EE-792 TEST-2 Summer-16

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- Open text book, and closed notes. One sheet of notes (A4 size, both sides) will be allowed to the exam.
- Time for this Test is one hour thirty minutes.
- Calculators are allowed for this test (any kind)
- All work in this exam must be your own, sharing of calculators, formula sheet or text book will not be allowed.

$\frac{91}{100}$

(1) A continues time invariant system is described by $\ddot{y} + 3\dot{y} + 2y = \dot{u} + u$ find the controllability and observability of the model by deriving the model by Jordan form. (25 points)

given $\ddot{y}+3\dot{y}+2y = \dot{u}+u$.

convert into s-form

$y(s^2+3s+2) = u(s+1)$

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$$y/u = \frac{s+1}{s^2+3s+2}$$

$$= \frac{s+1}{s^2+2s+s+2} = \frac{s+1}{s(s+2)+1(s+2)} = \frac{s+1}{(s+1)(s+2)}$$

Let $\frac{s+1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$, By partial fractions.

$AS+2A+BS+B = S+1$

$A+B=1$

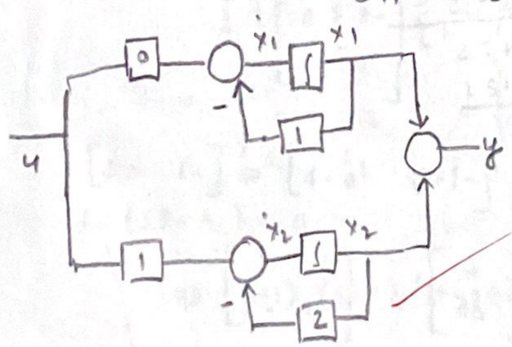
$2A+B=1$

$-A=0$

$A=0$

$B=1$

$$= \frac{(0)}{s+1} + \frac{B}{s+2}$$



$y = x_1 + x_2 + u$

$\dot{x}_1 = -x_1$

$\dot{x}_2 = -2x_2 + u$.

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [1]u.$$

Please show me your calculations for partial credit

Controllability of a model is verified as

$$P_c = [B \quad AB]$$

$$AB = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0+0 \\ 0-2 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix} \quad ; \quad P_c = [B \quad AB]$$

$$P_c = \begin{bmatrix} 0 & 0 \\ 1 & -2 \end{bmatrix}$$

for controllability $\det(P_c) = 0 \times -2 + 0 \times 1$

$$= 0$$

$\therefore$ The model is uncontrollable.

observability of the model is given as

$$P_o = \begin{bmatrix} C \\ CA \end{bmatrix}$$

$$CA = [1 \quad 1] \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$$

$$= [-1+0 \quad 0-2] = [-1 \quad -2]$$

$$P_o = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$$

for observability, find $\det(P_o)$

$$\det(P_o) = 1(-2) - 1(-1) = -2+1 = -1 \neq 0$$

$\therefore$ The model is observable.

(2) A model has $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$ $B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ Find the feedback gain such that the closed loop poles are at $-2 \pm j3$ *and* -1 (25 points)

given $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

we have feedback gain $= -Gx$

Let us say A_c, such that $A_c = A - BG$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix}$

$$A_c = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} [G_1 \ G_2 \ G_3]$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ G_1 & G_2 & G_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ -G_1 & -G_2 & -3-G_3 \end{bmatrix}$$

Now find $SI - A_c$

$$= \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ -G_1 & -G_2 & -G_3-3 \end{bmatrix}$$

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$$= \begin{bmatrix} s+1 & 0 & 0 \\ 0 & s+2 & 0 \\ G_1 & G_2 & s+G_3+3 \end{bmatrix}$$

$\det(sI - A_c) = 0$

$$\Rightarrow (s+1)(s+2)(s+G_3+3) = (s^2+3s+2)(s+G_3+3)$$

$$= s^3 + sG_3 + 3s + 3s^2 + 3sG_3 + 9s + 2s + 2G_3 + 6$$

$$= s^3 + s(G_3 + 3 + 3G_3 + 9 + 2) + 3s^2 + 2G_3 + 6 \rightarrow (1)$$

But given desired closed loop poles are at $(-1)(-2+j3)(-2-j3)$

$$= (s+1)(s+2-j3)(s+2+j3)$$

$$= (s+1)(\cancel{s^2} + 2s + sj3 + 2s + 4 + 6j - sj3 - 6j - j^2 9)$$

Please show me your calculations for partial credit

(2)

$$= s^3 + 2s^2 + 3s^2 j + 2s^2 + 4s + 6sj - s^2 j3 - 6sj - sj^2 9 + s^2 + 2s + 3sj + 2s$$
$$+ 4 + 6j - sj3 - 6j - j^2 9$$

we have ~~x put that~~ ① ≡ ②

$s^3 + s^2(2+3j+2.-3j) + s(4+6j-6i-j^2 9+2+3j+2-3j)$

~~Now equate the coefficients~~

$+ 4 + 6j - 6j - j^2 9.$

$= s^3 + 4s^2 + 17s + 4 + 9$ — ②

we have that ① ≡ ②.

$2G_3 + 6 = 13$

$2G_3 = 13 - 6 = 7$

$G_3 = 7/2 = 3.5$

$4G_3 + 14 = 17$

$4G_3 = 17 - 14 = 3$

$G_3 = 3/4 = 0.75$

∴ The results are conflicting here and hence no solution exists.

$G_3 = -1$

$G_3 = 2$

$G_3 = 7/2$

(3) $\begin{bmatrix}\dot{x_1}\\ \dot{x_2}\end{bmatrix} = \begin{bmatrix}-3 & 1\\ -2 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}1\\ 0\end{bmatrix}u,\ \ y = [0\ \ 1]\begin{bmatrix}x_1\\ x_2\end{bmatrix} + 3u$, Find the total response of the given model when $X_{1(0)} = 1$, $X_{2(0)} = -1$, & u=5 (25 points)

given $A = \begin{bmatrix}-3 & 1\\ 2 & 0\end{bmatrix}$, $B = \begin{bmatrix}1\\ 0\end{bmatrix}$, $C = [0\ \ 1]$, $D = [3]$

i) check the controllability of model

$P_c = \det[B\ \ AB]$

$AB = \begin{bmatrix}-3 & 1\\ -2 & 0\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix} = \begin{bmatrix}-3\\ -2\end{bmatrix}$

$P_c = \begin{bmatrix}1 & -3\\ 0 & -2\end{bmatrix}$, $\det(P_c) \neq 0$

ii) finding state transition matrix Q_s

$Q_s = [SI - A]^{-1}$

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$SI - A = \begin{bmatrix}s & 0\\ 0 & s\end{bmatrix} - \begin{bmatrix}-3 & 1\\ -2 & 0\end{bmatrix} = \begin{bmatrix}s+3 & -1\\ +2 & s\end{bmatrix}$

$(SI - A)^{-1} = \dfrac{1}{s(s+3)+2}\begin{bmatrix}s & 1\\ 2 & s+3\end{bmatrix} = \dfrac{1}{s^2+3s+2}\begin{bmatrix}s & 1\\ 2 & s+3\end{bmatrix}$

$= \dfrac{1}{(s+1)(s+2)}\begin{bmatrix}s & 1\\ 2 & s+3\end{bmatrix} = \begin{bmatrix}\dfrac{s}{(s+1)(s+2)} & 1/(s+1)(s+2)\\ 2/(s+1)(s+2) & \dfrac{s+3}{(s+1)(s+2)}\end{bmatrix}$

let us apply partial fractions

$\dfrac{s}{(s+1)(s+2)} = \dfrac{A}{s+1} + \dfrac{B}{s+2}$

$A + B = 1$
$2A + B = 0$
$B = -2A$

$A - 2A = 1$
$A = -1$
$B = 2$

$= \dfrac{-1}{s+1} + \dfrac{2}{s+2}$

Please show me your calculations for partial credit

$-s-2+2s+2$

$$\frac{1}{(s+1)(s+2)} = \frac{A}{(s+1)} + \frac{B}{(s+2)} \Rightarrow$$

$A+B=0$

$2A+B=1$

$A=-B$

$-2B+B=1$

$B=-1,\ A=1.$

$$= \frac{1}{s+1} - \frac{1}{s+2}.$$

$$\frac{2}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

$A+B=0 \Rightarrow A=-B$

$2A+B=2$

$-2B+B=2$

$-B=2 \Rightarrow B=-2,\ A=2$

$$\frac{2}{s+1} - \frac{2}{s+2}$$

$$\frac{s+3}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

$A+B=1$ $\quad A=$

$2A+B=3$

$-A=-2$ $\quad B=-1$

$$\frac{2}{s+1} - \frac{1}{s+2}$$

$$\therefore Q_s = \begin{bmatrix} \frac{-1}{s+1} + \frac{2}{s+2} & \frac{1}{s+1} - \frac{1}{s+2} \\ \frac{2}{s+1} - \frac{2}{s+2} & \frac{2}{s+1} - \frac{1}{s+2} \end{bmatrix}$$

$$Q_t = \begin{bmatrix} -e^{-t}+2e^{-2t} & +e^{-t} - e^{-2t} \\ -2e^{-t} - 2e^{-2t} & -2e^{-t} - e^{-2t} \end{bmatrix}$$

Input state conditions $= Q_t \cdot X_{o(t)}$

$$\begin{bmatrix} x_{o(1)} \\ x_{o(-2)} \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -e^{-t}+2e^{-2t} & e^{-t}-e^{-2t} \\ 2e^{-t}-2e^{-2t} & 2e^{-t}-e^{-2t} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

(4) $\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}-1\\ 1\end{bmatrix}u,\ \ y = [1\ \ 0]\begin{bmatrix}x_1\\ x_2\end{bmatrix} + 0u$, (a) find the system is stable or not? (b) Find the model is controlable or not? (c) if the model is uncontrollable explain why the model is uncontrollable and show me the calculations.

Find the solution if the desired poles are at -2 and -3 **(25 points)**

Find the solution if the desired poles are at -1 and -3 also write your obervation for the above example (5 bonus points)

given $A = \begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}$, $B = \begin{bmatrix}-1\\ 1\end{bmatrix}$, $C = [1\ \ 0]$

i) To find the System is Stable. find Transition Matrix poles ies

$\det(SI - A) = 0$

$SI - A = \begin{bmatrix}S & 0\\ 0 & S\end{bmatrix} - \begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix} = \begin{bmatrix}S & -1\\ 2 & S+3\end{bmatrix}$

$\det(SI - A) = S(S+3) + 2$

$S^2 + 3S + 2 = 0 \quad -① $

$S(S+2) + 1(S+2) = 0$

$S = -1, -2$

hence the System is Stable

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ii) To check whether the model is controllable find the value of $\det(P_c)$

$P_c = [B\ \ AB]$

$AB = \begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}\begin{bmatrix}-1\\ 1\end{bmatrix} = \begin{bmatrix}0+1\\ 2-3\end{bmatrix} = \begin{bmatrix}1\\ -1\end{bmatrix}$

$P_c = \begin{bmatrix}-1 & 1\\ 1 & -1\end{bmatrix}$

Please show me your calculations for partial credit

$\det(P_c) = 1 - 1$

$= 0$

$\therefore$ The model is not controllable as there are pole zero cancellations at one of the poles. at -2.

As pole zero cancellation exist, let us find

$$H_s = C \cdot (SI - A)^{-1} \cdot B$$

$$SI - A = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}$$

$$(SI - A)^{-1} = \frac{1}{s(s+3)+2} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

$$(SI - A)^{-1} \cdot B = \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \cdot \frac{1}{s(s+3)+2}$$

$$= \begin{bmatrix} -s-3+1 \\ 2+s \end{bmatrix} \cdot \frac{1}{s(s+3)+2}$$

$$= \begin{bmatrix} -s-2 \\ 2+s \end{bmatrix} \frac{1}{s(s+3)+2}$$

$$C \cdot (SI - A)^{-1} \cdot B = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} -s-2 \\ 2+s \end{bmatrix} \cdot \frac{1}{(s+1)(s+2)}$$

$$= \frac{-(s+2)}{(s+1)(s+2)}$$

Since we have roots are -1, -2 we get pole cancellations as it is not controllable at -2.

$$\begin{bmatrix} -e^{-t}+2e^{-2t} & -e^{-t}+e^{-2t} \\ 2e^{-t}-2e^{-2t} & -2e^{-t}+e^{-2t} \end{bmatrix}$$

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$$= \begin{bmatrix} -2e^{-t}+3e^{-2t} \\ -e^{-2t} \end{bmatrix}$$

ii) Initial conditions

$$X_{(t)} = Q_{(t)} \int_0^t Q_{(-\gamma)} \cdot B \cdot u_{(t)} \cdot d\gamma.$$

$$= \begin{bmatrix} -e^{-t}+2e^{-2t} & e^{-t}-e^{-2t} \\ 2e^{-t}-2e^{-2t} & 2e^{-t}-e^{-2t} \end{bmatrix} \int_0^t \begin{bmatrix} -e^{t}+2e^{2t} & e^{t}-e^{2t} \\ 2e^{t}-2e^{2t} & 2e^{t}-e^{2t} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot 3dt.$$

$$= \begin{bmatrix} -e^{-t}+2e^{-2t} & e^{-t}-e^{-2t} \\ 2e^{-t}-2e^{-2t} & 2e^{-t}-e^{-2t} \end{bmatrix} \int_0^t \begin{bmatrix} -e^{t}+2e^{2t} & e^{t}-e^{2t} \\ 2e^{t}-2e^{2t} & 2e^{t}-e^{2t} \end{bmatrix}_{2\times 2 \times 2 \times 1} \begin{bmatrix} 3 \\ 0 \end{bmatrix} d\gamma$$

$$= \begin{bmatrix} -e^{-t}+2e^{-2t} & e^{-t}-e^{-2t} \\ 2e^{-t}-2e^{-2t} & 2e^{-t}-e^{-2t} \end{bmatrix}_{2\times 2} \int_0^t \begin{bmatrix} -3e^{t}+2e^{2t} \\ 6e^{t}-6e^{2t} \end{bmatrix} d\gamma.$$

$$= \begin{bmatrix} -e^{-t}+2e^{-2t} & e^{-t}-e^{-2t} \\ 2e^{-t}-2e^{-2t} & 2e^{-t}-e^{2t} \end{bmatrix} \begin{bmatrix} -3(e^{t}-1) + 2(\frac{1}{2}e^{2t}-\frac{1}{2}) \\ 6(e^{t}-1) - 6(\frac{1}{2}e^{2t}-\frac{1}{2}) \end{bmatrix}$$

$$= \begin{bmatrix} -e^{-t}+2e^{-2t} & e^{-t}-e^{-2t} \\ 2e^{-t}-2e^{-2t} & 2e^{-t}-e^{2t} \end{bmatrix} \begin{bmatrix} -3(e^{t}-1) + e^{2t}-1 \\ 6(e^{t}-1) - 3(e^{2t}-1) \end{bmatrix}$$

Please show me your calculations for partial credit

$$= \begin{bmatrix} -e^{-t} + 2e^{-2t}\left[-3(e^{t}-1) + e^{2t}-1\right] + e^{-t} - e^{-2t}\left[6(e^{t}-1) - 3(e^{2t}-1)\right] \\ 2e^{-t} - 2e^{-2t}\left[-3(e^{t}-1) + e^{2t}-1\right] + 2e^{-t} - e^{2t}\left[6(e^{t}-1) - 3(e^{2t}-1)\right] \end{bmatrix}$$

$\therefore$ Total response of given model = zero input conditions + zero initial conditions.

$$= \begin{bmatrix} -2e^{-t} + 3e^{-2t} \\ -e^{-2t} \end{bmatrix} + \begin{bmatrix} -e^{-t} + 2e^{-2t}\left[-3(e^{t}-1) + e^{2t}-1\right] + e^{-t} - e^{-2t}\left[6(e^{t}-1) - 3(e^{2t}-1)\right] \\ 2e^{-t} - 2e^{-2t}\left[-3(e^{t}-1) + e^{2t}-1\right] + 2e^{-t} - e^{2t}\left[6(e^{t}-1) - 3(e^{2t}-1)\right] \end{bmatrix}$$

$$= \begin{bmatrix} -2e^{-t} + 3e^{-2t} - e^{-t} + 2e^{-2t}\left[-3(e^{t}-1) + e^{2t}-1\right] + e^{-t} - e^{-2t}\left[6(e^{t}-1) - 3(e^{2t}-1)\right] \\ -e^{-2t} + 2e^{-t} - 2e^{-2t}\left[-3(e^{t}-1) + e^{2t}-1\right] + 2e^{-t} - e^{2t}\left[6(e^{t}-1) - 3(e^{2t}-1)\right] \end{bmatrix} =$$

(1) A continues time invariant system is described by $\ddot{y} + 3\dot{y} + 2y = \ddot{u} + u$ find the controllability and observability of the model by deriving the model by Jordan form. (25 points)

given desired poles are at $-2, -3$

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$\therefore$ desired characteristic equation

$$= (s+2)(s+3)$$
$$= s^2 + 2s + 3s + 6$$
$$= s^2 + 5s + 6. \quad — (1)$$

We have that $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, B = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$A_c = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} - \begin{bmatrix} -\lambda_1 & -\lambda_2 \\ \lambda_1 & \lambda_2 \end{bmatrix} \quad \Bigg| \quad \begin{bmatrix} -1 \\ 1 \end{bmatrix} \begin{bmatrix} \lambda_1 & \lambda_2 \end{bmatrix}$$

$$\begin{matrix} -\lambda_1 & -\lambda_2 \\ \lambda_1 & \lambda_2 \end{matrix}$$

$$= \begin{bmatrix} \lambda_1 & 1+\lambda_2 \\ -2-\lambda_1 & -3-\lambda_2 \end{bmatrix}$$

$$sI - A_c = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} \lambda_1 & 1+\lambda_2 \\ -2-\lambda_1 & -3-\lambda_2 \end{bmatrix}$$

$$= \begin{bmatrix} s-\lambda_1 & s-1-\lambda_2 \\ 2-\lambda_1 & s+3+\lambda_2 \end{bmatrix}$$

$\det(sI - A_c) = 0$

$$\Rightarrow (s-\lambda_1)(s+3+\lambda_2) - (2-\lambda_1)(s-1-\lambda_2) = 0$$

$$s^2 + 3s + \lambda_2 s - \lambda_1 s - 3\lambda_1 - \lambda_1\lambda_2 - 2s + 2 + 2\lambda_2 + s\lambda_1 - \lambda_1 - \lambda_1\lambda_2 = 0$$

$$s^2 + s(3 - \cancel{\lambda_1} + \lambda_2 - 2 + \cancel{\lambda_1}) - 4\lambda_1 - 2\lambda_1\lambda_2 + 2 + 2\lambda_2 = 0. \quad — (2)$$

But comparing ① & ② we get

$$\lambda_2 + 1 = 5$$

$$\lambda_2 = 4.$$

$$-2\lambda_1\lambda_2 + 2 + 2\lambda_2 = 6.$$

$$-8\lambda_1 + 2 + 8 = 6$$

$$-8\lambda_1 = 6 - 10$$

$$\lambda_1 = \frac{-4}{-8} = 1/2$$

If the poles are at $-1, -3$, even then the pole cancellation exists But it will exist at -3 this time and we have uncontrollable pole at -3.