Dynamic programming

Chapter 15 from textbook

Algorithmic Paradigms

- Divide-and-conquer. Break up a problem into sub-problems, solve each sub-problem independently, and combine solution to sub-problems to form solution to original problem.
  - **Example:** Merge sort, Binary search
- Greedy. Build up a solution incrementally, myopically optimizing some local criterion.
  - **Example:** Shortest path using Dijkstra’s algorithm, MST using Prim’s/Kruskal’s algorithm, Hoffman encoding scheme
- Dynamic programming. Break up a problem into a series of overlapping sub-problems, and build up solutions to larger and larger sub-problems.
  - **Example:** Rod cutting, Matrix multiplication, Largest common subsequence, Edit distance, All pair shortest path

Dynamic Programming History

- Bellman. Pioneered the systematic study of dynamic programming in the 1950s.

- Etymology.
  - Dynamic programming = planning over time.
  - Secretary of Defense was hostile to mathematical research.
  - Bellman sought an impressive name to avoid confrontation.
    - "something not even a Congressman could object to"

Reference: Bellman, R. E. Eye of the Hurricane, An Autobiography.

Dynamic programming

- In Divide-and-conquer approach
  - We partition the problem into “independent” subproblems
  - Solve the subproblems recursively
  - Then combine the solutions to solve the original problem

- Dynamic programming is applicable when
  - We have “overlapping” subproblems
  - Naïve recursive method would solve these overlapping subproblems many many times resulting in exponential (in problem size) running time
  - Dynamic programming avoids this by solving subproblems in a particular order from smaller to larger subproblems such that each subproblem is solved exactly once and saves this in a table for later reuse

Introduction

- Dynamic Programming(DP) applies to optimization problems
  - Such problems can have many possible solutions
  - Each solution has a value and we wish to find a solution having optimal (minimum or maximum) value
  - We call such a solution as an optimal solution as oposed to the optimal solution, since there may be sevaral solutions that achieve the optimal value.
- When we develop dynamic programming algorithm we follow a sequence of four steps
  - Characterize the structure of an optimal solution
  - Recursively define the value of an optimal solution
  - Compute the value of an optimal solution in a bottom-up fashion
  - Construct an optimal solution from computed information

Dynamic Programming Applications

- Areas.
  - Bioinformatics.
  - Control theory.
  - Information theory.
  - Operations research.
  - Computer science: theory, graphics, AI, systems, ….
- Some famous dynamic programming algorithms.
  - Viterbi for hidden Markov models.
  - Unix diff for comparing two files.
  - Smith-Waterman for sequence alignment.
  - Ford-Marshall algorithm for all pair shortest path shortest path
  - Cocke-Kasami-Younger for parsing context free grammars.

Problem #1: Fibonacci numbers

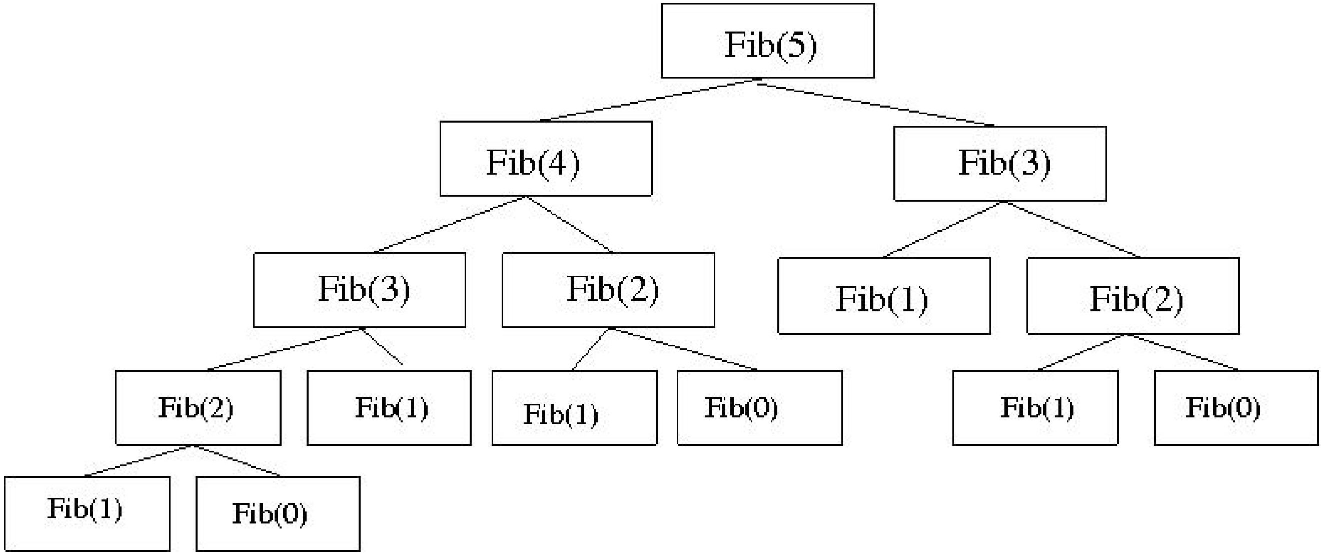
- Let’s consider calculating the Fibonacci numbers:
  - 0,1,1,2,3,5,8,13,21,44,65,…
  - Any number is sum of the previous two numbers
- A naïve recursive algorithm will look something like this:

```
Recursive Algorithm:
Fib(n)
{
    if (n == 0)
        return 0;
    if (n == 1)
        return 1;
   Return Fib(n-1)+Fib(n-2)
}
```

- What is the problem here?

Problem #1: Fibonacci numbers

- Same subproblems are solved repeatedly many times resulting in exponential running time
- T(n)=?



Problem #1: Fibonacci numbers

- Better solution is obtained by
  - Observing optimal substructure F(n)=F(n-1)+F(n-2)
  - And solving subproblems in particular order smaller subproblems first and reusing these results
- To compute F(n), maintain an array M[0,1,…,n]
  - Set M[0]=0, M[1]=1
  - For i=2 to n

    M[n]=M[n-1]+M[n-2]

- Note that this is not an optimization problem but illustrates the main idea

Problem #2: Rod cutting

- Suppose a company buys long rods and sells by cutting them into small pieces for a profit as per the following table

length i	1	2	3	4	5	6	7	8	9	10
price p_i	1	5	8	9	10	17	17	20	24	30

- **Question:** We are given a rod of length n, and want to maximize revenue, by cutting up the rod into pieces and selling each of the pieces.

- **Example:** we are given a 4 inches rod. What is the best solution to cut up?

Problem #2: Rod cutting

- **Example:** we are given a 4 inches rod. Best solution to cut up? We'll first list the solutions:

length i	1	2	3	4	5	6	7	8	9	10
price p_i	1	5	8	9	10	17	17	20	24	30

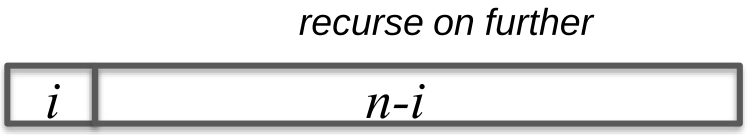
- **1.)** Cut into 2 pieces of length 2: $p_2 + p_2 = 5 + 5 = 10$
- **2.)** Cut into 4 pieces of length 1: $p_1 + p_1 + p_1 + p_1 = 1 + 1 + 1 + 1 = 4$
- **3-4.)** Cut into 2 pieces of length 1 and 3 (3 and 1): $p_3 + p_1 = 8 + 1 = 9$
- **5.)** Keep length 4: $p_4 = 9$
- **6-8.)** Cut into 3 pieces, length 1, 1 and 2 (and all the different orders)

  $p_1 + p_1 + p_2 = 7$ $p_1 + p_2 + p_1 = 7$ $p_2 + p_1 + p_1 = 7$

- **Total:** 8 cases for $n = 4$ (= 2^{n-1}). We can slightly reduce by always requiring cuts in non-decreasing order. But still a lot!

Problem #2: Rod cutting

- **Note:** We've computed a brute force solution; all possibilities for this simple small example. But we want more optimal solution!
  - One possible solution is as follows:



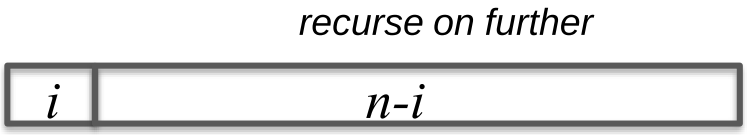
recurse on further

i n-i

  - What are we doing?
    - Cut rod into length i and n-i
    - Only remainder n-i can be further cut (recursed)
  - We need to define:
    - a.) **Maximum revenue** for rod of size n: $\boldsymbol{r_n}$ (that is the solution we want to find).
    - b.) **Revenue (price)** for single rod of length i: $\boldsymbol{p_i}$

Problem #2: Rod cutting

- **Note:** We’ve computed a brute force solution; all possibilities for this simple small example. But we want more optimal solution!
    - One possible solution is as follows:

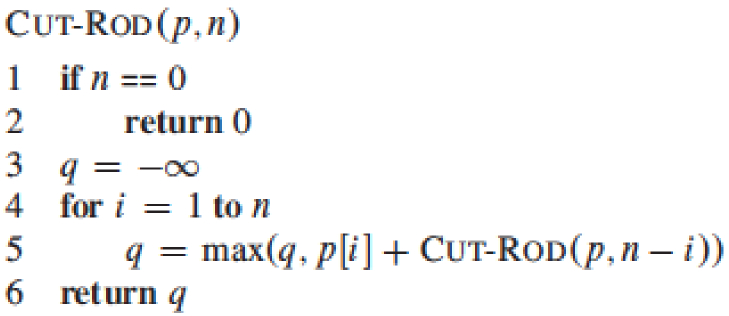


- Revenue: $p_i + r_{n-i}$ can be seen by recursing on n-i
- There many possible choices for i

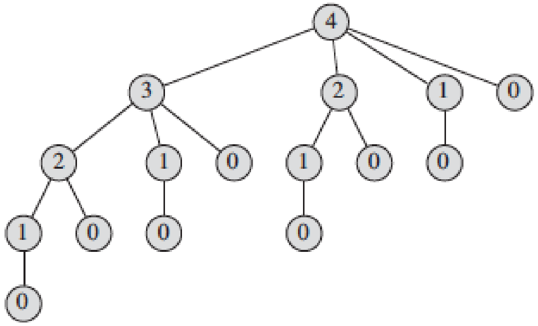
$$r_n = \max\begin{Bmatrix} p_1 + r_{n-1} \\ p_1 + r_{n-2} \\ \dots \\ p_n + r_0 \end{Bmatrix}$$

Problem #2: Rod cutting

- Consider naïve recursive (top-down approach)



```
CUT-ROD(p, n)
if $n == 0$
    return 0
$q = -\infty$
for $i = 1$ to $n$
    $q = \max(q, p[i] + \text{CUT-ROD}(p, n - i))$
return $q$
```



  - Problem? Slow runtime (it’s essential brute force)!
  - Why? Cut-rod calls itself repeatedly with the same parameter values (tree):
    - Node label: size of the subproblem called on
    - Can be seen by eye that many subproblems are called repeatedly (subproblem overlap)
    - Number of nodes exponential in *n* i.e., (2^n). therefore exponential number of calls.

Problem #2: Rod cutting

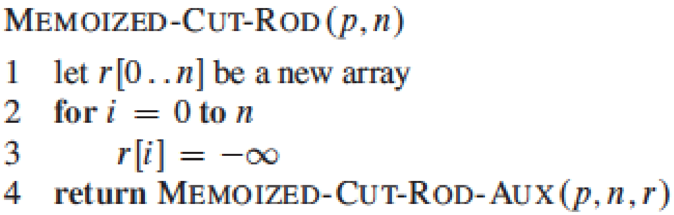
- We have seen, recursive solution is inefficient, since it repeatedly calculates a solution of the same subproblem (overlapping subproblem).
- Instead, solve each subproblem only once AND save its solution. Next time we encounter the subproblem look it up in a hash table or an array
  - This is called recursive **top-down solution with memoization**
- We will also talk about a second solution where we save the solution of subproblems of increasing size (i.e. in order) in an array.
  - Each time we will fall back on solutions that we obtained in previous steps and stored in an array (**bottom-up solution**).

Memoization

“In computing, **memoization** or memoisation is an optimization technique used primarily to speed up computer programs by storing the results of expensive function calls and returning the cached result when the same inputs occur again.”- Wikipedia

Problem #2: Rod cutting

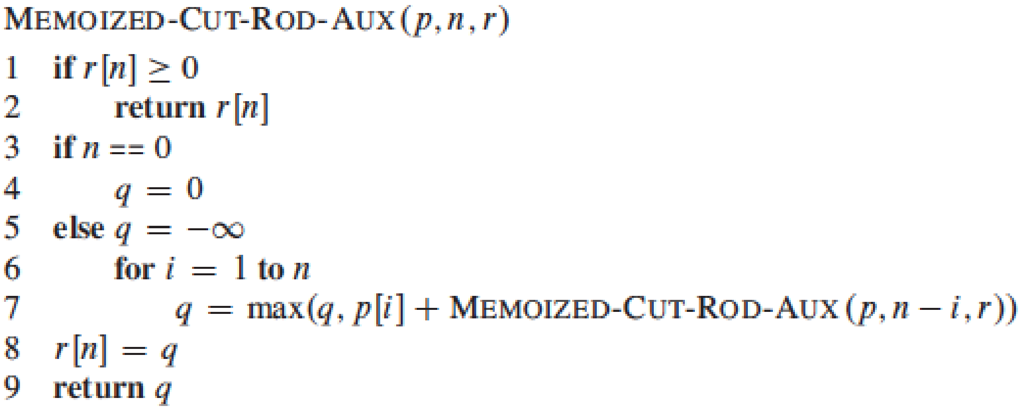
- Recursive top-down solution: **Cut-Rod with Memoization**
  - Step 1: Initialization



```
MEMOIZED-CUT-ROD(p, n)
let r[0..n] be a new array
for i = 0 to n
    r[i] = −∞
return MEMOIZED-CUT-ROD-AUX(p, n, r)
```

Creates array for holding memoized results, and initialized to minus infinity. Then calls the main auxiliary function.

  - Step 2: The main auxiliary function, which goes through the lengths, computes answers to subproblems and memoizes if subproblem not yet encountered:



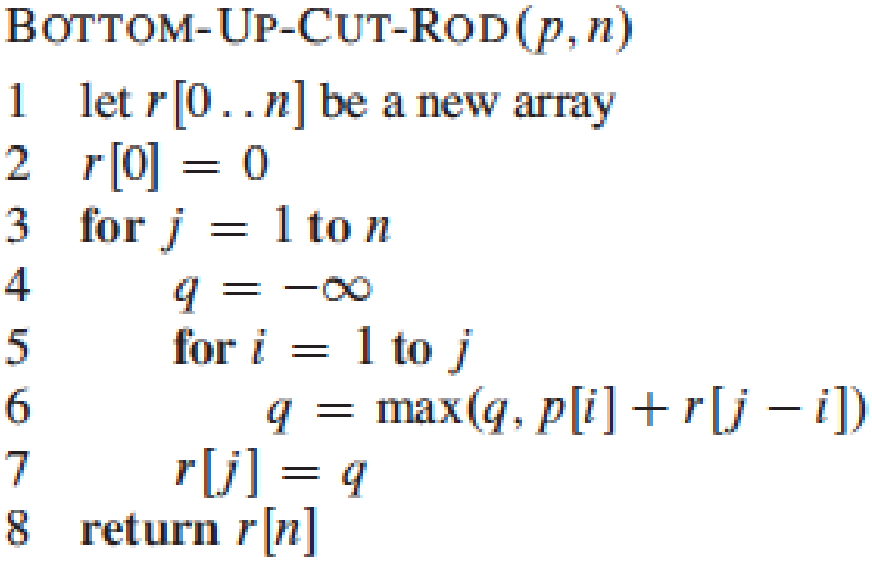
```
MEMOIZED-CUT-ROD-AUX(p, n, r)
if r[n] ≥ 0
    return r[n]
if n == 0
    q = 0
else q = −∞
    for i = 1 to n
        q = max(q, p[i] + MEMOIZED-CUT-ROD-AUX(p, n − i, r))
r[n] = q
return q
```

Problem #2: Rod cutting

- Running time of Recursive top-down solution with Memoization
  - Running time is $\theta(n^2)$
    - Recursive call to a previously solved problem runs immediately.
    - Memoized-Cut-Rod solves each subproblem just once
    - It solves subproblem of size 0,1,2,…,n
    - To solve a subproblem of size n, line 6-7 iterates n times
    - Thus, total number of iterations of this for loop, over all recursive calls of Memoized-Cut-Rod forms an arithmetic series giving a total $\theta(n^2)$ iterations

Problem #2: Rod cutting

- Bottom-up solution is even simpler
- Each time we reuse previously computed values stored in an array



```
BOTTOM-UP-CUT-ROD($p, n$)
let $r[0\,..\,n]$ be a new array
$r[0] = 0$
for $j = 1$ to $n$
    $q = -\infty$
    for $i = 1$ to $j$
        $q = \max(q, p[i] + r[j - i])$
    $r[j] = q$
return $r[n]$
```

Compute maximum revenue if it hasn't already been computed.

- Running time is $\theta(n^2)$

Problem #3: Matrix-Chain Multiplication

Problem: given a sequence $\langle A_1, A_2, \ldots, A_n \rangle$, compute the product:

$$A_1 \cdot A_2 \cdots A_n$$

- Matrix compatibility:

$C = A \cdot B$

$\text{col}_A = \text{row}_B$

$\text{row}_C = \text{row}_A$

$\text{col}_C = \text{col}_B$

$C = A_1 \cdot A_2 \cdots A_i \cdot A_{i+1} \cdots A_n$

$\text{col}_i = \text{row}_{i+1}$

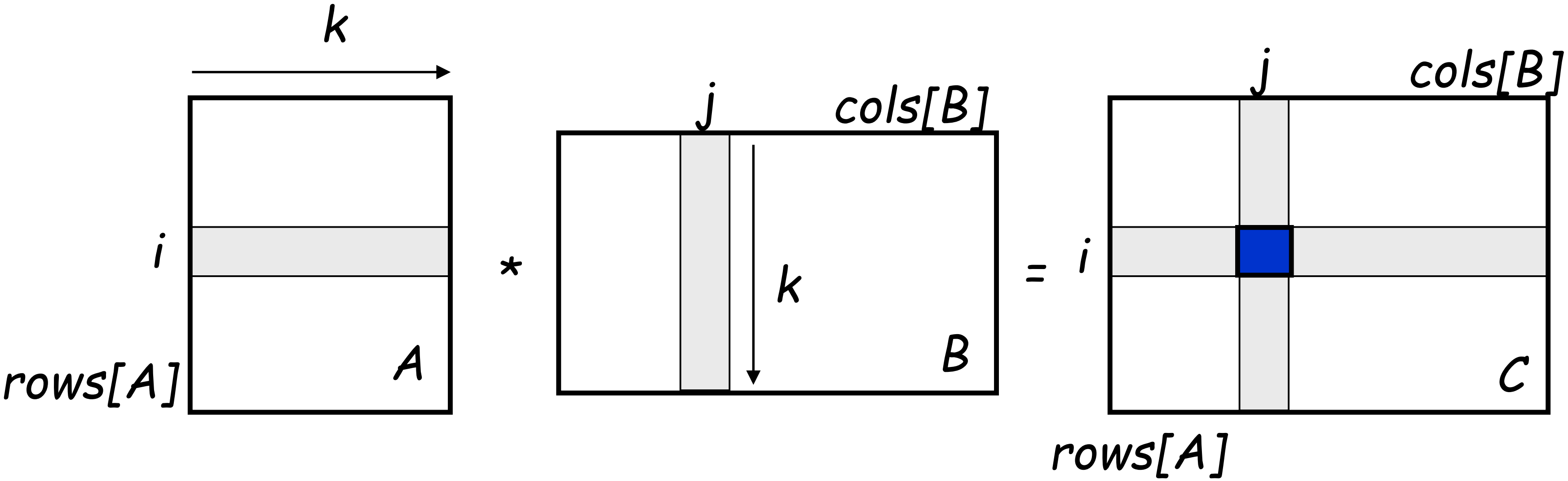
$\text{row}_C = \text{row}_{A1}$

$\text{col}_C = \text{col}_{An}$

MATRIX-MULTIPLY(A, B)

```
if columns[A] ≠ rows[B]
    then error "incompatible dimensions"
    else for i ← 1 to rows[A]
              do for j ← 1 to columns[B]
                      do C[i, j] = 0
                         for k ← 1 to columns[A]
                              do C[i, j] ← C[i, j] + A[i, k] B[k, j]
```

rows[A] · cols[A] · cols[B]
multiplications



Matrix-Chain Multiplication

- In what order should we multiply the matrices?

$$A_1 \cdot A_2 \cdots A_n$$

- Parenthesize the product to get the order in which matrices are multiplied

E.g.: $A_1 \cdot A_2 \cdot A_3 = ((A_1 \cdot A_2) \cdot A_3)$

$= (A_1 \cdot (A_2 \cdot A_3))$

- Which one of these orderings should we choose?
  - The order in which we multiply the matrices has a significant impact on the cost of evaluating the product

Example

$$A_1 \cdot A_2 \cdot A_3$$

- A_1: 10 x 100
- A_2: 100 x 5
- A_3: 5 x 50

1. $((A_1 \cdot A_2) \cdot A_3)$: $A_1 \cdot A_2 = 10 \times 100 \times 5 = 5{,}000$ (10 x 5)

   $((A_1 \cdot A_2) \cdot A_3) = 10 \times 5 \times 50 = 2{,}500$

   Total: 7,500 scalar multiplications

2. $(A_1 \cdot (A_2 \cdot A_3))$: $A_2 \cdot A_3 = 100 \times 5 \times 50 = 25{,}000$ (100 x 50)

   $(A_1 \cdot (A_2 \cdot A_3)) = 10 \times 100 \times 50 = 50{,}000$

   Total: 75,000 scalar multiplications

one order of magnitude difference!!

Matrix-Chain Multiplication: Problem Statement

- Given a chain of matrices $\langle A_1, A_2, \ldots, A_n \rangle$, where A_i has dimensions $p_{i-1} \times p_i$, fully parenthesize the product $A_1 \cdot A_2 \cdots A_n$ in a way that minimizes the number of scalar multiplications.

$$\begin{array}{ccccccccc} A_1 & \cdot & A_2 & \cdots & A_i & \cdot & A_{i+1} & \cdots & A_n \\ p_0 \text{ x } p_1 & & p_1 \text{ x } p_2 & & p_{i-1} \text{ x } p_i & & p_i \text{ x } p_{i+1} & & p_{n-1} \text{ x } p_n \end{array}$$

What is the number of possible parenthesizations?

- Exhaustively checking all possible parenthesizations is not efficient!
- It can be shown that the number of parenthesizations grows as $\Omega(4^n/n^{3/2})$

(see page 333 in your textbook)

1. The Structure of an Optimal Parenthesization

- Notation:

  $A_{i\ldots j} = A_i\, A_{i+1} \cdots A_j,\ i \le j$

- Suppose that an optimal parenthesization of $A_{i\ldots j}$ splits the product between A_k and A_{k+1}, where $i \le k < j$

$$
\begin{aligned}
A_{i\ldots j} &= A_i\, A_{i+1} \cdots A_j \\
&= A_i\, A_{i+1} \cdots A_k\, A_{k+1} \cdots A_j \\
&= A_{i\ldots k}\, A_{k+1\ldots j}
\end{aligned}
$$

Optimal Substructure

$$A_{i\ldots j} = A_{i\ldots k} A_{k+1\ldots j}$$

- The parenthesization of the “prefix” $A_{i\ldots k}$ must be an optimal parentesization
- If there were a less costly way to parenthesize $A_{i\ldots k}$, we could substitute that one in the parenthesization of $A_{i\ldots j}$ and produce a parenthesization with a lower cost than the optimum ⇒ contradiction!
- *An optimal solution to an instance of the matrix-chain multiplication contains within it optimal solutions to subproblems*

2. A Recursive Solution

- Subproblem: determine the minimum cost of parenthesizing $A_{i\dots j} = A_i A_{i+1} \cdots A_j$ for $1 \le i \le j \le n$
- Let $m[i, j]$ = the minimum number of multiplications needed to compute $A_{i\dots j}$
    - full problem ($A_{1..n}$): $m[1, n]$
    - $i = j$: $A_{i\dots i} = A_i \Rightarrow m[i, i] = 0$ for $i=1, 2, \dots, n$

2. A Recursive Solution

- Consider the subproblem of parenthesizing

$A_{i \ldots j} = A_i\, A_{i+1} \cdots A_j$ for $1 \le i \le j \le n$

$p_{i-1}p_kp_j$

$= A_{i \ldots k}\, A_{k+1 \ldots j}$ for $i \le k < j$

$m[i, k]$ $m[k+1,j]$

- Assume that the optimal parenthesization splits the product $A_i\, A_{i+1} \cdots A_j$ at k ($i \le k < j$)

$$m[i, j] = \underbrace{m[i, k]}_{\text{min \# of multiplications to compute } A_{i\ldots k}} + \underbrace{m[k+1, j]}_{\text{min \# of multiplications to compute } A_{k+1\ldots j}} + \underbrace{p_{i-1}p_kp_j}_{\text{\# of multiplications to compute } A_{i\ldots k}A_{k\ldots j}}$$

2. A Recursive Solution (cont.)

$$m[i, j] = m[i, k] \quad + \quad m[k+1, j] \quad + \quad p_{i-1}p_kp_j$$

- We do not know the value of k
  - There are j – i possible values for k: k = i, i+1, …, j-1
- Minimizing the cost of parenthesizing the product $A_i A_{i+1} \cdots A_j$ becomes:

$$m[i, j] = \begin{cases} 0 & \text{if } i = j \\ \min\limits_{i \le k < j} \{m[i, k] + m[k+1, j] + p_{i-1}p_kp_j\} & \text{if } i < j \end{cases}$$

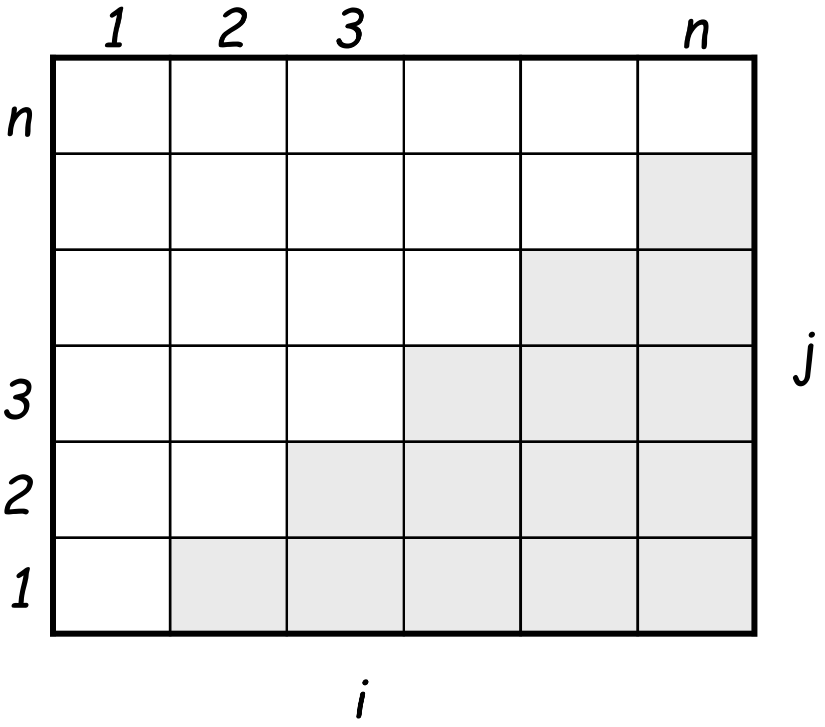
3. Computing the Optimal Costs

$$m[i, j] = \begin{cases} 0 & \text{if } i = j \\ \min\limits_{i \le k < j} \{m[i, k] + m[k+1, j] + p_{i-1}p_kp_j\} & \text{if } i < j \end{cases}$$

- Computing the optimal solution recursively takes exponential time!
- How many subproblems?

  $\Rightarrow \Theta(n^2)$

  - Parenthesize $A_{i \ldots j}$ for $1 \le i \le j \le n$
  - One problem for each choice of i and j



3. Computing the Optimal Costs (cont.)

$$m[i, j] = \begin{cases} 0 & \text{if } i = j \\ \min\limits_{i \le k < j} \{m[i, k] + m[k+1, j] + p_{i-1}p_kp_j\} & \text{if } i < j \end{cases}$$

- How do we fill in the tables m[1..n, 1..n]?
  - Determine which entries of the table are used in computing $m[i, j]$

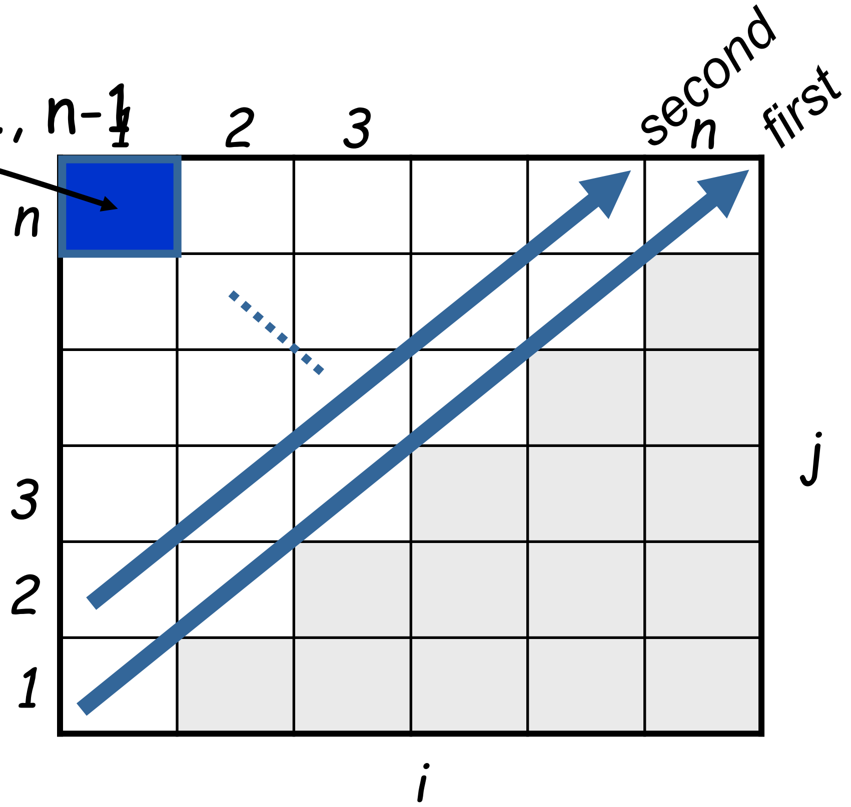
$$A_{i\ldots j} = A_{i\ldots k} A_{k+1\ldots j}$$

  - Subproblems’ size is one less than the original size
  - **Idea:** fill in m such that it corresponds to solving problems of increasing length

3. Computing the Optimal Costs (cont.)

$$m[i, j] = \begin{cases} 0 & \text{if } i = j \\ \min\limits_{i \le k < j} \{m[i, k] + m[k+1, j] + p_{i-1}p_kp_j\} & \text{if } i < j \end{cases}$$

- Length = 1: $i = j$, $i = 1, 2, \ldots, n$
- Length = 2: $j = i + 1$, $i = 1, 2, \ldots, n-1$

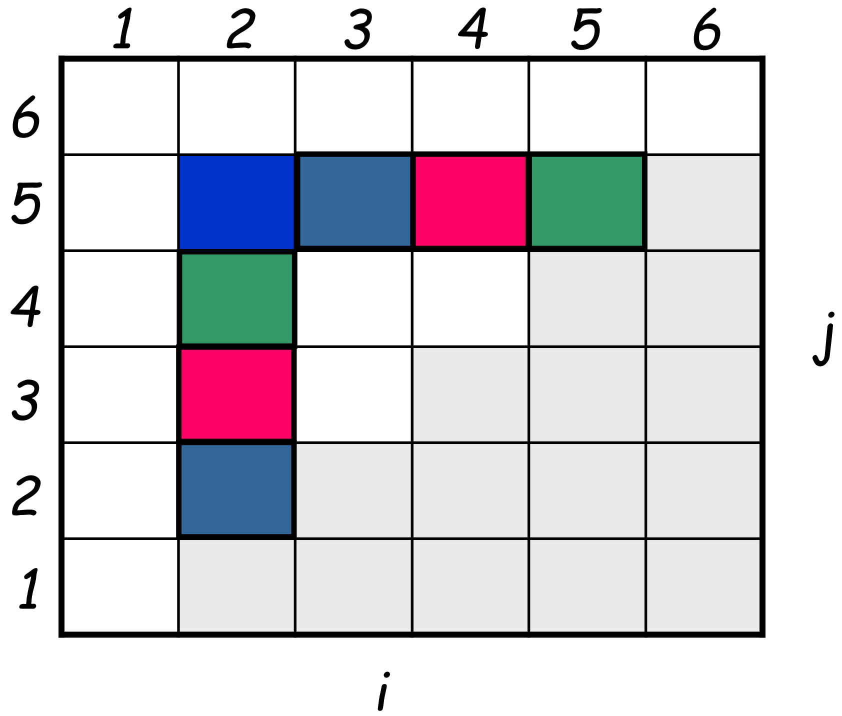


m[1, n] gives the optimal solution to the problem

Compute rows from bottom to top and from left to right

Example: $\min\{m[i, k] + m[k+1, j] + p_{i-1}p_kp_j\}$

$$m[2, 5] = \min \begin{cases} m[2, 2] + m[3, 5] + p_1p_2p_5 & k = 2 \\ m[2, 3] + m[4, 5] + p_1p_3p_5 & k = 3 \\ m[2, 4] + m[5, 5] + p_1p_4p_5 & k = 4 \end{cases}$$



- *Values $m[i, j]$ depend only on values that have been previously computed*

Example $\min\{m[i, k] + m[k+1, j] + p_{i-1}p_kp_j\}$

	1	2	3
3	2 7500	2 25000	0
2	1 5000	0	
1	0		

Compute $A_1 \cdot A_2 \cdot A_3$

- A_1: 10 x 100 $(p_0 \text{ x } p_1)$
- A_2: 100 x 5 $(p_1 \text{ x } p_2)$
- A_3: 5 x 50 $(p_2 \text{ x } p_3)$

$m[i, i] = 0$ for $i = 1, 2, 3$

$m[1, 2] = m[1, 1] + m[2, 2] + p_0p_1p_2$ (A_1A_2)

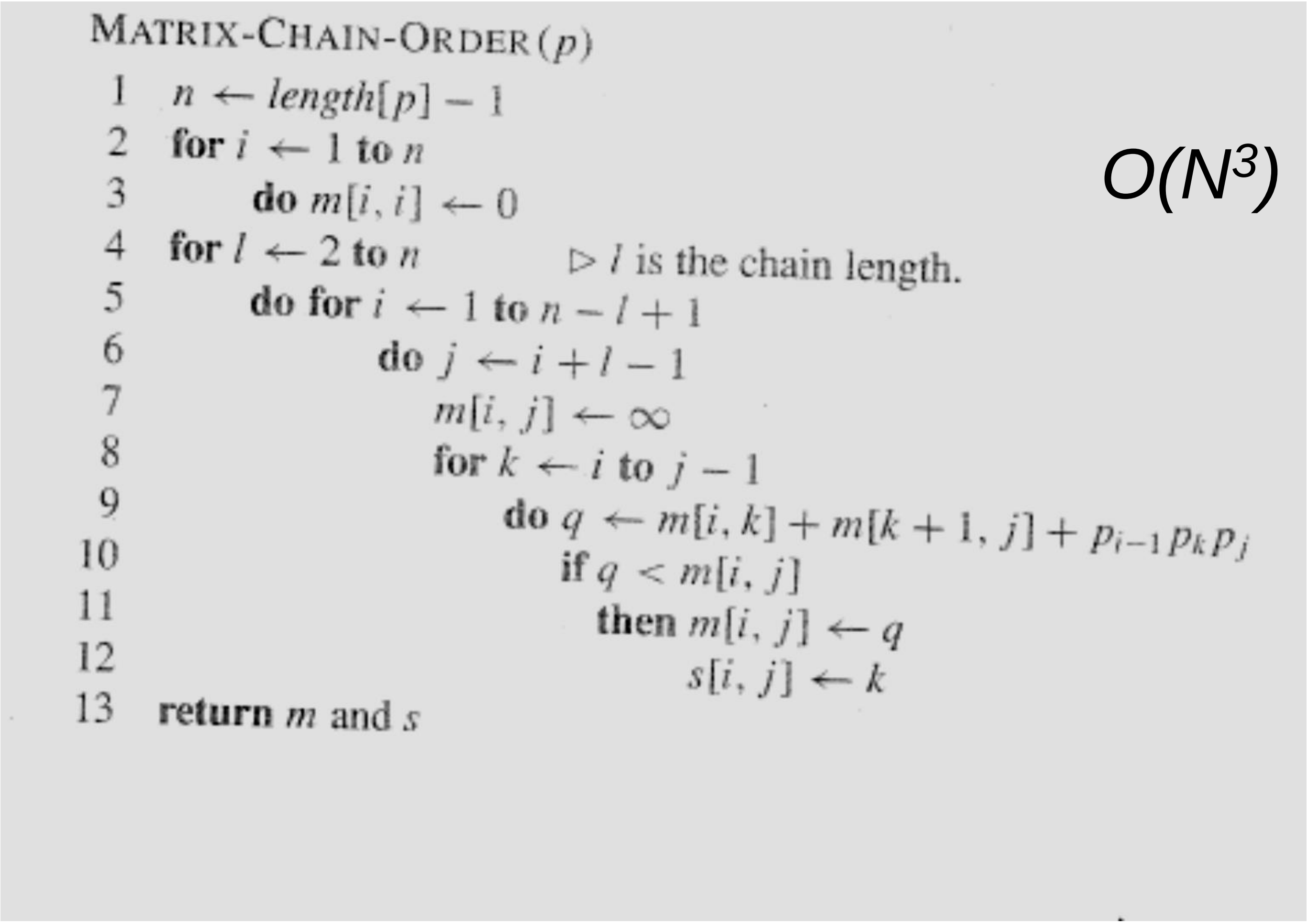
$= 0 + 0 + 10 * 100 * 5 = 5{,}000$

$m[2, 3] = m[2, 2] + m[3, 3] + p_1p_2p_3$ (A_2A_3)

$= 0 + 0 + 100 * 5 * 50 = 25{,}000$

$$m[1, 3] = \min \begin{cases} m[1, 1] + m[2, 3] + p_0p_1p_3 = 75{,}000 & (A_1(A_2A_3)) \\ m[1, 2] + m[3, 3] + p_0p_2p_3 = 7{,}500 & ((A_1A_2)A_3) \end{cases}$$

Matrix-Chain-Order(p)

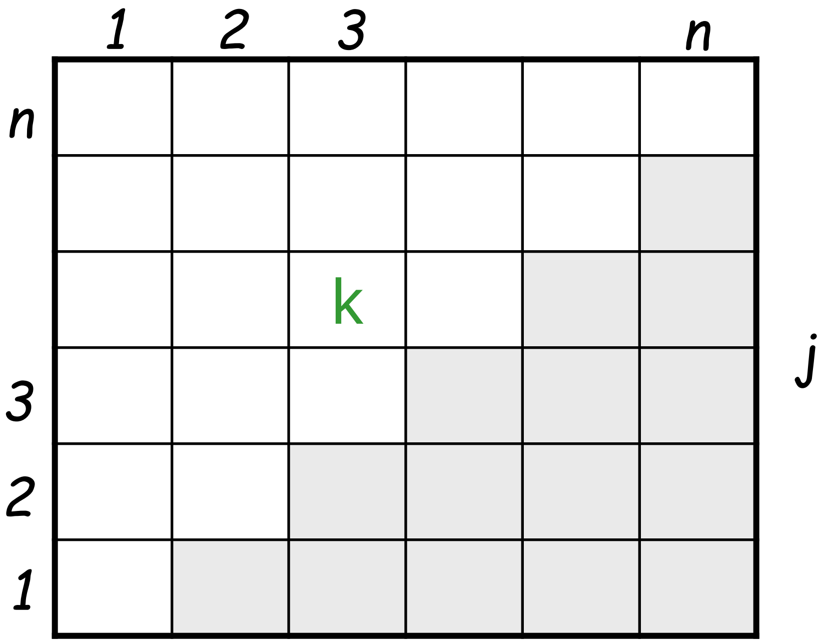


```
MATRIX-CHAIN-ORDER$(p)$
  $n \leftarrow length[p] - 1$
  for $i \leftarrow 1$ to $n$
      do $m[i, i] \leftarrow 0$
  for $l \leftarrow 2$ to $n$                ▷ $l$ is the chain length.
      do for $i \leftarrow 1$ to $n - l + 1$
             do $j \leftarrow i + l - 1$
                $m[i, j] \leftarrow \infty$
                for $k \leftarrow i$ to $j - 1$
                    do $q \leftarrow m[i, k] + m[k + 1, j] + p_{i-1} p_k p_j$
                       if $q < m[i, j]$
                          then $m[i, j] \leftarrow q$
                               $s[i, j] \leftarrow k$
  return $m$ and $s$
```

$O(N^3)$

4. Construct the Optimal Solution

- In a similar matrix s we keep the optimal values of k
- $s[i, j]$ = a value of k such that an optimal parenthesization of $A_{i..j}$ splits the product between A_k and A_{k+1}

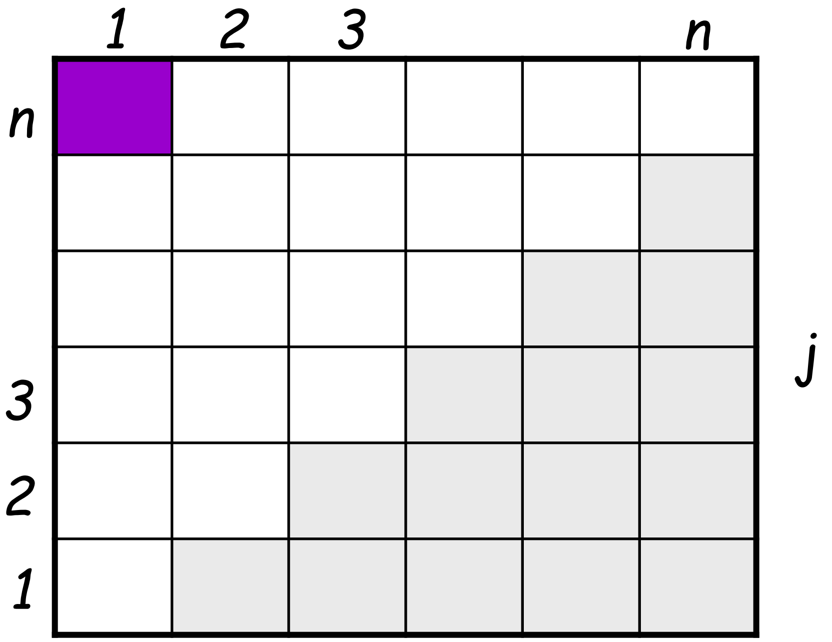


4. Construct the Optimal Solution

- $s[1, n]$ is associated with the entire product $A_{1..n}$
  - The final matrix multiplication will be split at k = $s[1, n]$

$$A_{1..n} = A_{1..s[1, n]} \cdot A_{s[1, n]+1..n}$$

  - For each subproduct recursively find the corresponding value of k that results in an optimal parenthesization



4. Construct the Optimal Solution

- $s[i, j]$ = value of k such that the optimal parenthesization of $A_i A_{i+1} \cdots A_j$ splits the product between A_k and A_{k+1}

	1	2	3	4	5	6
6	3	3	3	5	5	-
5	3	3	3	4	-	
4	3	3	3	-		
3	1	2	-			
2	1	-				
1	-					

j

i

- $s[1, 6] = 3 \Rightarrow A_{1..6} = A_{1..3}\, A_{4..6}$
- $s[1, 3] = 1 \Rightarrow A_{1..3} = A_{1..1}\, A_{2..3}$
- $s[4, 6] = 5 \Rightarrow A_{4..6} = A_{4..5}\, A_{6..6}$

4. Construct the Optimal Solution (cont.)

```
PRINT-OPT-PARENS(s, i, j)
if i = j
  then print "A"$_i$
  else print "("
        PRINT-OPT-PARENS(s, i, s[i, j])
        PRINT-OPT-PARENS(s, s[i, j] + 1, j)
        print ")"
```

	1	2	3	4	5	6
6	3	3	3	5	5	-
5	3	3	3	4	-	
4	3	3	3	-		
3	1	2	-			
2	1	-				
1	-					

j

i

Example: $A_1 \cdot \cdot \cdot A_6$ $((A_1 (A_2 A_3)) ((A_4 A_5) A_6))$

```
PRINT-OPT-PARENS(s, i, j)
if i = j
   then print “A”_i
   else print “(”
         PRINT-OPT-PARENS(s, i, s[i, j])
         PRINT-OPT-PARENS(s, s[i, j] + 1, j)
         print “)”
```

s[1..6, 1..6]

	1	2	3	4	5	6
6	3	3	3	5	5	-
5	3	3	3	4	-	
4	3	3	3	-		
3	1	2	-			
2	1	-				
1	-					

j

i

P-O-P(s, 1, 6)   s[1, 6] = 3

i = 1, j = 6   “(“   P-O-P (s, 1, 3)   s[1, 3] = 1

i = 1, j = 3   “(“   P-O-P(s, 1, 1)   ⇒ “A_1”

P-O-P(s, 2, 3) s[2, 3] = 2

i = 2, j = 3   “(“   P-O-P (s, 2, 2) ⇒ “A_2”

P-O-P (s, 3, 3) ⇒ “A_3”

“)”

“)”   …

Memoization

- Top-down approach with the efficiency of typical dynamic programming approach
- Maintaining an entry in a table for the solution to each subproblem
  - **memoize** the inefficient recursive algorithm
- When a subproblem is first encountered its solution is computed and stored in that table
- Subsequent “calls” to the subproblem simply look up that value

Memoized Matrix-Chain

Alg.: MEMOIZED-MATRIX-CHAIN(p)

```
n ← length[p] - 1
for i ← 1 to n
    do for j ← i to n
        do m[i, j] ← ∞
return LOOKUP-CHAIN(m, p, 1, n)
```

Initialize the m table with large values that indicate whether the values of m[i, j] have been computed

Top-down approach

Memoized Matrix-Chain

Running time is $O(n^3)$

```
Alg.: LOOKUP-CHAIN($m,p, i, j$)
    if $m[i, j] < \infty$
        then return $m[i, j]$
    if $i = j$
      then $m[i, j] \leftarrow 0$
      else for $k \leftarrow i$ to $j - 1$
               do $q \leftarrow$ LOOKUP-CHAIN($m, p, i, k$) +
                      LOOKUP-CHAIN($m, p, k+1, j$) + $p_{i-1}p_kp_j$
                 if $q < m[i, j]$
                     then $m[i, j] \leftarrow q$
  return $m[i, j]$
```

Running time of Memoized Matrix Chain

- Line 5 of MEMOIZED-MATRIX-CHAIN runs in $O(n^2)$ time
- We can categorize the calls of LOOKUP-CHAIN in two types
  - Calls in which $m[i,j]=\infty$, so that lines 3-9 execute (first type)
  - Calls in which $m[i,j]<\infty$, so that LLOKUP-Chain returns in line 2 (second type)
- There are $\Theta(n^2)$ calls of the first type, one per entry
- All calls of the second type are made as recursive calls by the calls of the first type
  - Whenever a given call of LOOKUP-CHAIN makes recursive calls it makes $O(n)$ of them
  - Therefore, there are $O(n^3)$ calls of the second type in all
  - Each of the second type calls take $O(1)$ time and each call of the first type takes $O(n)$ time plus time spent on its recursive calls
- Thus total run time is $O(n^3)$

Longest Common Subsequence

- Given two sequences

$$X = \langle x_1, x_2, \ldots, x_m \rangle$$

$$Y = \langle y_1, y_2, \ldots, y_n \rangle$$

find a maximum length common subsequence (LCS) of X and Y

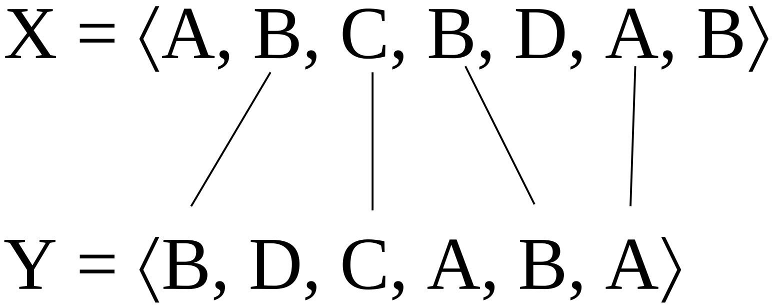
- *E.g.:*

$$X = \langle A, B, C, B, D, A, B \rangle$$

- Subsequences of X:
  - A subset of elements in the sequence taken in order

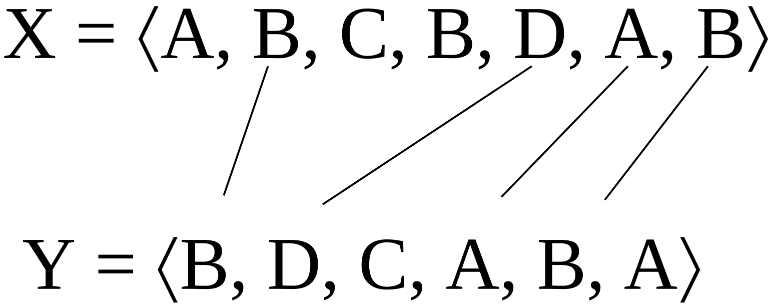
$\langle A, B, D \rangle$, $\langle B, C, D, B \rangle$, etc.

Example



X = ⟨A, B, C, B, D, A, B⟩

Y = ⟨B, D, C, A, B, A⟩



X = ⟨A, B, C, B, D, A, B⟩

Y = ⟨B, D, C, A, B, A⟩

- ⟨B, C, B, A⟩ and ⟨B, D, A, B⟩ are longest common subsequences of X and Y (length = 4)
- ⟨B, C, A⟩, however is not a LCS of X and Y

Brute-Force Solution

- For every subsequence of X, check whether it's a subsequence of Y
- There are 2^m subsequences of X to check
- Each subsequence takes $\Theta(n)$ time to check
  - scan Y for first letter, from there scan for second, and so on
- Running time: $\Theta(n2^m)$

Making the choice

X = ⟨A, B, D, E⟩

Y = ⟨Z, B, E⟩

- Choice: include one element into the common sequence (E) and solve the resulting subproblem

X = ⟨A, B, D, G⟩

Y = ⟨Z, B, D⟩

- Choice: exclude an element from a string and solve the resulting subproblem

Notations

- Given a sequence $X = \langle x_1, x_2, \ldots, x_m \rangle$ we define the i-th prefix of X, for $i = 0, 1, 2, \ldots, m$

  $$X_i = \langle x_1, x_2, \ldots, x_i \rangle$$

- $c[i, j]$ = the length of a LCS of the sequences $X_i = \langle x_1, x_2, \ldots, x_i \rangle$ and $Y_j = \langle y_1, y_2, \ldots, y_j \rangle$

A Recursive Solution

Case 1: $x_i = y_j$

e.g.: $X_i = \langle A, B, D, E\rangle$

$Y_j = \langle Z, B, E\rangle$

$$c[i, j] = c[i - 1, j - 1] + 1$$

- Append $x_i = y_j$ to the LCS of X_{i-1} and Y_{j-1}
- Must find a LCS of X_{i-1} and Y_{j-1} $\Rightarrow$ optimal solution to a problem includes optimal solutions to subproblems

A Recursive Solution

Case 2: $x_i \neq y_j$

e.g.: $X_i = \langle A, B, D, G\rangle$

$Y_j = \langle Z, B, D\rangle$

$$c[i, j] = \max \{ c[i-1, j], c[i, j-1] \}$$

- Must solve two problems
    - find a LCS of X_{i-1} and Y_j: $X_{i-1} = \langle A, B, D\rangle$ and $Y_j = \langle Z, B, D\rangle$
    - find a LCS of X_i and Y_{j-1}: $X_i = \langle A, B, D, G\rangle$ and $Y_j = \langle Z, B\rangle$

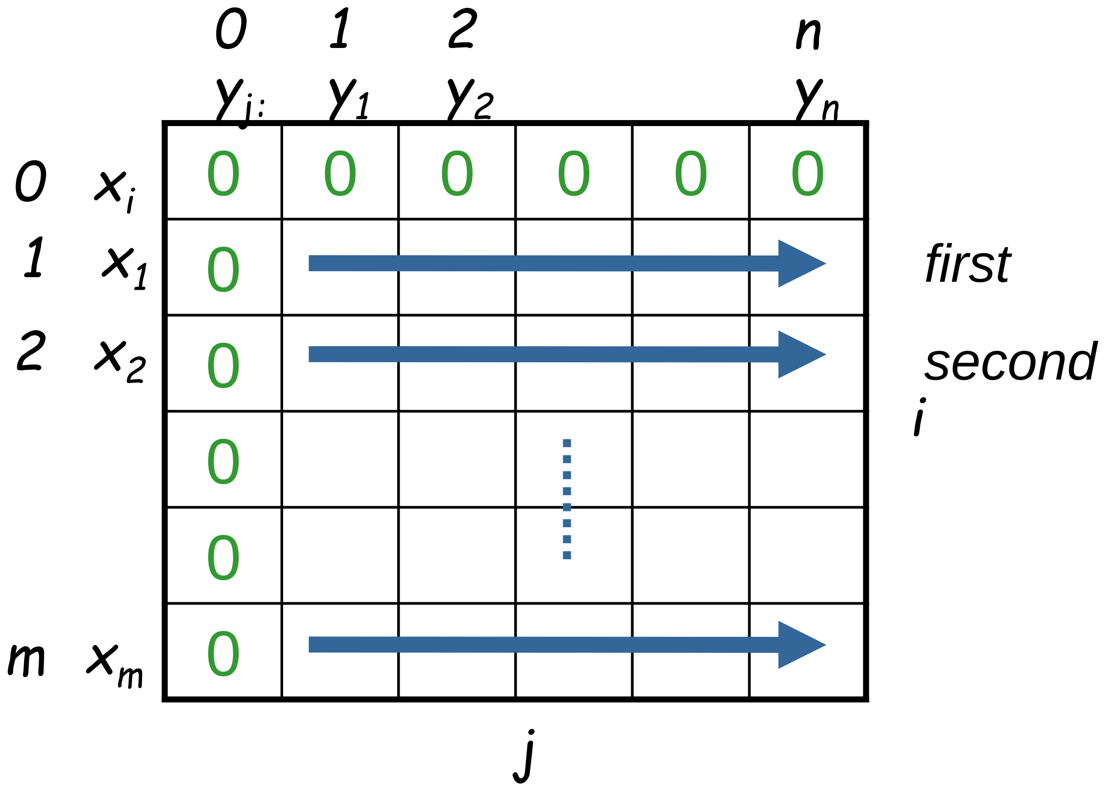
- Optimal solution to a problem includes optimal solutions to subproblems

Overlapping Subproblems

- To find a LCS of X and Y
  - We may need to find the LCS between X and Y_{n-1} and that of X_{m-1} and Y
  - Both the above subproblems has the subproblem of finding the LCS of X_{m-1} and Y_{n-1}
- Subproblems share subsubproblems

3. Computing the Length of the LCS

$$c[i, j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ c[i-1, j-1] + 1 & \text{if } x_i = y_j \\ \max(c[i, j-1], c[i-1, j]) & \text{if } x_i \neq y_j \end{cases}$$



Additional Information

$$c[i, j] = \begin{cases} 0 & \text{if } i,j = 0 \\ c[i-1, j-1] + 1 & \text{if } x_i = y_j \\ \max(c[i, j-1], c[i-1, j]) & \text{if } x_i \neq y_j \end{cases}$$

b & c:

		0	1	2	3		n
	y_j:		A	C	D		F
0	x_i	0	0	0	0	0	0
1	A	0					
2	B	0			c[i-1,j]		
3	C	0		↖ c[i,j-1]	↑		
		0					
m	D	0					

i

j

A matrix b[i, j]:

- *For a subproblem [i, j] it tells us what choice was made to obtain the optimal value*
- *If $x_i = y_j$*

  $b[i, j] =$ " ↖ "
- *Else, if $c[i - 1, j] \geq c[i, j-1]$*

  $b[i, j] =$ " ↑ "

  else

  $b[i, j] =$ " ← "

LCS-LENGTH(X, Y, m, n)

```
for i ← 1 to m
    do c[i, 0] ← 0
for j ← 0 to n
    do c[0, j] ← 0
for i ← 1 to m
    do for j ← 1 to n
        do if $x_i = y_j$
            then c[i, j] ← c[i - 1, j - 1] + 1
                b[i, j ] ← " ↖ "
            else if c[i - 1, j] ≥ c[i, j - 1]
                then c[i, j] ← c[i - 1, j]
                    b[i, j] ← "↑"
                else c[i, j] ← c[i, j - 1]
                    b[i, j] ← "←"
return c and b
```

The length of the LCS if one of the sequences is empty is zero

Case 1: $x_i = y_j$

Case 2: $x_i \neq y_j$

Running time: $\Theta(mn)$

Example

X = ⟨A, B, C, B, D, A⟩

Y = ⟨B, D, C, A, B, A⟩

$$c[i, j] = \begin{cases} 0 & \text{if } i = 0 \text{ or } j = 0 \\ c[i-1, j-1] + 1 & \text{if } x_i = y_j \\ \max(c[i, j-1], c[i-1, j]) & \text{if } x_i \neq y_j \end{cases}$$

If $x_i = y_j$

$b[i, j] =$ " ↖ "

Else if

$c[i - 1, j] \geq c[i, j-1]$

$b[i, j] =$ " ↑ "

else

$b[i, j] =$ " ← "

		0	1	2	3	4	5	6
		y_j	B	D	C	A	B	A
0	x_i	0	0	0	0	0	0	0
1	A	0	↑ 0	↑ 0	↑ 0	↖ 1	←1	↖ 1
2	B	0	↖ 1	←1	←1	↑ 1	↖ 2	←2
3	C	0	↑ 1	↑ 1	↖ 2	←2	↑ 2	↑ 2
4	B	0	↖ 1	↑ 1	↑ 2	↑ 2	↖ 3	←3
5	D	0	↑ 1	↖ 2	↑ 2	↑ 2	↑ 3	↑ 3
6	A	0	↑ 1	↑ 2	↑ 2	↖ 3	↑ 3	↖ 4
7	B	0	↖ 1	↑ 2	↑ 2	↑ 3	↖ 4	↑ 4

4. Constructing a LCS

- Start at b[m, n] and follow the arrows
- When we encounter a “↖“ in b[i, j] ⇒ $x_i = y_j$ is an element of the LCS

		0	1	2	3	4	5	6
		y_j	B	D	C	A	B	A
0	x_i	0	0	0	0	0	0	0
1	A	0	↑0	↑0	↑0	↖1	←1	↖1
2	B	0	↖1	←1	←1	↑1	↖2	←2
3	C	0	↑1	↑1	↖2	←2	↑2	↑2
4	B	0	↖1	↑1	↑2	↑2	↖3	←3
5	D	0	↑1	↖2	↑2	↑2	↑3	↑3
6	A	0	↑1	↑2	↑2	↖3	↑3	↖4
7	B	0	↖1	↑2	↑2	↑3	↖4	↑4

PRINT-LCS(b, X, i, j)

Running time: $\Theta(m + n)$

```
if $i = 0$ or $j = 0$
  then return
if $b[i, j]$ = "↖"
    then PRINT-LCS($b, X, i - 1, j - 1$)
            print $x_i$
elseif $b[i, j]$ = "↑"
            then PRINT-LCS($b, X, i - 1, j$)
            else PRINT-LCS($b, X, i, j - 1$)
```

Initial call: PRINT-LCS(b, X, $length[X]$, $length[Y]$)

Improving the Code

- What can we say about how each entry $c[i, j]$ is computed?
  - It depends only on $c[i -1, j - 1]$, $c[i - 1, j]$, and $c[i, j - 1]$
  - Eliminate table b and compute in $O(1)$ which of the three values was used to compute $c[i, j]$
  - We save $\Theta(mn)$ space from table b
  - However, we do not asymptotically decrease the auxiliary space requirements: still need table c

Improving the Code

- If we only need the length of the LCS
  - LCS-LENGTH works only on two rows of c at a time
    - The row being computed and the previous row
  - We can reduce the asymptotic space requirements by storing only these two rows