Solutions to Exercise 5A

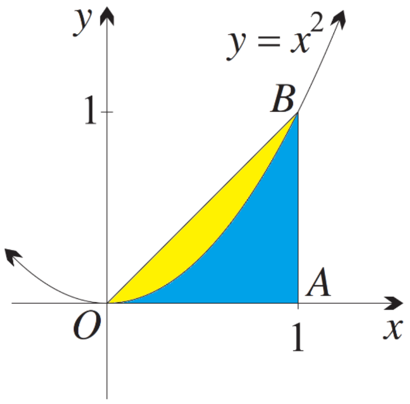
1a Area of triangle $= \frac{1}{2}bh$

For $\triangle OAB$,

$b = OA = 1, h = AB = 1$

Area of $\triangle OAB = \frac{1}{2} \times 1 \times 1 = \frac{1}{2}$ square units

1b



As $\int_0^1 x^2\,dx$ is defined as the area of the region between the x-axis and the curve between $x = 0$ and $x = 1$, (blue area), it is clear that the blue area is smaller than the area of the triangle $\triangle OAB$, therefore:

$$\int_0^1 x^2\,dx < \text{Area of } \triangle OAB$$

$$\int_0^1 x^2\,dx < \frac{1}{2}$$

2a Given that D is on the curve when $x = \frac{1}{2}, CD = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$

Area of triangle $= \frac{1}{2}bh$

For $\triangle OCD$,

$b = OC = \dfrac{1}{2}, h = CD = \dfrac{1}{4}$

Area of $\triangle OCD = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{4} = \frac{1}{16}$ square units

2b Given that B is on the curve when $x = 1, AB = 1^2 = 1$

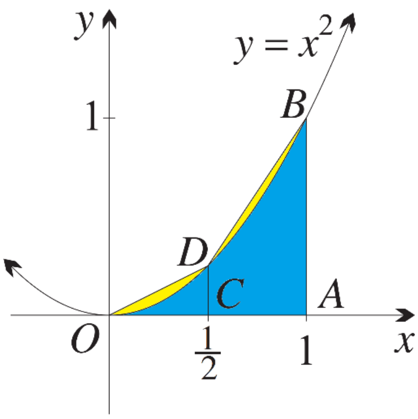
Area of trapezium $= \frac{1}{2}(a + b)h$

For $CADB$,

$a = CD = \dfrac{1}{4}, b = AB = 1, h = AC = \dfrac{1}{2}$

Area of $CABD = \frac{1}{2} \times \left(\frac{1}{4} + 1\right) \times \frac{1}{2} = \frac{5}{16}$ square units

2c



As $\int_0^1 x^2\,dx$ is defined as the area of the region between the x-axis and the curve between $x = 0$ and $x = 1$, (blue area), it is clear that there are gaps (see yellow region) between the total area of $\triangle\,OCD$ and trapezium $CABD$, therefore:

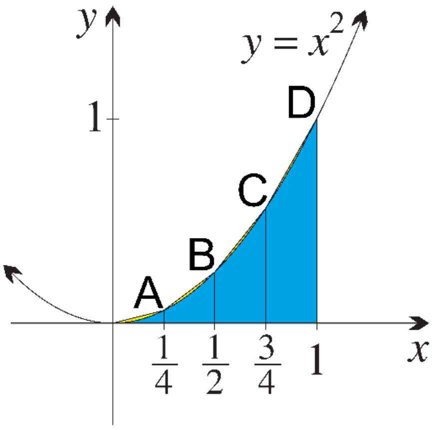
$\int_0^1 x^2\,dx <$ Area of $\triangle\,OCD$ + Area of $CABD$

$$\int_0^1 x^2\,dx < \frac{1}{16} + \frac{5}{16}$$

$$\int_0^1 x^2\,dx < \frac{6}{16}$$

$$\int_0^1 x^2\,dx < \frac{3}{8}$$

3a



y-coordinate of $A = \left(\frac{1}{4}\right)^2 = \frac{1}{16}$

y-coordinate of $B = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$

y-coordinate of $C = \left(\frac{3}{4}\right)^2 = \frac{9}{16}$

y-coordinate of $D = 1$

From the diagram, the sum of the area of the polygons must be greater than the area between the curve and x-axis.

The area of polygons left to right:

Polygon	Formula for area	Area of polygon
Triangle	$\frac{1}{2} \times \frac{1}{4} \times \frac{1}{16}$	$\frac{1}{128}$
Trapezium (left)	$\frac{1}{2} \times \left(\frac{1}{16} + \frac{1}{4}\right) \times \frac{1}{4}$	$\frac{5}{128}$
Trapezium (middle)	$\frac{1}{2} \times \left(\frac{1}{4} + \frac{9}{16}\right) \times \frac{1}{4}$	$\frac{13}{128}$
Trapezium (right)	$\frac{1}{2} \times \left(\frac{9}{16} + 1\right) \times \frac{1}{4}$	$\frac{25}{128}$

As $\int_0^1 x^2\, dx$ is defined as the area of the region between the x-axis and the curve between $x = 0$ and $x = 1$, (blue area), it is clear that there are gaps between the total area of the polygons (see yellow regions), therefore:

$\int_0^1 x^2\,dx <$ total area of polygons

$\int_0^1 x^2\,dx < \frac{1}{128} + \frac{5}{128} + \frac{13}{128} + \frac{25}{128}$

$\int_0^1 x^2\,dx < \frac{11}{32}$

3b Since $\int_0^1 x^2\,dx$ is less than both $\frac{11}{32}$ and $\frac{3}{8}$, and since $\frac{11}{32} < \frac{3}{8}$, it follows that:

$\frac{11}{32} - \int_0^1 x^2\,dx < \frac{3}{8} - \int_0^1 x^2\,dx.$

This statement means the difference is less between $\frac{11}{32}$ and $\int_0^1 x^2\,dx$ compared to $\frac{3}{8}$ and $\int_0^1 x^2\,dx$. This is evident by the difference in the areas above the curves (yellow regions) in the respective graphs getting smaller.

4a The definite integral $\int_0^2 3\,dx$ defines a rectangle between the curve and the x-axis, from $x = 0$ to $x = 2$.

Area of a rectangle $= bh$

$b = 2 - 0 = 2$

$h = 3 - 0 = 3$

$$\int_0^2 3\,dx = 2 \times 3 = 6$$

4b The definite integral $\int_0^3 4\,dx$ defines a rectangle between the curve and the x-axis, from $x = 0$ to $x = 3$.

Area of a rectangle $= bh$

$b = 3 - 0 = 3$

$h = 4 - 0 = 4$

$$\int_0^3 4\,dx = 3 \times 4 = 12$$

4c The definite integral $\int_0^4 x\,dx$ defines a triangle between the curve and the x-axis, from $x = 0$ to $x = 4$.

Area of a triangle $= \frac{1}{2}bh$

$b = 4 - 0 = 4$

$h = 4 - 0 = 4$

$$\int_0^4 x\,dx = \frac{1}{2} \times 4 \times 4 = 8$$

4d The definite integral $\int_0^3 2x\,dx$ defines a triangle between the curve and the x-axis, from $x = 0$ to $x = 3$

Area of a triangle $= \frac{1}{2}bh$

$b = 3 - 0 = 3$

$h = 6 - 0 = 6$

$$\int_0^3 2x\,dx = \frac{1}{2} \times 3 \times 6 = 9$$

4e The definite integral $\int_0^2 (2 - x)\,dx$ defines a triangle between the curve and the x-axis, from $x = 0$ to $x = 2$.

Area of a triangle $= \frac{1}{2}bh$

$b = 2 - 0 = 2$

$h = 2 - 0 = 2$

$$\int_0^2 (2 - x)\,dx = \frac{1}{2} \times 2 \times 2 = 2$$

4f The definite integral $\int_0^5 (5 - x)\,dx$ defines a triangle between the curve and the x-axis, from $x = 0$ to $x = 5$.

Area of a triangle $= \frac{1}{2}bh$

$b = 5 - 0 = 5$

$h = 5 - 0 = 5$

$$\int_0^5 (5 - x)\,dx = \frac{1}{2} \times 5 \times 5 = \frac{25}{2}$$

4g The definite integral $\int_0^2 (x + 2)\,dx$ defines a trapezium between the curve and the x-axis, from $x = 0$ to $x = 2$.

Area of a trapezium $= \frac{1}{2}(a + b)h$

$a = 2 - 0 = 2$

$b = 4 - 0 = 4$

$h = 2 - 0 = 2$

$$\int_0^2 (x + 2)\,dx = \frac{1}{2} \times (2 + 4) \times 2 = 6$$

4h The definite integral $\int_0^4 (x + 3)\,dx$ defines a trapezium between the curve and the x-axis, from $x = 0$ to $x = 4$.

Area of a trapezium $= \frac{1}{2}(a + b)h$

$a = 3 - 0 = 3$

$b = 7 - 0 = 7$

$h = 4 - 0 = 4$

$$\int_0^4 (x + 3)\,dx = \frac{1}{2} \times (3 + 7) \times 4 = 20$$

5a The definite integral $\int_{-1}^{3} 2\,dx$ defines a rectangle between the curve and the x-axis, from $x = -1$ to $x = 3$.

Area of a rectangle $= bh$

$b = 3 - (-1) = 4$

$h = 2 - 0 = 2$

$$\int_{-1}^{3} 2\,dx = 4 \times 2 = 8$$

5b The definite integral $\int_{-3}^{2} 5\,dx$ defines a rectangle between the curve and the x-axis, from $x = -3$ to $x = 2$.

Area of a rectangle $= bh$

$b = 2 - (-3) = 5$

$h = 5 - 0 = 5$

$$\int_{-3}^{2} 5\,dx = 5 \times 5 = 25$$

5c The definite integral $\int_{-2}^{1} (2x + 4)\,dx$ defines a triangle between the curve and the x-axis, from $x = -2$ to $x = 1$.

Area of a triangle $= \frac{1}{2}bh$

$b = 1 - (-2) = 3$

$h = 6 - 0 = 6$

$$\int_{-2}^{1} (2x + 4)\,dx = \frac{1}{2} \times 3 \times 6 = 9$$

5d The definite integral $\int_{-1}^{3} (3x + 3)\,dx$ defines a triangle between the curve and the x-axis, from $x = -1$ to $x = 3$.

Area of a triangle $= \frac{1}{2}bh$

$b = 3 - (-1) = 4$

$h = 12 - 0 = 12$

$$\int_{-1}^{3} (3x + 3)\,dx = \frac{1}{2} \times 4 \times 12 = 24$$

5e The definite integral $\int_{-1}^{5} (x + 4)\,dx$ defines a trapezium between the curve and the x-axis, from $x = -1$ to $x = 5$.

Area of a trapezium $= \frac{1}{2}(a + b)h$

$a = 3 - 0 = 3$

$b = 9 - 0 = 9$

$h = 5 - (-1) = 6$

$$\int_{-1}^{5} (x + 4)\,dx = \frac{1}{2} \times (3 + 9) \times 6 = 36$$

5f The definite integral $\int_{-2}^{2} (x + 6)\,dx$ defines a trapezium between the curve and the x-axis, from $x = -2$ to $x = 2$.

Area of a trapezium $= \frac{1}{2}(a + b)h$

$a = 4 - 0 = 4$

$b = 8 - 0 = 8$

$h = 2 - (-2) = 4$

$$\int_{-2}^{2} (x + 6)\,dx = \frac{1}{2} \times (4 + 8) \times 4 = 24$$

5g The definite integral $\int_{-3}^{3} |x|\,dx$ defines two equal triangles between the curve and the x-axis, from $x = -3$ to $x = 3$

Area of a triangle $= \frac{1}{2}bh$

$b = 3 - 0 = 3$

$h = 3 - 0 = 3$

$$\int_{-3}^{3} |x|\,dx = 2 \times \frac{1}{2} \times 3 \times 3 = 9$$

5h The definite integral $\int_{-2}^{2} |x|\,dx$ defines two equal triangles between the curve and the x-axis, from $x = -2$ to $x = 2$.

Area of a triangle $= \frac{1}{2}bh$

$b = 2 - 0 = 2$

$h = 4 - 0 = 4$

$$\int_{-2}^{2} |x|\,dx = 2 \times \frac{1}{2} \times 2 \times 4 = 8$$

6a Definite integral is the area of the region between the line $y = 5$ and the x-axis, from $x = 0$ to $x = 3$.

The area under the graph is a rectangle with:

$b = 3 - 0 = 3$

$h = 5 - 0 = 5$

$$\int_{0}^{3} 5\,dx = bh = 3 \times 5 = 15$$

6b Definite integral is the area of the region between the line $y = 5$ and the x-axis, from $x = -3$ to $x = 0$.

The area under the graph is a rectangle with:

$b = 0 - (-3) = 3$

$h = 5 - 0 = 5$

$$\int_{-3}^{0} 5\,dx = bh = 3 \times 5 = 15$$

6c Definite integral is the area of the region between the line $y = 5$ and the x-axis, from $x = -1$ to $x = 4$.

The area under the graph is a rectangle with:

$b = 4 - (-1) = 5$

$h = 5 - 0 = 5$

$$\int_{-1}^{4} 5\,dx = bh = 5 \times 5 = 25$$

6d Definite integral is the area of the region between the line $y = 5$ and the x-axis, from $x = -2$ to $x = 6$.

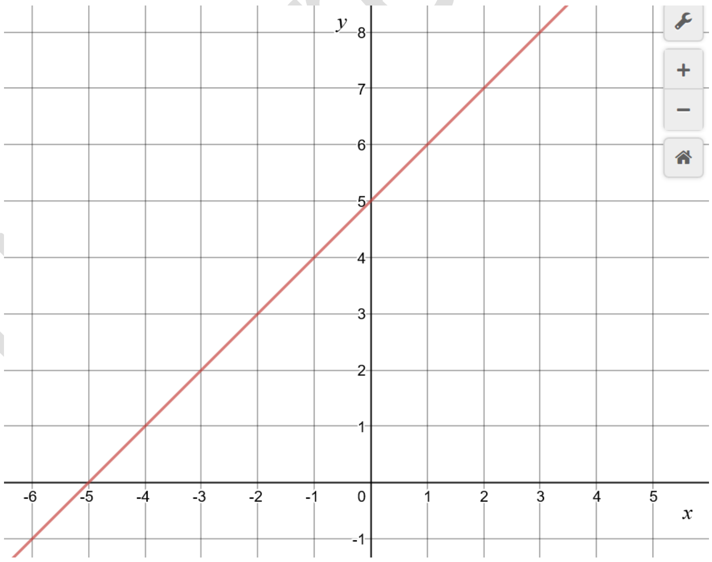
The area under the graph is a rectangle with:

$b = 6 - (-2) = 8$

$h = 5 - 0 = 5$

$$\int_{-2}^{6} 5\,dx = bh = 8 \times 5 = 40$$

For Q6e-h, the graph of $y = x + 5$ is shown below.



6e Definite integral is the area of the region between the line $y = x + 5$ and the x-axis, from $x = -5$ to $x = 0$.

The area under the graph is a triangle with:

$b = 0 - (-5) = 5$

$h = 5 - 0 = 5$

$$\int_{-5}^{0} (x + 5)\, dx = \frac{1}{2}bh = \frac{1}{2} \times 5 \times 5 = \frac{25}{2} = 12.5$$

6f Definite integral is the area of the region between the line $y = x + 5$ and the x-axis, from $x = 0$ to $x = 2$.

The area under the graph is a trapezium with:

$a = (0 + 5) - 0 = 5$

$b = (2 + 5) - 0 = 7$

$h = 2 - 0 = 2$

$$\int_{0}^{2} (x + 5)\, dx = \frac{1}{2}(a + b)h = \frac{1}{2} \times (5 + 7) \times 2 = 12$$

6g Definite integral is the area of the region between the line $y = x + 5$ and the x-axis, from $x = 2$ to $x = 4$.

The area under the graph is a trapezium with:

$a = (2 + 5) - 0 = 7$

$b = (4 + 5) - 0 = 9$

$h = 4 - 2 = 2$

$$\int_{2}^{4} (x + 5)\, dx = \frac{1}{2}(a + b)h = \frac{1}{2} \times (7 + 9) \times 2 = 16$$

6h Definite integral is the area of the region between the line $y = x + 5$ and the x-axis, from $x = -1$ to $x = 3$.

The area under the graph is a trapezium with:

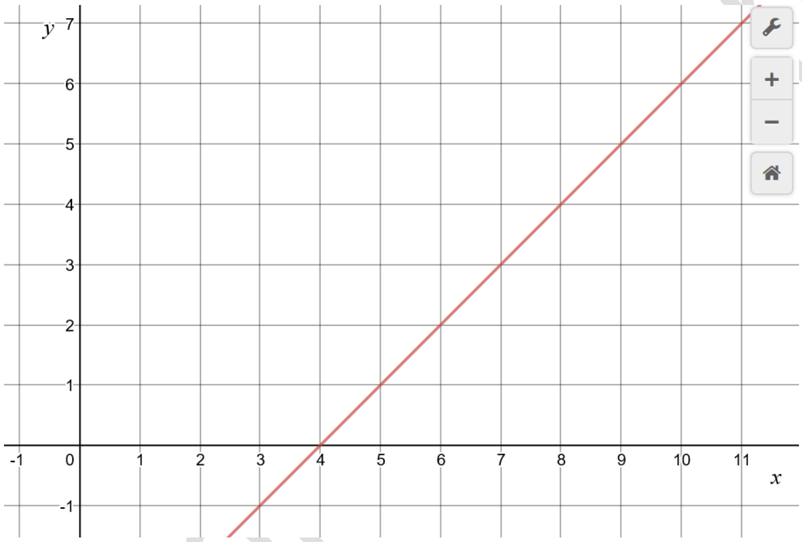
$a = (-1 + 5) - 0 = 4$

$b = (3 + 5) - 0 = 8$

$h = 3 - (-1) = 4$

$$\int_{-1}^{3} (x + 5)\, dx = \frac{1}{2}(a + b)h = \frac{1}{2} \times (4 + 8) \times 4 = 24$$

For Q6i-l, the graph of $y = x - 4$ is shown below.



6i Definite integral is the area of the region between the line $y = x - 4$ and the x-axis, from $x = 4$ to $x = 8$.

The area under the graph is a triangle with:

$b = (8 - 4) = 4$

$h = (8 - 4) - 0 = 4$

$$\int_{4}^{8} (x - 4)\, dx = \frac{1}{2}bh = \frac{1}{2} \times 4 \times 4 = 8$$

6j Definite integral is the area of the region between the line $y = x - 4$ and the x-axis, from $x = 4$ to $x = 10$.

The area under the graph is a triangle with:

$b = (10 - 4) = 6$

$h = (10 - 4) - 0 = 6$

$$\int_4^{10} (x - 4)\,dx = \frac{1}{2}bh = \frac{1}{2} \times 6 \times 6 = 18$$

6k Definite integral is the area of the region between the line $y = x - 4$ and the x-axis, from $x = 5$ to $x = 7$.

The area under the graph is a trapezium with:

$a = (5 - 4) - 0 = 1$

$b = (7 - 4) - 0 = 3$

$h = 7 - 5 = 2$

$$\int_5^{7} (x - 4)\,dx = \frac{1}{2}(a + b)h = \frac{1}{2} \times (1 + 3) \times 2 = 4$$

6l Definite integral is the area of the region between the line $y = x - 4$ and the x-axis, from $x = 6$ to $x = 10$.

The area under the graph is a trapezium with:

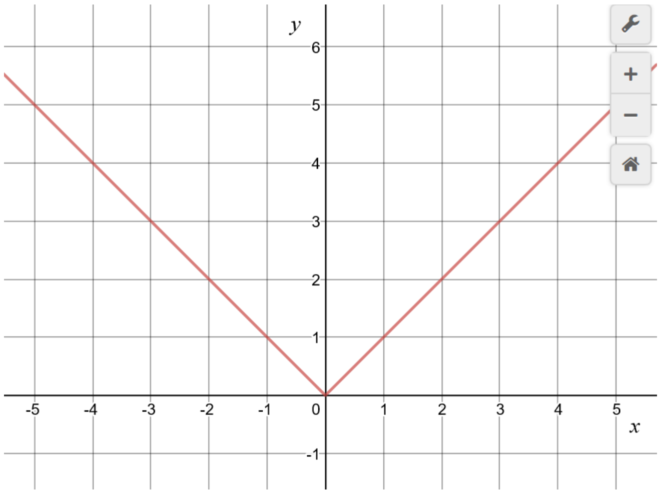
$a = (6 - 4) - 0 = 2$

$b = (10 - 4) - 0 = 6$

$h = 10 - 6 = 4$

$$\int_6^{10} (x - 4)\,dx = \frac{1}{2}(a + b)h = \frac{1}{2} \times (2 + 6) \times 4 = 16$$

For Q6m, n, the graph of $y = |x|$ is shown below.



6m Definite integral is the area of the region between the line $y = |x|$ and the x-axis, from $x = -2$ to $x = 2$.

The area under the graph is two equal triangles with:

$b = 2 - 0 = 2$

$h = |2| - 0 = 2$

$$\int_{-2}^{2} |x|\, dx = 2 \times \frac{1}{2} bh = 2 \times \frac{1}{2} \times 2 \times 2 = 4$$

6n Definite integral is the area of the region between the line $y = |x|$ and the x-axis, from $x = -4$ to $x = 4$

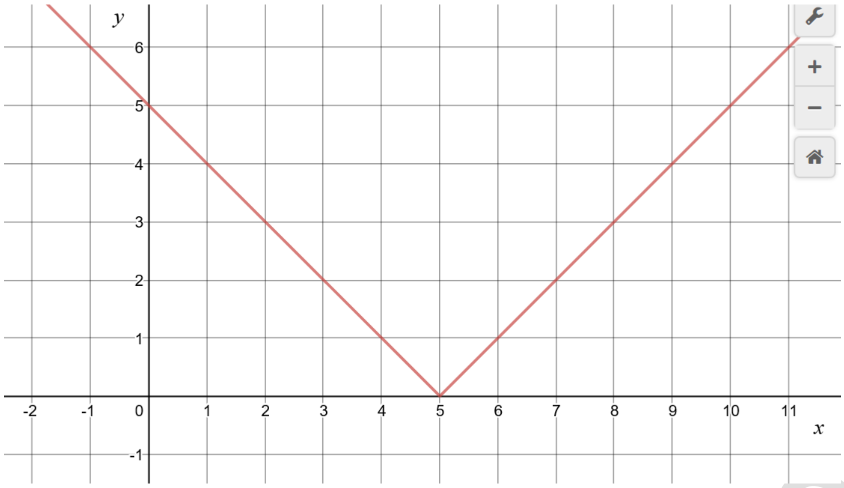
The area under the graph is two equal triangles with:

$b = 4 - 0 = 4$

$h = |4| - 0 = 4$

$$\int_{-4}^{4} |x|\, dx = 2 \times \frac{1}{2} bh = 2 \times \frac{1}{2} \times 4 \times 4 = 16$$

For Q6o, p, the graph of $y = |x - 5|$ is shown below.



6o Definite integral is the area of the region between the line $y = |x|$ and the x-axis, from $x = 0$ to $x = 5$.

The area under the graph is a triangle with:

$b = 5 - 0 = 5$

$h = |0 - 5| - 0 = 5$

$$\int_0^5 |x - 5|\, dx = \frac{1}{2}bh = \frac{1}{2} \times 5 \times 5 = \frac{25}{2} = 12.5$$

6p Definite integral is the area of the region between the line $y = |x|$ and the x-axis, from $x = 5$ to $x = 10$.

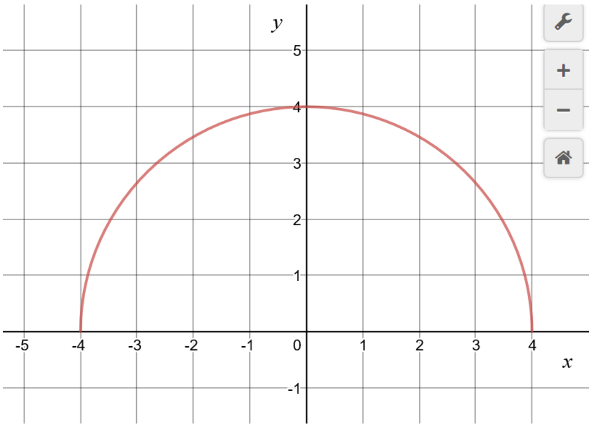
The area under the graph is a triangle with:

$b = 10 - 5 = 5$

$h = |10 - 5| - 0 = 5$

$$\int_5^{10} |x - 5|\, dx = \frac{1}{2}bh = \frac{1}{2} \times 5 \times 5 = \frac{25}{2} = 12.5$$

7a Graph of $y = \sqrt{16 - x^2}$

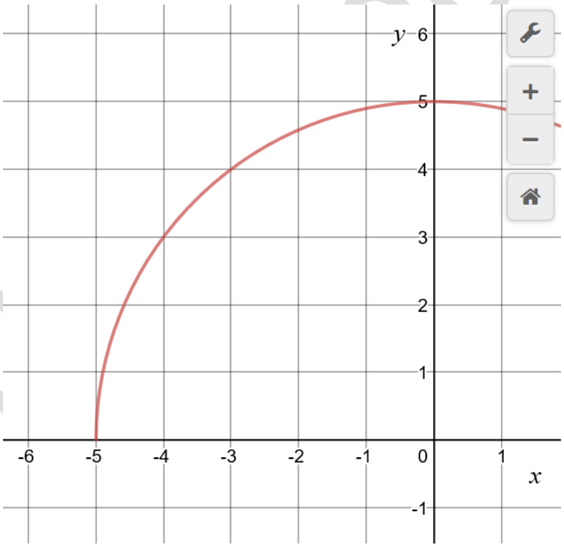


Definite integral is the area of the region between the curve $y = \sqrt{16 - x^2}$ and the x-axis from $x = -4$ to $x = 4$.

The area under the graph is the area of a semicircle with $r = 4$.

$$\int_{-4}^{4} \sqrt{16 - x^2}\, dx = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi(4)^2 = 8\pi$$

7b Graph of $y = \sqrt{25 - x^2}$

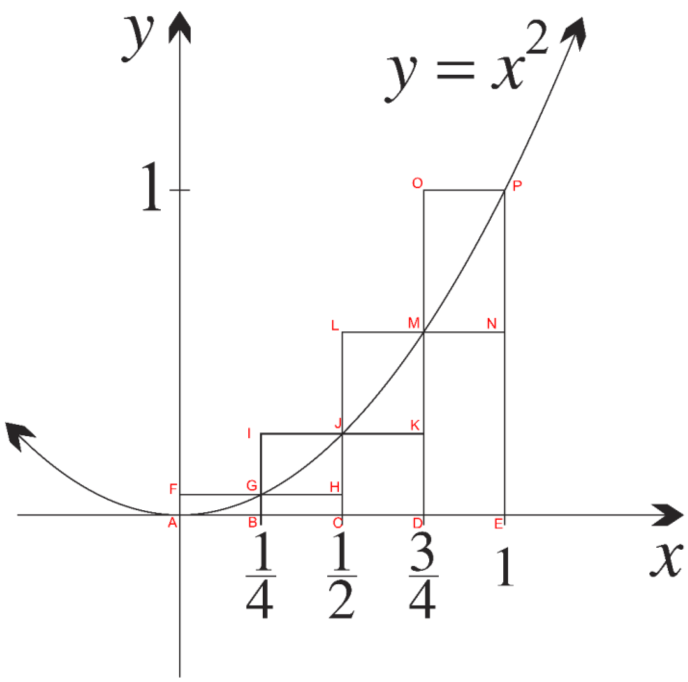


Definite integral is the area of the region between the curve $y = \sqrt{25 - x^2}$ and the x-axis from $x = -5$ to $x = 0$

The area under the graph is the area of a quadrant of a circle with $r = 5$.

$$\int_{-5}^{0} \sqrt{25 - x^2}\, dx = \frac{1}{4}\pi r^2 = \frac{1}{4}\pi(5)^2 = \frac{25\pi}{4}$$

8a



From the relabelled diagram above, the lower rectangles are $BCHG$, $CDKJ$ and $DENM$.

The area of rectangle can be determined from the formula bh, where the breadth of each rectangle is a constant step, $b = \frac{1}{4}$, and the heights, h corresponds to the y-coordinate of a point located at the top of each rectangle.

$$y\text{-coordinate of } G \;=\; \left(\frac{1}{4}\right)^2 = \frac{1}{16}$$

$$y\text{-coordinate of } J \;=\; \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

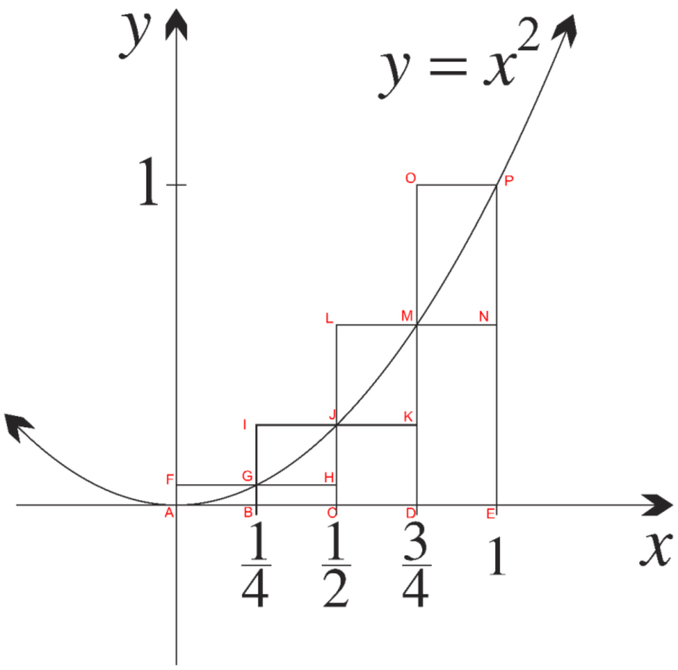
$$y\text{-coordinate of } M \;=\; \left(\frac{3}{4}\right)^2 = \frac{9}{16}$$

Therefore, the total area of the lower rectangles A_{lower} is:

$$A_{\text{lower}} = \frac{1}{4} \times \frac{1}{16} + \frac{1}{4} \times \frac{1}{4} + \frac{1}{4} \times \frac{9}{16}$$

$$= \frac{1}{4}\left(\frac{1}{16} + \frac{1}{4} + \frac{9}{16}\right)$$
$$= \frac{1}{4}\left(\frac{7}{8}\right)$$
$$= \frac{7}{32} \text{ square units}$$

8b



From the relabelled diagram above, the upper rectangles are $ABGF$, $BCJI$, $CDML$ and $DEPO$.

The area of rectangle can be determined from the formula bh, where the breadth of each rectangle is a constant step, $b = \frac{1}{4}$, and the heights, h corresponds to the y-coordinate of a point located at the top of each rectangle.

y-coordinate of G $= \left(\frac{1}{4}\right)^2 = \frac{1}{16}$

y-coordinate of J $= \left(\frac{1}{2}\right)^2 = \frac{1}{4}$

y-coordinate of M $= \left(\frac{3}{4}\right)^2 = \frac{9}{16}$

y-coordinate of P $= (1)^2 = 1$

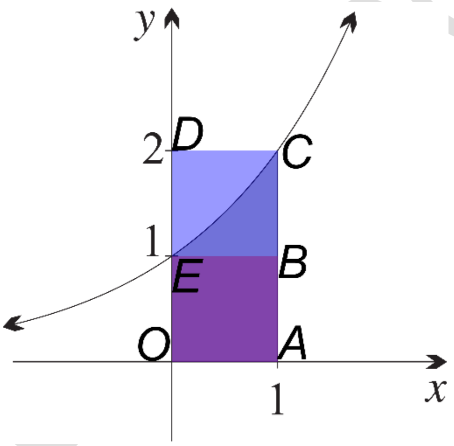
Therefore, the total area of the upper rectangles A_{upper} is:

$$\begin{aligned} A_{\text{upper}} &= \frac{1}{4}\times\frac{1}{16}+\frac{1}{4}\times\frac{1}{4}+\frac{1}{4}\times\frac{9}{16}+\frac{1}{4}\times 1 \\ &= \frac{1}{4}\left(\frac{1}{16}+\frac{1}{4}+\frac{9}{16}+1\right) \\ &= \frac{1}{4}\left(\frac{15}{8}\right) \\ &= \frac{15}{32}\ \text{ square units} \end{aligned}$$

8c As the lower and upper rectangles "trap" the integral from above and below between the x-coordinate values of 0 and 1, the total area of the lower rectangles serve as a lower bound for the unknown true value of the integral, while the total area of the upper rectangles serve as a upper bound. As the total area of the lower rectangles and the upper rectangles are $\frac{7}{32}$ and $\frac{15}{32}$ respectively, we can assert that:

$$\frac{7}{32} < \int_0^1 x^2\,dx < \frac{15}{32}$$

9a $y = 2^x$



The lower rectangle $OABE$ has an area of bh, where $b = OA = 1, h = OE = 1$.

Area of $OABE = bh = 1 \times 1 = 1$

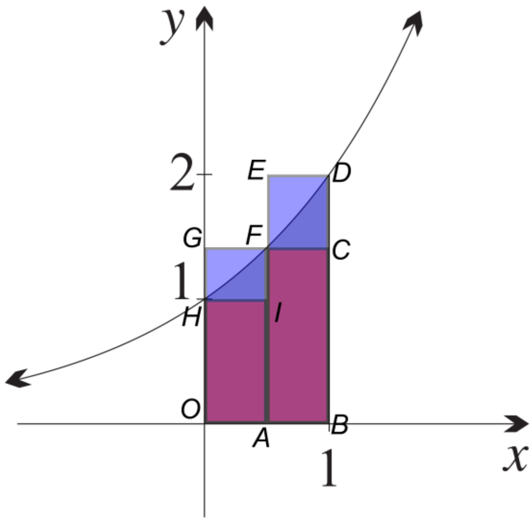
The upper rectangle $OACD$ has an area of bh, where $b = OA = 1, h = AC = 2$.

Area of $OACD = bh = 1 \times 2 = 2$

Since the curve is located between the two rectangles, it follows that:

$$1 < \int_0^1 2^x \, dx < 2$$

9b $y = 2^x$



$$OA = AB = \frac{1}{2}$$

The lower rectangle $OAIH$ has an area of bh, where $b = OA = \frac{1}{2}, h = OH = 1$.

Area of $OAIH = bh = \frac{1}{2} \times 1 = 0.5$

F is located at $\left(\frac{1}{2}, 2^{\frac{1}{2}}\right)$ so $AF = 2^{\frac{1}{2}} = \sqrt{2}$

The lower rectangle $ABCF$ has an area of bh, where $b = AB = \frac{1}{2}, h = AF = \sqrt{2}$.

Area of $ABCF = bh = \frac{1}{2} \times \sqrt{2} \doteqdot 0.7$

Total area of lower rectangles = area of $OAIH$ + area of $ABCF \doteqdot 0.5 + 0.7 = 1.2$

The upper rectangle $OAFG$ has an area of bh, where $b = OA = \frac{1}{2}, h = AF = \sqrt{2}$.

Area of $OAFG = bh = \frac{1}{2} \times \sqrt{2} \doteqdot 0.7$

The upper rectangle $ABDE$ has an area of bh, where $b = AB = \frac{1}{2}, h = AE = 2$.

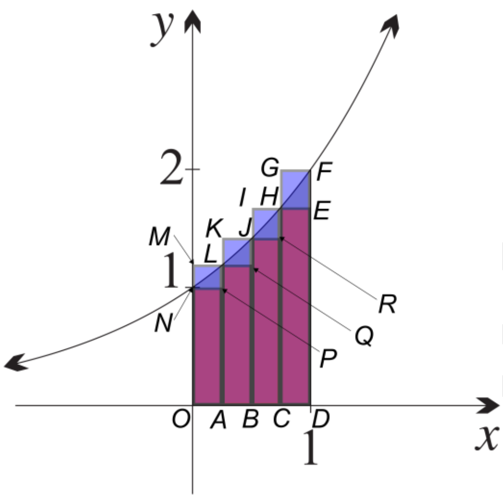
Area of $ABDE = bh = \frac{1}{2} \times 2 = 1$

Total area of upper rectangles = area of $OAFG$ + area of $ABDE \doteqdot 0.7 + 1 = 1.7$

Since the curve is located between the two rectangles, it follows that:

$$1.2 < \int_0^1 2^x \, dx < 1.7$$

9c $\quad y = 2^x$



$OA = AB = BC = CD = \frac{1}{4}$

The lower rectangle $OAPN$ has an area of bh, where $b = OA = \frac{1}{4}, h = ON = 1$.

Area of $OAPN = bh = \frac{1}{4} \times 1 = 0.25$

L is located at $\left(\frac{1}{4}, 2^{\frac{1}{4}}\right)$ so $AL = 2^{\frac{1}{4}}$

The lower rectangle $ABQL$ has an area of bh, where $b = AB = \frac{1}{4}, h = AL = 2^{\frac{1}{4}}$.

Area of $ABQL = bh = \frac{1}{4} \times 2^{\frac{1}{4}} \doteqdot 0.3$

J is located at $\left(\frac{1}{2}, 2^{\frac{1}{2}}\right)$ so $BJ = 2^{\frac{1}{2}}$

The lower rectangle $BCRJ$ has an area of bh, where $b = \frac{1}{4}, h = BJ = 2^{\frac{1}{2}}$

Area of $BCRJ = bh = \frac{1}{4} \times 2^{\frac{1}{2}} \doteqdot 0.35$

H is located at $\left(\frac{3}{4}, 2^{\frac{3}{4}}\right)$ so $CH = 2^{\frac{3}{4}}$

The lower rectangle $CDEH$ has an area of bh, where $b = \frac{1}{4}, h = CH = 2^{\frac{3}{4}}$

Area of $CDEH = bh = \frac{1}{4} \times 2^{\frac{3}{4}} \doteqdot 0.42$

Total area of lower rectangles

$=$ area of $OAPN$ $+$ area of $ABQL$ $+$ area of $BCRJ$ $+$ area of $CDEH$

$= \frac{1}{4} + \frac{1}{4} \times 2^{\frac{1}{4}} + \frac{1}{4} \times 2^{\frac{1}{2}} + \frac{1}{4} \times 2^{\frac{3}{4}}$

$\doteqdot 1.3$

The upper rectangle $OALM$ has an area of bh, where $b = OA = \frac{1}{4}, h = AL = 2^{\frac{1}{4}}$

Area of $OALM = bh = \frac{1}{4} \times 2^{\frac{1}{4}}$

The upper rectangle $ABJK$ has an area of bh, where $b = AB = \frac{1}{4}, h = BJ = 2^{\frac{1}{2}}$

Area of $ABJK = bh = \frac{1}{4} \times 2^{\frac{1}{2}}$

The upper rectangle $BCHI$ has an area of bh, where $b = BC = \frac{1}{4}, h = CH = 2^{\frac{3}{4}}$

Area of $BCHI = bh = \frac{1}{4} \times 2^{\frac{3}{4}}$

The upper rectangle $CDFG$ has an area of bh, where $b = CD = \frac{1}{4}, h = DF = 2$

Area of $CDFG = bh = \frac{1}{4} \times 2 = \frac{1}{2}$

Total area of upper rectangles

$=$ area of $OALM$ $+$ area of $ABJK$ $+$ area of $BCHI$ $+$ area of $CDFG$

$= \frac{1}{4} \times 2^{\frac{1}{4}} + \frac{1}{4} \times 2^{\frac{1}{2}} + \frac{1}{4} \times 2^{\frac{3}{4}} + \frac{1}{2}$

$\doteqdot 1.6$

Since the curve is located between the two rectangles, it follows that:

$$1.3 < \int_0^1 2^x \, dx < 1.6$$

9d

Number of rectangles	Interval
1	$2 - 1 = 1$
2	$1.7 - 1.2 = 0.5$
4	$1.6 - 1.3 = 0.3$

Based on the above, as the number of rectangles increases, the interval within which the exact area lies becomes smaller.

The exact value of the definite integral is:

$$\int_0^1 2^x \, dx = \left[\frac{1}{\ln 2} \times 2^x\right]_0^1 = \frac{2^1}{\ln 2} - \frac{2^0}{\ln 2} = \frac{1}{\ln 2} \doteqdot 1.44$$

10a For the two lower rectangles in the diagram, let the rectangles have an equal interval length of $= \frac{4-2}{2} = 1$.

The heights of the lower rectangles will correspond to the value of $y = \ln x$, when $x = 2$ and $x = 3$.

Area of lower rectangle 1, $A_{\text{lower1}} = bh = 1 \times \ln 2$

Area of lower rectangle 2, $A_{\text{lower2}} = bh = 1 \times \ln 3$

Total area of lower rectangles

$= A_{\text{lower1}} + A_{\text{lower2}}$

$= \ln 2 + \ln 3$

$\doteqdot 1.79$

For the two upper rectangles in the diagram, let the rectangles have an equal interval length of $= \frac{4-2}{2} = 1$.

The heights of the upper rectangles will correspond to the value of $y = \ln x$, when $x = 3$ and $x = 4$.

Area of upper rectangle 1, $A_{\text{upper1}} = bh = 1 \times \ln 3$

Area of upper rectangle 2, $A_{\text{upper2}} = bh = 1 \times \ln 4$

Total area of upper rectangles

$= A_{\text{upper1}} + A_{\text{upper2}}$

$= \ln 3 + \ln 4$

$\doteqdot 2.48$

As the lower and upper rectangles "trap" the integral from above and below between the x-coordinate values of 2 and 4, the total area of the lower rectangles serves as a lower bound for the unknown true value of the integral, while the total area of the upper rectangles serves as an upper bound. As the total area of the lower rectangles and the total area of the upper rectangles are 1.79 and 2.48 respectively, we can assert that:

$$1.79 < \int_2^4 \ln x \, dx < 2.48$$

10b For the four lower rectangles in the diagram, let the rectangles have an equal interval length of $b = \frac{4-2}{4} = \frac{1}{2}$.

The heights of the lower rectangles will correspond to the value of $y = \ln x$, when $x = 2, 2.5, 3$ and 3.5.

Area of lower rectangle 1, $A_{\text{lower1}} = bh = \frac{1}{2} \times \ln 2$

Area of lower rectangle 2, $A_{\text{lower2}} = bh = \frac{1}{2} \times \ln 2.5$

Area of lower rectangle 3, $A_{\text{lower3}} = bh = \frac{1}{2} \times \ln 3$

Area of lower rectangle 4, $A_{\text{lower4}} = bh = \frac{1}{2} \times \ln 3.5$

Total area of lower rectangles

$= A_{\text{lower1}} + A_{\text{lower2}} + A_{\text{lower3}} + A_{\text{lower4}}$

$= \dfrac{1}{2}(\ln 2 + \ln 2.5 + \ln 3 + \ln 3.5)$

$\doteqdot 1.98$

For the four upper rectangles in the diagram, let the rectangles have an equal interval length of $b = \frac{4-2}{4} = \frac{1}{2}$.

The heights of the upper rectangles will correspond to the value of $y = \ln x$, when $x = 2.5, 3, 3.5$ and 4.

Area of upper rectangle 1, $A_{\text{upper1}} = bh = \frac{1}{2} \times \ln 2.5$

Area of upper rectangle 2, $A_{\text{upper2}} = bh = \frac{1}{2} \times \ln 3$

Area of upper rectangle 3, $A_{\text{upper3}} = bh = \frac{1}{2} \times \ln 3.5$

Area of upper rectangle 4, $A_{\text{upper4}} = bh = \frac{1}{2} \times \ln 4$

Total area of upper rectangles

$$= A_{\text{upper1}} + A_{\text{upper2}} + A_{\text{upper3}} + A_{\text{upper4}}$$
$$= \frac{1}{2}(\ln 2.5 + \ln 3 + \ln 3.5 + \ln 4)$$
$$\doteqdot 2.33$$

As the lower and upper rectangles "trap" the integral from above and below between the x-coordinate values of 2 and 4, the total area of the lower rectangles serves as a lower bound for the unknown true value of the integral, while the total area of the upper rectangles serves as an upper bound. As the total area of the lower rectangles and the total area of the upper rectangles are 1.98 and 2.33 respectively, we can assert that:

$$1.98 < \int_2^4 \ln x \, dx < 2.33$$

10c For the eight lower rectangles in the diagram, let the rectangles have an equal interval length of $b = \frac{4-2}{8} = \frac{1}{4}$.

The heights of the lower rectangles will correspond to the value of $y = \ln x$, when $x = 2, 2.25, 2.5, 2.75, 3, 3.25, 3.5$ and 3.75.

Area of lower rectangle 1, $A_{\text{lower1}} = bh = \frac{1}{4} \times \ln 2$

Area of lower rectangle 2, $A_{\text{lower2}} = bh = \frac{1}{4} \times \ln 2.25$

Area of lower rectangle 3, $A_{\text{lower3}} = bh = \frac{1}{4} \times \ln 2.5$

Area of lower rectangle 4, $A_{\text{lower4}} = bh = \frac{1}{4} \times \ln 2.75$

Area of lower rectangle 5, $A_{\text{lower5}} = bh = \frac{1}{4} \times \ln 3$

Area of lower rectangle 6, $A_{\text{lower6}} = bh = \frac{1}{4} \times \ln 3.25$

Area of lower rectangle 7, $A_{\text{lower7}} = bh = \frac{1}{4} \times \ln 3.5$

Area of lower rectangle 8, $A_{\text{lower8}} = bh = \frac{1}{4} \times \ln 3.75$

Total area of lower rectangles,

$$= A_{\text{lower1}} + A_{\text{lower2}} + A_{\text{lower3}} + A_{\text{lower4}} + A_{\text{lower5}} + A_{\text{lower6}} + A_{\text{lower7}} + A_{\text{lower8}}$$
$$= \frac{1}{4}(\ln 2 + \ln 2.25 + \ln 2.5 + \ln 2.75 + \ln 3 + \ln 3.25 + \ln 3.5 + \ln 3.75)$$
$$\doteqdot 2.07$$

For the eight upper rectangles in the diagram, let the rectangles have an equal interval length of $b = \frac{4-2}{8} = \frac{1}{4}$.

The heights of the upper rectangles will correspond to the value of $y = \ln x$, when $x = 2.25, 2.5, 2.75, 3, 3.25, 3.5, 3.75$ and 4.

Area of upper rectangle 1, $A_{\text{upper1}} = bh = \frac{1}{4} \times \ln 2.25$

Area of upper rectangle 2, $A_{\text{upper2}} = bh = \frac{1}{4} \times \ln 2.5$

Area of upper rectangle 3, $A_{\text{upper3}} = bh = \frac{1}{4} \times \ln 2.75$

Area of upper rectangle 4, $A_{\text{upper4}} = bh = \frac{1}{4} \times \ln 3$

Area of upper rectangle 5, $A_{\text{upper5}} = bh = \frac{1}{4} \times \ln 3.25$

Area of upper rectangle 6, $A_{\text{upper6}} = bh = \frac{1}{4} \times \ln 3.5$

Area of upper rectangle 7, $A_{\text{upper7}} = bh = \frac{1}{4} \times \ln 3.75$

Area of upper rectangle 8, $A_{\text{upper8}} = bh = \frac{1}{4} \times \ln 4$

Total area of upper rectangles

$$= A_{\text{upper1}} + A_{\text{upper2}} + A_{\text{upper3}} + A_{\text{upper4}} + A_{\text{upper5}} + A_{\text{upper6}} + A_{\text{upper7}} + A_{\text{upper8}}$$
$$= \frac{1}{4}(\ln 2.25 + \ln 2.5 + \ln 2.75 + \ln 3 + \ln 3.25 + \ln 3.5 + \ln 3.75 + \ln 4)$$
$$\doteqdot 2.24$$

As the lower and upper rectangles “trap” the integral from above and below between the x-coordinate values of 2 and 4, the total area of the lower rectangles serves as a lower bound for the unknown true value of the integral, while the total area of the upper rectangles serves as an upper bound. As the total area of the lower rectangles and the total area of the upper rectangles are 2.07 and 2.24 respectively, we can assert that:

$$1.98 < \int_2^4 \ln x \, dx < 2.24$$

10d

Number of rectangles	Interval
2	$2.48 - 1.79 = 0.69$
4	$2.33 - 1.98 = 0.35$
8	$2.24 - 2.07 = 0.17$

Based on the above, as the number of rectangles increases, the interval within which the exact area lies becomes smaller.

11a Area of lower rectangle $= 1 \times \frac{1}{1+1} = \frac{1}{2}$ and underestimates A.

Area of upper rectangle has area $1 \times \frac{1}{0+1} = 1$ and overestimates A.

Hence $\frac{1}{2} < A < 1$.

11b Area of two lower rectangles

$$= \frac{1}{2} \times \frac{1}{0+1} + \frac{1}{2} \times \frac{1}{0.5+1}$$

$$= \frac{7}{12}$$

Area of two upper rectangles

$$= \frac{1}{2} \times \frac{1}{0.5+1} + \frac{1}{2} \times \frac{1}{1+1}$$

$$= \frac{5}{6}$$

Hence $\frac{7}{12} < A < \frac{5}{6}$ (or $0.58 < A < 0.83$).

11c Area of three lower rectangles

$$= \frac{1}{3} \times \frac{1}{0+1} + \frac{1}{3} \times \frac{1}{\frac{1}{3}+1} + \frac{1}{3} \times \frac{1}{\frac{2}{3}+1}$$

$$= \frac{37}{60}$$

Area of three upper rectangles

$$= \frac{1}{3} \times \frac{1}{\frac{1}{3}+1} + \frac{1}{3} \times \frac{1}{\frac{2}{3}+1} + \frac{1}{3} \times \frac{1}{1+1}$$

$$= \frac{47}{60}$$

Hence $\frac{37}{60} < A < \frac{47}{60}$ (or $0.62 < A < 0.78$).

11d Area of four lower rectangles

$$= \frac{1}{4} \times \frac{1}{0+1} + \frac{1}{4} \times \frac{1}{\frac{1}{4}+1} + \frac{1}{4} \times \frac{1}{\frac{2}{4}+1} + \frac{1}{4} \times \frac{1}{\frac{3}{4}+1}$$

$$= \frac{553}{840}$$

Area of four upper rectangles

$$= \frac{1}{4} \times \frac{1}{\frac{1}{4}+1} + \frac{1}{4} \times \frac{1}{\frac{2}{4}+1} + \frac{1}{4} \times \frac{1}{\frac{3}{4}+1} + \frac{1}{4} \times \frac{1}{1+1}$$

$$= \frac{319}{420}$$

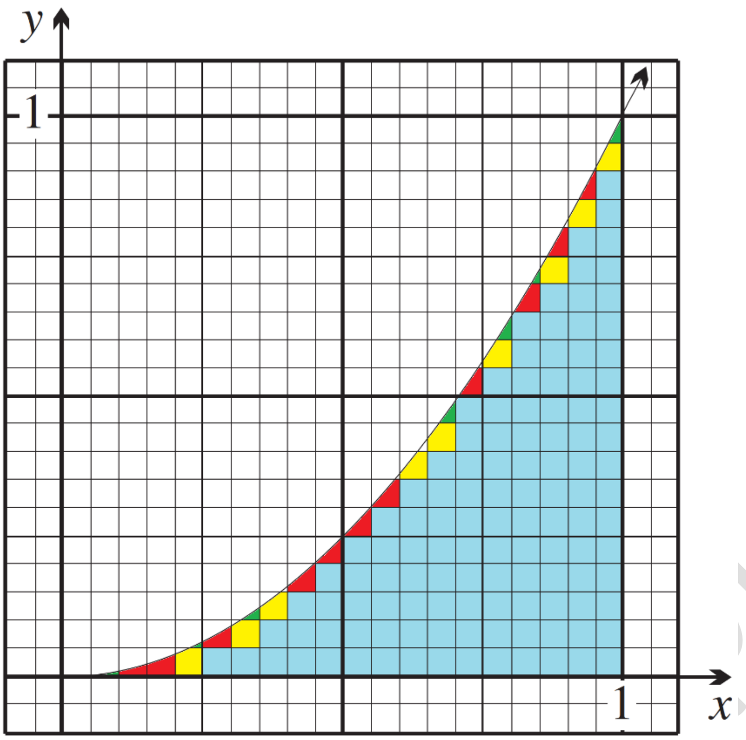
Hence $\frac{553}{840} < A < \frac{319}{420}$ (or $0.63 < A < 0.76$).

11e The interval is getting smaller.

11f Yes, they appear to be getting closer and closer to the exact value of 0.683 147….

12 This question uses technology to investigate some of the definite integrals from other questions in this exercise.

13a



The diagram shows different coloured segments for counting:

Number of BLUE (whole squares) $= 119$

Number of YELLOW (approximately $\frac{3}{4}$ of a square) $= 9$

Number of RED (approximately $\frac{1}{2}$ of a square) $= 11$

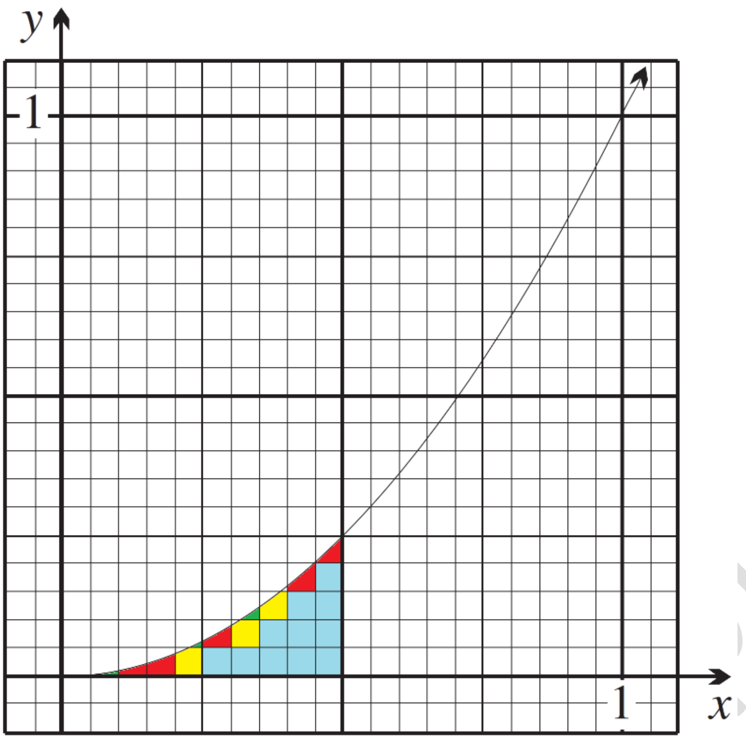
Number of GREEN (approximately $\frac{1}{4}$ of a square) $= 7$

Total number of whole squares $\doteqdot 119 \times 1 + 9 \times \frac{3}{4} + 11 \times \frac{1}{2} + 7 \times \frac{1}{4} = 133$

Area of each square $= \frac{1}{20} \times \frac{1}{20} = \frac{1}{400}$ square units

So $\int_0^1 x^2\,dx \doteqdot 133 \times \frac{1}{400} = 0.3325 \doteqdot 0.33$

13b i



The diagram shows different coloured segments for counting:

Number of BLUE (whole squares) $= 11$

Number of YELLOW (approximately $\frac{3}{4}$ of a square) $= 3$

Number of RED (approximately $\frac{1}{2}$ of a square) $= 4$

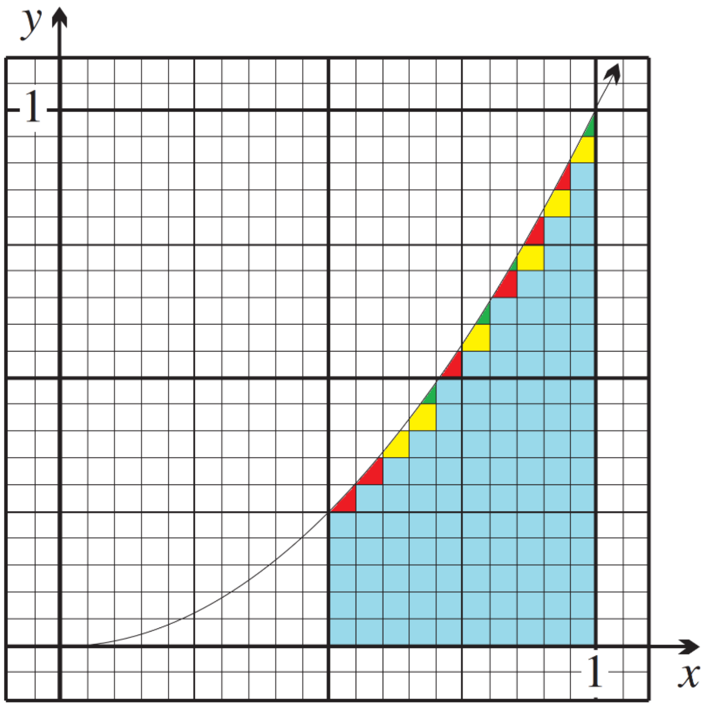
Number of GREEN (approximately $\frac{1}{4}$ of a square) $= 3$

Total number of whole squares $\doteqdot 11 \times 1 + 3 \times \frac{3}{4} + 4 \times \frac{1}{2} + 3 \times \frac{1}{4} = 16$

Area of each square $= \frac{1}{20} \times \frac{1}{20} = \frac{1}{400}$ square units

So $\int_0^{\frac{1}{2}} x^2\, dx \doteqdot 16 \times \frac{1}{400} = \frac{1}{25}$

13b ii



The diagram shows different coloured segments for counting:

Number of BLUE (whole squares) $= 108$

Number of YELLOW (approximately $\frac{3}{4}$ of a square) $= 6$

Number of RED (approximately $\frac{1}{2}$ of a square) $= 7$

Number of GREEN (approximately $\frac{1}{4}$ of a square) $= 4$

Total number of whole squares $\doteqdot 108 \times 1 + 6 \times \frac{3}{4} + 7 \times \frac{1}{2} + 4 \times \frac{1}{4} = 117$

Area of each square $= \frac{1}{20} \times \frac{1}{20} = \frac{1}{400}$ square units

So $\int_{\frac{1}{2}}^{1} x^2\, dx \doteqdot 117 \times \frac{1}{400} = 0.2925$

We can determine that $\frac{1}{25} + \frac{117}{400} = 0.3325 \doteqdot 0.33$, which confirms that the sum of the answers to parts i and ii is the answer to part a.

14a There are 315 little squares under the graph from $x = 0$ to $x = 1$.

14b $\int_0^1 \sqrt{1 - x^2}\, dx \doteqdot 315 \times \frac{1}{400} = 0.7875 \doteqdot 0.79$

14c $\frac{1}{4}\pi r^2 \doteqdot 0.79$

Since $r = 1$,

$\frac{1}{4}\pi \doteqdot 0.79$

$\pi \doteqdot 3.16$

15 We have the formula from chapter 2 that:

$1^3 + 2^3 + \cdots + n^3 = \frac{1}{4}n^2(n+1)^2$

Step A:

Divide the interval $0 \le x \le 1$ into n subintervals, each of width $\frac{1}{n}$.

On each subinterval form the upper and lower rectangle.

Thus,

(sum of lower rectangles) $\le A \le$ (sum of upper rectangles)

Step B:

The heights of successive upper rectangles are $\frac{1^3}{n^3}, \frac{2^3}{n^3}, \ldots, \frac{n^3}{n^3}$

Using the formula quoted above:

$$\begin{aligned}
\text{Sum of upper rectangles} &= \frac{1}{n}\left(\frac{1^3}{n^3} + \frac{2^3}{n^3} \ldots + \frac{n^3}{n^3}\right) \\
&= \frac{1}{n^4}(1^3 + 2^3 + \cdots + n^3) \\
&= \frac{1}{n^4} \times \frac{1}{4}n^2(n+1)^2 \\
&= \frac{1}{4} \times \frac{n^2}{n^2} \times \frac{(n+1)^2}{n^2} \\
&= \frac{1}{4} \times \left(1 + \frac{1}{n}\right)^2
\end{aligned}$$

Hence, the sum of upper rectangles has the limit $\frac{1}{4}$ as $n \to \infty$

Step C:

The heights of successive lower rectangles are $0, \frac{1^3}{n^3}, \frac{2^3}{n^3}, \dots, \frac{(n-1)^3}{n^3}$

Substituting $n-1$ for n in the formula quoted above:

$$\begin{aligned}\text{Sum of upper rectangles } &= \frac{1}{n}\left(0+\frac{1^3}{n^3}+\frac{2^3}{n^3}+\dots+\frac{(n-1)^3}{n^3}\right)\\ &= \frac{1}{n^4}(1^3+2^3+\dots+(n-1)^3)\\ &= \frac{1}{n^4}\times\frac{1}{4}n^2(n-1)^2\\ &= \frac{1}{4}\times\frac{n^2}{n^2}\times\frac{(n-1)^2}{n^2}\\ &= \frac{1}{4}\times(1-\frac{1}{n})^2\end{aligned}$$

Hence, the sum of lower rectangles has the limit $\frac{1}{4}$ as $n \to \infty$

Step D:

Finally, (sum of lower rectangles) $\le A \le$ (sum of upper rectangles)

And both these sums have same limit $\frac{1}{4}$ so it allows that $A = \frac{1}{4}$

So, $\int_0^1 x^3\,dx = \frac{1}{4}$

16a We have the formula from chapter 2 that

$1^2+2^2+\dots+n^2 = \frac{1}{6}n(n+1)(2n+1)$

Need to prove:

$\int_0^a x^2\,dx = \frac{a^3}{3}$

Step A:

Divide the interval $0 \le x \le a$ into n subintervals, each of width $\frac{a}{n}$

On each subinterval form the upper and lower rectangle.

Thus,

(sum of lower rectangles) $\le A \le$ (sum of upper rectangles)

Step B:

The heights of successive upper rectangles are $\frac{a^2}{n^2}, \frac{(2a)^2}{n^2}, \dots, \frac{(na)^2}{n^2}$

Using the formula quoted above

Sum of upper rectangles $= \frac{a}{n}\left(\frac{a^2}{n^2} + \frac{(2a)^2}{n^2} + \cdots + \frac{(na)^2}{n^2}\right)$

$= \frac{a^3}{n^3}(1^2 + 2^2 + \cdots + n^2)$

$= \frac{a^3}{3} \times \frac{1}{6}n(n+1)(2n+1)$

$= \frac{a^3}{3} \times \frac{n}{n} \times \frac{n+1}{n} \times \frac{2n+1}{2n}$

$= \frac{a^3}{3} \times (1 + \frac{1}{n}) \times (1 + \frac{1}{2n})$

Hence, the sum of upper rectangles has the limit $\frac{a^3}{3}$ as $n \to \infty$

Step C:

The heights of successive lower rectangles are $0, \frac{a^2}{n^2}, \frac{(2a)^2}{n^2}, \ldots, \frac{(n-1)^2}{n^2}$

Substituting $n - 1$ for n in the formula quoted above:

Sum of upper rectangles $= \frac{a}{n}\left(0 + \frac{a^2}{n^2} + \frac{(2a)^2}{n^2} + \cdots + \frac{(n-1)^2}{n^2}\right)$

$= \frac{a^3}{n^3}(1^2 + 2^2 + \cdots + (n-1)^2)$

$= \frac{a^3}{n^3} \times \frac{1}{6}(n-1)n(2n-1)$

$= \frac{a^3}{3} \times \frac{n}{n} \times \frac{n-1}{n} \times \frac{2n-1}{2n}$

$= \frac{a^3}{3} \times (1 - \frac{1}{n}) \times (1 - \frac{1}{2n})$

Hence, the sum of lower rectangles has the limit $\frac{a^3}{3}$ as $n \to \infty$

Step D:

Finally, (sum of lower rectangles) $\leq A \leq$ (sum of upper rectangles)

And both these sums have same limit $\frac{a^3}{3}$ so it allows that $A = \frac{a^3}{3}$.

So, $\int_0^a x^3\,dx = \frac{a^3}{3}$

16b We have the formula from chapter 2 that

$1^3 + 2^3 + \cdots + n^3 = \frac{1}{4}n^2(n+1)^2$

Need to prove:

$\int_0^a x^3\, dx = \frac{a^4}{3}$

Step A:

Divide the interval $0 \le x \le a$ into n subintervals, each of width $\frac{a}{n}$.

On each subinterval form the upper and lower rectangle.

Thus,

(sum of lower rectangles) $\le A \le$ (sum of upper rectangles)

Step B:

The heights of successive upper rectangles are $\frac{a^3}{n^3}, \frac{(2a)^3}{n^3}, \ldots, \frac{(na)^3}{n^3}$

Using the formula quoted above:

$$\begin{aligned}\text{Sum of upper rectangles } &= \frac{a}{n}\left(\frac{a^3}{n^3} + \frac{(2a)^3}{n^3} + \cdots + \frac{(na)^3}{n^3}\right)\\ &= \frac{a^4}{n^4}(1^3 + 2^3 + \cdots + n^3)\\ &= \frac{a^4}{n^4} \times \frac{1}{4}n^2(n+1)^2\\ &= \frac{a^4}{4} \times \frac{n^2}{n^2} \times \frac{(n+1)^2}{n^2}\\ &= \frac{a^4}{4} \times (1 + \frac{1}{n})^2\end{aligned}$$

Hence, the sum of upper rectangles has the limit $\frac{a^4}{4}$ as $n \to \infty$

Step C:

The heights of successive lower rectangles are $0, \frac{a^3}{n^3}, \frac{(2a)^3}{n^3}, \ldots, \frac{(n-1)^3}{n^3}$

Substituting $n - 1$ for n in the formula quoted above:

$$\begin{aligned}\text{Sum of upper rectangles } &= \frac{a}{n}\left(0 + \frac{a^3}{n^3} + \frac{(2a)^3}{n^3} + \cdots + \frac{(n-1)^3}{n^3}\right)\\ &= \frac{a^4}{n^4}(1^3 + 2^3 + \cdots + (n-1)^3)\end{aligned}$$

$$= \frac{a^4}{n^4} \times \frac{1}{4} n^2 (n-1)^2$$

$$= \frac{a^4}{4} \times \frac{n^2}{n^2} \times \frac{(n-1)^2}{n^2}$$

$$= \frac{a^4}{4} \times (1 - \frac{1}{n})^2$$

Hence, the sum of lower rectangles has the limit $\frac{a^4}{4}$ as $n \to \infty$

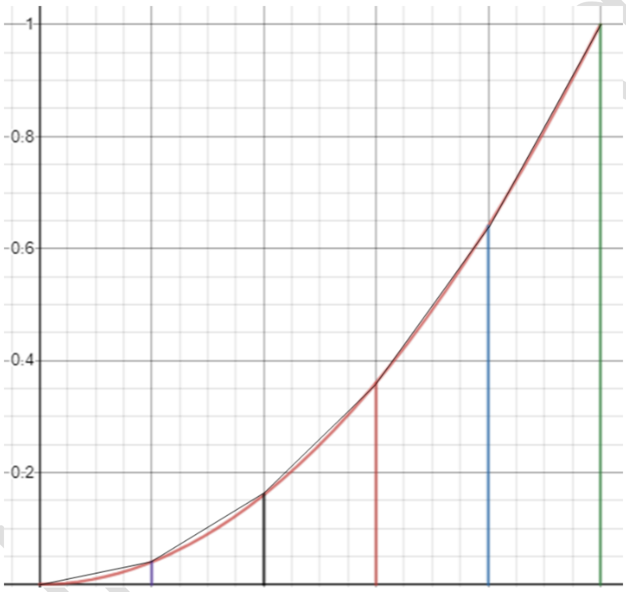
Step D:

Finally, (sum of lower rectangles) $\leq A \leq$ (sum of upper rectangles)

And both these sums have same limit $\frac{a^4}{4}$ so it allows that $A = \frac{a^4}{4}$.

So, $\int_0^a x^3 \, dx = \frac{a^4}{4}$

17



The area of a trapezium is $\frac{h}{2}(a+b)$, hence, the ith trapezia filling in the polygon will have area $\frac{x_i - x_{i-1}}{2}\left(f(x_{i-1}) + f(x_i)\right) = \frac{\frac{i}{n} - \frac{i-1}{n}}{2}\left(f\left(\frac{i-1}{n}\right) + f\left(\frac{i}{n}\right)\right)$. Summing these areas together, the total area of the polygon will be

$$A = \sum_{i=1}^{n} \frac{\frac{i}{n} - \frac{i-1}{n}}{2}\left(f\left(\frac{i-1}{n}\right) + f\left(\frac{i}{n}\right)\right)$$

$$=\sum_{i=1}^{n}\frac{\frac{i-(i-1)}{n}}{2}\left(f\left(\frac{i-1}{n}\right)+f\left(\frac{i}{n}\right)\right)$$
$$=\sum_{i=1}^{n}\frac{\frac{1}{n}}{2}\left(f\left(\frac{i-1}{n}\right)+f\left(\frac{i}{n}\right)\right)$$
$$=\sum_{i=1}^{n}\frac{1}{2n}\left(f\left(\frac{i-1}{n}\right)+f\left(\frac{i}{n}\right)\right)$$
$$=\sum_{i=1}^{n}\frac{1}{2n}\left(\left(\frac{i-1}{n}\right)^2+\left(\frac{i}{n}\right)^2\right)$$
$$=\sum_{i=1}^{n}\frac{1}{2n}\left(\frac{(i-1)^2}{n^2}+\frac{(i)^2}{n^2}\right)$$
$$=\sum_{i=1}^{n}\frac{1}{2n}\left(\frac{i^2-2i+1}{n^2}+\frac{i^2}{n^2}\right)$$
$$=\sum_{i=1}^{n}\frac{1}{2n}\left(\frac{2i^2-2i+1}{n^2}\right)$$
$$=\sum_{i=1}^{n}\frac{1}{2n^3}(2i^2-2i+1)$$
$$=\frac{1}{2n^3}\left(2\left(\sum_{i=1}^{n}i^2\right)-2\left(\sum_{i=1}^{n}i\right)+\left(\sum_{i=1}^{n}1\right)\right)$$
$$=\frac{1}{2n^3}\left(2\left(\frac{n(n+1)(2n+1)}{6}\right)-2\left(\frac{n(n+1)}{2}\right)+(n)\right)$$
$$=\frac{1}{2n^3}\left(\frac{n}{3}(2n^2+3n+1)-(n^2+n)+(n)\right)$$
$$=\frac{1}{2n^3}\left(\frac{2n^3}{3}+n^2+\frac{n}{3}-n^2-n+n\right)$$
$$=\frac{1}{2n^3}\left(\frac{2n^3}{3}+\frac{n}{3}\right)$$
$$=\frac{1}{3}+\frac{1}{6n^2}$$

Note that the values for $\sum_{i=1}^{n} i^2$, $\sum_{i=1}^{n} i$ and $\sum_{i=1}^{n} 1$ were found in chapter 2.

17b As the curve is concave up, the area trapezia will always overestimate the area of the curve. More specifically the lines $P_0P_1, P_1P_2, \ldots$ lie above the curve which means that the combined area of the trapezia is greater than the area of the curve.

17c $\lim_{n\to\infty} A = \lim_{n\to\infty}\left(\frac{1}{3}+\frac{1}{6n^2}\right)$

$= \lim_{n\to\infty}\left(\frac{1}{3}\right) + \lim_{n\to\infty}\left(\frac{1}{6n^2}\right)$

$= \lim_{n\to\infty}\left(\frac{1}{3}\right) + \frac{1}{6}\lim_{n\to\infty}\left(\frac{1}{n^2}\right)$

$= \frac{1}{3} + \frac{1}{6}(0)$

$= \frac{1}{3}$

Solutions to Exercise 5B

1a

$$\int_0^1 2x\ dx$$

$$= [x^2]_0^1$$

$$= 1^2 - 0^2$$

$$= 1$$

1b

$$\int_1^4 2x\ dx$$

$$= [x^2]_1^4$$

$$= 4^2 - 1^2$$

$$= 15$$

1c

$$\int_1^3 4x\ dx$$

$$= [2x^2]_1^3$$

$$= 2 \times 3^2 - 2 \times 1^2$$

$$= 16$$

1d

$$\int_2^5 8x\ dx$$

$$= [4x^2]_2^5$$

$$= 4 \times 5^2 - 4 \times 2^2$$

$$= 84$$

1e

$$\int_2^3 3x^2\ dx$$

$$= [x^3]_2^3$$

$$= 3^3 - 2^3$$

$$= 19$$

1f

$$\int_0^3 5x^4\ dx$$

$$= [x^5]_0^3$$

$$= 3^5 - 0^5$$

$$= 243$$

1g

$$\int_1^2 10x^4\ dx$$

$$= [2x^5]_1^2$$

$$= 2 \times 2^5 - 2 \times 1^5$$

$$= 62$$

1h

$$\int_0^1 12x^5\ dx$$

$$= [2x^6]_0^1$$

$$= 2 \times 1^6 - 2 \times 0^6$$

$$= 2$$

1i

$$\int_0^1 11x^{10}\ dx$$

$$= [x^{11}]_0^1$$

$$= 1$$

2a i

$$\int_0^1 4\ dx$$

$$= [4x]_0^1$$

$$= 4 \times 1 - 4 \times 0$$

$$= 4$$

2a ii

$$\int_2^7 5\ dx$$

$$= [5x]_2^7$$

$$= 5 \times 7 - 5 \times 2$$

$$= 25$$

2a iii

$$\int_4^5 1\, dx$$

$$= [x]_4^5$$

$$= 5 - 4$$

$$= 1$$

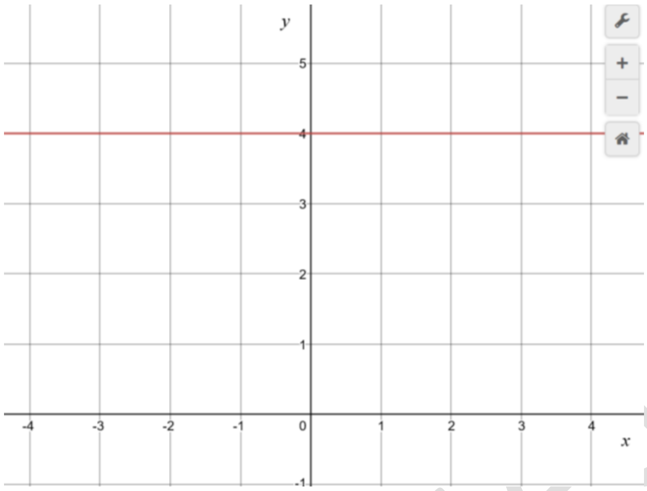
2b i The integral $\int_0^1 4 \ dx$ is defined by the area under the curve $y = 4$, which is a rectangle bounded by the lines $y = 4, x = 0, x = 1$ and the x-axis.

Area of the rectangle

$= bh$ where $b = 1, h = 4$

$= 1 \times 4$

$= 4$



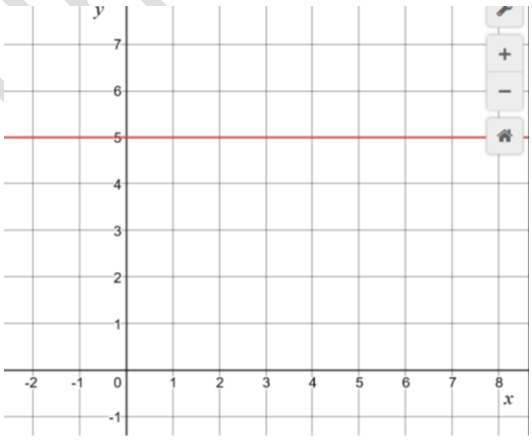
2b ii The integral $\int_2^7 5 \ dx$ is defined by the area under the curve $y = 5$, which is a rectangle bounded by the lines $y = 5, x = 2, x = 7$ and the x-axis.

Area of the rectangle

$= bh$ where $b = 5, h = 5$

$= 5 \times 5$

$= 25$



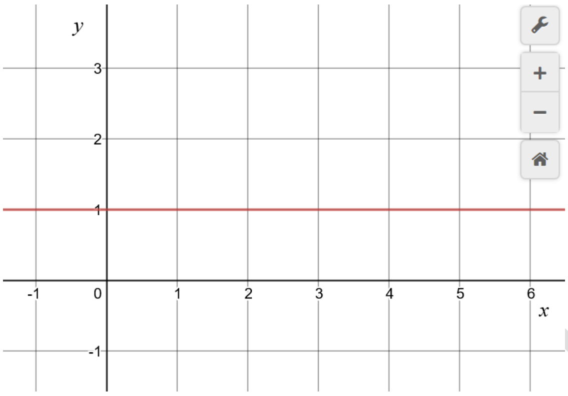
2b iii The integral $\int_4^5 1\, dx$ is defined by the area under the curve $y = 1$, which is a rectangle bounded by the lines $y = 1, x = 4, x = 5$ and the x-axis.

Area of the rectangle

$= bh$ where $b = 1, h = 1$

$= 1 \times 1$

$= 1$



3a

$$\int_3^6 (2x + 1)\, dx$$

$= [x^2 + x]_3^6$

$= (6^2 + 6) - (3^2 + 3)$

$= 42 - 12$

$= 30$

3b

$$\int_2^4 (2x - 3)\, dx$$

$= [x^2 - 3x]_2^4$

$= (4^2 - 3 \times 4) - (2^2 - 3 \times 2)$

$= 4 - (-2)$

$= 6$

3c

$$\int_0^3 (4x + 5)\,dx$$

$$= [2x^2 + 5x]_0^3$$

$$= (2 \times 3^2 + 5 \times 3) - (2 \times 0^2 + 5 \times 0)$$

$$= 33 - 0$$

$$= 33$$

3d

$$\int_2^3 (3x^2 - 1)\,dx$$

$$= [x^3 - x]_2^3$$

$$= (3^3 - 3) - (2^3 - 2)$$

$$= 24 - 6$$

$$= 18$$

3e

$$\int_1^4 (6x^2 + 2)\,dx$$

$$= [2x^3 + 2x]_1^4$$

$$= (2 \times 4^3 + 2 \times 4) - (2 \times 1^3 + 2 \times 1)$$

$$= 136 - 4$$

$$= 132$$

3f

$$\int_0^1 (3x^2 + 2x)\,dx$$

$$= [x^3 + x^2]_0^1$$

$$= (1^3 + 1^2) - (0^3 + 0^2)$$

$$= 2 - 0$$

$= 2$

3g

$\int_1^2 (4x^3 + 3x^2 + 1)\, dx$

$= [x^4 + x^3 + x]_1^2$

$= (2^4 + 2^3 + 2) - (1^4 + 1^3 + 1)$

$= 26 - 3$

$= 23$

3h

$\int_0^2 (2x + 3x^2 + 8x^3)\, dx$

$= [x^2 + x^3 + 2x^4]_0^2$

$= (2^2 + 2^3 + 2 \times 2^4) - (0^2 + 0^3 + 2 \times 0^4)$

$= 44 - 0$

$= 44$

3i

$\int_3^5 (3x^2 - 6x + 5)\, dx$

$= [x^3 - 3x^2 + 5x]_3^5$

$= (5^3 - 3 \times 5^2 + 5 \times 5) - (3^3 - 3 \times 3^2 + 5 \times 3)$

$= 75 - 15$

$= 60$

4a

$\int_{-1}^0 (1 - 2x)\, dx$

$= [x - x^2]_{-1}^0$

$= (0 - 0^2) - \left((-1) - (-1)^2\right)$

$= 0 - (-2)$

$= 2$

4b

$\int_{-1}^{0} (2x + 3)\, dx$

$= [x^2 + 3x]_{-1}^{0}$

$= (0^2 + 3 \times 0) - ((-1)^2 + 3 \times (-1))$

$= 0 - (-2)$

$= 2$

4c

$\int_{-2}^{1} 3x^2\, dx$

$= [x^3]_{-2}^{1}$

$= (1^3) - ((-2)^3)$

$= 1 - (-8)$

$= 9$

4d

$\int_{-1}^{2} (4x^3 + 5)\, dx$

$= [x^4 + 5x]_{-1}^{2}$

$= (2^4 + 5 \times 2) - \left((-1)^4 + 5 \times (-1)\right)$

$= 26 - (-4)$

$= 30$

4e

$$\int_{-2}^{2}(5x^4 + 6x^2)\,dx$$

$$= [x^5 + 2x^3]_{-2}^{2}$$

$$= (2^5 + 2 \times 2^3) - ((-2)^5 + 2 \times (-2)^3)$$

$$= 48 - (-48)$$

$$= 96$$

4f

$$\int_{-2}^{-1}(4x^3 + 12x^2 - 3)\,dx$$

$$= [x^4 + 4x^3 - 3x]_{-2}^{-1}$$

$$= \big((-1)^4 + 4 \times (-1)^3 - 3 \times (-1)\big) - \big((-2)^4 + 4 \times (-2)^3 - 3 \times (-2)\big)$$

$$= 0 - (-10)$$

$$= 10$$

5a

$$\int_{1}^{4}(x + 2)\,dx$$

$$= \left[\frac{x^2}{2} + 2x\right]_1^4$$

$$= \left(\frac{4^2}{2} + 2 \times 4\right) - \left(\frac{1^2}{2} + 2 \times 1\right)$$

$$= 16 - 2\frac{1}{2}$$

$$= 13\frac{1}{2}$$

5b

$$\int_0^2 (x^2 + x)\, dx$$

$$= \left[\frac{x^3}{3} + \frac{x^2}{2}\right]_0^2$$

$$= \left(\frac{2^3}{3} + \frac{2^2}{2}\right) - \left(\frac{0^3}{3} + \frac{0^2}{2}\right)$$

$$= 4\frac{2}{3} - 0$$

$$= 4\frac{2}{3}$$

5c

$$\int_0^3 (x^3 + x^2)\, dx$$

$$= \left[\frac{x^4}{4} + \frac{x^3}{3}\right]_0^3$$

$$= \left(\frac{3^4}{4} + \frac{3^3}{3}\right) - \left(\frac{0^4}{4} + \frac{0^3}{3}\right)$$

$$= 29\frac{1}{4} - 0$$

$$= 29\frac{1}{4}$$

5d

$$\int_{-1}^1 (x^3 - x + 1)\, dx$$

$$= \left[\frac{x^4}{4} - \frac{x^2}{2} + x\right]_{-1}^1$$

$$= \left(\frac{1^4}{4} - \frac{1^2}{2} + 1\right) - \left(\frac{(-1)^4}{4} - \frac{(-1)^2}{2} + (-1)\right)$$

$$= \frac{3}{4} - \left(-1\frac{1}{4}\right)$$

$$= 2$$

5e

$$\int_{-2}^{3} (2x^2 - 3x + 1)\, dx$$

$$= \left[\frac{2x^3}{3} - \frac{3x^2}{2} + x\right]_{-2}^{3}$$

$$= \left(\frac{2(3)^3}{3} - \frac{3(3)^2}{2} + (3)\right) - \left(\frac{2(-2)^3}{3} - \frac{3(-2)^2}{2} + (-2)\right)$$

$$= 7\frac{1}{2} - \left(-13\frac{1}{3}\right)$$

$$= 20\frac{5}{6}$$

5f

$$\int_{-4}^{-2} (16 - x^3 - x)\, dx$$

$$= \left[16x - \frac{x^4}{4} - \frac{x^2}{2}\right]_{-4}^{-2}$$

$$= \left(16(-2) - \frac{(-2)^4}{4} - \frac{(-2)^2}{2}\right) - \left(16(-4) - \frac{(-4)^4}{4} - \frac{(-4)^2}{2}\right)$$

$$= -38 - (-136)$$

$$= 98$$

6a

$$\int_{2}^{3} x(2 + 3x)\, dx$$

$$= \int_{2}^{3} (2x + 3x^2)\, dx$$

$= [x^2 + x^3]_2^3$

$= (3^2 + 3^3) - (2^2 + 2^3)$

$= 36 - 12$

$= 24$

6b

$\int_0^2 (x+1)(3x+1)\,dx$

$= \int_0^2 (3x^2 + 3x + x + 1)\,dx$

$= \int_0^2 (3x^2 + 4x + 1)\,dx$

$= [x^3 + 2x^2 + x]_0^2$

$= (2^3 + 2 \times 2^2 + 2) - (0^3 + 2 \times 0^2 + 0)$

$= 18 - 0$

$= 18$

6c

$\int_{-1}^1 x^2(5x^2 + 1)\,dx$

$= \int_{-1}^1 (5x^4 + x^2)\,dx$

$= \left[x^5 + \frac{x^3}{3}\right]_{-1}^1$

$= \left(1^5 + \frac{1^3}{3}\right) - \left((-1)^5 + \frac{(-1)^3}{3}\right)$

$= 1\frac{1}{3} - \left(-1\frac{1}{3}\right)$

$= 2\frac{2}{3}$

6d

$$\int_{-1}^{2}(x-3)^2\,dx$$

$$=\int_{-1}^{2}(x^2-6x+9)\,dx$$

$$=\left[\frac{x^3}{3}-3x^2+9x\right]_{-1}^{2}$$

$$=\left(\frac{2^3}{3}-3\times 2^2+9\times 2\right)-\left(\frac{(-1)^3}{3}-3\times(-1)^2+9\times(-1)\right)$$

$$=8\frac{2}{3}-\left(-12\frac{1}{3}\right)$$

$$=21$$

6e

$$\int_{-1}^{0}x(x-1)(x+1)\,dx$$

$$=\int_{-1}^{0}x(x^2-1)\,dx$$

$$=\int_{-1}^{0}(x^3-x)\,dx$$

$$=\left[\frac{x^4}{4}-\frac{x^2}{2}\right]_{-1}^{0}$$

$$=\left(\frac{0^4}{4}-\frac{0^2}{2}\right)-\left(\frac{(-1)^4}{4}-\frac{(-1)^2}{2}\right)$$

$$=0-\left(-\frac{1}{4}\right)$$

$$=\frac{1}{4}$$

6f

$$\int_{-1}^{0}(1-x^2)^2\,dx$$

$$=\int_{-1}^{0}(1-x^2)(1-x^2)\,dx$$

$$=\int_{-1}^{0}(1-2x^2+x^4)\,dx$$

$$=\left[x-\frac{2}{3}x^3+\frac{x^5}{5}\right]_{-1}^{0}$$

$$=\left(0-\frac{2}{3}\times 0^3+\frac{0^5}{5}\right)-\left((-1)-\frac{2}{3}\times(-1)^3+\frac{(-1)^5}{5}\right)$$

$$=0-\left(-\frac{8}{15}\right)$$

$$=\frac{8}{15}$$

7a

$$\int_{1}^{3}\frac{3x^3+4x^2}{x}dx$$

$$=\int_{1}^{3}\left(\frac{3x^3}{x}+\frac{4x^2}{x}\right)dx$$

$$=\int_{1}^{3}(3x^2+4x)\,dx$$

$$=[x^3+2x^2]_1^3$$

$$=(3^3+2\times 3^2)-(1^3+2\times 1^2)$$

$$=45-3$$

$$=42$$

7b

$$\int_1^2 \frac{4x^4 - x}{x}\,dx$$

$$= \int_1^2 \left(\frac{4x^4}{x} - \frac{x}{x}\right) dx$$

$$= \int_1^2 (4x^3 - 1)\,dx$$

$$= [x^4 - x]_1^2$$

$$= (2^4 - 2) - (1^4 - 1)$$

$$= 14 - 0$$

$$= 14$$

7c

$$\int_2^3 \frac{5x^2 + 9x^4}{x^2}\,dx$$

$$= \int_2^3 \left(\frac{5x^2}{x^2} + \frac{9x^4}{x^2}\right) dx$$

$$= \int_2^3 (5 + 9x^2)\,dx$$

$$= [5x + 3x^3]_2^3$$

$$= (5 \times 3 + 3 \times 3^3) - (5 \times 2 + 3 \times 2^3)$$

$$= 96 - 34$$

$$= 62$$

7d

$$\int_1^2 \frac{x^3 + 4x^2}{x}\,dx$$

$$= \int_1^2 \left(\frac{x^3}{x} + \frac{4x^2}{x}\right) dx$$

$$= \int_1^2 (x^2 + 4x)\,dx$$

$$= \left[\frac{x^3}{3} + 2x^2\right]_1^2$$

$$= \left(\frac{2^3}{3} + 2 \times 2^2\right) - \left(\frac{1^3}{3} + 2 \times 1^2\right)$$

$$= 10\frac{2}{3} - \left(2\frac{1}{3}\right)$$

$$= 8\frac{1}{3}$$

7e

$$\int_1^3 \frac{x^3 - x^2 + x}{x}\,dx$$

$$= \int_1^3 \left(\frac{x^3}{x} - \frac{x^2}{x} + \frac{x}{x}\right) dx$$

$$= \int_1^3 (x^2 - x + 1)\,dx$$

$$= \left[\frac{x^3}{3} - \frac{x^2}{2} + x\right]_1^3$$

$$= \left(\frac{3^3}{3} - \frac{3^2}{2} + 3\right) - \left(\frac{1^3}{3} - \frac{1^2}{2} + 1\right)$$

$$= 7\frac{1}{2} - \frac{5}{6}$$

$$= 6\frac{2}{3}$$

7f

$$\int_{-2}^{-1} \frac{x^3 - 2x^5}{x^2}\,dx$$

$$= \int_{-2}^{-1} \left(\frac{x^3}{x^2} - \frac{2x^5}{x^2}\right) dx$$

$$= \int_{-2}^{-1} (x - 2x^3)\,dx$$

$$= \left[\frac{x^2}{2} - \frac{x^4}{2}\right]_{-2}^{-1}$$

$$= \left(\frac{(-1)^2}{2} - \frac{(-1)^4}{2}\right) - \left(\frac{(-2)^2}{2} - \frac{(-2)^4}{2}\right)$$

$$= 0 - (-6)$$

$$= 6$$

8a

$$\int_0^{\frac{1}{2}} x^2\,dx$$

$$= \left[\frac{x^3}{3}\right]_0^{\frac{1}{2}}$$

$$= \left(\frac{\left(\frac{1}{2}\right)^3}{3}\right) - \left(\frac{0^3}{3}\right)$$

$$= \frac{1}{24} - 0$$

$$= \frac{1}{24}$$

8b

$$\int_0^{\frac{2}{3}} (2x + 3x^2)\,dx$$

$$= [x^2 + x^3]_0^{\frac{2}{3}}$$

$$= \left(\left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3\right) - (0^2 + 0^3)$$

$$= \frac{20}{27} - 0$$

$$= \frac{20}{27}$$

8c

$$\int_{\frac{1}{2}}^{\frac{3}{4}}(6-4x)\,dx$$

$$=[6x-2x^2]_{\frac{1}{2}}^{\frac{3}{4}}$$

$$=\left(6\left(\frac{3}{4}\right)-2\left(\frac{3}{4}\right)^2\right)-\left(6\left(\frac{1}{2}\right)-2\left(\frac{1}{2}\right)^2\right)$$

$$=3\frac{3}{8}-\left(2\frac{1}{2}\right)$$

$$=\frac{7}{8}$$

9a i

$$\int_5^{10} x^{-2}\,dx$$

$$=\left[\frac{x^{-1}}{-1}\right]_5^{10}$$

$$=\left[-\frac{1}{x}\right]_5^{10}$$

$$=\left(-\frac{1}{10}\right)-\left(-\frac{1}{5}\right)$$

$$=-\frac{1}{10}+\frac{1}{5}$$

$$=\frac{1}{10}$$

9a ii

$$\int_2^3 2x^{-3}\,dx$$

$$=\left[\frac{2x^{-2}}{-2}\right]_2^3$$

$$=[-x^{-2}]_2^3$$

$$= \left[-\frac{1}{x^2}\right]_2^3$$

$$= \left(-\frac{1}{3^2}\right) - \left(-\frac{1}{2^2}\right)$$

$$= -\frac{1}{9} - \left(-\frac{1}{4}\right)$$

$$= -\frac{1}{9} + \frac{1}{4}$$

$$= \frac{5}{36}$$

9a iii

$$\int_{\frac{1}{2}}^{1} 4x^{-5}\,dx$$

$$= \left[\frac{4x^{-4}}{-4}\right]_{\frac{1}{2}}^{1}$$

$$= [-x^{-4}]_{\frac{1}{2}}^{1}$$

$$= \left[-\frac{1}{x^4}\right]_{\frac{1}{2}}^{1}$$

$$= \left(-\frac{1}{1^4}\right) - \left(-\frac{1}{\left(\frac{1}{2}\right)^4}\right)$$

$$= -1 - \left(-\frac{1}{\left(\frac{1}{16}\right)}\right)$$

$$= -1 - (-16)$$

$$= 15$$

9b i

$$\int_1^2 \frac{1}{x^2}\,dx$$

$$= \int_1^2 x^{-2}\,dx$$

$$= \left[\frac{x^{-1}}{-1}\right]_1^2$$

$$= \left[-\frac{1}{x}\right]_1^2$$

$$= \left(-\frac{1}{2}\right) - \left(-\frac{1}{1}\right)$$

$$= -\frac{1}{2} - (-1)$$

$$= \frac{1}{2}$$

9b ii

$$\int_1^4 \frac{1}{x^3}\,dx$$

$$= \int_1^4 x^{-3}\,dx$$

$$= \left[\frac{x^{-2}}{-2}\right]_1^4$$

$$= \left[-\frac{1}{2x^2}\right]_1^4$$

$$= \left(-\frac{1}{2 \times 4^2}\right) - \left(-\frac{1}{2 \times 1^2}\right)$$

$$= -\frac{1}{32} - \left(-\frac{1}{2}\right)$$

$$= -\frac{1}{32} + \frac{1}{2}$$

$$= \frac{15}{32}$$

9b iii

$$\int_{\frac{1}{2}}^{1} \frac{3}{x^4}\,dx$$

$$= \int_{\frac{1}{2}}^{1} 3x^{-4}\,dx$$

$$= \left[\frac{3x^{-3}}{-3}\right]_{\frac{1}{2}}^{1}$$

$$= \left[-\frac{1}{x^3}\right]_{\frac{1}{2}}^{1}$$

$$= \left(-\frac{1}{1^3}\right) - \left(-\frac{1}{\left(\frac{1}{2}\right)^3}\right)$$

$$= -1 - (-8)$$

$$= 7$$

10a i

$$\int_{2}^{k} 3\,dx$$

$$= [3x]_2^k$$

$$= (3 \times k) - (3 \times 2)$$

$$= 3k - 6$$

10a ii Given that:

$$\int_{2}^{k} 3\,dx = 18$$

From 10a i, we have

$$\int_{2}^{k} 3\,dx = 3k - 6$$

$$3k - 6 = 18$$

$$k = 8$$

10b i

$$\int_0^k x\,dx$$

$$= \left[\frac{x^2}{2}\right]_0^k$$

$$= \left(\frac{k^2}{2}\right) - \left(\frac{0^2}{2}\right)$$

$$= \frac{k^2}{2}$$

10b ii Given that:

$$\int_0^k x\,dx = 18$$

From 10b i, we have

$$\int_0^k x\,dx = \frac{k^2}{2}$$

$$\frac{k^2}{2} = 18$$

$$k = \pm\sqrt{36}$$

The positive value of $k = 6$.

11a

$$\int_k^3 2\,dx = 4$$

$$[2x]_k^3 = 4$$

$$2 \times 3 - 2k = 4$$

$$-2k = 4 - 6$$

$$k = \frac{-2}{-2}$$

$$= 1$$

11b

$$\int_k^8 3\,dx = 12$$

$$[3x]_k^8 = 12$$

$$3 \times 8 - 3k = 12$$

$$-3k = 12 - 24$$

$$k = \frac{-12}{-3}$$

$$= 4$$

11c

$$\int_2^3 (k-3)\,dx = 5$$

$$[(k-3)x]_2^3 = 5$$

$$3(k-3) - 2(k-3) = 5$$

$$3k - 9 - 2k + 6 = 5$$

$$k = 5 + 9 - 6$$

$$= 8$$

11d

$$\int_3^k (x-3)\,dx = 0$$

$$\left[\frac{x^2}{2} - 3x\right]_3^k = 0$$

$$\left(\frac{k^2}{2} - 3k\right) - \left(\frac{3^2}{2} - 3 \times 3\right) = 0$$

$$\frac{k^2}{2} - 3k - \frac{9}{2} + 9 = 0$$

$$\frac{k^2}{2} - 3k + \frac{9}{2} = 0$$

$$k^2 - 6k + 9 = 0$$

$$(k-3)^2 = 0$$

$$k = 3$$

11e

$$\int_1^k (x+1)\,dx = 6$$

$$\left[\frac{x^2}{2} + x\right]_1^k = 6$$

$$\left(\frac{k^2}{2} + k\right) - \left(\frac{1^2}{2} + 1\right) = 6$$

$$\frac{k^2}{2} + k - \frac{1}{2} - 1 = 6$$

$$\frac{k^2}{2} + k - \frac{15}{2} = 0$$

$$k^2 + 2k - 15 = 0$$

$$(k+5)(k-3) = 0$$

$$k = -5 \text{ or } 3$$

As $k > 0$, $k = 3$

11f

$$\int_1^k (k+3x)\,dx = \frac{13}{2}$$

$$\left[kx + \frac{3x^2}{2}\right]_1^k = \frac{13}{2}$$

$$\left(k \times k + \frac{3k^2}{2}\right) - \left(k + \frac{3 \times 1^2}{2}\right) = \frac{13}{2}$$

$$k^2 + \frac{3k^2}{2} - k - \frac{3}{2} = \frac{13}{2}$$

$$\frac{5}{2}k^2 - k - 8 = 0$$

$$5k^2 - 2k - 16 = 0$$

$(5k+8)(k-2)=0$

$k=-\dfrac{8}{5}$ or 2

As $k>0$, $k=2$

12a

$$\int_0^4 f(x)\,dx=\int_0^1 f(x)\,dx+\int_1^2 f(x)\,dx+\int_2^3 f(x)\,dx+\int_3^4 f(x)\,dx$$

$\int_0^1 f(x)\,dx$ represents the area of a quadrant with $r=1$.

$$\int_0^1 f(x)\,dx=\frac{1}{4}\pi\times 1^2=\frac{\pi}{4}$$

$\int_1^2 f(x)\,dx$ represents the area of a triangle with $b=1, h=1$.

$$\int_1^2 f(x)\,dx=\frac{1}{2}\times 1\times 1=\frac{1}{2}$$

$\int_2^3 f(x)\,dx$ represents the area of a triangle with $b=1, h=1$.

$$\int_2^3 f(x)\,dx=\frac{1}{2}\times 1\times 1=\frac{1}{2}$$

$\int_3^4 f(x)\,dx$ represents the area of a quadrant with $r=1$.

$$\int_3^4 f(x)\,dx=\frac{1}{4}\pi\times 1^2=\frac{\pi}{4}$$

Therefore $\int_0^4 f(x)\,dx=\frac{\pi}{4}+\frac{1}{2}+\frac{1}{2}+\frac{\pi}{4}=1+\frac{\pi}{2}$

12b

$$\int_0^4 f(x)\,dx=\int_0^1 f(x)\,dx+\int_1^2 f(x)\,dx+\int_2^3 f(x)\,dx+\int_3^4 f(x)\,dx$$

$\int_0^1 f(x)\,dx$ represents the external area of a quadrant with $r=1$ inscribed in a square with side $b=1$.

$$\int_0^1 f(x)\,dx=1^2-\frac{1}{4}\pi\times 1^2=1-\frac{\pi}{4}$$

$\int_1^2 f(x)\,dx$ represents the area of a square with $b = 1$.

$$\int_1^2 f(x)\,dx = 1 \times 1 = 1$$

$\int_2^3 f(x)\,dx$ represents the area of a triangle with $b = 1, h = 1$.

$$\int_2^3 f(x)\,dx = \frac{1}{2} \times 1 \times 1 = \frac{1}{2}$$

$\int_3^4 f(x)\,dx$ represents the area of a quadrant with $r = 1$.

$$\int_3^4 f(x)\,dx = \frac{1}{4}\pi \times 1^2 = \frac{\pi}{4}$$

Therefore $\int_0^4 f(x)\,dx = 1 - \frac{\pi}{4} + 1 + \frac{1}{2} + \frac{\pi}{4} = 2\frac{1}{2}$

13a

$$\int_1^2 \frac{1 + x^2}{x^2}\,dx$$

$$= \int_1^2 \left(\frac{1}{x^2} + \frac{x^2}{x^2}\right) dx$$

$$= \int_1^2 (x^{-2} + 1)\,dx$$

$$= [-x^{-1} + x]_1^2$$

$$= \left[-\frac{1}{x} + x\right]_1^2$$

$$= \left(-\frac{1}{2} + 2\right) - \left(-\frac{1}{1} + 1\right)$$

$$= \frac{3}{2} - 0$$

$$= \frac{3}{2}$$

13b

$$\int_{-2}^{-1} \frac{1+2x}{x^3}\,dx$$

$$= \int_{-2}^{-1} \left(\frac{1}{x^3} + \frac{2x}{x^3}\right) dx$$

$$= \int_{-2}^{-1} (x^{-3} + 2x^{-2})\, dx$$

$$= \left[\frac{x^{-2}}{-2} + \frac{2x^{-1}}{-1}\right]_{-2}^{-1}$$

$$= \left[-\frac{1}{2x^2} - \frac{2}{x}\right]_{-2}^{-1}$$

$$= \left(-\frac{1}{2\times(-1)^2} - \frac{2}{(-1)}\right) - \left(-\frac{1}{2\times(-2)^2} - \frac{2}{(-2)}\right)$$

$$= \frac{3}{2} - \frac{7}{8}$$

$$= \frac{5}{8}$$

13c

$$\int_{-3}^{-1} \frac{1 - x^3 - 4x^5}{2x^2}\,dx$$

$$= \int_{-3}^{-1} \left(\frac{1}{2x^2} - \frac{x^3}{2x^2} - \frac{4x^5}{2x^2}\right) dx$$

$$= \int_{-3}^{-1} \left(\frac{1}{2x^2} - \frac{x}{2} - 2x^3\right) dx$$

$$= \left[-\frac{1}{2x} - \frac{x^2}{4} - \frac{x^4}{2}\right]_{-3}^{-1}$$

$$= \left(-\frac{1}{2\times(-1)} - \frac{(-1)^2}{4} - \frac{(-1)^4}{2}\right) - \left(-\frac{1}{2\times(-3)} - \frac{(-3)^2}{4} - \frac{(-3)^4}{2}\right)$$

$$= \left(\frac{1}{2} - \frac{1}{4} - \frac{1}{2}\right) - \left(\frac{1}{6} - \frac{9}{4} - \frac{81}{2}\right)$$

$$= -\frac{1}{4} - \left(-42\frac{7}{12}\right)$$

$$= 42\frac{1}{3}$$

14a

$$\int_1^3 \left(x + \frac{1}{x}\right)^2 dx$$

$$= \int_1^3 \left(x^2 + 2 + \frac{1}{x^2}\right) dx$$

$$= \left[\frac{x^3}{3} + 2x - \frac{1}{x}\right]_1^3$$

$$= \left(\frac{3^3}{3} + 2 \times 3 - \frac{1}{3}\right) - \left(\frac{1^3}{3} + 2 \times 1 - \frac{1}{1}\right)$$

$$= \left(9 + 6 - \frac{1}{3}\right) - \left(\frac{1}{3} + 2 - 1\right)$$

$$= 14\frac{2}{3} - 1\frac{1}{3}$$

$$= 13\frac{1}{3}$$

14b

$$\int_1^2 \left(x^2 + \frac{1}{x^2}\right)^2 dx$$

$$= \int_1^2 \left(x^4 + 2 + \frac{1}{x^4}\right) dx$$

$$= \left[\frac{x^5}{5} + 2x - \frac{1}{3x^3}\right]_1^2$$

$$= \left(\frac{2^5}{5} + 2 \times 2 - \frac{1}{3 \times 2^3}\right) - \left(\frac{1^5}{5} + 2 \times 1 - \frac{1}{3 \times 1^3}\right)$$

$$= \left(\frac{32}{5} + 4 - \frac{1}{24}\right) - \left(\frac{1}{5} + 2 - \frac{1}{3}\right)$$

$$= 10\frac{43}{120} - 1\frac{13}{15}$$

$$= 8\frac{59}{120}$$

14c

$$\int_{-2}^{-1} \left(\frac{1}{x^2} + \frac{1}{x}\right)^2 dx$$

$$= \int_{-2}^{-1} \left(\frac{1}{x^4} + \frac{2}{x^3} + \frac{1}{x^2}\right) dx$$

$$= \left[-\frac{1}{3x^3} - \frac{1}{x^2} - \frac{1}{x}\right]_{-2}^{-1}$$

$$= \left(-\frac{1}{3 \times (-1)^3} - \frac{1}{(-1)^2} - \frac{1}{(-1)}\right) - \left(-\frac{1}{3 \times (-2)^3} - \frac{1}{(-2)^2} - \frac{1}{(-2)}\right)$$

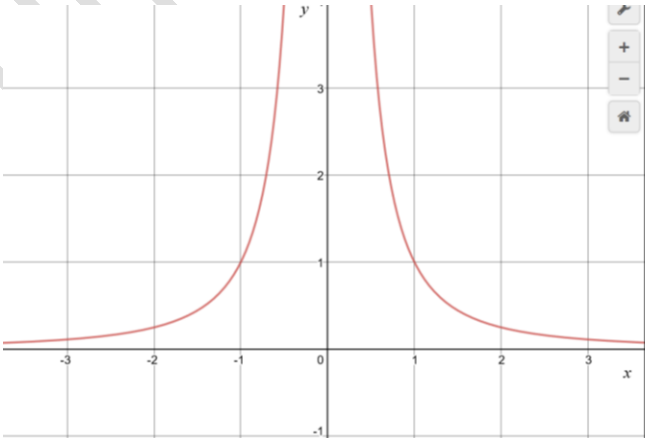
$$= \left(\frac{1}{3} - 1 + 1\right) - \left(\frac{1}{24} - \frac{1}{4} + \frac{1}{2}\right)$$

$$= \frac{1}{3} - \frac{7}{24}$$

$$= \frac{1}{24}$$

15a $x^2 \geq 0$ for all real x and hence $\frac{1}{x^2} > 0$ for all $x \neq 0$.

15b



The curve is discontinuous at $x = 0$, hence we cannot integrate over a domain containing $x = 0$. The integral is meaningless and hence use of the fundamental theorem is invalid.

15c Part ii is meaningless as it crosses the asymptote at $x = 3$. The rest are well defined.

16a i Here $f(t) = t^2$ and hence $f(x) = x^2$.

16a ii Here $f(t) = t^3 + 3t$ and hence $f(x) = x^3 + 3x$.

16a iii Here $f(t) = \frac{1}{t}$ and hence $f(x) = \frac{1}{x}$.

16a iv Here $f(t) = (t^3 - 3)^4$ and hence $f(x) = (x^3 - 3)^4$.

16b i

$$\frac{d}{dx}\int_1^x t^2\,dt$$

$$= \frac{d}{dx}\left[\frac{t^3}{3}\right]_1^x$$

$$= \frac{d}{dx}\left(\frac{x^3}{3} - \frac{1}{3}\right)$$

$$= x^2$$

16b ii

$$\frac{d}{dx}\int_2^x (t^3 + 3t)\,dt$$

$$= \frac{d}{dx}\left[\frac{t^4}{4} + \frac{3t^2}{2}\right]_2^x$$

$$= \frac{d}{dx}\left(\frac{x^4}{4} + \frac{3x^2}{2} - \left(\frac{2^4}{4} + \frac{3 \times 2^2}{2}\right)\right)$$

$$= x^3 + 3x$$

16b iii

$$\frac{d}{dx}\int_a^x \frac{1}{t}\,dt$$

$$= \frac{d}{dx}[\ln t]_a^x$$

$$= \frac{d}{dx}(\ln x - \ln a)$$

$$= \frac{1}{x}$$

16b iv

$$\frac{d}{dx}\int_a^x (t^3 - 3)^4\,dt$$

$$= \frac{d}{dx}\int_a^x (t^6 - 6t^3 + 9)(t^6 - 6t^3 + 9)\,dt$$

$$= \frac{d}{dx}\int_a^x (t^{12} - 12t^9 + 54t^6 - 108t^3 + 81)\,dt$$

$$= \frac{d}{dx}\left[\frac{t^{13}}{13} - \frac{12t^{10}}{10} + \frac{54t^7}{7} - \frac{108t^4}{4} + 81t\right]_a^x$$

$$= \frac{d}{dx}\left(\frac{x^{13}}{13} - \frac{12x^{10}}{10} + \frac{54x^7}{7} - \frac{108x^4}{4} + 81x - \left(\frac{a^{13}}{13} - \frac{12a^{10}}{10} + \frac{54a^7}{7} - \frac{108a^4}{4} + 81a\right)\right)$$

$$= x^{12} - 12x^9 + 54a^6 - 108a^3 + 81 - (0)$$

$= (x^3 - 3)^4$ using the reverse of the expansion shown above in the first 3 lines.

17a $V\,(x) = (a - x)\,U(x) + \int_0^x U(t)\,dt$

Given, $\frac{dU(x)}{dx} = u(x)$

$V'\,(x) = \frac{d}{dx}\left[(a - x)\,U(x) + \int_0^x U(t)\,dt\right]$

$V'\,(x) = \frac{d}{dx}(a - x)\,U(x) + \frac{d}{dx}\int_0^x U(t)\,dt$

As we know from the fundamental theorem that,

$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$

Hence,

$V'\,(x) = (a - x)\,u(x) - U(x) + U(x)$

$V'\,(x) = (a - x)\,u(x)$

17b Here we use the chain rule, let $U = U(x)$ and $dv = dx$. It follows that $\frac{dU}{dx} = u(x)$ and that $v = x$. Hence

$$\begin{aligned}\int_0^a U(x)\,dx &= [Uv]_0^a - \int_0^a v\,dU\\ &= [U(x)v]_0^a - \int_0^a x \cdot u(x)\,dx\\ &= [U(x)x]_0^a - \int_0^a x \cdot u(x)\,dx\\ &= U(a)\cdot a - U(0)\cdot 0 - \int_0^a x \cdot u(x)\,dx\\ &= aU(a) - \int_0^a x \cdot u(x)\,dx\end{aligned}$$

By the fundamental theorem of calculus $\int_0^a u(x) = U(a) - U(0)$ and so $U(a) = \int_0^a u(x) + U(0)$. Thus

$$\begin{aligned}\int_0^a U(x)\,dx &= a\left(\int_0^a u(x) + U(0)\right) - \int_0^a x \cdot u(x)\,dx\\ &= a\int_0^a u(x) + aU(0) - \int_0^a x \cdot u(x)\,dx\\ &= \int_0^a a\,u(x) + aU(0) - \int_0^a x \cdot u(x)\,dx\\ &= aU(0) + \int_0^a (a - x)u(x)\,dx\end{aligned}$$

Solutions to Exercise 5C

1

$$\int_4^5 (2x-3)\,dx$$

$$= \left[\frac{2x^2}{2} - 3x\right]_4^5$$

$$= \left(\frac{2\times 5^2}{2} - 3\times 5\right) - \left(\frac{2\times 4^2}{2} - 3\times 4\right)$$

$$= 10 - 4$$

$$= 6$$

Now inspect the reverse:

$$\int_5^4 (2x-3)\,dx$$

$$= \left[\frac{2x^2}{2} - 3x\right]_5^4$$

$$= \left(\frac{2\times 4^2}{2} - 3\times 4\right) - \left(\frac{2\times 5^2}{2} - 3\times 5\right)$$

$$= 4 - 10$$

$$= -6$$

It is observed that the values differ by a factor of -1.

2a

$$\text{LHS} = \int_0^1 6x^2\,dx$$

$$= [2x^3]_0^1$$

$$= 2\times 1^3 - 2\times 0^3$$

$$= 2$$

$$\text{RHS} = 6\int_0^1 x^2\,dx$$

$$= 6\left[\frac{x^3}{3}\right]_0^1$$

$$= 6 \times \left[\frac{1^3}{3} - \frac{0^3}{3}\right]$$

$$= 6 \times \left(\frac{1}{3} - 0\right)$$

$$= 6 \times \frac{1}{3}$$

$$= 2$$

So LHS = RHS.

2b

$$\text{LHS} = \int_{-1}^{2} (x^3 + x^2)\,dx$$

$$= \left[\frac{x^4}{4} + \frac{x^3}{3}\right]_{-1}^{2}$$

$$= \left(\frac{2^4}{4} + \frac{2^3}{3}\right) - \left(\frac{(-1)^4}{4} + \frac{(-1)^3}{3}\right)$$

$$= \left(\frac{20}{3}\right) - \left(-\frac{1}{12}\right)$$

$$= 6\frac{3}{4}$$

$$\text{RHS} = \int_{-1}^{2} x^3\,dx + \int_{-1}^{2} x^2\,dx$$

$$= \left[\frac{x^4}{4}\right]_{-1}^{2} + \left[\frac{x^3}{3}\right]_{-1}^{2}$$

$$= \left(\frac{2^4}{4}\right) - \left(\frac{(-1)^4}{4}\right) + \left[\left(\frac{2^3}{3}\right) - \left(\frac{(-1)^3}{3}\right)\right]$$

$$= 3\frac{3}{4} + 3$$

$$= 6\frac{3}{4}$$

So LHS = RHS.

2c

$$\text{LHS} = \int_0^3 (x^2 - 4x + 3)\,dx$$

$$= \left[\frac{x^3}{3} - 2x^2 + 3x\right]_0^3$$

$$= \left(\frac{3^3}{3} - 2 \times 3^2 + 3 \times 3\right) - \left(\frac{0^3}{3} - 2 \times 0^2 + 3 \times 0\right)$$

$$= 0 - 0$$

$$= 0$$

$$\text{RHS} = \int_0^2 (x^2 - 4x + 3)\,dx + \int_2^3 (x^2 - 4x + 3)\,dx$$

$$= \left[\frac{x^3}{3} - 2x^2 + 3x\right]_0^2 + \left[\frac{x^3}{3} - 2x^2 + 3x\right]_2^3$$

$$= \left[\left(\frac{2^3}{3} - 2 \times 2^2 + 3 \times 2\right) - \left(\frac{0^3}{3} - 2 \times 0^2 + 3 \times 0\right)\right] + \left[\left(\frac{3^3}{3} - 2 \times 3^2 + 3 \times 3\right) - \left(\frac{2^3}{3} - 2 \times 2^2 + 3 \times 2\right)\right]$$

$$= \left[\frac{2}{3} - 0\right] + \left[0 - \frac{2}{3}\right]$$

$$= \frac{2}{3} - \frac{2}{3}$$

$$= 0$$

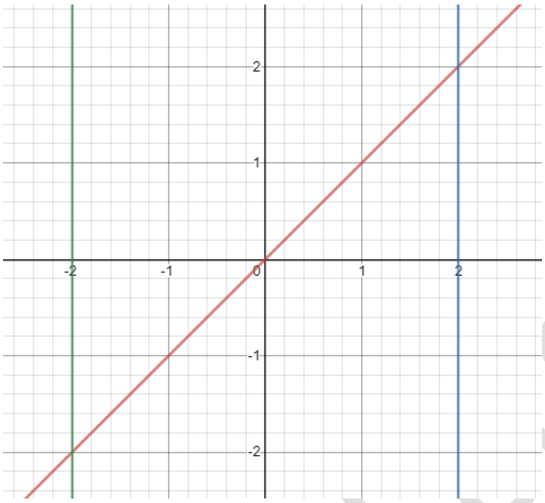
So LHS = RHS.

3a We have learned from the Fundamental Theorem of Calculus that a definite integral with a lower bound a and upper bound b evaluates to the following.

$$\int_a^b f(x)\ dx = F(b) - F(a), \text{where } F(x) \text{ is a primitive of } f(x)$$

If $a = b$, the integral will be over a zero-width interval, and the right-hand side of the equation, $F(b) - F(a)$ evaluates to 0 as $F(b) = F(a)$.

3b This expression is best visualised on the graph.



Since the line $y = x$ passes through the origin, the integral can be expressed as the sum of two integrals on either side of the origin, where $x = 0$.

$$\int_{-2}^{2} x\,dx = \int_{-2}^{0} x\,dx + \int_{0}^{2} x\,dx$$

Each of the integrals on the RHS of the equation above represents the area of a triangle between the line $y = x$ and the x-axis, and the lines $x = -2$ and $x = 2$ respectively. These are congruent triangles, rotated about the origin, therefore the areas are equal.

As we have previously learned that any interval over which a function is negative contributes negatively to the total value of the integral, we can therefore conclude that $\int_{-2}^{0} x\,dx$ is negative, and $\int_{0}^{2} x\,dx$ is positive. Therefore, the total value of the integral is equal to zero.

4a $\int_0^1 (x-1)\ dx$ is negative because in the interval $[0, 1]$, the function $y = x - 1$ is negative. So the area between the curve and the x-axis is negative for $[0, 1]$ as the region is below the x-axis.

4b $\int_1^2 (x-1)\ dx$ is positive because in the interval $[1, 2]$, the function $y = x - 1$ is positive. So the area between the curve and the x-axis is positive for $[1, 2]$ as the region is above the x-axis.

4c $\int_0^2 (x-1)\ dx$ can be expressed as the sum of the integrals $\int_0^1 (x-1)\,dx$ and $\int_1^2 (x-1)\,dx$. As these integrals each represent equal areas of triangles but of opposite sign, it can be concluded that $\int_0^2 (x-1)\ dx = 0$. The areas of the regions above and below the x-axis are equal.

4d The function $y = x - 1$ is negative in the interval $[-2, 1]$ and positive in the interval $[1, 2]$. If this were expressed in integral form:

$$\int_{-2}^{2} (x-1)\,dx = \int_{-2}^{1} (x-1)\,dx + \int_{1}^{2} (x-1)\,dx$$

Based on the graph, it is observed that the area of the triangle below the x-axis is greater than the area of the triangle above the x-axis. The resultant sum of the area of the triangles is negative.

Therefore the integral $\int_{-2}^{2} (x-1)\,dx$ is negative.

5a $\int_{-1}^{1} (1-x^2)\,dx$ is positive because in the interval $[-1, 1]$, the function $y = 1 - x^2$ is positive. So the area between the curve and the x-axis is positive for $[-1, 1]$ as the region is above the x-axis.

5b $\int_1^3 (1-x^2)\ dx$ is negative because in the interval $[1, 3]$, the function $y = 1 - x^2$ is negative. So the area between the curve and the x-axis is negative for $[1, 3]$ as the region is below the x-axis.

5c $\int_{-1}^{0} (1-x^2)\ dx$ represents the area between the curve and the x-axis, between $x = -1$ and $x = 0$. Since the curve between $x = 0$ and $x = 1$ is a reflection in the y-axis of the curve between $x = -1$ and $x = 0$, the area between the curve and the x-axis, between $x = 0$ and $x = 1$ is the same.

Hence $\int_{-1}^{0} (1-x^2)\ dx = \int_0^1 (1-x^2)\ dx$

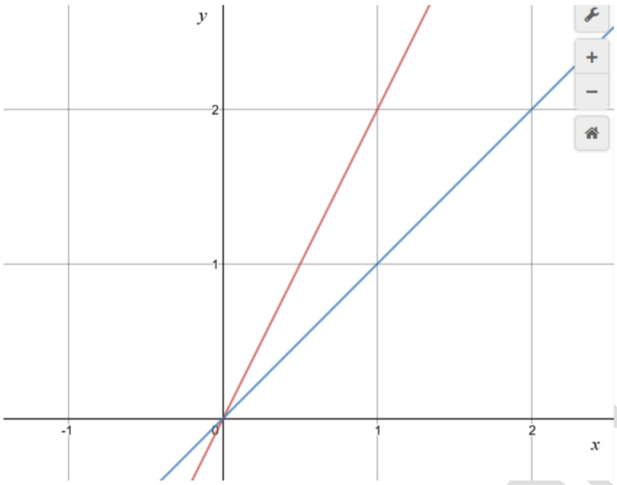
Thus, $y = 1 - x^2$ is an even function, so is symmetrical about the y-axis.

5d From the graph, the area under the function $y = 1 - x^2$ in the interval $\left[0, \frac{1}{2}\right]$ is greater than the area under the curve in the interval $\left[\frac{1}{2}, 1\right]$.

6a By reversing the integral, if $\int_1^3 f(x)\,dx = 7$, then $\int_3^1 f(x)\,dx = -7$

6b By reversing the integral, if $\int_2^1 g(x)\,dx = -5$, then $\int_1^2 g(x)\,dx = 5$

7 The graph below shows the lines for $y = 2x$ (red line) and $y = x$ (blue line).



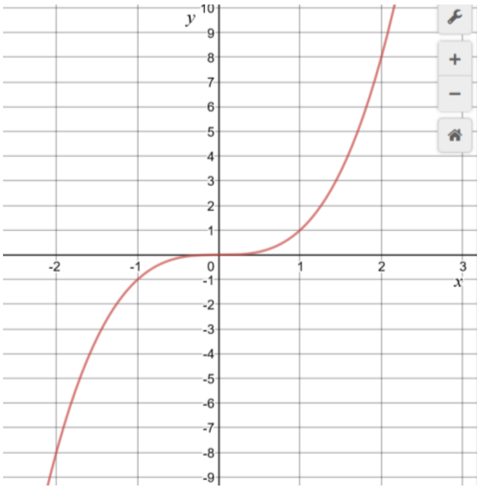
It is clear from the diagram above that the area under the graph of $y = 2x$ for $0 \le x \le 1$ is greater than the area under the graph of $y = x$ for $0 \le x \le 1$, therefore:

$$\int_0^1 2x\,dx > \int_0^1 x\,dx$$

8

$$\int_{-2}^0 x^3\,dx + \int_0^1 x^3\,dx = \int_{-2}^1 x^3\,dx$$

The graph of $y = x^3$ is shown below.



Based on the diagram, the area between the x-axis and the curve in the interval $-2 \le x \le 0$ is negative and the area between the x-axis and the curve in the interval $0 \le x \le 1$ is positive. Since the magnitude of the area between the x-axis and the curve in the interval $-2 \le x \le 0$ is greater than the magnitude of the area between the x-axis and the curve in the interval $0 \le x \le 1$, the definite integral $\int_{-2}^{1} x^3\, dx$ is negative.

9a i

$$\int_0^2 (3x^2 - 1)\, dx$$

$= [x^3 - x]_0^2$

$= (2^3 - 2) - (0^3 - 0)$

$= 6$

9a ii

$$\int_2^0 (3x^2 - 1)\, dx$$

$= [x^3 - x]_2^0$

$= (0^3 - 0) - (2^3 - 2)$

$= -6$

The integrals are opposites because the limits have been reversed.

9b i

$$\int_0^1 20x^3\,dx$$

$$= [5x^4]_0^1$$

$$= (5 \times 1^4) - (5 \times 0^4)$$

$$= 5$$

9b ii

$$20\int_0^1 x^3\,dx$$

$$= 20\left[\frac{x^4}{4}\right]_0^1$$

$$= 20\left(\frac{1^4}{4} - \frac{0^4}{4}\right)$$

$$= 20\left(\frac{1}{4}\right)$$

$$= 5$$

The integrals are equal. The constant factor of 20 was moved outside the integral in part ii.

9c i

$$\int_1^4 (4x + 5)\,dx$$

$$= [2x^2 + 5x]_1^4$$

$$= (2 \times 4^2 + 5 \times 4) - (2 \times 1^2 + 5 \times 1)$$

$$= 52 - 7$$

$$= 45$$

9c ii

$$\int_1^4 4x\,dx$$

$= [2x^2]_1^4$

$= (2 \times 4^2) - (2 \times 1^2)$

$= 32 - 2$

$= 30$

9c iii

$$\int_1^4 5\,dx$$

$= [5x]_1^4$

$= (5 \times 4) - (5 \times 1)$

$= 20 - 5$

$= 15$

From the values of the integrals, the relationship

$\int_1^4 (4x+5)\,dx = \int_1^4 (4x)\,dx + \int_1^4 (5)\,dx$ is observed. This shows that the integral of a sum is the sum of the integrals of each term in the expression.

9d i

$$\int_0^2 12x^3\,dx$$

$= [3x^4]_0^2$

$= (3 \times 2^4) - (3 \times 0^4)$

$= 48 - 0$

$= 48$

9d ii

$$\int_0^1 12x^3\,dx$$

$$= [3x^4]_0^1$$

$$= (3 \times 1^4) - (3 \times 0^4)$$

$$= 3 - 0$$

$$= 3$$

9d iii

$$\int_1^2 12x^3\,dx$$

$$= [3x^4]_1^2$$

$$= (3 \times 2^4) - (3 \times 1^4)$$

$$= 48 - 3$$

$$= 45$$

From the values of the definite integrals, the relationship

$\int_0^2 12x^3\,dx = \int_0^1 12x^3\,dx + \int_1^2 12x^3\,dx$ is observed. This is because the interval from $x = 0$ to $x = 2$ can be dissected into two successive intervals from $x = 0$ to $x = 1$ then from $x = 1$ to $x = 2$.

9e i

$$\int_3^3 (4 - 3x^2)\,dx$$

$$= [4x - x^3]_3^3$$

$$= (4 \times 3 - 3^3) - (4 \times 3 - 3^3)$$

$$= 0$$

9e ii

$$\int_{-2}^{-2} (4 - 3x^2)\, dx$$

$$= [4x - x^3]_{-2}^{-2}$$

$$= (4 \times -2 - (-2)^3) - (4 \times -2 - (-2)^3)$$

$$= 0$$

The values of the definite integrals are 0 as the integral is evaluated over an interval of zero width.

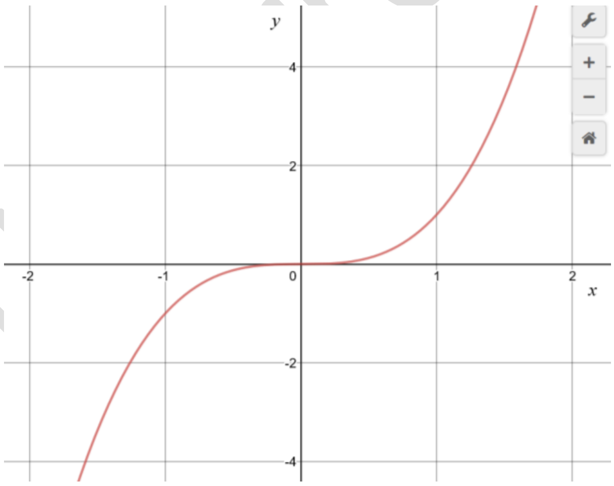
10a Since the lower limit = upper limit, the width of the interval is 0, and therefore the value of the integral is 0.

So $\int_3^3 \sqrt{9 - x^2}\, dx = 0$

10b Since the lower limit = upper limit, the width of the interval is 0, and therefore the value of the integral is 0.

So $\int_4^4 (x^3 - 3x^2 + 5x - 7)\, dx = 0$

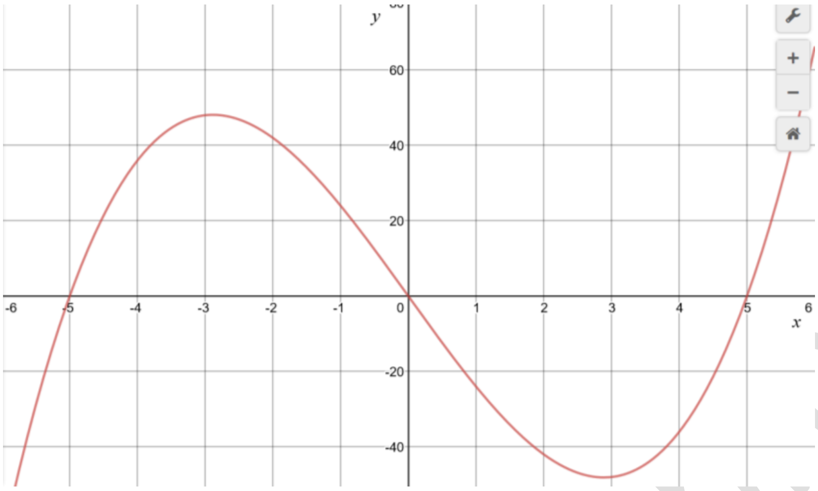
10c The graph of $y = x^3$ is shown below.



The integrand $y = x^3$ is an odd function with symmetry about the origin. For an odd function $f(x)$, $\int_{-a}^{a} f(x)\, dx = 0$.

So $\int_{-1}^{1} x^3\, dx = 0$

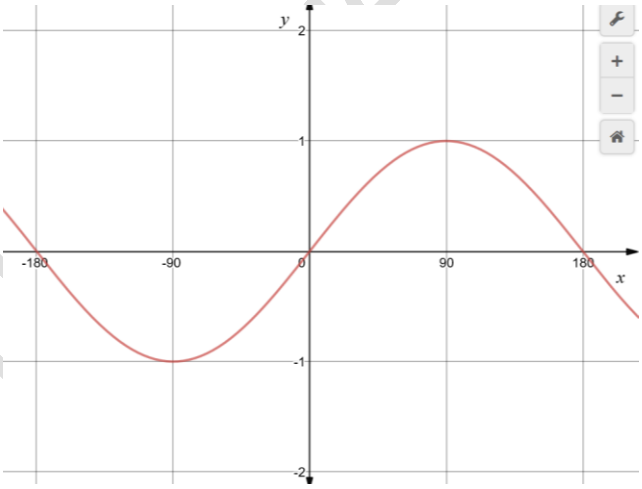
10d The graph of $y = x^3 - 25x$ is shown below.



The integrand $y = x^3 - 25x$ is an odd function with symmetry about the origin. For an odd function $f(x)$, $\int_{-a}^{a} f(x)\, dx = 0$.

So $\int_{-5}^{5} (x^3 - 25x)\, dx = 0$

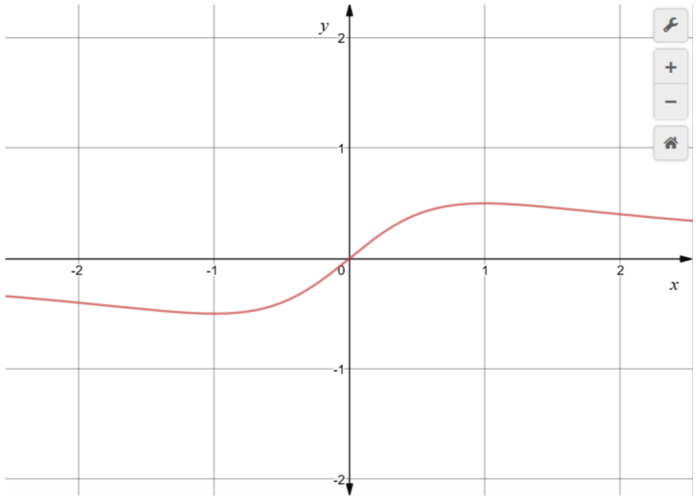
10e The graph of $y = \sin x$ is shown below.



The integrand $y = \sin x$ is an odd function with symmetry about the origin. For an odd function $f(x)$, $\int_{-a}^{a} f(x)\, dx = 0$.

So $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x\, dx = 0$

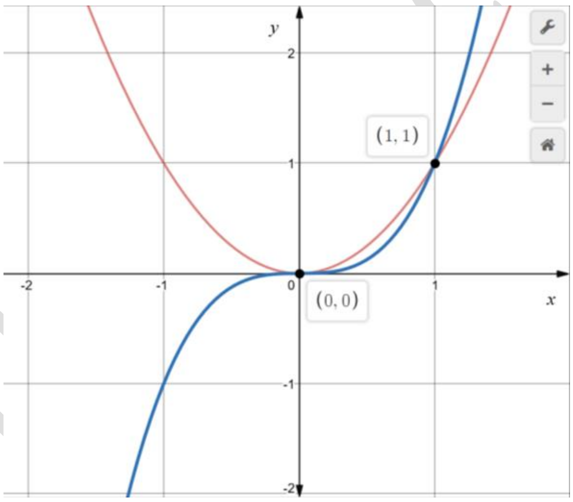
10f The graph of $y = \frac{x}{1+x^2}$ is shown below.



The integrand $y = \frac{x}{1+x^2}$ is an odd function with symmetry about the origin. For an odd function $f(x)$, $\int_{-a}^{a} f(x)\,dx = 0$.

So $\int_{-2}^{2} \frac{x}{1+x^2}\,dx = 0$

11a The graph below shows the curve for $y = x^2$ (red line) and the curve for $y = x^3$ (blue line). The points of intersection are (0, 0) and (1, 1).



11b If we employ the lower and upper rectangle trap method, we know that both $\int_0^1 x^3\,dx$ and $\int_0^1 x^2\,dx$ are between 0 and 1.

The area under the graph of $y = x^3$ to the x-axis is less than the area under the graph of $y = x^2$ to the x-axis in the interval $x = 0$ to $x = 1$.

Therefore, we know that:

$\int_0^1 x^3\,dx < \int_0^1 x^2\,dx$.

As they are both between 0 and 1, we can assert the following:

$$0 < \int_0^1 x^3\,dx < \int_0^1 x^2\,dx < 1$$

11c

$$\int_0^1 x^3\,dx$$

$$= \left[\frac{x^4}{4}\right]_0^1$$

$$= \left(\frac{1^4}{4}\right) - \left(\frac{0^4}{4}\right)$$

$$= \frac{1}{4}$$

$$\int_0^1 x^2\,dx$$

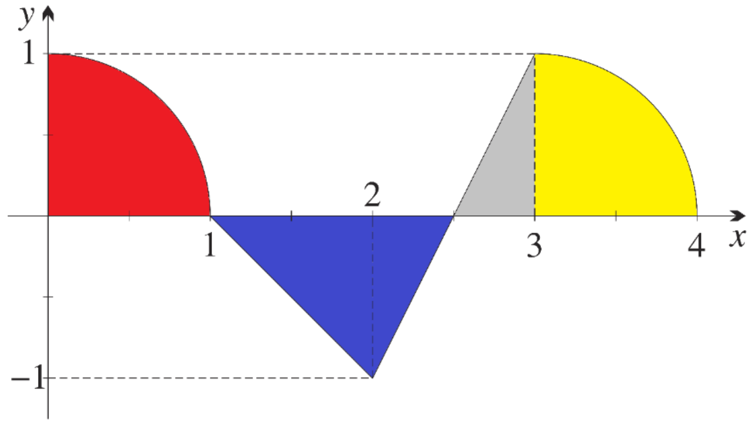
$$= \left[\frac{x^3}{3}\right]_0^1$$

$$= \left(\frac{1^4}{3}\right) - \left(\frac{0^4}{3}\right)$$

$$= \frac{1}{3}$$

Given that $0 < \frac{1}{4} < \frac{1}{3} < 1$, the inequality written in question 12b is true.

12a



In the interval $0 \leq x \leq 1$, the signed area is positive.

In the interval $1 \leq x \leq 2.5$, the signed area is negative.

In the interval $2.5 \leq x \leq 3$, the signed area is positive.

In the interval $3 \leq x \leq 4$, the signed area is positive.

The sum of the signed areas can be expressed by the following equation:

$$\int_0^4 f(x)\,dx = \int_0^1 f(x)\,dx - \int_1^{2.5} f(x)\,dx + \int_{2.5}^3 f(x)\,dx + \int_3^4 f(x)\,dx$$

The RHS of the equation above can be solved piecewise, as follows:

$\int_0^1 f(x)\,dx$ is the area of a quadrant of a circle of radius, $r = 1$ (shown in red).

$$\int_0^1 f(x)\,dx = \frac{1}{4}\pi \times 1^2 = \frac{\pi}{4}$$

$\int_1^{2.5} f(x)\,dx$ is the area of a triangle with base, $b = 2.5 - 1 = 1.5$, and height, $h = 1$ (shown in blue).

$$\int_1^{2.5} f(x)\,dx = \frac{1}{2} \times 1.5 \times 1 = \frac{3}{4}$$

$\int_{2.5}^3 f(x)\,dx$ is the area of a triangle with base, $b = 3 - 2.5 = 0.5$, and height, $h = 1$ (shown in grey).

$$\int_{2.5}^3 f(x)\,dx = \frac{1}{2} \times 0.5 \times 1 = \frac{1}{4}$$

$\int_3^4 f(x)\,dx$ is the area of a quadrant of a circle of radius, $r = 1$ (shown in yellow).

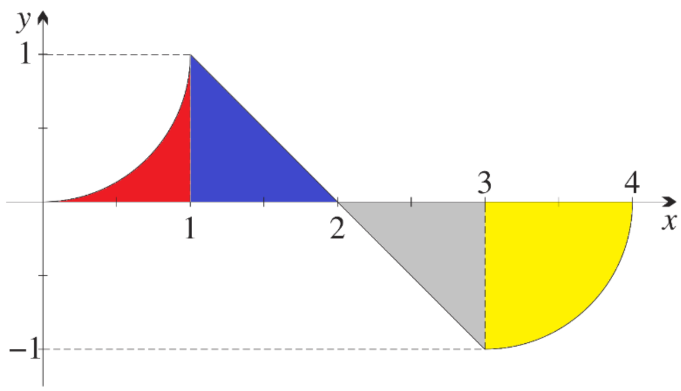
$$\int_3^4 f(x)\,dx = \frac{1}{4}\pi \times 1^2 = \frac{\pi}{4}$$

So $\int_0^4 f(x)\,dx$

$$= \frac{\pi}{4} - \frac{3}{4} + \frac{1}{4} + \frac{\pi}{4}$$

$$= \frac{\pi - 1}{2}$$

12b



In the interval $0 \le x \le 1$, the signed area is positive.

In the interval $1 \le x \le 2$, the signed area is positive.

In the interval $2 \le x \le 3$, the signed area is negative.

In the interval $3 \le x \le 4$, the signed area is negative.

The sum of the signed areas can be expressed by the following equation:

$$\int_0^4 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx - \int_2^3 f(x)\,dx - \int_3^4 f(x)\,dx$$

The RHS of the equation above can be solved piecewise, as follows:

$\int_0^1 f(x)\,dx$ is the area of the inverse of a quadrant of a circle of radius, $r = 1$ inscribed in a square of side, $b = 1$ (shown in red).

$$\int_0^1 f(x)\,dx = 1^2 - \frac{1}{4}\pi \times 1^2 = 1 - \frac{\pi}{4}$$

$\int_1^2 f(x)\,dx$ is the area of a triangle with base, $b = 2 - 1 = 1$, and height, $h = 1$ (shown in blue).

$$\int_1^2 f(x)\,dx = \frac{1}{2}\times 1\times 1 = \frac{1}{2}$$

$\int_2^3 f(x)\,dx$ is the area of a triangle with base, $b = 3 - 2 = 1$, and height, $h = 1$ (shown in grey).

$$\int_2^3 f(x)\,dx = \frac{1}{2}\times 1\times 1 = \frac{1}{2}$$

$\int_3^4 f(x)\,dx$ is the area of a quadrant of a circle of radius, $r = 1$ (shown in yellow).

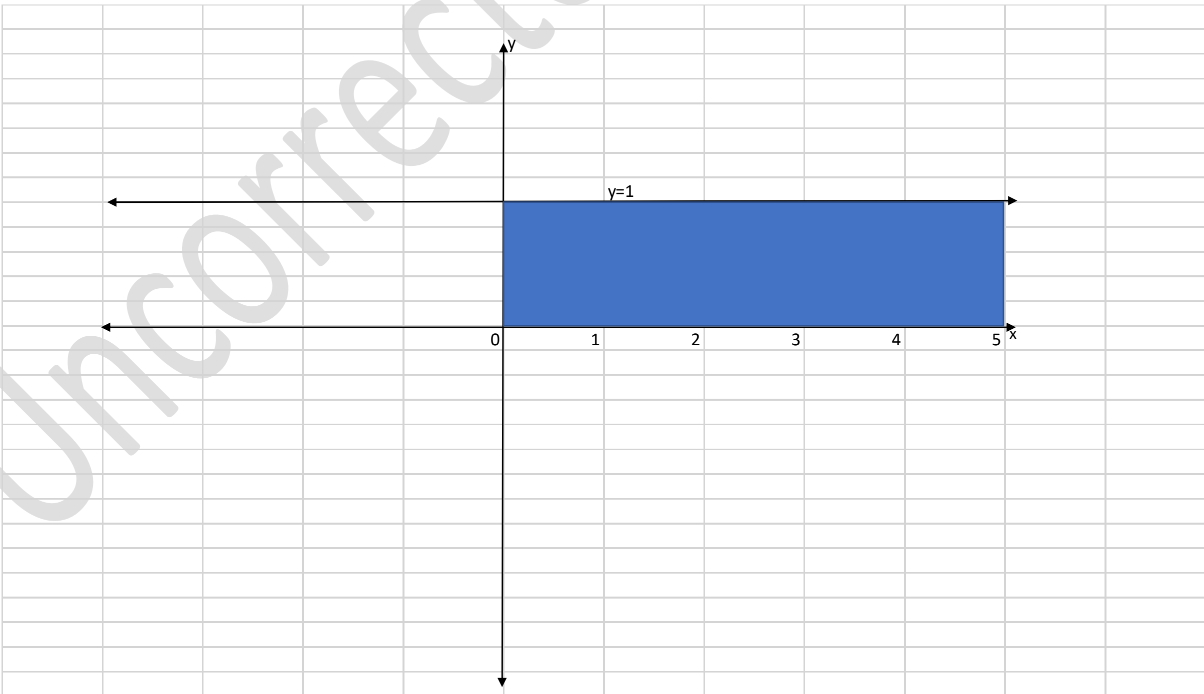
$$\int_3^4 f(x)\,dx = \frac{1}{4}\pi\times 1^2 = \frac{\pi}{4}$$

So $\int_0^4 f(x)\,dx$

$$= 1 - \frac{\pi}{4} + \frac{1}{2} - \frac{1}{2} - \frac{\pi}{4}$$

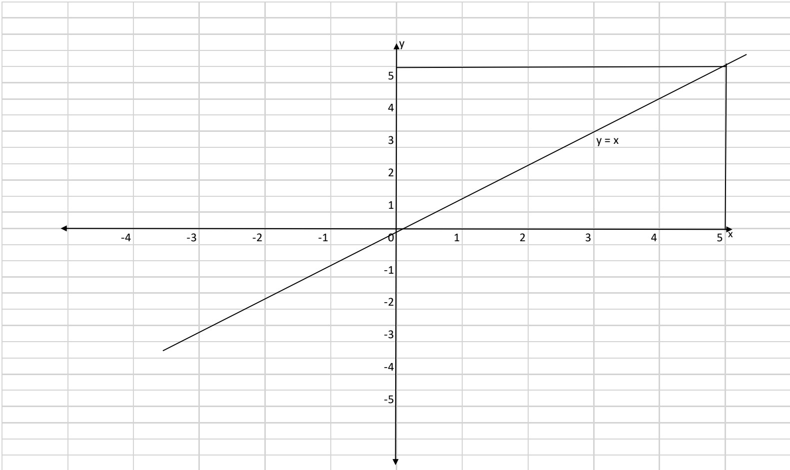
$$= 1 - \frac{\pi}{2}$$

13a i



Area of shaded region is $= l\times b = 1\times 5 = 5$

13a ii



Area of region by formula is $= \frac{1}{2} \times b \times h$

$= \frac{1}{2} \times 5 \times 5$

$= \frac{25}{2}$

13b i

$$\int_0^5 2x\,dx$$

$$= 2\int_0^5 x\,dx$$

$$= \left[\frac{2x^2}{2}\right]_0^5$$

$$= [x^2]_0^5$$

$$= (25 - 0)$$

$$= 25$$

13b ii

$$\int_0^5 x + 1\,dx$$

$$= \int_0^5 x\,dx + \int_0^5 1\,dx$$

$$= \left[\frac{x^2}{2}\right]_0^5 + [x]_0^5$$

$$= \left(\frac{25}{2} - 0\right) + (5 - 0)$$

$$= \frac{25}{2} + 5 = 17\frac{1}{2}$$

13b iii

$$\int_0^5 3x - 2\,dx$$

$$= \int_0^5 3x\,dx - \int_0^5 2\,dx$$

$$= 3\int_0^5 x\,dx - 2\int_0^5 1\,dx$$

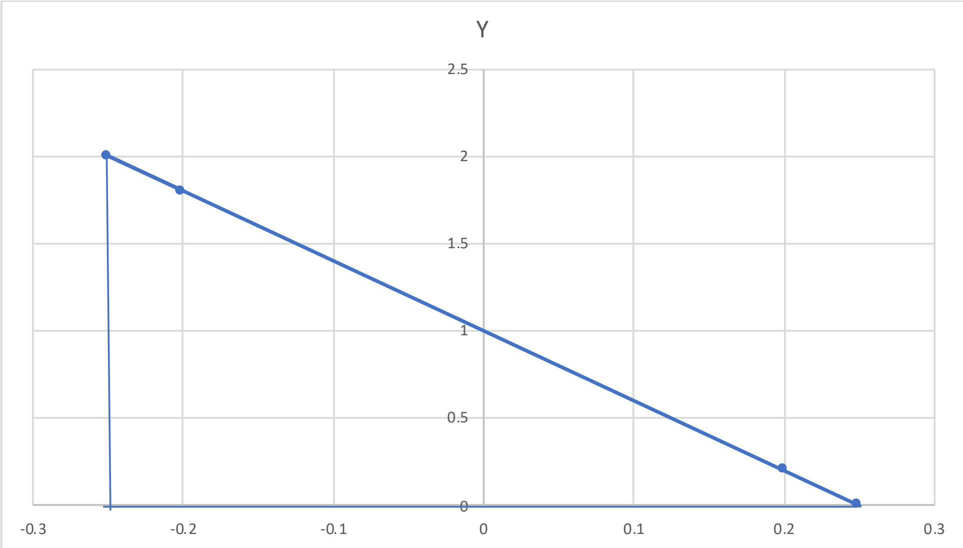
$$= \left[\frac{3x^2}{2}\right]_0^5 - [2x]_0^5$$

$$= \left(\frac{75}{2} - 0\right) - (10 - 0)$$

$$= \frac{75}{2} - 10$$

$$= 27\frac{1}{2}$$

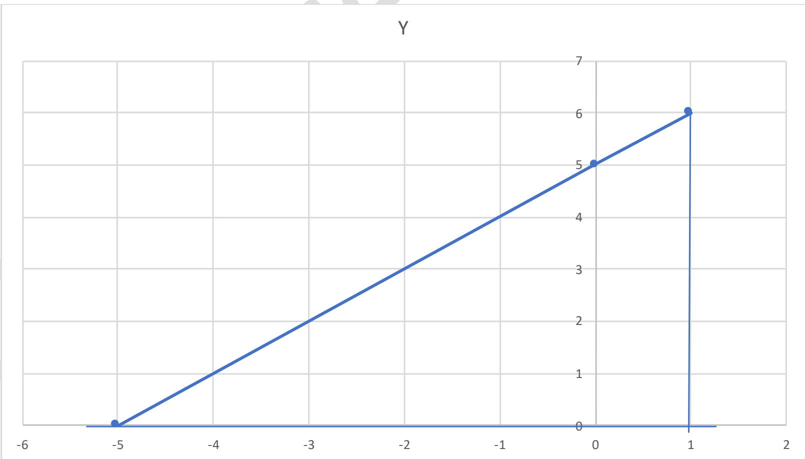
14a i



Area of region by formula is $= \frac{1}{2} \times \frac{1}{2} \times 2$

$= \frac{1}{2}$

14a ii



Definite integral is the area of the region between the line $y = |x + 5|$ and the x-axis, from $x = 1$ to $x = -5$.

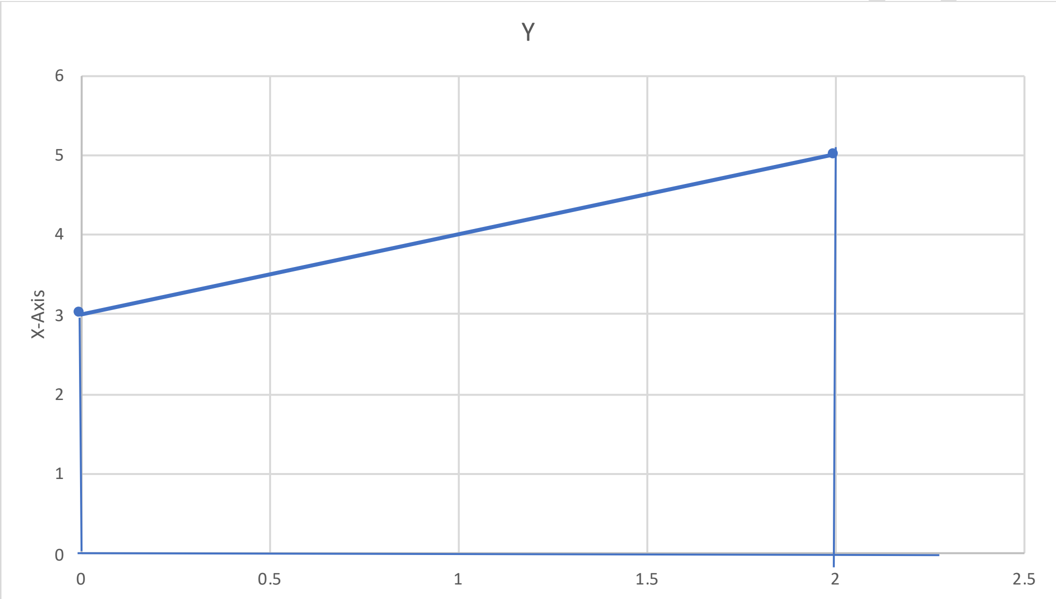
The area under the graph is a triangle with:

$b = 1 - (-5) = 6$

$h = 6 - 0 = 6$

$$\int_{-5}^{1} |x+5|\, dx = \frac{1}{2} bh = \frac{1}{2} \times 6 \times 6 = 18$$

14a iii



Definite integral is the area of the region between the line $y = |x| + 3$ and the x-axis, from $x = 0$ to $x = 2$.

The area under the graph is a trapezium with:

$a = 3 - 0 = 3$

$b = 5 - 0 = 5$

$h = 2 - 0 = 2$

Area of the region $= \frac{1}{2}(a+b)h = \frac{1}{2} \times (3+5) \times 2 = 8$

14b i According to the formula,

$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

Hence,

$\int_{\frac{1}{4}}^{-\frac{1}{4}}(1-4x)\,dx = -\int_{-\frac{1}{4}}^{\frac{1}{4}}(1-4x)\,dx = -\frac{1}{2}$

14b ii According to the formula,

$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

Hence,

$\int_1^{-5}|x+5|\,dx = -\int_{-5}^{1}|x+5|\,dx = -18$

14b iii According to the formula,

$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

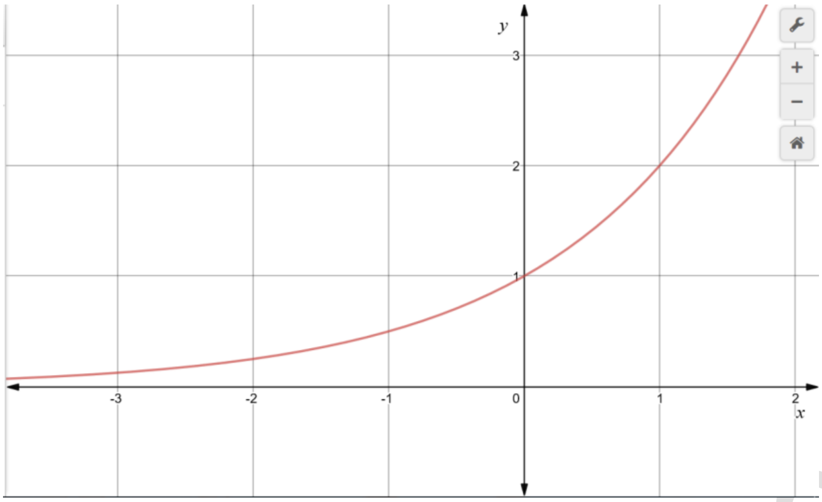
Hence,

$\int_2^0 |x| + 3\,dx = -\int_0^2 |x| + 3\,dx = -8$

15a The function is odd so the integral is zero.

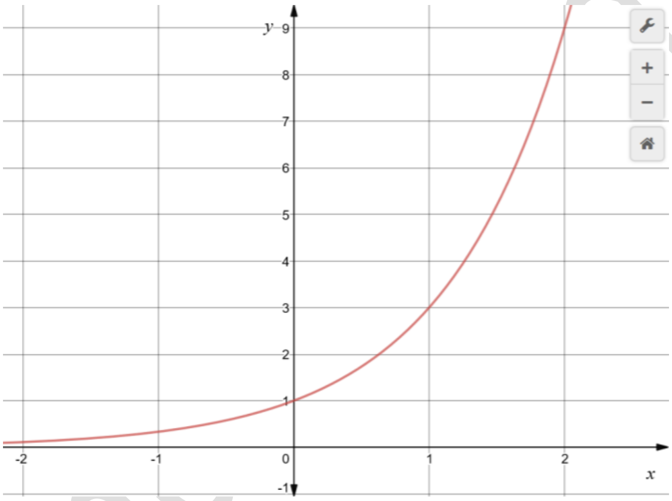
15b The function is even so its graph is symmetrical about the y-axis and is also an even function.

16a The graph of $y = 2^x$ is shown below.



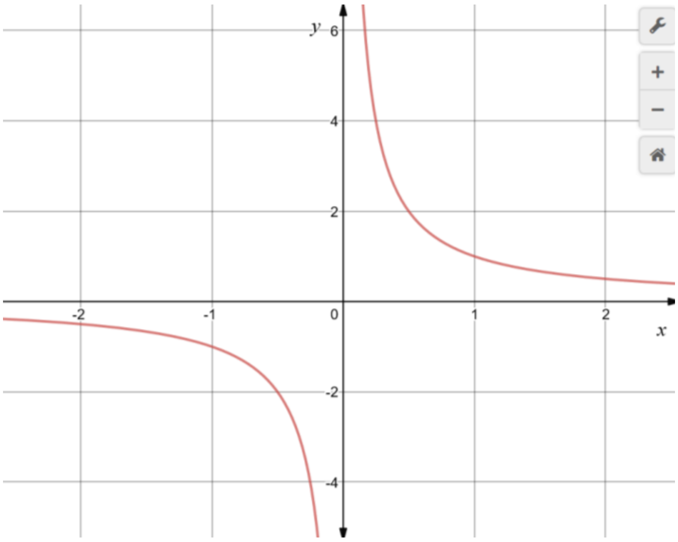
Since the function is not odd, the statement is false.

16b The graph of $y = 3^x$ is shown below.



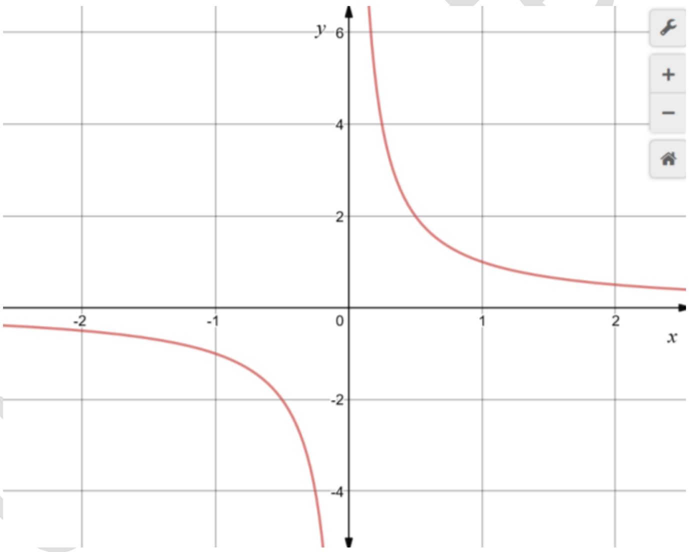
Since all of the required area from $x = 0$ to $x = 2$ is above the x-axis (and hence is positive), the statement is true.

16c The graph of $y = \frac{1}{x}$ is shown below.



Since all of the required area from $x = -2$ to $x = -1$ is below the x-axis (and hence is negative), the statement is false.

16d The graph of $y = \frac{1}{x}$ is shown below.



The area from $x = 1$ to $x = 2$ is above the x-axis (and is positive), so $\int_1^2 \frac{1}{x} dx > 0$.

Since the integral $\int_2^1 \frac{1}{x} dx$ has the reverse limits, $\int_2^1 \frac{1}{x} dx = -\int_1^2 \frac{1}{x} dx$.

This means that $\int_2^1 \frac{1}{x} dx < 0$ and hence the statement is false.

17a i

$$\int_3^4 dx = \int_3^4 1\, dx = [x]_3^4 = 4 - 3 = 1$$

$$\int_2^3 dx = \int_2^3 1\, dx = [x]_2^3 = 3 - 2 = 1$$

$$\int_1^2 dx = \int_1^2 1\, dx = [x]_1^2 = 2 - 1 = 1$$

Hence, $\int_3^4 dx = \int_2^3 dx = \int_1^2 dx = 1$

17a ii

$$\frac{2}{7}\int_3^4 x\, dx = \frac{2}{7}\left[\frac{x^2}{2}\right]_3^4 = \frac{2}{7}\times\left(\frac{16}{2} - \frac{9}{2}\right) = \frac{2}{7}\times\frac{7}{2} = 1$$

$$\frac{2}{5}\int_2^3 x\, dx = \frac{2}{5}\left[\frac{x^2}{2}\right]_2^3 = \frac{2}{5}\times\left(\frac{9}{2} - \frac{4}{2}\right) = \frac{2}{5}\times\frac{5}{2} = 1$$

$$\frac{2}{3}\int_1^2 x\, dx = \frac{2}{3}\left[\frac{x^2}{2}\right]_1^2 = \frac{2}{3}\times\left(\frac{4}{2} - \frac{1}{2}\right) = \frac{2}{3}\times\frac{3}{2} = 1$$

Hence, $\frac{2}{7}\int_3^4 x\, dx = \frac{2}{5}\int_2^3 x\, dx = \frac{2}{3}\int_1^2 x\, dx = 1$

17a iii

$$\frac{3}{37}\int_3^4 x^2\, dx = \frac{3}{37}\left[\frac{x^3}{3}\right]_3^4 = \frac{3}{37}\times\left(\frac{64}{3} - \frac{27}{3}\right) = \frac{3}{37}\times\frac{37}{3} = 1$$

$$\frac{3}{19}\int_2^3 x^2\, dx = \frac{3}{19}\left[\frac{x^3}{3}\right]_2^3 = \frac{3}{19}\times\left(\frac{27}{3} - \frac{8}{3}\right) = \frac{3}{19}\times\frac{19}{3} = 1$$

$$\frac{3}{7}\int_1^2 x^2\, dx = \frac{3}{7}\left[\frac{x^3}{3}\right]_1^2 = \frac{3}{7}\times\left(\frac{8}{3} - \frac{1}{3}\right) = \frac{3}{7}\times\frac{1}{3} = 1$$

Hence, $\frac{3}{37}\int_3^4 x^2\, dx = \frac{3}{19}\int_2^3 x^2\, dx = \frac{3}{7}\int_1^2 x^2\, dx = 1$

17b i

$$\int_1^4 dx = \int_1^4 1\, dx = [x]_1^4 = 4 - 1 = 3$$

17b ii

$$\int_1^3 x\, dx = \left[\frac{x^2}{2}\right]_1^3 = \left(\frac{9}{2} - \frac{1}{2}\right) = \frac{8}{2} = 4$$

17b iii

$$\int_2^1 x^2\,dx = \left[\frac{x^3}{3}\right]_2^1 = \left(\frac{1}{3}-\frac{8}{3}\right) = -\frac{7}{3}$$

17b iv

$$\int_1^2 (x^2+1)\,dx = \left[\frac{x^3}{3}+x\right]_1^2$$

$$= \left(\frac{8}{3}+2\right)-\left(\frac{1}{3}+1\right)$$

$$= \frac{14}{3}-\frac{4}{3} = \frac{10}{3}$$

17b v

$$\int_1^3 7x^2\,dx = 7\int_1^3 x^2\,dx$$

$$= 7\left[\frac{x^3}{3}\right]_1^3$$

$$= 7\times\left(\frac{27}{3}-\frac{1}{3}\right)$$

$$= 7\times\frac{26}{3} = 60\frac{2}{3}$$

17b vi

$$\int_1^4 (3x^2-6x+5)\,dx$$

$$= \int_1^4 (3x^2)\,dx - \int_1^4 (6x)\,dx + \int_1^4 5\,dx$$

$$= 3\int_1^4 x^2\,dx - 6\int_1^4 x\,dx + 5\int_1^4 1\,dx$$

$$= 3\left[\frac{x^3}{3}\right]_1^4 - 6\left[\frac{x^2}{2}\right]_1^4 + 5[x]_1^4$$

$$= 3\times\left(\frac{64}{3}-\frac{1}{3}\right) - 6\times\left(\frac{16}{2}-\frac{1}{2}\right) + 5\times(4-1)$$

$$= 3\times\frac{63}{3} - 6\times\frac{15}{2} + 5\times 3$$

$$= 64-45+15$$

$$= 33$$

18a The statement is true as, $\sin 90° = 1$ which makes the function odd and as per the fundamental theorem, if $f(x)$ is odd then $\int_{-a}^{a} f(x)\,dx = 0$.

18b The statement is true as, $\sin 120° = -\frac{\sqrt{3}}{2}$which is odd and $\cos 60° = \frac{1}{2}$ which is even makes the function odd and as per the fundamental theorem, if $f(x)$ is odd then $\int_{-a}^{a} f(x)\,dx = 0$

18c The statement is false as $2^{-x^2} > 0$ for all values of x.

18d The statement is true as $2^x < 3^x$ for all values of $0 < x < 1$

18e The statement is false as $2^x > 3^x$ for all values of $-1 > x > 0$

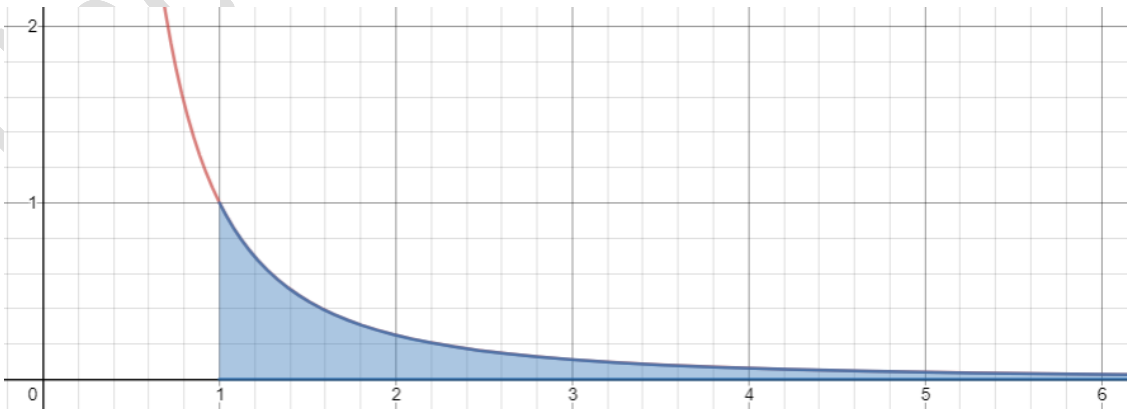
18f The statement is true as $t^n > t^{n+1}$ for all values of $0 \le t \le 1$

Hence,

$$\frac{1}{1+t^n} \le \frac{1}{1+t^{n+1}}$$

19a $\int_1^N \frac{1}{x^2}\,dx = \int_1^N x^{-2}\,dx = [-x^{-1}]_1^N = \left[-\frac{1}{x}\right]_1^N = -\frac{1}{N} - \left(-\frac{1}{1}\right) = 1 - \frac{1}{N}$

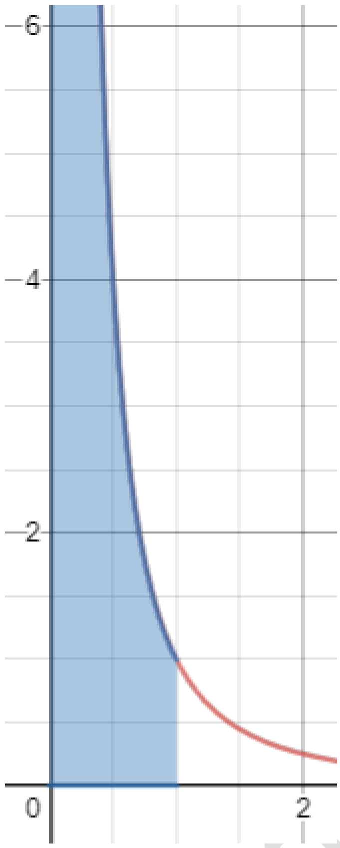
Thus as $N \to \infty$, $\int_1^N \frac{1}{x^2}\,dx = 1 - \frac{1}{N} \to 1 - 0 = 1$ so the integral converges to 1.



19b $\int_e^1 \frac{1}{x^2}\,dx = \int_e^1 x^{-2}\,dx = [-x^{-1}]_e^1 = \left[-\frac{1}{x}\right]_e^1 = -\frac{1}{1} - \left(-\frac{1}{e}\right) = \frac{1}{e} - 1$

Now, as $e \to 0^+, \frac{1}{e} \to \infty$, thus as $e \to 0^+$, $\int_e^1 \frac{1}{x^2}\,dx = \frac{1}{e} - 1 \to \infty$.

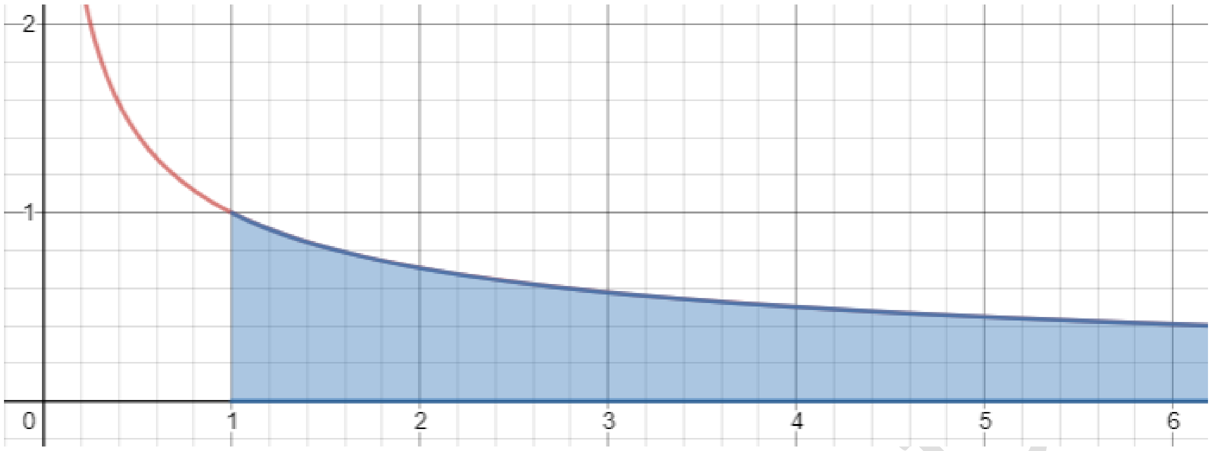
Thus, the integral diverges.



19c $\int_1^N \frac{1}{\sqrt{x}}\,dx = \int_1^N x^{-\frac{1}{2}}\,dx = \left[\frac{x^{\frac{1}{2}}}{\frac{1}{2}}\right]_1^N = \left[2\sqrt{x}\right]_1^N = 2\sqrt{N} - 2\sqrt{1} = 2\sqrt{N} - 2$

Now as $N \to \infty, \sqrt{N} \to \infty$, thus as $N \to \infty$, $\int_1^N \frac{1}{\sqrt{x}}\,dx = 2\sqrt{N} - 2 \to \infty$.

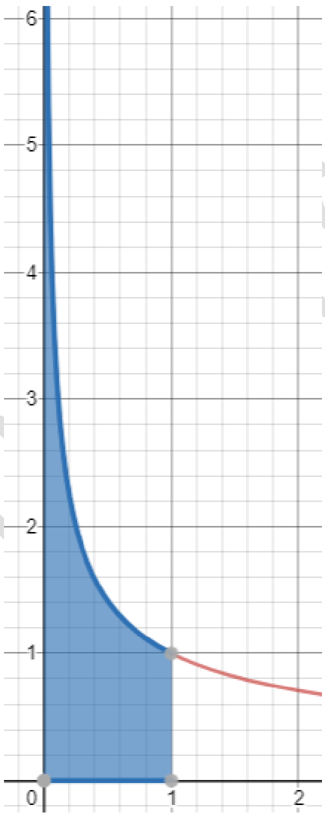
Thus, the integral diverges.



19d $\int_e^1 \frac{1}{\sqrt{x}}\,dx = \int_e^1 x^{-\frac{1}{2}}\,dx = \left[\frac{x^{\frac{1}{2}}}{\frac{1}{2}}\right]_e^1 = \left[2\sqrt{x}\right]_e^1 = 2\sqrt{1} - 2\sqrt{e} = 2 - 2\sqrt{e}$

Now as $e \to 0^+$, $\sqrt{e} \to 0^+$, thus as $e \to 0^+$, $\int_e^1 \frac{1}{\sqrt{x}}\,dx = 2 - 2\sqrt{e} \to 2 - 2(0) = 2$.

Thus, the integral converges to 2.



Solutions to Exercise 5D

1a The triangle formula for a triangle of base, b, and height, h, is $\frac{1}{2}bh$.

The function $A(x) = \int_0^x 3t\,dt$ describes a triangle of base $b = x$ and height $h = 3x$ for the values of x in the interval $[0,3]$.

$$A(x) = \int_0^x 3t\,dt$$
$$= \frac{1}{2} \times x \times 3x$$
$$= \frac{3}{2}x^2$$

1b As $A'(x) = \frac{d}{dx}(A(x))$,

$$A'(x) = \frac{d}{dx}\left(\frac{3}{2}x^2\right)$$
$$= 2 \times \frac{3}{2}x$$
$$= 3x$$

The function $A(x)$ is identical to $A'(x)$ apart from a change of letter.

2a This graph is the straight line $y = 3$.

The rectangle area formula, bh is required to calculate the signed area function.

The graph shows a rectangle of base $b = x$ and height $h = 3$.

$$A(x) = \int_0^x f(t)\,dt = x \times 3 = 3x$$

$$A'(x) = \frac{d}{dx}(A(x)) = \frac{d}{dx}(3x) = 3$$

The function $A'(x)$ is identical to the original function $y = 3$.

2b This graph is the straight line $y = 2t$.

The triangle area formula, $\frac{1}{2}bh$, is required to calculate the signed area function.

The graph shows a triangle of base $b = x$ and height $h = 2x$.

$$A(x) = \int_0^x f(t)\,dt = \frac{1}{2} \times x \times 2x = x^2$$

$$A'(x) = \frac{d}{dx}\big(A(x)\big) = \frac{d}{dx}(x^2) = 2x$$

The function $A'(x)$ is identical to the original function $y = 2t$, apart from the change of letter.

2c This graph is the straight line $y = t + 2$.

The trapezium area formula, $\frac{1}{2}(a + b)h$, is required to calculate the signed area function.

The graph shows a trapezium with dimensions $a = 2$, $b = x + 2$ and height $h = x$

$$A(x) = \int_0^x f(t)\,dt = \frac{1}{2}(2 + x + 2) \times x = \frac{1}{2}(4x + x^2) = 2x + \frac{1}{2}x^2$$

$$A'(x) = \frac{d}{dx}\big(A(x)\big) = \frac{d}{dx}\left(2x + \frac{1}{2}x^2\right) = 2 + x$$

The function $A'(x)$ is identical to the original function $y = t + 2$, apart from the change of letter.

2d This graph is the straight line $y = 5 - t$.

The trapezium area formula, $\frac{1}{2}(a + b)h$, is required to calculate the signed area function.

The graph shows a trapezium with dimensions $a = 5$, $b = 5 - x$ and height $h = x$

$$A(x) = \int_0^x f(t)\,dt = \frac{1}{2}(5 + 5 - x) \times x = \frac{1}{2}(10x - x^2) = 5x - \frac{1}{2}x^2$$

$$A'(x) = \frac{d}{dx}\big(A(x)\big) = \frac{d}{dx}\left(5x - \frac{1}{2}x^2\right) = 5 - x$$

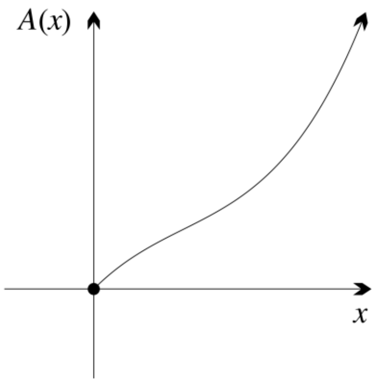
The function $A'(x)$ is identical to the original function $y = 5 - t$, apart from the change of letter.

3a The signed area function $A(x) = \int_0^x f(t)\, dx$ represents the area under the curve.

As x increases from 0, the value of $A(x)$ will increase. However, due to the decreasing nature of the curve $f(t)$ in the interval $[0, 2)$, the value of $A(x)$ will increase at a decreasing rate.

In the interval $(2, \infty)$, the curve of $f(t)$ is increasing, therefore the value of $A(x)$ will increase at an increasing rate.

While the value of $A(x)$ increases for all values of $x > 0$, it increases at a decreasing rate for $0 \le x < 2$ and increases at an increasing rate for $x > 2$ with an inflection point at $x = 2$.

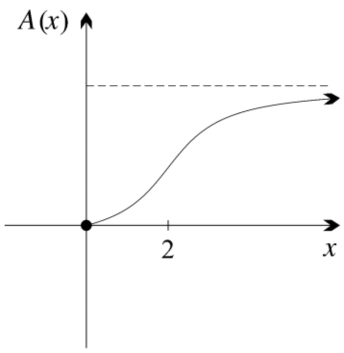


3b The signed area function $A(x) = \int_0^x f(t)\, dx$ represents the area under the curve.

As x increases from 0, the value of $A(x)$ will increase. However, due to the increasing nature of the curve $f(t)$ in the interval $[0, 2)$, the value of $A(x)$ will increase at an increasing rate.

In the interval $(2, \infty)$, the curve of $f(t)$ is decreasing, therefore the value of $A(x)$ will increase at an decreasing rate.

While the value of $A(x)$ increases for all values of $x > 0$, it increases at an increasing rate for $0 \le x < 2$ and increases at a decreasing rate for $x > 2$ with an inflection point at $x = 2$.



4a The differential form $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$, therefore

$$\frac{d}{dx}\int_1^x \frac{1}{t}\,dt = \frac{1}{x}$$

4b The differential form $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$, therefore

$$\frac{d}{dx}\int_0^x \frac{1}{1+t^3}\,dt = \frac{1}{1+x^3}$$

4c The differential form $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$, therefore

$$\frac{d}{dx}\int_0^x e^{-\frac{1}{2}t^2}\,dt = e^{-\frac{1}{2}x^2}$$

5a Based on the differential form of the fundamental theorem:

$$\frac{d}{dx}\int_1^x (3t^2 - 12)\,dt = 3x^2 - 12$$

By integrating first:

$$\int_1^x (3t^2 - 12)\,dt$$

$$= [t^3 - 12t]_1^x$$

$$= (x^3 - 12x) - (1^3 - 12 \times 1)$$

$$= x^3 - 12x + 11$$

Therefore, by differentiating:

$$\frac{d}{dx}\int_1^x (3t^2 - 12)\,dt$$

$$= \frac{d}{dx}(x^3 - 12x + 11)$$

$$= 3x^2 - 12$$

5b Based on the differential form of the fundamental theorem:

$$\frac{d}{dx}\int_2^x (t^3+4t)\,dt = x^3+4x$$

By integrating first:

$$\int_1^x (t^3+4t)\,dt$$

$$= \left[\frac{t^4}{4}+2t^2\right]_2^x$$

$$= \left(\frac{x^4}{4}+2x^2\right)-\left(\frac{2^4}{4}+2(2)^2\right)$$

$$= \frac{x^4}{4}+2x^2-12$$

Therefore, by differentiating:

$$\frac{d}{dx}\int_1^x (t^3+4t)\,dt$$

$$= \frac{d}{dx}\left(\frac{x^4}{4}+2x^2-12\right)$$

$$= x^3+4x$$

5c Based on the differential form of the fundamental theorem:

$$\frac{d}{dx}\int_2^x \left(\frac{1}{t^2}\right)dt = \frac{1}{x^2}$$

By integrating first:

$$\int_1^x \left(\frac{1}{t^2}\right)dt$$

$$= \int_1^x (t^{-2})\,dt$$

$$= \left[\frac{t^{-1}}{-1}\right]_2^x$$

$$= \left(-\frac{1}{x}\right)-\left(-\frac{1}{2}\right)$$

$$= \frac{1}{2} - \frac{1}{x}$$

Therefore, by differentiating:

$$\frac{d}{dx}\int_1^x \left(\frac{1}{t^2}\right) dt$$

$$= \frac{d}{dx}\left(\frac{1}{2} - \frac{1}{x}\right)$$

$$= \frac{1}{x^2}$$

6a i Using the fundamental theorem,

$$\frac{d}{dx}\int_1^x t^2\, dt = x^2$$

6a ii Using the fundamental theorem,

$$\frac{d}{dx}\int_2^x (t^3 + 3t)\, dt = x^3 + 3x$$

6a iii Using the fundamental theorem,

$$\frac{d}{dx}\int_a^x \frac{1}{t} dt = \frac{1}{x}$$

6a iv Using the fundamental theorem,

$$\frac{d}{dx}\int_a^x (t^3 - 3)^4\, dt = (x^3 - 3)^4$$

6b For 6a i, the solved part is as below:

$\frac{d}{dx}\int_1^x t^2\, dt$

Then, $\int_1^x t^2\, dt = \left[\frac{t^3}{3} - 1\right]_1^x$

$= \frac{x^3}{3} - 1$

Hence, $\frac{d}{dx}\left(\frac{x^3}{3}-1\right)=\frac{1}{3}\times 3x^2-0$

$=x^2$

For 6a ii, the solved part is as below

$\frac{d}{dx}\int_2^x (t^3+3t)\,dt$

Then, $\int_2^x (t^3+3t)\,dt=\left[\frac{t^4}{4}+\frac{3t^2}{2}\right]_2^x$

$=\left(\frac{x^4}{4}+\frac{3x^2}{2}\right)-\left(\frac{16}{4}+\frac{12}{2}\right)$

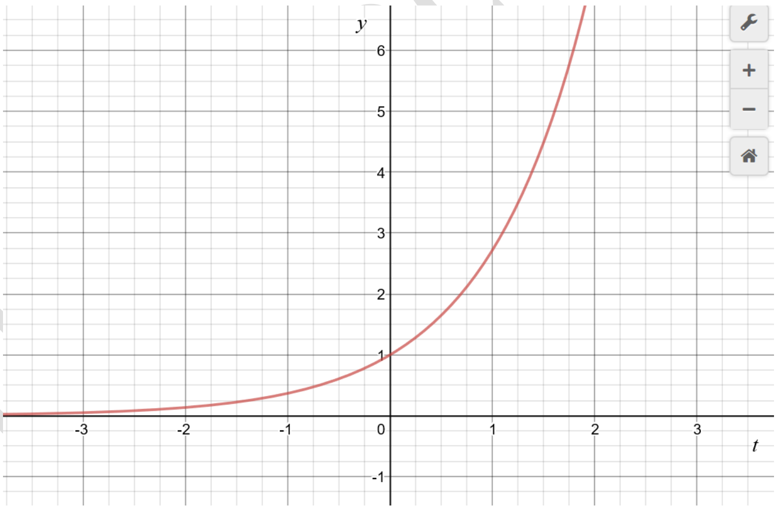
$=\frac{x^4}{4}+\frac{3x^2}{2}-10$

$=\frac{x^4+6x^2-40}{4}$

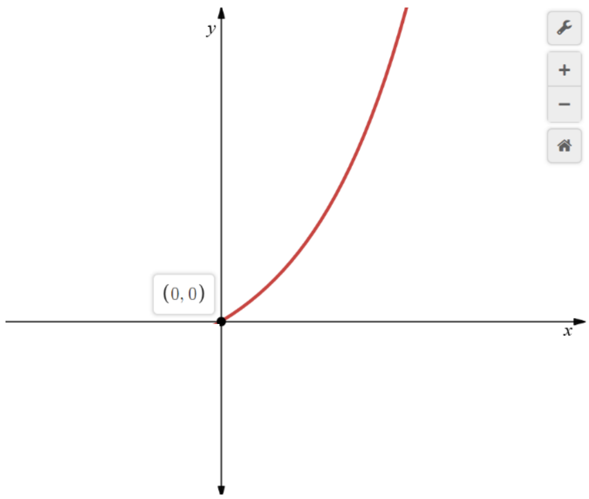
Hence, $\frac{d}{dx}\left(\frac{x^4+6x^2-40}{4}\right)=\frac{1}{4}\times(4x^3+12x)$

$=x^3+3x$

7a The graph of $y=e^t$ is shown below.

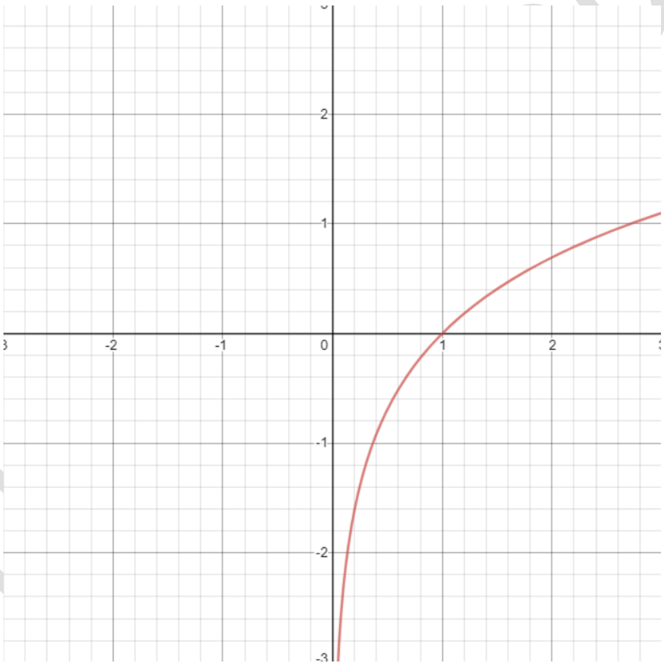


The sketch graph of $y = A(x)$ for $x \geq 0$ is shown below.

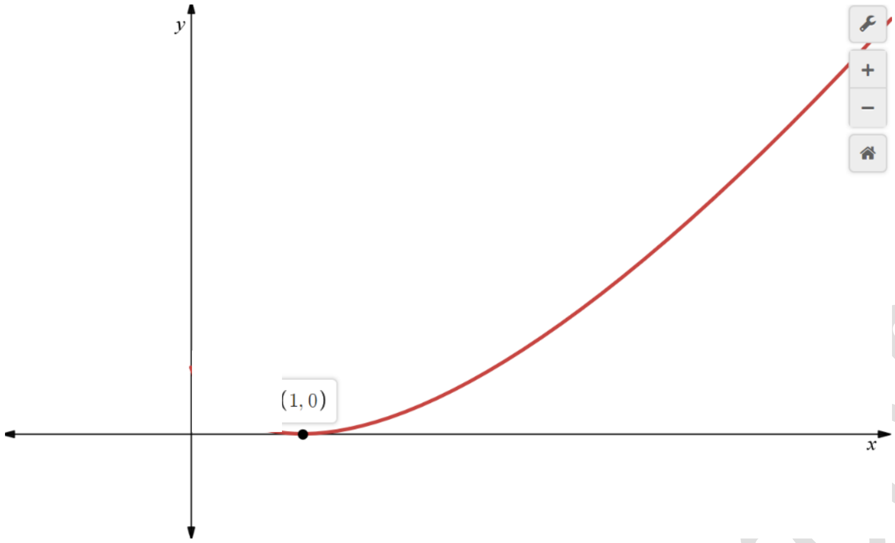


The function $y = A(x)$ is zero at $x = 0$ and is increasing at an increasing rate.

7b The graph of $y = \log_e t$ is shown below.

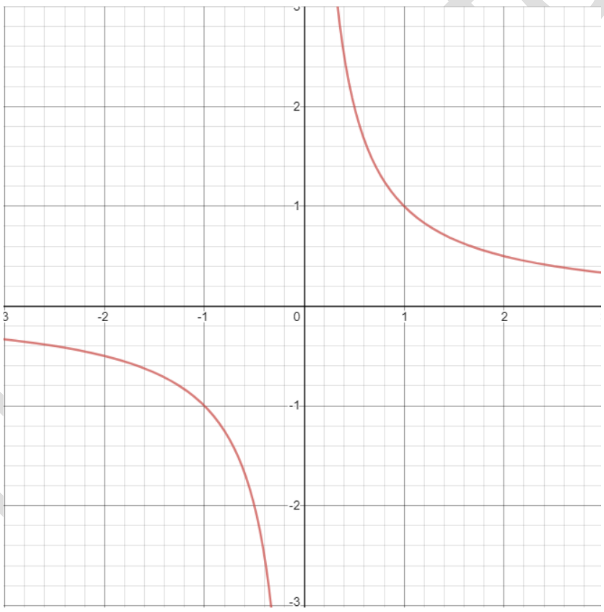


The sketch graph of $y = A(x)$ for $x \geq 1$ is shown below.

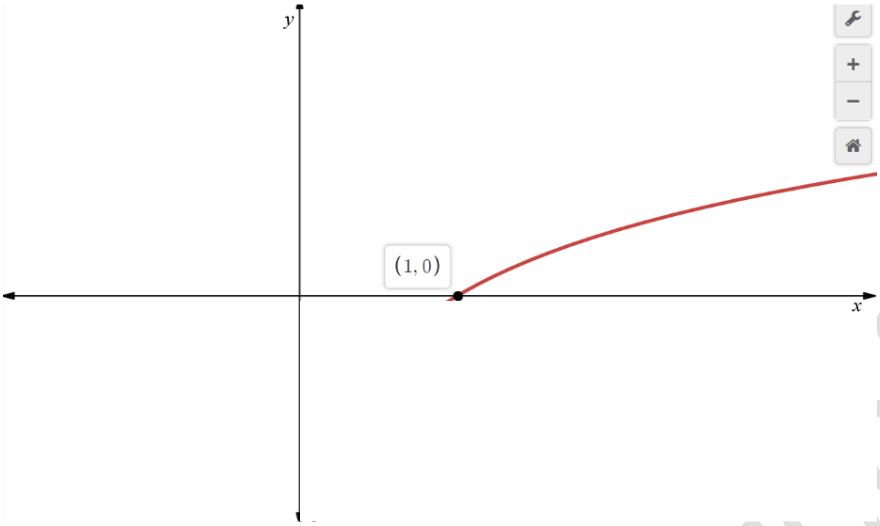


The function $y = A(x)$ is zero at $x = 1$ and is increasing at an increasing rate.

7c The graph of $y = \frac{1}{t}$ is shown below.



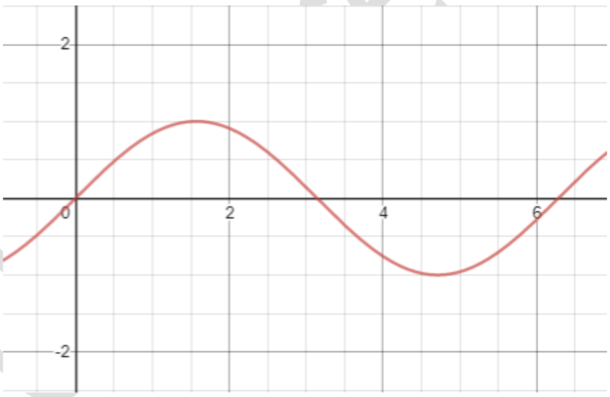
The graph of $y = A(x)$ for $x \geq 1$ is shown below.



The function $y = A(x)$ is zero at $x = 1$ and is increasing at a decreasing rate.

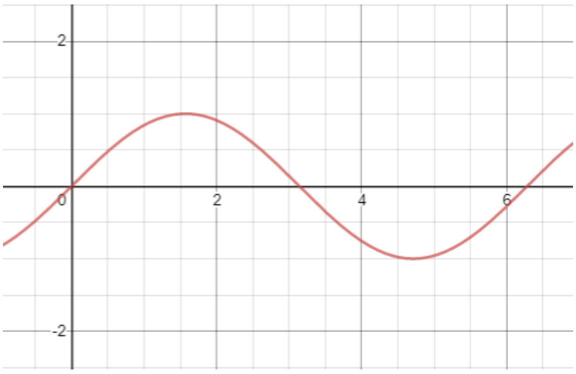
8a

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$A(x)$	0	1	0	-1	0



This looks like $y = \sin x$ which suggests that the derivative of $\sin x$ is $\cos x$.

8b The graph of $y = \sin t$ is shown below.



x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$A(x)$	0	1	2	1	0

$A(x)$

The graph looks like $1 - \cos x$ which suggests the derivative of $\cos x$ is $-\sin x$.

9a $A(x)$ is increasing when $f(t)$ is positive, that is, for $t < c$, and is decreasing for $t > c$.

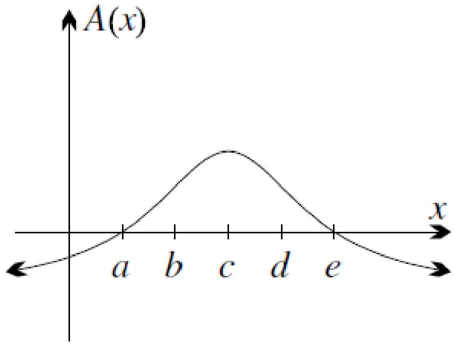
9b $A(x)$ has a maximum turning point when the total signed area is maximum, this is when $x = c$ as after that point the signed area starts decreasing. There are no minimum turning points which would be when the curve passes from below the x-axis (negative area) to above the x-axis (positive area).

9c $A(x)$ has inflections when $f'(t)$ changes in sign. That is at $x = b$ and $x = d$.

9d The zeroes of $A(x)$ occur when the total signed area is zero. This is at $x = a$ and $x = e$.

9e $A(x)$ is positive while the *total* signed area is positive. This is for $a < x < e$. We assume that the curve will follow a similar shape to what it currently presents and hence will be negative for $x < a$ and $x > e$.

9f



10a The function is continuous at every real number so it is a continuous function.

10b The domain is $x \neq 2$,and y is continuous at every value in its domain so it is a continuous function.

10c Zero lies in the domain, and y is not continuous at $x = 0$ so it is not a continuous function.

10d The domain is $x \geq 0$, and y is continuous at every value in its domain so it is a continuous function.

10e The domain is $x > 0$, and y is continuous at every value in its domain so it is a continuous function.

10f The domain is $x \geq 0$, and y is not continuous at $x = 0$ so it is not a continuous function.

Solutions to Exercise 5E

Let C be a constant.

1a $\int 4\,dx = 4x + C$

1b $\int 1\,dx = 1x + C = x + C$

1c $\int 0\,dx = 0x + C = C$

1d $\int (-2)\,dx = -2x + C$

1e $\int x\,dx = \frac{x^2}{2} + C$

1f $\int x^2\,dx = \frac{x^3}{3} + C$

1g $\int x^3\,dx = \frac{x^4}{4} + C$

1h $\int x^7\,dx = \frac{x^8}{8} + C$

2a $\int 2x\,dx$

$= \frac{2x^2}{2} + C$

$= x^2 + C$

2b $\int 4x\,dx$

$= \frac{4x^2}{2} + C$

$= 2x^2 + C$

2c $\int 3x^2\,dx$

$$=\frac{3x^3}{3}+C$$

$$=x^3+C$$

2d $\int 4x^3\,dx$

$$=\frac{4x^4}{4}+C$$

$$=x^4+C$$

2e $\int 10x^9\,dx$

$$=\frac{10x^{10}}{10}+C$$

$$=x^{10}+C$$

2f $\int 2x^3\,dx$

$$=\frac{2x^4}{4}+C$$

$$=\frac{x^4}{2}+C$$

2g $\int 4x^5\,dx$

$$=\frac{4x^6}{6}+C$$

$$=\frac{2x^6}{3}+C$$

2h $\int 3x^8\,dx$

$$=\frac{3x^9}{9}+C$$

$$=\frac{x^9}{3}+C$$

3a $\int (x + x^2)\, dx$

$= \frac{x^2}{2} + \frac{x^3}{3} + C$

3b $\int (x^4 - x^3)\, dx$

$= \frac{x^5}{5} - \frac{x^4}{4} + C$

3c $\int (x^7 + x^{10})\, dx$

$= \frac{x^8}{8} + \frac{x^{11}}{11} + C$

3d $\int (2x + 5x^4)\, dx$

$= \frac{2x^2}{2} + \frac{5x^5}{5} + C$

$= x^2 + x^5 + C$

3e $\int (9x^8 - 11)\, dx$

$= \frac{9x^9}{9} - 11x + C$

$= x^9 - 11x + C$

3f $\int (7x^{13} + 3x^8)\, dx$

$= \frac{7x^{14}}{14} + \frac{3x^9}{9} + C$

$= \frac{x^{14}}{2} + \frac{x^9}{3} + C$

3g $\int (4 - 3x)\, dx$

$= 4x - \frac{3x^2}{2} + C$

3h $\int (1 - x^2 + x^4)\, dx$

$= x - \frac{x^3}{3} + \frac{x^5}{5} + C$

3i $\int (3x^2 - 8x^3 + 7x^4)\, dx$

$= \frac{3x^3}{3} - \frac{8x^4}{4} + \frac{7x^5}{5} + C$

$= x^3 - 2x^4 + \frac{7x^5}{5} + C$

4a $\int x^{-2}\, dx$

$= \frac{x^{-1}}{-1} + C$

$= -x^{-1} + C$

4b $\int x^{-3}\, dx$

$= \frac{x^{-2}}{-2} + C$

$= -\frac{x^{-2}}{2} + C$

4c $\int x^{-8}\, dx$

$= \frac{x^{-7}}{-7} + C$

$= -\frac{x^{-7}}{7} + C$

4d $\int 3x^{-4}\, dx$

$= \frac{3x^{-3}}{-3} + C$

$= -x^{-3} + C$

4e $\int 9x^{-10}\,dx$

$= \dfrac{9x^{-9}}{-9} + C$

$= -x^{-9} + C$

4f $\int 10x^{-6}\,dx$

$= \dfrac{10x^{-5}}{-5} + C$

$= -2x^{-5} + C$

5a $\int x^{\frac{1}{2}}\,dx$

$= \dfrac{x^{\frac{3}{2}}}{\left(\frac{3}{2}\right)} + C$

$= \dfrac{2x^{\frac{3}{2}}}{3} + C$

5b $\int x^{\frac{1}{3}}\,dx$

$= \dfrac{x^{\frac{4}{3}}}{\left(\frac{4}{3}\right)} + C$

$= \dfrac{3x^{\frac{4}{3}}}{4} + C$

5c $\int x^{\frac{1}{4}}\,dx$

$= \dfrac{x^{\frac{5}{4}}}{\left(\frac{5}{4}\right)} + C$

$= \dfrac{4x^{\frac{5}{4}}}{5} + C$

5d $\int x^{\frac{2}{3}}\,dx$

$$= \frac{x^{\frac{5}{3}}}{\left(\frac{5}{3}\right)} + C$$

$$= \frac{3x^{\frac{5}{3}}}{5} + C$$

5e $\int x^{-\frac{1}{2}}\,dx$

$$= \frac{x^{\frac{1}{2}}}{\left(\frac{1}{2}\right)} + C$$

$$= 2x^{\frac{1}{2}} + C$$

5f $\int 4x^{\frac{1}{2}}\,dx$

$$= \frac{4x^{\frac{3}{2}}}{\left(\frac{3}{2}\right)} + C$$

$$= \frac{8x^{\frac{3}{2}}}{3} + C$$

6a $\int x(x+2)\,dx$

$$= \int (x^2 + 2x)\,dx$$

$$= \frac{x^3}{3} + x^2 + C$$

6b $\int x(4 - x^2)\,dx$

$$= \int (4x - x^3)\,dx$$

$$= 2x^2 - \frac{x^4}{4} + C$$

6c $\int x^2(5-3x)\,dx$

$$= \int (5x^2 - 3x^3)\,dx$$

$$= \frac{5x^3}{3} - \frac{3x^4}{4} + C$$

6d $\int x^3(x-5)\,dx$

$$= \int (x^4 - 5x^3)\,dx$$

$$= \frac{x^5}{5} - \frac{5x^4}{4} + C$$

6e $\int (x-3)^2\,dx$

$$= \int (x^2 - 6x + 9)\,dx$$

$$= \frac{x^3}{3} - 3x^2 + 9x + C$$

6f $\int (2x+1)^2\,dx$

$$= \int (4x^2 + 4x + 1)\,dx$$

$$= \frac{4x^3}{3} + 2x^2 + x + C$$

6g $\int (1-x^2)^2\,dx$

$$= \int (1 - 2x^2 + x^4)\,dx$$

$$= x - \frac{2x^3}{3} + \frac{x^5}{5} + C$$

6h $\int (2-3x)(2+3x)\,dx$

$$= \int (4-9x^2)\,dx$$

$$= 4x - 3x^3 + C$$

6i $\int (x^2-3)(1-2x)\,dx$

$$= \int (x^2 - 2x^3 - 3 + 6x)\,dx$$

$$= \frac{x^3}{3} - \frac{x^4}{2} - 3x + 3x^2 + C$$

7a

$$\int \frac{x^2+2x}{x}\,dx$$

$$= \int \left(\frac{x^2}{x} + \frac{2x}{x}\right) dx$$

$$= \int (x+2)\,dx$$

$$= \frac{x^2}{2} + 2x + C$$

7b

$$\int \frac{x^7+x^8}{x^6}\,dx$$

$$= \int \left(\frac{x^7}{x^6} + \frac{x^8}{x^6}\right) dx$$

$$= \int (x+x^2)\,dx$$

$$= \frac{x^2}{2} + \frac{x^3}{3} + C$$

7c

$$\int \frac{2x^3 - x^4}{4x}\,dx$$

$$= \int \left(\frac{2x^3}{4x} - \frac{x^4}{4x}\right) dx$$

$$= \int \left(\frac{x^2}{2} - \frac{x^3}{4}\right) dx$$

$$= \frac{x^3}{6} - \frac{x^4}{16} + C$$

8a

$$\int \frac{1}{x^2}\,dx$$

$$= \int x^{-2}\,dx$$

$$= \frac{x^{-1}}{-1} + C$$

$$= -\frac{1}{x} + C$$

8b

$$\int \frac{1}{x^3}\,dx$$

$$= \int x^{-3}\,dx$$

$$= \frac{x^{-2}}{-2} + C$$

$$= -\frac{1}{2x^2} + C$$

8c

$$\int \frac{1}{x^5}\,dx$$

$$= \int x^{-5}\,dx$$

$$= \frac{x^{-4}}{-4} + C$$

$$= -\frac{1}{4x^4} + C$$

8d

$$\int \frac{1}{x^{10}}\,dx$$

$$= \int x^{-10}\,dx$$

$$= \frac{x^{-9}}{-9} + C$$

$$= -\frac{1}{9x^9} + C$$

8e

$$\int \frac{3}{x^4}\,dx$$

$$= \int 3x^{-4}\,dx$$

$$= \frac{3x^{-3}}{-3} + C$$

$$= -x^{-3} + C$$

$$= -\frac{1}{x^3} + C$$

8f

$$\int \frac{5}{x^6}\,dx$$

$$= \int 5x^{-6}\,dx$$

$$= \frac{5x^{-5}}{-5} + C$$

$$= -\frac{1}{x^5} + C$$

8g

$$\int \frac{7}{x^8}\,dx$$

$$= \int 7x^{-8}\,dx$$

$$= \frac{7x^{-7}}{-7} + C$$

$$= -\frac{1}{x^7} + C$$

8h

$$\int \frac{1}{3x^2}\,dx$$

$$= \int \frac{x^{-2}}{3}\,dx$$

$$= \frac{x^{-1}}{3 \times -1} + C$$

$$= -\frac{1}{3x} + C$$

8i

$$\int \frac{1}{7x^5}\,dx$$

$$= \int \frac{x^{-5}}{7}\,dx$$

$$= \frac{x^{-4}}{7 \times -4} + C$$

$$= -\frac{1}{28x^4} + C$$

8j

$$\int -\frac{1}{5x^3}\,dx$$

$$= \int -\frac{x^{-3}}{5}\,dx$$

$$= -\frac{x^{-2}}{5 \times -2} + C$$

$$= \frac{1}{10x^2} + C$$

8k

$$\int \left(\frac{1}{x^2} - \frac{1}{x^5}\right) dx$$

$$= \int (x^{-2} - x^{-5})\,dx$$

$$= \frac{x^{-1}}{-1} - \frac{x^{-4}}{-4} + C$$

$$= -\frac{1}{x} + \frac{1}{4x^4} + C$$

8l

$$\int \left(\frac{1}{x^3}+\frac{1}{x^4}\right)dx$$

$$=\int (x^{-3}+x^{-4})\,dx$$

$$=\frac{x^{-2}}{-2}+\frac{x^{-3}}{-3}+C$$

$$=-\frac{1}{2x^2}-\frac{1}{3x^3}+C$$

9a

$$\int \sqrt{x}\,dx$$

$$=\int x^{\frac{1}{2}}\,dx$$

$$=\frac{x^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}+C$$

$$=\frac{2x^{\frac{3}{2}}}{3}+C$$

9b

$$\int \sqrt[3]{x}\,dx$$

$$=\int x^{\frac{1}{3}}\,dx$$

$$=\frac{x^{\frac{4}{3}}}{\left(\frac{4}{3}\right)}+C$$

$$=\frac{3x^{\frac{4}{3}}}{4}+C$$

9c

$$\int \frac{1}{\sqrt{x}}\,dx$$

$$= \int x^{-\frac{1}{2}}\,dx$$

$$= \frac{x^{\frac{1}{2}}}{\left(\frac{1}{2}\right)} + C$$

$$= 2x^{\frac{1}{2}} + C$$

$$= 2\sqrt{x} + C$$

9d

$$\int \sqrt[3]{x^2}\,dx$$

$$= \int x^{\frac{2}{3}}\,dx$$

$$= \frac{x^{\frac{5}{3}}}{\left(\frac{5}{3}\right)} + C$$

$$= \frac{3x^{\frac{5}{3}}}{5} + C$$

10a

$$\int_0^9 \sqrt{x}\,dx$$

$$= \left[\frac{2x^{\frac{3}{2}}}{3}\right]_0^9 \quad \text{(from question 9a)}$$

$$= \frac{2 \times 9^{\frac{3}{2}}}{3} - 0$$

$$= \frac{2 \times 27}{3}$$

$$= 18$$

10b

$$\int_0^8 \sqrt[3]{x}\,dx$$

$$= \left[\frac{3x^{\frac{4}{3}}}{4}\right]_0^8 \qquad \text{(from question 9b)}$$

$$= \frac{3 \times 8^{\frac{4}{3}}}{4} - 0$$

$$= \frac{3 \times 16}{4}$$

$$= 12$$

10c

$$\int_{25}^{49} \frac{1}{\sqrt{x}}\,dx$$

$$= \left[2\sqrt{x}\right]_{25}^{49} \qquad \text{(from question 9c)}$$

$$= 2\sqrt{49} - 2\sqrt{25}$$

$$= 14 - 10$$

$$= 4$$

10d

$$\int_0^1 \sqrt[3]{x^2}\,dx$$

$$= \left[\frac{3x^{\frac{5}{3}}}{5}\right]_0^1 \qquad \text{(from question 9d)}$$

$$= \frac{3}{5} - 0$$

$$= \frac{3}{5}$$

11a $\int (x+1)^5\,dx$

$$= \frac{(x+1)^6}{6} + C$$

$$= \frac{1}{6}(x+1)^6 + C$$

11b $\int (x+2)^3\,dx$

$$= \frac{(x+2)^4}{4} + C$$

$$= \frac{1}{4}(x+2)^4 + C$$

11c $\int (4-x)^4\,dx$

$$= \frac{(4-x)^5}{-1\times 5} + C$$

$$= -\frac{(4-x)^5}{5} + C$$

$$= -\frac{1}{5}(4-x)^5 + C$$

11d $\int (3-x)^2\,dx$

$$= \frac{(3-x)^3}{-1\times 3} + C$$

$$= -\frac{(3-x)^3}{3} + C$$

$$= -\frac{1}{3}(3-x)^3 + C$$

11e $\int (3x+1)^4\,dx$

$$= \frac{(3x+1)^5}{3\times 5} + C$$

$$= \frac{(3x+1)^5}{15} + C$$

$= \frac{1}{15}(3x+1)^5 + C$

11f $\int (4x-3)^7\,dx$

$= \frac{(4x-3)^8}{4\times 8} + C$

$= \frac{(4x-3)^8}{32} + C$

$= \frac{1}{32}(4x-3)^8 + C$

11g $\int (5-2x)^6\,dx$

$= \frac{(5-2x)^7}{-2\times 7} + C$

$= -\frac{(5-2x)^7}{14} + C$

$= -\frac{1}{14}(5-2x)^7 + C$

11h $\int (1-5x)^7\,dx$

$= \frac{(1-5x)^8}{-5\times 8} + C$

$= -\frac{(1-5x)^8}{40} + C$

$= -\frac{1}{40}(1-5x)^8 + C$

11i $\int (2x+9)^{11}\,dx$

$= \frac{(2x+9)^{12}}{2\times 12} + C$

$= \frac{(2x+9)^{12}}{24} + C$

$= \frac{1}{24}(2x+9)^{12} + C$

11j $\int 3(2x-1)^{10}\,dx$

$$=\frac{3(2x-1)^{11}}{2\times 11}+C$$

$$=\frac{3(2x-1)^{11}}{22}+C$$

$$=\frac{3}{22}(2x-1)^{11}+C$$

11k $\int 4(5x-4)^{6}\,dx$

$$=\frac{4(5x-4)^{7}}{5\times 7}+C$$

$$=\frac{4(5x-4)^{7}}{35}+C$$

$$=\frac{4}{35}(5x-4)^{7}+C$$

11l

$$\int 7(3-2x)^{3}\,dx$$

$$=\frac{7(3-2x)^{4}}{-2\times 4}+C$$

$$=-\frac{7(3-2x)^{4}}{8}+C$$

$$=-\frac{7}{8}(3-2x)^{4}+C$$

12a

$$\int\left(\frac{1}{3}x-7\right)^{4}dx$$

$$=\frac{\left(\frac{1}{3}x-7\right)^{5}}{\frac{1}{3}\times 5}+C$$

$$=\frac{3}{5}\left(\frac{1}{3}x-7\right)^{5}+C$$

12b

$$\int \left(\frac{1}{4}x - 7\right)^6 dx$$

$$= \frac{\left(\frac{1}{4}x - 7\right)^7}{\frac{1}{4} \times 7} + C$$

$$= \frac{4}{7}\left(\frac{1}{4}x - 7\right)^7 + C$$

12c

$$\int \left(1 - \frac{1}{5}x\right)^3 dx$$

$$= \frac{\left(1 - \frac{1}{5}x\right)^4}{-\frac{1}{5} \times 4} + C$$

$$= -\frac{5}{4}\left(1 - \frac{1}{5}x\right)^4 + C$$

13a

$$\int \frac{1}{(x+1)^3} dx$$

$$= \int (x+1)^{-3}\, dx$$

$$= \frac{(x+1)^{-2}}{1 \times -2} + C$$

$$= -\frac{1}{2(x+1)^2} + C$$

13b

$$\int \frac{1}{(x-5)^4} dx$$

$$= \int (x-5)^{-4}\, dx$$

$$= \frac{(x-5)^{-3}}{1 \times -3} + C$$

$$= -\frac{1}{3(x-5)^3} + C$$

13c

$$\int \frac{1}{(3x-4)^2}\,dx$$

$$= \int (3x-4)^{-2}\,dx$$

$$= \frac{(3x-4)^{-1}}{3 \times -1} + C$$

$$= -\frac{1}{3(3x-4)} + C$$

13d

$$\int \frac{1}{(2-x)^5}\,dx$$

$$= \int (2-x)^{-5}\,dx$$

$$= \frac{(2-x)^{-4}}{-1 \times -4} + C$$

$$= \frac{1}{4(2-x)^4} + C$$

13e

$$\int \frac{3}{(x-7)^6}\,dx$$

$$= \int 3(x-7)^{-6}\,dx$$

$$= \frac{3(x-7)^{-5}}{1 \times -5} + C$$

$$= -\frac{3}{5(x-7)^5} + C$$

13f

$$\int \frac{8}{(4x+1)^5}\,dx$$

$$= \int 8(4x+1)^{-5}\,dx$$

$$= \frac{8(4x+1)^{-4}}{4 \times -4} + C$$

$$= -\frac{1}{2(4x+1)^4} + C$$

13g

$$\int \frac{2}{(3-5x)^4}\,dx$$

$$= \int 2(3-5x)^{-4}\,dx$$

$$= \frac{2(3-5x)^{-3}}{-5 \times -3} + C$$

$$= \frac{2}{15(3-5x)^3} + C$$

13h

$$\int \frac{4}{5(1-4x)^2}\,dx$$

$$= \int \frac{4}{5}(1-4x)^{-2}\,dx$$

$$= \frac{\frac{4}{5}(1-4x)^{-1}}{-4 \times -1} + C$$

$$= \frac{1}{5(1-4x)} + C$$

$$= \frac{1}{5-20x} + C$$

13i

$$\int \frac{7}{8(3x+2)^5}\,dx$$

$$= \int \frac{7}{8}(3x+2)^{-5}\,dx$$

$$= \frac{\frac{7}{8}(3x+2)^{-4}}{3\times -4} + C$$

$$= -\frac{7}{96(3x+2)^4} + C$$

14a

$$\int \sqrt{x}\left(3\sqrt{x} - x\right)dx$$

$$= \int \left(3x - x\sqrt{x}\right)dx$$

$$= \int \left(3x - x^{\frac{3}{2}}\right)dx$$

$$= \frac{3}{2}x^2 - \frac{2}{5}x^{\frac{5}{2}} + C$$

14b

$$\int \left(\sqrt{x}-2\right)\left(\sqrt{x}+2\right)dx$$

$$= \int (x-4)\,dx$$

$$= \frac{x^2}{2} - 4x + C$$

$$= \frac{1}{2}x^2 - 4x + C$$

14c

$$\int \left(2\sqrt{x}-1\right)^2 dx$$

$$= \int \left(4x - 4\sqrt{x} + 1\right) dx$$

$$= \int \left(4x - 4x^{\frac{1}{2}} + 1\right) dx$$

$$= 2x^2 - \frac{8}{3}x^{\frac{3}{2}} + x + C$$

15a i

$$\int_0^1 x^{\frac{1}{2}}\, dx$$

$$= \left[\frac{x^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}\right]_0^1$$

$$= \left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^1$$

$$= \frac{2}{3}\times 1 - \frac{2}{3}\times 0$$

$$= \frac{2}{3}$$

15a ii

$$\int_1^4 x^{-\frac{1}{2}}\, dx$$

$$= \left[\frac{x^{\frac{1}{2}}}{\left(\frac{1}{2}\right)}\right]_1^4$$

$$= \left[2x^{\frac{1}{2}}\right]_1^4$$

$$= 2\left[x^{\frac{1}{2}}\right]_1^4$$

$$= 2\left(4^{\frac{1}{2}} - 1\right)$$

$$= 2(2 - 1)$$

$$= 2$$

15a iii

$$\int_0^8 x^{\frac{1}{3}}\, dx$$

$$= \frac{3}{4}\left[x^{\frac{4}{3}}\right]_0^8$$

$$= \frac{3}{4}\left(8^{\frac{4}{3}} - 0\right)$$

$$= \frac{3}{4}(2^4)$$

$$= 12$$

15b i

$$\int_0^4 \sqrt{x}\, dx$$

$$= \int_0^4 x^{\frac{1}{2}}\, dx$$

$$= \left[\frac{x^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}\right]_0^4$$

$$= \frac{2}{3}\left[x^{\frac{3}{2}}\right]_0^4$$

$$= \frac{2}{3}\left(4^{\frac{3}{2}} - 0\right)$$

$$= \frac{2}{3}(8 - 0)$$

$$= 5\frac{1}{3}$$

15b ii

$$\int_1^9 x\sqrt{x}\,dx$$

$$= \int_1^9 x^{\frac{3}{2}}\,dx$$

$$= \frac{2}{5}\left[x^{\frac{5}{2}}\right]_1^9$$

$$= \frac{2}{5}\left(9^{\frac{5}{2}} - 1\right)$$

$$= \frac{2}{5} \times 242$$

$$= 96\frac{4}{5}$$

15b iii

$$\int_1^9 \frac{1}{\sqrt{x}}\,dx$$

$$= \int_1^9 x^{-\frac{1}{2}}\,dx$$

$$= 2\left[x^{\frac{1}{2}}\right]_1^9$$

$$= 2\left[\sqrt{x}\right]_1^9$$

$$= 2\left(\sqrt{9} - \sqrt{1}\right)$$

$$= 2(3 - 1)$$

$$= 4$$

16a

$$\int_2^4 \left(2 - \sqrt{x}\right)\left(2 + \sqrt{x}\right)\,dx$$

$$= \int_2^4 (4 - x)\,dx$$

$$= \left[4x - \frac{x^2}{2}\right]_2^4$$

$$= (16 - 8) - (8 - 2)$$

$$= 2$$

16b

$$\int_0^1 \sqrt{x}\left(\sqrt{x} - 4\right) dx$$

$$= \int_0^1 \left(x - 4\sqrt{x}\right) dx$$

$$= \int_0^1 \left(x - 4x^{\frac{1}{2}}\right) dx$$

$$= \left[\frac{x^2}{2} - \frac{8}{3}x^{\frac{3}{2}}\right]_0^1$$

$$= \left(\frac{1}{2} - \frac{8}{3}\right) - (0 - 0)$$

$$= -\frac{13}{6}$$

16c

$$\int_4^9 \left(\sqrt{x} - 1\right)^2 dx$$

$$= \int_4^9 \left(x - 2\sqrt{x} + 1\right) dx$$

$$= \int_4^9 \left(x - 2x^{\frac{1}{2}} + 1\right) dx$$

$$= \left[\frac{x^2}{2} - \frac{4}{3}x^{\frac{3}{2}} + x\right]_4^9$$

$$= \left(\frac{9^2}{2} - \frac{4}{3} \times 9^{\frac{3}{2}} + 9\right) - \left(\frac{4^2}{2} - \frac{4}{3} \times 4^{\frac{3}{2}} + 4\right)$$

$$= 13\frac{1}{12} - 1\frac{1}{3}$$

$$= 12\frac{1}{6}$$

17 $\int x^{-1}dx = \frac{x^0}{0} + C$ is meaningless as dividing by zero is an invalid operation.

18a

$$\int \sqrt{2x-1}\,dx$$

$$= \int (2x-1)^{\frac{1}{2}}\,dx$$

$$= \frac{(2x-1)^{\frac{3}{2}}}{2 \times \frac{3}{2}} + C$$

$$= \frac{1}{3}(2x-1)^{\frac{3}{2}} + C$$

18b

$$\int \sqrt{7-4x}\,dx$$

$$= \int (7-4x)^{\frac{1}{2}}\,dx$$

$$= \frac{(7-4x)^{\frac{3}{2}}}{-4 \times \frac{3}{2}} + C$$

$$= -\frac{1}{6}(7-4x)^{\frac{3}{2}} + C$$

18c

$$\int \sqrt[3]{4x-1}\,dx$$

$$= \int (4x-1)^{\frac{1}{3}}\,dx$$

$$= \frac{(4x-1)^{\frac{4}{3}}}{4 \times \frac{4}{3}} + C$$

$$= \frac{3}{16}(4x-1)^{\frac{4}{3}} + C$$

18d

$$\int \frac{1}{\sqrt{3x+5}}\,dx$$

$$= \int (3x+5)^{-\frac{1}{2}}\,dx$$

$$= \frac{(3x+5)^{\frac{1}{2}}}{3 \times \frac{1}{2}} + C$$

$$= \frac{2}{3}\sqrt{3x+5} + C$$

19a

$$\int_0^2 (x+1)^4\,dx$$

$$= \left[\frac{(x+1)^5}{1 \times 5}\right]_0^2$$

$$= \frac{3^5}{5} - \frac{1^5}{5}$$

$$= \frac{242}{5}$$

19b

$$\int_2^3 (2x-5)^3\,dx$$

$$= \left[\frac{(2x-5)^4}{2 \times 4}\right]_2^3$$

$$= \frac{1^4}{8} - \frac{(-1)^4}{8}$$

$$= 0$$

19c

$$\int_{-2}^{2} (1-x)^5 \, dx$$

$$= \left[\frac{(1-x)^6}{-1 \times 6}\right]_{-2}^{2}$$

$$= \frac{(-1)^6}{-6} - \frac{3^6}{-6}$$

$$= -\frac{1}{6} + \frac{729}{6}$$

$$= 121\frac{1}{3}$$

19d

$$\int_{0}^{5} \left(1 - \frac{x}{5}\right)^4 dx$$

$$= \left[\frac{\left(1-\frac{x}{5}\right)^5}{-\frac{1}{5} \times 5}\right]_0^5$$

$$= -(1-1)^5 - (-(1-0)^5)$$

$$= 0 + 1$$

$$= 1$$

19e

$$\int_{0}^{1} \sqrt{9 - 8x} \, dx$$

$$= \int_{0}^{1} (9 - 8x)^{\frac{1}{2}} \, dx$$

$$= \left[\frac{(9-8x)^{\frac{3}{2}}}{-8\times\frac{3}{2}}\right]_0^1$$

$$= -\frac{1^{\frac{3}{2}}}{12} - \left(-\frac{9^{\frac{3}{2}}}{12}\right)$$

$$= -\frac{1}{12} + \frac{27}{12}$$

$$= \frac{13}{6}$$

19f

$$\int_2^7 \frac{1}{\sqrt{x+2}}\,dx$$

$$= \int_2^7 (x+2)^{-\frac{1}{2}}\,dx$$

$$= \left[\frac{(x+2)^{\frac{1}{2}}}{1\times\frac{1}{2}}\right]_2^7$$

$$= \left[2\sqrt{x+2}\right]_2^7$$

$$= 2\sqrt{9} - 2\sqrt{4}$$

$$= 6 - 4$$

$$= 2$$

19g

$$\int_{-2}^0 \sqrt[3]{x+1}\,dx$$

$$= \int_{-2}^0 (x+1)^{\frac{1}{3}}\,dx$$

$$= \left[\frac{(x+1)^{\frac{4}{3}}}{1\times\frac{4}{3}}\right]_{-2}^0$$

$$= \left[\frac{3}{4}(x+1)^{\frac{4}{3}}\right]_{-2}^{0}$$

$$= \frac{3}{4} \times 1^{\frac{4}{3}} - \frac{3}{4} \times (-1)^{\frac{4}{3}}$$

$$= \frac{3}{4} - \frac{3}{4}$$

$$= 0$$

19h

$$\int_{1}^{5} \sqrt{3x+1}\, dx$$

$$= \int_{1}^{5} (3x+1)^{\frac{1}{2}}\, dx$$

$$= \left[\frac{(3x+1)^{\frac{3}{2}}}{3 \times \frac{3}{2}}\right]_{1}^{5}$$

$$= \left[\frac{2}{9}(3x+1)^{\frac{3}{2}}\right]_{1}^{5}$$

$$= \frac{2}{9} \times 16^{\frac{3}{2}} - \frac{2}{9} \times 4^{\frac{3}{2}}$$

$$= \frac{2}{9} \times 64 - \frac{2}{9} \times 8$$

$$= \frac{112}{9}$$

$$= 12\frac{4}{9}$$

19i

$$\int_{-3}^{0} \sqrt{1-5x}\, dx$$

$$= \int_{-3}^{0} (1-5x)^{\frac{1}{2}}\, dx$$

$$= \left[\frac{(1-5x)^{\frac{3}{2}}}{-5\times\frac{3}{2}}\right]_{-3}^{0}$$

$$= \left[-\frac{2}{15}(1-5x)^{\frac{3}{2}}\right]_{-3}^{0}$$

$$= -\frac{2}{15}\times 1^{\frac{3}{2}} - \left(-\frac{2}{15}\times 16^{\frac{3}{2}}\right)$$

$$= -\frac{2}{15}+\frac{128}{15}$$

$$= 8\frac{2}{5}$$

20a if $y = uv$

Then, derivative of each side will be

$$\frac{dy}{dx} = \frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$u\frac{dv}{dx} = \frac{d(uv)}{dx} - v\frac{du}{dx}$$

Now integrating both the sides, we get

$$\int u\frac{dv}{dx}dx = \int \frac{d(uv)}{dx}dx - \int v\frac{du}{dx}dx$$

The first term on the right side is simply integrating the first derivative that we did hence,

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

Hence, proved.

20b i

Let $u = x$ and $\frac{dv}{dx} = (x-1)^4$. It follows that $\frac{du}{dx} = 1$ and that $v = \frac{1}{5}(x-1)^5$. Substituting this into the equation from 20a gives

$$\int x(x-1)^4\,dx = x\times\frac{1}{5}(x-1)^5 - \int\frac{1}{5}(x-1)^5\times 1\,dx$$
$$= \frac{x}{5}(x-1)^5 - \frac{1}{5}\int(x-1)^5\,dx$$

If we let $u = x - 1$, then $du = dx$ and

$$\int (x-1)^5\,dx = \int u^5\,du = \frac{u^6}{6} + C_1 = \frac{(x-1)^6}{6} + C_1$$

Hence

$$\int x(x-1)^4\,dx = \frac{x}{5}(x-1)^5 - \frac{1}{5}\left(\frac{(x-1)^6}{6} + C_1\right)$$
$$= \frac{x}{5}(x-1)^5 - \frac{1}{30}(x-1)^6 + C$$

20b ii

Let $u = x$ and $\frac{dv}{dx} = \sqrt{1+x}$. It follows that $\frac{du}{dx} = 1$ and that $v = \frac{(1+x)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)} = \frac{2}{3}(1+x)^{\frac{3}{2}}$.

Substituting this into the equation from 20a gives

$$\int x\sqrt{1+x}\,dx = x \times \frac{2}{3}(1+x)^{\frac{3}{2}} - \int \frac{2}{3}(1+x)^{\frac{3}{2}} \times 1\,dx$$
$$= \frac{2x}{3}(1+x)^{\frac{3}{2}} - \frac{2}{3}\int (1+x)^{\frac{3}{2}}\,dx$$

If we let $u = 1 + x$, then $du = dx$ and

$$\int (1+x)^{\frac{3}{2}}\,dx = \int u^{\frac{3}{2}}\,du = \frac{u^{\frac{5}{2}}}{\frac{5}{2}} + C_1 = \frac{2}{5}u^{\frac{5}{2}} + C_1 = \frac{2}{5}(1+x)^{\frac{5}{2}} + C_1$$

Hence

$$\int x\sqrt{1+x}\,dx = \frac{2x}{3}(1+x)^{\frac{3}{2}} - \frac{2}{3}\left(\frac{2}{5}(1+x)^{\frac{5}{2}} + C_1\right)$$
$$= \frac{2x}{3}(1+x)^{\frac{3}{2}} - \frac{4}{15}(1+x)^{\frac{5}{2}} + C$$

Solutions to Exercise 5F

1a Area of the shaded region

$= \int_0^2 2x\,dx$

$= [x^2]_0^2$

$= 2^2 - 0$

$= 4$ square units

1b Area of the shaded region

$= \int_1^3 3x^2\,dx$

$= [x^3]_1^3$

$= 3^3 - 1^3$

$= 27 - 1$

$= 26$ square units

1c Area of the shaded region

$= \int_0^3 4x^3\,dx$

$= [x^4]_0^3$

$= 3^4 - 0$

$= 81$ square units

1d Area of the shaded region

$= \int_{-1}^2 (3x^2 + 1)\,dx$

$= [x^3 + x]_{-1}^2$

$= (2^3 + 2) - ((-1)^3 - 1))$

$= 10 - (-2)$

$= 12$ square units

1e Area of the shaded region

$$= \int_0^3 x^2\,dx$$

$$= \left[\frac{x^3}{3}\right]_0^3$$

$$= \frac{3^3}{3} - 0$$

$= 9$ square units

1f Area of the shaded region

$$= \int_2^4 (x^2 - 2x)\,dx$$

$$= \left[\frac{x^3}{3} - x^2\right]_2^4$$

$$= \frac{4^3}{3} - 4^2 - \left(\frac{2^3}{3} - 2^2\right)$$

$$= \frac{64}{3} - 16 - \frac{8}{3} + 4$$

$= 6\frac{2}{3}$ square units

1g Area of the shaded region

$$= \int_0^{16} \sqrt{x}\,dx$$

$$= \int_0^{16} x^{\frac{1}{2}}\,dx$$

$$= \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^{16}$$

$= \left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^{16}$

$= \frac{2}{3} \times 16^{\frac{3}{2}} - \frac{2}{3} \times 0^{\frac{3}{2}}$

$= \frac{128}{3}$ square units

1h Area of the shaded region

$= \int_1^3 (5 - x)\, dx$

$= \left[5x - \frac{x^2}{2}\right]_1^3$

$= \left(15 - \frac{3^2}{2}\right) - \left(5 - \frac{1^2}{2}\right)$

$= 15 - \frac{9}{2} - 5 + \frac{1}{2}$

$= 6$ square units

1i Area of the shaded region

$= \int_{-1}^0 (x^3 - x)\, dx$

$= \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{-1}^0$

$= \left(\frac{0^4}{4} - \frac{0^2}{2}\right) - \left(\frac{(-1)^4}{4} - \frac{(-1)^2}{2}\right)$

$= 0 - 0 - \frac{1}{4} + \frac{1}{2}$

$= \frac{1}{4}$ square units

1j Area of the shaded region

$$= \int_{-4}^{3} (12 - x - x^2)\, dx$$

$$= \left[12x - \frac{x^2}{2} - \frac{x^3}{3}\right]_{-4}^{3}$$

$$= \left(12 \times 3 - \frac{3^2}{2} - \frac{3^3}{3}\right) - \left(12 \times (-4) - \frac{(-4)^2}{2} - \frac{(-4)^3}{3}\right)$$

$$= 36 - \frac{9}{2} - 9 + 48 + 8 - \frac{64}{3}$$

$$= 57\tfrac{1}{6} \text{ square units}$$

1k Area of the shaded region

$$= \int_{-1}^{2} 5x^4 + 1\, dx$$

$$= [x^5 + x]_{-1}^{2}$$

$$= (2^5 + 2) - \left((-1)^5 + (-1)\right)$$

$$= 32 + 2 + 1 + 1$$

$$= 36 \text{ square units}$$

1l Area of the shaded region

$$= \int_{1}^{27} \sqrt[3]{x}\, dx$$

$$= \int_{1}^{27} x^{\frac{1}{3}}\, dx$$

$$= \left[\frac{x^{\frac{4}{3}}}{\frac{4}{3}}\right]_{1}^{27}$$

$$= \frac{3}{4}\left[x^{\frac{4}{3}}\right]_{1}^{27}$$

$$= \frac{3}{4}\left[27^{\frac{4}{3}} - 1^{\frac{4}{3}}\right]$$

$= \frac{3}{4}(81 - 1)$

$= 60$ square units

2a The shaded area gives the area bounded by the curve and the y-axis between $x = 0$ and $x = 5$, hence the area will be given by:

$$A = \int_0^5 2y\, dy$$

$= [y^2]_0^5$

$= 5^2 - 0^2$

$= 25$ square units

Note that one could alternatively use the expression $5 \times \frac{5}{2} - \int_0^{\frac{5}{2}} \frac{x}{2}\, dx$ to obtain the area.

2b The shaded area gives the area bounded by the curve and the y-axis between $x = -2$ and $x = 0$, hence the area will be given by:

$$A = \int_{-2}^0 3y^2\, dy$$

$= [y^3]_{-2}^0$

$= 0^3 - (-2)^3$

$= 8$ square units

Note that one could alternatively use the expression $2 \times 3(2)^2 - |\int_0^{12} \sqrt{\frac{x}{3}}\, dx|$ to obtain the area.

2c

$$A = \int_2^4 (2y - 4)\, dy$$

$= [y^2 - 4y]_2^4$

$= (4^2 - 4 \times 4) - (2^2 - 4 \times 2)$

$= 4$ square units

2d

$$A = \int_{-3}^{3} (27 - 3y^2)\, dy$$

$$= [27y - y^3]_{-3}^{3}$$

$$= (27 \times 3 - 3^3) - (27 \times (-3) - (-3)^3)$$

$$= 81 - 27 + 81 - 27$$

$$= 108 \text{ square units}$$

2e

$$A = \int_{0}^{3} y\, dy$$

$$= \left[\frac{y^2}{2}\right]_0^3$$

$$= \frac{3^2}{2} - 0$$

$$= \frac{9}{2} \text{ square units}$$

2f

$$A = \int_{3}^{5} (y^2 + 1)\, dy$$

$$= \left[\frac{y^3}{3} + y\right]_3^5$$

$$= \left(\frac{5^3}{3} + 5\right) - \left(\frac{3^3}{3} + 3\right)$$

$$= \frac{125}{3} + 5 - \frac{27}{3} - 3$$

$$= 34\frac{2}{3} \text{ square units}$$

2g

$$A = \int_0^9 \sqrt{y}\, dy$$

$$= \int_0^9 y^{\frac{1}{2}}\, dy$$

$$= \left[\frac{y^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^9$$

$$= \frac{2}{3}\left[y^{\frac{3}{2}}\right]_0^9$$

$$= \frac{2}{3}\left(9^{\frac{3}{2}} - 0\right)$$

$$= \frac{2}{3} \times 27$$

$$= 18 \text{ square units}$$

2h

$$A = \int_1^4 \frac{1}{\sqrt{y}}\, dy$$

$$= \int_1^4 y^{-\frac{1}{2}}\, dy$$

$$= \left[\frac{y^{\frac{1}{2}}}{\frac{1}{2}}\right]_1^4$$

$$= 2\left[y^{\frac{1}{2}}\right]_1^4$$

$$= 2\left[\sqrt{y}\right]_1^4$$

$$= 2\left(\sqrt{4} - 1\right)$$

$$= 2(2 - 1)$$

$$= 2 \text{ square units}$$

3a Shaded region is below the x-axis so signed area will be negative.

$$\int_1^3 (x^2 - 4x + 3)\,dx$$

$$= \left[\frac{x^3}{3} - 2x^2 + 3x\right]_1^3$$

$$= \left(\frac{3^3}{3} - 2 \times 3^2 + 3 \times 3\right) - \left(\frac{1^3}{3} - 2 \times 1^2 + 3 \times 1\right)$$

$$= 9 - 18 + 9 - \frac{1}{3} + 2 - 3$$

$$= -\frac{4}{3}$$

Hence, the required area is $\frac{4}{3}$ square units.

3b Shaded region is below the x-axis so signed area will be negative.

$$\int_{-3}^0 3x\,dx$$

$$= \left[\frac{3x^2}{2}\right]_{-3}^0$$

$$= 0 - \frac{3(-3)^2}{2}$$

$$= -\frac{27}{2}$$

Hence, the required area is $\frac{27}{2}$ square units.

3c Shaded region is below the x-axis so signed area will be negative.

$$\int_{-3}^0 x^3\,dx$$

$$= \left[\frac{x^4}{4}\right]_{-3}^0$$

$$= 0 - \frac{(-3)^4}{4}$$

$$= -\frac{81}{4}$$

Hence, the required area is $\frac{81}{4}$ square units.

3d Shaded region is below the x-axis so signed area will be negative.

$$\int_1^3 (1 - x^4)\,dx$$

$$= \left[x - \frac{x^5}{5}\right]_1^3$$

$$= \left(3 - \frac{3^5}{5}\right) - \left(1 - \frac{1^5}{5}\right)$$

$$= 3 - \frac{243}{5} - 1 + \frac{1}{5}$$

$$= -46\frac{2}{5}$$

Hence, the required area is $46\frac{2}{5}$ square units.

4a Shaded region is to the left of the y-axis so signed area will be negative.

$$\int_1^4 (1 - y)\,dy$$

$$= \left[y - \frac{y^2}{2}\right]_1^4$$

$$= \left(4 - \frac{4^2}{2}\right) - \left(1 - \frac{1^2}{2}\right)$$

$$= 4 - 8 - 1 + \frac{1}{2}$$

$$= -\frac{9}{2}$$

Hence, the required area is $\frac{9}{2}$ square units.

4b Shaded region is to the left of the y-axis so signed area will be negative.

$$\int_2^4 (y^2 - 6y + 8)\,dy$$

$$= \left[\frac{y^3}{3} - 3y^2 + 8y\right]_2^4$$

$$= \left(\frac{4^3}{3} - 3 \times 4^2 + 8 \times 4\right) - \left(\frac{2^3}{3} - 3 \times 2^2 + 8 \times 2\right)$$

$$= \frac{64}{3} - 48 + 32 - \frac{8}{3} + 12 - 16$$

$$= -\frac{4}{3}$$

Hence, the required area is $\frac{4}{3}$ square units.

4c Shaded region is to the left of the y-axis so signed area will be negative.

$$\int_{-8}^{-1} \sqrt[3]{y}\,dy$$

$$= \int_{-8}^{-1} y^{\frac{1}{3}}\,dy$$

$$= \left[\frac{y^{\frac{4}{3}}}{\frac{4}{3}}\right]_{-8}^{-1}$$

$$= \frac{3}{4}\left[y^{\frac{4}{3}}\right]_{-8}^{-1}$$

$$= \frac{3}{4}\left((-1)^{\frac{4}{3}} - (-8)^{\frac{4}{3}}\right)$$

$$= \frac{3}{4}(1 - 16)$$

$$= -\frac{45}{4}$$

Hence, the required area is $\frac{45}{4}$ square units.

4d Shaded region is to the left of the y-axis so signed area will be negative.

$$\int_0^3 (-y^2)\,dy$$

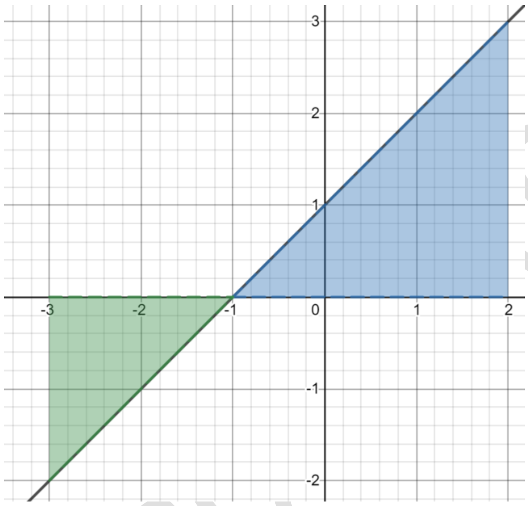
$$= \left[-\frac{y^3}{3}\right]_0^3$$

$$= -\frac{3^3}{3} - 0$$

$$= -9$$

Hence, the required area is 9 square units.

5a The required shading between the line $y = x + 1$ and the x-axis from $x = -3$ to $x = 2$ is shown below.



5b Shaded region is above the x-axis so signed area will be positive.

$$\int_{-1}^2 (x+1)\,dx$$

$$= \left[\frac{x^2}{2} + x\right]_{-1}^2$$

$$= (2+2) - \left(\frac{1}{2} - 1\right)$$

$$= 4\frac{1}{2}$$

Hence, the required area is $4\frac{1}{2}$ square units.

5c Shaded region is below the x-axis so signed area will be negative.

$$\int_{-3}^{-1} (x+1)\,dx$$

$$= \left[\frac{x^2}{2} + x\right]_{-3}^{-1}$$

$$= \left(\tfrac{1}{2} - 1\right) - \left(\tfrac{9}{2} - 3\right)$$

$$= -2$$

Hence, the required area is 2 square units.

5d Area of the entire shaded region $= 4\frac{1}{2} + 2 = 6\frac{1}{2}$ square units.

5e

$$\int_{-3}^{2} (x+1)\,dx$$

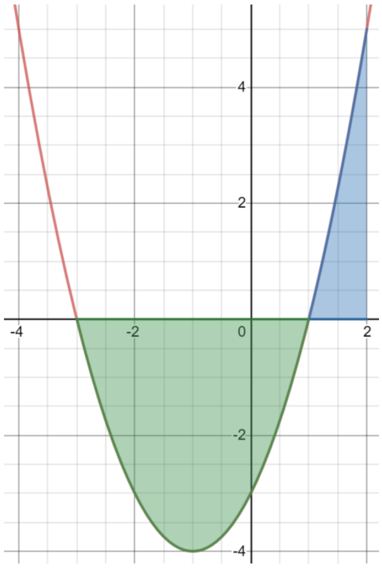
$$= \left[\frac{x^2}{2} + x\right]_{-3}^{2}$$

$$= (2+2) - \left(\tfrac{9}{2} - 3\right)$$

$$= 2\tfrac{1}{2}$$

This is the area above the x-axis minus the area below it. This is because when the integral is taken, areas below the x-axis are considered negative.

6a The required shading between the curve $y = x^2 + 2x - 3$ and the x-axis from $x = -3$ to $x = 2$ is shown below.



6b Shaded region is below the x-axis so signed area will be negative.

$$\int_{-3}^{1} (x^2 + 2x - 3)\,dx$$

$$= \left[\frac{x^3}{3} + x^2 - 3x\right]_{-3}^{1}$$

$$= \left(\frac{1^3}{3} + 1^2 - 3 \times 1\right) - \left(\frac{(-3)^3}{3} + (-3)^2 - 3 \times (-3)\right)$$

$$= \frac{1}{3} + 1 - 3 + 9 - 9 - 9$$

$$= -10\frac{2}{3}$$

Hence, the required area is $10\frac{2}{3}$ square units.

6c Shaded region is above the x-axis so signed area will be positive.

$$\int_{1}^{2} (x^2 + 2x - 3)\,dx$$

$$= \left[\frac{x^3}{3} + x^2 - 3x\right]_{1}^{2}$$

$$= \left(\frac{2^3}{3} + 2^2 - 3 \times 2\right) - \left(\frac{1^3}{3} + 1^2 - 3 \times 1\right)$$

$= \frac{8}{3} + 4 - 6 - \frac{1}{3} - 1 + 3$

$= 2\frac{1}{3}$

Hence, the required area is $2\frac{1}{3}$ square units.

6d Area of the entire shaded region $= 10\frac{2}{3} + 2\frac{1}{3} = 13$ square units.

6e

$$\int_{-3}^{2} (x^2 + 2x - 3)\, dx$$

$$= \left[\frac{x^3}{3} + x^2 - 3x\right]_{-3}^{2}$$

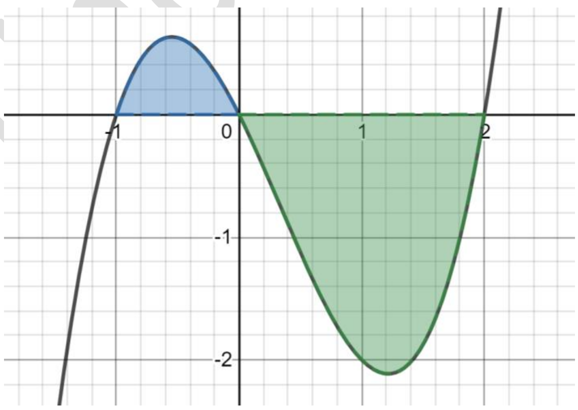
$$= \left(\frac{2^3}{3} + 2^2 - 3 \times 2\right) - \left(\frac{(-3)^3}{3} + (-3)^2 - 3 \times (-3)\right)$$

$$= \frac{8}{3} + 4 - 6 + 9 - 9 - 9$$

$$= -8\frac{1}{3}$$

This is the area above the x-axis minus the area below it.

7a The required shading between the curve $y = x^3 - x^2 + 2x$ and the x-axis from $x = -3$ to $x = 2$ is shown below.



7b Shaded region is below the x-axis so signed area will be negative.

$$\int_0^2 (x^3 - x^2 - 2x)\,dx$$

$$= \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_0^2$$

$$= \left(\frac{2^4}{4} - \frac{2^3}{3} - 2^2\right) - (0 - 0 - 0)$$

$$= 4 - \frac{8}{3} - 4$$

$$= -2\tfrac{2}{3}$$

Hence, the required area is $2\frac{2}{3}$ square units.

7c Shaded region is above the x-axis so signed area will be positive.

$$\int_{-1}^0 (x^3 - x^2 - 2x)\,dx$$

$$= \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{-1}^0$$

$$= (0 - 0 - 0) - \left(\frac{(-1)^4}{4} - \frac{(-1)^3}{3} - (-1)^2\right)$$

$$= -\frac{1}{4} - \frac{1}{3} + 1$$

$$= \frac{5}{12}$$

Hence, the required area is $\frac{5}{12}$ square units.

7d Area of the entire shaded region $= 2\frac{2}{3} + \frac{5}{12} = 3\frac{1}{12}$ square units

7e

$$\int_{-1}^{2}(x^3 - x^2 - 2x)\,dx$$

$$= \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{-1}^{2}$$

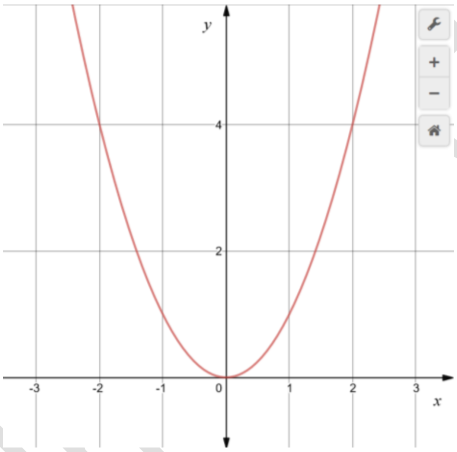
$$= \left(\frac{2^4}{4} - \frac{2^3}{3} - 2^2\right) - \left(\frac{(-1)^4}{4} - \frac{(-1)^3}{3} - (-1)^2\right)$$

$$= 4 - \frac{8}{3} - 4 - \frac{1}{4} - \frac{1}{3} + 1$$

$$= -2\tfrac{1}{4}$$

This is the area above the x-axis minus the area below it. This is because when the integral is taken, areas below the axis are considered negative.

8a The graph of $y = x^2$ is shown below. Between $x = -3$ and $x = 2$, the required region is above the x-axis and hence the signed area is positive.



$$\int_{-3}^{2} x^2\,dx$$

$$= \left[\frac{x^3}{3}\right]_{-3}^{2}$$

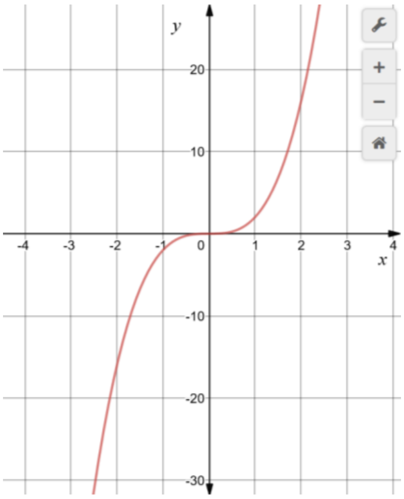
$$= \frac{2^3}{3} - \frac{(-3)^3}{3}$$

$$= \frac{8}{3} + 9$$

$= 11\frac{2}{3}$

Hence, the required area is $11\frac{2}{3}$ square units.

8b The graph of $y = 2x^3$ is shown below. Between $x = -4$ and $x = 0$, the region is below the x-axis (signed area is negative), and between $x = 0$ and $x = 1$, the region is above the x-axis (signed area is positive).



$\int_{-4}^{0} 2x^3\,dx$

$= \left[\frac{x^4}{2}\right]_{-4}^{0}$

$= 0 - \frac{(-4)^4}{2}$

$= -128$

$\int_{0}^{1} 2x^3\,dx$

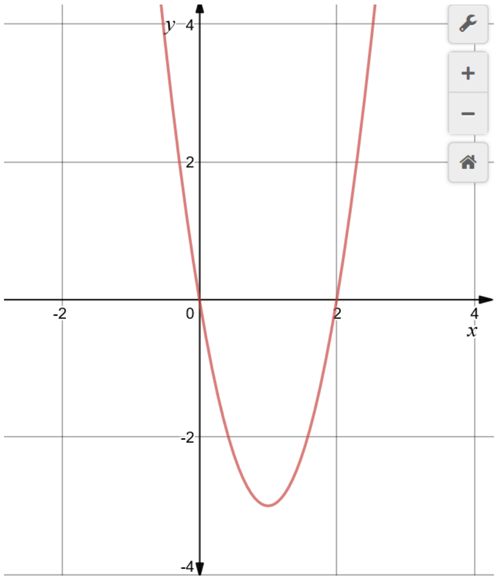
$= \left[\frac{x^4}{2}\right]_{0}^{1}$

$= \frac{1}{2} - 0$

$= \frac{1}{2}$

Area of the required region $= 128 + \frac{1}{2} = 128\frac{1}{2}$ square units.

8c The graph of $y = 3x(x - 2)$ is shown below. Between $x = 0$ and $x = 2$, the region is below the x-axis and hence the signed area is negative.



$$\int_0^2 3x(x-2)\,dx$$

$$= \int_0^2 (3x^2 - 6x)\,dx$$

$$= [x^3 - 3x^2]_0^2$$

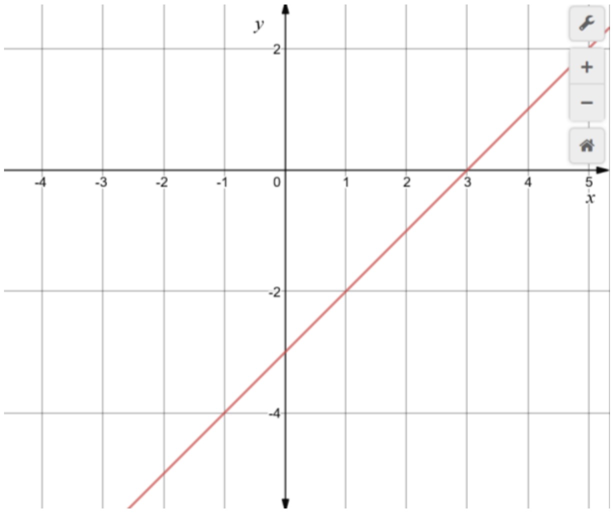
$$= (2^3 - 3 \times 2^2) - (0 - 0)$$

$$= 8 - 12$$

$$= -4$$

Area of the required region $= 4$ square units.

8d The graph of $y = x - 3$ is shown below. Between $x = -1$ and $x = 3$, the region is below the x-axis (signed area is negative), and between $x = 3$ and $x = 4$, the region is above the x-axis (signed area is positive).



$$\int_{-1}^{3}(x-3)\,dx$$

$$= \left[\frac{x^2}{2} - 3x\right]_{-1}^{3}$$

$$= \left(\frac{3^2}{2} - 3\times 3\right) - \left(\frac{(-1)^2}{2} - 3\times(-1)\right)$$

$$= \left(\frac{9}{2} - 9\right) - \left(\frac{1}{2} + 3\right)$$

$$= -8$$

$$\int_{3}^{4}(x-3)\,dx$$

$$= \left[\frac{x^2}{2} - 3x\right]_{3}^{4}$$

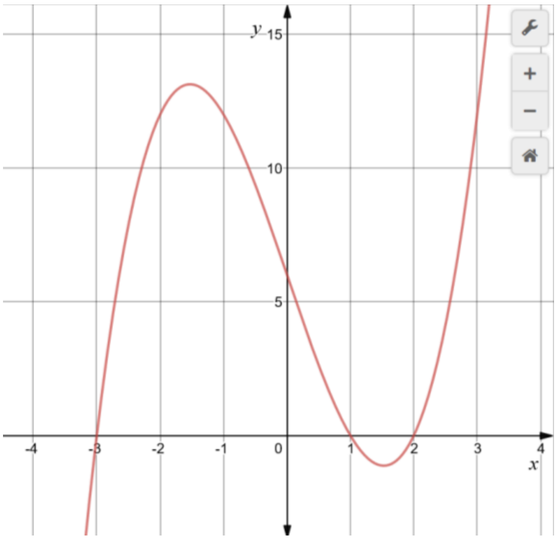
$$= \left(\frac{4^2}{2} - 3\times 4\right) - \left(\frac{3^2}{2} - 3\times 3\right)$$

$$= (8-12) - \left(\frac{9}{2} - 9\right)$$

$$= \frac{1}{2}$$

Area of the required region $= 8 + \frac{1}{2} = 8\frac{1}{2}$ square units

8e The graph of $y = (x-1)(x+3)(x-2)$ is shown below. Between $x = -3$ and $x = 1$, the region is above the x-axis (signed area is positive), and between $x = 1$ and $x = 2$, the region is below the x-axis (signed area is negative).



Hence the area is given by

$$\int_{-3}^{1} (x-1)(x+3)(x-2)\,dx$$

$$= \int_{-3}^{1} (x^3 - 7x + 6)\,dx$$

$$= \left[\frac{x^4}{4} - \frac{7x^2}{2} + 6x\right]_{-3}^{1}$$

$$= \left(\frac{1^4}{4} - \frac{7\times 1^2}{2} + 6\times 1\right) - \left(\frac{(-3)^4}{4} - \frac{7\times(-3)^2}{2} + 6\times(-3)\right)$$

$$= \left(\frac{1}{4} - \frac{7}{2} + 6\right) - \left(\frac{81}{4} - \frac{63}{2} - 18\right)$$

$$= 32$$

$$\int_{1}^{2} (x-1)(x+3)(x-2)\,dx$$

$$= \int_{1}^{2} (x^3 - 7x + 6)\,dx$$

$$= \left[\frac{x^4}{4} - \frac{7x^2}{2} + 6x\right]_{1}^{2}$$

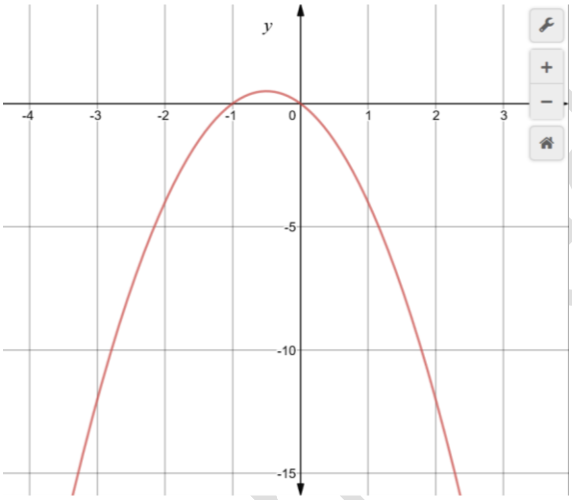
$$= \left(\frac{2^4}{4} - \frac{7 \times 2^2}{2} + 6 \times 2\right) - \left(\frac{1^4}{4} - \frac{7 \times 1^2}{2} + 6 \times 1\right)$$

$$= (4 - 14 + 12) - \left(\frac{1}{4} - \frac{7}{2} + 6\right)$$

$$= -\frac{3}{4}$$

Area of the required region $= 32 + \frac{3}{4} = 32\frac{3}{4}$ square units

8f The graph of $y = -2x(x + 1))$ is shown below. Between $x = -2$ and $x = -1$, region is below the x-axis (signed area is negative), between $x = -1$ and $x = 0$, the region is above the x-axis (signed area is positive), and between $x = 0$ and $x = 2$, the region is below the x-axis (signed area is negative).



$$\int_{-2}^{-1} -2x(x + 1)\, dx$$

$$= \int_{-2}^{-1} (-2x^2 - 2x)\, dx$$

$$= \left[-\frac{2}{3}x^3 - x^2\right]_{-2}^{-1}$$

$$= \left(-\frac{2}{3} \times (-1)^3 - (-1)^2\right) - \left(-\frac{2}{3} \times (-2)^3 - (-2)^2\right)$$

$$= \left(\frac{2}{3} - 1\right) - \left(\frac{16}{3} - 4\right)$$

$$= -1\frac{2}{3}$$

$$\int_{-1}^{0} -2x(x+1)\,dx$$

$$= \int_{-1}^{0} (-2x^2 - 2x)\,dx$$

$$= \left[-\frac{2}{3}x^3 - x^2\right]_{-1}^{0}$$

$$= \left(-\frac{2}{3}\times 0^3 - 0^2\right) - \left(-\frac{2}{3}\times(-1)^3 - (-1)^2\right)$$

$$= 0 - \left(\frac{2}{3} - 1\right)$$

$$= \frac{1}{3}$$

$$\int_{0}^{2} -2x(x+1)\,dx$$

$$= \int_{0}^{2} (-2x^2 - 2x)\,dx$$

$$= \left[-\frac{2}{3}x^3 - x^2\right]_{0}^{2}$$

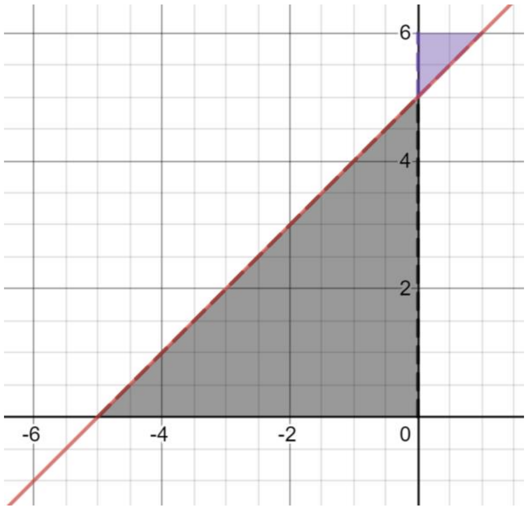
$$= \left(-\frac{2}{3}\times 2^3 - 2^2\right) - \left(-\frac{2}{3}\times 0^3 - 0^2\right)$$

$$= \left(-\frac{16}{3} - 4\right) - 0$$

$$= -9\frac{1}{3}$$

Area of the required region $= 1\frac{2}{3} + \frac{1}{3} + 9\frac{1}{3} = 11\frac{1}{3}$ square units

9a The graph of $x = y - 5$ (or $y = x + 5$) is shown below. Between $y = 0$ and $y = 5$, region is to the left of the y-axis (signed area is negative), and between $y = 5$ and $y = 6$, the region is to the right of the y-axis (signed area is positive).



$$\int_0^5 (y-5)\,dy$$

$$= \left[\frac{1}{2}y^2 - 5y\right]_0^5$$

$$= \left(\frac{1}{2}\times 5^2 - 5\times 5\right) - \left(\frac{1}{2}\times 0^2 - 5\times 0\right)$$

$$= \frac{25}{2} - 25 - 0$$

$$= -12\frac{1}{2}$$

$$\int_5^6 (y-5)\,dy$$

$$= \left[\frac{1}{2}y^2 - 5y\right]_5^6$$

$$= \left(\frac{1}{2}\times 6^2 - 5\times 6\right) - \left(\frac{1}{2}\times 5^2 - 5\times 5\right)$$

$$= 18 - 30 - \frac{25}{2} + 25$$

$$= \frac{1}{2}$$

Area of the required region $= 12\frac{1}{2} + \frac{1}{2} = 13$ square units

Alternatively, we can note that the area of each region is the area of a triangle.

$\int_0^6 (y-5)\,dy$

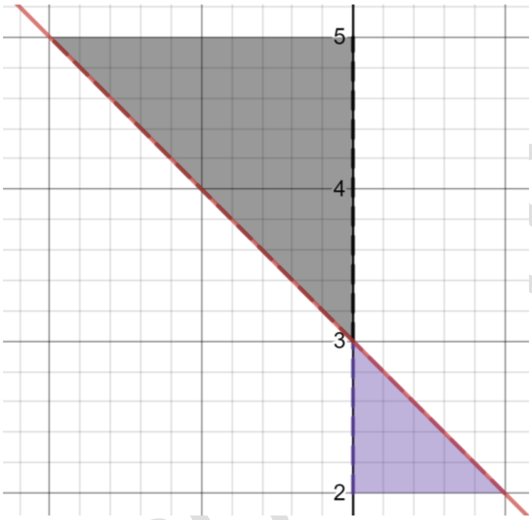
$= \text{Area}_{\text{black triangle}} + \text{Area}_{\text{purple triangle}}$

$= \frac{1}{2} \times 5 \times 5 + \frac{1}{2} \times 1 \times 1$

$= 12\frac{1}{2} + \frac{1}{2}$

$= 13$ square units

9b The graph of $x = 3 - y$ (or $y = 3 - x$) is shown below. Between $y = 2$ and $y = 3$, region is to the right of the y-axis (signed area is positive), and between $y = 3$ and $y = 5$, the region is to the left of the y-axis (signed area is negative).



We can note that the area of each region is the area of a triangle.

$\int_2^5 (3-y)\,dy$

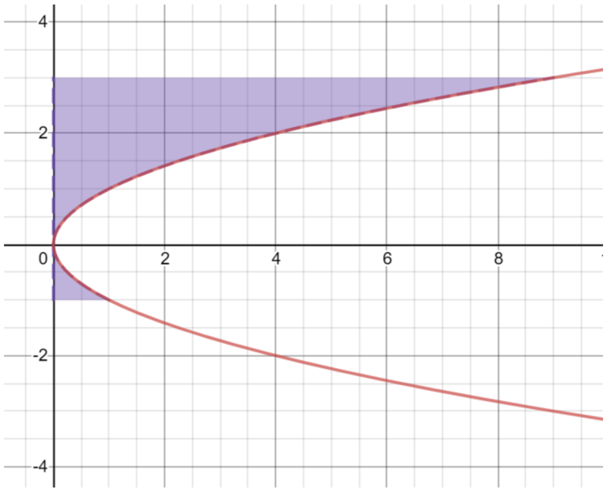
$= \text{Area}_{\text{black triangle}} + \text{Area}_{\text{purple triangle}}$

$= \frac{1}{2} \times 2 \times 2 + \frac{1}{2} \times 1 \times 1$

$= 2 + \frac{1}{2}$

$= 2\frac{1}{2}$ square units

9c The graph of $x = y^2$ (or $y = \pm\sqrt{x}$) is shown below. Between $y = -1$ and $y = 3$, region is to the right of the y-axis and hence the signed area is positive.



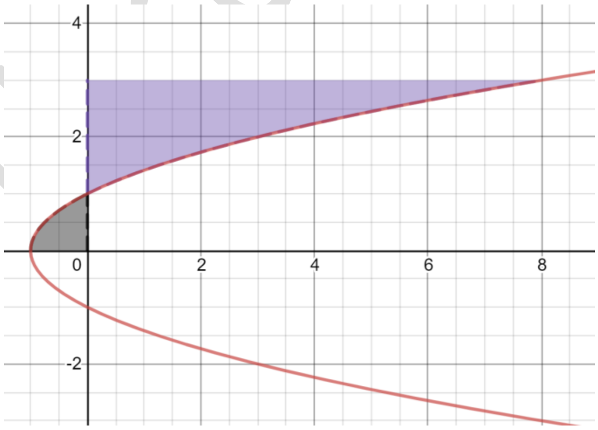
$$\int_{-1}^{3} y^2\, dy$$

$$= \left[\frac{y^3}{3}\right]_{-1}^{3}$$

$$= 9\frac{1}{3}$$

Area of the required region $= 9\frac{1}{3}$ square units

9d The graph of $x = (y - 1)(y + 1)$ is shown below. Between $y = 0$ and $y = 1$, region is to the left of the y-axis (signed area is negative), and between $y = 1$ and $y = 3$, the region is to the right of the y-axis (signed area is positive).



$$\int_0^1 (y-1)(y+1)\,dy$$

$$= \int_0^1 (y^2-1)\,dy$$

$$= \left[\frac{y^3}{3} - y\right]_0^1$$

$$= \left(\frac{1^3}{3} - 1\right) - \left(\frac{0^3}{3} - 0\right)$$

$$= -\frac{2}{3}$$

$$\int_1^3 (y-1)(y+1)\,dy$$

$$= \int_1^3 (y^2-1)\,dy$$

$$= \left[\frac{y^3}{3} - y\right]_1^3$$

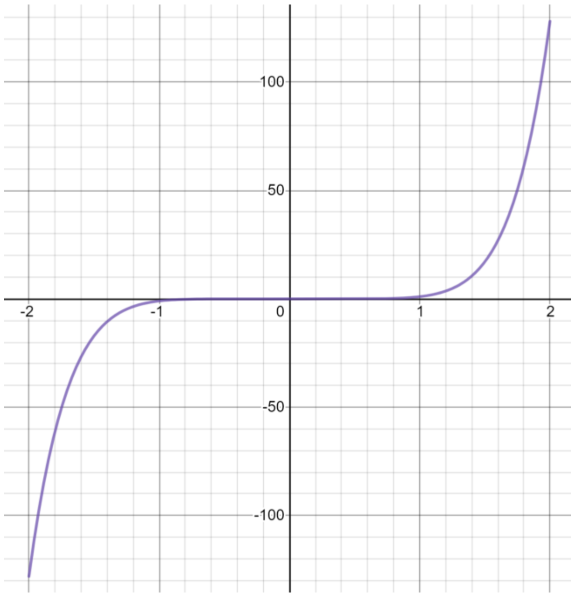
$$= \left(\frac{3^3}{3} - 3\right) - \left(\frac{1^3}{3} - 1\right)$$

$$= (9-3) - \left(\frac{1}{3} - 1\right)$$

$$= 6\frac{2}{3}$$

Area of the required region $= \frac{2}{3} + 6\frac{2}{3} = 7\frac{1}{3}$ square units

10a i The graph of $y = x^7$ is shown below. Between $x = -2$ and $x = 0$, the region is below the x-axis (signed area is negative), and between $x = 0$ and $x = 2$, the region is above the x-axis (signed area is positive).



Note that the function is odd and hence the area of the region between $x = -2$ and $x = 0$ is the same as the area of the region between $x = 0$ and $x = 2$.

Area of the required region

$= 2\int_0^2 x^7\,dy$

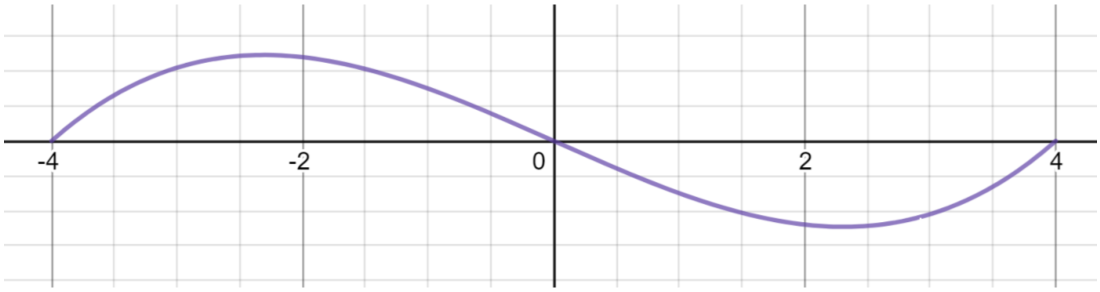
$= 2\left[\frac{x^8}{8}\right]_0^2$

$= \frac{1}{4}[x^8]_0^2$

$= \frac{2^8}{4} - 0$

$= 64$ square units

10a ii The graph of $y = x^3 - 16x$ is shown below. Between $x = -4$ and $x = 0$, the region is above the x-axis (signed area is positive), and between $x = 0$ and $x = 4$, the region is below the x-axis (signed area is negative).



Note that the function is odd and hence the area of the region between $x = -4$ and $x = 0$ is the same as the area of the region between $x = 0$ and $x = 4$.

Area of the required region

$= 2\int_{-4}^{0} (x^3 - 16x)\, dy$

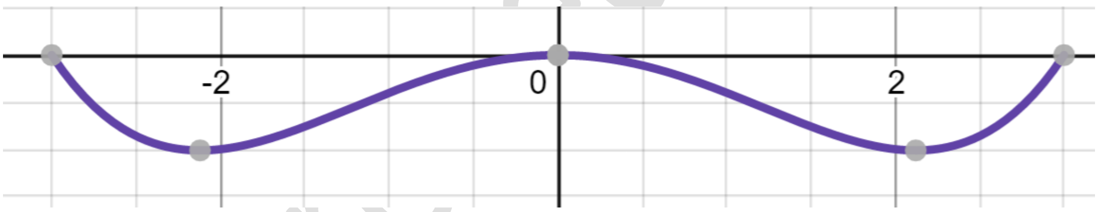
$= 2\left[\frac{x^4}{4} - 8x^2\right]_{-4}^{0}$

$= 2(0 - 0) - 2\left(\frac{(-4)^4}{4} - 8 \times (-4)^2\right)$

$= -2 \times -64$

$= 128$ square units

10a iii The graph of $y = x^4 - 9x^2$ is shown below. Between $x = -3$ and $x = 0$, the region is below the x-axis (signed area is negative), and between $x = 0$ and $x = 3$, the region is below the x-axis (signed area is negative).



Note that the function is even (reflection in y-axis) and hence the area of the region between $x = -3$ and $x = 0$ is the same as the area of the region between $x = 0$ and $x = 3$.

$2\int_{0}^{3} (x^4 - 9x^2)\, dx$

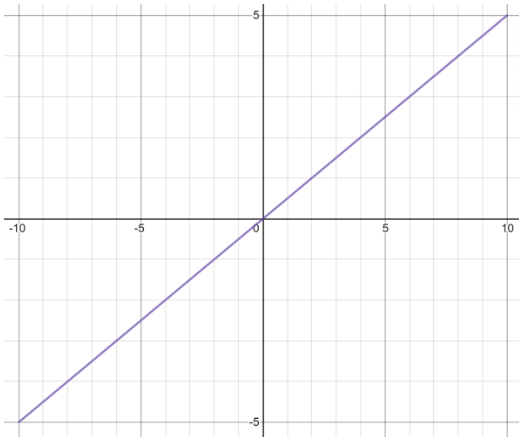
$= 2\left[\frac{x^5}{5} - 3x^3\right]_{0}^{3}$

$= 2\left(\frac{3^5}{5} - 3 \times 3^3\right) - 2(0)$

$= -64\frac{4}{5}$

Area of the required region $= 64\frac{4}{5}$ square units

10b i The graph of $x = 2y$ is shown below. Between $y = -5$ and $y = 0$, region is to the left of the y-axis (signed area is negative), and between $y = 0$ and $y = 5$, the region is to the right of the y-axis (signed area is positive).



Note that the total area will be given by the area of two identical triangles of width 10 units and height 5 units.

Area of the required region

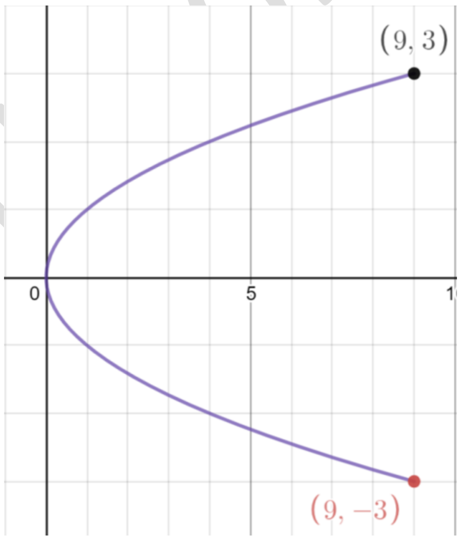
$= 2 \times \frac{1}{2}bh$

$= bh$

$= 10 \times 5$

$= 50$ square units

10b ii The graph of $x = y^2$ is shown below.



Note that the function is symmetric about the x-axis and hence the area of the region between $y = -3$ and $y = 0$ is the same as the area of the region between $y = 0$ and $y = 3$.

Area of the required region

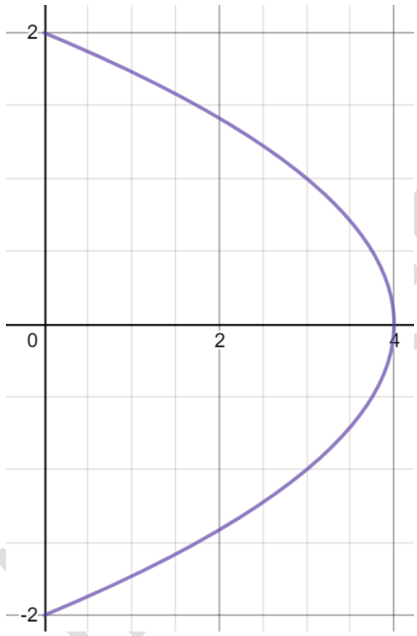
$$= \int_0^3 y^2 \, dy$$

$$= \left[\frac{y^3}{3}\right]_0^3$$

$$= \frac{3^3}{3} - 0$$

$= 9$ square units

10b iii The graph of $x = 4 - y^2$ is shown below.



Note that the function is symmetric about the x-axis and hence the area of the region between $y = -2$ and $y = 0$ is the same as the area of the region between $y = 0$ and $y = 2$.

Area of the required region

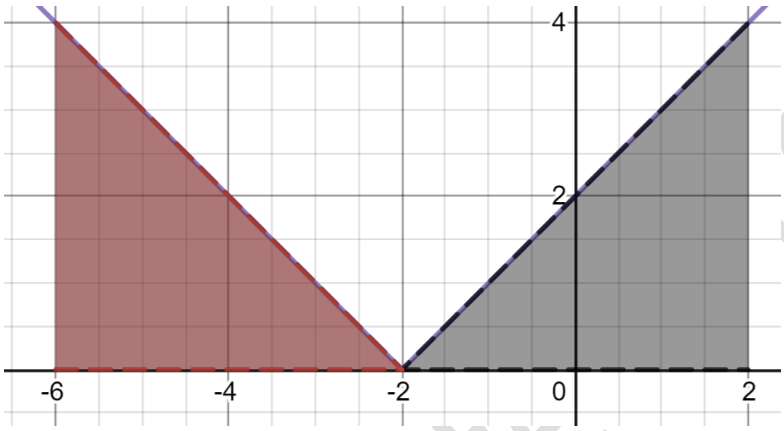
$$= 2\int_0^2 (4 - y^2) \, dy$$

$= 2\left[4y - \dfrac{y^3}{3}\right]_0^2$

$= 2\left(4 \times 2 - \dfrac{2^3}{3}\right) - 2\left(4 \times 0 - \dfrac{0^3}{3}\right)$

$= \frac{32}{3}$ square units

11 The graph of $y = |x + 2|$ is shown below.



The area between $x = -2$ and $x = 2$ needs to be calculated. It is same as the area of the triangle.

Area of the required region

$= \left(\dfrac{1}{2} \times 4 \times 4\right)$

$= 8$ square units

12 The x-intercepts occur when $y = 0$.

$0^2 = 16(2 - x)$

$(2 - x) = 0$

$x = 2$

Thus the x-intercept is $(2, 0)$.

The y-intercepts occur when $x = 0$.

$y^2 = 16(2 - 0)$

$y^2 = 2 \times 16$

$y = \pm 4\sqrt{2}$

Thus the y-intercepts are $(0, 4\sqrt{2})$, $(0, -4\sqrt{2})$.

12b i Area of the required region

$$= \int_0^2 4\sqrt{2-x}\,dx$$

$$= \int_0^2 4(2-x)^{\frac{1}{2}}\,dx$$

$$= \left[\frac{4(2-x)^{\frac{3}{2}}}{-1\times\frac{3}{2}}\right]_0^2$$

$$= 0 - \frac{4\times 2^{\frac{3}{2}}}{-\frac{3}{2}}$$

$$= \frac{2}{3}\times 4\times 2\times\sqrt{2}$$

$$= \frac{16\sqrt{2}}{3} \text{ square units}$$

12b ii Making x the subject:

$$\frac{y^2}{16} = 2 - x$$

$$x = 2 - \frac{y^2}{16}$$

Area of the required region

$$= \int_0^{4\sqrt{2}} \left(2 - \frac{y^2}{16}\right) dy$$

$$= \left[2y - \frac{y^3}{48}\right]_0^{4\sqrt{2}}$$

$$= \left(2\times 4\sqrt{2} - \frac{(4\sqrt{2})^3}{48}\right) - \left(2\times 0 - \frac{0^3}{48}\right)$$

$$= 8\sqrt{2} - \frac{128\sqrt{2}}{48} - 0$$

$$= \frac{384\sqrt{2} - 128\sqrt{2}}{48}$$

$$= \frac{256\sqrt{2}}{48}$$

$= \frac{16\sqrt{2}}{3}$ square units

13a

$$y = \int y'\,dx$$

$$= \int (x^2 - 4x + 3)\,dx$$

$$= \frac{1}{3}x^3 - 2x^2 + 3x + C$$

As the curve passes through the origin, substitute $(0, 0)$ into the equation.

$$0 = \frac{1}{3} \times 0^3 - 2 \times 0^2 + 3 \times 0 + C$$

$C = 0$

Hence $y = \frac{1}{3}x^3 - 2x^2 + 3x$

13b $y' = x^2 - 4x + 3$

For turning points, $y' = 0$.

$x^2 - 4x + 3 = 0$

$(x - 1)(x - 3) = 0$

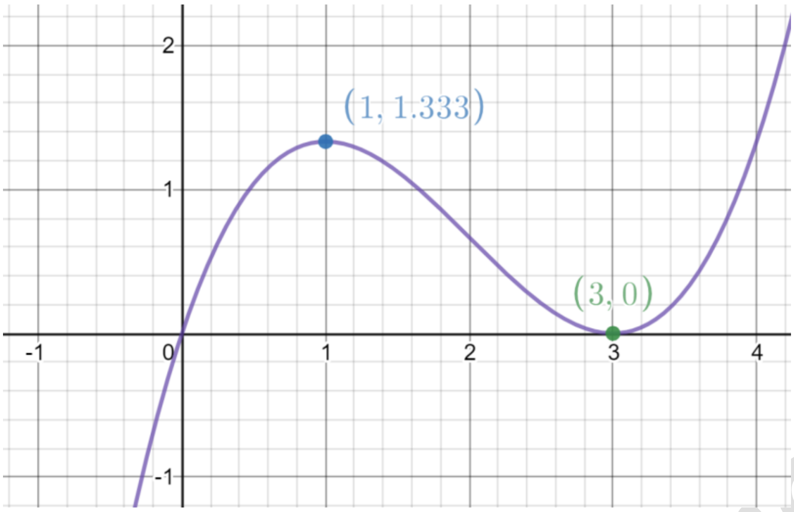
$x = 1$ or $x = 3$

When $x = 1, y = \frac{1}{3} - 2 + 3 = 1\frac{1}{3}$

When $x = 3, y = 0 - 0 + 0 = 0$

Hence there are turning points at $(1, 1\frac{1}{3})$ and $(3, 0)$.

The graph of $y = \frac{1}{3}x^3 - 2x^2 + 3x$ is shown below.



13c Area of the required region

$$= \int_1^3 \left(\frac{1}{3}x^3 - 2x^2 + 3x\right) dx$$

$$= \left[\frac{x^4}{12} - \frac{2}{3}x^3 + \frac{3}{2}x^2\right]_1^3$$

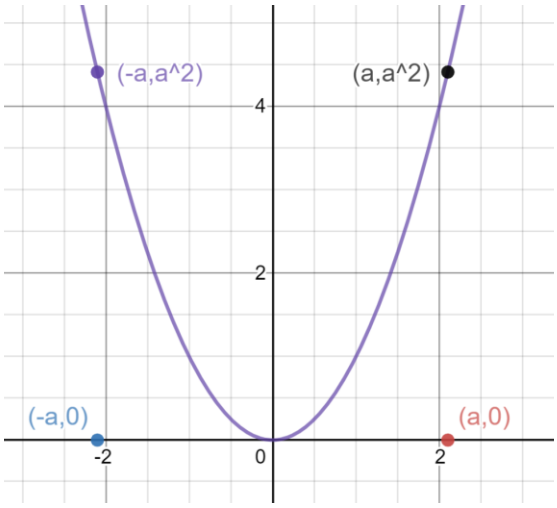
$$= \left(\frac{3^4}{12} - \frac{2}{3} \times 3^3 + \frac{3}{2} \times 3^2\right) - \left(\frac{1^4}{12} - \frac{2}{3} \times 1^3 + \frac{3}{2} \times 1^2\right)$$

$$= \left(\frac{81}{12} - 18 + \frac{27}{2}\right) - \left(\frac{1}{12} - \frac{2}{3} + \frac{3}{2}\right)$$

$$= 2\frac{1}{4} - \frac{11}{12}$$

$$= \frac{4}{3} \text{ square units}$$

14 The graph of $y = x^2$ is shown below with points marked for $A(a, a^2), B(-a, a^2)$, $P(a, 0)$ and $Q(-a, 0)$.



14a ΔOAP is a triangle of base length a and height a^2.

Area of $\Delta OAP = \frac{1}{2} \times a \times a^2 = \frac{a^3}{2}$

$$\int_0^a x^2\,dx$$

$$= \left[\frac{x^3}{3}\right]_0^a$$

$$= \frac{a^3}{3} - 0$$

$$= \frac{a^3}{3}$$

$$= \frac{2}{3}\left(\frac{a^3}{2}\right)$$

$$= \frac{2}{3}(\text{area } \Delta OAP)$$

14b $ABPQ$ is a rectangle with base length $2a$ and height a^2.

Area of rectangle $BPQ = 2a \times a^2 = 2a^3$.

$$\int_{-a}^{a} x^2 \, dx$$

$$= \left[\frac{x^3}{3}\right]_{-a}^{a}$$

$$= \frac{a^3}{3} - \left(-\frac{a^3}{3}\right)$$

$$= \frac{2a^3}{3}$$

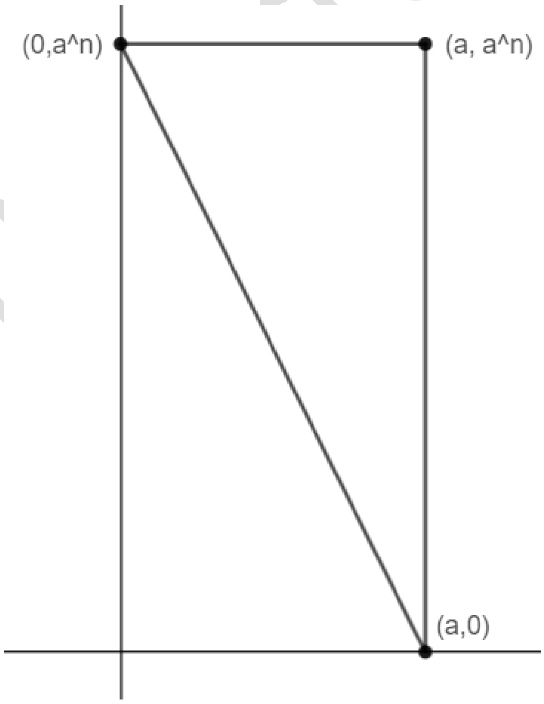
$$= \frac{1}{3}(2a^3)$$

$$= \frac{1}{3}(\text{area of rectangle } ABPQ)$$

15a $\int_0^a x^n = \left[\frac{x^{n+1}}{n+1}\right]_0^a = \frac{a^{n+1}}{n+1} - 0 = \frac{a^{n+1}}{n+1}$

Area of $\Delta AOP = \frac{1}{2}bh = \frac{1}{2}a \times a^n = a^{n+1}$ (see diagram below)

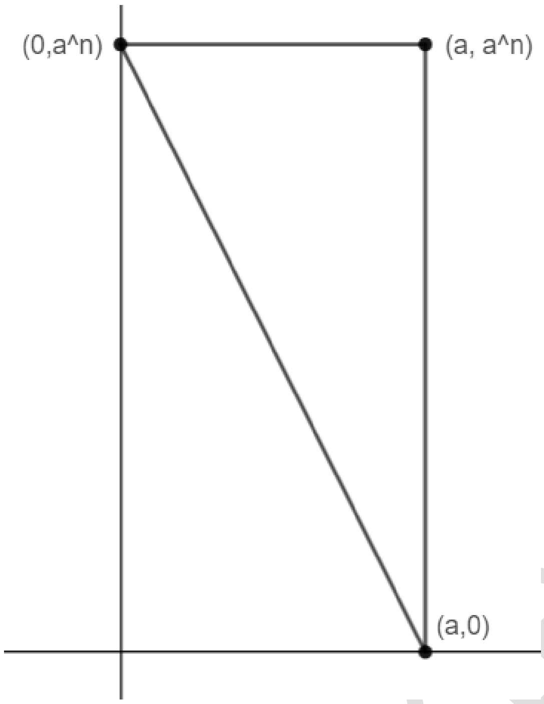
Thus $\int_0^a x^n : (\text{Area of } \Delta AOP) = \frac{a^{n+1}}{n+1} : \frac{1}{2}a^{n+1} = \frac{1}{n+1} : \frac{1}{2} = 2 : n+1$



15b $\int_0^a x^n = \left[\frac{x^{n+1}}{n+1}\right]_0^a = \frac{a^{n+1}}{n+1} - 0 = \frac{a^{n+1}}{n+1}$

Area of rectangle $OPAQ = bh = a \times a^n = a^{n+1}$ (see diagram below)

Thus $\int_0^a x^n$: (Area of rectangle $OPAQ$) $= \frac{a^{n+1}}{n+1} : a^{n+1} = \frac{1}{n+1} : 1 = 1 : n+1$



16a $x^4 - 2x^3 + x$

$= x(x^3 - 2x^2 + 1)$

$= x(x^3 - x^2 - x^2 + x - x + 1)$

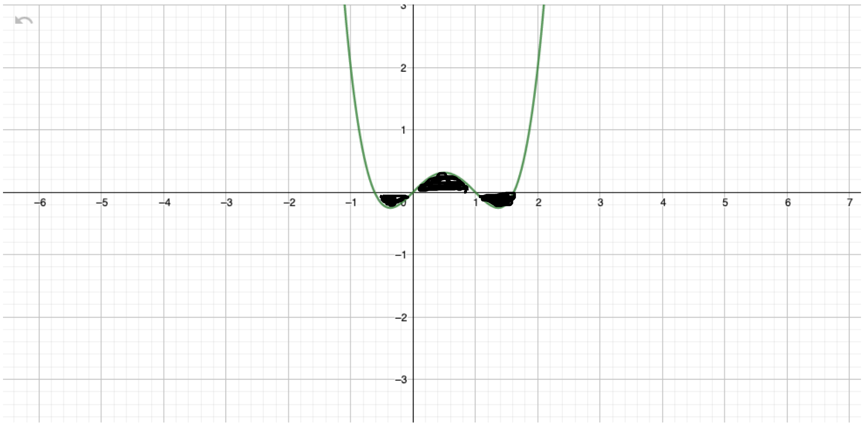
$= x(x^3 - x^2 - x - x^2 + x + 1)$

$= x[x(x^2 - x - 1) - 1(x^2 - x - 1)]$

$= x(x - 1)(x^2 - x - 1)$

Hence, proved

Required graph of $y = x^4 - 2x^3 + x$ with shaded regions bounded by the graph and the x-axis is shown below.



16b $a = \frac{1}{2}(1+\sqrt{5})$

$a^2 = \left[\frac{1}{2}(1+\sqrt{5})\right]^2$

$= \frac{1}{4}(1+5+2\sqrt{5})$

$= \frac{1}{4}(6+2\sqrt{5})$

$= \frac{1}{2}(3+\sqrt{5})$

$a^4 = \left[\frac{1}{2}(3+\sqrt{5})\right]^2$

$= \frac{9}{4} + \frac{5}{4} + \frac{3\sqrt{5}}{2}$

$= \frac{14}{4} + \frac{3\sqrt{5}}{2}$

$= \frac{7}{2} + \frac{3\sqrt{5}}{2}$

$= \frac{1}{2}(7+3\sqrt{5})$

$a^5 = a \times a^4$

$a^5 = \frac{1}{2}(1+\sqrt{5}) \times \frac{1}{2}(7+3\sqrt{5})$

$= \frac{1}{4}(7+3\sqrt{5}+7\sqrt{5}+15)$

$= \frac{1}{4}(22+10\sqrt{5})$

$= \frac{1}{2}(11+5\sqrt{5})$

16c The graph intercepts the axis when $x(x-1)(x^2-x-1)=0$, that is when $x=0, x=1$ or when $x^2-x-1=0$. Using the quadratic formula, $x^2-x-1=0$ when $x=\frac{-(-1)\pm\sqrt{(-1)^2-4(1)(-1)}}{2(1)}=\frac{1\pm\sqrt{5}}{2}$. So the intercepts are $x=\frac{1-\sqrt{5}}{2}, 0, 1, \frac{1+\sqrt{5}}{2}$.

The area of the leftmost shaded region is thus

$$A_1=\left|\int_{\frac{1-\sqrt{5}}{2}}^{0} x^4-2x^3+x\,dx\right|$$

$$=\left|\left[\frac{x^5}{5}-\frac{2x^4}{4}+\frac{x^2}{2}\right]_{\frac{1-\sqrt{5}}{2}}^{0}\right|$$

$$=\left|0-\left[\frac{\left(\frac{1-\sqrt{5}}{2}\right)^5}{5}-\frac{2\left(\frac{1-\sqrt{5}}{2}\right)^4}{4}+\frac{\left(\frac{1-\sqrt{5}}{2}\right)^2}{2}\right]\right|$$

$$=\frac{\left(\frac{1-\sqrt{5}}{2}\right)^5}{5}-\frac{2\left(\frac{1-\sqrt{5}}{2}\right)^4}{4}+\frac{\left(\frac{1-\sqrt{5}}{2}\right)^2}{2}$$

$$=\frac{1}{10}(11-5\sqrt{5})-\frac{1}{4}(7-3\sqrt{5})+\frac{1}{4}(3-\sqrt{5})$$

$$=\frac{1}{10}\text{ u}^2$$

The area of the central region is

$$A_2=\left|\int_0^1 x^4-2x^3+x\,dx\right|$$

$$=\left|\left[\frac{x^5}{5}-\frac{2x^4}{4}+\frac{x^2}{2}\right]_0^1\right|$$

$$=\left|\frac{1}{5}-\frac{2}{4}+\frac{1}{2}-0\right|$$

$$=\frac{1}{5}\text{ u}^2$$

The area of the far right region is

$$A_1 = \left|\int_1^{\frac{1+\sqrt{5}}{2}} x^4 - 2x^3 + x\,dx\right|$$

$$= \left|\left[\frac{x^5}{5} - \frac{2x^4}{4} + \frac{x^2}{2}\right]_1^{\frac{1+\sqrt{5}}{2}}\right|$$

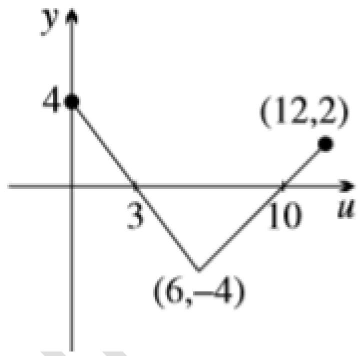
$$= \left|\left[\frac{\left(\frac{1+\sqrt{5}}{2}\right)^5}{5} - \frac{2\left(\frac{1+\sqrt{5}}{2}\right)^4}{4} + \frac{\left(\frac{1+\sqrt{5}}{2}\right)^2}{2}\right] - \left[1 - \frac{2}{4} + \frac{1}{2}\right]\right|$$

$$= \frac{1}{10}(11 + 5\sqrt{5}) - \frac{1}{4}(7 + 3\sqrt{5}) + \frac{1}{4}(3 + \sqrt{5})$$

$$= \frac{1}{10}\text{ u}^2$$

Thus $A_1 + A_3 = \frac{1}{10} + \frac{1}{10} = \frac{2}{10} = \frac{1}{5} = A_2$ so the area of one shaded region equals the sum of the areas of the other two.

17a



17b By the fundamental theorem of calculus $G'(x) = \frac{d}{dx}\int_0^x g(u)\,du = g(x)$.

Stationary points occur when $G'(x) = g(x) = 0$.

In the region where $0 \le x < 6$, the stationary points are when

$$4 - \frac{4}{3}x = 0$$

$4 = \frac{4}{3}x$

$x = 3$

$G(x) = \int_0^x 4 - \frac{4}{3}u\,du = \left[4u - \frac{2}{3}u^2\right]_0^x = 4x - \frac{2}{3}x^2$

Substituting in $x = 3$ gives $G(3) = 6$

$G''(x) = g'(x) = -\frac{4}{3} < 0$, hence this is a maximal stationary point.

In the region where $6 \le x \le 12$, the stationary points are when

$x - 10 = 0$

$x = 10$

$x = 3$

For $6 \le x \le 12$,

$G(x) = \int_0^x g(u)\,du$

$= \int_0^6 4 - \frac{4}{3}u\,du + \int_6^x u - 10\,du$

$= \left[4u - \frac{2}{3}u^2\right]_0^6 + \left[\frac{u^2}{2} - 10u\right]_6^x$

$= 4(6) - \frac{2}{3}(6)^2 - 0 + \left[\left(\frac{x^2}{2} - 10x\right) - \left(\frac{6^2}{2} - 10(6)\right)\right]$

$= 24 - 24 + \frac{x^2}{2} - 10x - 18 + 60$

$= \frac{x^2}{2} - 10x + 42$

Substituting in $x = 10$ gives $G(10) = -8$

$G''(x) = g'(x) = 10 > 0$, hence this is a minimal stationary point.

Thus, there is a maximum at $(3, 6)$ and a minimum at $(10, -8)$.

17c For $0 \le x < 6$, $G(x) = 0$ when

$$4x - \frac{2}{3}x^2 = 0$$

$$x\left(4 - \frac{2}{3}x\right) = 0$$

$x = 0$ or $4 - \frac{2}{3}x = 0$ so $x = 0$ or $4 = \frac{2}{3}x$ so $x = 0$ or $x = 6$

But 6 is outside of the domain so the only solution is $x = 0$.

For $6 \le x \le 12$

$$\frac{x^2}{2} - 10x + 42 = 0$$

$$x^2 - 20x + 84 = 0$$

Using the quadratic formula

$$x = \frac{-(-20) \pm \sqrt{(-20)^2 - 4(1)(84)}}{2}$$

$$= \frac{20 \pm \sqrt{64}}{2}$$

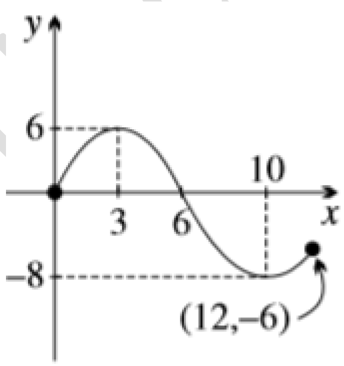
$$= \frac{20 \pm 8}{2}$$

$$= 6 \text{ or } 14$$

But 14 is outside of the domain so the only solution is $x = 6$.

Thus $G(x) = 0$ when $x = 0$ or $x = 6$.

17d



17e

$$A = \int_0^6 G(x)\,dx$$
$$= \int_0^6 4x - \frac{2}{3}x^2\,dx$$
$$= \left[2x^2 - \frac{2}{9}x^3\right]_0^6$$
$$= 2(6)^2 - \frac{2}{9}(6)^3 - 0$$
$$= 24 \text{ u}^2$$

18a $\int_1^N x^n\,dx = \left[\frac{x^{n+1}}{n+1}\right]_1^N = \frac{N^{n+1}}{n+1} - \frac{1}{n+1}$

Since $n < -1$, $n + 1 < 0$ and thus it follows that as $N \to \infty$, $N^{n+1} \to 0$.

Hence as $N \to \infty$, $\int_1^N x^n\,dx = \frac{N^{n+1}}{n+1} - \frac{1}{n+1} \to 0 - \frac{1}{n+1} = -\frac{1}{n+1}$.

18b $\int_e^1 x^n\,dx = \left[\frac{x^{n+1}}{n+1}\right]_e^1 = \frac{1}{n+1} - \frac{e^{n+1}}{n+1}$

Since $n > -1$, $n + 1 > 0$ and thus it follows that as $e \to 0^+$, $e^{n+1} \to 0^+$.

Hence as $e \to 0^+$, $\int_e^1 x^n\,dx = \frac{1}{n+1} - \frac{e^{n+1}}{n+1} \to \frac{1}{n+1} - 0 = \frac{1}{n+1}$.

18c The result from part a says that whenever $n < -1$, the area bounded by the x-axis, the line $x = 1$ and the curve x^n is finite when considering the area such that $x \geq 1$.

The result from part b says that whenever $n > -1$, the area bounded by the x-axis, the line $x = 1$ and the curve x^n is finite when considering the area such that $0 \leq x \leq 1$.

Solutions to Exercise 5G

1a Area of the shaded region

$$= \int_0^1 (x - x^2)\, dx$$

$$= \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$$

$$= \left(\frac{1}{2} - \frac{1}{3}\right) - 0$$

$= \frac{1}{6}$ square units

1b Area of the shaded region

$$= \int_0^1 (x - x^3)\, dx$$

$$= \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1$$

$$= \left(\frac{1}{2} - \frac{1}{4}\right) - 0$$

$= \frac{1}{4}$ square units

1c Area of the shaded region

$$= \int_0^1 (x - x^4)\, dx$$

$$= \left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1$$

$$= \left(\frac{1}{2} - \frac{1}{5}\right) - 0$$

$= \frac{3}{10}$ square units

1d Area of the shaded region

$$= \int_0^1 (x^2 - x^3)\,dx$$

$$= \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1$$

$$= \left(\frac{1}{3} - \frac{1}{4}\right) - 0$$

$$= \tfrac{1}{12} \text{ square units}$$

1e Area of the shaded region

$$= \int_0^1 (x^4 - x^6)\,dx$$

$$= \left[\frac{x^5}{5} - \frac{x^7}{7}\right]_0^1$$

$$= \left(\frac{1}{5} - \frac{1}{7}\right) - 0$$

$$= \tfrac{2}{35} \text{ square units}$$

1f Area of the shaded region

$$= \int_{-1}^4 \left((3x + 4) - x^2\right)dx$$

$$= \int_{-1}^4 (3x + 4 - x^2)\,dx$$

$$= \left[\frac{3x^2}{2} + 4x - \frac{x^3}{3}\right]_{-1}^4$$

$$= \left(\frac{3 \times 4^2}{2} + 4 \times 4 - \frac{4^3}{3}\right) - \left(\frac{3(-1)^2}{2} + 4 \times (-1) - \frac{(-1)^3}{3}\right)$$

$$= \left(24 + 16 - \frac{64}{3}\right) - \left(\frac{3}{2} - 4 + \frac{1}{3}\right)$$

$$= 20\tfrac{5}{6} \text{ square units}$$

1g Area of the shaded region

$$= \int_{-4}^{2} \left((9 - 2x) - (x^2 + 1)\right) dx$$
$$= \int_{-4}^{2} (8 - 2x - x^2)\, dx$$
$$= \left[8x - x^2 - \frac{x^3}{3}\right]_{-4}^{2}$$
$$= \left(8 \times 2 - 2^2 - \frac{2^3}{3}\right) - \left(8 \times (-4) - (-4)^2 - \frac{(-4)^3}{3}\right)$$
$$= \left(16 - 4 - \frac{8}{3}\right) - \left(-32 - 16 + \frac{64}{3}\right)$$
$$= 36 \text{ square units}$$

1h Area of the shaded region

$$= \int_{-3}^{2} \left(10 - x^2 - (x + 4)\right) dx$$
$$= \int_{-3}^{2} (6 - x^2 - x)\, dx$$
$$= \left[6x - \frac{x^3}{3} - \frac{x^2}{2}\right]_{-3}^{2}$$
$$= \left(6 \times 2 - \frac{2^3}{3} - \frac{2^2}{2}\right) - \left(6 \times (-3) - \frac{(-3)^3}{3} - \frac{(-3)^2}{2}\right)$$
$$= \left(12 - \frac{8}{3} - 2\right) - \left(-18 + 9 - \frac{9}{2}\right)$$
$$= 20\tfrac{5}{6} \text{ square units}$$

2a Area of the shaded region

$$= \int_{0}^{2} (2y - y^2)\, dy$$
$$= \left[y^2 - \frac{y^3}{3}\right]_{0}^{2}$$
$$= \left(2^2 - \frac{2^3}{3}\right) - 0$$
$$= \tfrac{4}{3} \text{ square units}$$

2b Area of the shaded region

$$= \int_1^2 (3y - 2 - y^2)\, dy$$
$$= \left[\frac{3y^2}{2} - 2y - \frac{y^3}{3}\right]_1^2$$
$$= \left(\frac{3 \times 2^2}{2} - 2 \times 2 - \frac{2^3}{3}\right) - \left(\frac{3 \times 1^2}{2} - 2 \times 1 - \frac{1^3}{3}\right)$$
$$= \left(6 - 4 - \frac{8}{3}\right) - \left(\frac{3}{2} - 2 - \frac{1}{3}\right)$$
$= \frac{1}{6}$ square units

2c Area of the shaded region

$$= \int_2^4 \big((5y - y^2 - 4) - (4 - y)\big)\, dy$$
$$= \int_2^4 (6y - y^2 - 8)\, dy$$
$$= \left[3y^2 - \frac{y^3}{3} - 8y\right]_2^4$$
$$= \left(3 \times 4^2 - \frac{4^3}{3} - 8 \times 4\right) - \left(3 \times 2^2 - \frac{2^3}{3} - 8 \times 2\right)$$
$$= \left(48 - \frac{64}{3} - 32\right) - \left(12 - \frac{8}{3} - 16\right)$$
$= \frac{4}{3}$ square units

2d Since $y = x - 4$ then $x = y + 4$.

Area of the shaded region

$$= \int_{-1}^2 \big(y + 4 - (y^2 + 2)\big)\, dy$$
$$= \int_{-1}^2 (-y^2 + y + 2)\, dy$$
$$= \left[-\frac{y^3}{3} + \frac{y^2}{2} + 2y\right]_{-1}^2$$
$$= \left(-\frac{2^3}{3} + \frac{2^2}{2} + 2 \times 2\right) - \left(-\frac{(-1)^3}{3} + \frac{(-1)^2}{2} + 2 \times (-1)\right)$$
$$= \left(-\frac{8}{3} + 2 + 4\right) - \left(\frac{1}{3} + \frac{1}{2} - 2\right)$$

$= 4\frac{1}{2}$ square units

3a Area of the shaded region

$= \int_{-2}^{0} (x-2)^2\,dx + \int_{0}^{2} (x+2)^2\,dx$

$= \left[\frac{(x-2)^3}{3}\right]_{-2}^{0} + \left[\frac{(x-3)^3}{3}\right]_{0}^{2}$

$= \frac{1}{3}[(0-2)^3 - (-2-2)^3] + \frac{1}{3}[(2+2)^3 - (0+2)^3]$

$= \frac{1}{3}(-8 + 64 + 64 - 8)$

$= 37\frac{1}{3}$ square units

3b Area of the shaded region

$= \int_{0}^{\frac{3}{2}} x^2\,dx + \int_{\frac{3}{2}}^{3} (x-3)^2\,dx$

$= \left[\frac{x^3}{3}\right]_{0}^{\frac{3}{2}} + \left[\frac{(x-3)^3}{3}\right]_{\frac{3}{2}}^{3}$

$= \left(\frac{\left(\frac{3}{2}\right)^3}{3} - 0\right) + \left(\frac{0^3}{3} - \frac{\left(\frac{3}{2} - 3\right)^3}{3}\right)$

$= \frac{27}{24} - 0 + 0 + \frac{27}{24}$

$= \frac{9}{4}$ square units

4a Area of the shaded region

$= \text{Area of rectangle} - \text{area under curve}$

$$= 3 \times 6 - \int_2^4 (6x - x^2 - 8)\, dx$$

$$= 18 - \left[3x^2 - \frac{x^3}{3} - 8x\right]_2^4$$

$$= 18 - \left[\left(3 \times 4^2 - \frac{4^3}{3} - 8 \times 4\right) - \left(3 \times 2^2 - \frac{2^3}{3} - 8 \times 2\right)\right]$$

$$= 18 - 48 + \frac{64}{3} + 32 + 12 - \frac{8}{3} - 16$$

$$= 16\tfrac{2}{3} \text{ square units}$$

4b Area of the shaded region

$$= \int_{-2}^{2} (4 - x^2)\, dx - \int_{-1}^{1} (1 - x^2)\, dx$$

$$= \left[4x - \frac{x^3}{3}\right]_{-2}^{2} - \left[x - \frac{x^3}{3}\right]_{-1}^{1}$$

$$= \left[\left(4 \times 2 - \frac{2^3}{3}\right) - \left(4 \times (-2) - \frac{(-2)^3}{3}\right)\right] - \left[\left(1 - \frac{1^3}{3}\right) - \left(1 \times (-1) - \frac{(-1)^3}{3}\right)\right]$$

$$= 8 - \frac{8}{3} + 8 - \frac{8}{3} - 1 + \frac{1}{3} - 1 + \frac{1}{3}$$

$$= 9\frac{1}{3} \text{ square units}$$

5a Equating the two equations gives:

$x^2 + 4 = x + 6$

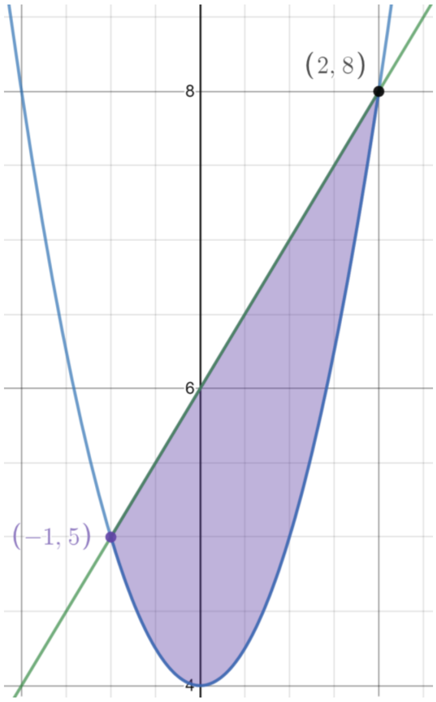
$x^2 - x - 2 = 0$

$(x + 1)(x - 2) = 0$

Hence the graphs intersect at the points where $x = -1$ or 2.

Substituting these values back into the equation $y = x + 6$ yields the points $(-1, 5)$ and $(2, 8)$ as the points of intersection of the two graphs.

5b



5c Area of the required region

$$= \int_{-1}^{2} \left((x+6) - (x^2+4)\right) dx$$

$$= \int_{-1}^{2} (x + 6 - x^2 - 4)\, dx$$

$$= \int_{-1}^{2} (x - x^2 + 2)\, dx$$

$$= \left[\frac{x^2}{2} - \frac{x^3}{3} + 2x\right]_{-1}^{2}$$

$$= \left(\frac{2^2}{2} - \frac{2^3}{3} + 2 \times 2\right) - \left(\frac{(-1)^2}{2} - \frac{(-1)^3}{3} + 2 \times (-1)\right)$$

$$= 2 - \frac{8}{3} + 4 - \frac{1}{2} - \frac{1}{3} + 2$$

$$= 4\frac{1}{2} \text{ square units}$$

6a Equating the two equations gives:

$x(3-x)=x$

$x(3-x)-x=0$

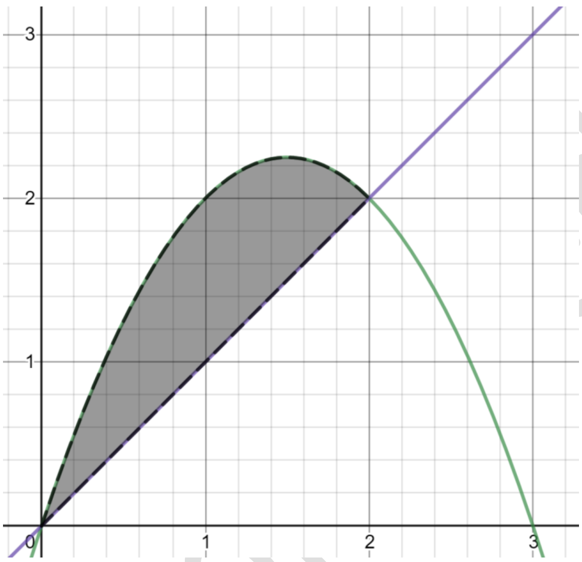
$x(3-x-1)=0$

$x(2-x)=0$

Hence $x=0$ or 2.

Substituting this back into the equation $y=x$ gives the points of intersection to be $(0,0)$ and $(2,2)$.

6b



6c Area of the required region

$$=\int_0^2 \left((3x-x^2)-x\right)dx$$

$$=\int_0^2 (3x-x^2-x)\,dx$$

$$=\int_0^2 (2x-x^2)\,dx$$

$$=\left[x^2-\frac{x^3}{3}\right]_0^2$$

$$=\left(2^2-\frac{2^3}{3}\right)-(0-0)$$

$= 4 - \frac{8}{3}$
$= \frac{4}{3}$ square units

7a Equating the two equations gives:

$(x-3)^2 = 14 - 2x$

$x^2 - 6x + 9 = 14 - 2x$

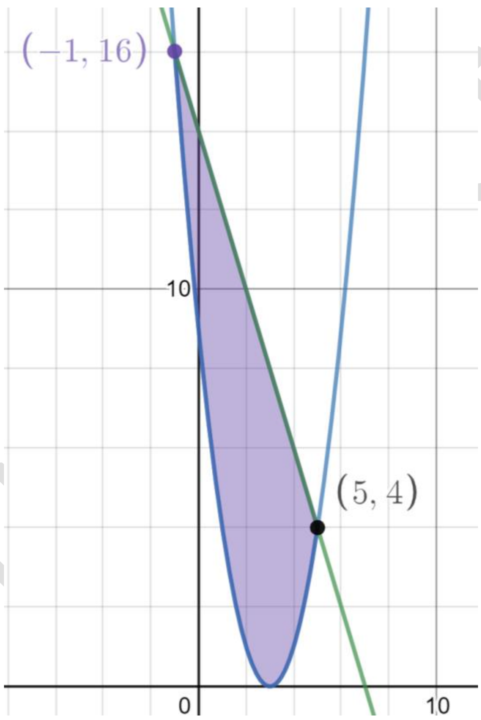
$x^2 - 4x - 5 = 0$

$(x+1)(x-5) = 0$

Hence the graphs intersect at the points where $x = -1$ or 5.

Substituting these values back into the equation $y = 14 - 2x$ yields the points $(-1, 16)$ and $(5, 4)$ as the points of intersection of the two graphs.

7b



7c Area of the required region

$$= \int_{-1}^{5} \left((14 - 2x) - (x - 3)^2\right) dx$$

$$= \int_{-1}^{5} \left((14 - 2x) - (x^2 - 6x + 9)\right) dx$$

$$= \int_{-1}^{5} (4x + 5 - x^2)\, dx$$

$$= \left[2x^2 + 5x - \frac{x^3}{3}\right]_{-1}^{5}$$

$$= \left(2 \times 5^2 + 5 \times 5 - \frac{5^3}{3}\right) - \left(2 \times (-1)^2 + 5 \times (-1) - \frac{(-1)^3}{3}\right)$$

$$= 50 + 25 - \frac{125}{3} - 2 + 5 - \frac{1}{3}$$

$= 36$ square units

8a Solving the equations simultaneously gives:

$x + 3 = x^2 + 1$

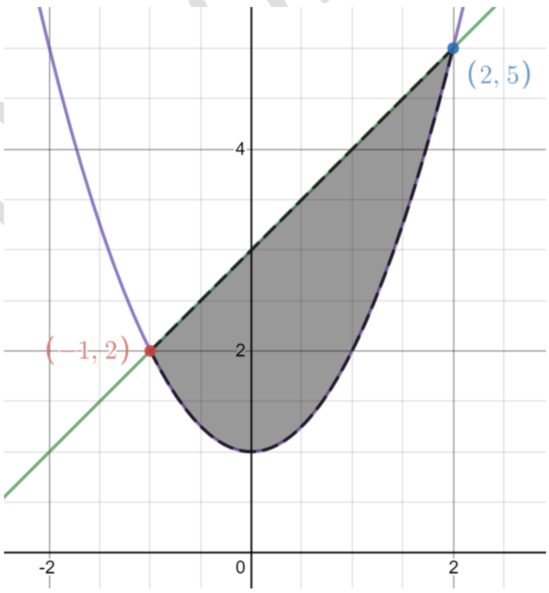
$x^2 - x - 2 = 0$

$(x + 1)(x - 2) = 0$

$x = -1$ or 2

Substituting this back into $y = x + 3$ gives the points of intersection to be $(-1, 2)$ and $(2, 5)$.

Sketching the graph gives:



Area enclosed between the two curves

$$= \int_{-1}^{2} \left((x+3) - (x^2+1)\right) dx$$

$$= \int_{-1}^{2} (x + 2 - x^2)\, dx$$

$$= \left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-1}^{2}$$

$$= \left(\frac{2^2}{2} + 2 \times 2 - \frac{2^3}{3}\right) - \left(\frac{(-1)^2}{2} + 2 \times (-1) - \frac{(-1)^3}{3}\right)$$

$$= 2 + 4 - \frac{8}{3} - \frac{1}{2} + 2 - \frac{1}{3}$$

$$= 4\tfrac{1}{2} \text{ square units}$$

8b Solving the equations simultaneously gives:

$9 - x^2 = 3 - x$

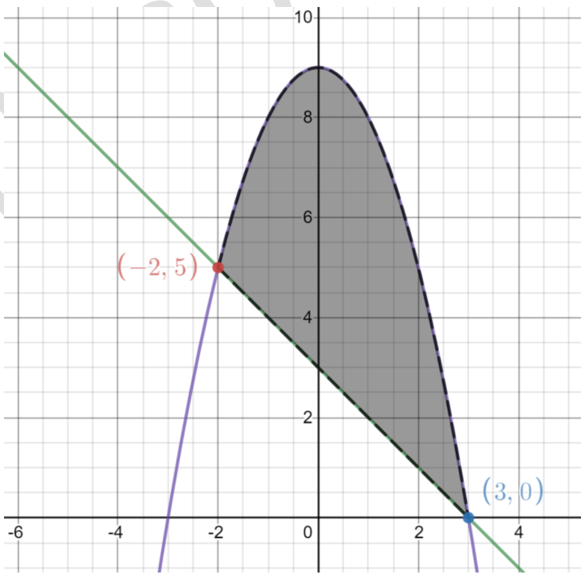
$x^2 - x - 6 = 0$

$(x+2)(x-3) = 0$

$x = -2$ or 3

Substituting this back into $y = 3 - x$ gives the points of intersection to be $(-2, 5)$ and $(3, 0)$.

Sketching the graph gives:



Area enclosed between the two curves

$$= \int_{-2}^{3} \left((9 - x^2) - (3 - x)\right) dx$$

$$= \int_{-2}^{3} (x - x^2 + 6)\, dx$$

$$= \left[\frac{x^2}{2} - \frac{x^3}{3} + 6x\right]_{-2}^{3}$$

$$= \left(\frac{3^2}{2} - \frac{3^3}{3} + 6 \times 3\right) - \left(\frac{(-2)^2}{2} - \frac{(-2)^3}{3} + 6 \times (-2)\right)$$

$$= \frac{9}{2} - 9 + 18 - 2 - \frac{8}{3} + 12$$

$$= 20\frac{5}{6} \text{ square units}$$

8c Solving the equations simultaneously gives:

$x^2 - x + 4 = -x^2 + 3x + 4$

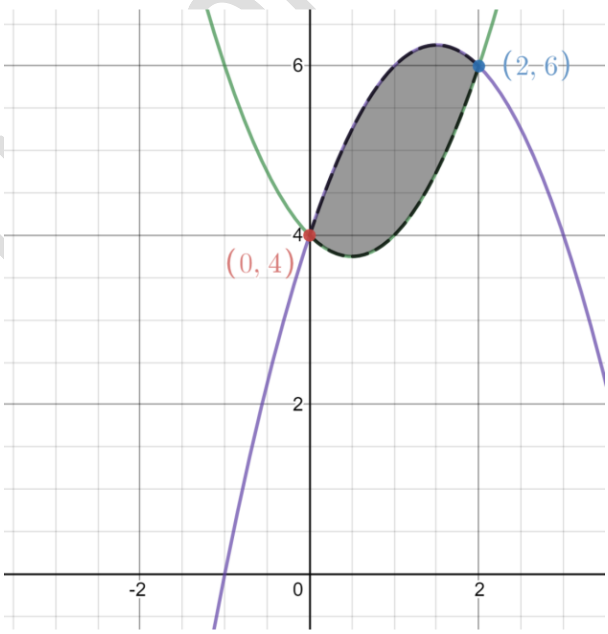
$2x^2 - 4x = 0$

$2x(x - 2) = 0$

$x = 0$ or 2

Substituting this back into $y = x^2 - x + 4$ gives the points of intersection to be $(0, 4)$ and $(2, 6)$.

Sketching the graph gives



Area enclosed between the two curves

$$= \int_0^2 \left((-x^2 + 3x + 4) - (x^2 - x + 4)\right) dx$$

$$= \int_0^2 (-2x^2 + 4x)\, dx$$

$$= \left[-\frac{2x^3}{3} + 2x^2\right]_0^2$$

$$= \left(-\frac{2 \times 2^3}{3} + 2 \times 2^2\right) - (0 + 0)$$

$$= -\frac{16}{3} + 8$$

$$= 2\frac{2}{3} \text{ square units}$$

9a Equating the two equations gives:

$x^2 + 2x - 8 = 2x + 1$

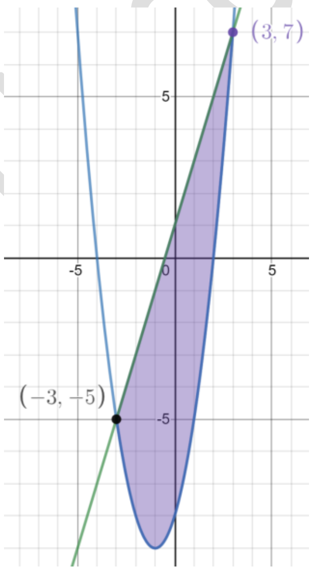
$x^2 - 9 = 0$

$(x + 3)(x - 3) = 0$

Hence the graphs intersect at the points where $x = -3$ or 3.

Substituting these values back into the equation $y = 2x + 1$ yields the points $(-3, -5)$ and $(3, 7)$ as the points of intersection of the two graphs.

9b



9c Area enclosed between the curves

$$= \int_{-3}^{3} \big((2x+1) - (x^2+2x-8)\big)\,dx$$
$$= \int_{-3}^{3} \big((2x+1) - x^2 - 2x + 8\big)\,dx$$
$$= \int_{-3}^{3} (9 - x^2)\,dx$$
$$= \left[9x - \frac{x^3}{3}\right]_{-3}^{3}$$
$$= \left(9\times 3 - \frac{3^3}{3}\right) - \left(9\times(-3) - \frac{(-3)^3}{3}\right)$$
$$= 27 - 9 + 27 - 9$$
$= 36$ square units

10a Simultaneously solving the equations gives:

$x^2 - x - 2 = x - 2$

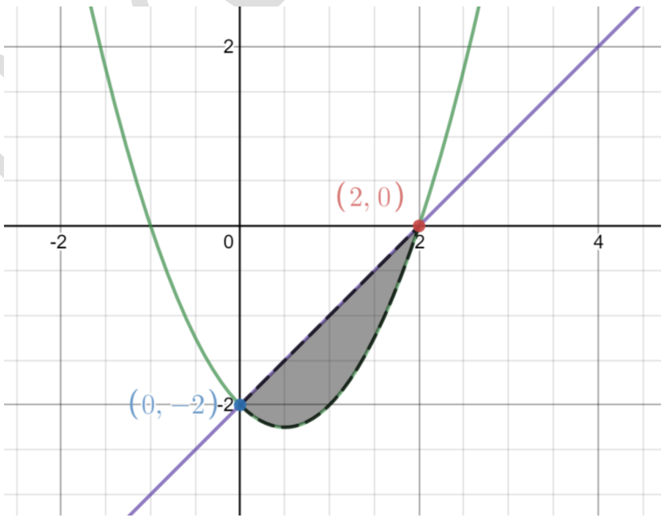
$x^2 - 2x = 0$

$x(x-2) = 0$

$x = 0$ or 2

Substituting this back into $y = x - 2$ gives the points of intersection to be $(0, -2)$ and $(2, 0)$.

10b



10c Area enclosed by the two curves

$$= \int_0^2 \left((x-2)-(x^2-x-2)\right)dx$$
$$= \int_0^2 (2x-x^2)\,dx$$
$$= \left[x^2-\frac{x^3}{3}\right]_0^2$$
$$= \left(2^2-\frac{2^3}{3}\right)-(0-0)$$
$$= 4-\frac{8}{3}$$
$$= \tfrac{4}{3} \text{ square units}$$

11a Simultaneously solving the equations gives:

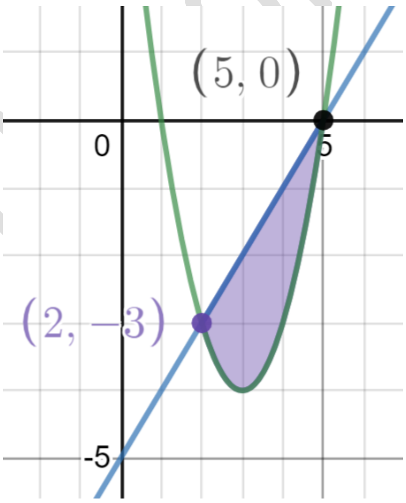
$x^2-6x+5=x-5$

$x^2-7x+10=0$

$(x-2)(x-5)=0$

$x=2$ or 5

Substituting this back into $y=x-5$ gives the points of intersection to be $(2,-3)$ and $(5,0)$.



Area enclosed by the two curves

$$= \int_2^5 \left((x-5) - (x^2 - 6x + 5)\right) dx$$
$$= \int_2^5 (-x^2 + 7x - 10)\, dx$$
$$= \left[-\frac{x^3}{3} + \frac{7x^2}{2} - 10x\right]_2^5$$
$$= \left[\left(-\frac{5^3}{3} + \frac{7 \times 5^2}{2} - 10 \times 5\right) - \left(-\frac{2^3}{3} + \frac{7 \times 2^2}{2} - 10 \times 2\right)\right]$$
$$= -\frac{125}{3} + \frac{175}{2} - 50 + \frac{8}{3} - 14 + 20$$
$$= 4\frac{1}{2} \text{ square units}$$

11b Simultaneously solving the equations gives:

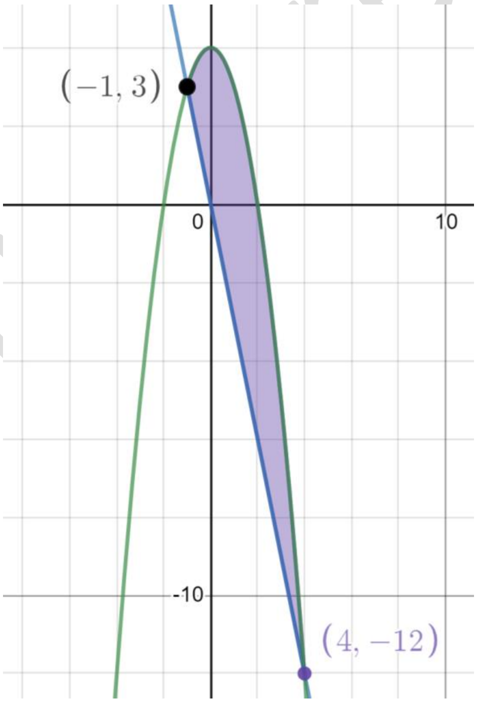
$-3x = 4 - x^2$

$x^2 - 3x - 4 = 0$

$(x+1)(x-4) = 0$

$x = -1 \text{ or } 4$

Substituting this back into $y = -3x$ gives the points of intersection to be $(-1, 3)$ and $(4, -12)$.



Area enclosed by the two curves

$= \int_{-1}^{4} \left(4 - x^2 - (-3x)\right) dx$

$= \int_{-1}^{4} (4 + 3x - x^2)\, dx$

$= \left[4x + \frac{3x^2}{2} - \frac{x^3}{3}\right]_{-1}^{4}$

$= \left(4 \times 4 + \frac{3 \times 4^2}{2} - \frac{4^3}{3}\right) - \left(4 \times (-1) + \frac{3 \times (-1)^2}{2} - \frac{(-1)^3}{3}\right)$

$= 16 + 24 - \frac{64}{3} + 4 - \frac{3}{2} - \frac{1}{3}$

$= 20\frac{5}{6}$ square units

11c Simultaneously solving the equations gives:

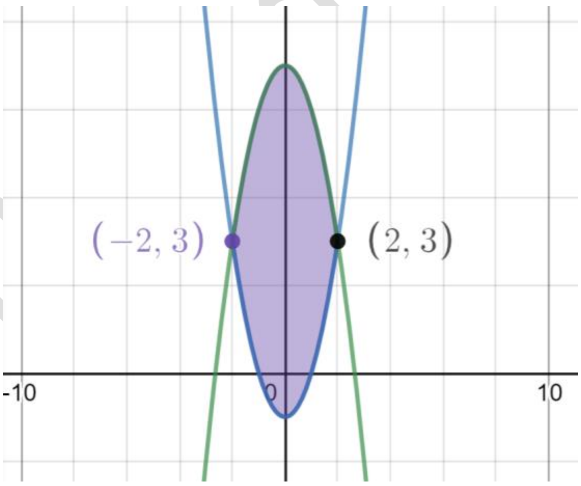
$x^2 - 1 = 7 - x^2$

$2x^2 = 8$

$x^2 = 4$

$x = -2$ or 2

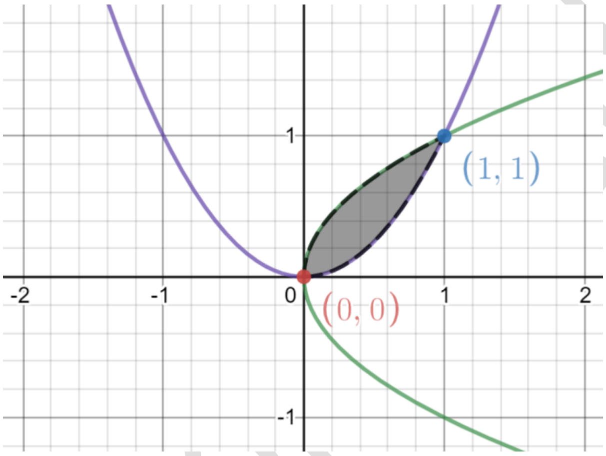
Substituting this back into $y = x^2 - 1$ gives the points of intersection to be $(-2, 3)$ and $(2, 3)$.



Area enclosed by the two curves

$$= \int_{-2}^{2} \left((7 - x^2) - (x^2 - 1)\right) dx$$
$$= \int_{-2}^{2} (8 - 2x^2)\, dx$$
$$= \left[8x - \frac{2x^3}{3}\right]_{-2}^{2}$$
$$= \left(8 \times 2 - \frac{2 \times 2^3}{3}\right) - \left(8 \times (-2) - \frac{2 \times (-2)^3}{3}\right)$$
$$= 16 - \frac{16}{3} + 16 - \frac{16}{3}$$
$$= 21\tfrac{1}{3} \text{ square units}$$

12a The graphs of $y = x^2$ and $x = y^2$ are shown below.



12b The region is enclosed by $y = x^2$ and $x = y^2$. As the curve is bounded by components of $x = y^2$ in the first quadrant, then the part of the curve we are considering is the same as $y = \sqrt{x}$.

Hence, the area of the enclosed region is given by the area under the curve $y = \sqrt{x}$ subtracted from the area under the curve $y = x^2$ between the x-values of 0 and 1.

Area of the required region $= \int_0^1 \left(\sqrt{x} - x^2\right) dx$

12c Area enclosed by the two curves

$$= \int_0^1 \left(\sqrt{x} - x^2\right) dx$$

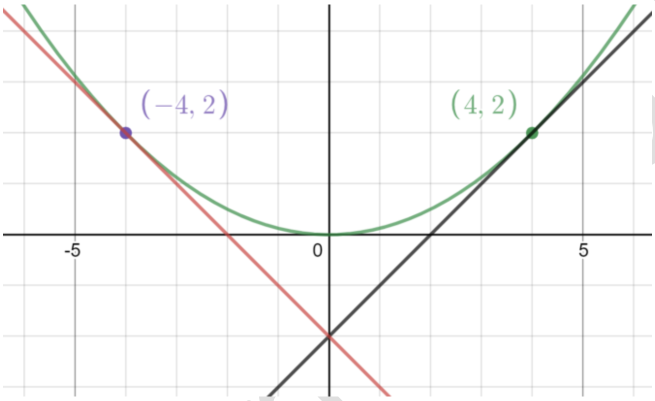
$$= \int_0^1 \left(x^{\frac{1}{2}} - x^2\right) dx$$

$$= \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{x^3}{3}\right]_0^1$$

$$= \left(\frac{2}{3} \times 1 - \frac{1}{3}\right) - (0 - 0)$$

$= \frac{1}{3}$ square units

13a



13b Note that the equation of the parabola is $y = \frac{x^2}{8}$, hence $\frac{dy}{dx} = \frac{2x}{8} = \frac{x}{4}$. Now, at the point A, $\frac{dy}{dx} = \frac{4}{4} = 1$. Hence the equation of the tangent is:

$$y - y_1 = m(x - x_1)$$

$$y - 2 = 1(x - 4)$$

$$y - 2 = x - 4$$

$$y = x - 2$$

13c Due to the symmetry, the area bounded by the curve and the two tangents is twice that bounded by the parabola, the tangent from A and the y-axis. Hence:

Area of required region

$$= 2\int_0^4 \left(\frac{x^2}{8} - (x-2)\right) dx$$

$$= 2\int_0^4 \left(\frac{x^2}{8} - x + 2\right) dx$$

$$= 2\left[\frac{x^3}{24} - \frac{x^2}{2} + 2x\right]_0^4$$

$$= 2\left(\frac{4^3}{24} - \frac{4^2}{2} + 2\times 4\right) - 2(0 - 0 + 0)$$

$$= 2\left(\frac{64}{24} - 8 + 8\right)$$

$$= 5\tfrac{1}{3} \text{ square units}$$

14a Since $y = x^3$, $\frac{dy}{dx} = 3x^2$. Hence when $x = 2$, $y = 8$ and $\frac{dy}{dx} = 12$.

The equation of the tangent is:

$$y - y_1 = m(x - x_1)$$

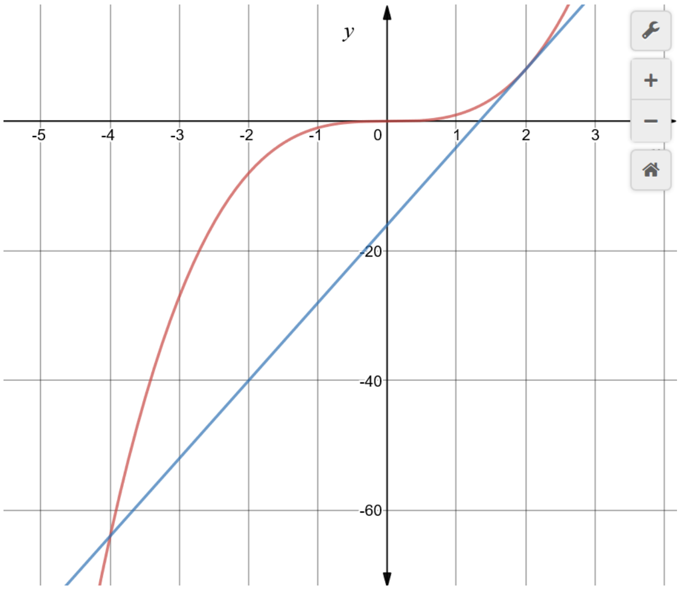
$$y - 8 = 12(x - 2)$$

$$y - 8 = 12x - 24$$

$$y = 12x - 16$$

14b Substituting $x = -4$ into $y = x^3$ gives $y = (-4)^3 = -64$.
Substituting $x = -4$ into $y = 12x - 16$ gives $y = -48 - 16 = -64$.

Hence both graphs pass through $(-4, -64)$ and hence they must intersect at that point.

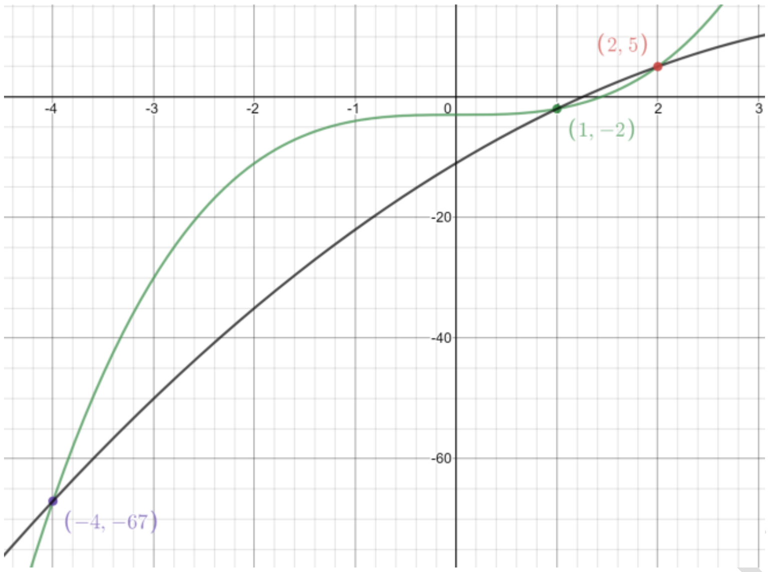


14c Area enclosed between curve and tangent

$$= \int_{-4}^{2} \left(x^3 - (12x - 16)\right) dx$$
$$= \int_{-4}^{2} (x^3 - 12x + 16)\, dx$$
$$= \left[\frac{x^4}{4} - 6x^2 + 16x\right]_{-4}^{2}$$
$$= \left(\frac{2^4}{4} - 6 \times 2^2 + 16 \times 2\right) - \left(\frac{(-4)^4}{4} - 6 \times (-4)^2 + 16 \times (-4)\right)$$
$$= 4 - 24 + 32 - 64 + 96 + 64$$
$= 108$ square units

15a When $x = -4$ both curves have $y = -67$, when $x = 1$ both curves have $y = -2$ and when $x = 2$, both curves have $y = 5$. So the points of intersection are $(-4, -67)$, $(1, -2)$ and $(2, 5)$.

15b The graphs of $y = x^3 - 3$ and $y = -x^2 + 10x - 11$ are shown below.



15c We must consider the two enclosed regions separately in our calculation.

Area of required region

$$= \int_{-4}^{1} \left(x^3 - 3 - (-x^2 + 10x - 11)\right) dx + \int_{1}^{2} \left((-x^2 + 10x - 11) - (x^3 - 3)\right) dx$$

$$= \int_{-4}^{1} (x^3 + x^2 - 10x + 8)\, dx + \int_{1}^{2} (-x^3 - x^2 + 10x - 8)\, dx$$

$$= \left[\frac{x^4}{4} + \frac{x^3}{3} - 5x^2 + 8x\right]_{-4}^{1} + \left[-\frac{x^4}{4} - \frac{x^3}{3} + 5x^2 - 8x\right]_{1}^{2}$$

$$= \left(\frac{1^4}{4} + \frac{1^3}{3} - 5 \times 1^2 + 8 \times 1\right) - \left(\frac{(-4)^4}{4} + \frac{(-4)^3}{3} - 5 \times (-4)^2 + 8 \times (-4)\right)$$

$$+ \left(-\frac{2^4}{4} - \frac{2^3}{3} + 5 \times 2^2 - 8 \times 2\right) - \left(-\frac{1^4}{4} - \frac{1^3}{3} + 5 \times 1^2 - 8 \times 1\right)$$

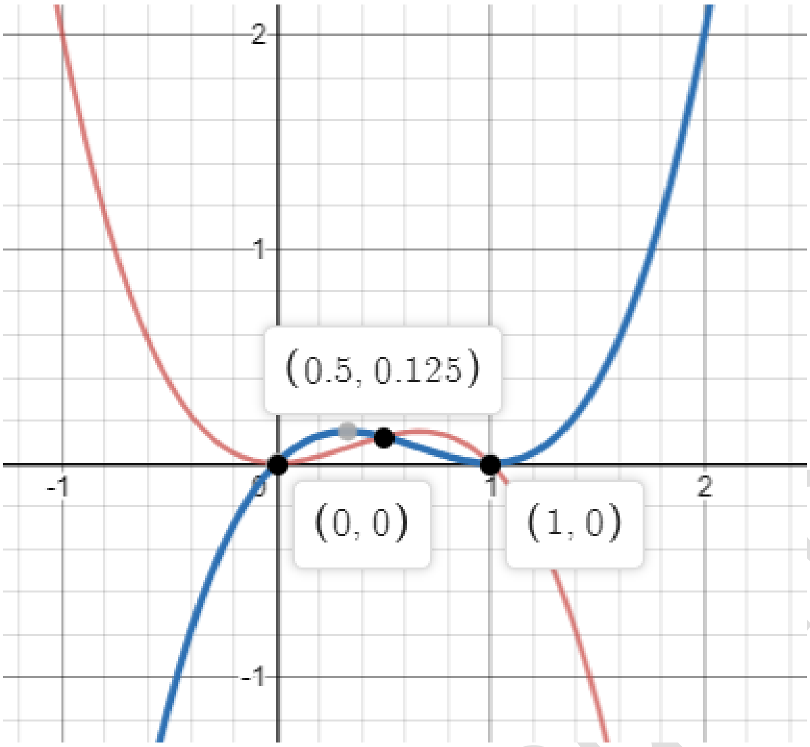
$$= \left(\frac{1}{4} + \frac{1}{3} - 5 + 8\right) - \left(64 - \frac{64}{3} - 80 - 32\right)$$

$$+ \left(-4 - \frac{8}{3} + 20 - 16\right) - \left(-\frac{1}{4} - \frac{1}{3} + 5 - 8\right)$$

$$= 3\frac{7}{12} + 69\frac{1}{3} - \frac{8}{3} + 3\frac{7}{12}$$

$$= 73\tfrac{5}{6} \text{ square units}$$

16a



When $x = 0$ both curves have $y = 0$, when $x = 1$ both curves have $y = 0$ and when $x = \frac{1}{2}$, both curves have $y = \frac{1}{8}$. So the points of intersection are $(0, 0)$, $(1, 0)$ and $\left(\frac{1}{2}, \frac{1}{8}\right)$.

16b We must consider the enclosed regions for calculation:

$= \left|\int_0^1 x^2(1 - x) - x(1 - x^2)\, dx\right|$

$= \left|\int_0^1 (x^2 - x^3 - x + x^3)\, dx\right|$

$= \left|\int_0^1 (x^2 - x)\, dx\right|$

$= \left|\left[\frac{x^3}{3} - \frac{x^2}{2}\right]_0^1\right|$

$= \left|\frac{1}{3} - \frac{1}{2}\right|$

$= \frac{1}{6}$ square unit

17a $f(x) = (x+1)(x-1)(x-3)$ and $g(x) = (x+1)(x-1)$

$(x+1)(x-1)(x-3) \geq (x+1)(x-1)$

$(x+1)(x-1)(x-3) - (x+1)(x-1) \geq 0$

$(x+1)(x-1)[(x-3)-1] \geq 0$

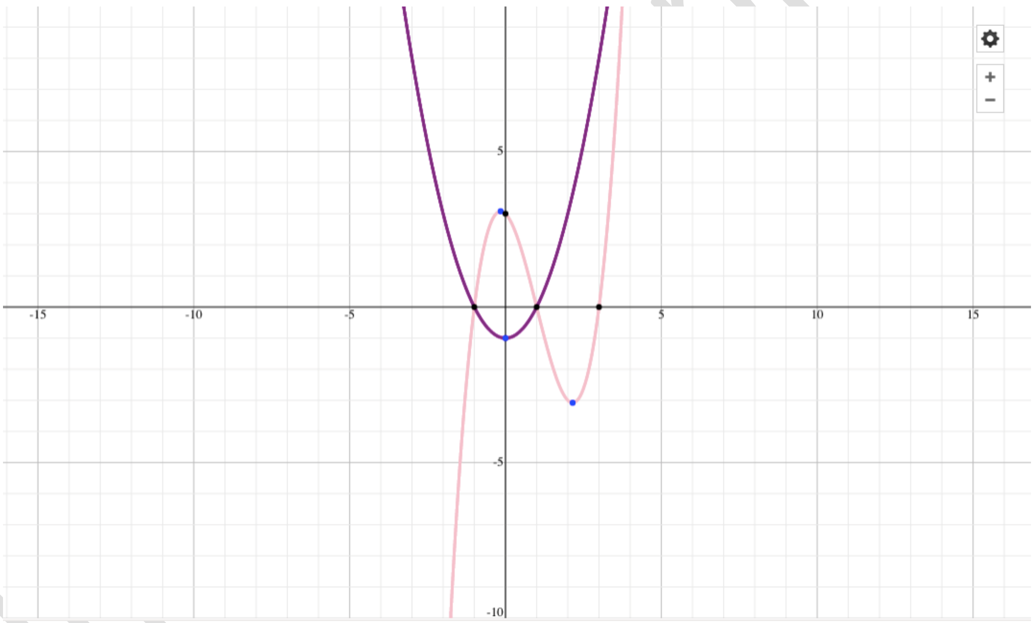
$(x+1)(x-1)[x-3-1] \geq 0$

$(x+1)(x-1)(x-4) \geq 0$

Hence, for $f(x) > g(x)$ the x value should be

$-1 < x < 1$ or $x > 4$

17b



From the graph, it is concluded that, the intersection points are at

$x = -1, 1$ and 4

Therefore, we must consider the enclosed regions for calculation:

$= \left|\int_{-1}^{1} (x+1)(x-1)(x-3) - (x+1)(x-1)\, dx\right|$

$+ \left|\int_{1}^{4} (x+1)(x-1)(x-3) - (x+1)(x-1)\, dx\right|$

$= \left|\int_{-1}^{1} (x^3 - 3x^2 - x + 3 - x^2 + 1)\, dx\right| + \left|\int_{1}^{4} (x^3 - 3x^2 - x + 3 - x^2 + 1)\, dx\right|$

$= \left|\int_{-1}^{1}(x^3 - 4x^2 - x + 4)\,dx\right| + \left|\int_{1}^{4}(x^3 - 4x^2 - x + 4)\,dx\right|$

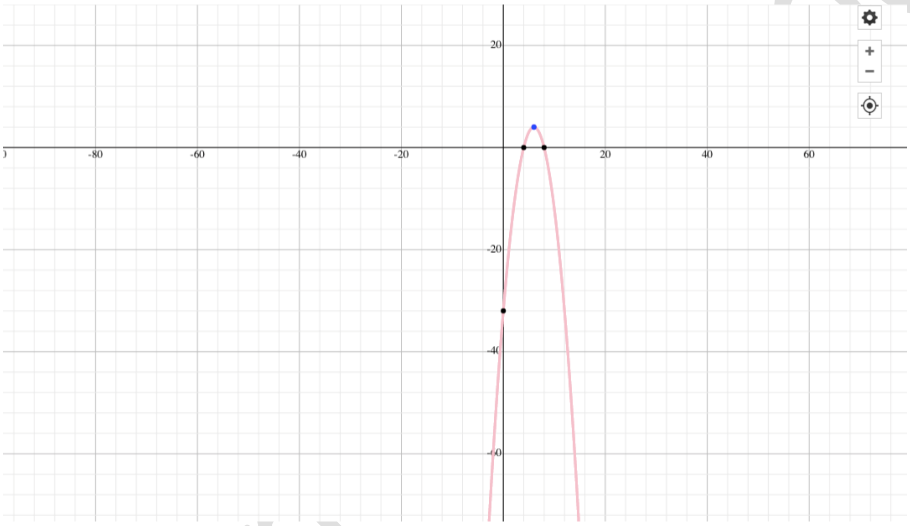
$= \left|\left[\frac{x^4}{4} - \frac{4x^3}{3} - \frac{x^2}{2} + 4x\right]_{-1}^{1}\right| + \left|\left[\frac{x^4}{4} - \frac{4x^3}{3} - \frac{x^2}{2} + 4x\right]_{1}^{4}\right|$

$= \frac{16}{3} + \left|-\frac{40}{3} - \frac{29}{12}\right|$

$= \dfrac{253}{12}$

$= 21\frac{1}{12}$ square units

18a



18b Note that $y' = 12 - 2x$. Hence when $x = 5, y = 3, y' = 2$.

Now the equation of a line is $y = mx + b$, substituting in $m = 2$ gives $y = 2x + b$. Since the line passes through $(5, 2)$ we have $3 = 2(5) + b$ and thus $b = 3 - 10 = -7$. Hence the equation of the tangent is $y = 2x - 7$.

18c The x-intercept occurs when $y = 0$, this is when $0 = 2x - 7$ and thus $x = \frac{7}{2}$. Hence $B = (\frac{7}{2}, 0)$. The curve intercepts the axis when $12x - 32 - x^2 = 0$

$x^2 - 12x + 32 = 0$

$(x - 8)(x - 4) = 0$

$x = 4, 8$

Hence, the x-intercept closest to the origin is $C = (4, 0)$.

Hence $A = (5, 3)$, $B = (\frac{7}{2}, 0)$, $C = (4, 0)$.

Thus, the area of the triangle is $A = \frac{1}{2}bh = \frac{1}{2}\left(4 - \frac{7}{2}\right)(3) = \frac{3}{4}$ square units.

19

Firstly, we find the point of intersection of the two curves, this will be when

$4(kx) = 4x - x^2$

$x^2 + 4kx - 4x = 0$

$x\big(x + 4(k - 1)\big) = 0$

$x = 0, 4(1 - k)$. Hence it follows that the area above the line $y = kx$ will be given by

$$A_1 = \int_0^{4(1-k)} \frac{1}{4}(4x - x^2) - kx\, dx$$

Whilst area on the other side of the line will be

$$A_2 = \int_0^{4(1-k)} kx\, dx + \int_{4(1-k)}^{4} \frac{1}{4}(4x - x^2)\, dx$$

In order for the line to bisect the area we must have

$A_1 = A_2$

$$\int_0^{4(1-k)} \frac{1}{4}(4x - x^2) - kx\, dx = \int_0^{4(1-k)} kx\, dx + \int_{4(1-k)}^{4} \frac{1}{4}(4x - x^2)\, dx$$

$$\int_0^{4(1-k)} \frac{1}{4}(4x - x^2) - kx\, dx - \int_0^{4(1-k)} kx\, dx = \int_{4(1-k)}^{4} \frac{1}{4}(4x - x^2)\, dx$$

$$\int_0^{4(1-k)} \frac{1}{4}(4x - x^2) - kx\, dx + \int_0^{4(1-k)} -kx\, dx = \int_{4(1-k)}^{4} \frac{1}{4}(4x - x^2)\, dx$$

$$\int_0^{4(1-k)} \frac{1}{4}(4x - x^2) - 2kx\,dx = \int_{4(1-k)}^{4} \frac{1}{4}(4x - x^2)\,dx$$

$$\left[\frac{1}{4}\left(2x^2 - \frac{x^3}{3}\right) - kx^2\right]_0^{4(1-k)} = \frac{1}{4}\left[2x^2 - \frac{x^3}{3}\right]_{4(1-k)}^{4}$$

$$4\left[\frac{1}{4}\left(2x^2 - \frac{x^3}{3}\right) - kx^2\right]_0^{4(1-k)} = \left[2x^2 - \frac{x^3}{3}\right]_{4(1-k)}^{4}$$

$$\left[\left(2x^2 - \frac{x^3}{3}\right) - 4kx^2\right]_0^{4(1-k)} = \left[2x^2 - \frac{x^3}{3}\right]_{4(1-k)}^{4}$$

$$\left[2x^2 - \frac{x^3}{3} - 4kx^2\right]_0^{4(1-k)} = \left[2x^2 - \frac{x^3}{3}\right]_{4(1-k)}^{4}$$

$$\left[2(1-2k)x^2 - \frac{x^3}{3}\right]_0^{4(1-k)} = \left[2x^2 - \frac{x^3}{3}\right]_{4(1-k)}^{4}$$

$$2(1-2k)[4(1-k)]^2 - \frac{[4(1-k)]^3}{3} = 32 - \frac{64}{3} - \left(2[4(1-k)]^2 - \frac{[4(1-k)]^3}{3}\right)$$

$$2(1-2k)[4(1-k)]^2 - \frac{[4(1-k)]^3}{3} = \frac{32}{3} - 2[4(1-k)]^2 + \frac{[4(1-k)]^3}{3}$$

$$0 = \frac{32}{3} - \big(2 + 2(1-2k)\big)[4(1-k)]^2 + \frac{2[4(1-k)]^3}{3}$$

$$0 = \frac{32}{3} - (4-4k)[4(1-k)]^2 + \frac{2[4(1-k)]^3}{3}$$

$$0 = \frac{32}{3} - 4(1-k)[4(1-k)]^2 + \frac{2[4(1-k)]^3}{3}$$

$$0 = \frac{32}{3} - [64(1-k)]^3 + \frac{2[4(1-k)]^3}{3}$$

$$0 = \frac{32}{3} - \frac{[4(1-k)]^3}{3}$$

$$[4(1-k)]^3 = 32$$

$$4(1-k) = \sqrt[3]{32}$$

$$1 - k = \frac{\sqrt[3]{32}}{4}$$

$$k = 1 - \frac{\sqrt[3]{32}}{4}$$

20

$$k = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$\int_a^b f(x) - k\,dx = \int_a^b f(x)\,dx - \int_a^b k\,dx$$

$$= \int_a^b f(x)\,dx - [kx]_a^b$$

$$= \int_a^b f(x)\,dx - (kb - ka)$$

$$= \int_a^b f(x)\,dx - (b-a)k$$

$$= \int_a^b f(x)\,dx - (b-a)\left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)$$

$$= \int_a^b f(x)\,dx - \int_a^b f(x)\,dx$$

$$= 0$$

Thus the total signed area between the curve and the line $y = k$ is equal to zero. This means that the area above the line $y = k$ must be exactly equal to the area below the line $y = k$.

Solutions to Exercise 5H

1a $\int_2^6 f(x)\,dx \doteqdot \frac{1}{2}(12+8)(6-2) = 40$

1b $\int_2^6 f(x)\,dx \doteqdot \frac{1}{2}(6.2+4.8)(6-2) = 22$

1c $\int_2^6 f(x)\,dx \doteqdot \frac{1}{2}(-4+(-9))(6-2) = -26$

2a $\int_2^{10} f(x)\,dx$

$$\doteqdot \frac{1}{2}(12+20)(6-2) + \frac{1}{2}(20+30)(6-2)$$

$$= 164$$

2b

$$\int_2^{10} f(x)\,dx$$

$$\doteqdot \frac{b-a}{2n}(f(a)+f(b)+2(f(x_1)+f(x_2)+\cdots))$$
$$= \frac{10-2}{2\times 2}\big(f(2)+f(10)+2f(6)\big)$$
$$= \frac{10-2}{4}(12+30+2\times 20)$$
$$= 164$$

3 $\int_{-5}^{5} f(x)$

$$\doteqdot \frac{1}{2}(2.4+2.6)\big(0-(-5)\big) + \frac{1}{2}(2.6+4.4)(5-0)$$

$$= 30$$

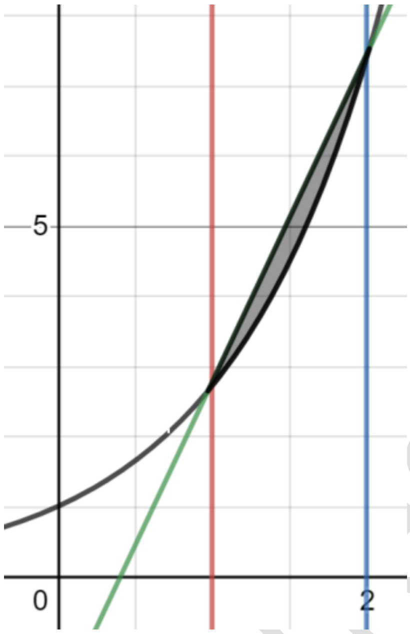
$$\int_{-5}^{5} f(x)\,dx$$

$$\doteqdot \frac{b-a}{2n}(f(a)+f(b)+2(f(x_1)+f(x_2)+\cdots))$$

$$= \frac{5-(-5)}{2\times 2}\big(f(-5)+f(5)+2f(0)\big)$$
$$= \frac{10}{4}(2.4+4.4+2\times 2.6)$$
$$= 30$$

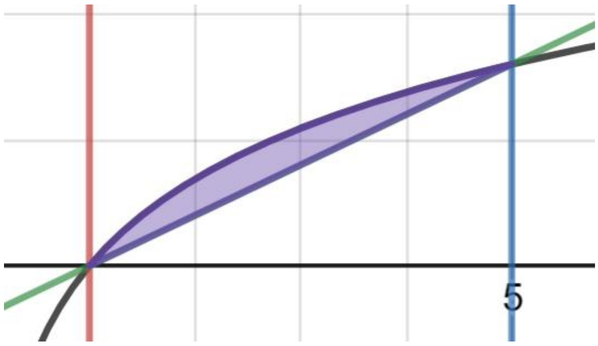
4a The curve is concave up, so the chord is above the curve, and the area under the chord will be greater than the area under the curve.

The diagram below shades the area that is overestimated in a typical concave up curve.



4b The curve is concave down, so the chord is underneath the curve, and the area under the chord will be less than the area under the curve.

The diagram below shades the area that is underestimated in a typical concave down curve.



5a

x	0	1	2	3	4
y	0	3	4	3	0

5b

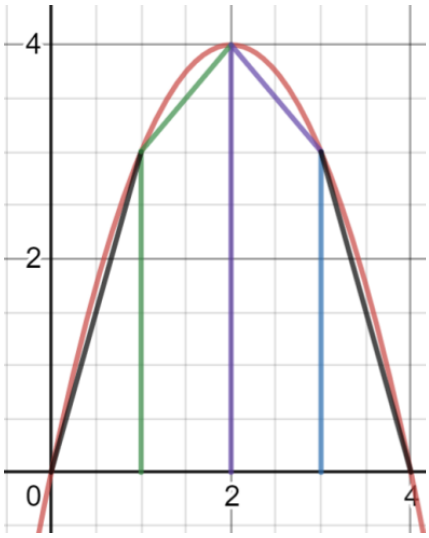
$$\int_0^4 f(x)\,dx$$
$$\doteqdot \frac{b-a}{2n}(f(a)+f(b)+2(f(x_1)+f(x_2)+\cdots))$$
$$= \frac{4-0}{2\times 4}\big(f(0)+f(4)+2\big(f(1)+f(2)+f(3)\big)\big)$$
$$= \frac{4}{8}\big(0+0+2(3+4+3)\big)$$
$$= 10$$

5c

$$\int_0^4 x(4-x)\,dx$$
$$= \int_0^4 (4x-x^2)\,dx$$
$$= \left[2x^2-\frac{x^3}{3}\right]_0^4$$
$$= \left(2\times 4^2-\frac{4^3}{3}\right)-\left(2\times 0^2-\frac{0^3}{3}\right)$$
$$= 32-\frac{64}{3}-0+0$$

$= 10\frac{2}{3}$

The curve is concave down and hence the trapezoidal estimate will not cover the entire region of the curve. This is shown in the diagram below



5d The error is $10\frac{2}{3} - 10 = \frac{2}{3}$, hence the percentage error is $\frac{2}{3} \div 10\frac{2}{3} = 6\frac{1}{4}\%$

6a

x	1	2	3	4	5
y	6	3	2	$\frac{3}{2}$	$\frac{6}{5}$

6b

$$\int_1^5 f(x)\,dx$$

$$\doteqdot \frac{b-a}{2n}(f(a) + f(b) + 2(f(x_1) + f(x_2) + \cdots))$$

$$= \frac{5-1}{2 \times 4}\big(f(1) + f(5) + 2(f(2) + f(3) + f(4))\big)$$

$$= \frac{4}{8}\left(6 + \frac{6}{5} + 2\left(3 + 2 + \frac{3}{2}\right)\right)$$

$$= 10\frac{1}{10}$$

6c

$$y = \frac{6}{x} = 6x^{-1}$$

$$\frac{dy}{dx} = -6x^{-2}$$

$$\frac{d^2y}{dx^2} = 12x^{-3} = \frac{12}{x^3}$$

Thus, for all $x > 0, \frac{d^2y}{dx^2} > 0$ so the curve is concave up over the entire region in which we are using the trapezoidal rule. This in turn means that we will overestimate the area of the curve.

7a

x	9	10	11	12	13	14	15
y	3	3.1627	3.3166	3.4641	3.6056	3.7416	3.8730

7b

$$\int_9^{16} f(x)\,dx$$

$$\doteqdot \frac{b-a}{2n}\left(f(a) + f(b) + 2(f(x_1) + f(x_2) + \cdots)\right)$$
$$= \frac{16-9}{2\times 8}\left(f(9) + f(15) + 2\left(f(10) + f(11) + f(12) + f(13) + f(14)\right)\right)$$
$$= \frac{7}{16}\left(3 + 3.8730 + 2(3.1627 + 3.3166 + 3.4641 + 3.6056 + 3.7416)\right)$$
$$\doteqdot 24.7$$

7c

$$\int_9^{16} \sqrt{x}\,dx$$

$$= \int_9^{16} x^{\frac{1}{2}}\,dx$$
$$= \frac{2}{3}\left[x^{\frac{3}{2}}\right]_9^{16}$$
$$= \frac{2}{3}\left[16^{\frac{3}{2}} - 9^{\frac{3}{2}}\right]$$

$= \frac{2}{3}[64 - 27]$
$= \frac{2}{3}[37]$
$= 24.67$

Now, if $y = \sqrt{x} = x^{\frac{1}{2}}$, then $\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$ and $\frac{d^2y}{dx^2} = -\frac{1}{4}x^{-\frac{3}{2}} = -\frac{1}{4x\sqrt{x}}$. This means that for all $x > 0$, $\frac{d^2y}{dx^2} < 0$ and hence the curve will be concave down over the region which we are using the trapezoidal rule to approximate area. This in turn means that the area will be under approximated.

8a

$$\int_0^1 2^{-x}\,dx$$

$\doteqdot \frac{b-a}{2n}(f(a) + f(b) + 2(f(x_1) + f(x_2) + \cdots))$
$= \frac{1-0}{2\times 2}\big(f(0) + f(1) + 2\big(f(0.5)\big)\big)$
$= \frac{1}{4}\left(1 + \frac{1}{2} + 2 \times \frac{1}{\sqrt{2}}\right)$
$\doteqdot 0.73$ (to two significant figures)

8b

$$\int_{-2}^0 2^{-x}\,dx$$

$\doteqdot \frac{b-a}{2n}(f(a) + f(b) + 2(f(x_1) + f(x_2) + \cdots))$
$= \frac{0-(-2)}{2\times 2}\big(f(-2) + f(0) + 2\big(f(-1)\big)\big)$
$= \frac{1}{2}(4 + 1 + 2 \times 2)$
$\doteqdot 4.5$ (to two significant figures)

8c

$$\int_1^3 \sqrt[3]{9-2x}\,dx$$

$$\doteqdot \frac{b-a}{2n}(f(a)+f(b)+2(f(x_1)+f(x_2)+\cdots))$$
$$=\frac{3-1}{2\times 2}\big(f(1)+f(3)+2\big(f(2)\big)\big)$$
$$=\frac{1}{2}(\sqrt[3]{7}+\sqrt[3]{3}+2\times\sqrt[3]{5})$$
$$\doteqdot 3.4 \text{ (to two significant figures)}$$

8d

$$\int_{-13}^{-1} \sqrt{3-x}\,dx$$

$$\doteqdot \frac{b-a}{2n}(f(a)+f(b)+2(f(x_1)+f(x_2)+\cdots))$$
$$=\frac{-1-(-13)}{2\times 2}(f(-13)+f(-1)+2\big(f(-7)\big)$$
$$=3\big(\sqrt{16}+\sqrt{4}+2\times\sqrt{10}\big)$$
$$\doteqdot 37 \text{ (to two significant figures)}$$

9a

$$\int_2^6 \frac{1}{x}\,dx$$

$$\doteqdot \frac{b-a}{2n}(f(a)+f(b)+2(f(x_1)+f(x_2)+\cdots))$$
$$=\frac{6-2}{2\times 4}(f(2)+f(6)+2(f(3)+f(4)+f(5)))$$
$$=\frac{4}{8}\left(\frac{1}{2}+\frac{1}{6}+2\left(\frac{1}{3}+\frac{1}{4}+\frac{1}{5}\right)\right)$$
$$\doteqdot 1.12 \text{ (to three significant figures)}$$

9b

$$\int_0^2 \frac{1}{2+\sqrt{x}}\,dx$$

$$\begin{aligned}
&\doteqdot \frac{b-a}{2n}\left(f(a)+f(b)+2(f(x_1)+f(x_2)+\cdots)\right)\\
&= \frac{2-0}{2\times 4}\left(f(0)+f(2)+2\left(f(0.5)+f(1)+f(1.5)\right)\right)\\
&= \frac{1}{4}\left(\frac{1}{2+\sqrt{0}}+\frac{1}{2+\sqrt{2}}+2\left(\frac{1}{2+\sqrt{0.5}}+\frac{1}{2+\sqrt{1}}+\frac{1}{2+\sqrt{1.5}}\right)\right)\\
&\doteqdot 0.705 \text{ (to three significant figures)}
\end{aligned}$$

9c

$$\int_4^8 \sqrt{x^2-3}\,dx$$

$$\begin{aligned}
&\doteqdot \frac{b-a}{2n}\left(f(a)+f(b)+2(f(x_1)+f(x_2)+\cdots)\right)\\
&= \frac{8-4}{2\times 4}\left(f(4)+f(8)+2\left(f(5)+f(6)+f(7)\right)\right)\\
&= \frac{1}{2}\left(\sqrt{4^2-3}+\sqrt{8^2-3}+2\left(\sqrt{5^2-3}+\sqrt{6^2-3}+\sqrt{7^2-3}\right)\right)\\
&\doteqdot 22.9 \text{ (to three significant figures)}
\end{aligned}$$

9d

$$\int_1^2 \log_{10} x\,dx$$

$$\begin{aligned}
&\doteqdot \frac{b-a}{2n}\left(f(a)+f(b)+2(f(x_1)+f(x_2)+\cdots)\right)\\
&= \frac{2-1}{2\times 4}\left(f(1)+f(2)+2\left(f(1.25)+f(1.5)+f(1.75)\right)\right)\\
&= \frac{1}{8}\left(\log_{10} 1+\log_{10} 2+2(\log_{10} 1.25+\log_{10} 1.5+\log_{10} 1.75)\right)\\
&\doteqdot 0.167 \text{ (to three significant figures)}
\end{aligned}$$

10 Distance travelled

$$= \int_0^5 v\, dt$$

$$\doteqdot \frac{b-a}{2n}\left(f(a) + f(b) + 2(f(x_1) + f(x_2) + \cdots)\right)$$
$$= \frac{5-0}{2 \times 5}\left(f(0) + f(5) + 2(f(1) + f(2) + f(3) + f(4))\right)$$
$$= \frac{1}{2}(1.5 + 2.7 + 2(1.3 + 1.4 + 2.0 + 2.4))$$
$$= 9.2 \text{ metres}$$

11 Surface area of water

$$= \int_0^{40} f(x)\; dx$$
$$\doteqdot \frac{b-a}{2n}\left(f(a) + f(b) + 2(f(x_1) + f(x_2) + \cdots)\right)$$
$$= \frac{40-0}{2 \times 4}\left(f(0) + f(40) + 2\left(f(10) + f(20) + f(30)\right)\right)$$
$$= \frac{40}{8}\left(0 + 0 + 2(20 + 18 + 17)\right)$$
$$= 550 \text{ m}^2$$

12 Area of vertical rock cutting

$$= \int_0^{300} f(x)\, dx$$

$$\doteqdot \frac{b-a}{2n}\left(f(a) + f(b) + 2(f(x_1) + f(x_2) + \cdots)\right)$$
$$= \frac{300-0}{2 \times 6}\left(f(0) + f(300) + 2(f(50) + f(100) + f(150) + f(200) + f(250))\right)$$
$$= 25(5 + 3 + 2(10 + 13 + 14 + 11 + 7))$$
$$= 2950 \text{ m}^2$$

13a

$$\int_0^1 \sqrt{1-x^2}\,dx$$

$$\doteqdot \frac{b-a}{2n}(f(a)+f(b)+2(f(x_1)+f(x_2)+\cdots))$$
$$= \frac{1-0}{2\times 4}\big(f(0)+f(1)+2(f(0.25)+f(0.5)+f(0.75))\big)$$
$$= \frac{1}{8}\left(\sqrt{1-0^2}+\sqrt{1-1^2}+2(\sqrt{1-0.25^2}+\sqrt{1-0.5^2}+\sqrt{1-0.75^2})\right)$$
$$\doteqdot 0.7489 \text{ (to four decimal places)}$$

13b $\int_0^1 \sqrt{1-x^2}$ is the area of $\frac{1}{4}$ of a circle with radius 1 unit (the right half of the semi-circle in the question). Hence $\int_0^1 \sqrt{1-x^2} = \frac{1}{4}(\pi \times 1^2) = \frac{\pi}{4}$.

Thus $\frac{\pi}{4} \doteqdot 0.7489$ so $\pi \doteqdot 4 \times 0.7489 \doteqdot 3.0$. Hence, the approximation is less than the integral, because the curve is concave down.

14

$$\int_1^5 \ln x\,dx$$

$$\doteqdot \frac{b-a}{2n}(f(a)+f(b)+2(f(x_1)+f(x_2)+\cdots))$$
$$= \frac{5-1}{2\times 4}\big(f(1)+f(5)+2(f(2)+f(3)+f(4))\big)$$
$$= \frac{1}{2}(\ln 1+\ln 5+2(\ln 2+\ln 3+\ln 4))$$
$$= \frac{1}{2}(\ln 1+\ln 5+2\ln 2+2\ln 3+2\ln 4)$$
$$= \frac{1}{2}(0+\ln 5+\ln 2^2+\ln 3^2+\ln 4^2)$$
$$= \frac{1}{2}\ln(5\times 2^2\times 3^2\times 4^2)$$
$$= \frac{1}{2}\ln 2880$$
$$= \ln 2880^{\frac{1}{2}}$$
$$= \ln\sqrt{2880}$$
$$= \ln 53.6656\ldots$$
$$\doteqdot \ln 54$$

15a $y = \sqrt{x} = x^{\frac{1}{2}}, y' = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$. For all $x > 0, \frac{1}{2\sqrt{x}} > 0$ and hence $y' > 0$ so the function is increasing for all $x > 0$.

15b Since the function is monotonically increasing, rectangles of width 1, with their right corners lying on the curve will overestimate the area underneath the curve

The area underneath the cure is

$$\int_0^n \sqrt{x}\,dx = \int_0^n x^{\frac{1}{2}}\,dx = \left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^n = \frac{2}{3}n^{\frac{3}{2}} = \frac{2n\sqrt{n}}{3}$$

And the total area under the rectangles is

$$1 \times \sqrt{1} + 1 \times \sqrt{2} + \cdots + 1 \times \sqrt{n} = \sqrt{1} + \sqrt{2} + \cdots + \sqrt{n}$$

Thus, it follows that

$$\sqrt{1} + \sqrt{2} + \cdots + \sqrt{n} \geq \int_0^n \sqrt{x}\,dx = \frac{2n\sqrt{n}}{3}$$

15c Since the curve is concave down, the trapezoidal rule will always underestimate the area under the curve, hence it follows that

$$\frac{b-a}{2n}(f(0) + f(n) + 2(f(1) + f(2) + \cdots + f(n-1) \leq \int_0^n \sqrt{x}\,dx = \frac{2n\sqrt{n}}{3}$$

$$\frac{n-0}{2n}\left(\sqrt{0} + \sqrt{n} + 2\left(\sqrt{1} + \sqrt{2} + \cdots + \sqrt{n-1}\right)\right) \leq \frac{2n\sqrt{n}}{3}$$

$$\sqrt{0} + \sqrt{n} + 2\left(\sqrt{1} + \sqrt{2} + \cdots + \sqrt{n-1}\right) \leq \frac{4n\sqrt{n}}{3}$$

$$2\left(\sqrt{1} + \sqrt{2} + \cdots + \sqrt{n-1}\right) \leq \frac{4n\sqrt{n}}{3} - \sqrt{n}$$

$$2\left(\sqrt{1} + \sqrt{2} + \cdots + \sqrt{n-1} + \sqrt{n}\right) \leq \frac{4n\sqrt{n}}{3} + \sqrt{n}$$

$$2\left(\sqrt{1} + \sqrt{2} + \cdots + \sqrt{n-1} + \sqrt{n}\right) \leq \frac{4n\sqrt{n}}{3} + \frac{3\sqrt{n}}{3}$$

$$2\left(\sqrt{1} + \sqrt{2} + \cdots + \sqrt{n-1} + \sqrt{n}\right) \leq \frac{\sqrt{n}(4n+3)}{3}$$

$$\left(\sqrt{1} + \sqrt{2} + \cdots + \sqrt{n-1} + \sqrt{n}\right) \leq \frac{\sqrt{n}(4n+3)}{6}$$

15d From parts b and c we have that

$$\frac{2(12000)\sqrt{12000}}{3} \le \sqrt{1}+\sqrt{2}+\sqrt{3}+\cdots+\sqrt{12000} \le \frac{\sqrt{12000}(4(12000)+3)}{6}$$

Thus

$$876356.092008 \le \sqrt{1}+\sqrt{2}+\sqrt{3}+\cdots+\sqrt{12000} \le 876410.86426$$

So to the nearest 100 it must be the case that $\sqrt{1}+\sqrt{2}+\sqrt{3}+\cdots+\sqrt{12000} \doteqdot 876400$

16a 5 subintervals: 2.66551

16b 10 subintervals: 2.47442

16c 20 subintervals: 2.41809

17 Investigation question – answers will vary

Solutions to Exercise 5I

Let C be a constant.

1a

$$\frac{d}{dx}(2x+3)^4$$

$$= 4(2x+3)^3 \times 2$$

$$= 8(2x+3)^3$$

1b i

$$\int 8(2x+3)^3\,dx$$

$$= \int \frac{d}{dx}(2x+3)^4\,dx$$

$$= (2x+3)^4 + C$$

1b ii

$$\int 16(2x+3)^3\,dx$$

$$= 2\int 8(2x+3)^3\,dx$$

$$= 2\int \frac{d}{dx}(2x+3)^4\,dx$$

$$= 2(2x+3)^4 + C$$

2a

$$\frac{d}{dx}(3x-5)^3$$

$$= 3(3x-5)^2 \times 3$$

$$= 9(3x-5)^2$$

2b i

$$\int 9(3x-5)^2\,dx$$

$$= \int \frac{d}{dx}(3x-5)^3\,dx$$

$$= (3x-5)^3 + C$$

2b ii

$$\int 27(3x-5)^2\,dx$$

$$= 3\int 9(3x-5)^2\,dx$$

$$= 3\int \frac{d}{dx}(3x-5)^3\,dx$$

$$= 3(3x-5)^3 + C$$

3a

$$\frac{d}{dx}(1+4x)^5$$

$$= 5(1+4x)^4 \times 4$$

$$= 20(1+4x)^4$$

3b i

$$\int 20(1+4x)^4\,dx$$

$$= \int \frac{d}{dx}(1+4x)^5\,dx$$

$$= (1+4x)^5 + C$$

3b ii

$$\int 10(1+4x)^4\,dx$$

$$=\frac{1}{2}\int 20(1+4x)^4\,dx$$

$$=\frac{1}{2}\int \frac{d}{dx}(1+4x)^5\,dx$$

$$=\frac{1}{2}(1+4x)^5+C$$

4a

$$\frac{d}{dx}(1-2x)^4$$

$$=4(1-2x)^3\times -2$$

$$=-8(1-2x)^3$$

4b i

$$\int -8(1-2x)^3\,dx$$

$$=\int \frac{d}{dx}(1-2x)^4\,dx$$

$$=(1-2x)^4+C$$

4b ii

$$\int -2(1-2x)^3\,dx$$

$$=\frac{1}{4}\int -8(1-2x)^3\,dx$$

$$=\frac{1}{4}\int \frac{d}{dx}(1-2x)^4\,dx$$

$$=\frac{1}{4}(1-2x)^4+C$$

5a

$$\frac{d}{dx}(4x+3)^{-1}$$

$$= -1(4x+3)^{-2} \times 4$$

$$= -4(4x+3)^{-2}$$

5b i

$$\int -4(4x+3)^{-2}\,dx$$

$$= \int \frac{d}{dx}(4x+3)^{-1}\,dx$$

$$= (4x+3)^{-1} + C$$

5b ii

$$\int (4x+3)^{-2}\,dx$$

$$= -\frac{1}{4}\int -4(4x+3)^{-2}\,dx$$

$$= -\frac{1}{4}(4x+3)^{-1} + C$$

6a

$$\frac{d}{dx}(2x-5)^{\frac{1}{2}}$$

$$= \frac{1}{2}(2x-5)^{-\frac{1}{2}} \times 2$$

$$= (2x-5)^{-\frac{1}{2}}$$

6b i

$$\int (2x-5)^{-\frac{1}{2}}\,dx$$

$$= \int \frac{d}{dx}(2x-5)^{\frac{1}{2}}\,dx$$

$$= (2x-5)^{\frac{1}{2}} + C$$

6b ii

$$\int \frac{1}{3}(2x-5)^{-\frac{1}{2}}\,dx$$

$$= \frac{1}{3}\int (2x-5)^{-\frac{1}{2}}\,dx$$

$$= \frac{1}{3}\int \frac{d}{dx}(2x-5)^{\frac{1}{2}}\,dx$$

$$= \frac{1}{3}(2x-5)^{\frac{1}{2}} + C$$

7a

$$\frac{d}{dx}(x^2+3)^4$$

$$= 4(x^2+3)^3 \times 2x$$

$$= 8x(x^2+3)^3$$

7b i

$$\int 8x(x^2+3)^3\,dx$$

$$= \int \frac{d}{dx}(x^2+3)^4\,dx$$

$$= (x^2+3)^4 + C$$

7b ii

$$\int 40x(x^2+3)^3\,dx$$

$$= 5\int 8x(x^2+3)^3\,dx$$

$$= 5\int \frac{d}{dx}(x^2+3)^4\,dx$$

$$= 5(x^2+3)^4 + C$$

8a

$$\frac{d}{dx}(x^3-1)^5$$

$$= 5(x^3-1)^4 \times 3x^2$$

$$= 15x^2(x^3-1)^4$$

8b i

$$\int 15x^2(x^3-1)^4\,dx$$

$$= \int \frac{d}{dx}(x^3-1)^5\,dx$$

$$= (x^3-1)^5 + C$$

8b ii

$$\int 3x^2(x^3-1)^4\,dx$$

$$= \frac{1}{5}\int 15x^2(x^3-1)^4\,dx$$

$$= \frac{1}{5}\int \frac{d}{dx}(x^3-1)^5\,dx$$

$$= \frac{1}{5}(x^3-1)^5 + C$$

9a

$$\frac{d}{dx}\sqrt{2x^2+3}$$

$$=\frac{d}{dx}(2x^2+3)^{\frac{1}{2}}$$

$$=\frac{1}{2}(2x^2+3)^{-\frac{1}{2}}\times 4x$$

$$=2x(2x^2+3)^{-\frac{1}{2}}$$

$$=\frac{2x}{\sqrt{2x^2+3}}$$

9b i

$$\int\frac{2x}{\sqrt{2x^2+3}}dx$$

$$=\int\frac{d}{dx}\sqrt{2x^2+3}\,dx$$

$$=\sqrt{2x^2+3}+C$$

9b ii

$$\int\frac{x}{\sqrt{2x^2+3}}dx$$

$$=\frac{1}{2}\int\frac{2x}{\sqrt{2x^2+3}}dx$$

$$=\frac{1}{2}\int\frac{d}{dx}\sqrt{2x^2+3}\,dx$$

$$=\frac{1}{2}\sqrt{2x^2+3}+C$$

10a

$$\frac{d}{dx}\left(\sqrt{x}+1\right)^3$$

$$= 3\left(\sqrt{x}+1\right)^2 \times \frac{1}{2}x^{-\frac{1}{2}}$$

$$= \frac{3\left(\sqrt{x}+1\right)^2}{2\sqrt{x}}$$

10b i

$$\int \frac{3\left(\sqrt{x}+1\right)^2}{2\sqrt{x}}$$

$$= \int \frac{d}{dx}\left(\sqrt{x}+1\right)^3 dx$$

$$= \left(\sqrt{x}+1\right)^3 + C$$

10b ii

$$\int \frac{\left(\sqrt{x}+1\right)^2}{\sqrt{x}}$$

$$= \frac{2}{3}\int \frac{3\left(\sqrt{x}+1\right)^2}{2\sqrt{x}}$$

$$= \frac{2}{3}\int \frac{d}{dx}\left(\sqrt{x}+1\right)^3 dx$$

$$= \frac{2}{3}\left(\sqrt{x}+1\right)^3 + C$$

11a

$$\frac{d}{dx}(x^3+3x^2+5)^4$$
$$= 4(x^3+3x^2+5)^3 \times (3x^2+6x)$$
$$= 12(x^2+2x)(x^3+3x^2+5)^3$$

11b i

$$\int 12(x^2+2x)(x^3+3x^2+5)^3\,dx$$
$$=\int \frac{d}{dx}(x^3+3x^2+5)^4\,dx$$
$$=(x^3+3x^2+5)^4+C$$

11b ii

$$\int (x^2+2x)(x^3+3x^2+5)^3\,dx$$
$$=\frac{1}{12}\int 12(x^2+2x)(x^3+3x^2+5)^3\,dx$$
$$=\frac{1}{12}\int \frac{d}{dx}(x^3+3x^2+5)^4\,dx$$
$$=\frac{1}{12}(x^3+3x^2+5)^4+C$$

12a

$$\frac{d}{dx}(5-x^2-x)^7$$
$$=7(5-x^2-x)^6\times(-2x-1)$$
$$=-7(2x+1)(5-x^2-x)^6$$

12b i

$$\int(-14x-7)(5-x^2-x)^6$$
$$=\int -7(2x+1)(5-x^2-x)^6$$
$$=\int \frac{d}{dx}(5-x^2-x)^7\,dx$$
$$=(5-x^2-x)^7+C$$

12b ii

$$\int (2x+1)(5-x^2-x)^6\,dx$$

$$= -\frac{1}{7}\int -7(2x+1)(5-x^2-x)^6\,dx$$

$$= -\frac{1}{7}\int \frac{d}{dx}(5-x^2-x)^7\,dx$$

$$= -\frac{1}{7}(5-x^2-x)^7 + C$$

13a Let $u = 5x+4$, then $\frac{du}{dx} = 5$ so $du = 5\,dx$

$$\int 5(5x+4)^3\,dx$$

$$= \int (5x+4)^3\,5\,dx$$

$$= \int u^3\,du$$

$$= \frac{u^4}{4} + C$$

$$= \frac{(5x+4)^4}{4} + C$$

13b Let $u = 1-3x$, then $\frac{du}{dx} = -3$ so $du = -3\,dx$

$$\int -3(1-3x)^5\,dx$$

$$= \int (1-3x)^5\,(-3\,dx)$$

$$= \int u^5\,du$$

$$= \frac{u^6}{6} + C$$

$$= \frac{(1-3x)^6}{6} + C$$

13c Let $u = x^2 - 5$, then $\frac{du}{dx} = 2x$ so $du = 2x\,dx$

$$\int 2x(x^2 - 5)^7\,dx$$
$$= \int (x^2 - 5)^7\,(2x\,dx)$$
$$= \int u^7\,du$$
$$= \frac{u^8}{8} + C$$
$$= \frac{(x^2 - 5)^8}{8} + C$$

13d Let $u = x^3 + 7$, then $\frac{du}{dx} = 3x^2$ so $du = 3x^2\,dx$

$$\int 3x^2(x^3 + 7)^4\,dx$$
$$= \int (x^3 + 7)^4\,(3x^2\,dx)$$
$$= \int u^4\,du$$
$$= \frac{u^5}{5} + C$$
$$= \frac{(x^3 + 7)^5}{5} + C$$

13e Let $u = 3x^2 + 2$, then $\frac{du}{dx} = 6x$ so $du = 6x\,dx$

$$\int \frac{6x}{(3x^2 + 2)^2}\,dx$$
$$= \int \frac{1}{(3x^2 + 2)^2}\,(6x\,dx)$$
$$= \int \frac{1}{u^2}\,du$$
$$= \int u^{-2}\,du$$
$$= -u^{-1} + C$$
$$= -\frac{1}{u} + C$$
$$= -\frac{1}{3x^2 + 2} + C$$

13f Let $u = 9 - 2x^3$, then $\frac{du}{dx} = -6x^2$ so $du = -6x^2\,dx$

$$\int \frac{-6x^2}{\sqrt{9-2x^3}}\,dx$$

$$= \int \frac{1}{\sqrt{9-2x^3}}\,(-6x^2\,dx)$$

$$= \int \frac{1}{\sqrt{u}}\,du$$

$$= \int u^{-\frac{1}{2}}\,du$$

$$= 2u^{\frac{1}{2}} + C$$

$$= 2\sqrt{u} + C$$

$$= 2\sqrt{9-2x^3} + C$$

14a Let $u = 5x^2 + 3$, then $\frac{du}{dx} = 10x$ so $du = 10x\,dx$

$$\int 10x(5x^2+3)^2\,dx$$

$$= \int (5x^2+3)^2\,(10x\,dx)$$

$$= \int u^2\,du$$

$$= \frac{u^3}{3} + C$$

$$= \frac{(5x^2+3)^3}{3} + C$$

14b Let $u = x^2 + 1$, then $\frac{du}{dx} = 2x$ so $du = 2x\,dx$

$$\int 2x(x^2+1)^3\,dx$$

$$= \int (x^2+1)^3\,(2x\,dx)$$

$$= \int u^3\,du$$

$$= \frac{u^4}{4} + C$$

$$= \frac{(x^2+1)^4}{4} + C$$

14c Let $u = 1 + 4x^3$, then $\frac{du}{dx} = 12x^2$ so $du = 12x^2\ dx$

$$\int 12x^2(1+4x^3)^5\ dx$$

$$= \int (1+4x^3)^5\ (12x^2\ dx)$$

$$= \int u^5\ du$$

$$= \frac{u^6}{6} + c$$

$$= \frac{(1+4x^3)^6}{6} + C$$

14d Let $u = 1 + 3x^2$, then $\frac{du}{dx} = 6x$ so $du = 6x\ dx$ and $\frac{1}{6}du = x\ dx$

$$\int x(1+3x^2)^4\ dx$$

$$= \int (1+3x^2)^4 \times x\ dx$$

$$= \int u^4 \times \frac{1}{6} du$$

$$= \frac{1}{6}\int u^4\ du$$

$$= \frac{\frac{1}{6}u^5}{5} + C$$

$$= \frac{u^5}{30} + C$$

$$= \frac{(1+3x^2)^5}{30} + C$$

14e Let $u = 1 - x^4$, then $\frac{du}{dx} = 4x^3$ so $du = 4x^3\,dx$ and $\frac{1}{4}du = x^3\,dx$

$$\int x^3(1-x^4)^7\,dx$$

$$= \int (1-x^4)^7 \times x^3 dx$$

$$= \int u^7 \times \frac{1}{4} du$$

$$= \frac{1}{4}\int u^7 du$$

$$= \frac{1}{4}\left(\frac{u^8}{8}\right) + C$$

$$= \frac{u^8}{32} + C$$

$$= \frac{(1-x^4)^8}{32} + C$$

14f Let $u = x^3 - 1$, then $\frac{du}{dx} = 3x^2$ so $du = 3x^2\,dx$

$$\int 3x^2\sqrt{x^3-1}\,dx$$

$$= \int \sqrt{x^3-1} \times 3x^2 dx$$

$$= \int \sqrt{u}\,du$$

$$= \int u^{\frac{1}{2}}\,du$$

$$= \frac{u^{\frac{3}{2}}}{\left(\frac{3}{2}\right)} + C$$

$$= \frac{2}{3}u^{\frac{3}{2}} + C$$

$$= \frac{2}{3}(x^3-1)^{\frac{3}{2}} + C$$

14g Let $u = 5x^2 - 1$, then $\frac{du}{dx} = 10x$ so $du = 10x\,dx$ and $\frac{1}{10}du = x\,dx$

$$\int x\sqrt{5x^2+1}\,dx$$

$$= \int \sqrt{5x^2+1} \times x\,dx$$

$$= \int \sqrt{u} \times \frac{1}{10}du$$

$$= \frac{1}{10}\int u^{\frac{1}{2}}\,du$$

$$= \frac{1}{10}\left(\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right) + C$$

$$= \frac{2}{30}u^{\frac{3}{2}} + C$$

$$= \frac{1}{15}(5x^2-1)^{\frac{3}{2}} + C$$

14h Let $u = x^2 + 3$, then $\frac{du}{dx} = 2x$ so $du = 2x\,dx$

$$\int \frac{2x}{\sqrt{x^2+3}}dx$$

$$= \int \frac{1}{\sqrt{x^2+3}} \times 2x\,dx$$

$$= \int \frac{1}{\sqrt{u}}\,du$$

$$= \int u^{-\frac{1}{2}}\,du$$

$$= 2u^{\frac{1}{2}} + C$$

$$= 2\sqrt{u} + C$$

$$= 2\sqrt{x^2+3} + C$$

14i Let $u = 4x^2 + 8x + 1$, then $\frac{du}{dx} = 8x + 8$ so $du = (8x + 8\,)dx$

and $\frac{1}{8}du = (x + 1)\,dx$

$$\int \frac{x+1}{\sqrt{4x^2+8x+1}}\,dx$$

$$= \int \frac{1}{\sqrt{4x^2+8x+1}} \times (x+1)\,dx$$

$$= \int \frac{1}{\sqrt{u}} \times \frac{1}{8}\,du$$

$$= \frac{1}{8}\int u^{-\frac{1}{2}}\,du$$

$$= \frac{1}{8}\left(\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right) + C$$

$$= \frac{1}{4}\sqrt{u} + C$$

$$= \frac{1}{4}\sqrt{4x^2+8x+1} + C$$

14j Let $u = x^2 + 5$, then $\frac{du}{dx} = 2x$ so $du = 2x\,dx$ and $\frac{1}{2}du = x\,dx$

$$\int \frac{x}{(x^2+5)^3}\,dx$$

$$= \int \frac{1}{(x^2+5)^3} \times x\,dx$$

$$= \int \frac{1}{u^3} \times \frac{1}{2}\,du$$

$$= \frac{1}{2}\int u^{-3}\,du$$

$$= \frac{1}{2} \times \frac{u^{-2}}{-2} + C$$

$$= -\frac{1}{4u^2} + C$$

$$= -\frac{1}{4(x^2+5)^2} + C$$

15a Let $u = x^3 + 1$, then $\frac{du}{dx} = 3x^2$ so $du = 3x^2\,dx$ and $\frac{1}{3}du = x^2\,dx$

When $x = 1, u = 2$ and when $x = -1, u = 0$.

$$\int_{-1}^{1} x^2(x^3+1)^4\,dx$$

$$= \int_{-1}^{1} (x^3+1)^4 \times x^2\,dx$$

$$= \int_{0}^{2} u^4 \times \frac{1}{3}du$$

$$= \frac{1}{3}\left[\frac{u^5}{5}\right]_0^2$$

$$= \frac{1}{3}\left(\frac{2^5}{5} - 0\right)$$

$$= \frac{32}{15}$$

15b Let $u = 5x^2 + 1$, then $\frac{du}{dx} = 10x$ so $du = 10x\,dx$ and $\frac{1}{10}du = x\,dx$

When $x = 1, u = 6$ and when $x = 0, u = 1$.

$$\int_{0}^{1} \frac{x}{(5x^2+1)^3}dx$$

$$= \int_{0}^{1} \frac{1}{(5x^2+1)^3} \times x\,dx$$

$$= \int_{1}^{6} \frac{1}{u^3} \times \frac{1}{10}du$$

$$= \frac{1}{10}\int_{1}^{6} u^{-3}\,du$$

$$= \frac{1}{10}\left[\frac{u^{-2}}{-2}\right]_1^6$$

$$= \frac{1}{10}\left[\frac{1}{-2u^2}\right]_1^6$$

$$= \frac{1}{10}\left(\frac{1}{-2\times 6^2} - \frac{1}{-2\times 1^2}\right)$$

$$= \frac{1}{10}\left(-\frac{1}{72} + \frac{1}{2}\right)$$

$= \dfrac{7}{144}$

15c Let $u = 1 - 4x^2$, then $\frac{du}{dx} = -8x$ so $du = -8x\,dx$ and $-\frac{1}{8}du = x\,dx$

When $x = \frac{1}{2}, u = 0$ and when $x = 0, u = 1$.

$$\int_0^{\frac{1}{2}} x\sqrt{1-4x^2}\,dx$$

$$= \int_0^{\frac{1}{2}} \sqrt{1-4x^2} \times x\,dx$$

$$= \int_1^0 \sqrt{u} \times -\frac{1}{8}du$$

$$= -\frac{1}{8}\int_1^0 u^{\frac{1}{2}}\,du$$

$$= -\frac{1}{8}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_1^0$$

$$= -\frac{1}{8}\left(\frac{2}{3} \times 0^{\frac{3}{2}} - \frac{2}{3} \times 1^{\frac{3}{2}}\right)$$

$$= -\frac{1}{8}\left(0 - \frac{2}{3}\right)$$

$$= \frac{1}{12}$$

15d Let $u = x^2 + 10x + 3$, then $\frac{du}{dx} = 2x + 10$ so $du = (2x + 10)\,dx$

and $\frac{1}{2}du = (x + 5\,)dx$

When $x = -1, u = -6$ and when $x = -3, u = -18$.

$$\int_{-3}^{-1} (x^2 + 10x + 3)^2 \times (x + 5)\,dx$$

$$= \int_{-18}^{-6} u^2 \times \frac{1}{2}du$$

$$= \frac{1}{2}\left[\frac{u^3}{3}\right]_{-18}^{-6}$$

$$= \frac{1}{2}\left(\frac{(-6)^3}{3} - \frac{(-18)^3}{3}\right)$$

$$= \frac{1}{2}(-72 + 1944)$$

$$= 936$$

16a Let $u = 1 - \frac{1}{x}$, then $\frac{du}{dx} = \frac{1}{x^2}$ so $du = \frac{1}{x^2}dx$

$$\int \frac{\left(1 - \frac{1}{x}\right)^5}{x^2} dx$$

$$= \int \left(1 - \frac{1}{x}\right)^5 \times \frac{1}{x^2} dx$$

$$= \int u^5 \, du$$

$$= \frac{1}{6} \times u^6 + C$$

$$= \frac{1}{6} \times \left(1 - \frac{1}{x}\right)^6 + C$$

16b Let $u = 1 + \sqrt{x}$, then $\frac{du}{dx} = \frac{1}{2} \times \frac{1}{\sqrt{x}}$ so $2du = \frac{1}{\sqrt{x}}\, dx$

When $x = 4, u = 3$ and when $x = 1, u = 2$.

$$\int_1^4 \frac{1}{\sqrt{x}\left(1 + \sqrt{x}\right)^2}\, dx$$

$$= \int_1^4 \frac{1}{\left(1 + \sqrt{x}\right)^2} \times \frac{1}{\sqrt{x}}\, dx$$

$$= \int_2^3 \frac{1}{u^2} \times 2\, du$$

$$= 2\int_2^3 u^{-2}\, du$$

$$= 2[-u^{-1}]_2^3$$

$$= 2\left[-\frac{1}{u}\right]_2^3$$

$$= 2\left(-\frac{1}{3} + \frac{1}{2}\right)$$

$$= 2\left(\frac{1}{6}\right)$$

$$= \frac{1}{3}$$

17a As the square root function is only defined for non-zero numbers, function is defined for all x such that $x^2 - 1 \geq 0$ which is $(x-1)(x+1) \geq 0$ and hence all x such that $x \leq -1$ or $x \geq 1$. Thus the domain is $x \in (-\infty, -1] \cup [1, \infty)$.

17b $f'(x) = \frac{d}{dx}\left(x\sqrt{x^2-1}\right)$

$$= \frac{d}{dx}(x)\sqrt{x^2-1} + x\frac{d}{dx}\left((x^2-1)^{\frac{1}{2}}\right)$$

$$= \sqrt{x^2-1} + x \cdot \frac{1}{2} \cdot 2x(x^2-1)^{-\frac{1}{2}}$$

$$= \sqrt{x^2-1} + \frac{x^2}{\sqrt{x^2-1}}$$

$$= \frac{\left(\sqrt{x^2-1}\right)^2}{\sqrt{x^2-1}} + \frac{x^2}{\sqrt{x^2-1}}$$

$$= \frac{x^2-1}{\sqrt{x^2-1}} + \frac{x^2}{\sqrt{x^2-1}}$$

$$= \frac{2x^2-1}{\sqrt{x^2-1}}$$

Stationary points occur when $f'(x) = 0$, that is when

$$\frac{2x^2-1}{\sqrt{x^2-1}} = 0$$

$$2x^2 - 1 = 0$$

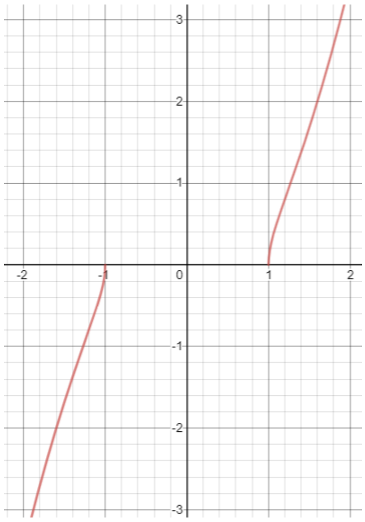
$$2x^2 = 1$$

$$x^2 = \frac{1}{2}$$

$$x = \pm\frac{1}{\sqrt{2}}$$

But this does not lie in the domain $x \in (-\infty, -1] \cup [1, \infty)$, hence there are no stationary points in the domain of the function.

17c $f(-x) = (-x)\sqrt{(-x)^2 - 1} = -x\sqrt{x^2 - 1} = -f(x)$. Hence the function is odd.



17d $A = \int_1^3 x\sqrt{x^2 - 1}\,dx$

Let $u = x^2 - 1$, $du = 2x\,dx$

When $x = 1, u = 0$

When $x = 3, u = 8$

Hence,

$$A = \int_0^8 \frac{1}{2}\sqrt{u}\,du = \int_0^8 \frac{1}{2}u^{\frac{1}{2}}\,du = \frac{1}{2}\frac{2}{3}\left[u^{\frac{3}{2}}\right]_0^8 = \frac{1}{3}\left[8^{\frac{3}{2}} - 0\right] = \frac{16}{3}\sqrt{2} \text{ square units}$$

18a
$$\begin{aligned} y' &= (7 - x^2)^3 + x(-2x)(3)(7 - x^2)^2 \\ &= (7 - x^2)^3 - 6x^2(7 - x^2)^2 \\ &= (7 - x^2)^2[(7 - x^2) - 6x^2] \\ &= (7 - x^2)^2[7 - 7x^2] \\ &= 7(7 - x^2)^2(1 - x^2) \end{aligned}$$

Stationary points occur when $y' = 0$, that is when

$7(7 - x^2)^2(1 - x^2) = 0$

$(7 - x^2) = 0$ or $(1 - x^2) = 0$

So the stationary points occur when $x = \pm 1, \pm\sqrt{7}$

Substituting this back into the equation for y gives the stationary points as $(-1, -216), (1, 216), (\pm\sqrt{7}, 0)$.

x	-10	$-\sqrt{7}$	-2	-1	0	1	2	$\sqrt{7}$	10
y	8043570	0	-54	-216	0	216	54	0	-8043570
y'	-5993757	0	-189	0	343	0	-189	0	-5993757
sign	$-$	0	$-$	0	$+$	0	$-$	0	$-$

From this we see that there are points of inflection at $(\pm\sqrt{7}, 0)$, a maximum at $(1, 216)$ and a minimum at $(-1, -216)$.

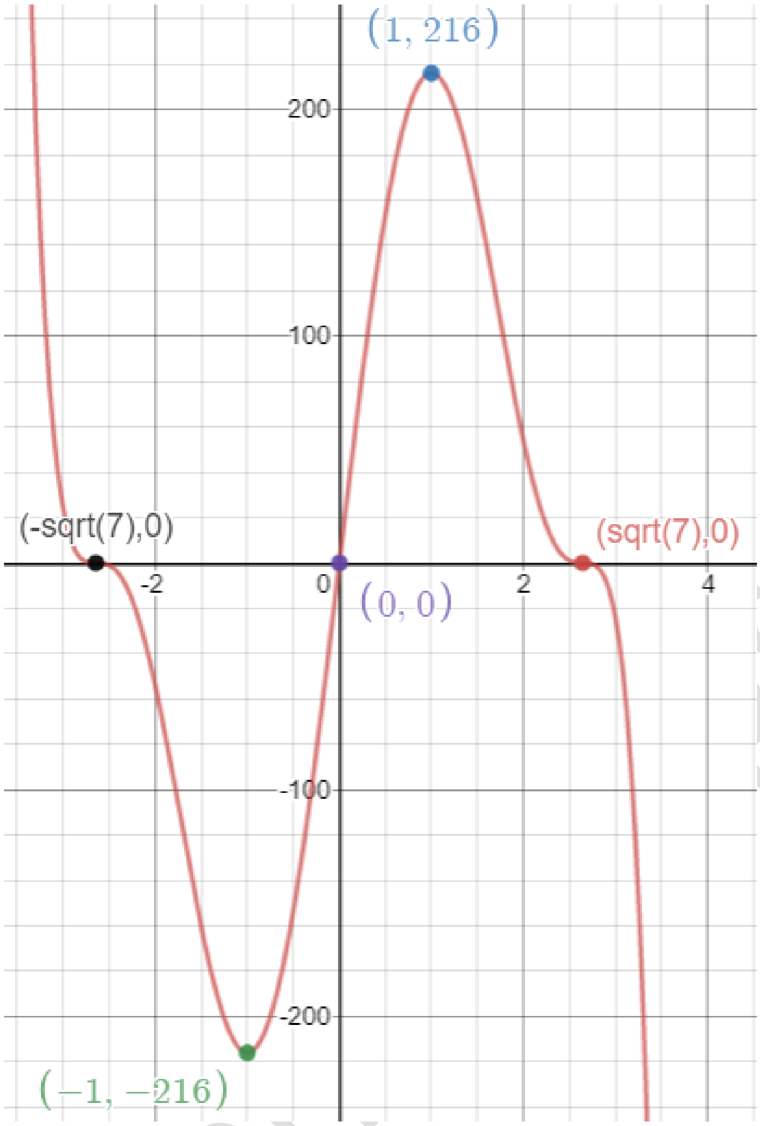
The x-axis interprets occur when $y = 0$

$x(7 - x^2)^3 = 0$

That is when $x = 0$, or $(7 - x^2)$ which is when $x = 0, \pm\sqrt{7}$

Substituting this back into the equation for y gives the points of intersection as as $(0,0)$, $(\pm\sqrt{7}, 0)$.

Hence drawing the graph gives



18b By observation, the total area enclosed by the graph and the x-axis will be double the area that is enclosed between the curve and the x-axis for $0 \leq x \leq \sqrt{7}$. Hence the total area enclosed is

$$A = 2\int_0^{\sqrt{7}} x(7 - x^2)^3\, dx = \int_0^{\sqrt{7}} 2x(7 - x^2)^3\, dx$$

Let $u = 7 - x^2$, it follows that $du = -2x\, dx$

When $x = 0, u = 7$

When $x = \sqrt{7}, u = 0$

Hence,

$$A = \int_7^0 -(u)^3\,du$$
$$= \int_0^7 u^3\,du$$
$$= \left[\frac{u^4}{4}\right]_0^7$$
$$= \frac{7^4}{4}$$
$$= 600\tfrac{1}{4} \text{ square units}$$

Solutions for Chapter review

1a

$$\int_0^1 3x^2\,dx$$

$$= [x^3]_0^1$$

$$= 1^3 - 0$$

$$= 1$$

1b

$$\int_1^2 x\,dx$$

$$= \left[\frac{x^2}{2}\right]_1^2$$

$$= \frac{2^2}{2} - \frac{1}{2}$$

$$= \frac{3}{2}$$

1c

$$\int_2^5 4x^3$$

$$= [x^4]_2^5$$

$$= 5^4 - 2^4$$

$$= 609$$

1d

$$\int_{-1}^{1} x^4\,dx$$

$$= \left[\frac{x^5}{5}\right]_{-1}^{1}$$

$$= \frac{1}{5} - \left(\frac{(-1)^5}{5}\right)$$

$$= \frac{1}{5} - \left(-\frac{1}{5}\right)$$

$$= \frac{1}{5} + \frac{1}{5}$$

$$= \frac{2}{5}$$

1e

$$\int_{-4}^{-2} 2x\,dx$$

$$= [x^2]_{-4}^{-2}$$

$$= (-2)^2 - (-4)^2$$

$$= 4 - 16$$

$$= -12$$

1f

$$\int_{-3}^{-1} x^2\,dx$$

$$= \left[\frac{x^3}{3}\right]_{-3}^{-1}$$

$$= \frac{(-1)^3}{3} - \frac{(-3)^3}{3}$$

$$= -\frac{1}{3} - \left(-\frac{27}{3}\right)$$

$$= -\frac{1}{3} + 9$$

$$= 8\frac{2}{3}$$

1g

$$\int_0^2 (x+3)\,dx$$

$$= \left[\frac{x^2}{2} + 3x\right]_0^2$$

$$= \left(\frac{2^2}{2} + 3 \times 2\right) - (0+0)$$

$$= 2 + 6$$

$$= 8$$

1h

$$\int_{-1}^4 (2x-5)\,dx$$

$$= [x^2 - 5x]_{-1}^4$$

$$= 16 - 20 - (1+5)$$

$$= -4 - 6$$

$$= -10$$

1i

$$\int_{-3}^1 (x^2 - 2x + 1)\,dx$$

$$= \left[\frac{x^3}{3} - x^2 + x\right]_{-3}^1$$

$$= \left(\frac{1^3}{3} - 1^2 + 1\right) - \left(\frac{(-3)^3}{3} - (-3)^2 - 3\right)$$

$$= \frac{1}{3} + 9 + 9 + 3$$

$$= 21\frac{1}{3}$$

2a

$$\int_1^3 x(x-1)\,dx$$

$$= \int_1^3 (x^2 - x)\,dx$$

$$= \left[\frac{x^3}{3} - \frac{x^2}{2}\right]_1^3$$

$$= \left(\frac{3^3}{3} - \frac{3^2}{2}\right) - \left(\frac{1}{3} - \frac{1}{2}\right)$$

$$= \left(9 - \frac{9}{2}\right) - \left(\frac{1}{3} - \frac{1}{2}\right)$$

$$= 4\frac{2}{3}$$

2b

$$\int_{-1}^0 (x+1)(x-3)\,dx$$

$$= \int_{-1}^0 (x^2 - 2x - 3)\,dx$$

$$= \left[\frac{x^3}{3} - x^2 - 3x\right]_{-1}^0$$

$$= \left(\frac{0^3}{3} - 0^2 - 3 \times 0\right) - \left(\frac{(-1)^3}{3} - (-1)^2 - 3 \times (-1)\right)$$

$$= 0 - \left(-\frac{1}{3} - 1 + 3\right)$$

$$= -1\frac{2}{3}$$

2c

$$\int_0^1 (2x-1)^2\,dx$$

$$= \int_0^1 (4x^2-4x+1)\,dx$$

$$= \left[\frac{4x^3}{3} - 2x^2 + x\right]_0^1$$

$$= \left(\frac{4\times 1^3}{3} - 2\times 1^2 + 1\right) - \left(\frac{0^3}{3} - 2\times 0^2 + 0\right)$$

$$= \frac{4}{3} - 2 + 1 - 0$$

$$= \frac{1}{3}$$

3a

$$\int_1^2 \frac{x^2-3x}{x}\,dx$$

$$= \int_1^2 \left(\frac{x^2}{x} - \frac{3x}{x}\right)dx$$

$$= \int_1^2 (x-3)\,dx$$

$$= \left[\frac{x^2}{2} - 3x\right]_1^2$$

$$= \left(\frac{2^2}{2} - 3\times 2\right) - \left(\frac{1^2}{2} - 3\times 1\right)$$

$$= 2 - 6 - \frac{1}{2} + 3$$

$$= -1\frac{1}{2}$$

3b

$$\int_2^3 \frac{3x^4 - 4x^2}{x^2}\,dx$$

$$= \int_2^3 \left(\frac{3x^4}{x^2} - \frac{4x^2}{x^2}\right) dx$$

$$= \int_2^3 (3x^2 - 4)\,dx$$

$$= [x^3 - 4x]_2^3$$

$$= (3^3 - 4 \times 3) - (2^3 - 4 \times 2)$$

$$= 27 - 12 - 8 + 8$$

$$= 15$$

3c

$$\int_{-2}^{-1} \frac{x^3 - 2x^4}{x^2}\,dx$$

$$= \int_{-2}^{-1} \left(\frac{x^3}{x^2} - \frac{2x^4}{x^2}\right) dx$$

$$= \int_{-2}^{-1} (x - 2x^2)\,dx$$

$$= \left[\frac{x^2}{2} - \frac{2x^3}{3}\right]_{-2}^{-1}$$

$$= \left(\frac{(-1)^2}{2} - \frac{2 \times (-1)^3}{3}\right) - \left(\frac{(-2)^2}{2} - \frac{2 \times (-2)^3}{3}\right)$$

$$= \frac{1}{2} + \frac{2}{3} - 2 - \frac{16}{3}$$

$$= -6\frac{1}{6}$$

4a i

$$\int_4^k 5\,dx$$

$$= [5x]_4^k$$

$$= 5k - 5 \times 4$$

$$= 5k - 20$$

4a ii

$$\int_4^k 5\,dx = 10$$

$$5k - 20 = 10$$

$$5k = 30$$

$$k = 6$$

4b i

$$\int_0^k (2x - 1)\,dx$$

$$= [x^2 - x]_0^k$$

$$= (k^2 - k) - (0 - 0)$$

$$= k^2 - k$$

4b ii

$$\int_0^k (2x - 1)\,dx = 6$$

$$k^2 - k = 6$$

$$k^2 - k - 6 = 0$$

$$(k + 2)(k - 3) = 0$$

$$k = -2 \text{ or } 3$$

Taking the positive solution, $k = 3$.

5a As upper and lower bound of this function are the same, the integral must be zero (you can think of this as area with 0 width has 0 area).

5b This is an odd function and hence the area under the curve between -2 and 0 is equal to that above the curve between 0 and 2. So the total signed area must be 0.

5c We know that x^3 is odd and that $-9x$ is odd. The sum of two odd functions gives another odd function. Hence $x^3 - 9x^2$ is odd. The area under the curve between -3 and 0 is equal to that above the curve between 0 and 3. So the total signed area must be 0.

6a

$$\int_0^3 f(x)$$

$$= \text{area of triangle} + \text{area of rectangle}$$

$$= \frac{1}{2} \times 2 \times 4 + 1 \times 1 \times 4$$

$$= 8$$

6b

$$\int_0^3 f(x)\,dx$$

$$= \text{area of triangle above } x\text{-axis (from } x = 0 \text{ to } 2)$$

$$+ \text{ area of triangle below } x\text{-axis (from } x = 2 \text{ to } 3)$$

$$= \frac{1}{2} \times 2 \times 2 + \frac{1}{2} \times 1 \times 1$$

$$= 2 - \frac{1}{2}$$

$$= \frac{3}{2}$$

7a i $A(x)$

$$= \int_{-2}^{x} (4 - t)\, dt$$

$$= \left[4t - \frac{t^2}{2}\right]_{-2}^{x}$$

$$= \left(4x - \frac{x^2}{2}\right) - \left(4 \times (-2) - \frac{(-2)^2}{2}\right)$$

$$= 4x - \frac{x^2}{2} + 8 + 2$$

$$= 4x - \frac{x^2}{2} + 10$$

7a ii $A(x)$

$$= \int_{2}^{x} t^{-2}\, dt$$

$$= [-t^{-1}]_2^x$$

$$= -x^{-1} - (-2^{-1})$$

$$= -\frac{1}{x} + \frac{1}{2}$$

$$= \frac{1}{2} - \frac{1}{x}$$

7b i

$$\frac{d}{dx}\int_{-2}^{x} (4 - t)\, dt$$

$$= \frac{d}{dx}\left(4x - \frac{x^2}{2} + 10\right)$$

$$= 4 - x$$

7b ii

$$\frac{d}{dx}\int_2^x t^{-2}\,dt$$

$$= \frac{d}{dx}\left(-x^{-1} - (-2^{-1})\right)$$

$$= x^{-2} - 0$$

$$= x^{-2}$$

7c i

$$\frac{d}{dx}\int_7^x (t^5 - 5t^3 + 1)\,dt = x^5 - 5x^3 + 1$$

by the fundamental theorem of calculus

7c ii

$$\frac{d}{dx}\int_3^x \frac{t^2+4}{t^2-1}\,dt = \frac{x^2+4}{x^2-1}$$

by the fundamental theorem of calculus

8a

$$\int (x+2)\,dx$$

$$= \frac{x^2}{2} + 2x + C$$

8b

$$\int (x^3 + 3x^2 - 5x + 1)\,dx$$

$$= \frac{x^4}{4} + x^3 - \frac{5x^2}{2} + x + C$$

8c

$$\int x(x-1)\,dx$$

$$= \int (x^2 - x)\,dx$$

$$= \frac{x^3}{3} - \frac{x^2}{2} + C$$

8d

$$\int (x-3)(2-x)\,dx$$

$$= \int (2x - x^2 - 6 + 3x)\,dx$$

$$= \int (5x - x^2 - 6)\,dx$$

$$= \frac{5x^2}{2} - \frac{x^3}{3} - 6x + C$$

8e

$$\int x^{-2}\,dx$$

$$= \frac{x^{-1}}{-1} + C$$

$$= -x^{-1} + C$$

$$= -\frac{1}{x} + C$$

8f

$$\int \frac{1}{x^7}\,dx$$

$$= \int x^{-7}\,dx$$

$$= \frac{x^{-6}}{-6} + C$$

$$= -\frac{1}{6x^6} + C$$

8g

$$\int \sqrt{x}\, dx$$

$$= \int x^{\frac{1}{2}}\, dx$$

$$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + C$$

$$= \frac{2}{3} x^{\frac{3}{2}} + C$$

8h

$$\int (x+1)^4\, dx$$

$$= \frac{(x+1)^5}{1 \times 5} + C$$

$$= \frac{1}{5}(x+1)^5 + C$$

8i

$$\int (2x-3)^5\, dx$$

$$= \frac{(2x-3)^6}{2 \times 6} + C$$

$$= \frac{1}{12}(2x-3)^6 + C$$

9a Area of shaded region

$$= \int_{-3}^{1} x^2 \, dx$$

$$= \left[\frac{x^3}{3}\right]_{-3}^{1}$$

$$= \frac{1}{3} - \frac{(-3)^3}{3}$$

$$= \frac{1}{3} + \frac{27}{3}$$

$$= 9\tfrac{1}{3} \text{ square units}$$

9b Area of shaded region

$$= \int_{-2}^{0} (x^3 - 4x) \, dx$$

$$= \left[\frac{x^4}{4} - 2x^2\right]_{-2}^{0}$$

$$= (0 - 0) - \left(\frac{(-2)^4}{4} - 2 \times (-2)^2\right)$$

$$= -4 + 8$$

$$= 4 \text{ square units}$$

9c Region is below the x-axis so the signed area is negative.

$$\int_{1}^{3} (x^2 - 4x + 3) \, dx$$

$$= \left[\frac{x^3}{3} - 2x^2 + 3x\right]_{1}^{3}$$

$$= \left(\frac{3^3}{3} - 2 \times 3^2 + 3 \times 3\right) - \left(\frac{1^3}{3} - 2 \times 1^2 + 3 \times 1\right)$$

$$= 9 - 18 + 9 - \frac{1}{3} + 2 - 3$$

$$= -\frac{4}{3}$$

Area of shaded region $= \frac{4}{3}$ square units

9d Area of shaded region

$= \int_3^4 (2y - 6)\,dy$

$= [y^2 - 6y]_3^4$

$= (4^2 - 6 \times 4) - (3^2 - 6 \times 3)$

$= 16 - 24 - 9 + 18$

$= 1$ square unit

9e Area of shaded region

$= \int_0^1 (x - x^2)\,dx$

$= \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$

$= \left(\frac{1}{2} - \frac{1}{3}\right) - (0 - 0)$

$= \frac{1}{6}$ square units

9f Area of shaded region

$= \int_{-1}^1 (x^2 - x^4)\,dx$

$= \left[\frac{x^3}{3} - \frac{x^5}{5}\right]_{-1}^1$

$= \left(\frac{1}{3} - \frac{1}{5}\right) - \left(-\frac{1}{3} - \left(-\frac{1}{5}\right)\right)$

$= \frac{1}{3} - \frac{1}{5} + \frac{1}{3} - \frac{1}{5}$

$= \frac{4}{15}$ square units

9g Area of shaded region

$$= \int_1^2 \left((3x-2)-x^2\right) dx$$

$$= \int_1^2 (3x-2-x^2)\, dx$$

$$= \left[\frac{3x^2}{2}-2x-\frac{x^3}{3}\right]_1^2$$

$$= \left(\frac{3\times 2^2}{2}-2\times 2-\frac{2^3}{3}\right)-\left(\frac{3\times 1^2}{2}-2\times 1-\frac{1^3}{3}\right)$$

$$= 6-4-\frac{8}{3}-\frac{3}{2}+2+\frac{1}{3}$$

$$= \frac{1}{6} \text{ square units}$$

9h Area of shaded region

$$= \int_{-2}^1 \left((1-x^2)-(x-1)\right) dx$$

$$= \int_{-2}^1 (2-x-x^2)\, dx$$

$$= \left[2x-\frac{x^2}{2}-\frac{x^3}{3}\right]_{-2}^1$$

$$= \left(2\times 1-\frac{1^2}{2}-\frac{1^3}{3}\right)-\left(2\times(-2)-\frac{(-2)^2}{2}-\frac{(-2)^3}{3}\right)$$

$$= 2-\frac{1}{2}-\frac{1}{3}+4+2-\frac{8}{3}$$

$$= 4\frac{1}{2} \text{ square units}$$

10a Solving the equations simultaneously gives:

$x^2-3x+5=x+2$

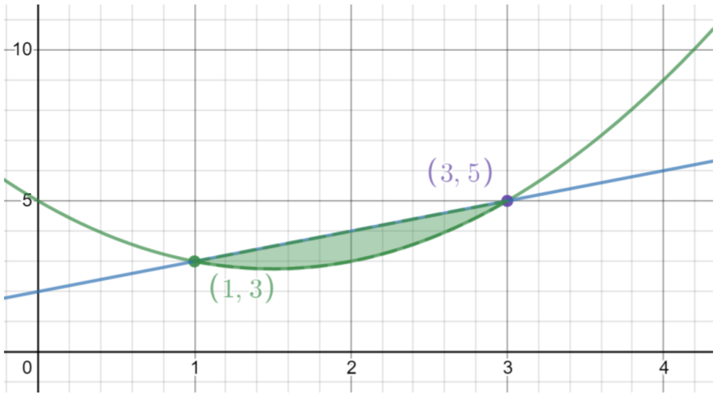
$x^2-4x+3=0$

$(x-1)(x-3)=0$

$x=1$ or 3

Substituting this back into $y=x+2$ gives the points of intersection as $(1,3)$ and $(3,5)$.

10b The graphs of $y = x^2 - 3x + 5$ and $y = x + 2$ are shown below.



Area enclosed by curves

$$= \int_1^3 \left(x + 2 - (x^2 - 3x + 5)\right) dx$$

$$= \int_1^3 (4x - 3 - x^2)\, dx$$

$$= \left[2x^2 - 3x - \frac{x^3}{3}\right]_1^3$$

$$= \left(2 \times 3^2 - 3 \times 3 - \frac{3^3}{3}\right) - \left(2 \times 1^2 - 3 \times 1 - \frac{1^3}{3}\right)$$

$$= 18 - 9 - 9 - 2 + 3 + \frac{1}{3}$$

$$= \tfrac{4}{3} \text{ square units}$$

11a

$$\int_1^3 2^x\, dx$$

$$\doteqdot \frac{b-a}{2n}\left(f(a) + f(b) + 2(f(x_1) + f(x_2) + \cdots)\right)$$

$$= \frac{3-1}{2 \times 2}\Big(f(1) + f(3) + 2\big(f(2)\big)\Big)$$

$$= \frac{2}{4}\left(2^1 + 2^3 + 2(2^2)\right)$$

$$= \frac{1}{2}(2 + 8 + 8)$$

$$= 9$$

11b

$$\int_1^3 \log_{10} x \, dx$$

$$\doteqdot \frac{b-a}{2n}(f(a)+f(b)+2(f(x_1)+f(x_2)+\cdots))$$

$$= \frac{3-1}{2\times 4}(f(1)+f(3)+2\big(f(1.5)+f(2)+f(2.5)\big))$$

$$= \frac{2}{8}\big(\log_{10} 1+\log_{10} 3+2(\log_{10} 1.5+\log_{10} 2+\log_{10} 2.5)\big)$$

$\doteqdot 0.56$ (to two significant figures)

12a

$$\frac{d}{dx}(3x+4)^6$$

$$= 6(3x+4)^5 \times 3$$

$$= 18(3x+4)^5$$

12b i

$$\int 18(3x+4)^5 \, dx$$

$$= \int \frac{d}{dx}(3x+4)^6 \, dx$$

$$= (3x+4)^6 + C$$

12b ii

$$\int 9(3x+4)^5 \, dx$$

$$= \frac{1}{2}\int 18(3x+4)^5 \, dx$$

$$= \frac{1}{2}\int \frac{d}{dx}(3x+4)^6 \, dx$$

$$= \frac{1}{2}(3x+4)^6 + C$$

13a

$$\frac{d}{dx}(x^2-1)^3$$

$$= 3(x^2-1)^2 \times 2x$$

$$= 6x(x^2-1)^2$$

13b i

$$\int 6x(x^2-1)^2\,dx$$

$$= \int \frac{d}{dx}(x^2-1)^3\,dx$$

$$= (x^2-1)^3 + C$$

13b ii

$$\int x(x^2-1)^2\,dx$$

$$= \frac{1}{6}\int 6x(x^2-1)^2\,dx$$

$$= \frac{1}{6}\int \frac{d}{dx}(x^2-1)^3\,dx$$

$$= \frac{1}{6}(x^2-1)^3 + C$$

14a Let $u = x^3 + 1$, then $\frac{du}{dx} = 3x^2$ so $du = 3x^2\,dx$

$$\int 3x^2(x^3+1)^4\,dx$$

$$= \int (x^3+1)^4 \times 3x^2\,dx$$

$$= \int u^4\,du$$

$$= \frac{u^5}{5} + C$$

$$= \frac{(x^3+1)^5}{5} + C$$

14b Let $u = x^2 - 5$, then $\frac{du}{dx} = 2x$ so $du = 2x\ dx$

$$\int \frac{2x}{(x^2-5)^3}\,dx$$

$$\int \frac{1}{(x^2-5)^3} \times 2x\ dx$$

$$= \int \frac{1}{u^3}\ du$$

$$= \int u^{-3}\ du$$

$$= \frac{u^{-2}}{-2} + C$$

$$= -\frac{1}{2u^2} + C$$

$$= -\frac{1}{2(x^2-5)^2} + C$$

15 Let $u = x^2 + 3$, then $\frac{du}{dx} = 2x$ so $du = 2x\ dx$ and $\frac{1}{2}\ du = x\ dx$

When $x = 0, u = 3$ and when $x = 1, u = 4$

$$\int_0^1 \frac{x}{\sqrt{x^2+3}}\,dx$$

$$= \int_0^1 \frac{1}{\sqrt{x^2+3}} \times x\ dx$$

$$= \int_3^4 \frac{1}{\sqrt{u}} \times \frac{1}{2}\ du$$

$$= \frac{1}{2}\int_3^4 u^{-\frac{1}{2}}\ du$$

$$= \frac{1}{2}\left[\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right]_3^4$$

$$= \frac{1}{2} \times 2 \times \left[\sqrt{u}\right]_3^4$$

$$= \left[\sqrt{u}\right]_3^4$$

$$= \sqrt{4} - \sqrt{3}$$

$$= 2 - \sqrt{3}$$

as required.