Implementation of an Iterative Linear Quadratic Regulator (iLQR)

Gabriel Desfrene Antoine Groudiev

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Plan

Problem statement

The iLQR algorithm

Our implementation

Demonstration time

Conclusion

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General formulation

- Dynamics function:

$$x_{t+1} = f(x_t, u_t)$$

- Goal: minimize a quadratic cost function
- Cost function:

$$J(u) = \sum_{t=0}^{T-1} \left(x_t^\top Q x_t + u_t^\top R u_t \right) + \frac{1}{2}(x_T - x^*)^\top Q_f (x_T - x^*)$$

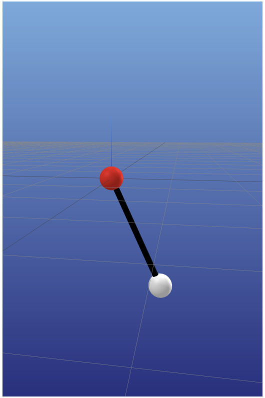
- Q: state cost matrix
- Q_f: final state cost matrix
- R: control cost matrix

Example: Simple Pendulum

- State: $x = [\theta \quad \dot{\theta}]$
- Control: u, torque applied to the pendulum
- Dynamics: physical laws (simulator)
- Target: $x = [0 \quad 0]$
- Cost function:

$$J(u) = \frac{1}{2}\left(\theta_f^2 + \dot{\theta}_f^2\right) + \frac{1}{2}\int_0^T r u^2(t)\mathrm{d}t$$

corresponding to $Q_f = I_2$, $Q = 0_2$, $R = rI_1$

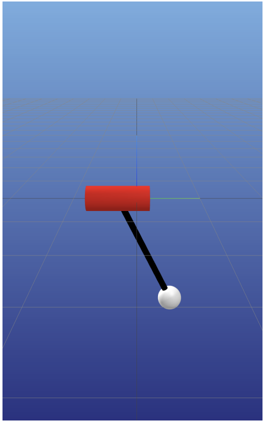


Example: Cartpole

- State: $x = [y \quad \theta \quad \dot{y} \quad \dot{\theta}]$
- Control: u, force applied to the cart
- Dynamics: physical laws (simulator)
- Target: $x = [0 \quad 0 \quad 0 \quad 0]$
- Cost function:

$$J(u) = \frac{1}{2}\left(\theta_f^2 + \dot{\theta}_f^2 + y_f^2 + \dot{y}_f^2\right) + \frac{1}{2}\int_0^T r u^2(t)\mathrm{d}t$$

corresponding to $Q_f = I_4$, $Q = 0_4$, $R = rI_1$



Plan

General idea

- iLQR is an iterative algorithm
- Start with an initial trajectory
- Iteratively improve it using a local linear approximation
- Stop when the trajectory converges

Linearizing the dynamics

The equation $x_{t+1} = f(x_t, u_t)$ is linearized (at each step) as:

$$\delta x_{t+1} = A_t \delta x_t + B_t \delta u_t$$

with:

- A_t: Jacobian of f with respect to x evaluated at (x_t, u_t)
- B_t: Jacobian of f with respect to u evaluated at (x_t, u_t)

We are in LQR (Linear Quadratic Regulator, cf. TP5) setup!

Trajectory refinement using LQR

1. **Forward pass**: compute the successive states (x_t) for the current controls (u_t), and the corresponding cost J
2. **Backward pass**: compute the gains, i.e. how much we should change the controls in each direction to minimize the cost
3. **Forward rollout**: apply the gains to the controls to obtain a new trajectory
4. Repeat until convergence

Computing the Jacobians

Finite differences method

We want to compute:

- $A_t = \frac{\partial f}{\partial x}(x_t, u_t)$, i.e. how much the state at time $t+1$ changes when we slightly change the state at time t
- $B_t = \frac{\partial f}{\partial u}(x_t, u_t)$, i.e. how much the state at time $t+1$ changes when we slightly change the control at time t

In a black box setting, we can use finite differences:

$$[A_t]_i \approx \frac{f(x_t + \varepsilon e_i, u_t) - f(x_t - \varepsilon e_i, u_t)}{2\varepsilon}$$
$$[B_t]_i \approx \frac{f(x_t, u_t + \varepsilon e_i) - f(x_t, u_t - \varepsilon e_i)}{2\varepsilon}$$

for some small ε and the canonical basis (e_i)

Computing the Jacobians

Using Pinocchio

Plan

What language to use?

Python

- Easy to use
- Bindings for many libraries
- Embarrassingly slow

C++

- Fast
- Not very funny

Rust

- Fast
- Very funny

From Rust to Python, and the other way around

API Basic usage

Plan

Demonstration time!

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