

Information Transfer and Behavioural Inertia in Starling Flocks

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Analyzed Works

- Attanasi, A; Cavagna, A; Del Castello, L; Giardina, I; Jelic, A; Melillo, S; Parisi, L; Shen, E; Viale, M. **Information transfer and behavioural inertia in starling flocks** *Nat. Phys.*, 10, 691–696 (2014), doi: 10.1038/nphys3035
- Cavagna, A; Del Castello, L; Giardina, I; et al. **Flocking and turning: a new model for self-organized collective motion** *Journal of Statistical Physics*, 158, 601-627 (2014), doi: 10.1007/s10955-014-1119-3
- Ling, H; Mclvor, GE; Westley, J; et al. **Collective turns in jackdaw flocks: kinematics and information transfer** *J. R. Soc. Interface*, 16, (2019), doi: 10.1098/rsif.2019.0450



Starling Flocks

Collective turns in flocks of starlings are a benchmark phenomena in the study of **collective decision-making**.

The mechanism with which information is transferred must be optimized in order to keep cohesion within the flock and not put it in a vulnerable condition.

The goal is to to understand how **information propagates** across the flock, namely how the collective decision to turn and a change of the global direction of the flock propagates.

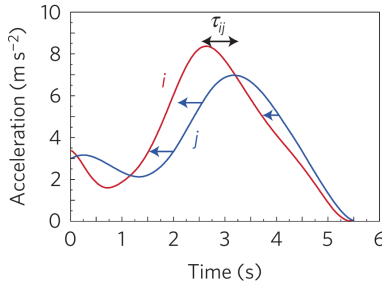


The paper studies a flock of *Sturnus vulgaris* under direct threat from a peregrine falcon.



Turning Delay

Trajectories of individual birds, for the entire duration of a turning event, are collected using stereo photography with three different view points.



The **mutual turning delay** τ_{ij} of bird i with respect to bird j is the time by which the curve of the radial acceleration $\mathbf{a}_j(t)$ has to be shift with respect to $\mathbf{a}_i(t)$ such that they are maximally overlapped.

This allows to reconstruct the ranking of the birds.



Turning Delay - Correlation function

The time shift τ_{ij} corresponds to the value of τ where the **correlation function** $G_{ij}(\tau)$ reaches its maximum:

$$G_{ij}(\tau) = \frac{\langle \mathbf{a}_i(t) \cdot \mathbf{a}_j(t - \tau) \rangle - \langle \mathbf{a}_i(t) \rangle \cdot \langle \mathbf{a}_j(t - \tau) \rangle}{\sigma_i \sigma_j} \quad (1)$$

where

$$\sigma_i = \sqrt{\langle \mathbf{a}_i(t)^2 \rangle - \langle \mathbf{a}_i(t) \rangle^2} \quad (2)$$

is the fluctuation of $\mathbf{a}_i(t)$ during the turn.

By convention:

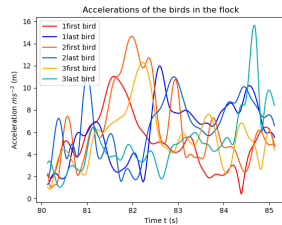
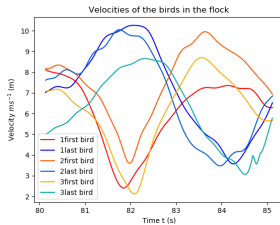
- $\tau_{ij} > 0$ means that j turns **before** i ;
- $\tau_{ij} < 0$ means that j turns **after** i .



Turning Delay - Correlation function

In our analysis, to measure τ_{ij} the correlation of velocities was used instead of the radial acceleration, since the velocity signal is less noisy than the acceleration one:

$$C_{ij}(\tau) = \frac{\langle \mathbf{v}_i(t) \cdot \mathbf{v}_j(t + \tau) \rangle}{\langle \|\mathbf{v}_i\| \|\mathbf{v}_j\| \rangle} \quad (3)$$

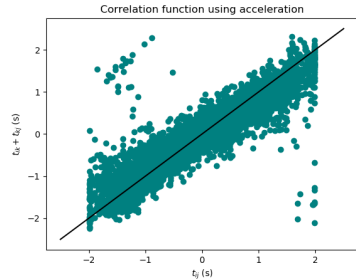
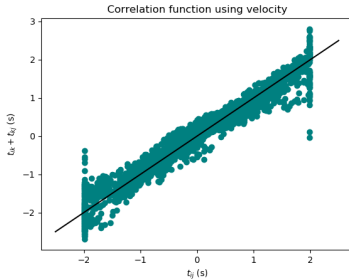


One possible reason for *jackdaw flocks* reducing their speed during the turn is because they make much sharper turns (e.g., a U-turn with small radius) compared to *starling flocks*. Birds have to slow down in order to make such sharp turn.



Turning Delay

In absence of noise, time ordering requires that $\tau_{ij} = \tau_{ik} + \tau_{kj}$, for each triplet i, j, k . However, noise can introduce some violations in the time ordering. Robustness of the relation was checked with respect to the noise and it was found that the spread is smaller considering the velocity correlation function rather than the acceleration one, for the data set used.



Through the turning delays the ranking of the birds can be calculated.

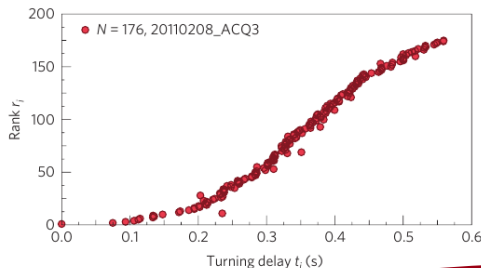


Ranking Curve

Starting from the mutual turning delays for each bird with respect to the top bird in the rank, it is possible to define a **ranking curve of the absolute turning times**. To reduce the statistical error, the absolute turning time was defined using the mutual delay τ_{ij} with respect to all bird j higher in rank than i

$$t_i = \frac{1}{r_i - 1} \sum_{r_j < r_i} (t_j + \tau_{ij}), \quad r_i > 1 \quad (4)$$

If there were no time-ordering violations, we would simply have $t_i = \tau_{i, \text{top}}$.



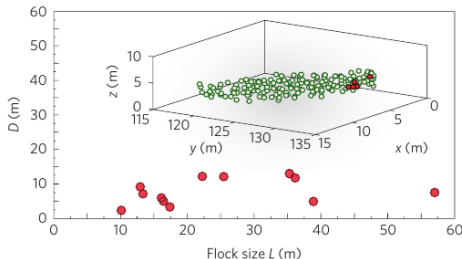


Localized origin of the turn

If the turn was originated by external factors or triggers which affect all the birds, the birds' responses would be independent and the nucleus of the turn would not be localized in space.

Instead:

- **the maximum mutual distance** between the top 5 birds in the rank does not increase with the linear size of the flock;
- **the first birds to turn are spatially close** to each other and this result does not change if we consider a different number of top birds.

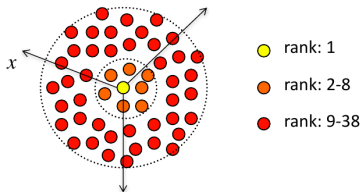




Ranking

If the turn origin is localized, then the distance the information travels in a time t , $x(t)$, can be obtained as the radius of the sphere containing the first $r(t)$ birds in the rank (before the border effects kick in):

$$\begin{aligned} \text{rank} &= (\text{density } \rho_{\text{flock}}) \cdot (\text{distance travelled by the information } x)^3 \longrightarrow \\ \longrightarrow \quad x(t) &= \left[\frac{r(t)}{\rho_{\text{flock}}} \right]^{\frac{1}{3}} \implies r(t) \sim x(t)^3 \end{aligned} \quad (5)$$

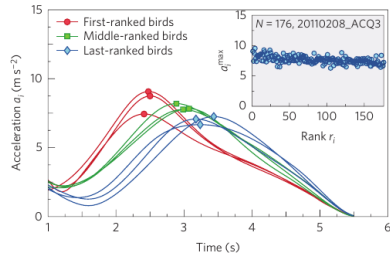
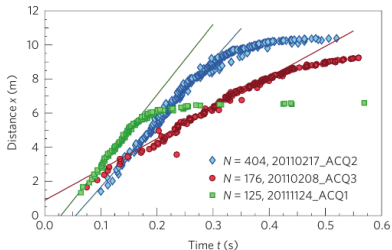


However, at the border, **the typical number of birds reached by the information scales with an exponent smaller than 3**. In general, for non-spherical systems at **later times**,

$$r(t) \sim x_{\text{late}}(t)^\alpha \quad \text{with } \alpha < 3 \quad (6)$$



Observations



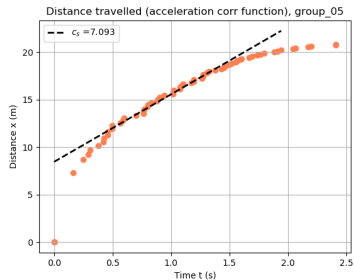
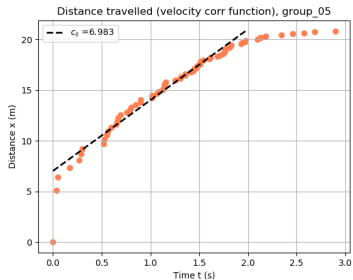
- linear dispersion law:

$$x(t) = c_s t \quad (7)$$

- very weak attenuation
- variability of the speed of propagation c_s : some flocks manage faster information transfer than others. If the flock is modelled as a fluid, a density rescaling does not lead the variability to disappear. Indeed, **what propagates during the turn are fluctuations of orientation.**



Jackdaw flock: Analysis



In our analysis we can identify linear growth for intermediate times, although the behaviour is less pronounced using the acceleration correlation function.



Dynamics of the velocities

As for all description of collective motion, the theoretical approach is based on **alignment dynamics**: each individual tends to keep its direction of motion as close as possible to that of its neighbours.

The update rule for the velocities reads

$$\mathbf{v}_i(t+1) = \mathbf{v}_i(t) + J \sum_{j \in \text{n. of } i} \mathbf{v}_j(t) \quad (8)$$

In continuous time, it leads to a **Langevin equation**, namely

$$\frac{d\mathbf{v}_i}{dt} = -\frac{\partial H}{\partial \mathbf{v}_i}, \quad H = -J \sum_{ij} n_{ij} \mathbf{v}_i \cdot \mathbf{v}_j \quad (9)$$

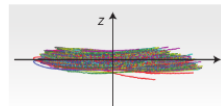
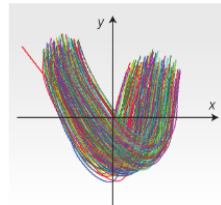
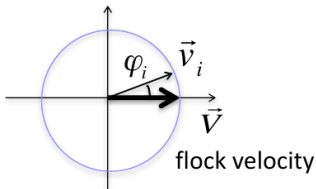
where n_{ij} is the *connectivity matrix* (being 1 if j interacts with i and 0 otherwise).

These relations tell us that each bird updates its velocity following a *social force* $\mathbf{F}_S^i = -\partial H / \partial \mathbf{v}_i$ produced by its direct neighbours.



Order Parameter

It has been seen that, during a turn, **the trajectories of birds lie on a plane**, hence **allowing the use of a 2D order parameter** $\mathbf{v}_i = (v_i^x, v_i^y) = v_0 e^{i\varphi_i}$, where φ_i is the angle between the direction of motion of i and that of the flock and v_0 is constant during the turn.



Trajectories of $N = 176$ birds in a flock from experimental data. Each trajectory lies approximately on a plane.



Highly Orderd Regime Hamiltonian

If the velocities \mathbf{v}_i differ little from the flock's one (**highly ordered regime**), the Hamiltonian can be expanded, thus obtaining

$$H = \frac{J}{2} \sum_{ij} n_{ij} (\varphi_i - \varphi_j)^2 \xrightarrow{\text{continuum limit in space}} \frac{1}{2} n_c a^2 J \int \frac{d^3x}{a^3} [\nabla \varphi(x, t)]^2 \quad (10)$$

where a is the typical distance between interacting neighbors and

$$n_c = \frac{1}{N} \sum_{i,j} n_{ij} \quad (11)$$

is the average number of interacting neighbors.



Diffusion-like predictions

The equation of motion associated to this new Hamiltonian reads:

$$\frac{\partial \varphi}{\partial t} = -\frac{\delta H}{\delta \varphi} = n_c a^2 J \nabla^2 \varphi \quad (12)$$

where $F_s = a n_c J \nabla^2 \varphi$ is the social force.

Eq (12) is a diffusion equation for the phase φ , with a dispersion law $\omega \sim i k^2$.

This result has two consequences in **stark disagreement with what was observed in flocks**.

- information travels much slower than linearly, indeed $x \sim \sqrt{t}$;
- the frequency is **imaginary**, meaning that this is a **non-propagating mode**. Transfer of information gets damped exponentially in space and time, however undamped propagation is observed in flocks.



Symmetry, Conservation Law and Inertia

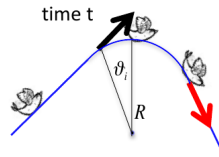
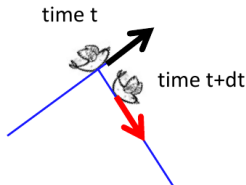
The model however does not represent well the turning of the flock. The two major problems in the previous model are:

- missing conservation law;

Hamiltonian (10) is invariant under a uniform rotation of the velocities \mathbf{v}_i , or in other word it is **$O(2)$ invariant** ($\varphi_i \rightarrow \varphi_i + \delta\varphi$). Hence, due to the *Noether's theorem*, one would expect to find a conservation law, which would affects the dynamics.

- absence of an inertial term;

In the standard flocking theory we have $\dot{\varphi} = F_s$: this means that the force acts on the velocity and the bird can turn instantaneously. In reality, turns are smooth, and the birds are subject to some constraints.





The new Hamiltonian

These considerations lead to a new formulation of the Hamiltonian, where both arguments are contained in a **new kinetic term** $s_z^2/2\chi$

$$H = \int \frac{d^3x}{a^3} \left[\frac{1}{2} n_c a^2 J(\nabla \varphi(x, t))^2 + \frac{s_z^2}{2\chi} \right], \quad (13)$$

where

- s_z is the momentum canonically conjugated to φ , and **is defined as the local generator of the rotations**, around the z axis, parametrized by φ ,
- χ is the **generalized moment of inertia**, an effective parameter which describes how much the birds resist changing their instantaneous radius of curvature when a social force is exerted by the neighbours



The meaning of the new kinetic term

The **new kinetic term** comes from the theory of constrained Hamiltonian dynamics.

The constraint on the velocities ($\|\mathbf{v}\| = v_0 = \text{constant}$) is analogous to the constraint on the positions of standard circular motion.

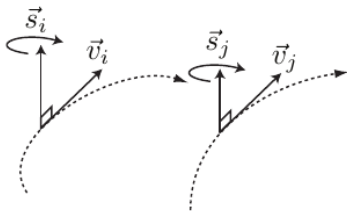
Standard Circular Motion	New Hamiltonian
<i>External space of coordinates</i>	<i>Internal space of velocities</i>
θ parametrizes the rotation in the (external) space of coordinates	φ parametrizes the rotation in the (internal) space of velocities
orbital angular momentum l	internal angular momentum s_z
external moment of inertia I	internal moment of inertia χ

Spin



The momentum s_z conjugated to the phase φ is **the intrinsic spin** of the bird. This can be seen exploiting the fact that it generates the symmetry parametrized by φ , which is encoded in the Poisson relation

$$\{\mathbf{v}, s_z\} = \frac{\partial \mathbf{v}}{\partial \varphi} = i\mathbf{v}. \quad (14)$$





Continuity Equation

When a generalized coordinate (φ) parametrizes a symmetry of the interaction, **its canonically conjugated momentum (s_z) is conserved.**

Indeed, the canonical equations of motion, namely

$$\frac{\partial \varphi}{\partial t} = \frac{s_z}{\chi}; \quad \frac{\partial s_z}{\partial t} = n_c a^2 J \nabla^2 \varphi \quad (15)$$

can be rewritten to obtain a continuity equation for s_z

$$\frac{\partial s_z}{\partial t} - \nabla \cdot \mathbf{j}_z = 0 \quad (16)$$

with $\mathbf{j}_z(x, t) = n_c a^2 J \nabla \varphi(x, t)$ being the *conserved current*.

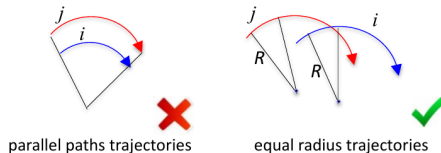
What is transported by $\mathbf{j}_z(x, t)$ is spin, in the form of fluctuations of the phase $\varphi(x, t)$.

Equation (16) is **the conservation law generated by the symmetry**, which was missing in the previous theory.



Curvature transport

The phase φ parametrizes rotations in *the internal space of the velocities*, hence corresponding to an **equal radius turn**, R .



As a consequence, there is a connection between the spin s_z and the *curvature* of the trajectory $\kappa = 1/R$, namely

$$s_z = \frac{v_0 \chi}{R} \sim \kappa. \quad (17)$$

Accordingly, $j_z = \rho_s \nabla \varphi$ is the spin current (**transport of curvature**) and $\rho_s = n_c a^2 J$ is the spin stiffness.



Spin conservation and information propagation

The continuity equation (16) defines a mechanism to carry out an excitation of the field $s_z(x, t)$, which translates into a propagating mode (**spin wave**). Indeed, rearranging the equations of motion it can be obtained

$$\frac{\partial^2 \varphi}{\partial t^2} - c_s^2 \nabla^2 \varphi = 0 \quad (18)$$

with $c_s^2 = n_c a^2 J / \chi$. The **D'Alembert's equation** (18) can be easily solved in Fourier space to get the dispersion relation

$$\omega = c_s k. \quad (19)$$

This translates in a **linear and undamped propagation**, which is in agreement to what was observed in turning flocks.

It is to remark that **the time scale of propagation in d'Alembert's equation must be shorter than the typical time scale of reshuffling of the network** n_{ij} , in order to justify the equations and it is indeed the case in natural flocks of starlings.



Behavioural Inertia

From the canonical point, the meaning of χ is **simply the inertia to changing $\dot{\varphi}$** . Indeed, the D'Alembert equation (18) can be rewritten as

$$\chi = \frac{aF_s}{\ddot{\varphi}} \quad (20)$$

where $F_s = an_c J \nabla^2 \varphi$ is the social force. This is the standard definition of inertia, however to better understand the meaning of χ , it may be useful to exploit the relation $\dot{\varphi} = s_z / \chi \sim \kappa / \chi$, namely

$$\chi = - \left(\frac{R^2}{v} \right) \frac{aF_s}{\dot{R}} = \left(\frac{1}{v} \right) \frac{aF_s}{\dot{\kappa}} \quad (21)$$

where $\kappa = 1/R$ is the *curvature*.

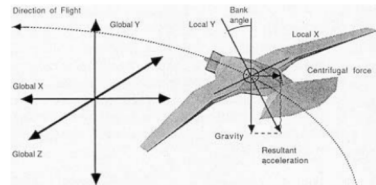
Hence, χ is the **resistance** of a bird to **change its instantaneous radius of curvature R** due to the social force F_s .



Behavioural Inertia - *Banking Angle*

A relation even more linked to the physics of birds flight can be obtained considering the **banking angle** γ , which to speak, is the angle formed between the axis of the wings and the horizontal plane.

With this maneuver, birds can redirect part of the total lift into a **centripetal force**, $F_c = mg\gamma$, making them turn ($\gamma \ll 1$).



It is straightfoward, then, to arrive at

$$\chi = \left(\frac{v}{g} \right) \frac{aF_s}{\dot{\gamma}} \quad (22)$$

Accordingly, the generalized moment of inertia χ is the **resistance** of a bird to **change its banking angle** γ .



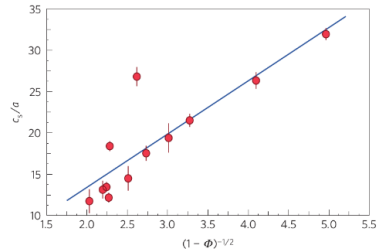
Variability of c_s and Polarization

During the turn what propagates are **phase fluctuations** and the **speed of propagation** $c_s^2 = n_s a^2 J / \chi$ has been observed to **vary across different flocks**. The reason of this can be found in the dependence of c_s on the alignment strength, J .

However, J cannot be measured experimentally, but it can be expressed through a measure of the **polarization**, namely

$$\Phi(J, \beta) = \left| \frac{1}{N} \sum_i \frac{\mathbf{v}_i}{\|\mathbf{v}_i\|} \right| \stackrel{(a)}{\sim} 1 - \frac{1}{\beta J}$$

$$\longrightarrow c_s = \frac{1}{\sqrt{\beta \chi}} \cdot \frac{a}{\sqrt{1 - \Phi}} \quad (23)$$





Variability of c_s and Polarization

The polarization measures the overall **degree of alignment** of the flock.

The approximation $\Phi(J, \beta) \sim 1 - 1/(\beta J)$ is verified for **small phase fluctuations**, which is the case for flocks.

This equation means that a large polarization ($\Phi \sim 1$) can be obtained either in:

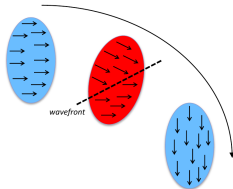
- the zero-noise limit $\beta \rightarrow \infty$;
- the case of infinitely strong alignment $J \rightarrow \infty$.

Indeed, real flocks are very ordered systems and the latter case is what is observed.



Polarization in natural groups

Loss of cohesion in birds can be a serious problem, since it can lead a spread of the flock. The fast propagation observed for the information in flocks keeps the **group's decoherence at minimum**, so it is probably the cause for this kind of mathematical result, and the corresponding collective movement in flocks.



Remembering

$$c_s \propto \frac{1}{\sqrt{1 - \Phi}} \quad (24)$$

to achieve large speed of propagation of the information, **strong polarization is necessary**.

Dissipation



Starting from the deterministic structure of the new theory introduced, it is then natural to add **noise** and **dissipation** terms, obtaining the final equations of the **inertial spin model** of collective motion.

Hence, the D'Alembert equation translates into

$$\chi \frac{\partial^2 \varphi}{\partial t^2} + \eta \frac{\partial \varphi}{\partial t} - \rho_s \nabla^2 \varphi - \xi^\perp = 0 \quad (25)$$

where η is a *generalized friction coefficient* and ξ^\perp is a *white noise*.

To better understand the effect of dissipation, it might be useful to consider the **zero-noise case**. The dispersion law, then, becomes

$$\chi \omega^2 - i\eta \omega - \rho_s k^2 = 0 \quad (26)$$

which in the overdamped limit ($\eta \gg \chi$) gives back the **diffusive results** as expected

$$\omega = i \left(\frac{\rho_s}{\eta} \right) k^2. \quad (27)$$



Dissipation

In general, however

$$\omega = i\frac{\eta}{2\chi} \pm c_s k \sqrt{1 - k_0^2/k^2} \quad \text{where} \quad c_s = \sqrt{\frac{\rho_s}{\chi}} \quad k_0 = \frac{\eta}{2\sqrt{\rho_s\chi}} \quad (28)$$

- For $k \geq k_0$ **Attenuated propagating waves**: the frequency has both a real and an imaginary part;
- For $k < k_0$ **Overdamped waves**: the frequency is purely imaginary.

With **small dissipation** ($\eta < \sqrt{\rho_s\chi}/L$) there is linear propagation and the time scale of the exponential decay is set by

$$\tau = 2\chi/\eta > L/c_s \quad (29)$$

The signal is **very weakly damped** across the **length scale L**, hence **propagation of information is qualitatively the same** as the zero dissipation theory.

Universality class



Interestingly, the same structure of equations and dispersion law that was introduced for flocks also hold in other kinds of systems.

In fact, the equations of motion (15) and the continuity equation (16) are formally identical and belong to the same **dynamical universality class** (**model F** in the Halperin-Hohenberg classification) as

- *the planar ferromagnet*
- *superfluid Helium*

However, it has to be remarked that **propagating modes** and a **linear dispersion law are not a generic feature of spin models**.

The essential ingredient for linear spin propagation is not simply the presence of a conserved quantity, but the fact that this conserved quantity must be the generator of a **spontaneously broken continuous symmetry**.



Conclusion

To summarize what was found

- Introduction of a **new model of collective motion** that describes information transfer in flocks of bird
- Inclusion of birds' **behavioural inertia**
- A **connection between the polarization** of the system and the **speed** with which information travels
- Similarities of the model with other more fundamental physical ones.

The results of the paper describe the behaviour of flocks of birds, compatible with the actual studied data, while overcoming some theoretical lacks that regarded previous studies.

Calculations



Hamiltonian Continuum Limit

From the update rule for the velocities of each individuals of the flock, one arrive at the Hamiltonian

$$H = -J \sum_{ij} \eta_{ij} \mathbf{v}_i \cdot \mathbf{v}_j \quad (30)$$

where η_{ij} is the connectivity matrix (being 1 if j is a particle interacting with i and 0 otherwise).

If we exploit the fact that the trajectories of the birds lie on a plane during a turn, justified by the data, we can introduce a 2D order parameter $\mathbf{v}_i = (v_i^x, v_i^y) = v_0 e^{i\varphi_i}$, where φ_i is the angle between the motion of i and that of the flock and v_0 is constant during the turn. In highly ordered regime the velocities of each individuals differ little from the flock's one, so that $\varphi_i \ll 1$ and

$$\mathbf{v}_i \cdot \mathbf{v}_j \propto \cos(\varphi_i - \varphi_j) \underset{\varphi_i - \varphi_j \approx 0}{\approx} 1 - \frac{1}{2}(\varphi_i - \varphi_j)^2 \quad (31)$$



Hamiltonian Continuum Limit

The Hamiltonian becomes

$$H = \frac{J}{2} \sum_{ij} \eta_{ij} (\varphi_i - \varphi_j)^2 \longrightarrow \frac{1}{2} n_c a^2 J \int \frac{d^3x}{a^3} [\nabla \varphi(x, t)]^2 \quad (32)$$

where the discrete variable φ_i has been promoted to a continuous variable $\varphi(x, t)$ and a is the typical distance a between interacting neighbors, therefore $(\varphi_i - \varphi_j)/a$ is actually the discrete derivative of φ_i that in the continuum becomes the gradient. Moreover, the term $1/a^3$ encodes the measure of the space on which we take the integral (it comes from the transition from a sum to an integral $\sum_{\langle ij \rangle} \longrightarrow \int d^3x / a^3$). The term $n_c = (1/N) \sum_i \sum_j \eta_{ij}$ is the average number of interacting neighbours.



Symmetry of the Hamiltonian

The Hamiltonian for the standard flocking theory is

$$H[\varphi] = \frac{1}{2} a^2 J \int \frac{d^3x}{a^3} [\nabla \varphi(x, t)]^2 \quad (33)$$

we want to show that it is invariant under a uniform rotation of the velocities, namely $\varphi \rightarrow \varphi + \delta\varphi$. It is sufficient to study what happens to the gradient term when a rotation is applied, as all others are already rotational invariant. A rotation is described by a matrix M belonging to the $SO(2)$ group. So

$$(\nabla \varphi)^2 = (\nabla \varphi)^T (\nabla \varphi) \rightarrow \quad (34)$$

$$\begin{aligned} (M(\nabla \varphi))^T (M(\nabla \varphi)) &= (\nabla \varphi)^T M^T M (\nabla \varphi) \\ &= (\nabla \varphi)^T (\nabla \varphi) = (\nabla \varphi)^2 \end{aligned} \quad (35)$$

Moreover, this proves also the occurrence of the spontaneous symmetry breaking. φ parametrizes the rotations: the rotational symmetry is spontaneously broken since a specific global direction of motion is selected by the system.



Functional Derivative

Given $H[\varphi] = \int dx h(x, \varphi, \nabla\varphi)$, the functional derivative w.r.t. φ is

$$\int \frac{\delta H}{\delta \varphi} \phi(x) dx = \left[\frac{d}{d\varepsilon} \int dx h(x, \varphi + \varepsilon\phi, \nabla\varphi + \varepsilon\nabla\phi) \right]_{\varepsilon=0} \quad (36)$$

$$= \int \left(\frac{\partial h}{\partial \varphi} \phi + \frac{\partial h}{\partial \nabla \varphi} \cdot \nabla \phi \right) dx \quad (37)$$

$$= \int \left(\frac{\partial h}{\partial \varphi} \phi + \nabla \cdot \left(\frac{\partial h}{\partial \nabla \varphi} \phi \right) - \left(\nabla \cdot \frac{\partial h}{\partial \nabla \varphi} \right) \phi \right) dx \quad (38)$$

$$= \int \left(\frac{\partial h}{\partial \varphi} - \nabla \cdot \frac{\partial h}{\partial \nabla \varphi} \right) \phi dx \quad (39)$$

$$\Rightarrow \frac{\delta H}{\delta \varphi} = \frac{\partial h}{\partial \varphi} - \nabla \cdot \frac{\partial h}{\partial \nabla \varphi} \quad (40)$$

In our case $H[\varphi] = \frac{1}{2} a^2 J \int \frac{d^3x}{a^3} [\nabla\varphi(x, t)]^2$, therefore

$$\frac{\delta H}{\delta \varphi} = -a^2 J \nabla^2 \varphi \quad (41)$$

Spin



We have that $\mathbf{v} = v_0 e^{i\varphi} = v_0 \sum_{n=0}^{\infty} \frac{(i\varphi)^n}{n!}$ and $\{\varphi, s_z\} = 1$ since they are conjugated variables in an hamiltonian description. Hence,

$$\{\mathbf{v}, s_z\} = i v_0 \{\varphi, s_z\} - \frac{1}{2} v_0 \{\varphi^2, s_z\} - \frac{i}{6} v_0 \{\varphi^3, s_z\} + \dots \quad (42)$$

$$= i v_0 \{\varphi, s_z\} - \frac{1}{2} v_0 (\varphi \{\varphi, s_z\} + \{\varphi, s_z\} \varphi) \quad (43)$$

$$- \frac{i}{6} v_0 (\varphi \{\varphi^2, s_z\} + \{\varphi, s_z\} \varphi^2) + \dots \quad (44)$$

$$= i v_0 - v_0 \varphi - \frac{i}{2} \varphi^2 + \dots \quad (45)$$

$$= i \mathbf{v} = \frac{\partial \mathbf{v}}{\partial \varphi} \quad (46)$$

Spin



$$\mathbf{v} = \dot{\mathbf{x}} = v_0 \cos \varphi(t) \hat{x} + v_0 \sin \varphi(t) \hat{y} \quad (47)$$

$$\mathbf{a} = \dot{\mathbf{v}} = -v_0 \dot{\varphi} \sin \varphi(t) \hat{x} + v_0 \dot{\varphi} \cos \varphi(t) \hat{y} \quad (48)$$

If we have a 2-dimensional curve given parametrically, namely $(x(t), y(t))$, the curvature radius is given by

$$R = \left| \frac{(\dot{x}^2 + \dot{y}^2)^{\frac{3}{2}}}{\dot{x}\ddot{y} - \ddot{x}y} \right| \quad (49)$$

Therefore,

$$R = \frac{(v_0^2 \cos^2 \varphi + v_0^2 \sin^2 \varphi)^{\frac{3}{2}}}{|v_0^2 \dot{\varphi} \cos^2 \varphi + v_0^2 \dot{\varphi} \sin^2 \varphi|} = \frac{v_0}{\dot{\varphi}} \quad (50)$$

Hence,

$$\dot{\varphi} = \frac{v_0}{R} \quad (51)$$



Generalized Moment of Inertia

$$\ddot{\varphi} - c_s^2 \nabla^2 \varphi = 0 \implies \chi = \frac{aF_s}{\ddot{\varphi}} \quad (52)$$

where we use that $c_s^2 = a^2 J / \chi$ and we have called $F_s = a J \nabla^2 \varphi$. Since $\dot{\varphi} = v_0 / R$, we have that $\ddot{\varphi} = -v_0 \dot{R} / R^2$, therefore

$$\chi = -\frac{R^2}{v_0} \frac{aF_s}{\dot{R}} = \frac{1}{v_0} \frac{aF_s}{\dot{\kappa}} \quad (53)$$

where $\kappa = 1/R$ is the curvature.

To express χ using the banking angle γ , we first have to remember that in this case, part of the total lift goes into a centripetal force, $F_c = mg\gamma$. The banking angle γ then satisfies the relation

$$\tan \gamma \underset{\gamma \ll 1}{\simeq} \gamma = \frac{v_0^2}{Rg} \implies \frac{\dot{R}}{R^2} = -\frac{\dot{\gamma}g}{v_0^2} \quad (54)$$

Hence, we find that

$$\chi = \frac{v_0}{g} \frac{aF_s}{\dot{\gamma}} \quad (55)$$



Relationship between Φ and J

The polarization is defined as the modulus of the magnetization vector

$$\Phi = \left\| \frac{1}{N} \sum_i \vec{v}_i \right\| \quad (56)$$

Using $\vec{v} = e^{i\varphi}(\|\vec{v}\| = 1)$ and expanding form small values of φ , we get

$$\Phi = 1 + i \sum_i \varphi_i - \frac{1}{2} \sum_i \varphi_i^2 = 1 - \frac{1}{2} \langle \varphi^2 \rangle \quad (57)$$

where $\langle \varphi \rangle = 0$ because it is the mean angle w.r.t. the direction of motion of the entire flocks, which should be zero.

We then calculate $\langle \varphi^2 \rangle$ in the spin-wave limit, where we have a Gaussian probability distribution

$$P(\varphi) \sim \exp \left\{ -\frac{1}{2} \beta \int \frac{d^3x}{a^3} \rho_s (\nabla \varphi)^2 \right\} \quad (58)$$

where β is the inverse temperature and $\rho_s = a^2 J$ is the strength of the alignment interaction.



Relationship between Φ and J

Using the Fourier transform we get

$$P(\varphi) = \exp \left\{ -\frac{1}{2} \beta \int \frac{d^3 k}{a^3} \rho_s k^2 \varphi_k \varphi_{-k} \right\} \quad (59)$$

which is gaussian in φ , hence since $\langle \varphi \rangle = 0$, we have that

$$\langle \varphi^2 \rangle = \text{Var}(\varphi) = 2 \int d^3 k \frac{a^3}{\beta \rho_s k^2}.$$

Therefore,

$$\Phi(J, \beta) = 1 - \int d^3 k \frac{a}{\beta J k^2} \sim 1 - \frac{1}{\beta J} \quad (60)$$

where in the last step we are not interested in the actual value of the integral, all we need to know is that it converges (of the form $\int d^d k \, k^{-\alpha}$ with $\alpha < d$)



Dissipation

$$\chi \frac{\partial^2 \varphi}{\partial t^2} + \eta \frac{\partial \varphi}{\partial t} - \rho_s \nabla^2 \varphi = 0 \quad (61)$$

with $\rho_s = a^2 J$ and η is a generalized friction coefficient. Writing

$$\varphi(x, t) = \int dk d\omega e^{i(k \cdot x + \omega t)} \tilde{\varphi}(k, \omega) \quad (62)$$

where $\tilde{\varphi}(k, \omega)$ is the Fourier transform of $\varphi(x, t)$, the equation 61 becomes

$$\left[-\chi \omega^2 + i\eta \omega + \rho_s k^2 \right] \tilde{\varphi}(k, \omega) = 0. \quad (63)$$

Hence, we are left with a second order algebraic equation $\chi \omega^2 - i\eta \omega - \rho_s k^2 = 0$



Dissipation

The general solutions are given by

$$\omega_{\pm} = \frac{i\eta}{2\chi} \pm \frac{1}{2\chi} \sqrt{-\eta^2 + 4\rho_s \chi k^2} \quad (64)$$

$$= \frac{i\eta}{2\chi} \pm \frac{2\sqrt{\rho_s \chi}}{2\chi} k \sqrt{1 - \frac{\eta^2}{4\rho_s \chi k^2}} \quad (65)$$

$$= \frac{i\eta}{2\chi} \pm c_s k \sqrt{1 - \frac{k_0^2}{k^2}} \quad (66)$$

$$= \frac{i}{\tau} \pm \omega_0 \sqrt{1 - \frac{k_0^2}{k^2}} \quad (67)$$

where $c_s = \sqrt{\rho_s/\chi}$, $k_0 = \eta/(2\sqrt{\rho_s \chi})$ and $\tau = 2\chi/\eta$, $\omega_0 = c_s k$.



General off-plane (3D) case

Even though a 2-dimensional description matches with what happens in real flocks, it is possible to carry out all the calculation also for the **3-dimensional case**.

$$H = \int \frac{d^3x}{a^3} \frac{1}{2} \rho_s \left[(\nabla \varphi_z)^2 + (\nabla \varphi_y)^2 \right] + \frac{1}{2\chi} \left[s_z^2 + s_y^2 \right] \quad (68)$$

As a consequence of *Goldstone's theorem* however, it simply produce **two separate propagating modes** (rather than one) along the transverse directions (y and z) of the one along which the information travels.



General off-plane (3D) case

The equations of motion remain the same, we only get separate sets of equations of motion for every symmetry direction broken:

$$\frac{\partial \varphi_\alpha}{\partial t} = \frac{s_\alpha}{\chi} \quad (69)$$

$$\frac{\partial s_\alpha}{\partial t} = a^2 J \nabla^2 \varphi_\alpha = \nabla \cdot \mathbf{j}_\alpha \quad (70)$$

with $\alpha = y, z$. Conversely, the D'Alembert equations are

$$\frac{\partial^2 \varphi_\alpha}{\partial t^2} - c_s^2 \nabla^2 \varphi_\alpha = 0, \quad c_s^2 = \rho_s / \chi \quad (71)$$