RedPRL

designing the people's refinement logic

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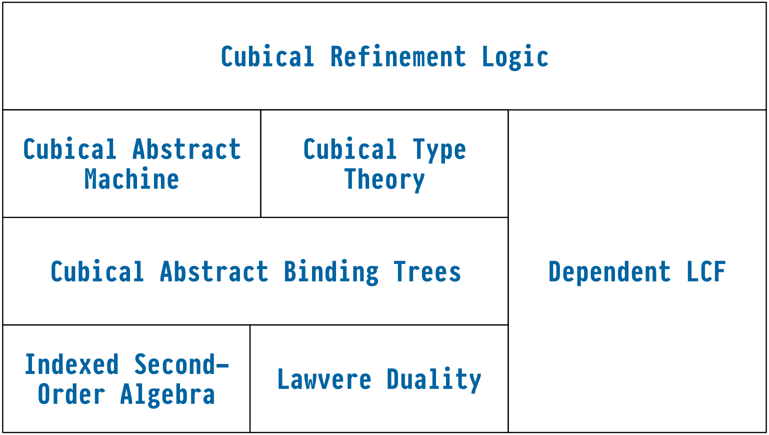
So, why bother?

the absolute idea

$$\begin{tikzcd}
\text{Practical Idea} \arrow[r, bend right, "\text{theory}"'] \arrow[r, phantom, "\perp"] & \text{Theoretical Idea} \arrow[l, bend right, "\text{praxis}"']
\end{tikzcd}$$

(With thanks to Hegel, Marx, Mao and Lawvere.)

overview of RedPRL



Cubical Refinement Logic

Cubical Abstract Machine

Cubical Type Theory

Cubical Abstract Binding Trees

Indexed Second-Order Algebra

Lawvere Duality

Dependent LCF

outline

1. The LCF Tradition
2. The Revisionists And Their Running Dogs
3. RedPRL: New Synthesis Of Proof Refinement

lcf architecture

```
type evd                                        (* evidence *)
type α state = α list ⊗ (evd list → evd)        (* proof state *)
type (α, β) tactic = α ⇀ β state
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(* proof state “monad” *)
val id : (α, α) tactic
val map : (α → β) → (α state, β) tactic
val mul : (α state state, α) tactic
val orelse : (α, β) tactic ⊗ (α, β) tactic → (α, β) tactic
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(* standard tacticals *)
val then : (α, β) tactic ⊗ (β, γ) tactic → (α, γ) tactic
fun then ($t_1$, $t_2$) = mul ∘ map  $t_2$ ∘ $t_1$
val thenl : (α, β) tactic ⊗ (β, γ) tactic list → (α, γ) tactic
```

lcf architecture

```
include LCF
```

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datatype prop =
    ∧ of prop ⊗ prop
    ∨ of prop ⊗ prop
    ⊃ of prop ⊗ prop
    ⊤, ⊥
```

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include LCF
datatype prop =
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datatype jdg = ⊢ of prop dict ⊗ prop
type rule = (jdg, jdg) tactic
```

lcf architecture

include LCF

datatype prop =

| $\wedge$ **of** prop $\otimes$ prop
| $\vee$ **of** prop $\otimes$ prop
| $\supset$ **of** prop $\otimes$ prop
| $\top, \bot$

datatype jdg = $\vdash$ **of** prop **dict** $\otimes$ prop

type rule = (jdg, jdg) **tactic**

val $\wedge_R, \vee_R, \supset_R, \top_R, \bot_R$: rule

val hyp : **string** $\to$ rule

val $\wedge_L$: **string** $\otimes$ **string** $\otimes$ **string** $\to$ rule

…

programs as evidence

Abstract (!!) type of evidence implemented as functional programming language:

```
datatype evd =
  | var of string
  | λ of string ⊗ evd
  | ap of evd ⊗ evd
  | pair of evd ⊗ evd
  | $\pi_1, \pi_2$ of evd
  | inl, inr of evd
  | split of evd ⊗ evd ⊗ evd
```

programs as evidence

Abstract **(!!)** type of evidence implemented as functional programming language:

$$
\begin{array}{l}
\textbf{datatype}\ \mathsf{evd} = \\
\quad \left|\ \begin{array}{l}
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\mathsf{split}\ \textbf{of}\ \mathsf{evd} \otimes \mathsf{evd} \otimes \mathsf{evd}
\end{array}\right.
\end{array}
$$

Other options possible: machine code, JavaScript, Perl, PHP, Julia ;-), etc.

inference rules

inference rule ⇔ ML function

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$$\frac{}{\Delta, x : P, \Xi \vdash P}\ hyp[x] \quad \Leftrightarrow \quad \begin{array}{l} \textbf{fun}\ \mathsf{hyp}\,(x)\ \left(\Gamma \vdash P\right) = \\ \quad \textbf{let} \quad (\Delta, Q, \Xi) = \mathsf{split}(\Gamma, x) \\ \quad \textbf{and} \quad \mathsf{true} = (P = Q) \\ \quad \textbf{in} \quad ([], \textbf{fn}\ [] \Rightarrow \mathsf{var}(x)) \end{array}$$

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$$\dfrac{\Gamma \vdash P \qquad \Gamma \vdash Q}{\Gamma \vdash P \wedge Q}\ \wedge_R \quad \Leftrightarrow \quad \begin{array}{l} \textbf{fun}\ \wedge_R \left(\Gamma \vdash P \wedge Q\right) = \\ \left(\begin{array}{l} \left[\Gamma \vdash P, \Gamma \vdash Q\right], \\ \textbf{fn}\ [M, N] \Rightarrow \mathsf{pair}(M, N) \end{array} \right) \end{array}$$

inference rules

$$\frac{\Delta, x : P \wedge Q, y : P, z : Q, \Xi \vdash R}{\Delta, x : P \wedge Q, \Xi \vdash R} \; \wedge_L[x, y, z]$$

$$\Updownarrow$$

$$\begin{array}{l}
\textbf{fun } \wedge_L (x, y, z) \left(\Gamma \vdash R \right) = \\
\quad \textbf{let } \; (\Delta, P \wedge Q, \Xi) = \mathtt{split}(\Gamma, x) \\
\quad\quad \textbf{in } \left(\begin{array}{l} \left[\Delta, x : P \wedge Q, y : P, z : Q, \Xi \vdash R \right], \\ \textbf{fn } [M] \Rightarrow [\pi_1(\mathtt{var}(x)), \pi_2(\mathtt{var}(x))/y, z]M \end{array} \right)
\end{array}$$

proofs and scripts

$$\frac{}{x : P \land Q \vdash P \land Q}$$

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$$\dfrac{\overline{x : P \wedge Q, y : P, z : Q \vdash P \wedge Q}}{x : P \wedge Q \vdash P \wedge Q}\ \wedge_L\,[x, y, z]$$

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$$\wedge_L\,(x, y, z)$$

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$\land_L\ (x, y, z)$
then $\land_R$

proofs and scripts

$$\dfrac{\dfrac{\dfrac{}{x : P \wedge Q, y : P, z : Q \vdash P}\ hyp[y] \qquad \dfrac{}{x : P \wedge Q, y : P, z : Q \vdash Q}\ hyp[z]}{x : P \wedge Q, y : P, z : Q \vdash P \wedge Q}\ \wedge_R}{x : P \wedge Q \vdash P \wedge Q}\ \wedge_L\ [x, y, z]$$

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free program invariants

A proof synthesizes a program ***(stop calling this "extraction"!).*** Depending on the structure of our logic, we can enforce many invariants!

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All this is possible, whilst generating efficient codes in an arbitrary language. Proof structure does not need to appear in programs.

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- ⋆ easy to check that a collection of rules is correct
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DECISIVELY SMASH THE FORMALIST CLIQUE!

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- ⋆ no existential variables and unification in core LCF framework (compromises soundness for some logics)
- ⋆ many sensible rules cannot be encoded (e.g. bidirectional typing)
- ⋆ complicated and brittle tactics are necessary for basic use

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3. RedPRL: New Synthesis Of Proof Refinement

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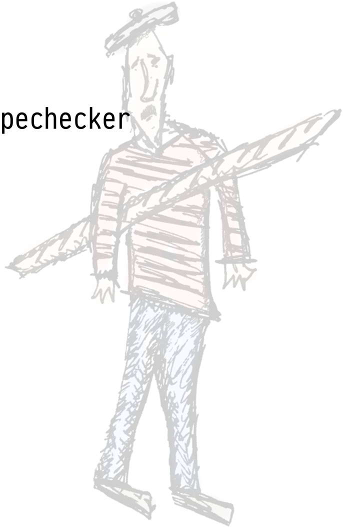

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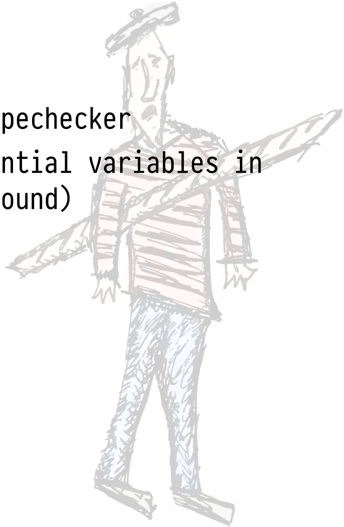

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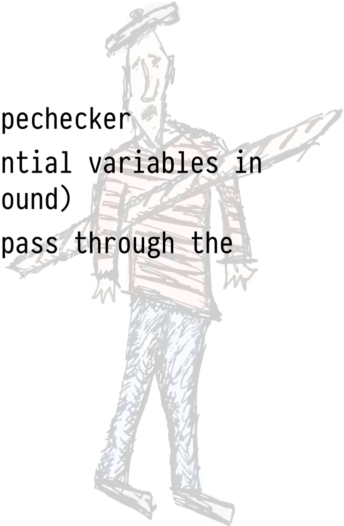

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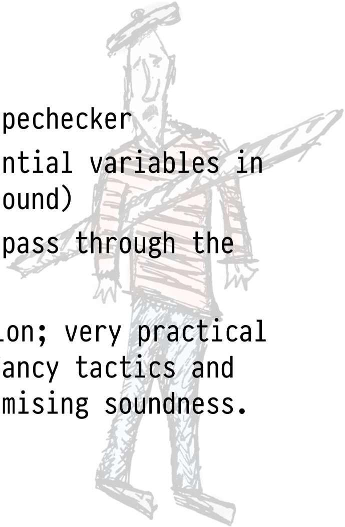

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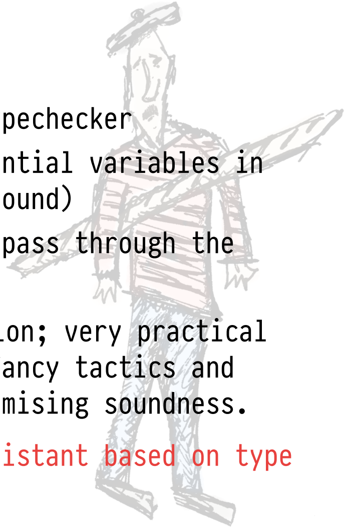

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Coq is the most successful proof assistant based on type theory in history.

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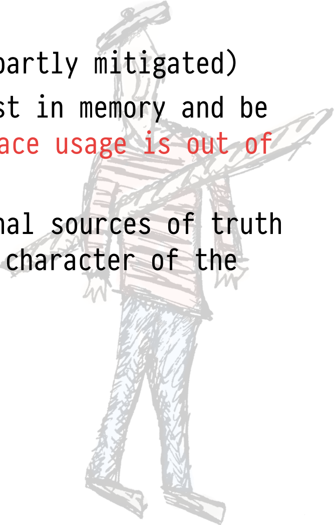

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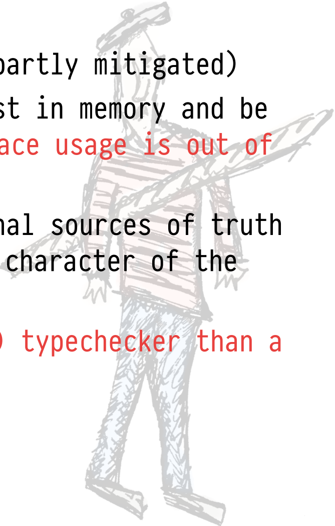

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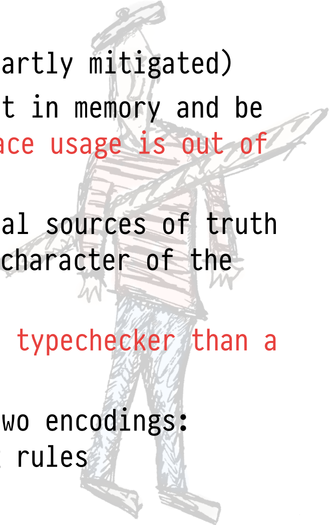

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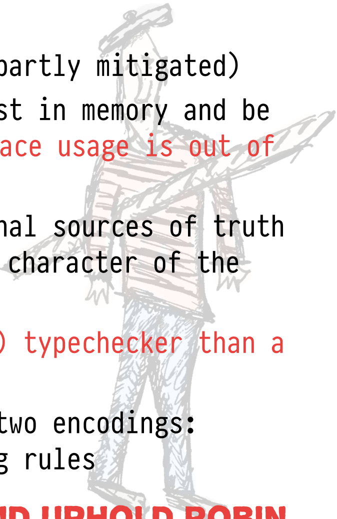

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RESIST FRENCH IMPERIALISM AND UPHOLD ROBIN MILNER THOUGHT!

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COQ

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- ⋆ **Metavariables can only be resolved *locally***, by refinement rules (NOT UNIFICATION).
- ⋆ Adds nothing to the object logic: just a means of incremental construction / refinement.
- ⋆ Precisely what is needed to encode existential instantiation, bidirectional typing rules.

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- ⋆ Works out fine in Coq because the refinement rules do not need to be sound.
- ⋆ Unification must be integrated as a judgment in your theory, not as part of a refinement framework. See Cockx/Devriese/Piessens ICFP 2016.

from classic lcf to *dependent lcf*

$$\frac{\begin{array}{l} \mathcal{J}_0 \rightsquigarrow \mathfrak{x}_0 \\ \mathcal{J}_1 \rightsquigarrow \mathfrak{x}_1 \\ \vdots \\ \mathcal{J}_n \rightsquigarrow \mathfrak{x}_n \end{array}}{\mathcal{J} \rightsquigarrow [\mathfrak{x}_0, \ldots, \mathfrak{x}_n].M} \; \texttt{my-rule}$$

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- ⋆ ***lax naturality*** ensures that rules commute with substitution up to approximation
- ⋆ gorgeous denotational semantics
- ⋆ EASY to implement. (maybe not super efficient! refinement machine future work.)

from classic lcf to *dependent lcf*

$$\dfrac{\begin{array}{l}[\Omega].\,\mathcal{J}_0 \rightsquigarrow \mathfrak{X}_0\\ [\Omega, \mathfrak{X}_0].\,\mathcal{J}_1 \rightsquigarrow \mathfrak{X}_1\\ \vdots\\ [\Omega, \mathfrak{X}_0, \ldots, \mathfrak{X}_{n-1}].\,\mathcal{J}_n \rightsquigarrow \mathfrak{X}_n\end{array}}{[\Omega].\,\mathcal{J} \rightsquigarrow [\mathfrak{X}_0, \ldots, \mathfrak{X}_n].\,M}\ \text{my-rule}$$

- ⋆ ***lax naturality*** ensures that rules commute with substitution up to approximation
- ⋆ gorgeous denotational semantics
- ⋆ EASY to implement. (maybe not super efficient! refinement machine future work.)
- ⋆ dependent refinement = maximum parallelism of proof acts

dependent lcf: examples

Enables a straightforward encoding of sophisticated dependent refinement rules which are not expressible in **Classic LCF** or Nuprl.

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wow!!

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A more sophisticated example: bidirectional typing rules (not just for typecheckers!—crucial for automation).

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$$\frac{}{[\Omega].\vartheta(M_0, \ldots, M_n)\ \mathit{match}\{i\}\ \vartheta \rightsquigarrow [\Omega].M_i}\ \text{match}$$

use *dependent lcf* today!

Implemented as a modular **Standard ML** library, which you can use in your own project today!

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Restricts automatically to **Classic LCF** when instantiated without dependency/substitution structure.

RedPRL's cubical type theory

$$\begin{tikzcd}
\sqcup^n_k \arrow[r] \arrow[d, hook] \arrow[rd, phantom, "\star"] & \chi \\
\square^n \arrow[ur, dashed] &
\end{tikzcd}$$

CUBICAL THOUGHT IS THE NEVER-SETTING SUN!

RedPRL's cubical type theory

Computational Higher-Dimensional Type Theory
[Angiuli/Harper/Wilson POPL 2017]

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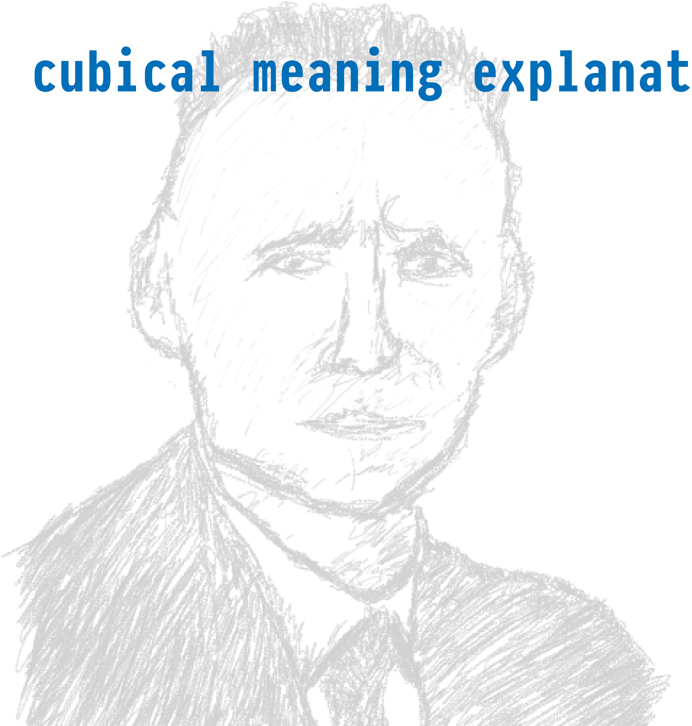
- a deterministic and type-free operational semantics, amenable to cost analysis.



cubical meaning explanation

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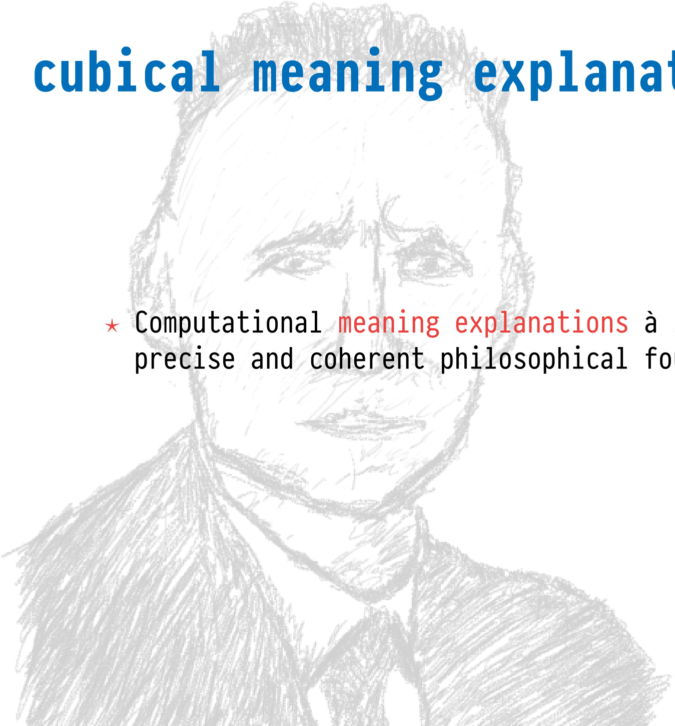
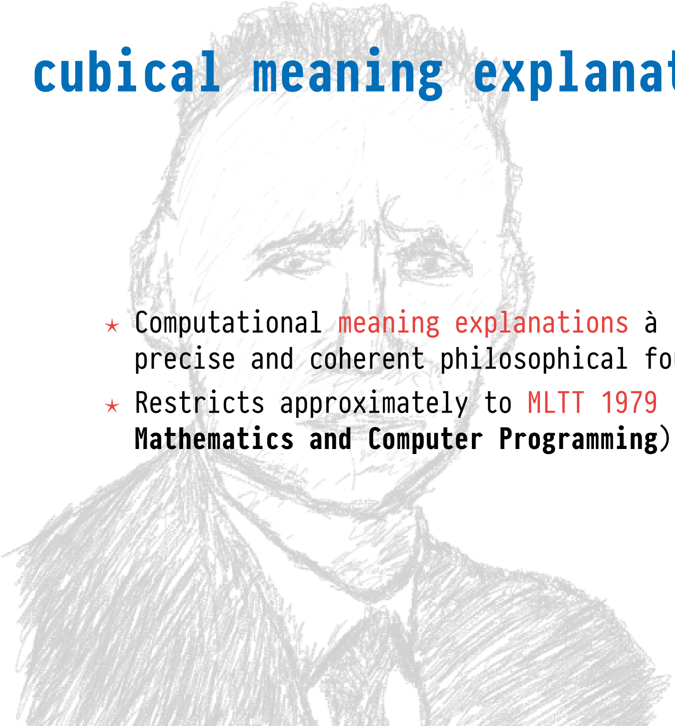


- Computational meaning explanations à la Martin-Löf: precise and coherent philosophical foundation.

cubical meaning explanation



⋆ Computational meaning explanations à la Martin-Löf: precise and coherent philosophical foundation.

⋆ Restricts approximately to MLTT 1979 (**Constructive Mathematics and Computer Programming**) at dimension 0.