Introduction to Computer Vision

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Contents

Abstract

This document is Antoine Groudiev's class notes while following the class *Introduction to Computer Vision* (Introduction à la vision artificielle) at the Computer Science Department of ENS Ulm. It is freely inspired by the class notes written by Jean Ponce and Gabriel Fiastre.

1 Introduction to Computer Vision

2 Camera Geometry

3 Camera Calibration

Camera calibration is the process of estimating the parameters of a camera model that relate the 3D world coordinates of a point to its 2D image coordinates. The camera parameters are represented in a *camera matrix* of shape 3 × 4, which essentially projects a 3D point (in homogeneous coordinates) to a 2D point.

The camera matrix is composed of two matrices: the *intrinsic matrix* and the *extrinsic matrix*. The intrinsic matrix contains the parameters that are specific to the camera, such as the focal length, the principal point, and the skew coefficient. The extrinsic matrix contains the parameters that describe the position and orientation of the camera in the world coordinate system.

3.1 Least squares calibration

Consider a set of n points $p_i \in \mathbb{R}^3$ and $P_i \in \mathbb{R}^4$, respectively the homogeneous coordinates of the 2D and 3D points that we will use for calibration:

$$p_i = \begin{bmatrix} u_i \\ v_i \\ 1 \end{bmatrix} \quad \text{and} \quad P_i = \begin{bmatrix} x_i \\ y_i \\ z_i \\ 1 \end{bmatrix}$$

Each point p_i corresponds to the 2D projection of the 3D point P_i in the camera frame. We want to estimate the camera matrix $M \in \mathscr{M}_{3,4}(\mathbb{R})$, such that:

$$\forall i \in [\![1, n]\!], \quad p_i = \frac{1}{z_i} M P_i \tag{3.1.1}$$

Let's denote m_1, m_2, and m_3 the rows of M, such that:

$$M = \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix}$$

We can rewrite (3.1.1) as:

$$\forall i \in [\![1, n]\!], \quad u_i = \frac{\frac{1}{z_i} m_1 P_i}{\frac{1}{z_i} m_3 P_i} \quad \text{and} \quad v_i = \frac{\frac{1}{z_i} m_2 P_i}{\frac{1}{z_i} m_3 P_i} \tag{3.1.2}$$

By clearing the denominators, we see that (3.1.2) is equivalent to:

$$\forall i \in [\![1, n]\!], \quad A_i X = 0_2 \tag{3.1.3}$$

where

$$A := \begin{bmatrix} P_i^\top & O_4^\top & -u_i P_i^\top \\ O_4^\top & P_i^\top & -v_i P_i^\top \end{bmatrix} \in \mathscr{M}_{2,12}(\mathbb{R}) \quad \text{and} \quad X := \begin{bmatrix} m_1^\top \\ m_2^\top \\ m_3^\top \end{bmatrix} \in \mathbb{R}^{12}$$

In practice, it is likely that (3.1.3) will not be exactly satisfied for all $i \in [\![1, n]\!]$, due to noise in the measurements. Therefore, we will not solve this exact problem, but instead solve the following minimisation problem:

$$\hat{X} = \underset{\|X\|^2=1}{\arg\min} \|\sum_i A_i X\|_2^2 = \underset{\|X\|^2=1}{\arg\min} \|AX\|_2^2 \tag{3.1.4}$$

where:

$$A := \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{bmatrix} = \begin{bmatrix} P_1^\top & O_4^\top & -u_1 P_1^\top \\ O_4^\top & P_1^\top & -v_1 P_1^\top \\ \vdots & \vdots & \vdots \\ P_n^\top & O_4^\top & -u_n P_n^\top \\ O_4^\top & P_n^\top & -v_n P_n^\top \end{bmatrix} \in \mathscr{M}_{2n,12}(\mathbb{R})$$

This can be solved using the *Singular Value Decomposition* (SVD) of A. If $A = U\Sigma V^\top$ is the SVD of A, then the solution $\hat{X}$ is given by the last column of V. Given $\hat{X}$, we can then extract the estimation of the camera matrix $\hat{M}$, by reshaping $\hat{X} \in \mathbb{R}^{12}$ into a 3×4 matrix.

Remark. *Estimating M does not give any explicit information on z_i; when projecting into the 2D plane, the u and v coordinates can be recovered by normalizing the projection by the third estimated value, which rescales by ensuring that the third coordinate is equal to 1.*

3.2 Parameter decomposition

Once the camera matrix M has been estimated, we can decompose it into its intrinsic and extrinsic parameters. One can show that any projection matrix M can be decomposed into the following form:

$$M = \mathcal{K}[R|t]$$

where:

$$\mathcal{K} = \begin{bmatrix} \alpha & -\alpha \cot\theta & u_0 \\ 0 & \beta/\sin\theta & v_0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{and} \qquad R = \begin{bmatrix} r_1^T \\ r_2^T \\ r_3^T \end{bmatrix}$$

Here, we denote by $r_1, r_2, r_2 \in \mathbb{R}^3$ the rows of the rotation matrix, and θ the skew rotation angle.

For a certain scale factor $\rho \in \mathbb{R}$, it is shown that:

$$\rho M = \begin{bmatrix} \alpha r_1 - \alpha \cot\theta r_2 & \alpha t_x - \alpha \cot\theta t_y + u_0 t_z \\ \frac{\beta}{\sin\theta} r_2 + v_0 r_3 & \frac{\beta}{\sin\theta} t_y + v_0 t_z \\ r_3 & t_z \end{bmatrix}$$

The normalisation by ρ comes from the fact that the Frobenius norm of the rotation matrix is equal to 1. Furthermore, we know from the rotation matrices properties that they are orthogonal ($R^\top R = I_3$), and that the determinant of a rotation matrix is equal to 1. Hence, we have:

$$\forall i \in [\![1, 3]\!], \quad \|r_i\|^2 = 1$$

If we write $M = [\mathcal{A}b]$ with $b \in \mathbb{R}^3$ the last column of M, and $a_1, a_2, a_3 \in \mathbb{R}^3$ the columns of $\mathcal{A}$, we have by identification that:

$$\rho a_3 = r_3$$

Hence, we can deduce the scale factor ρ:

$$\rho = \frac{\varepsilon}{\|a_3\|}$$

where $\varepsilon = \pm 1$. The choice of ε determines the orientation of the camera. In practical situations, the sign of t_z is often known, since it corresponds to knowing whether the origin of the world coordinate system is in front of or behind the camera.

Using the orthogonality of the rotation matrix, we have that:

$$r_i^\top \cdot r_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

Hence:

$$r_3^\top(\alpha r_1 - \alpha \cot\theta r_2 + u_0 r_3) = (\rho a_3) \cdot (\rho a_1)$$

Which gives us:

$$u_0 = \rho^2 a_3 \cdot a_1$$

Similarly, we can find that:

$$v_0 = \rho^2 a_3 \cdot a_2$$

It can also be shown that we can recover the skew rotation angle θ by computing:

$$\cos\theta = \frac{(a_1 \times a_3) \cdot (a_2 \times a_3)}{||a_1 \times a_3||||a_2 \times a_3||}$$

The skew rotation angle can later be used to retrieve the values of α and β:

$$\alpha = \rho^2 ||a_1 \times a_3|| \sin\theta$$

$$\beta = \rho^2 ||a_2 \times a_3|| \sin\theta$$

We then have all the elements to compute the lines of R:

$$\begin{cases} r_1 = \frac{a_2 \times a_3}{||a_2 \times a_3||} \\ r_3 = \rho a_3 \\ r_2 = r_3 \times r_1 \end{cases}$$

Finally, we can recover $t = \rho \mathcal{K}^{-1} b$. This method provides an implementation-ready way to decompose the camera matrix. In practice, the only unknown is the sign of the scale factor, ε.

4 Image Processing using filters and convolutions

An image can be interpreted either as a continuous function $f(x, y)$ or as a discrete array $F_{u,v}$. While many applications, especially in image processing, use the discrete array, the intuition and operations are directly derived from the continuous function setup.

4.1 Filters and convolution

4.1.1 Basic filters

An image can be blurred using a filter, by replacing a point by the average of its neighbors. Blurring an image gives a smoother image, making it easier to compute derivatives.

4.1.2 Convolutions

Given two integrable functions $f, g : \mathbb{R} \to \mathbb{R}$, we can define their convolution as:

$$\begin{aligned} f * g : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto \int_{-\infty}^{+\infty} f(x-t)g(t)dt \end{aligned}$$

Note that $f * g = g * f$ using a change of variable.

This is the definition of the convolution from a continuous perspective. When dealing with images, we want to apply the convolution to a discrete array. The definition becomes:

$$R_{i,j} = (F * G)_{i,j} = \sum_{u,v} F_{i-u,j-v} G_{u,v}$$

Convolution follow basic properties:

Commutativity $f * g = g * f$

Associativity $(f * g) * h = f * (g * h)$

Linearity $(af + bg) * h = af * h + bg * h$

Shift invariance $f_t * h = (f * h)_t$

where $f_t(x) = f(x - t)$. Note that is the only operator that is both linear and shift-invariant.

The convolution can be differentiated:

$$\frac{\partial}{\partial x}(f * g) = \frac{\partial f}{\partial x} * g \tag{4.1.1}$$

In practice, we are dealing with discrete and finite arrays; this causes border issues. When applying the convolution with a $K \times K$ kernel, the result is undefined for pixels closer than K pixels from the border of the image. There are multiple ways to solve this issue: *padding* the image with zeros, *cropping* the result, or *wraping around* the image.

4.1.3 Gaussian filters

Bluring images Gaussian filters are special filters that are used to blur images. Recall that in one dimension, the Gaussian function is defined as:

$$g(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}$$

In computer vision, we will mostly use the 2-D Gaussian function:

$$G(x, y) = \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}}$$

Note that in the following, we will denote by $1/C$ the normalization constants.

In the continuous setup, blurring a function is achieved by convoluting it with a Gaussian function. In the discrete setup, we can build a matrix kernel that approximates the Gaussian function. Note that the Gaussian function has infinite support, but in actual applications, we can truncate the kernel to a finite size.

Gaussian smoothing oftern provides better results than simple averaging. It is also quite effective to remove the noise in an image.

Properties of Gaussian filters Gaussian filters remove "high-frequency" components from the image; therefore, they are low-pass filters. The quantity of noise removed is proportional to the standard deviation σ of the Gaussian kernel. High values of σ will remove more noise but will also blur the image more.

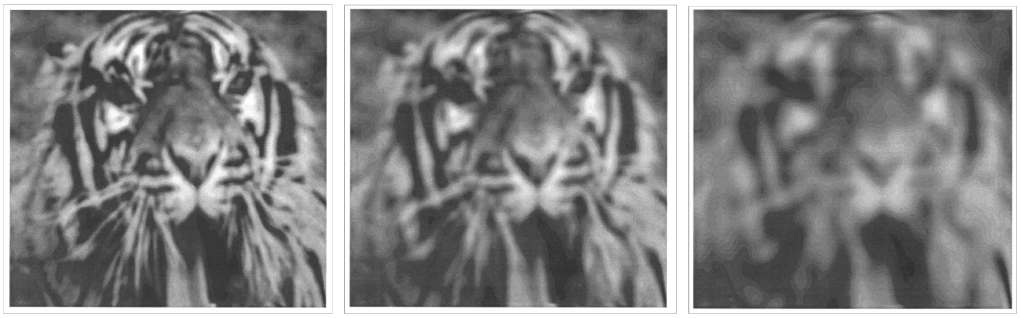


Figure 4.1: Effect of the standard deviation σ on the image.
The parameter σ is increased from left to right.

The combination of 2 Gaussian filters is a Gaussian filter:

$$G_{\sigma_1} * G_{\sigma_2} = G_{\sqrt{\sigma_1^2+\sigma_2^2}}$$

Each filter is separable, meaning that we can apply the filter in the x direction and then in the y direction:

$$G_\sigma * f = g_{\sigma\rightarrow} * g_{\sigma\uparrow} * f$$

This as a critical implication: filtering with a $n \times n$ Gaussian kernel can be implemented as two convolutions of size n, reducing the complexity from $O(n^2)$ to $O(n)$.

Oriented Gaussian Filters By default, G_σ smoothes the image by the same amount in all directions. This has the drawback of blurring edges in all directions, which might make edge detection harder later on in the image processing pipeline. If we have some information about preferred directions, we might want to smooth with some value σ_1 in the direction defined by the unit vector $\begin{bmatrix} a & b \end{bmatrix}$ and by σ_2 in the direction defined by $\begin{bmatrix} c & d \end{bmatrix}$. This can be achieved using:

$$G(x, y) = \frac{1}{C} \exp\left[-\frac{(ax+by)^2}{2\sigma_1^2} - \frac{(cx+dy)^2}{2\sigma_2^2}\right]$$

We can write this in a more compact form using the standard multivariate Gaussian notation:

$$G(x,y) = \frac{1}{C}\exp\left[-\frac{X^\top \Sigma^- 1 X}{2}\right] \quad \text{where} \quad X = \begin{bmatrix} x \\ y \end{bmatrix}$$

The two (orthogonal) directions of filtering are given by the eigenvectors of Σ, the amout of smoothing is given by the square root of the corresponding eigenvalues of Σ.

4.2 Image derivatives

We will see in the next chapter a variety of techniques to solve the *edge detection problem*. A building block of such methods are *image derivatives*: intuitively, we want to be able to measure how much the contrast of the image change locally. Peaks in contrast variation can be somehow interpreted as being close to edges, since this would be the point where the contours of the object contrast with the background.

4.2.1 Finite differences

Therefore, we want to compute at each pixel (x,y) the derivates. In the discrete case, we could take the difference between the left and right pixels:

$$\frac{\partial I}{\partial x} \simeq I[i+1,j] - I[i-1,j]$$

This is equivalent as convoluting the image by:

$$\partial_x = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}$$

The problem of this method is that it increases noise. Consider a noise model in which the actual image I can be decomposed as the sum of the true, noiseless, image $\hat{I}$, and a noise n, following for instance a normal distribution. When then have $I = \hat{I} + n$, and we obtain:

$$\underbrace{I[i+1,j] - I[i-1,j]}_{\text{Actual image values}} = \underbrace{\hat{I}[i+1,j] - \hat{I}[i-1,j]}_{\text{True difference}} + \underbrace{n_+ + n_-}_{\text{Noises}}$$

Where $n_+ - n_-$ follows a normal distribution of larger variance, providing therefore more noise on the derivate image.

4.2.2 Smooth derivatives

A solution is to first smooth the image by a Gaussian G_σ, and *then* take derivatives:

$$\frac{\partial f}{\partial x} \simeq \frac{\partial G_\sigma * f}{\partial x}$$

Applying the differentiation property of the convolution (4.1.1):

$$\frac{\partial f}{\partial x} \simeq \frac{\partial G_\sigma}{\partial x} * f$$

Therefore, taking the derivative in x of the image can be done by applying a convolution with the derivative of a Gaussian:

$$\frac{\partial G_\sigma}{\partial x} = \frac{1}{C} \cdot x \exp\left[-\frac{x^2+y^2}{2\sigma^2}\right]$$

Another crucial property is that the Gaussian derivative is also separable, reducing drastically the computational cost.

Smoothing before the derivative improves the results by reducing the noise, but still blurs away the edge information. In practice, there is always a tradeoff to find between smoothing and good edge localization.

4.2.3 Beyond smooth derivatives

Other methods are sometimes used in practice to overcome the limitations detailled above. *Directional derivatives* are the equivalent of directional smoothing; we output the following quantity

$$\cos\theta\frac{\partial G_\sigma}{\partial x}+\sin\theta\frac{\partial G_\sigma}{\partial y}$$

This allows to avoid the smoothing of the edges while keeping the differentiation in directions that matter.

Second-order methods can also prove effective. This is a non-separable method, approximated by a difference of Gaussians. The output of the convolution is the Laplacian of the image; zero-crossing correspond to edges:

$$\nabla^2 G_\sigma(x,y)=\frac{\partial^2 G_\sigma(x,y)}{\partial x^2}+\frac{\partial^2 G_\sigma(x,y)}{\partial y^2}$$

5 Edge Detection

The edge detection problem aims at identifying the *edges* inside an image; this requires a proper definition of “edge”, which often depends on the method used to compute it. In general, an edge is a place where the intensity of the image changes abruptly.

The edge detection problem is a fundamental problem in computer vision, as it is the first step in many image processing tasks. The main motivation behind edge detection is the idea that the edges of an image contain important information about the structure of the objects in the scene. If the brightness of an image changes abruptly, it is likely that other properties of the image also change abruptly at that point, and specifically higher-level properties. Edges define the boundaries of objects in the image; they are caused by change of texture, color, depth, or illumination, which all are important cues for the semantic interpretation of the image.

5.1 Gradient-based edge detection

Intuitively, an edge is a discontinuity of intensity in some direction; like explained previously, it could be detected by looking for place where the derivatives of the iamge have large values.

Gradient-based edge detectors run into three major issues:

Change of scale The gradient magnitudes at different scales are different: which one should we choose? This reflects a fundament problem in computer vision, which is the necessity to select a threshold under which edges are “too small” to be considered.

Thick countours The gradient magnitude is large along thick trails: how do we identify the significant points?

Continuity Simple edge detection algorithms will produce non-continuous lines, which does not fit our high-level understanding of edges. How do we link the relevant points up into curves?

Another way to detect an extremal first derivative is to look for a null second derivative. In practice, applying a Laplacian method always require smoothing with a Gaussian kernel first. The method goes as follows: smooth the image, apply the Laplacian, and mark the zero points where there is a sufficiently large derivative – that is enough contrast.

5.2 The Canny edge detector

5.2.1 Introduction

The Canny edge detector is a multi-step edge detection algorithm, providing a good trade-off between the three issues mentioned above. Instead of simply thresholding the gradient magnitude – which would lead to thick contours and discontinuous lines – the Canny edge detector track edges using a process called *hysteresis*.

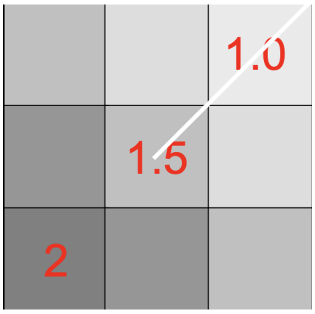
The starting point of Canny’s results is the observation that gradient magnitude does enhance the edges, but runs into two main issues:

- using a single threshold either leads to too many edges, or too few edges;
- even if we find a good threshold, the edges remain poorly localized, i.e. too thick.

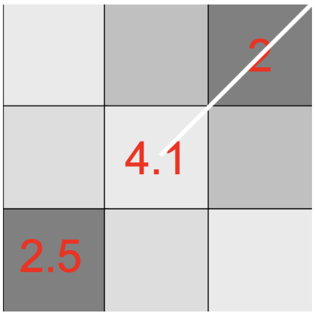
Therefore, Canny’s edge detector uses two standard tools to address these issues, *non-maximum suppression* and *hysteresis*.

5.2.2 Non-local maxima suppression

The goal of Non-Maximum Suppression (NMS) is to thin the edges; the idea is to keep only the pixels that correspond to a maximum in one of the directions of the gradient. If in all directions, the pixel is not a maximum, then it is likely to be a thick contour, and should be removed. Conversely, if the pixel is a maximum in one direction, it is likely to be at the center of the edge stroke, and should be kept.



Gradient magnitude at center is lower than a neighbor



Gradient magnitude is a local maximum

This creates a mask of the image, where only the local maxima are kept. Applying this mask to the gradient magnitude image will therefore thin the edges.

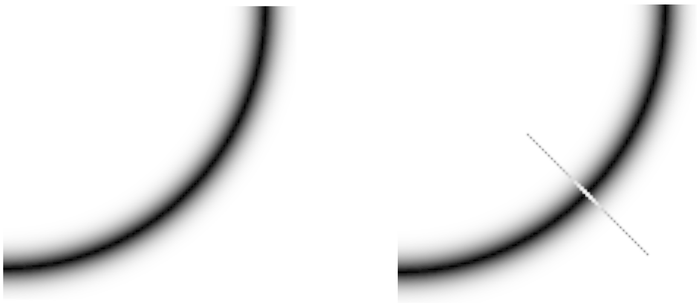
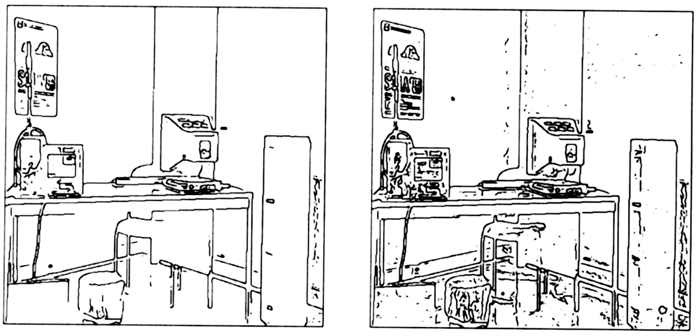


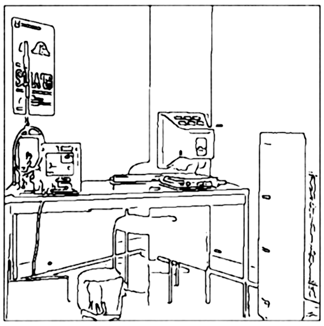
Figure 5.1: The left image shows the gradient magnitude of an image; on the right, the mask provided by NMS and its application to the gradient.

5.2.3 Hysteresis thresholding

As stated previously, lower thresholds keep important details, but also keep a lot of noise; higher thresholds remove the noise, but also remove important information about the edges. The idea of hysteresis thresholding is to use two thresholds, a lower and an upper one, to keep only the pixels that are above the upper threshold, or connected to a pixel above the upper threshold. This allows to keep the important details, while removing the noise (i.e. the pixels that are not connected to the edges). Hysteresis also tend to produce continuous edges, since we track the edges by following the pixels that are connected to the edges.



Two thresholds: $t = 15$ and $t = 5$



Hysteresis thresholding

Hysteresis thresholding can be implemented using a depth-first search algorithm, which will track the pixels connected to the edges. The algorithm goes as follows:

1. Keep a set E of the edges for which you have to visit neighbours;
2. Initialize E with the edges corresponding to the most discriminative threshold;
3. Until E is empty:
    - Extract an edge e from E; if e has already been visited, skip it;
    - For each neighbour e' of e:
        - If e' is a considered edge using the less discriminative threshold, add it to the output edges and to E
        - Otherwise, skip it

5.2.4 The algorithm

Putting all the steps together, the Canny edge detector goes as follows:

1. Compute x and y derivatives of the image I:
$$I_x = G_\sigma^x * I \quad \text{and} \quad I_y = G_\sigma^y * I$$
2. Compute the magnitude of the gradient at every pixel:
$$|\nabla I| = \sqrt{I_x^2 + I_y^2}$$
3. Eliminate the pixels that are not local maxima of the magnitude in the direction of the gradient:
$$|\nabla I|_{\text{NMS}} = \text{NMS}(|\nabla I|)$$
4. Apply hysteresis thresholding to the image:
$$\text{Canny}(I) = \text{Hysteresis}(|\nabla I|_{\text{NMS}}, t_h, t_l)$$

6 Image restoration

7 Radiometry and Color

8 Stereopsis

9 Two-view Geometry

Human see the world in binocular stereo vision, which allows them to have a good natural 3D understanding of the surrounding environments. Reproducing such multi-view image processing has the potential to enhance artificial vision algorithms, and has thus become a major challenge in computer vision.

The geometrical relationships between points visible in a two-view camera system can be represented in a matrix called the *Essential Matrix* (when the intrinsic parameters of the cameras are known) or the *Fundamental Matrix* (respectively when those parameters are unknown, which means that one has to compensate for it). In this chapter, we will focus on methods to estimate this matrix, and its use for aggregating geometrical information for stereo vision.

9.1 Epipolar Geometry

The basic setup of epipolar geometry is described in Figure 9.1.

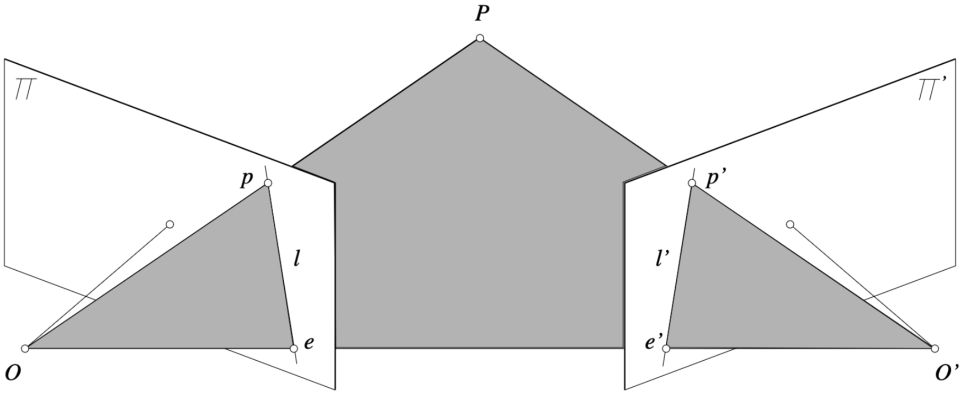


Figure 9.1: Epipolar geometry.

The scene is observed from two view points, caracterized by the two optical centers O and O', and their respective projected images (planes π and π'). The line OO' is called the *baseline*, and cuts the planes respectively on e and e' called the *epipoles*.

The plane formed by OPO' is called the *epipolar plane*. Its intersection with π is called the *epipolar line* associated to p', denote l. Respectively, its intersection with π' is called the epipolar line associated to p and is denoted l'.

Remark. *The epipolar line l is associated with p' and not p. This is due to the line l being known whenever p' is.*

Theorem (Epipolar constraint)**.** If a point P is observed in two views in p and p', the observation p **must belong** to the epipolar line associated to p', namely l. Respectively, p' must belong to the epipolar line l' associated to p.

The epipolar constraint (Theorem 9.1) and the geometrical relationship between the two viewpoints' projections can be written as an equation, defining the *Essential Matrix*, introduced by Longuet-Higgins and defined as follows (in the case of known intrinsic parameters):

$$p \cdot [t \times (\mathcal{R}p')] \tag{9.1.1}$$

where $p = \begin{bmatrix} u \\ v \\ 1 \end{bmatrix}$ and $p' = \begin{bmatrix} u' \\ v' \\ 1 \end{bmatrix}$ denote the homogeneous coordinates of the observed points in the two view, t is the coordinate vector of the translation $\overrightarrow{OO'}$ and $\mathcal{R}$ is the rotation matrix such that a free vector with coordinates w' in the second coordinate system has coordinates $\mathcal{R}w'$ in the first coordinate system.

Equation 9.1.1 can be rewritten as:

$$p^\top \mathcal{E} p' = 0 \tag{9.1.2}$$

where $\mathcal{E} = [t_\times]\mathcal{R}$.[1] The matrix $\mathcal{E}$ is called the *Essential Matrix.*

Remark. *Note that in practice, the expression of the essential matrix as $\mathcal{E} = [t_\times]\mathcal{R}$ does not matter, as we can directly estimate the matrix without computing the rotation and translation matrices. However, in the case of unknown intrinsic camera parameters, the full decomposition, and thus definition, would be different: one would have to compensate for the change of coordinate basis due to different camera parameters for the epipolar constraint to be verified. In this case, the fundamental matrix F is defined such that:*

$$(Kp)^\top F(K'p') = 0$$

with K and K' the intrinsic parameters of the two cameras.

9.2 Essential matrix estimation

We now assume that we are given a set of corresponding points from views π and π', with equal intrinsinc parameters, and denoted respectively as p and p'. We know that such points must satisfy Theorem 9.1. Therefore, we can use this information to estimate the essential matrix $\mathcal{E}$, and use it to make correspondences between new points.

Remark. *In practice, Longuet-Higgins showed that at least 8 correspondences were needed to have a reasonable estimate of $\mathcal{E}$. In these notes, the algorithm is given for one single pair of points (p, p'), but it can be easily extended to multiple pairs $(p_i, p'_i)_i$.*

Using Equation 9.1.1, we can write the epipolar constraint as:

$$\begin{bmatrix} u & v & 1 \end{bmatrix} \begin{bmatrix} \mathcal{E}_{11} & \mathcal{E}_{12} & \mathcal{E}_{13} \\ \mathcal{E}_{21} & \mathcal{E}_{22} & \mathcal{E}_{23} \\ \mathcal{E}_{31} & \mathcal{E}_{32} & \mathcal{E}_{33} \end{bmatrix} \begin{bmatrix} u' \\ v' \\ 1 \end{bmatrix} = 0 \tag{9.2.1}$$

which can be rewritten as:

$$\begin{bmatrix} uu' & uv' & u & vu' & vv' & v & u' & v' & 1 \end{bmatrix} \begin{bmatrix} \mathcal{E}_{11} & \mathcal{E}_{12} & \mathcal{E}_{13} & \mathcal{E}_{21} & \mathcal{E}_{22} & \mathcal{E}_{23} & \mathcal{E}_{31} & \mathcal{E}_{32} & \mathcal{E}_{33} \end{bmatrix}^\top = 0$$

Similarly to camera calibration, instead of solving this ill-posed problem, Longuet-Higgins proposed to solve the linear least-squares problem:

$$\underset{\|x\|^2=1}{\arg\min} \|Ax\|^2 \tag{9.2.2}$$

where A is the matrix of the 8 correspondences, and x is the vectorized form of $\mathcal{E}$. The solution to this approach is given by the singular value decomposition of A. Please refer to subsection 3.1 for more information.

[1] For any vector a, $[a_\times]$ denotes the skew-symmetric matrix such that $[a_\times]b = a \times b$ is the cross-product of the vectors a and b.

9.3 Epipolar lines

In practice, trying to find $\mathcal{E}$ such that every point satisfied the epipolar constraint is equivalent to saying that p should belong to the epipolar line associated to p', and vice-versa. In practice, p can be very close but will never exactly belong to the line, as the estimations introduce error. Thus, one might want to compute the average distance from the points to the epipolar lines associated to their respective corresponding points.

Once an estimate of $\mathcal{E}$ is obtained, the epipolar line associated to p' is given by:

$$l = \mathcal{E}p'$$

The distance of a point $x = \begin{bmatrix} u \\ v \\ 1 \end{bmatrix}$ to a line $l = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ can be written as:

$$d(x, l) = \frac{|au + bv + c|}{\sqrt{a^2 + b^2}}$$

This measure can give an idea of the quality of the estimate of $\mathcal{E}$. However, it is important to visualize the result, and it is necessary to draw the epipolar lines to have a better understanding of the quality of the estimate. For readability and interpretability, it is reasonable to draw segments instead of lines. They shall be centered in the projection of the associated point p on the epipolar line l; we denote this projection $\hat{p}$.

Given a point $p = \begin{bmatrix} u \\ v \\ 1 \end{bmatrix}$, the projection of p on the line $l = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ of direction $L := \begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$ is given by:

$$\hat{p} = p - p \cdot l \frac{L}{||L||^2} = p - \frac{p \cdot l}{a^2 + b^2} L = p - \frac{au + bv + c}{a^2 + b^2} \begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$$

Using these two formulae and basic linear algebra allow to recover the epipolar lines, to draw them, and to compute the distance with the points.

9.4 Hartley's normalization

The Longuet-Higgins approache described as Equation 9.2.2 is numerically poorly conditioned, and the estimations are often very inaccurate in practice. Hartley thus proposed an improvement of the algorithm, based on a rescaling of the data to ensure that the resulting matrix would satisfy the required properties of the essential or fundament matrix.

The idea is to center the data around 0, and to scale it such that the average distance to the origin is $\sqrt{2}$ (chosen empirically). This is done by applying the following transformation to the points p and p':

$$\begin{cases} \tilde{p} = Tp \\ \tilde{p'} = T'p' \end{cases}$$

Such matrices T and T' can be computed by:

$$T = \begin{bmatrix} \sqrt{\frac{2}{\sigma^2}} & 0 & -\sqrt{\frac{2}{\sigma^2}}\mu_x \\ 0 & \sqrt{\frac{2}{\sigma^2}} & -\sqrt{\frac{2}{\sigma^2}}\mu_y \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad T' = \begin{bmatrix} \sqrt{\frac{2}{\sigma'^2}} & 0 & -\sqrt{\frac{2}{\sigma'^2}}\mu'_x \\ 0 & \sqrt{\frac{2}{\sigma'^2}} & -\sqrt{\frac{2}{\sigma'^2}}\mu'_y \\ 0 & 0 & 1 \end{bmatrix}$$

where μ_x and μ_y are the average coordinate of p, and σ is the average distance to the mean (average norm of the vector) for the point p.

We can now estimate the solution $\tilde{\mathcal{E}}$ by solving the least-squares problem associated with the constraint:

$$\tilde{p}^\top \tilde{\mathcal{E}} \tilde{p}' = 0 \tag{9.4.1}$$

The data having been normalized, it is necessary to impose a constraint on the estimated matrix $\tilde{\mathcal{E}}$ such that it respects the properties of the essential matrix: in particular, it should be of rank 2. To enforce this rank-2 constraint, Hartley proposes to compute the SVD of $\tilde{\mathcal{E}}$, such that:

$$\tilde{\mathcal{E}} = U \Sigma V^\top$$

where $\Sigma = \operatorname{diag}(r, s, t)$ with $r \leqslant s \leqslant t$ the singular values. Replacing Σ by $\Sigma' = \operatorname{diag}(s, t, 0)$, guarantees that the rank of the matrix is 2. We therefore define:

$$\tilde{\mathcal{E}}^* = U \Sigma^* V^\top$$

which is a close estimate of $\tilde{\mathcal{E}}$ of rank 2.

From Equation 9.4.1, we can deduce that:

$$(Tp)^\top \tilde{\mathcal{E}}^* (T' p') \simeq 0$$

and thus, we can recover the final estimate of the essential matrix by projecting back the normalized data to the original coordinate basis:

$$\mathcal{E} = T^\top \tilde{\mathcal{E}}^* T'$$

In practice, the estimate using Hartley's normalization improves the results by a significant margin, especially for a small number of points (such as 8).