③ a) $h(t) = \frac{f(t)}{1-F(t)}$, $F(t) = \frac{1-e^{-(p+q)t}}{1+(q/p)\,e^{-(p+q)t}}$

prove that $h(t) = p + q\,F(t)$

$$f(t) = F'(t) = \frac{(p+q)\,e^{-(p+q)t}\cdot\left(1+(q/p)\,e^{-(p+q)t}\right) + \left(1-e^{-(q+p)t}\right)(p+q)(q/p)\,e^{-(p+q)t}}{\left(1+(q/p)\,e^{-(p+q)t}\right)^2}$$

$$= \frac{(p+q)\,e^{-(p+q)t} + (p+q)(q/p)\,e^{-(p+q)t}}{\left(1+(q/p)\,e^{-(p+q)t}\right)^2}$$

$$1-F(t) = \frac{\cancel{1}+(q/p)\,e^{-(p+q)t} - \cancel{1} + e^{-(p+q)t}}{1+(q/p)\,e^{-(p+q)t}} = \frac{e^{-(p+q)t}\,(q/p+1)}{1+(q/p)\,e^{-(p+q)t}}$$

$$\frac{f(t)}{1-F(t)} = \frac{(p+q)\,\cancel{e^{-(p+q)t}}\,(1+q/p)}{\left(1+(q/p)\,e^{-(p+q)t}\right)^{\cancel{2}}} \cdot \frac{\cancel{1+(q/p)\,e^{-(p+q)t}}}{\cancel{e^{-(p+q)t}}\,(q/p+1)} =$$

$$\Rightarrow \frac{(p+q)\left(\frac{p+q}{\cancel{p}}\right)}{\frac{q+p}{\cancel{p}}} = \boxed{\frac{(p+q)^2}{q+p}} \Rightarrow \frac{f(t)}{1-F(t)} = \frac{(p+q)^2}{\cancel{(q+p)}\left(1+(q/p)\,e^{-(p+q)t}\right)}$$

$$p+q\,F(t) = \frac{p+\cancel{q\,e^{-(p+q)t}}+q-\cancel{q\,e^{-(p+q)t}}}{1+(q/p)\,e^{-(p+q)t}} = \frac{p+q}{1+(q/p)\,e^{-(p+q)t}}$$

Or: $\cancel{q+}\ p+q\,F(t) = \frac{p+q}{1+(q/p)\,e^{-(p+q)t}} = h(t)$

Q.E.D.

Remove Watermark Now
pdfelement

c) $t_{peak} = \frac{\log(q) - \log(p)}{p+q}$

$$f(t) = \frac{(p+q)^2 e^{-(p+q)t}}{p\{1+(q/p)e^{-(p+q)t}\}^2}$$

wolfram

$$f'(t) = -\frac{p(p+q)^3 e^{t(p+q)}(pe^{t(p+q)} - q)}{(pe^{t(p+q)} + q)^3}$$

$$f'(t_{peak}) = 0 \overset{?}{=} pe^{t_{peak}(p+q)} - q$$

$$q = pe^{t_{peak}(p+q)}$$

$$q/p = e^{t_{peak}(p+q)}$$

$$\frac{\ln q - \ln q}{p+q} = t_{peak}$$

logistic: $f(t) = \frac{e^{-\frac{x-\mu}{s}}}{s(1+e^{-\frac{x-\mu}{s}})^2}$

~~wolfram~~

wolfram

$$f'(t_{peak}) = 0 = \cdots = e^{\mu/s} - qe^{t_{peak}/s}$$

$$e^{\frac{t_{peak}}{s}} = \frac{1}{q}e^{\mu/s}$$

$$\frac{t_{peak}}{s} = \ln\frac{1}{q} + \frac{\mu}{s}$$

$$t_{peak} = -s\ln q + \mu$$

exponential:

I'm maybe missing something, but it should be $t=0$, no?

as $f'(t) = -\lambda^2 e^{-\lambda t} \neq 0$

and obtains global minimum at 0.

pdfelement
Remove Watermark Now

4) a) There is no good reason for a person to purchase 2 iPhones or more for 1 individual unless something ~~real~~ quite rare happens, warranty renders replacement purchases almost obsolete, and when the warranty is over a better phone will exist, so no reason to buy the previous (Apple has this information)

b) If $m = 10{,}000{,}000$ as a given by Apple, I would say p has to be larger than q because I would predict the peak of sales to be very ~~near the~~ close to the initial sale. If with the VCR sales data most purchases where later, because of the imitation factor, here ~~it~~ is the reverse, although not quite, because imitation would still play a role.
With the VCR the stats where $q = 0.638$, $p = 0.00659$.
Here I would say around $q = 0.25$, $p = 0.75$.
So the answer: $p = 0.75 \pm 0.15$ $\quad q = 0.25 \pm 0.8$

$m = 10{,}000{,}000 \pm 3{,}000{,}000$

After modeling, much more reasonable values may be:
$p = 0.01 \pm 0.005$
$q = 0.001 \pm 0.0005$
$m = 20{,}000{,}000 \pm 6$mil.
because the first half is discounted

→ The higher the innovative value relative to the imitation value, the earlier the peak of sales would be, and visa-versa

→ The higher the p, q values generally ~~the more~~ the denser the distribution would be around 0.

→ m actually does not affect the continuous version of the model

c) You would take a ~~high~~ q (imitation) ~~and take a~~ much lower than your p value (innovation)

Remove Watermark Now
pdfelement

5) a) $S_n = a + ar + \dots + ar^{n-1}$

$$S_n - rS_n = a + \cancel{ar + \dots + ar^{n-1}} \cancel{- ar - \dots - ar^{n-1}} - ar^n$$

$$\Rightarrow S_n - rS_n = a - ar^n \Rightarrow \boxed{S_n = \frac{a(1-r^n)}{1-r}}$$

b) When $|r|<1$. Then: $\lim\limits_{n\to\infty} \frac{a(1-r^n)}{1-r} = \boxed{\frac{a}{1-r} = S_\infty}$

c) $a_1 = x_1 \qquad a_2 = \alpha x_2 + (1-\alpha)a_1 = \alpha x_2 + (1-\alpha)x_1$

$$a_3 = \alpha x_3 + \alpha(1-\alpha)x_2 + (1-\alpha)^2 a_1$$

$$\vdots$$

$$a_n = \alpha x_n + \alpha(1-\alpha)x_{n-1} + \dots + \alpha(1-\alpha)^{n-2} a_2 + (1-\alpha)^{n-1} a_1$$

$$S_n = \alpha + \alpha(1-\alpha) + \dots + \alpha(1-\alpha)^{n-2} = \frac{\alpha(1-(1-\alpha)^{n-1})}{\alpha} = 1-(1-\alpha)^{n-1}$$

$$\sum_{i=1}^{n} w_i = \alpha + \alpha(1-\alpha) + \dots + \alpha(1-\alpha)^{n-2} + (1-\alpha)^{n-1} =$$

$$= 1 - (1-\alpha)^{n-1} + (1-\alpha)^{n-1} = \boxed{1}$$

Well, the total sum is 1

d) $Var[a_n] = Var\{\alpha x_n + \alpha(1-\alpha)x_{n-1} + \dots + \alpha(1-\alpha)^{n-2}x_2 + (1-\alpha)^{n-1}x_1\}$

$$= \big(\underbrace{\alpha^2 + \alpha^2(1-\alpha)^2 + \dots + \alpha(1-\alpha)^{2(n-2)}}\, + (1-\alpha)^{2(n-1)}\big)\sigma^2$$

$$\sigma^2\ \frac{\alpha^2(1-((1-\alpha)^2)^{n-1})}{1-(1-\alpha)^2} = \frac{\alpha^2(1-(1-\alpha)^{2(n-1)})}{\cancel{(1-1+\alpha)}(1+1-\alpha)} =$$

$$= \frac{\alpha(1-(1-\alpha)^{2(n-1)})}{(2-\alpha)} = \frac{\alpha(1-(1-\alpha)^{n-1})(1+(1-\alpha)^{n-1})}{2-\alpha}$$

$$Var[a_n] = \left(\frac{\alpha(1-(1-\alpha)^{2(n-1)})}{2-\alpha} + (1-\alpha)^{2(n-1)}\right)\sigma^2 =$$

$$= \frac{\alpha(1-(1-\alpha)^{2(n-1)}) + 2(1-\alpha)^{2(n-1)} - \alpha(1-\alpha)^{2(n-1)}}{2-\alpha}\sigma^2 =$$

$$= \frac{-\alpha(1-\alpha)^{2(n-1)} + 2(1-\alpha)^{2(n-1)} - \alpha(1-\alpha)^{2(n-1)} + \alpha}{2-\alpha}\sigma^2 = \boxed{\frac{2(1-\alpha)^{2n-1} + \alpha}{2-\alpha}\sigma^2}$$

pdfelement
Remove Watermark Now