43. $5n^2(2n + 5)^2$
44. $8T(T - 7)^2$
45. $[(2R + 3r)(2R - 3r)]^2$
46. $[(6t - y)(6t + y)]^2$
47. $(x + y + 1)^2$
48. $(x + 2 + 3y)^2$
49. $(3 - x - y)^2$
50. $2(x - y + 1)^2$
51. $(5 - t)^3$
52. $(2s + 3)^3$
53. $(3L + 7R)^3$
54. $(2A - 5B)^3$
55. $(w + h - 1)(w + h + 1)$
56. $(2a - c + 2)(2a - c - 2)$
57. $(x + 2)(x^2 - 2x + 4)$
58. $(s - 3)(s^2 + 3s + 9)$
59. $(4 - 3x)(16 + 12x + 9x^2)$
60. $(2x + 3a)(4x^2 - 6ax + 9a^2)$
61. $(x + y)^2(x - y)^2$
62. $(x - y)(x + y)(x^2 + y^2)$

In Exercises 63–72, use the special products of this section to determine the products. Each comes from the technical area indicated.

63. $P_1(P_0c + G)$ (computers)
64. $akT(t_2 - t_1)$ (heat conduction)
65. $4(p + DA)^2$ (photography)
66. $(2J + 3)(2J - 1)$ (lasers)
67. $\frac{1}{2}\pi(R + r)(R - r)$ (architecture)
68. $K(T - T_0)(T + T_0)(T^2 + T_0^2)$ (radiation)
69. $\dfrac{L}{6}(x - a)^3$ (construction: beams)
70. $(1 - z)^2(1 + z)$ (motion: gyroscope)
71. $L_0[1 + a(T - T_0)]$ (thermal expansion)
72. $(s + 1 + j)(s + 1 - j)$ (electricity)

In Exercises 73–84, solve the given problems.

73. Multiply 49 by 51 by writing $(49)(51) = (50 - 1)(50 + 1)$ and using one of the special products.

74. Multiply 82 by 78 by writing $(82)(78) = (80 + 2)(80 - 2)$ and using one of the special products.

75. Is it true that $(1 - 2x)^2 = (2x - 1)^2$?

76. Is it true that $(1 - 2x)^3 = (2x - 1)^2(1 - 2x)$?

77. Is it true that $(1 - 2x)^4 = (2x - 1)^3(1 - 2x)$?

78. Using Eq. (2.8), find the expression in expanded from for the area of the isosceles trapezoidal window in Fig. 6.2.

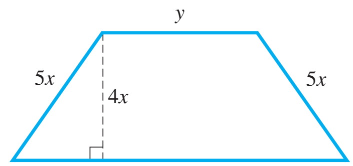

Fig. 6.2

79. The length of a piece of rectangular floor tile is 3 cm more than twice the side x of a second square piece of tile. The width of the rectangular piece is 3 cm less than twice the side of the square piece. Find the area of the rectangular piece in terms of x (in expanded form).

80. The radius of a circular oil spill is r. It then increases in radius by 40 m before being contained. Find the area of the oil spill at the time it is contained in terms of r (in expanded form). See Fig. 6.3.

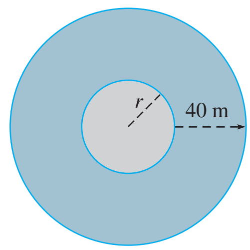

Fig. 6.3

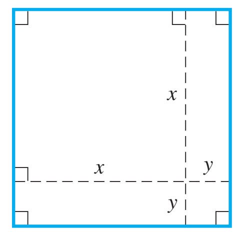

Fig. 6.4

81. Referring to Fig. 6.4, find (a) an expression for the area of the large square (the entire figure) as the power of a binomial; (b) the sum of the areas of the four figures within the large square as a trinomial. Noting that the results of (a) and (b) should be equal, what special product is shown geometrically?

82. If $(x + k)(x - 1) = x^2 + k(x + 2) - (x - 3)$, find the value of k.

83. Find the product $(2x - y - 2)(2x + y + 2)$ by first grouping terms as in Example 6. (*Hint:* This requires grouping the last two terms of each factor.)

W **84.** Verify Eqs. (6.8) and (6.9) by multiplication. Then explain why we may refer to Eqs. (6.1) to (6.10) as *identities*.

Answers to Practice Exercises

1. $9x^2 - 16$ **2.** $9x^2 + 24x + 16$ **3.** $3x^2 - 2x - 8$
4. $x^3 + 15x^2 + 75x + 125$

6.2 Factoring: Common Factor and Difference of Squares

Factoring • Prime Factors • Common Monomial Factors • Factoring Difference of Two Squares • Complete Factoring • Factoring by Grouping

We know that when an algebraic expression is the product of two quantities, each of these quantities is a ***factor*** of the expression (see Section 1.7). In practice, we often have an expression and need to find its factors. *Finding these factors, which is essentially reversing the process of finding a product, is called* **factoring.**

We will consider only factoring of polynomials (see Section 1.7) that have integers for all terms, and each factor will have integers for all coefficients. *A polynomial is called* **prime** *if it contains no factor other than* $+1$, -1, *and plus or minus itself.* Also, *an expression is* **factored completely** *if it is expressed as a product of its prime factors.*

EXAMPLE 1 Factoring completely

When we factor the expression $12x + 6x^2$ as

$$12x + 6x^2 = 2(6x + 3x^2)$$

we see that it has not been factored completely. The factor $6x + 3x^2$ is not prime, because it may be factored as

$$6x + 3x^2 = 3x(2 + x)$$

Therefore, the expression $12x + 6x^2$ is factored completely as

$$12x + 6x^2 = 6x(2 + x)$$

The factors x and $(2 + x)$ are prime. We can factor the numerical coefficient, 6, as $2(3)$, but it is standard not to write numerical coefficients in factored form. ■

To factor expressions easily, we must know how to do algebraic multiplication and really know the special products of the previous section.

NOTE ▸ ***The ability to factor algebraic expressions depends heavily on the proper recognition of the special products.***

The special products also give us a way of checking answers and deciding whether a given factor is prime.

COMMON MONOMIAL FACTORS

Often an expression contains a monomial that is common to each term of the expression. Therefore, *the first step in factoring any expression should be to factor out any* **common monomial factor** *that may exist.* To do this, we note the common factor by inspection and then use the reverse of the distributive law, Eq. (6.1), to show the factored form. The following examples illustrate factoring a common monomial factor out of an expression.

■ For reference, Eq. (6.1) is $a(x + y) = ax + ay$.

EXAMPLE 2 Common monomial factor

In factoring $6x - 2y$, we note each term contains a factor of 2:

$$6x - 2y = 2(3x) - 2y = 2(3x - y)$$

Here, 2 is the common monomial factor, and $2(3x - y)$ is the required factored form of $6x - 2y$. Once the common factor has been identified, it is not actually necessary to write a term like $6x$ as $2(3x)$. The result can be written directly.

NOTE ▸ *We check the result by multiplication.* In this case,

$$2(3x - y) = 6x - 2y$$

Since the result of the multiplication gives the original expression, the factored form is correct. ■

In Example 2, we determined the common factor of 2 by inspection. This is normally the way in which a common factor is found. Once the common factor has been found, the other factor can be determined by dividing the original expression by the common factor.

The next example illustrates the case where the common factor is the same as one of the terms, and special care must be taken to complete the factoring correctly.

EXAMPLE 3 Common factor same as term

Factor: $4ax^2 + 2ax$.

The numerical factor 2 and the literal factors a and x are common to each term. Therefore, the common monomial factor of $4ax^2 + 2ax$ is $2ax$. This means that

$$4ax^2 + 2ax = 2ax(2x) + 2ax(1) = 2ax(2x + 1)$$

Note the presence of the 1 in the factored form. When we divide $4ax^2 + 2ax$ by $2ax$, we get

Practice Exercises

Factor: **1.** $3cx^3 - 9cx$ **2.** $9cx^3 - 3cx$

$$\frac{4ax^2 + 2ax}{2ax} = \frac{4ax^2}{2ax} + \frac{2ax}{2ax}$$
$$= 2x + 1$$

NOTE ▸ noting very carefully that $2ax$ divided by $2ax$ is 1 and *does not simply cancel out leaving nothing.* In cases like this, where the common factor is the same as one of the terms, it is a common error to omit the 1. However,

CAUTION ▸ ***we must include the 1 in the factor $2x + 1$.***

Without the 1, when the factored form is multiplied out, we would not obtain the original expression.

Usually, the division shown in this example is done by inspection. However, we show it here to emphasize the actual operation that is being performed when we factor out a common factor. ■

EXAMPLE 4 Common factor by inspection

Factor: $6a^5x^2 - 9a^3x^3 + 3a^3x^2$.

After inspecting each term, we determine that each contains a factor of 3, a^3, and x^2. Thus, the common monomial factor is $3a^3x^2$. This means that

$$6a^5x^2 - 9a^3x^3 + 3a^3x^2 = 3a^3x^2(2a^2 - 3x + 1)$$ ■

In these examples, note that factoring an expression does not actually change the expression, although it does change the *form* of the expression. In equating the expression to its factored form, we write an *identity.*

It is often necessary to use factoring when solving an equation. This is illustrated in the following example.

EXAMPLE 5 Using factoring in solving an equation

An equation used in the analysis of FM reception is $R_F = \alpha(2R_A + R_F)$. Solve for R_F.

The steps in the solution are as follows:

■ FM radio was developed in the early 1930s.

$$\begin{aligned}
R_F &= \alpha(2R_A + R_F) && \text{original equation}\\
R_F &= 2\alpha R_A + \alpha R_F && \text{use distributive law}\\
R_F - \alpha R_F &= 2\alpha R_A && \text{subtract } \alpha R_F \text{ from both sides}\\
R_F(1 - \alpha) &= 2\alpha R_A && \text{factor out } R_F \text{ on left}\\
R_F &= \frac{2\alpha R_A}{1 - \alpha} && \text{divide both sides by } 1 - \alpha
\end{aligned}$$

We see that we collected both terms containing R_F on the left so that we could factor and thereby solve for R_F. ■

FACTORING THE DIFFERENCE OF TWO SQUARES

In Eq. (6.2), we see that the product of the sum and the difference of two numbers results in the difference between the squares of two numbers. Therefore, *factoring the difference of two squares gives factors that are the sum and the difference of the numbers.*

■ For reference, Eq. (6.2) is $(x + y)(x - y) = x^2 - y^2$.

EXAMPLE 6 Factoring difference of two squares

In factoring $x^2 - 16$, note that x^2 is the square of x and 16 is the square of 4. Therefore,

$$x^2 - 16 = \underbrace{\overbrace{x^2 - 4^2}^{\text{squares}}}_{\text{difference}} = \underbrace{(x + 4)}_{\text{sum}}\underbrace{(x - 4)}_{\text{difference}}$$ ■

■ Usually, in factoring an expression of this type, where it is very clear what numbers are squared, we do not actually write out the middle step as shown. However, if in doubt, write it out.

EXAMPLE 7 Factoring difference of two squares

(a) Because $4x^2$ is the square of $2x$ and 9 is the square of 3, we may factor $4x^2 - 9$ as

$$4x^2 - 9 = (2x)^2 - 3^2 = (2x + 3)(2x - 3)$$

(b) In the same way,

$$\begin{aligned}(y - 3)^2 - 16x^4 &= [(y - 3) + 4x^2][(y - 3) - 4x^2]\\ &= (y - 3 + 4x^2)(y - 3 - 4x^2)\end{aligned}$$

where we note that $16x^4 = (4x^2)^2$. ■

COMPLETE FACTORING

NOTE ▸ CAUTION ▸ As noted before, ***a common monomial factor should be factored out first.*** However, we must be careful to *see if the other factor can itself be factored.* It is possible, for example, that the other factor is the difference of squares. This means that ***complete factoring*** *often requires more than one step.* Be sure to include only prime factors in the final result.

EXAMPLE 8 Complete factoring—application

(a) In factoring $20x^2 - 45$, note a common factor of 5 in each term. Therefore, $20x^2 - 45 = 5(4x^2 - 9)$. However, the factor $4x^2 - 9$ itself is the difference of squares. Therefore, $20x^2 - 45$ is completely factored as

$$20x^2 - 45 = \overset{\text{common factor}}{5}(\underbrace{4x^2 - 9}_{\text{difference of squares}}) = 5\underbrace{(2x + 3)(2x - 3)}$$

Practice Exercises

Factor: **3.** $9c^2 - 64$ **4.** $18c^2 - 128$

(b) In factoring $x^4 - y^4$, note that we have the difference of two squares. Therefore, $x^4 - y^4 = (x^2 + y^2)(x^2 - y^2)$. However, the factor $x^2 - y^2$ is also the difference of squares. This means that

$$x^4 - y^4 = (x^2 + y^2)(x^2 - y^2) = (x^2 + y^2)(x + y)(x - y)$$

CAUTION ▸ ***The factor $x^2 + y^2$ is prime.*** It is **not** equal to $(x + y)^2$. (See Example 2 of Section 6.1.)

(c) In analyzing the energy collected by different circular solar cells, the expression $45R^2 - 20r^2$ arises. In factoring this expression, we note the factor 5 in each term. Therefore, $45R^2 - 20r^2 = 5(9R^2 - 4r^2)$. However, $9R^2 - 4r^2$ is the difference of squares. Therefore, factoring the original expression, we have

$$45R^2 - 20r^2 = 5(9R^2 - 4r^2) = 5(3R - 2r)(3R + 2r)$$ ■

FACTORING BY GROUPING

Terms in an expression can sometimes be grouped and then factored by methods of this section. The following example illustrates this method of *factoring by grouping*. In the next section, we discuss another type of expression that can be factored by grouping.

EXAMPLE 9 Factoring by grouping

Factor: $2x - 2y + ax - ay$.

We see that there is no common factor to all four terms, but that each of the first two terms contains a factor of 2, and each of the third and fourth terms contains a factor of a. Grouping terms this way and then factoring each group, we have

NOTE ▸

$$\begin{aligned} 2x - 2y + ax - ay &= (2x - 2y) + (ax - ay) \\ &= 2(x - y) + a(x - y) \quad \text{now note the common factor of } (x - y) \\ &= (x - y)(2 + a) \end{aligned}$$

■

EXERCISES 6.2

In Exercises 1–4, make the given changes in the indicated examples of this section and then solve the indicated problems.

1. In Example 3, change the + sign to − and then factor.
2. In Example 3, set the given expression equal to B and then solve for a.
3. In Example 8(a), change the coefficient of the first term from 20 to 5 and then factor.
4. In Example 9, change both − signs to + and then factor.

In Exercises 5–44, factor the given expressions completely.

5. $6x + 6y$ 6. $3a - 3b$ 7. $5a - 5$
8. $2x^2 + 2$ 9. $3x^2 - 9x$ 10. $20s + 4s^2$
11. $7b^2h - 28b$ 12. $5a^2 - 20ax$
13. $288n^2 + 24n$ 14. $90p^3 - 15p^2$
15. $2x + 4y - 8z$ 16. $23a - 46b + 69c$
17. $3ab^2 - 6ab + 12ab^3$ 18. $4pq - 14q^2 - 16pq^2$
19. $12pq^2 - 8pq - 28pq^3$ 20. $27a^2b - 24ab - 9a$
21. $2a^2 - 2b^2 + 4c^2 - 6d^2$ 22. $5a + 10ax - 5ay - 20az$
23. $x^2 - 4$ 24. $r^2 - 25$ 25. $100 - 9A^2$
26. $49 - Z^4$ 27. $36a^4 + 1$ 28. $324z^2 + 4$
29. $162s^2 - 50t^2$ 30. $36s^2 - 121t^4$
31. $144n^2 - 169p^4$ 32. $36a^2b^2 + 169c^2$
33. $(x + y)^2 - 9$ 34. $(a - b)^2 - 1$
35. $8 - 2x^2$ 36. $5a^4 - 125a^2$
37. $300x^2 - 2700z^2$ 38. $28x^2 - 700y^2$
39. $2(I - 3)^2 - 8$ 40. $a(x + 2)^2 - ay^2$
41. $x^4 - 16$ 42. $81 - y^4$
43. $x^{10} - x^2$ 44. $2x^4 - 8y^4$

In Exercises 45–50, solve for the indicated letter.

45. $2a - b = ab + 3$, for a 46. $n(x + 1) = 5 - x$, for x
47. $3 - 2s = 2(3 - st)$, for s
48. $k(2 - y) = y(2k - 1)$, for y
49. $(x + 2k)(x - 2) = x^2 + 3x - 4k$, for k
50. $(2x - 3k)(x + 1) = 2x^2 - x - 3$, for k

In Exercises 51–58, factor the given expressions by grouping as illustrated in Example 9.

51. $3x - 3y + bx - by$ 52. $am + an + cn + cm$
53. $a^2 + ax - ab - bx$ 54. $2y - y^2 - 6y^4 + 12y^3$
55. $x^3 + 3x^2 - 4x - 12$ 56. $S^3 - 5S^2 - S + 5$
57. $x^2 - y^2 + x - y$ 58. $4p^2 - q^2 + 2p + q$

In Exercises 59 and 60, evaluate the given expressions by using factoring. The results may be checked with a calculator.

59. $\dfrac{8^9 - 8^8}{7}$ 60. $\dfrac{5^9 - 5^7}{7^2 - 5^2}$

W *In Exercises 61 and 72, give the required explanations.*

61. Factor $n^2 + n$, and then explain why it represents a positive even integer if n is a positive integer.
62. Factor $n^3 - n$, and then explain why it represents a multiple of 6 if n is an integer greater than 1.

In Exercises 63–72, factor the expressions completely. In Exercises 69 and 70, it is necessary to set up the proper expression. Each expression comes from the technical area indicated.

63. $2Q^2 + 2$ (fire science)
64. $4d^2D^2 - 4d^3D - d^4$ (machine design)
65. $81s - s^3$ (rocket path)
66. $12(4 - x^2) - 2x(4 - x^2) - (4 - x^2)^2$ (container design)
67. $rR^2 - r^3$ (pipeline flow)
68. $p_1R^2 - p_1r^2 - p_2R^2 + p_2r^2$ (fluid flow)
69. As large as possible square is cut from a circular metal plate of radius r. Express in factored form the area of the metal pieces that are left.
70. A pipe of outside diameter d is inserted into a pipe of inside radius r. Express in factored form the cross-sectional area within the larger pipe that is outside the smaller pipe.

71. A spherical float has a volume of air within it of radius r_1, and the outer radius of the float is r_2. Express in factored form the difference in areas of the outer surface and the inner surface.

72. Add the areas of the rectangle and two triangles for the isosceles trapezodial window in Fig. 6.5. Express the result in factored form.

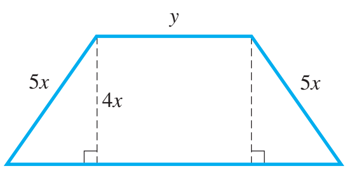

Fig. 6.5

In Exercises 73–78, solve for the indicated letter. Each equation comes from the technical area indicated.

73. $i_1R_1 = (i_2 - i_1)R_2$, for i_1 (electricity: ammeter)

74. $nV + n_1v = n_1V$, for n_1 (acoustics)

75. $3BY + 5Y = 9BS$, for B (physics: elasticity)

76. $Sq + Sp = Spq + p$, for q (computer design)

77. $ER = AtT_0 - AtT_1$, for t (energy conservation)

78. $R = kT_2^4 - kT_1^4$, for k (factor resulting denominator) (radiation)

Answers to Practice Exercises

1. $3cx(x^2 - 3)$ **2.** $3cx(3x^2 - 1)$
3. $(3c - 8)(3c + 8)$ **4.** $2(3c - 8)(3c + 8)$

6.3 Factoring Trinomials

When Coefficient of Square Term is 1 • Importance of the Middle Term • Factoring General Trinomials • Factoring by Grouping

In the previous section, we considered factoring based on the special products of Eqs. (6.1) and (6.2). The trinomials formed from Eqs. (6.3) to (6.6) are important expressions to be factored, and this section is devoted to them.

When factoring is based on Eq. (6.5), we start with the expression on the right and then find the factors on the left. By writing Eq. (6.5) with sides reversed, we have

■ For reference, Eq. (6.5) is $(x + a)(x + b) = x^2 + (a + b)x + ab$.

$$\text{coefficient} = 1 \longrightarrow x^2 + \underbrace{(a + b)}_{\text{sum}}x + \underbrace{ab}_{\text{product}} = (x + a)(x + b)$$

■ Factoring a trinomial for which the coefficient of x^2 is 1.

We are to find integers a and b, and they are found by noting that

1. *the coefficient of x^2 is 1,*
2. *the final constant is the product of the constants a and b in the factors, and*
3. ***the coefficient of x is the sum of a and b.*** CAUTION

As in Section 6.2, we consider only factors in which all terms have integral coefficients.

EXAMPLE 1 Factoring trinomial $x^2 + (a + b)x + ab$

In factoring $x^2 + 3x + 2$, we set it up as

$$x^2 + \underbrace{3x}_{\text{sum}} + \underbrace{2}_{\text{product}} = \underbrace{(x\,\boxed{})\ (x\,\boxed{})}_{\text{integers}}$$

The constant 2 tells us that the product of the required integers is 2. Thus, the only possibilities are 2 and 1 (or 1 and 2). The + sign before the 2 indicates that the sign before the 1 and 2 in the factors must be the same. The + sign before the 3, the sum of the integers, tells us that both signs are positive. Therefore,

$$x^2 + 3x + 2 = (x + 2)(x + 1)$$

In factoring $x^2 - 3x + 2$, the analysis is the same until we note that the middle term is negative. This tells us that both signs are negative in this case. Therefore,

$$x^2 - 3x + 2 = (x - 2)(x - 1)$$

For a trinomial with first term x^2 and constant $+2$ to be factorable, the middle term must be $+3x$ or $-3x$. No other middle terms are possible. This means, for example, the expressions $x^2 + 4x + 2$ and $x^2 - x + 2$ cannot be factored. ■