Guarded Computational Type Theory

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what is CTT$_{\text{🕒}}$?

a ***dependently typed*** program logic with guarded recursion, clocks, and coinductive types

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- ⋆ operational account of guarded recursion
- ⋆ immediate canonicity result
- ⋆ simple programming language (just λ-calculus)
- ⋆ defined by “operational sheaf model”

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Solve “guarded recursive equations”:

$$\mathsf{Stream} \cong \mathbb{N} \times \triangleright\mathsf{Stream}$$

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(Easy to interpret $\mathsf{fix}(x.M)$.)

causal (co)programming

the "later modality" $\triangleright$ enables ***causal*** programs on streams

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guarded recursion not enough for (co)programming!

local clocks for coinduction

idea: parameterize $\triangleright$ by a ***clock*** κ and add quantifier $\forall\kappa$.
due to McBride, Atkey

$$\frac{\Gamma \vdash_{\Delta,\kappa} M \in A[\kappa]}{\Gamma \vdash_{\Delta} \Lambda\kappa.M \in \forall\kappa.A[\kappa]} \qquad \frac{\Gamma \vdash_{\Delta} M \in \forall\kappa.A[\kappa] \quad \kappa \# \Delta}{\Gamma \vdash_{\Delta} M[\kappa'] \in A[\kappa']}$$

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local clocks for coinduction

require the following isomorphisms:

$$\forall\kappa.\, \triangleright_\kappa A \cong \forall\kappa.A$$

$$\forall\kappa.A \cong A \quad (\kappa \# A)$$

$$\forall\kappa.A \times B \cong (\forall\kappa.A) \times (\forall\kappa.B)$$

etc.

$\triangleright_\kappa$ + $\forall\kappa$ = coinduction!

Define streams in two parts (decompose productivity into causality and free use):

$$\mathsf{Stream}_\kappa \cong \mathbb{N} \times \triangleright_\kappa \mathsf{Stream}_\kappa$$
$$\mathsf{Stream} \triangleq \forall\kappa.\mathsf{Stream}_\kappa$$

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intensional version?

Both computation rule for $\mathsf{fix}_\kappa(x.M)$ and extensionality rule for next_κ incompatible with standard intensional type theory.

Birkedal et al resolve both problems in Guarded Cubical Type Theory. (Conjectured decidable typing result.)

canonicity currently unknown; no operational semantics
(but stay tuned)

resurrecting extensionality

In $\mathbf{CTT}_{\circledcirc}$ we contribute a version of Guarded Dependent Type Theory with simple operational semantics, immediate canonicity result, simpler term language, and new type equations.

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Hope to combine with Computational Higher Dimensional Type Theory (Angiuli, Harper, Wilson). Guarded recursion will be the first of many extensions.

preview of CTT$_{🕒}$

$$\frac{\Gamma \gg_{\Delta} M \in A}{\Gamma \gg_{\Delta} M \in \triangleright_{\kappa} A} \qquad \frac{\Gamma \gg_{\Delta} M \in \triangleright_{\kappa}(\Pi x : A.B[x]) \quad \Gamma \gg_{\Delta} M \in \triangleright_{\kappa} A}{\Gamma \gg_{\Delta} M(N) \in \triangleright_{\kappa} B[N]}$$

(look ma, no delayed substitutions!)

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preview of CTT$_{\circledcirc}$

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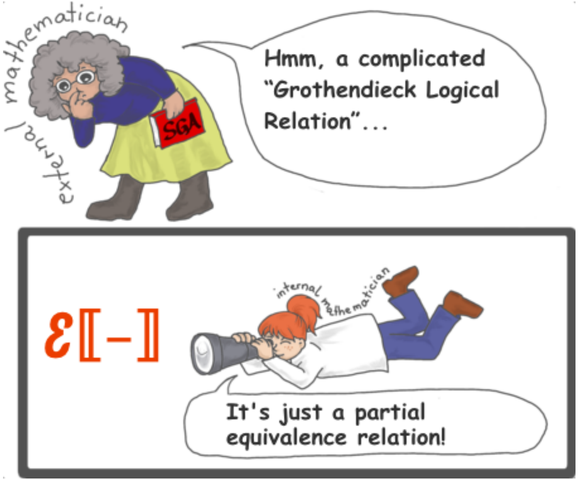
$$\frac{\Gamma \gg_{\Delta,\kappa} M \in A}{\Gamma \gg_{\Delta} M \in \cap\kappa.A} \qquad \frac{\Gamma \gg_{\Delta} M \in \cap\kappa.A}{\Gamma \gg_{\Delta} M \in A[\kappa]}$$

topos semantics

We develop the semantics (meaning explanation) for **CTT**$_{\circledcirc}$ internally to a suitable ***sheaf topos*** which provides all the machinery that we need ($\triangleright_\kappa$, $\forall\kappa$).

Easier to do real math than to wrestle with syntax! Do important proofs first, then implement PER semantics using the internal language of the topos.



(credit: Ingo Blechschmidt)