

Multiple integrals

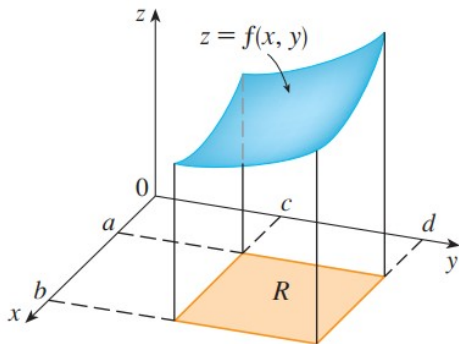
Dr. Nguyen Van Hoi

University of Information Technology

September 11, 2023



Double integrals over rectangles



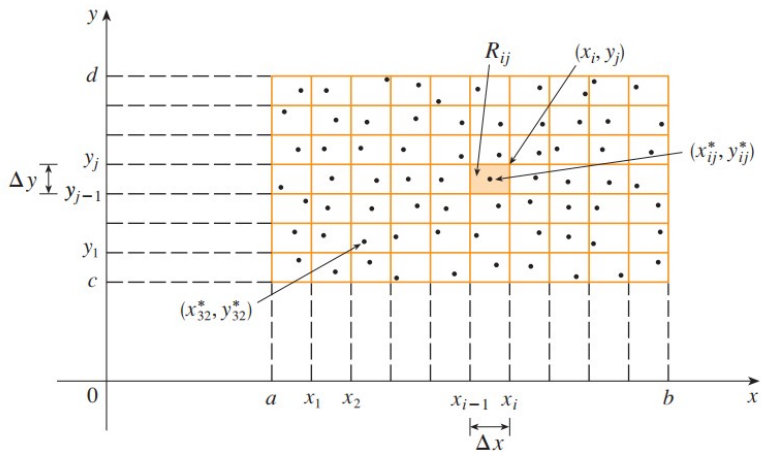
➡ Find $|S|$?

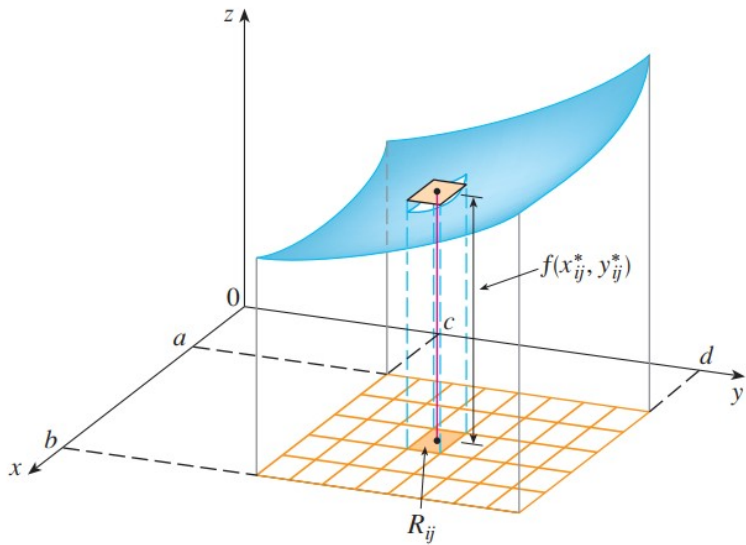
A closed rectangle is given by

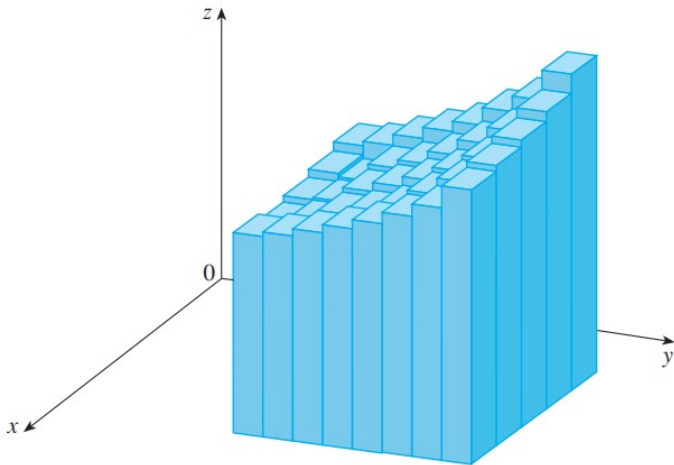
$$R = [a, b] \times [c, d] = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, c \leq y \leq d\}$$

The solid below f -a positive function, is

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \leq z \leq f(x, y), (x, y) \in R\}$$







⇒ The (volume) double integral of f over R is approximated by

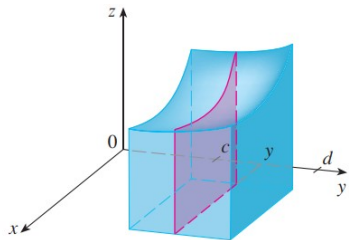
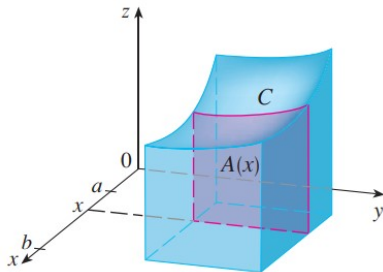
$$V = \lim_{m, n \rightarrow \infty} \sum_{i=1}^m \sum_{j=1}^n f(x_{ij}^*, y_{ij}^*) \Delta A := \iint_R f(x, y) dA.$$

If this is the case, f is called integrable.

Iterated integrals: Fubini's theorem

If f is bounded on R and is discontinuous at a finite number,

$$\iint_R f(x, y) dA = \int_a^b \int_c^d f(x, y) dy dx = \int_c^d \int_a^b f(x, y) dx dy.$$



Iterated integrals

If f is bounded on R and is discontinuous at a finite number,

$$\iint_R f(x, y) dA = \int_a^b \int_c^d f(x, y) dy dx = \int_c^d \int_a^b f(x, y) dx dy$$

If $f(x, y) = g(x)h(y)$ then

$$\iint_R f(x, y) dA = \int_a^b g(x) dx \int_c^d h(y) dy.$$

⇒ Evaluate $\iint_R (x - 3y^2) dA$ where $R = [0, 2] \times [1, 2]$.

$$\begin{aligned}\iint_R (x - 3y^2) dA &= \int_0^2 \int_1^2 (x - 3y^2) dy dx = \int_0^2 [xy - y^3]_{y=1}^{y=2} dx \\ &= \int_0^2 (x - 7) dx = \left[\frac{x^2}{2} - 7x \right]_{x=0}^{x=2} = -12.\end{aligned}$$

Another way,

$$\begin{aligned}\iint_R (x - 3y^2) dA &= \int_1^2 \int_0^2 (x - 3y^2) dx dy = \int_1^2 \left[\frac{x^2}{2} - 3xy^2 \right]_{x=0}^{x=2} dy \\ &= \int_1^2 (2 - 6y^2) dy = [2y - 2y^3]_{y=1}^{y=2} = -12.\end{aligned}$$

☞ If $R = [0, \pi/2] \times [0, \pi/2]$, then

$$\begin{aligned}\iint_R \sin x \cos y dA &= \int_0^{\pi/2} \sin x dx \int_0^{\pi/2} \cos y dy \\ &= [-\cos x]_0^{\pi/2} [\sin y]_0^{\pi/2} = 1 \cdot 1 = 1.\end{aligned}$$

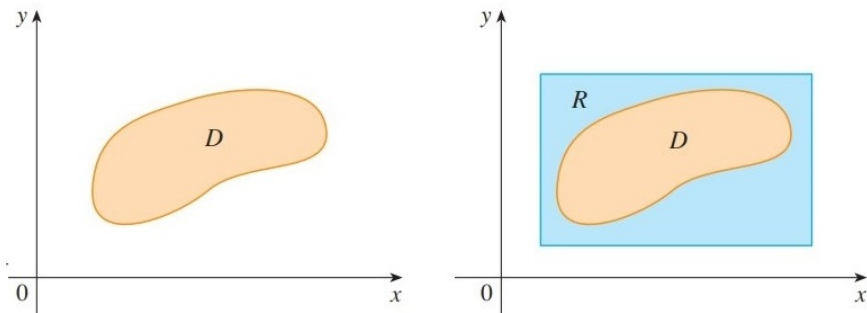
☞ Examples 3 and 4/pages 999 and 1000-Jame Stewart.

☞ Exercises: 3,7,13,19,25,26/pages 999 and 1000-Jame Stewart.

Double integrals over general domains

☞ If D is not a rectangle, what is $\iint_D f(x, y) dA$?

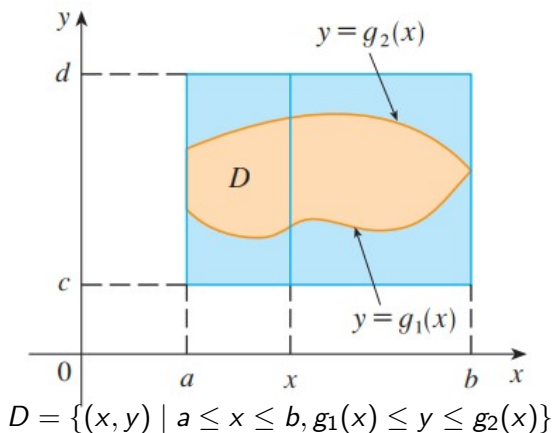
$$F(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \text{ is in } D \\ 0 & \text{if } (x, y) \text{ is in } R \text{ but not in } D \end{cases}$$



☞ If F is integrable over R , then we define

$$\iint_D f(x, y) dA = \iint_R F(x, y) dA.$$

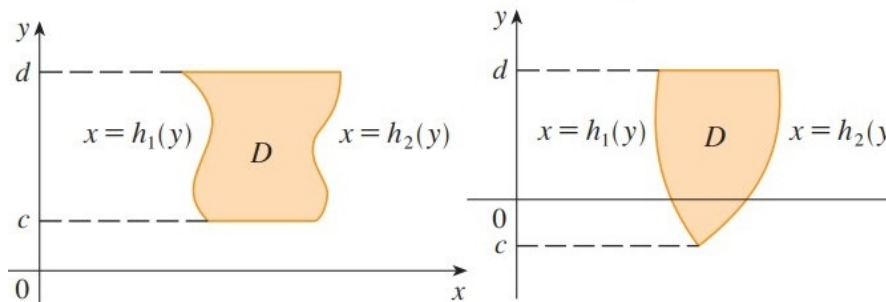
The integral over domain type I



☞ If f is continuous on D , then

$$\iint_D f(x, y) dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx.$$

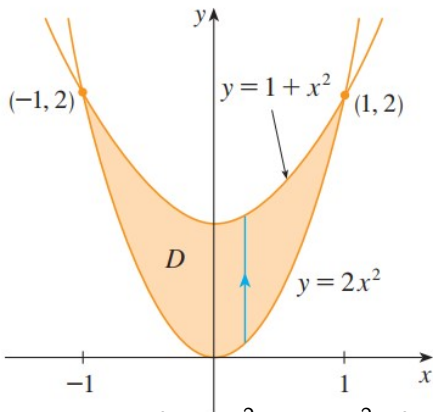
The integral over domain type II



$$D = \{(x, y) \mid c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\}$$

☞ If f is continuous on D , then

$$\iint_D f(x, y) dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) dx dy.$$



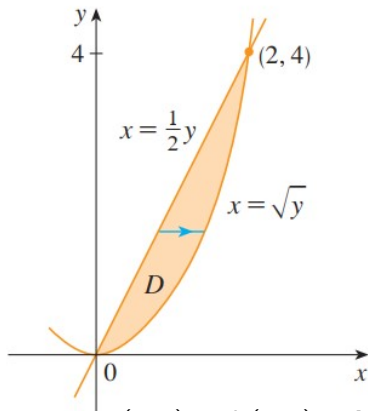
☞ The parabolas intersect when $2x^2 = 1 + x^2$, that is, $x = \pm 1$.

☞ Note that the region is a type I region but not a type II

$$D = \{(x, y) \mid -1 \leq x \leq 1, 2x^2 \leq y \leq 1 + x^2\}$$

☞ Since $f(x, y) = x + 2y$ is continuous on D , then

$$\iint_D (x + 2y) dA = \int_{-1}^1 \int_{2x^2}^{1+x^2} (x + 2y) dy dx = \text{exercise.}$$

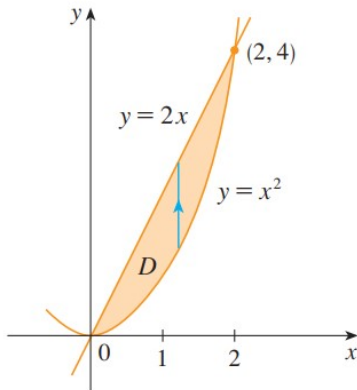


⌘ The intersect points are $(0,0)$ and $(2,4)$ ⌘ Note that the region is a type II

$$D = \{(x, y) \mid 0 \leq y \leq 4, \frac{1}{2}y \leq x \leq \sqrt{y}\}$$

⌘ Since $f(x, y) = x^2 + y^2$ is continuous on D , then

$$\iint_D (x^2 + y^2) dA = \int_0^4 \int_{\frac{1}{2}y}^{\sqrt{y}} (x^2 + y^2) dx dy = \text{exercise.}$$



☞ The region is also a type II

$$D = \{(x, y) \mid 1 \leq x \leq 2, x^2 \leq y \leq 2x\}$$

☞ Since $f(x, y) = x^2 + y^2$ is continuous on D , then

$$\iint_D (x^2 + y^2) dA = \int_0^2 \int_{2x}^{x^2} (x^2 + y^2) dy dx = \text{Exercise.}$$

☞ Exercises: 5,7,8,17,22/Section 15.3-Jame Stewart.

Properties of double integrals

- Sum:

$$\iint_R [f(x, y) + g(x, y)] dA = \iint_R f(x, y) dA + \iint_R g(x, y) dA.$$

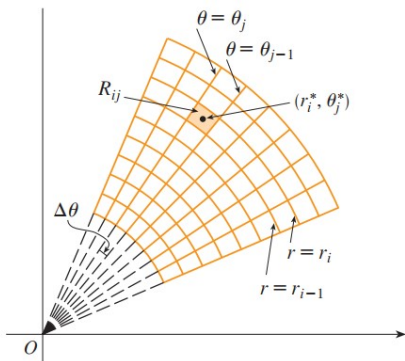
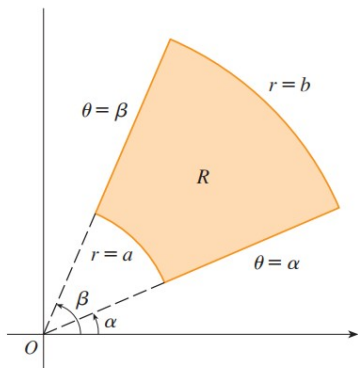
- Multiply with constant: $\iint_R cf(x, y) dA = c \iint_R f(x, y) dA.$
- If $f(x) \geq g(x)$ for all $(x, y) \in R$, then

$$\iint_R f(x, y) dA \geq \iint_R g(x, y) dA.$$

- Area of D : $A(D) = \iint_D 1 dA.$
- If $D = D_1 \cup D_2$ and they do not overlap,

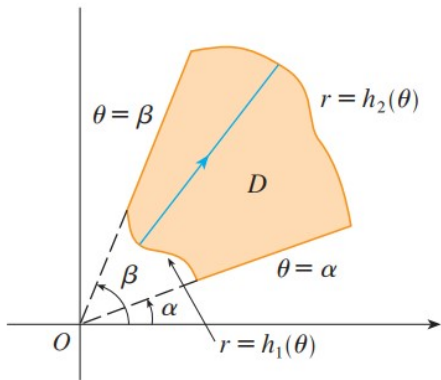
$$\iint_D f(x, y) dA = \iint_{D_1} f(x, y) dA + \iint_{D_2} f(x, y) dA.$$

Double integrals on polar coordinates



□ If f is continuous on a polar rectangle R given by $0 \leq a \leq r \leq b$, $\alpha \leq \theta \leq \beta$, where $0 \leq \beta - \alpha \leq 2\pi$, then

$$\iint_R f(x, y) dA = \int_{\alpha}^{\beta} \int_a^b f(r \cos \theta, r \sin \theta) r dr d\theta$$



□ If $D = \{(r, \theta) | \alpha \leq \theta \leq \beta, h_1(\theta) \leq r \leq h_2(\theta)\}$, then

$$\iint_R f(x, y) dA = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} f(r \cos \theta, r \sin \theta) r dr d\theta$$

☞ Evaluate $\iint_R (3x + 4y^2) dA$, where R is the region in the upper half-plane bounded by the circles $x^2 + y^2 = 1$ and $y^2 + y^2 = 4$.

$$R = \{(x, y) \mid y \geq 0, 1 \leq x^2 + y^2 \leq 4\}.$$

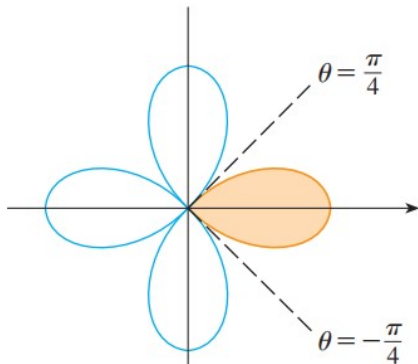
In the polar coordinates,

$$R = \{(r, \theta) \mid 1 \leq r \leq 2, 0 \leq \theta \leq \pi\}.$$

Then,

$$\iint_R (3x + 4y^2) dA = \int_0^\pi \int_1^2 (3r \cos \theta + 4r^2 \sin^2 \theta) r dr d\theta = \textbf{Exercise}.$$

☞ Exercises: 19-20-21-23-24/Section 15.4.



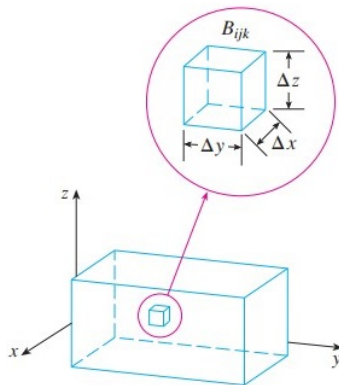
▮ Use a double integral to find the area enclosed by one loop of the four-leaved rose $r = \cos 2\theta$.

$$D = \{(r, \theta) \mid -\pi/4 \leq \theta \leq \pi/4, 0 \leq r \leq \cos 2\theta\}.$$

So the area is

$$\iint_D 1 dA = \int_{-\pi/4}^{\pi/4} \int_0^{\cos 2\theta} r dr d\theta = \mathbf{Exercise}.$$

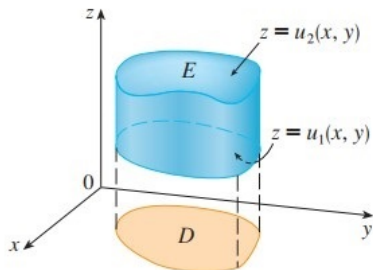
Triple integrals



3 Definition The **triple integral** of f over the box B is

$$\iiint_B f(x, y, z) \, dV = \lim_{l, m, n \rightarrow \infty} \sum_{i=1}^l \sum_{j=1}^m \sum_{k=1}^n f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*) \Delta V$$

if this limit exists, where the sample point $(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)$ is in B_{ijk} .



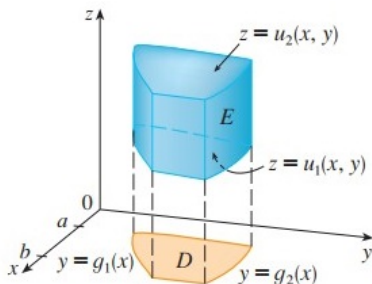
A type 1 solid region

$$\boxed{5} \quad E = \{(x, y, z) \mid (x, y) \in D, u_1(x, y) \leq z \leq u_2(x, y)\}$$

where D is the projection of E onto the xy -plane

$\boxed{6}$

$$\iiint_E f(x, y, z) \, dV = \iint_D \left[\int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z) \, dz \right] dA$$

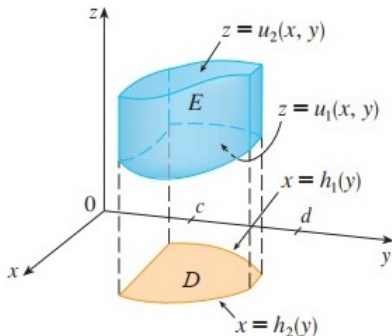


$$E = \{(x, y, z) \mid a \leq x \leq b, g_1(x) \leq y \leq g_2(x), u_1(x, y) \leq z \leq u_2(x, y)\}$$

Equation 6 becomes

7

$$\iiint_E f(x, y, z) dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z) dz dy dx$$



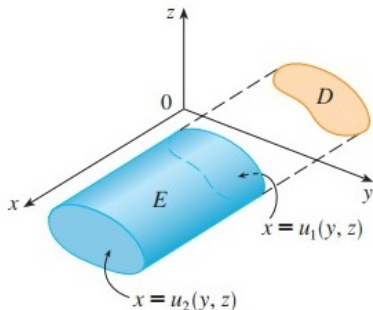
If, on the other hand, D is a type II plane region

$$E = \{(x, y, z) \mid c \leq y \leq d, h_1(y) \leq x \leq h_2(y), u_1(x, y) \leq z \leq u_2(x, y)\}$$

Equation 6 becomes

8

$$\iiint_E f(x, y, z) \, dV = \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z) \, dz \, dx \, dy$$



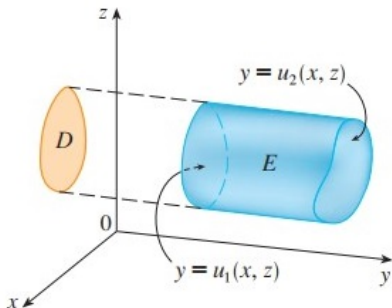
A solid region E is of **type 2** if it is of the form

$$E = \{(x, y, z) \mid (y, z) \in D, u_1(y, z) \leq x \leq u_2(y, z)\}$$

where, this time, D is the projection of E onto the yz -plane

10

$$\iiint_E f(x, y, z) \, dV = \iint_D \left[\int_{u_1(y, z)}^{u_2(y, z)} f(x, y, z) \, dx \right] dA$$



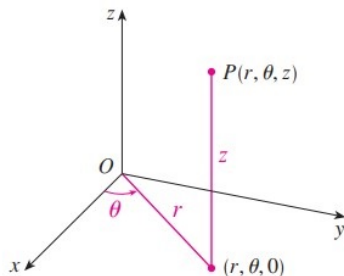
Finally, a **type 3** region is of the form

$$E = \{(x, y, z) \mid (x, z) \in D, u_1(x, z) \leq y \leq u_2(x, z)\}$$

where D is the projection of E onto the xz -plane,

$$\boxed{11} \quad \iiint_E f(x, y, z) \, dV = \iint_D \left[\int_{u_1(x, z)}^{u_2(x, z)} f(x, y, z) \, dy \right] dA$$

Integrals on cylindrical coordinates

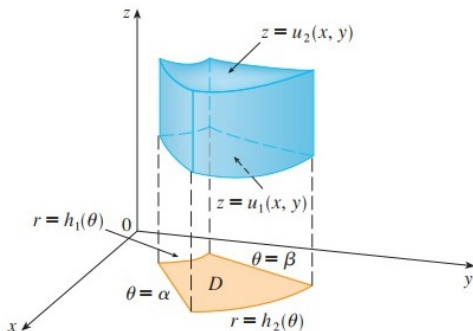


$$x = r \cos \theta \quad y = r \sin \theta \quad z = z$$

$$r^2 = x^2 + y^2 \quad \tan \theta = \frac{y}{x} \quad z = z$$

$$E = \{(x, y, z) \mid (x, y) \in D, u_1(x, y) \leq z \leq u_2(x, y)\}$$

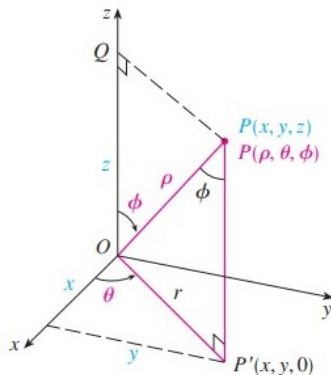
where $D = \{(r, \theta) \mid \alpha \leq \theta \leq \beta, h_1(\theta) \leq r \leq h_2(\theta)\}$



4

$$\iiint_V f(x, y, z) dV = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r \cos \theta, r \sin \theta)}^{u_2(r \cos \theta, r \sin \theta)} f(r \cos \theta, r \sin \theta, z) r dz dr d\theta$$

Integrals on spherical coordinates



1

$$x = \rho \sin \phi \cos \theta \quad y = \rho \sin \phi \sin \theta \quad z = \rho \cos \phi$$

2

$$\rho^2 = x^2 + y^2 + z^2$$

$$\begin{aligned}
 \text{3} \quad & \iiint_E f(x, y, z) \, dV \\
 &= \int_c^d \int_\alpha^\beta \int_a^b f(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi) \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi
 \end{aligned}$$

where E is a spherical wedge given by

$$E = \{(\rho, \theta, \phi) \mid a \leq \rho \leq b, \alpha \leq \theta \leq \beta, c \leq \phi \leq d\}$$

Thank you for listening!

Nguyen Van Hoi

hoinv@uit.edu.vn