Homework 1

Due on February 20 @ 4:00pm

CS 485/685 – Computer Vision

Problem 1

[20 pts] Self Assessment Questions: In this problem, you will review the lecture from unit 02- Math Review. After studying the topics covered, you will design the following questions. For each of your questions, you will provide the correct answers with the rationale behind each answer.

a) **[4 pts]** Design and answer four (4) multiple choice questions.

b) **[6 pts]** Design and answer two (2) fill in the blank questions.

c) **[10 pts]** Design and answer two (2) mathematical proof questions, mathematical computation questions, or a both.

Answer:

a) Design and answer four (4) multiple choice questions:

- 1. What are two main uses of Vectors?
  a) Picture, Pixel
  b) gradient, offsets
  c) Data, Geometric representation
  d) gradient, Geometric representation ← **Answer**
- 2. Dot product of two vectors is a
  a) vector
  b) integer
  c) point
  d) scalar ← **Answer**
- 3. Vectors can be expressed as what combinations of the unit vectors?
  a) linear ← **Answer**
  b) non-linear
  c) Geometric
  d) Gradient
- 4. When can we say a set of vectors are linearly dependent
  a) at least one is a non linear combination of the other
  b) at least one is a linear combination of the other ← **Answer**
  c) more than one is a Geometric representation of the other
  d) none of the above

b) Design and answer two (2) fill in the blank questions:

- 2D and 3D offsets can be represented by ______.
  vectors
- Points are ______ from the origin
  vectors

c) Design and answer two (2) mathematical proof questions, mathematical computation questions, or a both:

Problem 1 continued on next page. . .

- prove $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} + \vec{w}$

$$\begin{aligned}
\vec{u} \centerdot (\vec{v} + \vec{w}) &= \langle u_1, u_2, \ldots, u_n \rangle \centerdot (\langle v_1, v_2, \ldots, v_n \rangle + \langle w_1, w_2, \ldots, w_n \rangle) \\
&= \langle u_1, u_2, \ldots, u_n \rangle \centerdot \langle v_1 + w_1, v_2 + w_2, \ldots, v_n + w_n \rangle \\
&= u_1 \left(v_1 + w_1\right), u_2 \left(v_2 + w_2\right), \ldots, u_n \left(v_n + w_n\right) \\
&= u_1 v_1 + u_1 w_1, u_2 v_2 + u_2 w_2, \ldots, u_n v_n + u_n w_n \\
&= (u_1 v_1, u_2 v_2, \ldots, u_n v_n) + (u_1 w_1, u_2 w_2, \ldots, u_n w_n) \\
&= \langle u_1, u_2, \ldots, u_n \rangle \centerdot \langle v_1, v_2, \ldots, v_n \rangle + \langle u_1, u_2, \ldots, u_n \rangle \centerdot \langle w_1, w_2, \ldots, w_n \rangle \\
&= \vec{u} \centerdot \vec{v} + \vec{u} \centerdot \vec{w}
\end{aligned}$$

- prove $\vec{v} \cdot \vec{v} = 0 \rightarrowtail \vec{v} = 0$

$$\begin{aligned}
\vec{v} \centerdot \vec{v} &= \langle v_1, v_2, \ldots, v_n \rangle \centerdot \langle v_1, v_2, \ldots, v_n \rangle \\
&= v_1^2 + v_2^2 + \cdots + v_n^2 \\
&= 0
\end{aligned}$$

Problem 2

[20 pts]

$$\mathbf{A}^T = [1 \quad 2 \quad 3], \qquad \mathbf{B}^T = [2 \quad 1 \quad 3] \qquad \text{and} \qquad \mathbf{C}^T = [1 \quad 3 \quad 2] \tag{1}$$

Answer the following questions:

a) **[5 pts]** Do the vectors A, B, and C span $\mathbb{R}^3$? Why?

b) **[10 pts]** If the vectors above span $\mathbb{R}^3$, find the orthonormal basis vectors corresponding to each. If they don't span $\mathbb{R}^3$, modify one of the vectors and then find the orthonormal basis vectors corresponding to each.

c) **[5 pts]** Find the expansion of the vector $V^T = [6 \quad 8 \quad -4]$ in the orthonormal basis vector above.

Answer:

a) Calculating matrix rank

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -3 \\ 0 & 0 & -2 \end{bmatrix} \rightarrow \text{Rank } \rightarrow 3$$

Yes they span over R^3 it is clear the rank of the matrix is 3, so the vectors span a subspace of dimension 3

b)

$$\mathbf{u_k} = \mathbf{v_k} - \sum_{j=1}^{k-1} \text{proj}_{\mathbf{u_j}}(\mathbf{v_k}) \quad \text{where} \quad \text{proj}_{\mathbf{u}}(\mathbf{v}) = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{u} \cdot \mathbf{u}}\mathbf{u} \quad \text{Normalized vector is} \quad \mathbf{n_k} = \frac{\mathbf{u_k}}{\sqrt{\mathbf{u_k} \cdot \mathbf{u_k}}}$$

$$\mathbf{u_1} = \mathbf{v_1} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\mathbf{n_1} = \frac{\mathbf{u_1}}{\sqrt{\mathbf{u_1} \cdot \mathbf{u_1}}} = \begin{bmatrix} \frac{\sqrt{14}}{14} \\ \frac{\sqrt{14}}{7} \\ \frac{3\sqrt{14}}{14} \end{bmatrix} \leftarrow$$

$$\mathbf{u_2} = \mathbf{v_2} - \frac{\mathbf{u_1} \cdot \mathbf{v_2}}{\mathbf{u_1} \cdot \mathbf{u_1}}\mathbf{u_1} = \begin{bmatrix} \frac{15}{14} \\ -\frac{6}{7} \\ \frac{3}{14} \end{bmatrix}$$

$$\mathbf{n_2} = \frac{\mathbf{u_2}}{\sqrt{\mathbf{u_2} \cdot \mathbf{u_2}}} = \begin{bmatrix} \frac{5\sqrt{42}}{42} \\ \\ -\frac{2\sqrt{42}}{21} \\ \\ \frac{\sqrt{42}}{42} \end{bmatrix} \leftarrow$$

$$\mathbf{u_3} = \mathbf{v_3} - \frac{\mathbf{u_1} \cdot \mathbf{v_3}}{\mathbf{u_1} \cdot \mathbf{u_1}}\mathbf{u_1} - \frac{\mathbf{u_2} \cdot \mathbf{v_3}}{\mathbf{u_2} \cdot \mathbf{u_2}}\mathbf{u_2} = \begin{bmatrix} \frac{2}{3} \\ \\ \frac{2}{3} \\ \\ -\frac{2}{3} \end{bmatrix}$$

$$\mathbf{n_3} = \frac{\mathbf{u_3}}{\sqrt{\mathbf{u_3} \cdot \mathbf{u_3}}} = \begin{bmatrix} \frac{\sqrt{3}}{3} \\ \\ \frac{\sqrt{3}}{3} \\ \\ -\frac{\sqrt{3}}{3} \end{bmatrix} \leftarrow$$

Problem 3

[20 pts] Suppose you have n vectors $V_1, V_2, \cdots, V_n$ that form a basis for space $\mathbb{R}^n$. Any vector V in this space will have a unique expansion of the form below:

$$V = a_1V_1 + a_2V_2 + \cdots + a_nV_n \tag{2}$$

a) **[4 pts]** Compute the coefficients a_i of the vector V in the above equation, in terms of the basis vectors.

b) **[4 pts]** In order to find the expansion of a vector V, how many multiplications are required?

c) **[4 pts]** In order to find the expansion of a vector V, how many additions are required?

d) **[4 pts]** Write the number of multiplications and additions in the $O(\cdot)$ notation.

e) **[2 pts]** How many multiplications and how many additions are required to expand an arbitrary vector V, if the basis vectors were orthogonal? Show why.

f) **[2 pts]** How many multiplications and how many additions are required to expand an arbitrary vector V, if the basis vectors were orthonormal? Show why

Answer:

a)

$$V = a_1V_i \cdot V_1 + a_2V_i \cdot V_1 + \cdots + a_nV_i \cdot V_1$$

there dot product is zero hence

$$\vec{V}i \cdot \vec{V} = \vec{ai}$$

b) Number of multiplication is n^2 (n items)

c) Number of addition is $n^2 - n$ (n items)

d) $\mathrm{O}(n^2 + n^2 - n) = \mathrm{O}(n^2)$

e) It depends if basis were not considered or were irrelevant then we would have exactly n number of a_i

Problem 4

[20 pts] Find the determinant and rank of the following matrices:

$$A = \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & 4 & 2 \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \qquad C = \begin{bmatrix} 3 & 4 & 3 & 6 \\ 1 & 1 & 1 & 1 \\ 6 & 8 & 6 & 9 \\ 1 & 2 & 3 & 4 \end{bmatrix} \tag{3}$$

a) [3 pts] $det(A) =$

b) [2 pts] Rank $A =$

c) [4 pts] $det(B) =$

d) [3 pts] Rank $B =$

e) [5 pts] $det(C) =$

f) [3 pts] Rank $C =$

Answer:

a) $det(A) =$

$$\begin{vmatrix} 3 & 4 \\ 2 & 1 \end{vmatrix} = (3) \cdot (1) - (4) \cdot (2) = -5$$

b) Rank $A =$

$$\begin{bmatrix} 1 & \frac{4}{3} \\ 2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & \frac{4}{3} \\ 2 & -\frac{5}{3} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & \frac{4}{3} \\ 0 & 1 \end{bmatrix}$$

so rank is 2

c) $det(B) =$

$$\begin{vmatrix} 3 & 1 & -1 \\ 1 & 0 & 0 \\ 1 & 1 & 2 \end{vmatrix} = (1)\cdot(-1)^{2+1}\cdot \begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} + 0\cdot(-1)^{2+2}\cdot \begin{vmatrix} 3 & -1 \\ 1 & 2 \end{vmatrix} + 0\cdot(-1)^{2+3}\cdot \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} = - \begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix}$$

$$\begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} = (1) \cdot (2) - (-1) \cdot (1) = 3 \qquad (-1) \cdot (3) = -3$$

hence it is -3

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d) Rank $B =$

$$\begin{bmatrix} 1 & \frac{4}{3} & \frac{2}{3} \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & \frac{4}{3} & \frac{2}{3} \\ 0 & 1 & -1 \\ 0 & \frac{2}{3} & \frac{7}{3} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & \frac{4}{3} & \frac{2}{3} \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

3 non zero so rank is 3

e) $det(C) =$

$$\begin{vmatrix} 3 & 4 & 3 & 6 \\ 1 & 1 & 1 & 1 \\ 6 & 8 & 6 & 9 \\ 1 & 2 & 3 & 4 \end{vmatrix} = -6$$

the work was not shown since this was so long.

f) Rank $C =$

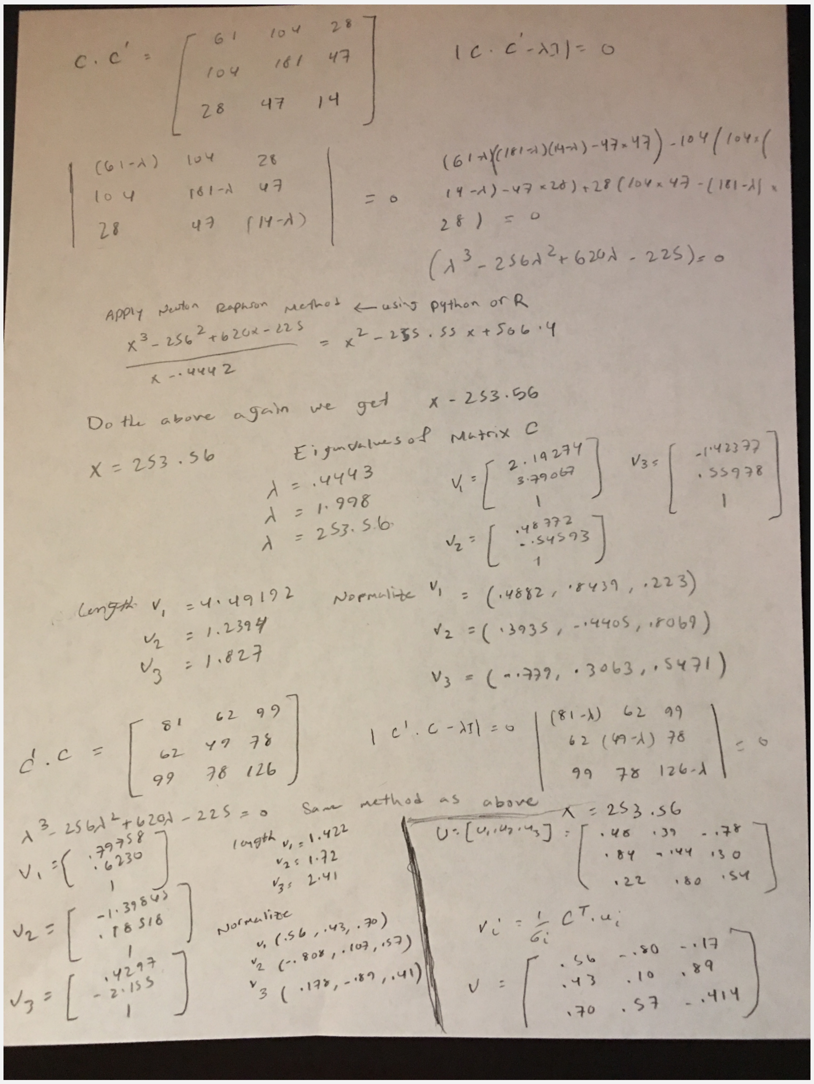
$$\begin{vmatrix} 1 & \frac{4}{3} & 1 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \rightarrow 4$$

Problem 5

[20 pts] Find the Support Vector Decomposition (SVD) of the following matrix. Show all your work.

$$C = \begin{bmatrix} 4 & 3 & 6 \\ 8 & 6 & 9 \\ 1 & 2 & 3 \end{bmatrix} \tag{4}$$

Answer:
This was a long problem and tedious to do on latex.



$$C \cdot C' = \begin{bmatrix} 61 & 104 & 28 \\ 104 & 181 & 47 \\ 28 & 47 & 14 \end{bmatrix} \qquad |C \cdot C' - \lambda I| = 0$$

$$\begin{vmatrix} (61-\lambda) & 104 & 28 \\ 104 & 181-\lambda & 47 \\ 28 & 47 & (14-\lambda) \end{vmatrix} = 0$$

$(61-\lambda)((181-\lambda)(14-\lambda)-47\times 47)-104(104\times(14-\lambda)-47\times 28)+28(104\times 47-(181-\lambda)\times 28)=0$

$(\lambda^3-256\lambda^2+620\lambda-225)=0$

Apply Newton Raphson method ← using python or R

$$\frac{x^3-256^2+620x-225}{x-.4442} = x^2-255.55x+506.4$$

Do the above again we get $x-253.56$

$x = 253.56$

Eigenvalues of Matrix C

$\lambda = .4443$
$\lambda = 1.998$
$\lambda = 253.56$

$$v_1 = \begin{bmatrix} 2.19274 \\ 3.79067 \\ 1 \end{bmatrix} \quad v_2 = \begin{bmatrix} .48772 \\ -.54593 \\ 1 \end{bmatrix} \quad v_3 = \begin{bmatrix} -1.42377 \\ .55978 \\ 1 \end{bmatrix}$$

Length $v_1 = 4.49192$
$v_2 = 1.2394$
$v_3 = 1.827$

Normalize $v_1 = (.4882, .8439, .223)$
$v_2 = (.3935, -.4405, .8069)$
$v_3 = (-.779, .3063, .5471)$

$$C' \cdot C = \begin{bmatrix} 81 & 62 & 99 \\ 62 & 49 & 78 \\ 99 & 78 & 126 \end{bmatrix} \qquad |C' \cdot C - \lambda I| = 0 \qquad \begin{vmatrix} (81-\lambda) & 62 & 99 \\ 62 & (49-\lambda) & 78 \\ 99 & 78 & 126-\lambda \end{vmatrix} = 0$$

$\lambda^3-256\lambda^2+620\lambda-225=0$ Same method as above $\lambda = 253.56$

$$v_1 = \begin{bmatrix} .79758 \\ .6230 \\ 1 \end{bmatrix} \quad v_2 = \begin{bmatrix} -1.39843 \\ .18516 \\ 1 \end{bmatrix} \quad v_3 = \begin{bmatrix} .4297 \\ -2.155 \\ 1 \end{bmatrix}$$

length $v_1 = 1.422$
$v_2 = 1.72$
$v_3 = 2.41$

Normalize
v_1 $(.56, .43, .70)$
v_2 $(-.808, .107, .57)$
v_3 $(.178, -.89, .41)$

$$U = [u_1, u_2, u_3] = \begin{bmatrix} .48 & .39 & -.78 \\ .84 & -.44 & .30 \\ .22 & .80 & .54 \end{bmatrix}$$

$$v_i = \frac{1}{\sigma_i} C^T \cdot u_i$$

$$V = \begin{bmatrix} .56 & -.80 & -.17 \\ .43 & .10 & .89 \\ .70 & .57 & -.414 \end{bmatrix}$$

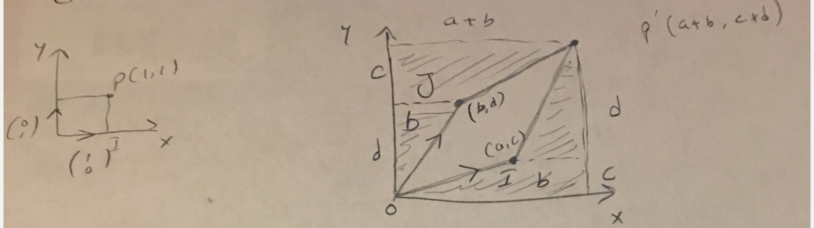
Graduate Only, Extra Credit for Undergraduate Students

[10 pts] Prove that the determinant of a matrix A is the area scaling factor of the geometric transformation represented by A.

Answer:

This was a long problem and tedious to do on latex, and I have no idea how to do the actual graphs on latex.

Extra credit



$$\underset{A}{\begin{bmatrix} a & b \\ c & d \end{bmatrix}} \begin{array}{c} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \\ O \quad I \quad P \quad J \end{array} = \begin{array}{c} O' \quad I' \quad P' \quad J' \\ \begin{bmatrix} 0 & a & a+b & b \\ 0 & c & c+d & d \end{bmatrix} \end{array}$$

$$\text{Area} = (a+b)(c+d) - 2\left[\frac{bd}{2} + \frac{(a+2b)c}{2}\right]$$

$$= ac + ad + bc + bd - bd - ac - 2bc$$

$$= ad - bc$$

$$= \text{Det}[A]$$

$$\text{Det}[A] = -(ad - bc)$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \leftarrow \text{changes the area by a factor of } |\text{Det}[A]|$$