2) $f(t)$ - ~~the~~ density of time T to purchase for randomly selected purchaser.

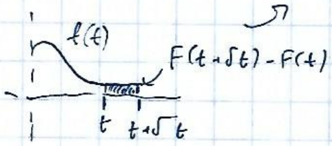
Show that:

a) $P(\text{Buys in next increment } \delta t \mid \text{no purchase by time } t) = h(t)\,\delta t$

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

But A is only possible if B already happened: $A \subset B$

$$P(A/B) = \frac{P(A)}{P(B)} = \frac{F(t+\delta t)-F(t)}{1-F(t)} \overset{\circledast}{\text{"="}} \frac{F'(t)\,dt}{1-F(t)} =$$



$$= \frac{f(t)}{1-F(t)}\,dt = h(t)$$

$\circledast$ $F(t+\delta t)-F(t) \text{"="} \frac{F(t+\delta t)-F(t)}{\delta t}\,\delta t = F'(t)\,dt = f(t)\,dt$

b) $S(t) = 1-F(t)$.

Show: $S(t) = e^{-\int_0^t h(u)\,du}$, $E[T] = \int_0^\infty S(t)\,dt$

a)
$$h(t) \overset{!}{=} \frac{f(t)}{1-F(t)} = \frac{F'(t)}{1-F(t)} = \frac{-S'(t)}{S(t)} = -(\ln S(t))'$$

$$\int_0^t h(u)\,du = -\ln S(t) \Rightarrow \boxed{S(t) = e^{-\int_0^t h(u)\,du}}$$

$$E[T] = \int_0^\infty t f(t)\,dt = \int_0^\infty t F'(t)\,dt = -\int_0^\infty t S'(t)\,dt =$$

$$= \lim_{n\to\infty} -\int_0^n \overset{f}{t}\, \overset{g'}{S'(t)}\,dt = \lim_{n\to\infty}\left(-t S(t)\Big|_0^n + \int_0^n S(t)\,dt\right) =$$

$$= \lim_{n\to\infty} -nS(n) + \int_0^n S(t)\,dt = \lim_{n\to\infty}\int_0^\infty S(t)-S(n)\,dt$$

$$= \lim_{n\to\infty}\int_0^\infty\int_t^n f(s)\,ds\,dt = \int_0^\infty\int_t^\infty f(s)\,ds\,dt =$$

$$= \int_0^\infty \text{"}F(\infty)\text{"} - F(t)\,dt = \int_0^\infty 1-F(t)\,dt = \int_0^\infty S(t)$$

c) (3.9) $h(t) = p + q\,F(t) \quad \mathrm{I}$

(3.6) $N_{t+1} = N_t + p(m - N_t) + \dfrac{q\,N_t(m - N_t)}{m} \quad \mathrm{II}$

$$h(t) = \frac{f(t)}{1-F(t)} \Rightarrow \mathrm{I} \Rightarrow \frac{f(t)}{1-F(t)} = p + q\,F(t)$$

Because ~~&~~ $F(t)$ is the CDF of the distribution of time until purchase we get: $mF(t)$ is the most likely buyer amount at given time t. Or N_t "=" $mF(t)$ - Cumulative sales at t

~~Replace that in II:~~

Also "t+1" counts as the next increment in time, so

$N_{t+1} - N_t \approx m f(t)$ ~~counts as the instantaneous rate of change~~. "Sales at t"

Swapping in II:

$$N_{t+1} - N_t = mf(t) = p(m - mF(t)) + \frac{q\,mF(t)(m - mF(t))}{m} \quad /: m$$

$$f(t) = p(1 - F(t)) + q\,F(t)(1 - F(t)) \quad /: (1 - F(t))$$

$t \neq \infty$

$$\frac{f(t)}{1-F(t)} = \boxed{h(t) = p + q\,F(t)}$$