DE assignment on numerical methods

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Problem:

y'=-2y+4x
y(0)=0, x ∈ [0,3]

Solution:

dy/dx + 2y = 4x
dy/dx + 2y = 0 => y = C*e^(-2x),
Change C with a function u(x):
u' * e^(-2x) = 4x
du = 4x*e^(2x) * dx,

$$\int du = \int 4x*e^{(2x)} * dx,$$

u = 2x*e^(2x) - e^(2x) + c
Substitute back and get:
y=(2x*e^2x - e^2x + c)*e^(-2x) ,
y=2x-1+c*e^(-2x),
y(0)=0 => c=1
So solution to ivp is y=2x-1+e^(-2x)

There are no points of discontinuity on given range

Implementation:

I have chosen Python programming language and used following libraries:
Math - for computing mathematical functions
Matplotlib - for plotting graphs
Numpy - for constructing array of values in some range with float step

Program:

Each method receives following arguments:

f - callable function of x and y such that y'=f(x,y),
xs - array of x-values
y0 - given ivp value for xs[0]
h - step

Euler method:

```
def euler(f, xs, y0, h):
  ys = [y0]
  for i in range(1, xs.size):
    ys += [ys[i - 1] + h * f(xs[i - 1], ys[i - 1])]
  return ys
```

Improved Euler:

```
def improved_euler(f, xs, y0, h):
  ys = [y0]
  for i in range(1, xs.size):
    k1 = f(xs[i - 1], ys[i - 1])
    k2 = ys[i - 1] + h * k1
    ys += [ys[i - 1] + h / 2 * (k1 + f(xs[i], k2))]
  return ys
```

Runge-Kutta:

```
def runge_khutta(f, xs, y0, h):
  ys = [y0]
  for i in range(1, xs.size):
    k1 = f(xs[i - 1], ys[i - 1])
    k2 = f(xs[i - 1] + h / 2, ys[i - 1] + h * k1 / 2)
    k3 = f(xs[i - 1] + h / 2, ys[i - 1] + h * k2 / 2)
    k4 = f(xs[i - 1] + h, ys[i - 1] + h * k3)
    ys += [ys[i - 1] + h / 6 * (k1 + 2 * k2 + 2 * k3 + k4)]
  return ys
```

- Each method is also wrapped with decorator(@calculate_error), which calculates local error for values got from method and returns it

Method for plotting each graph:

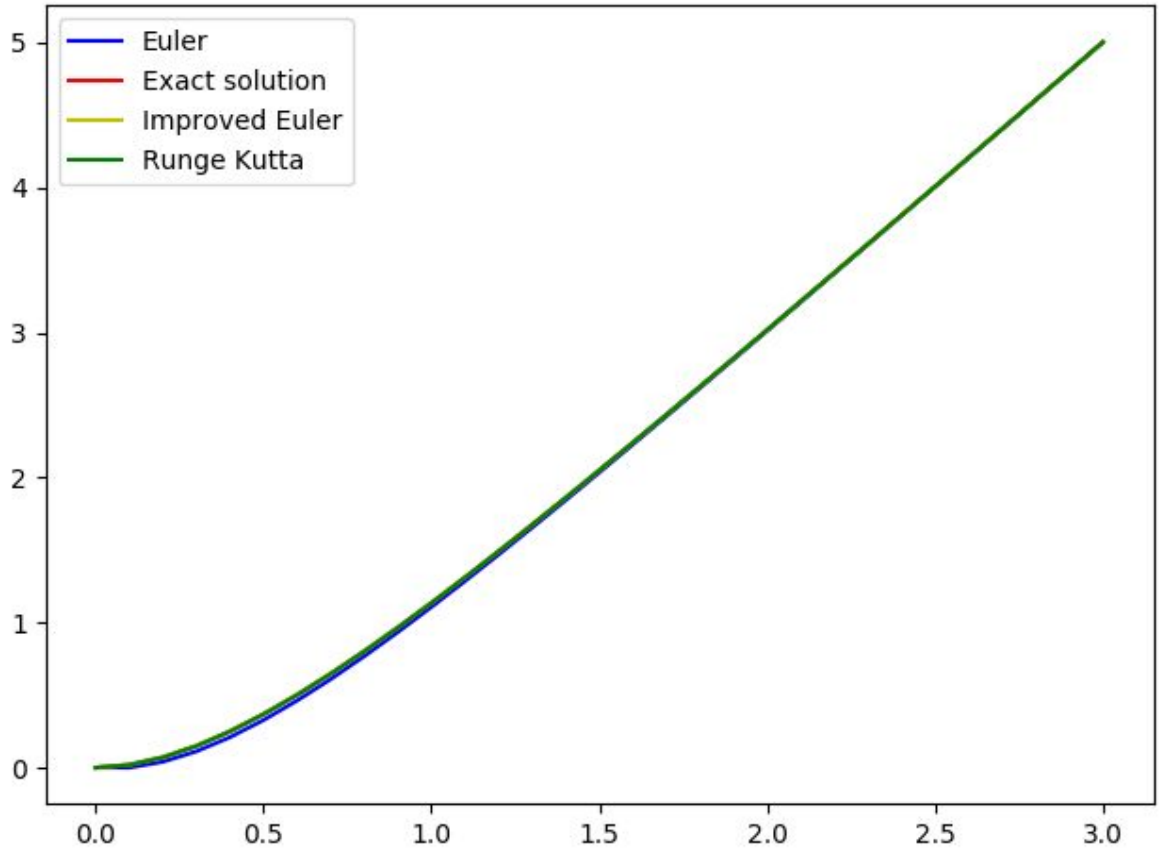
```
def plot(xs,y0,h,ysol):
  tup = (func, xs, y0, h, ysol)
  # draw euler
  y, error_euler = euler(*tup)
  plt.plot(xs, y, 'b', label=euler_)
  # draw exact solution
  plt.plot(xs, ysol, 'r', label=exact_)
  # draw improved euler
  y, error_improved = improved_euler(*tup)
  plt.plot(xs, y, 'y', label=improved_)
  # draw runge-khutta
  y, error_runge = runge_khutta(*tup)
  plt.plot(xs, y, 'g', label=runge_)
  plt.legend()
  # plot errors
print_errors(xs,  error_euler, error_improved, error_runge)
```

Method for plotting Global error graph:

```
def global_error(x0, x1, y0):
  error_euler = []
  error_improved_euler = []
  error_runge_kutta = []
  N = range(10, 2000)
  for n in N:
    h = (x1 - x0) / n
    xs = numpy.arange(x0, x1 + h, h)
    ysol = solution(xs)
    tup = (func, xs, y0, h, ysol)
    _, error1 = euler(*tup)
    _, error2 = improved_euler(*tup)
    _, error3 = runge_khutta(*tup)
    error_euler.append(max(error1))
    error_improved_euler.append(max(error2))
    error_runge_kutta.append(max(error3))
  print_errors(N, error_euler, error_improved_euler, error_runge_kutta)
```

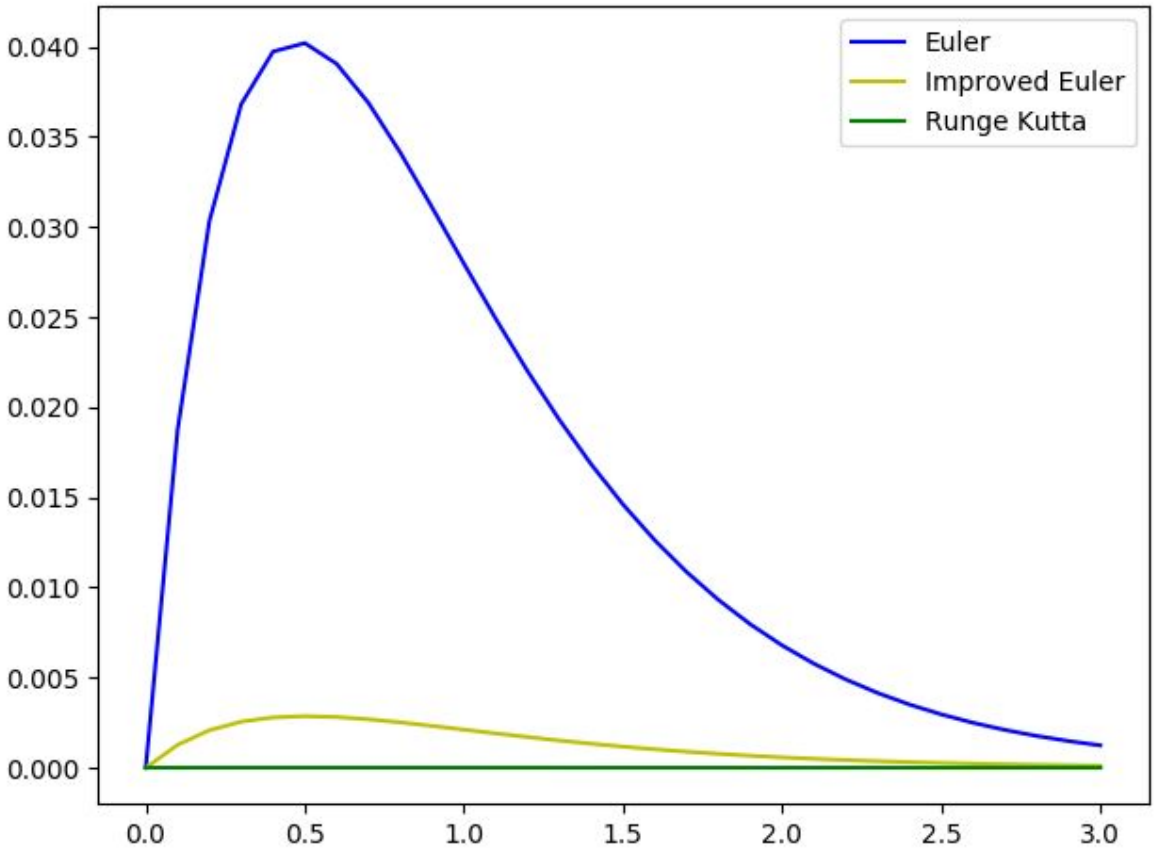
Results

Graphs for each method with step h=0.1



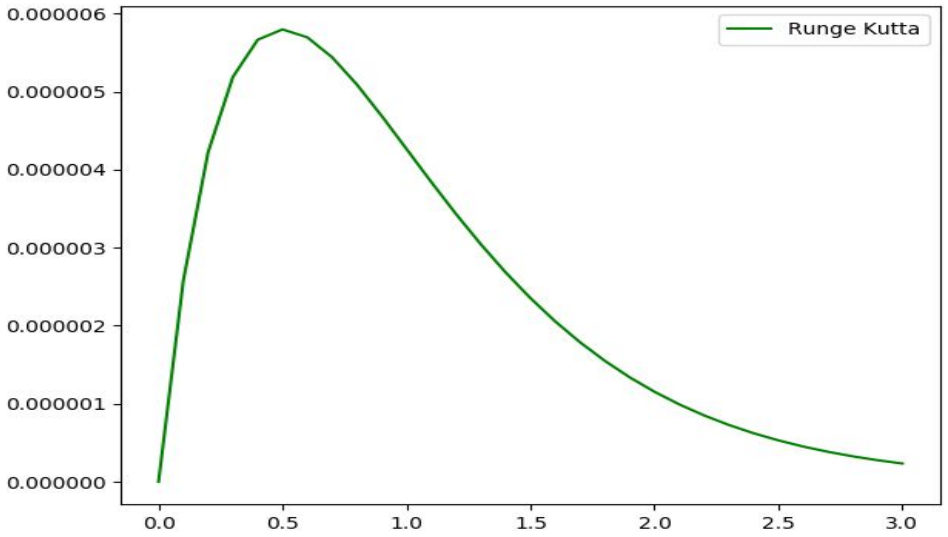
- As we see all three plots are very close to graph of an exact solution

Graph of local errors for each method

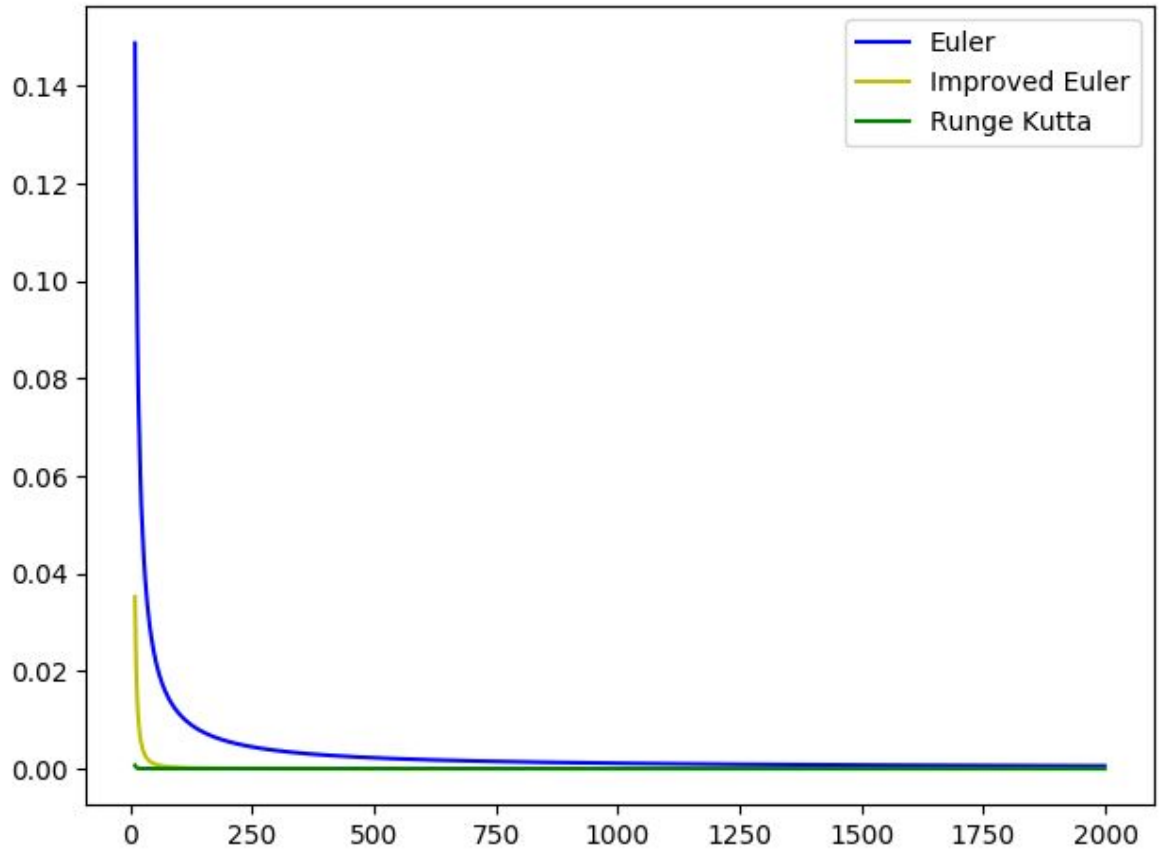


- In the beginning of calculations each method shows increasing local error near starting point location, however, as the number of calculated points increases accuracy increases too

Runge Kutta shows the best approximation, it’s local error graph closer:



Runge Kutta local error graph

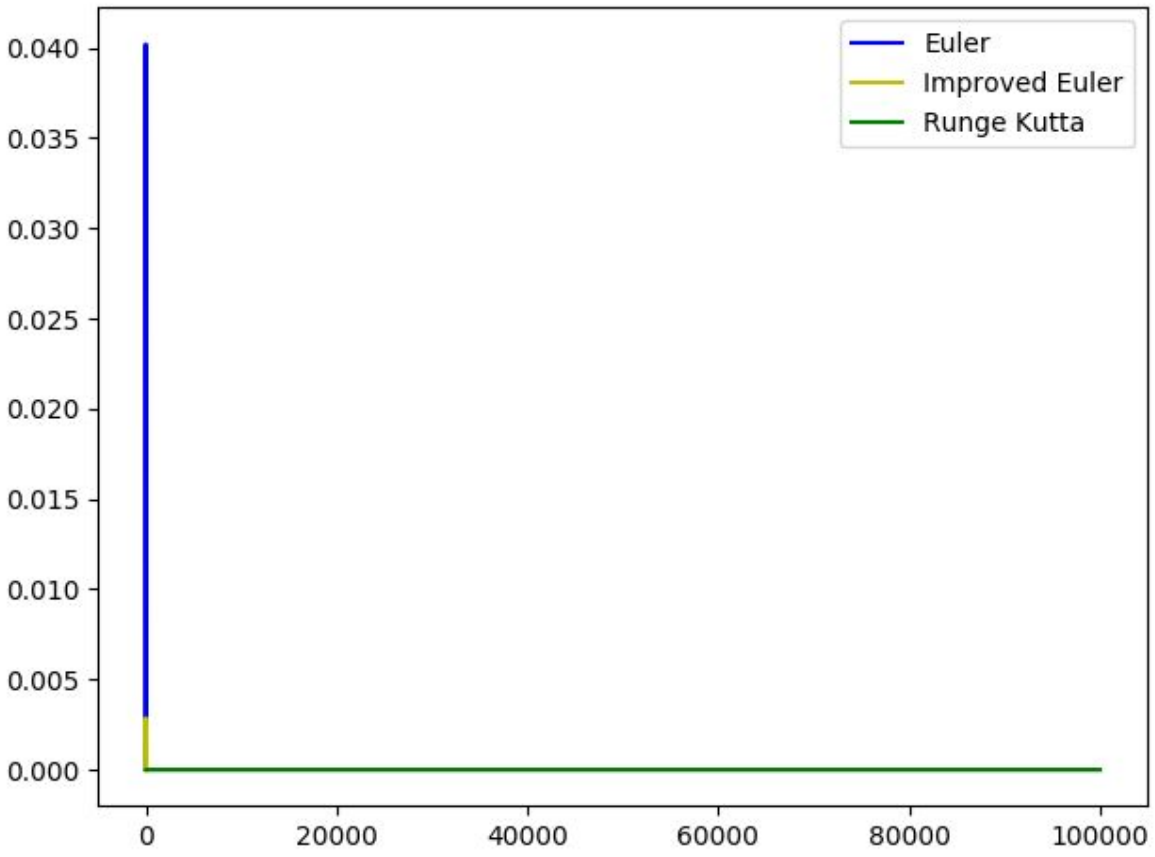


Graph of dependence of max local error on the number of approximated points

- Accuracy directly proportional to the number of calculated points

Convergence

We check that maximal approximation error of a method tends to zero with really large N, every approximation tend to zero so all methods converge



Graph of local error for each method for xn = 100 000