Convex Optimization

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Contents

Abstract

This document is Antoine Groudiev's class notes while following the class *Convex Optimization* (Optimisation Convexe) at the Computer Science Department of ENS Ulm. It is freely inspired by the lectures of Adrien Taylor.

1 Introduction

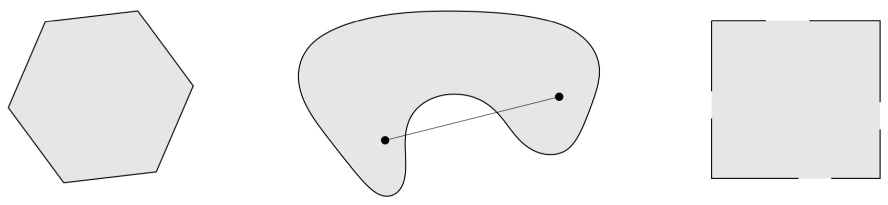
2 Convex sets

2.1 Definitions

Definition (Convex set)**.** A set C is a *convex set* if every segment that connects two points in C is in C. Formally:

$$\forall x, y \in C, \forall \theta \in [0,1], \quad \theta x + (1-\theta) y \in C$$

Example. Here are some examples of convex and non-convex sets:



In many cases, we will use proper (i.e. non-empty) convex sets, and closed convex sets.

Definition (Convex hull)**.** The *convex hull* of S, denoted $\mathrm{Conv}(S)$, is the smallest convex set that contains S.

Definition (Convex combinations)**.** The *convex combinations* of $x_1, \ldots, x_k$ are all the point x of the form:

$$x = \theta_1 x_1 + \cdots + \theta_k x_k$$

with $\theta_1, \ldots, \theta_k \geqslant 0$ and $\sum_{i=1}^k \theta_i = 1$.

Property 2.1. The convex hull of a set S is the set of all convex combinations of points in S:

$$\mathrm{Conv}(S) = \left\{ \sum_{i=1}^k \theta_i x_i \,\middle|\, (x_i) \in S^k, (\theta_i) \in \mathbb{R}_+^k, \sum_{i=1}^k \theta_i = 1 \right\}$$

2.2 Examples

2.2.1 Hyperplanes and halfspaces

Definition (Hyperplane)**.** A *hyperplane* is the set of the form:

$$H = \left\{ x \,\middle|\, a^\top x = b \right\}$$

for some $a \in \mathbb{R}^n \setminus \{0\}$ and $b \in \mathbb{R}$. a is called the *normal vector* of H. Hyperspaces are affine and convex.

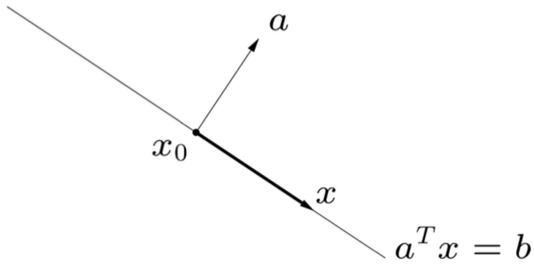


Figure 2.1: Hyperplane

Definition (Halfspace)**.** A *halfspace* is the set of the form:

$$H = \left\{ x \,\middle|\, a^\top x \leqslant b \right\}$$

for some $a \in \mathbb{R}^n \setminus \{0\}$ and $b \in \mathbb{R}$. a is called the *normal vector* of H. Halfspaces are convex.

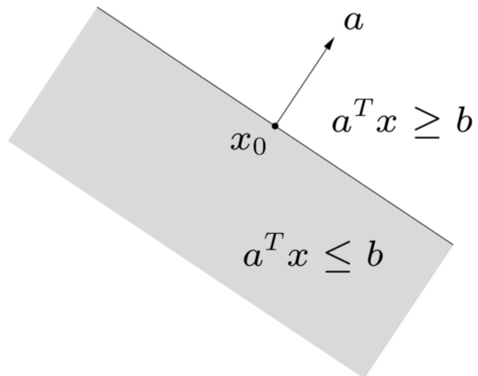


Figure 2.2: Halfspace

2.2.2 Euclidian balls and ellipsoids

Definition (Euclidian ball)**.** The *Euclidian ball* of center x_c and radius r is the set:

$$B(x_c, r) = \{ x \mid \|x - x_c\|_2 \leqslant r \} = \{ x_c + ru \mid \|u\|_2 \leqslant 1 \}$$

Euclidian balls are convex.

Definition (Ellipsoid)**.** An *ellipsoid* is the set of the form:

$$E = \left\{ x \,\middle|\, (x - x_c)^\top P^{-1}(x - x_c) \leqslant 1 \right\}$$

with $P \in \mathbb{S}^n_{++}$[1] and $x_c \in \mathbb{R}^n$. Ellipsoids are convex.

[1] $\mathbb{S}^n_{++}$ denotes the set of symmetric positive definite matrices of size n

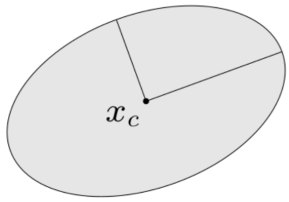


Figure 2.3: Ellipsoid

An alternative representation of an ellipsoid is:

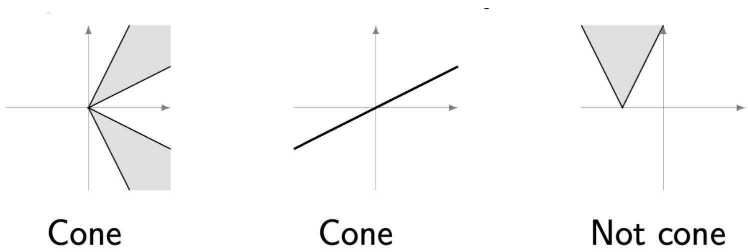
$$E = \{\, x_c + Au \mid \|u\|_2 \leqslant 1 \,\}$$

for some nonsingular matrix $A \in \mathrm{GL}_n(\mathbb{R})$. We can choose A symmetric and positive definite without loss of generality, for instance by choosing $A = P^{1/2}$.

2.2.3 Cones

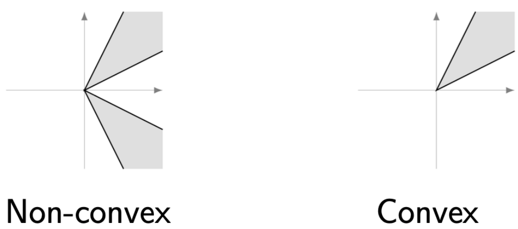
Definition (Cones)**.** A set K is a *cone*, or a *nonnegative homogeneous set*, if:

$$\forall x \in K, \forall \theta \in \mathbb{R}_+^*, \quad \theta x \in K$$



Definition (Convex cone)**.** A set K is a *convex cone* if:

$$\forall x_1, x_2 \in K, \forall \theta_1, \theta_2 \in \mathbb{R}_+^*, \quad \theta_1 x_1 + \theta_2 x_2 \in K$$



In the followings, we will denote by:

- $\mathbb{S}^n \subset \mathbb{R}^{n \times n}$ the set of symmetric matrices of size n
- $\mathbb{S}_+^n$ the set of positive semidefinite matrices of size n, that is matrices verifying:

  $$\forall z \in \mathbb{R}^n, z^\top X z \geqslant 0$$

  also denoted $X \succcurlyeq 0$.

- $\mathbb{S}^n_{++}$ the set of positive definite matrices of size n, that is matrices verifying:

$$\forall z \in \mathbb{R}^n, z^\top X z > 0$$

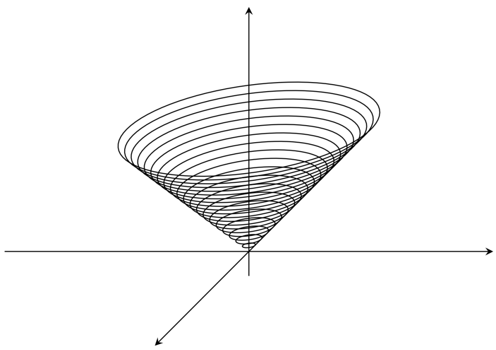
  also denoted $X \succ 0$.

$\mathbb{S}^n_+$ and $\mathbb{S}^n_{++}$ are convex cones.

Special cases of cones include:

Positive orthant $K = \mathbb{R}^n_+ = \{\, x \in \mathbb{R}^n \mid x_i \geqslant 0, \forall i \,\}$

Norm cones $K = \{\, (x,t) \in \mathbb{R}^n \times \mathbb{R} \mid \|x\| \leqslant t \,\}$. A particular case is the second-order cone (SOC), based on the ℓ_2 norm.



Positive polynomials $K_n = \{\, x \in \mathbb{R}^{n+1} \mid \forall t \in \mathbb{R}, \sum_{i=0}^n x_i t^i \geqslant 0 \,\}$

Positive semidefinite cone $\mathbb{S}^n_+ = \left\{ X \in \mathbb{S}^n \,\middle|\, \forall z \in \mathbb{R}^n, z^\top X z \geqslant 0 \right\}$

Co-positive cone $\mathbb{S}^n_+ = \left\{ X \in \mathbb{S}^n \,\middle|\, \forall z \in \mathbb{R}^n_+, z^\top X z \geqslant 0 \right\}$

Exponential cone $\left\{ (x,y,z) \in \mathbb{R} \times \mathbb{R}^*_+ \times \mathbb{R} \,\middle|\, z \geqslant y e^{x/y} \right\}$

Definition (Dual cones)**.** The *dual cone* to a convex cone K is the set:

$$K^* = \left\{ y \,\middle|\, \forall x \in K, \quad y^\top x \geqslant 0 \right\}$$

Convex cones and their duals are particularly useful for convex duality. A convex cone that satisfies $K = K^*$ is called *self-dual.*

Definition (Polar cones)**.** The *polar cone* to a convex cone K is the set:

$$K^\diamond = \left\{ y \,\middle|\, \forall x \in K, \quad y^\top x \leqslant 0 \right\}$$

We have the identity $K^\diamond = -K^*$.

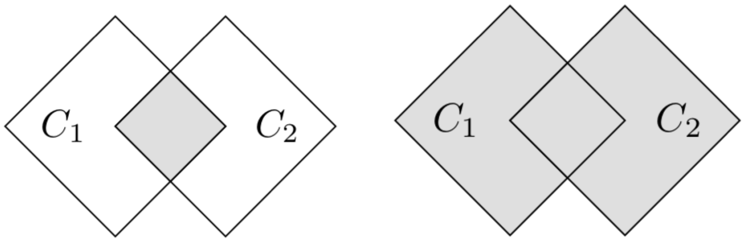
2.3 Convexity-preserving operations

To establish the convexity of a set C, the most basic approach is to apply the definition by proving that every segment that connects two points in C is in C. However, this can be tedious in practice. Instead, we can use operations that preserve convexity.

2.3.1 Intersection and union

Property 2.2 (Convexity is preserved by intersection)**.** For any convex sets C_1 and C_2, the intersection $C_1 \cap C_2$ is convex.

Likewise, the intersection of an arbitrary number of convex sets is convex.

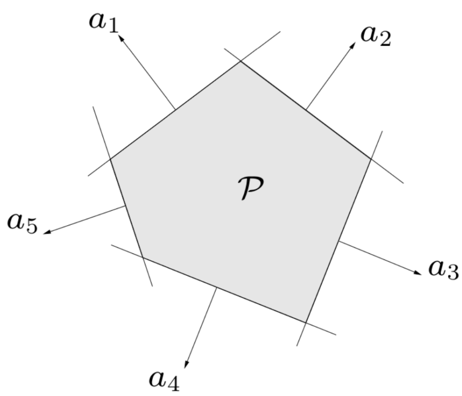


Remark. *The union of convex sets is not necessarily convex. For instance in* $\mathbb{R}$*, both* $[0, 1]$ *and* $[2, 3]$ *are convex, but their union* $[0, 1] \cup [2, 3]$ *is not.*

Definition (Polyhedron)**.** A *polyhedron* is the solution set of finitely many linear inequalities and equalities:

$$S = \{\, x \in \mathbb{R}^n \mid Ax \leqslant b, Cx = d \,\}$$

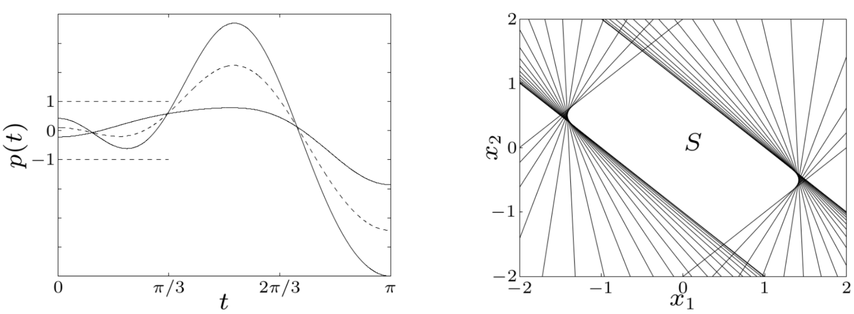
for $A \in \mathscr{M}_{m,n}(\mathbb{R})$ and $C \in \mathscr{M}_{p,n}(\mathbb{R})$. Polyhedra are convex, since they are the intersection of halfspaces and hyperplanes which are convex.



Example. Let:

$$S = \left\{ x \in \mathbb{R}^m \;\middle|\; \forall t \in \mathbb{R}, \quad |t| \leqslant \frac{\pi}{3} \implies \left| \sum_{k=1}^{m} x_k \cos(kt) \right| \leqslant 1 \right\}$$

S is convex, since it can be written as the intersection of convex sets.



Example. $\mathbb{S}^n_+$ is convex since it is the intersection of convex sets:

$$\mathbb{S}^n_+ = \left\{ X \in \mathbb{S}^n \,\middle|\, \forall z \in \mathbb{R}^n, z^\top X z \geqslant 0 \right\} = \mathbb{S}^n \cap \bigcap_{z \in \mathbb{R}^n} \left\{ X \in \mathscr{M}_n(\mathbb{R}) \,\middle|\, z^\top X z \geqslant 0 \right\}$$

Each set $\left\{ X \in \mathscr{M}_n(\mathbb{R}) \,\middle|\, z^\top X z \geqslant 0 \right\}$ being convex, their intersection is convex. In particular, it doesn't matter if the number of sets is finite, countable or uncountable.

2.3.2 Affine functions

Property 2.3 (The image of a convex set by an affine function is convex)**.** If $L : \mathbb{R}^n \to \mathbb{R}^m$ is an affine function, then if C is convex, $L(C)$ is convex.

More explicitly, let $A \in \mathscr{M}_{m,n}(\mathbb{R})$ and $b \in \mathbb{R}^m$. The affine function $L(x) = Ax + b$ maps C to $L(C) = \{ y \in \mathbb{R}^m \mid \exists x \in C, y = Ax + b \}$, which is convex if C is convex.

Property 2.4 (The pre-image of a convex set by an affine function is convex)**.** If $L : \mathbb{R}^n \to \mathbb{R}^m$ is an affine function, then $L^{-1}(C)$, the pre-image of C by L defined by:

$$L^{-1}(C) = \{ x \in \mathbb{R}^n \mid L(x) \in C \}$$

is convex if C is convex.

Example (Linear matrix inequalities)**.** Let $A_1, \dots, A_m \in \mathbb{S}^n(\mathbb{R})$. The set:

$$\left\{ x \in \mathbb{R}^m \,\middle|\, \sum_{i=1}^m x_i A_i \succcurlyeq 0 \right\}$$

is an affine pre-image of $\mathbb{S}^n_+$ for the mapping $L : \mathbb{R}^m \to \mathbb{S}^n$ defined by:

$$L(x) = \sum_{i=1}^m x_i A_i$$

$\mathbb{S}^n_+$ being convex, the set is convex. $\sum_{i=1}^m x_i A_i \succcurlyeq 0$ is called a *linear matrix inequality.*

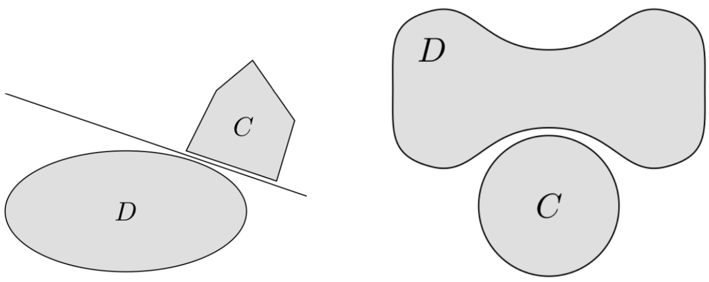
2.4 Geometric elements

2.4.1 Separating and supporting hyperplanes

Property 2.5 (Separating hyperplanes)**.** Soppose that $C, D \subseteq \mathbb{R}^n$ are two non-intersecting convex sets (that is $C \cap D = \varnothing$). Then there exists a hyperplane that separates C and D, that is:

$$\exists s \in \mathbb{R}^n \setminus \{0\}, \exists r \in \mathbb{R}, \quad \forall x \in C, s^\top x \leqslant r \quad \text{and} \quad \forall x \in D, s^\top x \geqslant r$$

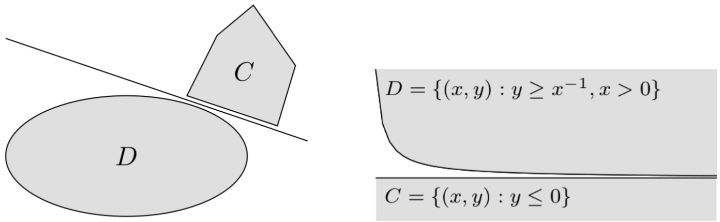
where $\left\{ x \in \mathbb{R}^n \,\middle|\, s^\top x = t \right\}$ is called the *separating hyperplane.*



Examples   Counter-example

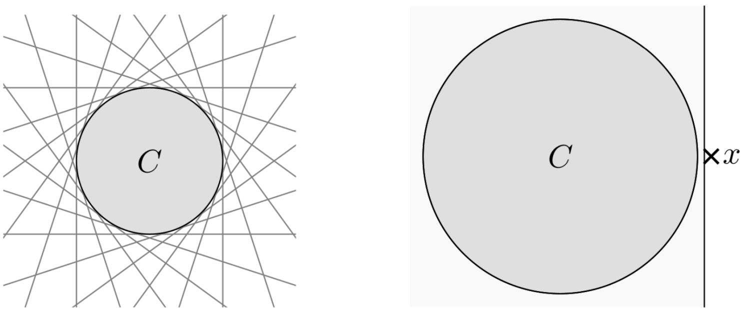
Property 2.6 (Strict separating hyperplanes)**.** Suppose that $C, D \subseteq \mathbb{R}^n$ are two non-intersecting **closed** convex sets, and that one of them is compact (closed and bounded in finite dimension). Then there exists a hyperplane that strictly separates C and D, that is:

$$\exists s \in \mathbb{R}^n \setminus \{0\}, \exists r \in \mathbb{R}, \quad \forall x \in C, s^\top x < r \quad \text{and} \quad \forall x \in D, s^\top x > r$$



Examples Counter-example

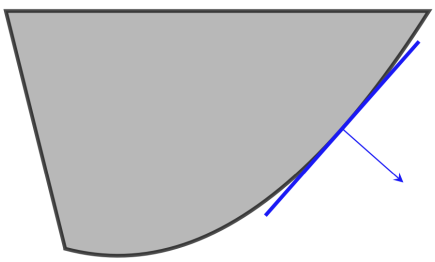
Note that a closed convex set C is the intersection of all halfspaces that contain it.



Definition (Supporting hyperplanes)**.** Supporting hyperplanes touch the boundary of a convex set, and have the entire set on one side. Formally, a hyperplane $H = \left\{ y \,\middle|\, s^\top y = r \right\}$ is a *supporting hyperplane* to a convex set C at a point $x \in \partial C$ if:

$$x^\top x = r \qquad \text{and} \qquad \forall y \in C, \quad a^\top y \leqslant r = s^\top x$$

We also say that H *supports* C at x.



Property 2.7. Let $C \subseteq \mathbb{R}^n$ be a non-empty convex set, and let $x \in \partial C$. Then there exists a supporting hyperplane to C at x.

2.4.2 Cone operators

Definition (Normal cone operator)**.** The *normal cone operator* to a set C at a point x is the set:

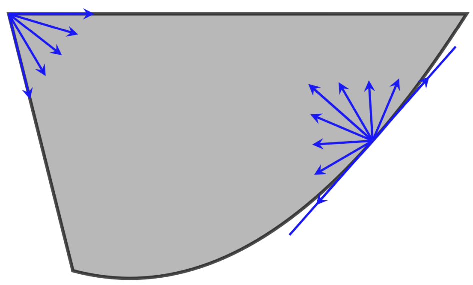
$$N_C(x) = \begin{cases} \left\{ g \in \mathbb{R}^n \;\middle|\; \forall y \in C, \quad g^\top(y-x) \leqslant 0 \right\} & \text{if } x \in C \\ \varnothing & \text{if } x \notin C \end{cases}$$

Intuitively, it is the set of vectors g that form obtuse angles for all $y - x$ with $y \in C$.

For $x \in \mathring{C}$, we have $N_c(x) = \{0\}$. For $x \in \partial C$, $N_C(x)$ is the set of the normal vectors to the supporting hyperplanes to C at x. If $x \notin C$, $N_C(x)$ is empty.

Definition (Tangent vector)**.** Let $C \subseteq \mathbb{R}^n$ be a convex set. A vector $d \in \mathbb{R}^n$ is tangent to C at x if:

$$\exists \{x_k\}_k \subseteq C, \exists \{\lambda_k\}_k \subset \mathbb{R}_+, \quad \lim_{k \to +\infty} \lambda_k(x_k - x) = d$$



Definition (Tangent cone)**.** The tangent cone of a convex set C at x is:

$$T_C(x) = N_C^\diamond(x)$$

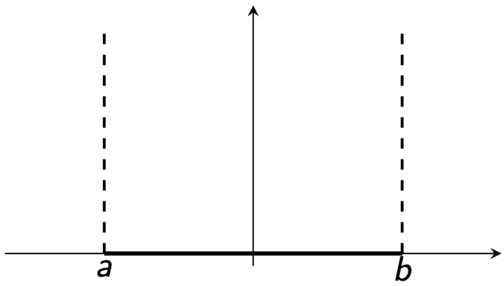
3 Convex functions

3.1 Extended-valued functions

Definition (Extended-valued function)**.** A function $f : \mathbb{R}^n \longrightarrow \bar{\mathbb{R}}$ is *extended-valued* if its domain is $\mathbb{R}^n$ and its range is $\bar{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$.

Example (Indicator function)**.** We consider the indicator function of interval $[a, b]$:

$$\mathbb{1}_{[a,b]}(x) := \begin{cases} 0 & \text{if } x \in [a, b] \\ +\infty & \text{otherwise} \end{cases}$$



Definition (Effective domain)**.** The *effective domain* of $f : \mathbb{R}^n \longrightarrow \bar{\mathbb{R}}$ is the set of points where f is finite:

$$\operatorname{dom} f := \{ x \in \mathbb{R}^n \mid f(x) < +\infty \} \tag{3.1.1}$$

A function is said to be *proper* if its effective domain is non-empty: $\operatorname{dom} f \neq \varnothing$.

3.2 Definition and first properties

Definition (Convex function)**.** A function $f : \mathbb{R}^n \longrightarrow \bar{\mathbb{R}}$ is *convex* if its graph is below any line connecting two points of the graph $(x, f(x))$ and $(y, f(y))$. That is:

$$\forall x, y \in \mathbb{R}^n, \forall \theta \in [0, 1], \quad f(\theta \cdot x + (1 - \theta) \cdot y) \leqslant \theta \cdot f(x) + (1 - \theta) \cdot f(y) \tag{3.2.1}$$

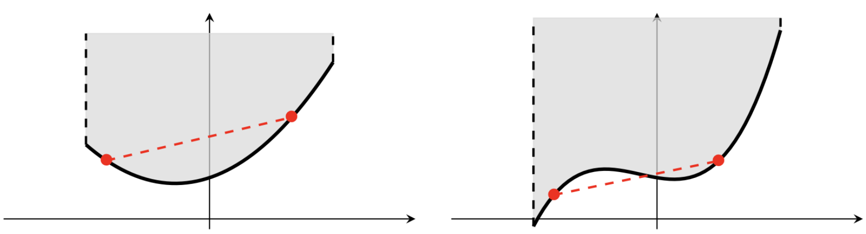
Definition (Concave function)**.** A function $f : \mathbb{R}^n \longrightarrow \bar{\mathbb{R}}$ is *concave* if $-f$ is convex. That is:

$$\forall x, y \in \mathbb{R}^n, \forall \theta \in [0, 1], \quad f(\theta \cdot x + (1 - \theta) \cdot y) \geqslant \theta \cdot f(x) + (1 - \theta) \cdot f(y)$$

Definition (Epigraph)**.** The *epigraph* of a function $f : \mathbb{R}^n \longrightarrow \bar{\mathbb{R}}$ is the set of points lying above the graph of f:

$$\operatorname{epi} f := \{ (x, t) \in \mathbb{R}^n \times \mathbb{R} \mid f(x) \leqslant t \} \tag{3.2.2}$$

Property 3.1 (Convexity and epigraph)**.** A function $f : \mathbb{R}^n \longrightarrow \bar{\mathbb{R}}$ is convex if and only if its epigraph is a convex set.



The following property allows to check the convexity of a multivariate function f by checking the convexity of functions of one variable.

Property 3.2. Let $f : \mathbb{R}^n \longrightarrow \bar{\mathbb{R}}$ be a function, and let $x \in \operatorname{dom} f$. We define:

$$\begin{aligned} g_{x,v} : \mathbb{R} &\longrightarrow \bar{\mathbb{R}} \\ t &\longmapsto f(x + tv) \end{aligned}$$

with $\operatorname{dom} g_{x,v} = \{\, t \in \mathbb{R} \mid x + tv \in \operatorname{dom} f \,\}$. Then, f is convex if and only if $g_{x,v}$ is convex in t for all $x \in \operatorname{dom} f$ and all $v \in \mathbb{R}^n$.

Definition (Sublevel sets)**.** Let $f : \mathbb{R}^n \longrightarrow \bar{\mathbb{R}}$ be a function. The *sublevel set* of f at level $\alpha \in \mathbb{R}$ is the set of points lying below the level α:

$$S_\alpha(f) = \{\, x \in \mathbb{R}^n \mid f(x) \leqslant \alpha \,\}$$

Property 3.3. If f is convex, then its sublevel sets are convex:

$$f \text{ is convex } \implies \forall \alpha \in \mathbb{R}, \quad S_\alpha(f) \text{ is convex}$$

The converse is not true.

3.3 First-order conditions

Property 3.4 (First-order condition for convexity)**.** Let $f : \mathbb{R}^n \longrightarrow \bar{\mathbb{R}}$ be a differentiable function, that is that $\nabla f(x)$ exists for all $x \in \operatorname{dom} f$. Then, f is convex if and only if $\operatorname{dom} f$ is convex and:

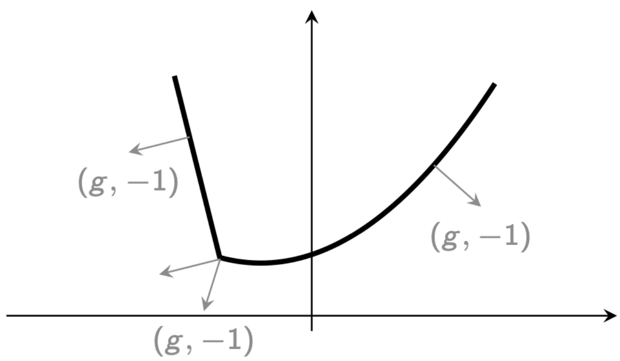
$$\forall x, y \in \operatorname{dom} f, \quad f(y) \geqslant f(x) + \nabla f(x)^\top (y - x)$$

In general, the function f might not be differentiable. In this case, we can use the subdifferential, a generalization of the local variation of a function, to characterize the convexity of f.

Recall that a supporting hyperplane $(g, -1)$ of $\operatorname{epi} f$ at $(x, f(x))$ is a hyperplane such that:

$$\forall y \in \mathbb{R}^n, \quad f(y) \geqslant f(x) + g^\top (y - x)$$

This motivates the following definition.



Definition (Subdifferential)**.** The *subdifferential* of a function $f : \mathbb{R}^n \longrightarrow \bar{\mathbb{R}}$ is the function associating to each point x the set of all supporting hyperplanes of $\operatorname{epi} f$ at $(x, f(x))$:

$$\begin{aligned} \partial f(x) : \mathbb{R}^n &\longrightarrow \mathcal{P}(\mathbb{R}^n) \\ x &\longmapsto \left\{ g \in \mathbb{R}^n \,\middle|\, \forall y \in \mathbb{R}^n, \quad f(y) \geqslant f(x) + g^\top (y - x) \right\} \end{aligned}$$

Any $g \in \partial f(x)$ is called a *subgradient* of f at x.

- If f is differentiable at x and $\partial f(x) \neq \varnothing$, then $\partial f(x) = \{\nabla f(x)\}$.
- If f is convex, and $\partial f(x)$ is a singleton, then $\partial f(x) = \{\nabla f(x)\}$.
- If f is convex but not differentiable at $x \in \operatorname{int} \operatorname{dom} f$, then:

$$\partial f(x) = \overline{\operatorname{Conv} S(x)} \tag{3.3.1}$$

  where $S(x) = \left\{ s \in \mathbb{R}^n \;\middle|\; \nabla f(x_k) \xrightarrow[x_k \to x]{} s \right\}$
- In general, for a convex function f:

$$\partial f(x) = \overline{\operatorname{Conv} S(x)} + N_{\operatorname{dom} f}(x) \tag{3.3.2}$$

Property 3.5 (Existence of subgradient)**.** For finite-valued convex functions, a subgradient exists for every x.

Property 3.6 (Existence of subgradient for extended-valued functions)**.** In the extendevalued setting, let $f : \mathbb{R}^n \longrightarrow \bar{\mathbb{R}}$ be a convex function. Then:

1. Subgradients exist for all x in the relative interior of $\operatorname{dom} f$.
2. Subgradients sometimes exist for x on the relative boundary of $\operatorname{dom} f$.
3. No subgradient exists for x ourside of $\operatorname{dom} f$.

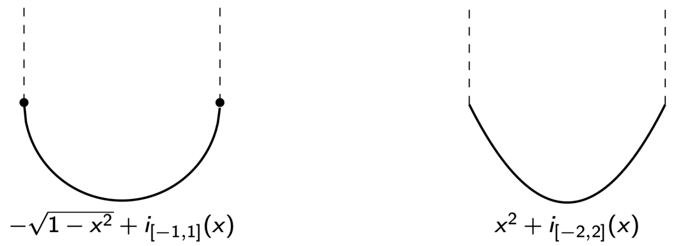


Figure 3.1: Examples for the second case, where boundary points exist on the relative boundary of $\operatorname{dom} f$. No subgradient (affine minorizer) exists for the left function at $x = \pm 1$.

3.4 Second-order conditions

Property 3.7 (Second-order condition for convexity)**.** Let $f : \mathbb{R}^n \longrightarrow \bar{\mathbb{R}}$ be a twice differentiable function (i.e. $\nabla^2 f(x)$ exists for all $x \in \operatorname{dom} f$ which is open). Then, f is convex if and only if $\operatorname{dom} f$ is convex and:

$$\forall x \in \operatorname{dom} f, \quad \nabla^2 f(x) \succcurlyeq 0 \tag{3.4.1}$$

3.5 Examples

In practice, we showed multiple practical ways to establish the convexity of a function:

- By definition, using the convexity criterion.
- By the existence of subgradients for all points of the domain.
- For twice differentiable functions, by checking the positive semidefiniteness of the Hessian.
- By decomposing the function into simpler functions through operations that preserve convexity.

3.5.1 One-dimensional examples

The following functions are convex:

- affine functions: $x \longmapsto ax+b$, $a, b \in \mathbb{R}$
- exponential functions: $x \longmapsto e^{ax}$, $a \in \mathbb{R}$
- power functions: $x : \mathbb{R}_+^* \longmapsto x^\alpha$, $a \geqslant 1$ or $\alpha \leqslant 0$
- powers of absolute value: $x \longmapsto |x|^p$, $p \geqslant 1$
- negative entropy: $x : \mathbb{R}_+^* \longmapsto x \log x$

The following functions are concave:

- affine functions: $x :\longmapsto ax+b$, $a, b \in \mathbb{R}$ (both convex and concave)
- power functions: $x : \mathbb{R}_+^* \longmapsto x^\alpha$, for $0 \leqslant \alpha \leqslant 1$
- logarithm: $x : \mathbb{R}_+^* \longmapsto \log x$

3.5.2 Examples on vectors

The following functions are convex on $\mathbb{R}^n$:

- affine functions $x \longmapsto a^\top x + b$, $a \in \mathbb{R}^n$, $b \in \mathbb{R}$
- norms: $x \longmapsto \|x\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$, $p \geqslant 1$
- quadratic functions:
$$f : x \longmapsto \frac{1}{2} x^\top P x + q^\top x + r$$
with $P \in \mathbb{S}^n$, $q \in \mathbb{R}^n$, $r \in \mathbb{R}$. Indeed, we have;
$$\nabla f(x) = Px + q \quad \text{and} \quad \nabla^2 f(x) = P \succcurlyeq 0$$
- least-squares objective:
$$f : x \longmapsto \|Ax - b\|_2^2$$
with $A \in \mathcal{M}_{m,n}(\mathbb{R})$, $b \in \mathbb{R}^m$. Indeed, we have:
$$\nabla f(x) = 2A^\top(Ax - b) \quad \text{and} \quad \nabla^2 f(x) = 2A^\top A \succcurlyeq 0$$

3.5.3 Examples on matrices

The following functions are convex on $\mathcal{M}_{m,n}(\mathbb{R})$:

- affine functions (convex and concave):
$$X \longmapsto \mathrm{Tr}(A^\top X) + b = \sum_{i=1}^m \sum_{j=1}^n A_{i,j} X_{i,j} + b$$
- spectral norm (maximum singular value):
$$X \longmapsto \|X\|_2 = \sigma_{\max}(X) = \sqrt{\lambda_{\max}(X^\top X)}$$
- in general, all norms are convex

3.5.4 Log-determinant function

The $\log\det$ function, defined on $\mathbb{S}^n$, is concave:
$$f : \mathbb{S}^n \longrightarrow \mathbb{R} X \quad \longmapsto \log\det X$$
with $\mathrm{dom}\, f = \mathbb{S}_{++}^n$. To show this, we will use Property 3.2; we define:
$$\begin{aligned} g_{X,V} : \mathbb{R} &\longrightarrow \mathbb{R} \\ t &\longmapsto \log\det(X + tV) \end{aligned}$$

Note that:

$$\begin{aligned}
g_{X,V}(t) &= \log\det(X + tV) \\
&= \log\det X + \log\det(I + tX^{-1/2}VX^{-1/2}) \\
&= \log\det X + \sum_{i=1}^{n} \log(1 + t\lambda_i)
\end{aligned}$$

where λ_i are the eigenvalues of $X^{-1/2}VX^{-1/2}$.

We then apply the second-order condition to $g_{X,V}$:

$$g''_{X,V}(t) = -\sum_{i=1}^{n} \frac{\lambda_i}{(1 + t\lambda_i)^2} \leqslant 0$$

Therefore, $g_{X,V}$ is concave for any X, V, hence f is concave.

3.5.5 Softmax function

The softmax function, defined on $\mathbb{R}^n$, is convex:

$$\begin{aligned}
f : \mathbb{R}^n &\longrightarrow \mathbb{R} \\
x &\longmapsto \log \sum_{i=1}^{n} e^{x_i}
\end{aligned}$$

If we denote by $z_i = e^{x_i} / \sum_j e^{x_j}$, then we get:

$$\nabla^2 f(x) = \mathrm{diag}(z) - zz^\top$$

with $z_i \geqslant 0$ and $\sum_i z_i = 1$. To show that $\nabla^2 f(x) \succcurlyeq 0$, we show that $\mathrm{diag}(z) - zz^\top$ is positive semidefinite. Let $v \in \mathbb{R}^n$, then:

$$\begin{aligned}
v^\top \nabla^2 f(x) v &= v^\top (\mathrm{diag}(z) - zz^\top) v \\
&= \sum_{i=1}^{n} z_i v_i^2 - \left(\sum_{i=1}^{n} z_i v_i\right)^2
\end{aligned}$$

According to the Cauchy-Schwarz inequality applied to $\sqrt{z_i} \times \sqrt{z_i} v_i$, we have:

$$\left(\sum_{i=1}^{n} z_i v_i\right)^2 \leqslant \sum_{i=1}^{n} z_i \sum_{i=1}^{n} z_i v_i^2 = \sum_{i=1}^{n} z_i v_i^2$$

Therefore, $v^\top \nabla^2 f(x) v \geqslant 0$, and f is convex.

3.6 Convexity-preserving operations

3.6.1 Nonnegative weighted sum

Property 3.8 (Nonnegative scaling)**.** Let $f : \mathbb{R}^n \longrightarrow \bar{\mathbb{R}}$ be a convex function, and $\alpha > 0$. Then, αf is convex.

Property 3.9 (Sum)**.** Let $f_1, f_2 : \mathbb{R}^n \longrightarrow \bar{\mathbb{R}}$ be convex functions. Then, $f_1 + f_2$ is convex; this extends to infinite sums and integrals.

Property 3.10 (Nonnegative weighted sum)**.** Let $f_1, f_2 : \mathbb{R}^n \longrightarrow \bar{\mathbb{R}}$ be convex functions, and $\alpha_1, \alpha_2 > 0$. Then, $\alpha_1 f_1 + \alpha_2 f_2$ is convex; this extends to infinite sums and integrals.

3.6.2 Compositions by an affine function

Property 3.11 (Composition by an affine function)**.** Let $f : \mathbb{R}^n \longrightarrow \bar{R}$ be a convex function and let $A \in \mathcal{M}_{m,}(\mathbb{R})$, $b \in \mathbb{R}^m$. Then:

$$x \longmapsto f(Ax + b) \text{ is convex}$$

Example. The log barrier function for linear inequalities:

$$f(x) = -\sum_{i=1}^{m} \log(b_i - a_i^\top x)$$

with $\operatorname{dom} f = \left\{ x \in \mathbb{R}^n \,\middle|\, \forall i \in [\![1, m]\!], \quad a_i^\top x < b_i \right\}$, is convex.

Example. Any norm of an affine function:

$$f(x) = \|Ax + b\|$$

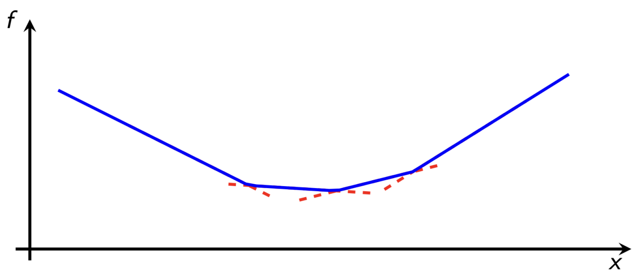
is convex.

3.6.3 Pointwise maximum

Property 3.12 (Pointwise maximum)**.** Let $f_1, f_2 : \mathbb{R}^n \longrightarrow \bar{\mathbb{R}}$ be convex functions. Then, $\max(f_1, f_2)$ is convex. This extends to the pointwise maximum of any finite number of convex functions.

Example. The following piecewise linear function is convex:

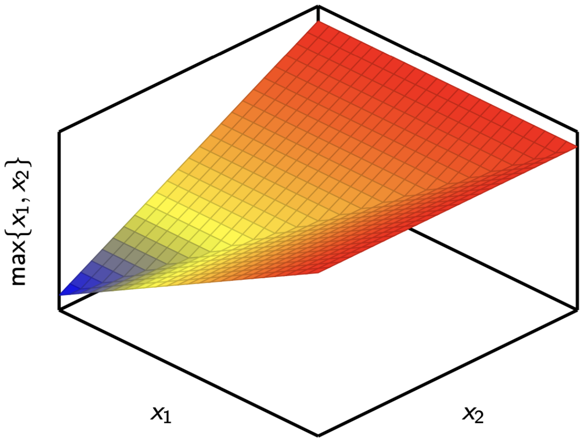
$$f(x) = \max_{i \in [\![1,m]\!]} a_i^\top x + b_i$$



Example (Sum of r largest components)**.** The sum of the r largest components of a vector $x \in \mathbb{R}^n$ is convex:

$$f(x) = x_{(1)} + \cdots + x_{(r)}$$

where $x_{(1)} \geqslant \ldots \geqslant x_{(n)}$ are the components of x sorted in decreasing order.



Indeed, we can write f as:

$$f(x) = \max\left\{ x_{i_1} + x_{i_2} + \cdots + x_{i_r} \mid 1 \leqslant i_1 < i_2 < \cdots < i_r \leqslant n \right\}$$

3.6.4 Pointwise supremum

Property 3.13 (Pointwise supremum)**.** If $\forall y \in \mathcal{A}, \quad x \longmapsto f(x, y)$ is convex, then:

$$g(x) = \sup_{y \in \mathcal{A}} f(x, y)$$

is convex.

Example (Support function)**.** The support function of a set C is convex:

$$S_C(x) = \sup_{y \in C} y^\top x$$

Example (Distance to farthest point)**.** The distance to the farthest point in a set C is convex:

$$f(x) = \sup_{y \in C} \|x - y\|$$

Example (Legendre-Fenchel conjugate)**.** Let $f : \mathbb{R}^n \longrightarrow \bar{\mathbb{R}}$ be a convex function. Then, its Legendre-Fenchel conjugate is convex:

$$f^*(x) = \sup_{y \in \mathbb{R}^n} x^\top y - f(y)$$

3.6.5 Eigenvalues

Property 3.14 (Maximum eigenvalue)**.** The function associating to a symmetric matrix $X \in \mathbb{S}_n$ its maximum eigenvalue is **convex** on $\mathbb{S}_n$:

$$\lambda_{\max}(X) = \sup_{\|y\|_2 = 1} y^\top X y$$

Property 3.15 (Minimum eigenvalue)**.** The function associating to a symmetric matrix $X \in \mathbb{S}_n$ its minimum eigenvalue is **concave** on $\mathbb{S}_n$:

$$\lambda_{\min}(X) = \inf_{\|y\|_2 = 1} y^\top X y$$

3.6.6 Composition with scalar functions

Property 3.16 (Composition with scalar functions)**.** Let $g : \mathbb{R}^n \to \bar{\mathbb{R}}$ and $h : \bar{\mathbb{R}} \to \bar{\mathbb{R}}$ be two functions. We define the composition:

$$f(x) = h(g(x))$$

If either:

- g is convex, h is convex and nondecreasing,
- g is concave, h is convex and nonincreasing,

then f is convex.

Proof. We will only prove the case where $n = 1$ and g, h are twice differentiable. We have:

$$f''(x) = h''(g(x))g'(x)^2 + h'(g(x))g''(x)$$

If h is convex, then $h''(g(x)) \geqslant 0$ and $h''(g(x))g'(x)^2 \geqslant 0$. In the first case, g convex implies that $g''(x) \geqslant 0$, and h nondecreasing implies that $h'(g(x)) \geqslant 0$. Therefore, $f''(x) \geqslant 0$ and f is convex. In the second case, g concave implies that $g''(x) \leqslant$, and h nonincreasing implies that $h'(g(x)) \leqslant$. Therefore, $f''(x) \geqslant 0$ and f is also convex.

Note that the monotonicity must hold for h on the whole domain of g, including the extended values. □

Example. This allows us to deduce the following properties:

- If g is convex then $\exp g$ is convex.
- If g is concave and positive, then $-\log g$ is convex.
- If g is concave and positive, then $1/g$ is convex.
- If g is convex and nonnegative, then for $\alpha \geqslant 1$ we have that g^α is convex.
- For $\alpha \geqslant 1$, then $\|\cdot\|^\alpha$ is convex (with $h = [\cdot]_+^\alpha$, $g = \|\cdot\|$).

Counter-example. The following counter-example shows the importance of the monotonicity of h:

$$g(x) = x^2 \qquad \text{and} \qquad h = \mathbb{1}_{[1,2]}$$

Then, we have the following composition, which is not convex:

$$h(g(x)) = \mathbb{1}_{[-\sqrt{2},-1]\cup[1,\sqrt{2}]}(x)$$

3.6.7 Vector composition

We derive a property similar to Property 3.16 for vector functions.

Property 3.17 (Vector composition)**.** Let $g : \mathbb{R}^n \to \bar{\mathbb{R}}^k$ and $h : \mathbb{R}^k \to \bar{\mathbb{R}}$ be two functions. We define the composition:

$$f(x) = h(g(x)) = h(g_1(x), \dots, g_k(x))$$

If either:

- the g_i are convex, h is convex and nondecreasing in each argument,
- the g_i are concave, h is convex and nonincreasing in each argument,

then f is convex.

Proof. A proof similar to the one of Property 3.16 can be done, by considering the second derivative of f:

$$f''(x) = g'(x)^\top \nabla^2 h(g(x)) g'(x) + \nabla h(g(x))^\top g''(x)$$

This function is positive for similar reasons as in the scalar case. $\square$

Example. This allows us to deduce the following properties:

- If the g_i are concave and positive, then $-\log \sum_{i=1}^m \log g_i$ is convex.
- If the g_i are convex, then $\log \sum_{i=1}^m \exp g_i$ is convex.

3.6.8 Partial minimization

Property 3.18 (Partial minimization)**.** If $f(x, y)$ is convex in (x, y) and C is a non-empty convex set, then the minimization over one variable is convex:

$$g(x) = \inf_{y \in C} f(x, y)$$

Example (Distance to a convex set)**.** The distance to a convex set S is convex:

$$\text{dist}(x, S) = \inf_{y \in S} \|x - y\|$$

4 Convex problems

4.1 Optimization problems in standard form

Definition (Optimization problem)**.** In its standard form, an optimization problem can be written as:

$$\text{minimize } f(x) \quad \text{subject to} \quad \begin{cases} \forall i \in [\![1, m]\!], & g_i(x) \leqslant 0 \\ \forall j \in [\![1, p]\!], & h_j(x) = 0 \end{cases}$$

where:

- $x \in \mathbb{R}^n$ is the optimization variable
- $f : \mathbb{R}^n \to \mathbb{R}$ is the *objectif* or *cost function*
- $g_i : \mathbb{R}^n \to \mathbb{R}$ are the inequality constraint functions
- $h_j : \mathbb{R}^n \to \mathbb{R}$ are the equality constraint functions

Remark. *This form can be generalized to support an infinity of constraints, and strict inequalities. Note that we can assume that the problem is subject only to inequations, without loss of generality: indeed, each equality $h_i(x) = 0$ can be expressed as two inequations $h_i(x) \leqslant 0$ and $-h_i(x) \leqslant 0$.*

Definition (Optimal value)**.** We define the optimal value associated to this optimization problem as:

$$p^* := \inf \{ f(x) \mid \forall i \in [\![1, m]\!],\ g_i(x) \leqslant 0 \quad \text{and} \quad \forall j \in [\![1, p]\!],\ h_j(x) = 0 \}$$

If $p^* = +\infty$, the problem is "infeasible": no x satisfies the constraints.
If $p^* = -\infty$, the problem is *unbounded below*.

Remark. *An optimization problem in standard form has an* implicit constraint *defined by the domain of the constraint functions:*

$$x \in \mathcal{D} := \bigcap_{i=0}^{m} \operatorname{dom} g_i \cap \bigcap_{j=0}^{p} \operatorname{dom} h_j$$

We call $\mathcal{D}$ the domain of the problem. *The constraints $g_i(x) \leqslant 0$ and $h_j(x) = 0$ are the explicit constraints, and the domain of the problem defines the implicit constraints. A problem is* unconstrained *if it has no explicit constraints ($m = p = 0$).*

Example. The following problem is unconstrained:

$$\text{minimize } -\sum_{i=1}^{k} \log(b_i - a_i^\top x)$$

The implit constraints are $a_i^\top x < b_i$ for all $i \in [\![1, k]\!]$.

Definition (Feasibility problem)**.** A feasibility problem is an optimization problem in which we seek a feasible point, i.e. a point that satisfies the constraints. It can be written as:

$$\text{find } x \quad \text{subject to} \quad \begin{cases} \forall i \in [\![1, m]\!], & g_i(x) \leqslant 0 \\ \forall j \in [\![1, p]\!], & h_j(x) = 0 \end{cases}$$

It can be considered a special case of the general problem with $f(x) = 0$:

$$\text{minimize } 0 \quad \text{subject to} \quad \begin{cases} \forall i \in [\![1, m]\!], & g_i(x) \leqslant 0 \\ \forall j \in [\![1, p]\!], & h_j(x) = 0 \end{cases}$$

If constraints are feasible, $p^* = 0$ and any feasible x is optimal.
If constraints are infeasible, $p^* = +\infty$.

4.2 Convex optimization problems

4.2.1 Definition

Definition (Convex Optimization problem)**.** In its standard form, a convex optimization problem can be written as:

$$\text{minimize } f(x) \quad \text{subject to} \quad \begin{cases} \forall i \in [\![1, m]\!], & g_i(x) \leqslant 0 \\ \forall j \in [\![1, p]\!], & a_j^\top x = b_j \end{cases}$$

where the g_i are convex, and the equality constraints are affine.

Such a problem is often written as:

$$\text{minimize } f(x) \quad \text{subject to} \quad \begin{cases} \forall i \in [\![1, m]\!], & g_i(x) \leqslant 0 \\ Ax = b \end{cases}$$

Remark. *The feasible set of a convex optimization problem is convex.*

Example. Consider the following optimization problem:

$$\text{minimize } x_1^2 + x_2^2 \quad \text{subject to} \quad \begin{cases} g_1(x) = x_1/(1 + x_2^2) \leqslant 0 \\ h_1(x) = (x_1 + x_2)^2 = 0 \end{cases}$$

The objective function $f(x) = x_1^2 + x_2^2$ is convex, and the feasible set

$$\{ (x_1, x_2) \mid x_1 = -x_2 \leqslant 0 \}$$

is convex. Nevertheless, this is not a convex problem according to Definition 4.2.1 because the constraint $g_1(x)$ is not convex and h_1 is not affine. We can rewrite this problem in an equivalent but not identical form:

$$\text{minimize } x_1^2 + x_2^2 \quad \text{subject to} \quad \begin{cases} x_1 \leqslant 0 \\ x_1 + x_2 = 0 \end{cases}$$

This problem is now convex according to Definition 4.2.1.

Remark. *One could ask why we enforce this definition for a convex optimization problem, and why we do not open it to more general forms. In general, recognizing a convex optimization problem is a difficult task, and this allows to provide a simple definition that is easy to check. Note that software tools exist to recognize convex optimization problems via composition rules, such as* Disciplined Convex Programming (DCP)*.*

4.2.2 Optimal and locally optimal points

Definition (Feasible point)**.** A point x is *feasible* if $x \in \operatorname{dom} f$ and it satisfies the constraints:

$$\forall i \in [\![1, m]\!],\ g_i(x) \leqslant 0 \quad \text{and} \quad \forall j \in [\![1, p]\!],\ h_j(x) = 0$$

Definition (Optimal point)**.** A feasible point x is *optimal* if $f(x) = p^*$. We denote X_{opt} the set of optimal points.

Definition (Locally optimal point)**.** A point x is *locally optimal* if there is an $R > 0$ such that x is optimal for the problem restricted to the ball $B(x, R)$:

$$\text{minimize } f(z) \quad \text{subject to} \quad \begin{cases} \forall i \in [\![1, m]\!], \quad g_i(x) \leqslant 0 \\ \forall j \in [\![1, p]\!], \quad h_j(z) = 0 \\ \|z - x\|_2 \leqslant R \end{cases}$$

Example. With $n = 1, m = p = 0$:

- $f(x) = x \log x$, we have $\operatorname{dom} f = \mathbb{R}_+^*$, $p^* = -1/e$, and $x = 1/e$ is optimal
- $f(x) = 1/x$, we have $\operatorname{dom} f = \mathbb{R}_+^*$, $p^* = 0$, but no optimal point
- $f(x) = -\log x$, we have $\operatorname{dom} f = \mathbb{R}_+^*$, $p^* = -\infty$
- $f(x) = x^3 - 3x$, we have $p^* = -\infty$ but a local optimum at $x = 1$

Theorem (Global optimality for convex problems)**.** Any locally optimal point of a convex problem is globally optimal.

Proof. Suppose that x is locally optimal and y is optimal with $f(y) < f(x)$. Since x is locally optimal, there is an $R > 0$ such that:

$$\forall z \in B(x, R), \quad z \text{ feasible} \implies f(z) \geqslant f(x)$$

Now consider $z = \theta y + (1 - \theta)x$ with $\theta = R/(2\|y - x\|_2)$. Since $\|y - x\|_2 > R$, we must have $0 < \theta < 1/2$. z is a combination of two feasible points, hence it is feasible since the problem is convex. Finally, $\|z - x\|_2 = R/2$ hence $z \in B(x, R)$, and:

$$f(z) \leqslant \theta f(x) + (1 - \theta) f(y) < f(x)$$

which contradicts the assumption that x is locally optimal. □

4.2.3 Equivalent convex problems

Two problems are informally equivalent if the solution of one is readily obtained from the solution of the other, and vice-versa. In the following, we will see multiple transformations that preserve both the solution and the convexity of an optimization problem.

Eliminating equality constraints Consider the problem:

$$\text{minimize } f(x) \quad \text{subject to} \quad \begin{cases} \forall i \in [\![1, m]\!], \quad g_i(x) \leqslant 0 \\ Ax = b \end{cases}$$

If we can find F and x_0 such that:

$$Ax = b \iff \exists z, \; x = Fz + x_0$$

then we can rewrite the problem as:

$$\text{minimize } f(Fz + x_0) \quad \text{subject to} \quad \forall i \in [\![1, m]\!], \; g_i(Fz + x_0) \leqslant 0$$

For instance, one can choose F such that $\operatorname{Im}(F) = \operatorname{Ker}(A)$ and x_0 such that $Ax_0 = b$.

Introducing equality constraints Reciprocally, the problem:

$$\text{minimize } f(A_0 z + b_0) \quad \text{subject to} \quad \forall i \in [\![1, m]\!],\ g_i(A_i z + b_i) \leqslant 0$$

Can be rewritten as:

$$\text{minimize } f(y_0) \quad \text{subject to} \quad \begin{cases} \forall i \in [\![1, m]\!], & g_i(y_i) \leqslant 0 \\ \forall i \in [\![0, m]\!], & y_i = A_i x + b_i \end{cases}$$

Introducing slack variables for linear inequalities The idea is to replace linear inequalities by linear equalities and non-negativity constraints. Formally, the problem:

$$\text{minimize } f(x) \quad \text{subject to} \quad \forall i \in [\![1, m]\!],\ a_i^\top x \leqslant b_i$$

is equivalent to:

$$\text{minimize } f(y_0) \quad \text{subject to} \quad \begin{cases} \forall i \in [\![1, m]\!],\ a_i^\top x + s_i = b_i \\ \forall i \in [\![1, m]\!],\ s_i \geqslant 0 \end{cases}$$

Epigraph form We saw previously that the epigraph of a convex function is a convex set. We can use this property to rewrite a convex optimization problem in its standard form as:

$$\text{minimize } t \quad \text{subject to} \quad \begin{cases} f(x) - t \leqslant 0 \\ \forall i \in [\![1, m]\!],\ g_i(x) \leqslant 0 \\ Ax = b \end{cases}$$

Minimizing over some variables Consider the problem:

$$\text{minimize } f(x_1, x_2) \quad \text{subject to} \quad \forall i \in [\![1, m]\!],\ g_i(x_1) \leqslant 0$$

This can be rewritten as:

$$\text{minimize } \tilde{f}(x_1) \quad \text{subject to} \quad \forall i \in [\![1, m]\!],\ g_i(x_1) \leqslant 0$$

where $\tilde{f}(x_1) = \inf_{x_2} f(x_1, x_2)$. Said otherwise, we can start by minimizing over the unconstrained variables, and then minimize over the constrained variables.

4.3 Special classes of convex problems

If methods exist to solve general convex optimization problems, some classes of problems have specific structures that can be exploited to design more efficient algorithms. We will present some of these classes in the following.

4.3.1 Linear programming (LP)

Definition (Linear programming problem)**.** A *linear programming* (LP) problem is an optimization problem in which the objective function is affine and the constraints are linear:

$$\text{minimize } c^\top x + d \quad \text{subject to} \begin{cases} Gx \leqslant h \\ Ax = b \end{cases}$$

The feasible set of a linear programming problem is a polyhedron $\mathcal{P}$.

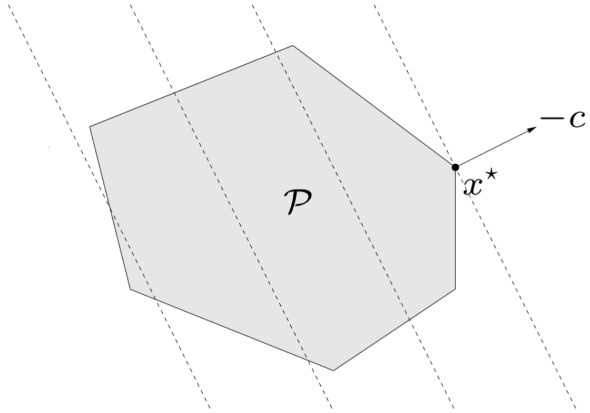


Figure 4.1: A polyhedron $\mathcal{P}$, the feasible set of a linear programming problem.

As an example, we consider the problem of finding the Chebyshev center of a polyhedron.

Definition (Chebyshev center)**.** Given a polyhedron $\mathcal{P}$ of the form:

$$\mathcal{P} = \left\{ x \,\middle|\, \forall i \in [\![1, m]\!],\ a_i^\top x \leqslant b_i, \right\}$$

its *Chebyshev center* is the center of the largest inscribed ball. Recall that the ball $B(x_c, r)$ of center x_c and radius r is defined as:

$$B(x_c, r) = \{ x \mid \|x - x_c\|_2 \leqslant r \}$$

Then, the Chebyshev center $\hat{x}$ is the point x_c that maximizes r:

$$\hat{x} = \underset{x_c, r}{\arg\min} \{ r \in \mathbb{R}_+ \mid B(x_c, r) \subseteq \mathcal{P} \} = \underset{x_c}{\arg\min} \max_{x \in \mathcal{P}} \|x - x_c\|_2$$

Property 4.1 (Chebyshev center as a linear programming problem)**.** The Chebyshev center of a polyhedron $\mathcal{P}$ can be computed as the solution of the following linear programming problem:

$$\text{maximize } r \quad \text{subject to} \quad \forall i \in [\![1, m]\!],\ a_i^\top x_c + r\|a_i\|_2 \leqslant b_i$$

4.3.2 Convex quadratic programming (QP)

Definition (Quadratic programming problem)**.** A *quadratic programming* (QP) problem is an optimization problem in which the objective function is quadratic and the constraints are linear:

$$\text{minimize } \frac{1}{2} x^\top P x + q^\top x + r \quad \text{subject to } \begin{cases} Gx \leqslant h \\ Ax = b \end{cases}$$

where $P \in \mathbb{S}_n^+(\mathbb{R})$ is positive semidefinite.

The feasible set of a quadratic programming problem is a still polyhedron $\mathcal{P}$ (since the constraints have the same form as an LP problem). Solving a QP problem corresponds to minimizing a quadratic function over a polyhedron.

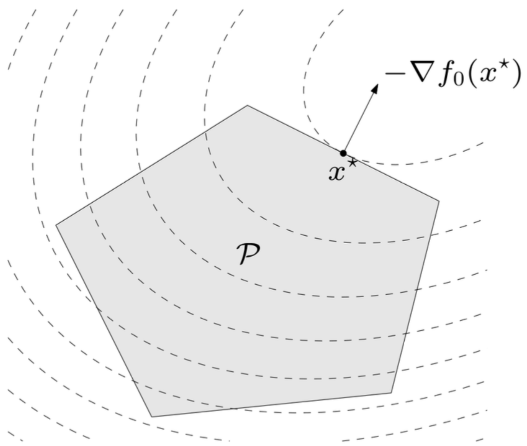


Figure 4.2: A quadratic function over a polyhedron.

Example (Least squares problem)**.** The least squares problem can be written as a QP problem:

$$\text{minimize } \|Ax - b\|_2^2$$

The analytical solution can be expressed using the Moore-Penrose pseudo-inverse of A, $A^\dagger$:

$$x^* = A^\dagger b = (A^\top A)^{-1} A^\top b$$

Linear constraints such as $I \leqslant x \leqslant u$ can be added.

Another common variant is the LASSO regularization:

$$\text{minimize } \|Ax - b\|_2^2 + \lambda \|x\|_1$$

Example (Linear program with random cost)**.** Consider a random vector $c \in \mathbb{R}^n$ with mean $\mathbb{E}[c] =: \bar{c}$ and covariance matrix $\mathbb{V}(c) =: \Sigma$. The following linear program is often used in economics and finance:

$$\text{minimize } \mathbb{E}[c^\top x] + \gamma \mathbb{V}(c^\top x) = \bar{c}^\top x + \gamma x^\top \Sigma x \quad \text{subject to} \quad \begin{cases} Gx \leqslant h \\ Ax = b \end{cases}$$

Where $\gamma > 0$ is a risk-aversion parameter: it controls how much we penalize the variance of the cost. Higher values of γ allow for solutions with higher expected cost but lower variance.

4.3.3 Quadratically constrained quadratic programming (QCQP)

Definition (Quadratically constrained quadratic programming problem)**.** A *quadratically constrained quadratic programming* (QCQP) problem is an optimization problem in which both the objective function and the constraints are quadratic:

$$\text{minimize } \frac{1}{2} x^\top P_0 x + q_0^\top x + r_0 \quad \text{subject to} \begin{cases} \frac{1}{2} x^\top P_i x + q_i^\top x + r_i \leqslant 0, \quad \forall i \in [\![1, m]\!] \\ Ax = b \end{cases}$$

where the objective and constraints matrices $P_i \in \mathbb{S}_n^+(\mathbb{R})$ are positive semidefinite. In the case where $P_1, \ldots, P_m \in \mathbb{S}_n^{++}(\mathbb{R})$ are positive definite, the feasible region is the intersection of m ellipsoids and an affine set.

4.3.4 Second-order cone programming (SOCP)

Definition (Second-order cone programming problem)**.** A *second-order cone programming* (SOCP) problem is an optimization problem in which the objective function is linear, and the constraints are second-order cones:

$$\text{minimize } f^\top x \quad \text{subject to } \begin{cases} \|A_i x + b_i\|_2 \leqslant c_i^\top x + d_i, & \forall i \in [\![1, m]\!] \\ Fx = g \end{cases}$$

where $f \in \mathbb{R}^n$, $A_i \in \mathscr{M}_{n_i,n}(\mathbb{R})$, $b_i \in \mathbb{R}^{n_i}$, $c_i \in \mathbb{R}^{n_i}$, $d_i \in \mathbb{R}$, $F \in \mathscr{M}_{p,n}(\mathbb{R})$, $g \in \mathbb{R}^p$.

Inequalities are second-order cone (SOC) constraints:

$$(A_i x + b_i, c_i^\top x + d_i) \in \text{second-order cone in } \mathbb{R}^{n_i+1}$$

For $n_i = 0$, this reduces to an LP problem. If $c_i = 0$, this reduces to a QCQP problem. In general, SOCP problems are more general than LP and QCQP problems.

4.4 Robust linear programming

4.4.1 Introduction

In many situations, the parameters of an optimization problems might be uncertain. For instance, in an LP problem such as:

$$\text{minimize } c^\top x \quad \text{subject to} \quad \forall i \in [\![1, m]\!],\ a_i^\top x \leqslant b_i$$

there can be uncertainty on the values of c, a_i, b_i, which can be modeled as taking any value in given intervals. There are two common approaches to handle this uncertainty: deterministic and stochastic models. Assume that the value of a_i can be any value in the set $\mathcal{E}_i$.

A **deterministic model** solves a harder problem, in which the constraints must hold for all $a_i \in \mathcal{E}_i$. This can be written as:

$$\text{minimize } c^\top x \quad \text{subject to} \quad \forall i \in [\![1, m]\!],\ \underbrace{\forall a_i \in \mathcal{E}_i}_{\text{uncertainty}},\ a_i^\top x \leqslant b_i$$

A **stochastic model** uses chance constraints: we consider that a_i is a random variable, and we require that the constraints hold with a given probability η. This can be written as:

$$\text{minimize } c^\top x \quad \text{subject to} \quad \forall i \in [\![1, m]\!],\ \underbrace{\mathbb{P}(a_i^\top x \leqslant b_i) \geqslant \eta}_{\text{probability constraint}}$$

In the following, we will develop both approaches using SOCP.

4.4.2 Deterministic approach via SOCP

We can model the uncertainty on a_i by choosing an ellipsoid as $\mathcal{E}_i$:

$$\mathcal{E}_i = \{\, \bar{a}_i + P_i u \mid \|u\|_2 \leqslant 1 \,\}$$

where $\bar{a}_i \in \mathbb{R}^n$ are the centers, and the semi-axes are determined by the singular values and vectors of $P_i \in \mathscr{M}_n(\mathbb{R})$. Therefore, the robust LP problem of the form:

$$\text{minimize } c^\top x \quad \text{subject to} \quad \forall i \in [\![1, m]\!],\ \forall a_i \in \mathcal{E}_i,\ a_i^\top x \leqslant b_i$$

can be written as the following SOCP problem:

$$\text{minimize } c^\top x \quad \text{subject to} \quad \forall i \in [\![1, m]\!],\ \bar{a}_i^\top x + \|P_i^\top x\|_2 \leqslant b_i$$

This comes from the fact that:

$$\sup_{\|u\|_2 \leqslant 1} (\bar{a}_i + P_i u)^\top x = \bar{a}_i^\top x + \|P_i^\top x\|_2$$

hence, if $\bar{a}_i^\top x + \|P_i^\top x\|_2 \leqslant b_i$ holds, then $a_i^\top x \leqslant b_i$ holds for all $a_i \in \mathcal{E}_i$.

4.4.3 Stochastic approach via SOCP

Assume that a_i is Gaussian with mean $\bar{a}_i$ and covariance matrix Σ_i:

$$a_i \sim \mathcal{N}(\bar{a}_i, \Sigma_i)$$

Therefore, $a_i^\top x$ is a Gaussian random vector with mean $\bar{a}_i^\top x$ and variance $x^\top \Sigma_i x$. We can model the probability constraint as:

$$\mathbb{P}(a_i^\top x \leqslant b_i) = \Phi\left(\frac{b_i - \bar{a}_i^\top x}{\|\Sigma_i^{1/2} x\|_2}\right)$$

where $\Phi(x) = (1/\sqrt{2\pi}) \int_{-\infty}^{x} e^{-t^2/2} dt$ is the *Cumulative Distribution Function* (CDF) of $\mathcal{N}(0, 1)$. The robust LP problem can be written as:

$$\text{minimize } c^\top x \quad \text{subject to} \quad \forall i \in [\![1, m]\!],\ \mathbb{P}(a_i^\top x \leqslant b_i) \geqslant \eta$$

This can be written as the following SOCP problem:

$$\text{minimize } c^\top x \quad \text{subject to} \quad \forall i \in [\![1, m]\!],\ \bar{a}_i^\top x + \Phi^{-1}(\eta) \|\Sigma_i^{1/2} x\|_2 \leqslant b_i$$

which is an SOCP problem when $\Phi^{-1}(\eta) \geqslant 0$, verified for $\eta \geqslant 1/2$. For values of $\eta < 1/2$, the problem is non-convex, as can be seen in the following visualization.

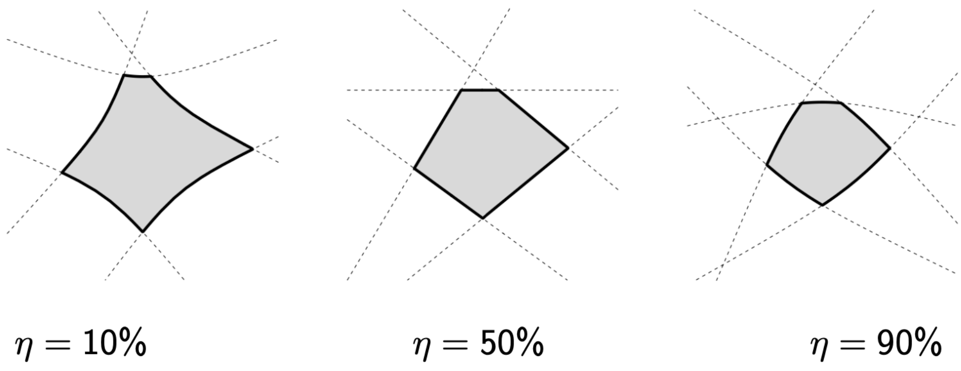


Figure 4.3: The set $\left\{ x \,\middle|\, \forall i \in [\![1, m]\!],\ \mathbb{P}(a_i^\top x \leqslant b_i) \geqslant \eta \right\}$ for multiple values of η. It is convex for $\eta \geqslant 1/2$ and in general non-convex for $\eta < 1/2$.

4.5 Generalized inequalities

For multiple reasons, it can be useful to consider more general inequalities than the standard ones, in the sense that the inequalities are not necessarily defined on the real line. This is mainly motivated by semi-definite programming, where the inequalities must hold on the cone of positive semidefinite matrices. We first define a partial ordering $\succcurlyeq$ on a cone K, and use it to introduce generalized inequalities.

4.5.1 Convex cone properties

Recall that a set K is a convex cone if:

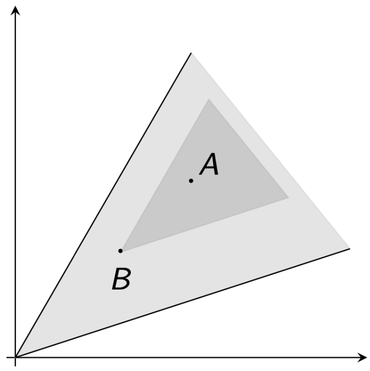
$$x_1, x_1 \in K \implies \forall \theta_1, \theta_2 > 0, \quad \theta_1 x_1 + \theta_2 x_2 \in K$$

Definition. A convex cone K can have the following properties:

- *pointed*: if it contains 0
- *salient*: if it contains no line, that is $x \in K \wedge -x \in K \implies x = 0$
- *solid*: if it has a non-empty interior ($\mathring{K} \neq \varnothing$)
- *closed*: if $K^{\complement}$ is an open set

Definition (Notation)**.** Let K be a pointed, salient and solid convex cone. Let A and B be two points of the ambient space.

- We note $A \succcurlyeq_K 0$ if and only if $A \in K$.
- We note $A \succcurlyeq_K B$ if and only if $A - B \succcurlyeq_K 0$.
- We note $A \succ_K 0$ if and only if $A \in \mathring{K}$ (which makes sense if K is solid).



Property 4.2. $\succcurlyeq_K$ defines a partial ordering.

Proof.

- K is pointed, hence $0 \in K$, hence for any A we have that $A - A \succcurlyeq_K 0$ and $A \succcurlyeq_K A$.
- $A \succcurlyeq_K B$ and $B \succcurlyeq_K A$ implies that $A - B \in K$ and $B - A \in K$. Since K is salient, $A - B = 0$ and $A = B$.
- $A \succcurlyeq_K B$ and $B \succcurlyeq_K C$ implies that $A - B \in K$ and $B - C \in K$. Therefore, $A - C = (A - B) + (B - C) \in K$ since K is a cone, and therefore $A \succcurlyeq_K C$.

□

Definition (Convex problem with generalized inequality constraints)**.** A *convex optimization problem with generalized inequality* constraints is a problem of the form:

$$\text{minimize } f(x) \quad \text{subject to} \quad \begin{cases} \forall i \in [\![1, m]\!],\ g_i(x) \preccurlyeq_{K_i} 0 \\ Ax = b \end{cases}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is convex, and the $g_i : \mathbb{R}^n \to \mathbb{R}^{k_i}$ are K_i-convex with respect to the proper cones K_i:

$$\forall i \in [\![1, m]\!],\ \forall x, y \in \mathbb{R}^n,\ \forall \theta \in [0, 1], \quad g_i(\theta x + (1 - \theta) y) \preccurlyeq_{K_i} \theta g_i(x) + (1 - \theta) g_i(y)$$

A convex optimization problem with generalized inequality constraints as the same properties as a standard convex problem: its feasible set is convex, any local optimum is a global optimum, etc. When the K_i are clear from the context, we can simply use $\preccurlyeq$ and omit the K_i.

Definition (Conic convex optimization problem)**.** A *conic convex optimization problem* is a special case of the generalized convex problem, where the objective and the constraints are affine:

$$\text{minimize } c^\top x \quad \text{subject to} \quad \begin{cases} Fx + g \preccurlyeq_K 0 \\ Ax = b \end{cases}$$

where $c \in \mathbb{R}^n$, $F \in \mathscr{M}_{p,n}(\mathbb{R})$, $g \in \mathbb{R}^p$, $A \in \mathscr{M}_{m,n}(\mathbb{R})$. This extends linear programming to non-polyedral cones. The linear programming case can be obtained by choosing $K = \mathbb{R}^m_+$.

4.5.2 Semidefinite programming (SDP)

Definition. A *semidefinite programming* (SDP) problem is a problem of the form:

$$\text{minimize } c^\top x \quad \text{subject to} \quad \begin{cases} x_1 G_1 + \cdots + x_n G_n + H \preccurlyeq 0 \\ Ax = b \end{cases}$$

where $G_i, H \in \mathbb{S}_k(\mathbb{R})$.

Remark. *The inequality constraint is called a* linear matrix inequality *(LMI). Note that problems with multiple LMIT constraints can be transformed into a single LMI constraint of higher dimension. Suppose that you have the constraints:*

$$x_1\hat{G}_1 + \cdots + x_n\hat{G}_n + \hat{H} \preccurlyeq 0 \quad \textit{and} \quad x_1\tilde{G}_1 + \cdots + x_n\tilde{G}_n + \tilde{H} \preccurlyeq 0$$

They can be rewritten as a single LMI:

$$x_1 \begin{bmatrix} \hat{G}_1 & 0 \\ 0 & \tilde{G}_1 \end{bmatrix} + \cdots + x_n \begin{bmatrix} \hat{G}_n & 0 \\ 0 & \tilde{G}_n \end{bmatrix} + \begin{bmatrix} \hat{H} & 0 \\ 0 & \tilde{H} \end{bmatrix} \preccurlyeq 0$$

Example (Largest eigenvalue minimization)**.** Consider the following problem:

$$\text{minimize } \lambda_{\max}(A(x))$$

where $A(x) = A_0 + x_1 A_x + \cdots + x_n A_n$ and $A_0, A_1, \ldots, A_n \in \mathbb{S}_k(\mathbb{R})$. This problem can be written as an equivalent SDP problem:

$$\text{minimize } t \quad \text{subject to} \quad A(x) \preccurlyeq tI$$

over variables $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$. This follows from:

$$\lambda_{\max}(A(x)) \leqslant t \iff A(x) \preccurlyeq tI$$

Example (Matrix norm minimization)**.** Consider the following problem:

$$\text{minimize } \|A(x)\|_2 = \sqrt{\lambda_{\max}(A(x)^\top A(x))}$$

where $A(x) = A_0 + x_1 A_x + \cdots + x_n A_n$ and $A_i \in \mathbb{S}_k(\mathbb{R})$. This problem can be written as an equivalent SDP problem:

$$\text{minimize } t \quad \text{subject to} \quad \begin{bmatrix} tI & A(x) \\ A(x)^\top & tI \end{bmatrix} \succeq 0$$

over variables $x \in \mathbb{R}^n$ and $t \in \mathbb{R}_+$. This follows from:

$$\|A(x)\|_2 \leqslant t \iff A^\top A \preccurlyeq t^2 I \iff \begin{bmatrix} tI & A(x) \\ A(x)^\top & tI \end{bmatrix} \succeq 0$$

where the last equivalence comes from the Schur complement.

4.5.3 LPs and SOCPs as SDPs

Property 4.3 (Any LP problem is equivalent to an SDP problem)**.** Consider the LP problem:

$$\text{minimize } c^\top x \quad \text{subject to} \quad Ax \leqslant b$$

This can be written as the following SDP problem:

$$\text{minimize } c^\top x \quad \text{subject to} \quad \operatorname{diag}(Ax - b) \preccurlyeq 0$$

Property 4.4 (Any SOCP problem is equivalent to an SDP problem)**.** Consider the SOCP problem:

$$\text{minimize } f^\top x \quad \text{subject to} \quad \forall i \in [\![1, m]\!],\ \|A_i x + b_i\|_2 \leqslant c_i^\top x + d_i$$

This can be written as the following SDP problem:

$$\text{minimize } f^\top x \quad \text{subject to} \quad \forall i \in [\![1, m]\!],\ \begin{bmatrix} c_i^\top x + d_i & A_i x + b_i \\ (A_i x + b_i)^\top & c_i^\top x + d_i \end{bmatrix} \succcurlyeq 0$$

4.6 Quasi-convex problems

4.6.1 Quasi-convex functions

4.6.2 Quasi-convex optimization

4.7 Examples

4.7.1 Regression

4.7.2 Classification