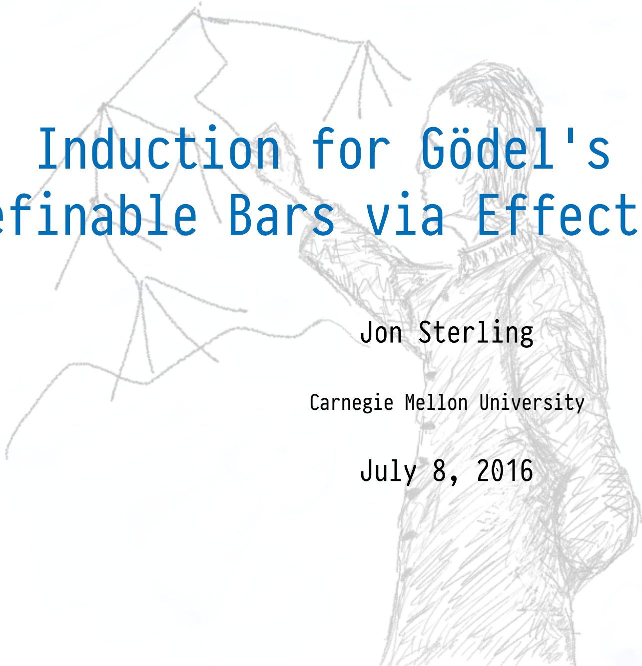

Induction for Gödel's **System** $\mathbb{T}$ Definable Bars via Effectful Forcing

Jon Sterling

Carnegie Mellon University

July 8, 2016

Brouwer's Bar Thesis

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- $\star$ $\Rightarrow$ all functions on the interval $\mathbb{I} \triangleq [0, 1]$ are uniformly continuous

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Usually, we will work implicitly with the ***universal spread***, which always says “yes”.

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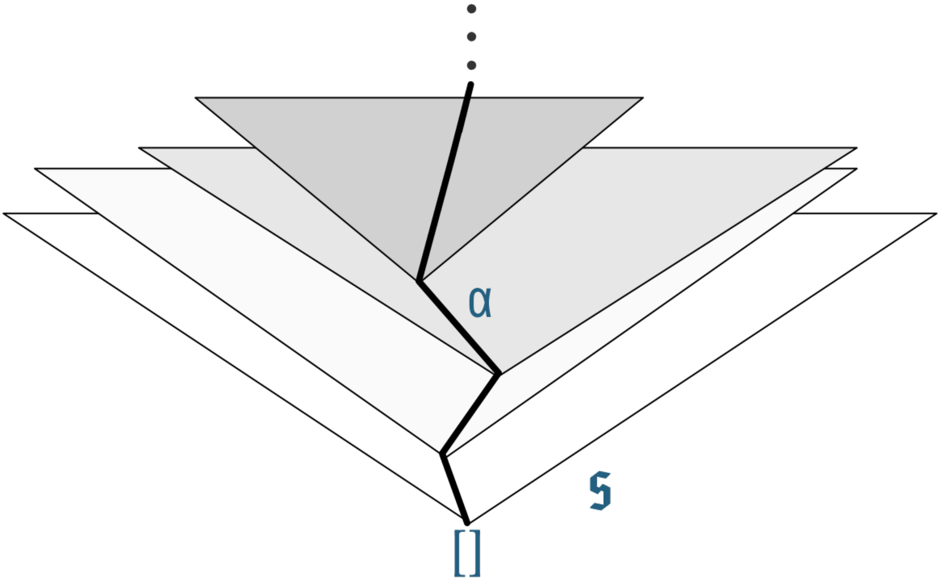
$$\vec{u} \prec \alpha \qquad (\vec{u} \text{ approximates } \alpha)$$

$$\alpha \in \vec{u} \qquad (\vec{u} \text{ is a neighborhood around } \alpha)$$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Bars and Securability

A bar $\mathfrak{B}$ is a predicate on neighborhoods such that every point “hits it”. More generally, $\mathfrak{B}$ bars a neighborhood $\vec{u}$ when every path through $\vec{u}$ ends up in $\mathfrak{B}$.

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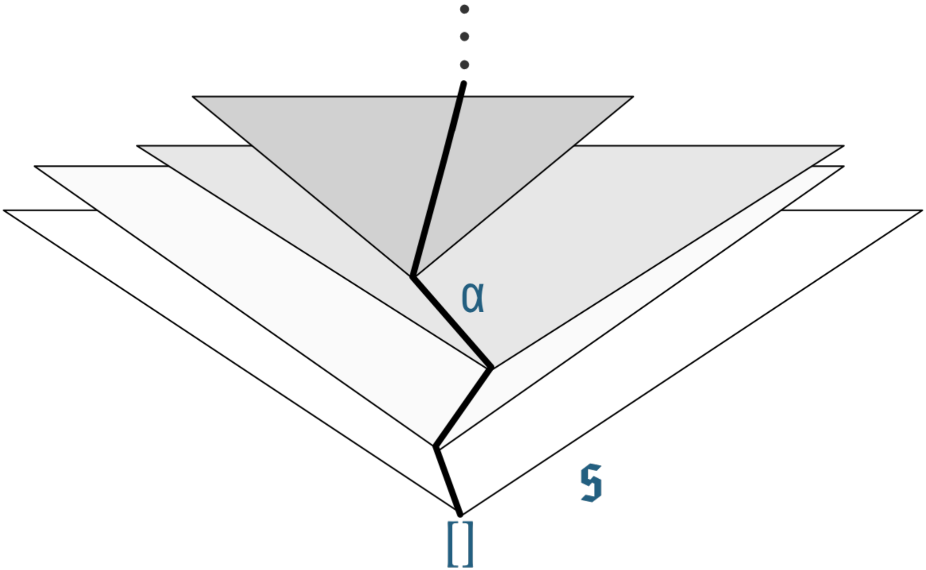
We say that a neighborhood is secured when it is in the bar ($\vec{u} \in \mathfrak{B}$), and that it is securable when every path out of it eventually hits the bar ($\vec{u} \lhd \mathfrak{B}$).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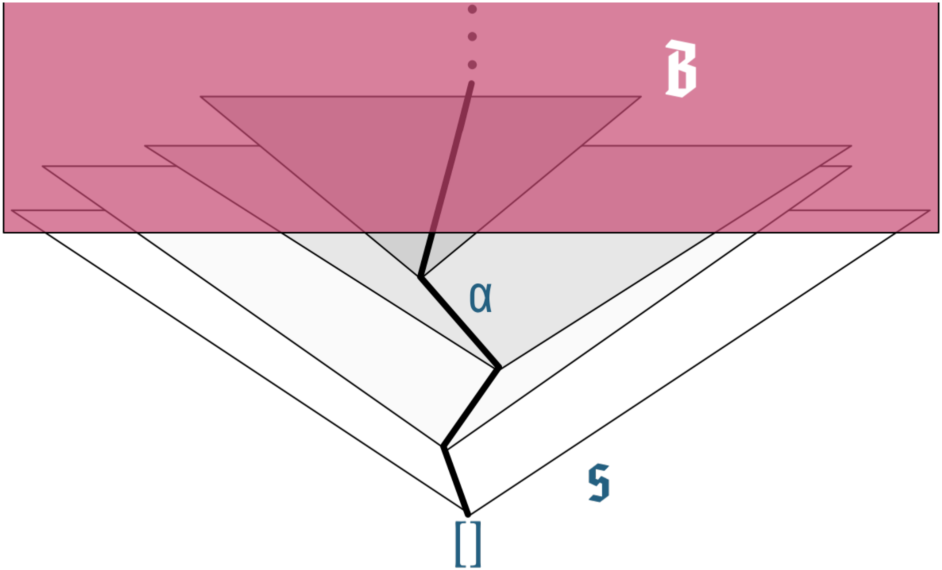
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Inductive Securability

“All demonstrations of securability can be analyzed into **inductive mental constructions.**”

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Let $\mathfrak{S}^{\natural}(\vec{u}) \triangleq \{ x \in \mathbb{N} \mid \vec{u} \frown x \in \mathfrak{S} \}$. Presupposing $\vec{u} \in \mathfrak{S}$ and $\mathfrak{B}$ *monotone*:

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Admissible (by monotonicity):

$$\frac{\vec{u} \blacktriangleleft_{ind} \mathfrak{B}}{\vec{u}^\frown x \blacktriangleleft_{ind} \mathfrak{B}}\ \zeta$$

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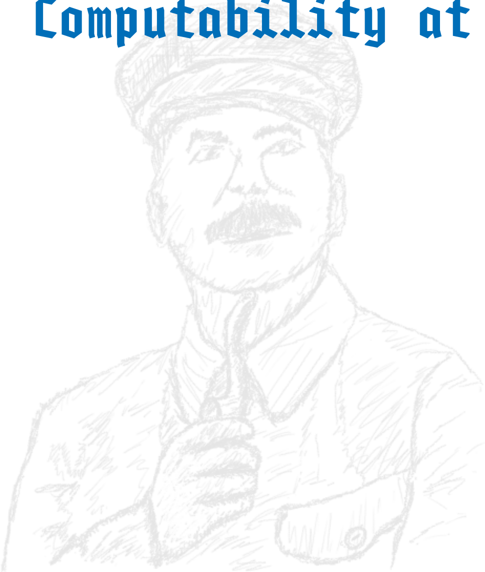
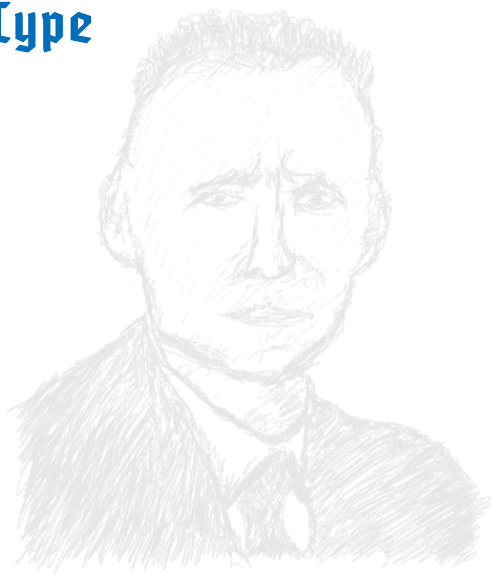
- ⋆ soundness is easy! Just count up the F-nodes.
- ⋆ completeness does not (generally) hold: procedure exists, but its termination requires Brouwer's Thesis!

Computability at Higher Type

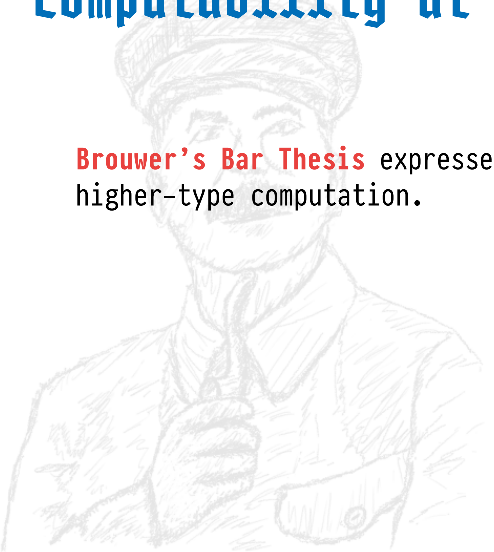


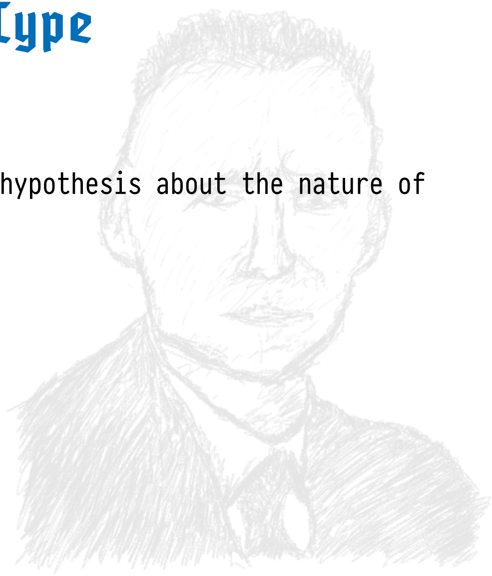


vulgar mechanicalism vs. phenomenology

Computability at Higher Type

Brouwer's Bar Thesis expresses a ***scientific*** hypothesis about the nature of higher-type computation.





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general recursive λ-calculus	wellfounded mental construction + oracles
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uniform continuity fails on $\mathbb{I}$	uniform continuity obtains

System $\mathbb{T}$ as a theory of constructions

Gödel's **System** $\mathbb{T}$ of primitive recursive functionals of finite type.

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Contexts

$$\frac{}{\cdot\ \mathit{ctx}} \qquad \frac{\Gamma\ \mathit{ctx} \quad \sigma\ \mathit{type}}{\Gamma, x : \sigma\ \mathit{ctx}}\ (x \notin \Gamma)$$

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$$\frac{\Gamma, x : \sigma \vdash m[x] : \tau}{\Gamma \vdash \lambda x.m[x] : \sigma \to \tau}\ \mathsf{lam} \qquad \frac{\Gamma \vdash m : \sigma \to \tau \quad \Gamma \vdash n : \sigma}{\Gamma \vdash m \bullet_{\sigma} n : \tau}\ \mathsf{ap}$$

Realizability of barhood

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Realizability of barhood

- $\star$ Use **System** $\mathbb{T}$ as a theory of constructions for primitive recursive arithmetic
- $\star$ $\vec{u} \lhd \mathcal{B}$ should be realized by a functional $\cdot \vdash \phi : (\mathsf{nat} \to \mathsf{nat}) \to \mathsf{nat}$
- $\star$ **Idea:** construct a model for **System** $\mathbb{T}$ in which we can read from the interpretation of ϕ a proof of $\vec{u} \blacktriangleleft_{ind} \mathcal{B}$

Standard semantics of System $\mathbb{T}$

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Standard semantics of System $\mathbb{T}$

Presupposing $\rho \in \mathcal{G}[\![\Gamma]\!]$ and $\Gamma \vdash m : \sigma$, define $[\![\Gamma \vdash m : \sigma]\!]_\rho \in \mathcal{U}[\![\sigma]\!]$ by recursion on m:

$$
\begin{aligned}
[\![\Gamma \vdash x : \sigma]\!]_\rho &\triangleq \rho(x) \\
[\![\Gamma \vdash \mathsf{z} : \mathsf{nat}]\!]_\rho &\triangleq 0 \\
[\![\Gamma \vdash \mathsf{s}(m) : \mathsf{nat}]\!]_\rho &\triangleq 1 + [\![\Gamma \vdash m : \mathsf{nat}]\!]_\rho \\
[\![\Gamma \vdash \mathsf{rec}_\sigma([x,y].s[x,y]; z; n) : \sigma]\!]_\rho &\triangleq \mathit{PrimRec}(S, Z, N)
\end{aligned}
$$

where

$$
\begin{aligned}
S(a,b) &\triangleq [\![\Gamma, x : \mathsf{nat}, y : \sigma \vdash s[x,y] : \sigma]\!]_{\rho, x \mapsto a, y \mapsto b} \\
Z &\triangleq [\![\Gamma \vdash z : \sigma]\!]_\rho \\
N &\triangleq [\![\Gamma \vdash n : \mathsf{nat}]\!]_\rho
\end{aligned}
$$

Standard semantics of System $\mathbb{T}$

$$\vdots$$

$$\llbracket \Gamma \vdash \lambda x.m[x] : \sigma \to \tau \rrbracket_\rho \triangleq a \mapsto \llbracket \Gamma, x : \sigma \vdash m[x] : \tau \rrbracket_{\rho, x \mapsto a}$$

$$\llbracket \Gamma \vdash m \bullet_\sigma n : \tau \rrbracket_\rho \triangleq \left(\llbracket \Gamma \vdash m : \sigma \to \tau \rrbracket_\rho \right) \left(\llbracket \Gamma \vdash n : \sigma \rrbracket_\rho \right)$$

Revising the Bar Thesis

A functional $\cdot \vdash \phi : (\mathsf{nat} \to \mathsf{nat}) \to \tau$ can be applied to a meta-level sequence α as follows:

$$\phi\,\langle\alpha\rangle \triangleq [\![\cdot \vdash \phi : (\mathsf{nat} \to \mathsf{nat}) \to \mathsf{nat}]\!]\,(\alpha)$$

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Escardo dialogues

An inductive encoding of functionals $Y^X \to Z$:

$$\frac{z \in Z}{\eta(z) \in \{\!\![Y^X, Z]\!\!\}}\ \text{return} \qquad \frac{x \in X \quad e \in Y \to \{\!\![Y^X, Z]\!\!\}}{\varphi\langle x\rangle(e) \in \{\!\![Y^X, Z]\!\!\}}\ \text{query}$$

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souped up neighborhood functions!

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We can execute dialogue trees against a sequence $\alpha \in Y^X$:

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$$\begin{tikzcd}
\{\!\![Y^X, A]\!\!\} \arrow[r, "{\{\!\![Y^X, f]\!\!\}}"] \arrow[d, "- \diamond \alpha"] & \{\!\![Y^X, B]\!\!\} \arrow[d, "- \diamond \alpha"] \\
A \arrow[r, "f"] & B
\end{tikzcd}$$

Escardo dialogues

We can execute dialogue trees against a sequence $\alpha \in Y^X$:

$$\eta(z) \diamond \alpha \triangleq z$$
$$\text{ϙ}\langle x\rangle(e) \diamond \alpha \triangleq e(\alpha(x)) \diamond \alpha$$

$$\begin{tikzcd}
\lBrace Y^X, A \rBrace \arrow[rr, "f^\star"] \arrow[d, "- \diamond \alpha"] & & \lBrace Y^X, B \rBrace \arrow[d, "- \diamond \alpha"] \\
A \arrow[r, "f"] & \lBrace Y^X, B \rBrace \arrow[r, "- \diamond \alpha"] & B
\end{tikzcd}$$

Dialectical semantics of System $\mathbb{T}$

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Dialectical semantics of System $\mathbb{T}$

Higher-Type Kleisli Extension
Presupposing σ *type* and $f \in X \to U\langle\!\langle\sigma\rangle\!\rangle$, define:

$$
\begin{aligned}
f^{\circledast}_{\sigma} &\in \{\![\mathbb{N}^{\mathbb{N}}, X]\!\} \to U\langle\!\langle\sigma\rangle\!\rangle \\
f^{\circledast}_{\iota}(e) &\triangleq f^{\star}(e) \\
f^{\circledast}_{\sigma\to\tau}(e) &\triangleq s \mapsto f(-)(s)^{\circledast}_{\tau}(e)
\end{aligned}
$$

Dialectical semantics of System $\mathbb{T}$

Presupposing $\rho \in \mathcal{G}\langle\!\langle\Gamma\rangle\!\rangle$ and $\Gamma \vdash m : \sigma$, we define the interpretation $\langle\!\langle\Gamma \vdash m : \sigma\rangle\!\rangle_\rho \in \mathcal{U}\langle\!\langle\sigma\rangle\!\rangle$:

$$
\begin{aligned}
\langle\!\langle\Gamma \vdash x : \sigma\rangle\!\rangle_\rho &\triangleq \rho(x) \\
\langle\!\langle\Gamma \vdash \mathsf{z} : \mathsf{nat}\rangle\!\rangle_\rho &\triangleq \eta(0) \\
\langle\!\langle\Gamma \vdash \mathsf{s}(m) : \mathsf{nat}\rangle\!\rangle_\rho &\triangleq \{\!\!\{\mathsf{N}^{\mathsf{N}}, 1 + -\}\!\!\}\left(\langle\!\langle\Gamma \vdash m : \mathsf{nat}\rangle\!\rangle_\rho\right) \\
\langle\!\langle\Gamma \vdash \mathsf{rec}_\sigma([x,y].s[x,y]; z; n) : \sigma\rangle\!\rangle_\rho &\triangleq \mathit{PrimRec}(S, Z, -)^{\star}_{\sigma}(N)
\end{aligned}
$$

where

$$
\begin{aligned}
S(a,b) &\triangleq \langle\!\langle\Gamma, x : \mathsf{nat}, y : \sigma \vdash s[x,y] : \sigma\rangle\!\rangle_{\rho, x\mapsto a, y\mapsto b} \\
Z &\triangleq \langle\!\langle\Gamma \vdash z : \sigma\rangle\!\rangle_\rho \\
N &\triangleq \langle\!\langle\Gamma \vdash n : \mathsf{nat}\rangle\!\rangle_\rho
\end{aligned}
$$

Dialectical semantics of System $\mathbb{T}$

$$\vdots$$

$$\langle\!\langle \Gamma \vdash \lambda x.m[x] : \sigma \to \tau \rangle\!\rangle_\rho \triangleq a \mapsto \langle\!\langle \Gamma, x : \sigma \vdash m[x] : \tau \rangle\!\rangle_{\rho, x \mapsto a}$$

$$\langle\!\langle \Gamma \vdash m \bullet_\sigma n : \tau \rangle\!\rangle_\rho \triangleq \left(\langle\!\langle \Gamma \vdash m : \sigma \to \tau \rangle\!\rangle_\rho \right) \left(\langle\!\langle \Gamma \vdash n : \sigma \rangle\!\rangle_\rho \right)$$

Coherence via Logical Relations

We need to show that the two interpretations cohere at base type, i.e. that the following diagram commutes:

$$\begin{tikzcd}
 & \cdot \vdash \imath \arrow[dl, "\langle\!\langle \cdot \vdash - : \imath \rangle\!\rangle"'] \arrow[dr, "[\![\cdot \vdash - : \imath]\!]"] & \\
U \langle\!\langle \imath \rangle\!\rangle \arrow[rr, "- \diamond \alpha"'] & & U [\![\imath]\!]
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SLOGAN: STRENGTHEN THE INDUCTIVE HYPOTHESIS WITH LOGICAL RELATIONS AS A WEAPON!

Coherence via Logical Relations

"logical relations" = family of relations defined by recursion on object-level types

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$$\frac{F = d \diamond \alpha}{F\ \mathcal{R}^{\alpha}_{\iota}\ d} \qquad \frac{\forall G \in \mathcal{U}\,[\![\sigma]\!], e \in \mathcal{U}\,\langle\!\langle\sigma\rangle\!\rangle.\ G\ \mathcal{R}^{\alpha}_{\sigma}\ e \implies F(G)\ \mathcal{R}^{\alpha}_{\tau}\ d(e)}{F\ \mathcal{R}^{\alpha}_{\sigma\to\tau}\ d}$$

$$\frac{\forall x \in |\Gamma|.\ \rho_0(x)\ \mathcal{R}^{\alpha}_{\Gamma(x)}\ \rho_1(x)}{\rho_0\ \overline{\mathcal{R}^{\alpha}_{\Gamma}}\ \rho_1}$$

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In the dialogue model, we can define a so-called “generic point” which is not definable in **System** $\mathbb{T}$:

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\mathit{generic} &\in \{\!\![Y^X, X]\!\!\} \to \{\!\![Y^X, Y]\!\!\} \\
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THE GENERIC POINT IS THE MAGIC WEAPON TO VICTORIOUSLY TRACE A FUNCTIONAL'S INTERACTION WITH THE AMBIENT CHOICE SEQUENCE!

— Quotations From Chairman Thierry Coquand

Taking stock

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- For any functional $\cdot \vdash \phi : (\mathsf{nat} \to \mathsf{nat}) \to \mathsf{nat}$, we can compute a dialogue tree $\langle\!\langle \cdot \vdash \phi : (\mathsf{nat} \to \mathsf{nat}) \to \mathsf{nat} \rangle\!\rangle\,(\mathit{generic}) \in \{\![\mathbb{N}^{\mathbb{N}}, \mathbb{N}]\!\}$.

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- Escardó's trees represent ***persistent inspection*** of a choice sequence; Brouwer's trees represent ***ephemeral consumption*** of a choice sequence.
- **Idea:** Normalize dialogue trees into Brouwerian mental constructions.

Brouwer's ephemeral dialectics

$$\frac{z \in Z}{\eta(z) \in (\!|Y^{\mathbb{N}}, Z|\!)}\ \texttt{spit} \qquad \frac{b \in Y \to (\!|Y^{\mathbb{N}}, Z|\!)}{\mathsf{F}(b) \in (\!|Y^{\mathbb{N}}, Z|\!)}\ \texttt{bite}$$

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Ephemeral execution:

$$\eta(z) \odot \alpha \triangleq z$$
$$\mathsf{F}(b) \odot \alpha \triangleq b(\mathsf{head}(\alpha)) \odot \mathsf{tail}(\alpha)$$

Normalizing dialogues

Presupposing $t \in \{\!\![Y^{\mathbb{N}}, Z]\!\!\}$, define total normalization relation $\vec{u} \Vdash t \rightsquigarrow b$ with $b \in (\!| Y^{\mathbb{N}}, Z |\!)$.

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Naïve Algorithm:

$$\mathit{norm}_{\vec{u}}(\eta(z)) \triangleq \eta(z)$$

$$\mathit{norm}_{\vec{u}}(\varrho\langle i\rangle(t)) \triangleq \begin{cases} \vec{u}_i & \text{if } i < |\vec{u}| \\ \mathsf{F}(y \mapsto \mathit{norm}_{\vec{u}\frown y}(\varrho\langle i\rangle(t))) & \text{if } i \geq |\vec{u}| \end{cases}$$

$$\frac{\mathit{norm}_{\vec{u}}(t) \equiv b}{\vec{u} \Vdash t \rightsquigarrow b}$$

Structurally recursive definition easy, but bureaucratic. See paper.

The Birds'-Eye View

$$
\begin{tikzcd}
\cdot \vdash \imath \arrow[r, "{[\![\cdot \vdash - : \imath]\!]}"] \arrow[d, "{\langle\!\langle \cdot \vdash - : \imath \rangle\!\rangle}"'] & U[\imath] \\
\{\![Y^{\mathbb{N}}, U[\imath]]\!\} \arrow[ur, "- \diamond \alpha"] \arrow[r, "norm_{\langle\rangle}(-)"'] & (\!| Y^{\mathbb{N}}, U[\imath] |\!) \arrow[u, "- \odot \alpha"']
\end{tikzcd}
$$

The Bar Thesis

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Summary of Results

- ⋆ The Bar Theorem is ***constructively valid*** in primitive recursive realizability (with correct/full interpretation of functional types).
- ⋆ Thence, we have the ***Bar Induction Principle*** and the ***Fan Theorem*** (constructive König's Lemma).
- ⋆ What is the status of Brouwer's Thesis as a ***scientific hypothesis***?

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- ⋆ What is the status of Brouwer's Thesis as a ***scientific hypothesis***?

Can this result be extended to general recursive realizability (assuming open interpretation of $\rightarrow$)?