Linear Systems

Exam - 3

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1) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & -2 & -1 \end{bmatrix}$ $B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $C = [1 \ 0 \ 1]$ $D = 3$

$$H(s) = C[sI - A]^{-1}B + D$$

$$sI - A = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & -2 & -1 \end{bmatrix} = \begin{bmatrix} s-1 & 0 & 0 \\ 0 & s-4 & 0 \\ 0 & 2 & s+1 \end{bmatrix}$$

$$[sI - A]^{-1} = \frac{1}{(s-1)(s-4)(s+1)} \begin{bmatrix} (s-4)(s+1) & 0 & 0 \\ 0 & (s-1)(s+1) & (s-1)2 \\ 0 & 0 & (s-1)(s-4) \end{bmatrix}^T$$

$$= \begin{bmatrix} 1/(s-1) & 0 & 0 \\ 0 & 1/s-4 & 0 \ \cancel{2/(s-4)(s+1)} \\ 0 & 2/(s-4)(s+1) & 1/s+1 \end{bmatrix}$$

$$[sI - A]^{-1}B = \begin{bmatrix} 1/s-1 & 0 & 0 \\ 0 & 1/s-4 & 0 \ \cancel{2/s^2-3s-4} \\ 0 & 2/s^2-3s-4 & 1/s+1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/s-1 \\ 0 \\ 1/s+1 \end{bmatrix}$$

$$= \begin{bmatrix} \cancel{1/s-1} \\ \cancel{2/s^2-3s-4} \\ \cancel{1/s+1} \end{bmatrix}$$

Now $C[sI - A]^{-1}B = [1 \ 0 \ 1] \begin{bmatrix} 1/s-1 \\ 0 \ \cancel{2/s^2-3s-4} \\ 1/s+1 \end{bmatrix}$

$\Rightarrow \frac{1}{s-1} + \frac{1}{s+1}$ Now $C[sI - A]^{-1}B + D$

$$= \frac{1}{s-1} + \frac{1}{s+1} + 3$$

$$\therefore \quad H(s) = \frac{1}{s-1} + \frac{1}{s+1} + 3$$

$$H(s) = \frac{y(s)}{u(s)} = \frac{(s+1)+(s-1)+3(s-1)(s+1)}{(s-1)(s+1)} = \frac{s+\cancel{1}+s-\cancel{1}+3(s^2-1)}{s^2-1}$$

$$\frac{y(s)}{u(s)} = \frac{3s^2+2s-3}{s^2-1}$$

$$\boxed{\ddot{y} - y = 3\ddot{u} + 2\dot{u} - 3u}$$

2) $A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 4 & 0 \\ 1 & -2 & 3 \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 4 & 0 \\ 1 & -2 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1-\lambda & 0 & -1 \\ 0 & 4-\lambda & 0 \\ 1 & -2 & 3-\lambda \end{bmatrix}$$

$$\det|A - \lambda I| = (1-\lambda)(4-\lambda)(3-\lambda) - 0 - 1(-(4-\lambda))$$

$$= (1-\lambda)(4-\lambda)(3-\lambda) + (4-\lambda)$$

$$= (4-\lambda)\left[(1-\lambda)(3-\lambda) + 1\right]$$

$$= (4-\lambda)\left[3 - 3\lambda - \lambda + \lambda^2 + 1\right]$$

$$\Rightarrow (4-\lambda)\left[\lambda^2 - 4\lambda + 4\right] = 0$$

$$(4-\lambda)\left[\lambda^2 - 2\lambda - 2\lambda + 4\right] = 0$$

$$(4-\lambda)\left[\lambda(\lambda-2) - 2(\lambda-2)\right] = 0$$

$$(4-\lambda)\left[\lambda-2\right]^2 = 0$$

$$\boxed{(4-\lambda)(\lambda-2)(\lambda-2) = 0}$$

$\therefore$ Eigen values are $2, 2, 4$ //

when $\lambda = 2$

$$[A - \lambda I] = \begin{bmatrix} 1-\lambda & 0 & -1 \\ 0 & 4-\lambda & 0 \\ 1 & -2 & 3-\lambda \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{21} \\ v_{31} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -1 & 0 & -1 \\ 0 & 2 & 0 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{21} \\ v_{31} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{aligned} -v_{11} - v_{31} &= 0 \\ 2v_{21} &= 0 \\ +v_{11} - 2v_{21} + v_{31} &= 0 \end{aligned}$$

Let $\boxed{v_{11} = 1}$ then $-v_{11} - v_{31} = 0$

$-1 - v_{31} = 0$

$-v_{31} = 1 \Rightarrow \boxed{v_{31} = -1}$

$v_{11} - 2v_{21} + v_{31} = 0$

$\cancel{1} - 2v_{21} \cancel{-1} = 0$ $\qquad \boxed{v_{21} = 0}$

$$= \begin{bmatrix} v_{11} \\ v_{21} \\ v_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} v_{12} \\ v_{22} \\ v_{32} \end{bmatrix}$$

when $\lambda = 4$

$$[A - \lambda I] = \begin{bmatrix} -3 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & -2 & -1 \end{bmatrix} \begin{bmatrix} v_{13} \\ v_{23} \\ v_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{aligned} -3v_{13} - v_{33} &= 0 \\ v_{13} - 2v_{23} - v_{33} &= 0 \end{aligned}$$

$\boxed{v_{33} = 3}$ now $-3v_{13} - 3 = 0$ $\quad \boxed{v_{13} = -1}$

$v_{13} - 2v_{23} - v_{33} = 0$

$-1 - 2v_{23} - 3 = 0$

$-2v_{23} = 4$

$\boxed{v_{23} = -2}$

$$\begin{bmatrix} v_{13} \\ v_{23} \\ v_{33} \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix}$$

$\therefore$ Eigen vector $= \begin{bmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & -2 \\ -1 & -1 & 3 \end{bmatrix} = T$ //

3 $A = \begin{bmatrix} -8 & 1 \\ 6 & 0 \end{bmatrix}$ $B = \begin{bmatrix} -5 \\ -6 \end{bmatrix}$ $C = \begin{bmatrix} -1 & 0 \end{bmatrix}$ $D = 3$

To Jordan form

$\cancel{AB} = \cancel{\begin{bmatrix} -8 & 1 \\ 6 & 0 \end{bmatrix}\begin{bmatrix} -5 \\ -6 \end{bmatrix}} = \cancel{\begin{bmatrix} 40-6 \end{bmatrix}}$

$A - \lambda I \Rightarrow \begin{bmatrix} -8 & 1 \\ 6 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} -8-\lambda & 1 \\ 6 & 0-\lambda \end{bmatrix}$

$\det |A - \lambda I| = \cancel{(-8-\lambda)(6-\lambda) - 6}$ $\quad (-8-\lambda)(-\lambda) - 6$

$\cancel{-48 - 6\lambda + 8\lambda + \lambda^2 - 6}$ $\quad +8\lambda + \lambda^2 - 6 = 0$

$\cancel{\lambda^2 + 2\lambda - 54 = 0}$ $\quad \lambda^2 + 8\lambda - 6 = 0$

$\lambda = -4 \pm \sqrt{22}$

$\boxed{\lambda = -8.69,\ 0.69}$

when $\lambda = 0.69$

$A - \lambda I = \begin{bmatrix} -8-0.69 & 1 \\ 6 & -0.69 \end{bmatrix} = \begin{bmatrix} -8.69 & 1 \\ 6 & -0.69 \end{bmatrix}\begin{bmatrix} v_{11} \\ v_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow -8.69\, v_{11} + v_{21} = 0 \quad - (1)$ $\qquad (1) \times 0.69 + (2)$

$6 v_{11} - 0.69\, v_{21} = 0 \quad - (2)$ $\qquad -5.99\, v_{11} + \cancel{0.69 v_{21}} + 6 v_{11} - \cancel{0.69 v_{21}} = 0$

(or)

let $v_{11} = 1 \quad \therefore \quad v_{21} = 8.69$ $\qquad \begin{bmatrix} v_{11} \\ v_{21} \end{bmatrix} = \begin{bmatrix} 1 \\ 8.69 \end{bmatrix}$

when $\lambda = -8.69$

$$A - \lambda I = \begin{bmatrix} -8+8.69 & 1 \\ 6 & +8.69 \end{bmatrix} \begin{bmatrix} v_{12} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad +0.69\, v_{12} + v_{22} = 0 \qquad\qquad v_{12} = 1$

$\qquad 6 v_{12} + v_{22} = 0 \qquad\qquad \therefore v_{22} = \cancel{16.69} - 0.69$

$$\begin{bmatrix} v_{12} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 1 \\ -0.69 \end{bmatrix}$$

$$\therefore \text{Eigen vector} = \begin{bmatrix} 1 & 1 \\ 8.69 & -0.69 \end{bmatrix} = T$$

$$T^{-1} = \frac{1}{-0.69 - 8.69} \begin{bmatrix} -0.69 & -1 \\ -8.69 & 1 \end{bmatrix} = \frac{1}{-9.38} \begin{bmatrix} -0.69 & -1 \\ -8.69 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.07 & 0.10 \\ 0.92 & -0.10 \end{bmatrix}$$

Now $\hat{A} = T^{-1} A T$

$$= \begin{bmatrix} 0.07 & 0.1 \\ 0.92 & -0.1 \end{bmatrix} \begin{bmatrix} -8 & 1 \\ 6 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 8.69 & -0.69 \end{bmatrix}$$

$$= \begin{bmatrix} 0.04 & 0.07 \\ -7.96 & 0.92 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 8.69 & -0.69 \end{bmatrix} = \begin{bmatrix} 0.65 & -0.008 \\ 0.034 & -8.594 \end{bmatrix}$$

$$\hat{B} = T^{-1} B = \begin{bmatrix} 0.07 & 0.1 \\ 0.92 & -0.1 \end{bmatrix} \begin{bmatrix} -5 \\ -6 \end{bmatrix} = \begin{bmatrix} -0.35 - 0.6 \\ -4 \end{bmatrix} = \begin{bmatrix} -0.95 \\ -4 \end{bmatrix}$$

$$\hat{C} = CT = [-1 \ \ 0]\begin{bmatrix} 1 & 1 \\ 8.69 & -0.69 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -1 \end{bmatrix}$$

$$\therefore \ \hat{\dot{x}} = \begin{bmatrix} 0.65 & -0.008 \\ 0.034 & -8.594 \end{bmatrix} x + \begin{bmatrix} -0.95 \\ -4 \end{bmatrix} u \ //$$

$$\hat{y} = [-1 \ -1] \, x + 3u \ //$$

4) $A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $C = [-1 \ \ 1]$

$$sI - A = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} s-3 & -1 \\ 0 & s-2 \end{bmatrix}$$

$$\det|sI - A| \Rightarrow (s-3)(s-2) = 0$$

$$s^2 - 3s - 2s + 6 = 0$$

$$s^2 - 5s + 6$$

$$\hat{A} = \begin{bmatrix} 5 & 1 \\ -6 & 0 \end{bmatrix} \qquad \hat{C} = [1 \ \ 0]$$

$$P_0 = \begin{bmatrix} C \\ CA \end{bmatrix} \qquad \text{Now} \quad CA = [-1 \ \ 1]\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} = \cancel{\begin{bmatrix} -3+0 \\ -1+2 \end{bmatrix}} = \cancel{\begin{bmatrix} -3 \\ 1 \end{bmatrix}}$$

$$= [-3 \ \ 1]$$

$$P_0 = \begin{bmatrix} -1 & 1 \\ -3 & 1 \end{bmatrix}$$

$$\hat{P}_0 = \begin{bmatrix} \hat{C} \\ \hat{C}\hat{A} \end{bmatrix} \qquad \hat{C}\hat{A} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ -6 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 1 \end{bmatrix}$$

$$\hat{P}_0 = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$$

$$P_0^{-1} = \begin{bmatrix} -1 & 1 \\ -3 & 1 \end{bmatrix}^{-1} = \frac{1}{-1-(-3)} \begin{bmatrix} 1 & -1 \\ 3 & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5 & -0.5 \\ 1.5 & -0.5 \end{bmatrix}$$

$$T = P_0^{-1} \hat{P}_0 = \begin{bmatrix} 0.5 & -0.5 \\ 1.5 & -0.5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & -0.5 \\ -1 & -0.5 \end{bmatrix}$$

$\cancel{\hat{B} =}$ Now $T^{-1} = \begin{bmatrix} -2 & -0.5 \\ -1 & -0.5 \end{bmatrix}^{-1} = \dfrac{1}{1 - \frac{1}{2}} \begin{bmatrix} -0.5 & +0.5 \\ 1 & -2 \end{bmatrix}$

$$= \begin{bmatrix} -1 & 1 \\ 2 & -4 \end{bmatrix}$$

$$\hat{B} = T^{-1} B = \begin{bmatrix} -1 & 1 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2+1 \\ 4-4 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \;//$$

5 $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & -2 & -1 \end{bmatrix}$ $B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $C = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$ $D = 3$

stability

$$SI - A = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & -2 & -1 \end{bmatrix} = \begin{bmatrix} s-1 & 0 & 0 \\ 0 & s-4 & 0 \\ 0 & 2 & s+1 \end{bmatrix}$$

$$\det |sI - A| = (s-1)(s+4)(s+1) - 0 + 0$$

$\Rightarrow s = 1, -4, -1$ $\therefore$ $s = 1$ lies right side of y axis

$\therefore$ It is unstable.

Controllability

$$P_c = [B \quad AB \quad A^2B] \qquad AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & -2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \phi \end{bmatrix}$$

$3 \times \boxed{3 \qquad 3} \times 1$

$$= \begin{bmatrix} 1 \\ 0 \\ -\phi \end{bmatrix}$$

$$P_c = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & -1 & 1 \end{bmatrix} \qquad A^2B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & -2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & -2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\det(P_c) = 1(0) - 1(0) + 1(0) \qquad = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & -6 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$= 0$

$\therefore$ It is uncontrollable.

observability

$$P_o = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} \qquad CA = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & -2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & -1 \end{bmatrix}$$

$$CA^2 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & -6 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 10 & 1 \end{bmatrix}$$

$$P_o = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & 10 & +1 \end{bmatrix} \qquad \det|P_o| = 1(2+10) - 1(1+1) + 1(10-2)$$

$$= \cancel{8 - 2 + 8} \quad 12 - 2 + 8 \neq 0$$

$$\cancel{-2 \neq 0}$$

$\therefore$ It is observable