CS-771
HW#6 (Solution)

[redacted]

2 Consider the following two random variables.

TP : Test positive

HD : Have disease.

$P(TP \mid HD) = 0.99$

$P(\neg TP \mid \neg HD) = 0.99$

$P(HD) = \frac{1}{10000} = 0.0001$

We want to find $P(HD \mid TP)$. From Bayes rule

$$P(HD \mid TP) = \frac{P(TP \mid HD)\, P(HD)}{P(TP)} \quad \text{—— ①}$$

Now

$$\begin{aligned} P(TP) &= P(TP \mid HD)\, P(HD) + P(TP \mid \neg HD)\, P(\neg HD) \\ &= 0.99 \times 0.0001 + (1 - P(\neg TP \mid \neg HD))\,(1 - P(HD)) \\ &= 0.99 \times 0.0001 + (1 - 0.99)\,(1 - 0.0001) \\ &= (0.99 \times 0.0001) + (0.01 \times 0.9999) \end{aligned}$$

$\therefore$ Plugging in these values for ①

$$P(HD \mid TP) = \frac{0.99 \times 0.0001}{(0.99 \times 0.0001) + (0.01 \times 0.9999)} = 0.0098$$

The good news is the following. Since the disease is really rare ($P(HD)$ is small), inspite of testing positive probability of actually having the disease is really small.

3

$$P(X,Y|Z) = P(X|Z)P(Y|Z)$$

(b) $$P(X|Y,Z) = \frac{P(X,Y|Z)}{P(Y|Z)} = \frac{P(X|Z)P(Y|Z)}{P(Y|Z)} = P(X|Z)$$

(c) $$P(Y|X,Z) = \frac{P(X,Y|Z)}{P(X|Z)} = \frac{P(X|Z)P(Y|Z)}{P(X|Z)} = P(Y|Z)$$

2

(a) $$P(B|LB) = \frac{P(LB|B)P(B)}{P(LB)} = \alpha\, P(LB|B)P(B) = \alpha \cdot 0.75\, P(B) \quad \text{— (1)}$$

$$P(\neg B|LB) = \frac{P(LB|\neg B)P(\neg B)}{P(LB)} = \alpha\, P(LB|\neg B)\, P(\neg B)$$

$$= \alpha\, [1 - P(\neg LB|\neg B)][1 - P(B)]$$

$$= \alpha\, (1 - 0.75)(1 - P(B))$$

$$= \alpha\, 0.25(1 - P(B)) \quad \text{— (2)}$$

from (1) and (2)

$$P(B|LB) + P(\neg B|LB) = \alpha\, 0.75\, P(B) + \alpha\, 0.25(1 - P(B)) = 1$$

$$\Rightarrow \alpha = \frac{1}{0.75\, P(B) + 0.25\,(1 - P(B))}$$

$$\therefore P(B|LB) = \frac{0.75\, P(B)}{0.75\, P(B) + 0.25\,[1 - P(B)]} = \frac{3P(B)}{3P(B) + [1 - P(B)]}$$

$$P(\neg B|LB) = \frac{0.25(1 - P(B))}{0.75\, P(B) + 0.25\,(1 - P(B))} = \frac{1 - P(B)}{3P(B) + [1 - P(B)]}$$

Unless we know $P(B)$, we can't conclude if

$$P(B|LB) > P(\neg B|LB) \quad \text{or} \quad P(B|LB) \leq P(\neg B|LB)$$

③ (b) $P(B) = 1/10 = 0.1$

$$\therefore\ P(B|LB) = \frac{3\times 0.1}{3\times 0.1 + (1-0.1)} = \frac{0.3}{0.12} = 0.25$$

$$P(\neg B|LB) = \frac{1-0.1}{3\times 0.1 + (1-0.1)} = \frac{0.9}{0.12} = 0.75$$

Since $P(\neg B|LB) > P(B|LB)$, most likely color of the taxi is green.

④ (a) There are 5 binary variables so $2^5 - 1 = 31$ variables are needed

(b) $1 + 4 + 1 + 8 + 2 = 16$

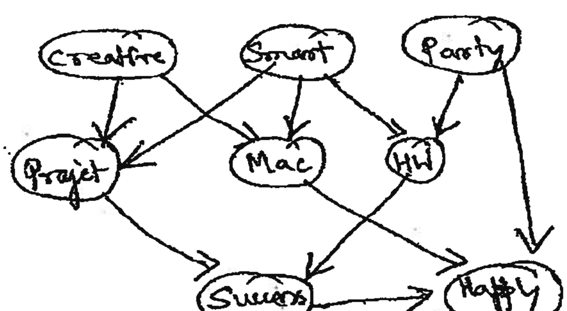
(c) $P(B, I, M, J, G) = P(B)\,P(M)\,P(I|B,M)\,P(G|B,I,M)\,P(J|G)$

(d) $P(B=t, I=t, M=f, G=t, J=t)$

$= P(B=t)\,P(M=f)\,P(I=t|B=t, M=f)\,P(G=t|B=t, I=t, M=f)$
$P(J=t|G=t)$

$= 0.9 \times (1-0.1) \times 0.5 \times 0.8 \times 0.9 = 0.2916.$

⑤ (a)

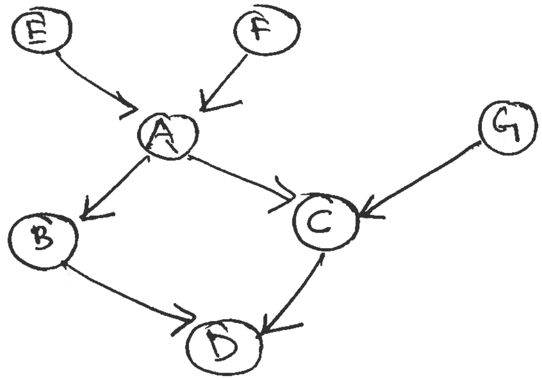


(b) ~~Project~~ $P(\text{Creative}, \text{Smart}, \text{Party}, \text{Project}, \text{Mac}, \text{HW}, \text{Success}, \text{Happy})$

$= P(\text{Creative})\,P(\text{Smart})\,P(\text{Party})\,P(\text{Project}|\text{Creative}, \text{Smart})\,P(\text{Mac}|\text{Creative}, \text{Smart})$

$\times\ P(\text{HW}|\text{Smart}, \text{Party})\,P(\text{Success}|\text{Project}, \text{HW})\,P(\text{Happy}|\text{Success}, \text{Mac}, \text{Party})$

Q (b)



A path is active if each triple in the path is active

— No active path → Independence

(a) $E \perp\!\!\!\perp G \mid A$ holds

~~Path~~ There are two paths from E to G

$E \rightarrow A \rightarrow C \leftarrow G$

$E \rightarrow A \rightarrow B \rightarrow D \leftarrow C \leftarrow G$

~~The first~~ The first path is inactive $E \rightarrow \cancel{A} \rightarrow C$

The second path is inactive $E \rightarrow \cancel{A} \rightarrow B$

~~Thus~~

(b) $E \perp\!\!\!\perp G \mid C$ does not hold.

Same two paths as before.

The first path is active since all triples are active.

Since there is atleast one active path

$E \perp\!\!\!\perp G \mid C$ Does not hold.

(c) $E \perp\!\!\!\perp D \mid A$. holds.

There are two paths from E to D

$E \rightarrow A \rightarrow C \rightarrow D$

$E \rightarrow A \rightarrow B \rightarrow D$

The first path is inactive because $E \rightarrow \cancel{A} \rightarrow C$ is inactive

" second path " " " $E \rightarrow \cancel{A} \rightarrow C$

(d) $E \perp\!\!\!\perp D \mid A, B.$ holds

Same as above.

(e) $E \perp\!\!\!\perp D \mid B, C$ holds.

Same as above.

(f) $B \perp\!\!\!\perp G \mid C.$ does not hold.

There are two paths.

$$B \leftarrow A \rightarrow C \leftarrow G$$

$$B \rightarrow D \leftarrow C \leftarrow G$$

The first path is active because each triple is active.

The second path is inactive because of $D \leftarrow C \leftarrow G$

Since there one active path $B \perp\!\!\!\perp G \mid C$ does not hold.

7 (a) $+c, +s, +r, +w$: $w = 0.1 \times 0.99 = 0.099$

(b) $+c, +s, -r, +w$: $w = 0.1 \times 0.9 = 0.09$

(c) $-c, +s, -r, +w$: $w = 0.5 \times 0.9 = 0.45$

(d) $-c, +s, +r, +w$: $w = 0.5 \times 0.99 = 0.495$