Algebraic Foundations of Proof Refinement

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***Abstract*—We contribute a general apparatus for *dependent* tactic-based proof refinement in the LCF tradition, in which the statements of subgoals may express a dependency on the proofs of other subgoals; this form of dependency is extremely useful and can serve as an *algorithmic* alternative to extensions of LCF based on non-local instantiation of schematic variables. Our framework, called Dependent LCF, is already deployed in the nascent RedPRL proof assistant for computational cubical type theory.**

I. Introduction

Interactive proof assistants are at their most basic level organized around some form of *proof refinement apparatus*, which defines the valid transitions between partial constructions, and induces a notion of *proof tactic*. The proof refinement tradition begins with Milner et al.'s *Logic for Computable Functions* [1], [2] and was further developed in Cambridge LCF, HOL and Isabelle [3], as well as the Nuprl family [4], [5], [6], [7]; tactic-based proof refinement is also used in the highly successful Coq proof assistant [8] as well as the new Lean theorem prover [9].

Notation I.1. Throughout this paper, we employ a notational scheme where the active elements of mathematical statements or judgments are colored according to their *mode*, i.e. whether they represent inputs to a mathematical construction or outputs. Terms in input-mode are colored with **blue**, whereas terms in output-mode are colored with **maroon**.

A. Proof refinement and evidence semantics

At the heart of LCF-style proof refinement is the coding of the inference rules of a formal logic into partial functions which take a conclusion and return a collection of premises (subgoals); this is called *backward inference*. Often, this collection of subgoals is equipped with a *validation*, a function that converts evidence of the premises into evidence of the conclusion (which is called *forward inference*).

An elementary example of the LCF coding of inference rules can be found in the right rule for conjunction in an intuitionistic sequent calculus:

$$\frac{\mathcal{D} \quad \mathcal{E}}{\Gamma \vdash P \quad \Gamma \vdash Q}{\scriptstyle \wedge_{\mathfrak{R}}}$$

$$\Gamma \vdash A \wedge B \Mapsto \left\langle \begin{array}{l} [\Gamma \vdash A, \Gamma \vdash B], \\ \lambda[\mathcal{D}, \mathcal{E}].\ \wedge_{\mathfrak{R}}(\mathcal{D}, \mathcal{E}) \end{array} \right\rangle$$
$$\_ \Mapsto \textbf{raise}\ \mathsf{Fail}$$

In the example above, the validation produces exactly a proof of $\Gamma \vdash A \wedge B$ in the intuitionistic sequent calculus, but in general it is not necessary for the meaning of the validations to correspond exactly with formal proofs in the logic being modeled.

In both Edinburgh LCF and Cambridge LCF, the implementation of a proof refinement logic was split between a trusted kernel which implemented *forward inference*, and a separate module which implemented backward inference (refinement) using the forward inference rules as validations [10]. Under this paradigm, every inference rule must be implemented twice, once in each direction.

However, it is also possible to view a collection of backward inference (refinement) rules as definitive in its own right, independent of any forward inference characterization of a logic; this approach, pioneered in the Nuprl proof assistant [4], enables a sharp distinction between *proof* and *evidence* (the former corresponding to *derivations* and the latter corresponding to *realizers*).

This flexibility is a major strength of the LCF design, because the notion of evidence can be varied independently from the notion of formal proof—the latter is defined by the assignment of premises to conclusions in the trusted refinement rules, whereas the former is defined by the validations of these rules.

For instance, in the Nuprl family of proof assistants, the *refinement logic* is a formal sequent calculus for a variant of Martin-Löf's type theory (see [11]), but the validations produce programs in an untyped language of realizers with computational effects [12]. Validations, rather than duplicating the existing refinement rules in forward inference, form the "program extraction" mechanism in Nuprl; this deep integration of extraction into the refinement logic lies in stark contrast with treatments of program extraction in other proof assistants, such as Coq and Agda.

This notion of *computational evidence semantics*, in which formal rules are related to an intended computational model by extraction, is due to Constable [13], [14]. In practice, this technique of separating the proof theory from the evidence semantics can be used to induce (sub)structural and behavioral invariants in programs from arbitrary languages *as they already exist*, regardless of whether these languages possess a sufficiently "proof-theoretic" character (embodied, for instance, in the much acclaimed "decidable typing" property).

This basic asymmetry between *proof* and *computational evidence* in refinement logics corresponds closely with the origin of the sequent calculus, which was designed as a theory of derivability for natural deduction. More generally, it reflects

the distinction between the demonstration of a judgment and the construction it effects [15], [16].

B. Dependency or barbarism!

An apparently essential part of the LCF apparatus is that each subgoal may be stated independently of evidence for the others; this characteristic, originally identified as part of the *constructible subgoals* property by Harper in his dissertation [17], allows proof refinement to proceed in parallel and in any order.

This restriction, however, raises difficulties in the definition of a refinement logic for dependent type theory, where the statement of one subgoal may depend on the evidence for another subgoal (a state of affairs induced by families of propositions which are indexed in proofs other another proposition). The most salient example of this problem is given by the introduction rule for the dependent sum of a family of types [17, p. 35]. First, consider the standard type *membership* rules, which pose no problem:

$$\frac{M \in A \quad N \in B[M]}{\langle M, N\rangle \in (x : A) \times B[x]}\ \times_{\mathfrak{J}}^{=}$$

The premises can be stated purely on the basis of the conclusion, because M appears in the statement of the conclusion. However, if we try to convert this to a *refinement* rule, in which we do not have to specify the exact member $\langle M, N\rangle$ in advance, we immediately run into trouble:

$$\frac{\begin{array}{cc} M & N \\ A\ \mathit{true} & B[M]\ \mathit{true}\end{array}}{\begin{array}{c}\langle M, N\rangle \\ (x : A) \times B[x]\ \mathit{true}\end{array}}\ \times_{\mathfrak{R}}$$

Suspending for the moment one's suspicions about the above notation, the fundamental problem is that this inference rule cannot be translated into an LCF rule, because the subgoal $B[M]\ \mathit{true}$ cannot even be stated until it is known what M is, i.e. until we have somehow run the validation for the completed proof of the first subgoal.

To paraphrase the immortal words of the German revolutionary Rosa Luxemburg, we stand at a crossroads: either we shall account for dependency, or we must regress into barbarism.

1) Ignore the problem!: The resolution adopted by the first several members of the Nuprl family was to defer the solution of this problem to a later year, thereby requiring that the user solve the first subgoal in advance of applying the introduction rule. This amounted to defining a countable family of rules $\times_{\mathfrak{R}}\{M\}$, with the term M a parameter:

$$\frac{\begin{array}{cc} \overset{*}{M} & N \\ M \in A & B[M]\ \mathit{true}\end{array}}{\begin{array}{c}\langle M, N\rangle \\ (x : A) \times B[x]\ \mathit{true}\end{array}}\ \times_{\mathfrak{R}}\{M\}$$

The subgoals of the above rule are independent of each other, and it can therefore easily be coded as an LCF rule. However, this has come at great cost, because the user of the proof assistant is no longer able to produce the proof of A using refinement, and must choose some witness before proceeding. This is disruptive to the interactive and incremental character of refinement proof, and we believe that the time is ripe to revisit this question.

2) Non-local unification: choosing barbarism: The most commonly adopted solution is to introduce a notion of *existential variable* and solve these non-locally in a proof via unification or spontaneous resolution. The basic idea is that you proceed with the proof of the second premise without yet knowing what M is, and then at a certain point, it will hopefully become obvious what it *must* be, on the basis of some goal in the subtree of this premise:

$$\frac{\begin{array}{cc} \overset{*}{?\mathfrak{m}} & N \\ ?\mathfrak{m} \in A & B[?\mathfrak{m}]\ \mathit{true}\end{array}}{\begin{array}{c}\langle ?\mathfrak{m}, N\rangle \\ (x : A) \times B[x]\ \mathit{true}\end{array}}$$

An essential characteristic of this approach is that when it is known what $?\mathfrak{m}$ must be, this knowledge propagates immediately to all nodes in the proof that mention it. This is a fundamentally *imperative* operation, and makes it very difficult to reason about proof scripts; moreover, it complicates the development of a clean and elegant semantics (denotational or otherwise) for proof refinement. At the very least, a fully formal presentation of the transition rules for such a system will be very difficult, both to state and to understand.

Another potential problem with using unification to solve goals is that one must be very cautious about how it is applied: using unification *uncritically* in the refinement apparatus can change the character of the object logic. One of the more destructive examples of this happening in practice was when overzealous use of unification in the Agda proof assistant [18] led to the injectivity of type constructors being derivable, whence the principle of the excluded middle could be refuted.

Uncritical use of unification in type theories like that of Nuprl might even lead to inconsistency, since greater care must be taken to negotiate the intricacies of subtyping and functionality which arise under the semantic typing discipline. As Cockx, Devriese and Piessens point out in their recent refit of Agda's unification theory, unification must be integrated closely with the object logic in order to ensure soundness [19].

3) Our solution: adopt a dependent refinement discipline: Taking a step back, there are two more things that should make us suspicious of the use of existential variables and unification in the above:

1) We *still* do not exhibit A using refinement rules, but rather simply hope that at some point, we shall happen upon (by chance) a suitable witness for which we can check membership. In many cases, it will be possible to inhabit $B[?\mathfrak{m}]$ regardless of whether $?\mathfrak{m}$ has any referent, which will leave us in much the same position as we were in Nuprl: we must cook up some arbitrary A independently without any help from the refinement rules.
2) It is a basic law of type-theoretic praxis that whatever structure exists at the propositional level should mirror a

form of construction that already exists at the judgmental level, adding to it only those characteristics which distinguish truth from the evidence of an arbitrary judgment (e.g. functionality, local character, fibrancy, etc.). In this case, the use of existential variables and unification seems to come out of nowhere, whereas we would expect a dependent sum at the type level to be defined in terms of some notion of dependent sum at the judgmental level.

With the above in mind, we are led to try and revise the old LCF apparatus to support a *dependent* refinement discipline, relaxing the constructible subgoals property in such a way as to admit a coding for the rule given at the beginning of this section, $\times_{\mathfrak{R}}$.

II. Survey of Related Works

A. Semantics of proof refinement in Nuprl

The most detailed denotational semantics for tactic-based proof refinement that we are aware of is contained in Todd Knoblock's PhD thesis [20] vis-à-vis the Nuprl refinement logic. Knoblock's purpose was to endow Nuprl with a tower of metalogics, each reflecting the contents of the previous ones, enabling internal reasoning about the proof refinement process itself; this involved specifying semantic domains for (reflected) Nuprl judgments, proofs and proof tactics in the Nuprl logic.

A detailed taxonomy of different forms of proof tactic was considered, including *search tactics* (analogous to *valid tactics* in LCF), *partial tactics* (tactics whose domain of applicability is circumscribed a priori), and *complete tactics* (partial tactics which produce no subgoals when applied in their domain of applicability).

Spiritually, our apparatus is most closely related to Knoblock's contributions, in light of the purely semantical and denotational approach which he pursued. The fact that Knoblock's semantic universe for proof refinement was the Nuprl logic itself enabled users of the system to prove internally the general effectiveness of a tactic on some class of Nuprl sequents once and for all; then, such a tactic could be applied without needing to be executed.

B. Isabelle as a meta-logical framework

Isabelle, a descendent of Cambridge LCF, is widely considered the gold standard in tactic-based proof refinement today; at its core, it is based on a version of Intuitionistic higher-order logic called Isabelle/Pure, which serves as the logical framework for all other Isabelle theories, including the famous Isabelle/HOL theory for classical higher-order logic.

In Isabelle, tactics generally operate on a full proof state (as opposed to the "local" style pioneered in LCF); a tactic is a partial function from proof states to *lazy sequences* of proof states. Note that the sequences here are used to accomodate sophisticated, possibly non-deterministic search schemata. In contrast to other members of the LCF family, the notion of *validation* has been completely eschewed; this has relieved Isabelle of the need to duplicate rules in both forward and backward inference, and simplifies the correctness conditions for a tactic.

Isabelle does not address the issue of dependent refinement, instead relying heavily on instantiation of schematic variables by higher-order unification. Because HOL is Isabelle's main theory, this is perhaps not so bad a state of affairs, since the most compelling uses of dependent refinement arise in proof systems for dependent type theory, in which a domain of discourse is identified with the inhabitants of a type or proposition. In non-type-theoretic approaches to logic, a proposition is proved using inference rules, whereas an element of the domain of discourse is constructed according to a *grammar*.

With that said, instantiation of schematic variables at higher type in Isabelle is not always best done by unification alone, and often requires manual guiding. A pathological case is the instantiation of predicates shaped like $?\mathfrak{m}[?\mathfrak{n}]$, where it is often difficult to proceed without further input [21, §4.2.2].

Under a dependent refinement discipline, however, the instantiation of schematic variables ranging over higher predicates can be pursued with the rules of the logic as a guide, in the same way that all other objects under consideration are constructed: in the validations of backward inference rules.

This insight, which is immanent in the higher-order propositions-as-types principle, is especially well-adapted for use in implementations of Martin-Löf-style dependent type theory, where it is often the case that the logic can guide the instantiation of a predicate variable in ways that pure unification cannot. This is essentially the difference between a direct and *algorithmic* treatment of synthetic judgment, and its *declarative* simulation as analytic judgment [22].

We stress that higher-order unification is an extremely useful technique, but it appears to *complement* rather than obviate a proper treatment of dependent refinement. Though it is not the topic of this paper, we believe that a combination of the two techniques would prove very fruitful.

C. OLEG & Epigram: tactics as admissible rules

Emanating from Conor McBride's dissertation work is a unique approach to proof elaboration which has been put to great use in several proof assistants for dependent type theory including OLEG, Epigram and Idris [23], [24], [25], [26], [27]. A more accessible introduction to McBride's methodology is given in [28].

Like ours, McBride's approach rests upon the specification of judgments in context which are stable under context substitution; crucially, McBride's apparatus was the first treatment of proof refinement to definitively rule out invalid scoping of schematic variables, a problem which plagued early implementations of "holes". In particular, McBride's framework is well suited to the development of type checkers, elaborators and type inferencers for formal type theories and programming languages.

One of the ways in which our contribution differs from McBride's system is that we treat rules of inference *algebraically*, i.e. as first-class entities in a semantic domain; then, following the LCF tradition, we develop a menagerie of combinators (tacticals) by which we can combine these

rules into composite proofs. In this sense, our development is a treatment of *derivability* relative to a trusted basis of (backward) inference rules.

McBride's approach is, on the contrary, to take the trusted basis of inference rules as given *ambiently* rather than algebraically, and then to develop a theory of proof tactic based on *admissibility* with respect to this basis theory.

Both approaches have been shown to be useful in practice, and we believe that it may be possible to view McBride's development calculi as particularly felicitous instantiations of our framework, which is very general and places few constraints on object theories.

D. Dependent subgoals in Coq 8.5

In his PhD thesis, Arnaud Spiwack addressed the lack of dependent refinement in the Coq proof assistant by redesigning its tactic apparatus [29]; Spiwack's efforts culminated in the release of Coq 8.5 in early 2016, which incorporates his new design, including support for "dependent subgoals" (which we call *dependent refinement*) and tactics that operate simultaneously on multiple goals (which we call *multitactics*).

Spiwack's work centered around a new formulation of LCF-style tactics which was powerful enough to support a number of useful features, including backtracking and subgoals expressing dependencies on other subgoals; the latter is effected through an imperative notion of existential variable (in contrast to the purely functional semantics for subgoal dependency that we give in this paper).

E. Our contributions

We take a very positive view of Spiwack's contributions in this area, especially in light of the successful concrete realization of his ideas in the Coq proof assistant. As far as engineering is concerned, we consider Spiwack to have *definitively* resolved the matter of dependent refinement for Coq.

At the same time, we believe that there is room for a mathematical treatment of dependent refinement which abstracts from the often complicated details of real-world implementations, and is completely decoupled from specific characteristics of a particular logic or proof assistant; our experience suggests that the development of a semantics for proof refinement along these lines can also lead to a cleaner, more reusable concrete realization.

Our contribution is a precise, compositional and purely functional semantics for dependent proof refinement which is also immediately suitable for implementation; our framework is called Dependent LCF, and our Standard ML implementation has already been used to great effect in the new **RedPRL** proof assistant for computational cubical type theory [7], [30].

III. Preliminaries

A. Lawvere Theories

We wish to study the algebraic structure of dependent proof refinement for a fixed language of constructions or evidence. To abstract away from the bureaucratic details of a particular encoding, we will work relative to some multi-sorted Lawvere theory $\mathbb{T}$, a strictly associative cartesian category whose objects can be viewed as sorts or contexts (finite products of sorts) and whose morphisms may be viewed as terms or substitutions.

Definition III.1 (Lawvere theory)**.** To define the notion of a multi-sorted Lawvere theory, we fix a set of sorts $\mathcal{S}$; let $\mathbb{T}_0$ be the free strict associative cartesian category on $\mathcal{S}$. Then, an $\mathcal{S}$-sorted Lawvere theory is a strictly associative cartesian category $\mathbb{T}$ equipped with a cartesian functor $\mathtt{k} : \mathbb{T}_0 \to \mathbb{T}$.

We will write $\Gamma, \Delta, \Xi : \mathbb{T}$ for the objects of $\mathbb{T}$ and $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} : \Gamma \Rightarrow \Delta$ for its morphisms. We will freely interchange "context" and "sort" (and "substitution" and "term") when one is more clear than the other. We will sometimes write $\Gamma, \mathtt{x} : \Delta$ for the context $\Gamma \times \Delta$, and then use $\mathtt{x}$ elsewhere as the canonical projection $\mathtt{p} : \Gamma \times \Delta \Rightarrow \Delta$.

Remark III.2 (Second-order theories). In the simplest case, a Lawvere theory $\mathbb{T}$ forms the category of contexts and substitutions for some *first-order* language. However, as Fiore and Mahmoud have shown, this machinery scales up perfectly well to the case of *second-order* theories (theories with binding) [31].

In that case, the objects are contexts of second-order variables associated to *valences* (a sort together with a list of sorts of bound variables), and the maps are second-order substitutions; when the output of a map is a single valence $\vec{\sigma}.\tau$, the map can be read as a term binder.

One of our reasons for specifying no more about $\mathbb{T}$ than we have done so far is to ensure that our apparatus generalizes well to the case of second-order syntax, which is what is necessary in nearly every concrete application of this work.

In what follows, we will often refer to variables as *schematic variables* in order to emphasize that these are variables which range over evidence in the proof refinement apparatus, as opposed to variables from the object logic. In the first-order case, all variables are schematic variables; in the second-order case, the second-order variables (called *metavariables* by Fiore et al) are the schematic variables, and the object variables are essentially invisible to our development.

B. Questions Concerning a Semantic Universe

Our main task is to define a semantic universe in which we can build objects indexed in $\mathbb{T}$, which respect substitutions of schematic variables. Some kind of presheaf category, then, seems to be what we want—and then proof refinement rules should be natural transformations in this presheaf category.

The question of which indexing category to choose is a subtle one; in order to construct our *proof states monad*, we will need to work with something like presheaves over $\mathbb{T}$, i.e. "variable sets" which implement all substitutions. However, most interesting *refinement rules* that we wish to define will not commute with substitutions. This corresponds to the fact that a refinement rule may fail to be applied if there is a schematic variable in a certain position, but may succeed if that variable is substituted for by some suitable term.

Essentially the same problem arises in the context of coalgebraic logic programming; several methods have been developed to deal with this behavior, including switching to an order-enriched semantic universe and using *lax natural transformations* for the operational semantics; another approach, called *saturation* involves trivializing naturality by treating $\mathbb{T}$ as a discrete category $|\mathbb{T}|$, and then "saturating" constructions along the adjunction $i^* : \widehat{\mathbb{T}} \to \widehat{|\mathbb{T}|} \dashv i_* : \widehat{|\mathbb{T}|} \to \widehat{\mathbb{T}}$. These techniques are discussed in [32], [33].

In the context of general dependent proof refinement, the lax semantics are the most convenient; we will apply a variation on this approach here.

Notation III.3. Following the notation of the French school [34, p. 25], we write $\widehat{\mathbb{X}}$ for the category of presheaves $\mathbb{SET}^{\mathbb{X}^{op}}$ on a category $\mathbb{X}$.

Notation III.4. We will write $\mathsf{Ctx} : \widehat{\mathbb{T}}$ for the constant presheaf of $\mathbb{T}$-objects, $\mathsf{Ctx}(\Gamma) \triangleq \mathsf{ob}(\mathbb{T})$. We may also write $\Gamma \Vdash X : F$ to mean $X \in F(\Gamma)$ when $F : \widehat{\mathbb{T}}$.

We will frequently have need for a presheaf of terms of an appropriate sort relative to a particular context, $(\Gamma \vdash \Delta) : \widehat{\mathbb{T}}$. This we can define informally as follows:

$$\frac{\Xi, \Gamma \vdash a : \Delta}{\Xi \Vdash a : (\Gamma \vdash \Delta)}$$

Formally, this is the exponential $\mathcal{H}(\Delta)^{\mathcal{H}(\Gamma)}$ with $\mathcal{H}(-) : \mathbb{T} \to \widehat{\mathbb{T}}$ the Yoneda embedding; this perspective is developed in Appendix **??**.

C. Presheaves and lax natural transformations

Let $\mathbb{SET}_{\preccurlyeq}$ be the order-enriched category whose objects are sets equipped with a partial order, and whose arrows are *all* maps between these sets; these arrows are then endowed with an order by pointwise approximation: $f \preccurlyeq g$ iff $\forall x. f(x) \preccurlyeq g(x)$.

Note especially that maps $f : \mathbb{SET}_{\preccurlyeq}(X, Y)$ are *not* required to be monotone. This choice may seem peculiar, especially in comparison with other work on lax semantics for logic programming, but we justify it in Section IV-D.

Definition III.5 (Presheaves and lax natural transformations)**.** A $\mathbb{SET}_{\preccurlyeq}$*-valued presheaf* on $\mathbb{C}$ is a functor from $\mathbb{C}^{op}$ into $\mathbb{SET}_{\preccurlyeq}$.

A *lax natural transformation* ϕ between two such presheaves P, Q is a collection of components whose naturality square commutes up to approximation in the following sense:

$$\begin{array}{ccc} P(d) & \xrightarrow{P(f)} & P(c) \\ {\scriptstyle \phi_d}\downarrow & \preccurlyeq & \downarrow{\scriptstyle \phi_c} \\ Q(d) & \xrightarrow[Q(f)]{} & Q(c) \end{array}$$

In other words, we need have only that $Q(f) \circ \phi_d \preccurlyeq \phi_c \circ P(f)$ in the above diagram.

We will write $\widehat{\mathbb{C}}_{lax}$ for the category of $\mathbb{SET}_{\preccurlyeq}$-valued presheaves and lax natural transformations on $\mathbb{C}$.

IV. A Framework for Proof Refinement

We will now proceed to develop the Dependent LCF theory by specifying the semantic objects under consideration, namely *judgment structures*, *proof states*, *refinement rules*, and *proof tactics*.

A. Judgment Structures

A *judgment* is an intention toward a particular form of construction; that is, a form of judgment is declared by specifying the $\mathbb{T}$-object (that is, sort or context) which classifies what it intends to construct. It is suggestive to consider this object the *output* of a judgment, in the sense that if the judgment is derived, it will emit a substitution of the appropriate sort which can be used elsewhere.

In the dependent proof refinement discipline, the statement of a judgment may be interrupted by a schematic variable, which ranges over the evidence of some other judgment, and may be substituted for by a term of the appropriate sort. This behavior captures the ubiquitous case of existential instantiation, where we have a predicate applied to a schematic variable which stands for an element of the domain of discourse, to be refined in the course of verifying another judgment.

To make this precise, we can define a notion of "judgment structure" as a collection of "judgments" which varies over contexts and substitutions, along with an assignment of sorts to judgments: the sort assigned to a judgment is intended to be the sort of the object that a judgment intends to construct. Then, a homomorphism between judgment structures would be a natural transformation of presheaves which preserves sort assignments.

Definition IV.1 (Judgment structures)**.** Formally, we define the category of judgment structures $\mathbb{J}$ on $\mathbb{T}$ as the slice category $\widehat{\mathbb{T}}_{lax}/\mathsf{Ctx}$; expanding definitions, an object $J : \mathbb{J}$ is a presheaf $J : \widehat{\mathbb{T}}_{lax}$ together with an assignment $\pi_J : J \to \mathsf{Ctx}$. Then, a map from J_0 to J_1 is a natural transformation that preserves π, in the sense that the following triangle commutes:

$$\begin{array}{ccc} J_0 & \xrightarrow{\phi} & J_1 \\ & {\scriptstyle \pi_{J_0}}\searrow \quad \swarrow{\scriptstyle \pi_{J_1}} & \\ & \mathsf{Ctx} & \end{array}$$

It will usually be most clear to define a judgment structure inductively in syntactic style, by specifying the (meta)judgment $\Gamma \Vdash X : J \rightsquigarrow \Delta$, pronounced "$X$ is a J-judgment in context Γ, constructing a substitution for Δ", which will mean $X \in J(\Gamma)$ and $\pi_J^\Gamma(X) = \Delta$.

Remark IV.2. It is easiest to understand this judgment in the special case where Δ is a unary context $x : \tau$; then the judgment means that the construction induced by the J-judgment X (i.e. it's "output") will be a term of sort τ; in general, we allow multiple outputs to a judgment, which corresponds to the case that Δ is a context with multiple elements.

The order on J-judgments will be specified using the (meta)judgment $\Gamma \Vdash X \preccurlyeq Y : J$ (pronounced "X approximates

Y as a J-judgment in context Γ"), which presupposes both $\Gamma \Vdash X : J \rightsquigarrow \Delta$ and $\Gamma \Vdash Y : J \rightsquigarrow \Delta$; unless otherwise specified, when defining a judgment structure, we usually assume the discrete order.

Example IV.3 (Cost dynamics of basic arithmetic)**.** A simple example of a judgment structure can be given by considering the cost dynamics for a small language of arithmetic expressions [35, Ch. 7.4].

We will fix two syntactic sorts, num and exp; num will be the sort of numerals, and exp will be the sort of arithmetic expressions; the Lawvere theory generated from these sorts and suitable operations that we define in Figure 1 will be called $\mathbb{A}$. We will write $\mathbb{J}_{\mathbb{A}}$ for the category of judgment structures over $\mathbb{A}$, namely the slice category $\widehat{\mathbb{A}_{lax}}/\mathsf{Ctxs}_{\mathbb{A}}$.

Then, we define a *judgment structure* $\mathsf{J}_{\mathbb{A}} : \mathbb{J}_{\mathbb{A}}$ for our theory by specifying the following forms of judgment:

1) $\mathsf{eval}(e)$ means that the arithmetic expression $e : \mathsf{exp}$ can be evaluated; its evidence is the numeral value of e and the cost k of evaluating e (i.e. the number of steps taken).
2) $\mathsf{add}(m; n)$ means that the numerals $m, n : \mathsf{num}$ can be added; the evidence of this judgment is the numeral which results from their addition.

The judgment structure $\mathsf{J}_{\mathbb{A}}$ summarized above is defined schematically in Figure 1.

We will use the above as our running example, and after we have defined a suitable notion of *refinement rule*, we will define the appropriate rules for the judgment structure $\mathsf{J}_{\mathbb{A}}$.

B. Telescopes and Proof States

1) Telescopes: An ordered sequence of judgments in which each induces a variable of the appropriate sort which the rest of the sequence may depend on is called a *telescope* [36]. The notion of a telescope will be the primary aspect in our definition of proof states later on, where it will specify the collection of judgments which still need to be proved.

Remark IV.4. If "dependent refinement" were replaced with "independent refinement", then the telescope data-structure could be replaced with lists. This design choice characterizes the LCF family of proof refinement apparatus.

We intend telescopes to themselves be an endofunctor on judgment structures, analogous to an iterated dependent sum; we will define the judgment structure endofunctor $\mathsf{Tl} : \mathbb{J} \to \mathbb{J}$ inductively.

For a judgment structure $J : \mathbb{J}$, a J-telescope is either $*$ (the empty telescope), or $x : X.\Psi$ with X a J-judgment and Ψ a J-telescope with x bound; the sort that is synthesized by a telescope is the product of the sorts synthesized by its constituent judgments. The precise rules for forming telescopes are given in Figure 2.

Note how in the above, we have used variable binding notation; formally, as can be seen in Appendix **??**, this corresponds to exponentiation by a representable functor, which is the usual way to account for variable binding in higher algebra. To be precise, given a presheaf $P : \widehat{\mathbb{T}}$ and a variable context $\Gamma : \mathbb{T}$, by exponentiation we can construct a new presheaf $P^{\mathcal{H}(\Gamma)} : \widehat{\mathbb{T}}$ (with $\mathcal{H}(-) : \mathbb{T} \to \widehat{\mathbb{T}}$ the Yoneda embedding) whose values are *binders* that close over the variables in Γ.

Henceforth, we are justified in adopting the *variable convention*, by which terms are identified up to renamings of their bound variables.

2) Proof States Monad: We define another endofunctor on judgment structures for proof states, $\mathsf{St} : \mathbb{J} \to \mathbb{J}$. Fixing a judgment structure $J : \mathbb{J}$, there are three ways to form a J-proof state:

1) When Ψ is a J-telescope that synthesizes sorts Ξ, and $\mathfrak{a}$ is a substitution from Ξ to Δ, then $\Psi \triangleright \mathfrak{a} :: \Delta$ is a proof state; as an object in a judgment structure, this proof state synthesizes Δ. Intuitively, Ψ is the collection of subgoals and $\mathfrak{a}$ is the *validation* which constructs evidence on the basis of the evidence for those subgoals. We will usually write $\Psi \triangleright \mathfrak{a}$ instead of $\Psi \triangleright \mathfrak{a} :: \Delta$ when it is clear from context.
2) For any Δ, $✠[\Delta]$ is a proof state that synthesizes Δ; this state represents *persistent failure*. We will always write $✠$ instead of $✠[\Delta]$.
3) For any Δ, $\bot[\Delta]$ is a proof state that synthesizes Δ; this state represents what we call *unsuccess*, or failure which may not be persistent. As above, we will always write $\bot$ instead of $\bot[\Delta]$.

Moreover, we impose the approximation ordering $\Gamma \Vdash \bot[\Delta] \preccurlyeq \Psi \triangleright \mathfrak{a} :: \Delta : \mathsf{St}(J)$ and $\Gamma \Vdash \bot[\Delta] \preccurlyeq ✠[\Delta] : \mathsf{St}(J)$.

The difference between *unsuccess* and *failure* is closely related to the difference between the statements "It is not the case that P is true" and "P is false" in constructive mathematics. In the context of proof refinement in the presence of schematic variables, it may be the case that a rule does not apply at first, but following a substitution, it does apply; capturing this case is the purpose of introducing the $\bot$ proof state.

Again, the precise rules for forming proof states are given in Figure 2.

Notation IV.5. We will write $\Psi \ldots \Psi'$ for the concatenation of two telescopes, where Ψ' may have variables from $\pi^{\Gamma}_{\mathsf{Tl}(J)}(\Psi)$ free. Likewise, we will write $\Psi' \ldots S$ to mean $\Psi' \ldots \Psi \triangleright \mathfrak{a}$ when $S \equiv \Psi \triangleright \mathfrak{a}$, and $✠$ when $S \equiv ✠$, and $\bot$ when $S \equiv \bot$.

We can instantiate a monad structure on proof states, which will abstractly implement the *identity* and *sequencing* tacticals from LCF.

$$\mathbf{1} \xrightarrow{\eta} \mathsf{St} \xleftarrow{\mu} \mathsf{St} \circ \mathsf{St}$$

The unit and multiplication operators are defined by the

$$\frac{\overline{n} \in \mathbb{N}}{\Gamma \vdash_{\mathbb{A}} \overline{n} : \mathsf{num}} \qquad \frac{\Gamma \vdash_{\mathbb{A}} n : \mathsf{num}}{\Gamma \vdash_{\mathbb{A}} \mathsf{num}[n] : \mathsf{exp}} \qquad \frac{\Gamma \vdash_{\mathbb{A}} e_1 : \mathsf{exp} \quad \Gamma \vdash_{\mathbb{A}} e_2 : \mathsf{exp}}{\Gamma \vdash_{\mathbb{A}} e_1 + e_2 : \mathsf{exp}}$$

$$\boxed{\mathsf{J}_{\mathbb{A}} : \mathbb{J}} \qquad \frac{\Gamma \vdash_{\mathbb{A}} e : \mathsf{exp}}{\Gamma \Vdash \mathsf{eval}(e) : \mathsf{J}_{\mathbb{A}} \rightsquigarrow x_c : \mathsf{num}, x_v : \mathsf{num}} \qquad \frac{\Gamma \vdash_{\mathbb{A}} m : \mathsf{num} \quad \Gamma \vdash_{\mathbb{A}} n : \mathsf{num}}{\Gamma \Vdash \mathsf{add}(m; n) : \mathsf{J}_{\mathbb{A}} \rightsquigarrow \mathsf{num}}$$

Fig. 1. An example theory $\mathbb{A}$ and judgment structure $\mathsf{J}_{\mathbb{A}}$.

$$\boxed{\mathsf{Tl} : \mathbb{J} \to \mathbb{J}} \qquad \frac{}{\Gamma \Vdash * : \mathsf{Tl}(\mathsf{J}) \rightsquigarrow \cdot} \qquad \frac{\Gamma \Vdash X : \mathsf{J} \rightsquigarrow \Delta \quad \Gamma, x : \Delta \Vdash \Psi_x : \mathsf{Tl}(\mathsf{J}) \rightsquigarrow \Xi}{\Gamma \Vdash x : X. \Psi_x : \mathsf{Tl}(\mathsf{J}) \rightsquigarrow x : \Delta, \Xi} \qquad \text{(Telescopes)}$$

$$\boxed{\mathsf{St} : \mathbb{J} \to \mathbb{J}} \qquad \frac{\Gamma \Vdash \Psi : \mathsf{Tl}(\mathsf{J}) \rightsquigarrow \Xi \quad \Gamma \Vdash \mathfrak{a} : (\Xi \vdash \Delta)}{\Gamma \Vdash (\Psi \triangleright \mathfrak{a} :: \Delta) : \mathsf{St}(\mathsf{J}) \rightsquigarrow \Delta} \qquad \frac{}{\Gamma \Vdash \maltese[\Delta] : \mathsf{St}(\mathsf{J}) \rightsquigarrow \Delta} \qquad \frac{}{\Gamma \Vdash \bot[\Delta] : \mathsf{St}(\mathsf{J}) \rightsquigarrow \Delta} \qquad \text{(Proof States)}$$

$$\frac{}{\Gamma \Vdash \bot[\Delta] \preccurlyeq \Psi \triangleright \mathfrak{a} :: \Delta : \mathsf{St}(\mathsf{J})} \qquad \frac{}{\Gamma \Vdash \bot[\Delta] \preccurlyeq \maltese[\Delta] : \mathsf{St}(\mathsf{J})}$$

$$\begin{aligned} \Psi \triangleright \mathfrak{a} &\triangleq \Psi \triangleright \mathfrak{a} :: \Delta \\ \maltese &\triangleq \maltese[\Delta] \\ \bot &\triangleq \bot[\Delta] \end{aligned} \qquad \text{(Notations)}$$

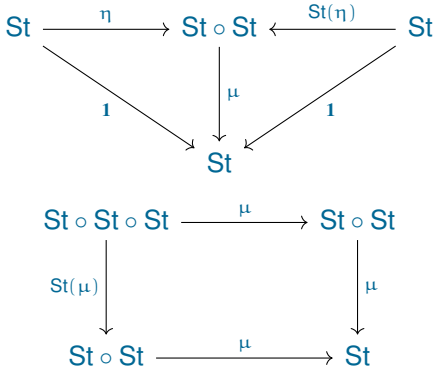
Fig. 2. Definitions of telescopes and proof states

following equations:

$$\begin{aligned} \eta_\Gamma(X) &= x : X. * \triangleright x \\ \mu_\Gamma(* \triangleright \mathfrak{a}) &= * \triangleright \mathfrak{a} \\ \mu_\Gamma(x : (\Psi_x \triangleright \mathfrak{a}_x). \underline{\Psi} \triangleright \mathfrak{a}) &= \Psi_x \ldots \mu_{\Gamma, \pi^\Gamma_{\mathsf{Tl}(\mathsf{J})}(\Psi_x)}(\underline{\Psi} \triangleright \mathfrak{a})[\mathfrak{a}_x/x] \\ \mu_\Gamma(x : \maltese. \underline{\Psi} \triangleright \mathfrak{a}) &= \maltese \\ \mu_\Gamma(x : \bot. \underline{\Psi} \triangleright \mathfrak{a}) &= \bot \\ \mu_\Gamma(\maltese) &= \maltese \\ \mu_\Gamma(\bot) &= \bot \end{aligned}$$

In the interest of clear notation, we have used Ψ to range over $\mathsf{Tl}(\mathsf{J})$ and $\underline{\Psi}$ to range over $\mathsf{Tl}(\mathsf{St}(\mathsf{J}))$.

Theorem IV.6. *Proof states form a monad on $\mathbb{J}$, i.e. the following diagrams commute:*



Proof. By nested induction; see Appendix **??**. □

C. Refinement Rules and Lax Naturality

We can now directly define the notions of *refinement rules*, *tactics* and *multitactics* as judgment structure homomorphisms.

Definition IV.7 (Refinement rules)**.** For judgment structures $\mathsf{J}_0, \mathsf{J}_1 : \mathbb{J}$, a *refinement rule* from J_0 to J_1 is a $\mathbb{J}$-homomorphism $\phi : \mathsf{Rule}(\mathsf{J}_0, \mathsf{J}_1) \triangleq \mathsf{J}_0 \to \mathsf{St}(\mathsf{J}_1)$. Unpacking definitions, ϕ is a *lax* natural transformation between the underlying presheaves of J and $\mathsf{St}(\mathsf{J}_1)$ which preserves the projection π.

Usually, one works with homogeneous refinement rules $\phi : \mathsf{J} \to \mathsf{St}(\mathsf{J})$, which can be called J-rules.

The ordered character of the $\mathsf{St}(\mathsf{J})$ judgment structure is crucial in combination with lax naturality; it is this which allows us to define a refinement rule which neither succeeds nor fails when it encounters a schematic variable that is blocking its applicability: that is, it does not commit to failure under all possible instantiations for that variable.

Full naturality would entail that a refinement rule commute with all substitutions from $\mathbb{T}$, whereas lax naturality only requires this square to commute up to approximation.

D. Why were maps in $\mathbb{SET}_{\preccurlyeq}$ not required to be monotone?

We can now explain in more detail the peculiarity that we chose the objects of $\mathbb{SET}_{\preccurlyeq}$ to be sets equipped with a partial order, but did not require maps between them to be monotone. In essence, the reason for this design choice is that we intend rules and proof tactics to be defined as families of $\mathbb{SET}_{\preccurlyeq}$-maps (subject to lax naturality), and *monotonicity is simply incompatible with the activity of tactic-based proof refinement*.

In particular, an essential aspect of tactic-based proof refinement in the presence of schematic variables is sensitivity to "unsuccess". One of the most common forms of tactic is one which tries to apply a collection of inference rules to a goal, by replacing one with the other in case it did not succeed.

Under a monotone tactic discipline, such a tactic would be permitted to branch on failure $\maltese$, but if a rule application was

merely "unsuccessful" (e.g. if the rule's applicability depended on the yet unknown referent of some schematic variable), this tactic would only be permitted to return the $\bot$ state.

This characteristic of monotone proof refinement is particularly destructive to proof automation, since the application of such a tactic would generally lead to most goals being reduced to $\bot$, a completely unrecoverable state of affairs. In order for our semantics to accurately model the activity of tactic-based proof automation, we must be able to retreat from the application of a tactic in case it has returned $\bot$.

Now, because we have not required monotonicity for maps in $\mathbb{SET}_{\preccurlyeq}$, it is possible to define tactic combinators which branch on either unsuccess or failure, or both. This facilitates a felicitous semantics for tactic-based proof automation in the presence of schematic variables.

Example IV.8 (Refinement rules for cost dynamics)**.** Resuming what we started in Example IV.3, we are now equipped to encode formal refinement rules for the judgment structure $\mathcal{J}_{\mathbb{A}} : \mathbb{J}$ defined in Figure 1.

We will implement the cost dynamics using two evaluation rules and one rule to implement the addition of numerals:

$$\begin{aligned} \mathbf{eval}_{\mathsf{num}} &: \mathcal{J}_{\mathbb{A}} \to \mathsf{St}(\mathcal{J}_{\mathbb{A}}) \\ \mathbf{eval}_{+} &: \mathcal{J}_{\mathbb{A}} \to \mathsf{St}(\mathcal{J}_{\mathbb{A}}) \\ \mathbf{add} &: \mathcal{J}_{\mathbb{A}} \to \mathsf{St}(\mathcal{J}_{\mathbb{A}}) \end{aligned}$$

First, let's consider what these rules would look like informally on paper, writing $e \Downarrow^{k} n$ for the statement that the judgment $\mathsf{eval}(e)$ obtains, synthesizing cost k and numeral n, and writing $m + n \equiv o$ for the statement that the judgment $\mathsf{add}(m;n)$ obtains, synthesizing numeral o:

$$\frac{}{\mathsf{num}[n] \Downarrow^{0} n}\ \mathbf{eval}_{\mathsf{num}}$$

$$\frac{\begin{array}{ll} e_1 \Downarrow^{k_1} n_1 & k_1 + k_2 \equiv k_{12} \\ e_2 \Downarrow^{k_2} n_2 & \overline{1} + k_{12} \equiv k \end{array} \qquad n_1 + n_2 \equiv n}{e_1 + e_2 \Downarrow^{k} n}\ \mathbf{eval}_{+}$$

$$\frac{}{\overline{m} + \overline{n} \equiv \overline{m+n}}\ \mathbf{add}$$

In keeping with standard practice and notation, in the informal definition of a refinement rule, clauses for failure and unsuccess are elided. When we code these rules as judgment structure homomorphisms, we add these clauses in the appropriate places, as can be seen from the formal definitions of $\mathbf{eval}_{\mathsf{num}}$, $\mathbf{eval}_{+}$ and **add** in Figure 3.

E. Tactics and Recursion

In keeping with standard usage, a *proof tactic* is a potentially diverging program that computes a proof on the basis of some collection of *refinement rules*. In order to define tactics precisely, we will first have to specify how we intend to interpret recursion.

Perhaps surprisingly, we will not be able to use the order-enriched apparatus to interpret recursion directly as in domain theory. The reason for this is that we will need to define tactic combinators which are not necessarily monotone or continuous with respect to a judgment structure's order; it is best to view the judgment structure's order as a device for accommodating unsuccessful (but not neccessarily failed) applications of rules in the presence of schematic variables.

However, a very lightweight way to interpret recursion is suggested by Capretta's *delay monad* [37] (the completely iterative monad on the identity functor), a coinductive representation of a process which *may eventually* return a value. We can define a variation on Capretta's construction as a monad $\infty : \mathbb{J} \to \mathbb{J}$ on judgment structures, defined as the *greatest* judgment structure closed under the rules in Figure 4.

To summarize, for a judgment structure $\mathcal{J} : \mathbb{J}$, there are two ways to construct a $\infty\mathcal{J}$-judgment:

1) $\lfloor X \rfloor$ is an $\infty\mathcal{J}$-judgment when X is a $\mathcal{J}$-judgment.
2) $\blacktriangleright X$ is an $\infty\mathcal{J}$-judgment when X is an $\infty\mathcal{J}$-judgment.

Lemma IV.9 (Delay monad)**.** $\infty : \mathbb{J} \to \mathbb{J}$ *forms a monad on* $\mathbb{J}$.

Notation IV.10. We will write $\eta_{\infty} : \mathbf{1}_{\mathbb{J}} \to \infty$ and $\mu_{\infty} : \infty \circ \infty \to \infty$ for the unit and multiplication operators respectively. We will also employ the following notational convention, inspired by the "do-notation" used in the Haskell programming language for monads:

$$x \leftarrow M; N(x) \triangleq \mu_{\infty}(\infty(x \mapsto N(x))(M))$$

Definition IV.11 (Tactics and multitactics)**.** A *tactic* for judgment structures $\mathcal{J}_0, \mathcal{J}_1 : \mathbb{J}$ is a $\mathbb{J}$-homomorphism $\phi : \mathsf{Tactic}(\mathcal{J}_0, \mathcal{J}_1) \triangleq \mathcal{J}_0 \to \infty\mathsf{St}(\mathcal{J}_1)$. Usually one works with homogeneous tactics $\phi : \mathsf{Tactic}(\mathcal{J}, \mathcal{J})$, which are called $\mathcal{J}$-tactics.

A $\mathcal{J}$-*multitactic* is a tactic for the judgment structure $\mathsf{St}(\mathcal{J})$.

F. Tacticals as Tactic Combinators

At this point we are equipped to begin defining a collection of standard "tacticals", or *tactic combinators*.

Note that the order enrichment which we have imposed on judgment structures is only for the purpose of enabling a merely procedural version of failure which is suitable for refinement rules in the presence of schematic variables; because we have not required arrows in $\mathbb{SET}_{\preccurlyeq}$ to be either monotone or continuous, we cannot hope to interpret fixed points using the underlying order.

1) Tactics from Rules: Every rule $\phi : \mathcal{J}_0 \to \mathsf{St}(\mathcal{J}_1)$ can be made into a tactic $\lfloor \phi \rfloor : \mathsf{Tactic}(\mathcal{J}_0, \mathcal{J}_1) \triangleq \eta_{\infty} \circ \phi$.

2) Conditional Tacticals: To begin with, we can define the join of two tactics $\phi, \psi : \mathsf{Tactic}(\mathcal{J}_0, \mathcal{J}_1)$, which implements **orelse** from LCF:

$$\begin{aligned} \phi \oplus \psi &: \mathsf{Tactic}(\mathcal{J}_0, \mathcal{J}_1) \\ (\phi \oplus \psi)_{\Gamma}(X) &\triangleq S \leftarrow \phi_{\Gamma}(X); \\ &\quad \begin{cases} \lfloor S \rfloor & \textbf{if } S = \Psi \triangleright \mathfrak{a} \\ \psi_{\Gamma}(X) & \textbf{otherwise} \end{cases} \end{aligned}$$

In $\phi \oplus \psi$ we have an example of a natural transformation whose components are not monotone with respect to the partial order on proof states; note that is irrelevant as far as recursion

$$
\begin{array}{rl|ll}
\mathbf{eval}^{\Gamma}_{\mathsf{num}} & \mathsf{eval}(\mathsf{num}[m]) & \Mapsto & * \triangleright [\overline{0}, m] \\
& \mathsf{eval}(x) & \Mapsto & \bot \\
& \_ & \Mapsto & \maltese \\
\hline
\mathbf{eval}^{\Gamma}_{+} & \mathsf{eval}(e_1 + e_2) & \Mapsto & \left[\begin{array}{l} [x_c, x_v] : \mathsf{eval}(e_1).\ [y_c, y_v] : \mathsf{eval}(e_2). \\ z_c : \mathsf{add}(x_c; y_c).\ z'_c : \mathsf{add}(\overline{1}; z_c).\ z_v : \mathsf{add}(x_v; y_v) \end{array}\right] \triangleright [z'_c, z_v] \\
& \mathsf{eval}(x) & \Mapsto & \bot \\
& \_ & \Mapsto & \maltese \\
\hline
\mathbf{add}_{\Gamma}(X) & \mathsf{add}(\overline{m}; \overline{n}) & \Mapsto & * \triangleright \overline{m+n} \\
& \mathsf{add}(\_;\_) & \Mapsto & \bot \\
& \_ & \Mapsto & \maltese
\end{array}
$$

Fig. 3. Defining refinement rules for $\mathsf{J}_{\mathbb{A}}$, the judgment structure of cost dynamics for arithmetic expressions.

$$
\boxed{\infty : \mathbb{J} \to \mathbb{J}} \qquad \frac{\Gamma \Vdash X : J \rightsquigarrow \Delta}{\Gamma \Vdash \lfloor X \rfloor : \infty J \rightsquigarrow \Delta} \qquad \frac{\Gamma \Vdash X : \infty J \rightsquigarrow \Delta}{\Gamma \Vdash \blacktriangleright_{\Delta} X : \infty J \rightsquigarrow \Delta} \qquad \text{(Delay Monad)}
$$

$$
\blacktriangleright X \triangleq \blacktriangleright_{\Delta} X \qquad \text{(Notation)}
$$

Fig. 4. Some auxiliary judgment structures, including Capretta's coinductive delay monad, the fiberwise product of two judgment structures, and the countable fiberwise product of a judgment structure with itself.

is concerned, since the latter is orthogonal to this partial order (recall that we are using ∞ to capture recursion).

In combination with the identity tactic $\mathbf{id} : \mathsf{Tactic}(J, J) \triangleq \lfloor \eta \rfloor$, we can define the **try** tactical which replaces a failure with the identity:

$$
\begin{aligned}
&\mathbf{try}(\phi) : \mathsf{Tactic}(J, J) \\
&\mathbf{try}(\phi) \triangleq \phi \oplus \mathbf{id}
\end{aligned}
$$

3) Multitacticals: We will factor the LCF sequencing tacticals **then** and **thenl** into a combination of a "multitactical" (a tactic that operates on $\mathsf{St}(J)$ instead of J) and a generic sequencing operation.

These multitacticals will be factored through tacticals that are sensitive to the *position* of a goal within a proof state, namely **all** and **each**. First, fixing a judgment structure $J : \mathbb{J}$, let us define a new judgment structure $J_{\times\mathbb{N}} : \mathbb{J}$ whose judgments are J-judgments labeled with a position:

$$
\boxed{J_{\times\mathbb{N}} : \mathbb{J}} \qquad \frac{\Gamma \Vdash X : J \rightsquigarrow \Delta \quad i \in \mathbb{N}}{\Gamma \Vdash \langle X, i \rangle : J_{\times\mathbb{N}} \rightsquigarrow \Delta}
$$

Let ϕ range over tactics $\mathsf{Tactic}(J, J)$, and let $\vec{\phi}$ range over a list of such tactics. We will now define some further tactics which work over labeled judgments:

$$
\begin{aligned}
&\mathbf{all}(\phi) : \mathsf{Tactic}(J_{\times\mathbb{N}}, J) \\
&\mathbf{all}(\phi)_{\Gamma}\langle X, i \rangle \triangleq \phi_{\Gamma}(X)
\end{aligned}
$$

$$
\begin{aligned}
&\mathbf{each}(\vec{\phi}) : \mathsf{Tactic}(J_{\times\mathbb{N}}, J) \\
&\mathbf{each}(\vec{\phi})_{\Gamma}\langle X, i \rangle \triangleq \begin{cases} \phi^{\Gamma}_i(X) & \textbf{if } i < |\vec{\phi}| \\ \lfloor \eta_{\Gamma}(X) \rfloor & \textbf{otherwise} \end{cases}
\end{aligned}
$$

Now, we need to show how to turn transform an operation on labeled judgments into a *multitactic*. We will need an operation to turn a telescope of potentially diverging computations into a potentially diverging computation of a telescope:

$$
\begin{array}{rl|ll}
\multicolumn{4}{l}{\mathsf{await} : \mathsf{Tl}(\infty J) \to \infty \mathsf{Tl}(J)} \\
\mathsf{await}_{\Gamma} & * & \Mapsto & \lfloor * \rfloor \\
& x : X^{\infty}.\, \Psi^{\infty} & \Mapsto & X \leftarrow X^{\infty}; \\
& & & \Psi \leftarrow \mathsf{await}_{\Gamma, x:\pi_J(X)}(\Psi^{\infty}); \\
& & & \lfloor x : X.\, \Psi \rfloor
\end{array}
$$

Next, define an operation to label the nodes of a telescope with their position:

$$
\begin{array}{rl|ll}
\multicolumn{4}{l}{\mathbf{lbl}_i : \mathsf{Tl}(J) \to \mathsf{Tl}(J_{\times\mathbb{N}})} \\
\mathbf{lbl}^{\Gamma}_i & * & \Mapsto & * \\
& x : X.\, \Psi & \Mapsto & x : \langle X, i \rangle.\, \mathbf{lbl}^{\Gamma, x:\pi^{\Gamma}_J(X)}_{i+1}(\Psi)
\end{array}
$$

Using this, we can transform χ into a multitactic $\mathsf{St}\{\chi\} : \mathsf{Tactic}(\mathsf{St}(J), \mathsf{St}(J))$:

$$
\begin{array}{rl|ll}
\multicolumn{4}{l}{\mathsf{St}\{\chi\} : \mathsf{Tactic}(\mathsf{St}(J), \mathsf{St}(J))} \\
\mathsf{St}\{\chi\}_{\Gamma} & \bot & \Mapsto & \lfloor \bot \rfloor \\
& \maltese & \Mapsto & \lfloor \maltese \rfloor \\
& \Psi \triangleright a & \Mapsto & \Psi' \leftarrow \mathsf{await}_{\Gamma}(\mathsf{Tl}(\chi)_{\Gamma}(\mathbf{lbl}^{\Gamma}_0(\Psi))); \\
& & & \lfloor \Psi' \triangleright a \rfloor
\end{array}
$$

4) Generic Sequencing: Fixing a tactic $\phi : \mathsf{Tactic}(J_0, J_1)$ and $\underline{\psi} : \mathsf{Tactic}(\mathsf{St}(J_1), \mathsf{St}(J_2))$ as above, we can define the *sequencing* of $\underline{\psi}$ after ϕ as the following composite, also displayed in Figure 5:

$$
\begin{aligned}
&\mathbf{seq}(\phi, \underline{\psi}) : \mathsf{Tactic}(J_0, J_2) \\
&\mathbf{seq}(\phi, \underline{\psi}) \triangleq \infty(\mu) \circ \mu_{\infty} \circ \infty(\underline{\psi}) \circ \phi
\end{aligned}
$$

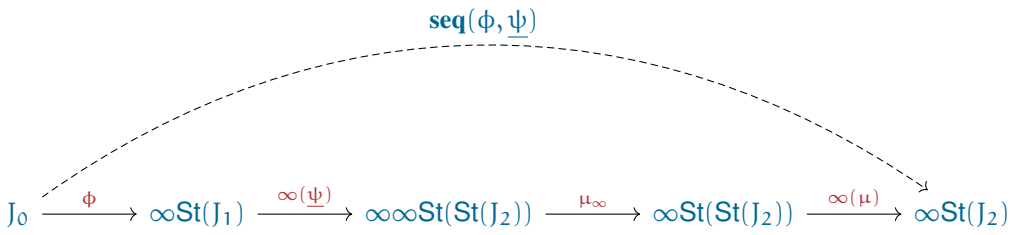


Fig. 5. The generic sequencing tactical displayed as a composite.

Now observe that the LCF sequencing tacticals can be defined in terms of the above combinators:

$$\mathbf{then}(\phi, \psi) \triangleq \mathbf{seq}(\phi, \mathsf{St}\{\mathbf{all}(\psi)\})$$
$$\mathbf{thenl}(\phi, \vec{\psi}) \triangleq \mathbf{seq}(\phi, \mathsf{St}\{\mathbf{each}(\vec{\psi})\})$$

5) Recursive Tacticals: We will now demonstrate how to take the fixed point of a tactical using Capretta's delay monad (Figure 4), and use it to construct the commonly-used **repeat** tactical from LCF. To begin with, we will need to define fiberwise products and ω-sequences as operations on judgment structures. The fiberwise product of two judgment structures is a judgment structure whose judgments consist in pairs of judgments that synthesize the same sort Δ:

$$\boxed{\otimes : \mathbb{J} \times \mathbb{J} \to \mathbb{J}} \qquad \frac{\Gamma \Vdash X : J_0 \rightsquigarrow \Delta \quad \Gamma \Vdash Y : J_1 \rightsquigarrow \Delta}{\Gamma \Vdash \langle X, Y \rangle : J_0 \otimes J_1 \rightsquigarrow \Delta}$$
(Fiberwise Product)

This is just the pullback of the two judgment structures when viewed as objects in the slice category $\widehat{\mathbb{T}}_{lax}/\mathsf{Ctx}$:

$$\begin{array}{ccc} J_0 \otimes J_1 & \longrightarrow & J_0 \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \pi_{J_0}} \\ J_1 & \xrightarrow{\pi_{J_1}} & \mathsf{Ctx} \end{array}$$

Next, we define an endofunctor on judgment structures that takes infinite sequences of judgments:

$$\boxed{-^{\omega} : \mathbb{J} \to \mathbb{J}} \qquad \frac{\Gamma \Vdash X : J_n \rightsquigarrow \Delta \quad (n \in \mathbb{N})}{\Gamma \Vdash \langle X_n \mid n \rangle : J^{\omega} \rightsquigarrow \Delta}$$
(ω-Sequence)

This too can be presented as the limit $\varprojlim_{n \in \mathbb{N}} J^n$ where J^n is the n-fold fiberwise product of J with itself.

Now that we have defined the objects we require, we will begin to define the operations by which we can take the fixed point of a tactical, following [37]. First, we have a way to "race" two delayed judgments, returning the one which resolves first:

$$\mathbf{race} : \infty J \otimes \infty J \to \infty J$$
$$\mathbf{race}_\Gamma \left| \begin{array}{lll} \langle \lfloor X \rfloor, Y_\infty \rangle & \Rrightarrow & \lfloor X \rfloor \\ \langle \blacktriangleright X_\infty, \lfloor Y \rfloor \rangle & \Rrightarrow & \lfloor Y \rfloor \\ \langle \blacktriangleright X_\infty, \blacktriangleright Y_\infty \rangle & \Rrightarrow & \blacktriangleright\, \mathbf{race}_\Gamma \langle X_\infty, Y_\infty \rangle \end{array} \right.$$

This construction can be lifted to an ω-sequence of judgments as follows:

$$\mathbf{search}_n : (\infty J)^{\omega} \otimes \infty J \to \infty J$$
$$\mathbf{search}_n^\Gamma \left| \begin{array}{lll} \langle F, \lfloor X \rfloor \rangle & \Rrightarrow & \lfloor X \rfloor \\ \langle F, \blacktriangleright X_\infty \rangle & \Rrightarrow & \blacktriangleright\, \mathbf{search}_{n+1}^\Gamma \langle F, \mathbf{race}_\Gamma \langle X_\infty, F_n \rangle \rangle \end{array} \right.$$

In the delay monad, it is possible to define an object which never resolves:

$$\mathbf{never}_\Gamma : \infty J(\Gamma)$$
$$\mathbf{never}_\Gamma = \blacktriangleright\, \mathbf{never}_\Gamma$$

Now, using this and our unbounded search operator, we can take the least upper bound of an ω-sequence of judgments:

$$\sqcup : (\infty J)^{\omega} \to \infty J$$
$$\sqcup_\Gamma(F) \triangleq \mathbf{search}_0^\Gamma \langle F, \mathbf{never}_\Gamma \rangle$$

Now fix a tactical $T : \mathsf{Tactic}(J, J) \to \mathsf{Tactic}(J, J)$; it is now easy to get the fixed point of T (if it exists) by taking the least upper bound of a sequence of increasingly many applications of T to itself:

$$\mathbf{fix}(T) : \mathsf{Tactic}(J, J)$$
$$\mathbf{fix}(T)_\Gamma(X) \triangleq \sqcup \langle T_\Gamma^n(X) \mid n \rangle$$

where

$$T^n : \mathsf{Tactic}(J, J)$$
$$T_\Gamma^0 = X \mapsto \mathbf{never}_\Gamma$$
$$T_\Gamma^{n+1} = T_\Gamma(T_\Gamma^n)$$

In practice, the following **repeat** tactical from LCF is a very useful example of a fixed point:

$$\mathbf{repeat}(\phi) : \mathsf{Tactic}(J, J)$$
$$\mathbf{repeat}(\phi) \triangleq \mathbf{fix}(\psi \mapsto \mathbf{try}(\mathbf{then}(\phi, \psi)))$$

Other recursive tacticals are possible, including recursive multitacticals.

Example IV.12 (Tactic for cost dynamics)**.** Returning to our running example (Examples IV.3, IV.8), we can now define a

useful tactic to discharge all $\mathrm{J}_{\mathbb{A}}$-judgments; our first cut can be defined in the following way:

$$\begin{aligned}&\mathbf{auto}_{aux} : \mathsf{Tactic}(\mathrm{J}_{\mathbb{A}}, \mathrm{J}_{\mathbb{A}})\\ &\mathbf{auto}_{aux} \triangleq \lfloor \mathbf{eval}_{\mathsf{num}} \rfloor \oplus \lfloor \mathbf{eval}_{+} \rfloor \oplus \lfloor \mathbf{add} \rfloor\end{aligned}$$

$$\begin{aligned}&\mathbf{auto} : \mathsf{Tactic}(\mathrm{J}_{\mathbb{A}}, \mathrm{J}_{\mathbb{A}})\\ &\mathbf{auto} \triangleq \mathbf{repeat}(\mathbf{auto}_{aux}) \qquad (*)\end{aligned}$$

This is not quite, however, what we want: the force of this tactic is to run all our rules repeatedly (until failure or completion) on each subgoal. This is fine, but because these processes are taking place independently on each subgoal, the instantiations induced in one subgoal will not propagate to an adjacent subgoal until the entire process has quiesced.

The practical result of this approach is that the **auto** tactic will terminate with unresolved subgoals, and must be run again; our intention was, however, for the tactic to discharge all subgoals through repetition.

What we defined above can be described as *depth-first repetition*; what we want is *breadth-first repetition*, in which we run all the rules once on each subgoal, repeatedly. Then, substitutions propagate along the subgoals telescope with *every* application of $\mathbf{auto}_{aux}$, instead of propagating only after *all* applications of $\mathbf{auto}_{aux}$.

The way to achieve this is to apply our repetition at the level of multitactics, instantiating the tactical as $\mathbf{repeat} : \mathsf{Tactic}(\mathsf{St}(\mathrm{J}_{\mathbb{A}}), \mathsf{St}(\mathrm{J}_{\mathbb{A}})) \to \mathsf{Tactic}(\mathsf{St}(\mathrm{J}_{\mathbb{A}}), \mathsf{St}(\mathrm{J}_{\mathbb{A}}))$ instead of $\mathbf{repeat} : \mathsf{Tactic}(\mathrm{J}_{\mathbb{A}}, \mathrm{J}_{\mathbb{A}}) \to \mathsf{Tactic}(\mathrm{J}_{\mathbb{A}}, \mathrm{J}_{\mathbb{A}})$. This we can accomplish as follows:

$$\begin{aligned}&\mathbf{auto}_{multi} : \mathsf{Tactic}(\mathsf{St}(\mathrm{J}_{\mathbb{A}}), \mathsf{St}(\mathrm{J}_{\mathbb{A}}))\\ &\mathbf{auto}_{multi} \triangleq \mathbf{repeat}(\mathbf{all}(\mathbf{auto}_{aux}))\end{aligned}$$

$$\begin{aligned}&\mathbf{auto} : \mathsf{Tactic}(\mathrm{J}_{\mathbb{A}}, \mathrm{J}_{\mathbb{A}})\\ &\mathbf{auto} \triangleq \mathbf{seq}(\mathbf{id}, \mathbf{auto}_{multi})\end{aligned}$$

V. Concrete Implementation in Standard ML

As part of the **RedPRL** project [7], we have built a practical implementation of the apparatus described above in the Standard ML programming language [38].[1]

RedPRL is an interactive proof assistant in the Nuprl tradition for computational cubical type theory [30], a higher dimensional variant of Martin-Löf's extensional type theory [39], [11]. Replacing LCF with Dependent LCF has enabled us to eliminate every last disruption to the proof refinement process in **RedPRL**'s refinement logic, including the introduction rule for dependent sums (as described in Section I-B), and dually, the elimination rule for dependent products.

Dependent LCF has also sufficed for us as a matrix in which to develop sophisticated type synthesis rules à la bidirectional typing, which has greatly simplified the proof obligations routinely incurred in an elaborator or refiner for extensional type theory, without needing to develop brittle and complex tactics for this purpose as was required in the Nuprl System.

Our experience suggests that, contrary to popularly-accepted folk wisdom, practical and usable implementations of extensional type theory are eminently possible, assuming that a powerful enough form of proof refinement apparatus is adopted.

[1] Our implementation is open-source and available at http://www.github.com/RedPRL/sml-dependent-lcf.

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References

[1] R. Milner, "Logic for computable functions: Description of a machine implementation." Stanford University, Tech. Rep., 1972. 1

[2] M. Gordon, R. Milner, and C. Wadsworth, "Edinburgh LCF: A mechanized logic of computation," in *Lecture Notes in Computer Science*, vol. 78. Springer-Verlag, 1979. 1

[3] M. Gordon, "From LCF to HOL: A short history," in *Proof, Language, and Interaction*, G. Plotkin, C. Stirling, and M. Tofte, Eds. Cambridge, MA, USA: MIT Press, 2000, pp. 169–185. [Online]. Available: http://dl.acm.org/citation.cfm?id=345868.345890 1

[4] R. L. Constable, S. F. Allen, H. M. Bromley, W. R. Cleaveland, J. F. Cremer, R. W. Harper, D. J. Howe, T. B. Knoblock, N. P. Mendler, P. Panangaden, J. T. Sasaki, and S. F. Smith, *Implementing Mathematics with the Nuprl Proof Development System*. Upper Saddle River, NJ, USA: Prentice-Hall, Inc., 1986. 1

[5] J. J. Hickey, "The MetaPRL logical programming environment," Ph.D. dissertation, Cornell University, Ithaca, NY, January 2001. 1

[6] J. Sterling, D. Gratzer, and V. Rahli, "JonPRL," 2015, http://www.jonprl.org/. 1

[7] J. Sterling, D. Gratzer, D. Christiansen, D. Morrison, E. Akentyev, and J. Wilcox, "RedPRL – the People's Refinement Logic," 2016, http://www.redprl.org/. 1, 4, 11

[8] The Coq Development Team, "The Coq Proof Assistant Reference Manual," 2016. 1

[9] L. de Moura, S. Kong, J. Avigad, F. van Doorn, and J. von Raumer, *The Lean Theorem Prover (System Description)*. Cham: Springer International Publishing, 2015, pp. 378–388. [Online]. Available: http://dx.doi.org/10.1007/978-3-319-21401-6_26 1

[10] L. C. Paulson, *Logic and computation : interactive proof with Cambridge LCF*, ser. Cambridge tracts in theoretical computer science. Cambridge, New York, Port Chester: Cambridge University Press, 1987, autre tirage : 1990 (br.). [Online]. Available: http://opac.inria.fr/record=b1086426 1

[11] P. Martin-Löf, *Intuitionistic type theory*, ser. Studies in Proof Theory. Bibliopolis, 1984, vol. 1. 1, 11

[12] V. Rahli and M. Bickford, "A nominal exploration of Intuitionism," in *Proceedings of the 5th ACM SIGPLAN Conference on Certified Programs and Proofs*, ser. CPP 2016. New York, NY, USA: ACM, 2016, pp. 130–141. 1

[13] R. L. Constable, "The semantics of evidence," Cornell University, Tech. Rep. TR85-684, 1985. 1

[14] ——, "Evidence semantics and refinement rules for first-order logics: Minimal, intuitionistic, and classical," 2012, course notes. 1

[15] P. Martin-Löf, "Truth of a proposition, evidence of a judgement, validity of a proof," *Synthese*, vol. 73, no. 3, pp. 407–420, 1987. 2

[16] D. Prawitz, "Truth and proof in intuitionism," in *Epistemology versus Ontology: Essays on the Philosophy and Foundations of Mathematics in Honour of Per Martin-Löf*, P. Dybjer, S. Lindström, E. Palmgren, and G. Sundholm, Eds. Dordrecht: Springer Netherlands, 2012, pp. 45–67. 2

[17] R. W. Harper, "Aspects of the implementation of type theory," Ph.D. dissertation, Cornell University, 1985. 2

[18] U. Norell, "Dependently typed programming in agda," in *Proceedings of the 4th International Workshop on Types in Language Design and Implementation*, ser. TLDI '09. New York, NY, USA: ACM, 2009, pp. 1–2. 2

[19] J. Cockx, D. Devriese, and F. Piessens, "Unifiers as equivalences: Proof-relevant unification of dependently typed data," in *ACM SIGPLAN International Conference on Functional Programming (ICFP 2016)*. ACM, September 2016. [Online]. Available: https://lirias.kuleuven.be/handle/123456789/544210 2

[20] T. B. Knoblock, "Metamathematical extensibility in type theory," Ph.D. dissertation, Cornell University, Ithaca, NY, USA, 1988. 3

[21] M. Wenzel, F. Haftmann, and L. Paulson, "The isabelle/isar implementation," 2016. 3

[22] P. Martin-Löf, *Analytic and Synthetic Judgements in Type Theory*. Dordrecht: Springer Netherlands, 1994, pp. 87–99. [Online]. Available: http://dx.doi.org/10.1007/978-94-011-0834-8_5 3

[23] C. McBride, "Dependently typed programs and their proofs," Ph.D. dissertation, University of Edinburgh, 1999. 3

[24] ——, "Epigram: Practical programming with dependent types," in *Proceedings of the 5th International Conference on Advanced Functional Programming*, ser. AFP'04. Berlin, Heidelberg: Springer-Verlag, 2005, pp. 130–170. [Online]. Available: http://dx.doi.org/10.1007/11546382_3 3

[25] E. Brady, "Practical Implementation of a Dependently Typed Functional Programming Language," Ph.D. dissertation, Durham University, 2005. 3

[26] ——, "Idris, a general-purpose dependently typed programming language: Design and implementation," *Journal of Functional Programming*, vol. 23, no. 5, p. 552593, Sep 2013. 3

[27] D. Christiansen and E. Brady, "Elaborator reflection: Extending Idris in Idris," in *Proceedings of the 21st ACM SIGPLAN International Conference on Functional Programming*, ser. ICFP 2016. New York, NY, USA: ACM, 2016, pp. 284–297. [Online]. Available: http://doi.acm.org/10.1145/2951913.2951932 3

[28] A. Gundry, C. McBride, and J. McKinna, "Type inference in context," in *Proceedings of the Third ACM SIGPLAN Workshop on Mathematically Structured Functional Programming*, ser. MSFP '10. New York, NY, USA: ACM, 2010, pp. 43–54. [Online]. Available: http://doi.acm.org/10.1145/1863597.1863608 3

[29] A. Spiwack, "Verified computing in homological algebra, a journey exploring the power and limits of dependent type theory," Ph.D. dissertation, École Polytechnique, 2011. 4

[30] C. Angiuli, R. Harper, and T. Wilson, "Computational higher-dimensional type theory," in *43nd ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages (POPL)*. ACM, Jan. 2016. 4, 11

[31] M. Fiore and O. Mahmoud, "Second-order algebraic theories," in *Mathematical Foundations of Computer Science 2010: 35th International Symposium, MFCS 2010, Brno, Czech Republic, August 23-27, 2010. Proceedings*, P. Hliněný and A. Kučera, Eds. Berlin, Heidelberg: Springer Berlin Heidelberg, 2010, pp. 368–380. 4

[32] F. Bonchi and F. Zanasi, *Saturated Semantics for Coalgebraic Logic Programming*. Berlin, Heidelberg: Springer Berlin Heidelberg, 2013, pp. 80–94. 5

[33] E. Komendantskaya and J. Power, "Category theoretic semantics for theorem proving in logic programming: embracing the laxness," in *Proceedings of Coalgebraic Methods in Computer Science*, ser. Lecture Notes in Computer Science, I. Hasuo, Ed. Springer, June 2016, pp. 94–113. [Online]. Available: http://opus.bath.ac.uk/49979/ 5

[34] S. Mac Lane and I. Moerdijk, *Sheaves in geometry and logic : a first introduction to topos theory*, ser. Universitext. New York: Springer, 1992. 5

[35] R. Harper, *Practical Foundations for Programming Languages*. New York, NY, USA: Cambridge University Press, 2016. 6

[36] N. de Bruijn, "Telescopic mappings in typed lambda calculus," *Information and Computation*, vol. 91, no. 2, pp. 189 – 204, 1991. [Online]. Available: http://www.sciencedirect.com/science/article/pii/089054019190066B 6

[37] V. Capretta, "General recursion via coinductive types," *Logical Methods in Computer Science*, vol. 1, no. 2, pp. 1–18, 2005. [Online]. Available: http://www.lmcs-online.org/ojs/viewarticle.php?id=55 8, 10

[38] R. Milner, M. Tofte, R. Harper, and D. Macqueen, *The Definition of Standard ML (Revised)*. MIT Press, 1997. 11

[39] P. Martin-Löf, "Constructive mathematics and computer programming," in *6th International Congress for Logic, Methodology and Philosophy of Science*, Hanover, August 1979, pp. 153–175, published by North Holland, Amsterdam. 1982. 11