Convex Optimization

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Abstract

This document is Antoine Groudiev's class notes while following the class *Deep Learning* at the Computer Science Department of ENS Ulm. It is freely inspired by the lectures of Adrien Taylor.

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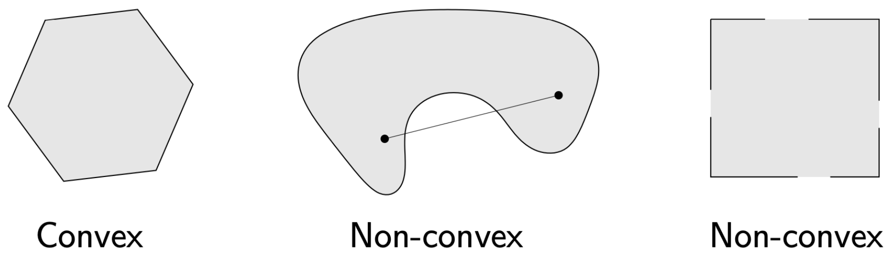
2.1 Definitions

Definition (Convex set)**.** A set C is a *convex set* if every segment that connects two points in C is in C. Formally:

$$\forall x, y \in C, \forall \theta \in [0,1], \quad \theta x + (1-\theta)y \in C$$

Example. Here are some examples of convex and non-convex sets:



In many cases, we will use proper (i.e. non-empty) convex sets, and closed convex sets.

Definition (Convex hull)**.** The *convex hull* of S, denoted $\mathrm{Conv}(S)$, is the smallest convex set that contains S.

Definition (Convex combinations)**.** The *convex combinations* of $x_1, \ldots, x_k$ are all the point x of the form:

$$x = \theta_1 x_1 + \cdots + \theta_k x_k$$

with $\theta_1, \ldots, \theta_k \geqslant 0$ and $\sum_{i=1}^k \theta_i = 1$.

Property 2.1. The convex hull of a set S is the set of all convex combinations of points in S:

$$\mathrm{Conv}(S) = \left\{ \sum_{i=1}^k \theta_i x_i \,\middle|\, (x_i) \in S^k, (\theta_i) \in \mathbb{R}_+^k, \sum_{i=1}^k \theta_i = 1 \right\}$$

2.2 Examples

2.2.1 Hyperplanes and halfspaces

Definition (Hyperplane)**.** A *hyperplane* is the set of the form:

$$H = \left\{ x \,\middle|\, a^\top x = b \right\}$$

for some $a \in \mathbb{R}^n \setminus \{0\}$ and $b \in \mathbb{R}$. a is called the *normal vector* of H. Hyperspaces are affine and convex.

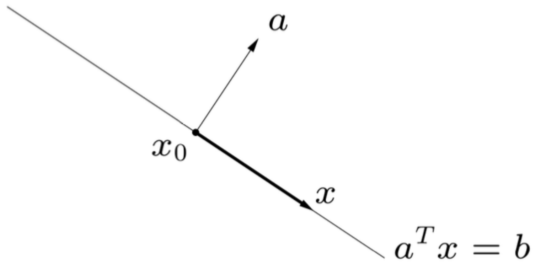


Figure 2.1: Hyperplane

Definition (Halfspace)**.** A *halfspace* is the set of the form:

$$H = \left\{ x \,\middle|\, a^\top x \leqslant b \right\}$$

for some $a \in \mathbb{R}^n \setminus \{0\}$ and $b \in \mathbb{R}$. a is called the *normal vector* of H. Halfspaces are convex.

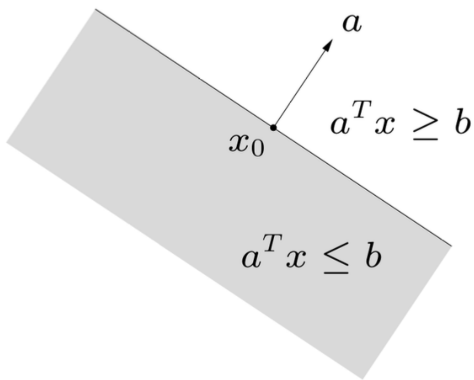


Figure 2.2: Halfspace

2.2.2 Euclidian balls and ellipsoids

Definition (Euclidian ball)**.** The *Euclidian ball* of center x_c and radius r is the set:

$$B(x_c, r) = \{ x \mid \|x - x_c\|_2 \leqslant r \} = \{ x_c + ru \mid \|u\|_2 \leqslant 1 \}$$

Euclidian balls are convex.

Definition (Ellipsoid)**.** An *ellipsoid* is the set of the form:

$$E = \left\{ x \,\middle|\, (x - x_c)^\top P^{-1}(x - x_c) \leqslant 1 \right\}$$

with $P \in \mathbb{S}^n_{++}$[1] and $x_c \in \mathbb{R}^n$. Ellipsoids are convex.

[1] $\mathbb{S}^n_{++}$ denotes the set of symmetric positive definite matrices of size n

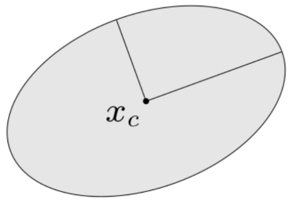


Figure 2.3: Ellipsoid

An alternative representation of an ellipsoid is:

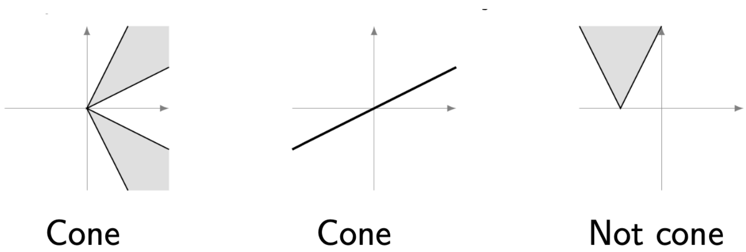
$$E = \{ x_c + Au \mid \|u\|_2 \leqslant 1 \}$$

for some nonsingular matrix $A \in \mathrm{GL}_n(\mathbb{R})$. We can choose A symmatric and positive definite without loss of generality, for instance by choosing $A = P^{1/2}$.

2.2.3 Cones

Definition (Cones)**.** A set K is a *cone*, or a *nonnegative homogeneous set*, if:

$$\forall x \in K, \forall \theta \in \mathbb{R}_+^*, \quad \theta x \in K$$



Definition (Convex cone)**.** A set K is a *convex cone* if:

$$\forall x_1, x_2 \in K, \forall \theta_1, \theta_2 \in \mathbb{R}_+^*, \quad \theta_1 x_1 + \theta_2 x_2 \in K$$

Special cases of cones include:

Positive orthant $K = \mathbb{R}_+^n = \{ x \in \mathbb{R}^n \mid x_i \geqslant 0, \forall i \}$

Norm cones $K = \{ (x,t) \in \mathbb{R}^n \times \mathbb{R} \mid \|x\| \leqslant t \}$. A particular case is the second-order cone (SOC), based on the ℓ_2 norm.

Positive polynomials $K_n = \{ x \in \mathbb{R}^{n+1} \mid \forall t \in \mathbb{R}, \sum_{i=0}^n x_i t^i \geqslant 0 \}$

Positive semidefinite cone $\mathbb{S}_+^n = \left\{ X \in \mathbb{S}^n \,\middle|\, \forall z \in \mathbb{R}^n, z^\top X z \geqslant 0 \right\}$

Co-positive cone $\mathbb{S}_+^n = \left\{ X \in \mathbb{S}^n \,\middle|\, \forall z \in \mathbb{R}_+^n, z^\top X z \geqslant 0 \right\}$

Exponential cone $\left\{ (x, y, z) \in \mathbb{R} \times \mathbb{R}_+^* \times \mathbb{R} \,\middle|\, z \geqslant y e^{x/y} \right\}$

2.3 Convexity-preserving operations

3 Convex functions

4 Convex problems