Guarded Computational Type Theory

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what is $\mathrm{CTT}_{\circlearrowleft}$?

a ***dependently typed*** program logic with guarded recursion, clocks, and coinductive types

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- ⋆ simple programming language (just λ-calculus)

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- ⋆ operational account of guarded recursion
- ⋆ immediate canonicity result
- ⋆ simple programming language (just λ-calculus)
- ⋆ defined by “operational sheaf model”

what is guarded recursion?

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$$\frac{M \in \triangleright(A \to B) \quad N \in \triangleright A}{M \circledast N \in \triangleright B}$$

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Solve “guarded recursive equations”:

$$\mathsf{Stream} \cong \mathbb{N} \times \triangleright \mathsf{Stream}$$

semantics in the topos of trees

internal (extensional) type theory of $\mathcal{S} \triangleq \widehat{\omega}$.

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(Easy to interpret $\mathsf{fix}(x.M)$.)

causal (co)programming

the “later modality” $\triangleright$ enables ***causal*** programs on streams

$$\begin{aligned}
&\mathsf{incr} : \mathsf{Stream} \to \mathsf{Stream} \\
&\mathsf{incr} \triangleq \mathsf{fix}(F.\ \lambda\langle n, \alpha\rangle.\ \langle n+1, F \circledast \alpha\rangle)
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guarded recursion not enough for (co)programming!

local clocks for coinduction

idea: parameterize $\triangleright$ by a ***clock*** κ and add quantifier $\forall\kappa$.
due to McBride, Atkey

$$\frac{\Gamma \vdash_{\Delta,\kappa} M \in A[\kappa]}{\Gamma \vdash_{\Delta} \Lambda\kappa.M \in \forall\kappa.A[\kappa]} \qquad \frac{\Gamma \vdash_{\Delta} M \in \forall\kappa.A[\kappa] \quad \kappa \# \Delta}{\Gamma \vdash_{\Delta} M[\kappa'] \in A[\kappa']}$$

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local clocks for coinduction

require the following isomorphisms:

$$\forall\kappa. \triangleright_\kappa A \cong \forall\kappa.A$$
$$\forall\kappa.A \cong A \quad (\kappa \# A)$$
$$\forall\kappa.A \times B \cong (\forall\kappa.A) \times (\forall\kappa.B)$$

etc.

$\triangleright_\kappa$ + $\forall\kappa$ = coindu
ction!

Define streams in two parts (decompose productivity into causality and free use):

$$\mathsf{Stream}_\kappa \cong \mathbb{N} \times \triangleright_\kappa \mathsf{Stream}_\kappa$$

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intensional version?

Both computation rule for $\mathsf{fix}_\kappa(x.M)$ and extensionality rule for next_κ incompatible with standard intensional type theory.

Birkedal et al resolve both problems in Guarded Cubical Type Theory. (Conjectured decidable typing result.)

canonicity currently unknown; no operational semantics
(but stay tuned)

resurrecting extensionality

In $\mathbf{CTT}_{\circlearrowleft}$ we contribute a version of Guarded Dependent Type Theory with simple operational semantics, immediate canonicity result, simpler term language, and new type equations.

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Hope to combine with Computational Higher Dimensional Type Theory (Angiuli, Harper, Wilson). Guarded recursion will be the first of many extensions.

preview of CTT$_{\text{🕒}}$

$$\frac{\Gamma \gg_{\Delta} M \in A}{\Gamma \gg_{\Delta} M \in \triangleright_{\kappa} A} \qquad \frac{\Gamma \gg_{\Delta} M \in \triangleright_{\kappa}(\Pi x : A.B[x]) \quad \Gamma \gg_{\Delta} M \in \triangleright_{\kappa} A}{\Gamma \gg_{\Delta} M(N) \in \triangleright_{\kappa} B[N]}$$

(look ma, no delayed substitutions!)

$$\frac{\Gamma, x : \triangleright_{\kappa} A \gg_{\Delta} M \in A}{\Gamma \gg_{\Delta} \mathsf{fix}(x.M) \in A}$$

preview of CTT$_{\circ}$

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preview of CTT$_{⏲}$

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use extensionality as a weapon!

$$\cap\kappa.\triangleright_{\kappa} A \doteq \cap\kappa.A \ \ \mathsf{type}$$

$$\cap\kappa.A \doteq A \ \ \mathsf{type} \quad (\kappa \# A)$$

$$\cap\kappa.A \times B \doteq (\cap\kappa.A) \times (\cap\kappa.B) \ \ \mathsf{type}$$

topos semantics

We develop the computational PER semantics (meaning explanation) for **CTT**$_{\circledcirc}$ internally to a suitable ***sheaf topos*** which provides all the machinery that we need ($\triangleright_\kappa$,$\forall\kappa$).

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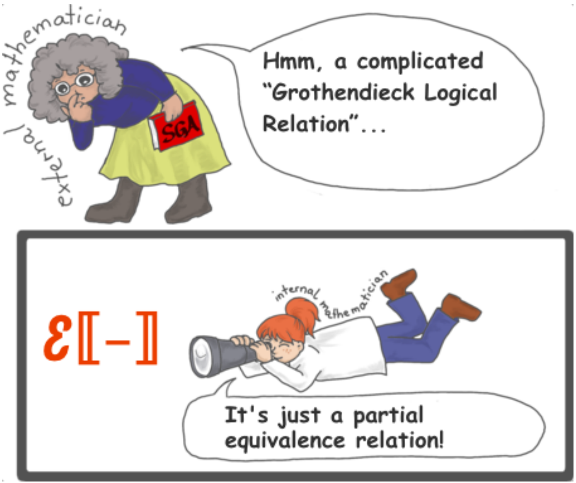
Easier to do real math than to wrestle with syntax! Do important proofs first, then implement PER semantics using the internal language of the topos.

CRITICIZE THE OLD WORLD AND BUILD A NEW ONE WITH BETH-KRIPKE-JOYAL AS A WEAPON!



(credit: Ingo Blechschmidt)

(time to use the blackboard)