LINEAR SYSTEMS
ASSIGNMENT #7

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1) $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 1 \end{bmatrix}$, $D = 3$.

Convert into Jordan form.

$\Rightarrow$ Controllability check $\Rightarrow$

$P_C = \begin{bmatrix} B & AB \end{bmatrix}$

$AB = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

$P_C = \begin{bmatrix} 0 & 1 \\ 1 & -3 \end{bmatrix}$. $\det |P_C| = 0 - 1 = -1 \neq 0$

$\therefore$ Model is controllable.

$\Rightarrow$ Find eigen values & eigen vectors $\Rightarrow$

$A - \lambda I = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{bmatrix}$

$\det(A - \lambda I) = 0 \quad \therefore$

$\therefore \quad \lambda^2 + 3\lambda + 2 = 0$

$\therefore \quad \lambda = -1, \quad \lambda = -2.$

$\therefore \quad A + 1I = \begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} V_{11} \\ V_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$V_{11} + V_{21} = 0 \qquad \therefore V_{11} = 1$

$V_{11} = -V_{21} \qquad V_{21} = -1$

$A + 2I = \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} V_{12} \\ V_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$2V_{12} + V_{22} = 0 \qquad \therefore V_{12} = 1$

$\therefore 2V_{12} = -V_{22} \qquad V_{22} = -2$

$\therefore V = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} = T$

Now,

$\hat{A} = T^{-1} A T = \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$

$$\therefore \hat{A} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} -1 & -2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$

$$\hat{B} = T^{-1} B = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\hat{C} = CT = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} -1 & -2 \end{bmatrix}$$

$$\hat{D} = 3$$

$$\therefore \dot{x} = \hat{A}\hat{x} + \hat{B}u$$

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u$$

$$y = Cx + Du$$

$$= \begin{bmatrix} -1 & -2 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} + 3u$$

2) $A = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 12 \\ 6 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 1 \end{bmatrix}$

Find controllable canonical form $\Rightarrow$

$$\Rightarrow sI - A = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} s+2 & 0 \\ 0 & s+1 \end{bmatrix}$$

$$\therefore s^2 + 3s + 2 = 0$$

$$\therefore \hat{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

To find $T \rightarrow$

$$P_C = \begin{bmatrix} B & AB \end{bmatrix}$$

$$AB = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 12 \\ 6 \end{bmatrix} = \begin{bmatrix} -24 \\ -6 \end{bmatrix}$$

$$P_C = \begin{bmatrix} 12 & -24 \\ 6 & -6 \end{bmatrix}$$

$$\hat{P}_C = \begin{bmatrix} \hat{B} & \hat{A}\hat{B} \end{bmatrix}$$

$$\hat{A}\hat{B} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

$$\therefore \hat{P}_C = \begin{bmatrix} 0 & 1 \\ 1 & -3 \end{bmatrix}$$

$$\therefore \hat{P}_C^{-1} = \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}$$

$T = P_c \cdot \hat{P}_c^{-1}$

$\therefore T = \begin{bmatrix} 12 & -24 \\ 6 & -6 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 12 & 12 \\ 12 & 6 \end{bmatrix}$

Now,

$\hat{C} = CT = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 12 & 12 \\ 12 & 6 \end{bmatrix} = \begin{bmatrix} 24 & 18 \end{bmatrix}$

$\therefore$ State response $\Rightarrow$

$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$

Output Response $\Rightarrow$

$y = \begin{bmatrix} 24 & 18 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} + 0u$

3) $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 1 \end{bmatrix}$.

convert into observable canonical form.

$\Rightarrow [sI - A] = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 \\ -2 & s+3 \end{bmatrix}$

$\therefore \quad \underrightarrow{s^2 + 3s + 2}$

$\therefore \hat{A} = \begin{bmatrix} -3 & +1 \\ -2 & -0 \end{bmatrix}$, $\hat{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}$

Now,

$P_o = \begin{bmatrix} C \\ CA \end{bmatrix}$

$CA = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} -2 & -3 \end{bmatrix}$

$\therefore P_o = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$, $P_o^{-1} = \frac{1}{2} \begin{bmatrix} -3 & -1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} -3/2 & -1/2 \\ 1 & 0 \end{bmatrix}$

$\hat{P}_o = \begin{bmatrix} \hat{C} \\ \hat{C}\hat{A} \end{bmatrix}$; $\hat{C}\hat{A} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} -3 & 1 \end{bmatrix}$

$\hat{P}_o = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$

$$T = P_0^{-1}\hat{P}_0 = \begin{bmatrix} -3/2 & -1/2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$$

$$T = \begin{bmatrix} 0 & -1/2 \\ 1 & 0 \end{bmatrix}$$

$$\therefore \hat{B} = T^{-1}B = 2\begin{bmatrix} 0 & +1/2 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$\therefore$ State Transition Matrix $\Rightarrow$

$$\dot{x} = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}u$$

Output Matrix $\Rightarrow$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} + 0u$$

4) $A = \begin{bmatrix} -6 & 1 \\ 0 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 1 \end{bmatrix}$

Convert into Jordan form $\Rightarrow$

$\Rightarrow$ Controllability Test $\Rightarrow$

$$P_C = \begin{bmatrix} B & AB \end{bmatrix}$$

$$AB = \begin{bmatrix} -6 & 1 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ -2 \end{bmatrix}$$

$$P_C = \begin{bmatrix} 1 & -5 \\ 1 & -2 \end{bmatrix}; \quad \det(P_C) = -2+5 = 3 \neq 0$$

Model is controllable.

To find Eigen values & Eigen Vectors

$$\Rightarrow A - \lambda I = \begin{bmatrix} -6 & 1 \\ 0 & -2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} -6-\lambda & 1 \\ 0 & -2-\lambda \end{bmatrix}$$

$\therefore \det(A - \lambda I) = 0$

$\therefore \lambda^2 + 8\lambda + 12 = 0$

$\therefore \lambda = -6, \ \lambda = -2$

$$\therefore A + 6I = \begin{bmatrix} 0 & 1 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} v_{11} \\ v_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$0 + v_{21} = 0$, ; $v_{21} = 0$

$0 + 4v_{21} = 0$; $v_{11} = 1$

$$A + 2I = \begin{bmatrix} -4 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_{12} \\ V_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore -4V_{12} + V_{22} = 0$ $\quad \therefore V_{12} = 1$

$-4V_{12} = -V_{22}$ $\quad V_{22} = 4$

$$\therefore V = \begin{bmatrix} 1 & 1 \\ 0 & 4 \end{bmatrix} = T$$

$$\hat{A} = T^{-1}AT = \frac{1}{4}\begin{bmatrix} 4 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -6 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 4 \end{bmatrix}$$

$$\hat{A} = \begin{bmatrix} 1 & -1/4 \\ 0 & 1/4 \end{bmatrix} \begin{bmatrix} -6 & -2 \\ 0 & -8 \end{bmatrix} = \begin{bmatrix} -6 & 0 \\ 0 & -2 \end{bmatrix}$$

$$\hat{B} = T^{-1}B = \begin{bmatrix} 1 & -1/4 \\ 0 & 1/4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3/4 \\ 1/4 \end{bmatrix}$$

$$\hat{C} = CT = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 4 \end{bmatrix}$$

State Transition Matrix $\Rightarrow$

$$\dot{X} = \begin{bmatrix} 1 & -1/4 \\ 0 & 1/4 \end{bmatrix} \begin{bmatrix} \hat{X}_1 \\ \hat{X}_2 \end{bmatrix} + \begin{bmatrix} 3/4 \\ 1/4 \end{bmatrix} u$$

Output Matrix $\Rightarrow$

$$Y = \begin{bmatrix} 0 & 4 \end{bmatrix} \begin{bmatrix} \hat{X}_1 \\ X_2 \end{bmatrix} + 0u \quad //$$