Linear Systems

Homework - 8

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① $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 5 & 0 \\ 2 & 0 & 7 \end{bmatrix}$ det $|A - \lambda I|$

$|A - \lambda I| = \begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 5-\lambda & 0 \\ 2 & 0 & 7-\lambda \end{bmatrix}$ Now det $|A - \lambda I| = ?$

$\Rightarrow 2(0 - 2(5-\lambda)) - 0 + (7-\lambda)((6-\lambda)(5-\lambda) - 4) = 0$

$\Rightarrow (35 - 12\lambda + \lambda^2)(6-\lambda) + 8\lambda - 48$

$\Rightarrow -\lambda^3 + 18\lambda^2 - 99\lambda + 162 = 0$

$\Rightarrow \lambda = 3, 6, 9$

If $\lambda = 9$ $A - \lambda I = \begin{bmatrix} -3 & -2 & 2 \\ -2 & -4 & 0 \\ 2 & 0 & -2 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{21} \\ v_{31} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$-3v_{11} - 2v_{21} + 2v_{31} = 0$ | $-2v_{11} - 4v_{21} = 0$ | $2v_{11} - 2v_{31} = 0$

If $v_{11} = 2$ then $v_{21} = -1$ then $v_{31} = 2$

If $\lambda = 3$ $A - \lambda I = \begin{bmatrix} 3 & -2 & 2 \\ -2 & 2 & 0 \\ 2 & 0 & 4 \end{bmatrix} \begin{bmatrix} v_{12} \\ v_{22} \\ v_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$3v_{12} - 2v_{22} + 2v_{32} = 0$ | $-2v_{12} + 2v_{22} = 0$ | $2v_{12} + 4v_{32} = 0$

If $v_{32} = 1$ then $v_{12} = -2$ then $v_{22} = -2$

$-2v_{12} + 2v_{22} = 0$

$-2(-2) + 2v_{22} = 0 \Rightarrow 2v_{22} = -4 \Rightarrow v_{22} = -2$

If $\lambda = 6$ $\quad A - \lambda I = \begin{bmatrix} 0 & -2 & 2 \\ -2 & -1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} V_{13} \\ V_{23} \\ V_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$-2V_{23} + 2V_{33} = 0$ | $-2V_{13} - V_{23} = 0$ | $2V_{13} + V_{33} = 0$

If $V_{33} = 2$ then $V_{13} = -1$ then $V_{23} = 2$

$\therefore$ Eigen vector $= \begin{bmatrix} 2 & -2 & 2 \\ -1 & -2 & 2 \\ 2 & 1 & -1 \end{bmatrix}$

② $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 12 \\ 6 \end{bmatrix} u, \quad y = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + u$

To CCF

$A = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 12 \\ 6 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix}$

$AB = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 12 \\ 6 \end{bmatrix} = \begin{bmatrix} -24 \\ -6 \end{bmatrix}$

$P_c = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 12 & -24 \\ 6 & -6 \end{bmatrix}$

$\det|P_c| = -72 + 24(6) \neq 0$ $\quad \therefore$ Controllable model

$sI - A \Rightarrow \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} s+2 & 0 \\ 0 & s+1 \end{bmatrix}$

$\det|sI - A| = (s+1)(s+2) \Rightarrow s^2 + 3s + 2 = 0$

$\hat{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \quad \hat{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad P_c = \begin{bmatrix} 12 & -24 \\ 6 & -6 \end{bmatrix}$

$$\hat{P}_c = \begin{bmatrix} \hat{B} & \hat{A}\hat{B} \end{bmatrix} \qquad \hat{A}\hat{B} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

$$\hat{P}_c = \begin{bmatrix} 0 & 1 \\ 1 & -3 \end{bmatrix} , \quad \hat{P}_c^{-1} = \frac{1}{-1} \begin{bmatrix} -3 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}$$

$$T = P_c \hat{P}_c^{-1} = \begin{bmatrix} 12 & -24 \\ 6 & -6 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 12 & 12 \\ 12 & 6 \end{bmatrix}$$

$$\hat{C} = CT = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 12 & 12 \\ 12 & 6 \end{bmatrix} = \begin{bmatrix} 24 & 18 \end{bmatrix}$$

3 $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$ $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $C = \begin{bmatrix} 0 & 1 \end{bmatrix}$

To OCF

$$sI - A = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}$$

$$\det |sI - A| \Rightarrow s^2 + 3s + 2 = 0$$

$$\hat{A} = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix} \quad \hat{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$P_o = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \qquad CA = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} -2 & -3 \end{bmatrix}$$

$$\hat{P}_o = \begin{bmatrix} \hat{C} \\ \hat{C}\hat{A} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \qquad \hat{C}\hat{A} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} -3 & 1 \end{bmatrix}$$

$$T = P_o^{-1} \hat{P}_o = \frac{1}{2} \begin{bmatrix} -3 & -1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -0.5 \\ 1 & 0 \end{bmatrix}$$

$$\hat{B} = T^{-1}B = \frac{1}{0.5}\begin{bmatrix} 0 & 0.5 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 2\begin{bmatrix} 0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

(4) $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$ $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $C = [0 \;\; 1]$ $D = 3$

To Jordan Form

* check Controllability $P_c = [B \;\; AB]$

$AB = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ $P_c = \begin{bmatrix} 0 & 1 \\ 1 & -3 \end{bmatrix}$

$\det|P_c| = -1 \neq 0$

We can proceed

$$A - \lambda I = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{bmatrix}$$

$$\det|A - \lambda I| = -\lambda(-3-\lambda) + 2 \Rightarrow 3\lambda + \lambda^2 + 2 = 0$$

$$\lambda = -1, -2$$

If $\lambda = -1$

$$\begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix}\begin{bmatrix} v_{12} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{l} v_{12} + v_{22} = 0 \\ -2v_{12} - 2v_{22} = 0 \end{array}$$

If $v_{12} = 1$, $v_{22} = -1$

If $\lambda = -2$

$$\begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} v_{11} \\ v_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{l} 2v_{11} + v_{21} = 0 \\ -2v_{11} - v_{21} = 0 \end{array}$$

$v_{11} = 1$ then $v_{21} = -2$

$$\therefore T = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \qquad T^{-1} = \frac{1}{-1+2}\begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ +2 & +1 \end{bmatrix}$$

$$\hat{A} = T^{-1}AT = \begin{bmatrix} -1 & -1 \\ +2 & +1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & -1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\hat{B} = T^{-1}B = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\hat{C} = CT = \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} -2 & -1 \end{bmatrix}$$