Linear Systems

~~Hom~~ Exam - 2

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1) $\ddot{y} + 3\dot{y} + 2y = \ddot{u} - 3\dot{u} + 2u$  Jordon form

$$s^2 y + 3sy + 2y = s^2 u - 3su + 2u$$

$$y(s^2 + 3s + 2) = u(s^2 - 3s + 2)$$

$$\frac{y}{u} = \frac{s^2 - 3s + 2}{s^2 + 3s + 2} = \frac{(s-1)(s-2)}{(s+1)(s+2)} \quad (\text{or}) \quad 1 + \frac{-6s}{(s+1)(s+2)}$$

$$\frac{-6s}{(s+1)(s+2)} = \frac{A}{(s+1)} + \frac{B}{(s+2)} \Rightarrow -6s = As + 2A + Bs + B$$

$$-6s = s(A+B) + 2A + B$$

$A + B = -6$ — ①  |  $2A + B = 0$ — ②

① × 2 - ② ⇒ $2A + 2B - \cancel{2}A - B = -12$

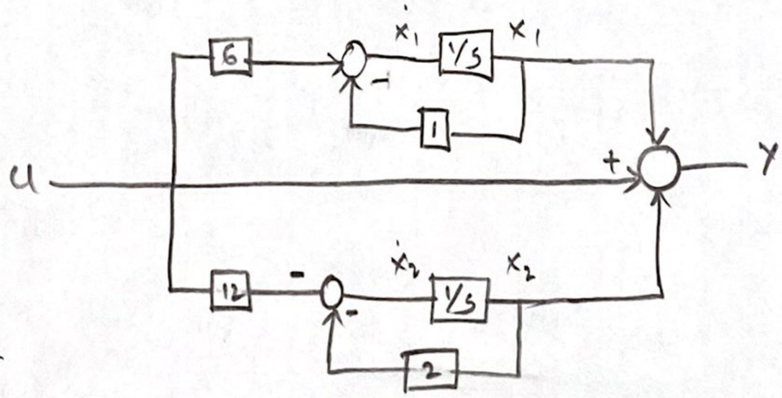
$$\boxed{B = -12}$$

① ⇒ $A + B = -6$

$A - 12 = -6$  $\boxed{A = 6}$

$A = -6 + 12 = 6$

$$\Rightarrow \frac{s^2 - 3s + 2}{s^2 + 3s + 2} \Rightarrow 1 + \frac{6}{s+1} + \frac{-12}{s+2}$$




$$\dot{x}_1 = 6u - x_1$$

$$\dot{x}_2 = -12u - 2x_2$$

$$Y = x_1 + x_2 + u$$

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \quad B = \begin{bmatrix} 6 \\ -12 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix} \quad D = 1$$

Controllability test

$$P_c = \begin{bmatrix} B & AB \end{bmatrix} \Rightarrow AB = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 6 \\ -12 \end{bmatrix} = \begin{bmatrix} -6 \\ +24 \end{bmatrix}$$

$$P_c = \begin{bmatrix} 6 & -6 \\ -12 & 24 \end{bmatrix} \quad |P_c| = 24(6) - (12)(6)$$

$$= 144 - 72 \neq 0$$

$\therefore$ It is Controllable

Observability test

$$P_O = \begin{bmatrix} C \\ CA \end{bmatrix} \quad CA = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} -1 & -2 \end{bmatrix}$$

$$P_O = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \quad |P_O| = -2 - (-1) \Rightarrow -2 + 1 = -1 \neq 0$$

$\therefore$ It is observable.

(2) $A = \begin{bmatrix} 0 & 1 \\ 2 & -3 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ closed loops poles at $-1 \pm j4$

Feedback gain = ?

$$A_c = A - BF$$

$$\Rightarrow \begin{bmatrix} 0 & 1 \\ 2 & -3 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} F_1 & F_2 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 1 \\ 2 & -3 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ F_1 & F_2 \end{bmatrix}$$

$$A_c = \begin{bmatrix} 0 & 1 \\ 2-F_1 & -3-F_2 \end{bmatrix}$$

$$SI - A_c \Rightarrow \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 2-F_1 & -3-F_2 \end{bmatrix} = \begin{bmatrix} S & -1 \\ -2+F_1 & S+3+F_2 \end{bmatrix}$$

$$\det|sI - A_c| = s(s+3+F_2) - (-1)(-2+F_1)$$

$$= s^2 + 3s + sF_2 - (2 - F_1)$$

$$= s^2 + 3s + sF_2 - 2 + F_1$$

$$= s^2 + s(F_2+3) + F_1 - 2 \quad \text{—} ①$$

Given that poles at $-1 \pm j4$ | $j^2 = -1$

$$\Rightarrow (s+1+4j)(s+1-4j) = 0$$

$$(s^2 + s + \cancel{4sj} + s + 1 - \cancel{4j} - \cancel{4sj} + \cancel{4j} - 16j^2) = 0$$

$$(s^2 + 2s + 1 - 16(-1)) = 0 \quad \Rightarrow \quad (s^2 + 2s + 17) = 0$$

$$s^2 + 2s + 17 = 0 \quad \text{—} ②$$

from ① & ②

$F_2 + 3 = 2$ | $F_1 - 2 = 17$

$\boxed{F_2 = -1}$ | $\boxed{F_1 = 19}$

$$\therefore u = -FX + r \quad \Rightarrow \quad u = -\begin{bmatrix} F_1 & F_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + r$$

$$= -\begin{bmatrix} 19 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + r$$

$$= \begin{bmatrix} -19 & +1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + r$$

$$u = -19x_1 + x_2 + r$$

(4) $A = \begin{bmatrix} 1 & 1 \\ -2 & -3 \end{bmatrix}$ $B = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ $C = [1 \ \ 0]$ $D = 0$

a) System is stable or not

$$sI - A = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} s-1 & -1 \\ 2 & s+3 \end{bmatrix}$$

$$\det|sI-A| = (s-1)(s+3) - 2(-1) \Rightarrow s^2 - s + 3s - 3 + 2$$

$$\Rightarrow s^2 + 2s - 1 = 0$$

$\sqrt{2} = 1.414$

$$\frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-2 \pm \sqrt{4-4(1)(-1)}}{2}$$

$$= \frac{-2 \pm 2\sqrt{2}}{2} \Rightarrow -1 \pm \sqrt{2}$$

$\therefore \boxed{s = -1 \pm \sqrt{2}}$

$s = -1 + 1.414$ | $s = -1 - 1.414$

$= 0.414, \ -2.414$

$\therefore$ System is unstable because $s = 0.414$ is right side of Y-axis

b) System is controllable or not

$$P_c = [B \ \ AB] \Rightarrow AB = \begin{bmatrix} 1 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$P_c = \begin{bmatrix} -1 & -1 \\ 0 & 2 \end{bmatrix} \quad \det|P_c| = -2 \neq 0$$

It is controllable (i.e Model)

Now

$$A_c = A - BF = \begin{bmatrix} 1 & 1 \\ -2 & -3 \end{bmatrix} - \begin{bmatrix} -1 \\ 0 \end{bmatrix} [F_1 \ \ F_2]$$

$$= \begin{bmatrix} 1 & 1 \\ -2 & -3 \end{bmatrix} - \begin{bmatrix} -F_1 & -F_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1+F_1 & 1+F_2 \\ -2 & -3 \end{bmatrix}$$

$$sI - A_c = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 1+F_1 & 1+F_2 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} s-1-F_1 & -1-F_2 \\ 2 & s+3 \end{bmatrix}$$

$$\det|sI - A_c| = (s-1-F_1)(s+3) - (2)(-1-F_2)$$

$$= s^2 - s - sF_1 + 3s - 3 - 3F_1 - [-2 - 2F_2]$$

$$= s^2 - s - sF_1 + 3s - 3 - 3F_1 + 2 + 2F_2$$

$$= s^2 + s(-1 - F_1 + 3) + (-3 - 3F_1 + 2 + 2F_2)$$

$$= s^2 + s(-F_1 + 2) + (-3F_1 + 2F_2 - 1) \quad \text{— ①}$$

If desired poles are at $-2, -3$

$$\Rightarrow (s+2)(s+3) = 0 \quad \Rightarrow \quad s^2 + 2s + 3s + 6 = 0$$

$$s^2 + 5s + 6 = 0 \quad \text{— ②}$$

From ① & ②

$-F_1 + 2 = 5$

$-F_1 = 5 - 2$

$-F_1 = 3$

$\boxed{F_1 = -3}$

$-3F_1 + 2F_2 - 1 = 6$

$-3(-3) + 2F_2 = 7$

$2F_2 = 7 - 9$

$2F_2 = -2$

$\boxed{F_2 = -1}$

$\therefore$ Solution exists and unique

If desired poles are at $-1, -3$

$$(s+1)(s+3) = 0 \quad \Rightarrow \quad s^2 + s + 3s + 3 = 0$$

$$s^2 + 4s + 3 = 0 \quad \text{— ③}$$

From ① & ③

$-F_1 + 2 = 4$

$-F_1 = 2$

$\boxed{F_1 = -2}$

$-3F_1 + 2F_2 - 1 = 3$

$-3(-2) + 2F_2 = 4$

$2F_2 = 4 - 6 \quad \Rightarrow \quad 2F_2 = -2$

$\boxed{F_2 = -1}$

$\therefore$ Solution exists and unique

(3) $A = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $C = \begin{bmatrix} 0 & 1 \end{bmatrix}$ $D = 3$

$x_1(0) = 1$ $x_2(0) = -1$ and $u = 5$

Now $sI - A = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} s+3 & -1 \\ 2 & s \end{bmatrix}$

$$[sI - A]^{-1} = \frac{1}{(s+3)s+2}\begin{bmatrix} s & 1 \\ -2 & s+3 \end{bmatrix} = \frac{1}{(s+1)(s+2)}\begin{bmatrix} s & 1 \\ -2 & s+3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{s}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s+3}{(s+1)(s+2)} \end{bmatrix}$$

Now, $\frac{s}{(s+1)(s+2)} = \frac{A}{(s+1)} + \frac{B}{(s+2)} \Rightarrow$

$s = As + Bs + 2A + B$

$A + B = 1$, $2A + B = 0$

$\cancel{2A} + 2B - \cancel{2A} - B = 2$

$\boxed{B = 2} \Rightarrow \boxed{A = -1}$

$\frac{-1}{s+1} + \frac{2}{s+2}$ — (1)

$\frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2} \Rightarrow 1 = As + Bs + 2A + B$

$A + B = 0$ | $2A + B = 1$

$A = -B$

$\Rightarrow \boxed{B = -1}$

$\boxed{A = 1}$

$$\Rightarrow \frac{1}{s+1} + \frac{(-1)}{s+2} \quad \text{—— (2)}$$

$$\frac{-2}{(s+1)(s+2)} = \frac{A}{(s+1)} + \frac{B}{(s+2)} \Rightarrow$$

$$\frac{-2}{s+1} + \frac{2}{s+2} \quad \text{—— (3)}$$

$$-2 = (A+B)s + 2A + B$$

$A = -B$ | $2A + B = -2$

$-B = -2$

$\boxed{A = -2}$ $\boxed{B = 2}$

$$\frac{s+3}{(s+1)(s+2)} = \frac{A}{(s+1)} + \frac{B}{(s+2)} \Rightarrow$$

$$\frac{2}{s+1} + \frac{-1}{s+2} \quad \text{—— (4)}$$

$$s+3 = As + 2A + Bs + B$$

$A + B = 1$ | $2A + B = 3$

$\boxed{B = -1}$ $\boxed{A = 2}$

$$\phi(s) = \begin{bmatrix} \frac{2}{s+2} - \frac{1}{s+1} & \frac{1}{s+1} - \frac{1}{s+2} \\ \frac{2}{s+2} - \frac{2}{s+1} & \frac{2}{s+1} - \frac{1}{s+2} \end{bmatrix}$$

$$\phi(t) = \begin{bmatrix} 2e^{-2t} - e^{-t} & e^{-t} - e^{-2t} \\ 2e^{-2t} - 2e^{-t} & 2e^{-t} - e^{-2t} \end{bmatrix}$$

Total Response

$$x(t) = \phi(t)\, x(0) + \phi(t) \int_0^t \phi(-\tau)\, B\, u(t)\, d\tau$$

I. Zero Input Response

$$x(t) = \phi(t)\, x(0) = \begin{bmatrix} 2e^{-2t} - e^{-t} & e^{-t} - e^{-2t} \\ 2e^{-2t} - 2e^{-t} & 2e^{-t} - e^{-2t} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2e^{-2t} - e^{-t} - e^{-t} + e^{-2t} \\ 2e^{-2t} - 2e^{-t} - 2e^{-t} + e^{-2t} \end{bmatrix} = \begin{bmatrix} 3e^{-2t} - 2e^{-t} \\ 3e^{-2t} - 4e^{-t} \end{bmatrix}$$

Zero Initial Condition Response

$$x(t) = \phi(t) \int_0^t \phi(-\tau)\, B\, u(t)\, d\tau = \phi(t) \int_0^t \begin{bmatrix} 2e^{2\tau} - e^{\tau} & e^{\tau} - e^{2\tau} \\ 2e^{2\tau} - 2e^{\tau} & 2e^{\tau} - e^{2\tau} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} 5\, d\tau$$

$$\Rightarrow \phi(t) \int_0^t \begin{bmatrix} 2e^{2\tau} - e^{\tau} \\ 2e^{2\tau} - 2e^{\tau} \end{bmatrix} 5\, d\tau \Rightarrow \phi(t) \int_0^t \begin{matrix} 10e^{2\tau} - 5e^{\tau} \\ 10e^{2\tau} - 10e^{\tau} \end{matrix}\, d\tau$$

we know that $\boxed{\int_0^t e^{a\tau}\, d\tau = \frac{1}{a}\left[e^{at} - 1\right]}$

$$x(t) = \phi(t) \begin{bmatrix} 10\left[\frac{1}{2}(e^{2t} - 1)\right] - 5\left[e^{t} - 1\right] \\ 10\left[\frac{1}{2}(e^{2t} - 1)\right] - 10\left[e^{t} - 1\right] \end{bmatrix} = \phi(t) \begin{bmatrix} 5e^{2t} - \cancel{5} - 5e^{t} + \cancel{5} \\ 5e^{2t} - 5 - 10e^{t} + 10 \end{bmatrix}$$

$$X(t) = \phi(t)\begin{bmatrix} 5e^{2t} - 5e^{t} \\ 5e^{2t} - 10e^{t} + 5 \end{bmatrix}$$

$$= \begin{bmatrix} 2e^{-2t} - e^{-t} & e^{-t} - e^{-2t} \\ 2e^{-2t} - 2e^{-t} & 2e^{-t} - e^{-2t} \end{bmatrix}\begin{bmatrix} 5e^{2t} - 5e^{t} \\ 5e^{2t} - 10e^{t} + 5 \end{bmatrix}$$

$$= \begin{bmatrix} (2e^{-2t} - e^{-t})(5e^{2t} - 5e^{t}) + (e^{-t} - e^{-2t})(5e^{2t} - 10e^{t} + 5) \\ (2e^{-2t} - 2e^{-t})(5e^{2t} - 5e^{t}) + (2e^{-t} - e^{-2t})(5e^{2t} - 10e^{t} + 5) \end{bmatrix}$$

$$= \begin{bmatrix} (\cancel{10} - \cancel{5e^{t}} - \cancel{10e^{-t}} + \cancel{5}) + (\cancel{5e^{t}} - \cancel{10} + 5e^{-t} - \cancel{5} + \cancel{10e^{-t}} - 5e^{-2t}) \\ (\cancel{10} - \cancel{10e^{t}} - \cancel{10e^{-t}} + \cancel{10} + \cancel{10e^{t}} - \cancel{20} + \cancel{10e^{-t}} - 5 + 10e^{-t} - 5e^{-2t}) \end{bmatrix}$$

$$= \begin{bmatrix} 5e^{-t} - 5e^{-2t} \\ 10e^{-t} - 5e^{-2t} - 5 \end{bmatrix}$$

$$X(t) = \text{Total resp} = \phi(t)\,X(0) + \phi_t \int_0^t \phi(-\tau)\, B\, u_t\, d\tau$$

$$= \text{Zero Input resp} + \text{zero initial condition resp}$$

$$X(t) = \begin{bmatrix} 3e^{-2t} - 2e^{-t} \\ 3e^{-2t} - 4e^{-t} \end{bmatrix} + \begin{bmatrix} 5e^{-t} - 5e^{-2t} \\ 10e^{-t} - 5e^{-2t} - 5 \end{bmatrix} = \begin{bmatrix} 3e^{-t} - 2e^{-2t} \\ 6e^{-t} - 2e^{-2t} - 5 \end{bmatrix}$$