PROBLEM N.3: ZERO DIVISORS OF $\mathbb{Z}_m$

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1. Definitions

To first understand how to determine if an element of $\mathbb{Z}_m$ is a zero divisor of m, it is important to define what a zero divisor is. A zero divisor is a non-zero integer a such that when multiplied by another non-zero integer b,

$$a \cdot b = mk$$

where k is some integer. In other words, it is two numbers a and b such that their product is a multiple of m. Knowing the definition, finding a zero divisor in the set $\mathbb{Z}_m$ is rather simple. Recall that the set of all residue classes is given by

$$\mathbb{Z}_m = 0, 1, 2, 3, 4, 5, ..., (m-1).$$

2. Determining Zero Divisors of $\mathbb{Z}_m$

Step 1.) If given a set of all residue classes for some number m, the first step to finding any zero divisors is to determine if m is itself the multiple of any other integers within $\mathbb{Z}_m$. For example, let $m = 18$. We know that the integers 2, 3, 6, 9 all divide perfectly into 18 and are elements of $\mathbb{Z}_{18}$. As such, that automatically makes those numbers zero divisors of 18 since a number m can be a multiple of itself:

$$2 \cdot 9 = 18 \cdot 1$$

$$3 \cdot 6 = 18 \cdot 1$$

Step 2.) However, we know something else. If a times another integer b yields a product of m, then a times a multiple of b should yield a multiple of m. For example, let us look at the element 3 of $\mathbb{Z}_{18}$. We know that 3 multiplied by 6 is 18, so 3 multiplied by a multiple of 6 should get a multiple of 18:

$$3 \cdot 12 = 3 \cdot 6 \cdot 2 = 18 \cdot 2$$

Therefore, 12 is a zero divisor of 18 since it can be multiplied by 3 to get twice of 18. If you know that two non-zero integers a and b can multiply into m, than any multiple of a or b that are within the set of $\mathbb{Z}_m$ will be zero divisors of $\mathbb{Z}_m$.

3. Conclusion

Knowing that zero divisors of $\mathbb{Z}_m$ have to be integers greater than 0 and less than m as well as multiples of other numbers that can divide into m means that m has to be a composite number in order to have zero divisors.

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