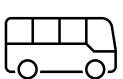




Đạo hàm



Mô tả sự biến thiên (chiều hướng và sự nhanh/chậm) của hàm số tại một điểm nào đó.

$$y'(x) = \frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{h \rightarrow 0} \frac{y(x + \Delta x) - y(x)}{\Delta x}$$



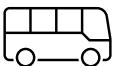
$y'(x_0) > 0$, đồng biến gần x_0 .



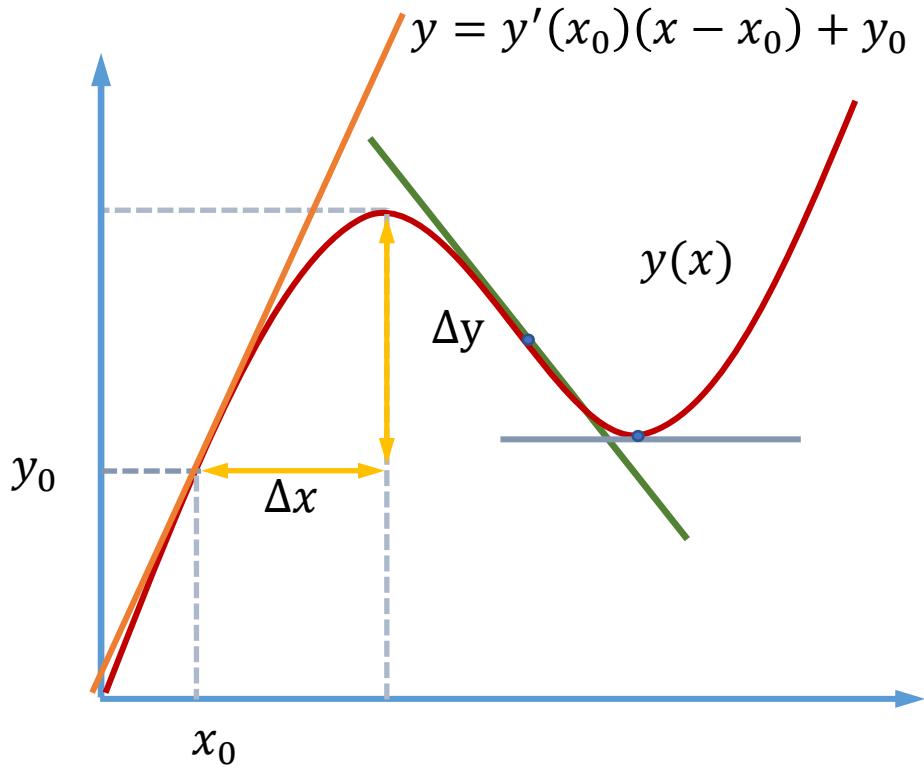
$y'(x_0) < 0$, nghịch biến gần x_0 .

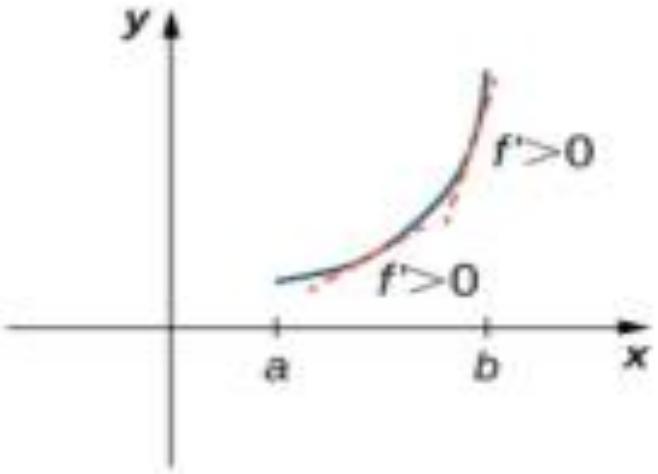


$y'(x_0) = 0$, không tăng/giảm gần x_0 .



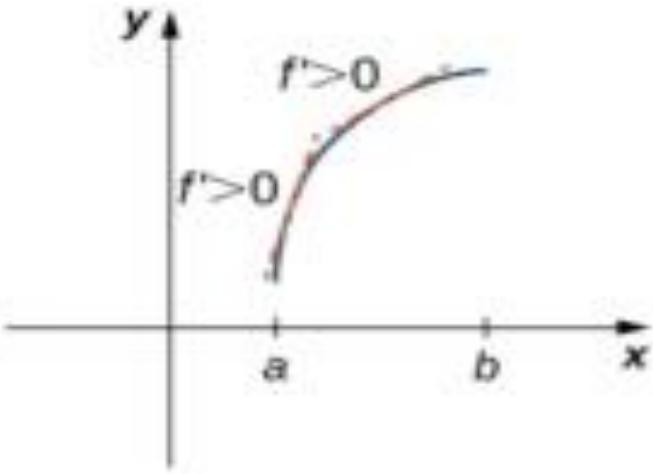
Ý nghĩa hình học: là hệ số góc của phương trình đường thẳng tiếp xúc với đồ thị tại điểm đó.





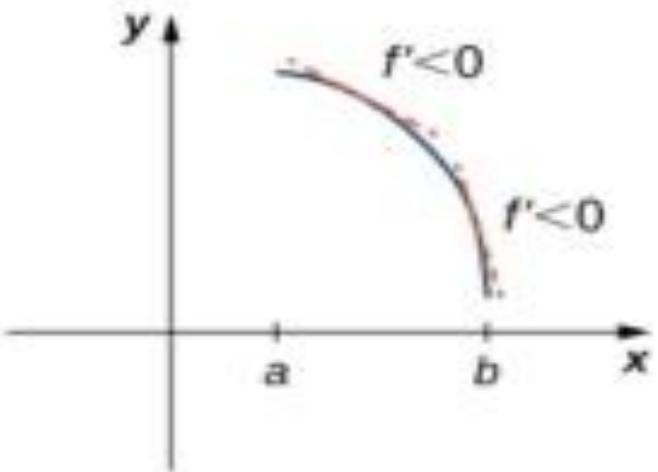
f is increasing

(a)



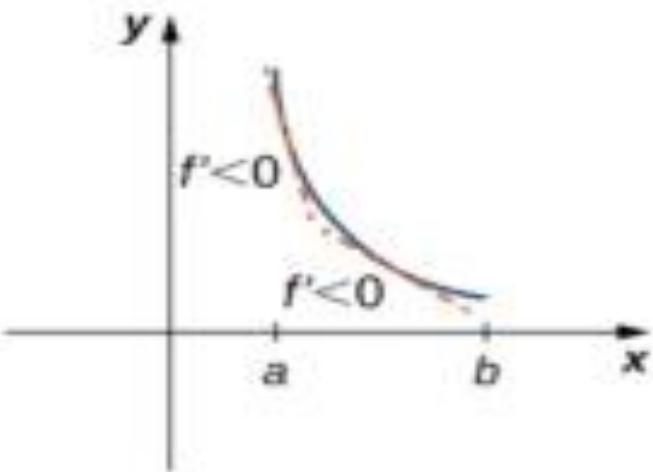
f is increasing

(b)



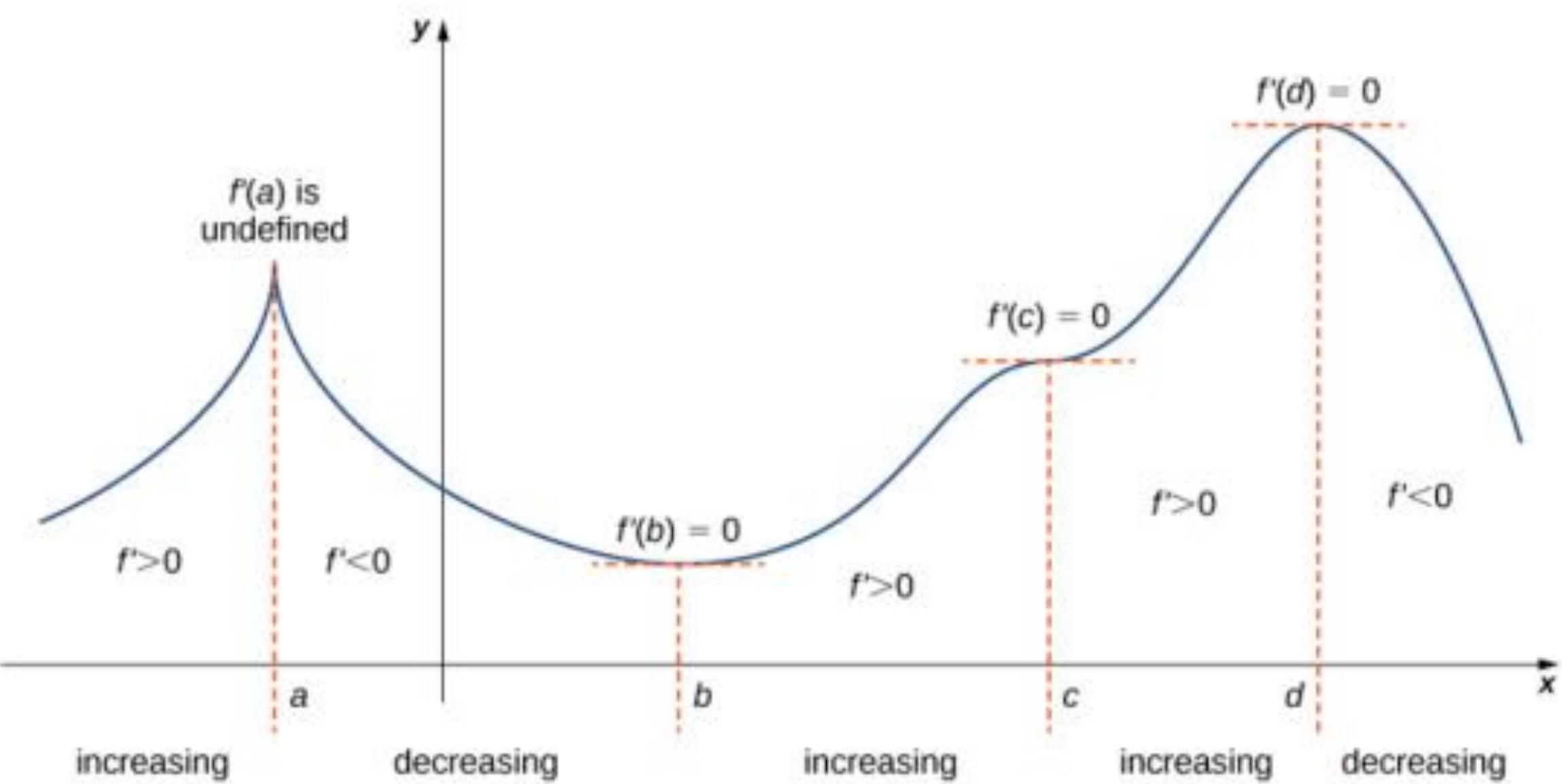
f is decreasing

(c)



f is decreasing

(d)





Cách tính đạo hàm

$$(ax + b)' = a$$

$$(f + g)' = f' + g'$$

$$(fg)' = f'g + fg'$$

$$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$$

$$f(g(x))' = f'(g(x))g'(x)$$



Đạo hàm thông dụng

$$(e^x)' = e^x$$

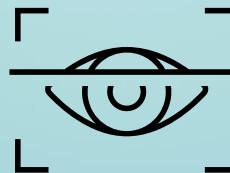
$$(\ln x)' = \frac{1}{x}$$

$$(\cos x)' = -\sin x$$

$$(\sin x)' = \cos x$$

$$(\tan x)' = \frac{1}{\cos^2 x}$$

$$(\cot x)' = -\frac{1}{\sin^2 x}$$





Tìm y' , y'' nếu

$$y = 4x^4 + 10x^3 - 2x^2 + x - 1$$

$$y = 3x^2 - 2\cos x$$

$$y = (2x^3 + 5)(2x^4 + x - 1)$$

$$y = \sin x \cos x$$

$$y = \frac{3x^3 + x + 1}{x - 2}$$

$$y = \frac{x \sin x}{1 + x}$$

$$y = \frac{2x + 1}{-4x + 3}$$

$$y = \sqrt{x \sin x}$$

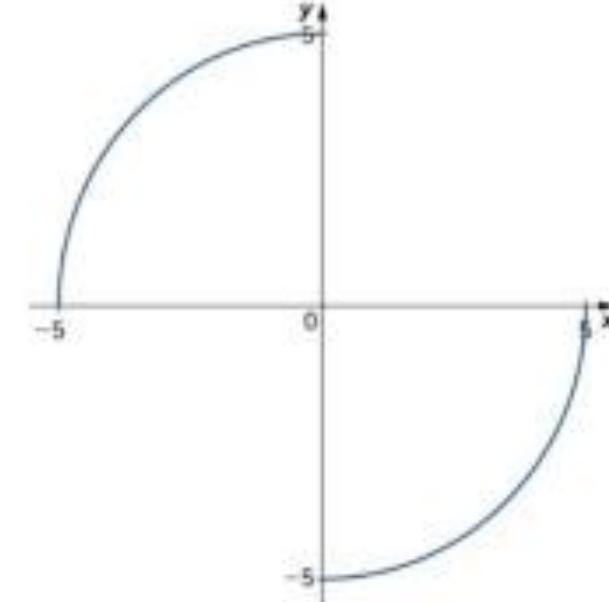
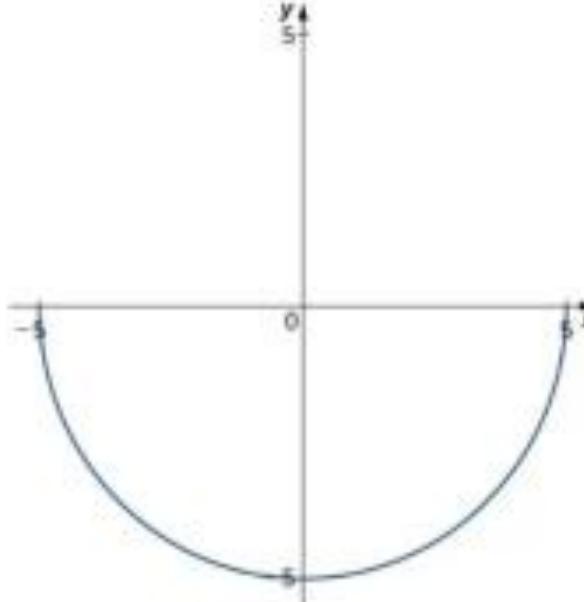
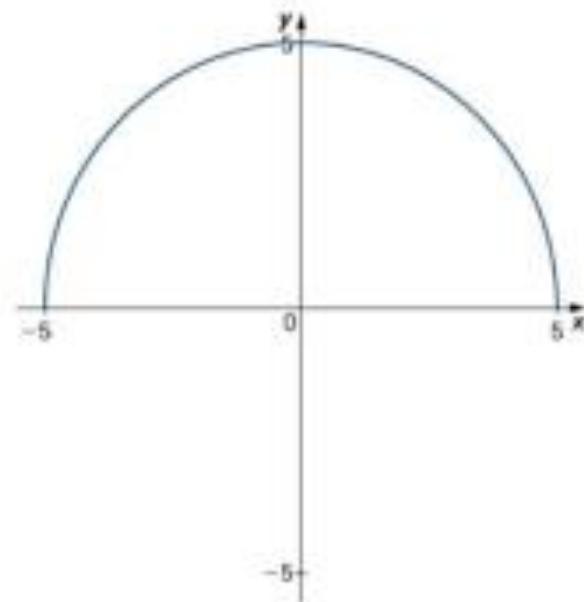
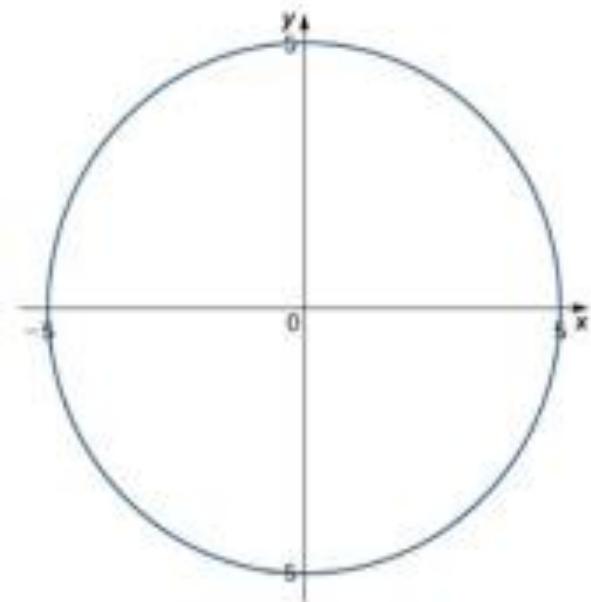
Tìm phương trình đường thẳng tiếp xúc với đồ thị $y = 3x^2 - 11$ tại $x = 2$

Tìm x sao cho phương trình đường thẳng tiếp xúc với đồ thị $y = x^3 - 7x^2 + 8x + 1$ tại $x = -2$, và song song với $y = 2x + 3$

$$y = \sqrt{25 - x^2}$$

$$y = -\sqrt{25 - x^2}$$

$$y = \begin{cases} \sqrt{25 - x^2}, & \text{if } -5 < x < 0 \\ -\sqrt{25 - x^2}, & \text{if } 0 < x < 5 \end{cases}$$



Tìm phương trình tiếp tuyến của đồ thị tại điểm $(3,4)$?



Take the derivative of both sides of the equation.

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(25)$$

$$2x + 2y \frac{dy}{dx} = 0$$

Rewrite the equation so that all terms containing $\frac{dy}{dx}$ are on the left and others are on the right

$$2y \frac{dy}{dx} = -2x$$

$$y \frac{dy}{dx} = -x$$

Factor out $\frac{dy}{dx}$ on the left

Solve for $\frac{dy}{dx}$ by dividing both sides of the equation by appropriate algebraic expression.

$$\frac{dy}{dx} = -\frac{x}{y}$$



Find the slope of the tangent line to the graph of $x^2 + y^2 = 25$ at the point $(3,4)$.

Solution: we already find out that

$$\frac{dy}{dx} = -\frac{x}{y}$$

Consequently, the slope of the tangent line is

$$\frac{dy}{dx} \Big|_{(3,-4)} = -\frac{3}{-4} = \frac{3}{4}$$

Using the point $(3, -4)$ and the slope $\frac{3}{4}$, we obtain the equation

$$y = \frac{3}{4}x - \frac{25}{4}$$



Assuming that y is defined implicitly by the equation $x^3 \sin y + y = 4x + 3$, find $\frac{dy}{dx}$

Take the derivative of both sides of the equation.

$$\frac{d}{dx}(x^3 \sin y + y) = \frac{d}{dx}(4x + 3)$$

$$\frac{d}{dx}(x^3 \sin y) + \frac{dy}{dx} = 4$$

$$\frac{d}{dx}(x^3) \sin y + x^3 \frac{d}{dx} \sin y + \frac{dy}{dx} = 4$$

$$3x^2 \sin y + x^3 \cos y \frac{dy}{dx} + \frac{dy}{dx} = 4$$



Rewrite the equation so that all terms containing $\frac{dy}{dx}$ are on the left and others are on the right

$$x^3 \cos y \frac{dy}{dx} + \frac{dy}{dx} = 4 - 3x^2 \sin y$$

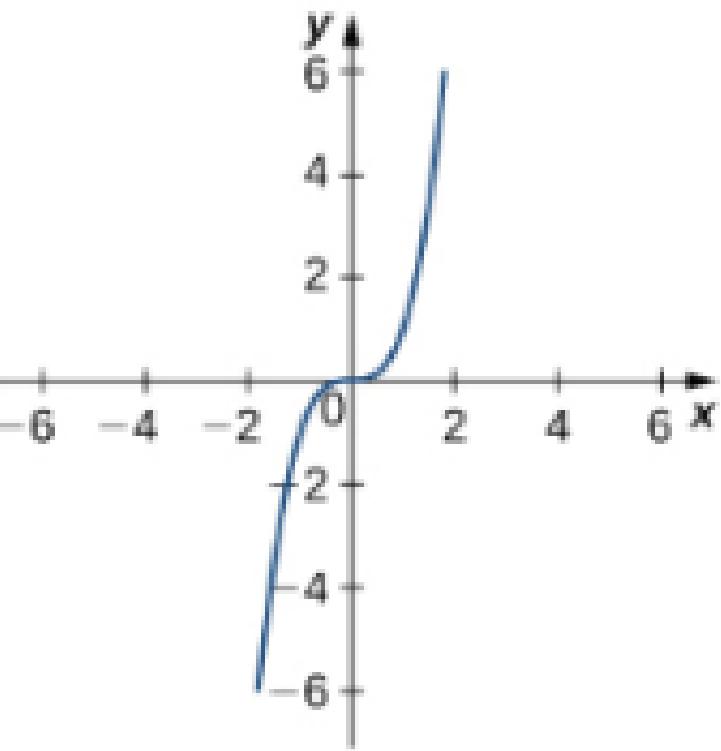
Factor out $\frac{dy}{dx}$ on the left

$$\frac{dy}{dx}(x^3 \cos y + 1) = 4 - 3x^2 \sin y$$

Solve for $\frac{dy}{dx}$ by dividing both sides of the equation by appropriate algebraic expression.

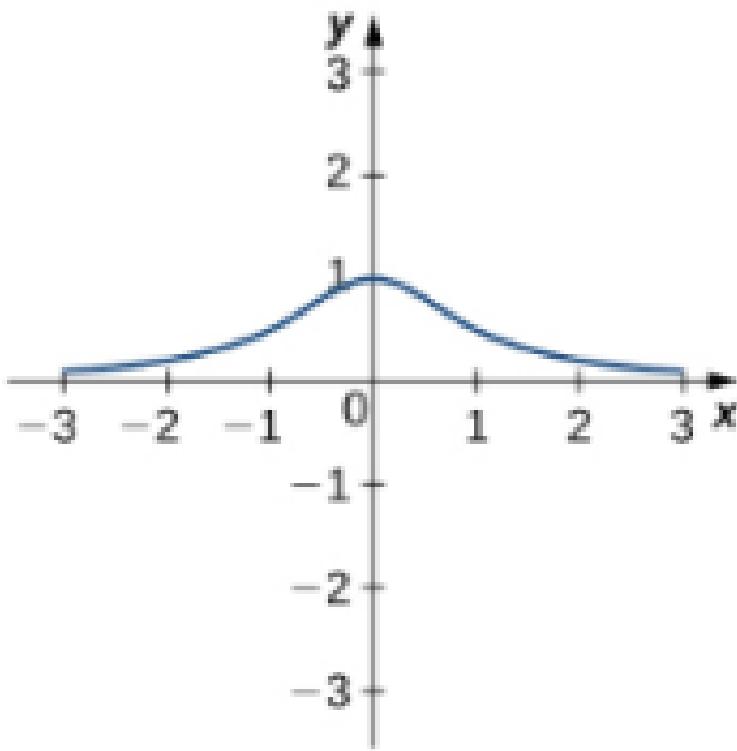
$$\frac{dy}{dx} = -\frac{4 - 3x^2 \sin y}{x^3 \cos y + 1}$$

Find the equation of the line tangent to the graph of $y^3 + x^3 - 3xy = 0$, at the point $\left(\frac{3}{2}, \frac{3}{2}\right)$



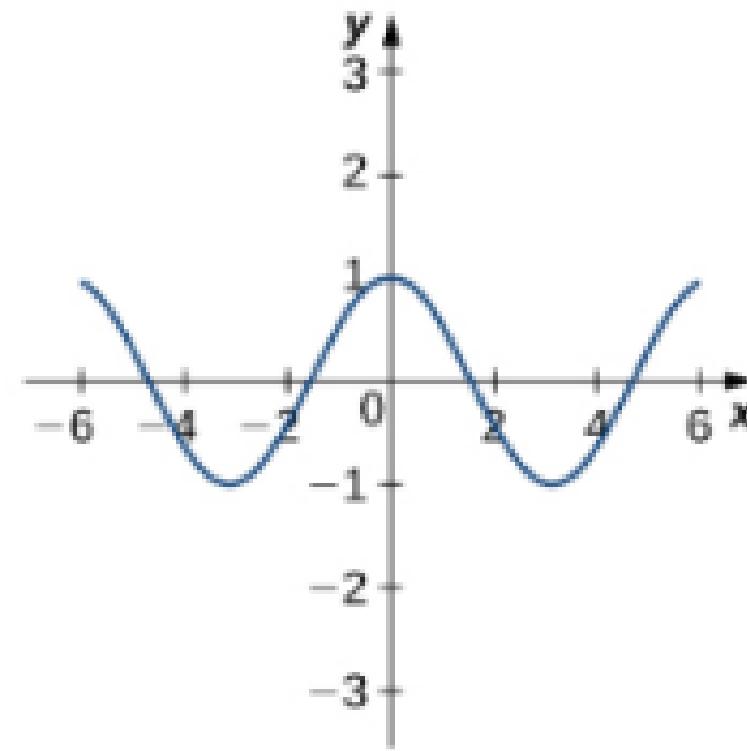
$f(x) = x^3$ on $(-\infty, \infty)$
No absolute maximum
No absolute minimum

(a)



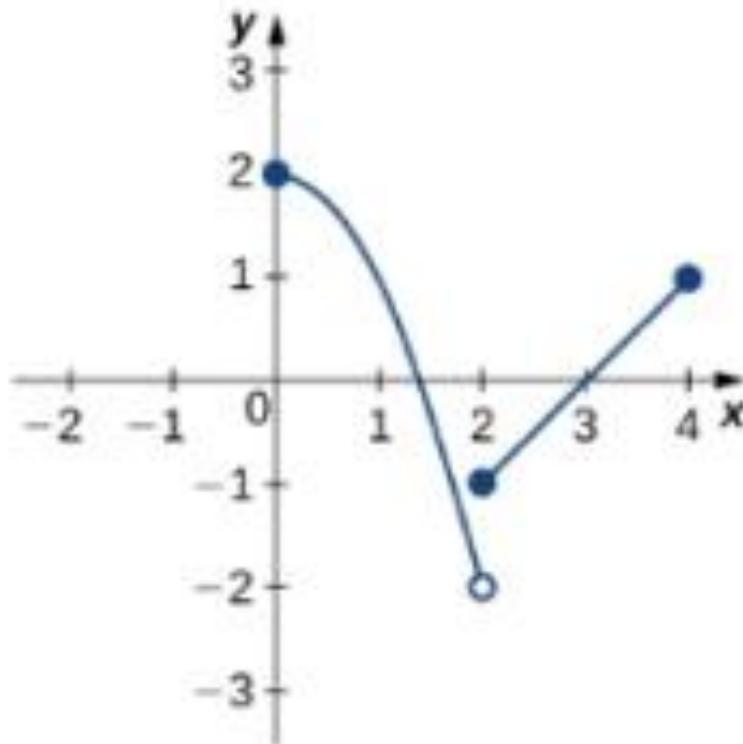
$f(x) = \frac{1}{x^2 + 1}$ on $(-\infty, \infty)$
Absolute maximum of 1 at $x = 0$
No absolute minimum

(b)



$f(x) = \cos(x)$ on $(-\infty, \infty)$
Absolute maximum of 1 at $x = 0, \pm 2\pi, \pm 4\pi\dots$
Absolute minimum of -1 at $x = \pm \pi, \pm 3\pi\dots$

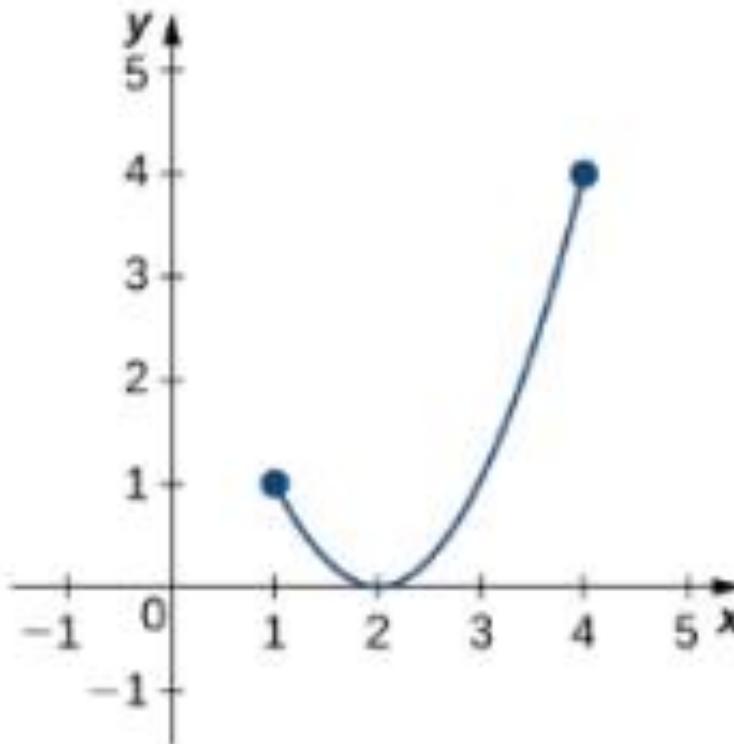
(c)



$$f(x) = \begin{cases} 2 - x^2 & 0 \leq x < 2 \\ x - 3 & 2 \leq x \leq 4 \end{cases}$$

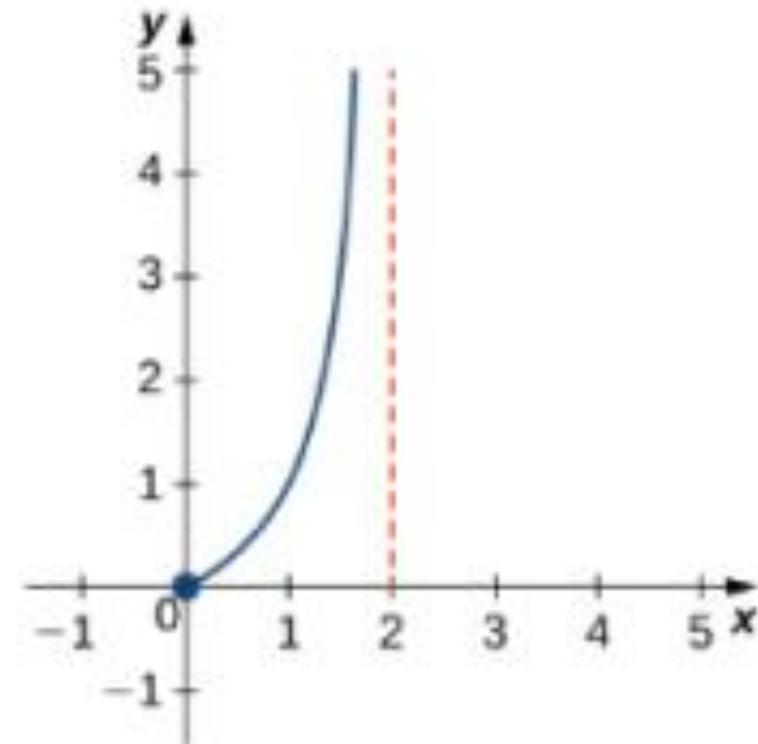
Absolute maximum of 2 at $x = 0$
No absolute minimum

(d)



$f(x) = (x - 2)^2$ on $[1, 4]$
Absolute maximum of 4 at $x = 4$
Absolute minimum of 0 at $x = 2$

(e)

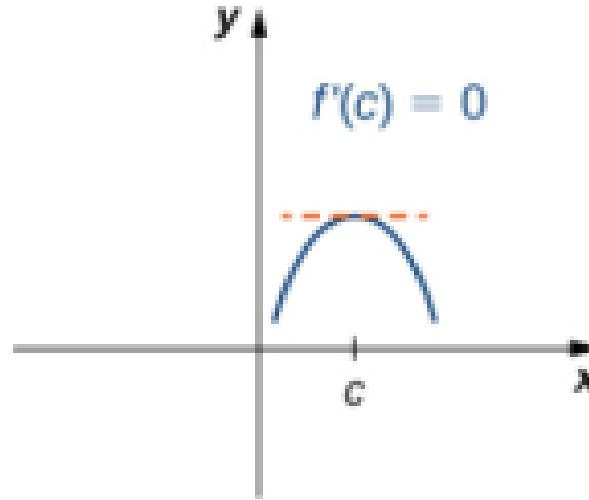


$f(x) = \frac{x}{2-x}$ on $[0, 2)$
No absolute maximum
Absolute minimum of 0 at $x = 0$

(f)

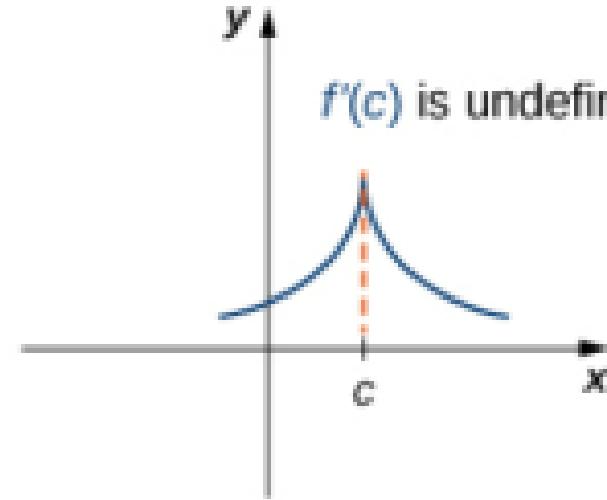
Theorem 1.2. If f is continuous on a closed interval $[a, b]$, then it attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.

Theorem 1.3. If f has a local maximum or minimum at c , and if $f'(c)$ exists, then $f'(c) = 0$.



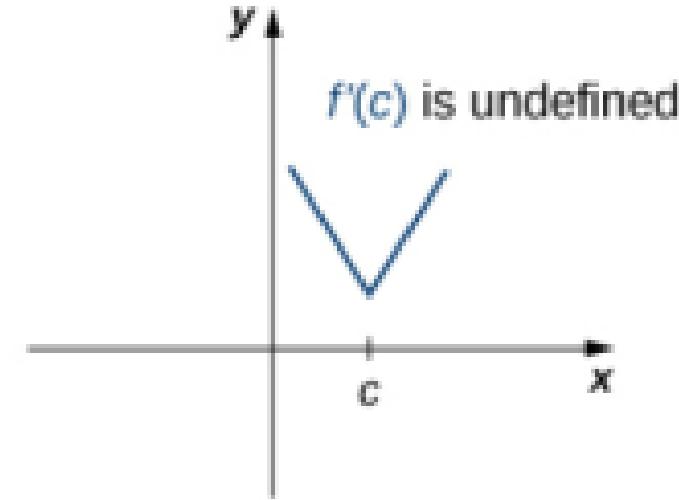
Local maximum at c

(a)



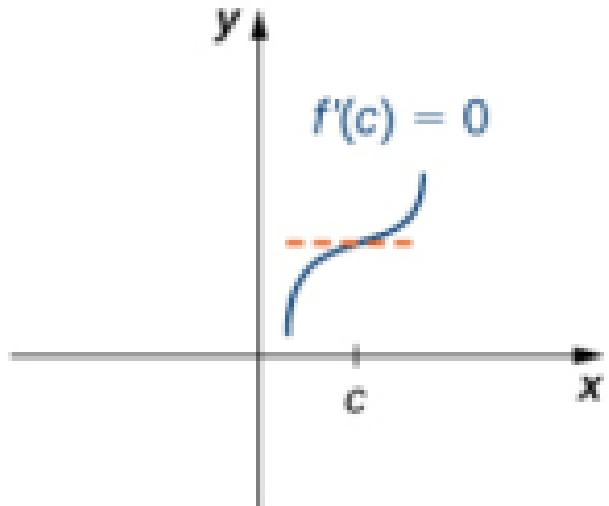
Local maximum at c

(b)



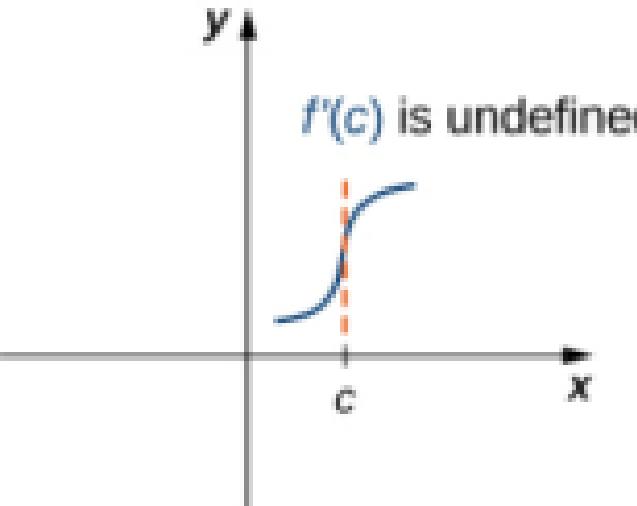
Local minimum at c

(c)



No local extremum at c

(d)



No local extremum at c

(e)

The Closed Interval Method To: find the absolute maximum and minimum values of a continuous function on a closed interval I :

- 1) Find the values of at the critical numbers of in I .
- 2) Find the values of at the endpoints of the interval.
- 3) The largest of the values from Steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

Question 1. For each of the following functions, find all critical points (i.e., is a point c such that $f'(c) = 0$ or $f'(c)$ does not exist).

1) $f(x) = \frac{1}{3}x^3 - \frac{5}{2}x^2 + 4x.$

2) $f(x) = (x^1 - 1)^3.$

3) $f(x) = \frac{4x}{1-x}.$

Question 2. For each of the following functions, find the absolute maximum and absolute minimum over the specified interval and state where those values occur.

1) $f(x) = -x^2 + 3x - 2$ over $[1, 3].$

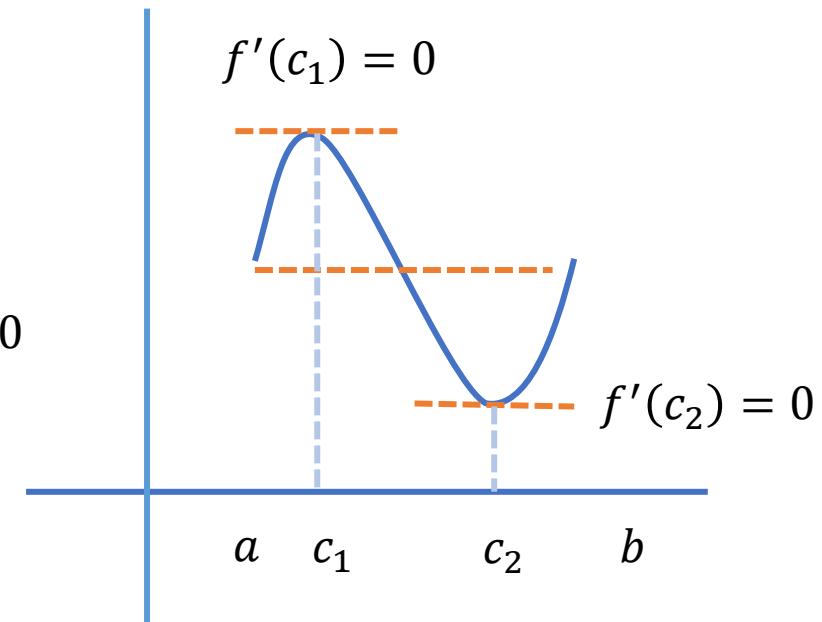
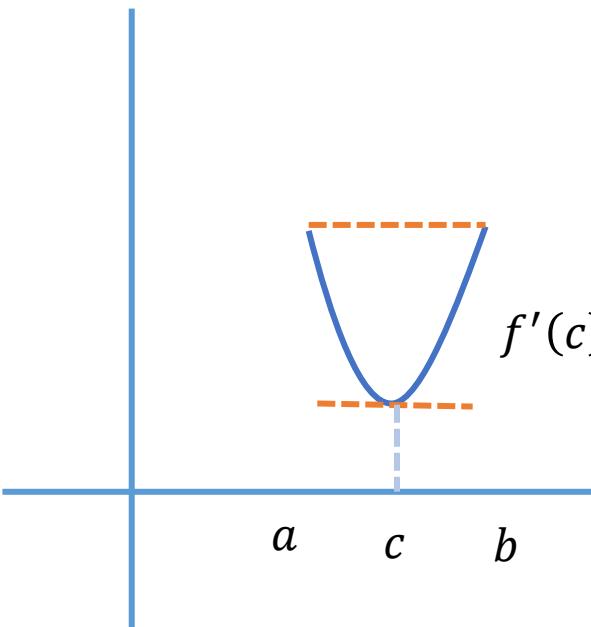
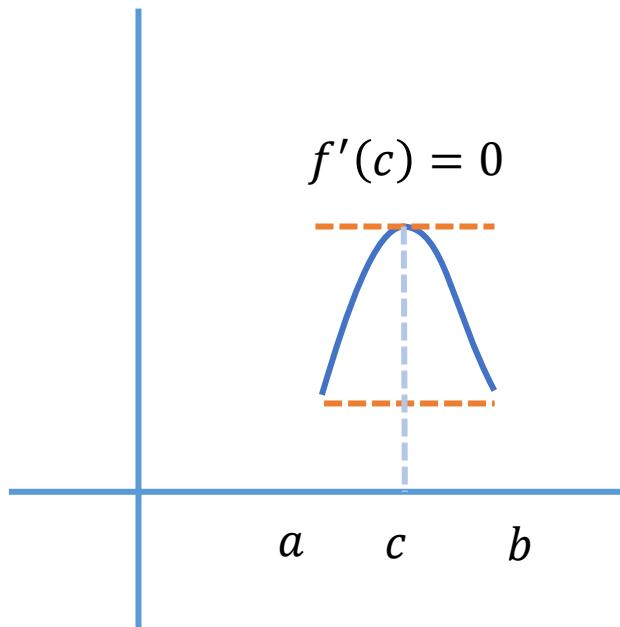
2) $f(x) = x^2 - 3x^{\frac{2}{3}}$ over $[0, 2].$

$f(t) = t\sqrt{4 - t^2}, \quad [-1, 2]$

$f(t) = \sqrt[3]{t}(8 - t), \quad [0, 8]$

$f(t) = 2\cos t + \sin 2t, \quad [0, \pi/2]$

$f(t) = t + \cot(t/2), \quad [\pi/4, 7\pi/4]$



☞ Nếu f liên tục trên $[a, b]$ và khả vi trên (a, b) thỏa $f(a) = f(b)$, thì có $c \in (a, b)$ thỏa $f'(c) = 0$.

☞ Nếu f liên tục trên $[a, b]$ và khả vi trên (a, b) , thì có $c \in (a, b)$ thỏa

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$



☞ Nếu f liên tục trên $[a, b]$ và khả vi trên (a, b) thỏa $f(a) = f(b)$, thì có $c \in (a, b)$ thỏa $f'(c) = 0$.

☞ Nếu f liên tục trên $[a, b]$ và khả vi trên (a, b) , thì có $c \in (a, b)$ thỏa

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

☞ Chứng minh phương trình sau có duy nhất một nghiệm $0 = 2x - 1 - \sin x$. $0 = 2x + \cos x$.

☞ Chứng minh phương trình $x^3 - 15x + c$ có nhiều nhất một nghiệm trong khoảng $[-2, 2]$.

☞ Chứng minh phương trình $x^4 + 4x + c$ có nhiều nhất hai nghiệm.

☞ Nếu $f(1) = 10, f'(x) \geq 2$ với $1 \leq x \leq 4$. Giá trị nhỏ nhất $f(4)$ có thể là bao nhiêu?

☞ Có tồn tại hàm f thỏa $f(0) = -1, f(2) = 4$ và $f'(2) \leq 2$ với mọi x hay không?



Nguyên lý Hopital: Giả sử f, g khả vi trên khoảng mở (b, c) chứa a , khi đó. Nếu

$$1) \lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x) = 0,$$

$$2) \text{ Hoặc } \lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x) = \pm\infty$$

$$3) \text{ Hoặc } \lim_{x \rightarrow a} f(x) = 0, \lim_{x \rightarrow a} g(x) = \infty.$$

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f'(x)}{\lim_{x \rightarrow a} g'(x)}$$

$$\lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3}$$

$$\lim_{x \rightarrow 0} \frac{x}{\tan x}$$

$$\lim_{x \rightarrow 0} \frac{3x + 5}{2x + 1}$$

$$\lim_{x \rightarrow 0} \frac{1 - \cos x}{x}$$

$$\lim_{x \rightarrow \pi} \frac{x - \pi}{\sin x}$$

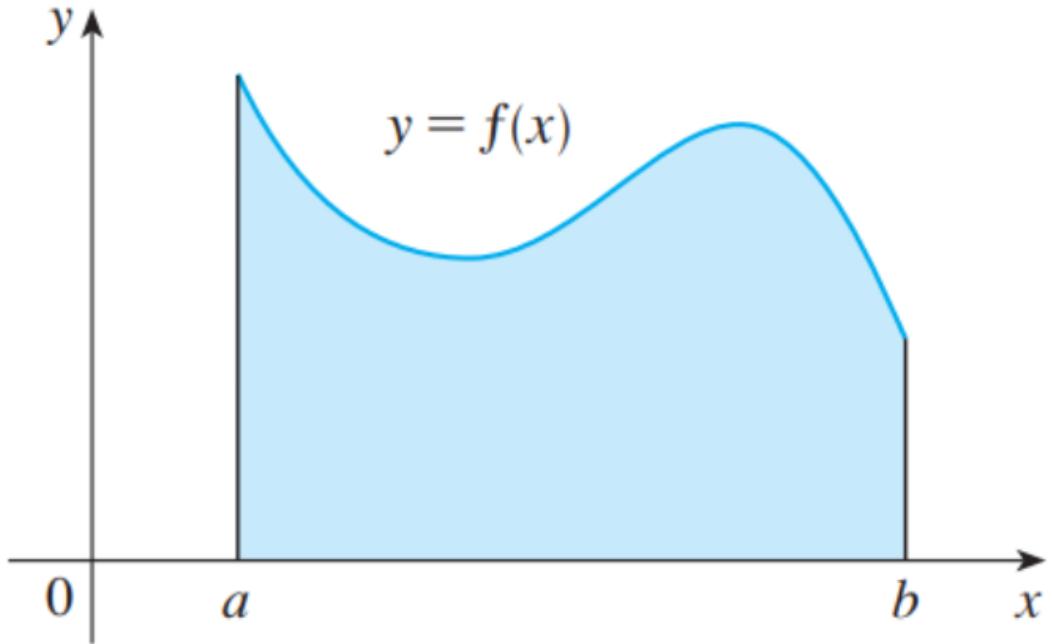
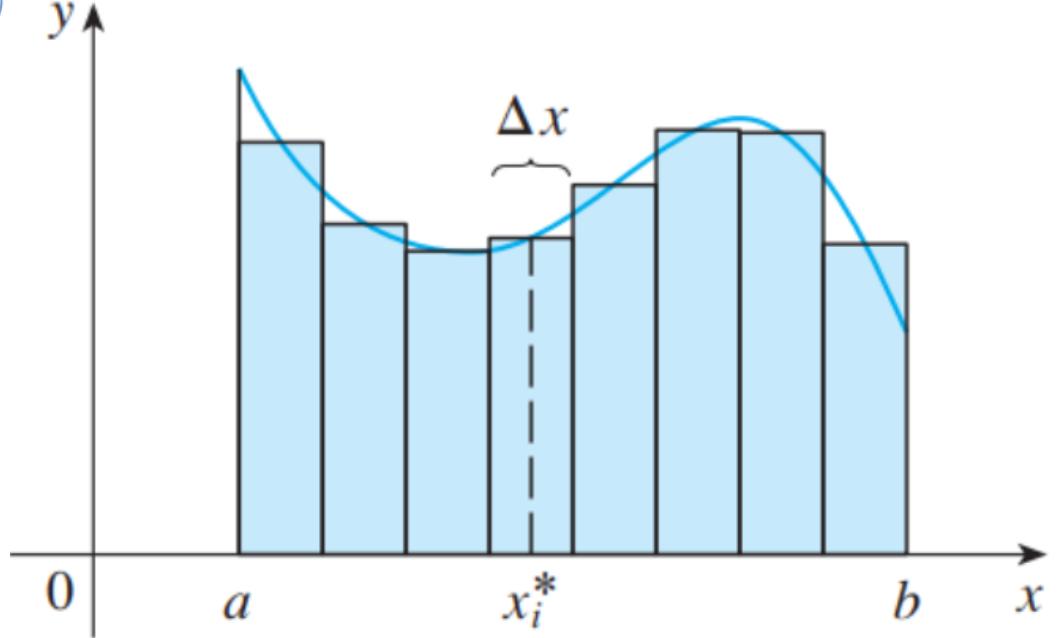
$$\lim_{x \rightarrow 0} \frac{\ln x}{x}$$

$$\lim_{x \rightarrow 2} \frac{x^3 - 7x^2 + 10x}{x^2 + x - 6}$$

$$\lim_{x \rightarrow -4} \frac{\sin(\pi x)}{x^2 - 16}$$

$$\lim_{x \rightarrow \infty} \frac{\ln(3x)}{x^2}$$

NOTE



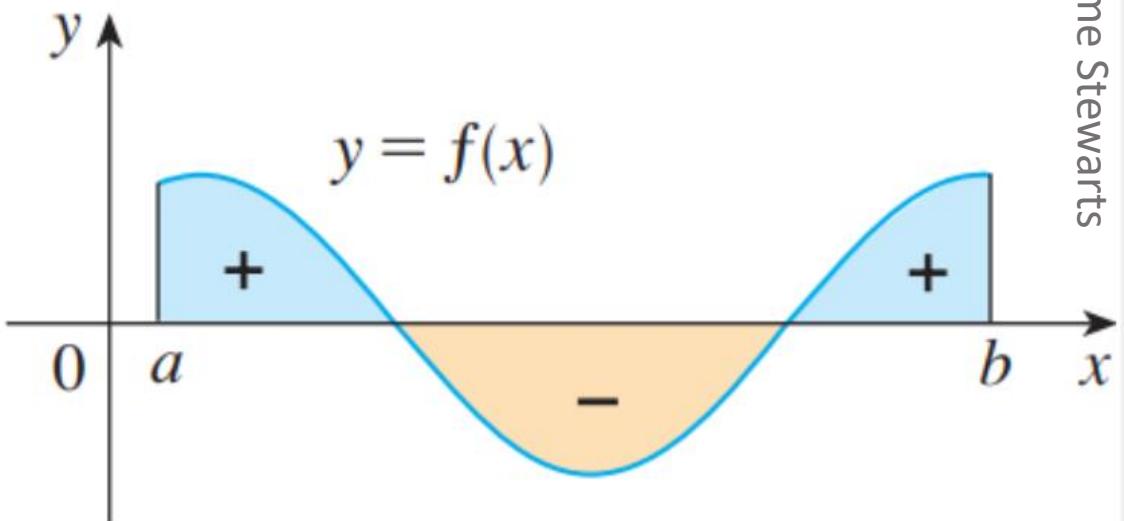
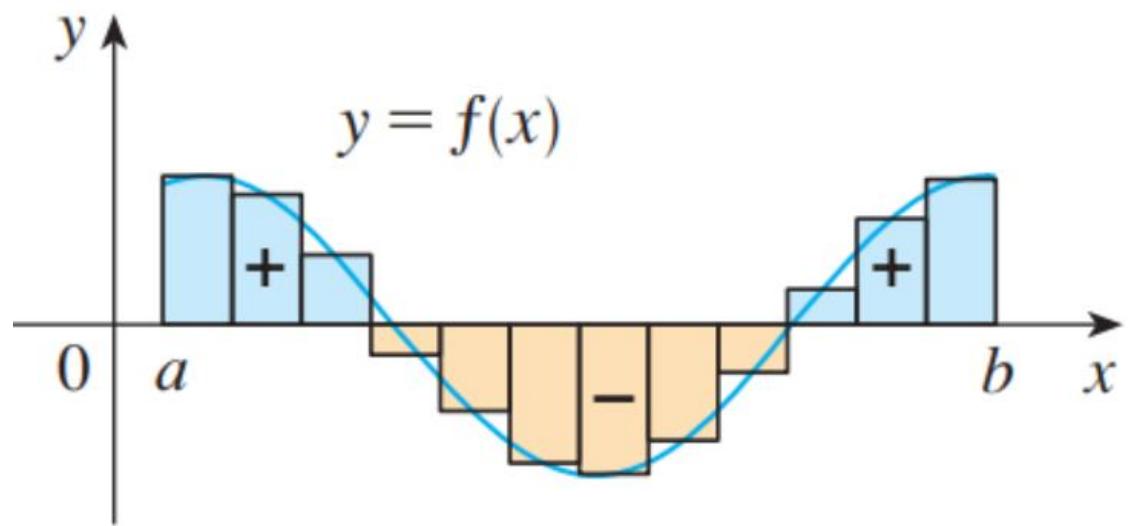
Bài toán: tính diện tích của miền giới hạn bởi phần dưới đồ thị $f \geq 0$ trên đoạn $[a, b]$

Chia $[a, b]$ thành n những hình chữ nhật có độ rộng bằng nhau. Khi đó, diện tích miền cần tính sấp xỉ

$$\sum_i f(x_i^*) \Delta x$$

Nếu n tiến đến vô cùng (nghĩa là ta càng chia nhỏ đoạn $[a, b]$), khi đó

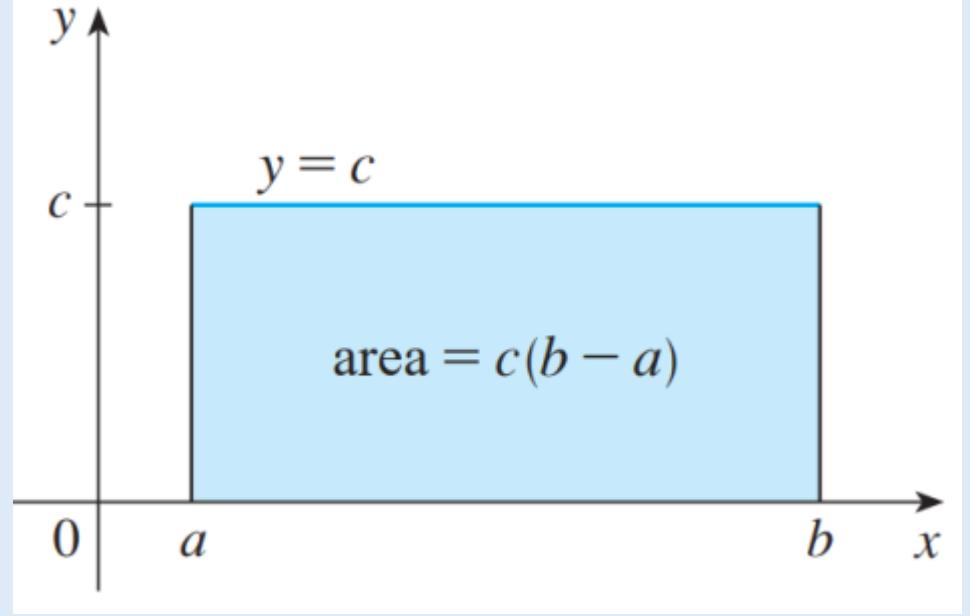
$$\sum_i f(x_i^*) \Delta x \rightarrow \text{số thực } A := \int_a^b f(x) dx$$



Tích phân $\int_a^b f(x)dx$ có thể âm hoặc bằng không, nó phụ thuộc vào dấu của hàm f

Diện tích, của miền giới hạn thì luôn dương và được tính bởi

$$\int_a^b |f(x)|dx$$



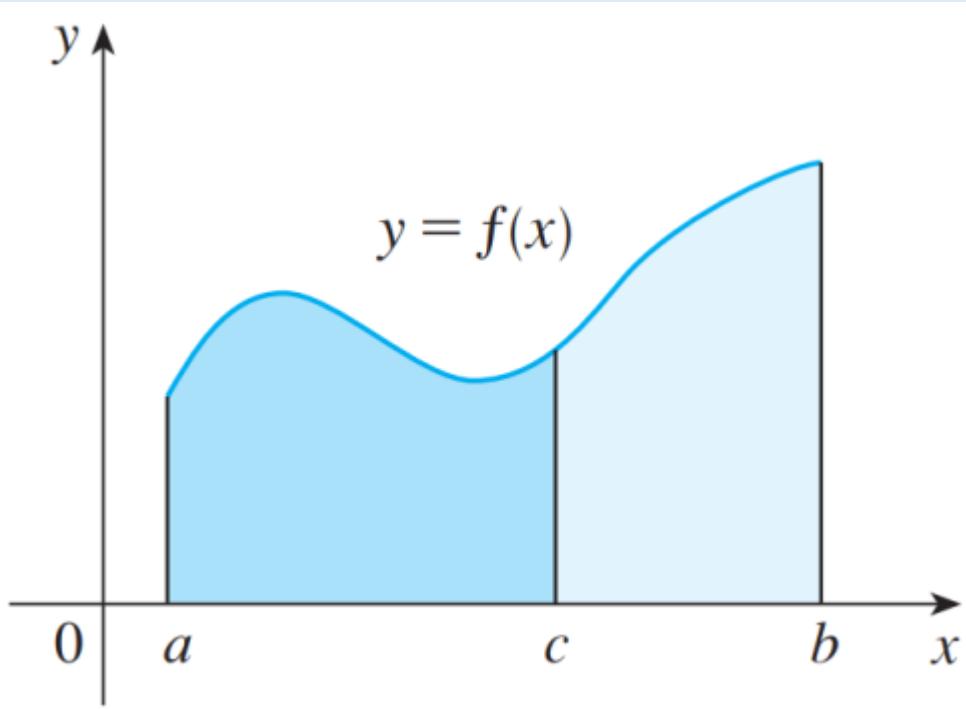
☞ $\int_a^b f(x)dx = - \int_b^a f(x)dx$

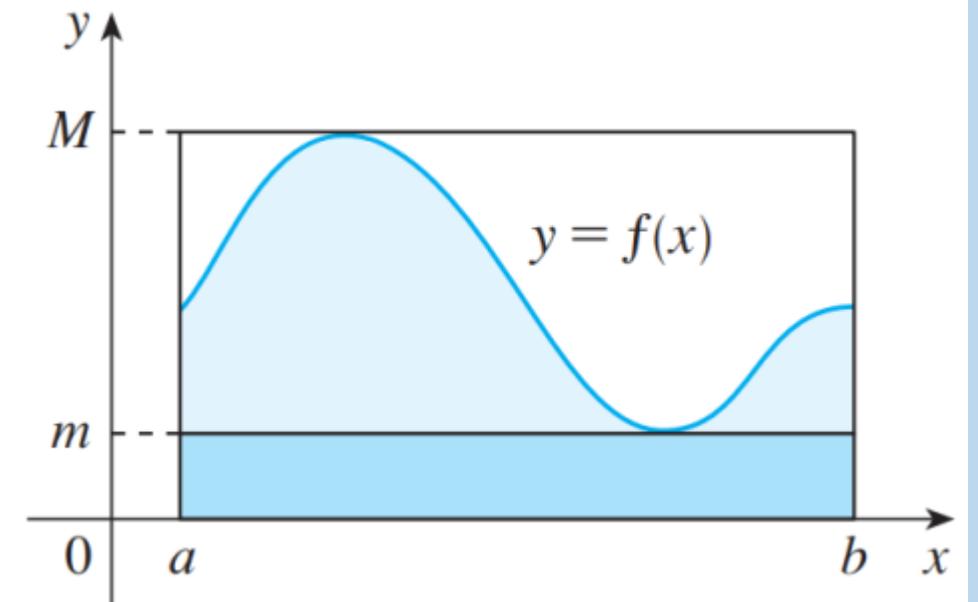
☞ $\int_a^b cf(x)dx = c \int_a^b f(x)dx$

☞ $\int_a^b cdx = c(b - a)$

☞ If $f(x) \leq g(x)$, then
 $\int_a^b f(x)dx \geq \int_a^b g(x)dx$

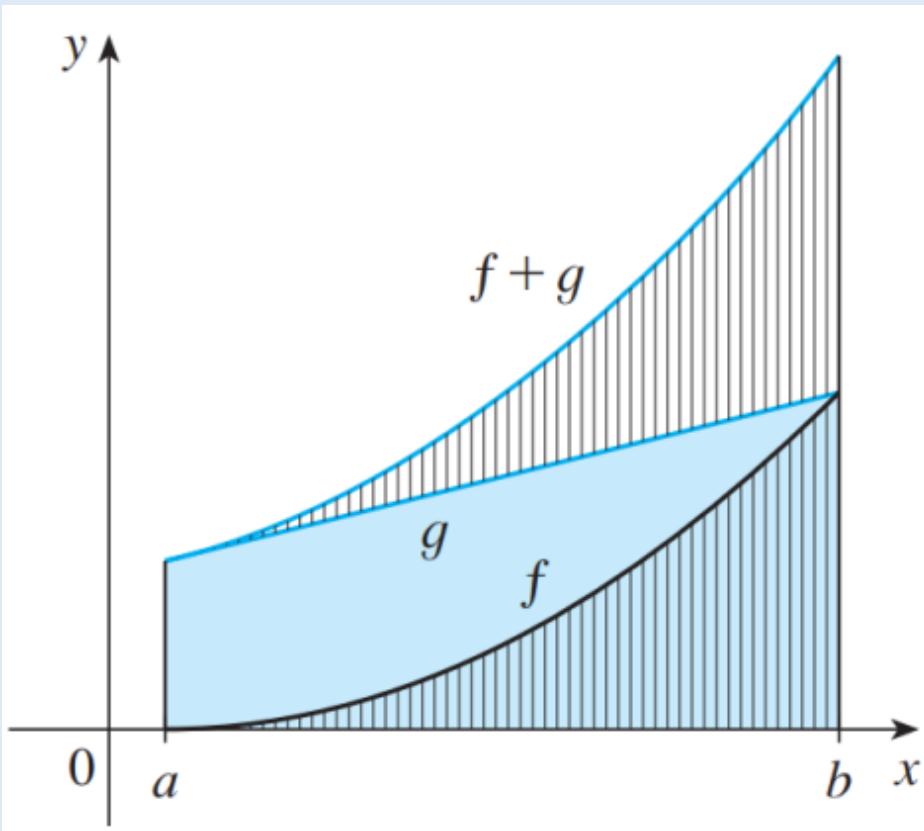
☞ $\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$





If $m \leq f(x) \leq M$, then
 $M(b - a) \geq \int_a^b f(x)dx \geq m(b - a)$

☞ $\int_a^b f(x) \pm g(x)dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$





Giả sử f liên tục trên $[a, b]$. Nếu

$$g(x) = \int_a^x f(x)dx, \quad \text{thì } g'(x) = f(x)$$

Và

$$\int_a^b f(x)dx = F(b) - F(a), \quad \text{nếu , } \quad F'(x) = f(x)$$

Tính các tích phân sau đây

$$\int_{-1}^2 (x^3 - 3x) dx, \quad \int_1^9 \sqrt{x} dx, \quad \int_1^9 \frac{x-1}{\sqrt{x}} dx$$

NOTE



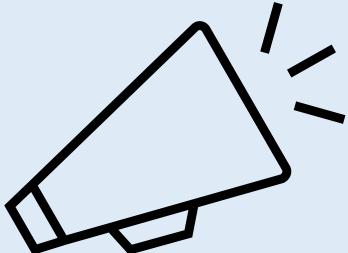
Cách tính tích phân khác

 g' liên tục trên $[a, b]$, f liên tục trên $Im(g)$. Vậy

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(t)dt,$$

 g', f' liên tục trên $[a, b]$. Khi đó

$$\int_a^b f(x)g'(x)dx = f(x)g(x) \Big|_a^b - \int_a^b g(x)f'(x)dx,$$



$$\int \sin^m x \cos^{2k+1} x dx = \int \sin^m x (1 - \sin^2 x)^k \cos x dx$$

 Đặt $u = \sin x$

$$\begin{aligned} \int \sin^{2k+1} x \cos^n x dx &= \int \sin^{2k} x \cos^n x \sin x dx \\ &= \int (1 - \cos^2 x)^k x \cos^n x \sin x dx \end{aligned}$$

 Đặt $u = \cos x$

$$\int \sin^{2k} x \cos^{2n} x dx = \int \frac{1}{4} (1 - \cos 2x)^k (1 + \cos 2x)^n dx$$



$$\int_0^4 \sqrt{2x+1} dx$$

$$\int_0^4 \frac{x}{\sqrt{1+2x}} dx$$

$$\int_0^{\frac{\pi}{2}} \cos^2 x \sin^5 x dx$$

$$\int_0^1 \tan^{-1} x dx$$

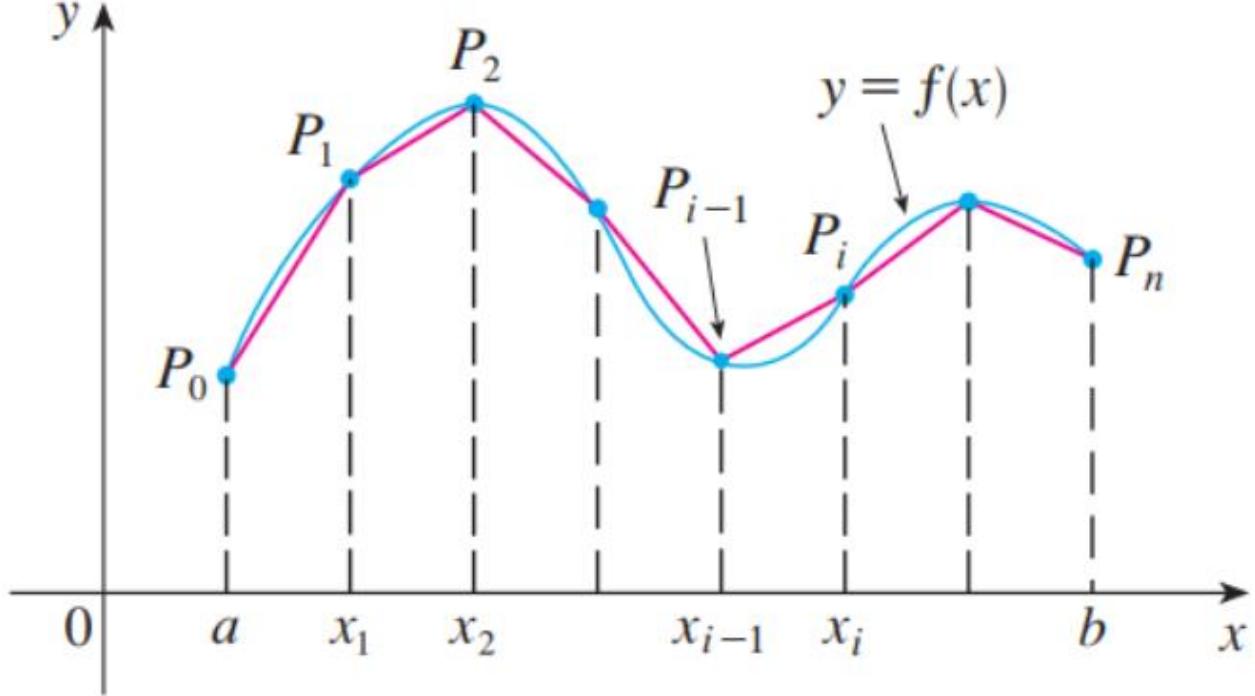
$$\int_0^{\frac{\pi}{2}} \cos^2 x dx$$

$$\int_0^2 x\sqrt{x-1} dx$$

$$\int_0^{\pi} \cos^3 x dx$$

$$\int_1^2 \ln x dx$$





$$L = \sum |P_{i-1}P_i|$$

$$|P_{i-1}P_i|^2 = (x_{i-1} - x_i)^2 + (y_{i-1} - y_i)^2$$

$$y_i - y_{i-1} = f'(x_i^*)(x_i - x_{i-1}), \text{ với } x_i^* \in (x_{i-1}, x_i)$$

$$L = \sum \sqrt{1 + [f'(x)]^2} \Delta x$$

Khi $n \rightarrow \infty$, ta có được công thức dưới đây

Giả sử f' liên tục trên $[a, b]$. Độ cung đường sinh bởi đồ thị $y = f(x)$ tính bởi

$$L = \int_a^b \sqrt{1 + [f'(x)]^2} dx,$$

$$y = x^2, \text{ với } 0 \leq x \leq 1.$$

$$y = 1 + 6x^{\frac{3}{2}}, \text{ với } 0 \leq x \leq 1.$$

$$y = \frac{x^3}{3} + \frac{1}{4x}, \text{ với } 0 \leq x \leq 1.$$