Reparametrizations of the Superformula

1) Introduction

Superformula is a function that can describe many different polar curves analytically. It is defined with polar coordinates, using polar angle as main parameter and radius described as a function of it. In this document, $r(\theta)$ will denote the superformula.

The reason to consider reparametrization is that, the set of points we get from the superformula with equidistant samples of θ is not that optimal, points don't represent the continuous superformula as best as they can. For example, they are being too dense in places near the pole, even when the curvature is very low. Or being too sparse in the places far from the pole, even if the curvature is high.

We can overcome this by taking lots and lots of samples, but we may need to do some heavy calculations on those points later, so this may not be the most viable option. We need to approximate the superformula as close as possible with the amount of points as low as possible to achieve acceptable levels of approximation errors with much less number of samples.

2) Definition of the curve as a function of new parameters

New parameters we define in this document are defined as functions of θ, with both domain and range being the interval $[\ -\pi,\ \pi\ ]$. This can easily be changed if neccesary, of course. $[\ -\pi,\ \pi\ ]$ is just an arbitrary interval, but all the parameters should be defined in the same interval to prevent extra complications.

These functions are one to one, always increasing functions, and can be invertible. We can generate equidistant samples of any new parameter, plug them into inverse functions to get corresponding θ samples, and then plug those θ samples into the superformula to get points on the curve corresponding to equidistant samples of the new parameter. Derivatives with respect to new parameters are a little bit trickier though, and we need first 3 derivatives in our other works.

Derivatives of the curve and inverse parameter functions with respect to new parameters do not change with each different kind of parameter, because they are defined with derivatives of the parameter functions. They'll only be written in this section and can be applied to all new parameters. μ will denote any new parameter in this section.

Superformula curve $\vec{P}\big(\theta(\mu)\big)$ in cartesian coordinates, and it's derivatives with respect to μ:

$$\vec{P}\big(\theta(\mu)\big) = < r\big(\theta(\mu)\big)\cos\big(\theta(\mu)\big), r\big(\theta(\mu)\big)\sin\big(\theta(\mu)\big) >$$

$$\vec{P}_\mu\big(\theta(\mu)\big) = \vec{P}_\theta\big(\theta(\mu)\big)\,\theta_\mu\big(\theta(\mu)\big)$$

$$\vec{P}_{\mu\mu}\big(\theta(\mu)\big) = \vec{P}_{\theta\theta}\big(\theta(\mu)\big)\,\theta_\mu\big(\theta(\mu)\big)^2 + \vec{P}_\theta\big(\theta(\mu)\big)\,\theta_{\mu\mu}\big(\theta(\mu)\big)$$

$$\vec{P}_{\mu\mu\mu}\big(\theta(\mu)\big) = \vec{P}_{\theta\theta\theta}\big(\theta(\mu)\big)\,\theta_\mu\big(\theta(\mu)\big)^3 + 3\vec{P}_{\theta\theta}\big(\theta(\mu)\big)\,\theta_\mu\big(\theta(\mu)\big)\,\theta_{\mu\mu}\big(\theta(\mu)\big) + \vec{P}_\theta\big(\theta(\mu)\big)\,\theta_{\mu\mu\mu}\big(\theta(\mu)\big)$$

Inverse parameter function $\theta(\mu)$ and it's derivatives with respect to μ:

$$\theta(\mu) = \mu^{-1}(\theta)$$

$$\theta_\mu(\theta(\mu)) = \frac{1}{\mu_\theta(\theta)}$$

$$\theta_{\mu\mu}(\theta(\mu)) = -\frac{\mu_{\theta\theta}(\theta)}{\mu_\theta(\theta)^3}$$

$$\theta_{\mu\mu\mu}(\theta(\mu)) = \frac{3\,\mu_{\theta\theta}(\theta)^2 - \mu_{\theta\theta\theta}(\theta)\,\mu_\theta(\theta)}{\mu_\theta(\theta)^5}$$

$\mu(\theta)$ can not be inverted to $\theta(\mu)$ analytically, but can be approximated using ~50 iterations of bisection method with relative error as small as 64-bit machine epsilon.

3) Fundamental parametrizations

Fundamental parametrizations are strictly defined parametrizations that are derived from a natural property of the curve. There exists only a single variety of each fundamental parametrization for each superformula curve, whereas non-fundamental mixed parametrization in the 4th section are not strictly defined and infinite variations of them exists for each curve.

For brevity, θ parameters of functions like $f(\theta)$ has been omitted and written as just f, starting from this line.

3.1) Arc length parameterization

Parametrization with arc length guarantees that, equidistant arc length parameter (t) samples will generate equidistant points on the superformula curve. In other words, if the total length of the curve is L, and N is the equidistant sample count of t, distance between each corresponding point would be L/N.

Arc length l at any θ can be defined as such:

$$l = \int_{-\pi}^{\theta} \sqrt{r_{\theta'}^2 + r^2}\, d\theta'$$

Total length of the curve is:

$$L = \int_{-\pi}^{\pi} \sqrt{r_\theta^2 + r^2}\, d\theta$$

Arc length parameter t is, arc length l adjusted from the interval $[\,0,\ L\,]$ to $[\,-\pi,\ \pi\,]$:

$$t = \frac{2\pi}{L} l - \pi = \frac{2\pi}{L} \int_{-\pi}^{\theta} \sqrt{r_{\theta'}^2 + r^2}\, d\theta' - \pi$$

Evaluating this integral is not possible in analytical form. But t is a smooth function, fast and accurate numerical integration is possible. This process doesn't take a noticable time even when N is around a million:
* Divide the integrand into N sub intervals.
* Calculate the trapezoidal area of each sub inteval.
* Sum them cumulatively.
* Build an interpolation from these points to make it continuous.
* Corresponding interval index for every θ is $floor\left(\frac{N\,(\theta+\pi)}{2\,\pi}\right)$, $O(1)$ complexity for lookup.

Derivatives of t can be described anaytically:

$$t_\theta = \frac{2\pi}{L}\sqrt{r^2 + r_\theta^2}$$

$$t_{\theta\theta} = \frac{2\pi}{L}\left(\frac{r_\theta(r + r_{\theta\theta})}{\sqrt{r^2 + r_\theta^2}}\right)$$

$$t_{\theta\theta\theta} = \frac{2\pi}{L}\left(\frac{r_\theta^2\,(r_\theta^2 - r\,r_{\theta\theta}) + r^2\,r_{\theta\theta}\,(r + r_{\theta\theta}) + r_{\theta\theta\theta}\,(\,r^2\,r_\theta + r_\theta^3\,)}{(r^2 + r_\theta^2)^{3/2}}\right)$$

3.2) Tangent angle parameterization

Tangent angle parameterization guarantees that, total absolute tangent angle change between samples are equal, similar to equal arc lengths in arc length parameter. If total absolute tangent angle change of the curve is K, and N is the equidistant sample count of Ψ (tangent angle parameter), absolute tangent angle change between each corresponding point would be K/N.

Tangent angle Φ can be defined with θ as:

$$\Phi = \theta + atan2(r, r_\theta) = \theta - i\,ln\left(\frac{r_\theta + i\,r}{\sqrt{r^2 + r_\theta^2}}\right)$$

Absolute of the derivative of Φ:

$$|\Phi_\theta| = \left|1 + \frac{r_\theta^2 - r\,r_{\theta\theta}}{r^2 + r_\theta^2}\right|$$

Total absolute tangent angle change over the whole curve:

$$K = \int_{-\pi}^{\pi} |\Phi_\theta|\,d\theta$$

Tangent angle parameter Ψ:

$$\Psi = \frac{2\pi}{K} \int_{-\pi}^{\theta} |\Phi_{\theta'}| \, d\theta' - \pi$$

We have the antiderivative of the non-absolute version of this integral, and we can turn it into the antiderivative of the absolute version with some numeric tricks. Essentially, we have to find increasing and decreasing intervals in Φ, and turn decreasing parts upside down to make it always increasing. Following algorithm can be used:

* Find critical points in Φ.
* Absolute differences between Φ values in each consecutive critical points are simply integrals of $|\Phi_\theta|$ in those intervals.
* Find integrals of $|\Phi_\theta|$ for each interval, and then sum them cumulatively to find integral values at each critical point.
* When θ is plugged into the function Ψ; find the closest previous critical point to θ, find their absolute differences of corresponding Φ values, then sum it with the cumulative integral value corresponding to that critical point. The result will be the value of Ψ function.

Derivatives of Ψ can be defined with some helper functions to reduce complexity:

$$A = r_\theta^2 - r\, r_{\theta\theta}$$
$$A_\theta = r_\theta\, r_{\theta\theta} - r\, r_{\theta\theta\theta}$$
$$A_{\theta\theta} = r_{\theta\theta}^2 - r\, r_{\theta\theta\theta\theta}$$

$$B = r^2 + r_\theta^2$$
$$B_\theta = 2\, r_\theta\, (r + r_{\theta\theta})$$
$$B_{\theta\theta} = 2\, (r_{\theta\theta}\, (r + r_{\theta\theta}) + r_\theta\, (r_\theta + r_{\theta\theta\theta}))$$

$$C = 1 + \frac{A}{B}$$
$$C_\theta = \frac{A_\theta B - B_\theta A}{B^2}$$
$$C_{\theta\theta} = \frac{(A_{\theta\theta}\, B - A\, B_{\theta\theta})B - 2\, (A_\theta\, B - A\, B_\theta)\, B_\theta}{B^3}$$

$$D = C^2$$
$$D_\theta = 2\, C\, C_\theta$$
$$D_{\theta\theta} = 2\, (C_\theta^2 + C\, C_{\theta\theta})$$

$$\Psi_\theta = \frac{2\pi}{K}\, D^{1/2}$$

$$\Psi_{\theta\theta} = \frac{\pi}{K}\, D^{-1/2}\, D_\theta$$

$$\Psi_{\theta\theta\theta} = \frac{\pi}{K} \left(D^{-1/2}\, D_{\theta\theta} - \frac{1}{2}\, D^{-\frac{3}{2}}\, D_\theta^2 \right)$$

4) Mixed parametrizations

It is possible to mix different parametrizations either by taking their superpositions or geometric means to derive new parametrizations. These new parametrizations are just like any other parametrizations, and can also be mixed again with other parametrizations to create a new mixed parameterization. Weights of each parameterization in the mixing process can be changed, for example if parameters A and B mixed with weights 1 and 10 respectively, B would be 10 times more dominant in the new parameter.

4.1) Superposition of parameters

I could have given the name "arithmatic mean of parameters" to this method, but I have chosen "superposition", because it really works like the superposition in physics. For example, if arc length and tangent angle parameters with equal weights are mixed with this method, the new parameter guarantees a maximum limit of $2\,L/N$ arc length distance and a maximum limit of $2\,K/N$ tangent angle change between samples, given that N equidistant samples of s are taken. I don't imagine something like this when I think about arithmatic mean.

In this section, s denotes the superposition parameter, ${}_{n}\mu$ denotes n'th parameter to be mixed, and ${}_{n}W$ denotes the weight of n'th parameter in the mixture. Left supscripts are being used to denote indices because right subscripts are reserved for derivatives in the scope of this document.

s and it's derivatives are defined as such:

$$M = \sum_{n} {}_{n}W$$

$$s = \frac{1}{M}\sum_{n}({}_{n}W\, {}_{n}\mu)$$

$$s_{\theta} = \frac{1}{M}\sum_{n}({}_{n}W\, {}_{n}\mu_{\theta})$$

$$s_{\theta\theta} = \frac{1}{M}\sum_{n}({}_{n}W\, {}_{n}\mu_{\theta\theta})$$

$$s_{\theta\theta\theta} = \frac{1}{M}\sum_{n}({}_{n}W\, {}_{n}\mu_{\theta\theta\theta})$$

4.2) Geometric mean of parameters

Parameters are multiplied instead of being summed in this parametrization. Main difference from the superposition parameter is very similar to the difference between OR and AND operations. Superposition parameter s changes when either one of it's sub parameters change. Geometric mean parameter g changes only when both of it's sub parameters change. This means that, when arc length and tangent angle parametrizations are mixed with this method, equidistant samples do not give too much unnecessary points lying on straight long lines, or curved but very small places. For this reason, this parametrization gives a very good area coverage. But this parametrization is still not the best, it loses some coverage when superformula has spiky and sharp edges. Maybe an extra mixing layer would solve this, something like superposition of geometric mean parameter with a little bit tangent angle parameter.

Here, g denotes the mixed parameter, ${}_n\mu$ denotes n'th parameter to be mixed, ${}_nW$ denotes the weight of n'th parameter in the mixture, ${}_nQ$ denotes normalized weights, T denotes the total number of parameters to be mixed, $\boldsymbol{F}$ denotes the set of integers ranging from 1 to T.

Definitions of g and it's derivatives:

$$ {}_nQ = \frac{{}_nW}{\sum_{k\in\boldsymbol{F}} {}_kW} $$

$$ G = \int_{-\pi}^{\pi} \prod_{n\in\boldsymbol{F}} {}_n\mu_\theta^{({}_nQ)} \ d\theta $$

$$ g = \frac{2\pi}{G} \int_{-\pi}^{\theta} \prod_{n\in\boldsymbol{F}} {}_n\mu_{\theta'}^{({}_nQ)} \ d\theta' - \pi $$

$$ g_\theta = \frac{2\pi}{G} \prod_{n\in\boldsymbol{F}} {}_n\mu_\theta^{({}_nQ)} $$

$$ g_{\theta\theta} = \frac{2\pi}{G} \sum_{n\in\boldsymbol{F}} \left({}_nQ \ {}_n\mu_\theta^{({}_nQ-1)} \ {}_n\mu_{\theta\theta} \prod_{k\in\boldsymbol{F}-n} {}_k\mu_\theta^{({}_kQ)} \right) $$

$$ \begin{aligned} g_{\theta\theta\theta} = \frac{2\pi}{G} \sum_{n\in\boldsymbol{F}} \Bigg({}_nQ \Bigg(& \left(({}_nQ - 1) \ {}_n\mu_\theta^{({}_nQ-2)} \ {}_n\mu_{\theta\theta}^2 + \ {}_n\mu_\theta^{({}_nQ-1)} \ {}_n\mu_{\theta\theta\theta} \right) \prod_{k\in\boldsymbol{F}-n} {}_k\mu_\theta^{({}_kQ)} \\ & + {}_n\mu_\theta^{({}_nQ-1)} \ {}_n\mu_{\theta\theta} \sum_{k\in\boldsymbol{F}-n} \left({}_kQ \ {}_k\mu_\theta^{({}_kQ-1)} \ {}_k\mu_{\theta\theta} \prod_{m\in\boldsymbol{F}-n-k} {}_m\mu_\theta^{({}_mQ)} \right) \Bigg) \Bigg) \end{aligned} $$

Sometimes, indices in those sums and products could point to an empty set, in those cases:

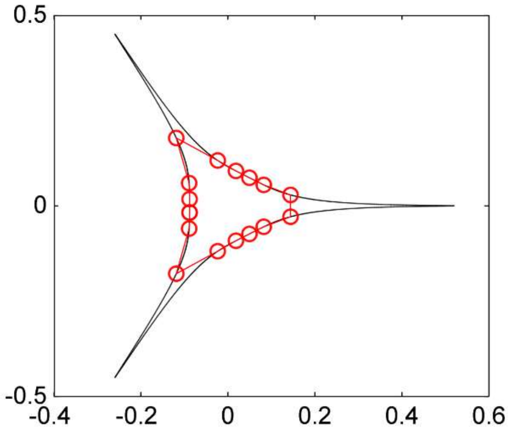
$$ \sum_{n\in\{\emptyset\}} {}_nA = 0 $$

$$ \prod_{n\in\{\emptyset\}} {}_nA = 1 $$

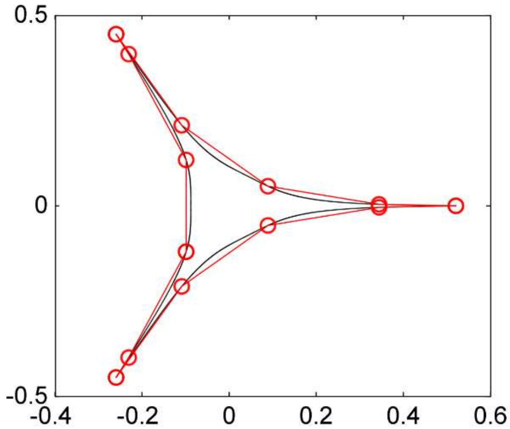
Integral inside the g function is not integrable analytically, but is easy to integrate with the same way described in arc length parametrization section.

5) Some examples of parameter behaviours

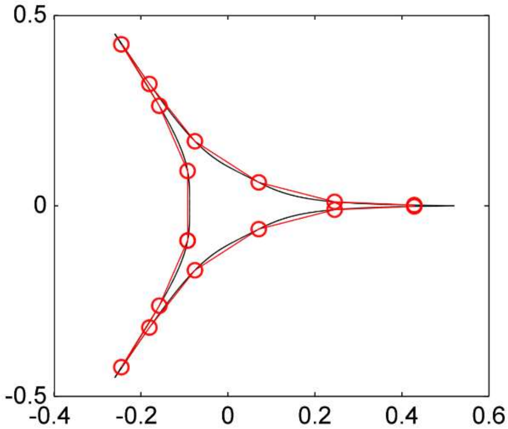
Spiky shape, sampled with 16 points:



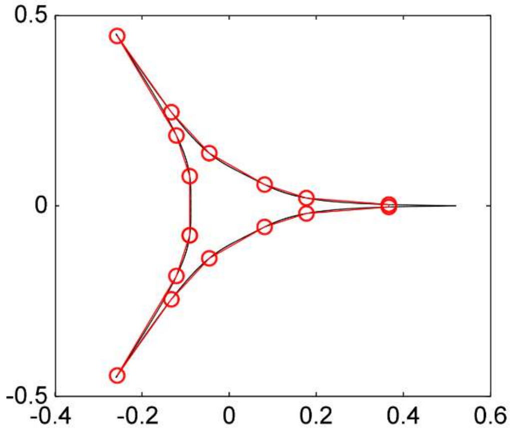
Pol



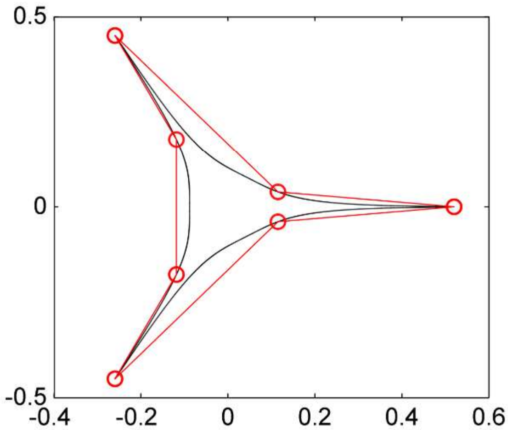
Sup, Arc + Tan



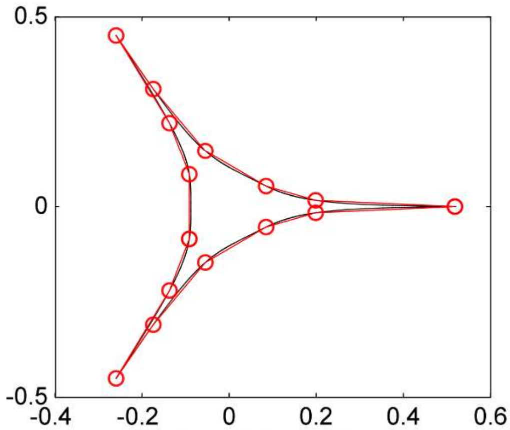
Arc



Gem, Arc * Tan

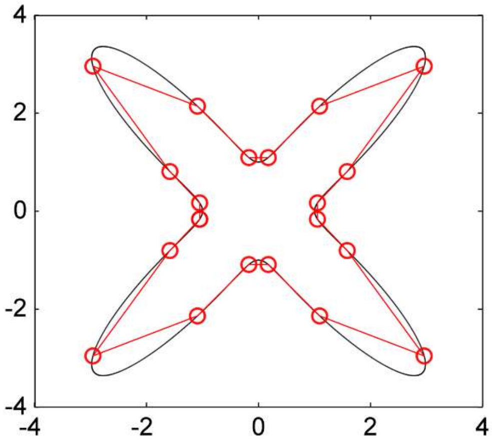


Tan



Sup, 4*Gem + Tan

Curvy shape, sampled with 20 points:



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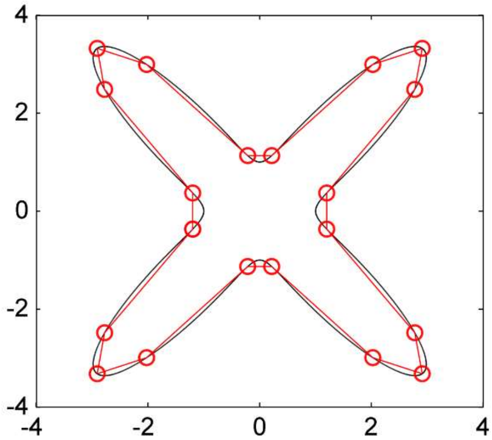
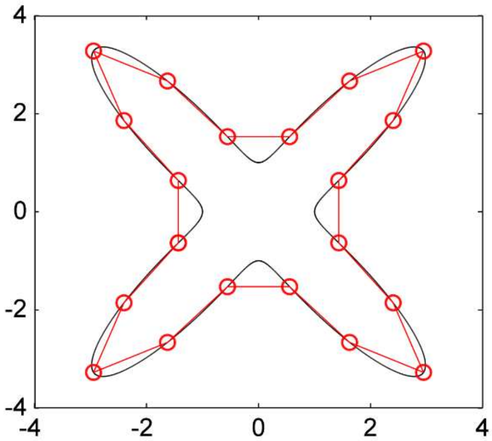
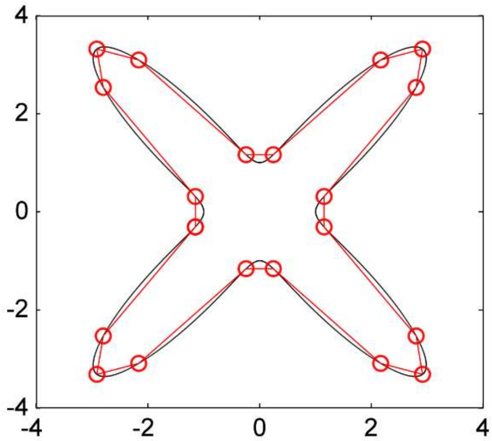


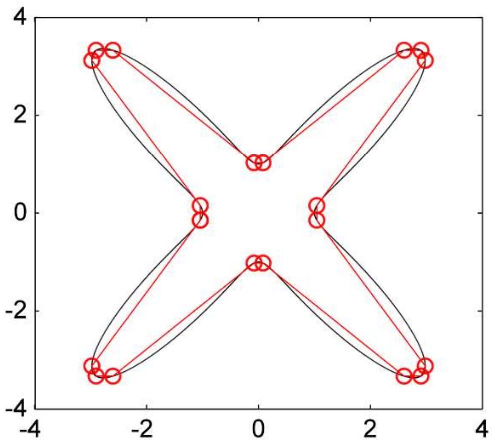
Sup, Arc + Tan



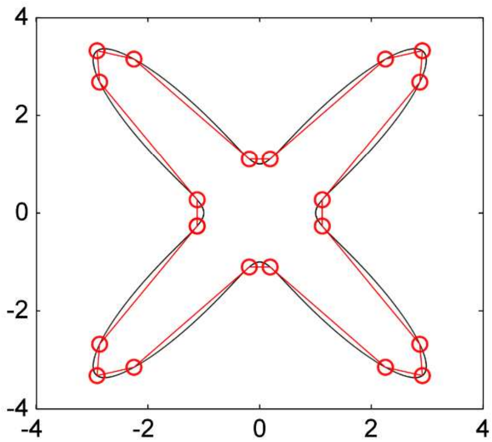
Arc



Gem, Arc * Tan



Tan



Sup, 4*Gem + Tan

Straight shape, sampled with 32 points: