Induction for Gödel's **System** $\mathbb{T}$ Definable Bars via Effectful Forcing

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Abstract

Using Martín Escardó's « effectful forcing » technique, we demonstrate the constructive validity of Brouwer's monotone Bar Theorem for any **System** $\mathbb{T}$-definable bar. We have not assumed any non-constructive (Classical or Brouwerian) principles in this proof, and have carried out the entire development formally in the Agda proof assistant [13] for Martin-Löf's Constructive Type Theory.

In 2013, Martín Escardó pioneered a technique called « effectful forcing » for demonstrating non-constructive (Brouwerian) principles for the definable functionals of Gödel's **System** $\mathbb{T}$ [7], including the continuity of functionals on the Baire space and uniform continuity of functionals on the Cantor space. Effectful forcing is a remarkably simpler alternative to standard sheaf-theoretic forcing arguments, using ideas from programming languages, including computational effects, monads and logical relations.

Following a suggestion from Thierry Coquand [2, 3], the author learnt that Brouwer's controversial Bar Theorem could in principle be validated in a Beth model by instantiating the premise of barhood at a « generic point », which would yield an inductive mental construction of barhood. In this paper, we put an analogous version of this idea into practice using Escardó's method.

1 Brouwer's Bar Thesis

There are many versions of the Bar Thesis and its corollary, the bar induction principle, but we will describe here a particularly perspicuous one. First we will define a point-free notion of topological space called a « spread ».

Definition 1.1. A *spread* consists in a set X and a species $\mathcal{S}$ of finite sequences (nodes) $\vec{u} \in X^\star$ such that the following hold:

$$\frac{}{\langle\rangle \in \mathcal{S}} \qquad \frac{\vec{u} \in \mathcal{S}}{\exists x \in X.\, \vec{u}\frown x \in \mathcal{S}} \qquad \frac{\vec{u} \in \mathcal{S} \quad \vec{v} \preccurlyeq \vec{u}}{\vec{v} \in \mathcal{S}}$$

Viewed as a topological space, the admitted finite sequences are the spread's open sets (neighborhoods), and its points are the infinite sequences $\alpha \in X^{\mathbb{N}}$ whose every prefix $\vec{u} \prec \alpha$ is admitted. The topology of a spread is given by the notion of a « bar » or « cover ». We say that a species of nodes $Q \subseteq \mathcal{S}$ covers (bars) a node $\vec{u}$ when every infinite sequence out of $\vec{u}$ has a prefix in Q. Formally:

$$\frac{\forall \alpha \succ \vec{u}.\, \exists k \in \mathbb{N}.\, \overline{\alpha}\,[k] \in Q}{\vec{u} \lhd Q}\ \textit{covering}$$

A species Q is called *monotone* when, if $\vec{u} \in Q$, we also have $\vec{u}\frown x \in Q$ for any $x \in X$.

Inductive Covers Separately, we define an inductive version of the covering relation for a *monotone* species of nodes Q, defined as the least relation closed under the following two rules of inference:

$$\frac{\vec{u} \in Q}{\vec{u} \blacktriangleleft Q}\ \eta \qquad \frac{\forall x \in X.\, \vec{u}\frown x \blacktriangleleft Q}{\vec{u} \blacktriangleleft Q}\ \mathsf{F}$$

Theorem 1.2. *Assuming Q is monotone, Brouwer's contested ζ inference is admissible:*

$$\frac{\vec{u} \blacktriangleleft Q}{\vec{u}\frown x \blacktriangleleft Q}\ \zeta$$

Proof. By case on the premise.

1. If the premise was η, then by monotonicity of Q and η.

2. If the premise was F, then we have for any $y \in X$, $\vec{u}\frown y \blacktriangleleft Q$. Choose $y \equiv x$.

□

Theorem 1.3. *The inductive relation $\vec{u} \blacktriangleleft Q$ is a sound characterization of covering, i.e. the following rule of inference is justified:*

$$\frac{\vec{u} \blacktriangleleft Q}{\vec{u} \lhd Q}\ \textit{soundness}$$

Proof. We have to show that for any $\alpha \succ \vec{u}$, there exists a $k \in \mathbb{N}$ such that $\overline{\alpha}[k] \in Q$. In fact, it suffices to show that for any $\alpha \succ \langle\rangle$, there exists a $k \in \mathbb{N}$ such that $\overline{\vec{u} \oplus \alpha}[|\vec{u}| + k] \in Q$. We proceed by cases on the premise.

1. If the premise was η, then choose $k \equiv 0$.

2. If the premise was $\digamma$, then we have for all $x \in X$, $\vec{u}\frown x \lhd Q$; by our inductive hypothesis, for any $x \in X$ and $\beta \succ \langle\rangle$, we have a $k' \in \mathbb{N}$ such that $\overline{\vec{u}\frown x \oplus \beta}[|\vec{u}| + 1 + k'] \in Q$. Let $x \equiv \mathsf{head}(\alpha)$ and $\beta \equiv \mathsf{tail}(\alpha)$; then choose $k \equiv k' + 1$.

$\square$

Proposition 1.4 (Brouwer's Bar Thesis)**.** *Brouwer's (monotone) Bar Thesis states that for any monotone species $Q \subseteq \mathbb{S}$, the inductive definition of covering is also complete, i.e. the following rule of inference is justified:*

$$\frac{\vec{u} \lhd Q}{\vec{u} \blacktriangleleft Q}\ \textit{completeness}$$

Instantiated at the Baire spread $\mathcal{B} \equiv \{\,\vec{u} \mid \vec{u} \in \mathbb{N}^\star\,\}$, Proposition 1.4 becomes the standard statement of Brouwer's Bar Thesis; at the Cantor spread $\mathcal{C} \equiv \{\,\vec{u} \mid \vec{u} \in \mathbb{2}^\star\,\}$, it proves the Fan Theorem.

Stated as above, Proposition 1.4 for the Baire and Cantor spreads is consistent with constructive foundations, but is not constructively valid. There is a computational procedure « bar recursion » to witness the validity of the completeness rule above, but the recognition of its effectiveness depends crucially on the assumption of the Bar Thesis itself.

As far as *computational* realizations of constructive foundations are concerned, then, this places the Bar Thesis at the same level as other axioms such as Markov's Principle, whose effectiveness can be assumed without disturbing the computational character of the framework.

2 Gödel's System $\mathbb{T}$ as a theory of constructions

In his famous *Dialectica* interpretation [11], Kurt Gödel introduced **System** $\mathbb{T}$ to serve as a formal theory of constructions for Heyting arithmetic, which we briefly reproduce in modern form here.

Types As a matter of convenience, we differentiate between atomic (base) types ι and types σ, τ:

$$\frac{}{\mathsf{nat}\ \mathit{atype}}$$

$$\frac{\iota\ \mathit{atype}}{\iota\ \mathit{type}} \qquad \frac{\sigma\ \mathit{type} \quad \tau\ \mathit{type}}{\sigma \to \tau\ \mathit{type}}$$

Contexts

$$\frac{}{\cdot\ \mathit{ctx}} \qquad \frac{\Gamma\ \mathit{ctx} \quad \sigma\ \mathit{type}}{\Gamma, x : \sigma\ \mathit{ctx}}\ (x \notin \Gamma)$$

Terms The well-typed terms of **System** $\mathbb{T}$ are defined using the sequent judgment form $\Gamma \vdash m : \sigma$, presupposing Γ *ctx* and σ *type*.

$$\frac{}{\Gamma, x : \sigma, \Delta \vdash x : \sigma}\ \mathtt{var}$$

$$\frac{}{\Gamma \vdash \mathsf{z} : \mathsf{nat}}\ \mathtt{zero} \qquad \frac{\Gamma \vdash m : \mathsf{nat}}{\Gamma \vdash \mathsf{s}(m) : \mathsf{nat}}\ \mathtt{succ}$$

$$\frac{\Gamma, x : \mathsf{nat}, y : \sigma \vdash s[x, y] : \sigma \quad \Gamma \vdash z : \sigma \quad \Gamma \vdash n : \mathsf{nat}}{\Gamma \vdash \mathsf{rec}_\sigma([x, y].s[x, y]; z; n) : \sigma}\ \mathtt{rec}$$

$$\frac{\Gamma, x : \sigma \vdash m[x] : \tau}{\Gamma \vdash \lambda x.m[x] : \sigma \to \tau}\ \mathtt{lam} \qquad \frac{\Gamma \vdash m : \sigma \to \tau \quad \Gamma \vdash n : \sigma}{\Gamma \vdash m \bullet_\sigma n : \tau}\ \mathtt{ap}$$

3 Denotational semantics of System $\mathbb{T}$

We will now proceed to develop two denotational semantics for **System** $\mathbb{T}$: a « standard » semantics and a « dialectical » semantics;[1] then we will prove that the two are coherent using a logical relations argument. This procedure is entirely due to Escardó [7].

Both semantics share the interpretation $\mathcal{V}[\![\iota]\!]$ of the atomic types ι *atype*, as follows:

$$\mathcal{V}[\![\mathsf{nat}]\!] \triangleq \mathbb{N} \tag{3.1}$$

[1]Not to be confused with a *Dialectica* interpretation.

3.1 Standard semantics of System $\mathbb{T}$

The standard semantics $\mathcal{V}[\![\sigma]\!]$ for the types σ *type* is as follows:

$$\mathcal{V}[\![\iota]\!] \triangleq \mathcal{V}[\iota] \tag{3.2}$$

$$\mathcal{V}[\![\sigma \to \tau]\!] \triangleq \mathcal{V}[\![\sigma]\!] \to \mathcal{V}[\![\tau]\!] \tag{3.3}$$

Contexts are Γ *ctx* interpreted as environments $\mathcal{G}[\![\Gamma]\!]$:

$$\mathcal{G}[\![\Gamma]\!] \triangleq \prod_{x \in |\Gamma|} \mathcal{V}[\![\Gamma(x)]\!] \tag{3.4}$$

Next, the interpretation of terms $[\![\Gamma \vdash m : \sigma]\!]_\rho \in \mathcal{V}[\![\sigma]\!]$ for $\rho \in \mathcal{G}[\![\Gamma]\!]$, presupposing $\Gamma \vdash m : \sigma$, is defined in the following way:

$$[\![\Gamma \vdash x : \sigma]\!]_\rho \triangleq \rho(x) \tag{3.5}$$

$$[\![\Gamma \vdash \mathsf{z} : \mathsf{nat}]\!]_\rho \triangleq 0 \tag{3.6}$$

$$[\![\Gamma \vdash \mathsf{s}(m) : \mathsf{nat}]\!]_\rho \triangleq 1 + [\![\Gamma \vdash m : \mathsf{nat}]\!]_\rho \tag{3.7}$$

To interpret primitive recursion, we define an auxiliary operation at the meta-level:

$$\begin{aligned} \mathit{rec}(s, z, 0) &\triangleq z \\ \mathit{rec}(s, z, n+1) &\triangleq s(n, \mathit{rec}(s, z, n)) \end{aligned}$$

$$[\![\Gamma \vdash \mathsf{rec}_\sigma([x, y].s[x, y]; z; n) : \sigma]\!]_\rho \triangleq \mathit{rec}(S, Z, N) \tag{3.8}$$

where

$$\begin{aligned} S(a, b) &\triangleq [\![\Gamma, x : \mathsf{nat}, y : \sigma \vdash s[x, y] : \sigma]\!]_{\rho, x \mapsto a, y \mapsto b} \\ Z &\triangleq [\![\Gamma \vdash z : \sigma]\!]_\rho \\ N &\triangleq [\![\Gamma \vdash n : \mathsf{nat}]\!]_\rho \end{aligned}$$

$$[\![\Gamma \vdash \lambda x.m[x] : \sigma \to \tau]\!]_\rho \triangleq a \mapsto [\![\Gamma, x : \sigma \vdash m[x] : \tau]\!]_{\rho, x \mapsto a} \tag{3.9}$$

$$[\![\Gamma \vdash m \bullet_\sigma n : \tau]\!]_\rho \triangleq \left([\![\Gamma \vdash m : \sigma \to \tau]\!]_\rho\right)\left([\![\Gamma \vdash n : \sigma]\!]_\rho\right) \tag{3.10}$$

3.2 Escardó dialogues: ideal codes for functionals

First, let us define the set $\mathfrak{E}^X_Y(Z)$ of « Escardó dialogues », which code functionals of type $Y^X \to Z$, as the least set closed under the following rules:

$$\frac{z \in Z}{\eta(z) \in \mathfrak{E}^X_Y(Z)}\ \texttt{return} \qquad \frac{x \in X \quad e \in Y \to \mathfrak{E}^X_Y(Z)}{\beta\langle x\rangle(e) \in \mathfrak{E}^X_Y(Z)}\ \texttt{query}$$

An Escardó dialogue is an idealized procedure or algorithm for computing a functional; leaf nodes $\eta(z)$ return a result z, and branch nodes $\beta\langle x\rangle(e)$ query for the xth element y of the input choice sequence and proceed with $e(y)$. Such interactions are essentially a hyper-intensional, syntactic representation of a *neighborhood function*.

$\mathfrak{E}^X_Y(-)$ is a monad on **Set**, natural in $X, Y \in \mathbf{Set}$. We define the action $\mathfrak{E}(f) \in \mathfrak{E}^X_Y(A) \to \mathfrak{E}^X_Y(B)$ of the functor as follows for $f \in A \to B$:

$$\begin{aligned} \mathfrak{E}(f)(\eta(a)) &\triangleq \eta(f(a)) \\ \mathfrak{E}(f)(\beta\langle x\rangle(e)) &\triangleq \beta\langle x\rangle(\mathfrak{E}(f) \circ e) \end{aligned}$$

η is the unit of the monad; we define the Kleisli extension $f^\star \in \mathfrak{E}^X_Y(A) \to \mathfrak{E}^X_Y(B)$ for $f \in A \to \mathfrak{E}^X_Y(B)$ as follows:

$$\begin{aligned} f^\star(\eta(a)) &\triangleq f(a) \\ f^\star(\beta\langle x\rangle(e)) &\triangleq \beta\langle x\rangle(f^\star \circ e) \end{aligned}$$

An Escardó dialogue $e \in \mathfrak{E}^X_Y(Z)$ may be executed on a choice sequence $\alpha \in Y^X$ to return a result $e \diamond \alpha \in Z$ as follows:

$$\begin{aligned} \eta(z) \diamond \alpha &\triangleq z \\ \beta\langle x\rangle(e) \diamond \alpha &\triangleq e(\alpha(x)) \diamond \alpha \end{aligned}$$

The following two lemmas are from Escardó [7].

Lemma 3.1. *For any $\alpha \in Y^X$, the execution map $- \diamond \alpha$ is a natural transformation $\mathfrak{E}^X_Y(-) \to 1_{\mathbf{Set}}$:*

$$\begin{tikzcd} \mathfrak{E}^X_Y(A) \arrow[r, "\mathfrak{E}(f)"] \arrow[d, "-\diamond\alpha"] & \mathfrak{E}^X_Y(B) \arrow[d, "-\diamond\alpha"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

Proof. Immediate by induction on the dialogue tree. □

Lemma 3.2. *Kleisli extension commutes with execution, in the following sense:*

$$\begin{tikzcd}
\mathfrak{E}_Y^X(A) \arrow[rr, "f^\star"] \arrow[d, "-\diamond\alpha"] & & \mathfrak{E}_Y^X(B) \arrow[d, "-\diamond\alpha"] \\
A \arrow[r, "f"] & \mathfrak{E}_Y^X(B) \arrow[r, "-\diamond\alpha"] & B
\end{tikzcd}$$

Proof. Immediate by induction on the dialogue tree. □

3.3 Dialectical semantics of System $\mathbb{T}$

Now we are prepared to begin interpreting well-typed **System** $\mathbb{T}$ terms into Escardó dialogues. The semantic domains are as follows:

$$\mathcal{V}\langle\!\langle \iota \rangle\!\rangle \triangleq \mathfrak{E}_{\mathbb{N}}^{\mathbb{N}}(\mathcal{V}[\iota]) \tag{3.11}$$

$$\mathcal{V}\langle\!\langle \sigma \to \tau \rangle\!\rangle \triangleq \mathcal{V}\langle\!\langle \sigma \rangle\!\rangle \to \mathcal{V}\langle\!\langle \tau \rangle\!\rangle \tag{3.12}$$

$$\mathcal{G}\langle\!\langle \Gamma \rangle\!\rangle \triangleq \prod_{x \in |\Gamma|} \mathcal{V}\langle\!\langle \Gamma(x) \rangle\!\rangle \tag{3.13}$$

We will need to lift the Kleisli extension $(-)^\star$ to apply at higher type; for a type σ *type* and a map $f \in X \to \mathcal{V}\langle\!\langle \sigma \rangle\!\rangle$, we have the lifted Kleisli extension $f_\sigma^{✪} \in \mathfrak{E}_{\mathbb{N}}^{\mathbb{N}}(X) \to \mathcal{V}\langle\!\langle \sigma \rangle\!\rangle$, defined as follows:

$$f_\iota^{✪}(e) \triangleq f^\star(e)$$

$$f_{\sigma\to\tau}^{✪}(e) \triangleq s \mapsto f(-, s)_\tau^{✪}(e)$$

The interpretation $\langle\!\langle \Gamma \vdash m : \sigma \rangle\!\rangle_\rho \in \mathcal{V}\langle\!\langle \sigma \rangle\!\rangle$ for an environment $\rho \in \mathcal{G}\langle\!\langle \Gamma \rangle\!\rangle$, presupposing $\Gamma \vdash m : \sigma$ is defined as follows:

$$\langle\!\langle \Gamma \vdash x : \sigma \rangle\!\rangle_\rho \triangleq \rho(x) \tag{3.14}$$

$$\langle\!\langle \Gamma \vdash \mathsf{z} : \mathsf{nat} \rangle\!\rangle_\rho \triangleq \eta(0) \tag{3.15}$$

$$\langle\!\langle \Gamma \vdash \mathsf{s}(m) : \mathsf{nat} \rangle\!\rangle_\rho \triangleq \mathfrak{E}(1 + -)\left(\langle\!\langle \Gamma \vdash m : \mathsf{nat} \rangle\!\rangle_\rho\right) \tag{3.16}$$

$$\langle\!\langle \Gamma \vdash \mathsf{rec}_\sigma([x,y].s[x,y]; z; n) : \sigma \rangle\!\rangle_\rho \triangleq \mathit{rec}(S, Z, -)_\sigma^{✪}(N) \tag{3.17}$$

where

$$S(a,b) \triangleq \langle\!\langle \Gamma, x : \mathsf{nat}, y : \sigma \vdash s[x,y] : \sigma \rangle\!\rangle_{\rho, x\mapsto a, y\mapsto b}$$

$$Z \triangleq \langle\!\langle \Gamma \vdash z : \sigma \rangle\!\rangle_\rho$$

$$N \triangleq \langle\!\langle \Gamma \vdash n : \mathsf{nat} \rangle\!\rangle_\rho$$

$$\langle\!\langle \Gamma \vdash \lambda x.m[x] : \sigma \to \tau \rangle\!\rangle_\rho \triangleq a \mapsto \langle\!\langle \Gamma, x : \sigma \vdash m[x] : \tau \rangle\!\rangle_{\rho, x \mapsto a} \tag{3.18}$$

$$\langle\!\langle \Gamma \vdash m \bullet_\sigma n : \tau \rangle\!\rangle_\rho \triangleq \left(\langle\!\langle \Gamma \vdash m : \sigma \to \tau \rangle\!\rangle_\rho\right)\left(\langle\!\langle \Gamma \vdash n : \sigma \rangle\!\rangle_\rho\right) \tag{3.19}$$

3.4 Coherence of interpretations

The standard semantics and the dialectical semantics cohere for closed terms of atomic type ι *atype* in the sense that the following diagram commutes for any $\alpha \in \mathbb{N}^\mathbb{N}$:

$$\begin{tikzcd}
& \cdot \vdash \iota \arrow[dl, "\langle\!\langle \cdot \vdash - : \iota \rangle\!\rangle"'] \arrow[dr, "\llbracket \cdot \vdash - : \iota \rrbracket"] & \\
\mathcal{V}\,\langle\!\langle \iota \rangle\!\rangle \arrow[rr, "- \diamond \alpha"'] & & \mathcal{V}\,\llbracket \iota \rrbracket
\end{tikzcd} \tag{3.20}$$

In order to prove this, we will need to actually prove a stronger lemma for open terms at higher type using logical relations. We define our logical relation $\mathcal{R}^\alpha_\sigma$ for $\alpha \in \mathbb{N}^\mathbb{N}$ between the (values, environments) of each interpretation by the following rules:

$$\mathcal{R}^\alpha_\sigma \subseteq \mathcal{V}\,\llbracket \sigma \rrbracket \times \mathcal{V}\,\langle\!\langle \sigma \rangle\!\rangle$$

$$\overline{\mathcal{R}^\alpha_\Gamma} \subseteq \mathcal{G}\,\llbracket \Gamma \rrbracket \times \mathcal{G}\,\langle\!\langle \Gamma \rangle\!\rangle$$

$$\frac{F = d \diamond \alpha}{F \; \mathcal{R}^\alpha_\iota \; d} \qquad \frac{\forall G \in \mathcal{V}\,\llbracket \sigma \rrbracket, e \in \mathcal{V}\,\langle\!\langle \sigma \rangle\!\rangle.\; G \; \mathcal{R}^\alpha_\sigma \; e \implies F(G) \; \mathcal{R}^\alpha_\tau \; d(e)}{F \; \mathcal{R}^\alpha_{\sigma \to \tau} \; d}$$

$$\frac{\forall x \in |\Gamma|.\; \rho_0(x) \; \mathcal{R}^\alpha_{\Gamma(x)} \; \rho_1(x)}{\rho_0 \; \overline{\mathcal{R}^\alpha_\Gamma} \; \rho_1}$$

It will be useful to prove an auxiliary lemma about $(-)^{\circledast}_\sigma$, following [7].

Lemma 3.3. *Fix ι atype, σ type, $F \in \mathcal{V}\,[\iota] \to \mathcal{V}\,\llbracket \sigma \rrbracket$, $d \in \mathcal{V}\,[\iota] \to \mathcal{V}\,\langle\!\langle \sigma \rangle\!\rangle$, $G \in \mathcal{V}\,[\iota]$, and $e \in \mathcal{V}\,\langle\!\langle \iota \rangle\!\rangle$. Then, we may infer*

$$\frac{G \; \mathcal{R}^\alpha_\iota \; e \qquad \forall k \in \mathcal{V}\,[\iota].\; F(k) \; \mathcal{R}^\alpha_\sigma \; d(k)}{F(G) \; \mathcal{R}^\alpha_\sigma \; d^{\circledast}_\sigma(e)}$$

Proof. By cases on σ *type*.

Case $\sigma \equiv \iota'$. By Lemma 3.2.

Case $\sigma \equiv \tau_1 \to \tau_2$. By the inductive hypothesis.

$\square$

Theorem 3.4. *The standard and dialectical interpretations of each **System** $\mathbb{T}$-definable term are related by $\mathcal{R}$, assuming environments related by $\overline{\mathcal{R}}$. More precisely, for any $\Gamma \vdash M : \sigma$, $\rho_0 \in \mathcal{G}\,[\![\Gamma]\!]$ and $\rho_1 \in \mathcal{G}\,\langle\!\langle\Gamma\rangle\!\rangle$ such that $\rho_0\ \overline{\mathcal{R}^{\alpha}_{\Gamma}}\ \rho_1$, we have $[\![\Gamma \vdash M : \sigma]\!]_{\rho_0}\ \mathcal{R}^{\alpha}_{\sigma}\ \langle\!\langle\Gamma \vdash M : \sigma\rangle\!\rangle_{\rho_1}$.*

Proof. By case on the term M.

Case $M \equiv x : \sigma$. Because $\rho_0\ \overline{\mathcal{R}^{\alpha}_{\Gamma}}\ \rho_1$, we also have $\rho_0(x)\ \mathcal{R}^{\alpha}_{\sigma}\ \rho_1(x)$.

Case $M \equiv \mathsf{rec}_{\sigma}([x,y].s[x,y]; z; n) : \sigma$. Let us begin with some auxiliary definitions:

$$S_0 \triangleq (a,b) \mapsto [\![\Gamma, x : \mathsf{nat}, y : \sigma \vdash s : \sigma]\!]_{\rho_0, x\mapsto a, y\mapsto b}$$
$$S_1 \triangleq (a,b) \mapsto \langle\!\langle\Gamma, x : \mathsf{nat}, y : \sigma \vdash s : \sigma\rangle\!\rangle_{\rho_1, x\mapsto a, y\mapsto b}$$

$$\begin{aligned} Z_0 &\triangleq [\![\Gamma \vdash z : \sigma]\!]_{\rho_0} & Z_1 &\triangleq \langle\!\langle\Gamma \vdash z : \sigma\rangle\!\rangle_{\rho_1} \\ N_0 &\triangleq [\![\Gamma \vdash n : \mathsf{nat}]\!]_{\rho_0} & N_1 &\triangleq \langle\!\langle\Gamma \vdash n : \mathsf{nat}\rangle\!\rangle_{\rho_1} \\ R_0 &\triangleq \mathit{rec}(S_0, Z_0, -) & R_1 &\triangleq \mathit{rec}(S_1 \circ \eta, Z_1, -) \end{aligned}$$

By backward chaining through Lemma 3.3, letting $\iota \equiv \mathsf{nat}$, $F \equiv R_0$, $d \equiv R_1$, $G \equiv N_0$ and $e \equiv N_1$, it suffices to show the following:

1. $N_0\ \mathcal{R}^{\alpha}_{\mathsf{nat}}\ N_1$: by the inductive hypothesis for n.
2. For any $k \in \mathbb{N}$, $R_0(k)\ \mathcal{R}^{\alpha}_{\sigma}\ R_1(k)$: by induction on k, applying the inductive hypotheses for z and s in the base case and inductive step respectively.

All remaining cases follow from their inductive hypotheses and Lemma 3.1. $\square$

Corollary 3.5. *Diagram 3.20 commutes for any ι atype and $\alpha \in \mathbb{N}^{\mathbb{N}}$.*

Proof. By Theorem 3.4, instantiated at the type ι and the empty environment. $\square$

4 Validity of the Bar Thesis for $\mathbb{T}$-definable bars

Recall the definition of a bar (cover) from Section 1:

$$\frac{\forall \alpha \succ \vec{u}.\ \exists k \in \mathbb{N}.\ \overline{\alpha}\,[k] \in Q}{\vec{u} \lhd Q}\ \textit{covering}$$

It is well known that we cannot hope to prove the Bar Thesis (Proposition 1.4) for this definition of covering, but our experience with the dialectical semantics of **System** $\mathbb{T}$ suggests that we might prove a slightly weaker rule, by requiring the premise to be *realized* by a **System** $\mathbb{T}$-definable functional.

If we interpret the quantifiers constructively and functionally, this is the same as to say that we have a functional $\cdot \vdash f : (\mathsf{nat} \to \mathsf{nat}) \to \mathsf{nat}$ which computes the length of an approximation that is in the bar. To apply such a functional to a meta-level choice sequence, let us exploit the standard semantics defined in Section 3.1:

$$f\left\langle \alpha \right\rangle \triangleq \llbracket \cdot \vdash f : (\mathsf{nat} \to \mathsf{nat}) \to \mathsf{nat} \rrbracket\,(\alpha)$$

Now, we can define a new **System** $\mathbb{T}$-centric notion of barhood or covering:

$$\frac{\exists \cdot \vdash f : (\mathsf{nat} \to \mathsf{nat}) \to \mathsf{nat}.\ \forall \alpha \succ \langle\rangle.\ \overline{\vec{u} \oplus \alpha}\,[f\left\langle \alpha \right\rangle + |\vec{u}|] \in Q}{\vec{u} \lhd_{\mathbb{T}} Q}\ \textit{covering}_{\mathbb{T}}$$

Proposition 4.1 (Bar Thesis for **System** $\mathbb{T}$)**.** *The Bar Thesis for **System** $\mathbb{T}$ states that for any monotone species $Q \subseteq \mathcal{B}$, the inductive definition of covering is complete in the sense that the following rule is justified:*

$$\frac{\vec{u} \lhd_{\mathbb{T}} Q}{\vec{u} \blacktriangleleft Q}\ \textit{completeness}$$

4.1 Brouwer's ephemeral dialectics

The content of Brouwer's purported (but failed) proof of his Bar Thesis was to assert that one can analyze the evidence for barhood into a well-founded mental construction [19, 6]; Escardó's translation of **System** $\mathbb{T}$ terms into dialogue trees is essentially a formalization of Brouwer's insight.

However, Escardó's dialogues differ from Brouwer's mental constructions of barhood, which are captured precisely by the judgment $\vec{u} \blacktriangleleft Q$, in one crucial respect: whereas Escardó's trees branch on an arbitrary query to the ambient choice sequence, queries in Brouwer's mental constructions must be made in order, i.e. with respect to the current moment in ideal time; moreover, the Brouwerian dialogues are *ephemeral*—with each query, the head of the ambient choice sequence is consumed and the remainder of the dialogue is interpreted with respect to the tail of the choice sequence.

Our task, then, will be to normalize Escardó's dialogues into Brouwer's ephemeral mental constructions, and then show how to massage these into a derivation of

$\vec{u} \blacktriangleleft Q$. Below we define the set of Brouwerian dialogues $\mathfrak{B}_Y(Z)$ (coding functionals $Y^{\mathbb{N}} \to Z$) as the least set closed under the following rules:

$$\frac{z \in Z}{\eta_{\mathfrak{B}}(z) \in \mathfrak{B}_Y(Z)}\ \texttt{spit} \qquad \frac{b \in Y \to \mathfrak{B}_Y(Z)}{\mathsf{F}(b) \in \mathfrak{B}_Y(Z)}\ \texttt{bite}$$

Naïvely, we can normalize the Escardó trees using the following general recursive procedure:

$$\begin{aligned} \mathit{norm}_{\vec{u}}(\eta(z)) &\triangleq \eta_{\mathfrak{B}}(z) \\ \mathit{norm}_{\vec{u}}(\beta\langle i\rangle(b)) &\triangleq \begin{cases} \vec{u}_i & \text{if } i < |\vec{u}| \\ \mathsf{F}(y \mapsto \mathit{norm}_{\vec{u}\frown y}(\beta\langle i\rangle(b))) & \text{if } i \geq |\vec{u}| \end{cases} \end{aligned}$$

Then we define $\mathit{norm}(e) \triangleq \mathit{norm}_{\langle\rangle}(e)$. To show that this procedure is effective and total, however, is not so easy. Instead, we design an inductive/proof-theoretic characterization of the normalizable Escardó dialogues, and then show that all Escardó dialogues can be coded as such. This yields a constructive and structurally recursive normalization algorithm.

To this end, we define below two mutually inductive forms of judgment:

1. $\vec{u} \Vdash d \rightsquigarrow b$ presupposes $\vec{u} \in Y^\star$ and $d \in \mathfrak{E}^{\mathbb{N}}_Y(Z)$, and guarantees $b \in \mathfrak{B}_Y(Z)$.
2. $\vec{u} \Vdash \beta\langle i\rangle\,(d) \ll \vec{v} \rightsquigarrow b$ presupposes $\vec{u}, \vec{v} \in Y^\star$, $i \in \mathbb{N}$ and $d \in Y \to \mathfrak{E}^{\mathbb{N}}_Y(Z)$, and guarantees $b \in \mathfrak{B}_Y(Z)$.

We will write $y :: \vec{u}$ for the « cons » of y to the front of the list $\vec{u}$.

$$\frac{}{\vec{u} \Vdash \eta(z) \rightsquigarrow \eta_{\mathfrak{B}}(z)}\ \texttt{norm}_\eta \qquad \frac{\vec{u} \Vdash \beta\langle i\rangle\,(d) \ll \vec{u} \rightsquigarrow b}{\vec{u} \Vdash \beta\langle i\rangle(d) \rightsquigarrow b}\ \texttt{norm}_\beta$$

$$\frac{\vec{u} \Vdash d(y) \rightsquigarrow b}{\vec{u} \Vdash \beta\langle 0\rangle\,(d) \ll y :: \vec{v} \rightsquigarrow b}\ \texttt{norm}^{::,\mathsf{z}}_\beta \qquad \frac{\vec{u} \Vdash \beta\langle i\rangle\,(d) \ll \vec{v} \rightsquigarrow b}{\vec{u} \Vdash \beta\langle i+1\rangle\,(d) \ll y :: \vec{v} \rightsquigarrow b}\ \texttt{norm}^{::,\mathsf{s}}_\beta$$

$$\frac{\forall y \in Y.\, \vec{u}\frown y \Vdash d(y) \rightsquigarrow b(y)}{\vec{u} \Vdash \beta\langle 0\rangle\,(d) \ll \langle\rangle \rightsquigarrow \mathsf{F}(b)}\ \texttt{norm}^{\langle\rangle,\mathsf{z}}_\beta$$

$$\frac{\forall y \in Y.\, \vec{u}\frown y \Vdash \beta\langle i\rangle\,(d) \ll \langle\rangle \rightsquigarrow b(y)}{\vec{u} \Vdash \beta\langle i+1\rangle\,(d) \ll \langle\rangle \rightsquigarrow \mathsf{F}(b)}\ \texttt{norm}^{\langle\rangle,\mathsf{s}}_\beta$$

Lemma 4.2. *The inductive characterization of normalization $\vec{u} \Vdash d \rightsquigarrow b$ is both complete and effective, i.e. for any $\vec{u} \in Y^\star$ and $d \in \mathfrak{C}_Y^{\mathbb{N}}(Z)$ there exists some unique $b \in \mathfrak{B}_Y(Z)$ such that $\vec{u} \Vdash d \rightsquigarrow b$.*

Proof. Simultaneously, we must also show that $\vec{u} \Vdash \beta\langle i\rangle\,(d) \ll \vec{v} \rightsquigarrow b$ is complete and effective. We will begin with $\vec{u} \Vdash d \rightsquigarrow b$, proceeding by case on $d \in \mathfrak{C}_Y^{\mathbb{N}}(Z)$.

Case $d \equiv \eta(z)$. By $\mathtt{norm}_\eta$, we have $b \equiv \eta_{\mathfrak{B}}(z)$.

Case $d \equiv \beta\langle i\rangle(e)$. By our inductive hypothesis, we have $\vec{u} \Vdash \beta\langle i\rangle\,(e) \ll \vec{u} \rightsquigarrow b$; apply $\mathtt{norm}_\beta$.

Next, we tackle $\vec{u} \Vdash \beta\langle i\rangle\,(d) \ll \vec{v} \rightsquigarrow b$ by simultaneous induction on $\vec{v} \in Y^\star$ (viewed as a *cons*-list) and $i \in \mathbb{N}$.

Case $\vec{v} \equiv \langle\rangle$, $i \equiv 0$. By our inductive hypothesis, we have $\vec{u}\frown y \Vdash d(y) \rightsquigarrow b(y)$ for any $y \in Y$; apply $\mathtt{norm}_\beta^{\langle\rangle,\mathsf{z}}$.

Case $\vec{v} \equiv \langle\rangle$, $i \equiv j+1$. By our inductive hypothesis, we have $\vec{u}\frown y \Vdash \beta\langle j\rangle\,(d) \ll \langle\rangle \rightsquigarrow b(y)$ for any $y \in Y$; apply $\mathtt{norm}_\beta^{\langle\rangle,\mathsf{s}}$.

Case $\vec{v} \equiv y :: w$, $i \equiv 0$. By our inductive hypothesis, we have $\vec{u} \Vdash d(y) \rightsquigarrow b$; apply $\mathtt{norm}_\beta^{::,\mathsf{z}}$.

Case $\vec{v} \equiv y :: w$, $i \equiv j+1$. By our inductive hypothesis, we have $\vec{u} \Vdash \beta\langle j\rangle\,(d) \ll \vec{v} \rightsquigarrow b$; apply $\mathtt{norm}_\beta^{::,\mathsf{s}}$.

$\square$

Corollary 4.3. *We have a structurally recursive function $norm_{\vec{u}}(d)$ such that for all $\vec{u} \in Y^\star$ and $d \in \mathfrak{C}_Y^{\mathbb{N}}(Z)$, $\vec{u} \Vdash d \rightsquigarrow norm_{\vec{u}}(d)$.*

Proof. This is the constructive content of Lemma 4.2. $\square$

4.2 Execution Semantics for $\mathfrak{B}_Y(Z)$

Just as we showed how to execute an Escardó dialogue against a choice sequence in Section 3.2, we can do the same for the Brouwerian, ephemeral version. For $b \in \mathfrak{B}_Y(Z)$ and $\alpha \in Y^{\mathbb{N}}$, we define $b \diamond_{\mathfrak{B}} \alpha \in Z$ by recursion on b as follows:

$$\begin{aligned} \eta_{\mathfrak{B}}(z) \diamond_{\mathfrak{B}} \alpha &\triangleq z \\ \mathsf{F}(b) \diamond_{\mathfrak{B}} \alpha &\triangleq b(\mathsf{head}(\alpha)) \diamond_{\mathfrak{B}} \mathsf{tail}(\alpha) \end{aligned}$$

Lemma 4.4. *Execution of dialogues coheres with normalization, as defined in Corollary 4.3; to be precise, the following diagram commutes for all $\vec{u} \in Y^\star$ and $\alpha \in Y^{\mathbb{N}}$.*

$$\begin{tikzcd} \mathfrak{C}_Y^{\mathbb{N}}(Z) \arrow[rr, "norm_{\vec{u}}(-)"] \arrow[dr, "- \diamond \vec{u} \oplus \alpha"'] & & \mathfrak{B}_Y(Z) \arrow[dl, "- \diamond_{\mathfrak{B}} \alpha"] \\ & Z & \end{tikzcd} \tag{4.1}$$

When $\vec{u} \equiv \langle\rangle$, this becomes the statement that $- \diamond \alpha = - \diamond_{\mathfrak{B}} \alpha \circ norm(-)$.

Proof. We have to show that for any $\vec{u} \in Y^\star$, $\alpha \in Y^{\mathbb{N}}$ and $d \in \mathfrak{C}_Y^{\mathbb{N}}(Z)$, we have $d \diamond \vec{u} \oplus \alpha = norm_{\vec{u}}(d) \diamond_{\mathfrak{B}} \alpha$. This follows by straightforward induction on the normalization of d. □

We can compose Diagrams 3.20, 4.1 to see a birds' view of the state of affairs concerning interpretation, normalization and execution. For any ι *atype* and $\alpha \in Y^{\mathbb{N}}$, the following diagram commutes:

$$\begin{tikzcd} \cdot \vdash \iota \arrow[r, "{[\![\cdot \vdash - : \iota]\!]}"] \arrow[d, "{\langle\!\langle \cdot \vdash - : \iota \rangle\!\rangle}"'] & \mathcal{V}[\iota] \\ \mathfrak{C}_Y^{\mathbb{N}}(\mathcal{V}[\iota]) \arrow[ur, "- \diamond \alpha"] \arrow[r, "norm(-)"'] & \mathfrak{B}_Y(\mathcal{V}[\iota]) \arrow[u, "- \diamond_{\mathfrak{B}} \alpha"'] \end{tikzcd} \tag{4.2}$$

4.3 The Generic Point

In the dialogue model, we can define a so-called « generic point » which is not definable in **System** $\mathbb{T}$:

$$\begin{aligned} &generic \in \mathfrak{C}_Y^X(X) \to \mathfrak{C}_Y^X(Y) \\ &generic \triangleq (\beta\langle - \rangle(\eta))^\star \end{aligned}$$

Intuitively, by applying the dialogue interpretation of a functional $\cdot \vdash \phi : (\mathsf{nat} \to \mathsf{nat}) \to \mathsf{nat}$ to this generic point, we get a dialogue tree $\langle\!\langle \cdot \vdash \phi : (\mathsf{nat} \to \mathsf{nat}) \to \mathsf{nat} \rangle\!\rangle\,(generic) \in \mathfrak{C}_{\mathbb{N}}^{\mathbb{N}}(\mathbb{N})$ which is precisely the trace of ϕ's interaction with the ambient choice sequence. Then, assuming that ϕ witnesses $\vec{u} \lhd_{\mathbb{T}} Q$, we can compute the derivation of $\vec{u} \blacktriangleleft Q$ by induction on this trace.

Lemma 4.5. *The generic point commutes with dialogue execution in the following sense:*

$$\begin{tikzcd}
\mathfrak{C}^X_Y(X) \arrow[r, "\mathit{generic}"] \arrow[d, "- \diamond \alpha"] & \mathfrak{C}^X_Y(Y) \arrow[d, "- \diamond \alpha"] \\
X \arrow[r, "\alpha"] & Y
\end{tikzcd}$$

Proof. Immediate by induction on the dialogue tree. □

4.4 Brouwer's Bar Theorem

We may now prove the Bar Theorem for **System** $\mathbb{T}$-definable bars.

Theorem 4.6 (Proposition 4.1)**.** *For any monotone species $Q \subseteq \mathcal{B}$, the following inference is justified:*

$$\frac{\vec{u} \lhd_{\mathbb{T}} Q}{\vec{u} \blacktriangleleft Q}\ \textit{completeness}$$

Proof. By inversion on the premise, we must have some $\cdot \vdash f : (\mathsf{nat} \to \mathsf{nat}) \to \mathsf{nat}$ such that for all $\alpha \succ \langle\rangle$, we know $\overline{\vec{u} \oplus \alpha}\,[[\![\cdot \vdash f : (\mathsf{nat} \to \mathsf{nat}) \to \mathsf{nat}]\!]\,\alpha + |\vec{u}|] \in Q$. Let $d \triangleq \langle\!\langle \cdot \vdash f : (\mathsf{nat} \to \mathsf{nat}) \to \mathsf{nat} \rangle\!\rangle\,(\mathit{generic})$ and $b \triangleq \mathit{norm}(d)$; then, by coercing along Diagram 4.2 and Lemma 4.5, we have a proof $p(\alpha)$ that $\overline{\vec{u} \oplus \alpha}\,[b \diamond_{\mathfrak{B}} \alpha + |\vec{u}|] \in Q$.

We proceed by induction on $b \in \mathfrak{B}_{\mathbb{N}}(\mathbb{N})$.

Case $b \equiv \eta_{\mathfrak{B}}(0)$. In this case, we are already in the bar. Let $0 \cdots$ be the choice sequence $\alpha(i) \triangleq 0$; from $p(0 \cdots)$, we know $\overline{\vec{u} \oplus 0 \cdots}\,[0 + |\vec{u}|] \in Q$, which is the same as $\vec{u} \in Q$; therefore, apply η. Note that the choice of $0 \cdots$ was completely arbitrary, since at this stage, we have stopped consuming from the choice sequence.

Case $b \equiv \eta_{\mathfrak{B}}(k+1)$. We have not yet reached the bar, and may step in any direction to approach it. Apply F, fixing $y \in \mathbb{N}$; then, we want to apply our inductive hypothesis at $\eta_{\mathfrak{B}}(k)$. It suffices to show that for any $\alpha \succ \langle\rangle$, we have $\overline{\vec{u}^\frown y \oplus \alpha}\,[k + 1 + |\vec{u}|] \in Q$; this follows from $p(y :: \alpha)$, and from the fact that $\vec{u}^\frown y \oplus \alpha = \vec{u} \oplus y :: \alpha$.

Case $b \equiv \mathsf{F}(b')$. Apply F, fixing $y \in \mathbb{N}$; apply the inductive hypothesis at $b'(y)$. Now, fixing $\alpha \succ \langle\rangle$, it suffices to show that $\overline{\vec{u}^\frown y \oplus \alpha}\,[b'(y) \diamond_{\mathfrak{B}} \alpha + 1 + |\vec{u}|] \in Q$. By $p(y :: \alpha)$, we have $\overline{\vec{u} \oplus y :: \alpha}\,[b'(y) \diamond_{\mathfrak{B}} \alpha + |\vec{u}|] \in Q$; because $\vec{u} \oplus y :: \alpha = \vec{u}^\frown y \oplus \alpha$, we only have to show that we will remain in Q if we take one more element from the composite choice sequence. But this is precisely what it means for Q to be monotone, and so we are done.

□

5 Formalization in Agda

This paper amounts to an « unformalization » of a completely formal development[2] in the Agda proof assistant [13]. We owe a lot to Martín Escardó's original formalizations of effectful forcing in Agda [7]. It is important to emphasize that the proof has been completely effected within the intensional dialect of Martin-Löf's Constructive Type Theory as implemented in Agda, without postulating any further principles.

Many of the fine details of the proof, in particular several instances of equational reasoning on choice sequences (the details of which have been suppressed in this paper) would have been much cleaner in a type theory with exact equality, such as Nuprl's or RedPRL's [1, 17]. However, no instance of extensionality proved essential.

6 Related Work

Forcing in type theory Aside from Escardó, whose results we have discussed in detail already, there has been a great deal of work related to (traditional) forcing in type theory over the past several years. Coquand and Jaber have combined forcing with realizability to obtain a version of type theory which validates the uniform continuity of functionals on the Cantor space [4]; Coquand and Mannaa have also used a similar technique to demonstrate the independence of Markov's Principle from dependent type theory [5].

Howe's computational open-endedness In his remarkable paper, *On Computational Open-Endedness in Martin-Löf's Type Theory* [12], Douglas Howe demonstrated that the computation systems of Martin-Löf-style type theories may be extended with infinitary rules, for instance, to endow type theory with oracles or classical / set-theoretic functions, Brouwerian free choice sequences, etc.

Howe's result is crucial for demonstrating the adequacy of Type Theory as a semi-formal theory of constructions in which to perform Brouwer's ontological descriptivist mathematics, in which the (at least) potential existence of non-constructive (non-algorithmic) operations is absolutely essential (see [8, 9, 10]).

Bar induction in Nuprl A weak form of the bar induction principle has been added as a basic rule in Nuprl's proof theory; consequently, via a bootstrapping

[2] The full development is available here: https://github.com/jonsterling/agda-effectful-forcing

technique, stronger forms of the bar principle also hold in Nuprl, including both the monotone and decidable bar principles [14, 16]. As a result, the Fan Theorem is true in Nuprl, and has been used to establish the uniform continuity of all functionals on the Cantor spread [15].

Traditionally, the soundness of Nuprl's proof theory with respect to its partial equivalence relation semantics has been established by completely constructive means. With the addition of the bar induction principle, however, Nuprl's semantics must be performed in a classical metatheory. We are curious whether the technique used here can be extended to a language that supports universal computation, like Nuprl; if so, this may provide a path toward recovering the constructive character of Nuprl's semantics.

In order to prove the validity of their bar induction rule, all (classical) sequences of numerals have been added directly to Nuprl's computation system; this is essentially an iteration of what has been done in the present work, and by Escardó [7], and is justified by Howe's open-endedness result [12]. It corresponds to the well-known fact that the bar principle cannot hold if functions are restricted to the computable (recursive, algorithmic) operations [18].

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