# RedPRL

designing the people's refinement logic

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# what is RedPRL?

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So, why bother?

# the absolute idea

$$
\begin{tikzcd}
\text{Practical Idea} \arrow[r, bend right, "\textit{theory}"'] \arrow[r, phantom, "\perp"] & \text{Theoretical Idea} \arrow[l, bend right, "\textit{praxis}"']
\end{tikzcd}
$$

(With thanks to Hegel, Marx, Mao and Lawvere.)

# overview of RedPRL

Cubical Refinement Logic

Cubical Abstract Machine

Cubical Type Theory

Cubical Abstract Binding Trees

Dependent LCF

Indexed Second-Order Algebra

Lawvere Duality

# outline

1. The LCF/PRL Tradition
2. The Revisionists And Their Running Dogs
3. RedPRL: New Synthesis Of Proof Refinement

# lcf architecture

```
type evd                                       (* evidence *)
type α state = α list ⊗ (evd list → evd)       (* proof state *)
type (α, β) tactic = α ⇀ β state
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(* proof state “monad” *)
val id : (α, α) tactic
val map : (α → β) → (α state, β) tactic
val mul : (α state state, α) tactic
val orelse : (α, β) tactic ⊗ (α, β) tactic → (α, β) tactic
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(* standard tacticals *)
val then : (α, β) tactic ⊗ (β, γ) tactic → (α, γ) tactic
fun then (t1, t2) = mul ∘ map t2 ∘ t1
val thenl : (α, β) tactic ⊗ (β, γ) tactic list → (α, γ) tactic
```

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datatype prop =
  | ∧ of prop ⊗ prop
  | ∨ of prop ⊗ prop
  | ⊃ of prop ⊗ prop
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datatype jdg = ⊢ of prop dict ⊗ prop
type rule = (jdg, jdg) tactic
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datatype jdg = ⊢ of prop dict ⊗ prop
type rule = (jdg, jdg) tactic

val ∧R, ∨R, ⊃R, ⊤R, ⊥R : rule
val hyp : string → rule
val ∧L : string ⊗ string ⊗ string → rule
...
```

# programs as evidence

**Abstract** **(!!)** type of evidence implemented as functional programming language:

$$
\begin{aligned}
&\textbf{datatype}\ \mathtt{evd} = \\
&\quad\mid\ \mathtt{var}\ \textbf{of}\ \textbf{string} \\
&\quad\mid\ \lambda\ \textbf{of}\ \textbf{string} \otimes \mathtt{evd} \\
&\quad\mid\ \mathtt{ap}\ \textbf{of}\ \mathtt{evd} \otimes \mathtt{evd} \\
&\quad\mid\ \mathtt{pair}\ \textbf{of}\ \mathtt{evd} \otimes \mathtt{evd} \\
&\quad\mid\ \pi_1, \pi_2\ \textbf{of}\ \mathtt{evd} \\
&\quad\mid\ \mathtt{inl}, \mathtt{inr}\ \textbf{of}\ \mathtt{evd} \\
&\quad\mid\ \mathtt{split}\ \textbf{of}\ \mathtt{evd} \otimes \mathtt{evd} \otimes \mathtt{evd}
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\end{array}
$$

Other options possible: machine code, JavaScript, Perl, PHP, Julia ;-), etc.

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inference rule $\Leftrightarrow$ ML function

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$$\frac{}{\Delta, x : P, \Xi \vdash P}\ hyp[x] \quad \Leftrightarrow \quad \begin{array}{l} \textbf{fun}\ \mathsf{hyp}\,(x)\ \big(\Gamma \vdash P\big) = \\ \quad \textbf{let}\quad (\Delta, Q, \Xi) = \mathsf{split}(\Gamma, x) \\ \quad \textbf{and}\quad \mathsf{true} = (P = Q) \\ \quad \textbf{in}\quad ([\,], \textbf{fn}\ [\,] \Rightarrow \mathsf{var}(x)) \end{array}$$

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$$\frac{\Gamma \vdash P \quad \Gamma \vdash Q}{\Gamma \vdash P \wedge Q}\ \wedge_R \quad \Leftrightarrow \quad \begin{array}{l} \textbf{fun}\ \wedge_R \left(\Gamma \vdash P \wedge Q\right) = \\ \left( \begin{array}{l} \left[\Gamma \vdash P, \Gamma \vdash Q\right], \\ \textbf{fn}\ [M, N] \Rightarrow \mathsf{pair}(M, N) \end{array} \right) \end{array}$$

# inference rules

$$\frac{\Delta, x : P \wedge Q, y : P, z : Q, \Xi \vdash R}{\Delta, x : P \wedge Q, \Xi \vdash R} \wedge_L[x, y, z]$$

$$\Updownarrow$$

$$
\begin{array}{l}
\textbf{fun } \wedge_L (x, y, z) \; \left( \Gamma \vdash R \right) = \\
\quad \textbf{let} \;\; (\Delta, P \wedge Q, \Xi) = \mathtt{split}(\Gamma, x) \\
\quad \textbf{in} \;\; \left( \begin{array}{l} \left[ \Delta, x : P \wedge Q, y : P, z : Q, \Xi \vdash R \right], \\ \textbf{fn } [M] \Rightarrow [\pi_1(\mathtt{var}(x)), \pi_2(\mathtt{var}(x))/y, z]M \end{array} \right)
\end{array}
$$

# proofs and scripts

$$\overline{x : P \wedge Q \vdash P \wedge Q}$$

$$\Updownarrow$$

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$$\dfrac{\overline{x : P \wedge Q, y : P, z : Q \vdash P \wedge Q}}{x : P \wedge Q \vdash P \wedge Q} \;\wedge_L\, [x, y, z]$$

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$$\wedge_L\,(x, y, z)$$

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All this is possible, whilst generating efficient codes in an arbitrary language. Proof structure does not appear in programs.

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**DECISIVELY SMASH THE FORMALIST CLIQUE!**

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- ⋆ no existential variables and unification in core LCF framework (compromises soundness for some logics)
- ⋆ many sensible rules cannot be encoded (e.g. bidirectional typing)
- ⋆ complicated and brittle tactics are necessary for basic use

# outline

1. The LCF/PRL Tradition

2. **The Revisionists And Their Running Dogs**

3. RedPRL: New Synthesis Of Proof Refinement

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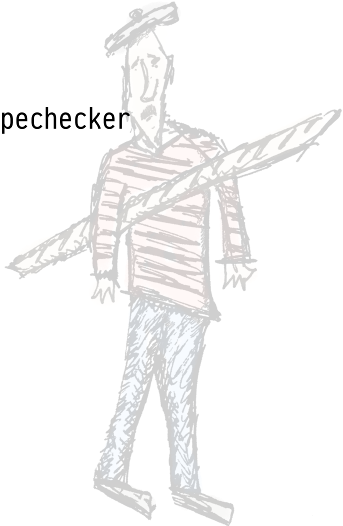

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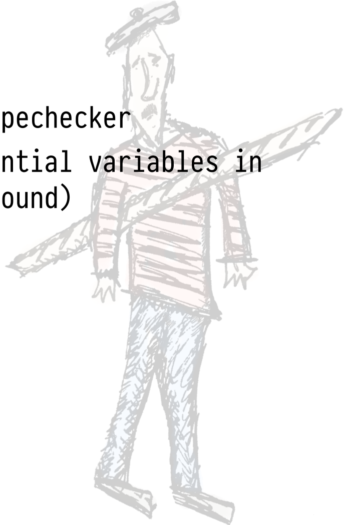

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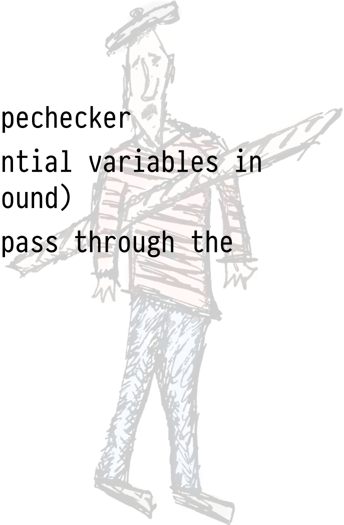

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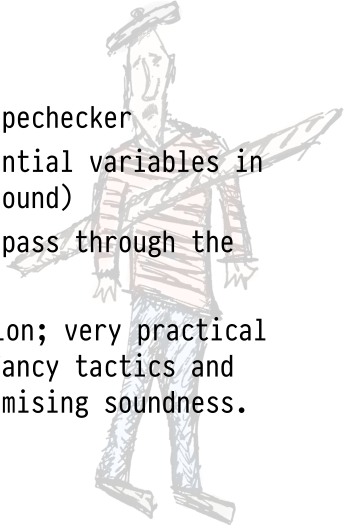

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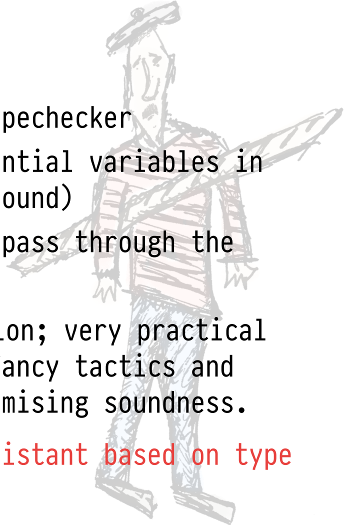

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Coq is the most successful proof assistant based on type theory in history.

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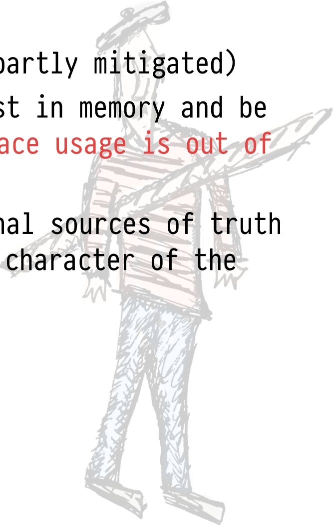

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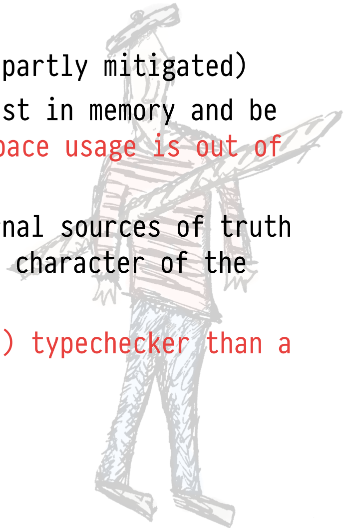

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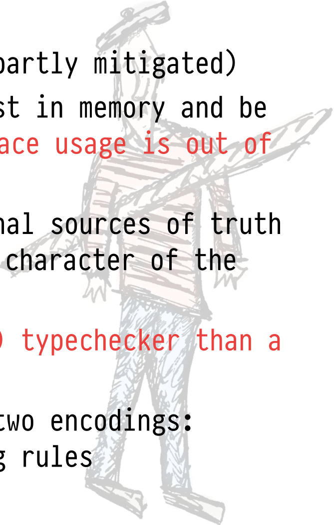

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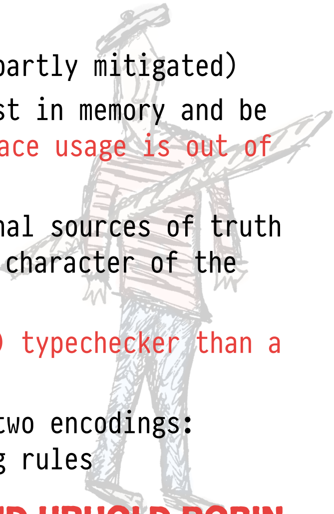

**RESIST FRENCH IMPERIALISM AND UPHOLD ROBIN MILNER THOUGHT!**

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- ⋆ **Metavariables can only be resolved *locally***, by refinement rules (NOT UNIFICATION).
- ⋆ Strict $1 \leftrightarrow 1$ relation between goals and metavariables in context.
- ⋆ Precisely what is needed to encode bidirectional typing rules.

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- ⋆ Works out fine in Coq because the refinement rules do not need to be sound.
- ⋆ Unification must be integrated as a judgment in your theory, not as part of a refinement framework. See Cockx/Devriese/Piessens ICFP 2016.

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Enables a straightforward encoding of sophisticated dependent refinement rules which are not expressible in Classic LCF or Nuprl.

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$$\frac{\Gamma \vdash \overset{(M)}{A} \ \textit{true} \quad \Gamma \vdash [M/x]B_x \ \textit{true}}{\Gamma \vdash (x : A) \times B_x \ \textit{true}} \text{ intro/}\Sigma$$

# use *dependent lcf* today!

Implemented as a modular **Standard ML** library, which you can use in your own project today!

https://github.com/RedPRL/sml-dependent-lcf

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Restricts automatically to **Classic LCF** when instantiated without dependency/substitution structure.

# RedPRL's cubical type theory

$$
\begin{tikzcd}
\sqcup^n_k \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\star"] & \chi \\
\square^n \arrow[ur, dashed] &
\end{tikzcd}
$$

# RedPRL's cubical type theory

**Computational Higher-Dimensional Type Theory**
[Angiuli/Harper/Wilson POPL 2017]

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- ⋆ higher inductive types: the circle
- ⋆ strict types: strict booleans (new)
- ⋆ computational canonicity (previous Licata/Harper result established canonicity up to ***judgmental equality*** for 2D type theory)

# RedPRL's cubical type theory

**Computational Higher-Dimensional Type Theory**
[Angiuli/Harper/Wilson POPL 2017]

- a type theory with both extensional ***equality*** and intensional ***identification*** (paths)
- higher inductive types: the circle
- strict types: strict booleans (new)
- computational canonicity (previous Licata/Harper result established canonicity up to ***judgmental equality*** for 2D type theory)
- an instance of univalence, $\mathsf{not}_x$.

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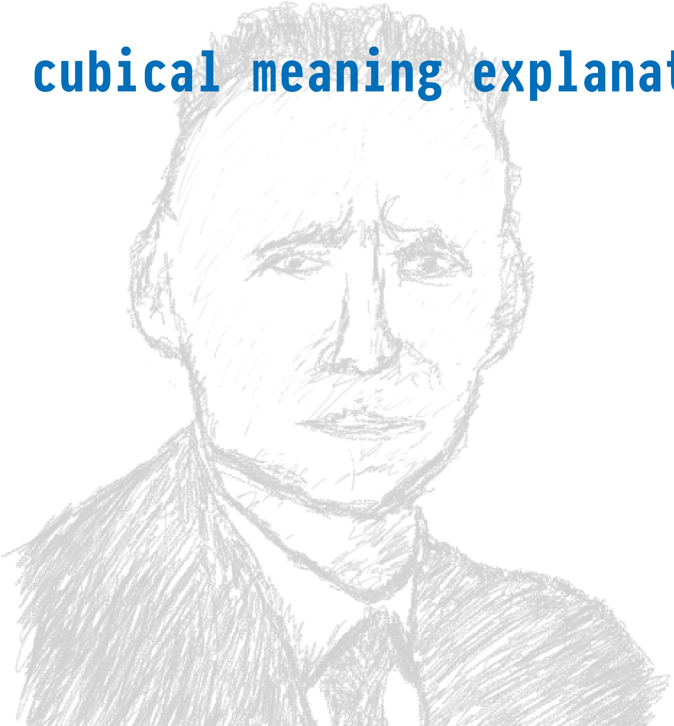
- ⋆ higher inductive types: the circle
- ⋆ strict types: strict booleans (new)
- ⋆ computational canonicity (previous Licata/Harper result established canonicity up to ***judgmental equality*** for 2D type theory)
- ⋆ an instance of univalence, $\mathrm{not}_x$.
- ⋆ a deterministic and type-free operational semantics, amenable to cost analysis.

# cubical meaning explanation



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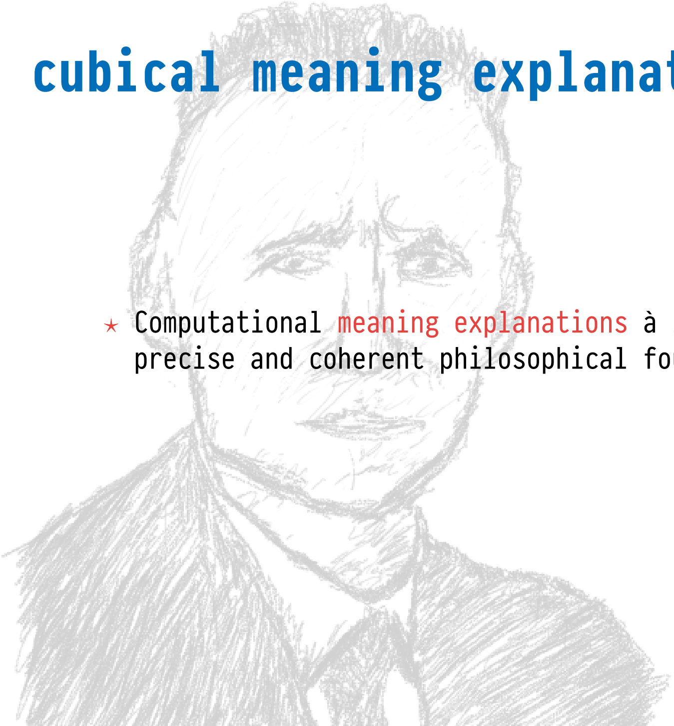
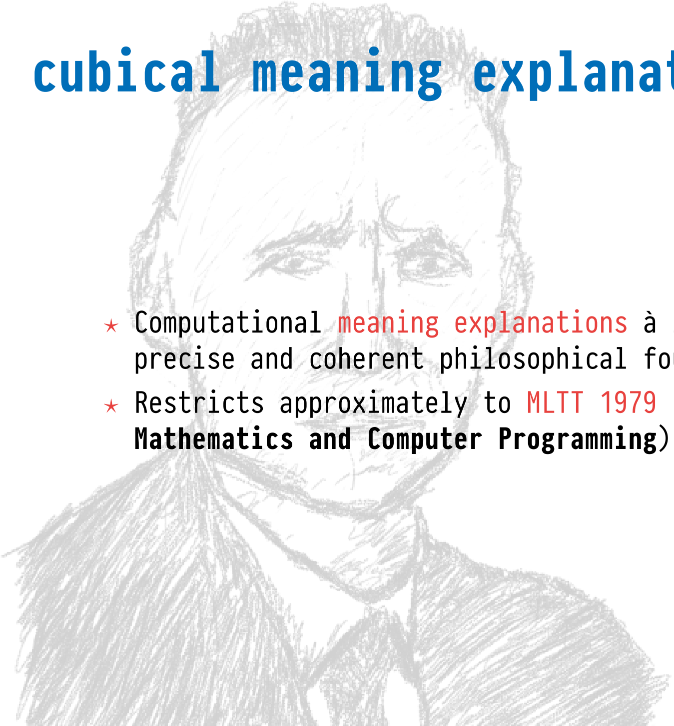


- ⋆ Computational meaning explanations à la Martin-Löf: precise and coherent philosophical foundation.



# cubical meaning explanation

- ⋆ Computational meaning explanations à la Martin-Löf: precise and coherent philosophical foundation.
- ⋆ Restricts approximately to MLTT 1979 (**Constructive Mathematics and Computer Programming**) at dimension 0.


COQ