MATH 3406 Notes

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1 Preliminaries

Note that these are assumed unless otherwise specified. The specific section's definition of these variables take priority over these definitions. If something seems unclear please contact me.

1.1 U, V, W

Denotes a vector space.

1.2 u, v, w

Denotes a vector in its corresponding vector space.

1.3 $\mathbb{F}$

Denotes the field V is over, usually $\mathbb{R}$ or $\mathbb{C}$.

1.4 x_i

Used to refer to a list; i ranges from 1 or 0 to some arbitrary natural number.

1.5 T

Denotes a linear map, usually from V to W.

2 Vector Spaces

Corresponding to Chapter 1, sections B and C of Axler.

2.1 Properties of a Vector Space

V is a vector space iff $\forall \lambda, \lambda_1, \lambda_2 \in \mathbb{F}$ and $\forall u, v, w \in V$:

1. $u + v \in V$
2. $\lambda v \in V$
3. $u + v = v + u$
4. $(u + v) + w = u + (v + w)$
5. $\exists 0$ s.t. $v + 0 = v$
6. $\exists(-v)$ s.t. $v + (-v) = 0$
7. $\exists 1$ s.t. $1v = v$
8. $(\lambda_1 + \lambda_2)v = \lambda_1 v + \lambda_2 v$, $\lambda(u + v) = \lambda u + \lambda v$

Sometimes written as V.S. in shorthand.

2.2 Subspace

U is a subspace of V iff $\forall \lambda \in \mathbb{F}$, $\forall u, w \in U$:

1. $0 \in U$
2. $u + w \in U$
3. $\lambda u \in U$

Note that it is *usually* more efficient to show something is a subspace of another V.S. than to show it is a V.S. directly.

2.3 Sums

If U_i are subsets of V then $\sum U_i = \{\sum u_i | u_i \in U_i\}$. Note this is similar to the union of sets in set theory.

2.4 Direct Sums

Denoted $U_1 \oplus \cdots \oplus U_m$, a direct sum is said to be when each element of $\sum U_i$ can be uniquely written as a sum of u_i.

2.4.1 Condition for a Direct Sum

$\sum U_i$ is a direct sum iff $\sum u_i = 0$ only when $\forall u_i, u_i = 0$.

2.4.2 Condition for a Direct Sum

$U + W$ is a direct sum iff $U \cap W = \{0\}$.

3 Span

Corresponding to Chapter 2, Section A, first half of Axler.

3.1 Span

The *span* of a set of vectors v_i is $\{\sum a_i v_i | a_i \in \mathbb{F}\}$, denoted $\text{span}(v_i)$. Sometimes defined as the set of all linear combinations of v_i; a *linear combination* of a set of vectors v_i is simply $\sum a_i v_i$ for some $a_i \in \mathbb{F}$.

3.1.1 Span and Vector Spaces

We say v_i *spans* a V.S. V if V is the smallest V.S. that contains every vector in $\text{span}(v_i)$.

3.2 Finite-Dimensional Vector Space

V is *finite-dimensional* if $\exists v_i$ that spans V. **Note:** by definition, a list has finite length.

3.3 Polynomials

The definition of a polynomial is assumed, and is denoted $p(z)$. However, note that some polynomials may be over a different field. $p(z) = (2i + 7)z^3 - (3i - 11)z^2 + 12$ is a polynomial over $\mathbb{C}$, for example.

3.3.1 $\mathcal{P}(\mathbb{F})$

The set of all polynomials with coefficients in $\mathbb{F}$.'

3.3.2 Degree of a Polynomial

The *degree* of a polynomial is the highest degree m s.t. $p(z)$ can be expressed as

$$p(z) = \sum_{i=0}^{m} a_i p^i, a_i \in \mathbb{F}.$$

Then we say deg $p = m$. If a polynomial is identically 0, then its degree is $-\infty$.

3.3.3 $\mathcal{P}_m(\mathbb{F})$

The set of all polynomials of degree m, coefficients $\in \mathbb{F}$.

3.4 Infinite-Dimensional Vector Space

A V.S. that is not finite-dimensional.

4 Linear (In)Dependence

Corresponding to Chapter 2, Section A, second half of Axler.

4.1 Linear Independence

v_i is *linearly independent* if there exists a unique solution to $\sum a_i v_i = 0$ for $a_i \in F$. The solution is then all $a_i = 0$. Note the empty list () is also linearly independent.

4.2 Linear Dependence

v_i is *linearly dependent* if it is not linearly independent. Thus there exists a_i not all 0 such that $\sum a_i v_i = 0$.

4.2.1 Linear Dependence Lemma

Suppose $v_i, i \in [m]$ is linearly dependent. Then $\exists j \in [m]$ s.t.

1. $v_j \in \text{span}(v_1, \ldots v_{j-1})$
2. $\text{span}(v_i) = \text{span}(v_i - v_j)$. Note that $v_i - v_j$ denotes the original list of v_i with v_j removed.

Note that this implies that in a finite-dimensional V.S., the length of every linearly independent list of vectors is $\leq$ the length of every spanning list of vectors.

4.3 Finite-Dimensional Subspaces

Every subspace of a finite-dimensional V.S. is finite-dimensional.

5 Bases

Corresponding to Chapter 2, Section B of Axler.

5.1 Basis

A list of vectors in V that is linearly independent and spans V.

5.2 Criterion for Basis

v_i is a basis for V iff $\forall v \in V$, $v = \sum a_i v_i$.

5.3 Spanning Lists and Bases

Every spanning list is a superlist of a basis.

5.4 Basis of Finite-Dimensional Vector Spaces

$\exists$ a basis for every finite-dimensional V.S.

5.5 Linearly Independent Lists and Bases

Every linearly independent list in a finite-dimensional V.S. is a sublist of a basis.

5.6 Existence of Subspaces in Direct Sums

If V is finite-dimensional, and $U \subseteq V$, then $\exists W \subseteq V$ s.t. $V = U \oplus W$.

6 Dimension

Corresponding to Chapter 2, Section C of Axler.

6.1 Dimension

The length a basis of the V.S.; denoted $\dim V$.

6.2 Dimension of a Subspace

Given finite-dimensional V, $U \subseteq V$, $\dim U \leq \dim V$.

6.3 Linearly Independent Lists and Bases (and Dimension)

Every linearly independent list in V with length $\dim V$ is a basis of V.

6.4 Spanning Lists and Bases (and Dimension)

Every spanning list in V with length $\dim V$ is a basis of V.

6.5 Dimension of a Sum

Given $U, W \subseteq V$, then $\dim (U + W) = \dim U + \dim V - \dim (U \cap W)$. Note for direct sums $\dim (U + W) = \dim U + \dim V$, since $(U \cap W) = \{0\}$, and hence $\dim (U \cap W) = 0$

7 Vector Space of Linear Maps

Corresponding to Chapter 3, Section A of Axler.

7.1 Linear Map

The function $T : V \to W$ s.t. $\forall \lambda \in \mathbb{F}, \forall u, v \in V$:

1. $T(u + v) = Tu + Tv$
2. $T(\lambda v) = \lambda(Tv)$

Note that $T(v) = Tv$, and usually parenthesis are removed.

7.1.1 Zero Map

The *zero map*, or 0, is defined as $\forall v \in V, 0v = 0$.

7.1.2 Identity Map

The *identity map*, or I, is defined as $\forall v \in V, Iv = v$.

7.2 $\mathcal{L}(V, W)$

The set of all linear maps from V to W.

7.3 Linear Maps and Bases

If v_i is a basis of V and w_i is a basis of W, then $\exists T \in \mathcal{L}(V, W)$ s.t. $\forall j, Tv_j = w_j$.

7.4 Addition, Scalar Multiplication on $\mathcal{L}(V, W)$

For $S, T \in \mathcal{L}(V, W), v \in V, \lambda \in \mathbb{F}$, we define $(S+T)(v) = Sv+Tv$, and $(\lambda T)(v) = \lambda(Tv)$. Note that this implies $\mathcal{L}(V, W)$ is a V.S.

7.5 Product of Linear Maps

Given $T \in \mathcal{L}(U, V)$, $S \in \mathcal{L}(V, W)$, $u \in U$, define $ST \in \mathcal{L}(U, W)$ s.t. $(ST)(u) = S(Tu)$.

7.6 Algebraic Properties of Linear Maps

The following are some notable properties of linear maps. Given $T, T_i \in \mathcal{L}(U, V)$, $S, S_i \in \mathcal{L}(V, W)$:

1. $(T_1T_2)T_3 = T_1(T_2T_3)$
2. $TI = IT = T$
3. $(S_1 + S_2)T = S_1T + S_2T$, $S(T_1 + T_2) = ST_1 + ST_2$
4. $T(0) = 0$

8 Null Spaces and Ranges

Corresponding to Chapter 3, Section B of Axler.

8.1 Null Space

Denoted null T, defined as $\{v \in V | Tv = 0\}$. This is a subspace of V.

8.2 Injective

T is injective if $Tu = Tv \Rightarrow u = v$. This is equivalent to null $T = \{0\}$

8.2.1 Dimension and Injectivity

If $T \in \mathcal{L}(V, W)$ where dim $V >$ dim W, then T is not injective.

8.3 Range

Denoted range T, defined as $\{Tv | v \in V\}$. This is a subspace of V.

8.4 Surjective

T is surjective if range $T = W$.

8.4.1 Dimension and Surjectivity

If $T \in \mathcal{L}(V, W)$ where dim $V <$ dim W, then T is not surjective.

8.5 The Fundamental Theorem of Linear Maps

dim $V =$ dim null $T +$ dim range T

8.6 (In)Homogeneous Systems of Linear Equations

Not covered in Hannah Turner's Section of MATH 3406. Please contact me if you have questions regarding this section of Axler, preferably when I don't have any exams coming up.

9 Matrices

Corresponding to Chapter 3, Section C of Axler. I don't feel like doing the latex for this one right before my exam. You can read the textbook; this stuff isn't too hard.

10 Invertibility and Isomorphisms

Corresponding to Chapter 3, Section D of Axler. See above. I should probably just review my practice midterm. I'll type these up some time after my graph theory exam on Monday, Oct 2, 2023.

11 Invariant Subspaces

Corresponding to Chapter 5, Section A of Axler. See above.