ as $e^{-\frac{1}{2}x^2}$ tends towards zero at a rate faster than that at which x increases.

Hence we can draw the derivative graph $y' = f(x) = -xe^{-\frac{1}{2}x^2}$ as follows.



The graph shows $f(x)$ is greatest at $x = -1$ and least at $x = 1$.

13b ii

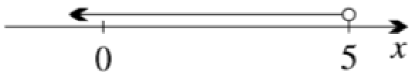
$$
\begin{aligned}
f'(x) &= \frac{d}{dx}\left(-xe^{-\frac{1}{2}x^2}\right)\\
&= \frac{d}{dx}(-x)e^{-\frac{1}{2}x^2} + (-x)\frac{d}{dx}\left(e^{-\frac{1}{2}x^2}\right)\\
&= -\,e^{-\frac{1}{2}x^2} + x^2e^{-\frac{1}{2}x^2}\\
&= (x^2-1)\,e^{-\frac{1}{2}x^2}
\end{aligned}
$$

Thus to have $f'(x) = 0$, we require $(x^2-1)e^{-\frac{1}{2}x^2} = 0$, $x^2 - 1 = 0$, $x^2 = 1$ and thus $x^2 = \pm 1$. Thus, the function of the gradient of y has its extreme values when $x = 1$ and $x = -1$. By observation of the graph above, $f(x)$ is greatest at $x = -1$ and least at $x = 1$. This implies that the graph is steepest (in the positive direction) at $x = -1$ and steepest (in the negative direction) when $x = 1$.

Solutions to Exercise 3D

1a $\quad x - 2 < 3$

$x < 5$



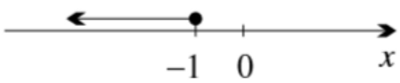
1b $\quad 3x \geq -6$

$x \geq -2$

1c $\quad 4x - 3 \leq -7$

$4x \leq -4$

$x \leq -1$



1d $\quad 6x - 5 < 3x - 17$

$6x - 3x < 5 - 17$

$3x < -12$

$x < -4$

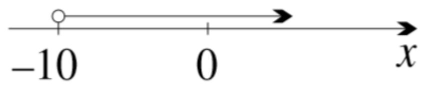
1e $\quad \frac{1}{5}x - \frac{1}{2}x < 3$

$2x - 5x < 30$

$-3x < 30$

$3x > -30$

$x > -10$



1f $\frac{1}{6}(2-x) - \frac{1}{3}(2+x) \geq 2$

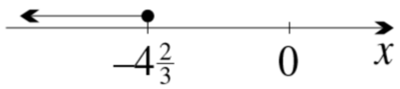
$(2-x) - 2(2+x) \geq 12$

$-2 - 3x \geq 12$

$-3x \geq 14$

$x \leq -\frac{14}{3}$

$x \leq -4\frac{2}{3}$



2a $3 - 2x > 7$

$3 - 7 > 2x$

$-4 > 2x$

$-2 > x$

$x < -2$

Solution is $(-\infty, -2)$

2b $3 - 3x \leq 19 + x$

$3 - 19 \leq 3x + x$

$-16 \leq 4x$

$-4 \leq x$

$x \geq -4$

Solution is $[-4, \infty)$

2c $12 - 7x > -2x - 18$

$12 + 18 > 7x - 2x$

$30 > 5x$

$6 > x$

$x < 6$

Solution is $(-\infty, 6)$

3 The line $5x - 4$ is below the line $7 - \frac{1}{2}x$ when:

$5x - 4 < 7 - \frac{1}{2}x$

$\frac{11}{2}x < 11$

$x < 2$

4a $-1 \leq 2x \leq 3$

$-\frac{1}{2} \leq x \leq 1\frac{1}{2}$

So, the solution in interval notation is $\left[-\frac{1}{2}, 1\frac{1}{2}\right]$

4b $-4 < -2x < 8$

$-8 < 2x < 4$

$-4 < x < 2$

In interval notation, solution is $(-4, 2)$

4c $-7 \leq 5 - 3x < 4$

$-12 \leq -3x < -1$

$1 < 3x \leq 12$

$\frac{1}{3} < x \leq 4$

In interval notation, solution is $\left(\frac{1}{3}, 4\right]$

4d $-2 < x - 3 \leq 4$

$-2 < x \leq 7$

In interval notation, solution is $(-2, 7]$

4e $-7 < 5x + 3 \leq 3$

$-10 < 5x \leq 0$

$-2 < 5x \leq 0$

In interval notation, solution is $[-2, 0)$

4f $-4 < 1 - \frac{1}{3}x \leq 3$

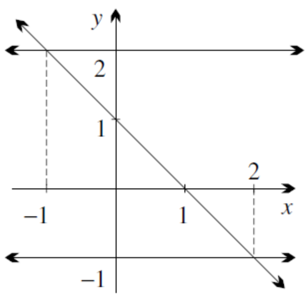
$-5 < -\dfrac{1}{3}x \leq 2$

$-2 \leq \dfrac{1}{3}x < 5$

$-6 \leq x < 15$

In interval notation, solution is $[-6, 15)$

5a



Points of intersection are $(-1, 2)$ and $(2, -1)$ by observation.

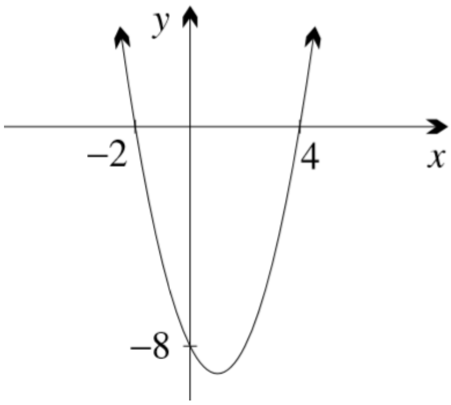
5b $-1 < 1 - x \le 2$

$-2 < -x \le 1$

$-1 \le x < 2$

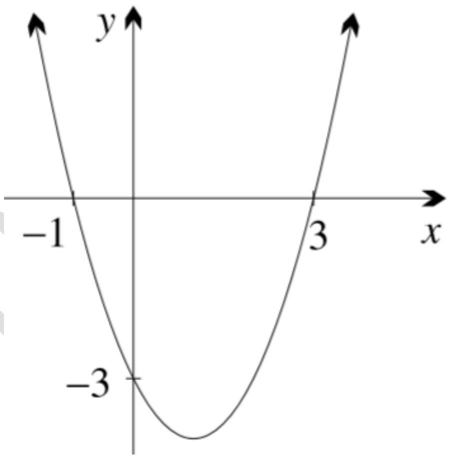
The solution to the inequation is where the diagonal line ($y = 1 - x$) lies between the horizontal lines ($y = -1$ and $y = 2$).

6a Factorising the equation gives $(x - 4)(x + 2) < 0$, hence the parabola is



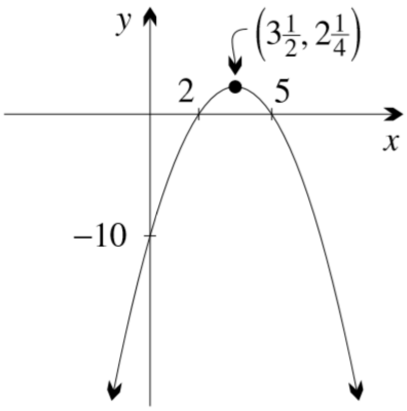
The solutions are when the graph is below the x-axis. This is when $-2 < x < 4$.

6b Factorising the equation gives $(x - 3)(x + 1) > 0$, hence the parabola is



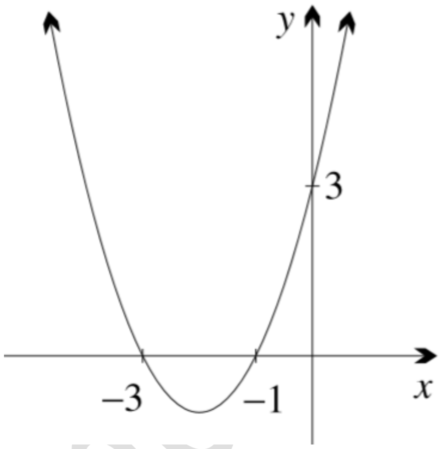
The solutions are when the graph is above the x-axis. This is when $x < -1$ or $x > 3$.

6c Factorising the equation gives $(2-x)(x-5)>0$, hence the parabola is



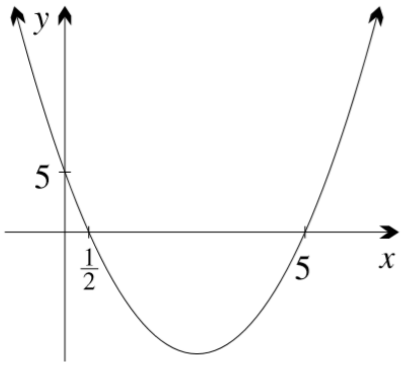
The solutions are when the graph is on or above the x-axis. This is when $2 \le x \le 5$.

6d Factorising the equation gives $(x+1)(x+3)>0$, hence the parabola is



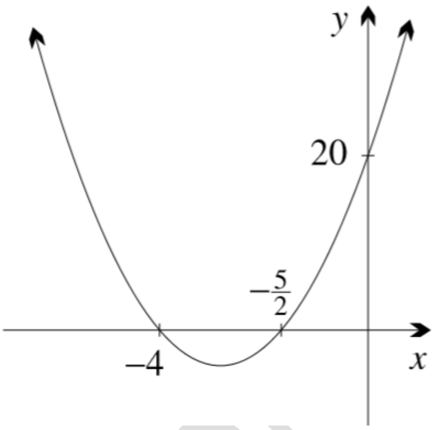
The solutions are when the graph is above or on the x-axis. This is when $x \le -3$ or $x \ge -1$.

6e Factorising the equation gives $(2x-1)(x-5) > 0$, hence the parabola is



The solutions are when the graph is above the x-axis. This is when $x < \frac{1}{2}$ and $x > 5$.

6f Factorising the equation gives $(x+4)(2x+5) > 0$, hence the parabola is



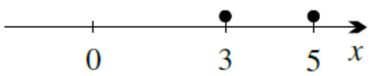
The solutions are when the graph is on or below the x-axis. This is when $-4 \leq x \leq -\frac{5}{2}$.

7a $|x-4| = 1$

$x - 4 = 1$ or $x - 4 = -1$

$x = 5$ or $x = 3$

$x = 3$ or 5



7b $|2x-3|=7$

$2x-3=7$ or $2x-3=-7$

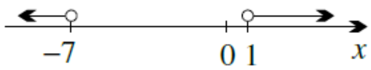
$x=-2$ or $x=5$

$x=-2$ or 5

7c $|x+3|>4$

$x+3>4$ or $x+3<-4$

$x>1$ or $x<-7$

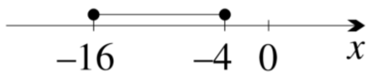


7d $|-x-10|\le 6$

$-x-10\le 6$ and $-x-10\ge -6$

$-16\le x$ and $-4\ge x$

$-16\le x\le -4$



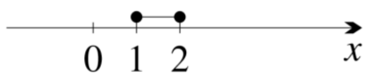
7e $|3-2x|\le 1$

$3-2x\le 1$ and $3-2x\ge -1$

$2\le 2x$ and $4\ge 2x$

$1\le x$ and $2\ge x$

$1\le x\le 2$

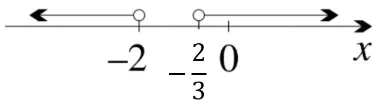


7f $|3x + 4| > 2$

$3x + 4 < -2$ or $3x + 4 > 2$

$3x < -6$ or $3x > -2$

$x < -2$ or $x > -\frac{2}{3}$



8a $\frac{1}{x} \le 2$

$\frac{1}{x} \times x^2 \le 2x^2$

$x \le 2x^2$

$0 \le 2x^2 - x$

$0 \le x(2x - 1)$

This is a concave up parabola, so the regions greater than zero will be those to the left of the first intersection and those to the right of the right intersection. Noting that $x \neq 0$, we have that $x < 0$ or $x \ge \frac{1}{2}$.

8b $\frac{3}{2-x} > 1$

$\frac{3}{2-x}(2-x)^2 > (2-x)^2$

$3(2-x) > 4 - 4x + x^2$

$6 - 3x > 4 - 4x + x^2$

$0 > x^2 - x - 2$

$0 > (x-2)(x+1)$

This is a concave up parabola, so the regions less than zero will be those in the region $-1 < x < 2$.

8c $\frac{4}{3-2x} < 1$

$\frac{4}{3-2x}(3-2x)^2 < (3-2x)^2$

$4(3-2x) < 9 - 12x + 4x^2$

$12 - 8x < 9 - 12x + 4x^2$

$0 < -3 - 4x + 4x^2$

$0 < (2x-3)(2x+1)$

This is a concave up parabola, so the regions greater than zero will be those to the left of the first intersection and those to the right of the right intersection. Hence $x < -\frac{1}{2}$ or $x > \frac{3}{2}$.

8d $\frac{5}{4x-3} \geq -2$

$\frac{5}{4x-3}(4x-3)^2 \geq -2(4x-3)^2$

$5(4x-3) \geq -2(16x^2 - 24x + 9)$

$20x - 15 \geq -32x^2 + 48x - 18$

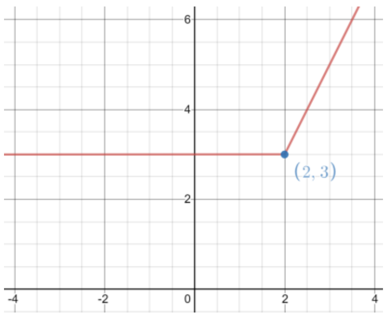
$32x^2 - 28x + 3 \geq 0$

$(8x-1)(4x-3) \geq 0$

This is a concave up parabola, so the regions greater than zero will be those to the left of the first intersection and those to the right of the right intersection. Also note that the original curve is undefined at $x = \frac{3}{4}$ and hence this point cannot be included in our final answer. Hence $x \leq \frac{1}{8}$ or $x > \frac{3}{4}$.

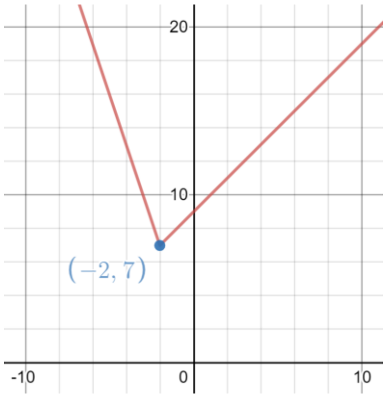
9a For $x \geq 2$, $y = |x-2| + x + 1 = x - 2 + x + 1 = 2x - 1$

For $x < 2$, $y = |x-2| + x + 1 = -(x-2) + x + 1 = 3$



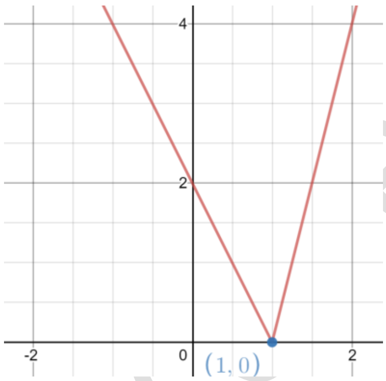
9b For $x \geq -2$, $y = |2x + 4| - x + 5 = 2x + 4 - x + 5 = x + 9$

For $x < -2$, $y = |2x + 4| - x + 5 = -(2x + 4) - x + 5 = -3x + 1$



9c For $x \geq -1$, $y = 3|x - 1| + x - 1 = 3(x - 1) + x - 1 = 4x - 4$

For $x < -1$, $y = 3|x - 1| + x - 1 = -3(x - 1) + x - 1 = -2x + 2$



10a $3^x \geq 27$

$x \geq \log_3 27$

$x \geq \dfrac{\ln 27}{\ln 3}$

$x \geq 3$

10b $1 < 5^x \leq 125$

$\log_5 1 < x \leq \log_5 125$

$\frac{\ln 1}{\ln 5} < x \leq \frac{\ln 125}{\ln 5}$

$0 < x \leq 3$

10c $\frac{1}{16} \leq 2^x \leq 16$

$\log_2 \frac{1}{16} \leq x \leq \log_2 16$

$\frac{\ln \frac{1}{16}}{\ln 2} \leq x \leq \frac{\ln 16}{\ln 2}$

$-4 \leq x \leq 4$

10d $2^{-x} > 16$

$-x > \log_2 16$

$-x > \frac{\ln 16}{\ln 2}$

$-x > 4$

$x < -4$

10e $\log_2 x < 3$

$0 < x < 2^3$

$0 < x < 8$

10f $-2 \leq \log_5 x \leq 4$

$5^{-2} \leq x \leq 5^4$

$\frac{1}{25} \leq x \leq 625$

11a The parabola $y = x^2 - 2x$ is below the line $y = x$ when $x^2 - 2x < x$. Rearranging gives:

$x^2 - 2x < x$

$x^2 - 3x < 0$

$x(x - 3) < 0$

This is a concave up parabola, so the region less than zero will be that between the two points of intersection. Hence $0 < x < 3$.

11b The parabola $y = x^2 - 2x - 3$ is below the line $y = 3 - 3x$ when $x^2 - 2x - 3 > 3 - 3x$. Rearranging gives:

$x^2 + x - 6 > 0$

$(x + 3)(x - 2) > 0$

This is a concave up parabola, so the regions greater than zero will be those to the left of the first intersection and those to the right of the second intersection. Hence $x < -3$ or $x > 2$.

11c The hyperbola $y = \frac{1}{x}$ is below the line $y = -x$ when $\frac{1}{x} < -x$. Rearranging gives:

$\frac{1}{x} \times x^2 < -x \times x^2$

$x < -x^3$

$x^3 + x < 0$

$x(x^2 + 1) < 0$

Note that $x^2 + 1$ is always positive, so the sign of the function will only depend on the sign of x. The function will be less than zero whenever $x < 0$. Thus $y = \frac{1}{x}$ is below $y = -x$ when $x < 0$.

11d The hyperbola $y = \frac{2}{x-1}$ is above the line $y = \frac{1}{2}x - 2$ when $\frac{2}{x-1} > \frac{1}{2}x - 2$. Rearranging gives:

$\frac{2}{x - 1}(x - 1)^2 > \left(\frac{1}{2}x - 2\right)(x - 1)^2$

$2(x-1) > \left(\frac{1}{2}x-2\right)(x-1)^2$

$2(x-1) - \left(\frac{1}{2}x-2\right)(x-1)^2 > 0$

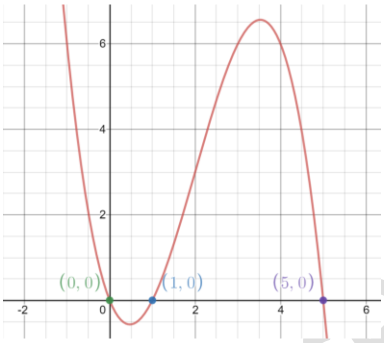
$(x-1)\left(2-\left(\frac{1}{2}x-2\right)(x-1)\right) > 0$

$(x-1)\left(2-\left(\frac{1}{2}x^2-\frac{5}{2}x+2\right)\right) > 0$

$(x-1)\left(\frac{5}{2}x-\frac{1}{2}x^2\right) > 0$

$\frac{1}{2}x(x-1)(5-x) > 0$

The graph of this is



By observation of the graph, the inequality is satisfied when $x < 0$ and when $1 < x < 5$. Thus $y = \frac{2}{x-1}$ is above the line $y = \frac{1}{2}x - 2$ when $x < 0$ and when $1 < x < 5$.

11e The parabola $y = x^2 + 2x - 3$ is below the parabola $y = 5 - 4x - x^2$ when $x^2 + 2x - 3 < 5 - 4x - x^2$. Rearranging gives:

$x^2 + 2x - 3 < 5 - 4x - x^2$

$2x^2 + 6x - 8 < 0$

$x^2 + 3x - 4 < 0$

$(x+4)(x-1) < 0$

This is a concave up parabola, so the region less than zero will be that between the two points of intersection. Hence $-4 < x < 1$.

12ai For $x \geq 0, y = |2x| + x = 2x + x = 3x$

For $x < 0, y = |2x| + x = -2x + x = -x$

12aii Solving $|2x| + x > 1$

For the region $x \geq 0$, we solve the inequality $3x > 1$ which gives $x > \frac{1}{3}$

For the region $x < 0$, we solve the inequality $-x > 1$ which gives $x < -1$

So $x < -1$ or $x > \frac{1}{3}$

12bi For $x \geq 2, y = 3|x - 2| + x - 2 = 3(x - 2) + x - 2 = 4x - 8$

For $x < 2, y = 3|x - 2| + x - 2 = -3(x - 2) + x - 2 = -2x + 4$

Solving $3|x - 2| + x - 2 \leq 2$

For the region $x \geq 2$, we solve the inequality $4x - 8 \leq 2$ which gives $4x \leq 10$ and so $x \leq 2\frac{1}{2}$

For the region $x < 2$, we solve the inequality $-2x + 4 \leq 2$ which gives $-2x \leq -2$ and so $x \geq 1$

So $1 \leq x \leq 2\frac{1}{2}$

12bii For $x \geq -1, y = |x + 1| - \frac{1}{2}x = (x + 1) - \frac{1}{2}x = \frac{1}{2}x + 1$

For $x < -1, y = y = |x + 1| - \frac{1}{2}x = -(x + 1) - \frac{1}{2}x = -\frac{3}{2}x - 1$

Solving $|x + 1| - \frac{1}{2}x < 3$

For the region $x \geq -1$, we solve the inequality $\frac{1}{2}x + 1 < 3$ which gives $\frac{1}{2}x < 2$ and so $x \leq 4$

For the region $x < -1$, we solve the inequality $-\frac{3}{2}x - 1 < 3$ which gives $-\frac{3}{2}x < 4$ and so $x > -\frac{8}{3}$

So $-\frac{8}{3} < x < 4$

13a $\frac{2x+1}{x-3} > 1$

$$\frac{2x+1}{x-3}(x-3)^2 > (x-3)^2$$

$$(2x+1)(x-3) > x^2 - 6x + 9$$

$$2x^2 - 5x - 3 > x^2 - 6x + 9$$

$$x^2 + x - 12 > 0$$

$$(x+4)(x-3) > 0$$

This is a concave up parabola, so the regions greater than zero will be those to the left of the first intersection and those to the right of the second intersection. Hence $x < -4$ or $x > 3$.

13b $\frac{x-1}{x+1} \leq 2$

$$\frac{x-1}{x+1} \leq 2$$

$$\frac{x-1}{x+1}(x+1)^2 \leq 2(x+1)^2$$

$$(x-1)(x+1) \leq 2(x+1)^2$$

$$x^2 - 1 \leq 2(x^2 + 2x + 1)$$

$$x^2 - 1 \leq 2x^2 + 4x + 2$$

$$0 \leq x^2 + 4x + 3$$

$$0 \leq (x+3)(x+1)$$

This is a concave up parabola, so the regions greater than zero will be those to the left of the first intersection and those to the right of the second intersection. Hence $x \leq -3$ or $x > -1$. Note that the original equation is not defined at the point $x = -1$, hence it is not included in our solution.

13c $\frac{3x}{2x-1} \geq 4$

$$\frac{3x}{2x-1}(2x-1)^2 \geq 4(2x-1)^2$$

$$3x(2x-1) \geq 4(2x-1)^2$$

$0 \geq 4(2x-1)^2 - 3x(2x-1)$

$0 \geq (2x-1)(4(2x-1)-3x)$

$0 \geq (2x-1)(8x-4-3x)$

$0 \geq (2x-1)(5x-4)$

This is a concave up parabola, so the region less than or equal to zero will be that between or on the two points of intersection. Hence $\frac{1}{2} < x \leq \frac{4}{5}$. Note that the original equation is not defined at the point $x = \frac{1}{2}$, hence it is not included in our solution.

14a $\cos x > 1 - \sin^2 x$

$\cos x > \cos^2 x$

$0 > \cos^2 x - \cos x$

$0 > \cos x\,(\cos x - 1)$

This is satisfied when $\cos x < 0$ and $\cos x - 1 > 0$ ($\cos x > 1$) or when $\cos x > 0$ and $\cos x - 1 < 0$ ($\cos x < 1$).

$\cos x < 0$ and $\cos x > 1$ has no solutions.

$\cos x > 0$ and $\cos x < 1$ when $0 < \cos x < 1$. This is when $0 < x < \frac{\pi}{2}$ and $\frac{3\pi}{2} < x < 2\pi$.

14b $\tan x \leq \sec^2 x - 1$

$\tan x \leq \tan^2 x$

$0 \leq \tan^2 x - \tan x$

$0 \leq \tan x\,(\tan x - 1)$

This is when $\tan x \geq 0$ and $\tan x \geq 1$ or when $\tan x \leq 0$ and $\tan x - 1 \leq 0$ making $\tan x \leq 1$.

The first solution is when $\tan x \geq 1$, this is when $\frac{\pi}{4} \leq x < \frac{\pi}{2}$.

The second solution is when $\tan x \leq 0$, this is when $-\frac{\pi}{2} < x \leq 0$.

15a $1 < |x + 2| \leq 3$

For $x \geq -2$, this inequality becomes $1 < x + 2 \leq 3$ which reduces to $-1 < x \leq 1$.

For $x < -2$, this inequality becomes $1 < -x - 2 \leq 3$ which reduces to $3 < -x \leq 5$ and is this $-5 \leq x < -3$. This the solutions are $-5 \leq x < -3$ or $-1 < x \leq 1$.

15b $1 \leq |2x - 3| < 4$

For $x \geq \frac{3}{2}$, this inequality becomes $1 \leq 2x - 3 < 4$ which reduces to $4 \leq 2x < 7$ and is thus $2 \leq x < 3\frac{1}{2}$.

For $x < \frac{3}{2}$, this inequality becomes $1 \leq -2x + 3 < 4$ which reduces to $-2 \leq 2x < 1$ and is this $-\frac{1}{2} < x \leq 1$. This the solutions are $-\frac{1}{2} < x \leq 1$ or $2 \leq x < 3\frac{1}{2}$.

16a This statement is false with a counterexample given by **ii** with $x = 2$ and $y = -2$ yielding LHS $= |2 - 2| = 0$ and RHS $= |2| + |-2| = 4$ so LHS $\neq$ RHS. A second counterexample is given by **iii** with $x = -2$ and $y = 2$ yielding LHS $= |-2 + 2| = 0$ and RHS $= |-2| + |2| = 4$ so LHS $\neq$ RHS.

16b If $x > 0$ and $y > 0$ then LHS $= x + y$ and RHS $= x + y$ so LHS $=$ RHS.

If $x > 0$ and $y < 0$ then LHS $= |x + y|$ and RHS $= x - y$ so LHS $<$ RHS.

If $x < 0$ and $y > 0$ then LHS $= |x + y|$ and RHS $= -x + y$ so LHS $<$ RHS.

If $x < 0$ and $y < 0$ then LHS $= -x - y$ and RHS $= -x - y$ so LHS $=$ RHS.

Hence the inequality is true.

This is known as the triangle inequality.

16c This statement is false with a counterexample given by **ii** with $x = 2$ and $y = -2$ yielding LHS $= |-2 - 2| = 4$ and RHS $= |-2| - |2| = 0$ so LHS $\geq$ RHS. A second counterexample is given by **iii** with $x = -2$ and $y = 2$ yielding LHS $= |- - 2 - 2| = 0$ and RHS $= |-2| - |2| = 4$ so LHS $\geq$ RHS.

16d If $x > 0$ and $y > 0$ then LHS $= x - y$ or $y - x$ and RHS $= x + y$ so LHS $<$ RHS.

If $x > 0$ and $y < 0$ then LHS $= x - y$ and RHS $= x + (-y)$ so LHS $=$ RHS.

If $x < 0$ and $y > 0$ then LHS $= y - x$ and RHS $= (-x) + y$ so LHS $=$ RHS.

If $x < 0$ and $y < 0$ then LHS $= x - y$ or $y - x$ and RHS $= -x - y$ so LHS $<$ RHS.

Hence this is true.

16e If $x > 0$ and $y > 0$ then LHS $= x - y$ or $y - x$ and RHS $= x - y$ or $y - x$ so LHS $=$ RHS.

If $x > 0$ and $y < 0$ then LHS $= x - y$ and RHS $= x - (-y)$ so LHS $>$ RHS.

If $x < 0$ and $y > 0$ then LHS $= y - x$ and RHS $= (-x) + y$ so LHS $>$ RHS.

If $x < 0$ and $y < 0$ then LHS $= x - y$ or $y - x$ and RHS $= -x - (-y)$ so LHS $=$ RHS.

Hence this inequality is true.

16f This is false, consider the case where $x = -2$. LHS $= 2^{|-2|} = 2^2 = 4$ and RHS $= 2^{-2} = \frac{1}{2^2} = \frac{1}{4}$. Thus LHS $=$ RHS and so the equality is not true.

17a $\sqrt{5-x}$ is strictly positive but $x + 1$ can be negative, so squaring both sides could make the inequality reverse direction.

17b Solving for a point of intersections of the graphs gives

$\sqrt{5-x} = (x + 1)$

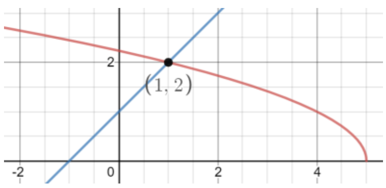
$5 - x = (x + 1)^2$

$5 - x = x^2 + 2x + 1$

$x^2 + 3x - 4 = 0$

$(x + 4)(x - 1) = 0$

$x = -4$ or $x = 1$. By observation of the graph of the two curves, $x = -4$ is not a solution, and the inequality is satisfied to the left of the point where $x = 1$. Thus, the solution to the inequation is $x < 1$.

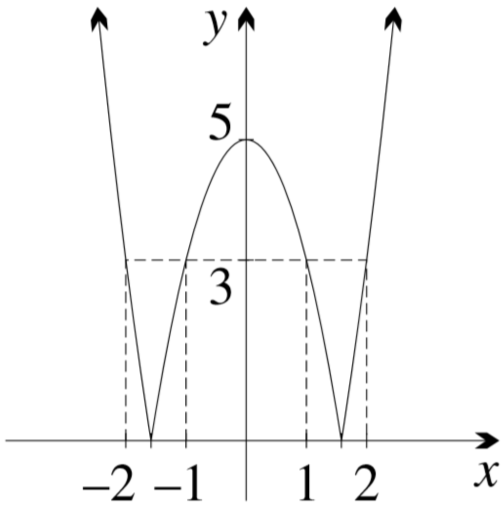


18a

When $x < -\frac{\sqrt{5}}{\sqrt{2}}$, $x^2 > \frac{5}{2}$ and hence $f(x) = 2x^2 - 5$.

When $-\frac{\sqrt{5}}{\sqrt{2}} < x < \frac{\sqrt{5}}{\sqrt{2}}$, $x^2 < \frac{5}{2}$ and hence $f(x) = 5 - 2x^2$.

When $x > \frac{\sqrt{5}}{\sqrt{2}}$, $x^2 > \frac{5}{2}$ and hence $f(x) = 2x^2 - 5$.



The x-intercepts occur when $|5 - 2x^2| = 0$. This is when $5 - 2x^2 = 0$ and hence when $5 = 2x^2$, this is when $x^2 = \frac{5}{2}$ and thus the x-intercepts are when $x = \pm\sqrt{\frac{5}{2}}$.

18b By observation of the graph, the inequation is true when $2x^2 - 5 \geq 3$ and when $5 - 2x^2 \geq 3$.

Solving $2x^2 - 5 \geq 3$:

$2x^2 \geq 8$

$x^2 \geq 4$

$x \geq 2$ or $x \leq -2$

Solving $5 - 2x^2 \geq 3$:

$2 \geq 2x^2$

$1 \geq x^2$

$-1 \leq x \leq 1$

Hence the inequation is true when $x \leq -2, -1 \leq x \leq 1$ or $x \geq 2$.

18c Suppose that $|g(x)| \geq k$. Since $|g(x)|$ is equal to either $g(x)$ or $-g(x)$, this means that either $g(x) \geq k$, or $-g(x) \geq k$.

If $-g(x) \geq k$, we multiply both sides by -1 and reverse the inequality to get $g(x) \leq -k$.

Thus, if $|g(x)| \geq k$, then either $g(x) \geq k$, or $g(x) \leq -k$.

19 By completing the square, we have

$x^2 + xy + y^2$

$= x^2 + 2\left(x\dfrac{y}{2}\right) + \left(\dfrac{y}{2}\right)^2 + y^2 - \left(\dfrac{y}{2}\right)^2$

$= \left(x + \dfrac{y}{2}\right)^2 + \dfrac{3y^2}{4}$

Since $a^2 \geq 0$ for all a, we know that $\left(x + \frac{y}{2}\right)^2 + \frac{3y^2}{4} \geq 0$ for all x and y.

If $\left(x + \frac{y}{2}\right)^2 + \frac{3y^2}{4} = 0$, it follows that $x = y = 0$, and therefore $x^2 + xy + y^2 > 0$ for all non-zero x and y.

20a $(x - y)^2 \geq 0$

$x^2 - 2xy + y^2 \geq 0$

$x^2 + 4xy - 2xy + y^2 \geq 4xy$

$x^2 + 2xy + y^2 \geq 4xy$

$(x + y)^2 \geq 4xy$

20b $(x + y)^2 \geq 4xy$

$x^2 + 2xy + y^2 \geq 4xy$

$x^2 + y^2 \geq 2xy$

$\dfrac{x^2 + y^2}{2} \geq xy$

$\dfrac{1}{xy} \geq \dfrac{2}{x^2 + y^2}$

Using this, we obtain

$$\frac{1}{x^2}+\frac{1}{y^2}=\frac{y^2}{x^2y^2}+\frac{x^2}{x^2y^2}$$
$$=\frac{y^2+x^2}{x^2y^2}$$
$$=\frac{y^2+2xy+x^2-2xy}{x^2y^2}$$
$$=\frac{(x+y)^2-2xy}{x^2y^2}$$
$$\geq\frac{4xy-2xy}{x^2y^2}$$
$$=\frac{2xy}{x^2y^2}$$
$$=\frac{2}{xy}$$
$$=2\left(\frac{1}{xy}\right)$$
$$\geq 2\left(\frac{2}{x^2+y^2}\right)$$
$$=\frac{4}{x^2+y^2}$$

21a $(a-b)^2+(b-c)^2+(a-c)^2$
$=a^2-2ab+b^2+b^2-2bc+c^2+a^2-2ac+c^2$
$=2a^2+2b^2+2c^2-2ab-2bc-2ac$
$=2(a^2+b^2+c^2-ab-bc-ac)$

We know that the square of any real number must be positive hence

$(a-b)^2\geq 0$

$(b-c)^2\geq 0$

$(a-c)^2\geq 0$

Thus

$(a-b)^2+(b-c)^2+(a-c)^2\geq 0$

$2(a^2+b^2+c^2-ab-bc-ac)\geq 0$

$a^2+b^2+c^2-ab-bc-ac\geq 0$

$a^2+b^2+c^2\geq ab+bc+ac$

21b $(a+b+c)((a-b)^2+(b-c)^2+(a-c)^2)$

$$\begin{aligned}&= (a+b+c)\times 2(a^2+b^2+c^2-ab-bc-ac)\\ &= 2(a^3+ab^2+ac^2-a^2b-abc-a^2c+a^2b+b^3+c^2b-ab^2-b^2c-abc\\ &\qquad +a^2c+b^2c+c^3-abc-bc^2-ac^2)\\ &= 2(a^3+b^3+c^3-3abc)\end{aligned}$$

Now if a, b and c are positive, then $a+b+c>0$ and for all positive a, b and c we know that $(a-b)^2+(b-c)^2+(a-c)^2>0$. Thus, it follows that $(a+b+c)((a-b)^2+(b-c)^2+(a-c)^2)>0$ and hence $2(a^3+b^3+c^3-3abc)>0$ so it follows that $a^3+b^3+c^3>3abc$.

Solutions to Exercise 3E

1a 1 point of intersection, hence 1 solution

1b 2 points of intersection, hence 2 solutions

1c 3 points of intersection, hence 3 solutions

1d 2 points of intersection, hence 2 solutions

1e 2 points of intersection, hence 2 solutions

1f 3 points of intersection, hence 3 solutions

2a $x = \frac{1}{2}$

2b $x = -\frac{3\pi}{4}$ or $\frac{\pi}{4}$

2c $x \doteqdot -2.1, 0.3$ or 1.9

2d $x = 1$ or $x \doteqdot 3.5$

2e $x = 1$ or $x \doteqdot -1.9$

2f $x = 0$ or $x \doteqdot -1.9$ or 1.9

3a $x \leq -3$

3b $0 \leq x \leq 2$

3c $x = 1$

4a $x < -2$ or $x > 1$

4b $0 \leq x \leq 1$

4c $-1 < x < 0$ or $x > 1$

5a We read off the values by noting that when $y = 3$

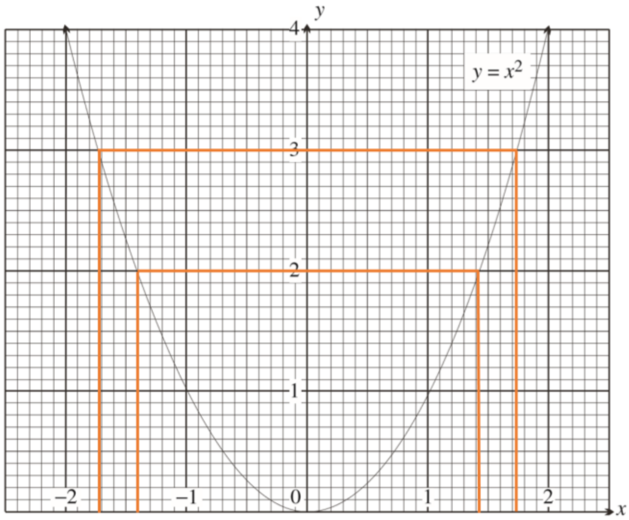
$3 = x^2$

$x = \pm\sqrt{3}$

and when $y = 2$

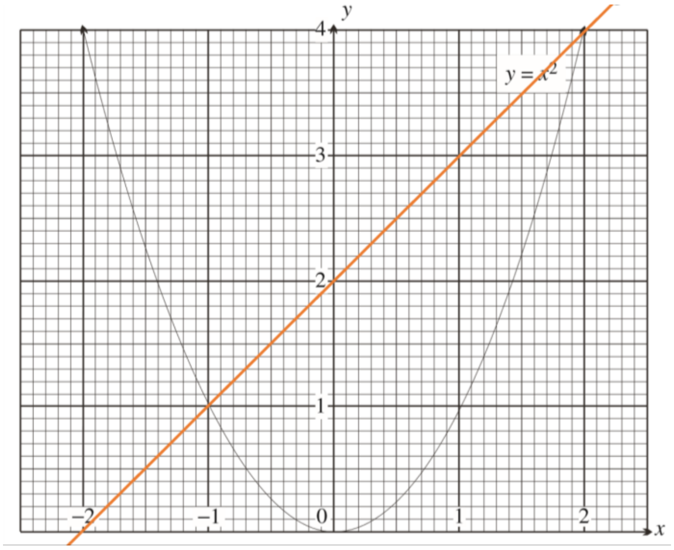
$2 = x^2$

$x = \pm\sqrt{2}$



Now as we want the positive values, we read off the values from the right of the axis. This gives $\sqrt{2} \doteqdot 1.4, \sqrt{3} \doteqdot 1.7$.

5b



By observation the intercepts are at $(-1, 1)$ and $(2, 4)$.

Hence the solutions are $x = -1$ or $x = 2$.

Solving the equation algebraically gives

$x^2 = x + 2$

$x^2 - x - 2 = 0$

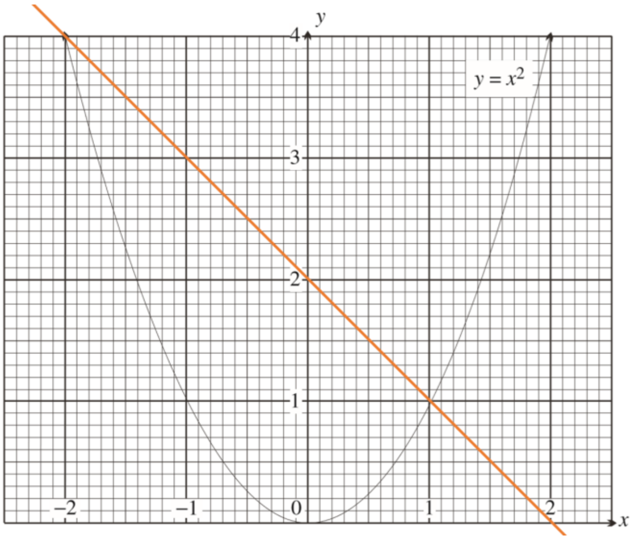
$(x - 2)(x + 1) = 0$

$x = -1$ or 2

When $x = -1$, $y = 1$ and when $x = 2$, $y = 4$. This confirms that the intercepts are at $(-1, 1)$ and $(2, 4)$.

5c Reading off the graph, $x^2 > x + 2$ when $x < -1$ and $x > 2$.

5d



By observation the intercepts are at $(-2, 4)$ and $(1, 1)$. Hence the solutions are $x = -2$ or $x = 1$.

Solving the equation algebraically gives:

$x^2 = 2 - x$

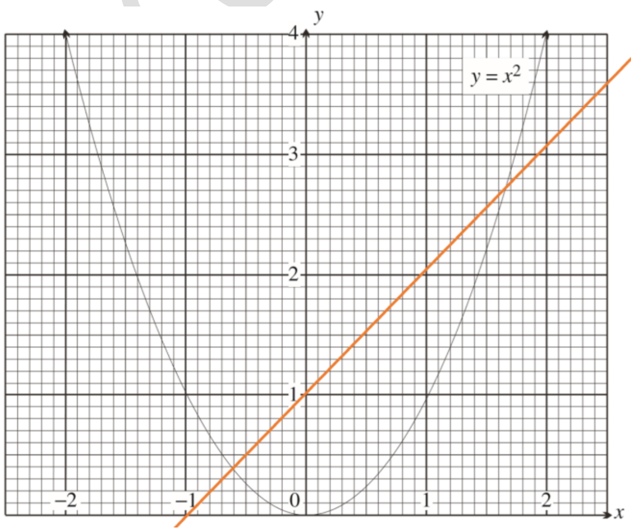
$x^2 + x - 2 = 0$

$(x + 2)(x - 1) = 0$

$x = -2$ or $x = 1$

Reading off the graph, $x^2 \leq 2 - x$ when $-2 \leq x \leq 1$.

5e



From the graph, $x \doteqdot 1.62$ or $x \doteqdot -0.62$.

Solving the equation algebraically gives:

$x^2 = x + 1$

$x^2 - x - 1 = 0$

$$x = \frac{1 \pm \sqrt{(1)^2 - 4(1)(-1)}}{2}$$

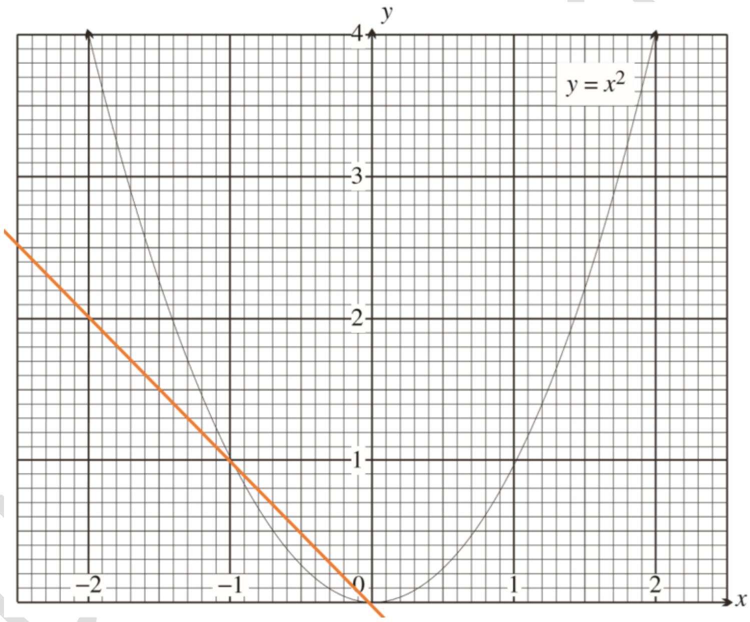
$$= \frac{1 \pm \sqrt{5}}{2}$$

$\doteqdot 1.62$ or -0.62

5f i $\quad x^2 + x = 0$

$x^2 = -x$

Draw the line for $y = -x$ on the graph.

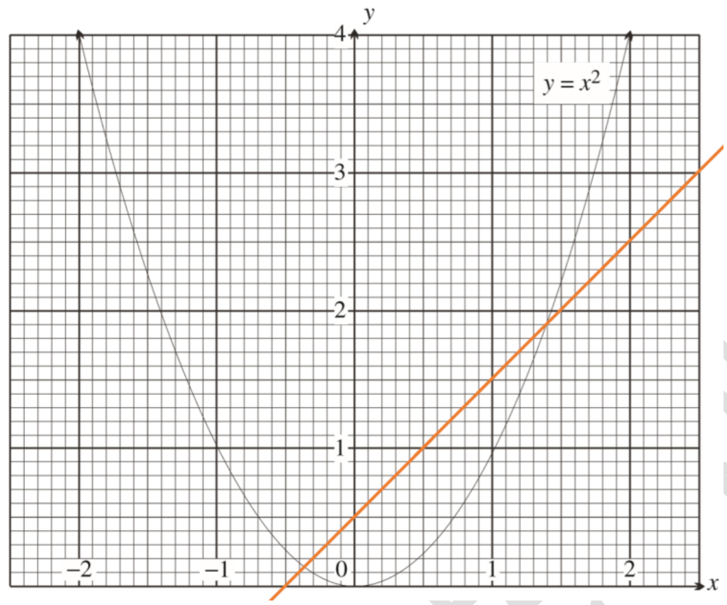


From the graph, $x = 0$ or $x = -1$.

5f ii $x^2 - x - \frac{1}{2} = 0$

$$x^2 = x + \frac{1}{2}$$

Draw the line for $y = x + \frac{1}{2}$ on the graph.



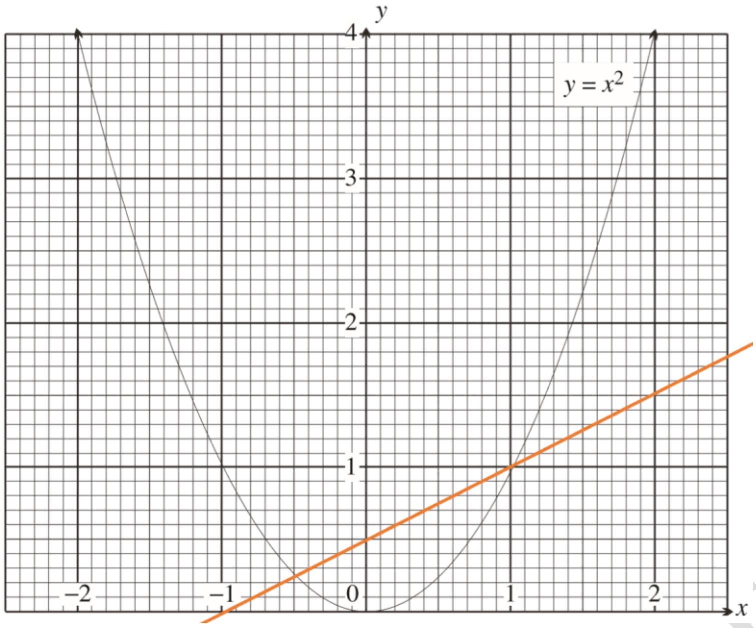
From the graph, $x \doteqdot 1.37$ or $x \doteqdot -0.37$.

5f iii $2x^2 - x - 1 = 0$

$2x^2 = x + 1$

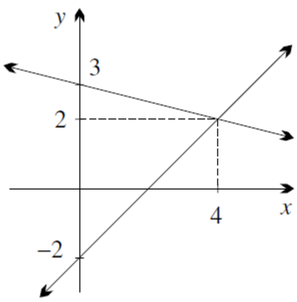
$x^2 = \frac{1}{2}x + \frac{1}{2}$

Draw the line for $y = \frac{1}{2}x + \frac{1}{2}$ on the graph.



From the graph, $x = 1$ or $x = -\frac{1}{2}$.

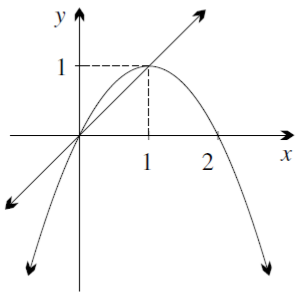
6a i $y = x - 2$ and $y = 3 - \frac{1}{4}x$



6a ii Point of intersection is $(4, 2)$.

6a iii $x - 2 = 3 - \frac{1}{4}x$

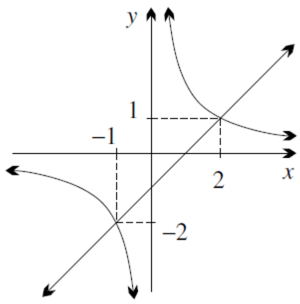
6b i $y = x$ and $y = 2x - x^2$



6b ii Points of intersection are $(0, 0)$ and $(1, 1)$.

6b iii $x = 2x - x^2$

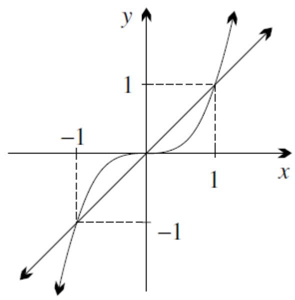
6c i $y = \frac{2}{x}$ and $y = x - 1$



6c ii Points of intersection are $(-1, -2)$ and $(2, 1)$.

6c iii $\frac{2}{x} = x - 1$

6d i $y = x^3$ and $y = x$



6d ii Points of intersection are $(-1,-1)$, $(0,0)$ and $(1,1)$.

6d iii $x^3 = x$

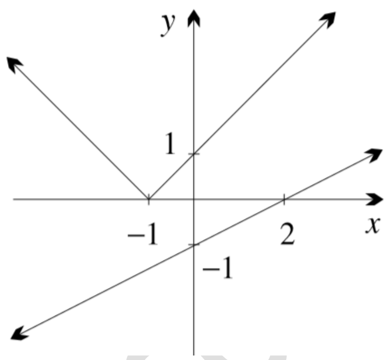
7a $x \geq 4$

7b $0 < x < 1$

7c $x < -1$ or $0 < x < 2$

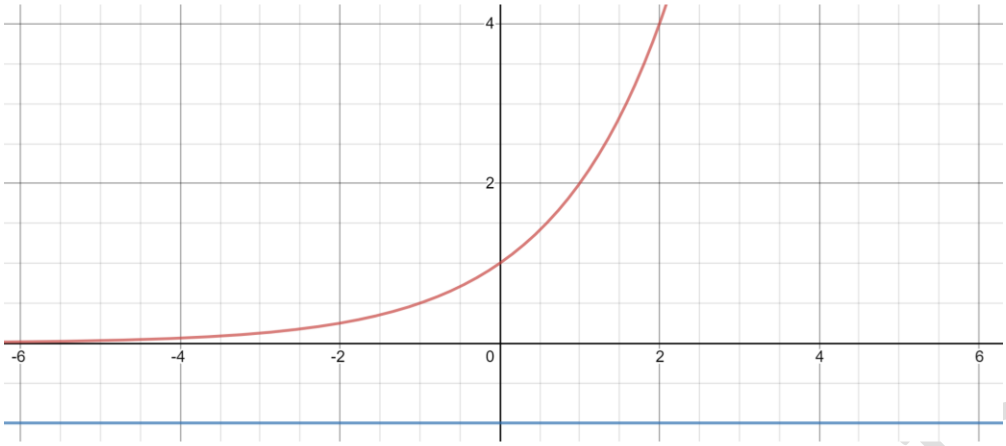
7d $-1 < x < 0$ or $x > 1$

8a i



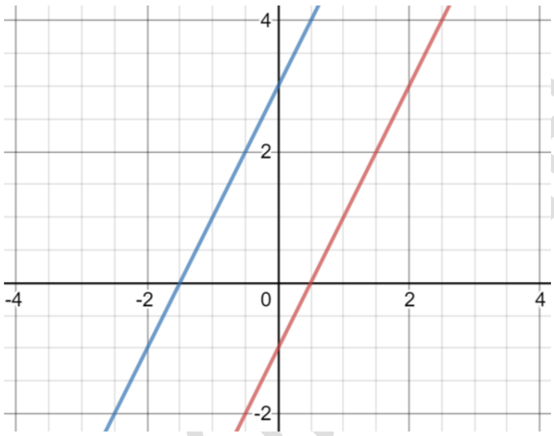
8a ii By observation of the above diagram, the graph of $y = |x+1|$ always lies above the graph of $y = \frac{1}{2}x - 1$, hence $|x+1| > \frac{1}{2}x - 1$ for all real x.

8b Draw the graphs of $y = 2^x$ and $y = -1$.



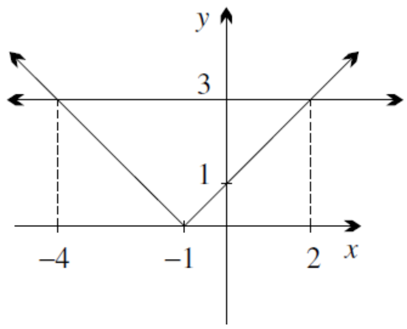
We see that the graph of $y = 2^x$ (shown in red) is always above that of $y = -1$ (shown in blue). Hence it follows that $2^x > -1$ for all x. Thus, it must be the case that $2^x \leq -1$ has no solutions.

8c



The equation $y = 2x + 3$ (shown in blue) is always above the equation $y = 2x - 1$ (shown in red). Note that these lines never intersect because they have the same gradient and are hence parallel. Thus, it follows that $2x - 1 \leq 2x + 3$ for all values of x.

9a

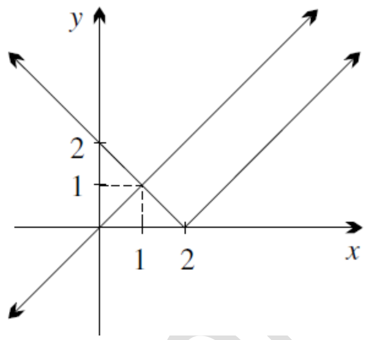


When $x < -1$, we find the point of intersection by solving $-x - 1 = 3$ which gives $x = -4$.

When $x > -1$, we find the point of intersection by solving $x + 1 = 3$ which gives $x = 2$.

Points of intersection are at $(-4, 3)$ and $(2, 3)$.

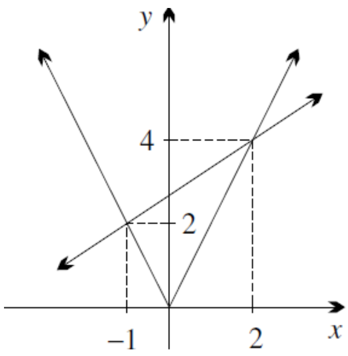
9b



When $x < 2$, we find points of intersection by solving the equation $-x + 2 = x$ which gives $2x = 2$ and thus $x = 1$. Substituting this into $y = x$ gives $y = 1$. Observation of the graph reveals that there are no other points of intersection.

Point of intersection is at $(1, 1)$.

9c



When $x < 0$, we find the points of intersection by solving the equations $y = -2x$ and $2x - 3y + 8 = 0$. Substituting $y = -2x$ into $2x - 3y + 8 = 0$ gives

$2x - 3(-2x) + 8 = 0$

$2x + 6x + 8 = 0$

$8x + 8 = 0$

$8x = -8$

$x = -1$

Substituting this into $y = 2x$ gives $y = -2$.

When $x > 0$, we find the points of intersection by solving the equations $y = 2x$ and $2x - 3y + 8 = 0$. Substituting $y = 2x$ into $2x - 3y + 8 = 0$ gives

$2x - 3(2x) + 8 = 0$

$2x - 6x + 8 = 0$

$-4x + 8 = 0$

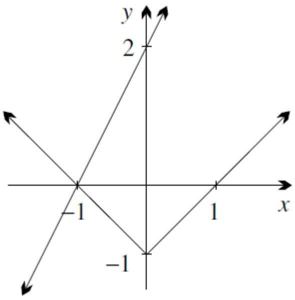
$4x = 8$

$x = 2$

Substituting this into $y = 2x$ gives $y = 4$.

Points of intersection are at $(-1, 2)$ and $(2, 4)$.

9d



When $x < 0$, we find the points of intersection by solving the equations $y = -x - 1$ and $y = 2x + 2$. Substituting $y = -x - 1$ into $y = 2x + 2$ gives

$-x - 1 = 2x + 2$

$-3 = 3x$

$x = -1$

Substituting this into $y = -x - 1$ gives $y = 0$.

By observation of the graph, there are no other solutions.

Point of intersection is at $(-1, 0)$.

10 We can either use the answers from the previous question or solve these algebraically as follows.

10a $|x + 1| \leq 3$

$-3 \leq x + 1 \leq 3$

$-4 \leq x \leq 2$

10b $|x - 2| > x$

$x - 2 > x$ or $x - 2 < -x$

$-2 > 0$ (no solution) or $2x < 2$

$x < 1$

10c $|2x| \geq \frac{2x+8}{3}$

$2x \geq \frac{2x+8}{3}$ or $2x \leq -\frac{2x+8}{3}$

Rearranging the first inequality gives:

$6x \geq 2x + 8$

$4x \geq 8$

$x \geq 2$

Rearranging the second inequality gives:

$6x \leq -2x - 8$

$8x \leq -8$

$x \leq -1$

Solution is $x \leq -1$ or $x \geq 2$.

10d $|x| > 2x + 3$

$x > 2x + 3$ or $x < -2x - 3$

Rearranging the first inequality gives:

$0 > x + 3$

$x < -3$

Rearranging the second inequality gives:

$3x < -3$

$x < -1$

Solution is $x < -1$.

11a Divide by e^x to get $e^x = e^{1-x}$

11b Multiply by $\cos x$ to get $\sin x = \cos x$

11c Subtract 1 then divide by x to get $x^2 - 4 = -\frac{1}{x}$

12a The table below traps the solution between $x = -1.690$ and $x = -1.6905$, so it is $x = -1.690$, correct to three decimal places.

x	-2	-1.7	-1.6	-1.68
2^x	0.25	0.3078	0.3299	0.3121
$x + 2$	0	0.3	0.4	0.32

x	-1.69	-1.691	-1.6905
2^x	0.3099	0.3097	0.3098
$x + 2$	0.31	0.309	0.3095

12b Part c:

x	-2	-2.1	-2.11	-2.114	-2.115	-2.116
$x^2 - 4$	0	0.41	0.4521	0.468996	0.473225	0.477456
$-\frac{1}{x}$	0.5	0.47619047	0.47393364	0.47303689	0.47281323	0.47258979

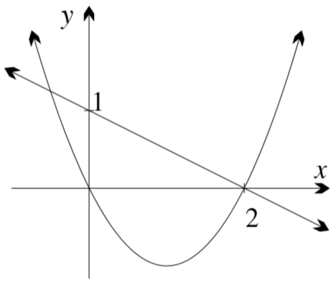
$x \doteqdot -2.115$

Part e:

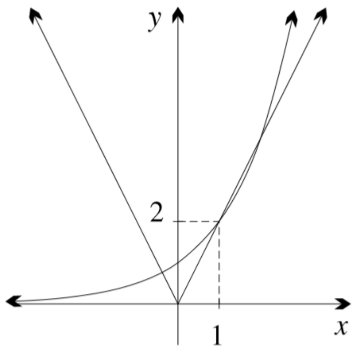
x	-1.9	-1.89	-1.88	-1.875	-1.872	-1.871
3^x	0.12401	0.12538	0.12677	0.12747	0.12788	0.12803
$x + 2$	0.1	0.11	0.12	0.125	0.128	0.129

$x \doteqdot -1.872$

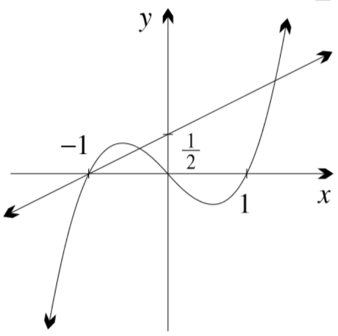
13a 2 solutions



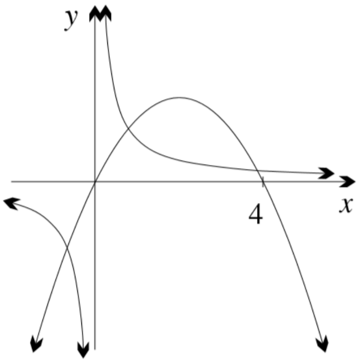
13b 3 solutions



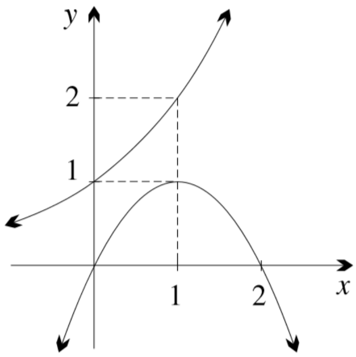
13c 3 solutions



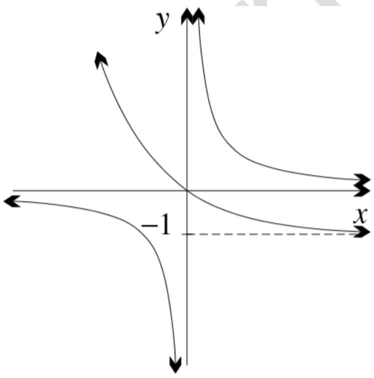
13d 3 solutions



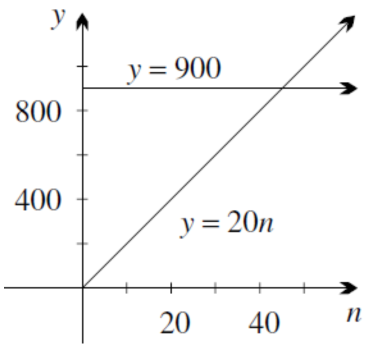
13e no solutions



13f no solutions

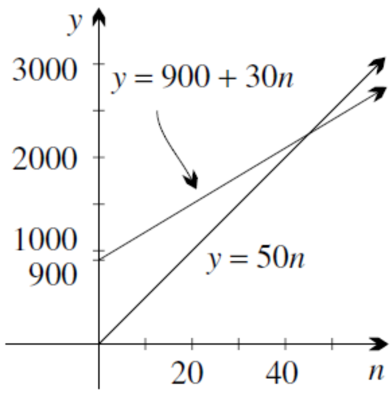


14a



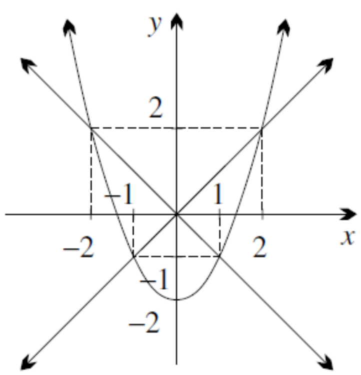
The break-even point is when $900 = 20n$ which is when $n = 45$. Hence the break-even point is $n = 45$. Total sales are \$2250 at that point.

14b



The break-even point is when $900 + 30n = 50n$, this reduces to $900 = 20n$ which is hence when $n = 45$. Hence the break-even point is $n = 45$. Total sales are \$2250 at that point.

15a



By observation, points of intersection are $(-2, 2), (-1, 1), (1, 2)$ and $(2, 2)$.

15b The graph of $y = |x|$ has 0 or positive values only for y.

So the points of intersection of $y = |x|$ and $y = x^2 - 2$ are $(-2, 2)$ and $(2, 2)$.

Hence the solutions for $x^2 - 2 = |x|$ are $x = 2$ or $x = -2$.

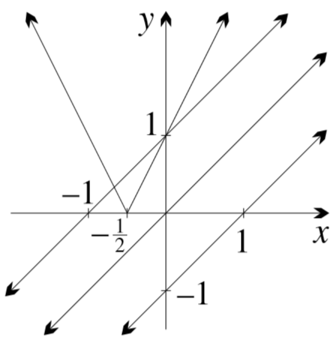
15c Consider where the graph of $y = x^2 - 2$ is above the graph of $y = |x|$.

Solution is $x < -2$ or $x > 2$.

15d Consider where the graph of $y = x^2 - 2$ is below or intersecting the graph of $y = -|x|$.

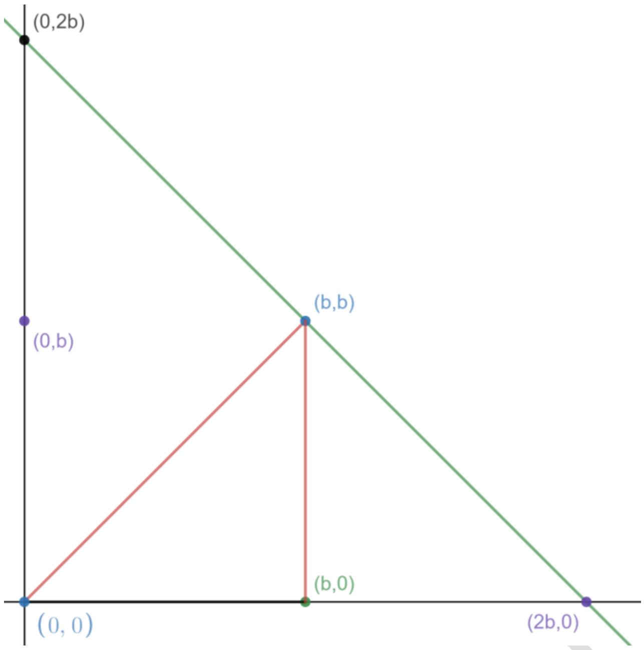
Solution is $-1 \le x \le 1$.

16a, b



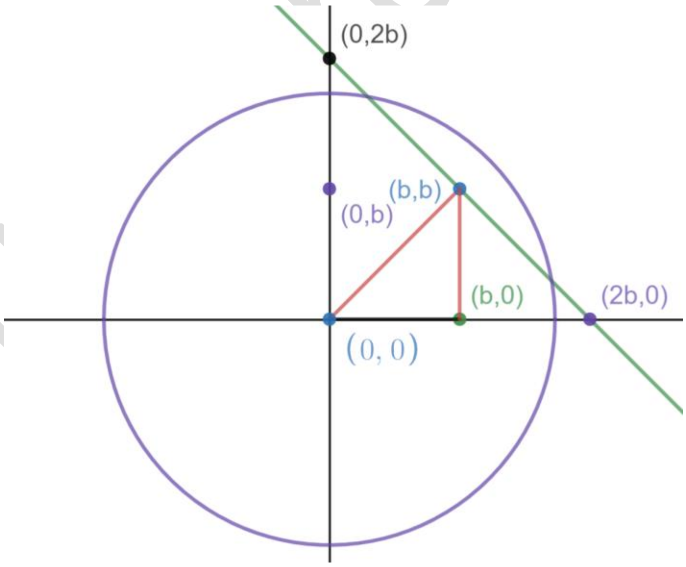
16c By observation of the above graph, there are two solutions when the line $y = x$ is shifted up by more than $\frac{1}{2}$ units, that is, when $c > \frac{1}{2}$.

17a



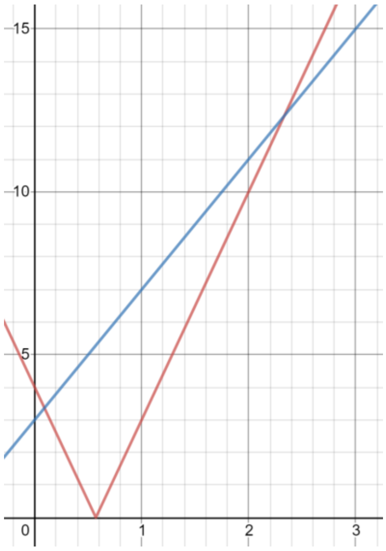
Observe, that the perpendicular distance from the line $x + y = 2b$ is the hypotenuse of a right angled triangle with the opposite and adjacent sides both of length b. Hence it follows that the length of the hypotenuse is $\sqrt{b^2 + b^2} = \sqrt{2b^2} = b\sqrt{2}$.

17b



There will be two solutions when the perpendicular distance of the line from the origin is less than the radius of the circle. That is when $b\sqrt{2} < 3$. Thus the line intersects the circle twice when $b < \frac{3}{\sqrt{2}}$ and is thus $b < \frac{3\sqrt{2}}{2}$. Note there will also be solutions when the line $x + y = 2b$ lies below the origin, this will be when $b > -\frac{3\sqrt{2}}{2}$. Thus, the solutions are $-\frac{3\sqrt{2}}{2} < b < \frac{3\sqrt{2}}{2}$.

18a



By observation of the graph there are two solutions.

18b As the solutions are not integers (whole numbers) it is hard to accurately determine the solution from the graph.

18c Firstly, consider the branch $y = 7x - 4$. Solving $7x - 4 = 4x + 3$ gives

$3x = 7$

$x = \frac{7}{3}$

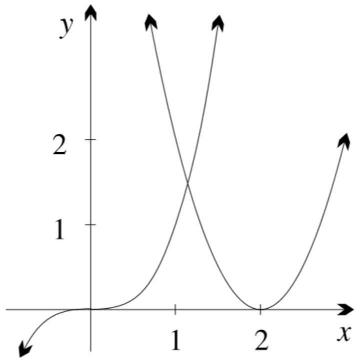
Now consider the branch $y = -7x + 4$. Solving $-7x + 4 = 4x + 3$ gives

$-11x = -1$

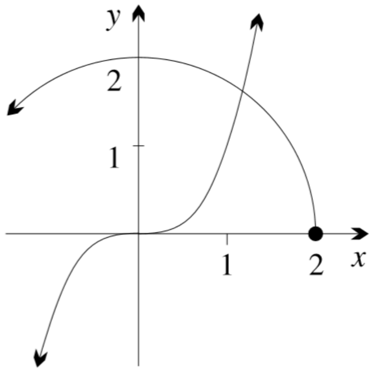
$x = \frac{1}{11}$

Hence the solutions are $x = \frac{7}{3}$ and $x = \frac{1}{11}$.

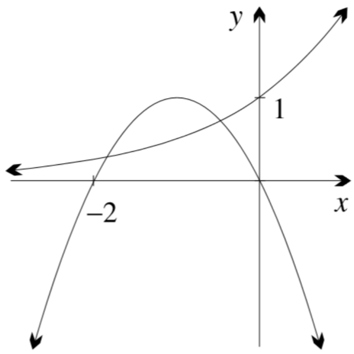
19a $x \doteqdot 1.1$



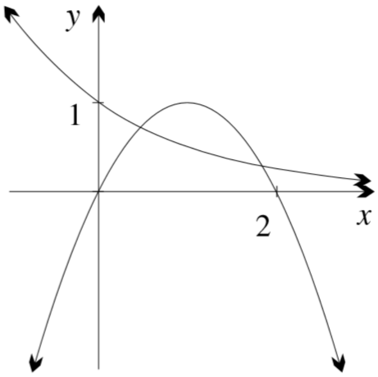
19b $x \doteqdot 1.2$



19c $x \doteqdot -0.5$ or $x \doteqdot -1.9$



19d $x \doteqdot 0.5$ or $x \doteqdot 1.9$

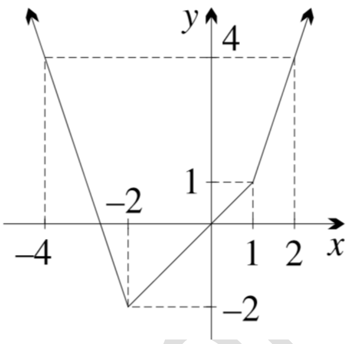


20a When $x < -2$ the equation is $y = -2x - 4 + (-x + 1) - 5 = -3x - 8$.

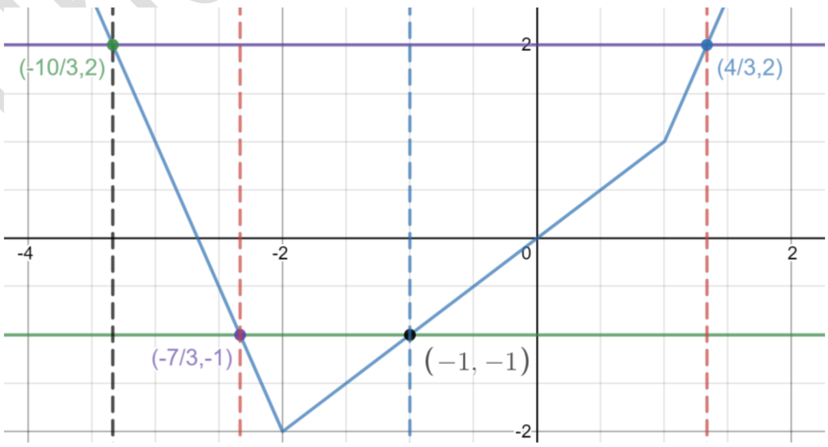
When $-2 \le x < 1$ the equation is $y = 2x + 4 + (-x + 1) - 5 = x$.

When $x \ge 1$ the equation is $y = 2x + 4 + x - 1 - 5 = 3x - 2$.

Hence we can plot the graph as:



20b



By observation, the equation intersects the line $y = -1$ when it is described by the equation $y = x$ and $y = -3x - 8$.

Solving with $y = x$, gives the intersection as $(-1, -1)$ and solving with $y = -3x - 8$ gives $-1 = -3x - 8$ and thus $3x = -7$ giving $x = -\frac{7}{3}$ giving the point of intersection as $(-\frac{7}{3}, -1)$.

By observation, the equation intersects the line $y = 2$ when it is described by the equation $y = 3x - 2$ and $y = -3x - 8$.

Solving with $y = 3x - 2$ gives $2 = 3x - 2$, $3x = 4$ and hence $x = \frac{4}{3}$.

Solving with $y = -3x - 8$ gives $2 = -3x - 8$, $3x = 10$ and hence $x = \frac{10}{3}$.

By observation of the graph, the solution to the inequality

$-1 \le |2x + 4| + |x - 1| - 5 \le 2$. Is thus $-3\frac{1}{3} \le x \le -2\frac{1}{3}$ or $-1 \le x \le 1\frac{1}{3}$

21a When $m > 1$:

If $b \ge 1$ then consider $y = |x + 1|$ when $x < -1$. In this case we have that $y = -x - 1$ for the equation and hence for a point of intersection $mx + b = -x - 1$. Solving gives

$x(m + 1) = -b - 1$

$$x = -\frac{b + 1}{m + 1}$$

Which instead lies in the region where $x < -1$ and thus we have a solution.

If $b < 1$ then consider $y = |x + 1|$ when $x \ge 1$. In this case we have that $y = x + 1$ for the equation and hence for a point of intersection $mx + b = x + 1$. Solving gives

$x(m - 1) = 1 - b$

$$x = \frac{1 - b}{m - 1}$$

Which instead lies in the region where $x \ge 1$ and thus we have a solution.

When $m < -1$:

If $b < 1$ then consider $y = |x + 1|$ when $x < -1$. In this case we have that $y = -x - 1$ for the equation and hence for a point of intersection $mx + b = -x - 1$. Solving gives

$x(m + 1) = -b - 1$

$x = -\dfrac{b+1}{m+1}$

Which instead lies in the region where $x < -1$ and thus we have a solution.

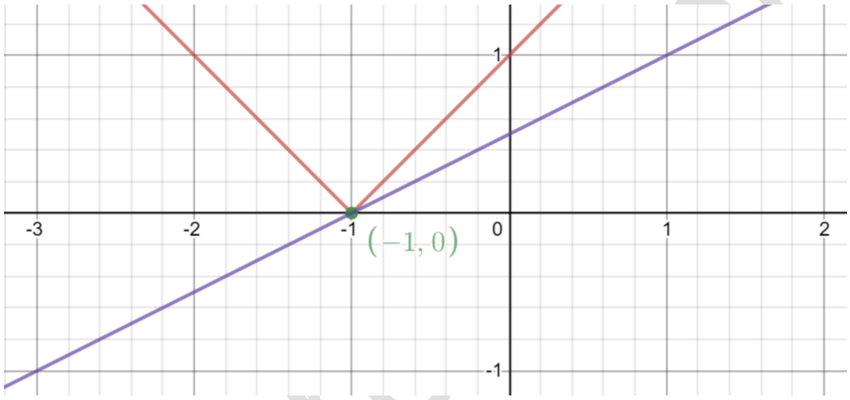
If $b \geq 1$ then consider $y = |x+1|$ when $x \geq 1$. In this case we have that $y = x + 1$ for the equation and hence for a point of intersection $mx + b = x + 1$. Solving gives

$x(m-1) = 1 - b$

$x = \dfrac{1-b}{m-1}$

Which instead lies in the region where $x \geq 1$ and thus we have a solution.

21b



Note that there will be no solutions provided that the line passes below the point $(-1, 0)$. Considering the equation $y = mx + b$, we must have $0 > m(-1) + b$ which is when $b < m$.

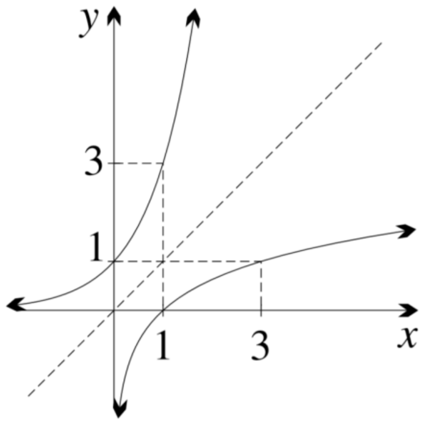
21c Applying the same principle as before, there will always be one solution when $m < -p$ or when $m > p$. Thus, the first condition that we require for no solutions is that $-p \leq m \leq p$.

With this condition satisfied, we then require that the line $y = mx + b$ must pass below the point $\left(\frac{q}{p}, 0\right)$, which is the turning point of $y = |px - q|$.

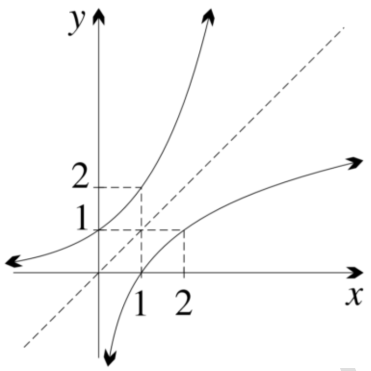
Thus $m\left(\frac{q}{p}\right) + b < 0$

$b < -\dfrac{mq}{p}$

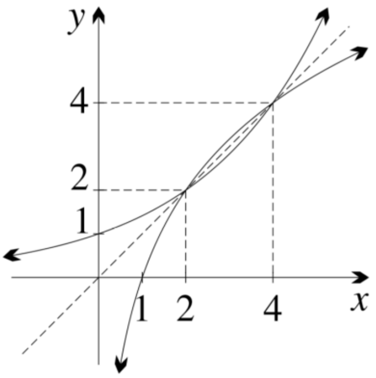
22a i



22a ii



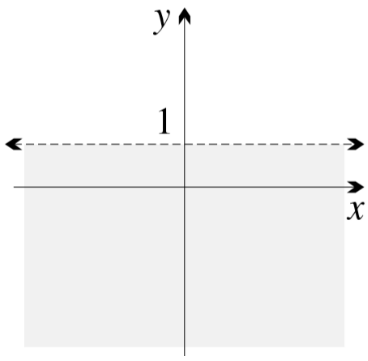
22a iii



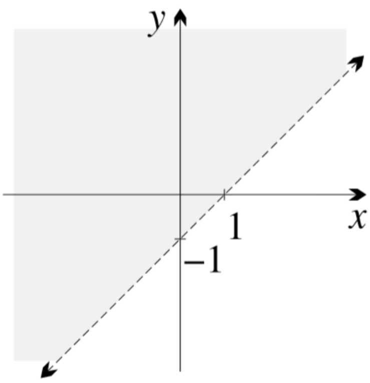
22b There are either no solutions (like in part **ai**), 1 solution (when the curves touch) or 2 solutions (like in part **aiii**). There can be no more solutions as both curves have a constant concavity and thus cannot intersect one another again.

Solutions to Exercise 3F

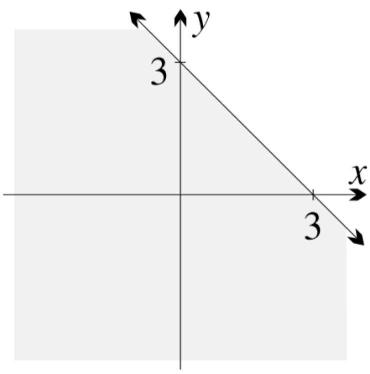
1a Substituting $(0, 0)$ into the inequality gives $0 < 1$. which is true. Hence, we shade the region containing the origin.



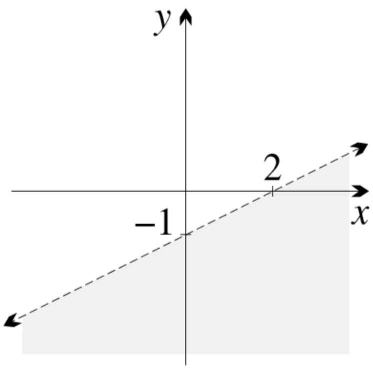
1b Substituting $(0, 0)$ into the inequality gives $0 > -1$. which is true. Hence, we shade the region containing the origin.



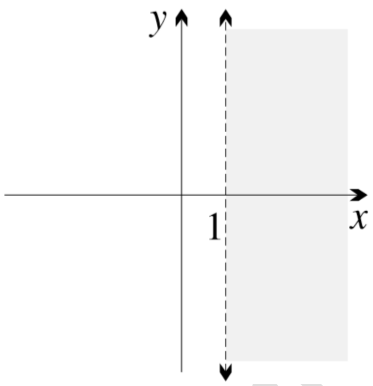
1c Substituting $(0, 0)$ into the inequality gives $0 \leq 3$. which is true. Hence, we shade the region containing the origin.



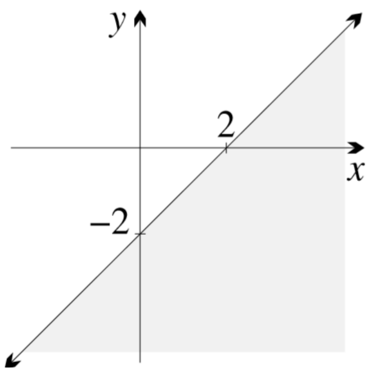
1d Substituting $(0, 0)$ into the inequality gives $0 < -1$. which is not true. Hence, we shade the region that does not contain the origin.



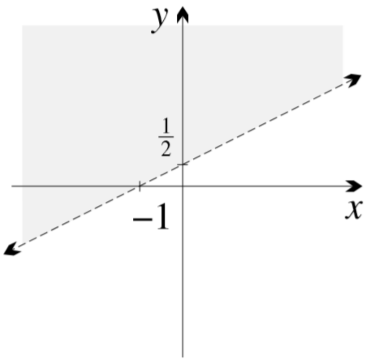
2a Substituting $(0, 0)$ into the inequality gives $0 < -1$. which is not true. Hence, we shade the region that does not contain the origin.



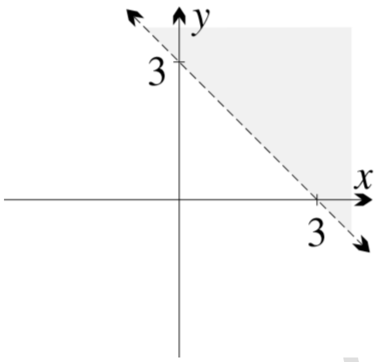
2b Substituting $(0, 0)$ into the inequality gives $0 \geq 2$. which is not true. Hence, we shade the region that does not contain the origin.



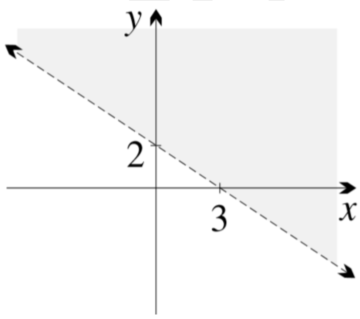
2c Substituting $(0, 0)$ into the inequality gives $0 < -1$. which is not true. Hence, we shade the region that does not contain the origin.



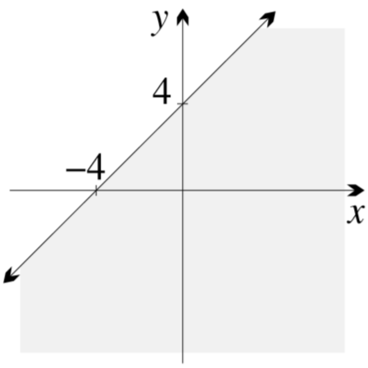
2d Substituting $(0, 0)$ into the inequality gives $0 > 3$. which is not true. Hence, we shade the region that does not contain the origin.



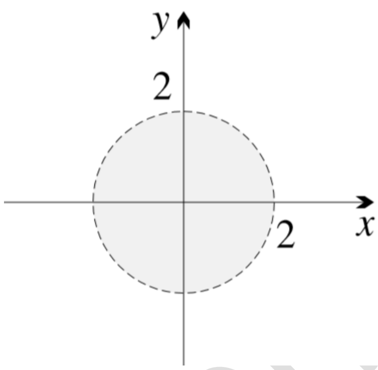
3a Testing at $(0, 0)$ we obtain $-6 > 0$ which is false. Hence, we shade the region not containing $(0, 0)$.



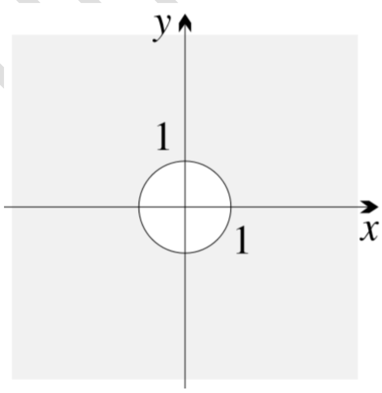
3b Testing at $(0, 0)$ we obtain $4 \geq 0$ which is true. Hence, we shade the region containing $(0, 0)$.



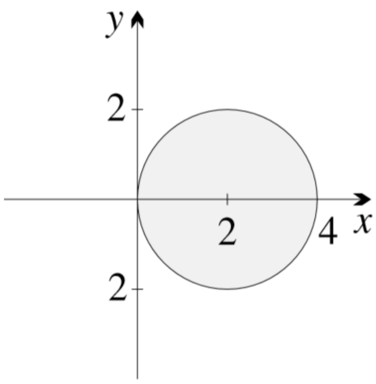
4a Testing at $(0, 0)$ we obtain $0 < 4$ which is true. Hence, we shade the region containing $(0, 0)$.



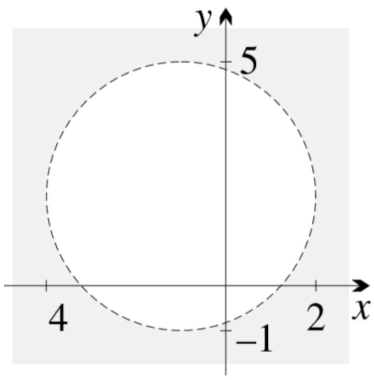
4b Testing at $(0, 0)$ we obtain $0 \geq 1$ which is false. Hence, we shade the region not containing $(0, 0)$.



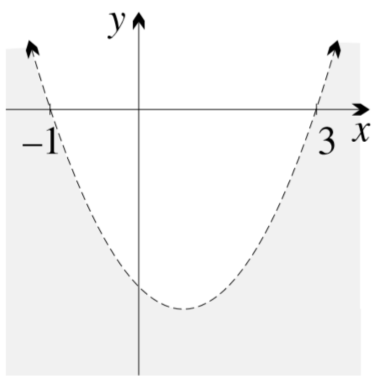
4c Testing at $(2, 0)$ we obtain $0 \leq 4$ which is true. Hence, we shade the region containing $(0, 0)$.



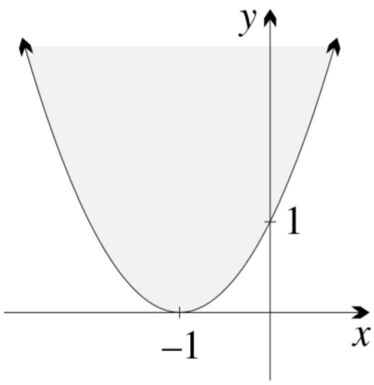
4d Testing at $(0, 0)$ we obtain $5 > 9$ which is false. Hence, we shade the region not containing $(0, 0)$.



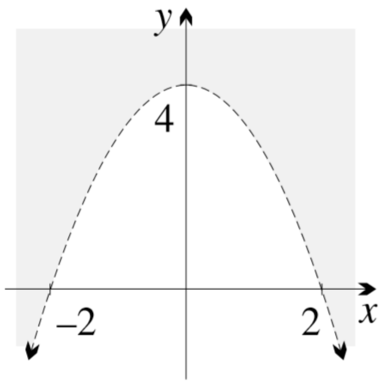
5a Factoring $y < x^2 - 2x - 3$ gives $y < (x - 3)(x + 1)$.



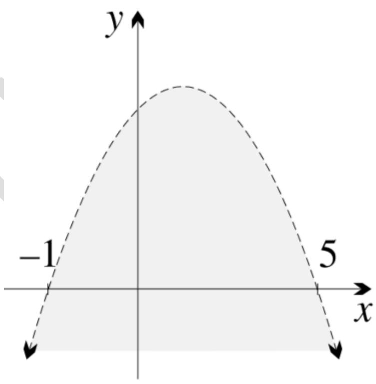
5b Factoring $y \geq x^2 + 2x + 1$ gives $y \geq (x+1)^2$



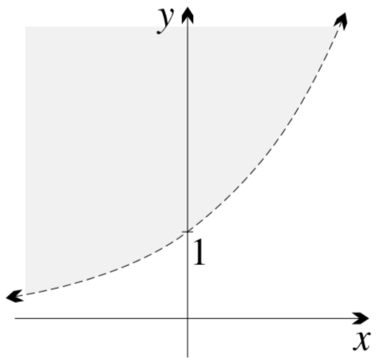
5c Factoring $y > 4 - x^2$ gives $y > (2-x)(2+x)$



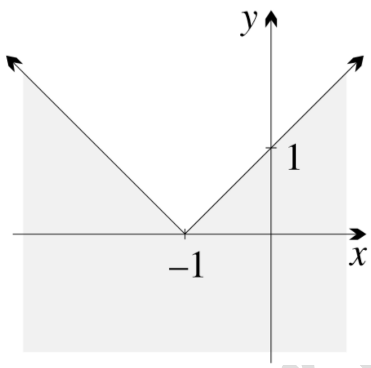
5d Drawing $y < (5-x)(1+x)$ gives



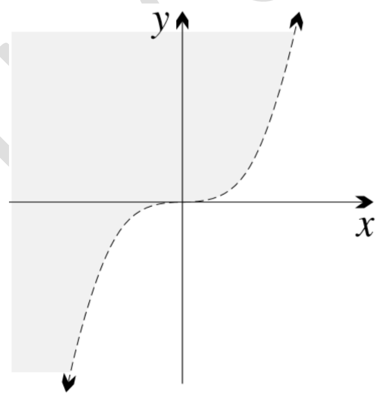
6a Testing at $(0, 0)$ we obtain $0 > 1$ which is false. Hence, we shade the region not containing $(0, 0)$.



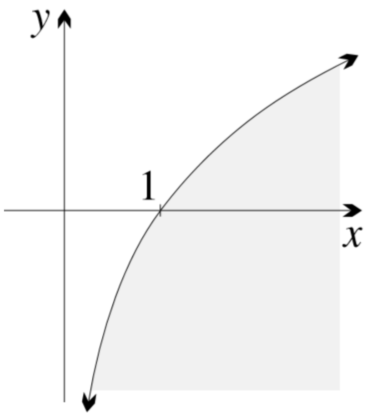
6b Testing at $(0, 0)$ we obtain $0 \leq 1$ which is true. Hence, we shade the region containing $(0, 0)$.



6c Testing at $(0, 1)$ we obtain $1 > 0$ which is true. Hence, we shade the region containing $(0, 1)$.

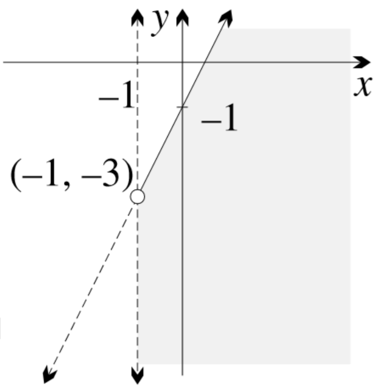


6d Testing at $(2, 0)$ we obtain $0 \leq 1$ which is true. Hence, we shade the region containing $(2, 0)$.

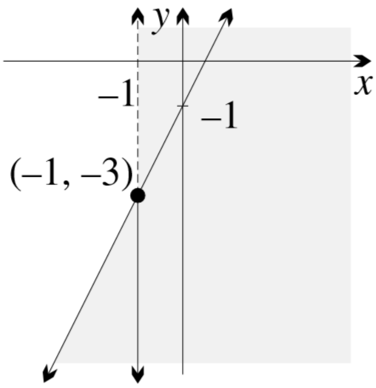


7a The point of intersection occurs when $x = -1$ and $y = 2(-1) - 1 = -2 - 1 = -3$ which is at $(-1, -3)$.

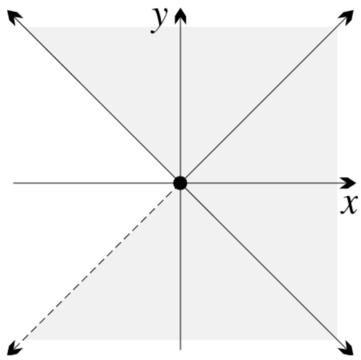
7b Consider the point $(1, -1)$. This satisfies both the inequality $x > -1$ and $y \leq 2x - 1$ so we shade the region bounded by the two curves that contains the point $(-1, 1)$.



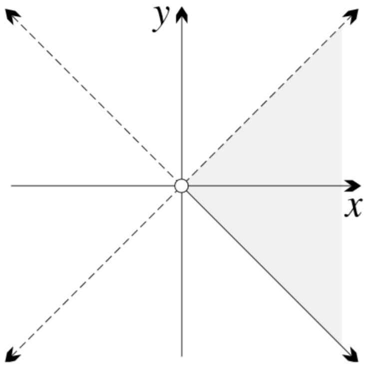
7c Consider the point $(1, -1)$. This satisfies both the inequality $x > -1$ and $y \leq 2x - 1$ so we shade the region bounded by the each of the curves that contains the point $(-1, 1)$.



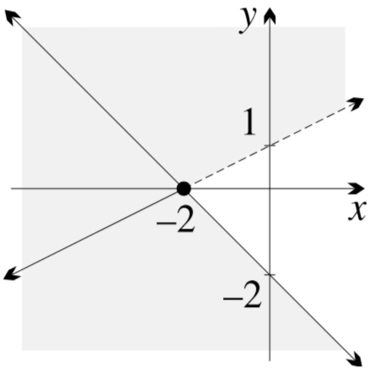
8ai Both inequalities are satisfied by the point $(1, 0)$. Thus, for the union, we shade the region such that the it lies on the same side as at least one of the two lines. We include the point $(0, 0)$ as it satisfies the inequality $y \geq -x$.



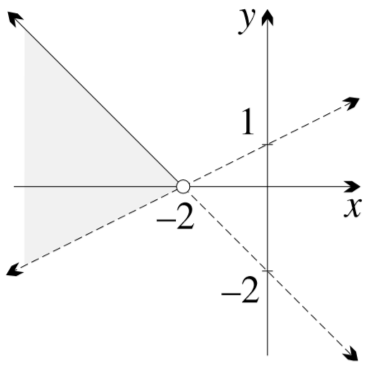
8aii Both inequalities are satisfied by the point $(1, 0)$. Thus, for the union, we shade the region bounded that contains that point. We do not include the point $(0, 0)$ as it does not satisfy the inequality $y < x$.



8bi Both inequalities are satisfied by the point $(-3, 0)$. Thus, for the union, we shade the region such that the it lies on the same side as at least one of the two lines. We include the point $(-2, 0)$ as it satisfies the inequality $y \leq -x - 2$.



8bii Both inequalities are satisfied by the point $(-3, 0)$. Thus, for the union, we shade the region bounded that contains that point. We do not include the point $(-2, 0)$ as it does not satisfy the inequality $y > \frac{1}{2}x + 1$.



9a $x \geq 0$ and $y \geq 0$

9b $x \le 0$ and $y \ge 0$

9c $x \le 0$ and $y \le 0$

9d $x \ge 0$ and $y \le 0$

9e $x \ge 0$ or $y \ge 0$

9f $x \ge 0$ or $y \le 0$

10a Note that the point $(1, 0)$ must satisfy the inequalities, this gives
$y < x$ and $y \le 2 - x$

10b Note that the point $(3, -3)$ must satisfy the inequalities, this gives
$y \le -\frac{1}{2}x - 1$ or $y \ge 2 - 2x$

10c Note that the point $(0, 0)$ must satisfy the inequalities, this gives
$y < x + 2$ or $y > 4x - 1$

11a Substituting $(-2, -1)$ into $y = x + 1$ gives $-1 = -2 + 1$ which is true. Hence this point lies on the line.

Substituting $(1, 2)$ into $y = x + 1$ gives $2 = 1 + 1$ which is true. Hence this point lies on the line.

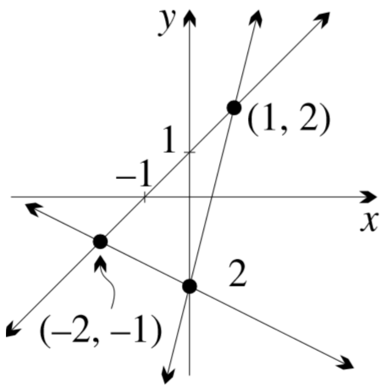
Substituting $(-2, -1)$ into $y = -\frac{1}{2}x - 2$ gives $-1 = -\frac{1}{2}(-2) - 2$ which is true. Hence this point lies on the line.

Substituting $(0, -2)$ into $y = -\frac{1}{2}x - 2$ gives $-2 = -\frac{1}{2}(0) - 2$ which is true. Hence this point lies on the line.

Substituting $(0, -2)$ into $y = 4x - 2$ gives $-2 = 4(0) - 2$ which is true. Hence this point lies on the line.

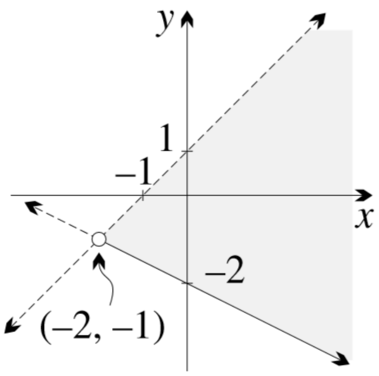
Substituting $(1, 2)$ into $y = 4x - 2$ gives $2 = 4(1) - 2$ which is true. Hence this point lies on the line.

Thus $y = x + 1$ and $y = -\frac{1}{2}x - 2$ both pass through and hence intersect at $(-2, -1)$, $y = x + 1$ and $y = 4x - 2$ both pass through and hence intersect at $(0, -2)$, $y = -\frac{1}{2}x - 2$ and $y = 4x - 2$ both pass through and hence intersect at $(1, 2)$.

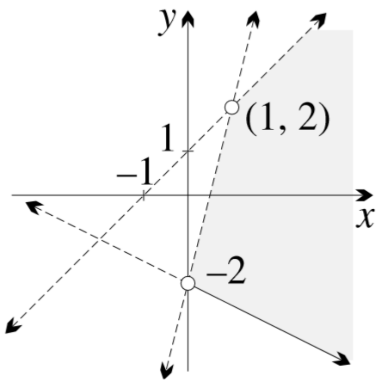


11 The following graphs are obtained by following the method shown at the start of Section 3F.

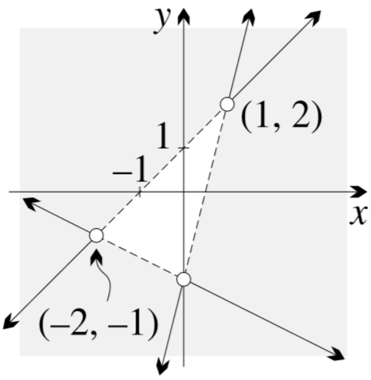
11b i



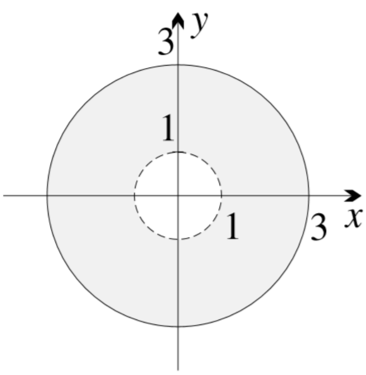
11b ii



11biii

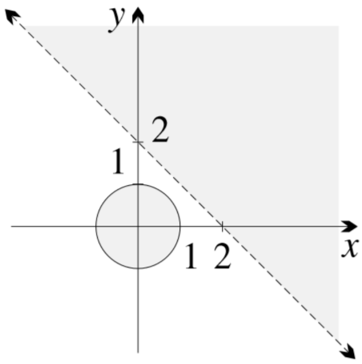


12a



12b The whole plane is covered by the union of the two regions.

13a



13b By observation of the above graph, the regions to not overlap and hence there is no intersection.

14a The curves intersect when $x^2 + (4 - x)^2 = 16$ which is when

$x^2 + 16 - 8x + x^2 = 16$

$2x^2 - 8x = 0$

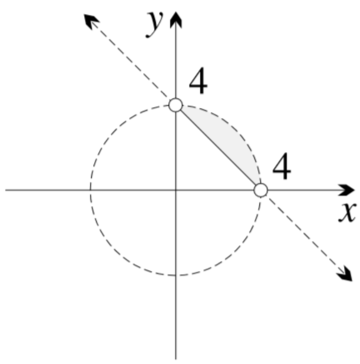
$2x(x - 4) = 0$

$x = 0, 4$

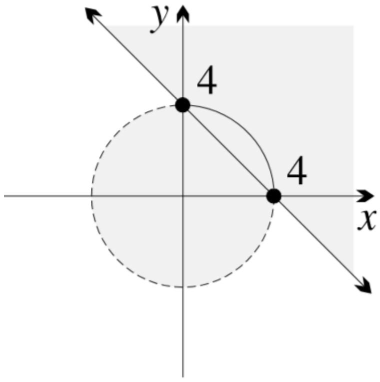
Substituting this back into $y = 4 - x$ gives the points of intersection as $(0, 4)$ and $(4, 0)$.

14b Now that we have the points where the curves intersect, follow the method at the start of Section 3F to obtain the following graphs.

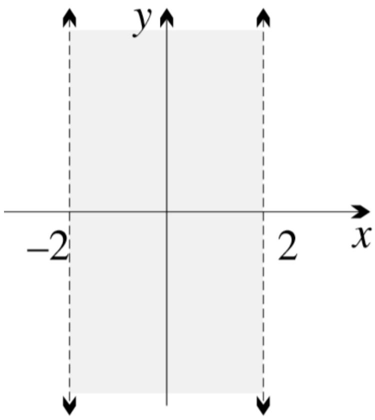
14b i



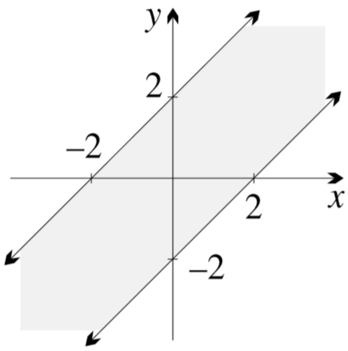
14b ii



15a $|x| < 2$ means that $x < 2$ and $x > -2$, that is $-2 < x < 2$



15b $|x - y| \leq 2$ gives $-2 \leq x - y \leq 2$ which is $x - y \leq 2$ and $x - y \geq -2$ or alternatively $y \geq x - 2$ and $y \leq x + 2$.



16 Finding points of intersection of $x^2 + y^2 = 5$ and $x = -1$ gives $1 + y^2 = 5$, $y^2 = 4$ and hence $y = \pm 2$. Thus the intersections are at $(1, 2)$ and $(1, -2)$.

Finding points of intersection of $x^2 + y^2 = 5$ and $y = 1$ gives $1 + x^2 = 5$, $x^2 = 4$ and hence $x = \pm 2$. Thus the intersections are at $(-1, -2)$ and $(-1, 2)$.

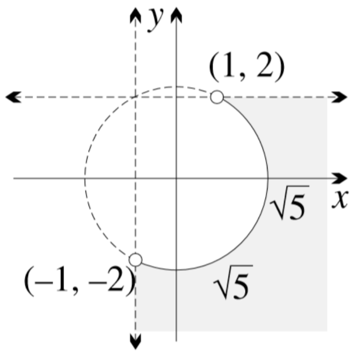
Finding intersections of $x^2 + y^2 = 5$ with the axis.

When $x = 0$, $y^2 = 5$ and hence $y = \pm\sqrt{5}$

When $y = 0$, $x^2 = 5$ and hence $x = \pm\sqrt{5}$

So the points of intersection of the circle with the axis are $(0, \pm\sqrt{5})$ and $(\pm\sqrt{5}, 0)$.

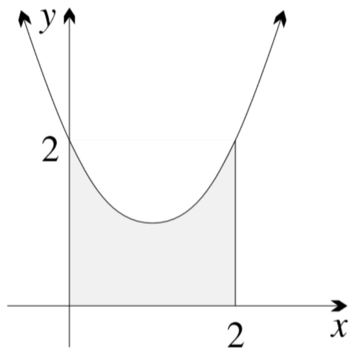
From this we can draw the graph using the method on page 95.



17 Finding the y-intercept of $y = x^2 - 2x + 2$.

When $x = 0$, $y = 0 - 0 + 2 = 2$ and thus the intercept is at $(0, 2)$.

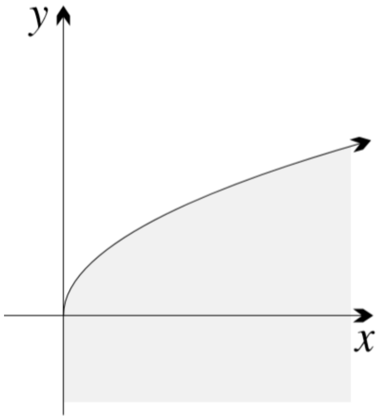
Using this information, we can now sketch the graph using the method on page 95 of the textbook.



18 Refer to the graph in 18c

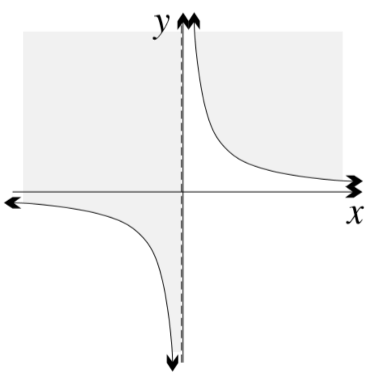
18b The curve is undefined for $x < 0$ (as you cannot take the square root of a negative number) and hence the inequality is undefined in this region.

18c

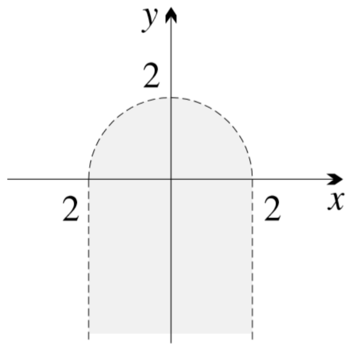


19a The curve is undefined when $x = 0$ (as you cannot divide by zero) and hence $x = 0$ is a boundary.

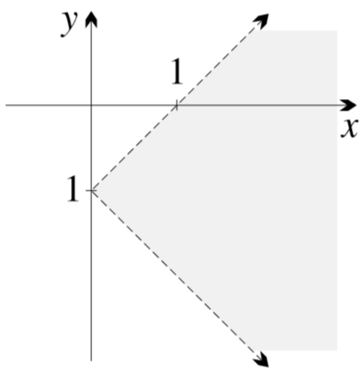
19b Begin by drawing the graph of $x = 0$ and the graph of $y = \frac{1}{x}$. Note that there are 4 separate regions which need to be considered. Select a point in each of the regions. For example, $(-2, -2)$, $(-1, 0)$, $(1, 0)$ and $(2, 2)$ are points in each of the four regions. Substituting each of the points into the inequality $y \geq \frac{1}{x}$, we find that $(-1, 0)$ and $(2, 2)$ are the only two regions satisfying the inequality so we shade the regions containing these points.



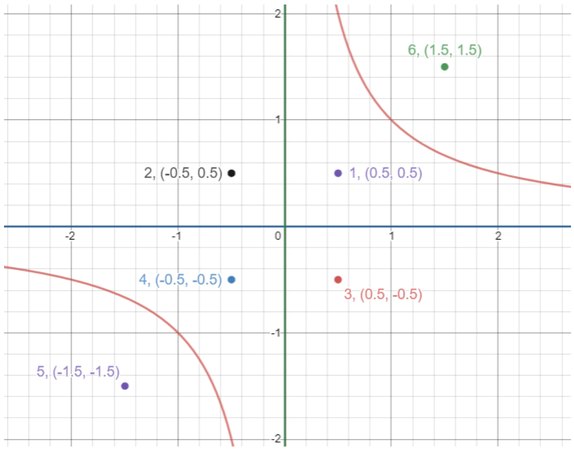
20 Begin by sketching the curve $y = \sqrt{4 - x^2}$ (note that you will need to draw a broken line as we are dealing with a strict inequality). Since the function is only defined for $-2 \leq x \leq 2$, only points in this domain are able to satisfy the inequality. Now, we have two regions consider, that above the semicircle and that below the semicircle. Select a point in one of the two regions, for ease we select $(0, 0)$. This satisfies the inequality and hence we shade all points in the region $-2 < x < 2$ below the semicircle. Note that we do not include any points such that $x = \pm 2$ due to the fact that we have a strict inequality.



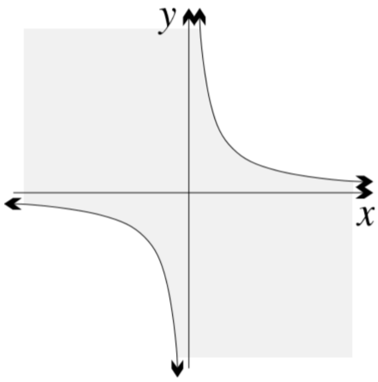
21 Note that the inverse of $x = |y + 1|$ is $y + 1 = \pm x$ and is thus $y = -1 \pm x$. Hence the region is bounded by the two lines $y = -1 - x$ and $y = -1 + x$. The point $(3, 1)$ satisfies the inequality $x > |y + 1|$. Hence, we shade the region enclosed by the two lines that contains $(3, 1)$.



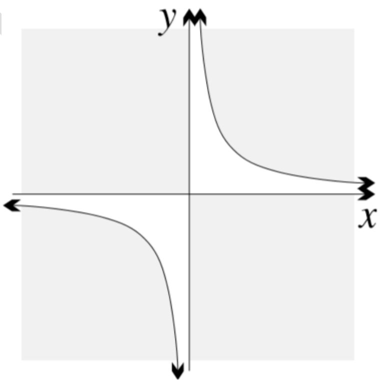
22a There are 6 distinct regions which are labelled in the graph below.



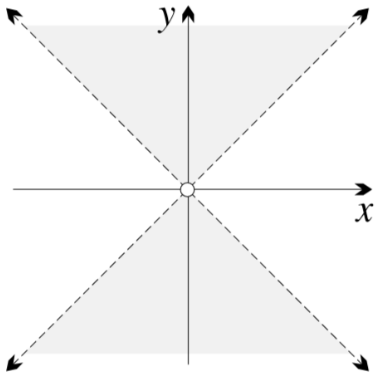
22b i Substituting each of the points given in **22a** into the inequality, we find that the points $(\pm 0.5, \pm 0.5)$ all satisfy the inequality, hence we shade each of the regions containing these points.



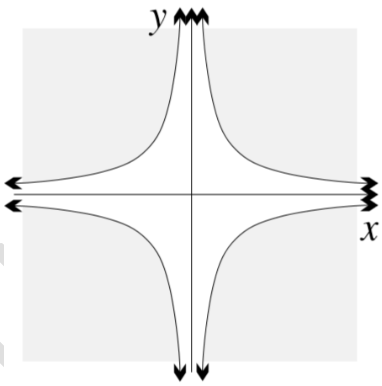
22b ii Substituting each of the points given in **22a** into the inequality, we find that the points $(-0.5, 0.5)$, $(0.5, -0.5)$, $(1.5, 1.5)$ and $(-1.5, -1.5)$ all satisfy the inequality, hence we shade each of the regions containing these points.



23a Note that the point $(0, 0)$ does not satisfy the inequality as $0 > 0$ is false. Now, in the first quadrant, $|y| = y$ and $|x| = x$ so we shade the region such that $y > x$. In the second quadrant, $|y| = -y$ and $|x| = x$ so shade the region such that $-y > x$. In the third quadrant, $|y| = -y$ and $|x| = -x$ so shade the region such that $-y > -x$, that is where $y < x$. In the fourth quadrant, $|y| = y$ and $|x| = -x$ so shade the region such that $y > -x$.



23b Begin by drawing the graphs $xy = 1$ and $xy = -1$. Note that the graph is broken into 8 distinct regions. Select 1 point in each region, for example $(\pm 0.5, \pm 0.5), (\pm 2, \pm 2)$. Substituting these values into the inequality, we find that the only points that satisfy the inequality are $(\pm 2, \pm 2)$ so shade the region that contains these points.



23c Note that the inequality is undefined at $x = 0$ and at $y = 0$ so include these as boundaries for the graph.

In the first and third quadrant, $x < y$ at all points above the line $y = x$.

In the fourth quadrant, $x < 0$ and $y > 0$ so the inequality $x < y$ is always true.

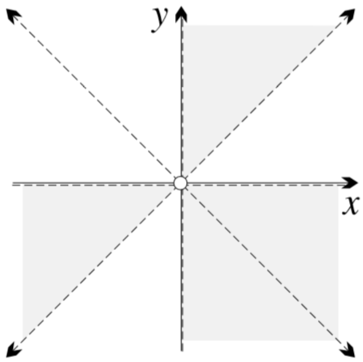
In the second quadrant, $x > 0$ and $y < 0$ so the inequality $x < y$ is never true.

In the first and third quadrants, $\frac{1}{x} > \frac{1}{y}$ at all points below the line $y = x$.

In the second quadrant, $\frac{1}{x} < 0$ and $\frac{1}{y} > 0$ so the inequality $\frac{1}{x} > \frac{1}{y}$ is false at all points.

In the fourth quadrant, $\frac{1}{x} > 0$ and $\frac{1}{y} < 0$ so the inequality $\frac{1}{x} > \frac{1}{y}$ is true at all points.

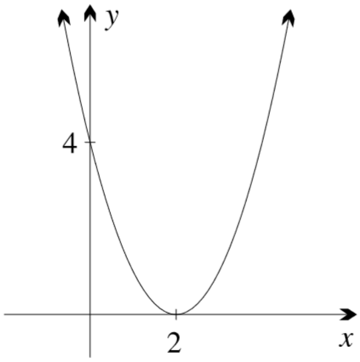
As such, we can sketch the region satisfied by the inequality as follows:



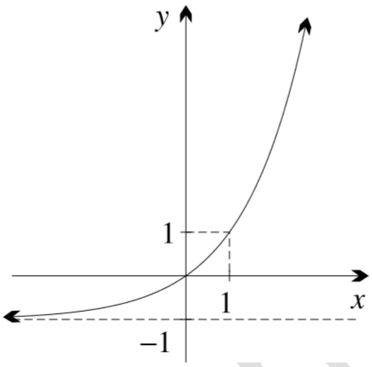
Solutions to Exercise 3G

Solutions to Exercise 2F

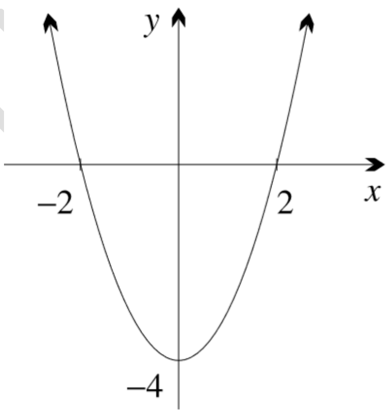
1a $\quad y = (x-2)^2$



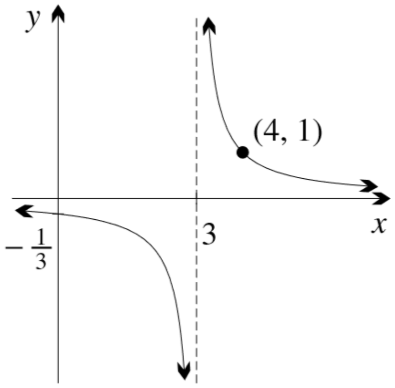
1b $\quad y = 2^x - 1$



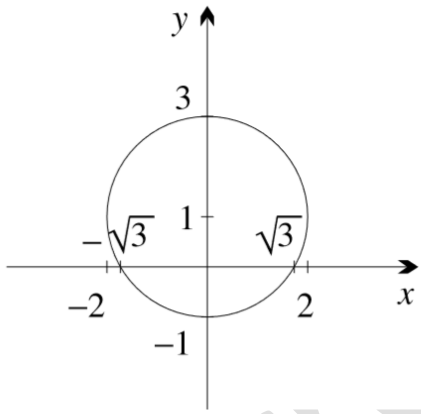
1c $\quad y = x^2 - 4$



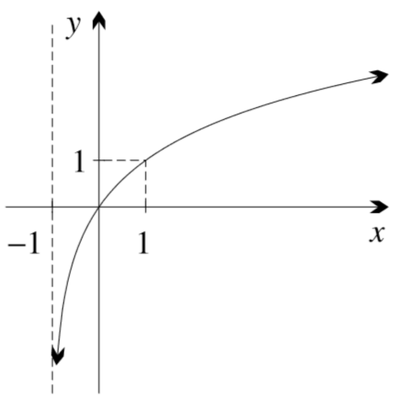
1d $y = \frac{1}{x-3}$



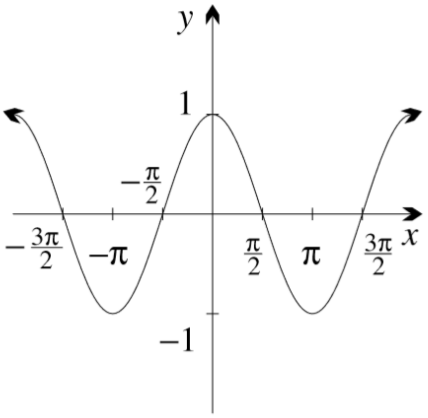
1e $x^2 + (y-1)^2 = 4$



1f $y = \log_2(x+1)$

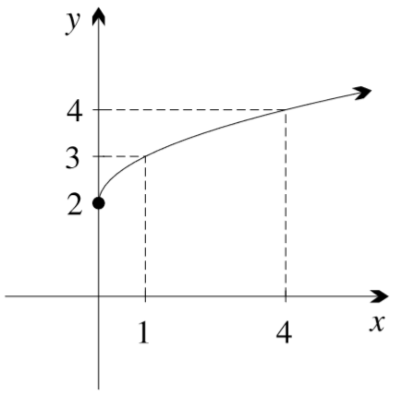


1g $y = \sin\left(x + \frac{\pi}{2}\right)$



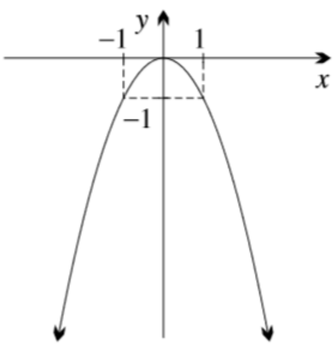
This is also $y = \cos x$.

1h $y = \sqrt{x} + 2$

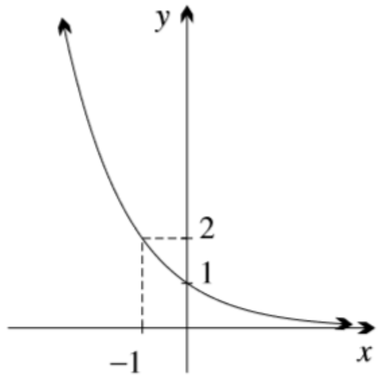


2 For this question, note that a 180° rotation is equivalent to flipping in the x- and y-axes.

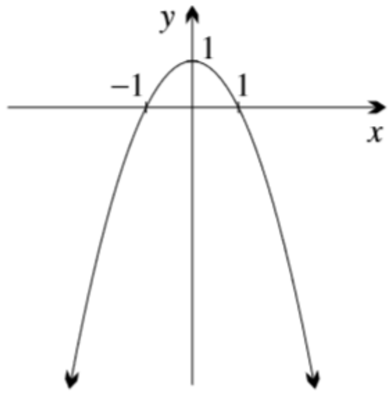
2a $y = -x^2$



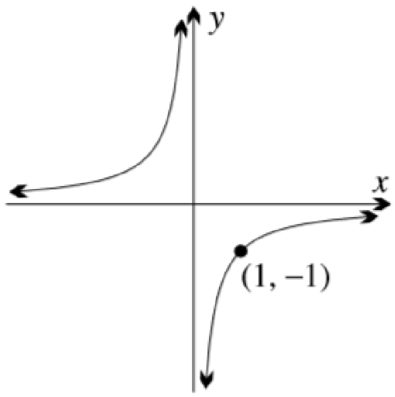
2b $y = 2^{-x}$



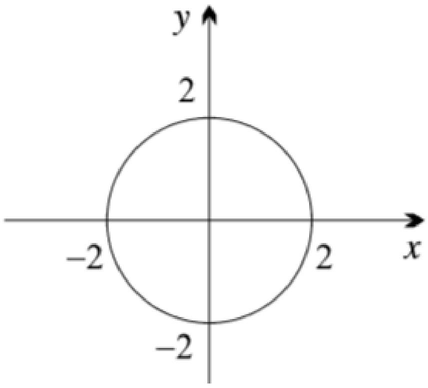
2c $y = 1 - x^2$



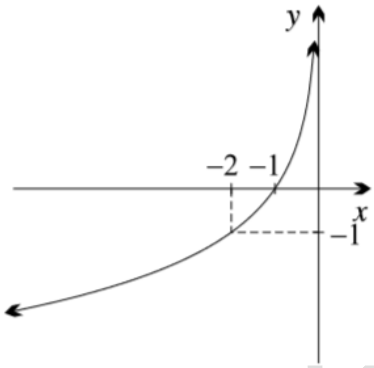
2d $y = -\frac{1}{x}$



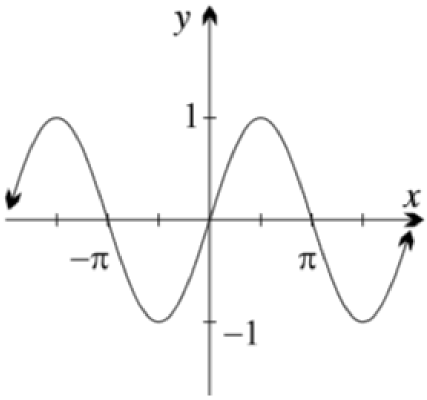
2e $x^2 + y^2 = 4$



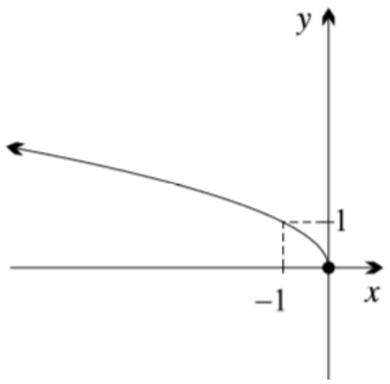
2f $y = -\log_2(-x)$



2g $y = \sin x$



2h $y = \sqrt{-x}$



3 In part **e**, the circle is symmetric in the y-axis. In part **g**, $y = \sin x$ is an odd function, and so is unchanged by a rotation of 180°.

4 Recall that the equation for a circle is $(x-h)^2 + (y-k)^2 = r^2$, where the circle has centre (h, k) and radius r.

4a Circle with radius $r = 2$ and centre at $(-1, 0)$

4b Circle with radius $r = 1$ and centre at $(1, 2)$

4c Firstly, completing the square:

$x^2 - 4x + y^2 = 0$

$x^2 - 4x + 4 + y^2 = 4$

$(x-2)^2 + y^2 = 4$

Circle with radius $r = 2$ and centre at $(2, 0)$.

4d Firstly, completing the square:

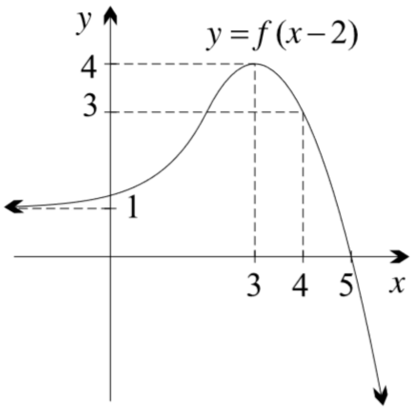
$x^2 + y^2 - 6y = 16$

$x^2 + y^2 - 6y + 9 = 16 + 9$

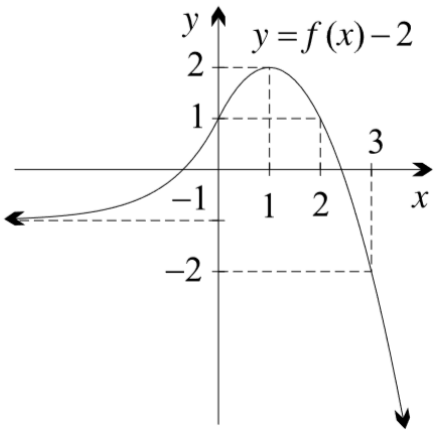
$x^2 + (y-3)^2 = 25$

Circle with radius $r = 5$ and centre at $(0, 3)$.

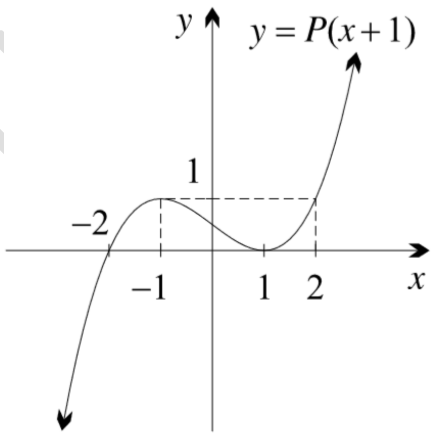
5a i This is given by a translation 2 units to the right.



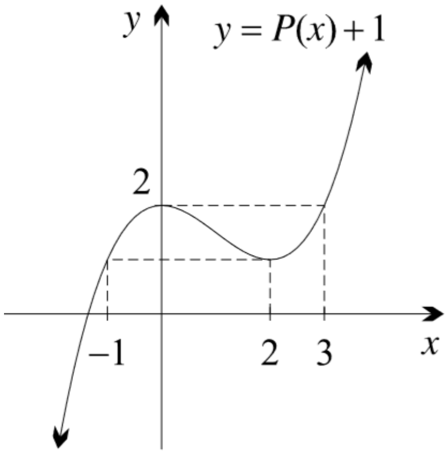
5a ii This is given by a translation 2 units downwards.



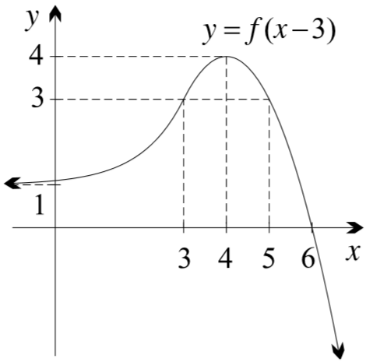
5b i This is given by a translation 1 unit to the left



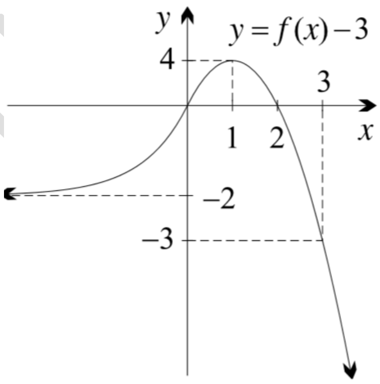
5b ii This is given by a translation 1 unit up.



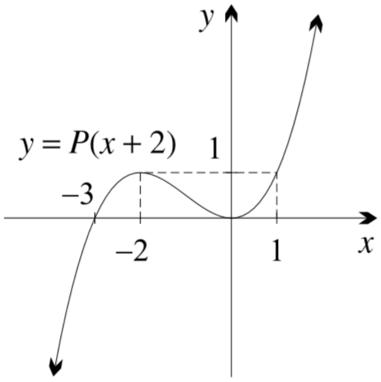
6a i $y = f \circ h(x) = f\big(h(x)\big) = f(x-3)$



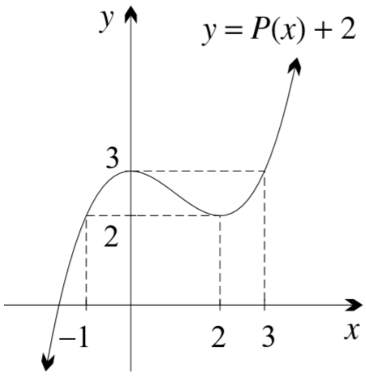
6a ii $y = h \circ f(x) = h\big(f(x)\big) = f(x) - 3$



6b i $y = P \circ k(x) = P\big(k(x)\big) = P(x+2)$



6b ii $y = k \circ P(x) = k\big(P(x)\big) = P(x) + 2$



7a Left 1 unit: $y = (x + 1)^2$

Then up 2 units: $y = (x + 1)^2 + 2$

7b Right 2 units: $y = \frac{1}{x-2}$

Then up 3 units: $y = \frac{1}{x-2} + 3$

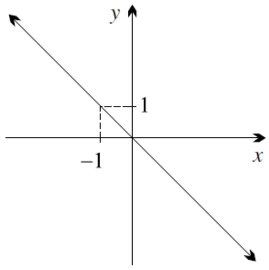
7c Right $\frac{\pi}{3}$ units: $y = \cos\left(x - \frac{\pi}{3}\right)$

Then down 2 units: $y = \cos\left(x - \frac{\pi}{3}\right) - 2$

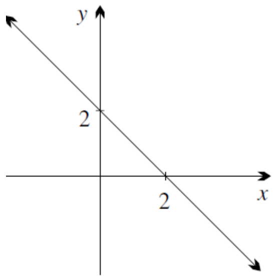
7d Left 2 units: $y = e^{x+2}$

Then down 1 unit: $y = e^{x+2} - 1$

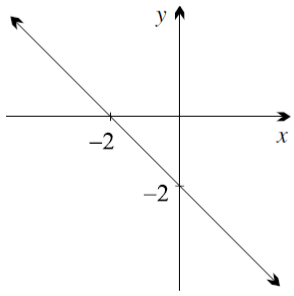
8a From $y = -x$



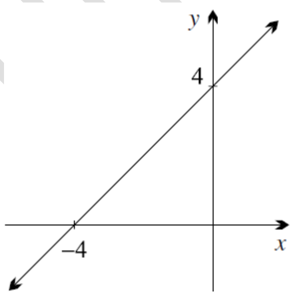
8a i Shift up 2 (or right 2).



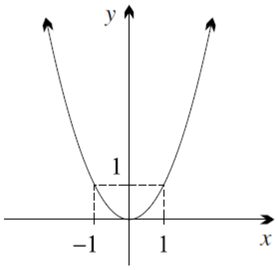
8a ii Shift down 2 (or left 2).



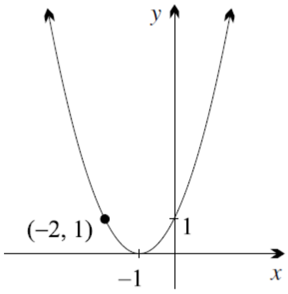
8a iii Reflect in x-axis or (y-axis) and shift up 4 (or left 4).



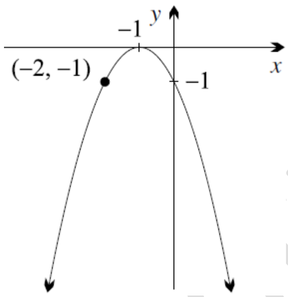
8b From $y = x^2$:



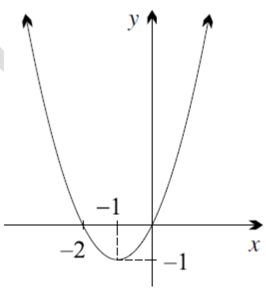
8bi Shift 1 unit left.



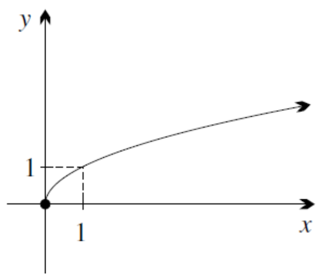
8b ii Shift 1 unit left and reflect in the x-axis.



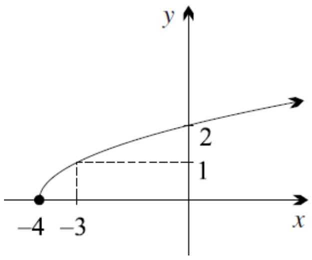
8b iii Shift 1 unit left and shift down 1 unit.



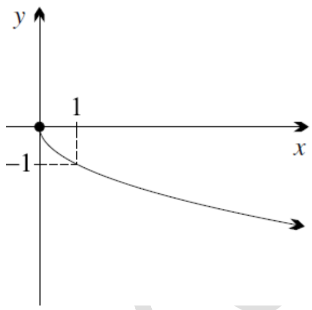
8c From $y = \sqrt{x}$:



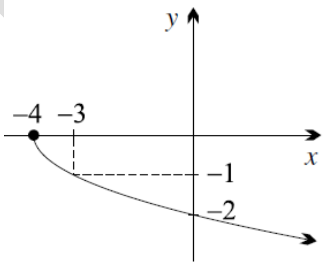
8c i Shift 4 units left.



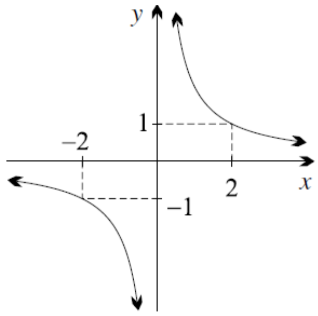
8c ii Reflect in the x-axis.



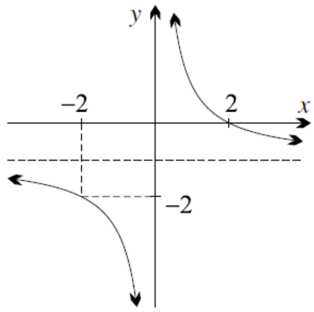
8c iii Shift 4 units left and reflect in the x-axis.



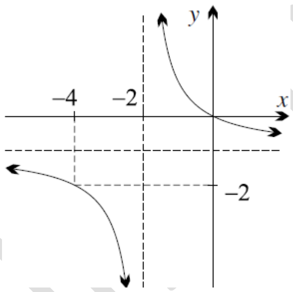
8d From $y = \frac{2}{x}$:



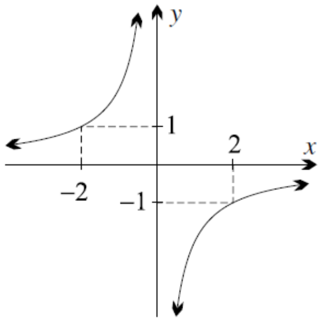
8d i Shift down 1 unit.



8d ii Shift down 1 unit and left 2 units.



8d iii Reflect in the x-axis or in the y-axis.



9a

$$\frac{dy}{dx} = 3x^2 - 3$$

The tangent is horizontal when $\frac{dy}{dx} = 0$.

$3x^2 - 3 = 0$

$3(x^2 - 1) = 0$

$x^2 - 1 = 0$

$x^2 = 1$

$x = \pm 1$

When $x = -1, y = 2$ and $x = 1, y = -2$.

Hence the coordinates are $(-1, 2)$ and $(1, -2)$.

9b i The equation of a cubic shifted up one unit is given by adding a constant to the right-hand side of the equation $y = x^3 - 3x + 1$.

9b ii

$$\frac{dy}{dx} = 3x^2 - 3$$

The tangent is horizontal when $\frac{dy}{dx} = 0$.

$3x^2 - 3 = 0$

$3(x^2 - 1) = 0$

$x^2 - 1 = 0$

$x^2 = 1$

$x = \pm 1$

When $x = -1, y = 3$ and $x = 1, y = -1$.

Hence the coordinates are $(-1, 3)$ and $(1, -1)$.

Thus the x-coordinates where the tangent is horizontal have not changed.

9c i The equation of this third cubic is given by substituting $x = x + 1$ into the equation, as this shifts all coordinates 1 unit left.

$y = (x + 1)^3 - 3(x + 1)$

$y = x^3 + 3x^2 + 3x + 1 - 3x - 3$

$y = x^3 + 3x^2 - 2$

9c ii

$\frac{dy}{dx} = 3x^2 + 6x$

The tangent is horizontal when $\frac{dy}{dx} = 0$.

$3x(x + 2) = 0$

$x = 0, -2$

When $x = -2, y = 2$ and when $x = 0, y = -2$.

Hence the coordinates are $(0, -2)$ and $(-2, 2)$.

Thus the y-coordinates where the tangent is horizontal have not changed.

10a Finding the intercepts.

When $x = 0$,

$y^2 - 8y = 0$

$y(y - 8) = 0$

Hence the y-intercepts are $(0, 0)$ and $(0, 8)$.

When $y = 0$,

$x^2 + 4x = 0$

$x(x + 4) = 0$

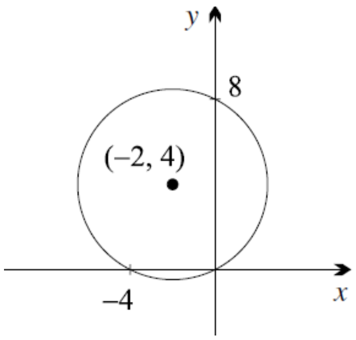
Hence the x-intercepts are $(0, 0)$ and $(-4, 0)$.

$x^2 + 4x + y^2 - 8y = 0$

$x^2 + 4x + 4 + y^2 - 8y + 16 = 20$

$(x + 2)^2 + (y - 4)^2 = (2\sqrt{5})^2$

The centre is $(-2, 4)$ and the radius is $r = 2\sqrt{5}$.



10b Finding the intercepts.

When $x = 0$,

$y^2 + 4y = -1$

$y^2 + 4y + 1 = 0$

$$y = \frac{-4 \pm \sqrt{4^2 - 4 \times 1 \times 1}}{2} = \frac{-4 \pm \sqrt{12}}{2} = \frac{-4 \pm 2\sqrt{3}}{2} = -2 \pm \sqrt{3}$$

Hence the y-intercepts are $(0, -2 + \sqrt{3})$ and $(0, -2 - \sqrt{3})$.

When $y = 0$,

$x^2 - 2x + 1 = 0$

$(x - 1)^2 = 0$

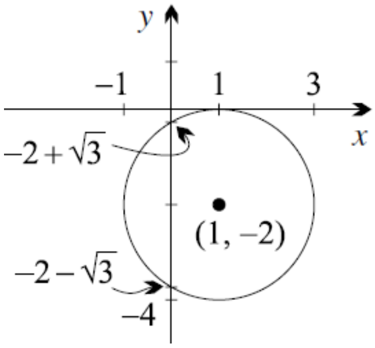
Hence the x-intercept is $(1, 0)$.

$x^2 - 2x + y^2 + 4y = -1$

$x^2 - 2x + 1 + y^2 + 4y + 4 = 4$

$(x - 1)^2 + (y + 2^2) = 2^2$

The centre is $(1, -2)$ and the radius is $r = 2$.



11a The parabola $y = x^2$ shifted left 2 units and down 1 unit.

Equation is $y + 1 = (x + 2)^2$

11b The hyperbola $xy = 1$ shifted right 2 units and down 1 unit.

Equation is $y + 1 = \frac{1}{x-2}$

11c The exponential $y = 2^x$ reflected in the x-axis and shifted 1 unit up.

Equation is $y = 1 - 2^x$

11d The curve $y = \cos x$ reflected in the x-axis and shifted 1 unit up.

Equation is $y = 1 - \cos x$

12a The parabola $y = x^2$ reflected in the x-axis, then shifted 3 units right and 1 unit up.

Equation is $y = -(x - 3)^2 + 1$

12b The curve $y = \log_2 x$ reflected in the y-axis, then shifted right 2 units and down 1 unit.

Equation is $y = -\log_2(x - 2) - 1$

12c The half parabola $y = \sqrt{x}$ reflected in the x-axis, then shifted left 4 units and 2 units up.

Equation is $y = -\sqrt{x+4} + 2$

13a i Let $y = f(x)$. Applying $\mathcal{I}$ gives $x = f(y)$, then $\mathcal{H}$ gives $x = f(-y)$, then $\mathcal{I}$ gives $-y = f(x)$ then $\mathcal{H}$ gives $-y = f(-x)$. This is equivalent to a 180° rotation around $(0, 0)$. The functions unchanged by the transformation all have the property $f(-x) = -f(x)$ and are thus odd functions.

13a ii Consider the function $y = x^2$.

Applying $\mathcal{I}$ gives $x = y^2$ and then $\mathcal{H}$ gives $x = (-y)^2$ which is just $x = y^2$.

Alternatively, first applying $\mathcal{H}$ gives $-y = x^2$ which is $y = -x^2$ and then $\mathcal{I}$ gives $x = -y^2$.

From this we see that applying the two reflections in different orders results in two different functions. Thus $\mathcal{I}$ and $\mathcal{H}$ do not commute.

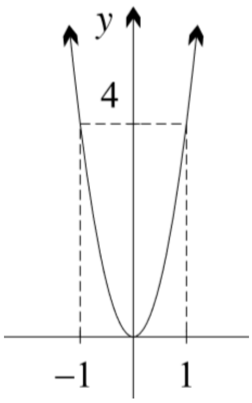
13b i Shifting left by a gives $y = f(x - a)$, reflecting in the y-axis gives $y = f(-(x - a)) = f(-x + a)$. Shifting right then gives $y = f(-x + a + a) = f(2a - x)$.

13b ii A reflection in the line $x = a$

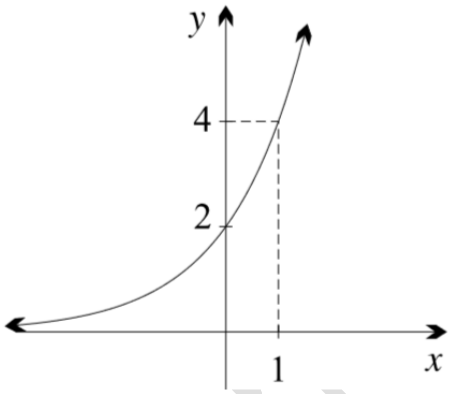
13b iii Reflecting $g(a + t)$ around $x = a$ gives $g\big(2a - (a + t)\big) = g(a - t)$. Thus we require that $g(a + t) = g(a - t)$ and hence we require that $g(x)$ is symmetric in $x = a$.

Solutions to Exercise 3H

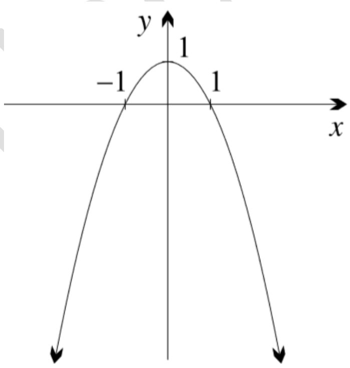
1a $y = 4x^2$



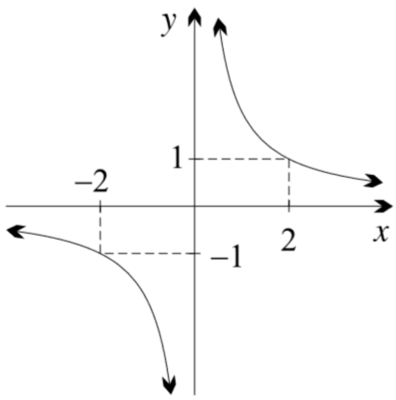
1b $y = 2 \times 2^x = 2^{x+1}$



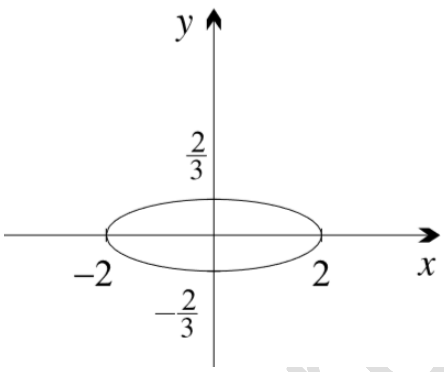
1c $y = 1 - x^2$



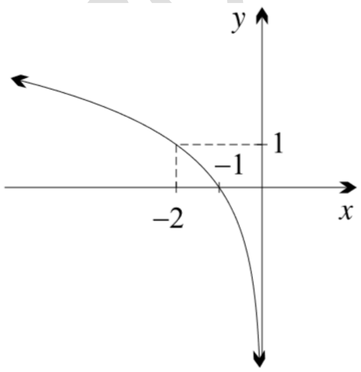
1d $y = \frac{2}{x}$



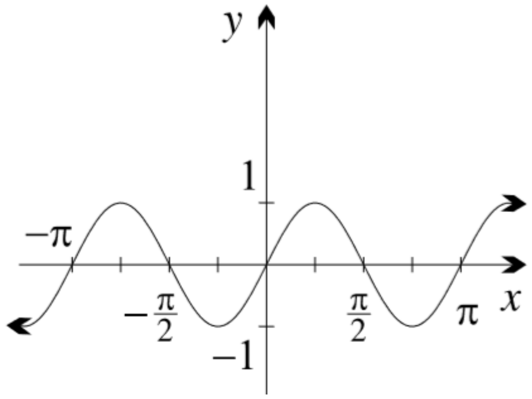
1e $x^2 + 9y^2 = 4$



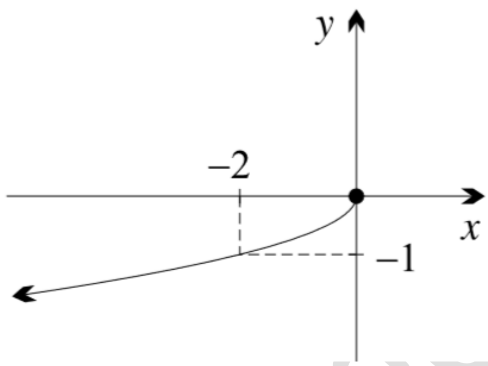
1f $y = \log_2(-x)$



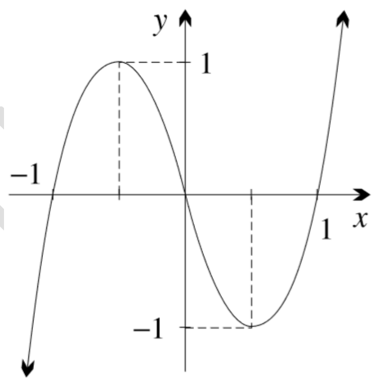
1g $y = \sin 2x$



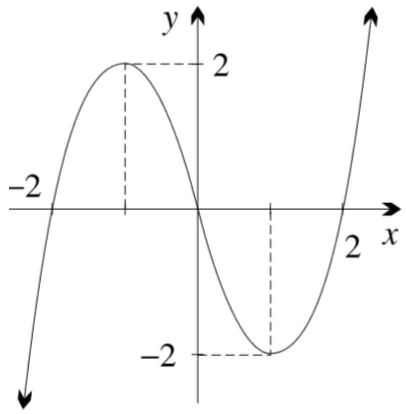
1h $y = -2\sqrt{x}$



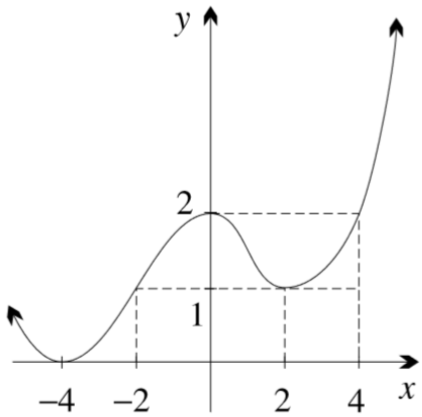
2a i Dilate by a factor of $\frac{1}{2}$ from the x-axis.



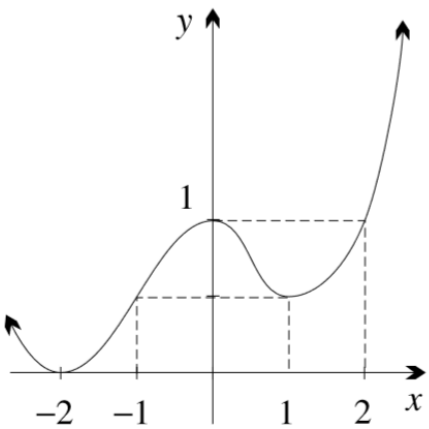
2a ii Dilate by a factor of 2 from the y-axis.



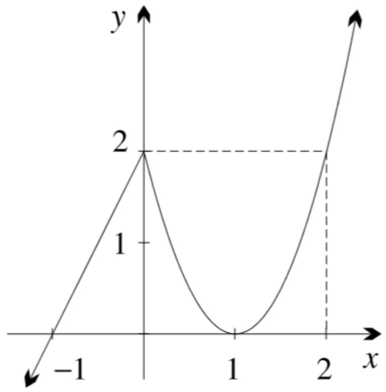
2b i Dilate by a factor of 2 from the x-axis.



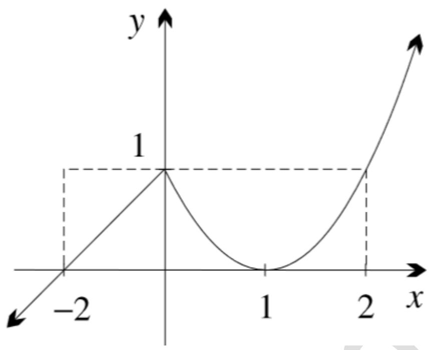
2b ii Dilate by a factor of $\frac{1}{2}$ from the y-axis.



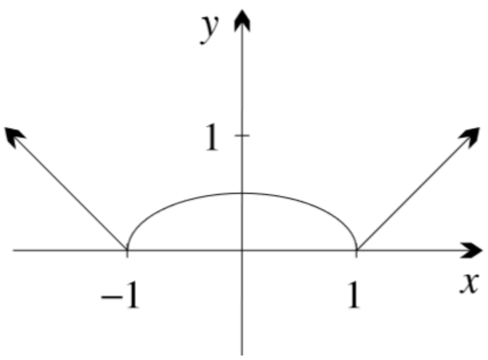
2c i Dilate by a factor of 2 from the x-axis (note $y = 2h(x)$).



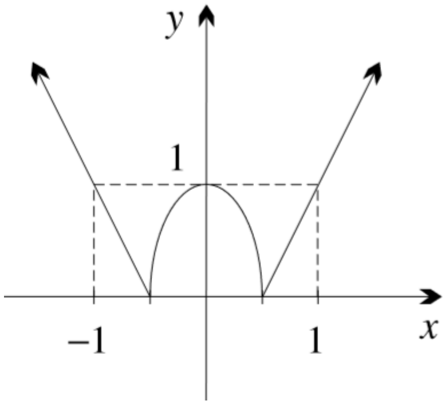
2c ii Dilate by a factor of 2 from the y-axis.



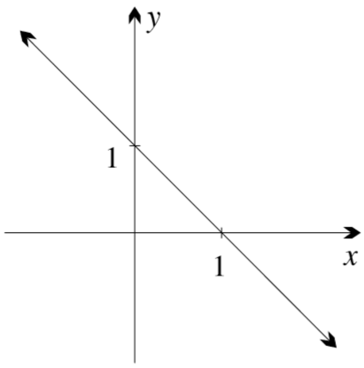
2di Dilate by a factor of $\frac{1}{2}$ from the x-axis (note $y = \frac{1}{2}g(x)$).



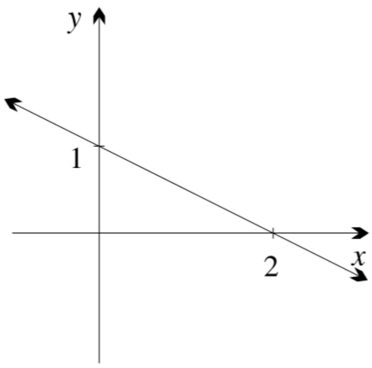
2d ii Dilate by a factor of $\frac{1}{2}$ from the y-axis.



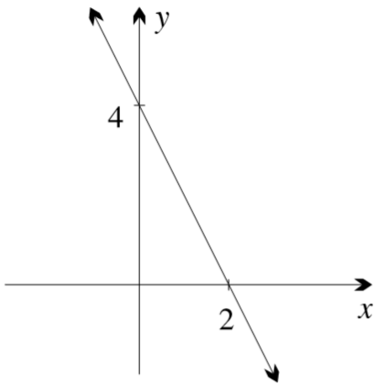
3 Original sketch of graph $x + y = 1$.



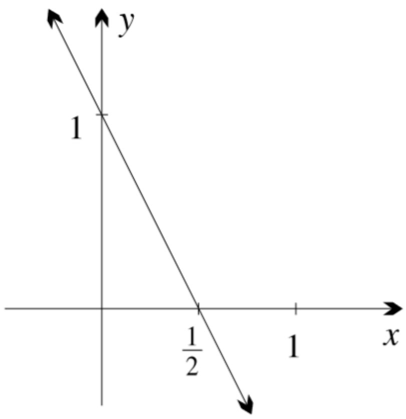
3a Stretch horizontally by a factor of 2.



3b Stretch horizontally by a factor of 2 and vertically by a factor of 4.



3c Stretch horizontally by a factor of $\frac{1}{2}$.



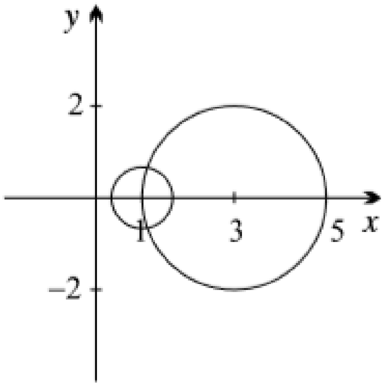
4a If the graph is enlarged by a factor of $\frac{1}{3}$ then x is replaced with $3x$ and y is replaced with $3y$.

Hence the new equation is:

$(3x-3)^2+(3y)^2=4$

$9(x-1)^2+9y^2=4$

$(x-1)^2+y^2=\frac{4}{9}$

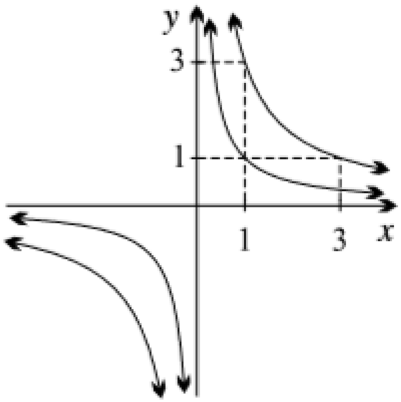


4b If the graph is enlarged by a factor of $\sqrt{3}$ then x is replaced with $\frac{1}{\sqrt{3}}x$ and y is replaced with $\frac{1}{\sqrt{3}}y$.

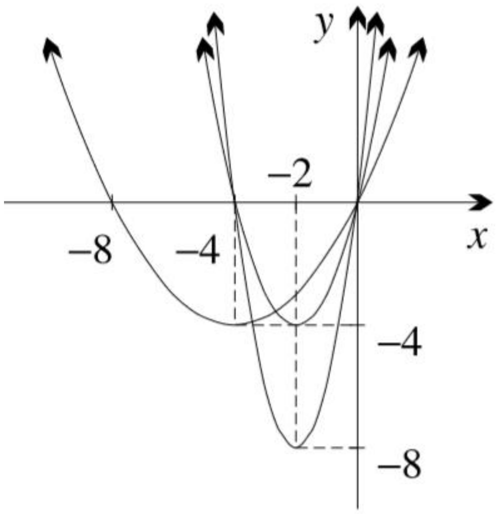
Hence the new equation is:

$$\frac{1}{\sqrt{3}}y = \frac{1}{\frac{1}{\sqrt{3}}x}$$

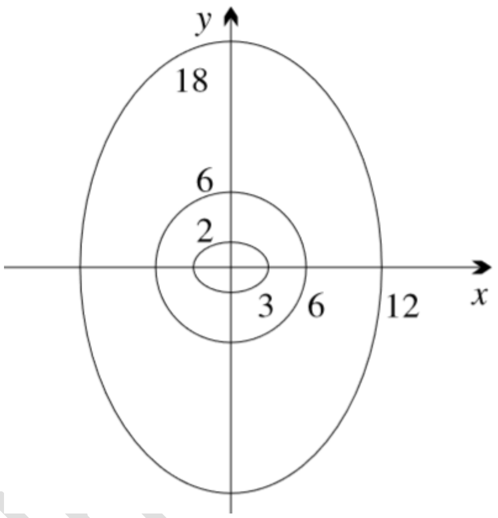
$$y = \frac{3}{x}$$



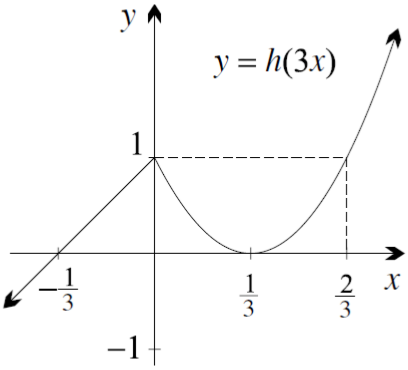
5a



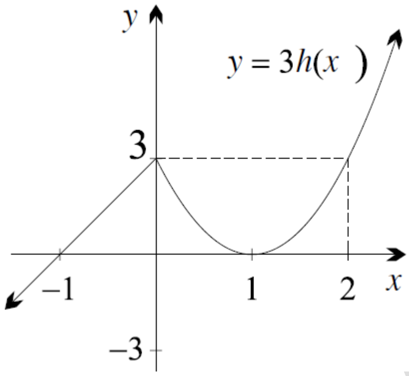
5b



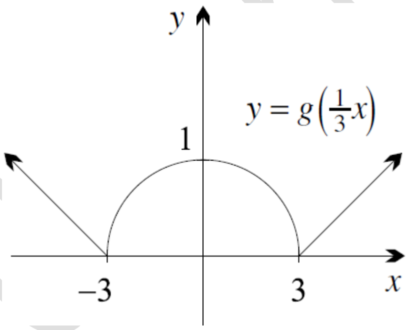
6a i


y
$y = h(3x)$
1
$-\frac{1}{3}$
$\frac{1}{3}$
$\frac{2}{3}$
x
−1

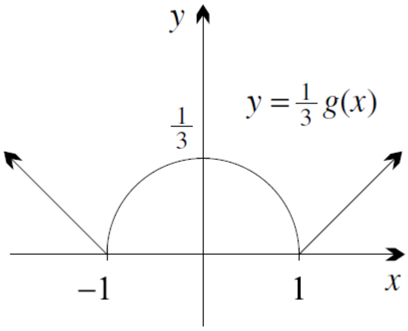
6a ii


y
$y = 3h(x)$
3
−1
1
2
x
−3

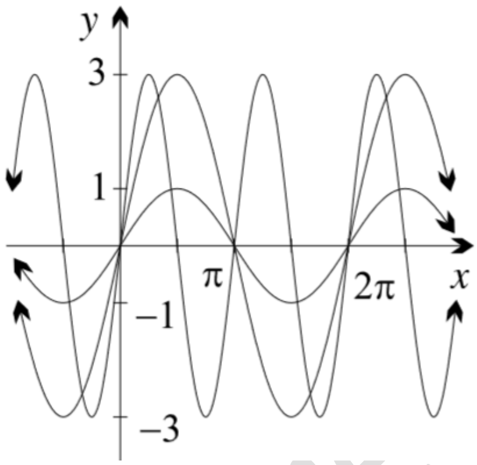
6b i


y
$y = g\left(\frac{1}{3}x\right)$
1
−3
3
x

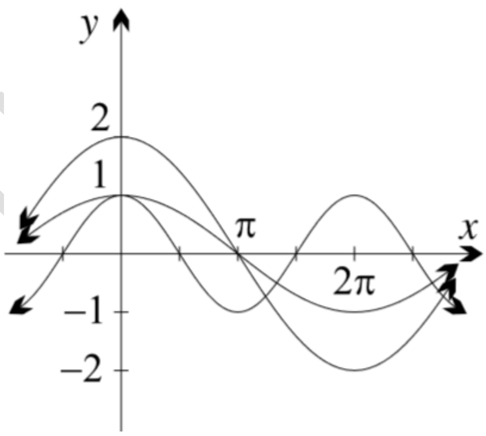
6b ii



7a



7b



8a $y = x^3 - 3x$

$$\frac{dy}{dx} = 3x^2 - 3$$

The tangent is horizontal when $\frac{dy}{dx} = 0$.

$3x^2 - 3 = 0$

$x^2 - 1 = 0$

$(x - 1)(x + 1) = 0$

$x = \pm 1$

Substituting this back into the equation $y = x^3 - 3x$ gives that the tangent is horizontal at $(1, -2)$ and $(-1, 2)$.

8b i Replacing y with $\frac{y}{2}$:

$$\frac{y}{2} = x^3 - 3x$$

$y = 2x^3 - 6x$

8b ii $y = 2x^3 - 6x$

$$\frac{dy}{dx} = 6x^2 - 6$$

The tangent is horizontal when $\frac{dy}{dx} = 0$.

$x^2 - 1 = 0$

$(x - 1)(x + 1) = 0$

$x = \pm 1$

Substituting this back into the equation $y = 2x^3 - 6x$ gives that the tangent is horizontal at $(1, -4)$ and $(-1, 4)$.

So the x-coordinates where the tangent is horizontal have not changed.

8c i Replacing x with $\frac{x}{3}$:

$$y = \left(\frac{x}{3}\right)^3 - 3\left(\frac{x}{3}\right) = \frac{1}{27}x^3 - x$$

8c ii $y = \frac{1}{27}x^3 - x$

$$\frac{dy}{dx} = \frac{x^2}{9} - 1$$

The tangent is horizontal when $\frac{dy}{dx} = 0$.

$$\frac{x^2}{9} - 1 = 0$$

$$x^2 - 9 = 0$$

$$(x - 3)(x + 3) = 0$$

$$x = \pm 3$$

Substituting this back into the equation $y = \frac{1}{27}x^3 - x$ gives that the tangent is horizontal at $(3, -2)$ and $(-3, 2)$.

9a Rearranging the second equation gives $\frac{y}{3} = x^2 - 2x$ so the transformation occurs by replacing y with $\frac{y}{3}$ which means the graph is scaled vertically by a factor of 3.

9b The transformation occurs by replacing x with $2x$ which means the graph is scaled horizontally by a factor of $\frac{1}{2}$.

9c The transformation occurs by replacing x with $\frac{4}{x}$ which means the graph is scaled horizontally by a factor of 4.

9d Rearranging the second equation gives $\frac{y}{2} = \frac{1}{x+1}$ so the transformation occurs by replacing y with $\frac{y}{2}$ which means the graph is scaled vertically by a factor of 2.

10a Replacing y with $\frac{y}{2}$:

$$\frac{y}{2} = \frac{1}{x}$$

$$y = \frac{2}{x}$$

10b Replacing x with $\frac{x}{2}$:

$$y = \frac{1}{\left(\frac{x}{2}\right)} = \frac{2}{x}$$

10c Both dilations give the same graph.

10d Yes, by a factor of $\sqrt{2}$.

10e Answers will vary (hint: look at functions with symmetries).

11a In order to dilate horizontally by a factor of $\frac{1}{2}$ we must have replace x with $2x$ so the new equation is $y = (2x)^2 = 4x^2$.

11b In order to dilate vertically by a factor of 4 we must replace y with $\frac{y}{4}$ so the new equation is $\frac{y}{4} = x^2$ and this means the new equation is $y = 4x^2$.

11c We notice that these two different transformations produce the same graph.

11d The parabolas in parts a and b cannot be produced by an enlargement since they are not similar to $y = x^2$ anymore.

11e Answers will vary.

12a $M(0) = 3 \times 2^{-\frac{1}{53}(0)} = 3 \times 1 = 3$ grams

12b Half of mass is half of 3 grams.

$$M(t) = \frac{3}{2}$$

$3 \times 2^{-\frac{1}{53}t} = \frac{3}{2}$

$2^{-\frac{1}{53}t} = 2^{-1}$

$-\frac{1}{53}t = -1$

$t = 53$

Hence the half-life is 53 years.

12c i The mass has been dilated by a factor of 2, so $M = 6 \times 2^{-\frac{1}{53}t}$.

12c ii $M(0) = 6 \times 2^{-\frac{1}{53}(0)} = 6$ grams

For the half life:

$M(t) = \frac{6}{2}$

$6 \times 2^{-\frac{1}{53}t} = \frac{6}{2}$

$2^{-\frac{1}{53}t} = 2^{-1}$

$-\frac{1}{53}t = -1$

$t = 53$

Hence the half-life is still 53 years.

13 Any enlargement will replace x with ax and y with ay.

Thus, the equation becomes $ay = amx$.

Dividing both sides by a gives $y = mx$ so the line is unchanged by all enlargements.

14a The unit circle $x^2 + y^2 = 1$, dilated horizontally by a factor of 3 and vertically by a factor of 2.

Hence the new equation is:

$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$$

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

14b The exponential $y = 3^x$, dilated vertically by a factor of -2.

Hence the new equation is:

$$y = -2 \times 3^x$$

14c The curve $y = \tan x$, dilated horizontally by a factor of 3 and vertically by a factor of 2.

Hence the new equation is:

$$y = 2\tan\frac{x}{3}$$

15a i Note that $y = 2^{x+1} = 2 \times 2^x$

Hence to obtain this curve, stretch vertically by a factor of 2, $\frac{y}{2} = 2^x$, or translate left by 1 unit, $y = 2^{(x+1)}$.

15a ii Stretch along both axes by a factor of k, $\frac{y}{k} = \frac{1}{\frac{x}{k}}$, or stretch horizontally by a factor of k^2, $y = \frac{1}{\left(\frac{x}{k^2}\right)}$.

15a iii Reciprocal, $y = \frac{1}{3^x}$, or reflect in the y-axis, $y = 3^{-x}$.

16 The horizontal stretch causes the curve to become $y = \left(\frac{x}{a}\right)^2 = \frac{x^2}{a^2}$.

Hence stretching vertically by a factor of a^2 gives $\frac{y}{a^2} = \frac{x^2}{a^2}$ and hence $y = x^2$.

17 Begin with $y = x^3 - x$, stretch horizontally by a factor of $\sqrt{3}$ to give $y = \left(\frac{x}{\sqrt{3}}\right)^3 - \frac{x}{\sqrt{3}} = \frac{x^3}{3\sqrt{3}} - \frac{x}{\sqrt{3}}$, then stretch vertically by a factor of $3\sqrt{3}$ to give $\frac{y}{3\sqrt{3}} = \frac{x^3}{3\sqrt{3}} - \frac{x}{\sqrt{3}}$ and hence $y = x^3 + 3x$.

Solutions to Exercise 3I

1 Note that the vertex occurs at $x = -\frac{b}{2a} = -\frac{(-2)}{2(1)} = 1$.

The x-intercepts occur when $y = 0$.

$0 = x^2 - 2x$

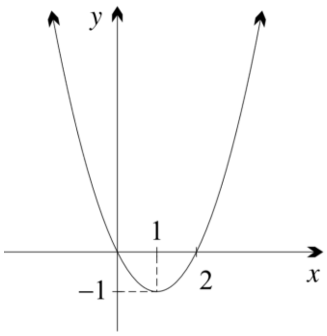
$0 = x(x-2)$

$x = 0, 2$

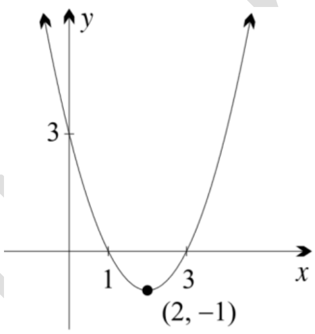
The y-intercepts occur when $x = 0$.

$y = 0^2 - 2(0) = 0$

Hence the graph is:



1a i



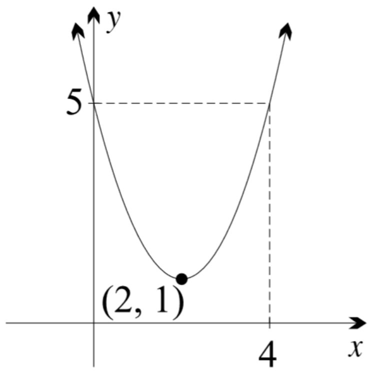
In order to find the new equation, we must replace x with $x - 1$:

$y = (x-1)^2 - 2(x-1)$

$= x^2 - 2x + 1 - 2x + 2$

$= x^2 - 4x + 3$

1a ii

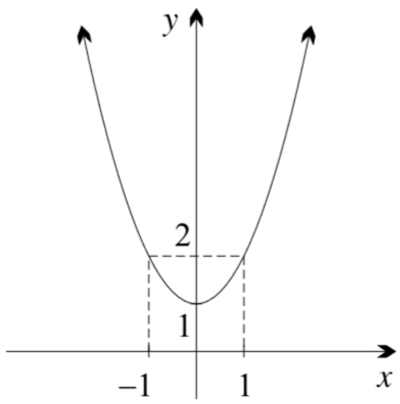


In order to find the new equation, we must replace y with $y - 2$:

$y - 2 = x^2 - 4x + 3$

$y = x^2 - 4x + 5$

1b i

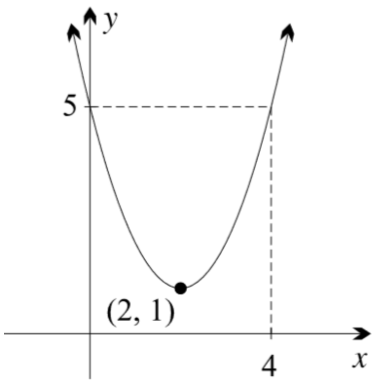


In order to find the new equation, we must replace y with $y - 2$:

$y - 2 = x^2 - 2x$

$y = x^2 - 2x + 2$

1b ii



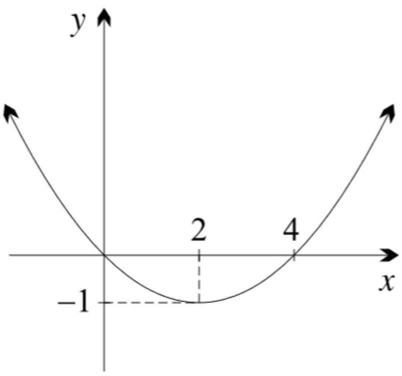
In order to find the new equation, we must replace x with $x - 1$:

$$y = (x-1)^2 - 2(x-1) + 2$$
$$= x^2 - 2x + 1 - 2x + 2 + 2$$
$$= x^2 - 4x + 5$$

1c Yes, they commute (can be applied in any order and still produce the same result) as the equations and graphs produced by parts a and b are the same. Furthermore, note that all translational transformations are commutative as we are simply adding and subtracting linear coordinates.

2a i Replacing x with $\frac{1}{2}x$ gives:

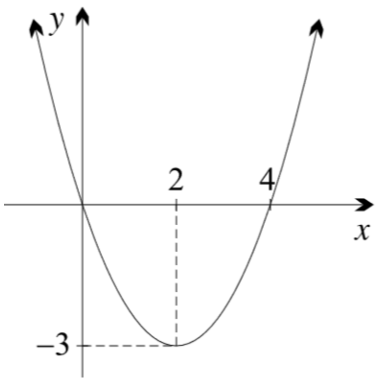
$$y = \left(\frac{1}{2}x\right)^2 - 2\left(\frac{1}{2}x\right)$$
$$= \frac{1}{4}x^2 - x$$



2a ii Replacing y with $\frac{y}{3}$ gives:

$\frac{y}{3} = \frac{1}{4}x^2 - x$

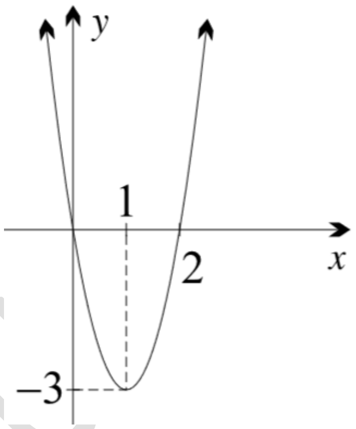
$y = \frac{3}{4}x^2 - 3x$



2b i Replacing y with $\frac{y}{3}$ gives:

$\frac{y}{3} = x^2 - 2x$

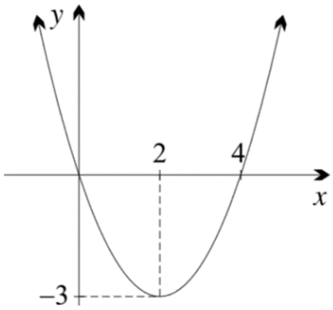
$y = 3x^2 - 6x$



2b ii Replacing x with $\frac{1}{2}x$ gives:

$$y = 3\left(\frac{1}{2}x\right)^2 - 6\left(\frac{1}{2}x\right)$$

$$= \frac{3}{4}x^2 - 3x$$

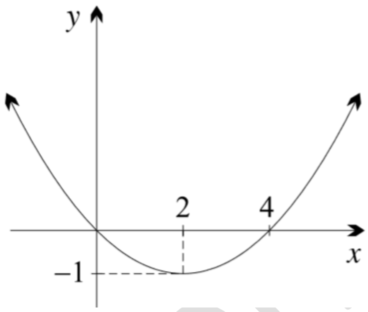


2c Yes, they are commutative (can be applied in any order and still produce the same result) as they produce the same equation and graph.

3a i Replacing x with $\frac{1}{2}x$ gives:

$$y = \left(\frac{1}{2}x\right)^2 - 2\left(\frac{1}{2}x\right)$$

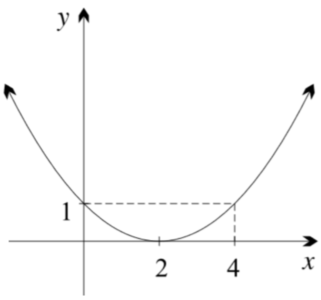
$$= \frac{1}{4}x^2 - x$$



3a ii Replacing y with $y - 1$ gives:

$$y - 1 = \frac{1}{4}x^2 - x$$

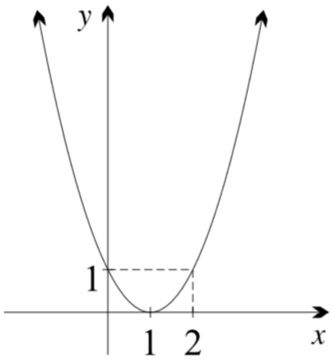
$$y = \frac{1}{4}x^2 - x + 1$$



3b i Replacing y with $y-1$ gives:

$y - 1 = x^2 - 2x$

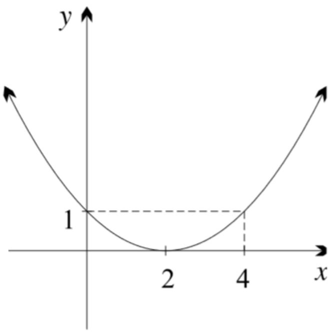
$y = x^2 - 2x + 1$



3b ii Replacing x with $\frac{1}{2}x$ gives:

$y = \left(\frac{1}{2}x\right)^2 - 2\left(\frac{1}{2}x\right) + 1$

$= \frac{1}{4}x^2 - x + 1$



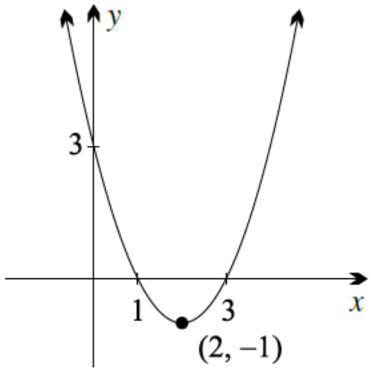
3c Yes, they are commutative (can be applied in any order and still produce the same result) as they produce the same equation and graph.

4a i Replacing x with $x-1$ gives:

$y = (x-1)^2 - 2(x-1)$

$= x^2 - 2x + 1 - 2x + 2$

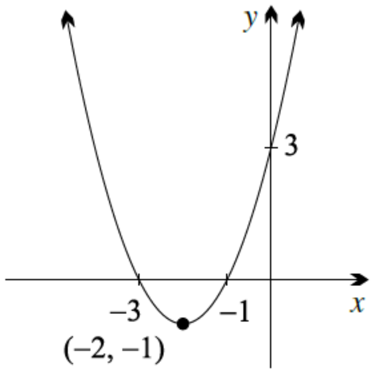
$= x^2 - 4x + 3$



4a ii Replacing x with $-x$ gives:

$y = (-x)^2 - 4(-x) + 3$

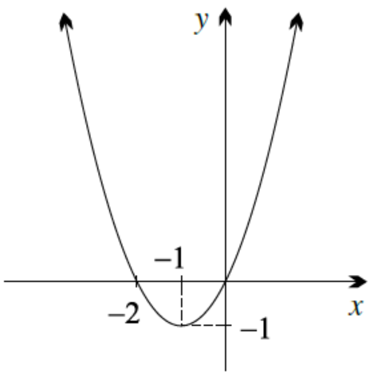
$= x^2 + 4x + 3$



4b i Replacing x with $-x$ gives:

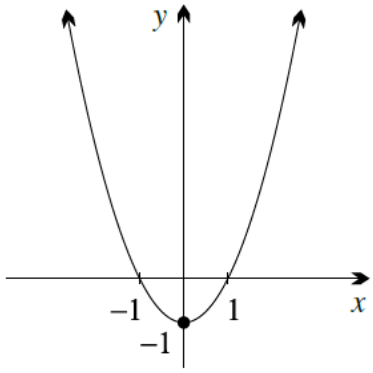
$y = (-x)^2 - 2(-x)$

$= x^2 + 2x$



4b ii Replacing x with $x-1$ gives:

$$y = (x-1)^2 + 2(x-1)$$
$$= x^2 - 2x + 1 + 2x - 2$$
$$= x^2 - 1$$



4c No, they are not commutative as the order in which the operations are applied affects the resultant equation/graph.

5a No

Consider $y = x^2$, if we reflect and then translate, we first replace x with $-x$ to get $y = x^2$ and then replace x with $x - c$ to get $y = (x-c)^2 = x^2 - 2xc + c^2$.

If we translate then reflect, we first replace x with $x - c$ to get $y = (x-c)^2 = x^2 - 2xc + c^2$ and then replace x with $-x$ to get $y = x^2 + 2xc + c^2$ which differs from the original result.

5b No

Consider $y = x^2$, if we dilate and then translate, we first replace y with $2y$ to get $2y = x^2$ or $y = \frac{1}{2}x^2$ and then replace y with $y - c$ to get $y - c = \frac{1}{2}x^2$ or $y = \frac{1}{2}x^2 + c$.

If we translate then dilate, we first replace y with $y - c$ to get $y - c = x^2$ or $y = x^2 + c$ and then replace y with $2y$ to get $2y = x^2 + c$ or $y = \frac{1}{2}x^2 + \frac{1}{2}c$ which differs from the original result.

5c Yes

Reflection in the x-axis can be considered 'negative' dilation and generally operations of the same form are commutative (although this is not always the case for some operations).

5d Yes

All translational transformations are commutative as we are simply adding and subtracting linear coordinates.

5e No

Consider $y = x^2$, if we reflect and then translate, we first replace x with $2x$ to get $y = 4x^2$ and then replace x with $x - c$ to get $y = 4(x - c)^2 = 4x^2 - 8xc + 4c^2$.

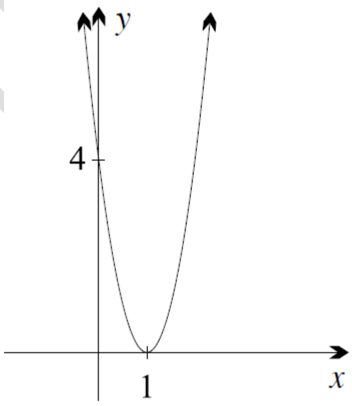
If we translate then reflect, we first replace x with $x - c$ to get $y = (x - c)^2 = x^2 - 2xc + c^2$ and then replace x with $2x$ to get $y = 4x^2 + 4xc + c^2$ which differs from the original result.

5f Yes

Reflection on the x-axis affects the y-component of functions, whilst horizontal translation affects the x-component. We can safely say that these operations will not interfere with one another and hence they will commute.

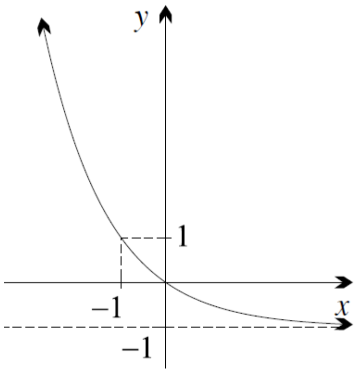
6a Translate right 2 units: $y = (x - 1)^2$

Then dilate horizontally by a factor of $\frac{1}{2}$: $y = 4(x - 1)^2$



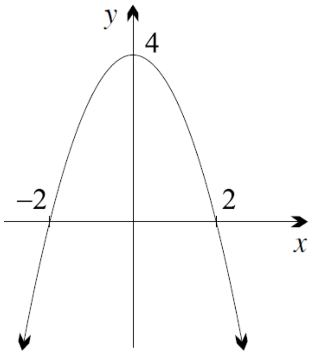
6b Translate down 1 unit: $y = 2^x - 1$

Then reflect in the y-axis: $y = 2^{-x} - 1$



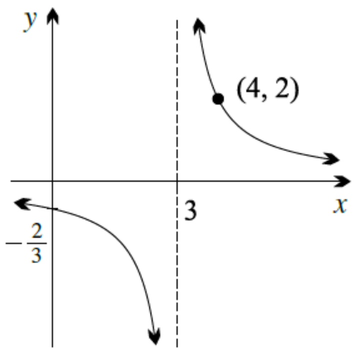
6c Translate down 3 units: $y = x^2 - 4$

Then dilate vertically by a factor of -1: $y = 4 - x^2$



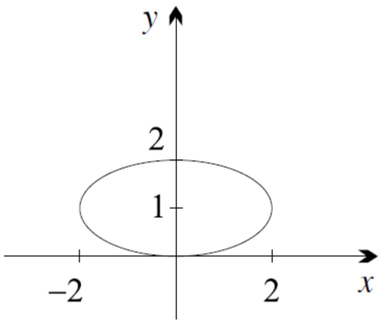
6d Translate right 3 units: $y = \frac{1}{x-3}$

Then dilate vertically by a factor of $\frac{1}{2}$: $y = \frac{2}{x-3}$



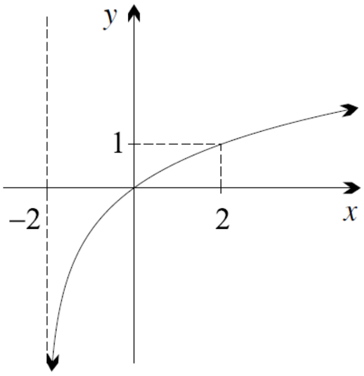
6e Translate up 2 units: $x^2 + (y - 2)^2 = 4$

Then dilate vertically by a factor of $\frac{1}{2}$: $x^2 + (2y - 2)^2 = 4$ or $x^2 + 4(y - 1)^2 = 4$



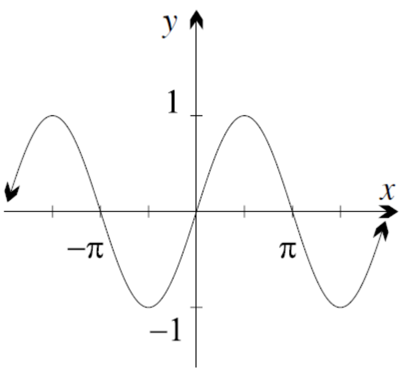
6f Translate left 1 unit: $y = \log_2(x + 1)$

Then dilate horizontally by a factor of 2: $y = \log_2\left(\frac{1}{2}x + 1\right)$



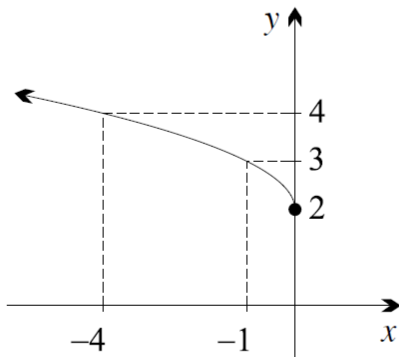
6g Translate left π units: $y = \sin(x + \pi)$

Then reflect in the x-axis: $y = -\sin(x + \pi)$

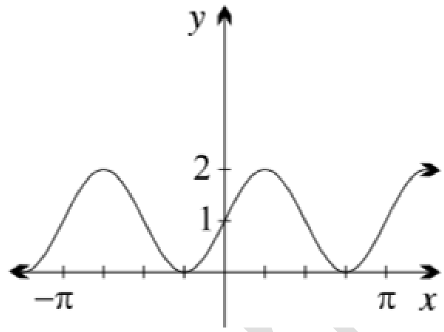


6h Translate up 2 units: $y = \sqrt{x} + 2$

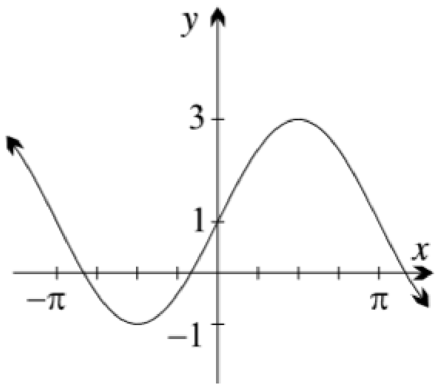
Then dilate horizontally by a factor of -1: $y = -\sqrt{x} + 2$



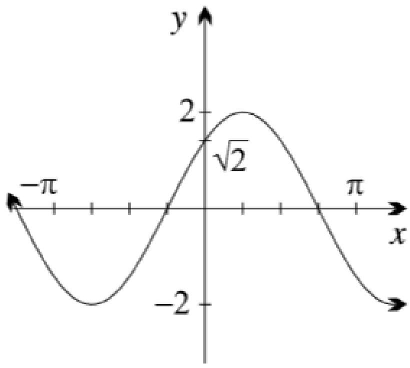
7a Take the graph of $y = \sin x$, dilate it horizontally by a factor of $\frac{1}{2}$ and then translate it up 1 unit.



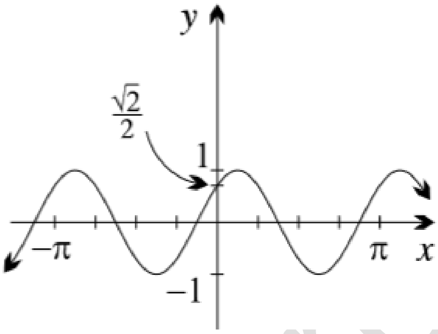
7b Take the graph of $y = \sin x$, dilate it vertically by a factor of 2 and then translate it up 1 unit.



7c Take the graph of $y = \sin x$, dilate it vertically by a factor of 2 and then translate it left $\frac{\pi}{4}$ units.



7d Take the graph of $y = \sin x$, dilate it horizontally by a factor of $\frac{1}{2}$ and then translate it left $\frac{\pi}{4}$ units. (Note for this particular example, order does matter).



8a Applying the transformations in order gives the following.

Translate left 1 unit:

$y = (x + 1)^2$

Down 4 units:

$y = (x + 1)^2 - 4$

Dilate horizontally by factor of 2:

$y = \left(\frac{x}{2} + 1\right)^2 - 4$

$y = \frac{1}{4}(x + 2)^2 - 4$

8b Applying the transformations in order gives the following.

Translate down 4 units:

$y = x^2 - 4$

Dilate horizontally by factor of 2:

$y = \left(\frac{x}{2}\right)^2 - 4$

$y = \frac{x^2}{4} - 4$

Translate left 1 unit:

$y = \frac{1}{4}(x + 1)^2 - 4$

8c Applying the transformations in order gives the following.

Translate down 1 unit:

$y = 2^x - 1$

Translate right 1 unit:

$y = 2^{x-1} - 1$

Dilate vertically by factor of -2:

$\frac{y}{-2} = 2^{x-1} - 1$

$y = -2 \times 2^{x-1} + 2$

$y = -2^{x-1+1} + 2$

$y = 2 - 2^x$

8d Applying the transformations in order gives the following.

Translate right 2 units:

$y = \frac{1}{x - 2}$

Dilate vertically by factor of 2:

$\frac{y}{2} = \frac{1}{x - 2}$

$$y = \frac{2}{x-2}$$

Translate up 1 unit:

$$y = \frac{2}{x-2} + 1$$

9a Shifting 2 units left:

$y = (x - 1 + 2)^2$

$y = (x + 1)^2$

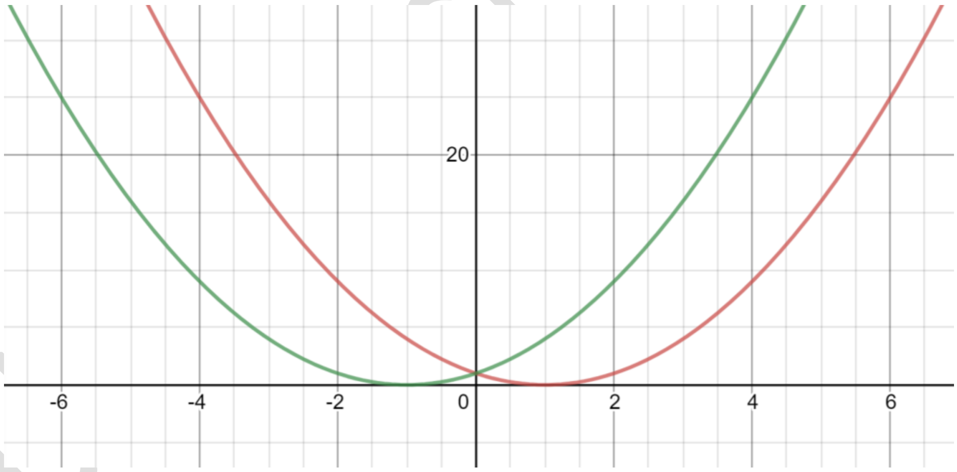
Reflecting in the y-axis:

$y = (-x + 1)^2$

$y = (x - 1)^2$

Hence the new parabola has the same equation.

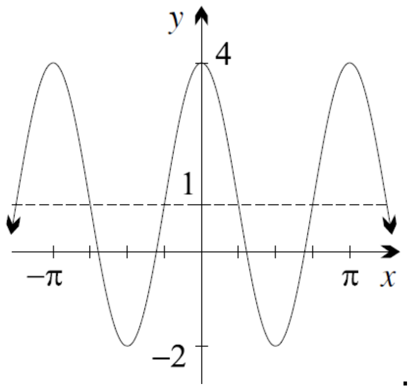
9b This can be explained by the graph below. Note that the red curve represents the graph $y = (x - 1)^2$ and the green curve represents $y = (x + 1)^2$.



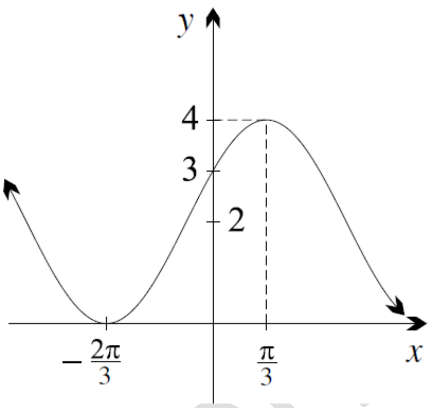
Note that reflections on the y-axis switches the two curves whilst a shift left moves the red curve 2 units left into the position of the green curve. Hence the shift and the reflection in the order they are performed have the effect of cancelling one another out to return to the original curve.

10 Note that the transformations you give may vary from this in the solutions.

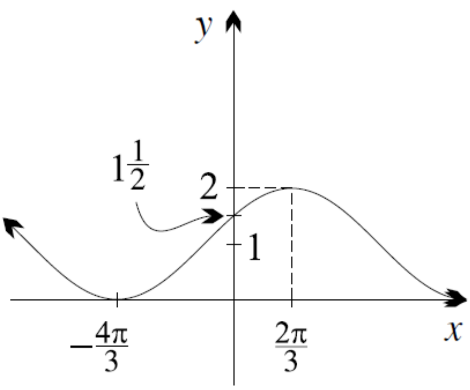
10a Take the graph of $y = \cos x$, dilate vertically by a factor of 3, shift up 1 unit and then dilate horizontally by a factor of $\frac{1}{2}$.



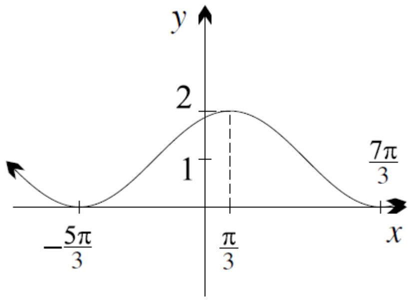
10b Take the graph of $y = \cos x$, shift $\frac{\pi}{3}$ units right, move 1 unit up and then dilate vertically by a factor of 2.



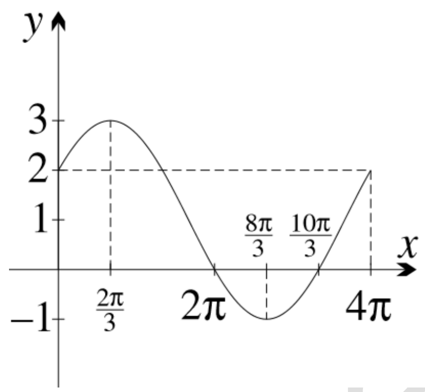
10c Take the graph of $y = \cos x$, dilate horizontally by a factor of 2, shift $\frac{\pi}{3}$ units right and then shift up by 1 unit.



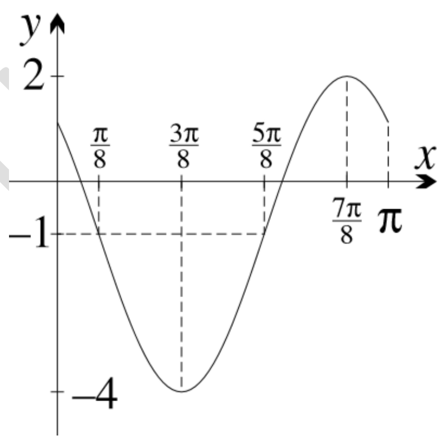
10d Take the graph of $y = \cos x$, shift $\frac{\pi}{3}$ units right, dilate horizontally by a factor of 2 and then shift up by 1 unit.



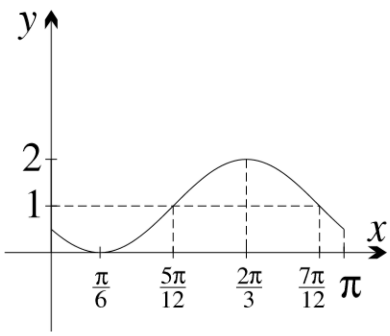
11a Plot the graph by taking $y = \sin x$, begin by dilating it by a factor of 2 from the x-axis, and then shift it up by a unit of 1. Shift $\frac{\pi}{6}$ units left and then dilate it by a factor of 2 from the y-axis.



11b Plot the graph by taking $y = \cos x$, begin by dilating it by a factor of 3 from the x-axis, and then shift it down by a unit of 1. Shift $\frac{\pi}{4}$ units left and then dilate it by a factor of $\frac{1}{2}$ from the y-axis.



11c Plot the graph by taking $y = \cos x$, begin by reflecting it about the x-axis, and then shift it up by a unit of 1. Shift $\frac{\pi}{3}$ units right and then dilate it by a factor of 2 from the y-axis.



12a Let $y = f(x)$.

First applying H gives $y = f(x - a)$ and then V gives $y = f(x - a) + b$.

Alternatively, first applying V gives $y = f(x) + b$ and then H gives $y = f(x - a) + b$.

Thus, the operations H and V commute.

12b Let $y = f(x)$.

First applying E gives $y = f(\frac{1}{a}x)$ and then U gives $\frac{y}{b} = f\left(\frac{1}{a}x\right)$.

Alternatively, first applying U gives $\frac{y}{b} = f(x)$ and then E gives $\frac{y}{b} = f(\frac{1}{a}x)$.

Thus, the operations E and U commute.

12c Let $y = f(x)$.

First applying F gives $y = f(-x)$ and then L gives $-y = f(-x)$.

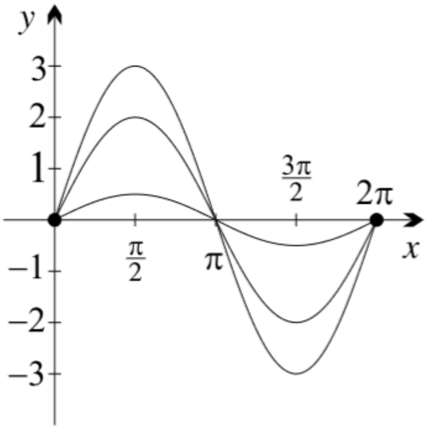
Alternatively, first applying L gives $-y = f(x)$ and then F gives $-y = f(-x)$.

Thus, the operations L and F commute.

12d If the transformations are dilations or reflections then they commute. A reflection is a special type of dilation, and any pair of dilations commute. If one of the transformations is a translation in a different direction to the other transformation, for example $\mathcal{H}$ and $\mathcal{U}$, then they commute.

Solutions to Exercise 3J

1a



Note $-1 \le \sin x \le 1$ and hence $-a \le a\sin x \le a$, this means that the amplitude of each graph in the question will be given by the coefficient of the sine function.

1a i $\frac{1}{2}$

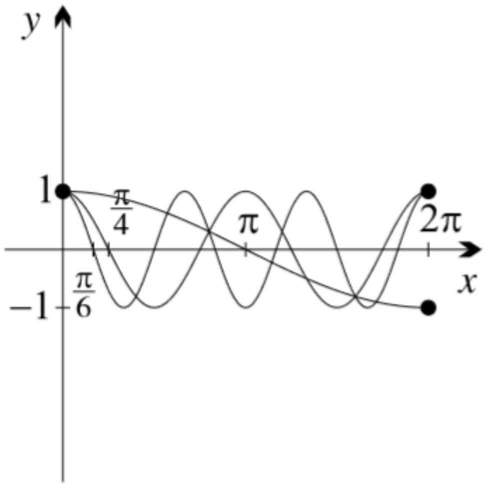
1a ii 2

1a iii 3

1b In order to obtain this transformation, begin with the graph of $y = \sin x$ and then replace y with $\frac{y}{k}$ (which corresponds to a vertical stretch by a factor of k) to yield $\frac{y}{k} = \sin x$ which is $y = k\sin x$.

1c Recall that as k is the coefficient of the sine function, it gives the amplitude of the graph. Hence it follows that as k in increases, the amplitude increases. The bigger the amplitude, the steeper the wave.

2a



2a i $T = \frac{2\pi}{\frac{1}{2}} = 4\pi$

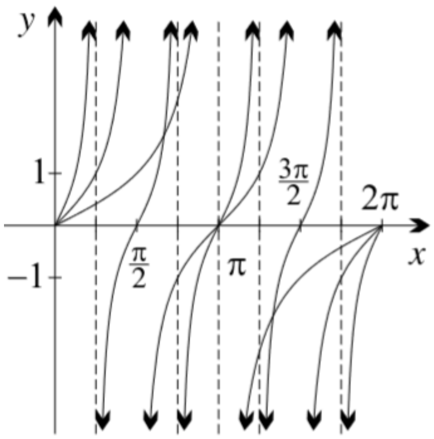
2a ii $T = \frac{2\pi}{2} = \pi$

2a iii $T = \frac{2\pi}{3}$

2b The graph of $y = \cos x$ is stretched horizontally by a factor of $\frac{1}{n}$.

2c Noting that $T = \frac{2\pi}{n}$, we see that T is inversely proportional to n. Thus it follows that as n increases, the period decreases.

3a



3a i $T = \frac{\pi}{1} = \pi$

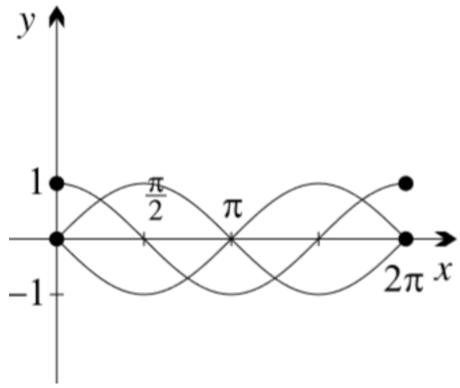
3a ii $T = \frac{\pi}{\frac{1}{2}} = 2\pi$

3a iii $T = \frac{\pi}{2}$

3b The graph $y = \tan x$ is stretched horizontally by a factor of $\frac{1}{a}$.

3c Noting that $T = \frac{\pi}{a}$, we see that T is inversely proportional to a. Thus, it follows that as n increases, the period decreases.

4a



4a i $\frac{\pi}{2}$

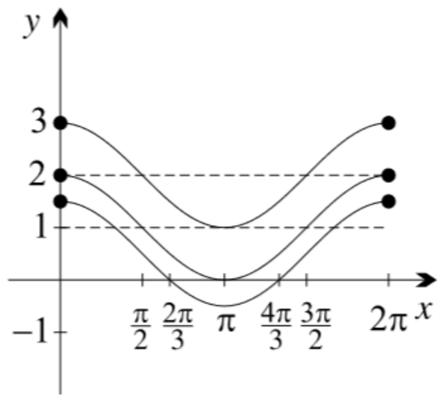
4a ii π

4a iii 2π or 0. Note that $y = \sin(x + 2\pi) = \sin(x)$.

4b The graph $y = \sin x$ is shifted a units to the left.

4c The graph is always the same, because $y = \sin x$ has period 2π.

5a



5a i Range: $0 \le y \le 2$ or $[0, 2]$, mean value: 1

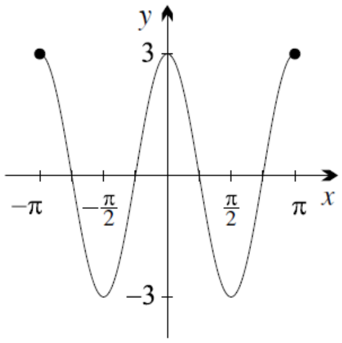
5a ii Range: $[1, 3]$, mean value: 2

5a iii Range: $[-\frac{1}{2}, \frac{3}{2}]$, mean value: $\frac{1}{2}$

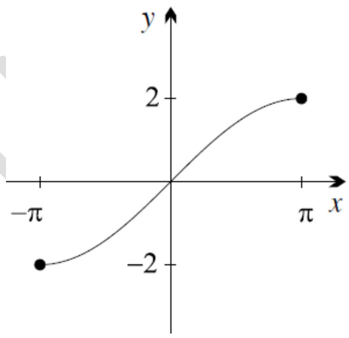
5b The graph $y = \cos x$ is shifted c units up and the mean value is c.

5c As c increases, the graph of $y = \cos x$ moves up and the mean value increases.

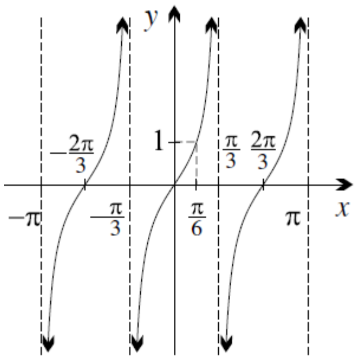
6a period $= \pi$, amplitude $= 3$



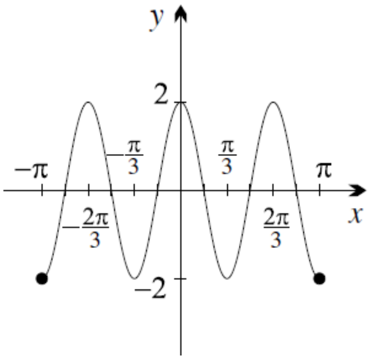
6b period $= 4\pi$, amplitude $= 2$



6c period $= \frac{2\pi}{3}$, no amplitude

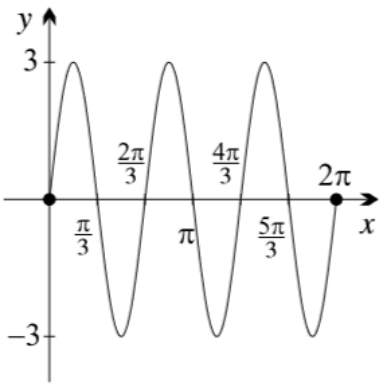


6d period $= \frac{2\pi}{3}$, amplitude $= 2$



7a Stretch horizontally by a factor of $\frac{1}{3}$: $y = \sin 3x$

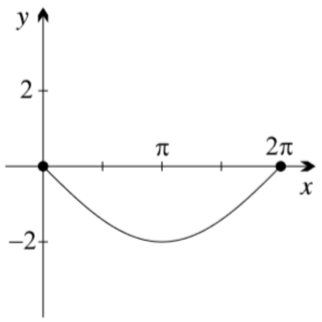
Then stretch vertically by a factor of 3: $y = 3\sin 3x$



7b Stretch horizontally by a factor of 2: $y = \sin\frac{x}{2}$

Then stretch vertically by a factor of 2: $y = 2\sin\frac{x}{2}$

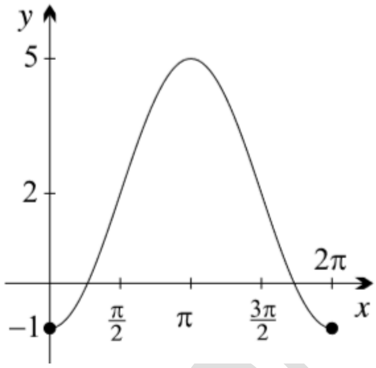
Then reflect in the x-axis: $y = -2\sin\frac{x}{2}$



7c Shift $\frac{\pi}{2}$ units right: $y = \sin\left(x - \frac{\pi}{2}\right)$

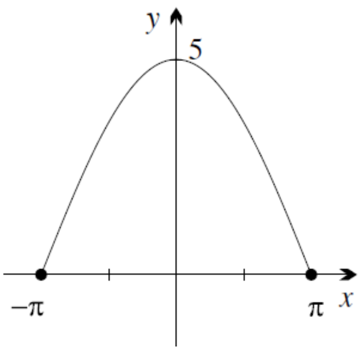
Then stretch vertically by a factor of 3: $y = 3\sin\left(x - \frac{\pi}{2}\right)$

Then shift 2 units up: $y = 3\sin\left(x - \frac{\pi}{2}\right) + 2$



8a Stretch horizontally by a factor of 2: $y = \cos\frac{1}{2}x$

Then stretch vertically by a factor of 5: $y = 5\cos\frac{1}{2}x$

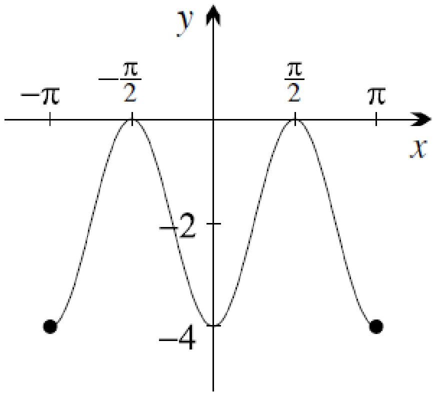


8b Stretch horizontally by a factor of $\frac{1}{2}$: $y = \cos 2x$

Then stretch vertically by a factor of 2: $y = 2\cos 2x$

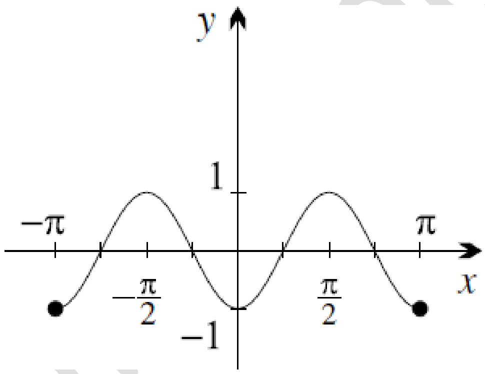
Then reflect in the x-axis: $y = -2\cos 2x$

Then shift 2 units down: $y = -2\cos 2x - 2$



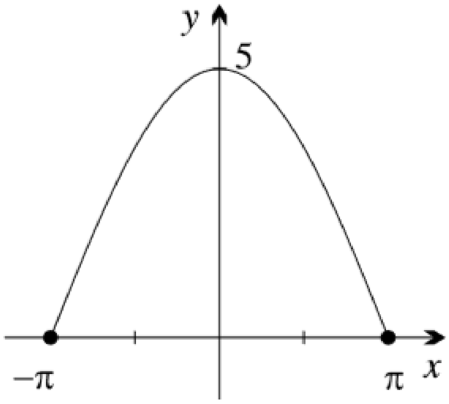
8c Stretch horizontally by a factor of $\frac{1}{2}$: $y = \cos 2x$

Then shift $\frac{\pi}{2}$ units right: $y = \cos\left(2\left(x - \frac{\pi}{2}\right)\right)$



9a Stretch horizontally by a factor of $\frac{1}{3}$: $y = \sin 3x$

Then shift $\frac{\pi}{6}$ units left: $y = \sin\left(3\left(x + \frac{\pi}{6}\right)\right)$ or $y = \sin\left(3x + \frac{\pi}{2}\right)$

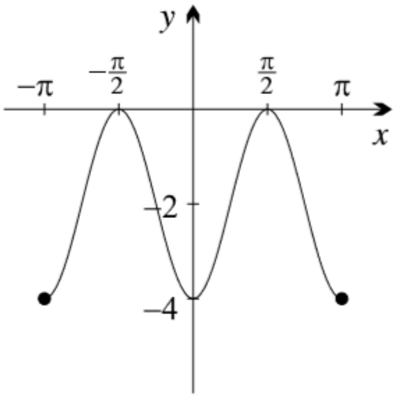


9b Stretch horizontally by a factor of $\frac{1}{4}$: $y = \sin 4x$

Then shift $\frac{\pi}{4}$ units right: $y = \sin\left(4\left(x - \frac{\pi}{4}\right)\right)$ or $y = \sin(4x - \pi)$

Then stretch vertically by a factor of $\frac{1}{4}$: $y = \frac{1}{4}\sin(4x - \pi)$

Then shift 4 units down: $y = \frac{1}{4}\sin(4x - \pi) - 4$

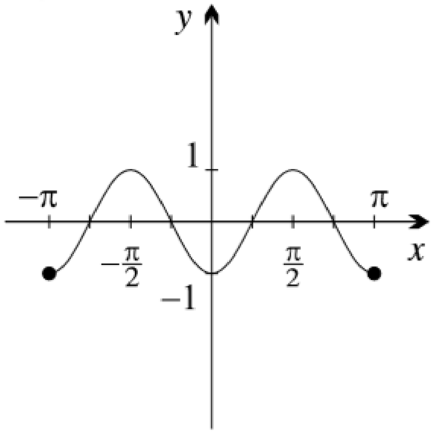


9c Stretch horizontally by a factor of 2: $y = \sin\frac{1}{2}x$

Then shift $\frac{\pi}{2}$ units left: $y = \sin\left(\frac{1}{2}\left(x + \frac{\pi}{2}\right)\right)$ or $y = \sin\left(\frac{x}{2} + \frac{\pi}{4}\right)$

Then stretch vertically by a factor of 6: $y = 6\sin\left(\frac{x}{2} + \frac{\pi}{4}\right)$

Then reflect in the x-axis: $y = -6\sin\left(\frac{x}{2} + \frac{\pi}{4}\right)$



10a Part **a:** period $= \frac{2\pi}{3}$, phase $= 0 + \frac{\pi}{2} = \frac{\pi}{2}$

Part **b:** period $= \frac{2\pi}{4} = \frac{\pi}{2}$, phase $= -\pi$ (but this is twice the period, so we can also say phase $= 0$.

Part **c:** period $= 4\pi$, phase $= \frac{\pi}{4}$

10b i period $= \pi$, phase $= 2\left(0 - \frac{\pi}{3}\right) = -\frac{2\pi}{3}$

10b ii period $= 6\pi$, phase $= \frac{\pi}{3}$

10b iii period $= \frac{\pi}{3}$, phase $= \frac{3\pi}{8}$

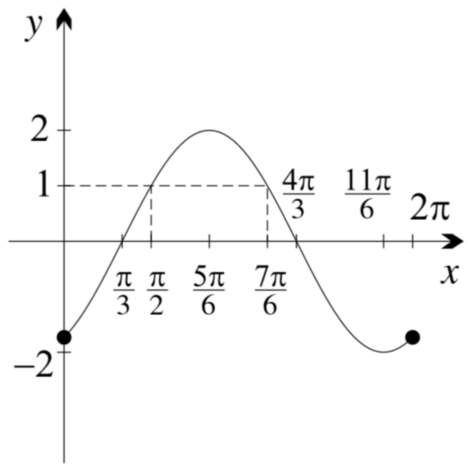
11a $2\sin\left(x - \frac{\pi}{3}\right) = 1$

$$\sin\left(x - \frac{\pi}{3}\right) = \frac{1}{2}$$

Since $0 \le x \le 2\pi$, $-\frac{\pi}{3} \le x - \frac{\pi}{3} \le \frac{5\pi}{3}$. Hence:

$$x - \frac{\pi}{3} = \frac{\pi}{6}, \frac{5\pi}{6}$$

So $x = \frac{\pi}{2}$ or $\frac{7\pi}{6}$



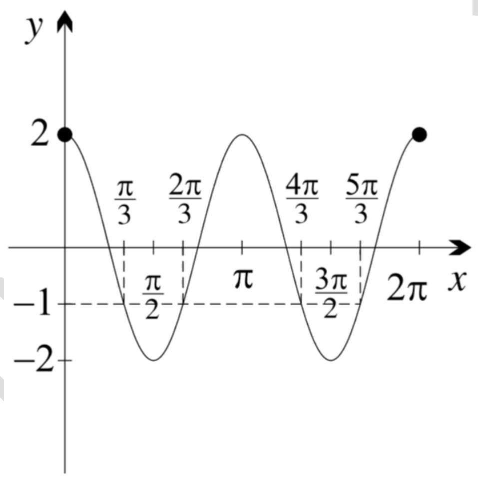
11b $2\cos 2x = -1$

$\cos 2x = -\dfrac{1}{2}$

Since $0 \le x \le 2\pi$, $0 \le 2x \le 4\pi$. Hence:

$2x = \dfrac{2\pi}{3}, \dfrac{4\pi}{3}, \dfrac{8\pi}{3}, \dfrac{10\pi}{3}$

$x = \dfrac{\pi}{3}, \dfrac{2\pi}{3}, \dfrac{4\pi}{3}$ or $\dfrac{5\pi}{3}$



12a $\cos(x + 0.2) = -0.3$ $\quad (0 \le x \le \pi$ so $0.2 \le x + 0.2 \le \pi + 0.2)$

$x + 0.2 = \cos^{-1}(-0.3)$

$x = \cos^{-1}(-0.3) - 0.2$

$x \doteqdot 1.675$

12b $\tan 2x = 0.5 \quad (0 \le x \le \pi \text{ so } 0 \le 2x \le 2\pi)$

$2x = \tan^{-1}(0.5)$

$2x = \tan^{-1}(0.5) \text{ or } \pi + \tan^{-1}(0.5)$

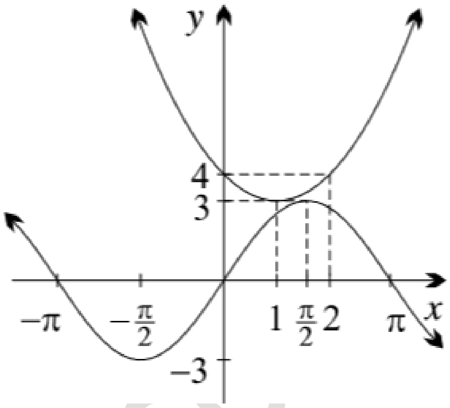
$x = \frac{1}{2}\tan^{-1}(0.5) \text{ or } \frac{1}{2}(\pi + \tan^{-1}(0.5))$

$x \doteqdot 0.232 \text{ or } 1.803$

13a The vertex of the parabola occurs when $x = -\frac{b}{2a} = -\frac{-2}{2} = 1$.

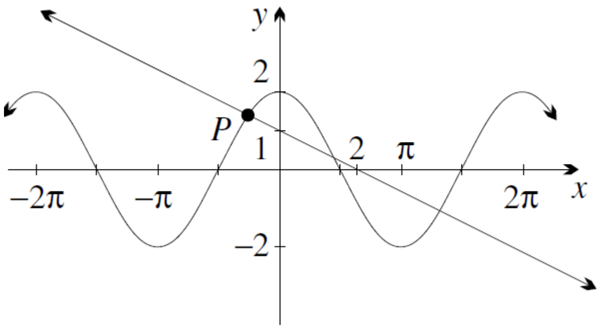
Thus $y = 1^2 - 2(1) + 4 = 3$. The vertex is at $(1, 3)$.

13b



Since the curve for $y = x^2 - 2x + 4$ is always above the curve for $y = 3\sin x$, $x^2 - 2x + 4 > 3\sin x$ for all real values of x.

14a,b



14c There are three points of intersection, so $2\cos x = 1 - \frac{1}{2}x$ has three solutions.

14d P is in the second quadrant (recall that quadrant 1 is the top right and the rest follow anticlockwise).

14e We know that $-1 \leq \cos x \leq 1$ and hence:

$-2 \leq 2\cos x \leq 2$

If $2\cos x = 1 - \frac{1}{2}x$ then

$$-2 \leq 1 - \frac{1}{2}x \leq 2$$

$$-3 \leq -\frac{1}{2}x \leq 1$$

$$-1 \leq \frac{1}{2}x \leq 3$$

$-2 \leq x \leq 6$

Hence if $x = n$ is a solution, $-2 \leq n \leq 6$.

15a $T = \frac{2\pi}{\left(\frac{2}{\pi}\right)} = 4$

15b

15c The equation describes any straight line passing through the origin. Hence, the only fixed point it passes through for varying values of m is the origin.

15d One solution is when $m = 0$ as it is tangential to the curve at the point $x = 3$. Otherwise, it must have positive gradient that is steep enough to be above the point $(4, 1)$ (otherwise it will re-intersect the curve). This is when $m > \frac{1}{4}$.

16a The maximum value of $2\cos(\frac{\pi}{7}t)$ is 2 and hence the greatest depth will be $y = 2 + 8 = 10$ metres.

16b The minimum value of $2\cos(\frac{\pi}{7}t)$ is -2 and hence the lowest depth will be $y = -2 + 8 = 6$ metres.

16c This will occur when:

$$2\cos\left(\frac{\pi}{7}t\right) = -2$$

$$\cos\left(\frac{\pi}{7}t\right) = -1$$

$$\frac{\pi}{7}t = \pi$$

$$t = 7$$

This is 7 hours after 7 am which is 2 pm.

16d $2\cos\left(\frac{\pi}{7}t\right) + 8 = 9$

$$2\cos\left(\frac{\pi}{7}t\right) = 1$$

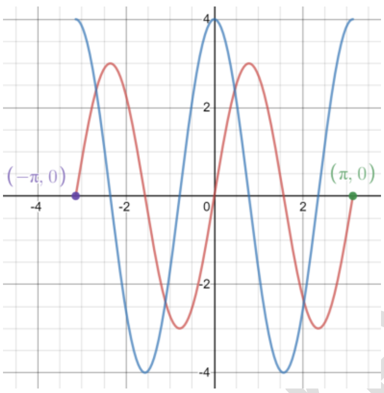
$$\cos\left(\frac{\pi}{7}t\right) = \frac{1}{2}$$

$$\frac{\pi}{7}t = \frac{\pi}{3}$$

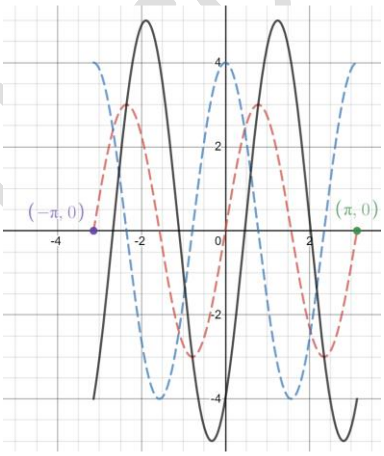
$$t = \frac{7}{3}$$

This is $\frac{7}{3}$ hours or 2 hours 20 minutes after 7 am which is 9: 20 am.

17a



17b



17c Amplitude $= 5$

18a i

18a ii 1 (the tangent is the line $y = x$)

18a iii By observation, $0 < k < 1$.

18b i For the arc:

$l = r(2\theta)$

$400 = 2r\theta$

$r = \frac{200}{\theta}$

For the chord:

$d = \sqrt{r^2 + r^2 - 2 \times r \times r \times \cos 2\theta}$

$\quad = r\sqrt{2 - 2\cos\theta}$

$300 = r\sqrt{2 - 2\cos 2\theta}$

$\quad = r\sqrt{2 - 2(1 - 2\sin^2\theta)}$

$\quad = r\sqrt{4\sin^2\theta}$

$\quad = 2r\sin\theta$

$r = \dfrac{150}{\sin\theta}$

Hence:

$\dfrac{200}{\theta} = \dfrac{150}{\sin\theta}$

And thus

$$\sin\theta = \frac{150}{200}\theta = \frac{3}{4}\theta$$

18b ii

Drawing the two graphs, their point of intersection appears to be at $\theta \doteqdot 1.3$.

18b iii $\angle AOB = 2\theta \doteqdot 2.6$ radians

Substituting for θ in $r = \frac{200}{\theta}$ gives $r \doteqdot 154$ metres.

18c i For the arc:

$$l = r(2\alpha) \text{ so } r = \frac{l}{2\alpha}$$

For the chord:

$$d = \sqrt{r^2 + r^2 - 2 \times r \times r \times \cos 2\alpha} = r\sqrt{2 - 2\cos\alpha}$$

$$\begin{aligned} 300 &= r\sqrt{2 - 2\cos 2\alpha} \\ &= r\sqrt{2 - 2(1 - 2\sin^2\alpha)} \\ &= r\sqrt{4\sin^2\alpha} \\ &= 2r\sin\alpha \end{aligned}$$

$$r = \frac{150}{\sin\alpha}$$

Equating the expressions for r:

$$\frac{l}{2\alpha} = \frac{150}{\sin\alpha}$$

$$\sin\alpha = \frac{150 \times 2\alpha}{l} = \frac{300\alpha}{l}$$

18c ii Chord length, $d = 300$ and $l > d$. Hence $l > 300$.

19a At $x = \frac{\pi}{3}$

$$\text{LHS} = \frac{1}{1 + \cos\frac{\pi}{3}}$$
$$= \frac{1}{1 + \frac{1}{2}}$$
$$= \frac{1}{\frac{3}{2}}$$
$$= \frac{2}{3}$$

$$\text{RHS} = \frac{2\left(\frac{\pi}{3}\right)}{\pi}$$
$$= \frac{2\pi}{3\pi}$$
$$= \frac{2}{3}$$

Thus $x = \frac{\pi}{3}$ satisfies the equation.

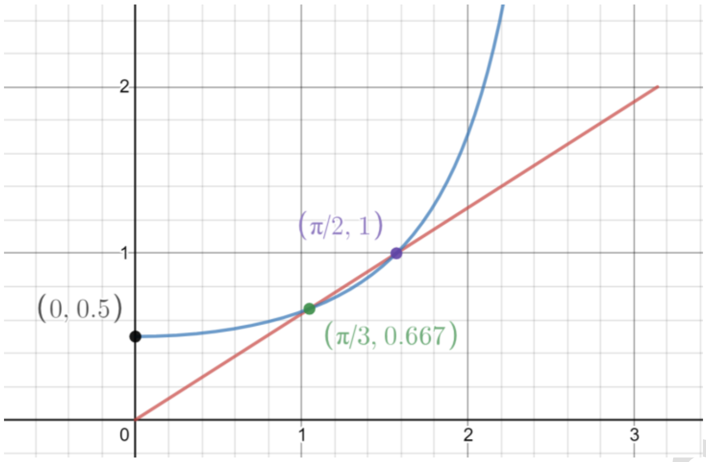
At $x = \frac{\pi}{2}$

$$\text{LHS} = \frac{1}{1 + \cos\frac{\pi}{2}}$$
$$= \frac{1}{1 + 0}$$
$$= \frac{1}{1}$$
$$= 1$$

$$\text{RHS} = \frac{2\left(\frac{\pi}{2}\right)}{\pi}$$
$$= \frac{2\pi}{2\pi}$$
$$= 1$$

Thus $x = \frac{\pi}{2}$ satisfies the equation

19b



19c For $\frac{\pi}{3} < x < \frac{\pi}{2}$ we have that $\frac{1}{1+\cos x} < \frac{2x}{\pi}$ from the diagram in part b. Hence in this domain we have.

$$\frac{1}{1+\cos x} < \frac{2x}{\pi}$$
$$\pi < 2x(1+\cos x)$$
$$\pi < 2x\left(1+\left(2\cos^2\frac{1}{2}x - 1\right)\right)$$
$$\pi < 2x\left(2\cos^2\frac{1}{2}x\right)$$
$$\pi < 4x\cos^2\frac{1}{2}x$$

And thus $4x\cos^2\frac{1}{2}x < \pi$ for $\frac{\pi}{3} < x < \frac{\pi}{2}$

Solutions to Chapter review

1a i $-1 < x < 2$

1a ii $(-1, 2)$

1b i $-1 \leq x < 2$

1b ii $[-1, 2)$

1c i $x \leq 2$

1c ii $(-\infty, 2]$

2a i $f \circ g(-2) = f\big(g(-2)\big) = f(-2+1) = f(-1) = (-1)^2 - 1 = 1 - 1 = 0$

2a ii $g \circ f(-2) = g\big(f(-2)\big) = g((-2)^2 - 1) = g(4-1) = g(3) = 3 + 1 = 4$

2a iii $f \circ f(-2) = f\big(f(-2)\big) = f((-2)^2 - 1) = f(4-1) = f(3) = 3^2 - 1 = 9 - 1 = 8$

2a iv $g \circ g(-2) = g\big(g(-2)\big) = g(-2+1) = g(-1) = -1 + 1 = 0$

2b i $f \circ g(x) = f\big(g(x)\big) = (x+1)^2 - 1 = x^2 + 2x + 1 - 1 = x^2 + 2x$

2b ii $g \circ f(x) = g\big(f(x)\big) = (x^2 - 1) + 1 = x^2$

2b iii $f \circ f(x) = f\big(f(x)\big) = (x^2 - 1)^2 - 1 = x^4 - 2x^2 + 1 - 1 = x^4 - 2x^2$

2b iv $g \circ g(x) = g\big(g(x)\big) = (x+1)+1 = x+2$

3a

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{1}{x+2} = \lim_{x\to\infty} \frac{\frac{1}{x}}{1+\frac{2}{x}} = \frac{0}{1+0} = 0^+$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{1}{x+2} = \lim_{x\to-\infty} \frac{\frac{1}{x}}{1+\frac{2}{x}} = \frac{0}{1-0} = 0^-$$

Hence the horizontal asymptote occurs when $y = 0$.

3b

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x-3}{2x+5} = \lim_{x\to\infty} \frac{1-\frac{3}{x}}{2+\frac{5}{x}} = \frac{1-0}{2+0} = \frac{1}{2}^-$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x-3}{2x+5} = \lim_{x\to-\infty} \frac{1-\frac{3}{x}}{2+\frac{5}{x}} = \frac{1+0}{2-0} = \frac{1}{2}^+$$

Hence the horizontal asymptote occurs when $y = \frac{1}{2}$.

3c

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x}{x^2+1} = \lim_{x\to\infty} \frac{\frac{1}{x}}{x+\frac{1}{x^2}} = \frac{0}{x+0} = 0^+$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x}{x^2+1} = \lim_{x\to-\infty} \frac{\frac{1}{x}}{x+\frac{1}{x^2}} = \frac{0}{x+0} = 0^-$$

Hence the horizontal asymptote occurs when $y = 0$.

4a $-\infty < x < \infty$

4b Note that $y = x(x - 3)(x + 3)$

The y-intercepts occur when $x = 0$. This is when:

$y = 0(0 - 3)(0 + 3) = 0$

Hence there is a y-intercept at $(0, 0)$.

The x-intercepts occur when $y = 0$. This is when:

$0 = x(x - 3)(x + 3)$

$x = 0, \pm 3$

Hence there are x-intercepts at $(0, 0)$, $(3, 0)$ and $(6, 0)$.

4c The function is defined at all points so there are no vertical asymptotes. The function diverges as x tends towards $\pm\infty$ and hence there are no horizontal asymptotes.

4d

x	-1	0	1	2	3	4	5	6	7
y	-28	0	10	8	0	-8	-10	0	28

(See the diagram in part e)

4e $y' = 3x^2 - 18x + 18$

$= 3(x^2 - 6x + 6)$ so $y' = 0$ when:

$3(x^2 - 6x + 6) = 0$

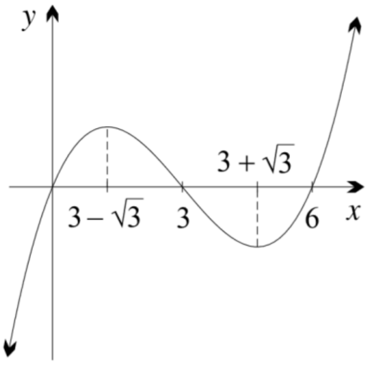
$x^2 - 6x + 6 = 0$

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(6)}}{2 \times 1}$$

$$= \frac{6 \pm \sqrt{12}}{2}$$

$$= \frac{6 \pm 2\sqrt{3}}{2}$$

$= 3 \pm \sqrt{3}$



5a $-6 < -3x \le 12$

$-12 \le 3x < 6$

$-4 \le x < 2$

$[-4, 2)$

5b $-2 < 2x + 1 < 1$

$-3 < 2x < 0$

$-\frac{3}{2} < x < 0$

$(-\frac{3}{2}, 0)$

5c $-7 \le 5 + 4x < 7$

$-12 \le 4x < 2$

$-3 \le x < \frac{1}{2}$

$[-3, \frac{1}{2})$

5d $-4 \le 1 - \frac{1}{2}x \le 3$

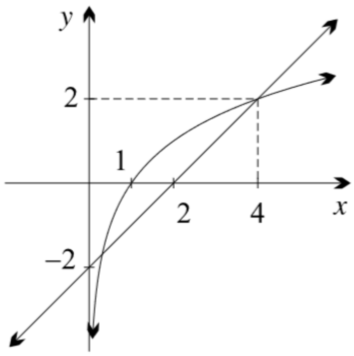
$-3 \le \frac{1}{2}x - 1 \le 4$

$$-2 \le \frac{1}{2}x \le 5$$

$$-4 \le x \le 10$$

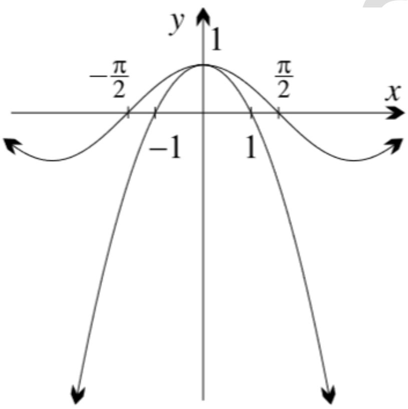
$$[-4, 10]$$

6a Sketch showing $y = x - 2$ and $y = \log_2 x$:



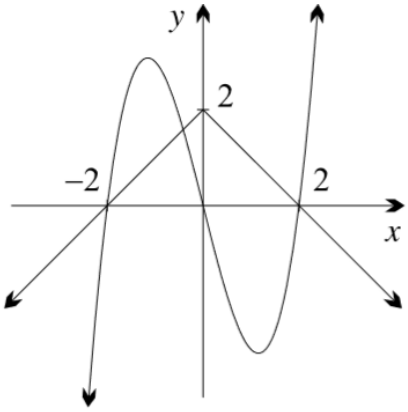
There are two points of intersection so there are 2 solutions.

6b Sketch showing $y = \cos x$ and $y = 1 - x^2$:



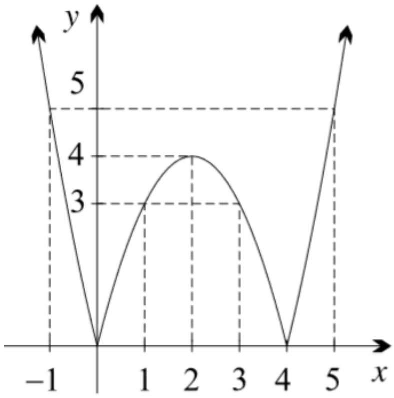
There is one point of intersection so there is 1 solution.

6c Sketch showing $y = x(x-2)(x+2)$ and $y = 2 - |x|$:

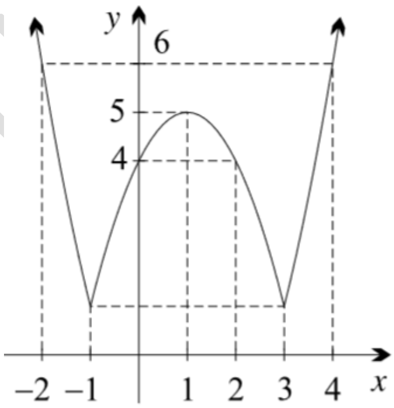


There are three points of intersection so there are 3 solutions.

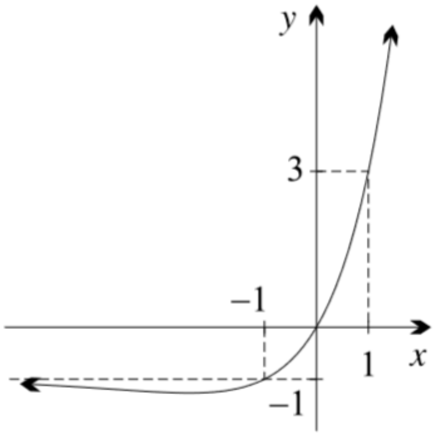
7a i Shift the diagram 1 unit to the right



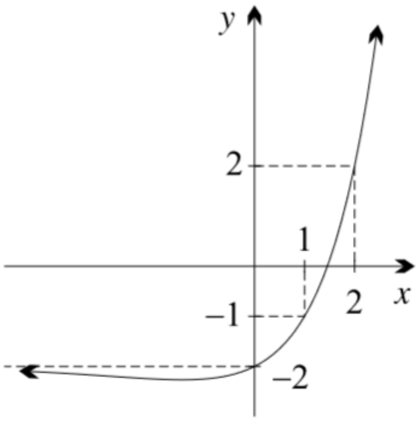
7a ii Shift the diagram 1 unit up.



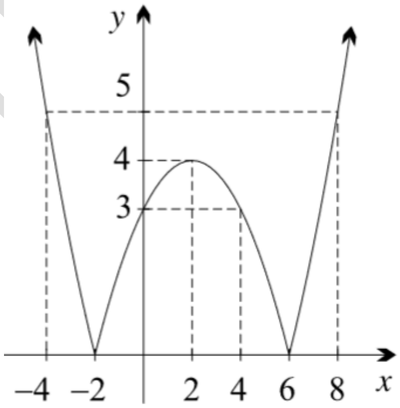
7b i Shift the diagram 1 unit left.



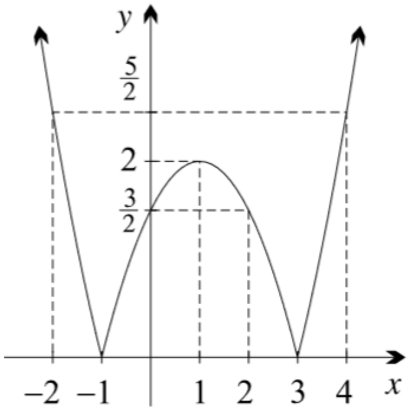
7b ii Shift the diagram 1 unit down.



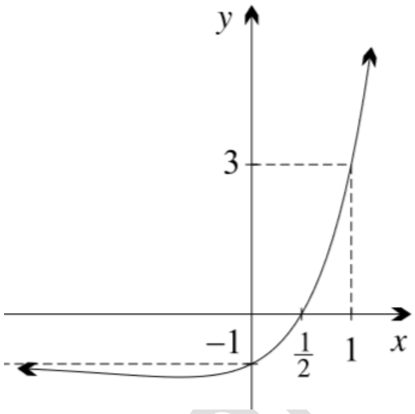
8a i Dilate the graph horizontally by a factor of 2.



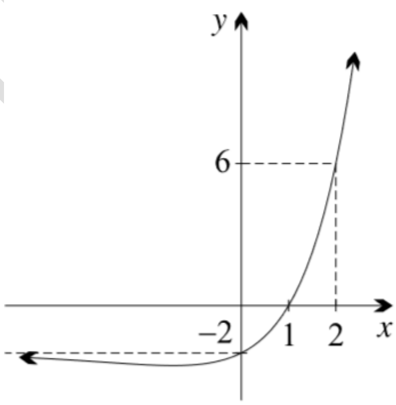
8a ii Dilate the graph vertically by a factor of $\frac{1}{2}$.



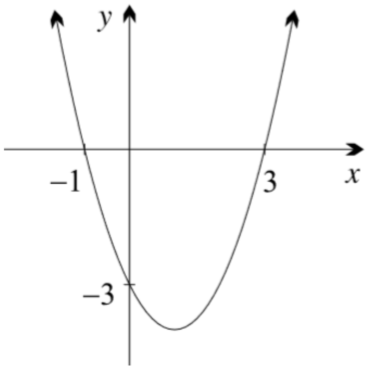
8bi Dilate the graph horizontally by a factor of $\frac{1}{2}$.



8b ii Dilate the graph vertically by a factor of 2.

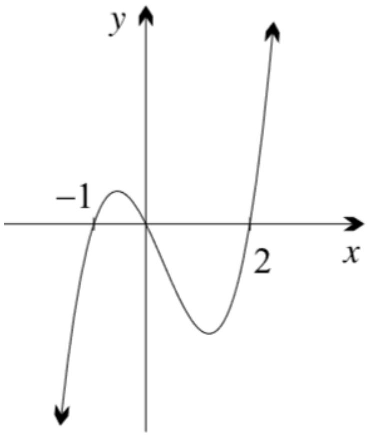


9a $f(x) = (x + 1)(x - 3)$



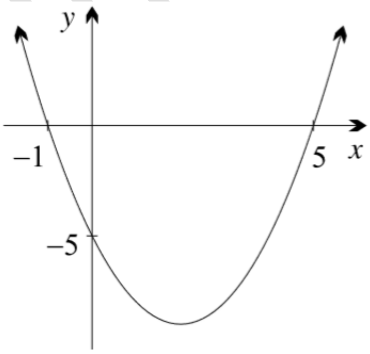
$f(x) \leq 0$ for $-1 \leq x \leq 3$

9b $f(x) = x(x - 2)(x + 1)$



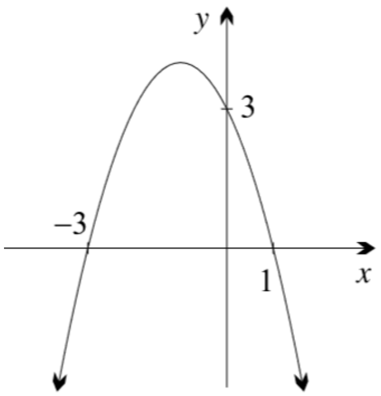
$f(x) \leq 0$ for $x \leq -1$ or $0 \leq x \leq 2$

9c $f(x) = x^2 - 4x - 5 = (x - 5)(x + 1)$



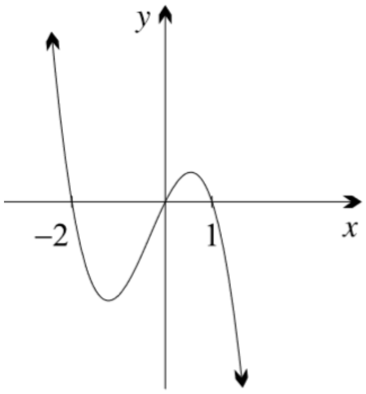
$f(x) \leq 0$ for $-1 \leq x \leq 5$

9d $f(x) = 3 - 2x - x^2 = -(x+3)(x-1)$



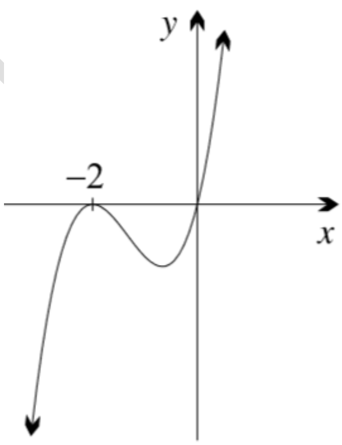
$f(x) \le 0$ for $x \le -3$ or $x \ge 1$

9e $f(x) = 2x - x^2 - x^3 = -x(x+2)(x-1)$



$f(x) \le 0$ for $-2 \le x \le 0$ or $x \ge 1$

9f $f(x) = x^3 + 4x^2 + 4x = x(x+2)^2$



$f(x) \le 0$ for $x \le 0$

10a The curve is defined for all x such that the denominator is non-zero. That is all x such that $(x-2)(x+2) \ne 0$ which is all x such that $x \ne -2, 2$.

10b The y-intercept occurs when $x = 0$.

$$y = \frac{4}{(0+2)(2-0)} = \frac{4}{4} = 1$$

y-intercept is at $(0, 1)$.

10c $\lim_{x \to \pm\infty} y$

$$= \lim_{x \to \pm\infty} \frac{4}{(x+2)(2-x)}$$

$$= \lim_{x \to \pm\infty} \frac{4}{4 - x^2}$$

$$= \lim_{x \to \pm\infty} \frac{\frac{4}{x^2}}{\frac{4}{x^2} - 1}$$

$$= \frac{0}{0-1}$$

$$= 0$$

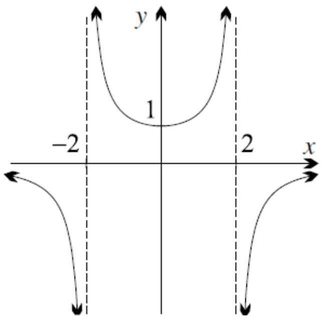
Hence $y = 0$ is a horizontal asymptote.

10d

x	-3	-2	0	2	3
y	-5	*	1	*	-5
sign	$-$	*	$+$	*	$-$

10e Vertical asymptotes occur when this curve is undefined. This is at $x = 2$ and $x = -2$.

10f



By observation, the range is $(-\infty, 0) \cup [1, \infty)$

11a $y = \frac{3(x+1)}{(x+3)(x-1)}$

11b domain: $x \neq 1$ and $x \neq -3$

The x-intercept occurs when $y = 0$.

$3(x + 1) = 0$

$x + 1 = 0$

$x = -1$

x-intercept is at $(-1, 0)$.

The y-intercept occurs when $x = 0$.

$$y = \frac{3(1)}{(0 + 3)(0 - 1)} = \frac{3}{-3} = -1$$

y-intercept is at $(0, -1)$.

So the intercepts are $(-1, 0)$ and $(0, -1)$.

11c The domain is not symmetric around $x = 0$.

11d Vertical asymptotes occur when $(x + 3)(x - 1) = 0$.

So the vertical asymptotes are $x = -3$ and $x = 1$.

$\lim_{x \to \pm\infty} y$

$$= \lim_{x \to \pm\infty} \frac{3x + 3}{x^2 + 2x - 3}$$

$$= \lim_{x \to \pm\infty} \frac{\frac{3}{x} + \frac{3}{x^2}}{1 + \frac{2}{x} - \frac{3}{x^2}}$$

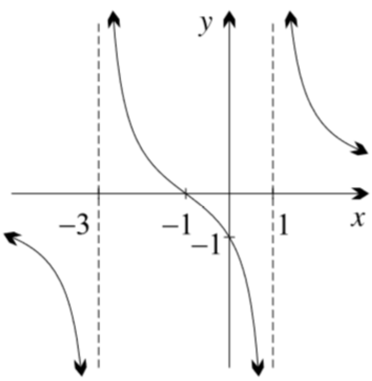
$$= \frac{0 + 0}{1 + 0 - 0}$$

$$= 0$$

Hence $y = 0$ is a horizontal asymptote.

Asymptotes are $x = -3$, $x = 1$, and $y = 0$.

11e



12a $|2x| = 7$

$2x = \pm 7$

$x = \pm \frac{7}{2}$

$x = 3\frac{1}{2}$ or $x = -3\frac{1}{2}$.

12b $|3x - 2| = 1$

$3x - 2 = \pm 1$

$3x = 2 \pm 1$

$x = \frac{1}{3}(2 \pm 1) = \frac{2}{3} \pm \frac{1}{3}$

$x = 1$ or $x = \frac{1}{3}$.

12c $|3x + 5| \leq 4$

$-4 \leq 3x + 5 \leq 4$

$-9 \leq 3x \leq 1$

$-3 \leq x \leq \dfrac{1}{3}$

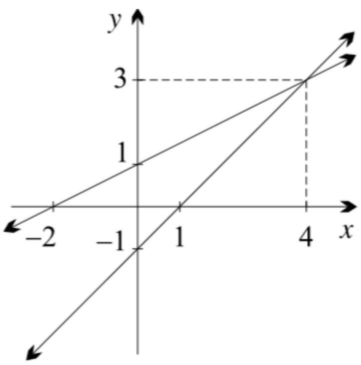
12d $|6x + 7| > 5$

$6x + 7 > 5$ or $6x + 7 < -5$

$6x > -2$ or $6x < -12$

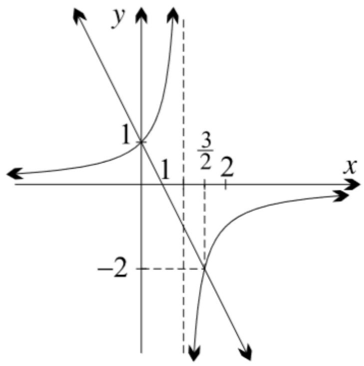
$x > -\frac{1}{3}$ or $x < -2$

13a Sketch showing $y = x - 1$ and $y = 1 + \frac{1}{2}x$:



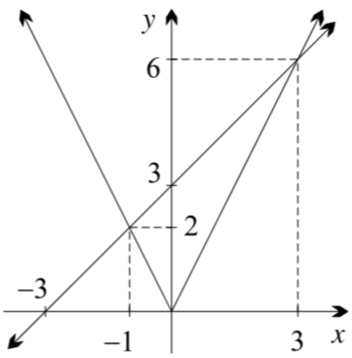
$x - 1 \geq 1 + \dfrac{1}{2}x$ when $x \geq 4$

13b Sketch showing $y = \frac{1}{1-x}$ and $y = 1 - 2x$:



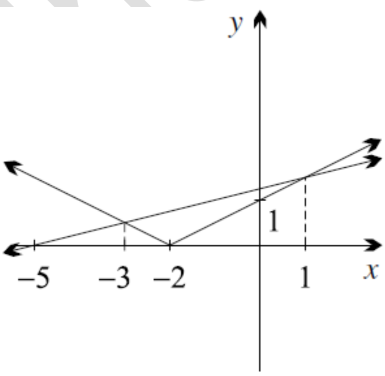
$\frac{1}{1-x} > y = 1 - 2x$ when $0 < x < 1$ or $x > 1\frac{1}{2}$

13c Sketch showing $y = |2x|$ and $y = x + 3$:



$|2x| \le y = x + 3$ when $-1 \le x \le 3$

13d Sketch showing $y = \left|\frac{1}{2}x + 1\right|$ and $y = \frac{1}{4}(x + 5)$:



$\left|\frac{1}{2}x + 1\right| > y = \frac{1}{4}(x + 5)$ when $x < -3$ or $x > 1$

14a We find the point of intersections by solving the equation

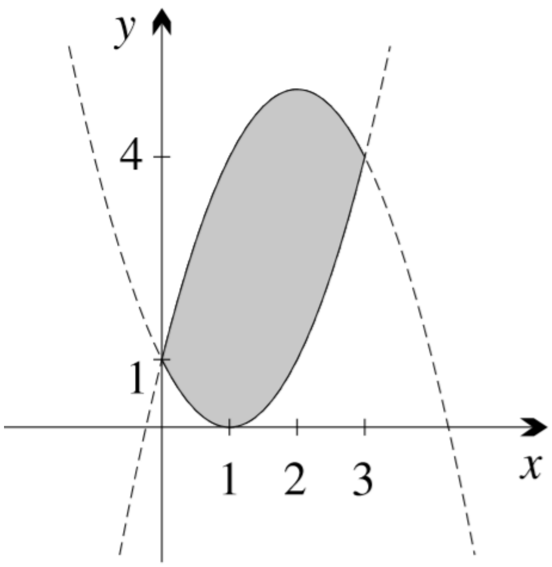
$x^2 - 2x + 1 = 1 + 4x - x^2$

$2x^2 - 6x = 0$

$2x(x - 3) = 0$

Hence the solutions are $x = 0$ and $x = 3$. Substituting this back into the equations gives the points of intersection as $(0, 1)$ and $(3, 4)$.

14b



15a Shift right 2 units:

$y = (x - 2)^2$

Then shift up 1 unit:

$y = (x - 2)^2 + 1$

15b Shift left 2 units:

$$y = \frac{1}{x + 2}$$

Then shift down 3 units:

$$y = \frac{1}{x+2} - 3$$

15c Shift left $\frac{\pi}{6}$ units:

$$y = \sin\left(x + \frac{\pi}{6}\right)$$

Then shift down 1 unit:

$$y = \sin\left(x + \frac{\pi}{6}\right) - 1$$

15d Shift right 2 units:

$$y = e^{x-2}$$

Then shift up 1 unit:

$$y = e^{x-2} + 1$$

16a Dilate horizontally by a factor of 2:

$$y = \left(\frac{x}{2}\right)^2 - 2\left(\frac{x}{2}\right)$$

$$= \frac{1}{4}x^2 - x$$

16b Dilate vertically by a factor of $\frac{1}{2}$:

$$2y = \frac{1}{x-4}$$

$$y = \frac{1}{2x-8}$$

16c Dilate vertically by a factor of $\frac{1}{3}$:

$$3y = \cos x$$

$$y = \frac{1}{3}\cos x$$

16d Dilate horizontally by a factor of 2:

$$y = \frac{1}{\frac{x}{2} + 2}$$

$$= \frac{2}{x + 2}$$

17a Yes, consider $y = f(x)$.

Perform a reflection in the y-axis:

$y = f(-x)$

Then perform a reflection in the x-axis:

$y = -f(-x)$

Now in the other order.

Perform a reflection in the x-axis:

$y = -f(x)$

Then perform a reflection in the y-axis:

$y = -f(-x)$

Both orders produce the same result so the operations commute.

17b No, consider $y = x$.

Perform a vertical reflection:

$y = -x$

Then translate 2 units up:

$y = -x + 2$

Now in the other order.

Translate 2 units up:

$y = x + 2$

Then perform a vertical reflection:

$-y = x + 2$

$y = -x - 2$

Hence we see that the operations do not commute.

17c No, consider $y = x$.

Translate 2 units left:

$y = x + 2$

Then dilate horizontally by a factor of $\frac{1}{2}$:

$y = 2x + 2$

Now in the other order.

Dilate horizontally by a factor of $\frac{1}{2}$:

$y = 2x$

Then translate 2 units left:

$y = 2(x + 2)$

$y = 2x + 4$

Hence we see that the operations do not commute.

17d Yes, consider the function $y = f(x)$.

Perform a vertical translation of a units:

$y = f(x) + a$

Then dilate horizontally by a factor of b:

$y = f\left(\frac{x}{b}\right) + a$

Now in the other order.

Dilate horizontally by a factor of b:

$y = f\left(\frac{x}{b}\right)$

Then perform a vertical translation of a units:

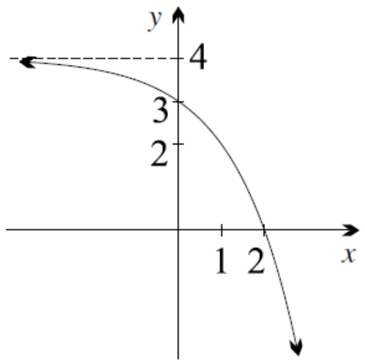
$y = f\left(\frac{x}{b}\right) + a$

Hence both orders produce the same result so they commute.

18a Start with $y = 2^x$.

Reflect in the y-axis: $y = -2^x$

Then shift up 4 units: $y = -2^x + 4 = 4 - 2^x$

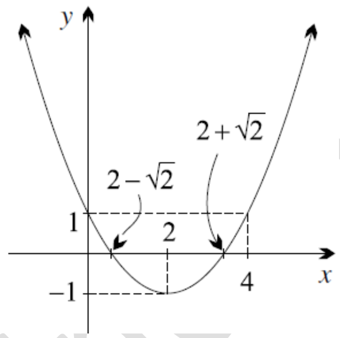


18b Start with $y = x^2$.

Shift right 2 units: $y = (x - 2)^2$

Then dilate vertically by a factor of $\frac{1}{2}$: $y = \frac{1}{2}(x - 2)^2$

Then shift 1 unit down: $y = \frac{1}{2}(x - 2)^2 - 1$

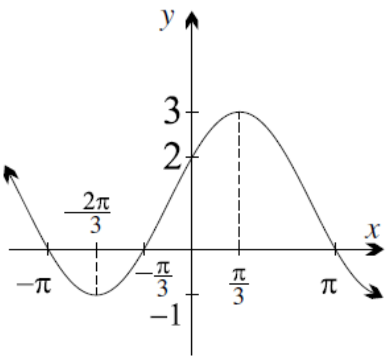


18c Start with $y = \sin x$.

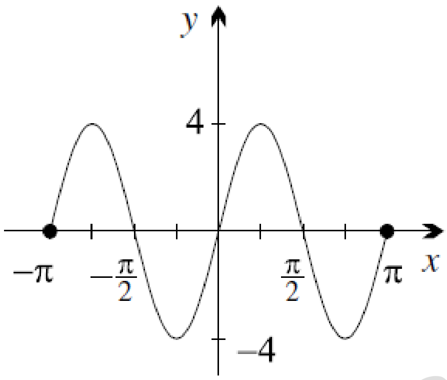
Shift $\frac{\pi}{6}$ units left: $y = \sin\left(x + \frac{\pi}{6}\right)$

Then dilate vertically by a factor of 2: $y = 2\sin\left(x + \frac{\pi}{6}\right)$

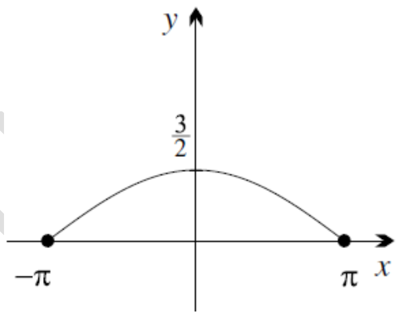
Then shift 1 unit up: $y = 2\sin\left(x + \frac{\pi}{6}\right) + 1$



19a amplitude is 4, period is $\frac{2\pi}{2}$ or π



19b amplitude is $\frac{3}{2}$, period is $\frac{2\pi}{\frac{1}{2}}$ or 4π

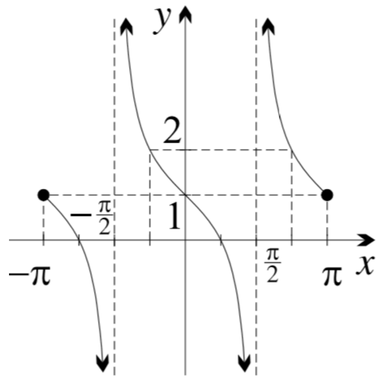


20a Start with $y = \tan x$.

Reflect in the x-axis: $y = -\tan x$

Then shift up 1 unit: $y = -\tan x + 1 = 1 - \tan x$

20b



21a Start with $y = \cos x$.

Reflect in the y-axis: $y = \cos(-x)$

Then stretch vertically with a factor of 3: $y = 3\cos(-x)$

Then shift down 2 units: $y = 3\cos(-x) - 2$

Actually, the first transformation, reflect in the y-axis, is unnecessary because $y = \cos x$ is even.

21b Start with $y = \cos x$.

Stretch horizontally with a factor of $\frac{1}{4}$: $y = \cos 4x$

Then dilate vertically by a factor of 4: $y = 4\cos 4x$

Then shift left $\frac{\pi}{2}$ units: $y = 4\cos\left(4\left(x + \frac{\pi}{2}\right)\right)$

There is no need to shift left $\frac{\pi}{2}$ units because the period is 2.

21c Start with $y = \cos x$.

Stretch horizontally with a factor of $\frac{1}{2}$: $y = \cos 2x$

Then shift right $\frac{\pi}{6}$ units: $y = \cos\left(2\left(x - \frac{\pi}{6}\right)\right) = \cos\left(2x - \frac{\pi}{3}\right)$

22a 0

22b $4\left(0+\frac{\pi}{2}\right)=2\pi$ or more simply 0.

22c $0-\frac{\pi}{3}=-\frac{\pi}{3}$

23a 3

23b 3 solutions as there are 3 points of intersection. 1 positive solution as one solution lies above the x-axis.

23c The range of the sine curve is $-1 \leq y \leq 1$. Outside the the domain, the line is beyond this range and thus there are no solutions.