Modèles avancés de la courbe des taux (cours VI)

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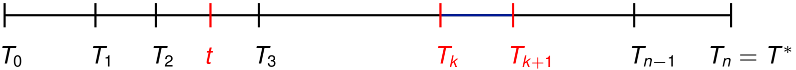
On the Libor Market Model (LMM) and its extensions

Libor market models –notation

Discrete tenor structure: $0 = T_0 < T_1 < \ldots < T_n = T^*$, with $\delta_k = T_{k+1} - T_k$



Zero coupon bonds: $B_t(T_1), \ldots, B_t(T_n)$

Forward Libor rate at time $t \leq T_k$ for the accrual period $[T_k, T_{k+1}]$

$$L(t, T_k) = \frac{1}{\delta_k}\left(\frac{B_t(T_k)}{B_t(T_{k+1})} - 1\right)$$

Libor market models

- The main idea of the Libor market model is to model directly the (forward) Libor rates, which are the underlying rates for caps/floors, under the corresponding forward measures
- Assuming deterministic volatilities, the caplet/floorlet prices is this model are given by the Black formula, which is the formula used by the markets (hence the name “market model”)
- The model was first proposed in the papers by Brace, Gatarek and Musiela (1997) (BGM model) and Miltersen, Sandmann and Sondermann (1997) and provides a theoretical framework justifying the use of the Black formula for caplet/floorlet prices

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Remark: In view of the current reform of the interbank rates (Libor, Euribor and other rates), the existing models should and will be revised and modified in order to take into account these changes, see e.g. the paper
A. Lyashenko and F. Mercurio (2019). Looking forward to backward-looking rates: A modeling framework for term rates replacing LIBOR. Preprint (available at `https://doi.org/10.2139/ssrn.3330240`)

Black formula

Consider a caplet with maturity T_{k+1} and strike K whose pay-off at T_{k+1} is given by

$$\delta_k (L(T_k, T_k) - K)^+$$

If we assume that the dynamics of the Libor rate $L(\cdot, T_k)$ is given by

$$dL(t, T_k) = L(t, T_k)\sigma dW_t$$

under *some* measure used for pricing, we get that the caplet price is given by the Black formula

$$\pi_t^{caplet}(K, T_k) = \delta_k B_t(T_{k+1})(L(t, T_k)\mathcal{N}(d_1) - K\mathcal{N}(d_2)),$$

where $\mathcal{N}(\cdot)$ denotes the cumulative distribution function of the standard normal random variable $N(0, 1)$ and

$$d_{1,2} = \frac{\log\left(\frac{L(t,T_k)}{K}\right) \pm \frac{1}{2}\sigma^2(T_k - t)}{\sigma\sqrt{T_k - t}}$$

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$\Rightarrow$ pour plus de détails voir Brigo et Mercurio (2005), p. 196, Section 6.2

Black formula

- The Black formula has been the market formula used for pricing of caplets/floorlets

- In the standard Gaussian interest rate models such as the Vasiček (Vasiček extended Hull-White) short rate model and the Gaussian HJM model the above assumption on the dynamics of the forward Libor rate is not satisfied. Recall that in all these models we have that the forward price process

$$d\left(\frac{B_t(T_k)}{B_t(T_{k+1})}\right) = \frac{B_t(T_k)}{B_t(T_{k+1})}\Sigma(t, T_k, T_{k+1})dW_t^{T_{k+1}}$$

  is a log-normal $\mathbb{Q}_{T_{k+1}}$-martingale, which means that

$$L(t, T_k) = \frac{1}{\delta_k}\left(\frac{B_t(T_k)}{B_t(T_{k+1})} - 1\right)$$

  does not have log-normal distribution under $\mathbb{Q}_{T_{k+1}}$

- In the Libor market model (with deterministic volatilities) the forward Libor rates indeed have log-normal dynamics under the corresponding forward measures and the Black formula is valid

Libor market model - probabilistic setup

- Let $(\Omega, \mathcal{F}_{T_n}, \mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T_n}, \mathbb{Q}_{T_n})$ be a complete stochastic basis
- The measure $\mathbb{Q}_{T_n}$ will be used as the forward measure associated with maturity T_n and numéraire $B_t(T_n)$
- Let W^{T_n} be a d-dimensional $\mathbb{Q}_{T_n}$-standard Brownian motion, $\mathbb{F}$-adapted

Construction of Libor rates - backward induction

Begin by specifying the dynamics of the most distant Libor rate under $\mathbb{Q}_{T_n}$ (the forward measure associated with date T_n)

$$L(t, T_{n-1}) = L(0, T_{n-1}) \exp\left(\int_0^t b^L(s, T_{n-1}) \mathrm{d}s + \int_0^t \lambda(s, T_{n-1}) \mathrm{d}W_s^{T_n} \right),$$

where the volatility $\lambda(\cdot, T_{n-1})$ is a deterministic function of time and the drift is chosen in such a way that $L(\cdot, T_{n-1})$ becomes a $\mathbb{Q}_{T_n}$-martingale:

$$b^L(s, T_{n-1}) = -\frac{1}{2} ||\lambda(s, T_{n-1})||^2$$

$$\Rightarrow \quad dL(t, T_{n-1}) = L(t, T_{n-1}) \lambda(t, T_{n-1}) dW_t^{T_n}$$

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$$b^L(s, T_{n-1}) \;=\; -\frac{1}{2}||\lambda(s, T_{n-1})||^2$$

$$\Rightarrow \quad dL(t, T_{n-1}) = L(t, T_{n-1})\lambda(t, T_{n-1})dW_t^{T_n}$$

Next, define on $\mathcal{F}_{T_{n-1}}$ the forward measure $\mathbb{Q}_{T_{n-1}}$ associated with date T_{n-1} via the Radon-Nikodym derivative

$$\left.\frac{\mathrm{d}\mathbb{Q}_{T_{n-1}}}{\mathrm{d}\mathbb{Q}_{T_n}}\right|_{\mathcal{F}_t} = \frac{1+\delta_{n-1}L(t, T_{n-1})}{1+\delta_{n-1}L(0, T_{n-1})} = \frac{B_t(T_{n-1})}{B_t(T_n)}\,\frac{B_0(T_n)}{B_0(T_{n-1})}$$

Construction of Libor rates - backward induction

The explicit expression for this density is obtained as follows. Note that from

$$dL(t, T_{n-1}) = L(t, T_{n-1})\lambda(t, T_{n-1})dW_t^{T_n},$$

we immediately get

$$\begin{aligned} d(1 + \delta_{n-1}L(t, T_{n-1})) &= \delta_{n-1}dL(t, T_{n-1}) \\ &= (1 + \delta_{n-1}L(t, T_{n-1}))\frac{\delta_{n-1}L(t, T_{n-1})}{1 + \delta_{n-1}L(t, T_{n-1})}\lambda(t, T_{n-1})dW_t^{T_n} \\ &= (1 + \delta_{n-1}L(t, T_{n-1}))\ell(t, T_{n-1})\lambda(t, T_{n-1})dW_t^{T_n}, \end{aligned}$$

with

$$\ell(t, T_{n-1}) := \frac{\delta_{n-1}L(t, T_{n-1})}{1 + \delta_{n-1}L(t, T_{n-1})}$$

and thus,

$$\left.\frac{\mathrm{d}\mathbb{Q}_{T_{n-1}}}{\mathrm{d}\mathbb{Q}_{T_n}}\right|_{\mathcal{F}_t} = \exp\left(\int_0^t \ell(s, T_{n-1})\lambda(s, T_{n-1})dW_s^{T_n} - \frac{1}{2}\int_0^t (\ell(s, T_{n-1}))^2 ||\lambda(s, T_{n-1})||^2 ds\right)$$

Construction of Libor rates - backward induction

From Girsanov theorem we get that the process $W^{T_{n-1}}$ given by

$$W_t^{T_{n-1}} = W_t^{T_n} - \int_0^t \ell(s, T_{n-1})\lambda(s, T_{n-1})ds$$

is a Brownian motion with respect to the forward measure $\mathbb{Q}_{T_{n-1}}$.

Now we continue with the modeling of the dynamics of the forward Libor rate $L(\cdot, T_{n-2})$ under the measure $\mathbb{Q}_{T_{n-1}}$ and set

$$L(t, T_{n-2}) = L(0, T_{n-2}) \exp\left(\int_0^t b^L(s, T_{n-2})\mathrm{d}s + \int_0^t \lambda(s, T_{n-2})\mathrm{d}W_s^{T_{n-1}}\right),$$

where the volatility $\lambda(\cdot, T_{n-2})$ is a deterministic function of time and the drift is chosen in such a way that $L(\cdot, T_{n-2})$ becomes a $\mathbb{Q}_{T_{n-1}}$-martingale:

$$b^L(s, T_{n-2}) = -\frac{1}{2}||\lambda(s, T_{n-2})||^2$$

Construction of Libor rates - backward induction

General step: for each T_k

(i) define the forward measure $\mathbb{Q}_{T_{k+1}}$ via

$$\left.\frac{\mathrm{d}\mathbb{Q}_{T_{k+1}}}{\mathrm{d}\mathbb{Q}_{T_n}}\right|_{\mathcal{F}_t} = \prod_{l=k+1}^{n-1} \frac{1+\delta_l L(t,T_l)}{1+\delta_l L(0,T_l)} = \frac{B_t(T_{k+1})}{B_t(T_n)} \frac{B_0(T_n)}{B_0(T_{k+1})}$$

(ii) the dynamics of the Libor rate $L(\cdot, T_k)$ under this measure

$$L(t,T_k) = L(0,T_k)\exp\left(\int_0^t b^L(s,T_k)\mathrm{d}s + \int_0^t \lambda(s,T_k)\mathrm{d}W_s^{T_{k+1}}\right), \tag{1}$$

where

$$W_t^{T_{k+1}} = W_t^{T_n} - \sum_{i=k+1}^{n-1}\int_0^t \ell(s,T_i)\lambda(s,T_i)ds$$

is a $\mathbb{Q}_{T_{k+1}}$-Brownian motion by Girsanov theorem with

$$\ell(t,T_i) := \frac{\delta_i L(t,T_i)}{1+\delta_i L(t,T_i)}, \quad i = k+1,\dots,n-1$$

The drift term $b^L(s,T_k)$ is chosen such that $L(\cdot, T_k)$ becomes a $\mathbb{Q}_{T_{k+1}}$-martingale:

$$b^L(s,T_k) = -\frac{1}{2}||\lambda(s,T_k)||^2$$

- This construction guarantees that the forward bond price processes

$$\left(\frac{B_t(T_j)}{B_t(T_k)}\right)_{0\leq t\leq T_j\wedge T_k}$$

are martingales for all $j = 1,\dots,n$ under the forward measure $\mathbb{Q}_{T_k}$ associated with the date T_k ($k = 1,\dots,n$).

(*see Lemma 11.2 in Filipović (2009) for the proof*)

- The arbitrage-free price π_t^X at time t of a contingent claim with payoff X at maturity T_k is given by

$$\pi_t^X = B_t(T_k)\mathbb{E}^{\mathbb{Q}_{T_k}}[X|\mathcal{F}_t].$$

Exercise: Let $k, l \in \{1,\dots,n-1\}$. Write the dynamics of the forward Libor rate $L(\cdot, T_k)$ under the forward measure $\mathbb{Q}_{T_{l+1}}$ (consider three separate cases: $k < l$, $k = l$ and $k > l$).

(*see Lemma 11.1 in Filipović (2009)*)

Caplet pricing

Proposition

The price $\pi_t^{caplet}(K, T_k)$ at time $t \le T_k$ of a caplet with maturity T_{k+1} and strike K with pay-off at T_{k+1}

$$\delta_k (L(T_k, T_k) - K)^+$$

is given by the following Black formula

$$\pi_t^{caplet}(K, T_k) = \delta_k B_t(T_{k+1})(L(t, T_k)\mathcal{N}(d_1) - K\mathcal{N}(d_2)),$$

where $\mathcal{N}(\cdot)$ denotes the cumulative distribution function of the standard normal random variable $N(0, 1)$ and

$$d_{1,2} = \frac{\log\left(\frac{L(t,T_k)}{K}\right) \pm \frac{1}{2}\int_t^{T_k} ||\lambda(s, T_k)||^2 ds}{\sqrt{\int_t^{T_k} ||\lambda(s, T_k)||^2 ds}}$$

Caplet pricing

Remark: The Black caplet implied volatility is thus given by

$$(\sigma^{Black})^2 = \frac{1}{T_k - t} \int_t^{T_k} ||\lambda(s, T_k)||^2 ds$$

and does not depend on the strike $K \Rightarrow$ no volatility smile in the LMM model

Proof of the Proposition:

The time-t price of the caplet is given under the forward measure $\mathbb{Q}_{T_{k+1}}$ by

$$\pi_t^{caplet}(K, T_k) = \delta_k B_t(T_{k+1}) \mathbb{E}^{\mathbb{Q}_{T_{k+1}}}[(L(T_k, T_k) - K)^+ | \mathcal{F}_t]$$

Thus, we have to price a call option with strike K and underlying which is the forward Libor rate $L(\cdot, T_k)$.

Caplet pricing

Recalling that the forward Libor rate $L(\cdot, T_k)$ has the following dynamics under the forward measure $\mathbb{Q}_{T_{k+1}}$

$$L(t, T_k) = L(0, T_k) \exp\left(-\frac{1}{2} \int_0^t ||\lambda(s, T_k)||^2 \mathrm{d}s + \int_0^t \lambda(s, T_k) \mathrm{d}W_s^{T_{k+1}} \right)$$

we note that it is a $\mathbb{Q}_{T_{k+1}}$-martingale and the random variable $L(T_k, T_k)$ has log-normal distribution under $\mathbb{Q}_{T_{k+1}}$.

Therefore, to calculate the conditional expectation above, we can apply the generalized Black-Scholes formula with strike K, underlying asset price $L(t, T_k)$, interest rate $r = 0$ and integrated volatility

$$\int_t^{T_k} ||\lambda(s, T_k)||^2 ds.$$

Swaption pricing

Consider a swap defined on the set of equidistant dates $T_1 < \cdots < T_n$, with $\delta = T_{k+1} - T_k$ and a related swaption with exercise date $T = T_1$ and strike rate R Recall that the payoff of the swaption is given by

$$\left(\pi_{T_1}^{swap}(T_1, T_n, R)\right)^+ = \left(1 - B_{T_1}(T_n) - R\delta \sum_{k=2}^{n} B_{T_1}(T_k)\right)^+$$
$$= \delta \sum_{k=2}^{n} B_{T_1}(T_k)(S(T_1, T_1, T_n) - R)^+,$$

where $S(T_1, T_1, T_n)$ is the swap rate at time T_1 defined by

$$S(T_1, T_1, T_n) = \frac{1 - B_{T_1}(T_n)}{\delta \sum_{k=2}^{n} B_{T_1}(T_k)}$$

Swaption pricing

The time-t price of the swaption $\pi_t^{swaption}(T_1, T_n, R)$ is thus given under the forward measure $\mathbb{Q}_{T_1}$ by

$$\pi_t^{swaption}(T_1, T_n, R) = \delta B_t(T_1)\mathbb{E}^{\mathbb{Q}_{T_1}}\left[\sum_{k=2}^{n} B_{T_1}(T_k)(S(T_1, T_1, T_n) - R)^+ | \mathcal{F}_t\right]$$

To evaluate this expectation we need the joint law of $B_{T_1}(T_k), k = 2, \ldots, n$, under the measure $\mathbb{Q}_{T_1}$, or equivalently the joint law of the Libor rates $L(T_1, T_k), k = 1, \ldots, n-1$ under $\mathbb{Q}_{T_1}$.

$\Rightarrow$ We will try to simplify the calculation by changing the numéraire

Swaption pricing

Define the process

$$A_t := \sum_{k=2}^{n} B_t(T_k), \quad t \leq T_1$$

This is a sum of prices of traded assets (ZC bonds) and thus a valid numéraire.
Note that

$$\left(\frac{A_t}{B_t(T_1)}\right)_{t \leq T_1}$$

is a $\mathbb{Q}_{T_1}$-martingale (simply by definition of the forward measure).

Then we can introduce a new measure $\mathbb{Q}^{swap}$ on $\mathcal{F}_{T_1}$, equivalent to $\mathbb{Q}_{T_1}$, by setting the Radon-Nikodym derivative

$$\left.\frac{\mathrm{d}\mathbb{Q}^{swap}}{\mathrm{d}\mathbb{Q}_{T_1}}\right|_{\mathcal{F}_t} = \frac{A_t}{B_t(T_1)} \frac{B_0(T_1)}{A_0}$$

Swaption pricing

It is easy to show using Bayes formula that the swap rate

$$\left(S(t, T_1, T_n) = \frac{B_t(T_1) - B_t(T_n)}{\delta \sum_{k=2}^{n} B_t(T_k)}\right)_{t \le T_1}$$

is a $\mathbb{Q}^{swap}$-martingale.

The measure $\mathbb{Q}^{swap}$ is called the forward swap measure associated with the numéraire A_t.

Once again by Bayes formula we get

$$\begin{aligned} \pi_t^{swaption}(T_1, T_n, R) &= \delta B_t(T_1)\mathbb{E}^{\mathbb{Q}_{T_1}}\left[A_{T_1}(S(T_1, T_1, T_n) - R)^+ | \mathcal{F}_t\right] \\ &= \delta A_t \mathbb{E}^{\mathbb{Q}^{swap}}\left[(S(T_1, T_1, T_n) - R)^+ | \mathcal{F}_t\right] \end{aligned}$$

$\Rightarrow$ we reduced the problem to pricing a call option with strike R and underlying $S(t, T_1, T_n)$

Swaption pricing

It can be shown that the swap rate $S(\cdot, T_1, T_n)$ being a positive, $\mathbb{Q}^{swap}$-martingale has the following representation

$$dS(t, T_1, T_n) = S(t, T_1, T_n)\varrho^{swap}(t)dW_t^{swap},$$

where W^{swap} is a Brownian motion with respect to the measure $\mathbb{Q}^{swap}$ and ϱ^{swap} is the swap volatility process.

$\Rightarrow$ If the volatility ϱ^{swap} *would be deterministic*, then the swaption price would be given by the Black formula

Swaption pricing

$\Rightarrow$ In the Libor market model ϱ^{swap} *cannot be deterministic* and thus we have the following choices:

1. Stick to the Libor market model, have the Black formula for caplets/floorlets and price swaptions using Monte Carlo methods or via analytic approximations ("Rebonato formula")

   (*see Filipovic(2009) Section 11.5.2 for approximate swaption pricing using Rebonato formula and Section 11.6 for some hints on Monte Carlo simulation*)

2. Model the swap rate $S(\cdot, T_1, T_n)$ directly and postulate that it is log-normal under the forward swap measure $\mathbb{Q}^{swap}$ $\Rightarrow$ swap market model, developed by Jamshidian (1997)

   Note that in the swap market model, the swaption prices are given by the Black formula, but the forward Libor rate volatilities cannot be deterministic in this model and we do not have the Black formula for caplets/floorlets. There is no model where the Black formula is available both for caplets and swaptions

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$\Rightarrow$ voir Brigo et Mercurio (2005), p. 203, Section 6.2 et p. 244, Section 6.8

Extensions of the LMM

In order to produce a caplet volatility smile, the LMM has been extended in several ways. Recalling the dynamics of the forward Libor rate $L(\cdot, T_k)$ under the forward measure $\mathbb{Q}_{T_{k+1}}$

$$dL(t, T_k) = L(t, T_k)\lambda(t, T_k)\mathrm{d}W_t^{T_{k+1}}$$

- Replace the Brownian motion $W^{T_{k+1}}$ by a Lévy process (or another semimartingale with jumps) $X^{T_{k+1}}$ (as in e.g. the Lévy Libor model)
- Replace the deterministic volatility $\lambda(t, T_k)$ by a stochastic volatility, as in e.g. the SABR forward rate model given by

$$\begin{aligned} dL(t, T_k) &= L(t, T_k)^{\beta_k}\lambda(t, T_k)\mathrm{d}W_t^{T_{k+1}} \\ d\lambda(t, T_k) &= \alpha_k\lambda(t, T_k)\mathrm{d}Z_t^{T_{k+1}} \end{aligned}$$

  where $Z^{T_{k+1}}$ is a $\mathbb{Q}_{T_{k+1}}$-Brownian motion correlated with $W_t^{T_{k+1}}$ such that $d\langle W^{T_{k+1}}, Z^{T_{k+1}}\rangle_t = \rho_k dt$, and β_k, α_k are constants such that $0 \leq \beta_k \leq 1$ and $\alpha_k \geq 0$.

Multiple curve LMM

- Similarly to other interest rate models, after the financial crisis and the appearance of the multiple interest rate curves in the market, the LMM was extended to take this phenomenon into account (*see Mercurio (2009, 2010)*).

  The OIS forward rates are defined and modeled as in the classical LMM framework

  $$dF^{OIS}(t, T_k) = F^{OIS}(t, T_k)\lambda^{OIS}(t, T_k)\mathrm{d}W_t^{T_{k+1}}$$

  under the OIS-forward measure $\mathbb{Q}_{T_{k+1}}$ and the risky Libor forward rates are modeled using the spreads $S(\cdot, T_k)$ above the OIS forward curve

  $$L(t, T_k) = F^{OIS}(t, T_k) + S(t, T_k)$$

  The model for $S(\cdot, T_k)$ is chosen such that it ensures positivity (market data shows that the LIBOR-OIS spreads are positive) and analytical tractability of the model (easy manipulation of the sum $F^{OIS}(t, T_k) + S(t, T_k)$ appearing in the option price calculations)

Shifted LMM for negative rates

- In order to take into account the negative rates observed in the markets, the LMM is modified by shifting the rates by a deterministic shift γ (the shifted LMM). Under the forward measure $\mathbb{Q}_{T_{k+1}}$ the dynamics of the Libor rate is given by

$$dL(t, T_k) = (L(t, T_k) + \gamma)\lambda(t, T_k)\mathrm{d}W_t^{T_{k+1}}$$

  The parameter $\gamma > 0$ is obtained from calibration to market data. The lower boundary for the values of the Libor rates is given by $-\gamma$.

- The shifted model allows moreover for better calibration in the presence of very low interest rates

- A very popular model in practice is a shifted SABR model for forward rates

$$\begin{aligned} dL(t, T_k) &= (L(t, T_k) + \gamma)^{\beta_k}\lambda(t, T_k)\mathrm{d}W_t^{T_{k+1}} \\ d\lambda(t, T_k) &= \alpha_k\lambda(t, T_k)\mathrm{d}Z_t^{T_{k+1}} \end{aligned}$$