μ Reparametrization

Superformula curve is defined with polar angle parameter. Due to the curve's curvy nature, samples taken using polar angle parameter do not capture the information about the shape when the shape is highly curved in the places away from the pole. Derivatives with respect to polar angle are also being too big and spiky at those places, which is bad for the convergence in the frequency domain.

Arc length parametrization captures the shape of the curve regardless of the distance from the pole, but derivatives with respect to it turns too fast in highly curved places, thus, generating spiky derivatives.

Parametrization with tangent angle works good at capturing the information in highly curved places, but derivatives are being too big at linear places.

For this reason, a new parametrization technique has been developed by taking both polar angle, arc length and tangent angle into account. Polar angle parameter θ and new parameter μ are always in the same range, generally: $\theta, \mu \in [\,-\pi,\ \pi\,]$. Relation between θ and μ can be defined with a monotonically increasing function $\mu(\theta)$, and it's inverse be $\theta(\mu)$.

$r(\theta)$ is the superformula, $l(\theta)$ is the arc length, and $a(\theta)$ is the tangent angle in formulas below:

$$\mu(\theta) = \frac{2\pi}{M} \int_{-\pi}^{\theta} \frac{dm(\theta')}{d\theta'} d\theta' - \pi \qquad M = \int_{-\pi}^{\pi} \frac{dm(\theta)}{d\theta} d\theta$$

$$\frac{dm(\theta)}{d\theta} = \left(\left(\frac{dl(\theta)}{d\theta} \right)^{L} \left(\left(\frac{da(\theta)}{d\theta} \right)^{2} + P^{2} \right)^{1/2} \right)^{1/abs(L)+1}$$

$$\frac{dl(\theta)}{d\theta} = \sqrt{r(\theta)^2 + \left(\frac{dr(\theta)}{d\theta} \right)^2} \qquad \frac{da(\theta)}{d\theta} = 1 + \frac{\left(\frac{dr(\theta)}{d\theta} \right)^2 + r(\theta) \frac{dr^2(\theta)}{d\theta^2}}{r(\theta)^2 + \left(\frac{dr(\theta)}{d\theta} \right)^2}$$

Values of L and P in $\frac{dm(\theta)}{d\theta}$ represents the dominance of arc length and polar angle parameter respectively, relative to the tangent angle parameter. Their values change for every different superformula shape, and best values that generate smallest derivatives can be found with stochastic hill climbing algorithms.

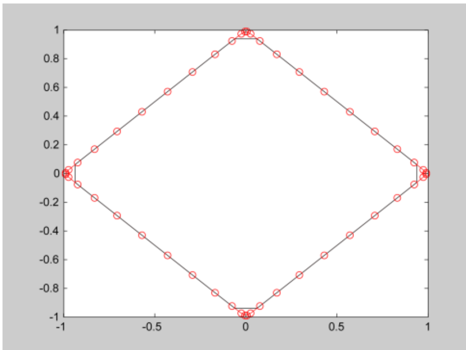
Importance of the μ parameter is that, in some cases where θ parameter is not parametrizing the superformula very well, it increases the convergence dramatically in the frequency domain of some values used in electromagnetic calculations. Computational complexity of electromagnetic calculations are not linear. Reducing the number of samples needed for certain level of approximation can be very effective for reducing execution time of electromagnetic calculations.

4 cases are presented in the figures below. μ parametrization is very effective in the first case, "Square" shape. Moderately effective in "5-Star" shape, and not making much difference in "Track" and "Plus" shape. Red colored graphs represent μ parametrization results, and black colored graphs represent θ parametrization results.

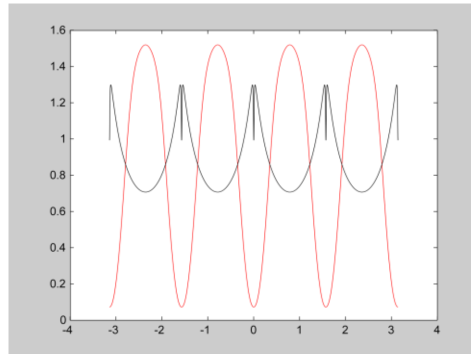
“Square”

$a = 1$ $b = 1$ $m = 4$ $n_1 = 1$ $n_2 = 1$ $n_3 = 1$

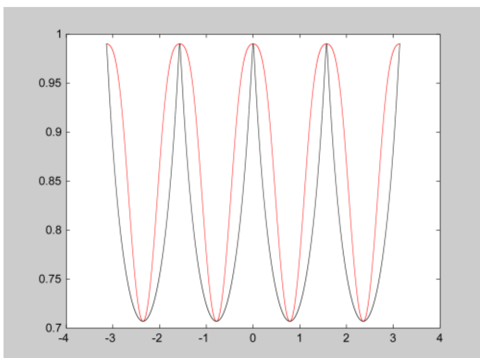
$\alpha = 0.01$ $\beta = 0.01$ $L = -2.436539500262644$ $P = 0$



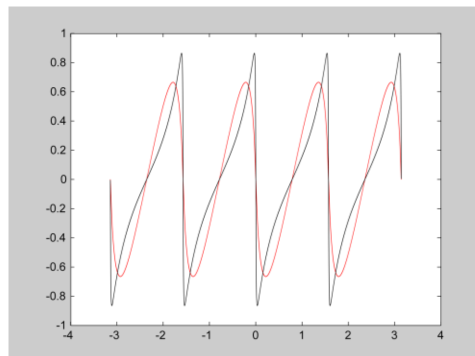
Contour



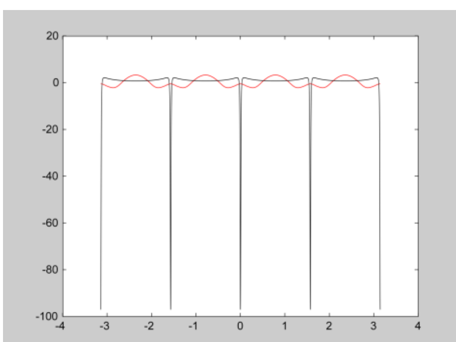
Derivative of Arc Length



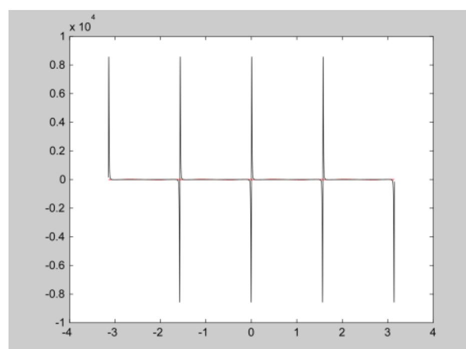
Superformula Function



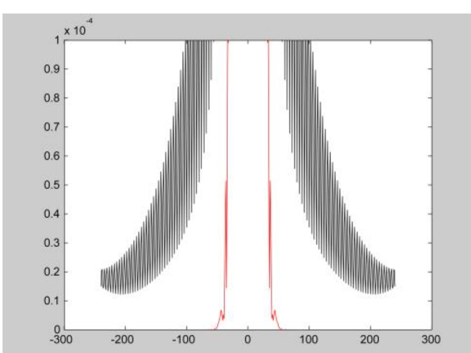
First Derivative



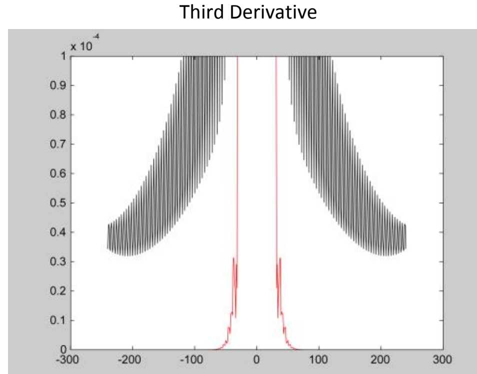
Second Derivative



Third Derivative



Convergence of Z

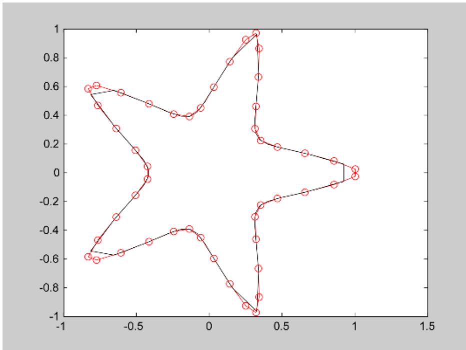


Convergence of J2

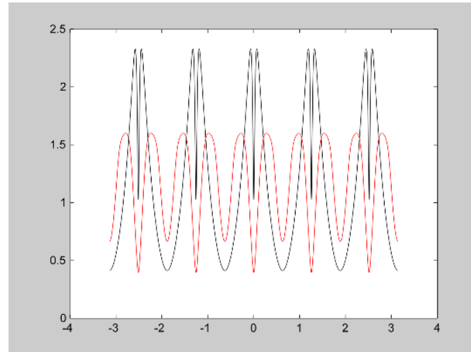
“5-Star”

$a = 5.08$ $b = 6.90$ $m = 10.0$ $n_1 = 0.139$ $n_2 = 0.139$ $n_3 = 0.972$

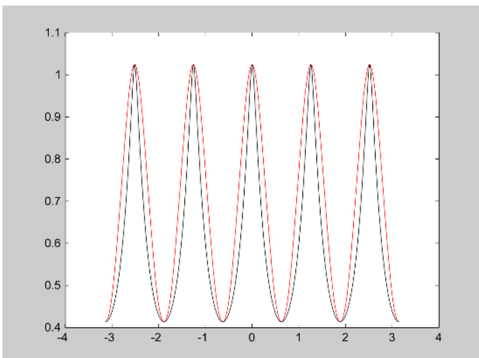
$\alpha = 4.29$ $\beta = 0.10$ $L = 2.553424164133645$ $P = 0.744234858869572$



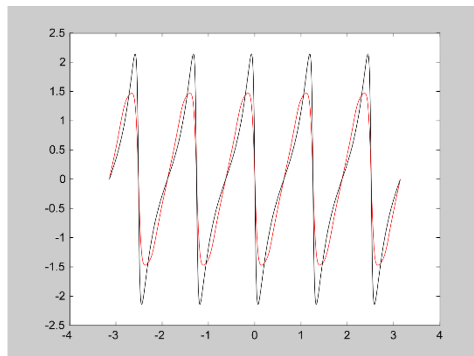
Contour



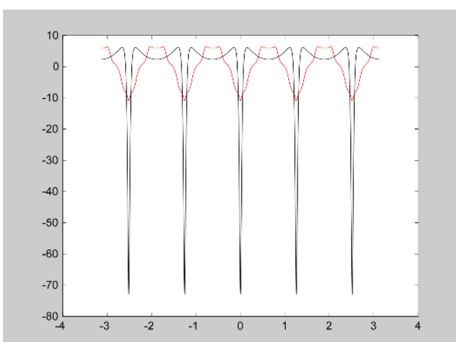
Derivative of Arc Length



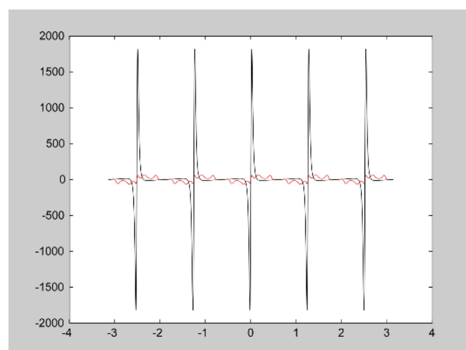
Superformula Function



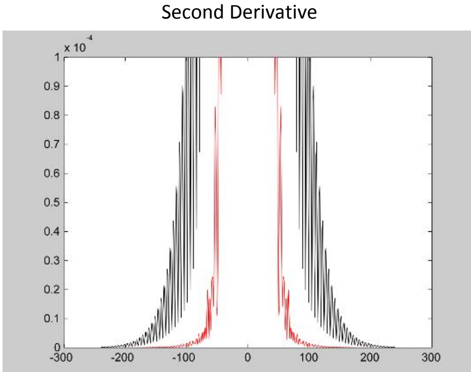
First Derivative



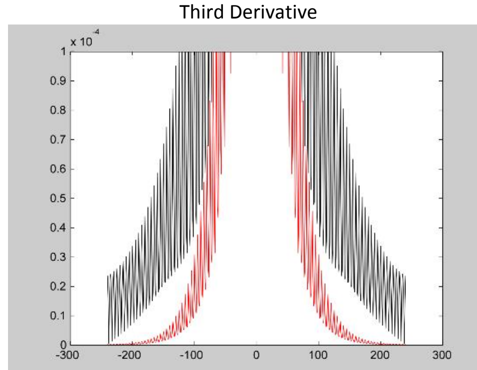
Second Derivative



Third Derivative



Convergence of Z

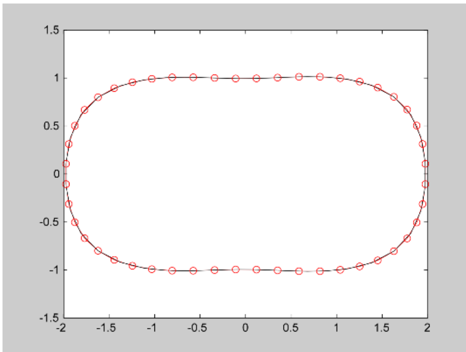


Convergence of J2

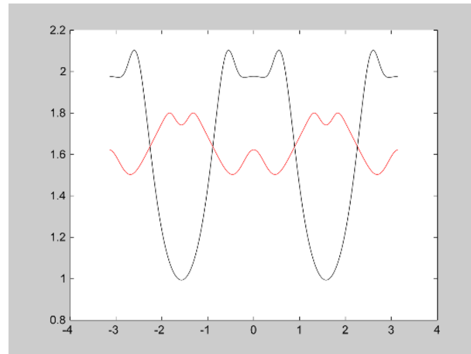
"Track"

$a = 8.10$ $b = 1.05$ $m = 3.99$ $n_1 = 3.28$ $n_2 = 3.22$ $n_3 = 4.52$

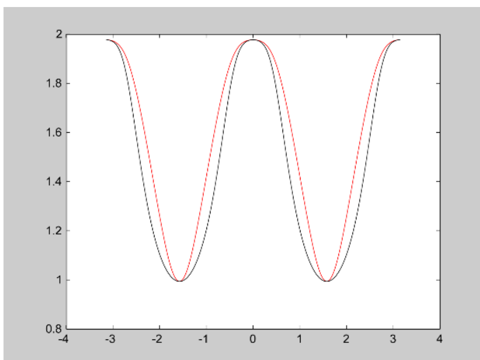
$\alpha = 3.90$ $\beta = 0.255$ $L = 8.773131436327327$ $P = 0.284255520628371$



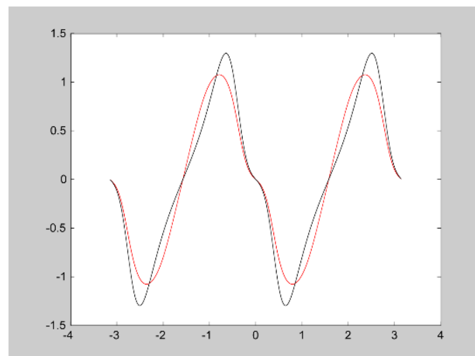
Contour



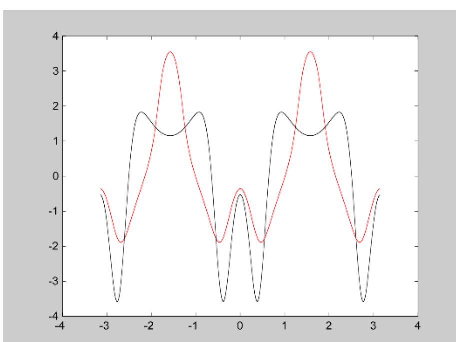
Derivative of Arc Length



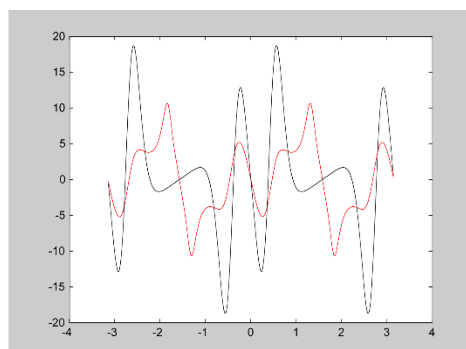
Superformula Function



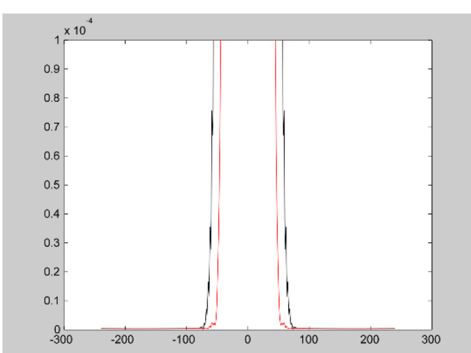
First Derivative



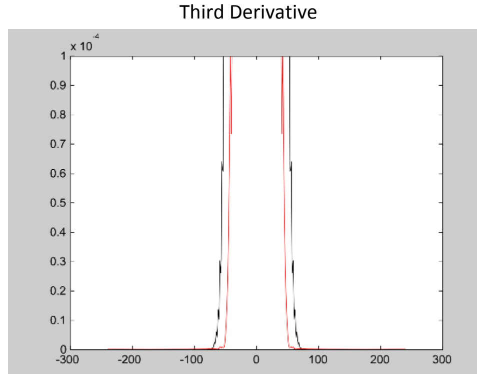
Second Derivative



Third Derivative



Convergence of Z

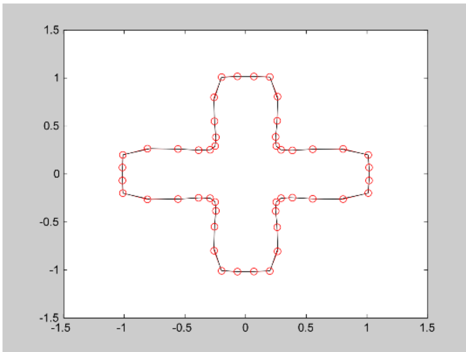


Convergence of J2

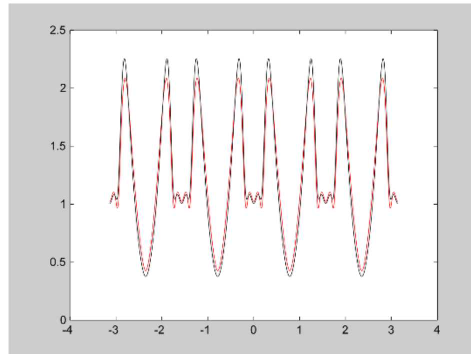
“Plus”

$a = 1.05$ $b = 1.13$ $m = 8.01$ $n_1 = 2.35$ $n_2 = 10.0$ $n_3 = 10.0$

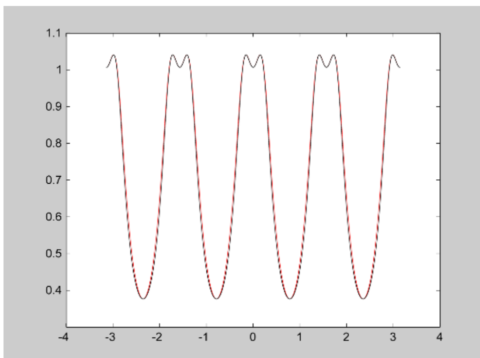
$\alpha = 0.119$ $\beta = 1.01$ $L = 0.151738055200703$ $P = 25.547567331539245$



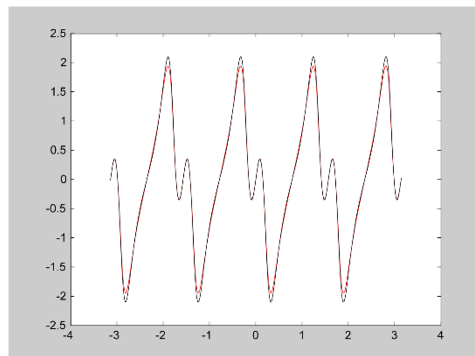
Contour



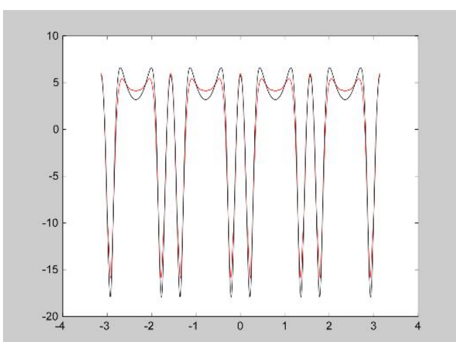
Derivative of Arc Length



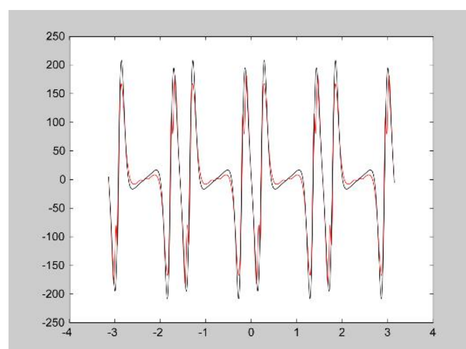
Superformula Function



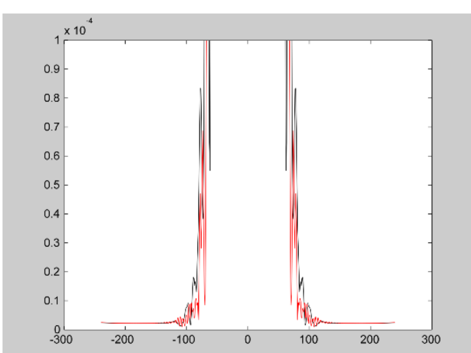
First Derivative



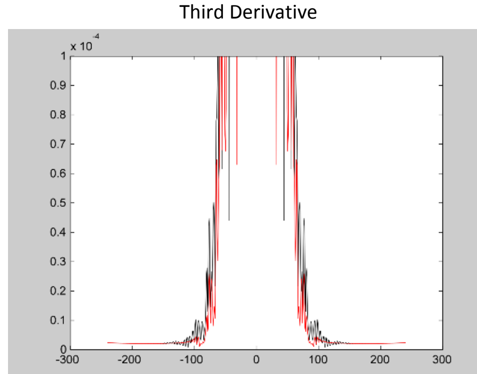
Second Derivative



Third Derivative



Convergence of Z



Convergence of J2