Circuit Theory and Electronics Fundamentals

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T1 Laboratory Report

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Contents

1 Introduction

In this laboratory assignment, we study the circuit ilustrated in the figure (1). Our objective is to better understand how an RC circuit works and how it responds to certain stimuli. More specifically, we studied the circuit in its stationary state ($t < 0\,s$), as well as how it responded to a sine wave with frequency f: $v_s(t) = \sin(2\pi f t)$, when $t > 0\,s$.
In section (2), we performed a theoretical analysis of the circuit. Using the Nodal Method, we described its behavior when $t < 0\,s$, computed the equivalent resistor as seen by the capacitor to obtain the natural solution after $t > 0\,s$, and through the computation of phasor voltages we obtained the forced solution and studied the frequency response.
In section (3), we present the results derived from various simulations of the circuit that were obtained from *Ngspice* scripts. We simulated the linear circuit when $t < 0\,s$, the total solution when the circuit is subjected to the stimulus and its frequency response. Moreover, we also compare the values determined through theoretical methods and the values derived from the circuit simulations side by side, explaining why they did or did not differ from one another.
Finally, in section (4), we sumarize the results of this assignment, namely the comparison between the theoretical analysis and the simulation.

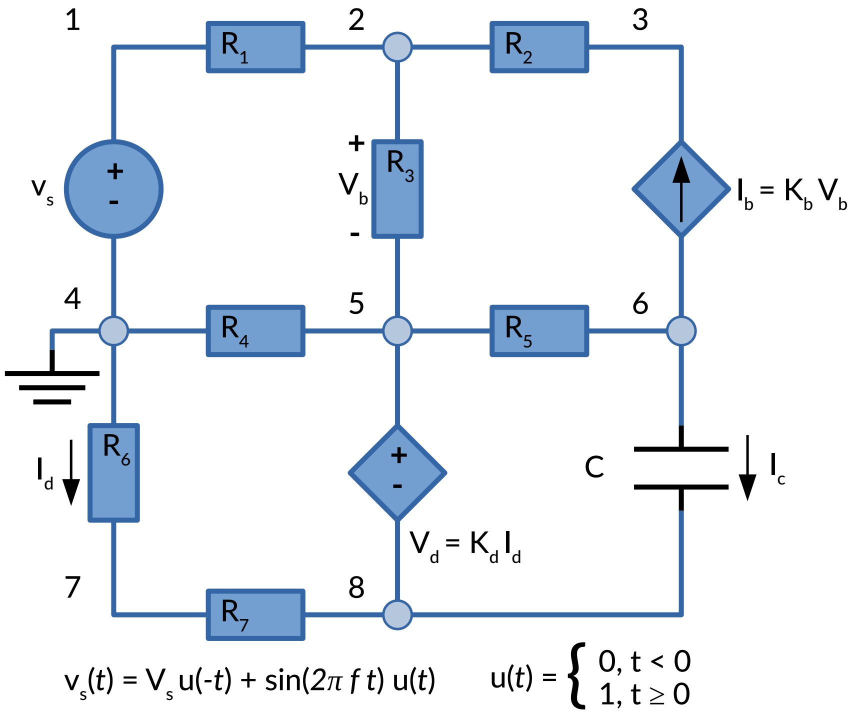

Figure 1: RC Circuit

2 Theoretical Analysis

In this section we analyse the circuit shown in Figure (1) before and after $t = 0\,s$ using the theoretical models studied in class.

2.1 Nodal Analysis Method - $t < 0\,s$

We start by analysing the stationary regime of the circuit, or in other words, we assume enough time has passed so that the capacitor is fully charged. The capacitor is described by equation (1).

$$i_c = C\frac{dv_c}{dt} \tag{1}$$

Where i_c and v_c are, respectively, the current passing through the capacitor and the voltage at it's terminals. Therefore if we are in a stationary situation, that is $v_c = constant$, then $i_c = 0\,A$. The capacitor has an open circuit behavior which implies the circuit is equivalent to the one in figure (2).

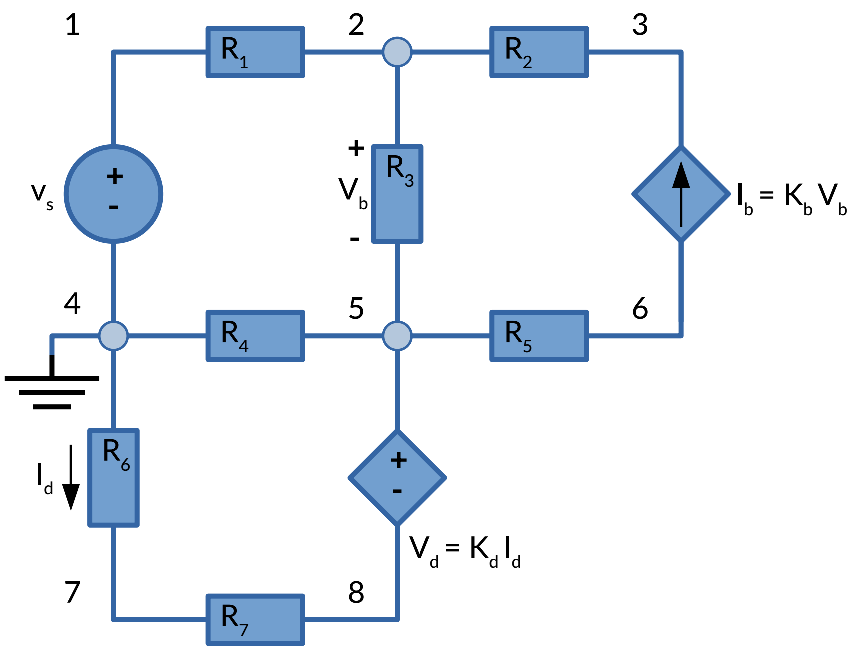

Figure 2: RC Circuit to be analysed with Node Voltages conventions

Using the nodal voltage analysis method we can obtain 7 linear equations for V_1, V_2, V_3, V_5, V_6, V_7 and V_8, which we now present in matricial form.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{R_1} & \frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3} & -\frac{1}{R_2} & -\frac{1}{R_3} & 0 & 0 & 0 \\ 0 & \frac{1}{R_2}+K_b & -\frac{1}{R_2} & -K_b & 0 & 0 & 0 \\ 0 & -\frac{1}{R_3} & 0 & \frac{1}{R_3}+\frac{1}{R_4}+\frac{1}{R_5} & -\frac{1}{R_5} & -\frac{1}{R_7} & \frac{1}{R_7} \\ 0 & K_b & 0 & -\frac{1}{R_5}-K_b & \frac{1}{R_5} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{R_6}+\frac{1}{R_7} & -\frac{1}{R_7} \\ 0 & 0 & 0 & 1 & 0 & \frac{K_d}{R_6} & -1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_5 \\ V_6 \\ V_7 \\ V_8 \end{bmatrix} = \begin{bmatrix} V_s \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{2}$$

By solving this equations using *Octave* we obtained the values in table (1). In this table we also present the values for the currents in all branches, I_{Rn}, in the direction defined as from the lower to the higher numbered node, in order to match the ones calculated in *Ngspice*, as well as V_b and V_d.

Quantity	Value
I_d (mA)	0.980272
I_b (mA)	-0.271033
I_{R1} (mA)	0.258827
I_{R2} (mA)	0.271033
I_{R3} (mA)	-0.012206
I_{R4} (mA)	-1.239099
I_{R5} (mA)	-0.271033
I_{R6} (mA)	0.980272
I_{R7} (mA)	0.980272
V_1 (V)	5.243570
V_2 (V)	4.981487
V_3 (V)	4.435588
V_4 (V)	0
V_5 (V)	5.019873
V_6 (V)	5.844139
V_7 (V)	-1.989217
V_8 (V)	-2.979997
V_d (V)	7.999870
V_b (V)	-0.038386

Table 1: Nodal Analysis Results - $t < 0\ s$

2.2 Equivalent Resistance, R_{eq}

In this section we want do determine the equivalent resistance, R_{eq}, of the circuit as seen from the capacitor terminals. To do this we consider the Thévenin equivalent of the circuit as seen in figure (3).

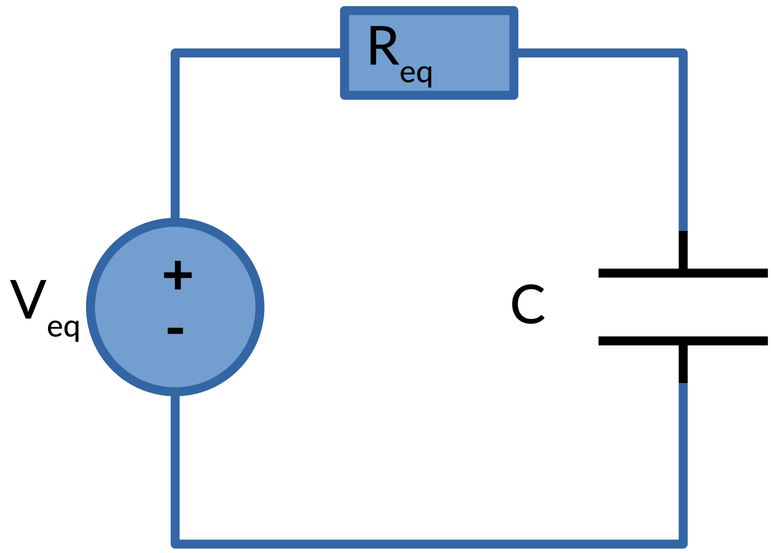

Figure 3: Thévenin Equivalent Circuit

To calculate the equivalent resistance we turn off all independent sources in the circuit. This corresponds to replacing the voltage source in the Thévenin equivalent with a short circuit. Therefore, by replacing the capacitor with a voltage source that imposes a voltage $V_x = V_6 - V_8$, where V_6 and V_8 are the nodal voltages determined in the previous section we have a simple way to calculate R_{eq}. We only need to determine I_x, that is, the current that is supplied by the voltage source V_x to the equivalent resistance. By doing this, we can use Ohm's Law to determine $R_{eq} = \frac{V_x}{I_x}$.
All this considered, to calculate I_x we use nodal analysis in the circuit presented in figure (4).

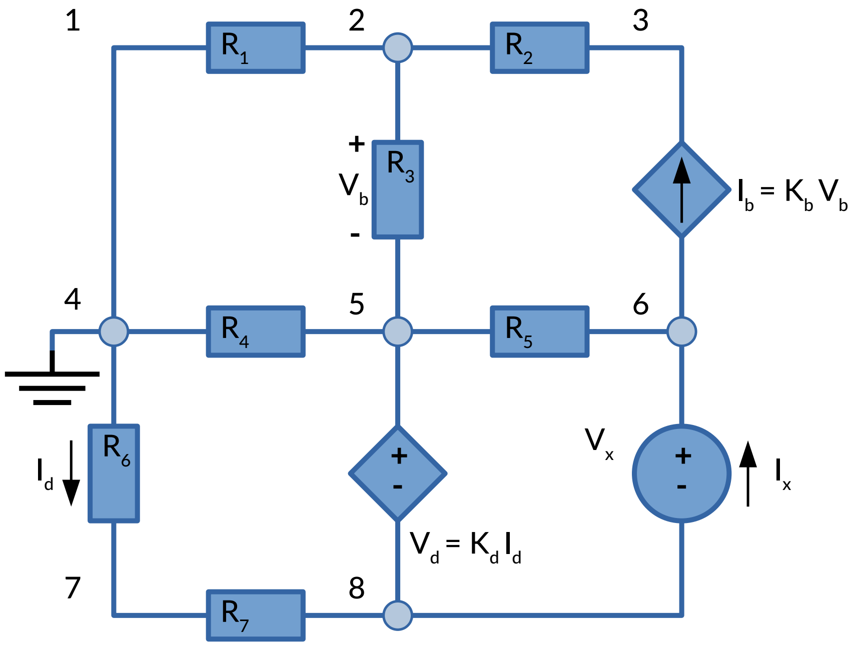

Figure 4: Circuit to be analysed with Node Voltages conventions

This whole procedure is necessary due to the fact that the circuit in question has several resistances and two dependent sources which complicates the calculation of R_{eq}. Furthermore it is useful to use $V_x = V_6 - V_8$ because by doing so, we will not only determine the equivalent resistance, R_{eq}, but also calculate the nodal voltages of all nodes in instant $t = 0^+\ s$. This is a result of the fact that $v_s(0^+) = 0\,V$ (at that instant the voltage source is equivalent to a wire) and that the voltage of the capacitor is continuous at $t = 0$ which implies that $v_6(0^+) - v_8(0^+) = V_x$. The equations correspondent to the nodal analysis are presented in matricial form bellow:

$$\begin{bmatrix} \frac{1}{R_2}+\frac{1}{R_3}+\frac{1}{R_4} & -\frac{1}{R_2} & -\frac{1}{R_3} & 0 & 0 & 0 \\ \frac{1}{R_2}+K_b & -\frac{1}{R_2} & -K_b & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{K_d}{R_6} & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & \frac{1}{R_6}+\frac{1}{R_7} & -\frac{1}{R_7} \\ -\frac{1}{R_3}+K_b & 0 & \frac{1}{R_3}+\frac{1}{R_4}-K_b & 0 & -\frac{1}{R_7} & \frac{1}{R_7} \end{bmatrix} \begin{bmatrix} V_2 \\ V_3 \\ V_5 \\ V_6 \\ V_7 \\ V_8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ V_x \\ 0 \\ 0 \end{bmatrix} \tag{3}$$

The equivalent current can be calculated using expression (4).

$$I_x = \frac{V_6 - V_5}{R_5} + I_b \tag{4}$$

We can also calculate the time constant associated to the circuit:

$$\tau = R_{eq}C \tag{5}$$

Where C is the capacitor's capacity. Solving the system of equations above and by using expressions (4) and (5) we obtained the values in table (2).

Quantity	Value
I_d (mA)	0.000000
I_b (mA)	0.000000
I_{R1} (mA)	-0.000000
I_{R2} (mA)	0.000000
I_{R3} (mA)	0.000000
I_{R4} (mA)	-0.000000
I_{R5} (mA)	-2.901531
I_{R6} (mA)	0.000000
I_{R7} (mA)	0.000000
V_1 (V)	0
V_2 (V)	0.000000
V_3 (V)	0.000000
V_4 (V)	0
V_5 (V)	0.000000
V_6 (V)	8.824136
V_7 (V)	-0.000000
V_8 (V)	-0.000000
V_b (V)	0.000000
V_d (V)	0.000000
V_x (V)	8.824136
I_x (mA)	2.901531
R_{eq} (Ω)	3041.200103
τ (ms)	3.146672

Table 2: Nodal Analysis Results - $t = 0^+ \, s$

2.3 Natural Solution, $v_{6n}(t)$ when $t > 0 \, s$

In this section we determine the natural solution for the voltage at node 6 after $t > 0 \, s$. From the theory classes we know the general formula for the voltage of an RC circuit. Therefore considering the circuit in figure (1) we know that the natural solution for the voltage in node 6 will be given by an expression of the form:

$$v_{6n} = v_{6n}(0)e^{-\frac{t}{\tau}} \tag{6}$$

Considering the results obtained in the previous section we know that $v_6(0^+) - v_8(0^+) = V_x$ and that $v_8(0^+) = 0 \, V$.
Therefore the natural solution for the voltage in node 6 is given by:

$$v_{6n} = 8.824136e^{-\frac{t}{3.146672\times10^{-3}}} \tag{7}$$

The solution is plotted in figure (5) for the interval $[0, 20] \, ms$.

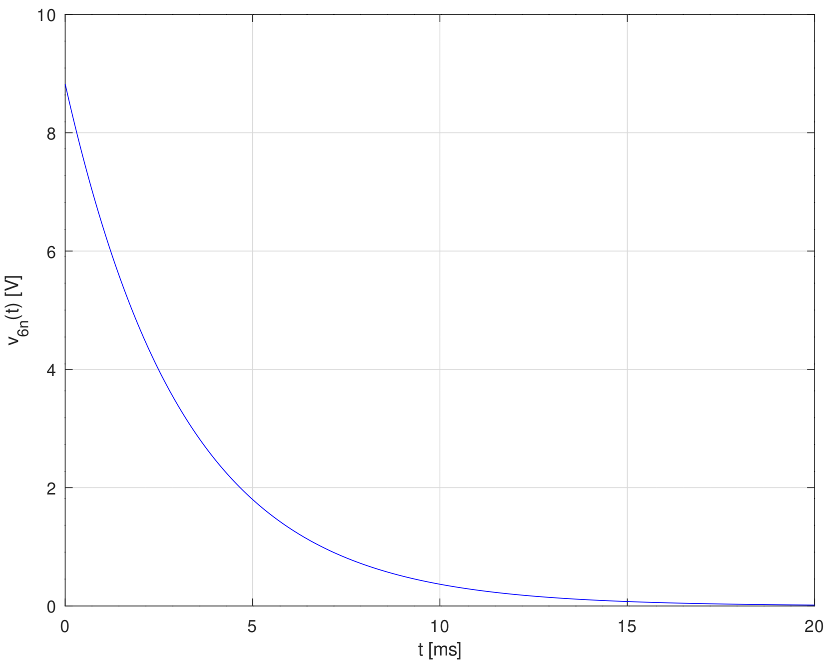

Figure 5: Natural Solution of the Voltage at Node 6, $v_{6n}(t)$

2.4 Forced Solution, $v_{6f}(t)$ when $t > 0\ s$

In this section we determine the forced solution for the voltage at node 6 with $t > 0\ s$ To do so we consider once again the circuit shown in figure (1).

Using node analysis with phasors we can determine the node amplitude and phase for each one of the nodes in the circuit considering only the forced regime. Taking into acount Ohm's Law using impedances and that $Z_C = \frac{1}{j\omega C}$ for capacitors and $Z_R = R$ for resistances we obtain the following equations in matricial form:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{R_1} & \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} & -\frac{1}{R_2} & -\frac{1}{R_3} & 0 & 0 & 0 \\ 0 & \frac{1}{R_2} + K_b & -\frac{1}{R_2} & -K_b & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \frac{K_d}{R_6} & -1 \\ 0 & K_b & 0 & -K_b - \frac{1}{R_5} & \frac{1}{R_5} + j\omega C & 0 & -j\omega C \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{R_6} + \frac{1}{R_7} & -\frac{1}{R_7} \\ 0 & -\frac{1}{R_3} & 0 & \frac{1}{R_3} + \frac{1}{R_4} + \frac{1}{R_5} & -j\omega C - \frac{1}{R_5} & -\frac{1}{R_7} & \frac{1}{R_7} + j\omega C \end{bmatrix} \begin{bmatrix} \widetilde{V_1} \\ \widetilde{V_2} \\ \widetilde{V_3} \\ \widetilde{V_5} \\ \widetilde{V_6} \\ \widetilde{V_7} \\ \widetilde{V_8} \end{bmatrix} = \begin{bmatrix} \widetilde{V_s} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{8}$$

Where $\widetilde{V_s} = -j$ is the phasor of the voltage source v_s.
Using *Octave* we can solve the system above and obtain the corresponding amplitudes for the nodal phasors which are presented in table (3).

Quantity	Value
$\lvert\widetilde{V_1}\rvert$ (V)	1.000000
$\lvert\widetilde{V_2}\rvert$ (V)	0.950018
$\lvert\widetilde{V_3}\rvert$ (V)	0.845910
$\lvert\widetilde{V_4}\rvert$ (V)	0
$\lvert\widetilde{V_5}\rvert$ (V)	0.957339
$\lvert\widetilde{V_6}\rvert$ (V)	0.570374
$\lvert\widetilde{V_7}\rvert$ (V)	0.379363
$\lvert\widetilde{V_8}\rvert$ (V)	0.568314

Table 3: Forced Solution - Magnitude of Voltage Node Phasors

Considering $\omega = 2\pi f$ with a linear frequency of $f = 1\ kHz$ we can determine the forced solution on node 6, v_{6f}, by taking the real part of $\widetilde{V_6} \cdot e^{\omega t}$. In figure (6) we plot the forced solution in the interval $[0, 20]\ ms$.

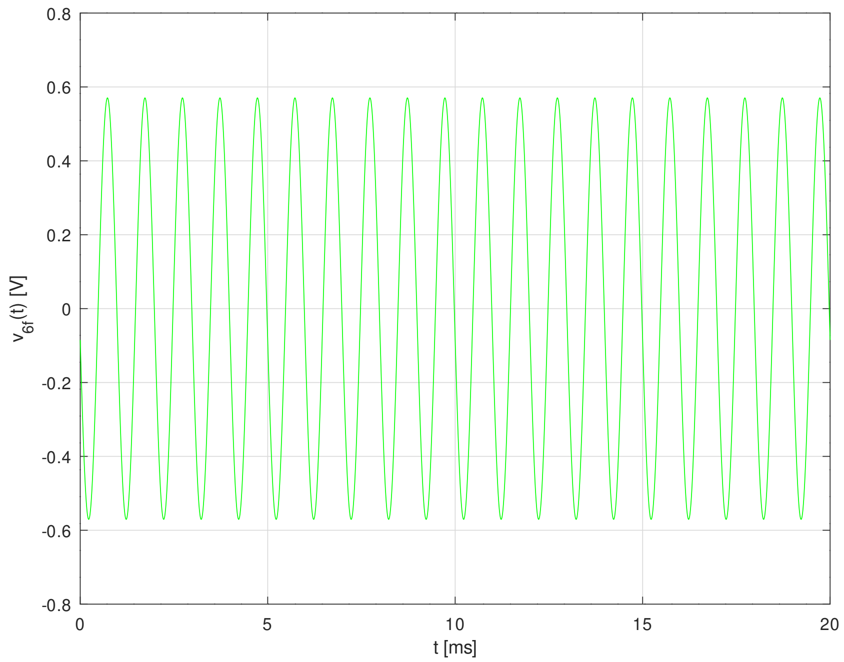

Figure 6: Forced Solution of the Voltage at Node 6, $v_{6f}(t)$

2.5 Total Solution

In this section we determine the total solution of the voltage at node 6 for values $t > 0\ s$ by superimposing the natural and forced solutions calculated previously. To compare the results with the behavior of the voltage source v_s we plot both $v_s(t)$ and $v_6(t)$ in the interval $[-5, 20]\ ms$. The values of $v_6(t)$ when $t < 0\ s$ correspond to the ones calculated in section (2.1). The plot in question is presented in figure (7).
We can see that the difference between the phases of the two graphs is approximately π, this result will be confirmed in the next section.

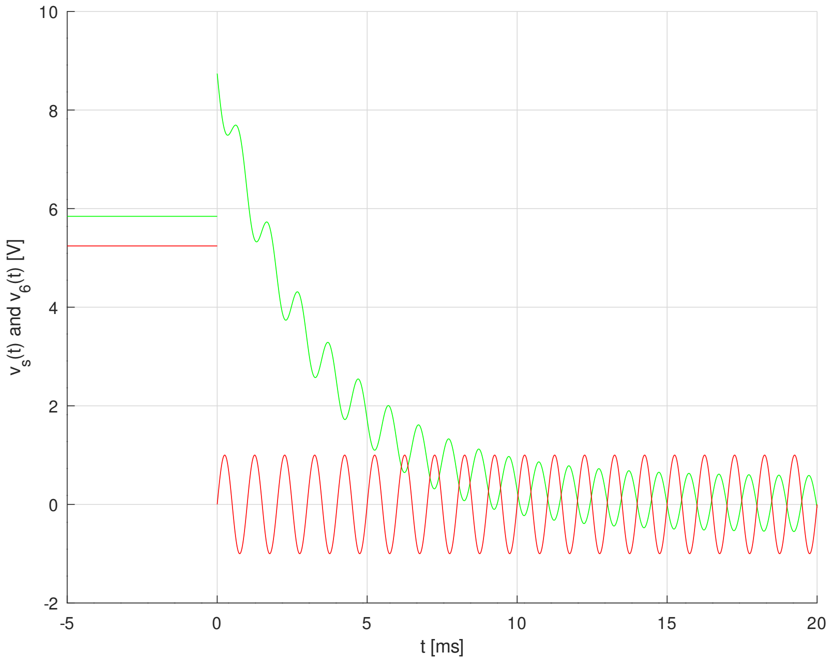

Figure 7: Green - Total Solution of the Voltage at Node 6 ($v_6(t)$) and Red - Voltage Source ($v_s(t)$) plots with $t \in [-5, 20]\, ms$

2.6 Frequency Responses

In this section we analyse the frequency responses of $v_c(f) = v_6(f) - v_8(f)$, $v_6(f)$ and $v_s(f)$ for the frequency range $f \in [0.1, 10^6]\, Hz$.
Using *Octave* we solved symbolically the nodal method analysis equations for the forced regime. By doing so we obtained expressions for each nodal voltage phasor as a function of f. From there we just divided the phasor in question ($\widetilde{V_c}$, $\widetilde{V_6}$ or $\widetilde{V_s}$) by $\widetilde{V_s}$ to obtain the transformation function.
The absolute value of the transformation functions in decibels and their phase's in degrees are plotted in graph (8).
When analysing the magnitudes we can see that the transformation function yields a magnitude of 1 ($0\, dB$) for v_s, this is obviously a trivial case. On the other hand, we can see that the transformation function regarding the voltage of the capacitator goes to 0 ($-\infty\, dB$) as the frequency goes to infinity. This means that the capacitor only let's through lower frequencies which is coherent with what we have seen in theory classes.
As the frequency tends to infinity the capacitor starts to have the behaviour of a short-circuit therefore the capacitor acts in a similar fashion to a regular wire which explains why the magnitude of $v_6(f)$ assumes a stationary value.

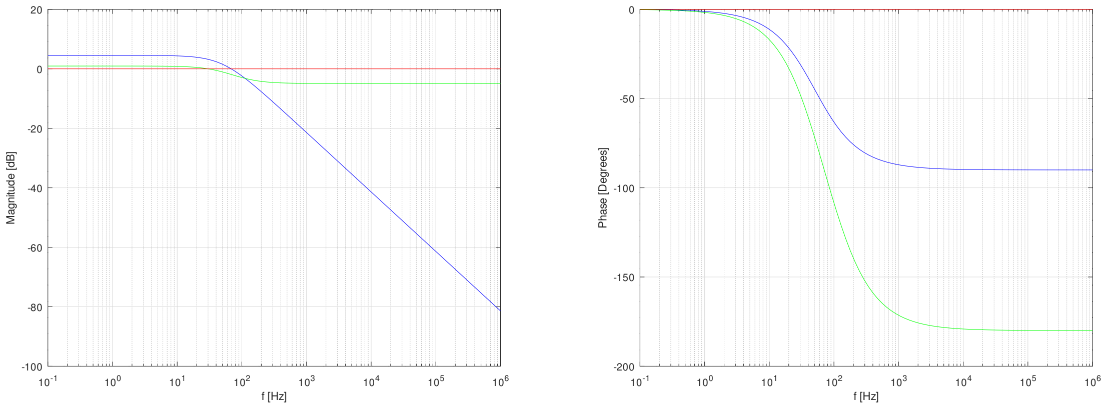

Figure 8: Frequency Response - Magnitude and Phase: Red - Voltage Source; Green - Node 6; Blue - Capacitor

3 Simulation Analysis

In this section, we simulate the RC circuit in its different stages in time using *Ngspice*, performing Operating Point Analysis to get the branch currents and node voltages in the first two subsections. We also perform a Transient Analysis to get the voltage in node 6 and compare it with the stimulus in node 1 in the third and fourth subsections. Finally we do an AC Frequency Responce Analysis, in order to compare the Magnitudes and Phases of the input source, node 6 and the capacitor.

3.1 .OP: Circuit when $t < 0\ s$

In the table (4), we display all the branch currents and node voltages obtained from the simulation of the circuit ilustrated in figure (2). We also present, side by side, the table from section (2.1) to facilitate comparing results.

Table 4: Nodal Method Analysis - $t < 0\ s$

Table 5: Theoretical Results

Quantity	Value
I_d (mA)	0.980272
I_b (mA)	-0.271033
I_{R1} (mA)	0.258827
I_{R2} (mA)	0.271033
I_{R3} (mA)	-0.012206
I_{R4} (mA)	-1.239099
I_{R5} (mA)	-0.271033
I_{R6} (mA)	0.980272
I_{R7} (mA)	0.980272
V_1 (V)	5.243570
V_2 (V)	4.981487
V_3 (V)	4.435588
V_4 (V)	0
V_5 (V)	5.019873
V_6 (V)	5.844139
V_7 (V)	-1.989217
V_8 (V)	-2.979997
V_d (V)	7.999870
V_b (V)	-0.038386

Table 6: Simulated Results

Quantity	Value [A or V]
i(vaux)	9.802716e-04
@gb[i]	-2.71033e-04
@r1[i]	2.588267e-04
@r2[i]	2.710327e-04
@r3[i]	-1.22060e-05
@r4[i]	-1.23910e-03
@r5[i]	-2.71033e-04
@r6[i]	9.802716e-04
@r7[i]	9.802716e-04
v(1)	5.243570e+00
v(2)	4.981487e+00
v(3)	4.435588e+00
v(5)	5.019871e+00
v(6)	5.844136e+00
v(7a)	-1.98922e+00
v(7b)	-1.98922e+00
v(8)	-2.98000e+00
v(5,8)	7.999867e+00
v(2,5)	-3.83845e-02

Comparing the values derived from the theoretical analysis with the simulated ones, we can observe that they are very similar, only slightly differing from each other probably due to the program's precision.

3.2 .OP: Circuit when $t = 0^+\ s$

In the table below, (7), we display all the branch currents and node voltages obtained from the simulation of the circuit ilustrated in figure (4). The need for this step is explained in the subsection (2.2) and in this subsection we look to confirm the values previously obtained with the theoretical analysis. We once again compare both theoretical results and simulation results side by side.

Table 7: Nodal Method Analysis - $t = 0^+ \ s$

Table 8: Theoretical Results

Quantity	Value
I_d (mA)	0.000000
I_b (mA)	0.000000
I_{R1} (mA)	-0.000000
I_{R2} (mA)	0.000000
I_{R3} (mA)	0.000000
I_{R4} (mA)	-0.000000
I_{R5} (mA)	-2.901531
I_{R6} (mA)	0.000000
I_{R7} (mA)	0.000000
V_1 (V)	0
V_2 (V)	0.000000
V_3 (V)	0.000000
V_4 (V)	0
V_5 (V)	0.000000
V_6 (V)	8.824136
V_7 (V)	-0.000000
V_8 (V)	-0.000000
V_b (V)	0.000000
V_d (V)	0.000000
V_x (V)	8.824136
I_x (mA)	2.901531
R_{eq} (Ω)	3041.200103
τ (ms)	3.146672

Table 9: Simulated Results

Quantity	Value [A or V]
i(vaux)	0.000000e+00
@gb[i]	0.000000e+00
@r1[i]	0.000000e+00
@r2[i]	0.000000e+00
@r3[i]	0.000000e+00
@r4[i]	0.000000e+00
@r5[i]	-2.90153e-03
@r6[i]	0.000000e+00
@r7[i]	0.000000e+00
v(1)	0.000000e+00
v(2)	0.000000e+00
v(3)	0.000000e+00
v(5)	0.000000e+00
v(6)	8.824136e+00
v(7a)	0.000000e+00
v(7b)	0.000000e+00
v(8)	0.000000e+00
v(5,8)	0.000000e+00
v(2,5)	0.000000e+00
v(6,8)	8.824136e+00
i(vx)	-2.90153e-03

Similarly to the previous subsection, we can observe that the values from the theoretical analysis and the simulated ones are very similar, only slightly differing from each other due to the program's precision.

3.3 .TRAN: Natural Solution of the Circuit

In figure (9), we ilustrate the natural solution of the circuit in node 6 in the interval $t \in [0, 20] \, ms$. We can notice both graphs obtained from the theoretical and simulated analysis look pretty much the same as expected.

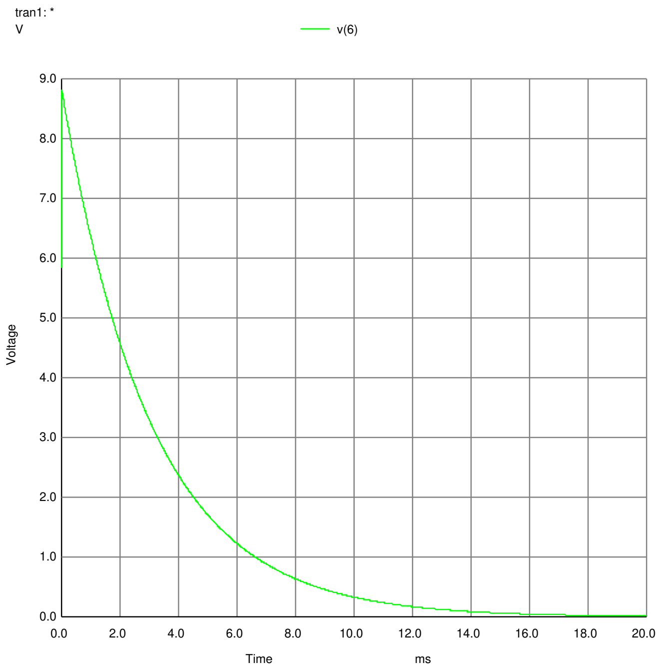

Figure 9: Natural Solution of the Voltage at Node 6, $v_{6n}(t)$

3.4 .TRAN: Total Solution of the Circuit

In figure (10), we display the first $20\ ms$ of the voltage in node 6 of the circuit, colored green, when subjected to the stimulus, colored red. Once again, both graphs obtained from the theoretical and simulated analysis look the same as expected.

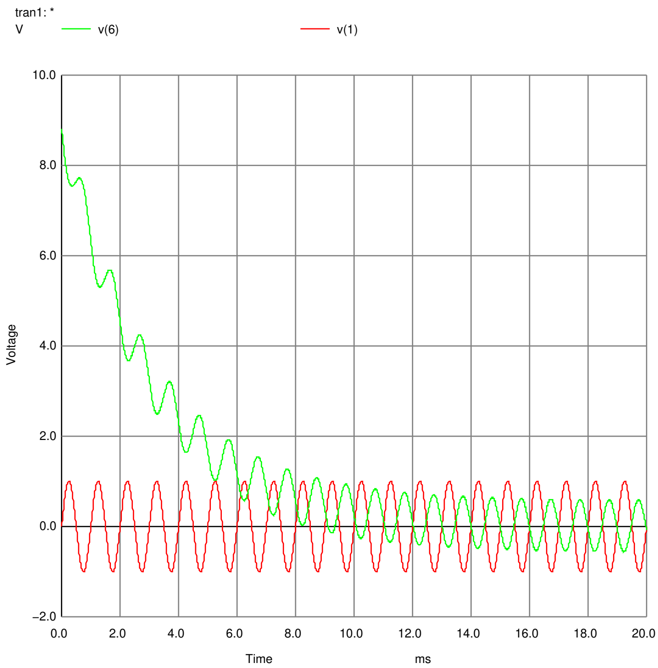

Figure 10: Total Solution: Red - Voltage Source; Green - Node 6

3.5 .AC: Circuit Frequency Response

In this section, we study the frequency response obtained from an AC analysis between $0.1Hz$ and $1MHz$, similar to what was done in section (2.6).
In figure (11), we show the magnitudes in decibels (dB) and the phases in degrees of the input source voltage (red), of the capacitor voltage (green) and of the voltage in node 6 (blue), as a function of the frequency, which is displayed logarithmically in the x axis.

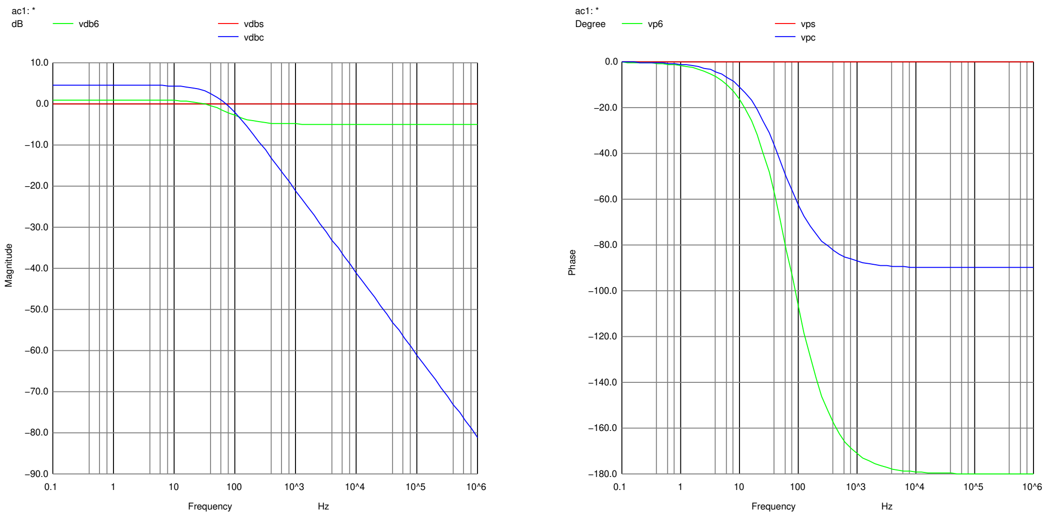

Figure 11: Frequency Response - Magnitude and Phase: Red - Voltage Source; Green - Node 6; Blue - Capacitor

By comparing the graphs obtained by simulated and theoretical means, we can once again come to the conclusion that both methods produce the same overall results, never producing a noticeable discrepancy.
Therefore, because the plots are basically the same, the reasons on why the responses of the elements plotted differ from each other can be found in subsection (2.6).

4 Conclusion

In this report, we have thouroughly analysed and simulate a multi-branched RC circuit, figure (1).
We managed to describe the circuit while in its stationary state using the Nodal Method, when $t < 0\ s$, as well as how it responded to a sine wave stimulus, when $t > 0\ s$, obtaining the total solution by superimposing the natural and forced solutions, which we got by computing the equivalent resistance and the voltage phasors, respectively. We also analysed how the circuit responded to a range of frequencies established by the input source and explained the derived results. Furthermore, both the theoretical and simulated analysis produced almost identical data, which further emphasizes the exactness of the obtained results.
In conclusion, all the established objectives in section (1) have been accomplished since we could accurately describe the behaviour of the RC circuit given.