# MATH1141: Higher Mathematics 1A (Calculus)

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# Contents

## 0.1 Content Overview

- Functions: continuity, differentiability, invertibility (inverse functions)
- Curve sketching
- Integration: area, Riemann sum (approximation), Fundamental theorems of calculus
- Logarithm: exponential, hyperbolic

# Chapter 1

# Sets, inequalities and functions

## 1.1 Sets of numbers

A set is a collection of distinct objects. The objects in a set are elements or members of the set. Commonly used sets:

- The set of natural numbers: $\mathbb{N} = \{0, 1, 2, 3, 4, ...\}$
- The set of integers: $\mathbb{Z} = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}$
- The set of rational numbers: $\mathbb{Q} = \{\frac{p}{q} : p, q$ are integers and $q \neq 0\}$
- The set of $\mathbb{R}$ real numbers may be represented as the collection of points lying on the number line

If $A$ is a set of numbers and the number $x$ is a member of the set $A$, then we write:

$$x \in A$$

If $x$ is not a member of $A$, then we write

$$x \notin A$$

### 1.1.1 Intervals

- Parenthesis: excludes endpoints
- Bracket: includes endpoints
- Combination: neither open nor closed interval

### 1.1.2 Solving Inequalities

For $x, y, z \in \mathbb{R}$:

- if $x > y$, then $x + z > y + z$
- if $x > y$ and $z > 0$, then $xz > yz$
- if $x > y$ and $z < 0$, then $xz < yz$

Example

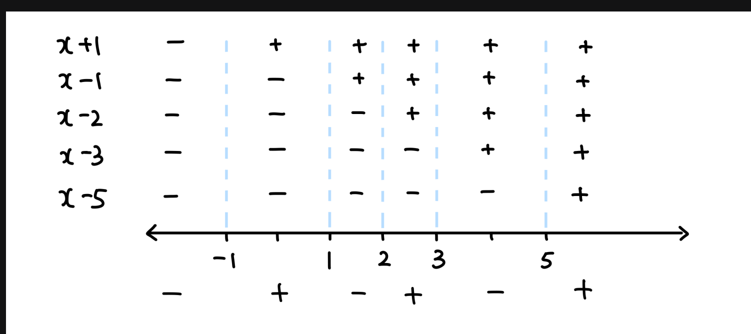
Solve the quadratic inequality $x^2 + 4x > 21$

$$\begin{aligned} x^2 + 4x > 21 &\iff x^2 + 4x - 21 > 0 \\ &\iff (x+7)(x-3) > 0 \\ \text{Solution set: } & x < -7 \text{ or } x > 3 \\ & x \in (-\infty, -7) \cup (3, \infty) \end{aligned}$$

Example

Solve the rational inequality $\frac{1}{x+1} < \frac{1}{(x-2)(x-2)}$

$$\begin{aligned} &\text{Multiply both sides by } (x+1)^2(x-2)^2(x-3)^2 \quad (>= 0), \\ \frac{1}{x+1} < \frac{1}{(x-2)(x-3)} &\iff (x+1)(x-2)^2(x-3)^2 < (x+1)^2(x-2)(x-3) \\ &\iff (x+1)(x-2)^2(x-3)^2 - (x+1)^2(x-2)(x-3) < 0 \\ &\iff (x+1)(x-2)(x-3)(x-4)(x-5) < 0 \\ \text{Solution set: } & (-\infty, -1) \cup (1, 2) \cup (3, 5) \end{aligned}$$




### 1.1.3 Absolute Values

The absolute value (or magnitude) of a real number $x$ is

$$|x| = \begin{cases} x & \text{if } x \le 0, \\ -x & \text{if } x < 0 \end{cases}$$

Example

Solve the inequality $|3x + 1| \le 4$

$$\begin{aligned} |3x+1| \le 4 &\iff |x + \frac{1}{3}| \le \frac{4}{3} \\ &\iff x + \frac{1}{3} \le \frac{-4}{3} \quad \text{or} \quad x + \frac{1}{3} \le \frac{4}{3} \\ &\iff x \le \frac{-5}{3} \quad \text{or} \quad x \le 1 \\ \text{Solution set: } (-\infty, \frac{-5}{3}] \cup [1, \infty) \end{aligned}$$

Example

Solve the inequality $\frac{|x+5|}{|x-11|} < 1$

$$\begin{aligned}\frac{|x+5|}{|x-11|} < 1 &\iff |x+5| < |x-11| \quad \text{(multiply both sides by |x-11|)}\\ &\iff |x+5|^2 < |x-11|^2\\ &\iff x^2+10x+25 < x^2-22x+121\\ &\iff 32x < 96\\ &\iff x < 3\end{aligned}$$

Solution set: $(-\infty, 3)$

Note: 11 is not in the solution set

### 1.1.4 Functions

**Domain, codomain and range**

Let $A$ and $B$ be subsets of $\mathbb{R}$: A function with domain $A$ and codomain $B$ is a rule which assigns to every $x \in A$ exactly one number $f(x) \in B$

$$f : A \to B$$
$$f : A \ni x \mapsto f(x) \in B$$

- $A = \mathrm{Dom}(f)$: set of all inputs
- $B = \mathrm{Codom}(f)$: set that contains all the outputs
- Range of $f$: set of actual outputs, defined by:

$$\mathrm{Range}(f) = \{f(x) \in B : x \in A\}$$

- Hence, $\mathrm{Range}(f) \subseteq \mathrm{Codom}(f)$
- The codomain does not have to be unique
- The codomain is useful when the range is unknown

Example

Given: $f : [0, \infty) \to \mathbb{R}, f(x) = 2 + \sqrt{x}, \forall x \in [0, \infty)$
Then, $\mathrm{Dom}(f) = [0, \infty)$, $\mathrm{Codom}(f) = \mathbb{R}$, and $\mathrm{Range}(f) = [2, \infty)$

Natural domain (or maximal domain) of $f$ is the largest possible domain for which the rule makes sense (for real numbers)

**Forming new functions**

Let $A$, $B$, $C$ and $D$ be subsets of $\mathbb{R}$. Given two functions $f : A \to B$ and $g : A \to B$, then the functions

$$f+g, \quad f-g, \quad fg, \quad \frac{f}{g}$$

are defined by the rules

$$
\begin{aligned}
(f+g)(x) &= f(x)+g(x), \quad \forall x \in A \\
(f-g)(x) &= f(x)-g(x), \quad \forall x \in A \\
(fg)(x) &= f(x)g(x), \quad \forall x \in A \\
\left(\frac{f}{g}\right)(x) &= \frac{f(x)}{g(x)}, \quad \forall x \in \text{A provided that } g(x)=0.
\end{aligned}
$$

To form these new functions, the domains of both $f$ and $g$ must be the same
Suppose that $f : C \to D$ and $g : A \to B$ are functions such that

$$\text{Range}(g) \subseteq \text{Dom}(f) = C$$

Then the composition function $(f \circ g)(x) = f(g(x)), \quad \forall x \in A$

### 1.1.5 The Elementary Functions

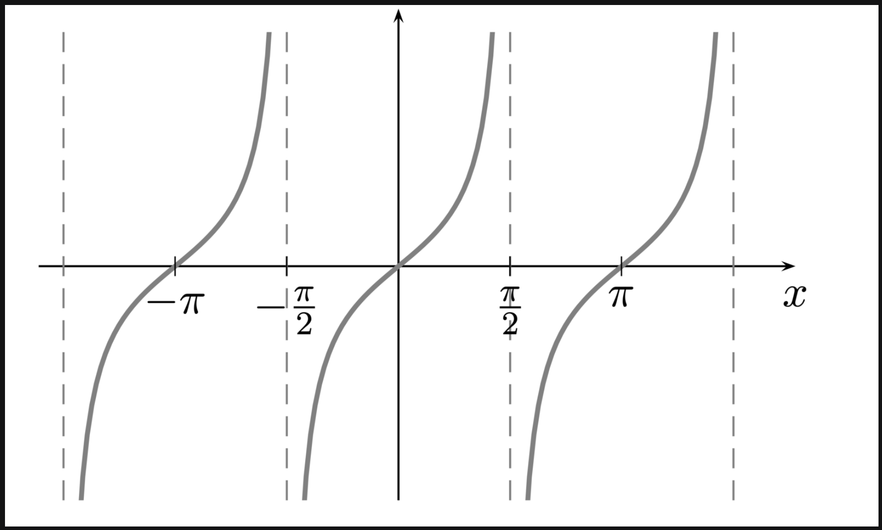
Elementary functions are those that can be constructed by combining a finite number of polynomials, exponentials, logarithms, roots, trigonometric and inverse trigonometric functions via function composition ($\circ$), $+$, $-$, $\times$, and $\div$.

### 1.1.6 Implicitly Defined Functions

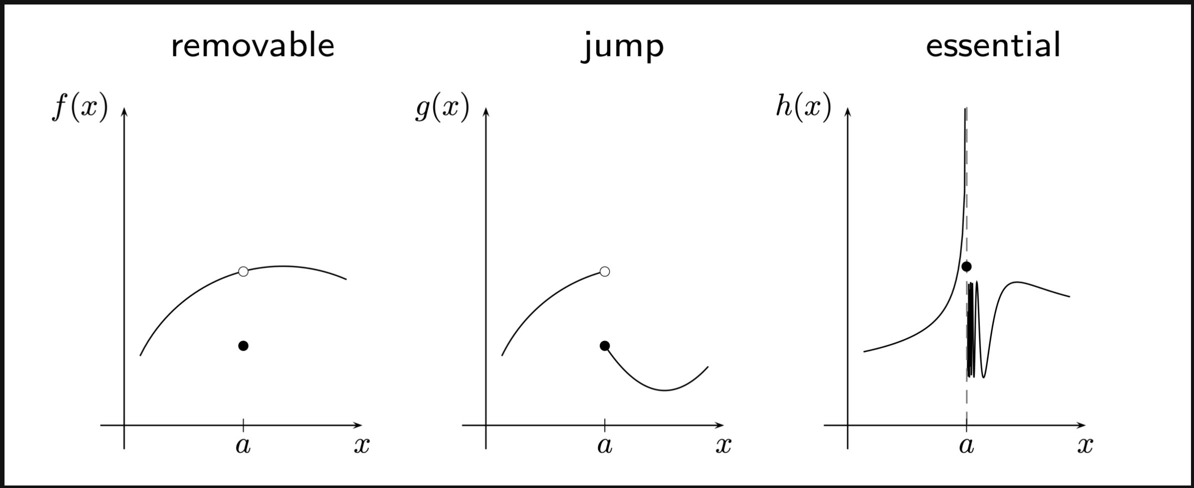
Many curves on the plane can be described by the points $(x, y)$ that satisfy some equation involving x and y. However, not all these curves are necessarily functions. Some cannot be expressed with a single $y$ term ($y = \dots$). In this case, the curve may be decomposed into two or more functions that are implicitly defined by the curve.

### 1.1.7 Continuous Functions

In the below example, the $\tan x$ graph breaks at $x = -\frac{\pi}{2}, \frac{\pi}{2}, \dots$.




However, this break in the domain does not make it a discontinuity in the graph. Discontinuities can be categorised as following:


removable
jump
essential
$f(x)$
$g(x)$
$h(x)$
$a$
$x$
$a$
$x$
$a$
$x$

# Chapter 2

# Limits

## 2.1 Limits of functions at infinity

- If $f(x)$ gets arbitrarily close to some $L \in \mathbb{R}$ as $x$ tends to $\infty$, then we write:

$$\lim_{x\to\infty} f(x) = L \quad \text{or} \quad f(x) \to L \text{ as } x \to L$$

- If $f(x)$ gets arbitrarily large as $x$ tends to $\infty$, then we write:

$$f(x) \to \infty \text{ as } x \to \infty$$

  - In this case, we state that the limit of $f(x)$ as $x$ tends to $\infty$ does not exist

$$\lim_{x\to\infty} \frac{1}{f(x)} = 0$$

- The limit notation can only be used when the limit has a constant definition (do not write $\lim_{x\to\infty} = \infty$)

### 2.1.1 Basic rules for limits

If $\lim_{x\to\infty} f(x) = L_1$ and $\lim_{x\to\infty} g(x) = L_2$ then

- $\lim_{x\to\infty} (f(x) + g(x)) = L_1 + L_2$
- $\lim_{x\to\infty} (f(x) - g(x)) = L_1 - L_2$
- $\lim_{x\to\infty} f(x)g(x) = L_1 L_2$
- $\lim_{x\to\infty} \dfrac{f(x)}{g(x)} = \dfrac{L_1}{L_2}$, provided that $L_2 \neq 0$

Example

Find the limit of $f(x) = \frac{1+\frac{2}{3x+4}}{5-6e^{-x}}$ as $x \to \infty$ (if it exists)

$$\begin{aligned}
\lim_{x\to\infty} f(x) &= \frac{\lim_{x\to\infty}(1+\frac{2}{3x+4})}{\lim_{x\to\infty}(5-6e^{-x})} \\
&= \frac{\lim_{x\to\infty} 1 + \lim_{x\to\infty} \frac{2}{3x+4}}{\lim_{x\to\infty} 5 - \lim_{x\to\infty} 6e^{-x}} \\
&= \frac{1+0}{5-0} \\
&= \frac{1}{5}
\end{aligned}$$

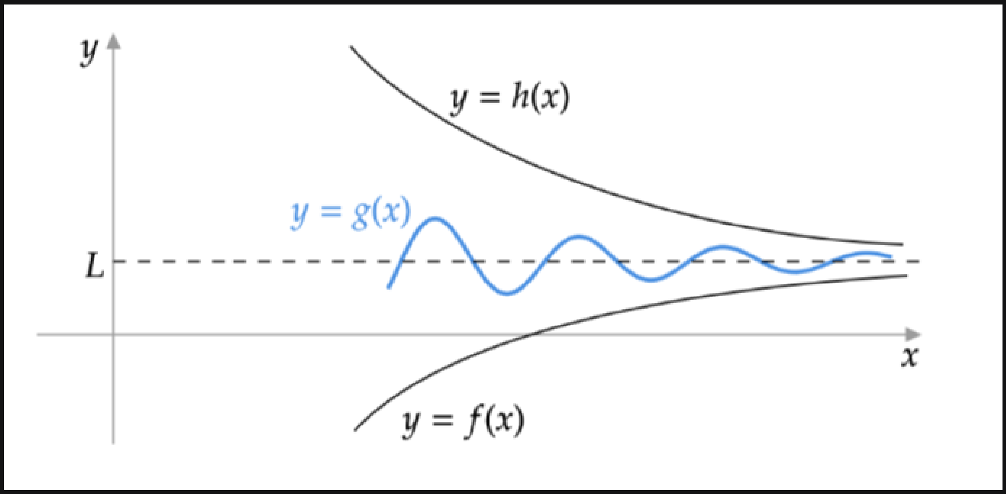
### 2.1.2 The pinching theorem

Suppose that $f$ and $h$ have the same limit as $x \to \infty$. If the graph of $g$ always lies between the graphs of $f$ and $h$, then $g$ has the same limit.

**Theorem** Let $f$, $g$, and $h$ be functions defined on $(b,\infty)$ for some $b \in \mathbb{R}. If$ :

- $f(x) \leq g(x) \leq h(x), \quad \forall x \in (b,\infty)$, and
- $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} h(x) = L$, then

$$\lim_{x\to\infty} g(x) = L$$



Example

Find the limit of $g(x) = e^{-x^2} 2^{\sin 3x}$ as $x \to \infty$ (if it exists).

$$\begin{aligned}
&\sin x \in [-1,1], \quad \forall x \in \mathbb{R} \\
&\Rightarrow \frac{1}{2} \leq 2^{\sin 3x} \leq 2, \quad \forall x \in \mathbb{R} \\
&\Rightarrow \frac{e^{-x^2}}{2} \leq e^{x^2} 2^{\sin 3x} \leq 2e^{-x^2} \\
\text{Since } &\lim_{x\to\infty} \frac{e^{-x^2}}{2} = \lim_{x\to\infty} 2e^{-x^2} = 0, \\
&\therefore \lim_{x\to\infty} e^{-x^2} 2^{\sin 3x} = 0
\end{aligned}$$

### 2.1.3 Limits of the form $f(x)/g(x)$

When calculating $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ when both $f(x)$ and $g(x)$ tend to $\infty$ as $x \to \infty$, the previous rules cannot be applied. The main goal in this situation is to **divide both $f$ and $g$ by the fastest growing term in $g$**.

Example

Find the limit $\lim_{x\to\infty} \frac{5x^3 + 6x^2 - 4\sin x}{\cos 3x + 5x - 2x^3}$ (if it exists).

$$\begin{aligned}
\frac{5x^3 + 6x^2 - 4\sin x}{\cos 3x + 5x - 2x^3} &= \frac{5 + \frac{6}{x} - 4\frac{\sin x}{x^3}}{\frac{\cos 3x}{x^3} + \frac{5}{x^2} - 2} \\
&= \frac{5+0+0}{0+0-2} \\
&= -\frac{5}{2} \qquad \text{as } x \to \infty
\end{aligned}$$

Example

Find the limit $\lim_{x\to\infty} \frac{x^2 + 3x}{\sqrt{2x^4+3} - 4x}$ (if it exists).

$$\begin{aligned}
\frac{x^2 + 3x}{\sqrt{2x^4+3} - 4x} &= \frac{1 + \frac{3}{x}}{\frac{1}{x^2}\sqrt{2x^4+3} - \frac{4}{x}} \\
&= \frac{1 + \frac{3}{x}}{\sqrt{2 + \frac{3}{x^4}} - \frac{4}{x}} \\
&\to \frac{1+0}{\sqrt{2+0} - 0} \\
&= \frac{1}{\sqrt{2}} \qquad \text{as } x \to \infty
\end{aligned}$$

### 2.1.4 Limits of the form $\sqrt{f(x)} - \sqrt{g(x)}$

In this case, the leading term may not help. Instead the lower order term will usually determine the limit. This can be achieved by **multiplying the numerator and denominator by the conjugate**.

Example

Find the limit of $\lim_{x\to\infty} \sqrt{2x^2+3x} - \sqrt{2x^2-x}$.

$$\begin{aligned}
\sqrt{2x^2+3x} - \sqrt{2x^2-x} &= \frac{\left(\sqrt{2x^2+3x} - \sqrt{2x^2-x}\right)\left(\sqrt{2x^2+3x} + \sqrt{2x^2-x}\right)}{\left(\sqrt{2x^2+3x} + \sqrt{2x^2-x}\right)} \\
&= \frac{2x^2+3x-\left(2x^2-x\right)}{\sqrt{2x^2+3x} + \sqrt{2x^2-x}} \\
&= \frac{4x}{\sqrt{2x^2+3x} + \sqrt{2x^2-x}} \\
&= \frac{x}{\sqrt{2+\frac{3}{x}} + \sqrt{2-\frac{1}{x}}} \\
&\to \frac{4}{\sqrt{2}+\sqrt{2}} \\
&= \sqrt{2} \quad \text{as } x \to \infty
\end{aligned}$$

### 2.1.5 Indeterminate forms

Some limits such as the form $\frac{\infty}{\infty}$ cannot be detemined in advance and is hence called an indeterminate form. Other indeterminate forms include:

- $\infty - \infty$ if $f(x) \to \infty$ and $g(x) \to \infty$ as $x \to \infty$, $\quad f(x) - g(x) \to ?$
- $\frac{0}{0}$ if $f(x) \to 0$ and $g(x) \to 0$ as $x \to \infty$, $\quad \frac{f(x)}{g(x)} \to ?$
- $0 \times \infty$ if $f(x) \to 0$ and $g(x) \to \infty$ as $x \to \infty$, $\quad f(x)g(x) \to ?$

## 2.2 The definition of $\lim_{x\to\text{inf}} f(x)$

Suppose that $b, L \in \mathbb{R}$, and $f$ is a real-valued function defined on $(b, \infty)$. It is said:

$$\lim_{x\to\infty} f(x) = L$$

if for every $\epsilon > 0$, there is $M \in \mathbb{R}$ such that:

$$x > M \Rightarrow |f(x) - L| < \epsilon$$

## 2.3 Proving that $\lim_{x\to\infty} f(x) = L$ using the limit definition

Example

Prove that $\lim_{x\to\infty} \frac{5x}{x+3} = 3$.

$$
\begin{aligned}
&|f(x)-L| < \epsilon \text{ if ?}\\
&|f(x)-L| = \left|\frac{5x}{x+3} - 5\right| = \left|\frac{5x-5(x+3)}{x+2}\right| = \left|\frac{-15}{x+3}\right|\\
&\qquad = \frac{15}{x+3} \quad \text{for } x>3\\
&\qquad < \frac{15}{x} \quad \text{for } x>0\\
&\Rightarrow |f(x)-L| < \epsilon \quad \text{if} \quad \frac{15}{x} < \epsilon \text{ and } x>0\\
&\quad (\text{i.e. } x > \frac{15}{\epsilon} > 0)\\
&\Rightarrow \text{For any } \epsilon>0, \exists M = \frac{15}{\epsilon} \text{ such that}\\
&\quad |f(x)-L| < \epsilon \text{ whenever } x > M
\end{aligned}
$$

**Strategy for finding $M$**

1. Find a good upper bound for $|f(x)-L|$
2. Find a simple condition on $x$ such that this upper bound is less than $\epsilon$
3. Use this condition to state an appropriate value for M

Note: The value of $M$ is not unique. If $M_0$ is a value for $M$, then any upper bound of $M_0$ is also a value for $M$

Example

Show that $\lim_{x\to\infty} \frac{x^2-\cos x}{x^2+1} = 1$

$$
\begin{aligned}
&\left|\frac{x^2-\cos x}{x^2+1} - 1\right| = \left|\frac{x^2-\cos x-(x^2+1)}{x^2+1}\right| = \left|\frac{-\cos x - 1}{x^2+1}\right| = \left|\frac{\cos x+1}{x^2+1}\right|\\
&\qquad \le \frac{2}{x^2+1}\\
&\qquad \le \frac{2}{x^2}\\
&\Rightarrow |f(x)-1| < \epsilon \quad \text{if} \quad \frac{2}{x^2} < \epsilon\\
&\quad \because \frac{2}{x^2} < \epsilon \Leftrightarrow \sqrt{\frac{2}{\epsilon}} < x \quad \text{for} \quad x>0, \quad M \triangleq \sqrt{\frac{2}{\epsilon}}
\end{aligned}
$$

### 2.3.1 Proofs of basic limit results

Not assessable.

## 2.4 Limits of functions at a point

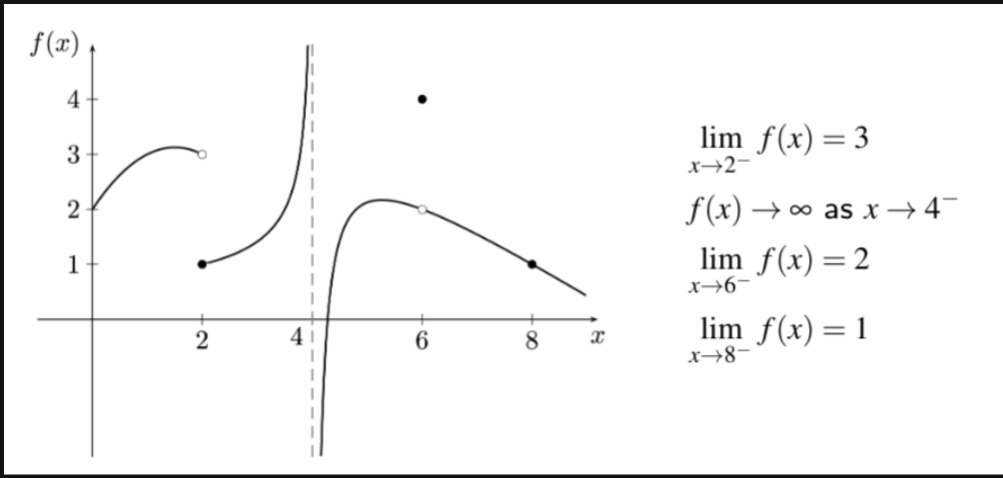
### 2.4.1 Left-hand, right-hand and two-sided limits

- If $f(x)$ approaches $L$ as $x$ approaches $a$ from the left-hand side, then:

$$\lim_{x \to a^-} f(x) = L \qquad \text{(left-hand limit)}$$

- If $f(x)$ gets arbitrarily large as $x$ approaches $a$ from the left side, then:

$$f(x) \to \infty \quad \text{as} \quad x \to a^-$$

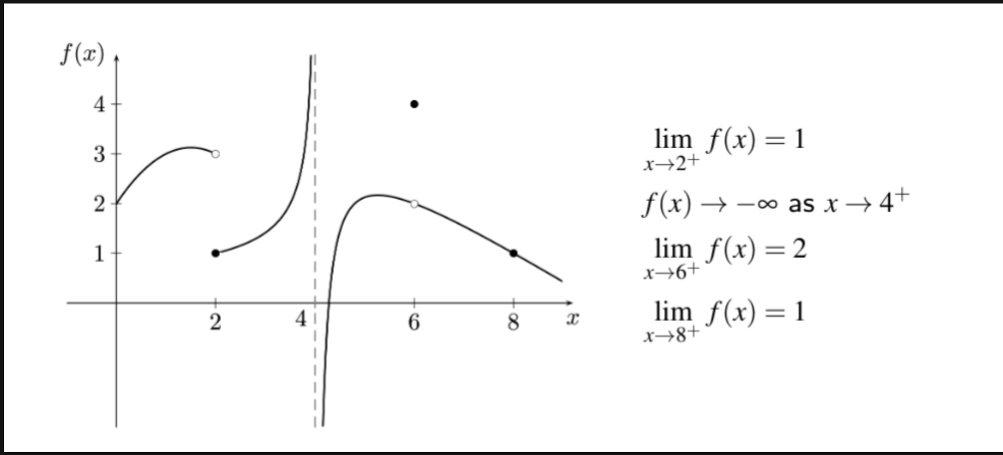



- If $f(x)$ approaches $L$ as $x$ approaches $a$ from the right-hand side, then:

$$\lim_{x \to a^+} f(x) = L \qquad \text{(right-hand limit)}$$

- If $f(x)$ gets arbitrarily large as $x$ approaches $a$ from the right side, then:

$$f(x) \to \infty \quad \text{as} \quad x \to a^+$$



- Left $I$ be an open interval that contains the point $a$. For a real-valued function $f$ defined on $I$ except possibly at $a$, if both one-sided limits exist and

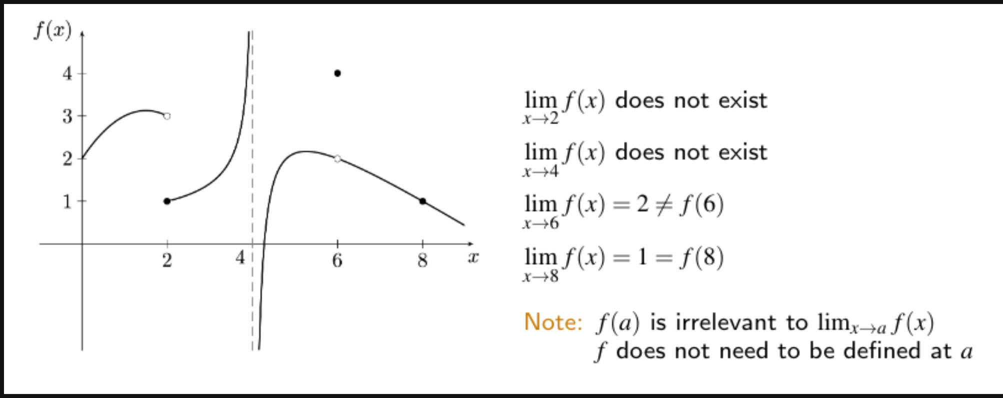
$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L,$$

then we say that thw (two-sided) limit of $f(x)$ as $x \to a$ exists and is equal to $L$, and we write

$$\lim_{x\to a} f(x) = L.$$

Otherwise, $\lim_{x\to a} f(x)$ does not exist.

- Note: $f(a)$ is irrelevant to $\lim_{x\to a} f(x)$. $f$ does not need to be defined at $a$



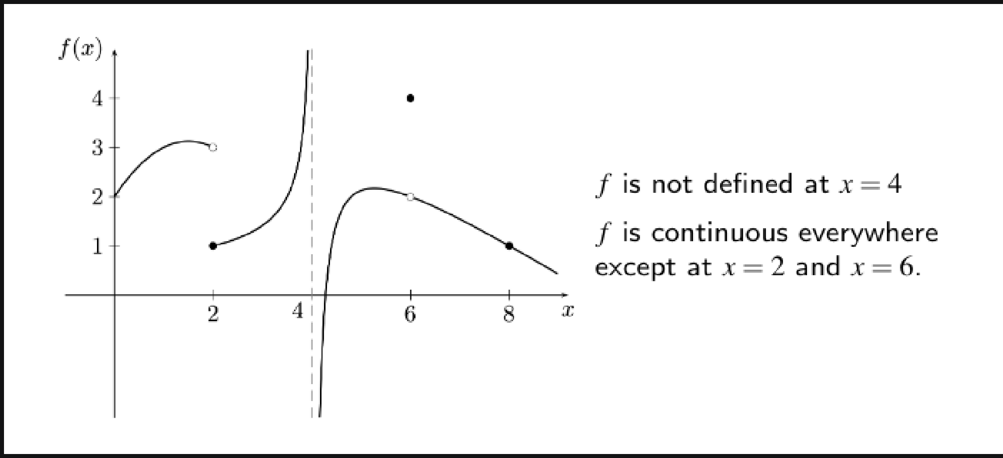

### 2.4.2 Limits and continuous functions

Let $I$ be an open interval that contains the point $a$. A function $f : I \to \mathbb{R}$ is continuous at $a$ if

$$\lim_{x\to a} f(x) = f(a)$$

Otherwise, we say that $f$ is discontinuous at $a$.

If $f$ is continuous at every point of $\mathrm{Dom}(f)$, then we say that $f$ is continuous everywhere.



### 2.4.3 Rules for limits at a point

**Arithmetic**

Suppose that $a \in \mathbb{R}$, and $\lim_{x\to a} f(x) = L_1$ and $\lim_{x\to a} g(x) = L_2$. Then

- $\lim_{x\to a}(f(x)+g(x)) = L_1 + L_2$
- $\lim_{x\to a}(f(x)-g(x)) = L_1 - L_2$
- $\lim_{x\to a} f(x)g(x) = L_1 L_2$
- $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{L_1}{L_2}$

**Function composition**

If $\lim_{x\to a} f(x) = L$ and $g$ is continuous at $L$, then

$$\lim_{x\to a} g(f(x)) = g\left(\lim_{x\to a} f(x)\right) = g(L)$$

Example

Find the limit $\lim_{x\to 5}\sqrt{2x+\cos\pi x^2}$.

$$\begin{aligned}
\lim_{x\to 5}\sqrt{2x+\cos\pi x^2} &= \sqrt{\lim_{x\to 5} 2x+\cos\pi x^2} && \left(\sqrt{\phantom{x}} \text{ is continuous}\right)\\
&= \sqrt{\lim_{x\to 5} 2x+\lim_{x\to 5}\cos\pi x^2}\\
&= \sqrt{\lim_{x\to 5} 2x+\cos\lim_{x\to 5}\pi x^2} && (\cos \text{ is continuous})\\
&= \sqrt{10+\cos 5^2\pi} && \left(\cos 5^2\pi = -1\right)\\
&= 3
\end{aligned}$$

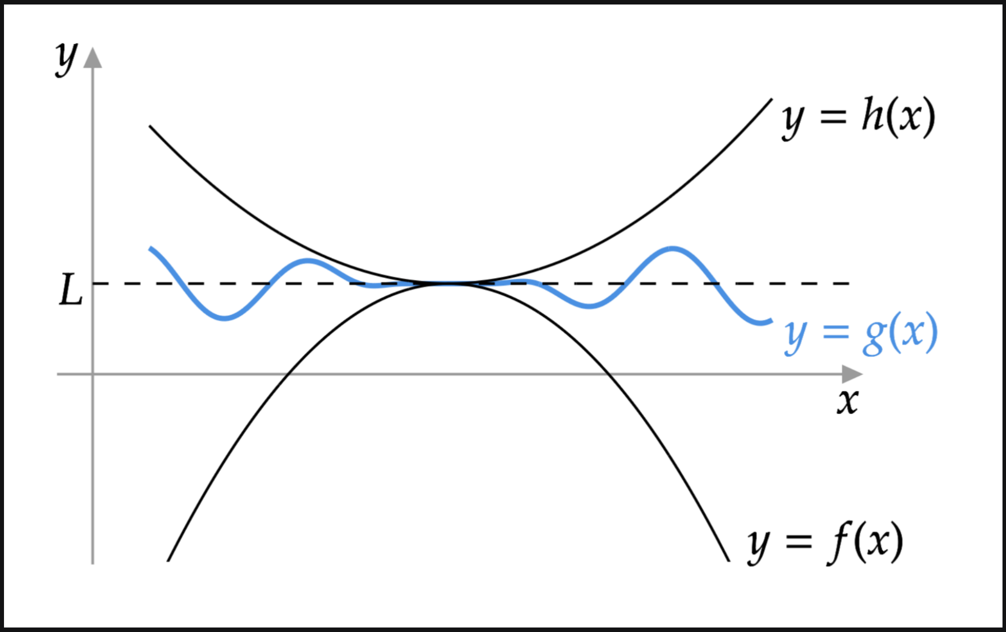
**Pinching theorem (two-sided limit)**

Let $I$ be an open interval that contains the point $a$. Suppose that $f$, $g$, and $h$ are functions defined on $I$. If

- $f(x) \le g(x) \le h(x), \quad \forall x \in I$, and
- $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$

then

$$\lim_{x\to a} g(x) = L$$




Example

Suppose that $f : \mathbb{R} \to \mathbb{R}$ is defined by the formula

$$f(x) = \begin{cases} \frac{|x^2-9|}{x-3} & \text{if } x \neq 3 \\ -6 & \text{if } x = 3 \end{cases}$$

Discuss the limiting behaviour of $f(x)$ as $x$ approaches 3.

$$\text{For } x \in (-3, 3), \quad \frac{|x^2-9|}{x-3} = \frac{|x+3||x-3|}{x-3} = -(x+3)$$
$$\Rightarrow \lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} -(x+3) = -6 = f(3)$$

$$\text{For } x > 3, \quad \frac{|x^2-9|}{x-3} = x+3$$
$$\Rightarrow \lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} x+3 = 6$$

Since $\lim_{x\to 3^+} f(x) \neq \lim_{x\to 3^-} f(x)$, two-sided limit does not exist at $x = 3$

So, $f$ is not continuous at $x = 3$