Solutions to Exercise 9A

1a $x = t^2 - 4$ then,

$x = (0)^2 - 4 = -4$ when $t = 0$

$x = (1)^2 - 4 = -3$ when $t = 1$

$x = (2)^2 - 4 = 0$ when $t = 2$

$x = (3)^2 - 4 = 5$ when $t = 3$

Hence,

t	0	1	2	3
x	-4	-3	0	5

1b i Average velocity during the first second

$= \dfrac{x_1 - x_0}{1 - 0}$

$= \dfrac{(-3) - (-4)}{1}$

$= 1$ m/s

1b ii Average velocity during the first two seconds

$= \dfrac{x_2 - x_0}{2 - 0}$

$= \dfrac{(0) - (-4)}{2}$

$= 2$ m/s

1b iii Average velocity during the first three seconds

$= \dfrac{x_3 - x_0}{3 - 0}$

$= \dfrac{(5) - (-4)}{3}$

$= 3$ m/s

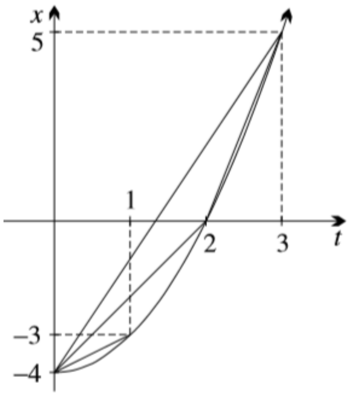
1b iv Average velocity during the third second

$= \dfrac{x_3 - x_2}{3 - 2}$

$= \dfrac{(5) - (0)}{1}$

$= 5 \text{ m/s}$

1c



2a $x = 4t - t^2$ then,

$x = 4 \times (0) - (0)^2 = 0$ when $t = 0$

$x = 4 \times (1) - (1)^2 = 3$ when $t = 1$

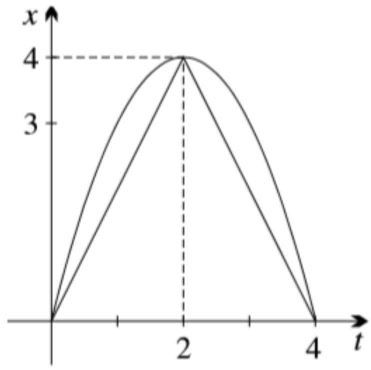
$x = 4 \times (2) - (2)^2 = 4$ when $t = 2$

$x = 4 \times (3) - (3)^2 = 3$ when $t = 3$

$x = 4 \times (4) - (4)^2 = 0$ when $t = 4$

t	0	1	2	3	4
x	0	3	4	3	0

2b



2c Total distance travelled is $4 + 4 = 8$ metres (4 metres when ascending, 4 metres when descending).

Average speed

$$= \frac{\text{total distance travelled}}{\text{time taken}}$$

$$= \frac{8}{4}$$

$$= 2 \text{ m/s}$$

2d i Average velocity

$$= \frac{x_2 - x_0}{2 - 0}$$

$$= \frac{4 - 0}{2}$$

$$= 2 \text{ m/s}$$

2d ii Average velocity

$$= \frac{x_4 - x_2}{4 - 2}$$

$$= \frac{0 - 4}{2}$$

$$= -2 \text{ m/s}$$

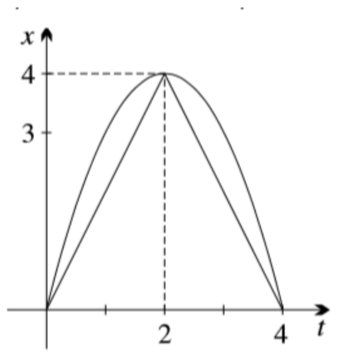
2d iii Average velocity

$$= \frac{x_4 - x_0}{4 - 0}$$

$$= \frac{0 - 0}{4}$$

$$= 0 \text{ m/s}$$

2e



3a

t	0	4	8	12
x	0	120	72	0

3b 120 metres when ascending and 120 metres when descending, so the total distance travelled by the cardboard is 240 metres.

3c Average speed

$$= \frac{\text{total distance travelled}}{\text{time taken}}$$

$$= \frac{240}{12}$$

$$= 20 \text{ m/s}$$

3d i Average velocity

$$= \frac{x_4 - x_0}{4 - 0}$$

$$= \frac{120 - 0}{4}$$

$$= 30 \text{ m/s}$$

3d ii Average velocity

$$= \frac{x_{12} - x_4}{12 - 4}$$

$$= \frac{0 - 120}{8}$$

$$= -15 \text{ m/s}$$

3d iii Average velocity

$$= \frac{x_{12} - x_0}{12 - 0}$$

$$= \frac{0 - 0}{12}$$

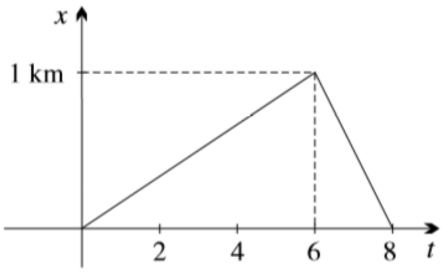
$$= 0 \text{ m/s}$$

4a i $V = \frac{x}{t}$ then $t = \frac{x}{V}$

Hence, $t_{up} = \frac{x_{up}}{V_{up}} = \frac{1}{10}$ hour $= 6$ minutes.

4a ii $t_{down} = \frac{x_{down}}{V_{down}} = \frac{1}{30}$ hour $= 2$ minutes.

4b



4c Average speed

$$= \frac{\text{total distance travelled (km)}}{\text{time taken (hours)}}$$

$$= \frac{2}{\frac{1}{10} + \frac{1}{30}}$$

$$= \frac{2}{\frac{4}{30}}$$

$$= 15 \text{ km/h}$$

4d uphill speed$= \frac{\text{total distance travelled (km)}}{\text{time taken (hours)}} = \frac{1}{\frac{1}{10}} = 10 \text{ km/h}$

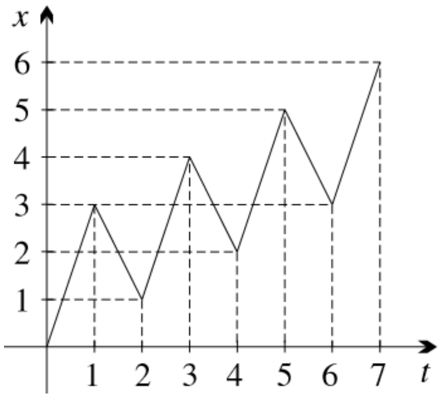
downhill speed$= \frac{\text{total distance travelled (km)}}{\text{time taken (hours)}} = \frac{1}{\frac{1}{30}} = 30 \text{ km/h}$

average of up and downhill speed$= \frac{10+30}{2} = 20 \text{ km/h}$

5a

t	0	1	2	3	4	5	6	7
x	0	3	1	4	2	5	3	6

5b



5c 7 hours

5d Total distance $= 3 + 2 + 3 + 2 + 3 + 2 + 3 = 18$ metres

Average speed $= \frac{18}{7} = 2\frac{4}{7}$ m/hr

5e Average velocity $= \frac{x_{final} - x_{initial}}{t_{final} - t_{initial}} = \frac{6-0}{7-0} = \frac{6}{7}$ m/hr

5f Those between 1 and 2 metres high or between 4 and 5 metres high (Drawing horizontal lines and observing how many times the horizontal line cuts the graph may help.)

6a Given that $x = 2\sqrt{t}$

If $x = 0$ then $0 = 2\sqrt{t}$ and $t = 0$

If $x = 2$ then $2 = 2\sqrt{t}$ and $t = 1$

If $x = 4$ then $4 = 2\sqrt{t}$ and $t = 4$

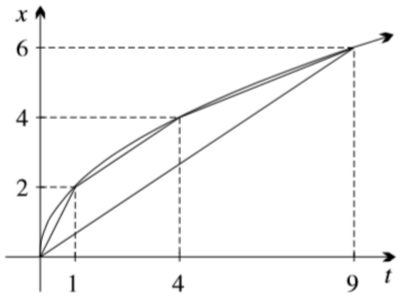
If $x = 6$ then $6 = 2\sqrt{t}$ and $t = 9$

If $x = 8$ then $8 = 2\sqrt{t}$ and $t = 16$

Hence,

t	0	1	4	9	16
x	0	2	4	6	8

Therefore,



6b i Average velocity

$$= \frac{x_1 - x_0}{1 - 0}$$

$$= \frac{2 - 0}{1}$$

$$= 2 \text{ cm/s}$$

6b ii Average velocity

$$= \frac{x_2 - x_1}{4 - 1}$$

$$= \frac{4 - 2}{3}$$

$$= \frac{2}{3} \text{ cm/s}$$

6b iii Average velocity

$$= \frac{x_3 - x_2}{9 - 4}$$

$$= \frac{6 - 4}{5}$$

$= \frac{2}{5}$ cm/s

6b iv Average velocity

$= \frac{x_3 - x_0}{9 - 0}$

$= \frac{6 - 0}{9}$

$= \frac{2}{3}$ cm/s

6c The chords are parallel.

7a i Average velocity

$= \frac{x_8 - x_0}{8 - 0}$

$= \frac{0 - 8}{8}$

$= -1$ m/s

7a ii Average velocity

$= \frac{x_{17} - x_{12}}{17 - 12}$

$= \frac{20 - 0}{5}$

$= 4$ m/s

7a iii Average velocity

$= \frac{x_{30} - x_{24}}{30 - 24}$

$= \frac{8 - 20}{6}$

$= -2$ m/s

7b The total distance travelled $= 8 + 20 + 12 = 40$ m

Average speed $= \dfrac{40}{30} = \dfrac{4}{3} = 1\dfrac{1}{3}$ m/s

7c Displacement is 0 metres. Hence, the average velocity is 0 m/s.

7d The total time she paused is: $4 + 7 = 11$ s

Average speed $= \dfrac{40}{19} = 2\dfrac{2}{19}$ m/s

8a i The weight is 3 metres above the surface of the water, once.

8a ii The weight is 1 metre above the surface of the water, three times.

8a iii The weight is $\frac{1}{2}$ metre below the surface of the water, twice.

8b i The weight is at the water surface when $t = 4$ seconds and $t = 14$ seconds.

8b ii The weight is above the water surface when $0 \le t < 4$ seconds and $4 < t < 14$ seconds.

8c The weight touches the water at $t = 4$ seconds and after that first touch, it rises 2 Metres, when $t = 8$ seconds.

8d The greatest depth of the weight under the water surface is 1 metre at $t = 17$ seconds.

8e As $t \to \infty$, $x \to 0$, meaning that the weight eventually ends up at the surface.

8f i Average velocity

$$= \frac{x_4 - x_0}{4 - 0}$$

$$= \frac{0 - 4}{4}$$

$$= -1 \text{ m/s}$$

8f ii Average velocity

$$= \frac{x_8 - x_4}{8 - 4}$$

$$= \frac{2 - 0}{4}$$

$$= \tfrac{1}{2} \text{ m/s}$$

8f iii Average velocity

$$= \frac{x_{17} - x_8}{17 - 8}$$

$$= \frac{-1 - 2}{9}$$

$$= -\tfrac{1}{3} \text{ m/s}$$

8g i The weight travels 4 metres over the first 4 seconds.

8g ii The weight travels $4 + 2 = 6$ metres over the first 8 seconds.

8g iii The weight travels $4 + 2 + 2 + 1 = 9$ metres over the first 17 seconds.

8g iv The weight travels $4 + 2 + 2 + 1 + 1 \doteqdot 10$ metres eventually.

8h i Average speed

$= \dfrac{\text{total distance travelled}}{\text{time taken}}$

$= \dfrac{4\text{ m}}{4\text{ s}}$

$= 1\text{ m/s}$

8h ii Average speed

$= \dfrac{\text{total distance travelled}}{\text{time taken}}$

$= \dfrac{6\text{ m}}{8\text{ s}}$

$= \dfrac{3}{4}\text{ m/s}$

8h iii Average speed

$= \dfrac{\text{total distance travelled}}{\text{time taken}}$

$= \dfrac{9\text{ m}}{17\text{ s}}$

$= \dfrac{9}{17}\text{ m/s}$

9a $T = \frac{2\pi}{n}$ and $n = \frac{\pi}{8}$ then $T = \frac{2\pi}{\frac{\pi}{8}} = 16$ seconds.

9b The maximum value of displacement is $x = 3$ cm and the minimum value of displacement is $x = -3$ cm

9c $x = 3$ when $3\sin\left(\frac{\pi}{8}t\right) = 3$ or $\sin\left(\frac{\pi}{8}t\right) = 1$

Or $\frac{\pi}{8}t = \frac{\pi}{2} + 2m\pi$ where m is a natural number.

Hence, $t = \frac{\frac{\pi}{2}+2m\pi}{\frac{\pi}{8}} = 4 + 16m$

Therefore, the displacement reaches its maximum value for the first time when $t = 4$ seconds when $m = 0$ and for the second time when $t = 20$ seconds when $m = 1$.

9d $x = 0$ when $3\sin\left(\frac{\pi}{8}t\right) = 0$ or $\sin\left(\frac{\pi}{8}t\right) = 0$

$\frac{\pi}{8}t = 0 + 2m\pi$ or $\frac{\pi}{8}t = \pi + 2m\pi$, where m is a natural number

Hence, $t = \frac{2m\pi}{\frac{\pi}{8}} = 16m$ or $t = \frac{\pi+2m\pi}{\frac{\pi}{8}} = 8 + 16m$

Therefore, the particle returns its initial position for the first time when

$t = 8$ seconds when $m = 0$

and for the second time when $t = 16$ seconds when $m = 1$

9e Between the 8th and 16th seconds, the particle is travelling in the negative direction. Therefore, the answer is $8 < t < 16$.

9f The total distance travelled in the first 16 seconds is $3 + 3 + 3 + 3 = 12$ cm.

The average speed is $|V_{ave}| = \frac{\text{distance travelled}}{\text{time taken}} = \frac{12}{16} = 0.75$ cm/s

10a The amplitude is 4 metres and the period, T, is: $T = \frac{2\pi}{n}$ and $n = \frac{\pi}{6}$

Then $T = \frac{2\pi}{\frac{\pi}{6}} = 12$ seconds.

10b The particle is at $x = 0$ when $t = 6, 12, 18, 24, 30, 36, 42, 48, 54, 60$.

Therefore, the particle is at its initial position ten times in the first minute.

10c $x = 4$ when $4\sin\left(\frac{\pi}{6}t\right) = 4$ or $\sin\left(\frac{\pi}{6}t\right) = 1$ or

$\frac{\pi}{6}t = \frac{\pi}{2} + 2m\pi$ where m is a natural number.

$t = \frac{\frac{\pi}{2}+2m\pi}{\frac{\pi}{6}} = 3 + 12m$. Hence,

$t = 3$ when $m = 0$

$t = 15$ when $m = 1$

$t = 27$ when $m = 2$

$t = 39$ when $m = 3$

$t = 51$ when $m = 4$

Therefore, the particle visits $x = 4$ metres when $t = 3, 15, 27, 39$ and 51 seconds.

10d The particle is at the origin when $t = 12$ seconds.

However, it has travelled $4 + 4 + 4 + 4 = 16$ metres in the first 12 seconds.

The average speed is $|V_{ave}| = \frac{\text{distance travelled}}{\text{time taken}} = \frac{16}{12} = 1\frac{1}{3}$ cm/s

10e $x = 4\sin\left(\frac{\pi}{6} \times (0)\right) = 0$ when $t = 0$

$x = 4\sin\left(\frac{\pi}{6} \times (1)\right) = 2$ when $t = 1$

$x = 4\sin\left(\frac{\pi}{6} \times (3)\right) = 4$ when $t = 3$

The average speed in the first second is $|V_{ave}| = \frac{\text{distance travelled}}{\text{time taken}} = \frac{2-0}{2-1} = 2$ cm/s

The average speed is $|V_{ave}| = \frac{\text{distance travelled}}{\text{time taken}} = \frac{4-2}{3-1} = 1$ cm/s

Therefore, the average speed in the first second is twice the average speed in the following 2 seconds.

11a $x = 10\cos\left(\frac{\pi}{12}t\right)$ then the amplitude is 10 metres because the range of the function is $[-10, 10]$

$T = \frac{2\pi}{n}$ then $T = \frac{2\pi}{\frac{\pi}{12}} = 24$ seconds.

11b $x = 10\cos\left(\frac{\pi}{12}t\right) = 0$ when $\frac{\pi}{12}t = \frac{\pi}{2} + 2\pi$ or $\frac{\pi}{12}t = \frac{3\pi}{2} + 2\pi$

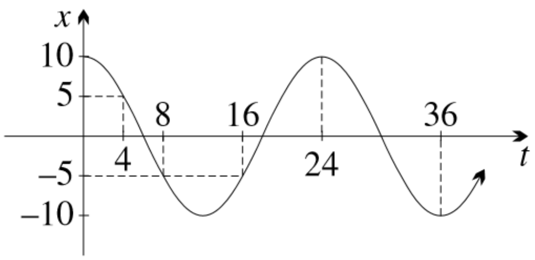
Or when $t = 6, 18$ or 30

$x = 10\cos\left(\frac{\pi}{12}t\right) = 10$ when $\frac{\pi}{12}t = 0 + 2\pi$

Or when $t = 0$ or 24

$x = 10\cos\left(\frac{\pi}{12}t\right) = -10$ when $\frac{\pi}{12}t = \pi + 2\pi$

Or when $t = 12$ or 36



11c The particle is at the origin when

$x = 10\cos\left(\frac{\pi}{12}t\right) = 0$ when $\frac{\pi}{12}t = \frac{\pi}{2} + 2\pi$ or $\frac{\pi}{12}t = \frac{3\pi}{2} + 2\pi$

Or when $t = 6, 18$ or 30 seconds

11d When $t = 0$, the particle is $x = 10\cos\left(\frac{\pi}{12} \times (0)\right) = 10$ metres away from the origin.

Since 10 metres is the amplitude, and the particle starts its motion at $(0, 10)$ the maximum distance this particle travels is 20metres.

The particle reaches this maximum distance twice in 36 minutes as shown below:

$x = 10\cos\left(\frac{\pi}{12}t\right) = -10$ when $\frac{\pi}{12}t = \pi + 2\pi$

Or when $t = 12$ or 36 seconds.

11e As it can be observed from the graph, the particle travels 60 metres in 36 seconds. The average speed in this time interval is:

Average speed

$$= \frac{\text{total distance travelled}}{\text{time taken}}$$

$$= \frac{60\text{ m}}{36\text{ s}}$$

$$= 1\frac{2}{3}\text{ m/s}$$

11f $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$ then

$$x = 10\cos\left(\frac{\pi}{12} \times 4\right) = 10 \times \frac{1}{2} = 5$$

$$x = 10\cos\left(\frac{\pi}{12} \times 8\right) = 10 \times -\frac{1}{2} = -5$$

$$x = 10\cos\left(\frac{\pi}{12} \times 12\right) = 10 \times -1 = -10$$

$$x = 10\cos\left(\frac{\pi}{12} \times 16\right) = 10 \times -\frac{1}{2} = -5$$

$$x = 10\cos\left(\frac{\pi}{12} \times 20\right) = 10 \times \frac{1}{2} = 5$$

$$x = 10\cos\left(\frac{\pi}{12} \times 24\right) = 10 \times 1 = 10$$

t	4	8	12	16	20	24
x	5	-5	-10	-5	5	10

11g i Average velocity

$$= \frac{x_4 - x_0}{4 - 0}$$

$$= \frac{5 - 10}{4}$$

$$= -1\frac{1}{4}\text{ m/s}$$

11g ii Average velocity

$$= \frac{x_8 - x_4}{8 - 4}$$

$$= \frac{-5 - 5}{4}$$

$$= -\frac{10}{4}$$

$$= -2\frac{1}{2} \text{ m/s}$$

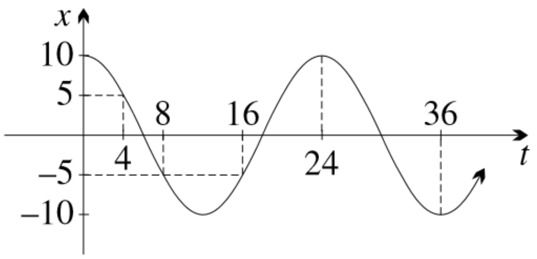
11g iii Average velocity

$$= \frac{x_{12} - x_8}{12 - 8}$$

$$= \frac{-10 - (-5)}{4}$$

$$= -1\frac{1}{4} \text{ m/s}$$

11h



From the graph, it can be observed that the particle is more than 15 metres from its initial position when $x < -5$ or when $8 < t < 16$ in the first 24 seconds.

12a $h = 8000\left(1 - e^{-0.06\times(0)}\right) = 0$ when $t = 0$ and

$h \to 8000$ as $t \to \infty$, because $e^{-0.06\times t}$ converges to 0 as t approaches to infinity.

Hence, $h = 8000(1 - 0) = 8000$

12b $h = 8000\left(1 - e^{-0.06\times(0)}\right) = 0$ when $t = 0$

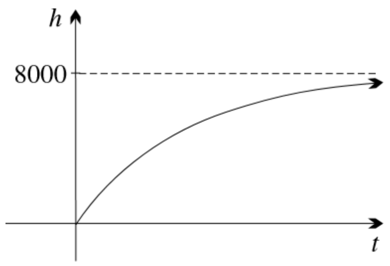
$h = 8000\left(1 - e^{-0.06\times(10)}\right) \doteqdot 3609$ when $t = 10$

$h = 8000\left(1 - e^{-0.06\times(20)}\right) \doteqdot 5590$ when $t = 20$

$h = 8000\left(1 - e^{-0.06\times(30)}\right) \doteqdot 6678$ when $t = 30$

t	0	10	20	30
h	0	3609	5590	6678

12c Since $h \to 8000$ as $t \to \infty$, there is a horizontal asymptote at $y = 8000$ metres and the function is increasing for all $t \geq 0$ starting from $(0, 0)$.



12d During the first ten minutes,

$|V_{ave}| = \frac{\text{distance travelled}}{\text{time taken}} = \frac{3609.1-0}{10-0} \doteqdot 361$ m/s

During the second ten minutes,

$|V_{ave}| = \frac{\text{distance travelled}}{\text{time taken}} = \frac{5590.45-3609.1}{20-10} \doteqdot 198$ m/s

During the third ten minutes,

$|V_{ave}| = \frac{\text{distance travelled}}{\text{time taken}} = \frac{6677.61-5590.45}{30-20} \doteqdot 109$ m/s

12e $h = 8000\left(1 - e^{-0.06\times(76)}\right) \doteqdot 7916.3$ when $t = 76$

$\frac{7916.3}{8000} \times 100\% = 98.9538$

Therefore, the balloon has not reached 99% of its final height when $t = 76$ min.

$h = 8000\left(1 - e^{-0.06\times(77)}\right) \doteqdot 7921.18$ when $t = 77$

$\frac{7921.18}{8000} \times 100\% = 99.014$

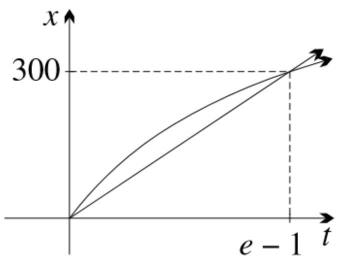
Therefore, the balloon has reached 99% of its final height when $t = 77$ min.

13a $x = kt$ is a straight line and equals zero when $t = 0$.

$x = 300\log_e(t+1)$ is an increasing function,

$x = 300\log_e(0+1) = 300 \times 0 = 0$ when $t = 0$ and

they intersect at $(e-1, 300)$.



13b If $(e-1, 300)$ is on $x = kt$ then $300 = k \times (e-1)$. Therefore, $k = \frac{300}{e-1}$

13c The distance (D) between Thomas and Henry is $D = 300\log_e(t+1) - \frac{300}{e-1}t$

The maximum distance is when $D' = 0$.

$D' = \frac{300}{t+1} - \frac{300}{e-1} = 0$ or when $t = e - 2 \doteqdot 0.718$ minutes $\doteqdot 43$ seconds

Therefore, the maximum distance is

$D = 300\log_e\big((e-2)+1\big) - \frac{300}{e-1} \times (e-2) \doteqdot 37$ metres.

14a Let t_1 be the time taken to travel from A to B (the distance x) and

t_2 be the time taken to travel from B to C (the distance x).

$W = \frac{2x}{t_1+t_2} = \frac{2x}{\frac{x}{U}+\frac{x}{V}} = \frac{2x}{\frac{x(U+V)}{UV}} = \frac{2UV}{U+V}$

Since $\frac{U+V}{2UV} = \frac{1}{2V} + \frac{1}{2U} = \frac{1}{2}\left(\frac{1}{V} + \frac{1}{U}\right)$, $\frac{U+V}{2UV}$ is the arithmetic mean of $\frac{1}{U}$ and $\frac{1}{V}$.

Therefore, W is the harmonic mean of U and V.

14b i Let t_1 be the time taken to travel from A to B (the distance x_1) and

t_2 be the time taken to travel from B to C (the distance x_2).

If $W = \frac{U+V}{2}$ then $t_1 = t_2$ because the average velocity, $W = \frac{U+V}{2}$ is equal to

$W = \frac{x_1+x_2}{t_1+t_2} = \frac{Ut_1+Vt_2}{t_1+t_2}$ only when $t_1 = t_2$.

Hence, $\frac{x_1}{x_2} = \frac{Ut_1}{Vt_2} = \frac{Ut}{Vt} = \frac{U}{V}$

14b ii Let t_1 be the time taken to travel from A to B (the distance x_1) and

t_2 be the time taken to travel from B to C (the distance x_2).

If $W = \sqrt{UV}$ then $U = V$ because $W = \sqrt{UV} = \frac{x_1+x_2}{t_1+t_2} = \frac{Ut_1+Vt_2}{t_1+t_2}$ only when $U = V$.

Hence, when $W = \sqrt{UV}$, $U = V$ and

$x_1 t_1 = x_2 t_2$

$$\frac{x_1}{x_2} = \frac{t_2}{t_1}$$

$$\frac{{x_1}^2}{{x_2}^2} = \frac{x_1 \times t_2}{x_2 \times t_1}$$

$$\frac{x_1}{x_2} = \sqrt{\frac{x_1 \times t_2}{x_2 \times t_1}}$$

$$\frac{x_1}{x_2} = \frac{\sqrt{\frac{x_1}{t_1}}}{\sqrt{\frac{x_2}{t_2}}}$$

$$\frac{x_1}{x_2} = \frac{\sqrt{U}}{\sqrt{V}}$$

Therefore, $x_1 : x_2 = \sqrt{U} : \sqrt{V}$

Solutions to Exercise 9B

1a If $x = 20 - t^2$ then $v = -2t$

1b If $v = -2t$ then $a = -2$

1c When $t = 3$,

$x = 20 - (3)^2 = 11$ m

$v = -2 \times (3) = -6$ m/s

$a = -2$ m/s^2

1d The distance from the origin is $x = 20 - (3)^2 = 11$ metres and its speed is $v = -2 \times (3) = -6$. Therefore, the speed of the particle is 6 metres per second.

2a $x = t^2 - 10t$ then $v = 2t - 10$

2b When $t = 3$, $x = (3)^2 - 10 \times (3) = -21$.

Therefore, the displacement is -21 cm and the distance from the origin is 21 cm.

$v = 2 \times (3) - 10 = -4$.

Therefore, the velocity is -4 cm/s and the speed is 4 cm/s.

2c The particle is stationary when its velocity is zero. Therefore,

$v = 2t - 10 = 0$ when $t = 5$ seconds.

When $t = 5$ seconds, the particle is at $x = (5)^2 - 10 \times (5) = -25$.

3a $x = t^3 - 6t^2$ then $v = 3t^2 - 12t$

and $a = 6t - 12$

3b When $t = 0, x = (0)^3 - 6(0)^2 = 0$, the particle is at the origin.

$|v| = 3(0)^2 - 12 \times (0) = 0$ cm/s

$a = 6 \times (0) - 12 = -12$ cm/s^2

3c When $t = 3, x = (3)^3 - 6(3)^2 = -27$. Therefore, the particle is on the left of the origin.

3d When $t = 3, v = 3(3)^2 - 12 \times (3) = -9$ cm/s. Therefore, the particle is travelling to the left.

3e When $t = 3, a = 6 \times (3) - 12 = 6$ cm/s^2. Therefore, the particle is accelerating to the right.

3f $v = 3(4)^2 - 12 \times (4) = 0$ cm/s when $t = 4$. Therefore, the particle is stationary when $t = 4$ seconds. When $t = 4, x = (4)^3 - 6(4)^2 = -32$ cm

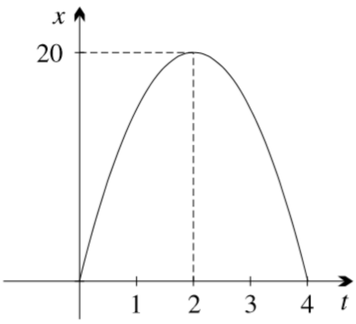
3g $x = t^3 - 6t^2 = t^2(t - 6) = 0$ when $t = 0$ or $t = 6$. Therefore, the particle is at the origin when $t = 6$ seconds and v$= 3(6)^2 - 12 \times (6) = 36$ cm/s.

Hence, $|v| = 36$ cm/s.

4a $x = 20t - 5t^2$ then $v = 20 - 10t$ and $a = -10$ m/s^2

Since $a < 0$ for all $t \geq 0$, the ball is always accelerating downwards.

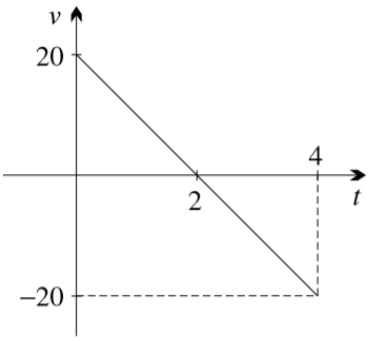
Displacement function:



$x = 0$ when $t = 0$ and $t = 4$. Therefore, $(0, 0)$ and $(4, 0)$ are the x-intercepts.

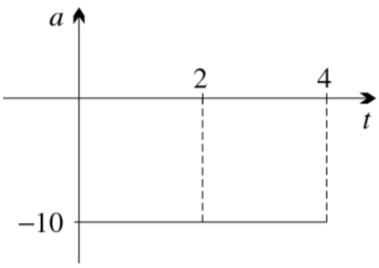
x has a turning point at $t = 2$ because it is the x-coordinate of the axis of symmetry and $x = 20$ at $t = 2$.

Velocity function:



x-intercept is $(2, 0)$ because the velocity is zero at $t = 2$ and it is the turning point of the graph of displacement function. y-intercept: $(0, 20)$ and $v = -20$ when $t = 4$.

Acceleration function:



The acceleration is constant and equal to -10 for all t.

4b $|v| = |20 - 10 \times (0)| = 20$ m/s when the the ball was thrown (when $t = 0$).

4c $x = 20t - 5t^2 = 5t(4 - t) = 0$ when $t = 0$ or $t = 4$ seconds.

The speed of the ball at $t = 0$ (answer 7b)

and $t = 4$ is $|v| = |20 - 10 \times (4)| = 20$ m/s.

4d Maximum height is reached when $v = 0$ and $v = 20 - 10t = 0$ when $t = 2$.

Therefore, the maximum height is $x = 20 \times (2) - 5(2)^2 = 20$ metres after 2 seconds.

4e The acceleration at $t = 2$ is $a = -10$ m/s^2 and it exists because there exists gravitational acceleration and the particle's velocity is changing for all $t \geq 0$ even though $v = 0$ at $t = 2$.

5a $x = e^{-4t}$ then $\dot{x} = -4e^{-4t}$ and $\ddot{x} = 16e^{-4t}$.

None of the above functions can ever change sign, because $e^{-4t} > 0$ for all t.

$x > 0$ for all t.

$\dot{x} < 0$ for all t.

$\ddot{x} > 0$ for all t.

5b i $x = e^{-4(0)} = 1$ when $t = 0$. Therefore, the particle is initially at $x = 1$.

5b ii $x \to 0$ as $t \to \infty$. Therefore, the particle gets closer and closer to $x = 0$.

5c i $\dot{x} = -4e^{-4(0)} = -4$. Therefore, the velocity of the particle is initially -4.

$\ddot{x} = 16e^{-4(0)} = 16$. Therefore, the acceleration of the particle is initially 16.

5c ii $\dot{x} = -4e^{-4t} \rightarrow 0$ as $t \rightarrow \infty$.

Therefore, the velocity of the particle is going to get closer and closer to 0.

$\ddot{x} = 16e^{-4t} \rightarrow 0$ as $t \rightarrow \infty$.

Therefore, the acceleration of the particle is going to get closer and closer to 0.

6a $x = 2\sin(\pi t)$ then $= 2\pi\cos(\pi t)$, $a = -2\pi^2 \sin(\pi t)$

When $t = 1$,

$x = 2\sin\big(\pi \times (1)\big) = 0$. Hence, the particle is at the origin.

$v = 2\pi\cos\big(\pi \times (1)\big) = -2\pi$

$a = -2\pi^2 \sin\big(\pi \times (1)\big) = 0$

6b i When $= \frac{1}{3}$,

$v = 2\pi\cos\left(\pi \times \left(\frac{1}{3}\right)\right) = \pi$. Therefore, the particle is moving towards right.

6b ii When $= \frac{1}{3}$,

$a = -2\pi^2 \sin\left(\pi \times \left(\frac{1}{3}\right)\right) = -\sqrt{3}\,\pi^2$. Therefore, the particle is accelerating towards left.

7a $x = t^2 - 8t + 7$ then

$\dot{x} = 2t - 8 = 2(t - 4)$

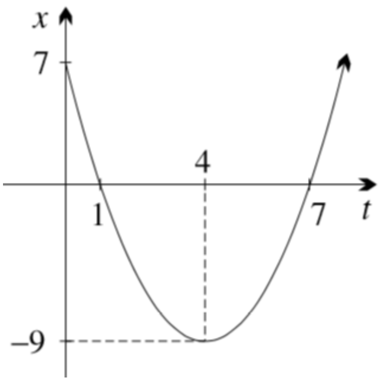
$\ddot{x} = 2$

7b $x = t^2 - 8t + 7 = (t - 7)(t - 1)$

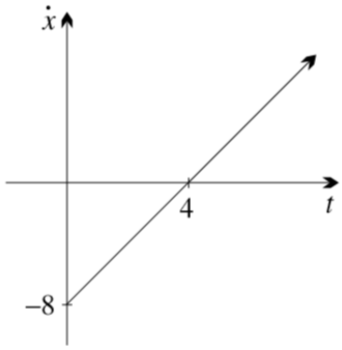
$(t - 7)(t - 1) = 0$ when $t = 1$ and $t = 7$. Therefore, $(1, 0)$ and $(7, 0)$ are the x-intercepts. The y-intercept is $(0, 7)$ (Substitute 0 in the function for t)

To find the x-coordinate of the turning point, solve $\dot{x} = 2(t - 4) = 0$.

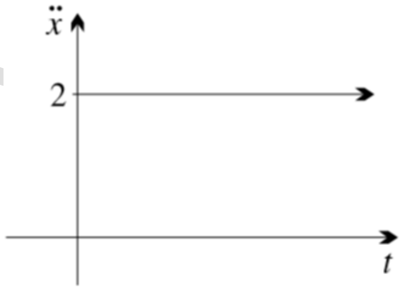
$2(t - 4) = 0$ when $t = 4$ and $x = (4)^2 - 8 \times (4) + 7 = -9$. Therefore, $(4, -9)$ is the turning point.



$\dot{x} = 2(t - 4) = 0$ when $t = 4$ and the y-intercept is $(0, -8)$ (Substitute 0 for t in the function and sketch the straight-line graph.



$\ddot{x} = 2$ is the graph of $y = 2$ which is a horizontal line.



None of these graphs are defined for $t < 0$ because the motion starts when $t = 0$.

7c i $x = (t-7)(t-1) = 0$ when $t = 1$ or $t = 7$. Therefore, the particle is at the origin when $t = 1$ and $t = 7$.

7c ii $\dot{x} = 2(t-4) = 0$ when $t = 4$. Therefore, the particle is stationary when $t = 4$.

7d i There is no turning point in the interval $[0, 2]$.

Thus, by substituting $t = 0$ and $t = 2$ we can find when the particle is further away from the origin.

$x = (0)^2 - 8 \times (0) + 7 = 7$ when $t = 0$ and $x = (2)^2 - 8 \times (2) + 7 = -5$ when $t = 2$. Therefore, the particle further away when $t = 0$.

7d ii Since the turning point is in the interval $[0, 6]$, and $x = (4)^2 - 8 \times (4) + 7 = 9$ the particle is furthest from the origin when $t = 4$.

7d iii In the interval $[0, 6]$, $x = (10)^2 - 8 \times (10) + 7 = 27$ metres. Therefore, the particle is furthest from the origin when $t = 10$.

7e Average velocity$= \frac{x_7 - x_0}{7-0} = \frac{0-7}{7-0} = -1$ m/s

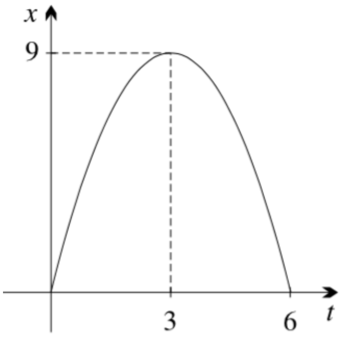
The instantaneous velocity is $\dot{x} = 2(t-4) = -1$ when $2t = 7$ or $t = 3.5$ seconds and the particle is $x = (3.5)^2 - 8 \times (3.5) + 7 = -8.75 = -8\frac{3}{4}$ metres away from the origin.

7f The first 7 minutes, particle moves 9 metres away.

The average speed is$= \frac{\text{distance travelled}}{\text{time taken}} = \frac{7+9+9}{7} = \frac{25}{7} = 3\frac{4}{7}$ m/s

8a $x = 6t - t^2$ then $v = 6 - 2t$ and $a = -2$

8b The graph of x is shown below:



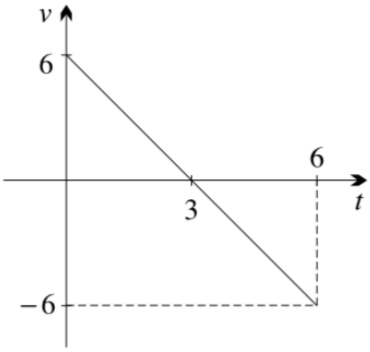
y-intercept: $(0, 0)$

x-intecepts: $(0, 0)$ and $(6, 0)$

and $v = 6 - 2t = 0$ when $t = 3$

Hence, $x = 6 \times (3) - (3)^2 = 9$ when x is maximum.

The graph of v is shown below:



and $v = 6 - 2t = 0$ when $t = 3$. Therefore, the graph cuts the x-axis at $x = 3$.

y-intercept is: $(0, 6)$ and since $x = 6t - t^2 = 0$ when both

$t = 0$ and $t = 6$ seconds, $(6, -6)$ is on the graph since $v = 6 - 2 \times (6) = -6$ m/s.

8c i When $t = 2$,

$v = 6 - 2 \times (2) = 2$ m/s. The ice is moving upwards.

and $a = -2$. The ice is accelerating downwards.

8c ii When $t = 4$,

$v = 6 - 2 \times (4) = -2$ m/s. The ice is moving downwards.

and $a = -2$. The ice is accelerating downwards.

8d $v = 6 - 2t = 0$ when $t = 3$. Therefore, the ice is stationary at the end of third second, for an instant. Since $x = 6(3) - (3)^2 = 9$ when $t = 3$, it is 9 metres up the surface at the end of the third second.

The acceleration is constant and is $a = -2$ m/s^2

8e $v_{average} = \frac{x_2 - x_0}{2-0} = \frac{(6\times(2)-(2)^2)-(6\times(0)-(0)^2)}{2}$

$v_{average} = 4$ m/s

When $v = 4$ m/s, $6 - 2t = 4, t = 1, x = 6 \times (1) - (1)^2 = 5$ m

8f $|v_{average}| = \left|\frac{x_3 - x_0}{3-0}\right| = \left|\frac{(6\times(3)-(3)^2)-(6\times(0)-(0)^2)}{3}\right| = 3$

$|v_{average}| = \left|\frac{x_6 - x_3}{6-3}\right| = \left|\frac{(6\times(6)-(6)^2)-(6\times(3)-(3)^2)}{3}\right| = 3$

$|v_{average}| = \left|\frac{x_6 - x_0}{6-0}\right| = \left|\frac{(6\times(6)-(6)^2)-(6\times(0)-(0)^2)}{6}\right| = 3$

9a 45 metres, 3 seconds, average speed$= \frac{\text{distance travelled}}{\text{time taken}} = \frac{45-0}{3-0} = 15$ m/s

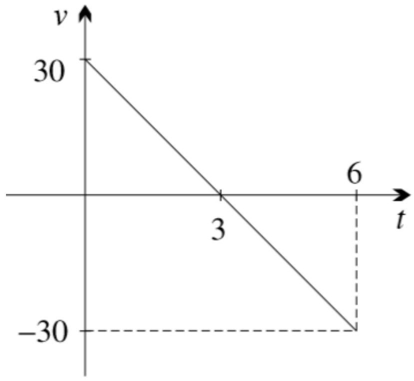
9b 30 m/s, 20 m/s, 10 m/s, 0 m/s, −10 m/s, −20 m/s, −30 m/s

Draw tangent lines and use the squares to determine the slope, which is equal to the ratio: $\frac{\text{rise}}{\text{run}}$.

9c 0 seconds. Its velocity is instantaneously zero.

9d The acceleration is always negative.

9e The velocity decreased at a constant rate of 10 m/s every second until $t = 3$, it was equal to zero when $t = 3$ for an instant and then it increased at a constant rate of 10 m/s every second until $t = 6$.



10a The maximum distance from the origin is 8 metres at the end of the third second.

10b i The gradient of the displacement function is zero when $t = 3$ and $t = 9$. Therefore, the particle is stationary when $t = 3$ and $t = 9$.

10b ii The gradient of the displacement function is positive when $0 < t < 3$ and $t > 9$. Therefore, the particle is moving to the right in these intervals.

10b iii The gradient of the displacement function is negative when $3 < t < 9$. Therefore, the particle is moving to the left in this interval.

10c It returns to the origin at $t = 9$. Its velocity is zero at $t = 9$ because the particle is changing direction at that instant. It is accelerating towards right, because the cavity is upwards.

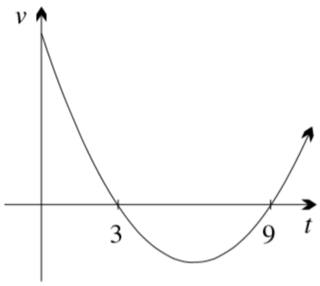
10d At $t = 6$ (at the point of inflection the second derivative is zero) and it is accelerating to the right (because the concavity is upwards)

10e The particle's acceleration is negative for $0 \leq t < 6$.

10f i When $t = 2$, the displacement is close to 7. The other t-values, where the displacement is 7, are $t = 4$ and $t = 12$.

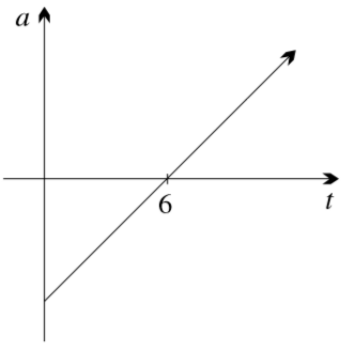
10f ii When $t = 2$, the displacement is increasing at a certain rate. Another t-value, where the velocity is similar is $t = 10$.

10g The velocity function is shown below:



t-intecepts are $(3, 0)$ and $(9, 0)$ because the velocity is zero (turning points in the displacement function) at $t = 3$ and $t = 9$.

The acceleration function is shown below:



x-intercept is $(6, 0)$ because it is assumed to be the point of inflection.

11a $v = 4 \times -\sin\left(\frac{\pi}{4}t\right) \times \frac{\pi}{4} = -\pi \sin\left(\frac{\pi}{4}t\right)$

$a = -\pi \cos\left(\frac{\pi}{4}t\right) \times \frac{\pi}{4} = -\frac{1}{4}\pi^2 \cos\left(\frac{\pi}{4}t\right)$

11b Maximum displacement is 4 metres when $t = 0$ or $t = 8$ seconds.

Maximum velocity is π m/s when $t = 0$ or $t = 8$ seconds.

Maximum acceleration is $\frac{\pi^2}{4}$ m/s^2 when $t = 4$ seconds.

11c The particle travels 8 metres every 4 seconds. Therefore, it travels $8 \times 5 = 40$ metres in the first 20 seconds.

The average velocity in this time interval is $\frac{40}{20} = 2$ m/s

11d When $t = 1\frac{1}{3}$, $x = 4\cos\left(\frac{\pi}{4} \times \left(\frac{4}{3}\right)\right) = 4\cos\left(\frac{\pi}{3}\right) = 4 \times \frac{1}{2} = 2$ metres.

When $t = 6\frac{2}{3}$, $x = 4\cos\left(\frac{\pi}{4} \times \left(\frac{20}{3}\right)\right) = 4\cos\left(\frac{5\pi}{3}\right) = 4 \times \frac{1}{2} = 2$ metres.

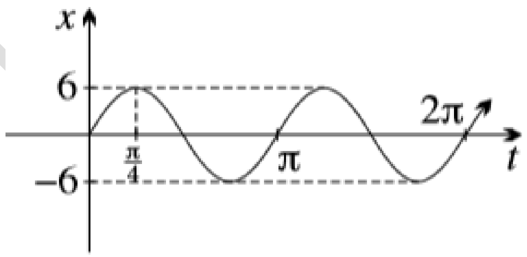
11e i $v = -\pi\sin\left(\frac{\pi}{4}t\right) = 0$ when $\sin\left(\frac{\pi}{4}t\right) = 0$ or $t = 0$, $t = 4$ and $t = 8$

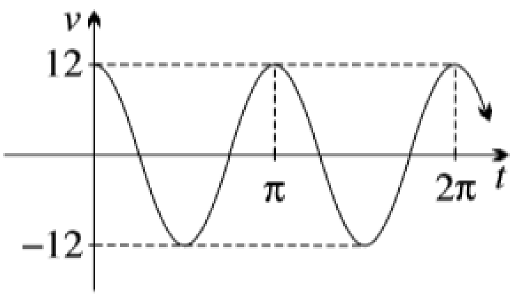
11e ii $v > 0$ when $-\pi\sin\left(\frac{\pi}{4}t\right) > 0$ or when $4 < t < 8$.

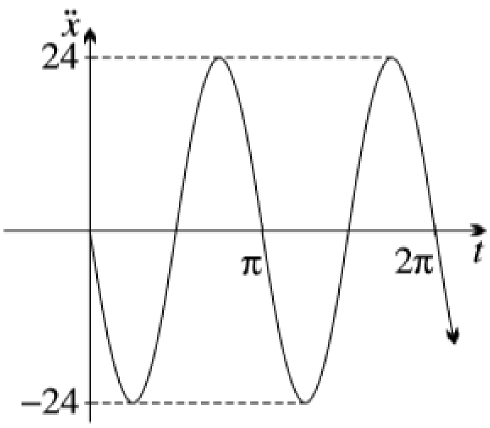
12a Height of the oscillating particle is $x = 6\sin 2t$ cm

$v = 6 \times 2 \times \cos 2t = 12\cos 2t$

$\ddot{x} = 12 \times (-2\sin 2t) = -24\sin 2t$







12b $\ddot{x} = -24 \sin 2t$

$\ddot{x} = -4 \times 6 \sin 2t = -4x$

Comparing with $\ddot{x} = -kx, k = 4$

12c i Particle is at origin when x = 0, i.e., when $t = 0, \frac{\pi}{2}$ or π

12c ii Particle is stationary when v = 0, i.e., when $t = \frac{\pi}{4}$ or $\frac{3\pi}{4}$

12c iii Particle is at origin when $\ddot{x} = 0$, i.e., when $t = 0, \frac{\pi}{2}$ or π

12d i The particle is below the origin when $x < 0$, i.e., when $\frac{\pi}{2} < t < \pi$

12d ii The particle is moving downwards when $v < 0$. That is, when $\frac{\pi}{4} < t < \frac{3\pi}{4}$

12d iii The particle is accelerating downwards when $\ddot{x} < 0$. That is, when $0 < t < \frac{\pi}{2}$.

12e i Substitute $x = 3$ in the equation $x = 6 \sin 2t$

$3 = 6 \sin 2t$

$2t = \sin^{-1}\frac{1}{2}$

$\therefore t = \frac{\pi}{12}$

12e ii Substitute $v = 6$ in the equation $v = 12 \cos 2t$

$6 = 12 \cos 2t$

$2t = \cos^{-1}\frac{1}{2}$

$\therefore t = \frac{\pi}{6}$

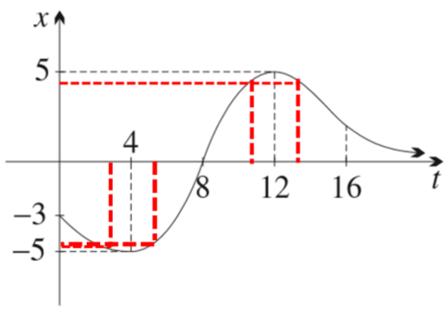
13a i The particle is below the origin when $0 \leq t < 8$.

13a ii The particle is moving downwards when $0 < t < 4$ and when $t > 12$, because the particle is travelling in the negative direction in these intervals.

13a iii The particle is accelerating downwards roughly when $8 < t < 16$, because the graph is concave down in this interval.

13b The speed of the particle is greatest at about $t = 8$, because the rate of change in the distance travelled is the steepest at $t = 8$.

13c i As shown below, at about $t = 5, 11$ and 13, the distance from the origin is the same as at $t = 3$.



13c ii At $t = 13$ and $t = 20$, the velocity is close to the velocity at $t = 3$,

because the slopes of the tangent lines are approximately the same where $t = 3$

$t = 13$ and $t = 20$.

13d $V_{ave} = \frac{x_{final} - x_{initial}}{t_{final} - t_{initial}} = \frac{5-(-5)}{12-4} = \frac{10}{8} = 1.25$

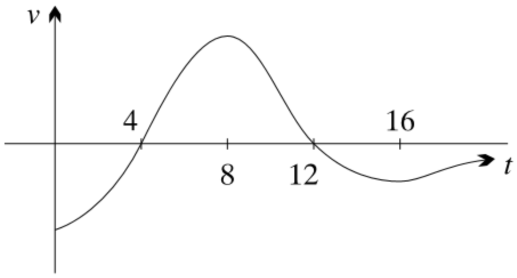
Velocity is vectoral and is positive in the interval $4 < t < 12$.

The velocity of the particle increases from zero to a certain number and then decreases back to zero. Therefore, the instantaneous velocity is equal to 1.25 m/s twice, in the interval $4 < t < 12$.

13e The particle travels, 2 units in $0 < t < 4$, 10 units in $4 < t < 12$, approximately 5 units in $t > 12$. Therefore, the total distance travelled will eventually be approximately 17 units.

13f The initial velocity is negative because the particle is moving in the negative direction initially, the graph will cut the x-axis at $(4, 0)$ and $(12, 0)$, the velocity is maximum at $t = 8$ because the rate of change in distance is the highest at $t = 8$

and the velocity eventually gets close to zero.



14a $\dot{x} = -12 \times (-0.5) \times e^{-0.5t} = 6e^{-0.5t}$

$\ddot{x} = 6 \times (-0.5) \times e^{-0.5t} = -3e^{-0.5t}$

14b The stone is travelling downwards (downwards is positive here)

14c As $t \to \infty$, $x \to 12$ metres below ground level, $v \to 0$ m/min and $a \to 0$ m/min^2

14d $e^{-0.5t} = \frac{1}{2}$

$-0.5t = \log_e\left(\frac{1}{2}\right) = -0.693147$

$t = 1.38629$ minutes

The initial speed of the stone is, $\dot{x} = 6e^{-0.5\times(0)} = 6$ m/min and the speed at $t = 1.38629$ is $\dot{x} = 6e^{-0.5\times(1.38629)} = 3$ m/min. Therefore, the velocity of the stone when $e^{-0.5t} = \frac{1}{2}$ is half of its initial velocity.

The initial acceleration of the stone is, $\ddot{x} = -3e^{-0.5\times(0)} = -3$ m/min^2 and the acceleration at $t = 1.386\,29$ is $\ddot{x} = -3e^{-0.5\times(1.38629)} = -1.5$ m/min^2. Therefore, the acceleration of the stone when $e^{-0.5t} = \frac{1}{2}$ is half of its initial acceleration.

14e $x = 12 - 12e^{-0.5\times(18)} \doteqdot 11.9985$ metres when $t = 18$ minutes.

11.9985 metres = 11 998.5 mm. Therefore, when $t = 18$ minutes, the stone is within 2 mm of its final position which is 12000 mm from the ground level.

$x = 12 - 12e^{-0.5\times(19)} \doteqdot 11.9991$ metres when $t = 19$ minutes.

11.9991 metres = 11 999.1 mm. Therefore, when $t = 19$ minutes, the stone is within 1 mm of its final position which is 12000 mm from the ground level.

15a The instantaneous length of PA that depends on the angle θ can be calculated by the cosine theorem.

$PA^2 = r^2 + (2r)^2 - r \times (2r) \times \cos\theta$

$PA^2 = 5r^2 - 4r^2 \cos\theta$

$PA^2 = r^2(5 - 4\cos\theta)$

$PA = r\sqrt{5 - 4\cos\theta}$

$PA - r$ is the distance x that the mass M has been pulled.

Therefore, $x = r\sqrt{5 - 4\cos\theta} - r$ or $x = -r + r\sqrt{5 - 4\cos\theta}$

The minimum value of x is 0 when $\cos\theta = 1$ and the maximum value is $2r$ when $\cos\theta = -1$. Therefore, the range of x is $[0, 2r]$

15b i $\frac{dx}{d\theta} = r \times \frac{4\sin\theta}{2\sqrt{5-4\cos\theta}} = \frac{2r\sin\theta}{\sqrt{5-4\cos\theta}}$

$\frac{dx}{d\theta} > 0$ when $\sin\theta > 0$ or $0 < \theta < \pi$.

Therefore, M is travelling upwards when $0 < \theta < \pi$

15b ii $\frac{dx}{d\theta} < 0$ when $\sin\theta < 0$ or $\pi < \theta < 2\pi$.

Therefore, M is travelling downwards when $\pi < \theta < 2\pi$

15c

$$\frac{d^2x}{d\theta^2} = \frac{2r\cos\theta\sqrt{5-4\cos\theta} - 2r\sin\theta\dfrac{2\sin\theta}{\sqrt{5-4\cos\theta}}}{5-4\cos\theta}$$

$$= \frac{2r\cos\theta\,(5-4\cos\theta) - 4r\sin^2\theta}{(5-4\cos\theta)^{\frac{3}{2}}}$$

$$= \frac{2r(5\cos\theta - 4\cos^2\theta - 2r\sin^2\theta)}{(5-4\cos\theta)^{\frac{3}{2}}}$$

$$= \frac{2r(5\cos\theta - 2\cos^2\theta - 2)}{(5-4\cos\theta)^{\frac{3}{2}}}$$

$$= -\frac{2r(2\cos^2\theta - 5\cos\theta + 2)}{(5-4\cos\theta)^{\frac{3}{2}}}$$

$\frac{d^2x}{d\theta^2} = 0$ when

$2\cos^2\theta - 5\cos\theta + 2 = 0$

$(2\cos\theta - 1)(\cos\theta - 2) = 0$ $(\cos\theta \neq 2$ for any value of $\theta)$

$\cos\theta = \frac{1}{2}$ or when $\theta = \frac{\pi}{3}$ or $\theta = \frac{5\pi}{3}$

θ		$\frac{\pi}{3}$		$\frac{5\pi}{3}$	
$\frac{d^2x}{d\theta^2}$	+	0	−	0	+
$\frac{dx}{d\theta}$	/	Maximum turning point	\	Minimum turning point	/

Therefore, the speed is maximum when $\theta = \frac{\pi}{3}$ and it is $\frac{dx}{d\theta} = \frac{2r\sin\frac{\pi}{3}}{\sqrt{5-4\cos\frac{\pi}{3}}} = \frac{r\sqrt{3}}{\sqrt{3}} = r$

and the speed is minimum when $\theta = \frac{5\pi}{3}$ and it is $\frac{dx}{d\theta} = \frac{2r\sin\frac{5\pi}{3}}{\sqrt{5-4\cos\frac{5\pi}{3}}} = \frac{-r\sqrt{3}}{\sqrt{3}} = -r$

15d When $\theta = \frac{\pi}{3}$ or $\theta = \frac{5\pi}{3}$, $\angle APC$ is a right angle, so AP is a tangent to the circle. At these places, P is moving directly towards A or directly away from A, and so the distance AP is changing at the maximum rate. Again because AP is a tangent, $\frac{dx}{d\theta}$ at these points must equal the rate of change of arc length with respect to θ, which is r or $-r$ when $\theta = \frac{\pi}{3}$ or $\frac{5\pi}{3}$ respectively.

16 The velocity of the particle on the inclined surface is $v = 6 - 2t$ and the initial velocity is $v = 6$ when $t = 0$. The vertical velocity is $v_{vertical} = 6\sin\alpha - gt$ where g is the gravitational acceleration and t is time.

Since $v = 0$ when $t = 3$, $v_{vertical} = 6\sin\alpha - gt = 0$ when $t = 3$.

Therefore,

$6\sin\alpha - g \times 3 = 0$

$6\sin\alpha = 3g$

$sin\ \alpha = \frac{2}{g} \doteqdot 0.204\,08$, $\alpha \doteqdot 11°\ 47\,'$

Solutions to Exercise 9C

Let C be a constant.

1a $x = \int (3t^2 - 6t)dt = t^3 - 3t^2 + C$

If $x = 4$ when $t = 0$ then $(0)^3 - 3(0)^2 + C = 4$ and $C = 4$

Therefore, $x = t^3 - 3t^2 + 4$

1b $x = (2)^3 - 3(2)^2 + 4 = 0$. Therefore, the particle is at the origin when $t = 2$ and $v = 3(2)^2 - 6 \times (2) = 0$ m/s when the particle is at the origin.

1c If $v = 3t^2 - 6t$ then $a = 6t - 6$.

1d $a = 6 \times (1) - 6 = 0$ m/s^2 when $t = 1$.

$x = (1)^3 - 3(1)^2 + 4 = 2$ m

2a If $a = 10$ then $v = \int 10\, dt = 10t + C$. Since $v = 0$ when $t = 0$, $C = 0$.

Therefore, $v = 10t$

If $v = 10t$ then $x = \int 10t\, dt = 5t^2 + C$. Since $x = 0$ when $t = 0$, $C = 0$.

Therefore, $x = 5t^2$

2b If $x = 5t^2 = 80$ then $t = 4$. Therefore, it takes 4 seconds the particle to fall 80 metres. Hence, $|v| = |10 \times (4)| = 40$ m/s when $t = 4$ seconds.

2c $x = 5(2)^2 = 20$ metres. Therefore, the particle is $80 - 20 = 40$ metres above the ground when it is halfway through its flight time and its speed is $v = 10 \times 2 = 20$ m/s.

2d $x = 5(t)^2 = 40$ when $t = 2\sqrt{2}$. Therefore, it takes $2\sqrt{2}$ seconds the particle to travel halfway and $v = 10 \times \left(2\sqrt{2}\right) = 20\sqrt{2}$ m/s when $t = 2\sqrt{2}$.

3a $a = -10$ then $v = \int -10\, dt = -10t + C$.

Given that $v = -25$ when $t = 0$,

$-25 = -10 \times (0) + C$ and $C = -25$.

Therefore, $v = -10t - 25$

Then $x = \int(-10t - 25)dt = -5t^2 - 25t + C$.

Given that $x = 120$ when $t = 0$,

$120 = -5(0)^2 - 25 \times (0) + C$ and $C = 120$.

Therefore, $x = -5t^2 - 25t + 120$.

3b $x = -5t^2 - 25t + 120 = -5(t + 8)(t - 3) = 0$

Therefore, $t = 3$ seconds when the particle reaches the ground.

3c $|v| = |-10 \times (3) - 25| = 55$ m/s when it hits the ground.

3d $\left|v_{average}\right| = \left|\frac{x_3 - x_0}{3-0}\right| = \left|\frac{120-0}{3}\right| = 40$ m/s

4a i $\dot{x} = 3t^2 + C$ and initial velocity is zero, then $C = 0$. Therefore, $\dot{x} = 3t^2$. Hence, the displacement function is $x = t^3 + C$ and since displacement is zero when $t = 0$, $C = 0$. Therefore, the displacement function is: $x = t^3$.

4a ii $\dot{x} = -\frac{1}{3}e^{-3t} + C$ and initial velocity is zero, then $C = \frac{1}{3}$.

Therefore, $\dot{x} = -\frac{1}{3}e^{-3t} + \frac{1}{3}$.

Hence, the displacement function is $x = \frac{1}{9}e^{-3t} + \frac{t}{3} + C$ and since displacement is zero when $t = 0$, $C = -\frac{1}{9}$.

Therefore, the displacement function is $x = \frac{1}{9}e^{-3t} + \frac{t}{3} - \frac{1}{9}$.

4a iii $\dot{x} = \frac{\sin(\pi t)}{\pi} + C$ and initial velocity is zero, then $C = 0$.

Therefore, $\dot{x} = \frac{\sin(\pi t)}{\pi}$.

Hence, the displacement function is $x = -\frac{1}{\pi^2}\cos(\pi t) + C$ and since displacement is zero when $t = 0$, $C = \frac{1}{\pi^2}$.

Therefore, the displacement function is $x = -\frac{1}{\pi^2}\cos(\pi t) + \frac{1}{\pi^2}$

4a iv $\dot{x} = -12(t+1)^{-1} + C$ and initial velocity is zero, then $C = 12$.

Therefore, $\dot{x} = -12(t+1)^{-1} + 12$.

Hence, the displacement function is $x = -12\log_e(t+1) + 12t + C$ and since displacement is zero when $t = 0$, $C = 0$. Therefore, the displacement function is:

$x = -12\log_e(t+1) + 12t$.

4b i If $v = -4$ then $a = 0$

and $x = \int -4\, dt = -4t + C$.

Since $x = -2$ when $t = 0$, $-4 \times (0) + C = -2$ and $C = -2$.

Therefore, $x = -4t - 2$.

4b ii If $v = e^{\frac{1}{2}t}$ then $a = \frac{1}{2}e^{\frac{1}{2}t}$

and $x = \int e^{\frac{1}{2}t}\, dt = \frac{e^{\frac{1}{2}t}}{\frac{1}{2}} + c = 2e^{\frac{1}{2}t} + C$.

Since $x = -2$ when $t = 0$, $2e^{\frac{1}{2}\times(0)} + C = -2$ and $C = -4$.

Therefore, $x = 2e^{\frac{1}{2}t} - 4$.

4b iii If $v = 8\sin 2t$ then $a = 16\cos 2t$

and $x = \int 8\sin 2t\,dt = 8 \times \left(\frac{-\cos 2t}{2}\right) + C = -4\cos 2t + C$

Since $x = -2$ when $t = 0$, $-4\cos\big(2 \times (0)\big) + C = -2$ and $C = 2$.

Therefore, $x = -4\cos 2t + 2$

4b iv If $v = \sqrt{t}$ then $a = \frac{1}{2\sqrt{t}}$ or $a = \frac{1}{2}t^{-\frac{1}{2}}$

and $x = \int \sqrt{t}\,dt = \int t^{\frac{1}{2}} = \frac{t^{\frac{3}{2}}}{\frac{3}{2}} + C = \frac{2}{3}t^{\frac{3}{2}} + C$

Since $x = -2$ when $t = 0$, $\frac{2}{3}(0)^{\frac{3}{2}} + C = -2$ and $c = -2$.

Therefore, $x = \frac{2}{3}t^{\frac{3}{2}} - 2$

5a $\dot{x} = 6t^2 + C$. Since $\dot{x} = -24$ when $t = 0$, $C = -24$.

Therefore, $\dot{x} = 6t^2 - 24$

$x = 2t^3 - 24t + C$ and since $x = 20$ when $t = 0$, $C = 20$

$x = 2t^3 - 24t + 20$

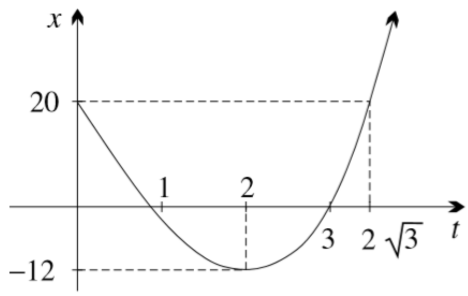
5b $x = 2t^3 - 24t + 20 = 0$ when $t = 2\sqrt{3}$. Its speed at that time is:

$\dot{x} = 6\big(2\sqrt{3}\big)^2 - 24 = 48$ m/s

5c $\dot{x} = 6t^2 - 24 = 0$ when $t = 2$. Therefore, the minimum displacement occurs when $t = 2$ and the displacement at $t = 2$ is

$x = 2(2)^3 - 24(2) + 20 = -12$ metres.

5d



6a $4 < t < 14$ (when $\dot{x} > 0$)

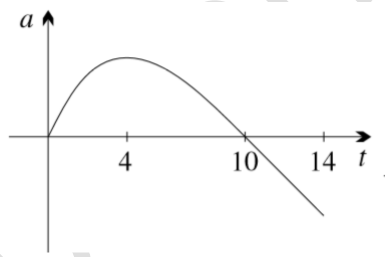
6b $0 < t < 10$ (when the function is increasing)

6c $t = 14$

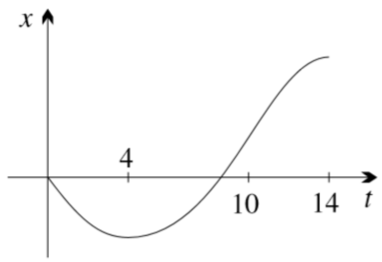
6d $t = 14$ (Starts going in the negative direction at $t = 10$)

6e $t \doteqdot 8$

6f The graph of acceleration



The graph of displacement



7a $a = 2$ then $v = \int 2\, dt = 2t + C$.

Since the car is initially at rest, $v = 0$ when $t = 0$.

Hence, $2 \times (0) + C = 0$ then $c = 0$.

Therefore, $v_1 = 2t$ (where v_1 is the speed in the first 10 seconds)

The speed at the end of first 10 seconds is $|v| = |2 \times 10| = 20$ m/s

Since the car does not accelerate the following 30 seconds, its speed remains constant. Therefore, the speed of the car when $t = 20$ is 20 m/s.

7b i $x = \int v_1 dt = \int 2t\, dt = t^2 + C$

Since the car is initially at the front gate of the house, $x = 0$ when $t = 0$.

$(0)^2 + C = 0$ then $C = 0$. Thus, $x = t^2$

Therefore, $x = (10)^2 = 100$ metres when $t = 10$ seconds.

7b ii The car travels with $v = 20t$ the next 30 seconds.

Therefore, it travels $20 \times 30 = 600$ metres, as $x = v \times t$.

7b iii The velocity of the car at the end of the first 40 seconds is $v = 20$ m/s.

When the car starts decelerating, it has an initial velocity of $v = 20$ m/s.

Hence, $v = \int -1\, dt = -t + C$ and $C = 20$.

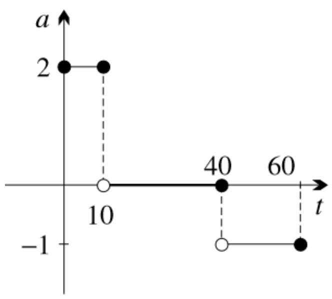
Therefore, $v = -t + 20$ after the 40th second.

If $v = -t + 20$ then $x = \int (-t + 20)\, dt = -\frac{t^2}{2} + 20t + C$ is the displacement function of the last 20 seconds.

Hence, the displacement between $t = 0$ and $t = 20$ is:

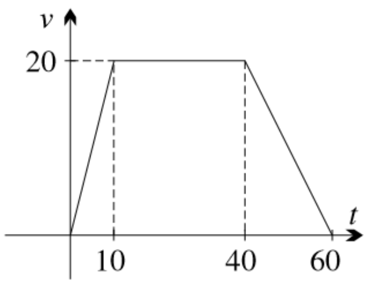
$\left(-\frac{(20)^2}{2} + 20 \times (20) + C\right) - \left(-\frac{(0)^2}{2} + 20 \times (0) + C\right) = 200$ metres.

7c The graph of acceleration is:



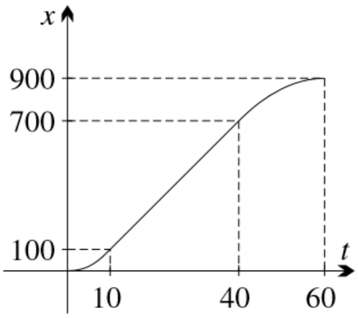
The car accelerates with $a = 2$ m/s² the first ten seconds, does not accelerate the next 30 and decelerates with $a = -1$ m/s² in the last 20 minutes.

The graph of velocity is:



The car reaches 20 m/s velocity by the end of the first 10 seconds, then the velocity remains constant for 30 seconds and then decelerates until the velocity is 0 again, which takes 20 more seconds.

The graph of displacement is:



As shown in 13b, the car accelerates and travels 100 metres the first ten seconds, then the velocity remains constant and travels 600 metres, and finally decelerates and travels a distance of 200 metres.

8a $\ddot{x} = -4$ cm/s² and $x = 16t - 2t^2 + C$

8b $x = C$ when $t = 0$. The particle is at $x = C$ initially. It is again at $x = C$, when $t = 8$. $(16t - 2t^2 = 2t(8 - t)$ and $2t(8 - t) = 0$ when $t = 0$ or $t = 8)$

Its speed when it is at $x = 8$ is $|\dot{x}| = |16 - 4 \times 8| = |-16| = 16$ cm/s

8c $\dot{x} = 16 - 4t = 0$ when $t = 4$. Therefore, the particle is stationary when $t = 4$. Maximum distance right is 32 cm when $t = 4$, maximum distance left is 40 cm when $t = 10$. The acceleration is -4 cm/s² at all times.

8d distance travelled$= \int_0^4 (16t - 2t^2)\,dx - \int_4^{10} (16t - 2t^2)\,dx = 104$ cm

Average speed $= \dfrac{\text{total distance travelled}}{\text{time taken}} = \dfrac{104}{10} = 10.4$ cm/s

9a $v = 4t(t - 3)(t - 6) = 0$ when $t = 0$, $t = 3$ or $t = 6$ seconds. The mouse turns back to its hole when $t = 6$ seconds and its velocity is zero in the hole.

9b $x = \int(4t^3 - 36t^2 + 72t)dt = t^4 - 12t^3 + 36t^2 + c$

Since $x = 0$ when $t = 0$, $c = 0$.

Therefore, $x = t^4 - 12t^3 + 36t^2$

The maximum distance of the mouse from the whole is

$x = (3)^4 - 12(3)^3 + 36(3)^2 = 81$ cm.

The distance the mouse travels in 6 seconds is then $2 \times 81 = 162$ cm.

The average speed is $|V_{Ave}| = \left|\frac{distance\ travelled}{time\ taken}\right| = \left|\frac{162}{6}\right| = 27$ cm/s

9c $\ddot{x} = \frac{d(4t^3-36t^2+72t)}{dt} = 12t^2 - 72t + 72 = 12(t^2 - 6t + 6)$ and

$\ddot{x} = 0$ when $12(t^2 - 6t + 6) = 0$ or

when $t = 3 + \sqrt{3}$ or $t = 3 - \sqrt{3}$

t		$3 - \sqrt{3}$		$3 + \sqrt{3}$	
$\ddot{x}$	$+$	0	$-$	0	$+$
v	/	Maximum turning point	\	Minimum turning point	/

The maximum velocity is reached when $t = 3 - \sqrt{3}$ and it is

$v = 4(3 - \sqrt{3})\left((3 - \sqrt{3}) - 3\right)\left((3 - \sqrt{3}) - 6\right) = 24\sqrt{3}$ cm/s

9d The graphs of x, v and $\ddot{x}$ are all unchanged by reflection in $t = 3$, but the mouse would be running backwards!

10a $\ddot{x} = kt$ then $v = \int kt\, dt = \frac{1}{2}kt^2 + C$.

Since $(1, -6)$ and $(2, 3)$ are on the graph of v,

$-6 = \frac{1}{2}k(1)^2 + C$ and $3 = \frac{1}{2}k(2)^2 + C$. Solving these two equations together,

$-6-\frac{1}{2}k(1)^2 = 3-\frac{1}{2}k(2)^2$

$-6-\frac{k}{2} = 3-2k$

$2k-\frac{k}{2} = 9$

$\frac{3k}{2} = 9$

$k = 6$ and substituting the value of k in $3 = \frac{1}{2}k(2)^2 + C$,

$3 = \frac{1}{2}(6)(2)^2 + C$

$C = -9$

Therefore, $\ddot{x} = 6t$ and $v = 3t^2 - 9$

10b $x = \int(3t^2-9)\,dt = t^3 - 9t + C_1$

11 $x = \int \frac{1}{t+1}dt = \log_e(t+1) + C$

Given that $x = -1$ when $t = 0$,

$\log_e\left((0)+1\right) + C = -1$ and $C = -1$

Therefore, $x = \log_e(t+1) - 1$

$a = \frac{dv}{dt} = \frac{d\left((t+1)^{-1}\right)}{dt} = -(t+1)^{-2} = -\frac{1}{(t+1)^2}$

11 $x = 0$ when $\log_e(t+1) - 1 = 0$ or $t+1 = e$ or $t = e-1$

Therefore, $x = 0$ when $t = e-1$.

$v = \frac{1}{(e-1)+1} = \frac{1}{e}$ when $t = e-1$

$a = -\frac{1}{\left((e-1)+1\right)^2} = -\frac{1}{e^2}$ when $t = e-1$

11 As $t \to \infty$, $x \to \infty$ because $x = \log_e(t+1) - 1$ is increasing when $t > 0$,

$v \to 0$ because $v = \frac{1}{t+1}$ is decreasing when $t > 0$ and has a horizontal asymptote at $y = 0$, $a \to 0$ because $a = -\frac{1}{(t+1)^2}$ is decreasing when $t > 0$ and has a horizontal asymptote at $y = 0$.

Therefore, the velocity and acceleration approach zero, but the particle moves to infinity.

12a $\dot{x} = \int -40\, e^{-2t} dt = -40 \times \frac{e^{-2t}}{-2} + c = 20e^{-2t} + C$

Since the initial velocity is $\dot{x} = 15$ m/s, $20e^{-2\times(0)} + C = 15$ when $t = 0$, $C = -5$

Therefore, $\dot{x} = 20e^{-2t} - 5$.

$x = \int (20e^{-2t} - 5)\, dt = 20 \times \frac{e^{-2t}}{-2} - 5t + C = -10e^{-2t} - 5t + C$

Since the body is initially at the origin, $x = 0$ when $t = 0$

Then $-10e^{-2\times(0)} - 5 \times (0) + C = 0$ and $C = 10$

Therefore, $x = -10e^{-2t} - 5t + 10$

The body is stationary when its velocity is zero and $\dot{x} = 20e^{-2t} - 5 = 0$ when

$e^{-2t} = \frac{1}{4}$

$-2t = \log_e\left(\frac{1}{4}\right)$

$-2t = \log_e(2^{-2})$

$-2t = -2 \times \log_e 2$ or

$t = \log_e 2$ seconds.

12b When $t = \log_e 2$ seconds,

$x = -10e^{-2\times(\log_e 2)} - 5 \times (\log_e 2) + 10 = 7\frac{1}{2} - 5\log_e 2$

$\ddot{x} = -40e^{-2\times(\log_e 2)} = -10$ m/s^2 (which is 10 m/s^2 downwards)

12c As $t \to \infty$, $\dot{x} \to -5$ m/s (which is 5 m/s downwards) because as $t \to \infty$, $e^{-2t} \to 0$.

13a $v = \int -2\cos t \, dt = -2 \times \sin t + c$

Given that $v = 1$ m/s when $t = 0$ seconds, $-2 \times \sin(0) + c = 1$ and $c = 1$.

Therefore, $v = -2\sin t + 1$ or $v = 1 - 2\sin t$

$x = \int(-2\sin t + 1)\, dt = -2 \times (-\cos t) + t + C$

Given that $x = 2$ metres when $t = 0$ seconds,

$-2 \times (-\cos(0)) + (0) + C = 2$ and $C = 0$

Therefore, $x = 2\cos t + t$ or $x = t + 2\cos t$

13b $a = -2\cos t$ and $a > 0$ when $\cos t < 0$ which is when $\frac{\pi}{2} < t < \frac{3\pi}{2}$

Therefore, the acceleration is positive when $\frac{\pi}{2} < t < \frac{3\pi}{2}$

13c The particle is stationary when the velocity is zero. $v = 1 - 2\sin t = 0$ when $\sin t = \frac{1}{2}$. Hence, the particle is stationary when

$t = \frac{\pi}{6}$, when $x = \frac{\pi}{6} + 2\cos\left(\frac{\pi}{6}\right) = \frac{\pi}{6} + \sqrt{3}$

or $= \frac{5\pi}{6}$, when $x = \frac{5\pi}{6} + 2\cos\left(\frac{5\pi}{6}\right) = \frac{5\pi}{6} - \sqrt{3}$

13d The maximum and minimum velocity of the particle is when $\frac{dv}{dt} = 0$

Thus, $a = -2\cos t = 0$ when $\cos t = 0$ or $t = \frac{\pi}{2}$ or $t = \frac{3\pi}{2}$

t		$\frac{\pi}{2}$		$\frac{3\pi}{2}$	
a	$-$	0	$+$	0	$-$
v	\	Minimum turning point	/	Maximum turning point	\

Therefore, the minimum velocity is $v_{min} = 1 - 2\sin\left(\frac{\pi}{2}\right) = -1$ m/s when $t = \frac{\pi}{2}$

and the maximum velocity is $v_{max} = 1 - 2\sin\left(\frac{3\pi}{2}\right) = 3$ m/s when $t = \frac{3\pi}{2}$ seconds

14a $v_T = \frac{20}{t+1}$ and $v_H = 5$ then $v_T = \frac{20}{(0)+1} = 20$ and $v_H = 5$.

Therefore, initially, v_T is 15 m/s faster than v_H.

14b $x_T = \int \frac{20}{t+1} dt = 20 \log_e(t+1) + C_1$ and $(0, 0)$ is on x_T.

Therefore, $x_T = 20 \log_e(t+1)$

$x_H = \int 5\, dt = 5t + C_2$ and $(0, 0)$ is on x_H. Therefore, $x_H = 5t$

14c $x_T = x_H$ or $20 \log_e(t+1) = 5t$ when $t = 9.346\,65$ seconds which is during the 10th second.

By the end of this second $(9.346\,65)$, $t = 10$ seconds and $v_T = \frac{20}{(10)+1} = \frac{20}{11}$ m/s and $v_H = 5$. Therefore, the trains are drawing apart from each other by $5 - \frac{20}{11} = 3\frac{2}{11}$ m/s

14d $x_T - x_H = 20 \log_e(t+1) - 5t$ is the distance function. To find the time when the distance between the trains is the maximum, the roots of the first derivative should be found.

$\frac{d(x_T - x_H)}{dt} = \frac{20}{t+1} - 5 = 0$ when $t = 3$ seconds.

t		3	
$\frac{d(x_T - x_H)}{dt}$	$+$	0	$-$
Distance	/	Maximum turning point	\

As shown in the table above, the maximum distance between the trains is

$x_T - x_H = 20 \log_e((3)+1) - 5(3) = 12.7259 \doteqdot 13$ m when $t = 3$ seconds.

15a The initial distance between the ball and the stone is 180 metres and the distance travelled by the ball is $x_b = \frac{1}{2}gt^2$ or $x_b = \frac{1}{2} \times 10 \times t^2 = 5t^2$

Since the ball is dropped from 180 metres, it takes 6 seconds for the ball to travel this distance, because $180 = 5t^2$ then $t = 6$ seconds. At the time when they collide, their height from the ground is the same.

Let the height of the ball be h_b then $h_b = 180 - 5t^2$

Let the height of the stone be h_s then $h_s = Vt - 5t^2$

Thus, $180 - 5t^2 = Vt - 5t^2$ and $180 = Vt$.

Since the maximum value of $t = 6$ seconds, the minimum value of $V = 30$ m/s.
Therefore, $V \geq 30$ m/s and V is the speed of the collusion.

In terms of V, they collide when $t = \frac{180}{V}$ (because $180 = Vt$ as shown above)

And when $t = \frac{180}{V}$, the height is $h_b = 180 - 5\left(\frac{180}{V}\right)^2 = \frac{180}{V^2}(V^2 - 900)$

15b Since they collide halfway up the cliff,

$h_b = 180 - 5\left(\frac{180}{V}\right)^2 = 90$ and $V = 30\sqrt{2}$ m/s

$5\left(\frac{180}{V}\right)^2 = 90$

$\left(\frac{180}{V}\right)^2 = 18$

and $V = 30\sqrt{2}$ m/s

Therefore, when $V = 30\sqrt{2}$ m/s and $t = \frac{180}{V}$, $t = \frac{180}{30\sqrt{2}} = 3\sqrt{2}$ seconds.

16a $\ddot{x} = \frac{dv}{dt} = -10 - 2v$

$\frac{dt}{dv} = \frac{1}{-10-2v}$

$dt = \frac{1}{-10-2v} \times dv$

$\int dt = \int \frac{-1}{10+2v} dv$

$t = -\frac{1}{2} \times \log_e(2v + 10) + C$

$v = 0$ when $t = 0$ then $0 = -\frac{1}{2} \times \log_e(2 \times 0 + 10) + C$ and $C = \frac{1}{2}\log_e 10$

Therefore, $t = -\frac{1}{2}\log_e(2v + 10) + \frac{1}{2}\log_e 10$

$t = \frac{1}{2}\log_e 10 - \frac{1}{2}\log_e(2v + 10)$ and $t = \frac{1}{2}\left(\log_e \frac{10}{2v+10}\right)$

Hence, $2t = \left(\log_e \frac{10}{2v+10}\right)$ and $e^{2t} = \frac{10}{2v+10}$

$2v + 10 = 10e^{-2t}$

$2v = 10e^{-2t} - 10$

$v = 5e^{-2t} - 5 = 5(e^{-2t} - 1)$

If $v = 5(e^{-2t} - 1)$ then $x = \int v\,dt = 5\int(e^{-2t} - 1)\,dt$ and $x = 5\left(\frac{e^{-2t}}{-2} - t\right) + C$

Since $x = 0$ when $t = 0$, $0 = 5\left(\frac{e^{-2\times 0}}{-2} - 0\right) + C$ and $C = \frac{5}{2}$

Therefore, $x = 5\left(\frac{e^{-2t}}{-2} - t\right) + \frac{5}{2} = \frac{5}{2}(1 - e^{-2t}) - 5t$

16b $\lim_{t\to\infty} 5(e^{-2t} - 1) = 5(0 - 1) = -5$ m/s.

Therefore, the speed gradually increases with limit 5 m/s (the terminal velocity).

17a i $v = \int a\,dt = at + C$

Since the initial velocity of the particle is u, $v = u$ when $t = 0$.

Hence, $v = a \times (0) + C = u$ and $C = u$.

Therefore, $v = at + u$ or $v = u + at$

17a ii $s = \int(u + at)\,dt = ut + \frac{at^2}{2} + C$

Since $s = 0$ when $t = 0$, $s = u \times (0) + \frac{a\times(0)^2}{2} + C = 0$ and $C = 0$.

Therefore, $s = ut + \frac{1}{2}at^2$

17a iii $v^2 = (u + at)^2 = u^2 + 2uat + a^2t^2$

$$= u^2 + 2a\left(ut + \frac{1}{2}at^2\right) \quad \left(s = ut + \frac{1}{2}at^2 \text{ from 18b}\right)$$

$$= u^2 + 2as$$

18a $x_1 = \int v_1\, dt = \int (6 + 2t)\, dt = 6t + t^2 + C_1$

Since P_1 is at $x = 2$ when $t = 0$, $x_1 = 6 \times (0) + (0)^2 + C_1 = 2$

$C_1 = 2$ and $x_1 = 2 + 6t + t^2$

$x_2 = \int v_2\, dt = \int (4 - 2t)\, dt = 4t - t^2 + C_2$

Since P_2 is at $x = 1$ when $t = 0$, $x_2 = 4 \times (0) - (0)^2 + C_2 = 1$

$C_2 = 1$ and $x_2 = 1 + 4t - t^2$

$D = x_1 - x_2 = 2 + 6t + t^2 - (1 + 4t - t^2) = 1 + 2t + 2t^2$

18b When particles meet, $D = 0$

$D = 1 - 2t + 2t^2 = 0$ and $\Delta = b^2 - 4ac = (-2)^2 - 4 \times 2 \times 1 = 4 - 8 = -4$

Since $\Delta < 0$, $D = 0$ has no solution. Therefore, the particles never meet.

18c Let the distance between the midpoint between the particles and the initial position be D_M.

$$D_M = \frac{x_1 + x_2}{2} = \frac{2 + 6t + t^2 + 1 + 4t - t^2}{2} = \frac{3 + 10t}{2}$$

$\frac{dD_M}{dt} = 5$ m/s. Therefore, the velocity of the midpoint is constant.

When $t = 3$ seconds, $x_1 = 2 + 6 \times 3 + 3^2 = 29$ m and $D_M = \frac{3+10\times3}{2} = \frac{33}{2}$

The distance between the particle and the midpoint is $29 - \frac{33}{2} = 12\frac{1}{2}$ m.

Solutions to Exercise 9D

1a $V = 20t$ then there will be $V = 20 \times (4) = 80$ tonnes of grain after 4 minutes.

1b $V = 20 \times (0) = 0$ when $t = 0$. Therefore, the silo was empty at the beginning.

1c If the silo is filled in 18 mins then $V = 20 \times (18) = 360$ tonnes is its capacity.

1d $\frac{dV}{dt} = 20$. Therefore, the rate at which the silo is being filled is 20 tonnes/minute

2a $F = 200\big(20 - (0)\big)^2 = 80\,000$ litres when $t = 0$.

2b $F = 200\big(20 - (15)\big)^2 = 5000$ litres when $t = 15$ mins

2c $F = 200(20 - t)^2 = 0$ when $t = 20$. Thus, it takes 20 minutes for the tank to empty. Therefore, the domain of F is $0 \le t \le 20$.

2d $\frac{dF}{dt} = 200 \times 2 \times (20 - t) \times (-1) = -400(20 - t)$

$\frac{dF}{dt} = -400\big(20 - (5)\big) = -6000.$

Therefore, the tank is emptying at the rate 6000 L/min when $t = 5$.

2e $\frac{dF}{dt} = -400(20 - t)$

t	$t < 20$	20	$t > 20$
$\frac{dF}{dt}$	$-$	0	$+$

Since $\frac{dF}{dt}$ is a linear function, and $\frac{dF}{dt} > 0$ for all values bigger than 20,

$\frac{dF}{dt} < 0$ for $0 \le t \le 20$.

The tank is emptying, so F is decreasing.

3a $\frac{dV}{dt} = 300$ then $V = \int 300\, dt = 300t + C$.

Since the tank has 1500 L when $t = 0$ minutes, $300 \times (0) + C = 1500$

and $C = 1500$

3b $V = \left(\frac{dV}{dt}\right) t + C$ and $\frac{dV}{dt} = 300$. Therefore, $k = 300$.

3c The tank is full when $V = 6000$ L and

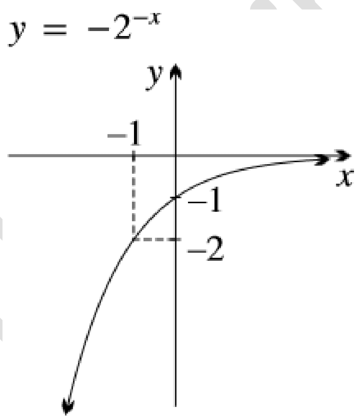
$V = 300t + 1500 = 6000$ when $t = 15$ mins

3d Average rate of flow

$$= \frac{(300 \times (15) + 1500) - (300 \times (0) + 1500)}{15 - 0}$$

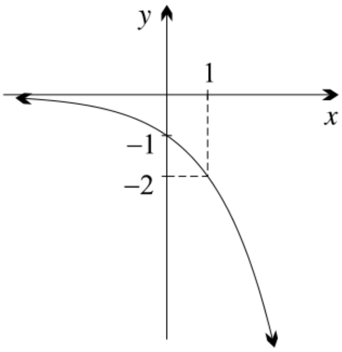
$= 300$ L/min

4a



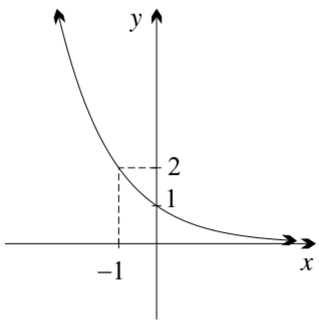
4b

$y = -2^x$



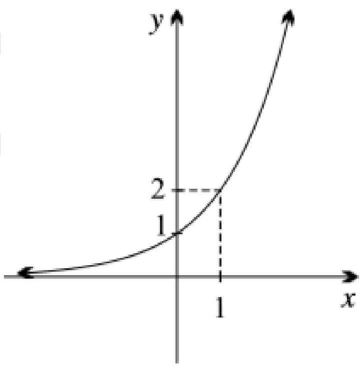
4c

$y = 2^{-x}$

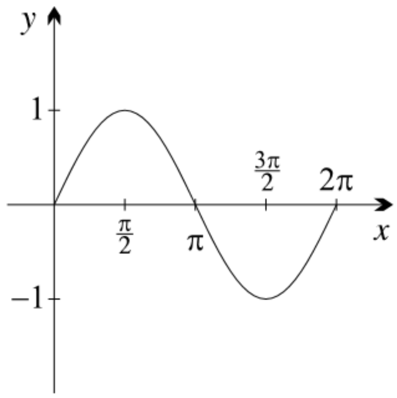


4d

$y = 2^x$



5a The graph of $y = \sin x$ with domain $0 \le x \le 2\pi$ is shown below:



$\sin 0 = 0$, $\sin \pi = 0$, $\sin 2\pi = 0$, $\sin \frac{\pi}{2} = 1$ and $\sin \frac{3\pi}{2} = -1$

5a i y is increasing at a decreasing rate in the interval $0 \le x \le \frac{\pi}{2}$

5a ii y is decreasing at an increasing rate in the interval $\frac{\pi}{2} \le x \le \pi$

5a iii y is decreasing at a decreasing rate in the interval $\pi \le x \le \frac{3\pi}{2}$

5a iv y is increasing at an increasing rate in the interval $\frac{3\pi}{2} \le x \le 2\pi$

5b i y is concave up in the interval $\pi \le x \le 2\pi$

5b ii y is concave down in the interval $0 \le x \le \pi$

6a $h = 180\left(1 - e^{-\frac{1}{3}t}\right) - 30t$ then

$$\frac{dh}{dt} = -180 \times \left(-\frac{1}{3}\right) \times e^{-\frac{1}{3}t} - 30$$

$$\frac{dh}{dt} = 60e^{-\frac{1}{3}t} - 30$$

6b $v = \frac{dh}{dt} = 60e^{-\frac{1}{3}t} - 30$

Then, $60e^{-\frac{1}{3}\times(0)} - 30 = 30$ m/s upwards when $t = 0$

6c $v = \frac{dh}{dt} = 60e^{-\frac{1}{3}t} - 30 = 0$ when $e^{-\frac{1}{3}t} = \frac{1}{2}$ or $t = \frac{\ln\left(\frac{1}{2}\right)}{-\frac{1}{3}} = 3\ln 2$ seconds. Thus, the object reaches its maximum height ans stops for an instant at $T = 3\ln 2$

Therefore,

the maximum height is $h = 180\left(1 - e^{-\frac{1}{3}(3\ln 2)}\right) - 30 \times (3\ln 2) \doteqdot 27.62$ m

when $T \doteqdot 2.079$ seconds.

6d When $t = 2T = 6\ln 2$,

$h = 180\left(1 - e^{-\frac{1}{3}(6\ln 2)}\right) - 30 \times (6\ln 2) \doteqdot 10.23$ m

$v = 60e^{-\frac{1}{3}\times(6\ln 2)} - 30 = -15$. Therefore, 15 m/s downwards.

6e As $x \to \infty$ $|v| = \left|60e^{-\frac{1}{3}\times(\infty)} - 30\right|$

$|v| = \left|60 \times \frac{1}{e^{\infty}} - 30\right|$

$|v| = |0 - 30|$

$|v| = 30$ m/s downwards.

7a i If $R = 10 + \frac{10}{1+2t}$ then

$R = 10 + \frac{10}{1+2\times(2)} = 12$ kg/min when $t = 2$ min

7a ii $R = 10 + \frac{10}{1+2\times(7)} = 10\frac{2}{3}$ kg/min when $t = 7$ min

7b As $t \longrightarrow \infty$, $R = 10 + \frac{10}{1+2\times\infty} = 10$ kg/min

7c $\frac{dR}{dt} = -20(1+2t)^{-2} = \frac{-20}{(1+2t)^2}$

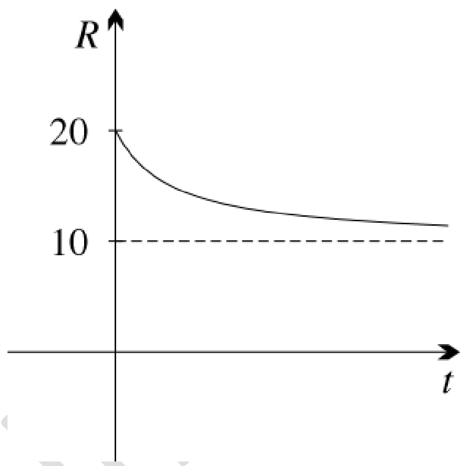
Since $(1+2t)^2 \geq 0$ for all t, $\frac{dR}{dt} = \frac{-20}{(1+2t)^2} < 0$ for all t

$\frac{d^2R}{dt^2} = 40 \times (1+2t)^{-3} \times (2) = 80(1+2t)^{-3}$

Since $80(1+2t)^{-3} = \frac{80}{(1+2t)^3} > 0$ for $t \geq 0$, $\frac{d^2R}{dt^2} > 0$ for $t = 0$ and for all positive values of t.

7d As it can be observed from the graph in 7e, the function is decreasing at a decreasing rate.

7e The graph of R starts decreasing from its initial value 20kg at $t = 0$ $\left(R = 10 + \frac{10}{1+2\times(0)} = 20\right)$ and decreasing at a decreasing rate while approaching the limiting value $R = 10$kg (as found in 7b).



8a $M = 9 \times (0) \times e^{-(0)} = 0$ when $t = 0$ and

$M = 9 \times (9) \times e^{-(9)} = 0.000001 \doteqdot 0.0$ when $t = 9$

8b $\frac{dM}{dt} = 9(e^{-t} - te^{-t}) = 9e^{-t}(1-t)$

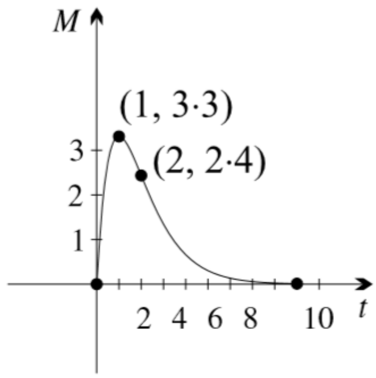
$9e^{-t}(1-t) = 0$ when $t = 1$ and $M(1) = 9e^{-1}$.

Therefore, the turning point is approximately $(1, 3.3)$

8c $\frac{d^2M}{dt^2} = 9(-e^{-t}(1-t) - e^{-t}) = 9e^{-t}(t-2)$ and $9e^{-t}(t-2) = 0$ when $t = 2$

$M(2) = 18e^{-2}$. Therefore, the stationary point is approximately $(2, 2.4)$

8d



8e $t = 1$ (refer to the graph)

8f $t = 0$ (when $\frac{dM}{dt} = 0$)

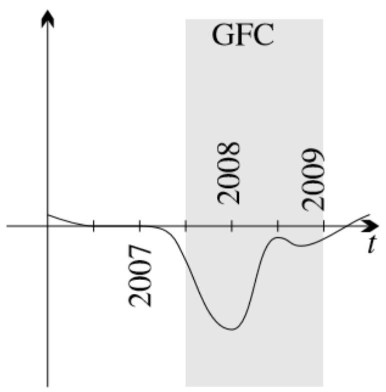
8g $t = 2$ (when $\frac{d^2M}{dt^2} = 0$)

9a The graph is decreasing steeply in 2008. Therefore, the crisis was at its most frightening in 2008.

9b The graph stops decreasing and stabilises in January 2009. Therefore, the stationary trend around January 2009 indicates the end of the crisis.

9c In 2008, the decrease of the graph slows down (this is when the decrease has a decreasing rate) and this may be the reason why an economist might have been optimistic, thinking that the crisis was going to end.

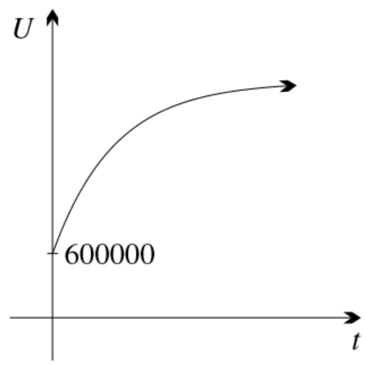
9d



10a The unemployment was increasing

10b The rate of increase in unemployment was decreasing

10c



11a $N(0) = \frac{A}{2+e^{-(0)}} = 30000$ then $A = 30000 \times 3 = 9 \times 10^5$

11b $N(1) = \frac{9\times10^5}{2+e^{-(1)}} \doteqdot 380\ 087$

11c When t is large, N is close to 4.5×10^5

11d

$$\frac{dN}{dt} = \frac{0 \times (2 + e^{-t}) - 9 \times 10^5(-e^{-t})}{(2 + e^{-t})^2} = \frac{9 \times 10^5 e^{-t}}{(2 + e^{-t})^2}$$

11e

$$\frac{N(A - 2N)}{A}$$

$$= \frac{NA - 2N^2}{A}$$

$$= N - \frac{2N^2}{A}$$

$$= \frac{A}{2 + e^{-t}} - \frac{2\left(\frac{A}{2 + e^{-t}}\right)^2}{A}$$

$$= \frac{A}{2 + e^{-t}} - \frac{2A}{(2 + e^{-t})^2}$$

$$= \frac{A(2 + e^{-t})}{(2 + e^{-t})^2} - \frac{2A}{(2 + e^{-t})^2}$$

$$= \frac{9 \times 10^5 e^{-t}}{(2 + e^{-t})^2}$$

Therefore, $\frac{dN}{dt} = \frac{N(A-2N)}{A}$

12a If $I = \frac{100}{C} \times \frac{dC}{dt}\%$ and $C(t) = -\frac{1}{5}t^3 + 3t^2 + 200$ then

$$I = \frac{100}{-\frac{1}{5}t^3 + 3t^2 + 200} \times \left(-\frac{3}{5}t^2 + 6t\right)\% \text{ or}$$

$$I = \frac{300t\left(2 - \frac{1}{5}t\right)}{-\frac{1}{5}t^3 + 3t^2 + 200}\%$$

12b

$$I = \frac{300 \times (4) \times \left(2 - \frac{1}{5} \times (4)\right)}{-\frac{1}{5}(4)^3 + 3(4)^2 + 200}\%$$

$$= \frac{1440}{-\frac{64}{5} + 48 + 200}\%$$

$$= \frac{1440}{248 - \frac{64}{5}}\%$$

$$= \frac{1440}{\frac{1176}{5}}\%$$

$$= \frac{300}{49}\%$$

$$\doteqdot 6.12\%$$

12c $I = \frac{300t\left(2-\frac{1}{5}t\right)}{-\frac{1}{5}t^3+3t^2+200} = 0$ when $= 0$, $2 - \frac{1}{5}t = 0$ or $t = 10$. $t = 10$ must be rejected because the model is demonstrating 8 years only.

13a $\phi(-x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(-x)^2}$

$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$

$= \phi(x)$

Since $\phi(-x) = \phi(x)$, $\phi(x)$ is an even function.

13b $\phi(x) > 0$ for all $x \in \mathbb{R}$ because $\frac{1}{\sqrt{2\pi}} > 0$ and $e^{-\frac{1}{2}x^2} > 0$ for all $x \in \mathbb{R}$

13c $\phi(0) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(0)^2} = \frac{1}{\sqrt{2\pi}}$ when $x = 0$ and

$$\lim_{x\to\infty} \phi(x) = \lim_{x\to\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} = \frac{1}{\sqrt{2\pi}} \times \lim_{x\to\infty} e^{-\frac{1}{2}x^2} = \frac{1}{\sqrt{2\pi}} \times 0 = 0$$

13d $\phi'(x) = \frac{1}{\sqrt{2\pi}} \times \left(-\frac{1}{2} \times 2x\right) \times e^{-\frac{1}{2}x^2}$

$= -\frac{1}{\sqrt{2\pi}} x e^{-\frac{1}{2}x^2}$

$= -x \times \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\right)$

$= -x\phi(x)$

The function $\phi(x)$ is decreasing when $\phi'(x) < 0$. Since $\phi(x) > 0$ for all $x \in \mathbb{R}$,

$\phi'(x) = -x\phi(x)$ is negative when $x > 0$.

13e $\phi''(x) = -1 \times \phi(x) + (-x) \times \phi'(x)$

$= -\phi(x) - x(-x\phi(x))$

$= -\phi(x) + x^2\phi(x)$

$= (x^2 - 1)\phi(x)$

$\phi''(x) = (x^2 - 1)\phi(x) = 0$ when $x = -1$ or $x = 1$ because $\phi(x) > 0$ for all $x \in \mathbb{R}$

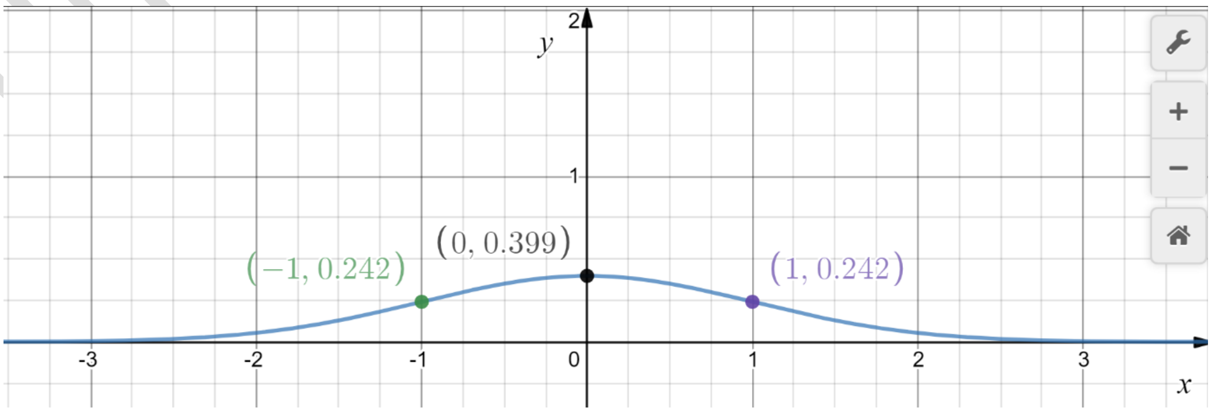
Therefore, there are points of inflection at $x = -1$ and $x = 1$.

13f $\phi(0) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(0)^2} \doteqdot 0.399$. Therefore, $(0, 0.399)$ is the y-intercept.

$\phi(-1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(-1)^2} \doteqdot 0.242$. Therefore, $(-1, 0.242)$ is an inflection point.

$\phi(1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(1)^2} \doteqdot 0.242$, Therefore, $(1, 0.242)$ is an inflection point.

And $\lim\limits_{x\to\infty} \phi(x) = \lim\limits_{x\to-\infty} \phi(x) = 0$



13g $\phi'(x) < 0$ and $\phi''(x) < 0$ in the interval $0 < x < 1$.

Therefore, $\phi(x)$ is decreasing at an increasing rate in the interval $0 \le x \le 1$.

$\phi'(x) < 0$ and $\phi''(x) > 0$ in the interval $x > 1$.

Therefore, $\phi(x)$ is decreasing at a decreasing rate in the interval $x \ge 1$.

13h The curve approaches the horizontal asymptote more slowly for larger x.

14a $y = 2e^{-a\times(0)}\cos(0) = 2 \times 1 \times 1 = 2$. Therefore, the y-intercept is $(0, 2)$

$y = 2e^{-ax}\cos x = 0$ when $\cos x = 0$ or $x = \frac{\pi}{2}$ or $x = \frac{3\pi}{2}$ or $x = \frac{5\pi}{2}$, etc.

Therefore, $\left(\frac{\pi}{2}, 0\right)$, $\left(\frac{3\pi}{2}, 0\right)$, $\left(\frac{5\pi}{2}, 0\right)$, etc are the x-intercepts.

14b $y = 2e^{-ax}\cos x$ then

$y' = 2 \times (-a) \times e^{-ax}\cos x + 2e^{-ax} \times (-\sin x)$

$y' = -2ae^{-ax}\cos x - 2e^{-ax}\sin x$

$y' = -2e^{-ax}(a\cos x + \sin x)$

14c $y' = -2e^{-ax}(a\cos x + \sin x)$ then

$y'' = -2 \times (-a) \times e^{-ax}(a\cos x + \sin x) + (-2e^{-ax}) \times (-a\sin x + \cos x)$

$y'' = -2e^{-ax}(-a^2\cos x - a\sin x - a\sin x + \cos x)$

$y'' = -2e^{-ax}\left((1 - a^2)\cos x - 2a\sin x\right)$

$y'' = 2e^{-ax}\left((a^2 - 1)\cos x + 2a\sin x\right)$

14d If $y' = -2e^{-ax}(a\cos x + \sin x)$ and $a = \tan\left(\frac{\pi}{12}\right)$ then

$y' = -2e^{-\tan\left(\frac{\pi}{12}\right)x}\left(\tan\left(\frac{\pi}{12}\right)\cos x + \sin x\right) = 0$ when

$x = n\pi - \frac{\pi}{12}$ where n is a natural number. (calculator)

Therefore, y has a stationary point at $x = \frac{11\pi}{12}$ when $n = 1$, $x = \frac{23\pi}{12}$ when $n = 2$,

$x = \frac{35\pi}{12}$ when $n = 3$, $x = \frac{47\pi}{12}$ when $n = 4$, etc.

14e If $a = \tan\left(\frac{\pi}{12}\right) = 2 - \sqrt{3}$ and $y'' = 2e^{-ax}\left((a^2 - 1)\cos x + 2a \sin x\right)$ then

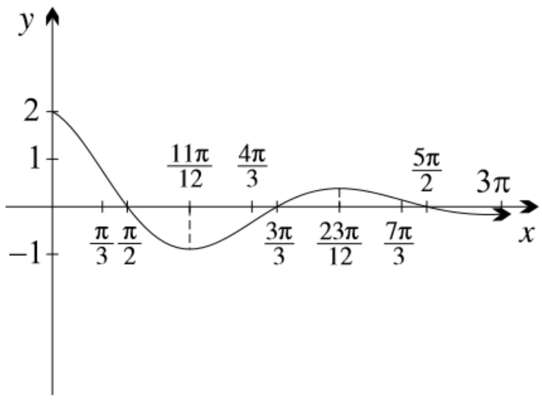
$y'' = 0$ when $2e^{-(2-\sqrt{3})x}\left(\left((2-\sqrt{3})^2 - 1\right)\cos x + 2(2-\sqrt{3})\sin x\right) = 0$

Or when $x = n\pi + \frac{\pi}{3}$ where n is a natural number. (calculator)

Therefore, y has an inflection point at $x = \frac{\pi}{3}$ when $n = 0$, $x = \frac{4\pi}{3}$ when $n = 1$,

$x = \frac{7\pi}{3}$ when $n = 2$, $x = \frac{10\pi}{3}$ when $n = 3$, $x = \frac{13\pi}{3}$ when $n = 4$, etc.

14f



Solutions to Exercise 9E

1a Side length $= x$ and $\frac{dx}{dt} = 0.1$ m/s

Area $= A = (x)^2$ and $\frac{dA}{dt} = 2x \times \frac{dx}{dt} = 2x \times 0.1 = 0.2x$ m²/s

1b $\frac{dA}{dt} = 0.2 \times 5 = 1$ m²/s when $x = 5$ metres.

1c $\frac{dA}{dt} = 1.4 = 0.2x$ then $x = \frac{1.4}{0.2} = 7$ metres.

1d $\frac{dA}{dt} = 0.6 = 0.2x$ then $x = \frac{0.6}{0.2} = 3$ metres and the area is $A = 9$ m²

2a $\frac{dl}{dt} = -\frac{1}{2}$ m/s where l is the diagonal of a square with side length $\frac{l}{\sqrt{2}}$

Then the area, A, of the square is $A = \frac{1}{2}l^2$

2b $\frac{dA}{dt} = 2 \times \frac{1}{2}l \times \frac{dl}{dt} = l \times -\frac{1}{2} = -\frac{1}{2}l$ m²/s

2c i $\frac{dA}{dt} = -\frac{1}{2} \times 10 = -5$ m²/s .

Therefore, the area is decreasing by 5 m²/s when $l = 10$ metres.

2c ii Since $A = \frac{1}{2}l^2$, $18 = \frac{1}{2}l^2$ then $l = 6$ metres when $A = 18$ m²

Hence, $\frac{dA}{dt} = -\frac{1}{2} \times 6 = -3$ m²/s .

Therefore, the area is decreasing by 3 m²/s when $A = 18$ m²

2d $\frac{dA}{dt} = -17 = -\frac{1}{2} \times l$ then $l = 34$ metres.

3a $\frac{dr}{dt} = 0.3$ m/s

$V = \frac{4}{3}\pi r^3$ then $\frac{dV}{dt} = 4\pi r^2 \times \frac{dr}{dt} = 4\pi r^2 \times 0.3 = 1.2\pi r^2$ m³/s

$\frac{dV}{dt} = 1.2\pi(2)^2 = 4.8\pi \doteqdot 15.1$ m³/s

3b The surface area of the sphere is $S = 4\pi r^2$ and

$\frac{dS}{dt} = 8\pi r \times \frac{dr}{dt} = 8\pi r \times 0.3 = 2.4\,\pi r$ m²/s

$\frac{dS}{dt} = 2.4\,\pi \times (4) = 9.6\pi \doteqdot 30.2$ m²/s

4a $V = \frac{4}{3}\pi r^3$ then $\frac{dV}{dt} = 3 \times \frac{4}{3}\pi r^2 \times \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}$

4b $\frac{dV}{dt} = 4\pi(15)^2 \frac{dr}{dt} = 200$ when $r = 15$ cm and $\frac{dV}{dt} = 200$ cm³/s

Therefore, $\frac{dr}{dt} = \frac{200}{4\pi(15)^2} = \frac{2}{9\pi}$ cm/s

4c $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$, $\frac{dr}{dt} = 0.5$ cm/s and $\frac{dV}{dt} = 200$ cm³/s, then

$200 = 4\pi r^2 \times 0.5$

$r^2 = \frac{100}{\pi}$

$r = \frac{10}{\sqrt{\pi}}$ cm. Therefore, $V = \frac{4}{3}\pi\left(\frac{10}{\sqrt{\pi}}\right)^3 = \frac{4000}{3\sqrt{\pi}}$ cm³

5a Volume of a cone is $V = \frac{1}{3}\pi r^2 \times h$ and

When $h = 2r$, $V = \frac{1}{3}\pi r^2 \times (2r) = \frac{2}{3}\pi r^3$.

5b $\frac{dV}{dt} = 3 \times \frac{2}{3}\pi r^2 \times \frac{dr}{dt} = 2\pi r^2 \times \frac{dr}{dt}$

When $\frac{dV}{dt} = 5$ cm³/s, $h = 10$ cm, $h = 2r$, and $r = 5$ cm

Then $5 = 2\pi(5)^2 \times \frac{dr}{dt}$ and $\frac{dr}{dt} = \frac{1}{10\pi}$ cm/s

6a $\tan\theta = \frac{1.5}{x}$ then $x = \frac{1.5}{\tan\theta}$ or $x = \frac{3}{2\tan\theta}$ or $x = \frac{3}{2\frac{\sin\theta}{\cos\theta}} = \frac{3}{2} \times \frac{\cos\theta}{\sin\theta}$

Hence, $\frac{dx}{d\theta} = \frac{d\left(\frac{3}{2}\times\frac{\cos\theta}{\sin\theta}\right)}{d\theta} = \frac{3}{2} \times \frac{-\sin\theta\times\sin\theta - \cos\theta\times\cos\theta}{\sin^2\theta} = \frac{3}{2} \times \frac{-1}{\sin^2\theta} = -\frac{3}{2\sin^2\theta}$

6b When $\theta = \frac{\pi}{3}$, $\frac{dx}{d\theta} = -\frac{3}{2\sin^2\left(\frac{\pi}{3}\right)} = -\frac{3}{2\left(\frac{\sqrt{3}}{2}\right)^2} = -\frac{3}{2\times\frac{3}{4}} = -2$ km/h

Hence, $\frac{dx}{d\theta} \times \frac{d\theta}{dt} = \frac{dx}{dt}$

$-2 \times \frac{d\theta}{dt} = 650$

$\frac{d\theta}{dt} = -325.$

Therefore, the angle θ is changing 325 radians per hour, anti-clockwise.

Converting radians to degrees and hours to seconds,

$\frac{d\theta}{dt} = \frac{\frac{180}{\pi}\times 325}{3600} = 5.17254 \doteqdot 5$ degrees per second

7 Let the distance between the ship and the cliff be x metres.

Then $\tan\theta = \frac{100}{x}$, where θ is the angle of depression.

Thus, $x = \frac{100}{\tan\theta}$ and $\frac{dx}{d\theta} = \frac{-100}{\sin^2\theta}$

Since $\frac{dx}{dt} = 50$ m/min, $\frac{dx}{d\theta} \times \frac{d\theta}{dt} = \frac{dx}{dt}$ and $\frac{d\theta}{dt} = \frac{\frac{dx}{dt}}{\frac{dx}{d\theta}}$

$\frac{d\theta}{dt} = \frac{50}{\frac{-100}{\sin^2\theta}}$. when $\theta = 15°$, $\frac{d\theta}{dt} = -0.033494$ radians per minute.

Converting the rate to degrees, $\frac{d\theta}{dt} = -0.033494 \times \frac{180}{\pi} = -1.91904 \doteqdot -2$ degrees. Therefore, the angle of depression is changing 2 degrees per minute.

8a $V = \frac{1}{2}(2h) \times h \times 100x = 100h^2x$ cm³

8b $2h \times 100x = 200hx$ is the surface area and the rate of change of the volume is 10% of the surface area. Therefore, $\frac{dV}{dt} = \frac{200hx}{10} = 20hx$ cm³/day

From 8a, $V = 100h^2x$ then $\frac{dV}{dt} = 200hx \times \frac{dh}{dt}$

Hence, $20hx = 200hx \times \frac{dh}{dt}$ and $\frac{dh}{dt} = \frac{1}{10} = 0.1$ cm/day. Therefore, the height of the water is changing at a constant rate.

9 Volume of a sphere is $V = \frac{4}{3}\pi r^3$ and the surface area of a sphere is $S = 4\pi r^2$ where r is the radius of the sphere.

$\frac{dV}{dt} = 3 \times \frac{4}{3}\pi r^2 \times \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}$ as r is increasing in time.

$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$ ....(1)

If at instant t, the increase in volume is equal to the surface area, then

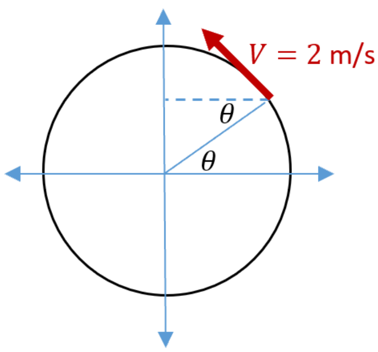
$\frac{dV}{dt} = 4\pi r^2$ ....(2)

Hence from (1) and (2), $4\pi r^2 = 4\pi r^2 \frac{dr}{dt}$. Therefore, $\frac{dr}{dt} = 1$.

10a $x^2 + y^2 = 1$ then $y^2 = 1 - x^2$

$y = \sqrt{1 - x^2}$

The x-component of the velocity is $V_x = V \times \cos\left(\frac{\pi}{2} - \theta\right) = 2 \times \sin\theta$



Since $\sin\theta = y$, $V_x = 2\sqrt{1 - x^2}$

Therefore, the rate of change in the x-coordinate is $-2\sqrt{1 - x^2}$.

10b The rate of change when $x = 0$ is $-2\sqrt{1-(0)^2} = -2$ m/s – as the point is crosses the y-axis, it is travelling horizontally at a speed of 2 m/s.

11a The length of the truck is C metres.

$L = Vt$ (the distance from the truck to the overtaking lane)

$2C + L = (V + at)t$ (the distance from the car to the overtaking lane)

$2C + L = Vt + at^2$

$2C + L = L + at^2$

$2C = at^2$

$a = \frac{2C}{t^2}$

Since $L = Vt$, $t = \frac{L}{V}$

Therefore,

$$a = \frac{2C}{\left(\frac{L}{V}\right)^2}$$

$$= \frac{2CV^2}{L^2} \text{ m/s}^2$$

11b $V_c = V + at = V + \frac{2CV^2}{L^2}t$ where $t = \frac{L}{V}$

Therefore,

$$V_c = V + \frac{2CV^2}{L^2} \times \frac{L}{V}$$

$$= V + \frac{2CV}{L}$$

$$= V\left(1 + \frac{2C}{L}\right)$$

11c As L decreases, the speed passing the truck increases, so the driver should wait if possible before beginning to accelerate. A similar result is obtained if the distance between car and truck is increased. Optimally, the driver should allow both L to decrease and C to increase.

11d To spend minimum time alongside the truck, the car should pass the truck with a maximum speed.

The speed of the car when it passes the truck is $V\left(1+\frac{2C}{L}\right)$

Since the upper speed limit is 100 km/h,

$V\left(1+\dfrac{2C}{L}\right) = 100$

$90\left(1+\frac{100}{L}\right) = 100 \quad (\text{since } C = 50 \text{ m})$

$1+\dfrac{100}{L} = \dfrac{10}{9}$

$\dfrac{100}{L} = \dfrac{10}{9} - 1$

$\dfrac{100}{L} = \dfrac{1}{9}$

$L = 900$

$L + C = 900 + 50 = 950$

Therefore, should the car begin to accelerate at least 950 metres before the overtaking lane if applying the objective in part c.

12a $x = r\cos\theta$ and the length of the chord is $2 \times r\sin\theta$

Therefore, the area, A_T, of the triangle (unshaded region in the sector)

is $A_T = \frac{1}{2} \times x \times 2r\sin\theta = x\,r\sin\theta$

Hence, $A_T = r^2\cos\theta\sin\theta$

The area of the sector is $\pi r^2 \times \frac{2\theta}{2\pi} = \theta r^2$

Therefore, the area of the segment is,

$A_s = \theta r^2 - r^2\cos\theta\sin\theta = r^2(\theta - \cos\theta\sin\theta)$

12b This is just two applications of the chain rule.

12c $\sin\theta = \frac{\sqrt{r^2-x^2}}{r}$

Differentiating both sides with respect to x,

$\cos\theta \times \frac{d\theta}{dx} = \frac{-2x}{2r\sqrt{r^2-x^2}}$

$\cos\theta \times \frac{d\theta}{dx} = \frac{-x}{r\sqrt{r^2-x^2}}$

Since $\cos\theta = \frac{x}{r}$, $\frac{x}{r} \times \frac{d\theta}{dx} = \frac{-x}{r\sqrt{r^2-x^2}}$

Therefore, $\frac{d\theta}{dx} = \frac{-1}{\sqrt{r^2-x^2}}$

12d $A = \theta r^2 - r^2\cos\theta\sin\theta = \theta r^2 - r^2\frac{\sin 2\theta}{2}$

$\frac{dA}{d\theta} = r^2 - r^2\frac{\cos 2\theta}{2} \times 2 = r^2 - r^2\cos 2\theta$

Since $\frac{dA}{dt} = \frac{dA}{d\theta} \times \frac{d\theta}{dx} \times \frac{dx}{dt}$, $r = 2$ and $\frac{dx}{dt} = -\sqrt{3}$ when $x = 1$,

$\frac{dA}{dt} = (r^2 - r^2\cos 2\theta) \times \frac{-1}{\sqrt{r^2-x^2}} \times -\sqrt{3}$

Moreover, when $r = 2$ and $x = 1$, $\cos\theta = \frac{1}{2}$. Thus, $\theta = \frac{\pi}{3}$.

$\frac{dA}{dt} = \left(2^2 - 2^2\cos\frac{2\pi}{3}\right) \times \frac{-1}{\sqrt{2^2-1^2}} \times -\sqrt{3} = \left(4 - 4 \times \frac{-1}{2}\right) \times \frac{-1}{\sqrt{3}} \times -\sqrt{3}$

$\frac{dA}{dt} = 6$

13a $\tan\alpha = \frac{h}{x+100}$ and $\tan\beta = \frac{h}{x}$

Hence, $h = \tan\alpha \times (x + 100)$ and $h = \tan\beta \times x$

Therefore, $x\tan\beta = \tan\alpha\,(x + 100)$

13b $x\tan\beta = \tan\alpha\,(x+100)$ (from 13a)

$x = \frac{\tan\alpha(x+100)}{\tan\beta}$ then $\frac{dx}{dt} = \frac{d\left(\frac{\tan\alpha(x+100)}{\tan\beta}\right)}{dt}$

$$\frac{dx}{dt} = \frac{\left(\dot{\alpha}\sec^2\alpha\,(x+100) + \tan\alpha\frac{dx}{dt}\right)\tan\beta - \tan\alpha\,(x+100)\sec^2\beta \times \dot{\beta}}{\tan^2\beta}$$

(from part a, $x\tan\beta = \tan\alpha\,(x+100)$)

$$\frac{dx}{dt} = \frac{\left(\dot{\alpha}\sec^2\alpha\,(x+100) + \tan\alpha\frac{dx}{dt}\right)\tan\beta - x\tan\beta\sec^2\beta \times \dot{\beta}}{\tan^2\beta}$$

$$\frac{dx}{dt} = \frac{\left(\dot{\alpha}\sec^2\alpha\,(x+100) + \tan\alpha\frac{dx}{dt}\right) - x\sec^2\beta \times \dot{\beta}}{\tan\beta}$$

$\tan\beta\frac{dx}{dt} = \dot{\alpha}(x+100)\sec^2\alpha + \tan\alpha\frac{dx}{dt} - \dot{\beta}x\sec^2\beta$

$\tan\beta\frac{dx}{dt} - \tan\alpha\frac{dx}{dt} = \dot{\alpha}(x+100)\sec^2\alpha - \dot{\beta}x\sec^2\beta$

$\frac{dx}{dt}(\tan\beta - \tan\alpha) = \dot{\alpha}(x+100)\sec^2\alpha - \dot{\beta}x\sec^2\beta$

$\frac{dx}{dt} = \frac{\dot{\alpha}(x+100)\sec^2\alpha - \dot{\beta}x\sec^2\beta}{\tan\beta - \tan\alpha}$

13c When $\alpha = \frac{\pi}{6}$ and $\beta = \frac{\pi}{4}$, and $x\tan\beta = \tan\alpha\,(x+100)$,

$x\tan\left(\frac{\pi}{4}\right) = \tan\left(\frac{\pi}{6}\right)(x+100)$

$x = \frac{\sqrt{3}}{3}(x+100)$

$x = 50\sqrt{3} + 50 = 50\left(\sqrt{3}+1\right)$

Since $\beta = \frac{\pi}{4}$, $h = x = 50\left(\sqrt{3}+1\right)$

13d Given that $\alpha = \frac{\pi}{6}$, $\beta = \frac{\pi}{4}$, $\frac{d\alpha}{dt} = \frac{5}{36}\left(\sqrt{3}-1\right)$, $\frac{d\beta}{dt} = \frac{5}{18}\left(\sqrt{3}-1\right)$

and $x = 50\left(\sqrt{3}+1\right)$ from part 13c,

$$\frac{dx}{dt} = \frac{\dot{\alpha}(x+100)\sec^2\alpha - \dot{\beta}x\sec^2\beta}{\tan\beta - \tan\alpha}$$

$$= \frac{\frac{5}{36}(\sqrt{3}-1)\times\left(\left(50(\sqrt{3}+1)\right)+100\right)\times\frac{4}{3}-\frac{5}{18}(\sqrt{3}-1)\times\left(50(\sqrt{3}+1)\right)\times 2}{1-\frac{\sqrt{3}}{3}}$$

$\frac{dx}{dt} \doteqdot -55.6$ km/h and the speed is approximately 55.6 km/h

14a $AP^2 = a^2 + x^2$ and $PB^2 = y^2 + b^2$ (Pythagoras's Theorem)

Hence, $AP = \sqrt{a^2 + x^2}$ and $PB = \sqrt{y^2 + b^2}$

Therefore, $APB = s = \sqrt{a^2 + x^2} + \sqrt{y^2 + b^2}$

14b $\frac{d(s)}{dx} = \frac{d(vt)}{dx} = \frac{d\left(\sqrt{a^2+x^2}+\sqrt{y^2+b^2}\right)}{dx}$, $v = \frac{s}{t}$ and $y = c - x$

Then, $t = \frac{s}{v}$ and

$$\frac{dt}{dx} = \frac{1}{v}\left(\frac{2x}{2\sqrt{a^2+x^2}} + \frac{2y\times\frac{dy}{dx}}{2\sqrt{(y)^2+b^2}}\right)$$

$$= \frac{1}{v}\left(\frac{x}{\sqrt{a^2+x^2}} - \frac{y}{\sqrt{(y)^2+b^2}}\right)$$

$$= \frac{1}{v}\left(\frac{x}{\sqrt{a^2\left(1+\frac{x^2}{a^2}\right)}} - \frac{y}{\sqrt{b^2\left(\frac{y^2}{b^2}+1\right)}}\right)$$

$$= \frac{1}{v}\left(\frac{x}{a\sqrt{1+\frac{x^2}{a^2}}} - \frac{y}{b\sqrt{\frac{y^2}{b^2}+1}}\right)$$

$$= \frac{1}{v}\left(\frac{\frac{x}{a}}{\sqrt{1+\left(\frac{x}{a}\right)^2}} - \frac{\frac{y}{b}}{\sqrt{1+\left(\frac{y}{b}\right)^2}}\right)$$

Therefore, $\frac{dt}{dx} = \frac{1}{v}\left(\frac{\frac{x}{a}}{\sqrt{1+\left(\frac{x}{a}\right)^2}} - \frac{\frac{y}{b}}{\sqrt{1+\left(\frac{y}{b}\right)^2}}\right)$ and $v\frac{dt}{dx} = \frac{\frac{x}{a}}{\sqrt{1+\left(\frac{x}{a}\right)^2}} - \frac{\frac{y}{b}}{\sqrt{1+\left(\frac{y}{b}\right)^2}}$

14c $v\frac{dt}{dx} = 0$ when $\frac{\frac{x}{a}}{\sqrt{1+\left(\frac{x}{a}\right)^2}} - \frac{\frac{y}{b}}{\sqrt{1+\left(\frac{y}{b}\right)^2}} = 0$ or

$$\frac{\frac{x}{a}}{\sqrt{1+\left(\frac{x}{a}\right)^2}} = \frac{\frac{y}{b}}{\sqrt{1+\left(\frac{y}{b}\right)^2}}$$

$$\frac{x}{a} \times \sqrt{1+\left(\frac{y}{b}\right)^2} = \frac{y}{b} \times \sqrt{1+\left(\frac{x}{a}\right)^2}$$

$$\left(\frac{x}{a}\right)^2 \times \left(1+\left(\frac{y}{b}\right)^2\right) = \left(\frac{y}{b}\right)^2 \times \left(1+\left(\frac{x}{a}\right)^2\right)$$

$$\left(\frac{x}{a}\right)^2 = \left(\frac{y}{b}\right)^2$$

$$\frac{x}{a} = \frac{y}{b}$$

14d

x		$\frac{x}{a} = \frac{y}{b}$	
$\frac{dt}{dx}$		0	
t	\	Minimum turning point	/

Because when the light hits the surface at P, it changes direction.

14e $\frac{x}{a} = \frac{y}{b}$ then $\cot\alpha = \cot\beta$. Therefore, $\alpha = \beta$.

Solutions to Exercise 9F

1a $\frac{dP}{dt} = 12t - 3t^2$ then $P = \int(12t - 3t^2)\,dt = 6t^2 - t^3 + C$

If $P = 25$ when $t = 0$, $6(0)^2 - (0)^3 + C = 25$ and $C = 25$

Therefore, $P = 6t^2 - t^3 + 25$

1b $\frac{dP}{dt} = 12t - 3t^2 = 0$ when $3t(4 - t) = 0$, $t = 0$ or $t = 4$.

x	-1	0	3	4	5
$\frac{dP}{dt}$	$-$	0	$+$	0	$-$
P	\	—	/	—	\

Therefore, the population reaches its maximum when $t = 4$ years.

1c $P = 6(4)^2 - (4)^3 + 25 = 57$ wallabies when $t = 4$

1d $\frac{d^2P}{dt^2} = 12 - 6t = 0$ when $t = 2$ years.

2a $\frac{dV}{dt} = 10t - 250 = 0$ when $t = 25$. Therefore, the water stops flowing after 25 minutes.

2b If $\frac{dV}{dt} = 10t - 250$ then $V = \int(10t - 250)\,dt = 5t^2 - 250t + C$

If $V = 20$ when $t = 25$ (when the water flow stops)

then $5(25)^2 - 250 \times (25) + C = 20$ and $C = 3145$.

Hence, $V = 5t^2 - 250t + 3145$

2c When $t = 0$ (initially) there was $5(0)^2 - 250 \times (0) + 3145 = 3145$ L of water

3a $\frac{dP}{dt} = -\frac{2}{t+1}$ then $P = -2\log_e(t+1) + C$

Given that $P = 6.8$ when $t = 0$,

$6.8 = -2\log_e\left((0)+1\right) + C$

$C = 6.8$. Therefore, $P = -2\log_e(t+1) + 6.8$

3b $0 = -2\log_e(t+1) + 6.8$

$2\log_e(t+1) = 6.8$

$\log_e(t+1) = 3.4$

$t + 1 = e^{3.4}$

$t = e^{3.4} - 1 \doteqdot 29$ days

4a $\frac{dV}{dt} = -2 + \frac{1}{10}(0) = -2$ when $t = 0$. Therefore, the initial flow rate is -2 m^3/s.

4b $\frac{dV}{dt} = -2 + \frac{1}{10}t = 0$ when $t = 20$.

Therefore, it takes 20 seconds to turn the tap off.

4c $\frac{dV}{dt} = -2 + \frac{1}{10}t$ then

$$V = \int \left(-2 + \frac{1}{10}t\right) dt$$

$$= -2t + \frac{1}{10} \times \frac{t^2}{2} + C$$

$$= -2t + \frac{t^2}{20} + C$$

If $V = 500$ when $t = 20$, then $V = -2 \times (20) + \frac{(20)^2}{20} + C = 500$ and $C = 520$

Therefore, $V = -2t + \frac{t^2}{20} + 520$

4d $V = -2 \times (0) + \frac{(0)^2}{20} + 520 = 520$ when $t = 0$.

Thus, initially, there were 520 m^3 water in the tank.

Since there are 500 m^3 of water in the tank at the end of 20 seconds,

$520 - 500 = 20$ m^3 of water is released during the time it takes to turn the tap off.

4e Initially, there is 520 m^3 water in the tank and 20 m^3 of water is released during the time it takes to turn the tap off. If 300 m^3 of water is going to be released, than $300 - 20 = 280$ m^3 of water should be released before gradually turning it off. Since the initial flow rate is $\frac{dV}{dt} = -2$ m^3/s and its speed is

$\left|\frac{dV}{dt}\right| = |-2| = 2$ m^3/s

$V = 2 \times t$, $280 = 2 \times t$

Hence, $t = 140$ seconds. Therefore, to release 300 m^3 water, the tap should be left fully on, for 2 minutes and 20 seconds, before gradually turning it off.

5a It does not, because $e^{-0.4t}$ is never equal to zero.

5b $\frac{dx}{dt} = e^{-0.4t}$ then $x = \frac{e^{-0.4t}}{-0.4} = -\frac{5}{2}e^{-0.4t} + C$

If the particle is at the origin initially, $x = 0$ when $t = 0$.

Hence, $0 = -\frac{5}{2}e^{-0.4\times(0)} + C$

$0 = -\frac{5}{2} + C$

$C = \frac{5}{2}$

Therefore, $x = \frac{5}{2} - \frac{5}{2}e^{-0.4t} = \frac{5}{2}(1 - e^{-0.4t})$

5c $1 = \frac{5}{2}(1 - e^{-0.4t})$

$\frac{2}{5} = 1 - e^{-0.4t}$

$e^{-0.4t} = \frac{3}{5}$

$\log_e\left(\frac{3}{5}\right) = -0.4t$

$t = \frac{\log_e\left(\frac{3}{5}\right)}{-0.4} \doteqdot 1.28$

5d For large values of t, x gets closer and closer to $\frac{5}{2}$.

6a The initial speed is when $t = 0$.

Hence, $\frac{dx}{dt} = 250\left(e^{-0.2(0)} - 1\right) = 0$ and $\frac{dx}{dt} = 0$.

6b The eventual speed is when t is a large number.

Thus, $\left|\frac{dx}{dt}\right| = \left|250\left(e^{-0.2(\infty)} - 1\right)\right| = 250$ m/s

6c $\frac{dx}{dt} = 250(e^{-0.2t} - 1)$

then $x = \int\left(250(e^{-0.2t} - 1)\right) dt = 250 \times \left(\frac{e^{-0.2t}}{-0.2}\right) - 250t + C$

If $x = 200$ when $t = 0$, then $250 \times \left(\frac{e^{-0.2\times(0)}}{-0.2}\right) - 250 \times (0) + C = 200$

$250 \times \left(\frac{1}{-\frac{1}{5}}\right) - 250 \times (0) + C = 200$

$(250 \times -5) + C = 200$

$-1250 + C = 200$

$C = 1450$

Therefore, $x = 250 \times \left(\frac{e^{-0.2t}}{-0.2}\right) - 250t + 1450$

7a $\frac{dI}{dt} = -5 + 4\cos\left(\frac{\pi}{12}t\right)$ then

$I = \int\left(-5 + 4\cos\left(\frac{\pi}{12}t\right)\right)dt = -5t + \frac{4}{\frac{\pi}{12}}\sin\left(\frac{\pi}{12}t\right) + C$

$I = -5t + \frac{48}{\pi}\sin\left(\frac{\pi}{12}t\right) + C$

Since $I = 18\,000$ when $t = 0$,

$-5 \times (0) + \frac{48}{\pi}\sin\left(\frac{\pi}{12} \times (0)\right) + C = 18\,000$. Hence, $C = 18\,000$

Therefore, $I = 18\,000 - 5t + \frac{48}{\pi}\sin\left(\frac{\pi}{12}t\right)$

7b $\frac{dI}{dt} = -5 + 4\cos\left(\frac{\pi}{12}t\right)$ is negative for all $t \in \mathbb{R}$ because

$-1 \le \cos\left(\frac{\pi}{12}t\right) \le 1$ for all $t \in \mathbb{R}$. Hence, $-4 \le 4\cos\left(\frac{\pi}{12}t\right) \le 4$ and

$-9 \le -5 + 4\cos\left(\frac{\pi}{12}t\right) \le -1$

7c $I = 18\,000 - 5 \times (120 \times 24) + \frac{48}{\pi}\sin\left(\frac{\pi}{12} \times (120 \times 24)\right)$

$= 18\,000 - 14\,400 + \frac{48}{\pi} \times 0$

$= 18\,000 - 14\,400$

$= 3\,600$ tonnes when $t = 120 \times 24$ hours.

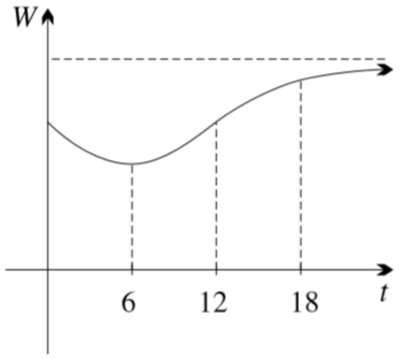
8a It was decreasing for the first 6 months and increasing thereafter.

8b after 6 months

8c after 12 months

8d It appears to have stabilised, increasing towards a limiting value.

8e



9a

$$\frac{d\theta}{dt} = \frac{1}{1+t^2}$$

$$\theta = \int \frac{1}{1+t^2}\,dt$$

$$\theta = \tan^{-1} t + C$$

When $t = 0, \theta = \frac{\pi}{4}$,

$$\frac{\pi}{4} = \tan^{-1} 0 + C$$

$$\frac{\pi}{4} = 0 + C$$

$$C = \frac{\pi}{4}$$

Hence

$$\theta = \tan^{-1} t + \frac{\pi}{4}$$

9b

$$\theta = \tan^{-1} t + \frac{\pi}{4}$$

$$\theta - \frac{\pi}{4} = \tan^{-1} t$$

$$t = \tan\left(\theta - \frac{\pi}{4}\right)$$

9c

$\theta = \tan^{-1} t + \frac{\pi}{4}$

For $t = 0$, we know that $\theta = \frac{\pi}{4}$.

As $t \to \infty$, $\tan^{-1} t \to \frac{\pi}{2}$ so $\theta \to \frac{\pi}{2} + \frac{\pi}{4}$ or $\theta \to \frac{3\pi}{4}$

Hence $\frac{\pi}{4} \le \theta < \frac{3\pi}{4}$.

This means that θ never moves through an angle of more than $\left(\frac{3\pi}{4} - \frac{\pi}{4}\right)$ or $\frac{\pi}{2}$.

10a $\frac{dW}{dt} = 1.2 - \cos^2\left(\frac{\pi}{12}t\right)$ then

$\frac{d^2W}{dt^2} = -2 \times \cos\left(\frac{\pi}{12}t\right) \times \left(-\sin\left(\frac{\pi}{12}t\right)\right) \times \frac{\pi}{12}$

$= \frac{\pi}{12}\sin\left(\frac{\pi}{6}t\right)$ (using the identity $2\sin a \cos a = \sin 2a$)

$\frac{d^2W}{dt^2} = 0$ when $\frac{\pi}{12}\sin\left(\frac{\pi}{6}t\right) = 0$ or $t = 0$ or $t = 6$ months.

Therefore, the flow rate is maximum at the beginning of July.

10b $W = \int\left(1.2 - \cos^2\left(\frac{\pi}{12}t\right)\right)dt = \int\left(1.2 - \left(\frac{\cos\left(\frac{\pi}{6}t\right)+1}{2}\right)\right)dt$

(Here, use the identity $\cos 2a = \cos^2 a - 1$ or $\frac{\cos 2a+1}{2} = \cos^2 a$)

$= \int 1.2\,dt - \frac{1}{2}\int\left(\cos\left(\frac{\pi}{6}t\right) + 1\right)dt$

$= \int 1.2\,dt - \frac{1}{2}\int \cos\left(\frac{\pi}{6}t\right)dt - \frac{1}{2}\int 1\,dt$

$= 1.2t - \frac{\frac{1}{2}}{\frac{\pi}{6}}\sin\left(\frac{\pi}{6}t\right) - \frac{1}{2}t + C$

$= 0.7t - \frac{3}{\pi}\sin\left(\frac{\pi}{6}t\right) + C$

Given that $W = 0$ when $t = 0$

$W = 0.7 \times (0) - \frac{3}{\pi}\sin\left(\frac{\pi}{6} \times (0)\right) + C = 0$ then $C = 0$

Therefore, $W = 0.7t - \frac{3}{\pi}\sin\left(\frac{\pi}{6}t\right)$

10c $W = 0.7 \times (3 \times 12) - \frac{3}{\pi}\sin\left(\frac{\pi}{6} \times (3 \times 12)\right) = 25.2$ tonnes $= 25\,200$ m^3

Therefore, the dam will be full in 3 years.

11a

$$\frac{dr}{dt} = -k$$

$$r = \int -k\,dt$$

$$r = -kt + C$$

When $t = 0, r = \frac{5}{2}$,

$$\frac{5}{2} = 0 + C$$

$$C = \frac{5}{2}$$

Hence

$$r = -kt + \frac{5}{2}$$

$$r = \frac{5}{2} - kt$$

11b When $t = 12, r = 0$, so $r = \frac{5}{2} - kt$ becomes

$$0 = \frac{5}{2} - k \times 12$$

$$12k = \frac{5}{2}$$

$$k = \frac{5}{24}$$

12a Volume of a cone with radius r and height h is $V_c = \frac{1}{3}\pi r^2 h$ and if the apex angle is 90° then $h = r$. Volume of a sphere with radius r is $V_s = \frac{4}{3}\pi r^3$

The ratio $\frac{V_s}{V_c} = \frac{\frac{4}{3}\pi r^3}{\frac{1}{3}\pi r^3} = 4$. Therefore, V_c is one quarter of V_s

12b $\frac{dV_C}{dt} = 3 \times \frac{1}{3}\pi r^2 \times \frac{dr}{dt}$

$0.5 = \pi r^2 \times \frac{dr}{dt}$

$\frac{dr}{dt} = \frac{1}{2\pi r^2}$

12c $\frac{dr}{dt} = \frac{1}{2\pi r^2}$

$\frac{dt}{dr} = 2\pi r^2$

$dt = 2\pi r^2\, dr$

$\int dt = \int 2\pi r^2\, dr$

$t = 2\pi \frac{r^3}{3} + C$

When $t = 0, r = 10$ then $0 = 2\pi \frac{(10)^3}{3} + C$ and $C = -2\pi \frac{(10)^3}{3}$

Therefore, $t = 2\pi \frac{r^3}{3} - 2\pi \frac{(10)^3}{3} = \frac{2\pi}{3}(r^3 - 1000)$

12d Since $r = h$,

$\frac{2\pi}{3}((12)^3 - 1000) - \frac{2\pi}{3}((10)^3 - 1000) = t_{final} - t_{initial}$

$t_{final} - t_{initial} = 1524.72$ seconds

$t_{final} - 0 = 25.412$ minutes

Time taken $=$ 25 minutes and 25 seconds

13a $y^2 = 16 - x^2$

$V = \pi \int_{-4}^{-h} (16 - x^2)\, dx = \left[\pi\left(16 \times (-h) - \frac{(-h)^3}{3}\right)\right] - \left[\pi\left(16 \times (-4) - \frac{(-4)^3}{3}\right)\right]$

$V = \frac{\pi}{3}(128 - 48h + h^3)$

13b i $y^2 = 16 - x^2$ and $r^2 = 16 - h^2$ when $x = -h$ and $y = r$

Therefore, $A = \pi(16 - h^2)$

13b ii $V = \frac{\pi}{3}(128 - 48h + h^3)$

$\frac{dV}{dt} = \frac{\pi}{3}(-48 + 3h^2)\frac{dh}{dt} = \pi(-16 + h^2)\frac{dh}{dt} = -\pi(16 - h^2)\frac{dh}{dt}$

Given that $\frac{dV}{dt} = -kA = -k\pi(16 - h^2)$,

$-k\pi(16 - h^2) = -\pi(16 - h^2)\frac{dh}{dt}$

$\frac{dh}{dt} = k$ Since it is decreasing, the rate is $-k$

13b iii If the initial height is 2 cm and the rate that it decreases is 0.025 cm/min,

then the water evaporates in $\frac{2}{0.025} = 80$ minutes (1 hr and 20 mins)

Solutions to Chapter review

Let C be a constant.

1a $x = 20 + t^2$ then

$x = 20 + (2)^2 = 24$ when $t = 2$ and $x = 20 + (4)^2 = 36$ when $t = 4$

t	2	4
x	24	36

Average velocity

$= \dfrac{36 - 24}{4 - 2}$

$= 6$ cm/s

1b $x = (t + 2)^2$ then

$x = \big((2) + 2\big)^2 = 16$ when $t = 2$ and $x = \big((4) + 2\big)^2 = 36$ when $t = 4$

t	2	4
x	16	36

Average velocity

$= \dfrac{36 - 16}{4 - 2}$

$= 10$ cm/s

1c $x = t^2 - 6t$ then

$x = (2)^2 - 6 \times (2) = -8$ when $t = 2$ and $x = (4)^2 - 6 \times (4) = -8$ when $t = 4$

t	2	4
x	-8	-8

Average velocity

$= \dfrac{-8-(-8)}{4-2}$

$= 0 \text{ cm/s}$

1d $\quad x = 3^t$ then

$x = 3^{(2)} = 9$ when $t = 2$ and $x = 3^{(4)} = 81$ when $t = 4$

t	2	4
x	9	81

Average velocity

$= \dfrac{81-9}{4-2}$

$= 36 \text{ cm/s}$

2a $\quad x = 40t - t^2$, $x = 175$ m when $t = 5$ s

$\dot{x} = 40 - 2t$, $\dot{x} = 30$ m/s when $t = 5$ s

$\ddot{x} = -2$, $\ddot{x} = -2 \text{ m/s}^2$ when $t = 5$ s

2b $\quad x = t^3 - 25t$, $x = 0$ m when $t = 5$ s

$\dot{x} = 3t^2 - 25$, $\dot{x} = 50$ m/s when $t = 5$ s

$\ddot{x} = 6t$, $\ddot{x} = 30 \text{ m/s}^2$ when $t = 5$ s

2c $\quad x = 4(t-3)^2$, $x = 16$ m when $t = 5$ s

$\dot{x} = 8(t-3)$, $\dot{x} = 16$ m/s when $t = 5$ s

$\ddot{x} = 8$, $\ddot{x} = 8 \text{ m/s}^2$ when $t = 5$ s

2d $\quad x = 50 - t^4$, $x = -575$ m when $t = 5$ s

$\dot{x} = -4t^3$, $\dot{x} = -500$ m/s when $t = 5$ s

$\ddot{x} = -12t^2$, $\ddot{x} = -300$ m/s² when $t = 5$ s

2e $x = 4\sin\pi t$, $x = 0$ m when $t = 5$ s

$\dot{x} = 4\pi\cos\pi t$, $\dot{x} = -4\pi$ m/s when $t = 5$ s

$\ddot{x} = -4\pi^2\sin\pi t$, $\ddot{x} = 0$ m/s² when $t = 5$ s

2f $x = 7e^{3t-15}$, $x = 7$ m when $t = 5$ s

$\dot{x} = 21e^{3t-15}$, $\dot{x} = 21$ m/s when $t = 5$ s

$\ddot{x} = 63e^{3t-15}$, $\ddot{x} = 63$ m/s² when $t = 5$ s

3a $x = 16t - t^2$ then $v = \frac{dx}{dt} = 16 - 2t$ and $a = \frac{dv}{dt} = -2$ m/s²

3b When $t = 10$ seconds,

$x = 16 \times (10) - (10)^2 = 60$ m

$v = \frac{dx}{dt} = 16 - 2 \times (10) = -4$ m/s

$|v| = \left|\frac{dx}{dt}\right| = |16 - 2 \times (10)| = |-4| = 4$ m/s

$a = \frac{dv}{dt} = -2$ m/s²

3c $x = 16t - t^2 = 0$ when $= t(16 - t) = 0$, $t = 0$ or $t = 16$ seconds. Thus, the ball is back at its starting point at $t = 16$ seconds.

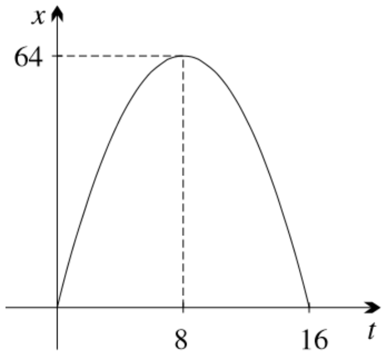
At $t = 16$ seconds, $v = 16 - 2(16) = -16$ m/s.

3d $v = 16 - 2t = 0$ when $t = 8$.

Therefore, the ball is farthest up the plane after 8 seconds and it is

$x = 16 \times (8) - (8)^2 = 64$ metres.

3e



4a $a = \frac{dv}{dt} = 0$

$\int 7dt = 7t + C$

$7 \times (0) + C = 4$ when $t = 0$, then $C = 4$

Therefore, $x = 7t + 4$

4b $a = \frac{dv}{dt} = -18t$

$\int (4 - 9t^2)dt = 4t - 3t^3 + C$

$4(0) - 3(0)^3 + C = 4$ when $t = 0$, then $C = 4$

Therefore, $x = 4t - 3t^3 + 4$

4c $a = \frac{dv}{dt} = 2(t - 1)$

$\int (t - 1)^2 dt = \frac{1}{3}(t - 1)^3 + C$

$\frac{1}{3}\left((0) - 1\right)^3 + C = 4$ when $t = 0$, then $C = 4\frac{1}{3}$

Therefore, $x = \frac{1}{3}(t - 1)^3 + 4\frac{1}{3}$

4d $a = \frac{dv}{dt} = 0$

$\int 0\, dt = 0 \times t + C$

$C = 4$ when $t = 0$

Therefore, $x = 4$

4e $a = \frac{dv}{dt} = -24\sin\ (2t)$

$\int 12\cos(2t)dt = 6\sin(2t) + C$

$6\sin(2t) + C = 4$ when $t = 0$, then $C = 4$

Therefore, $x = 6\sin(2t) + 4$

4f $a = \frac{dv}{dt} = -36e^{-3t}$

$\int 12e^{-3t}dt = -4e^{-3t} + C$

$-4e^{-3t} + C = 4$ when $t = 0$, then $C = 8$

Therefore, $x = -4e^{-3t} + 8$

5a $a = 6t + 2$ then $v = \int(6t + 2)\, dt = 3t^2 + 2t + C$

If $v = 0$ when $t = 0$ then $3(0)^2 + 2 \times (0) + C = 0$

Hence, $C = 0$. Therefore, $v = 3t^2 + 2t$.

$v = 3t^2 + 2t$ then $x = \int(3t^2 + 2t)\, dt = t^3 + t^2 + C$

If $x = 2$ when $t = 0$ then $(0)^3 + (0)^2 + C = 2$

Hence, $C = 2$. Therefore, $x = t^3 + t^2 + 2$.

5b $a = -8$ then $v = \int(-8)\, dt = -8t + C$

If $v = 0$ when $t = 0$ then $-8 \times (0) + C = 0$

Hence, $C = 0$. Therefore, $v = -8t$.

$v = -8t$ then $x = \int(-8t)\, dt = -4t^2 + c$

If $x = 2$ when $t = 0$ then $-4(0)^2 + c = 2$

Hence, $c = 2$. Therefore, $x = -4t^2 + 2$.

5c $a = 36t^2 - 4$ then $v = \int (36t^2 - 4)\,dt = 12t^3 - 4t + C$

If $v = 0$ when $t = 0$ then $12(0)^3 - 4 \times (0) + C = 0$

Hence, $C = 0$. Therefore, $v = 12t^3 - 4t$.

$v = 12t^3 - 4t$ then $x = \int (12t^3 - 4t)\,dt = 3t^4 - 2t^2 + C$

If $x = 2$ when $t = 0$ then $3(0)^4 - 2(0)^2 + C = 2$

Hence, $C = 2$. Therefore, $x = 3t^4 - 2t^2 + 2$.

5d $a = 0$ then $v = \int (0)\,dt = 0t + C$

If $v = 0$ when $t = 0$ then $0 \times (0) + C = 0$

Hence, $C = 0$. Therefore, $v = 0$.

$v = 0$ then $x = \int (0\,)\,dt = 0t + C$

If $x = 2$ when $t = 0$ then $0 \times (0) + C = 2$

Hence, $C = 2$. Therefore, $x = 2$.

5e $a = 5\cos(t)$ then $v = \int 5\cos(t)\ dt = 5\sin(t) + C$

If $v = 0$ when $t = 0$ then $v = -5\sin(0) + C = 0$

Hence, $C = 0$. Therefore, $v = -5\sin(t)$.

$v = 5\sin(t)$ then $x = \int 5\sin(t)\,dt = -5\cos(t) + C$

If $x = 2$ when $t = 0$ then $-5\cos(0) + C = 2$

Hence, $C = 7$. Therefore, $x = 7 - 5\cos(t)$.

5f $a = 7e^t$ then $v = \int 7e^t\,dt = 7e^t + c$

If $v = 0$ when $t = 0$ then $7e^{(0)} + C = 0$

Hence, $C = -7$. Therefore, $v = 7e^t - 7$.

$v = 7e^t - 7$ then $x = \int (7e^t - 7\,)\,dt = 7e^t - 7t + C$

If $x = 2$ when $t = 0$ then $7e^{(0)} - 7 \times (0) + C = 2$

Hence, $C = -5$. Therefore, $x = 7e^t - 7t - 5$.

6a $\ddot{x} = 6t$ then $\dot{x} = 3t^2 + C$.

$\dot{x} = -12$ when $t = 0$ then $C = -12$.

$\dot{x} = 3t^2 - 12$ then $x = t^3 - 12t + C$ (initially at the origin, then $c = 0$)

Therefore, $x = t^3 - 12t$

6b $\dot{x} = 3(2)^2 - 12 = 0$ when $t = 2$

6c $x = (2)^3 - 12(2) = 8 - 24 = -16$.

16 cm on the negative side of the origin.

6d $x = t^3 - 12t = 0$ when $t(t^2 - 12) = 0$. Therefore, $x = 0$ when $t = 2\sqrt{3}$ seconds.

$\dot{x} = 3\left(2\sqrt{3}\right)^2 - 12 = 24$ cm/s

$\ddot{x} = 6\left(2\sqrt{3}\right) = 12\sqrt{3}$ cm/s^2

6e As $t \to \infty, x \to \infty$ and $v \to \infty$.

7a The acceleration function is $a = -10$ because the gravitational acceleration on earth is close to the number 9.8 and since it is stated that the upwards motion is positive, $a = -10$.

7b When $= 0$, $v = 40$ m/s , $x = 45$ metres and

$v = \int(-10)\,dt = -10t + c$

Thus, $40 = -10 \times (0) + c$ and $c = 40$. Therefore, $v = -10t + 40$

$x = \int(-10t + 40)\,dt = -5t^2 + 40t + c$

Thus, $45 = -5(0)^2 + 40 \times (0) + c$ and $c = 45$. Therefore, $x = -5t^2 + 40t + 45$

7c The stone reaches its maximum height when its velocity is zero and

$v = -10t + 40 = 0$ when $t = 4$ seconds.

Thus, its maximum height is: $x = -5(4)^2 + 40 \times (4) + 45 = 125$ metres

7d $x = -5t^2 + 40t + 45 = 0$ when $-5(t^2 - 8t - 9) = 0$

$-5(t + 1)(t - 9) = 0$ or $t = 9$. Therefore, the flight time before the stone hits the ground is 9 seconds.

7e $|v| = |-10 \times (9) + 40| = |-50| = 50$ when $t = 9$.

Therefore, the speed of the stone is 50 m/s when it hits the ground.

7f $x = -5(1)^2 + 40 \times (1) + 45 = 80$ metres is the height of the stone after 1 second.

$x = -5(2)^2 + 40 \times (2) + 45 = 105$ metres is the height of the stone after 2 seconds.

7g The average velocity during the 2nd second is $\frac{105-80}{2-1} = 25$ m/s

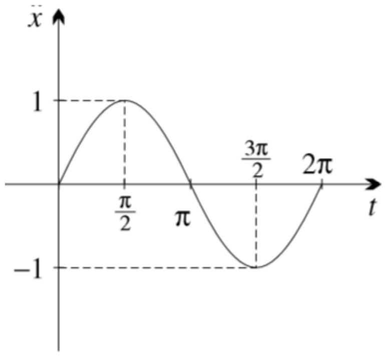
8a $\ddot{x} = \sin(0) = 0$ when $t = 0$

$\ddot{x} = \sin\left(\frac{\pi}{2}\right) = 1$ when $t = \frac{\pi}{2}$

$\ddot{x} = \sin(\pi) = 0$ when $t = \pi$

$\ddot{x} = \sin\left(\frac{3\pi}{2}\right) = -1$ when $t = \frac{3\pi}{2}$

$\ddot{x} = \sin(2\pi) = 0$ when $t = 2\pi$



8b $t = \pi$ and $t = 2\pi$

8c $\dot{x} = \int \sin(t)\, dt = -\cos(t) + c$

$\dot{x} = -\cos(0) + c = -1$ when $t = 0$

Thus, $c = 0$.

Therefore, $\dot{x} = -\cos(t)$

8d $\dot{x} = -\cos(t) = 0$ for the first time, when $t = \frac{\pi}{2}$ seconds

8e i $x = \int -\cos(t)\, dt = -\sin(t) + c$

$x = -\sin(0) + c = 5$ (initially at $x = 5$)

Then $c = 5$ and therefore, $x = -\sin(t) + 5$

8e ii When $t = \frac{\pi}{2}$, the body is at $x = -\sin\left(\frac{\pi}{2}\right) + 5 = 4$ metres away from the origin, in the positive direction.

9a $v = 20\, e^{-(0)} = 20$ m/s when $t = 0$ seconds.

9b Because $v = 20\, e^{-t} > 0$ for all $t \in \mathbb{R}$

9c $a = \frac{dv}{dt} = \frac{d(20\,e^{-t})}{dt} = -20e^{-t}$

9d $a = -20e^{-(0)} = -20\text{ m/s}^2$ at $t = 0$

9e $x = \int 20e^{-t}\,dt = -20e^{-t} + c$ and $x = 0$ when $t = 0$. Then,

$-20e^{-(0)} + c = 0$ and $c = 20$

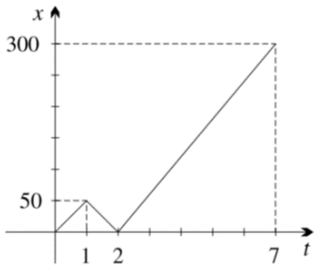
Therefore, $x = -20e^{-t} + 20$

9f As t increases, a converges to zero.

v converges to zero because the acceleration is negative

x converges to 20 metres.

10a



10b $50 + 50 + 300 = 400$ km.

10c Average speed

$= \frac{\text{total distance travelled}}{\text{time taken}}$

$= \frac{400}{7}$

$= 57\frac{1}{7}$ km/hr

11a He started at $x = 20$ when $t = 0$ and his initial speed was 0 m/s because the graph has a minimum turning point at $x = 0$.

11b i Average velocity

$$= \frac{80 - 40}{10 - 5}$$

$= 8$ m/s

11b ii Average velocity

$$= \frac{100 - 100}{25 - 15}$$

$= 0$ m/s

11b iii Average velocity

$$= \frac{0 - 80}{40 - 30}$$

$= -8$ m/s

11c i north of the oak tree

11c ii south of the oak tree

11c iii south of the oak tree

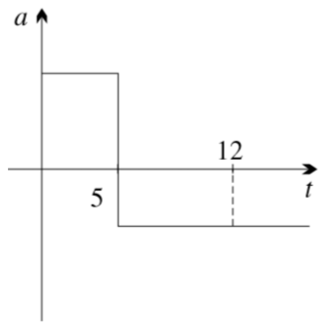
12a at $t = 5$

12b at $t = 0$ and $t = 12$ seconds, because the velocity is zero. The motor moves upwards in the interval $0 < t < 12$ and downwards when $t > 12$.

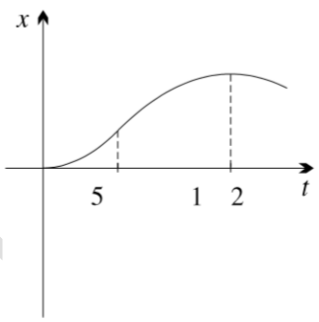
12c The motor accelerates upwards in the interval $0 < t < 5$ and downwards when $t > 5$.

12d at $t = 12$, when the velocity was zero.

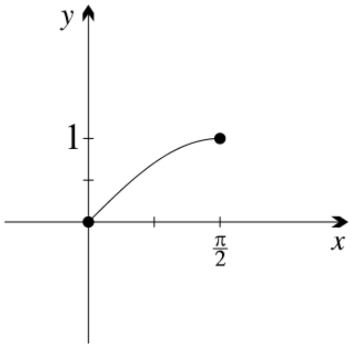
12e The motor has a constant acceleration throughout its motion. $\ddot{x} > 0$ the first 5 seconds and $\ddot{x} < 0$ the rest of the time.



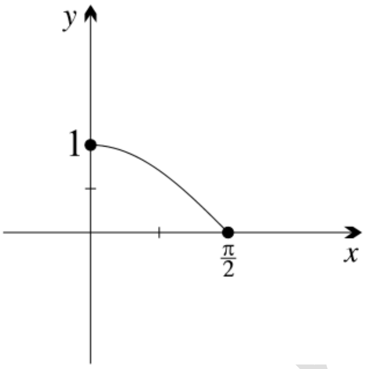
12f Since the motor goes in the positive direction until $t = 12$ seconds, it gets further away from the origin even though it slows down after the 5th second.



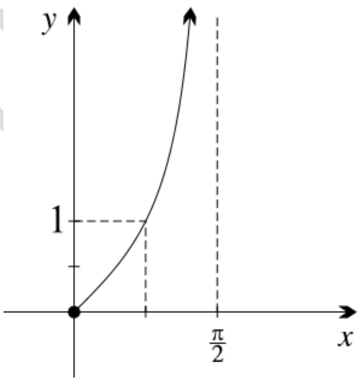
13a i $\sin(0) = 0$ and $\sin\left(\frac{\pi}{2}\right) = 1$



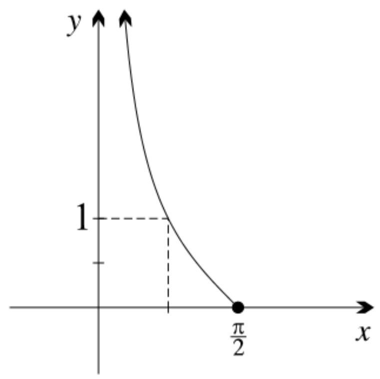
13a ii $\cos(0) = 1$ and $\cos\left(\frac{\pi}{2}\right) = 0$



13a iii $\tan(0) = 0,\ \tan\left(\frac{\pi}{4}\right) = 1$ and $\tan\left(\frac{\pi}{2}\right) = \text{undefined}$



13a iv $\cot(0) = \text{undefined},\ \cot\left(\frac{\pi}{4}\right) = 1$ and $\cot\left(\frac{\pi}{2}\right) = 0$



13b i $y = \sin(x)$

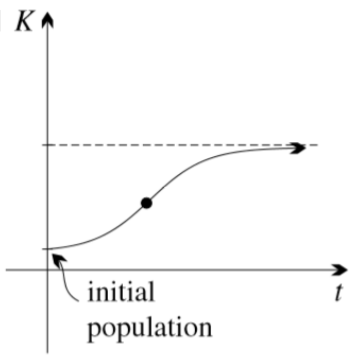
13b ii $y = \cos(x)$

13b iii $y = \cot(x)$

13b iv $y = \tan(x)$

14a Initially K increases at an increasing rate so the graph is concave up. Then K increases at a decreasing rate so is concave down. The change in concavity coincides with the inflection point.

14b



15a $V = 3\left(50 - 2 \times (0)\right)^2 = 7\,500$ L when $t = 0$.

15b $\frac{dV}{dt} = 6\,(50 - 2t) \times (-2) = -12\,(50 - 2t)$

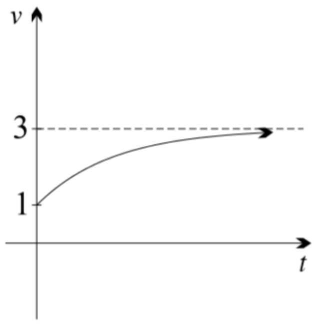
15c $\frac{dV}{dt} < 0$ in the given domain.

15d $\frac{d^2V}{dt^2} = 24 > 0$ for all t. Therefore, the outflow decreases.

16a The initial velocity of the particle is $3 - 2e^{-\frac{1}{5}\times(0)} = 1$ when $t = 0$.

16b explains why the graph is increasing when $t > 0$

16c explains why there is a horizontal asymptote at $y = 3$.



16b $\frac{dx}{dt} = 3 - 2e^{-\frac{1}{5}t}$ then $\frac{d^2x}{dt^2} = 0.4e^{-\frac{1}{5}t}$.

$\frac{d^2x}{dt^2} > 0$ for all t. Thus, $\frac{d^2x}{dt^2}$ is increasing for all t.

Therefore, $\ddot{x}$ increases so it accelerates.

16c $\frac{dx}{dt} = 3 - 2e^{-\frac{1}{5}t}$ As t gets larger and larger, $e^{-\frac{1}{5}t} \to 0$.

Hence, $t \to \infty$ then $\frac{dx}{dt} \to 3$.

Therefore, the particle reaches the velocity 3 m/s eventually,

and the graph has a horizontal asymptote at $y = 3$.

16d $x = \int \frac{dx}{dt} = \int \left(3 - 2e^{-\frac{1}{5}t}\right) dt = 3t + 10e^{-\frac{1}{5}t} + c$

The particle is at the origin initially. Therefore, $3 \times (0) + 10e^{-\frac{1}{5}\times(0)} + c = 0$,

$c = -10$ and $x = 3t + 10e^{-\frac{1}{5}t} - 10$ or $x = 3t + 10\left(e^{-\frac{1}{5}t} - 1\right)$

17a

$\frac{dV}{dt} = \frac{2}{5}t - 20$

$V = \frac{1}{5}t^2 - 20t + C$

At $t = 0, V = 500$

$500 = 0 - 0 + C$

$C = 500$

So $V = \frac{1}{5}t^2 - 20t + 500$

17b Consider when $V = 0$

$0 = \frac{1}{5}t^2 - 20t + 500$

$0 = t^2 - 100t + 2500$

$0 = (t - 50)^2$

$t = 50$

So it took James 50 seconds to drink the contents of the bottle.

17c Consider when $V = 250$

$$250 = \frac{1}{5}t^2 - 20t + 500$$

$$1250 = t^2 - 100t + 2500$$

$$0 = t^2 - 100t + 1250$$

$$t = \frac{100 \pm \sqrt{(-100)^2 - 4(1)(1250)}}{2}$$

$$t = 50 \pm \frac{\sqrt{5000}}{2}$$

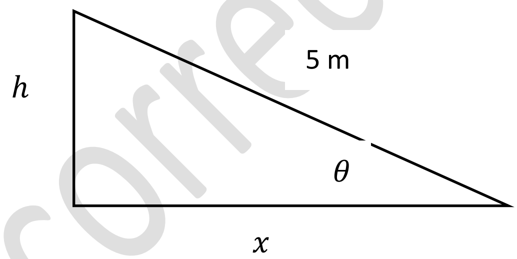
$$t = 50 \pm \sqrt{1250}$$

$$t = 50 \pm 25\sqrt{2}$$

$$t \doteqdot 15 \text{ or } 85 \text{ seconds}$$

Since the bottle is empty after 50 seconds, we can discard the value of 85 seconds. Hence, James would take 15 seconds to drink half the contents of the bottle.

18a



$$\frac{dx}{dt} = 5 \text{ cm/s}$$

Using Pythagoras' theorem with the right-angled triangle above,

$$x^2 + h^2 = 5^2$$

$$h^2 = 25 - x^2$$

$$h = \sqrt{25 - x^2} \quad \text{(since } h \text{ is positive)}$$

$$h = (25 - x^2)^{\frac{1}{2}}$$

$$\frac{dh}{dx} = \frac{1}{2}(25 - x^2)^{-\frac{1}{2}} \times -2x$$

$$\frac{dh}{dx} = -\frac{x}{\sqrt{25 - x^2}}$$

Using the chain rule,

$$\frac{dh}{dx} = \frac{dh}{dt} \times \frac{dt}{dx}$$

$$-\frac{x}{\sqrt{25 - x^2}} = \frac{dh}{dt} \times \frac{1}{5}$$

$$\frac{dh}{dt} = -\frac{5x}{\sqrt{25 - x^2}}$$

When $x = 1.4$ or $\frac{7}{5}$,

$$\frac{dh}{dt} = -\frac{5x}{\sqrt{25 - x^2}}$$

$$= -\frac{5 \times \frac{7}{5}}{\sqrt{25 - \left(\frac{7}{5}\right)^2}}$$

$$= -\frac{7}{\sqrt{25 - \frac{49}{25}}}$$

$$= -\frac{7}{\sqrt{\frac{576}{25}}}$$

$$= -\frac{7 \times 5}{\sqrt{576}}$$

$$= -\frac{35}{24}$$

The rate at which the height is changing is $-\frac{35}{24}$ cm/s.

18b From the right-angled triangle above,

$$\cos\theta = \frac{x}{5}$$

$$x = 5\cos\theta$$

$$\frac{dx}{d\theta} = -5\sin\theta$$

Using the chain rule,

$$\frac{dx}{dt} = \frac{dx}{d\theta} \times \frac{d\theta}{dt}$$

$$5 = -5\sin\theta \times \frac{d\theta}{dt}$$

$$\frac{d\theta}{dt} = -\frac{1}{\sin\theta}$$

When $x = 1.4$ or $\frac{7}{5}$,

$$\cos\theta = \frac{7}{25}$$

$$\sin\theta = \sqrt{1 - \left(\frac{7}{25}\right)^2} \qquad \text{(using the identity } \sin^2\theta + \cos^2\theta = 1\text{)}$$

$$= \sqrt{1 - \frac{49}{625}}$$

$$= \sqrt{\frac{576}{625}}$$

$$= \frac{24}{25}$$

$$\frac{d\theta}{dt} = -\frac{1}{\frac{24}{25}}$$

$$= -\frac{25}{24}$$

The rate at which the angle of inclination is changing is $-\frac{25}{24}$ radians per second.