# Operations Research, Spring 2018
# Homework 2

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## 1 Problems

1. (25 points) Ikuta is traveling from Japan to Taiwan and have bought a bag that can carry up to 20 kg of furniture. The weight and degree of importance of each item that she is considering carrying are given in the table below. Ikuta wants to maximize the total degree of importance of the items she carry while satisfying the capacity constraint.

| Item | Importance | Weight (kg) |
|---|---|---|
| 1 | 7 | 8 |
| 2 | 4 | 6 |
| 3 | 1 | 3 |
| 4 | 3 | 4 |
| 5 | 1 | 2 |
| 6 | 2 | 7 |
| 7 | 5 | 7 |

   (a) (5 points) Formulate an integer program that solves her problem.

   (b) (5 points) Find the dual LP of the linear relaxation of the IP in Part (a).

   (c) (10 points) Apply complementary slackness to find which dual constraints must be binding at a dual optimal solution.

   (d) (5 points) For the linear relaxation of the IP in Part (a), find the shadow price of the capacity constraint.

   **Hint.** You may, but you do not need to solve the primal LP by the simplex method or solve the dual LP. Use the definition of shadow prices!

2. (20 points) Consider the following integer program, which represents an instance of the "two-copy knapsack problem:"

$$\begin{aligned} \max \quad & 5x_1 + 2x_2 + 4x_3 + 3x_4 + 8x_5 \\ \text{s.t.} \quad & 2x_1 + 3x_2 + 2x_3 + 6x_4 + 3x_5 \leq 13 \\ & x_i \in \{0, 1, 2\} \quad \forall i = 1, ..., 5. \end{aligned}$$

   (a) (5 points) Formulate the linear relaxation of the integer program.

   **Hint.** What should be the upper bound of each $x_i$?

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(b) (5 points) Use the greedy algorithm introduced in class to solve the linear relaxation of this integer program.

(c) (10 points) Use the branch-and-bound algorithm to solve the original integer program. Depict the full branch-and-bound tree. Do not write down the solution process of each node; write down just an optimal solution and its objective value of each node. Choose any branching strategy you like.

3. (30 points; 10 points each) A city is divided into eight districts. The time (in minutes) it takes an ambulance to travel from one district to another is shown in the table below. The population of each district (in thousands) is as follows: district 1, 40; district 2, 30; district 3, 35; district 4, 20; district 5, 15; district 6, 50; district 7, 45; district 8, 60.

| District | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 3 | 4 | 6 | 8 | 9 | 8 | 10 |
| 2 | 3 | 0 | 5 | 4 | 8 | 6 | 12 | 9 |
| 3 | 4 | 5 | 0 | 2 | 2 | 3 | 5 | 7 |
| 4 | 6 | 4 | 2 | 0 | 3 | 2 | 5 | 4 |
| 5 | 8 | 8 | 2 | 3 | 0 | 2 | 2 | 4 |
| 6 | 9 | 6 | 3 | 2 | 2 | 0 | 3 | 2 |
| 7 | 8 | 12 | 5 | 5 | 2 | 3 | 0 | 2 |
| 8 | 10 | 9 | 7 | 4 | 4 | 2 | 2 | 0 |

The city has two ambulances and wants to locate them to two of the districts. For each district, the *population-weighted firefighting time* is defined as the product of the district population times the amount of time it takes for the closest ambulance to travel to it.

(a) Formulate an integer program that can minimize the maximum population-weighted firefighting time among the eight districts.

(b) Continue from the previous problem. Now suppose that (1) we must locate an ambulance to at least one of Districts 2, 3, and 4, (2) if we locate an ambulance to District 5, we cannot locate one to District 6, and (3) we cannot locate more than one ambulance to Districts 1, 3, 6, and 8. Augment your previous formulation by adding more constraints (and variables, if needed).

(c) Invoke a solver to solve the above two problems and find an optimal solution for each problem. Do not submit your computer programs. Just write down your optimal solutions, report their objective values, and make managerial suggestions.

4. (25 points; 5 points each) You are producing $n$ products by using $m$ kinds of materials. Each unit of product $j$ requires $A_{ij}$ unit of materials $i$. The maximum possible supply of materials $i$ is $K_i$ units. The unit price of product $j$ is $P_j$. All products that are produced may be sold.

(a) Formulate a linear program that can determine the production quantity of all products to maximize the total revenue.

(b) Continue from the previous problem. Suppose that $n = 4$, $m = 3$, and

$$A = \begin{bmatrix} 3 & 4 & 2 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & 2 \end{bmatrix}, K = \begin{bmatrix} 50 \\ 40 \\ 60 \end{bmatrix}, \text{ and } P = \begin{bmatrix} 7 \\ 8 \\ 5 \\ 6 \end{bmatrix}.$$

Using whatever method you like to find an optimal solution. You may use a solver if you prefer.

(c) Continue from the previous problem. Using whatever method you like to find the shadow prices of the three material supply constraints.

(d) Continue from the previous problem. Suppose that a material supplier approaches you and offer more supplies of materials $i$ at unit price $C_i$. Should you take the offer and buy any unit of material $i$ from it? Why or why not?

(e) Continue from Part (a). Suppose that producing a positive amount of product $j$ requires a setup cost $S_j$. Reformulate your linear program in Part (a) to a mixed integer linear program to find a production plan that maximizes the total profit (sales revenue minus setup cost). Please assume that your production quantity may be fractional.

## 2 Submission rules

The deadline of this assignment is ***2:00 pm, April 29***. Works submitted between 2:00 pm and 3:00 pm will get 10 points deducted as a penalty. Submissions later than 3:00 pm will not be accepted. Please submit a PDF file containing your answers for all problems. In your PDF file, you should summarize the solutions generated by solvers; you should not copy and paste your computer programs into the PDF file. Include your student ID and name in the first page of your PDF file. Please submit both files to NTU COOL.