Linear Systems

Homework - 7

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(1) $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $C = \begin{bmatrix} -1 & 1 \end{bmatrix}$

transfer function $H(s) = C[sI - A]^{-1} B + D$

$$sI - A = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} s-2 & 0 \\ 0 & s-1 \end{bmatrix}$$

$$[sI-A]^{-1} = \frac{1}{(s-1)(s-2)} \begin{bmatrix} s-1 & 0 \\ 0 & s-2 \end{bmatrix}, \quad C[sI-A]^{-1} = \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} s-1 & 0 \\ 0 & s-2 \end{bmatrix} \frac{1}{(s-1)(s-2)}$$

$$C[sI-A]^{-1} = \begin{bmatrix} -s+1 & s-2 \end{bmatrix} \frac{1}{(s-1)(s-2)}$$

$$C[sI-A]^{-1}B = \begin{bmatrix} -s+1 & s-2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{1}{(s-1)(s-2)} = \frac{-\cancel{s}+1+\cancel{s}-2}{s^2-3s+2} = \frac{-1}{s^2-3s+2}$$

$$H(s) = \frac{-1}{s^2-3s+2} = \frac{A}{s-1} + \frac{B}{s-2} \Rightarrow As - 2A + Bs - B = -1$$

$$s(A+B) - 2A - B = -1$$

$A + B = 0$ | $-2A - B = -1$

$A = -B$ | $+2B - B = -1$

$$\boxed{B = -1} \Rightarrow \boxed{A = 1}$$

$$H(s) = \frac{1}{(s-1)} - \frac{1}{(s-2)}$$

$$A_c = A - BF = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} F_1 & F_2 \end{bmatrix} = \begin{bmatrix} 2-F_1 & -F_2 \\ -F_1 & 1-F_2 \end{bmatrix}$$

$$sI - A_c = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 2-F_1 & -F_2 \\ -F_1 & 1-F_2 \end{bmatrix} = \begin{bmatrix} s-2+F_1 & -F_2 \\ -F_1 & s-1+F_2 \end{bmatrix}$$

$$\det|sI - A_c| = (s-2+F_1)(s-1+F_2) - F_1 F_2$$

$$= s^2 - s + sF_2 - 2s + 2 - 2F_2 + sF_1 - F_1 + \cancel{F_1 F_2} - \cancel{F_1 F_2}$$

$$= s^2 + s(-3 + F_2 + F_1) + 2 - 2F_2 - F_1 \quad \text{— (1)}$$

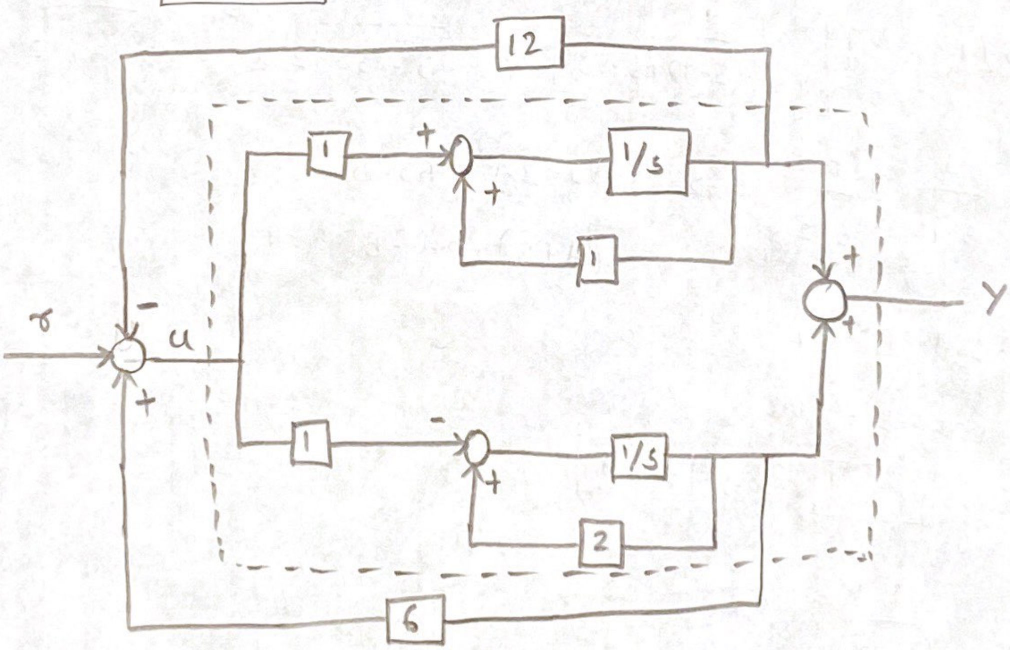
Desired poles at $-1, -2 \Rightarrow (s+1)(s+2) = s^2 + 3s + 2$ — (2)

$$-3 + F_1 + F_2 = 3 \qquad \Big| \qquad -F_1 - 2F_2 + \cancel{2} = \cancel{2}$$

$$F_1 + F_2 = 6 \qquad \qquad F_1 = -2F_2$$

$$-2F_2 + F_2 = 6 \qquad \boxed{F_1 = 12}$$

$$\boxed{F_2 = -6}$$



$$u = -FX + r = \begin{bmatrix} -F_1 & -F_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + r$$

$$u = \begin{bmatrix} -12 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + r$$

(2) $A = \begin{bmatrix} -4 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

$\phi(t) = e^{At} = L^{-1}[SI - A]^{-1}$

$[SI - A] = \begin{bmatrix} S & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & S \end{bmatrix} - \begin{bmatrix} -4 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} S+4 & 0 & 0 \\ 0 & S+3 & 0 \\ 0 & 0 & S+2 \end{bmatrix}$

$[SI - A]^{-1} = \dfrac{1}{(S+4)(S+3)(S+2)} \begin{bmatrix} (S+3)(S+2) & 0 & 0 \\ 0 & (S+4)(S+2) & 0 \\ 0 & 0 & (S+4)(S+3) \end{bmatrix}^T$

$\phi(S) = \dfrac{1}{(S+4)(S+3)(S+2)} \begin{bmatrix} (S+3)(S+2) & 0 & 0 \\ 0 & (S+4)(S+2) & 0 \\ 0 & 0 & (S+4)(S+3) \end{bmatrix}$

$\phi(S) = \begin{bmatrix} 1/S+4 & 0 & 0 \\ 0 & 1/S+3 & 0 \\ 0 & 0 & 1/S+2 \end{bmatrix} = [SI - A]^{-1}$

$\phi(t) = \begin{bmatrix} e^{-4t} & 0 & 0 \\ 0 & e^{-3t} & 0 \\ 0 & 0 & e^{-2t} \end{bmatrix}$ is the state transition matrix.

3) $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \quad , \quad y = [0 \quad 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 3u$

$x_1(0) = 1 \quad , \quad x_2(0) = -1 \quad , \quad u = 5$

$A = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad C = [0 \quad 1] \quad D = 3$

$$sI - A = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} s+3 & -1 \\ 2 & s \end{bmatrix}$$

$$[sI - A]^{-1} = \frac{1}{(s+3)s+2} \begin{bmatrix} s & 1 \\ -2 & s+3 \end{bmatrix} = \frac{1}{(s+1)(s+2)} \begin{bmatrix} s & 1 \\ -2 & s+3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{s}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s+3}{(s+1)(s+2)} \end{bmatrix}$$

$$\frac{s}{(s+1)(s+2)} = \frac{A}{(s+1)} + \frac{B}{(s+2)} = \frac{-1}{s+1} + \frac{2}{s+2} \quad \text{— (1)}$$

$$s = As + Bs + 2A + B$$

$$A + B = 1, \quad 2A + B = 0$$

$$\cancel{2A} + 2B - \cancel{2A} - B = 2$$

$$\boxed{B = 2} \Rightarrow \boxed{A = -1}$$

$$\frac{1}{(s+1)(s+2)} = \frac{A}{(s+1)} + \frac{B}{(s+2)} \Rightarrow 1 = As + Bs + 2A + B$$

$$A + B = 0 \;\Big|\; 2A + B = 1$$

$$2(-B) + B = 1$$

$$B = -1 \;\Big|\; A = 1$$

$$= \frac{1}{s+1} + \frac{(-1)}{s+2} \quad \text{— (2)}$$

$$\frac{-2}{(s+1)(s+2)} = \frac{A}{(s+1)} + \frac{B}{(s+2)} = \frac{-2}{(s+1)} + \frac{2}{(s+2)} \quad \text{— (3)}$$

$$-2 = (A+B)s + 2A + B$$

$$A = -B \;\Big|\; 2A + B = -2$$

$$-B = -2$$

$$B = 2$$

$$A = -2$$

$$\frac{s+3}{(s+1)(s+2)} \Rightarrow s + 3 = (A+B)s + 2A + B = \frac{2}{s+1} + \frac{-1}{s+2} \quad \text{— (4)}$$

$$A + B = 1 \;\Big|\; 2A + B = 3$$

$$B = -1 \;\Big|\; A = 2$$

$$\phi_{(s)} = \begin{bmatrix} \frac{2}{s+2} - \frac{1}{s+1} & \frac{1}{s+1} - \frac{1}{s+2} \\ \frac{2}{s+2} - \frac{2}{s+1} & \frac{2}{s+1} - \frac{1}{s+2} \end{bmatrix} \quad \phi_{(t)} = \begin{bmatrix} 2e^{-2t} - e^{-t} & e^{-t} - e^{-2t} \\ 2e^{-2t} - 2e^{-t} & 2e^{-t} - e^{-2t} \end{bmatrix}$$

Total Response

$$x(t) = \phi_{(t)} x_{(0)} + \phi_{(t)} \int_0^t \phi_{(-\tau)} B\, u_{(t)}\, d\tau$$

Zero Input Response

$$x(t) = \phi(t) x(0) = \begin{bmatrix} 2e^{-2t} - e^{-t} & e^{-t} - e^{-2t} \\ 2e^{-2t} - 2e^{-t} & 2e^{-t} - e^{-2t} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2e^{-2t} - e^{-t} - e^{-t} + e^{-2t} \\ 2e^{-2t} - 2e^{-t} - 2e^{-t} + e^{-2t} \end{bmatrix} = \begin{bmatrix} 3e^{-2t} - 2e^{-t} \\ 3e^{-2t} - 4e^{-t} \end{bmatrix}$$

Zero state Response

$$x(t) = \phi_{(t)} \int_0^t \phi_{(-\tau)} B\, u_{(t)}\, d\tau = \phi_{(t)} = \int_0^t \begin{bmatrix} 2e^{+2\tau} - e^{+\tau} & e^{\tau} - e^{2\tau} \\ 2e^{2\tau} - 2e^{\tau} & 2e^{\tau} - e^{2\tau} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} 5\, d\tau$$

$$x(t) = \phi(t) \int_0^t \begin{bmatrix} 2e^{2\tau} - e^{\tau} \\ 2e^{2\tau} - 2e^{\tau} \end{bmatrix} 5\, d\tau = \phi(t) = \int_0^t \begin{matrix} 10e^{2\tau} - 5e^{\tau} \\ 10e^{2\tau} - 10e^{\tau} \end{matrix}\, d\tau$$

$$* \int_0^t e^{a\tau} d\tau = \frac{1}{a}[e^{at} - 1]$$

$$x(t) = \phi(t) \times \begin{bmatrix} 10\left[\frac{1}{2}[e^{2t} - 1]\right] - 5[e^{at} - 1] \\ 10\left[\frac{1}{2}[e^{2t} - 1]\right] - 10[e^{t} - 1] \end{bmatrix} = \phi(t) \begin{bmatrix} 5e^{2t} - \cancel{5} - 5e^{t} + \cancel{5} \\ 5e^{2t} - 5 - 10e^{t} + 10 \end{bmatrix}$$

$$x(t) = \phi(t)\begin{bmatrix} 5e^{2t} - 5e^{t} \\ 5e^{2t} - 10e^{t} + 5 \end{bmatrix}$$

$$= \begin{bmatrix} 2e^{-2t} - e^{-t} & e^{-t} - e^{-2t} \\ 2e^{-2t} - 2e^{-t} & 2e^{-t} - e^{-2t} \end{bmatrix} \begin{bmatrix} 5e^{2t} - 5e^{t} \\ 5e^{2t} - 10e^{t} + 5 \end{bmatrix}$$

$$= \begin{bmatrix} \cancel{10}(2e^{-2t} - e^{-t})(5e^{2t} - 5e^{t}) + (e^{-t} - e^{-2t})(5e^{2t} - 10e^{t} + 5) \\ (2e^{-2t} - 2e^{-t})(5e^{2t} - 5e^{t}) + (2e^{-t} - e^{-2t})(5e^{2t} - 10e^{t} + 5) \end{bmatrix}$$

$$= \begin{bmatrix} (\cancel{10} - \cancel{5e^{t}} - \cancel{10e^{-t}} + \cancel{5}) + (\cancel{5e^{t}} - \cancel{10} + 5e^{-t} - \cancel{5} + \cancel{10e^{-t}} - 5e^{-2t}) \\ \cancel{10} - \cancel{10e^{t}} - \cancel{10e^{-t}} + \cancel{10} + \cancel{10e^{t}} - \cancel{20} + \cancel{10e^{-t}} - 5 + 10e^{-t} - 5e^{-2t} \end{bmatrix}$$

$$= \begin{bmatrix} 5e^{-t} - 5e^{-2t} \\ 10e^{-t} - 5e^{-2t} - 5 \end{bmatrix}$$

Total Response $= x(t) = \begin{bmatrix} 3e^{-2t} - 2e^{-t} \\ 3e^{-2t} - 4e^{-t} \end{bmatrix} + \begin{bmatrix} 5e^{-t} - 5e^{-2t} \\ 10e^{-t} - 5e^{-2t} - 5 \end{bmatrix}$

$$x(t) = \begin{bmatrix} 3e^{-t} - 2e^{-2t} \\ 6e^{-t} - 2e^{-2t} - 5 \end{bmatrix}$$

4) $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$ $B = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $C = [1 \ 0]$ $D = 0$

$sI - A = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}$

$\det|sI - A| = s^2 + 3s + 2 = (s+1)(s+2) \Rightarrow s = -2, \ s = -1$

$\therefore$ $-1, -2$ are in left side of y-axis

$\therefore$ System is stable

$P_C = [B \quad AB]$ $AB = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$P_C = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ $|P_C| = 1 - 1 = 0$

$\therefore$ ~~System~~ Model is uncontrollable

We expect to have pole zero cancellation

$[sI - A]^{-1} = \dfrac{1}{s(s+3) - (-1)(2)} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} = \dfrac{1}{s^2 + 3s + 2} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$

$C[sI - A]^{-1} = [1 \ 0] \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} \dfrac{1}{s^2 + 3s + 2} = \dfrac{1}{s^2 + 3s + 2} \begin{bmatrix} s+3 & 1 \end{bmatrix}$

$C[sI - A]^{-1} B = \dfrac{1}{s^2 + 3s + 2} [s+3 \ \ 1] \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \dfrac{-(s+2)}{(s+2)(s+1)}$

$s = -2$ is a pole that matches a zero. Since, the model was ~~stable and~~ uncontrollable and system is stable.

If the desired poles are at $-2, -3$

$$A_C = A - BF = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} - \begin{bmatrix} -1 \\ 1 \end{bmatrix} \begin{bmatrix} F_1 & F_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} - \begin{bmatrix} -F_1 & -F_2 \\ F_1 & F_2 \end{bmatrix}$$

$$= \begin{bmatrix} F_1 & 1+F_2 \\ -2-F_1 & -3-F_2 \end{bmatrix}$$

$\det|sI - A_C| \Rightarrow$

$$sI - A_C = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} F_1 & 1+F_2 \\ -2-F_1 & -3-F_2 \end{bmatrix} = \begin{bmatrix} s-F_1 & -1-F_2 \\ 2+F_1 & s+3+F_2 \end{bmatrix}$$

$$\Delta_E(s) = s^2 + (-F_1+3+F_2)s + 2-2F_1+2F_2 \quad \text{— (1)}$$

$$\Delta_d(s) = (s+2)(s+3) = s^2+5s+6$$

$-F_1+3+F_2 = 5$	$2-2F_1+2F_2 = 6 \Rightarrow 1-F_1+F_2 = 3$
$-F_1+F_2 = 2$	$-F_1+F_2 = +2$

$\therefore$ Both the equations give the same information. Solution exists but not unique.

If desired poles are at $-1, -3 \Rightarrow (s+1)(s+3)$

$$= s^2+4s+3$$

$\Rightarrow -F_1+3+F_2 = 4$	$2-2F_1+2F_2 = 3$
$-F_1+F_2 = 1$	$-2F_1+2F_2 = 1$
	$-F_1+F_2 = 1/2$

There is no solution.

From the above problem, If we try to change pole zero cancellation pole then system will fail. If we change the other pole then, solution exists but not unique solution.