# Guarded Computational Type Theory

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# what is $\mathrm{CTT}_{⏲}$?

a ***dependently typed*** program logic with guarded recursion, clocks, and coinductive types

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**COMPUTATION IS THE NEVER-SETTING SUN!**

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$$\frac{M \in \triangleright(A \to B) \qquad N \in \triangleright A}{M \circledast N \in \triangleright B}$$

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Solve “guarded recursive equations”:

$$\mathsf{Stream} \cong \mathbb{N} \times \triangleright\mathsf{Stream}$$

# semantics in the topos of trees

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(Straightforward to interpret $\mathsf{fix}(x.M)$.)

# causal (co)programming

the “later modality” $\triangleright$ enables ***causal*** programs on streams

$$\mathsf{incr} : \mathsf{Stream} \to \mathsf{Stream}$$
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guarded recursion not enough for (co)programming!

# local clocks for coinduction

idea: parameterize $\triangleright$ by a ***clock*** $\kappa$ and add quantifier $\forall\kappa$.
due to McBride, Atkey

$$\frac{\Gamma \vdash_{\Delta,\kappa} M \in A[\kappa]}{\Gamma \vdash_{\Delta} \Lambda\kappa.M \in \forall\kappa.A[\kappa]} \qquad \frac{\Gamma \vdash_{\Delta} M \in \forall\kappa.A[\kappa] \quad \kappa \mathrel{\#} \Delta}{\Gamma \vdash_{\Delta} M[\kappa'] \in A[\kappa']}$$

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$$\frac{\Gamma \vdash_{\Delta} A \ \text{type} \quad \kappa \in \Delta}{\Gamma \vdash_{\Delta} \triangleright_{\kappa} A \ \text{type}} \qquad \frac{\Gamma \vdash_{\Delta} M \in A}{\Gamma \vdash_{\Delta} \mathsf{next}_{\kappa}(M) \in \triangleright_{\kappa} A}$$

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# local clocks for coinduction

require the following isomorphisms:

$$\forall\kappa.\, \triangleright_{\kappa} A \cong \forall\kappa.A$$
$$\forall\kappa.A \cong A \quad (\kappa \# A)$$
$$\forall\kappa.A \times B \cong (\forall\kappa.A) \times (\forall\kappa.B)$$

etc.

# $\triangleright_\kappa$ + $\forall\kappa$ = coinduction!

Define streams in two parts (decompose productivity into causality and free use):

$$\mathsf{Stream}_\kappa \cong \mathbb{N} \times \triangleright_\kappa \mathsf{Stream}_\kappa$$

$$\mathsf{Stream} \triangleq \forall\kappa.\mathsf{Stream}_\kappa$$

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$$(\forall\kappa.\mathbb{N}) \times (\forall\kappa. \triangleright_\kappa \mathsf{Stream}_\kappa)$$

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# intensional version?

Both computation rule for $\mathsf{fix}_\kappa(x.M)$ and extensionality rule for $\mathsf{next}_\kappa$ incompatible with standard intensional type theory.

Birkedal et al resolve both problems in Guarded Cubical Type Theory. (Conjectured decidable typing result.)

**canonicity currently unknown; no operational semantics**
(but stay tuned)

# resurrecting extensionality

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Hope to combine with Computational Higher Dimensional Type Theory (Angiuli, Harper, Wilson). Guarded recursion will be the first of many extensions.

# preview of CTT$_{\circlearrowright}$

$$\frac{\Gamma \gg_{\Delta} M \in A}{\Gamma \gg_{\Delta} M \in \triangleright_{\kappa} A} \qquad \frac{\Gamma \gg_{\Delta} M \in \triangleright_{\kappa}(\Pi x : A.B[x]) \quad \Gamma \gg_{\Delta} M \in \triangleright_{\kappa} A}{\Gamma \gg_{\Delta} M(N) \in \triangleright_{\kappa} B[N]}$$

(look ma, no delayed substitutions!)

$$\frac{\Gamma, x : \triangleright_{\kappa} A \gg_{\Delta} M \in A}{\Gamma \gg_{\Delta} \mathsf{fix}(x.M) \in A}$$

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use extensionality as a weapon!

$$\cap\kappa.\triangleright_\kappa A \doteq \cap\kappa.A \;\; \mathsf{type}$$

$$\cap\kappa.A \doteq A \;\; \mathsf{type} \quad (\kappa \# A)$$

$$\cap\kappa.A \times B \doteq (\cap\kappa.A) \times (\cap\kappa.B) \;\; \mathsf{type}$$

# topos semantics

We develop the computational PER semantics (meaning explanation) for **CTT**$_{\circleddash}$ internally to a suitable ***sheaf topos*** which provides all the machinery that we need ($\triangleright_\kappa$, $\forall\kappa$).

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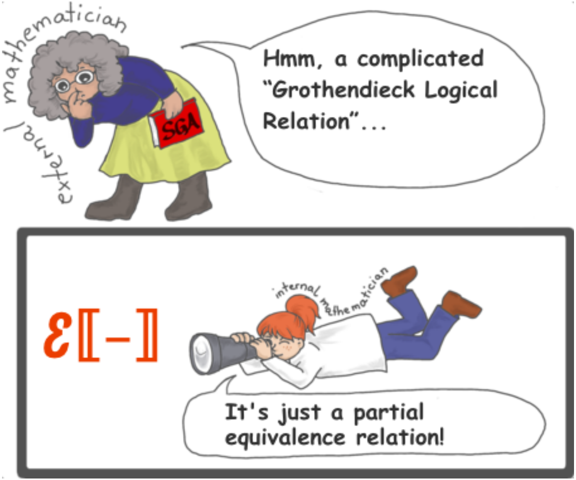
Please use these slogans:

- ⋆ **CRITICIZE THE OLD WORLD AND BUILD A NEW ONE WITH INTERNAL THOUGHT AS A WEAPON!**
- ⋆ **UPHOLD BETH-KRIPKE-JOYAL THOUGHT!**




(credit: Ingo Blechschmidt)

(time to use the blackboard)