Convex Optimization

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ENS | PSL

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Contents

Abstract

This document is Antoine Groudiev's class notes while following the class *Deep Learning* at the Computer Science Department of ENS Ulm. It is freely inspired by the lectures of Adrien Taylor.

1 Introduction

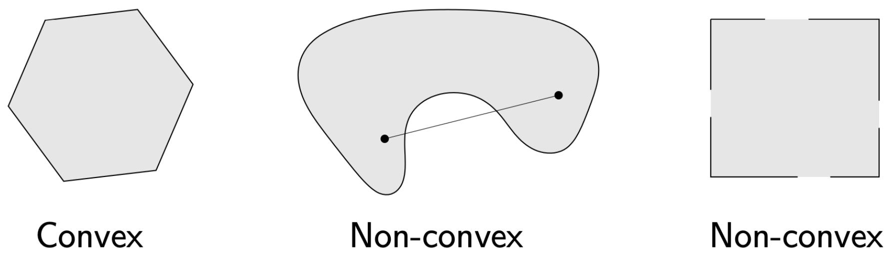
2 Convex sets

2.1 Definitions

Definition (Convex set)**.** A set C is a *convex set* if every segment that connects two points in C is in C. Formally:

$$\forall x, y \in C, \forall \theta \in [0,1], \quad \theta x + (1-\theta) y \in C$$

Example. Here are some examples of convex and non-convex sets:



In many cases, we will use proper (i.e. non-empty) convex sets, and closed convex sets.

Definition (Convex hull)**.** The *convex hull* of S, denoted $\mathrm{Conv}(S)$, is the smallest convex set that contains S.

Definition (Convex combinations)**.** The *convex combinations* of $x_1, \dots, x_k$ are all the point x of the form:

$$x = \theta_1 x_1 + \cdots + \theta_k x_k$$

with $\theta_1, \dots, \theta_k \geqslant 0$ and $\sum_{i=1}^k \theta_i = 1$.

Property 2.1. The convex hull of a set S is the set of all convex combinations of points in S:

$$\mathrm{Conv}(S) = \left\{ \sum_{i=1}^k \theta_i x_i \,\middle|\, (x_i) \in S^k, (\theta_i) \in \mathbb{R}_+^k, \sum_{i=1}^k \theta_i = 1 \right\}$$

2.2 Examples

2.2.1 Hyperplanes and halfspaces

Definition (Hyperplane)**.** A *hyperplane* is the set of the form:

$$H = \left\{ x \,\middle|\, a^\top x = b \right\}$$

for some $a \in \mathbb{R}^n \setminus \{0\}$ and $b \in \mathbb{R}$. a is called the *normal vector* of H. Hyperspaces are affine and convex.

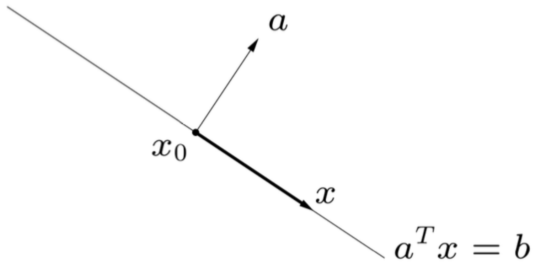


Figure 2.1: Hyperplane

Definition (Halfspace)**.** A *halfspace* is the set of the form:

$$H = \left\{ x \;\middle|\; a^\top x \leqslant b \right\}$$

for some $a \in \mathbb{R}^n \setminus \{0\}$ and $b \in \mathbb{R}$. a is called the *normal vector* of H. Halfspaces are convex.

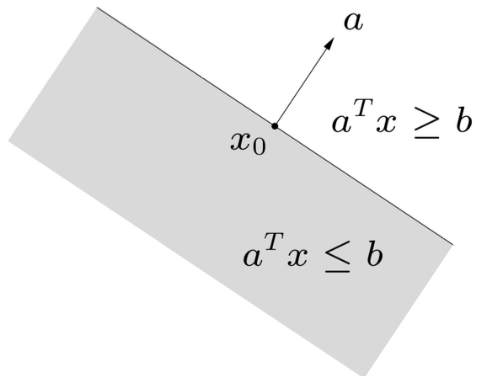


Figure 2.2: Halfspace

2.2.2 Euclidian balls and ellipsoids

Definition (Euclidian ball)**.** The *Euclidian ball* of center x_c and radius r is the set:

$$B(x_c, r) = \{\, x \mid \|x - x_c\|_2 \leqslant r \,\} = \{\, x_c + ru \mid \|u\|_2 \leqslant 1 \,\}$$

Euclidian balls are convex.

Definition (Ellipsoid)**.** An *ellipsoid* is the set of the form:

$$E = \left\{ x \;\middle|\; (x - x_c)^\top P^{-1}(x - x_c) \leqslant 1 \right\}$$

with $P \in \mathbb{S}^n_{++}$[1] and $x_c \in \mathbb{R}^n$. Ellipsoids are convex.

[1] $\mathbb{S}^n_{++}$ denotes the set of symmetric positive definite matrices of size n

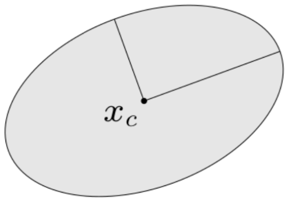


Figure 2.3: Ellipsoid

An alternative representation of an ellipsoid is:

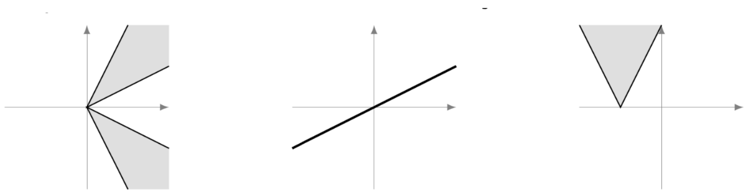
$$E = \{\, x_c + Au \mid \|u\|_2 \leqslant 1 \,\}$$

for some nonsingular matrix $A \in \mathrm{GL}_n(\mathbb{R})$. We can choose A symmatric and positive definite without loss of generality, for instance by choosing $A = P^{1/2}$.

2.2.3 Cones

Definition (Cones)**.** A set K is a *cone*, or a *nonnegative homogeneous set*, if:

$$\forall x \in K, \forall \theta \in \mathbb{R}_+^*, \quad \theta x \in K$$



Cone Cone Not cone

Definition (Convex cone)**.** A set K is a *convex cone* if:

$$\forall x_1, x_2 \in K, \forall \theta_1, \theta_2 \in \mathbb{R}_+^*, \quad \theta_1 x_1 + \theta_2 x_2 \in K$$

Non-convex Convex

In the followings, we will denote by:

- $\mathbb{S}^n \subset \mathbb{R}^{n \times n}$ the set of symmetric matrices of size n
- $\mathbb{S}_+^n$ the set of positive semidefinite matrices of size n, that is matrices verifying:

$$\forall z \in \mathbb{R}^n, z^\top X z \geqslant 0$$

  also denoted $X \succcurlyeq 0$.

- $\mathbb{S}^n_{++}$ the set of positive definite matrices of size n, that is matrices verifying:

$$\forall z \in \mathbb{R}^n, z^\top X z > 0$$

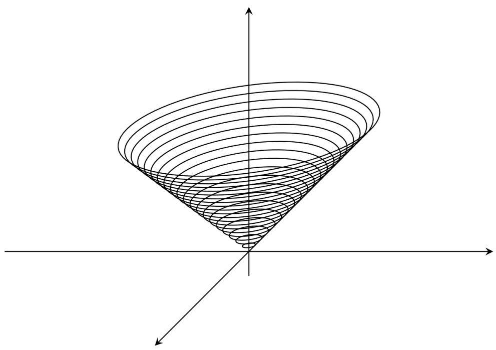
also denoted $X \succ 0$.

$\mathbb{S}^n_+$ and $\mathbb{S}^n_{++}$ are convex cones.

Special cases of cones include:

Positive orthant $K = \mathbb{R}^n_+ = \{\, x \in \mathbb{R}^n \mid x_i \geqslant 0, \forall i \,\}$

Norm cones $K = \{\, (x,t) \in \mathbb{R}^n \times \mathbb{R} \mid \|x\| \leqslant t \,\}$. A particular case is the second-order cone (SOC), based on the ℓ_2 norm.



Positive polynomials $K_n = \{\, x \in \mathbb{R}^{n+1} \mid \forall t \in \mathbb{R}, \sum_{i=0}^n x_i t^i \geqslant 0 \,\}$

Positive semidefinite cone $\mathbb{S}^n_+ = \left\{ X \in \mathbb{S}^n \,\middle|\, \forall z \in \mathbb{R}^n, z^\top X z \geqslant 0 \right\}$

Co-positive cone $\mathbb{S}^n_+ = \left\{ X \in \mathbb{S}^n \,\middle|\, \forall z \in \mathbb{R}^n_+, z^\top X z \geqslant 0 \right\}$

Exponential cone $\left\{ (x,y,z) \in \mathbb{R} \times \mathbb{R}^*_+ \times \mathbb{R} \,\middle|\, z \geqslant y e^{x/y} \right\}$

Definition (Dual cones)**.** The *dual cone* to a convex cone K is the set:

$$K^* = \left\{ y \,\middle|\, \forall x \in K, \quad y^\top x \geqslant 0 \right\}$$

Convex cones and their duals are particularly useful for convex duality. A convex cone that satisfies $K = K^*$ is called *self-dual.*

Definition (Polar cones)**.** The *polar cone* to a convex cone K is the set:

$$K^\diamond = \left\{ y \,\middle|\, \forall x \in K, \quad y^\top x \leqslant 0 \right\}$$

We have the identity $K^\diamond = -K^*$.

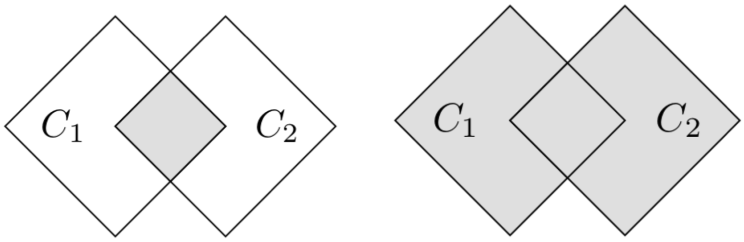
2.3 Convexity-preserving operations

To establish the convexity of a set C, the most basic approach is to apply the definition by proving that every segment that connects two points in C is in C. However, this can be tedious in practice. Instead, we can use operations that preserve convexity.

2.3.1 Intersection and union

Property 2.2 (Convexity is preserved by intersection)**.** For any convex sets C_1 and C_2, the intersection $C_1 \cap C_2$ is convex.

Likewise, the intersection of an arbitrary number of convex sets is convex.

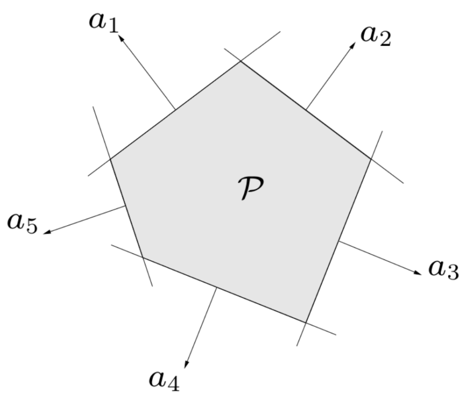


Remark. *The union of convex sets is not necessarily convex. For instance in* $\mathbb{R}$*, both* $[0,1]$ *and* $[2,3]$ *are convex, but their union* $[0,1] \cup [2,3]$ *is not.*

Definition (Polyhedron)**.** A *polyhedron* is the solution set of finitely many linear inequalities and equalities:

$$S = \{\, x \in \mathbb{R}^n \mid Ax \leqslant b, Cx = d \,\}$$

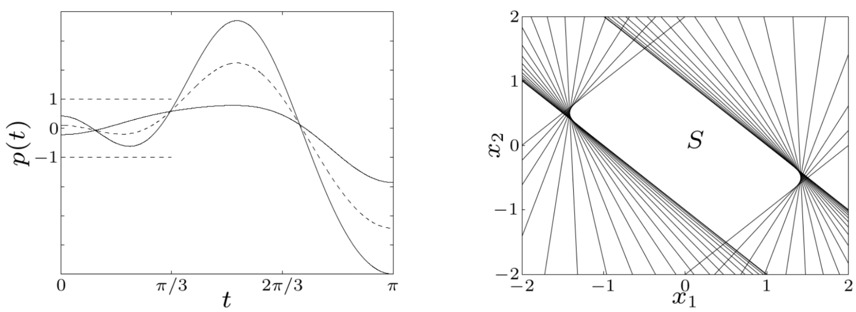
for $A \in \mathscr{M}_{m,n}(\mathbb{R})$ and $C \in \mathscr{M}_{p,n}(\mathbb{R})$. Polyhedra are convex, since they are the intersection of halfspaces and hyperplanes which are convex.



Example. Let:

$$S = \left\{ x \in \mathbb{R}^m \;\middle|\; \forall t \in \mathbb{R}, \quad |t| \leqslant \frac{\pi}{3} \implies \left| \sum_{k=1}^{m} x_k \cos(kt) \right| \leqslant 1 \right\}$$

S is convex, since it can be written as the intersection of convex sets.



Example. $\mathbb{S}_+^n$ is convex since it is the intersection of convex sets:

$$\mathbb{S}_+^n = \left\{ X \in \mathbb{S}^n \,\middle|\, \forall z \in \mathbb{R}^n, z^\top X z \geqslant 0 \right\} = \mathbb{S}^n \cap \bigcap_{z \in \mathbb{R}^n} \left\{ X \in \mathscr{M}_n(\mathbb{R}) \,\middle|\, z^\top X z \geqslant 0 \right\}$$

Each set $\left\{ X \in \mathscr{M}_n(\mathbb{R}) \,\middle|\, z^\top X z \geqslant 0 \right\}$ being convex, their intersection is convex. In particular, it doesn't matter if the number of sets is finite, countable or uncountable.

2.3.2 Affine functions

Property 2.3 (The image of a convex set by an affine function is convex)**.** If $L : \mathbb{R}^n \to \mathbb{R}^m$ is an affine function, then if C is convex, $L(C)$ is convex.

More explicitly, let $A \in \mathscr{M}_{m,n}(\mathbb{R})$ and $b \in \mathbb{R}^m$. The affine function $L(x) = Ax + b$ maps C to $L(C) = \{ y \in \mathbb{R}^m \mid \exists x \in C, y = Ax + b \}$, which is convex if C is convex.

Property 2.4 (The pre-image of a convex set by an affine function is convex)**.** If $L : \mathbb{R}^n \to \mathbb{R}^m$ is an affine function, then $L^{-1}(C)$, the pre-image of C by L defined by:

$$L^{-1}(C) = \{ x \in \mathbb{R}^n \mid L(x) \in C \}$$

is convex if C is convex.

Example (Linear matrix inequalities)**.** Let $A_1, \ldots, A_m \in \mathbb{S}^n(\mathbb{R})$. The set:

$$\left\{ x \in \mathbb{R}^m \,\middle|\, \sum_{i=1}^m x_i A_i \succcurlyeq 0 \right\}$$

is an affine pre-image of $\mathbb{S}_+^n$ for the mapping $L : \mathbb{R}^m \to \mathbb{S}^n$ defined by:

$$L(x) = \sum_{i=1}^m x_i A_i$$

$\mathbb{S}_+^n$ being convex, the set is convex. $\sum_{i=1}^m x_i A_i \succcurlyeq 0$ is called a *linear matrix inequality.*

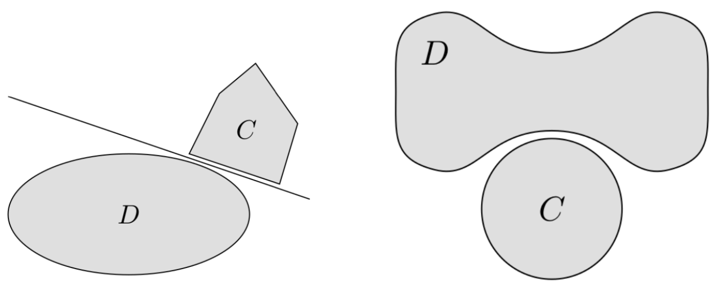
2.4 Geometric elements

2.4.1 Separating and supporting hyperplanes

Property 2.5 (Separating hyperplanes)**.** Soppose that $C, D \subseteq \mathbb{R}^n$ are two non-intersecting convex sets (that is $C \cap D = \varnothing$). Then there exists a hyperplane that separates C and D, that is:

$$\exists s \in \mathbb{R}^n \setminus \{0\}, \exists r \in \mathbb{R}, \quad \forall x \in C, s^\top x \leqslant r \quad \text{and} \quad \forall x \in D, s^\top x \geqslant r$$

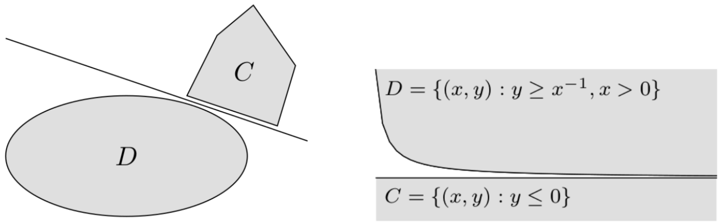
where $\left\{ x \in \mathbb{R}^n \,\middle|\, s^\top x = t \right\}$ is called the *separating hyperplane.*



Examples Counter-example

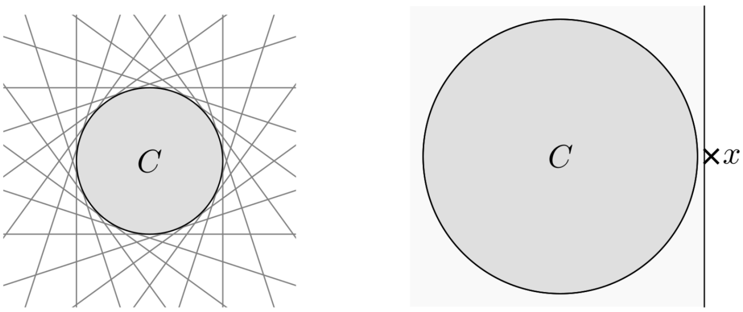
Property 2.6 (Strict separating hyperplanes)**.** Suppose that $C, D \subseteq \mathbb{R}^n$ are two non-intersecting **closed** convex sets, and that one of them is compact (closed and bounded in finite dimension). Then there exists a hyperplane that strictly separates C and D, that is:

$$\exists s \in \mathbb{R}^n \setminus \{0\}, \exists r \in \mathbb{R}, \quad \forall x \in C, s^\top x < r \quad \text{and} \quad \forall x \in D, s^\top x > r$$



Examples Counter-example

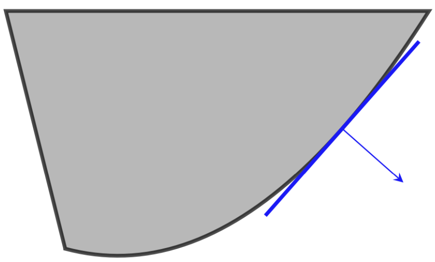
Note that a closed convex set C is the intersection of all halfspaces that contain it.



Definition (Supporting hyperplanes)**.** Supporting hyperplanes touch the boundary of a convex set, and have the entire set on one side. Formally, a hyperplane $H = \left\{ y \,\middle|\, s^\top y = r \right\}$ is a *supporting hyperplane* to a convex set C at a point $x \in \partial C$ if:

$$x^\top x = r \qquad \text{and} \qquad \forall y \in C, \quad a^\top y \leqslant r = s^\top x$$

We also say that H *supports* C at x.



Property 2.7. Let $C \subseteq \mathbb{R}^n$ be a non-empty convex set, and let $x \in \partial C$. Then there exists a supporting hyperplane to C at x.

2.4.2 Cone operators

Definition (Normal cone operator)**.** The *normal cone operator* to a set C at a point x is the set:

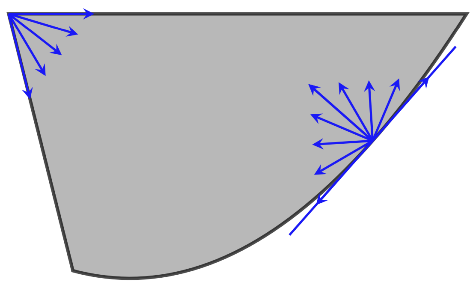
$$N_C(x) = \begin{cases} \left\{ g \in \mathbb{R}^n \,\middle|\, \forall y \in C, \quad g^\top (y - x) \leqslant 0 \right\} & \text{if } x \in C \\ \varnothing & \text{if } x \notin C \end{cases}$$

Intuitively, it is the set of vectors g that form obtuse angles for all $y - x$ with $y \in C$.

For $x \in \mathring{C}$, we have $N_c(x) = \{0\}$. For $x \in \partial C$, $N_C(x)$ is the set of the normal vectors to the supporting hyperplanes to C at x. If $x \notin C$, $N_C(x)$ is empty.

Definition (Tangent vector)**.** Let $C \subseteq \mathbb{R}^n$ be a convex set. A vector $d \in \mathbb{R}^n$ is tangent to C at x if:

$$\exists \{x_k\}_k \subseteq C, \exists \{\lambda_k\}_k \subset \mathbb{R}_+, \quad \lim_{k \to +\infty} \lambda_k (x_k - x) = d$$



Definition (Tangent cone)**.** The tangent cone of a convex set C at x is:

$$T_C(x) = N_C^\diamond(x)$$

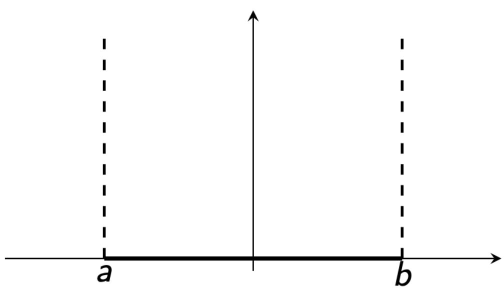
3 Convex functions

3.1 Extended-valued functions

Definition (Extended-valued function)**.** A function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is *extended-valued* if its domain is $\mathbb{R}^n$ and its range is $\bar{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$.

Example (Indicator function)**.** We consider the indicator function of interval $[a, b]$:

$$\mathbb{1}_{[a,b]}(x) := \begin{cases} 0 & \text{if } x \in [a, b] \\ +\infty & \text{otherwise} \end{cases}$$



Definition (Effective domain)**.** The *effective domain* of $f : \mathbb{R}^n \longrightarrow \bar{\mathbb{R}}$ is the set of points where f is finite:

$$\operatorname{dom} f := \{\, x \in \mathbb{R}^n \mid f(x) < +\infty \,\}$$

A function is said to be *proper* if its effective domain is non-empty: $\operatorname{dom} f \neq \varnothing$.

3.2 Definition and first properties

Definition (Convex function)**.** A function $f : \mathbb{R}^n \longrightarrow \bar{\mathbb{R}}$ is *convex* if its graph is below any line connecting two points of the graph $(x, f(x))$ and $(y, f(y))$. That is:

$$\forall x, y \in \mathbb{R}^n, \forall \theta \in [0,1], \quad f(\theta \cdot x + (1-\theta)\cdot y) \leqslant \theta \cdot f(x) + (1-\theta) \cdot f(y)$$

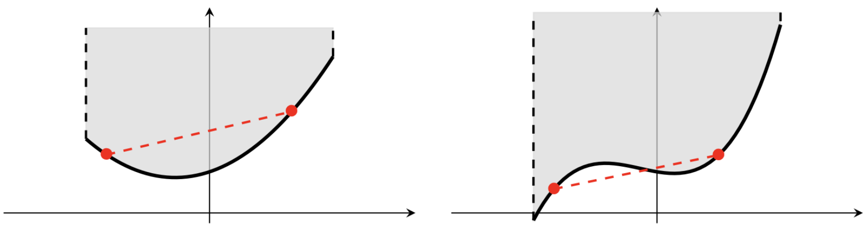
Definition (Concave function)**.** A function $f : \mathbb{R}^n \longrightarrow \bar{\mathbb{R}}$ is *concave* if $-f$ is convex. That is:

$$\forall x, y \in \mathbb{R}^n, \forall \theta \in [0,1], \quad f(\theta \cdot x + (1-\theta)\cdot y) \geqslant \theta \cdot f(x) + (1-\theta) \cdot f(y)$$

Definition (Epigraph)**.** The *epigraph* of a function $f : \mathbb{R}^n \longrightarrow \bar{\mathbb{R}}$ is the set of points lying above the graph of f:

$$\operatorname{epi} f := \{\, (x,t) \in \mathbb{R}^n \times \mathbb{R} \mid f(x) \leqslant t \,\}$$

Property 3.1 (Convexity and epigraph)**.** A function $f : \mathbb{R}^n \longrightarrow \bar{\mathbb{R}}$ is convex if and only if its epigraph is a convex set.



The following property allows to check the convexity of a multivariate function f by checking the convexity of functions of one variable.

Property 3.2. Let $f : \mathbb{R}^n \longrightarrow \bar{\mathbb{R}}$ be a function, and let $x \in \operatorname{dom} f$. We define:

$$\begin{aligned} g_{x,v} : \mathbb{R} &\longrightarrow \bar{\mathbb{R}} \\ t &\longmapsto f(x + tv) \end{aligned}$$

with $\operatorname{dom} g_{x,v} = \{\, t \in \mathbb{R} \mid x + tv \in \operatorname{dom} f \,\}$. Then, f is convex if and only if $g_{x,v}$ is convex in t for all $x \in \operatorname{dom} f$ and all $v \in \mathbb{R}^n$.

Definition (Sublevel sets)**.** Let $f : \mathbb{R}^n \longrightarrow \bar{\mathbb{R}}$ be a function. The *sublevel set* of f at level $\alpha \in \mathbb{R}$ is the set of points lying below the level α:

$$S_\alpha(f) = \{\, x \in \mathbb{R}^n \mid f(x) \leqslant \alpha \,\}$$

Property 3.3. If f is convex, then its sublevel sets are convex:

$$f \text{ is convex} \implies \forall \alpha \in \mathbb{R}, \quad S_\alpha(f) \text{ is convex}$$

The converse is not true.

3.3 First-order conditions

Property 3.4 (First-order condition for convexity)**.** Let $f : \mathbb{R}^n \longrightarrow \bar{\mathbb{R}}$ be a differentiable function, that is that $\nabla f(x)$ exists for all $x \in \operatorname{dom} f$. Then, f is convex if and only if $\operatorname{dom} f$ is convex and:

$$\forall x, y \in \operatorname{dom} f, \quad f(y) \geqslant f(x) + \nabla f(x)^\top (y - x)$$

4 Convex problems