All pair Shortest Path

Chapter 25 from textbook

All Pairs Shortest Paths (APSP)

- *given : directed graph $G = (V, E)$,*
  weight function $\omega : E \rightarrow R$, $|V| = n$

- *goal : create an $n \times n$ matrix $D = (d_{ij})$ of shortest path distances*
  i.e., $d_{ij} = \delta(v_i, v_j)$

- *trivial solution : run a SSSP (single source shortest path) algorithm n times, one for each vertex as the source.*

All Pairs Shortest Paths (APSP)

- All edge weights are nonnegative : use Dijkstra's algorithm
  - Implementation using binary heap :
    - Running time: $O(V(V+E)\log V) = O(V^2\log V + EV\lg V)$
      - For dense graph: $O(V^3\lg V)$
      - For sparse graph: $O(V^2\log V)$

- Negative edge weights : use Bellman-Ford algorithm
  - Running time: $(V(VE)) = O(V^2E)$
    - For dense graph: $O(V^4)$
    - For sparse graph: $O(V^3)$
- Can we do better?

Adjacency Matrix Representation of Graphs

- ► n x n matrix W = (ω_{ij}) of edge weights :

$$\omega_{ij} = \begin{cases} \omega(v_i, v_j) & \text{if } (v_i, v_j) \in E \\ \infty & \text{if } (v_i, v_j) \notin E \end{cases}$$

- ► assume $\omega_{ii} = 0$ for all $v_i \in V$, because
  - no neg-weight cycle

    $\Rightarrow$ shortest path to itself has no edge,

    i.e., $\delta(v_i, v_i) = 0$

(δ denotes the shortest path)

Dynamic Programming

(1) Characterize the structure of an optimal solution.

(2) Recursively define the value of an optimal solution.

(3) Compute the value of an optimal solution in a bottom-up manner.

(4) Construct an optimal solution from information constructed in (3).

Shortest Paths and Matrix Multiplication

Assumption : negative edge weights may be present, but no negative weight cycles.

(1) Structure of a Shortest Path :

- Consider a shortest path p_{ij}^{m} from v_i to v_j such that $|p_{ij}^{m}| \leq m$
  - i.e., path p_{ij}^{m} has at most m edges.
- no negative-weight cycle $\Rightarrow$ all shortest paths are simple
  - $\Rightarrow$ m is finite $\Rightarrow m \leq n - 1$
- $i = j \Rightarrow |p_{ii}| = 0$ & $\omega(p_{ii}) = 0$
- $i \neq j \Rightarrow$ decompose path p_{ij}^{m} into p_{ik}^{m-1} & $v_k \rightarrow v_j$, where $|p_{ik}^{m-1}| \leq m - 1$
  - p_{ik}^{m-1} should be a shortest path from v_i to v_k by optimal substructure property. (why?)
  - Therefore, $\delta(v_i, v_j) = \delta(v_i, v_k) + \omega_{kj}$

Shortest Paths and Matrix Multiplication

(2) A Recursive Solution to All Pairs Shortest Paths Problem :

- d_{ij}^{m} = minimum weight of any path from v_i to v_j that contains at most "*m*" edges.

- *m = 0* : There exist a shortest path from v_i to v_j with no edges $\leftrightarrow$ *i = j* .

$$d_{ij}^{0} = \begin{cases} 0 & \text{if } i = j \\ \infty & \text{if } i \neq j \end{cases}$$

- $m \geq 1$: $d_{ij}^{m} = \min \{ d_{ij}^{m-1} , \min_{1 \leq k \leq n \wedge k \neq j} \{ d_{ik}^{m-1} + \omega_{kj} \}\}$

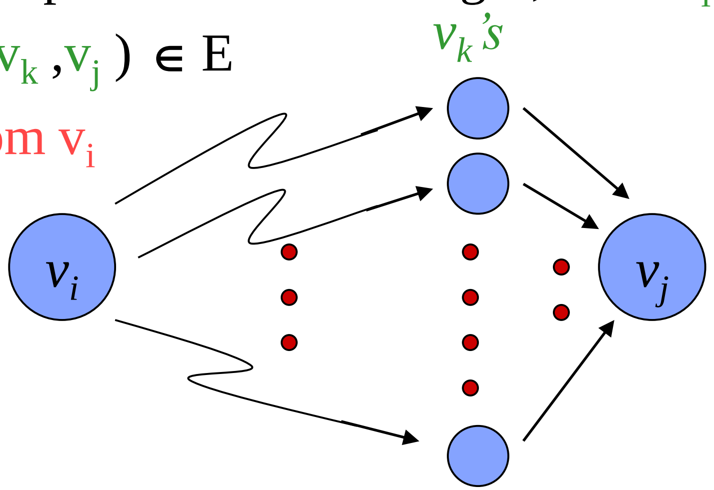
$$= \min_{1 \leq k \leq n} \{ d_{ik}^{m-1} + \omega_{kj} \} \text{ for all } v_k \in V,$$

since $\omega_{j\,j} = 0$ for all $v_j \in V$.

Shortest Paths and Matrix Multiplication

- to consider all possible shortest paths with $\leq m$ edges from v_i to v_j
  - ► consider shortest path with $\leq m$ -1 edges, from v_i to v_k, where $v_k \in R_{v_i}$ and $(v_k, v_j) \in E$

v_k is reachable from v_i



- note : $\delta(v_i, v_j) = d_{ij}^{n-1} = d_{ij}^{n} = d_{ij}^{n+1}$, since $m \leq n - 1 = |V| - 1$

Shortest Paths and Matrix Multiplication

(3) Computing the shortest-path weights bottom-up :

- *given* $W = D^1$ *, compute a series of matrices* $D^2, D^3, \ldots, D^{n-1}$ *, where* $D^m = (d_{ij}{}^m)$ *for* $m = 1, 2, \ldots, n-1$
  - ► *final matrix* D^{n-1} *contains actual shortest path weights, i.e.,* $d_{ij}{}^{n-1} = \delta (v_i , v_j)$

- *SLOW-APSP(W)*

  $D^1 \leftarrow W$

  for $m \leftarrow 2$ *to* n-1 *do*

  $\quad D^m \leftarrow$ *EXTEND*(D^{m-1} , W)

  return D^{n-1}

Shortest Paths and Matrix Multiplication

EXTEND (D , W)

► $D = (d_{ij})$ is an n x n matrix

for $i \leftarrow 1$ **to** n **do**

for $j \leftarrow 1$ **to** n **do**

$d_{ij} \leftarrow \infty$

for $k \leftarrow 1$ **to** n **do**

$d_{ij} \leftarrow \mathbf{min}\{d_{ij} , d_{ik} + \omega_{kj}\}$

return D

MATRIX-MULT (A , B)

► $\mathbf{C} = (c_{ij})$ *is an n x n result matrix*

for $i \leftarrow 1$ ***to*** n ***do***

for $j \leftarrow 1$ ***to*** n ***do***

$c_{ij} \leftarrow 0$

for $k \leftarrow 1$ ***to*** n ***do***

$c_{ij} \leftarrow c_{ij} + a_{ik} \times b_{kj}$

return C

Shortest Paths and Matrix Multiplication

×

- relation to matrix multiplication C = A B : $\mathbf{c}_{ij} = \sum_{1\le k\le n} \mathbf{a}_{ik} \times \mathbf{b}_{kj}$,
  - ► $D^{m-1} \leftrightarrow A$ & $\mathbf{W} \leftrightarrow B$ & $D^{m} \leftrightarrow C$
    "min" ↔ "addition" & "addition" ↔ "x" & "∞" ↔ "0"

- Thus, we compute the sequence of matrix products

$$D^1 = D^0 \times W = W \;;\; \text{note } D^0 = \text{identity matrix},$$

$$D^2 = D^1 \times W = W^2 \qquad \text{i.e., } d_{ij}^{\,0} = \begin{cases} 0 & \text{if } i = j \\ \infty & \text{if } i \neq j \end{cases}$$

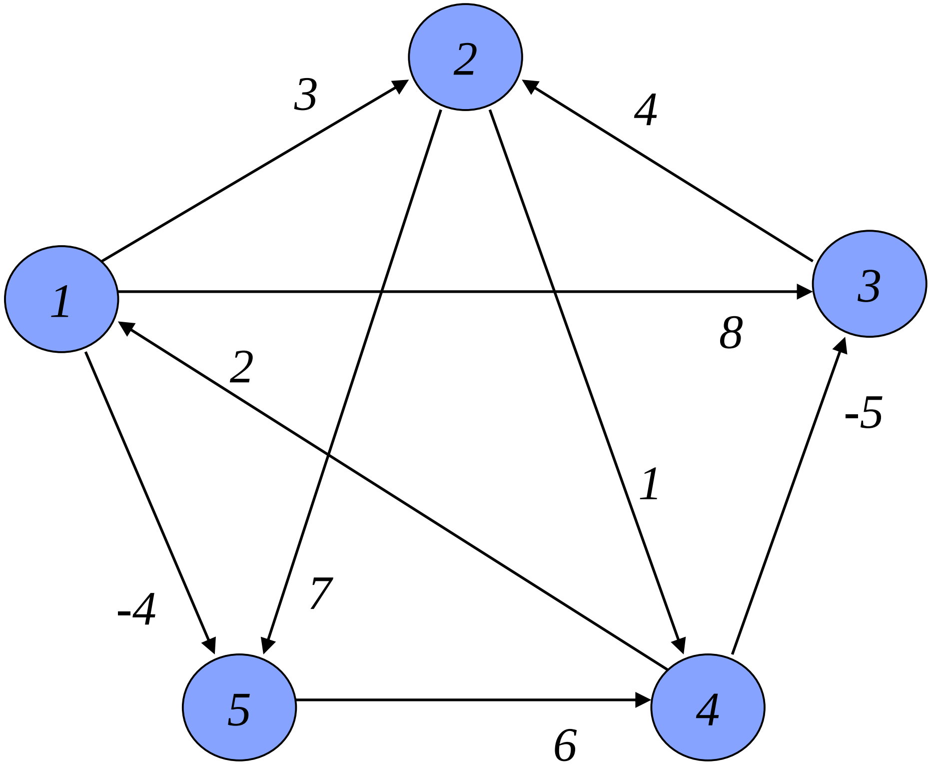
$$D^3 = D^2 \times W = W^3$$

$$D^{n-1} = D^{n-2} \times W = W^{n-1}$$

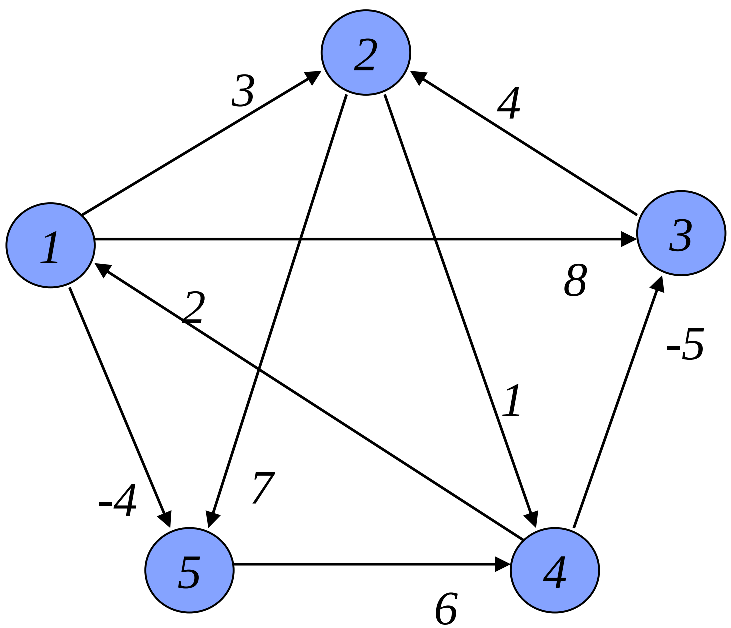
- running time : $\Theta(n^4) = \Theta(V^4)$
  - ► each matrix product : $\Theta(n^3)$
  - ► number of matrix products : n-1

Shortest Paths and Matrix Multiplication

- *Example*



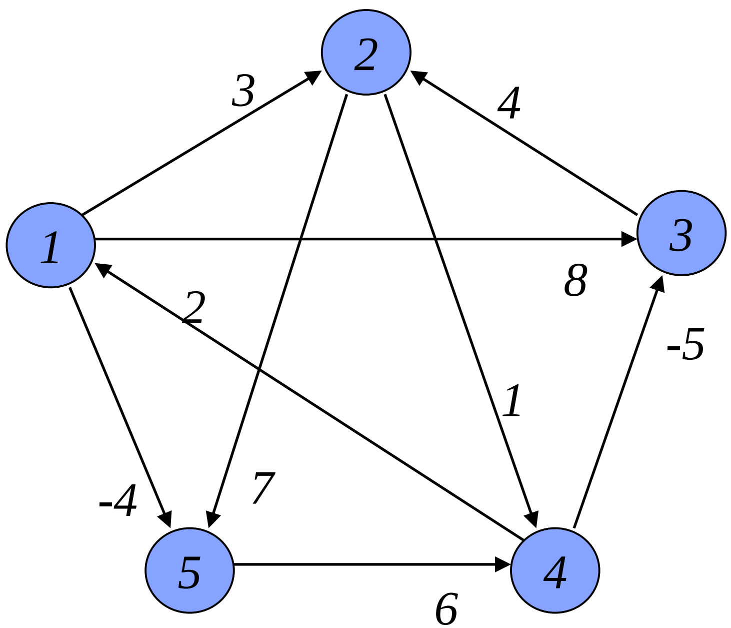
Shortest Paths and Matrix Multiplication



	1	2	3	4	5
1	0	3	8	∞	-4
2	∞	0	∞	1	7
3	∞	4	0	∞	∞
4	2	∞	-5	0	∞
5	∞	∞	∞	6	0

$D^1 = D^0 W$

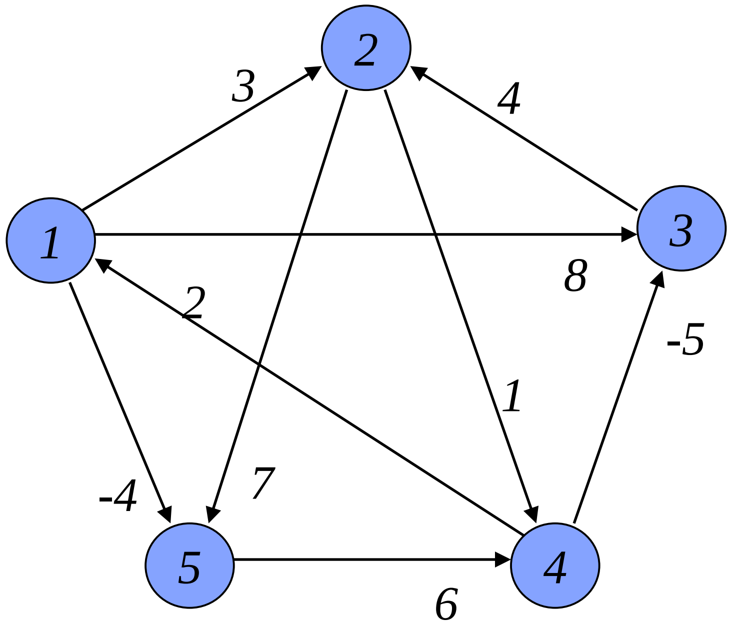
Shortest Paths and Matrix Multiplication



	1	2	3	4	5
1	0	3	8	2	-4
2	3	0	-4	1	7
3	∞	4	0	5	11
4	2	-1	-5	0	-2
5	8	∞	1	6	0

$D^2 = D^1 W$

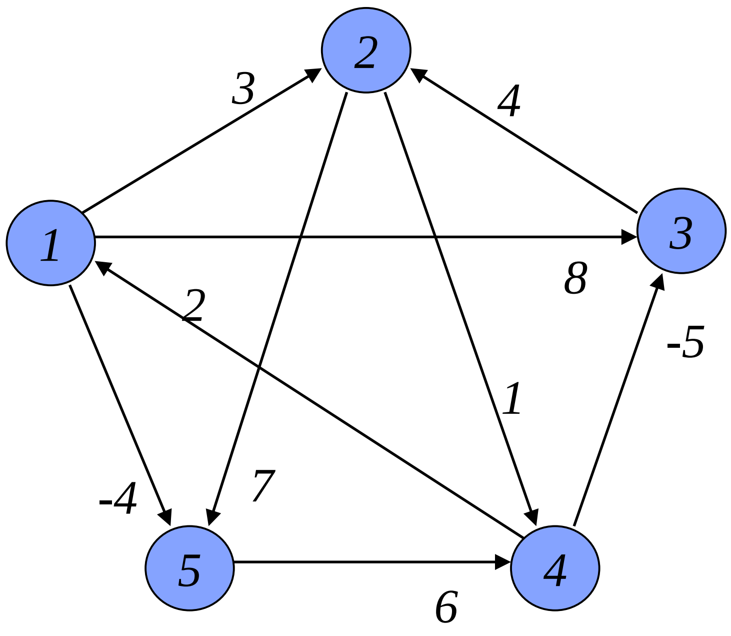
Shortest Paths and Matrix Multiplication



	1	2	3	4	5
1	0	3	-3	2	-4
2	3	0	-4	1	-1
3	7	4	0	5	11
4	2	-1	-5	0	-2
5	8	5	1	6	0

$D^3 = D^2 W$

Shortest Paths and Matrix Multiplication



	1	2	3	4	5
1	0	1	-3	2	-4
2	3	0	-4	1	-1
3	7	4	0	5	3
4	2	-1	-5	0	-2
5	8	5	1	6	0

$D^4 = D^3 W$

Improving Running Time Through Repeated Squaring

- idea : goal is not to compute all D^m matrices
  - ► we are interested only in matrix D^{n-1}
- recall : no negative-weight cycles $\Rightarrow D^m = D^{n-1}$ for all $m \geq n-1$
- we can compute D^{n-1} with only $\lceil \lg(n-1) \rceil$ matrix products as

$$D^1 = W$$

$$D^2 = W^2 = W \times W$$

$$D^4 = W^4 = W^2 \times W^2$$

$$D^8 = W^8 = W^4 \times W^4$$

$$D^{2^{\lceil \lg(n-1) \rceil}} = W^{2^{\lceil \lg(n-1) \rceil}} = W^{2^{\lceil \lg(n-1) \rceil - 1}} \times W^{2^{\lceil \lg(n-1) \rceil - 1}}$$

- This technique is called repeated squaring.

Improving Running Time Through Repeated Squaring

- FASTER-APSP (W)
  - $D^1 \leftarrow W$
  - $m \leftarrow 1$
  - while $m < n\text{-}1$ do
    - $D^{2m} \leftarrow$ EXTEND (D^m , D^m)
    - $m \leftarrow 2m$
  - return D^m

- final iteration computes D^{2m} for some $n-1 \le 2m \le 2n-2 \Rightarrow D^{2m} = D^{n-1}$

- running time : $\Theta(n^3 \lg n) = \Theta(V^3 \lg V)$

  - ► each matrix product : $\Theta(n^3)$
  - ► # of matrix products : $\lceil \lg(n-1) \rceil$
  - ► simple code, no complex data structures, small hidden constants in Θ-notation.

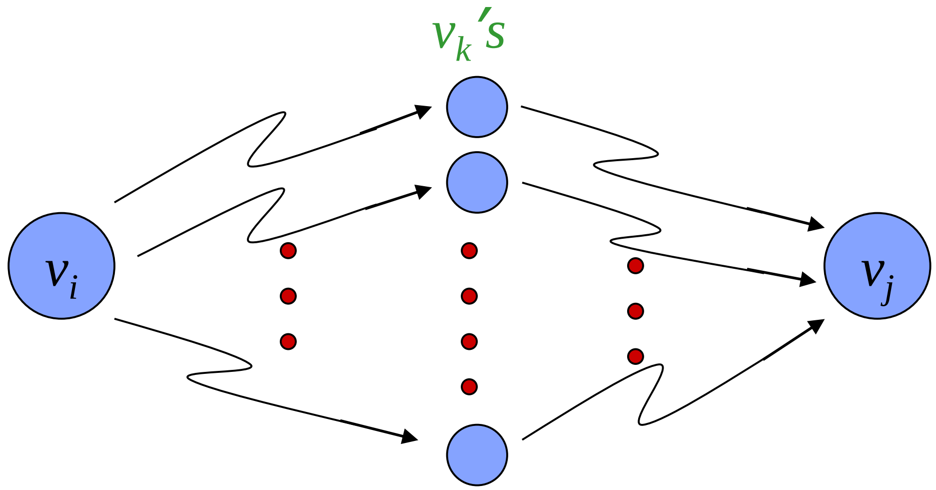
Idea Behind Repeated Squaring

- decompose p_{ij}^{2m} as p_{ik}^{m} & p_{kj}^{m}, where

  p_{ij}^{2m} : $v_i \rightsquigarrow v_j$

  p_{ik}^{m} : $v_i \rightsquigarrow v_k$

  p_{kj}^{m} : $v_k \rightsquigarrow v_j$



Floyd-Warshall Algorithm

- Assumption : negative-weight edges, but no negative-weight cycles

(1) The Structure of a Shortest Path :

- Definition : intermediate vertex of a path $p = < v_1 , v_2 , v_3 , \ldots , v_k >$ is the set of vertices appearing on the path p other than v_1 or v_k .
- p_{ij}^{m} : a shortest path from v_i to v_j with all intermediate vertices from $V_m = \{ v_1 , v_2 , \ldots , v_m \}$
- Relationship between p_{ij}^{m} and p_{ij}^{m-1} will depends on whether v_m is an intermediate vertex of p_{ij}^{m} or not

- case 1: v_m is not an intermediate vertex of p_{ij}^{m}

$\Rightarrow$ all intermediate vertices of p_{ij}^{m} are in V_{m-1}

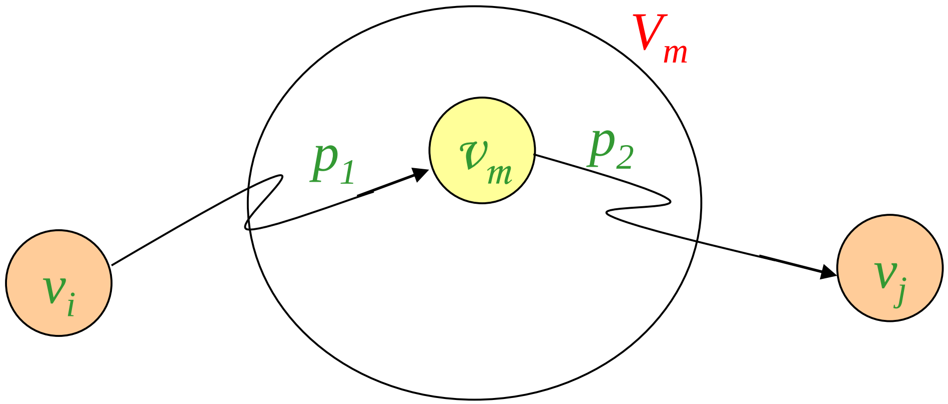
$\Rightarrow p_{ij}^{m} = p_{ij}^{m-1}$

Floyd-Warshall Algorithm

- case 2 : v_m is an intermediate vertex of p_{ij}^{m}
  - decompose path as $v_i \leadsto v_m \leadsto v_j$

    $\Rightarrow p_1 : v_i \leadsto v_m$ & $p_2 : v_m \leadsto v_j$
  - by opt. structure property both p_1 & p_2 are shortest paths.
  - v_m is not an intermediate vertex of p_1 & p_2

    $\Rightarrow p_1 = p_{im}^{m-1}$ & $p_2 = p_{mj}^{m-1}$



Floyd-Warshall Algorithm

(2) A Recursive Solution to APSP Problem :

- $d_{ij}^{m} = \omega(p_{ij})$: weight of a shortest path from v_i to v_j with all intermediate vertices from

  $V_m = \{ v_1 , v_2 , \ldots , v_m \}$.

- note : $d_{ij}^{n} = \delta(v_i, v_j)$ since $V_n = V$
  - ► i.e., all vertices are considered for being intermediate vertices of p_{ij}^{n} .

Floyd-Warshall Algorithm

- compute d_{ij}^{m} in terms of d_{ij}^{k} with smaller $k < m$
- $m = 0$: V_0 = empty set

  $\Rightarrow$ path from v_i to v_j with no intermediate vertex.

  i.e., v_i to v_j paths with at most one edge

  $\Rightarrow d_{ij}^{0} = \omega_{i\,j}$

- $m \geq 1$: $d_{ij}^{m} = \min\{d_{ij}^{m-1}, d_{im}^{m-1} + d_{mj}^{m-1}\}$

v_m *is an not intermediate vertex of* p_{ii}^{m}

v_m *is an intermediate vertex of* p_{ij}^{m}

Floyd-Warshall Algorithm

(3) Computing Shortest Path Weights Bottom Up :

FLOYD-WARSHALL(W)

▶ $D^0, D^1, \ldots, D^n$ are n x n matrices

for $m \leftarrow 1$ to n do

for $i \leftarrow 1$ to n do

for $j \leftarrow 1$ to n do

$d_{ij}^{m} \leftarrow \min \{d_{ij}^{m-1}, d_{im}^{m-1} + d_{mj}^{m-1}\}$

return D^n

Floyd-Warshall Algorithm

```
FLOYD-WARSHALL ( W )
    ▶ D  is an n x n matrix
    D ← W
    for m ← 1 to n do
      for i ← 1 to n do
         for j ← 1 to n do
            if  d_ij > d_im + d_mj  then
               d_ij ← d_im + d_mj
    return D
```

Floyd-Warshall Algorithm

- maintaining *n* D matrices can be avoided by dropping all superscripts.
  - *m-th* iteration of outermost for-loop

    begins with $D = D^{m-1}$

    ends with $D = D^m$
  - computation of d_{ij}^{m} depends on d_{im}^{m-1} and d_{mj}^{m-1} .

    no problem if d_{im} & d_{mj} are already updated to d_{im}^{m} & d_{mj}^{m}
    since $d_{im}^{m} = d_{im}^{m-1}$ & $d_{mj}^{m} = d_{mj}^{m-1}$.

- running time : $\Theta(n^3) = \Theta(V^3)$

  simple code, no complex data structures, small hidden constants