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Emanuela Rosazza Gianin
Carlo Sgarra

# Mathematical Finance

## Theory Review and Exercises

*Second Edition*

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Emanuela Rosazza Gianin
Dipartimento di Statistica e Metodi Quantitativi
Università di Milano-Bicocca
Milano, Italy

Carlo Sgarra
Dipartimento di Matematica
Politecnico di Milano
Milano, Italy

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*To my family*
*To Francesco and Teresa*

# Preface to the Second Edition

This new edition of the textbook includes some new exercises both solved and proposed. Some new material has been added to the theoretical introduction of a few chapters. A couple of exercises has been removed since they were looking too similar to other exercises already illustrated. Most important is the correction of several misprints present in the previous edition in spite of the detailed checking we tried to perform before sending our manuscript for printing. We detected all these misprints during lectures, often thanks to our students or to our teaching assistants, who pointed out some notational or numerical inconsistencies. We express our gratitude to all of them, they are too many to mention them explicitly.
We included also some new references related to subjects developed quite recently and to other textbooks published almost at the same time of ours or afterwards.
We hope this textbook will be useful to colleagues looking for auxiliary application-oriented material for their courses on Mathematical Finance and to their students.

Milano, Italy
October 2022

Emanuela Rosazza Gianin
Carlo Sgarra

# Preface to the First Edition

This exercise textbook is the result of several years of experience in teaching courses on Quantitative Finance at our Universities. Although part of the content was already published in an Italian edition a few years ago (2007), this new version has been substantially modified: the exercise collection is much more consistent and three new chapters have been included: one chapter on Arbitrage Theory for discrete-time models, which is a relevant issue in many courses on Mathematical Finance, another on Risk Measures, and one on option pricing in models beyond the classical Black-Scholes framework. The subjects covered by the textbook include option-pricing methods based both on Stochastic Calculus and Partial Differential Equations, and some basic notions about portfolio optimization and risk theory. Many textbooks on option pricing focus *either* on stochastic methods *or* on PDE methods. The aim of the present collection of exercises is to provide examples of both approaches and try to illustrate their mutual interplay with the aid of the fundamental notions. Some of the exercises can be solved by students with a rather basic mathematical background (essentially, we mean an introductory course on Calculus and a basic course in Probability), while others require a more developed mathematical knowledge (a course on Differential Equations and an introductory course on stochastic processes). The book can be used in courses with mathematical finance contents aimed at graduate students of Engineering, Economics, and Mathematics.

The material proposed has been organized in 12 chapters. The first chapter reviews the basic notions of probability and stochastic processes through specific examples, while the second chapter illustrates the fundamental results of (static) portfolio optimization. Chapter 3 presents applications of the basic option-pricing techniques in a discrete-time framework, mainly in the binomial setting. Chapter 4 provides examples of the fundamental results of arbitrage theory in a discrete-time setting. Chapter 5 focuses on the main results of the Black-Sholes model related to European option-pricing and hedging; examples involving both static and dynamic hedging strategies are illustrated. Chapter 6 presents exercises on some standard methods in partial differential equations, which turn out to be helpful in solving valuation problems of derivatives in continuous-time models. Chapter 7 deals with more complex derivative products, namely American options: valuation and hedging problems for these options do not admit explicit solutions, except in very few (somehow trivial) cases; approximate solutions can be found by applying suitable numerical techniques. We present a few examples in a discrete-time setting and simple applications of the basic notions. Chapter 8 deals with valuation and hedging of Exotic options; a huge number of different kind of derivative contracts belong to this class, but we focused on the most common type of contracts, in particular those for which an explicit solution for the valuation problem exists in a diffusion setting: Barrier, Lookback, geometric Asian options; moreover, we provide several examples of valuation in a binomial setting. Chapter 10 provides applications of the valuation results for interest rate derivatives: the concern is mainly on short-rate models, but a few examples on the so-called *change of numéraire* technique are included. Chapter 11 attempts to illustrate how the derivatives valuation problem can be attacked in models that drop some of the main assumptions underlying the Black-Scholes model: a few examples, mainly involving affine stochastic volatility models, are provided together with simple examples of jump-diffusion models; all these are typically incomplete market models, and some specific assumptions about the risk-neutral measure adopted by the market in order to assign an arbitrage-free price to contingent claim must be made; the theoretical issues arising in this framework go far beyond the purpose of the present textbook, so we decided to limit our description to the most basic (and popular) models, in which these problems can be avoided by making simple, but reasonable assumptions. Chapter 12 presents some applications of the most important notions related to risk measures; these seem to play an increasingly relevant role in many introductory courses in mathematical finance, so we decided to include some examples in our exercise collection.

We have to thank several people for reading the manuscript and providing useful comments on the material included: we thank Fabio Bellini, Marco Frittelli, Massimo Morini, Paolo Verzella for reading the first draft of the textbook, and Andrea Cosso, Daniele Marazzina, Lorenzo Mercuri for reading the final version and suggesting important modifications. We thank our colleague Giovanni Cutolo for the invaluable help and assistance on LaTeX and the graphical packages necessary to edit the manuscript. We thank Francesca Ferrari and Francesca Bonadei of Springer-Verlag for providing highly qualified editorial support. Finally we want to thank all our colleagues and students who offered any kind of help or comment in support and encouragement to the present work.

Milano, Italy — Emanuela Rosazza Gianin
May 2013 — Carlo Sgarra

# Contents

# Chapter 1
# Short Review of Probability and of Stochastic Processes

## 1.1 Review of Theory

Given a probability space $(\Omega, \mathscr{F}, P)$, where $\Omega$ denotes a non-empty set, $\mathscr{F}$ a $\sigma$-algebra and $P$ a probability measure on $\Omega$:

- a *random variable* (r.v.) is a function $X : \Omega \to \mathbb{R}$ such that $\{\omega \in \Omega : X(\omega) \in A\} \in \mathscr{F}$ for any Borel set $A$ in $\mathbb{R}$;
- a *stochastic process* is a family $(X_t)_{t \geq 0}$ of random variables defined on $(\Omega, \mathscr{F}, P)$.

  The stochastic process $(X_t)_{t \geq 0}$ is said to be *a discrete-time stochastic process* if $t$ takes values in $\mathbb{N}$ and *a continuous-time stochastic process* if $t$ takes values in $\mathbb{R}^+$.
- A *filtration* on $\Omega$ is a family $(\mathscr{F}_t)_{t \geq 0}$ of $\sigma$-algebras on $\Omega$ such that $\mathscr{F}_u \subseteq \mathscr{F}_v$ for any $u \leq v$.

A stochastic process $(X_t)_{t \geq 0}$ is called *adapted* to the filtration $(\mathscr{F}_t)_{t \geq 0}$ if, for any $s \geq 0$, $X_s$ is $\mathscr{F}_s$-measurable.

Given a stochastic process $(X_t)_{t \geq 0}$, a filtration $(\mathscr{F}_t)_{t \geq 0}$ is said to be *generated by* $(X_t)_{t \geq 0}$ if, for any $s \geq 0$, $\mathscr{F}_s$ is the smallest $\sigma$-algebra that makes $X_s$ measurable.

A filtration can be interpreted as the evolution of the information available up to a given time.

For a more detailed and exhaustive treatment of the notions recalled here, we refer to the books of Mikosch [32] and Ross [39].

The following families of random variables and stochastic processes are widely used in Mathematical Finance.

*Bernoulli Random Variable*

A random variable $X$ has a Bernoulli distribution if it assumes only two values (typically 1 and 0) with probability $p$ and $(1 - p)$ respectively. In such a case, we write $X \sim B(p)$.



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As a consequence, the expected value and the variance of $X$ are equal, respectively, to $E[X] = p$ and $V(X) = p(1 - p)$.

*Binomial Random Variable and Binomial Process*

A binomial random variable $Y_n$ counts the number of successes in a series of $n$ independent trials, where $p$ is the probability of success in any one trial. In such a case, $Y_n \sim Bin(n; p)$.

$Y_n$ can be written as a sum $Y_n = \sum_{i=1}^{n} X_i$ where $(X_i)_{i=1,\ldots,n}$ are independent and identically distributed (i.i.d.), $X_i \sim B(p)$ and $X_i = 1$ denotes a success in trial $i$. The expected value and the variance of a binomial random variable $Y_n \sim Bin(n; p)$ are, respectively, $E[Y_n] = np$ and $V(Y_n) = np(1 - p)$.

The sequence $(Y_n)_{n\in\mathbb{N}}$ is called a binomial process.

*Poisson Random Variable and Poisson Process*

A random variable $Z$ has a Poisson distribution with parameter $\lambda > 0$ (in symbols, $Z \sim Poi(\lambda)$) if it takes values in $\mathbb{N}$ and its probability mass function is given by

$$P(Z = k) = \frac{e^{-\lambda}\lambda^k}{k!}, \quad \forall k = 0, 1, 2, \ldots, n, \ldots.$$

for any $n \in \mathbb{N}$. Consequently, the expected value and the variance of $Z$ are equal, respectively, to $E[Z] = \lambda$ and to $V(Z) = \lambda$.

Furthermore, recall that a Poisson random variable can be obtained by taking the limit of a sequence of binomial random variables as $p \to 0$, $n \to +\infty$ and with $pn = \lambda$.

Setting $\lambda = \nu t$ (for $t \geq 0$), $\nu$ can be understood as the rate or average number of arrivals per unit of time.

A process $(Z_t)_{t\geq 0}$ is said to be a Poisson process of rate $\nu$ if $Z_0 = 0$ and if all the increments $Z_t - Z_s$ (for $0 \leq s \leq t$) are independent and identically distributed as a Poisson with parameter $\nu(t - s)$.

If $(Z_t)_{t\geq 0}$ is a Poisson process of rate $\nu$ counting the number of arrivals and $T$ denotes the time between two arrivals, then $T$ has an *exponential distribution* with parameter $\nu > 0$ ($T \sim Exp(\nu)$). The density function of $T$ is given by

$$f_T(t) = \nu e^{-\nu t}, \ \forall t > 0.$$

The expected value and the variance of $T$ are then equal to $E[T] = \frac{1}{\nu}$ and $V(T) = \frac{1}{\nu^2}$, respectively.

*Pareto Random Variable*

A random variable $X$ has a Pareto distribution with parameters $x_0 > 0$ and $a > 0$ if its cumulative distribution function is given by

$$P(X \leq x) = \left(1 - \left(\frac{x_0}{x}\right)^a\right) \mathbf{1}_{[x_0,+\infty)}(x).$$

For $a > 1$, $X$ has finite expected value $E[X] = \frac{a}{a-1} x_0$.

*Normal (or Gaussian) Random Variable*

A random variable $X$ taking values in $\mathbb{R}$ has a Gaussian or normal distribution with parameters $\mu \in \mathbb{R}$ and $\sigma^2 > 0$ ($X \sim N(\mu, \sigma^2)$) if its cumulative distribution function (denoted by $N$) is given by

$$N(x) \triangleq P(X \leq x) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{x} e^{-\frac{(y-\mu)^2}{2\sigma^2}} dy.$$

As a consequence, its expected value and variance are equal, respectively, to $E[X] = \mu$ and $V(X) = \sigma^2$. When $\mu = 0$ and $\sigma^2 = 1$, the random variable $X$ is called standard normal. Tables providing the values of $N$ are widely available in the literature.

Furthermore, a normal random variable $X \sim N(\mu, \sigma^2)$ can be transformed into a standard normal $Z \sim N(0, 1)$ by taking $Z = (X - \mu)/\sigma$.

The *Central Limit Theorem* guarantees that the sum of $n$ random variables, that are independent and identically distributed (i.i.d.) with finite expected values and variances, converges in distribution to a standard normal. More precisely: given a sequence $(X_n)_{n\in\mathbb{N}}$ of i.i.d. random variables with $E(X_n) = \mu$ and $V(X_n) = \sigma^2$ for any $n \in \mathbb{N}$, where $\mu \in \mathbb{R}$ and $\sigma^2 > 0$, it follows that $\frac{\sum_{i=1}^n X_i - n\mu}{\sigma\sqrt{n}} \xrightarrow{d}_n N(0, 1)$.

In particular, if $X_n$ is a binomial random variable with parameters $n$, $p$ (i.e. the sum of $n$ independent Bernoulli random variables with parameter $p$), then $\frac{X_n - np}{\sqrt{np(1-p)}}$ can be approximated by a standard normal.

By the Central Limit Theorem the following stochastic processes can be obtained as limits of processes discussed previously.

*Brownian Motion*

A *Brownian Motion* $(X_t)_{t\geq 0}$ is a continuous-time stochastic process with continuous trajectories such that

- $X_0 = 0$;
- its increments are stationary and independent;
- for any $t \geq 0$, $X_t \sim N(\mu t, \sigma^2 t)$.

$\mu$ and $\sigma$ are called *drift* and *diffusion*, respectively. When $\mu = 0$ and $\sigma^2 = 1$, the Brownian motion is called standard and denoted by $(W_t)_{t\geq 0}$.

By definition of Brownian motion it follows that, for any $0 \leq s \leq t$,

$$X_t - X_s \sim N(\mu\,(t-s)\,,\sigma^2\,(t-s)).$$

For a standard Brownian motion we have

$$W_t - W_s \sim N(0, t-s).$$

A standard Brownian motion can be obtained as limit of a suitable binomial process. Suppose that, at any interval $\Delta t$ of time, the random variable at the previous time may have a positive or negative increment $\Delta x$ with the same probability $p$. After $n = [t/\Delta t]$ periods, we have $Y_n = \sum_{i=1}^{n} Y_i(\Delta x)$, where $Y_i$ takes values in $\{-1, 1\}$. Such a process is called binomial *random walk*. By taking the limit of $Y_n$ (under the assumption that $\Delta x \to 0$, $\Delta t \to 0$ and $\lim[(\Delta x)^2/\Delta t] = \sigma^2$ as $n \to +\infty$) and applying the Central Limit Theorem, one obtains a Brownian motion with mean 0 and variance $\sigma^2 t$. In order to obtain a standard Brownian motion, it is sufficient to have $\lim(\Delta x)^2/\Delta t = 1$.

*Log-Normal Stochastic Process*

A stochastic process $(S_t)_{t\geq 0}$ such that

$$S_t = \exp\left(X_t\right),$$

where $(X_t)_{t\geq 0}$ is a Brownian motion, is said to be log-normal. It follows, therefore, that for any $0 \leq s \leq t$

$$\ln\left(\frac{S_t}{S_0}\right) \sim N(\mu t, \sigma^2 t)$$
$$\ln\left(\frac{S_t}{S_s}\right) \sim N(\mu\,(t-s)\,,\sigma^2\,(t-s)).$$

As for the Brownian motion, also the log-normal process can be seen as the limit of a suitable stochastic process. It can be obtained, indeed, as the limit (for $t_k - t_{k-1} = \Delta t \to 0$) of a discrete process of the form $S_k = S_{k-1}X$, where $X$ is a Bernoulli random variable taking values $u = \exp(\sigma\sqrt{\Delta t})$ and $d = \exp(-\sigma\sqrt{\Delta t})$ with probabilities $p_u = (1+\mu\sqrt{\Delta t}/\sigma)/2$ and $p_d = (1-\mu\sqrt{\Delta t}/\sigma)/2$, respectively. By passing to the limit, the process $(S_t)_{t\geq 0}$ satisfies

$$\ln\left(\frac{S_t}{S_0}\right) \sim N(\mu t, \sigma^2 t), \tag{1.1}$$

so that $(\ln(S_t)/\ln(S_0))_{t\geq 0}$ is a Brownian motion with drift $\mu$ and diffusion $\sigma$. Such a process $(S_t)_{t\geq 0}$ is also called *geometric Brownian motion* with drift $\mu$ and diffusion $\sigma$.

Among the different families of stochastic processes, a remarkable one is the family of *martingales*.

*Discrete-Time Martingales*

A stochastic process $(X_n)_{n\geq 0}$ is said to be a discrete-time martingale with respect to the filtration $(\mathscr{F}_n)_{n\geq 0}$ if $(X_n)_{n\geq 0}$ is $(\mathscr{F}_n)_{n\geq 0}$-adapted and if, for any $n \geq 0$, we have $E[|X_n|] < +\infty$ and

$$E[X_{n+1}|\mathscr{F}_n] = X_n.$$

*Continuous-Time Martingales*

A stochastic process $(X_t)_{t\geq 0}$ is said to be a continuous-time martingale with respect to the filtration $(\mathscr{F}_t)_{t\geq 0}$ if $(X_t)_{t\geq 0}$ is $(\mathscr{F}_t)_{t\geq 0}$-adapted and, for any $0 \leq s \leq t$, we have $E[|X_t|] < +\infty$ and

$$E[X_t|\mathscr{F}_s] = X_s.$$

## 1.2 Solved Exercises

**Exercise 1.1** Let $(W_t)_{t\geq 0}$ be a standard Brownian motion.

1. Calculate $P(W_6 - W_2 < 0; W_1 > 0)$ and $V((W_6 - W_2)W_1)$.
2. Consider the stochastic process $(X_t)_{t\geq 0}$ defined as $X_t = \mu t + W_t$. Establish if there exists a drift $\mu > 0$ such that the probability $P(X_6 - X_2 < 0; X_1 > 0)$ is at least equal to 20%.
3. Set $(Y_t)_{t\geq 0}$ to be the process $Y_t = \mu t + \sigma W_t$. Find the distribution of $Y_t$ for $\mu = 0$ and for $\mu \neq 0$.
4. Compute $P(Y_6 - Y_2 < 0; Y_1 > 0)$, $P(Y_6 - Y_2 < 4\sigma)$ and $E[(Y_6 - Y_2)Y_1]$ for $\mu = 0.1$ and $\sigma = 0.4$.

**Solution**

1. Since $(W_6 - W_2)$ and $W_1$ are independent random variables (by definition of Brownian motion), we deduce that

$$P(W_6 - W_2 < 0; W_1 > 0) = P(W_6 - W_2 < 0) \cdot P(W_1 > 0).$$

By $(W_6 - W_2) \sim N(0; 4)$ (hence, $\frac{W_6-W_2}{\sqrt{4}} \sim N(0,1)$) and $W_1 \sim N(0; 1)$, it follows that

$$\begin{aligned} P(W_6 - W_2 < 0; W_1 > 0) &= P(W_6 - W_2 < 0) \cdot P(W_1 > 0) \\ &= P\left(\frac{W_6 - W_2}{\sqrt{4}} < 0\right) \cdot P(W_1 > 0) \\ &= N(0) \cdot [1 - N(0)] = \frac{1}{2} \cdot \frac{1}{2} = 0.25, \end{aligned}$$

where $N$ denotes the cumulative distribution function of a standard normal.

Furthermore,

$$V((W_6 - W_2) W_1) = E\left[(W_6 - W_2)^2 W_1^2\right] - (E[(W_6 - W_2) W_1])^2, \quad (1.2)$$

and because of the properties of standard Brownian motions we get

$$\begin{aligned} &V((W_6 - W_2) W_1) \\ &= E\left[(W_6 - W_2)^2\right] E\left[W_1^2\right] - (E[W_6 - W_2])^2 (E[W_1])^2 \\ &= V(W_6 - W_2) V(W_1) = 4 \cdot 1 = 4. \end{aligned}$$

2. We have to establish if there exists $\mu > 0$ such that

$$P(X_6 - X_2 < 0; X_1 > 0) \geq 0.2.$$

To this end we rewrite the left-hand side of the inequality above in terms of $\mu$ and of the Brownian motion. This gives

$$\begin{aligned} P(X_6 - X_2 < 0; X_1 > 0) &= P(6\mu + W_6 - (2\mu + W_2) < 0; \mu + W_1 > 0) \\ &= P(W_6 - W_2 < -4\mu; W_1 > -\mu) \\ &= P(W_6 - W_2 < -4\mu) \cdot P(W_1 > -\mu) \\ &= P\left(\frac{W_6 - W_2}{2} < -2\mu\right) \cdot P(W_1 > -\mu) \\ &= N(-2\mu) \cdot [1 - N(-\mu)] = N(-2\mu) \cdot N(\mu), \end{aligned}$$

where the last inequality is due to the symmetry of the normal distribution.

The initial problem is, therefore, equivalent to establish if there exists a drift $\mu > 0$ so that $N(-2\mu) \cdot N(\mu) \geq 0.2$. The answer is yes, because taking $\mu = 0.1$, for instance, gives $N(-2\mu) \cdot N(\mu) = 0.227 \geq 0.2$.

3. Consider, first, the case where $Y_t = \mu t + \sigma W_t$ with $\mu = 0$.

Recall that $W_t \sim N(0; t)$ for any $t > 0$. It follows that $\sigma W_t$ is distributed as a normal with mean

$$E[\sigma W_t] = \sigma E[W_t] = 0$$

and variance

$$V(\sigma W_t) = \sigma^2 V(W_t) = \sigma^2 t.$$

Hence, $\sigma W_t \sim N\left(0; \sigma^2 t\right)$.

In general, for an arbitrary $\mu \in \mathbb{R}$ and for any $t > 0$, $Y_t = \mu t + \sigma W_t$ is distributed as a normal with mean

$$E[Y_t] = E[\mu t + \sigma W_t] = \mu t + \sigma E[W_t] = \mu t$$

and variance

$$V(Y_t) = V(\mu t + \sigma W_t) = V\left(\sigma^2 W_t\right) = \sigma^2 t.$$

Hence, for any $t > 0$, $Y_t \sim N\left(\mu t; \sigma^2 t\right)$ either for $\mu = 0$ or $\mu \neq 0$.

4. Proceeding as above, we get

$$\begin{aligned} P(Y_6 - Y_2 < 0; Y_1 > 0) &= P(\sigma(W_6 - W_2) < -4\mu; \sigma W_1 > -\mu) \\ &= P\left(\frac{W_6 - W_2}{2} < \frac{-2\mu}{\sigma}\right) \cdot P\left(W_1 > -\frac{\mu}{\sigma}\right) \\ &= N\left(-\frac{2\mu}{\sigma}\right)\left[1 - N\left(-\frac{\mu}{\sigma}\right)\right] \\ &= N(-0.5)[1 - N(-0.25)] = 0.185, \end{aligned}$$

and also

$$\begin{aligned} P(Y_6 - Y_2 < 4\sigma) &= P(\sigma(W_6 - W_2) < -4\mu + 4\sigma) \\ &= P\left(\frac{W_6 - W_2}{2} < \frac{-2\mu}{\sigma} + 2\right) \\ &= N\left(\frac{3}{2}\right) = 0.933. \end{aligned}$$

By the properties of Brownian motions, we also find

$$
\begin{aligned}
&E\left[(Y_6 - Y_2)\,Y_1\right] \\
&\quad = E\left[(4\mu + \sigma\,(W_6 - W_2))\,(\mu + \sigma W_1)\right] \\
&\quad = E\left[4\mu^2 + 4\mu\sigma W_1 + \mu\sigma\,(W_6 - W_2) + \sigma^2\,(W_6 - W_2)\,W_1\right] \\
&\quad = 4\mu^2 + 4\mu\sigma E\left[W_1\right] + \mu\sigma E\left[W_6 - W_2\right] + \sigma^2 E\left[(W_6 - W_2)\,W_1\right] \\
&\quad = 4\mu^2 + \sigma^2 E\left[(W_6 - W_2)\,W_1\right] = 4\mu^2,
\end{aligned}
$$

where the last inequality can be deduced in two different ways. The first is based on the independence of $(W_6 - W_2)$ and $W_1$. Then $E\left[(W_6 - W_2)\,W_1\right] = E\left[W_6 - W_2\right]E\left[W_1\right] = 0$. The second comes from $E\left[W_t W_s\right] = \min\{s; t\}$; so $E\left[(W_6 - W_2)\,W_1\right] = E\left[W_6 W_1\right] - E\left[W_2 W_1\right] = 1 - 1 = 0$.

**Exercise 1.2** Consider two geometric Brownian motions $\left(S_t^1\right)_{t \geq 0}$ and $\left(S_t^2\right)_{t \geq 0}$, representing the prices of two stocks, with drifts $\mu_1$, $\mu_2$ and diffusions $\sigma_1 = \sigma > 0$, $\sigma_2 = 2\sigma$ respectively, and with the same initial price $S_0^1 = S_0^2 = S_0 > 0$.

1. Compute $P\left(S_t^1 \geq S_t^2\right)$.
2. For $\mu_1 = 4\mu_2$ and $t = 4$ years, find under which conditions on $\mu_2/\sigma$ one has $P\left(S_4^1 \geq S_4^2\right) \geq \frac{1}{4}$.
3. Compute $E\left[S_t^1 - S_t^2\right]$ and $V\left(\frac{S_t^1}{S_t^2}\right)$.

**Solution** Recall that for a geometric Brownian motion $(X_t)_{t \geq 0}$ with

$$
X_t = X_0 e^{\mu t + \sigma W_t},
$$

one has

$$
\ln\left(\frac{X_t}{X_s}\right) \sim N\left(\mu\,(t - s)\,;\sigma^2\,(t - s)\right) \tag{1.3}
$$

for any $0 \leq s \leq t$.

1. By (1.3) and $S_0^1 = S_0^2 = S_0 > 0$ we deduce that

$$
\begin{aligned}
P\left(S_t^1 \geq S_t^2\right) &= P\left(S_0 e^{\mu_1 t + \sigma W_t} \geq S_0 e^{\mu_2 t + 2\sigma W_t}\right) \\
&= P\left(\mu_1 t + \sigma W_t \geq \mu_2 t + 2\sigma W_t\right) \\
&= P\left(\sigma W_t \leq (\mu_1 - \mu_2)\,t\right) \\
&= P\left(\frac{W_t}{\sqrt{t}} \leq \frac{(\mu_1 - \mu_2)\sqrt{t}}{\sigma}\right) \\
&= N\left(\frac{(\mu_1 - \mu_2)\sqrt{t}}{\sigma}\right).
\end{aligned}
$$

2. By substituting $\mu_1 = 4\mu_2$ and $t = 4$ in the expression of $P\left(S_t^1 \geq S_t^2\right)$ just obtained, we get

$$P\left(S_4^1 \geq S_4^2\right) = N\left(\frac{(\mu_1 - \mu_2)\sqrt{4}}{\sigma}\right) = N\left(\frac{6\mu_2}{\sigma}\right).$$

In order to have $P\left(S_4^1 \geq S_4^2\right) = N\left(\frac{6\mu_2}{\sigma}\right) \geq 0.25$, the inequality $\frac{6\mu_2}{\sigma} \geq N^{-1}(0.25) \cong -0.67$ should hold (see the tables of the cumulative distribution function of the standard normal). This forces $\frac{\mu_2}{\sigma} \geq -0.11$.

3. First of all, we remind that $E\left[e^Y\right] = e^{m+\frac{s^2}{2}}$ for $Y \sim N\left(m, s^2\right)$. Hence

$$\begin{aligned} E\left[S_t^1 - S_t^2\right] &= E\left[\frac{S_t^1}{S_0} \cdot S_0\right] - E\left[\frac{S_t^2}{S_0} \cdot S_0\right] \\ &= S_0 \cdot E\left[e^{\mu_1 t + \sigma W_t}\right] - S_0 \cdot E\left[e^{\mu_2 t + 2\sigma W_t}\right] \\ &= S_0\left[e^{\mu_1 t + \frac{\sigma^2 t}{2}} - e^{\mu_2 t + 2\sigma^2 t}\right], \end{aligned}$$

since $\mu_1 t + \sigma W_t \sim N\left(\mu_1 t; \sigma^2 t\right)$ and $\mu_2 t + 2\sigma W_t \sim N\left(\mu_2 t; 4\sigma^2 t\right)$.

As for the variance, we obtain that

$$\begin{aligned} & V\left(\frac{S_t^1}{S_t^2}\right) = V\left(\frac{S_t^1}{S_0} \cdot \frac{S_0}{S_t^2}\right) = V\left(e^{\mu_1 t + \sigma W_t} \cdot e^{-(\mu_2 t + 2\sigma W_t)}\right) \\ & = V\left(e^{(\mu_1 - \mu_2)t - \sigma W_t}\right) = E\left[\left(e^{(\mu_1 - \mu_2)t - \sigma W_t}\right)^2\right] - \left[E\left(e^{(\mu_1 - \mu_2)t - \sigma W_t}\right)\right]^2 \\ & = E\left[e^{2(\mu_1 - \mu_2)t - 2\sigma W_t}\right] - \left[e^{(\mu_1 - \mu_2)t + \frac{\sigma^2 t}{2}}\right]^2 \\ & = e^{2(\mu_1 - \mu_2)t + 2\sigma^2 t} - e^{2(\mu_1 - \mu_2)t + \sigma^2 t} = e^{2(\mu_1 - \mu_2)t + \sigma^2 t}\left[e^{\sigma^2 t} - 1\right], \end{aligned}$$

where the previous equalities are based on the fact that $W_t$ and $(-W_t)$ have the same distribution.

**Exercise 1.3** Suppose that the number of shares of a given stock bought over time (measured in minutes) follows a Poisson process of rate $\lambda = 12$ per minute.

1. Establish how many minutes are needed so that more than 36 shares are bought with probability of at least 94.8%. Denote by $n^*$ this minimum number of minutes.
2. Compute the average waiting time for the purchase of 40 shares.
3. Assume that the same stock is also sold on a different market and that the number of its shares bought in time (measured in minutes) in such a market follows a Poisson process of rate $\mu = 8$ per minute, independent of the first one.

Compute the probability that the total number of shares (bought in the two markets combined) in $n^*$ minutes is greater than 72.

**Solution**

1. Denote by $X_t$ the number of shares bought in $t$ minutes on the first market. We have to find how many minutes $t$ are needed so that $P(X_t > 36) \geq 0.948$. Denote by $n^*$ the smallest following integer, namely $n^* = [t] + 1$. We know that

$$X_t \sim Poi(\lambda t).$$

Hence,

$$P(X_t > 36) = 1 - P(X_t \leq 36) = 1 - \sum_{\kappa=0}^{36} e^{-\lambda t} \cdot \frac{(\lambda t)^k}{k!}$$
$$= 1 - \sum_{\kappa=0}^{36} e^{-12t} \cdot \frac{(12t)^k}{k!}.$$

Since one has $P(X_t > 36) = 1.44 \cdot 10^{-7}$ for $t = 1$, $P(X_t > 36) = 0.008$ for $t = 2$, $P(X_t > 36) = 0.456$ for $t = 3$ and $P(X_t > 36) = 0.956$ for $t = 4$, we deduce that the minimum number of minutes to wait in order for $P(X_t > 36) \geq 0.948$ is $n^* = 4$.

2. Let $T_i$ be the waiting time (in minutes) of the $i$-th purchase. It is well known that

$$T_{i+1} - T_i \sim Exp(\lambda).$$

Hence, the average time to wait for the purchase of 40 shares (measured in minutes) is given by

$$E[T_{40} - T_0] = E[T_{40} - T_{39}] + E[T_{39} - T_{38}] + \ldots + E[T_1 - T_0]$$
$$= \frac{1}{\lambda} + \frac{1}{\lambda} + \ldots + \frac{1}{\lambda} = \frac{40}{\lambda} = \frac{40}{12} = 3.33.$$

3. Denote now by $Y_t$ the number of shares bought in $t$ minutes on the second market.

Since $Y_t$ is assumed to be a Poisson process of intensity $\mu$ (so, $Y_t \sim Poi(\mu t)$) and $X_t$ and $Y_t$ are assumed to be independent,

$$X_t + Y_t \sim Poi((\lambda + \mu)t)$$

for any $t \geq 0$.

Since $n^* = 4$ (by item 1.), it follows that

$$\begin{aligned} P\left(X_{n^*} + Y_{n^*} > 72\right) &= 1 - P\left(X_4 + Y_4 \le 72\right) \\ &= 1 - \sum_{k=0}^{72} e^{-4(\lambda+\mu)} \cdot \frac{\left(4\left(\lambda+\mu\right)\right)^k}{k!} \\ &= 1 - \sum_{k=0}^{72} e^{-80} \cdot \frac{(80)^k}{k!} \cong 0.80. \end{aligned}$$

**Exercise 1.4** Consider a stock with current price of 8 euros. In each of the following 2 years the stock price may increase by 20% (with probability 40%) or decrease by 20% (with probability 60%).

Denote with $(S_n)_{n=0,1,2}$ the process representing the evolution of the stock price in time. $S_1$ may then take the values $S_1^u = S_0 u$ and $S_1^d = S_0 d$, while $S_2$ the values $S_2^{uu} = S_0 u^2$, $S_2^{ud} = S_0 ud$ and $S_2^{dd} = S_0 d^2$, where $u$ is the growth factor and $d$ the decreasing factor.

1. Is $(S_n)_{n=0,1,2}$ a martingale with respect to the above probability and with respect to the filtration generated by $(S_n)_{n=0,1,2}$?
2. Consider the stochastic process $(\tilde{S}_n)_{n\geq 0}$ defined as

$$\begin{aligned} &\tilde{S}_0 \triangleq S_0 \\ &\tilde{S}_1^u \triangleq S_1^u - k; \quad \tilde{S}_1^d \triangleq S_1^d + k \\ &\tilde{S}_2 \triangleq S_2. \end{aligned}$$

   Establish whether $\left(\tilde{S}_n\right)_{n=0,1,2}$ can be a martingale with respect to the probability and the filtration of the previous item for a suitable $k > 0$.
3. Discuss if there exists a probability measure $Q$ such that the probability that the stock price increases (respectively, decreases) in the first year is equal to the probability that the stock price increases (respectively, decreases) in the second year, and such that $(S_n)_{n=0,1,2}$ is a martingale with respect to $Q$.
4. Establish if there exist $\hat{u} > 1$ and $\hat{d} > 0$ such that the new stock price process is a martingale with respect to the probability measure of item 1.

**Solution** Since the growth factor is $u = 1.2$ and the decreasing factor is $d = 0.8$, the stock prices evolves as follows.

$$
\begin{array}{ccccc}
 & & & & 11.52 = S_0u^2 \\
 & & & \nearrow & \\
 & & S_0u = 9.6 & & \\
 & \nearrow & & \searrow & \\
S_0 = 8 & & & & 7.68 = S_0ud \\
 & \searrow & & \nearrow & \\
 & & S_0d = 6.4 & & \\
 & & & \searrow & \\
 & & & & 5.12 = S_0d^2 \\
\text{– – ––} & & \text{– – – – – ––} & & \text{– – – – – ––} \\
0 & & 1\text{ year} \quad (S_1) & & 2\text{ years} \quad (S_2)
\end{array}
$$

It is also easy to check that

$$
\begin{aligned}
&P\,(S_1 = 9.6) = 0.4; \quad P\,(S_1 = 6.4) = 0.6 \\
&P\,(S_2 = 11.52) = 0.16; \quad P\,(S_2 = 7.68) = 0.48; \quad P\,(S_2 = 5.12) = 0.36 \\
&P\,(\,S_2 = 11.52|\,S_1 = 9.6) = P\,(\,S_2 = 7.68|\,S_1 = 6.4) = 0.4 \\
&P\,(\,S_2 = 7.68|\,S_1 = 9.6) = P\,(\,S_2 = 5.12|\,S_1 = 6.4) = 0.6,
\end{aligned}
$$

and so on.

1. Let us verify if $(S_n)_{n=0,1,2}$ is a martingale with respect to $P$.
   From

$$
\begin{aligned}
E\,[\,S_1|\,S_0] &= S_1^u p + S_1^d\,(1-p) = 9.6 \cdot 0.4 + 6.4 \cdot 0.6 \\
&= 7.68 \neq S_0
\end{aligned}
$$

   it follows immediately that $(S_n)_{n=0,1,2}$ is not a martingale with respect to $P$.
2. We need to verify if the following equalities hold for some $k > 0$:

$$
\begin{cases}
E\left[\tilde{S}_1 \middle| \tilde{S}_0\right] = \tilde{S}_0 \\
E\left[\tilde{S}_2 \middle| \tilde{S}_1\right] = \tilde{S}_1
\end{cases}.
$$

   The first equality is equivalent to

$$
\begin{aligned}
&E\left[\tilde{S}_1 \middle| \tilde{S}_0\right] = \tilde{S}_0 \\
&(9.6 - k) \cdot 0.4 + (6.4 + k) \cdot 0.6 = 8 \\
&7.68 + 0.2 \cdot k = 8 \\
&k = 1.6,
\end{aligned}
$$

hence $\tilde{S}_1^u = \tilde{S}_1^d = 8$. This implies that

$$E\left[\tilde{S}_2 \middle| \tilde{S}_1 = \tilde{S}_1^u\right] = 11.52 \cdot 0.4 + 7.68 \cdot 0.6 = 9.216 \neq \tilde{S}_1^u,$$

so $\left(\tilde{S}_n\right)_{n=0,1,2}$ is not a martingale with respect to $P$.

3. Define now a probability measure $Q$ as follows:

$$\begin{aligned}
&Q(S_1 = 9.6) = q; \quad Q(S_1 = 6.4) = 1 - q \\
&Q(S_2 = 11.52) = q^2; \quad Q(S_2 = 7.68) = 2q(1-q) \\
&Q(S_2 = 5.12) = (1-q)^2 \\
&Q(S_2 = 11.52 | S_1 = 9.6) = Q(S_2 = 7.68 | S_1 = 6.4) = q \\
&Q(S_2 = 7.68 | S_1 = 9.6) = Q(S_2 = 5.12 | S_1 = 6.4) = 1 - q.
\end{aligned}$$

We look now for $q \in (0, 1)$ such that

$$\begin{cases} E_Q[S_1 | S_0] = S_0 \\ E_Q[S_2 | S_1] = S_1 \end{cases},$$

or, equivalently,

$$E_Q[S_1 | S_0] = S_0 \tag{1.4}$$

$$E_Q\left[S_2 | S_1 = S_1^u\right] = S_1^u \tag{1.5}$$

$$E_Q\left[S_2 | S_1 = S_1^d\right] = S_1^d. \tag{1.6}$$

By Eq. (1.4) it follows that

$$\begin{aligned}
&S_0 uq + S_0 d(1-q) = S_0 \\
&q = \tfrac{1-d}{u-d} = \tfrac{1}{2}.
\end{aligned}$$

There remains therefore to check if such a $q$ also satisfies equations (1.5) and (1.6) or not.

Since $uq + d(1-q) = 1$, we obtain that

$$\begin{aligned}
E_Q\left[S_2 | S_1 = S_1^u\right] &= S_0 u^2 q + S_0 ud(1-q) \\
&= S_0 u[uq + d(1-q)] \\
&= S_0 u = S_1^u,
\end{aligned}$$

hence (1.5) is satisfied. In a similar way, it is easy to check that also (1.6) is true.

Consequently, $(S_n)_{n=0,1,2}$ is a martingale with respect to the probability measure $Q$ defined above with $q = 0.5$.

4. We need to establish if there exist $\hat{u} > 1$ and $\hat{d} > 0$ such that the process $\left(\hat{S}_n\right)_{n=0,1,2}$ defined below is a martingale with respect to $P$ (and to the natural filtration):

$$
\begin{array}{ccccc}
 & & & & S_0\hat{u}^2 \\
 & & & \nearrow & \\
 & & S_0\hat{u} & & \\
 & \nearrow & & \searrow & \\
\hat{S}_0 = S_0 = 8 & & & & S_0\hat{u}\hat{d} \\
 & \searrow & & \nearrow & \\
 & & S_0\hat{d} & & \\
 & & & \searrow & \\
 & & & & S_0\hat{d}^2 \\
 & & & & \\
------- & & ------- & & ------- \\
 & & 1\,\text{year} \quad (\hat{S}_1) & & 2\,\text{years} \quad (\hat{S}_2)
\end{array}
$$

Proceeding as above, we need to check if there exist $\hat{u} > 1$ and $\hat{d} > 0$ satisfying $E\left[\hat{S}_1 \middle| \hat{S}_0\right] = \hat{S}_0$ and $E\left[\hat{S}_2 \middle| \hat{S}_1\right] = \hat{S}_1$. Or, equivalently, if the following system in the unknowns $\left(\hat{u}, \hat{d}\right)$ admits solutions:

$$
\begin{cases}
S_0\hat{u}p + S_0\hat{d}\,(1-p) = S_0 \\
S_0\hat{u}^2 p + S_0\hat{u}\hat{d}\,(1-p) = S_0\hat{u} \\
S_0\hat{u}\hat{d}p + S_0\hat{d}^2\,(1-p) = S_0\hat{d}
\end{cases}
$$

Since the previous system is equivalent to $\hat{u}p + \hat{d}\,(1-p) = 1$, we find

$$
\begin{array}{l}
\hat{u}p + \hat{d} - \hat{d}p = 1 \\
\hat{u} = \frac{1-\hat{d}+\hat{d}p}{p}.
\end{array}
$$

Taking, for instance, $\hat{d} = 0.8$, we obtain $\hat{u} = 1.3 > 1$. The pair $\left(\hat{u} = 1.3; \hat{d} = 0.8\right)$ is therefore one among the (infinitely many) pairs of factors for which the corresponding process $\left(\hat{S}_n\right)_{n=0,1,2}$ is a martingale with respect to $P$.

**Exercise 1.5** Consider the stock of Exercise 1.4 and assume that in each period the price may increase with probability $p$ and decrease with probability $(1-p)$.

Let $X_1$ and $X_2$ be two random variables defined as

$$X_1 = \begin{cases} 1; \text{ if } S_1 = S_1^u \\ 0; \text{ if } S_1 = S_1^d \end{cases}$$

$$X_2 = \begin{cases} 2; \text{ if } S_2 = S_2^{uu} \\ 1; \text{ if } S_2 = S_2^{ud} \\ 0; \text{ if } S_2 = S_2^{dd} \end{cases}$$

1. Write $P\left(S_2 = S_2^{uu}\right)$ as a function of $P\left(X_2 = 2\right)$.
2. Establish if $(X_n)_{n=1,2}$ is a binomial process.

**Solution**

1. By the definition of $X_2$ it follows that

$$P\left(X_2 = 2\right) = P\left(S_2 = S_2^{uu}\right) = p^2$$
$$P\left(X_2 = 1\right) = P\left(S_2 = S_2^{ud}\right) = 2p\left(1-p\right)$$
$$P\left(X_2 = 0\right) = P\left(S_2 = S_2^{dd}\right) = \left(1-p\right)^2.$$

2. We have to check if $(X_n)_{n=1,2}$ is a binomial process, i.e. if $X_1 \sim Bin\left(1; p\right)$ and $X_2 \sim Bin\left(2; p\right)$.

   It is immediate to see that $X_1 \sim Bin\left(1; p\right)$. By definition of a binomial variable with parameters $n = 2$ and $p$ and using the probabilities computed above, it follows that $X_2 \sim Bin\left(2; p\right)$.

   Consequently, $(X_n)_{n=1,2}$ is a binomial process.

## 1.3 Proposed Exercises

**Exercise 1.6** Let $(W_t)_{t\geq 0}$ be a standard Brownian motion.

Consider two stocks whose associated gains (difference between current stock price and buying price) evolve as

$$X_t = \mu t + \sigma W_t$$
$$Y_t = \mu_1 t + \sigma_1 W_t,$$

with $X_0 = 1$, $Y_0 = 10$, $\mu = 20$, $\mu_1 = 10$, $\sigma = 10$ and $\sigma_1 = 20$ euros per year.

Taking into account the stocks above, consider also a derivative of value

$$Z_t = X_t^2 Y_t$$

at time $t$.

1. Suppose we buy the derivative only if in 2 years its expected value will exceed $10X_0^2 Y_0$ (that is, 10 times its current price). Decide whether we eventually buy the derivative or not.
2. Compute the probability of having a positive net gain $(Z_1 - Z_0)$ in 1 year or the probability of having a net gain $(Z_2 - Z_1)$ between the first and the second years greater than 100 euros.
3. Compute the probability of both events occurring ($Z_1 - Z_0 > 0$ and $Z_2 - Z_1 > 100$). Are such events independent?

**Exercise 1.7** Let $(W_t)_{t\geq 0}$ be a standard Brownian motion. Consider a stock whose price evolves as the following stochastic process $(Y_t)_{t\geq 0}$:

$$Y_t = Y_0 e^{\mu t + \sigma W_t}.$$

1. Compute $P\left(Y_t^2 \geq Y_t\right)$, when $Y_0 = 4$, $t = 2$, $\mu = 0.1$ and $\sigma = 0.4$.
2. Establish if $Y_{2t}$ and $Y_t^2$ have the same distribution when $Y_0 = 1$.
3. Compute $E\left[\ln\left(\frac{Y_8^2}{Y_{16}}\right)\right]$ and $V\left(\ln\left(\frac{Y_8^2}{Y_{16}}\right)\right)$ when $Y_0 = 1$.

   [Hint: remember that $E\left[W_t W_s\right] = \min\left(s; t\right)$.]

# Chapter 2
# Portfolio Optimization in Discrete-Time Models

## 2.1 Review of Theory

We restrict our interest to one-period models, and in this case the “portfolio”, which we are going to define formally below, is characterized through its composition at the initial time, and the returns of different assets are assumed to be random variables. A systematic exposition of the notions briefly summarized below can be found in the textbooks by Barucci [3], Capiński and Zastawniak [10] and Luenberger [30]. A rich collection of examples and exercises on multi-period models is provided in the textbook [36].

Let us consider a market model where $n$ assets are traded, and whose values at a prescribed date are represented by $n$ random variables $S_1, S_2, \ldots, S_n$. We shall denote by $\mathbf{S}$ the random vector $\mathbf{S} = (S_1, S_2, \ldots, S_n)$ with components $S_1, S_2, \ldots, S_n$.

We define *portfolio* a vector $\mathbf{v} = (v_{1,} v_2, \ldots v_n)$ with each component $v_i$ identifying the number of units of the asset with value $S_i$. The portfolio value $V$ is given by the scalar product of the two vectors $\mathbf{v}$ and $\mathbf{S}$, i.e.:

$$V = \mathbf{v} \cdot \mathbf{S}^T = \sum_{j=1}^{n} v_j S_j .$$

More often a portfolio is characterized by the vector of its relative weights $\mathbf{w} = (w_1, w_2, \ldots w_n)$, where the components $w_i$ are defined by:

$$w_i \triangleq \frac{v_i S_i}{V} = \frac{v_i S_i}{\sum_{j=1}^{n} v_j S_j} . \tag{2.1}$$



E. Rosazza Gianin, C. Sgarra, *Mathematical Finance*,
La Matematica per il 3+2 149, https://doi.org/10.1007/978-3-031-28378-9_2

One can immediately verify that the weights $w_i$ sum up to 1. A negative value of the component $w_i$ denotes a short position in the asset with value $S_i$. The set of portfolios satisfying the normalization condition on the weights $w_i$ is called the set of admissible portfolios.

The *portfolio optimization problem* consists, briefly, in finding a portfolio whose expected utility is the maximum possible among all admissible portfolios. The *utility function* whose expectation must be maximized should be chosen so to describe the investors preferences, and it should take into account their risk-aversion attitude.

A *utility function* is a nondecreasing, concave function $U : \mathbb{R} \to [-\infty, +\infty)$ of class $C^2$. Because of their simplicity, the most commonly adopted utility functions are the following:

- *exponential utility:*

$$U(x) = 1 - \exp(-\alpha x), \quad \alpha > 0; \tag{2.2}$$

- *logarithmic utility:*

$$U(x) = \begin{cases} -\infty, & x \leq 0 \\ \ln(x), & x > 0 \end{cases}; \tag{2.3}$$

- *power utility:*

$$U(x) = \begin{cases} -\infty, & x \leq 0 \\ x^\alpha, & x > 0 \end{cases}, \quad 0 < \alpha < 1. \tag{2.4}$$

In the simplest formulation, the portfolio optimization problem can be reduced to the problem of finding a portfolio with the maximum expected return and the minimal variance, assuming the latter is adopted as a risk measure associated to the portfolio under consideration. A portfolio with minimum variance with respect to all other portfolios with the same expected return, or with maximum expected return with respect to all portfolios with the same variance, is called *efficient*.

If we denote by $r_K$ the return of portfolio $K$, $r_i$ $(i = 1, \ldots, n)$ the return of asset $S_i$, $\mu_i$ its expected value and $\sigma_{ij}$ $(i, j = 1, \ldots, n)$ the covariance between $r_i$ and $r_j$, then the expected return and the variance[1] of portfolio $K$ can be easily proved to be given by:

$$E(r_K) = \sum_{i=1}^{n} w_i E(r_i) = \sum_{i=1}^{n} w_i \mu_i \tag{2.5}$$

$$Var(r_K) = \sum_{i=1}^{n} w_i^2 Var(r_i) + \sum_{\substack{i=1,\dots,n \\ i \neq j}} \sum_{j=1}^{n} w_i w_j Cov(r_i, r_j) = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{ij}. \tag{2.6}$$

The portfolio with minimum variance (with no assigned expected return) can be determined via the following formula:

$$\mathbf{w} = \frac{\mathbf{u}\mathbf{C}^{-1}}{\mathbf{u}\mathbf{C}^{-1}\mathbf{u}^T}, \tag{2.7}$$

where $\mathbf{u}$ is the $n$-dimensional vector with components equal to 1 and $\mathbf{C}$ is the covariance matrix of returns.

The portfolio with minimum variance among those with expected return $\mu_V$ can be determined via the following formula:

$$\mathbf{w} = \frac{\det\begin{pmatrix} 1 & \mathbf{u}\mathbf{C}^{-1}\mathbf{m}^T \\ \mu_V & \mathbf{m}\mathbf{C}^{-1}\mathbf{m}^T \end{pmatrix} \mathbf{u}\mathbf{C}^{-1} + \det\begin{pmatrix} \mathbf{u}\mathbf{C}^{-1}\mathbf{u}^T & 1 \\ \mathbf{m}\mathbf{C}^{-1}\mathbf{u}^T & \mu_V \end{pmatrix} \mathbf{m}\mathbf{C}^{-1}}{\det\begin{pmatrix} \mathbf{u}\mathbf{C}^{-1}\mathbf{u}^T & \mathbf{u}\mathbf{C}^{-1}\mathbf{m}^T \\ \mathbf{m}\mathbf{C}^{-1}\mathbf{u}^T & \mathbf{m}\mathbf{C}^{-1}\mathbf{m}^T \end{pmatrix}}, \tag{2.8}$$

where $\mathbf{m} = (\mu_1, \mu_2, \dots \mu_n)$ is the vector of the expected returns of the $n$ assets.

The set of efficient portfolios with expected return $\mu_V$ can also be characterized as the solution of the following:

$$\begin{cases} \sum_{j=1}^{n} \sigma_{ij} w_j - \lambda_1 \mu_i - \lambda_2 = 0, \quad i = 1, \dots, n \\ \sum_{i=1}^{n} w_i \mu_i = \mu_V \\ \sum_{i=1}^{n} w_i = 1 \end{cases}$$

or, equivalently,

$$\begin{cases} \mathbf{C} \cdot \mathbf{w}^T - \lambda_1 \mathbf{m} - \lambda_2 = 0 \\ \mathbf{w} \cdot \mathbf{m}^T = \mu_V \\ \mathbf{w} \cdot \mathbf{1}^T = 1 \end{cases},$$

[1] In contrast to the other chapters, to avoid confusion with the notation adopted for the portfolio value we shall denote the variance of a random variable by $Var(\cdot)$ and not by $V(\cdot)$.

where $\lambda_1, \lambda_2$ are the Lagrange multipliers relative to the last two constraint equations. The set of efficient portfolios obtained by varying expected returns is called the *efficient frontier*.

Sometimes the portfolio optimization problem includes some constraints on the portfolio weights; the most typical example of this situation is the *no-short-selling* constraint that requires all the portfolio weights to be nonnegative, i.e. $w_i \geq 0$ for $i = 1, \ldots, n$. When the portfolio optimization problem consists in variance minimization with a prescribed expected return, and the no-short-selling constraints are imposed, the problem can be identified as a *quadratic programming* problem. This kind of problem, in general, cannot be solved explicitly without the support of a computer program (there are several available on the market), unless the dimension of the problem is small enough to allow explicit calculations.

The *Mutual Funds Theorem* is another fundamental result of mathematical portfolio theory: it states that given a market with $n$ assets, it is always possible to determine two *funds* (i.e. two portfolios) such that every efficient portfolio (as far as the expected return and variance are concerned) can be expressed as a linear combination of them. In order to compute these two portfolios, it is sufficient to find two solutions of the optimization problem for two arbitrary values of the Lagrange multipliers $\lambda_1, \lambda_2$. In general the two solutions determined by this procedure will violate the normalization condition on the weights, but this can be easily amended by simply dividing the weights obtained by their sum. We shall provide an example in order to illustrate the method just outlined.

The *Capital Asset Pricing Model* (CAPM) assumes as a starting point the linear dependence of the return of a generic portfolio $K$ (the model postulates the same relation for all different returns of each asset) on the return $r_M$ of a benchmark portfolio, called *market portfolio*:

$$r_K = r_f + \beta_K (r_M - r_f) + \varepsilon_i, \tag{2.9}$$

where $r_f$ is the risk-free interest rate and $\varepsilon_i$ is a zero-mean Gaussian random variable. By simply taking the expectation of each side of the previous equality, one obtains:

$$\mu_K = E(r_K) = r_f + \beta_K (\mu_M - r_f). \tag{2.10}$$

The (straight) regression line for the present model is identified by the coefficients $\beta_K$, $\alpha_K$, where $\beta_K$ and $\alpha_K$ can be obtained by the following formulas:

$$\beta_K = \frac{Cov(r_M, r_K)}{\sigma_M^2} \tag{2.11}$$

$$\alpha_K = \mu_K - \beta_K \mu_M. \tag{2.12}$$

## 2.2 Solved Exercises

**Exercise 2.1** Determine the expected return and the variance of the portfolio formed by the two assets $S_1$, $S_2$ with weights $w_1 = 0.6$ and $w_2 = 0.4$. The assets' returns are described by the following scheme:

| Scenario | Probability | $r_1$ | $r_2$ |
|---|---|---|---|
| $\omega_1$ | 0.1 | −20% | −10% |
| $\omega_2$ | 0.4 | 0% | 20% |
| $\omega_3$ | 0.5 | 20% | 40% |

**Solution** Since the expected returns of the two assets are given by:

$$E(r_1) = 0.1 \cdot (-0.2) + 0.4 \cdot 0 + 0.5 \cdot 0.2 = 0.08$$
$$E(r_2) = 0.1 \cdot (-0.1) + 0.4 \cdot 0.2 + 0.5 \cdot 0.4 = 0.27,$$

the expected return of the portfolio $K$ with weights $w_1 = 0.6$ and $w_2 = 0.4$ for the assets $S_1$ and $S_2$ is then:

$$E(r_K) = w_1 E(r_1) + w_2 E(r_2) = 0.6 \cdot 0.08 + 0.4 \cdot 0.27 = 0.156.$$

In order to calculate the portfolio variance it is necessary to know the covariance matrix of the two returns. In the present case, we obtain:

$$\begin{aligned}
Var(r_1) &= E\left(r_1^2\right) - (E(r_1))^2 \\
&= 0.1 \cdot (-0.2)^2 + 0.4 \cdot 0^2 + 0.5 \cdot (0.2)^2 - (0.08)^2 = 0.0176 \\
Var(r_2) &= E\left(r_2^2\right) - (E(r_2))^2 \\
&= 0.1 \cdot (-0.1)^2 + 0.4 \cdot (0.2)^2 + 0.5 \cdot (0.4)^2 - (0.27)^2 = 0.0241 \\
Cov(r_1, r_2) &= E(r_1 r_2) - E(r_1)E(r_2) \\
&= 0.1 \cdot (-0.2)(-0.1) + 0.4 \cdot 0 \cdot 0.2 + 0.5 \cdot 0.2 \cdot 0.4 - 0.08 \cdot 0.27 \\
&= 0.0204
\end{aligned}$$

Hence

$$\begin{aligned}
Var(r_K) &= w_1^2 Var(r_1) + w_2^2 Var(r_2) + 2w_1 w_2 Cov(r_1 r_2) \\
&= (0.6)^2 \cdot 0.0176 + (0.4)^2 \cdot 0.0241 + 2 \cdot 0.6 \cdot 0.4 \cdot 0.0204 = 0.019984.
\end{aligned}$$

**Exercise 2.2** Suppose that the returns $r_1, r_2$ of the two assets $S_1, S_2$ are as in the following scheme:

| Scenario | Probability | $r_1$ | $r_2$ |
|---|---|---|---|
| $\omega_1$ | 0.2 | −10% | 10% |
| $\omega_2$ | 0.3 | 5% | −2% |
| $\omega_3$ | 0.5 | 20% | 15% |

1. Compare the risk (assume the variance as a risk measure) of a portfolio $K$ composed by the two assets above with weights $w_1 = 0.3$, $w_2 = 0.7$, with the risk of the two single assets considered separately.
2. By using the same table, determine the composition of a portfolio $K^*$ still consisting of the two previous assets, but with expected return equal to 9%. Calculate the variance of the portfolio $K^*$.

**Solution**

1. In strict analogy with Exercise 2.1, one obtains easily

$$\begin{aligned} &E(r_1) = 0.095; \qquad && E(r_2) = 0.089 \\ &Var(r_1) = 0.0137; \qquad && Var(r_2) = 0.0054 \\ &Cov(r_1, r_2) = 0.0042. \end{aligned}$$

Then

$$\rho_{12} = \frac{Cov(r_1, r_2)}{\sqrt{Var(r_1)Var(r_2)}} = 0.49.$$

The variance of the portfolio $K$ is then given by:

$$\begin{aligned} Var(r_K) &= w_1^2 Var(r_1) + w_2^2 Var(r_2) + 2\rho_{12} w_1 w_2 \sqrt{Var(r_1)Var(r_2)} \\ &= (0.3)^2 \cdot 0.0137 + (0.7)^2 \cdot 0.0054 \\ &\quad +2 \cdot 0.49 \cdot 0.3 \cdot 0.7 \cdot \sqrt{0.0137 \cdot 0.0054} \\ &= 0.0056. \end{aligned}$$

2. We have to find a new portfolio $K^*$ with expected return equal to 9%. More precisely, we have to compute the weights corresponding to the two assets in the new portfolio $K^*$.

Reminding that $E(r_1) = 0.095$ and $E(r_2) = 0.089$, the two required weights $w_1^*, w_2^*$ must satisfy the following system of equations:

$$\begin{cases} w_1^* \cdot 0.095 + w_2^* \cdot 0.089 = 0.09 \\ w_1^* + w_2^* = 1 \end{cases}.$$

The solution is then given by

$$w_1^* = \frac{1}{6}, \quad w_2^* = \frac{5}{6}.$$

The weights for the two assets $S_1$ and $S_2$ in the new portfolio $K^*$ are then $w_1^* = \frac{1}{6}$ and $w_2^* = \frac{5}{6}$.

The variance of the return of $K^*$ is then given by:

$$\begin{aligned} Var(r_{K^*}) &= \left(w_1^*\right)^2 Var(r_1) + \left(w_2^*\right)^2 Var(r_2) + 2w_1^* w_2^* Cov(r_1 r_2) \\ &= (1/6)^2 \cdot 0.0137 + (5/6)^2 \cdot 0.0054 + 2 \cdot \frac{5}{36} \cdot 0.0042 = 0.0053. \end{aligned}$$

**Exercise 2.3** Let three risky assets be given. Their returns have expectations and covariance matrix described in the following scheme:

| | | |
|---|---|---|
| $\mu_1 = 0.20$ | $\sigma_1 = 0.25$ | $\rho_{12} = -0.20$ |
| $\mu_2 = 0.12$ | $\sigma_2 = 0.30$ | $\rho_{23} = 0.50$ |
| $\mu_3 = 0.15$ | $\sigma_3 = 0.22$ | $\rho_{13} = 0.30$ |

Find the minimum variance portfolio and compute its expected return and variance.

**Solution** Let us recall the formula providing the weight vector $\mathbf{w}$ of the minimum variance portfolio:

$$\mathbf{w} = \frac{\mathbf{u}\mathbf{C}^{-1}}{\mathbf{u}\mathbf{C}^{-1}\mathbf{u}^T},$$

where $\mathbf{C}$ is the returns covariance matrix and $\mathbf{u}$ is the vector with all components equal to 1. In the present case, we have:

$$\mathbf{C} = \begin{pmatrix} 0.0625 & -0.015 & 0.0165 \\ -0.015 & 0.09 & 0.033 \\ 0.0165 & 0.033 & 0.048 \end{pmatrix}.$$

Since the inverse matrix $\mathbf{C}^{-1}$ of $\mathbf{C}$ is:

$$\mathbf{C}^{-1} = \begin{pmatrix} 21.497 & 8.4132 & -13.174 \\ 8.4132 & 18.1487 & -15.369 \\ -13.174 & -15.369 & 35.928 \end{pmatrix},$$

we can easily obtain the weights of the minimum variance portfolio:

$$w_1 = 0.474, \quad w_2 = 0.317, \quad w_3 = 0.209.$$

The expected return and the variance of this portfolio are then

$$\begin{aligned}\mu_V &= w_1\mu_1 + w_2\mu_2 + w_3\mu_3 = 0.164\\ \sigma_V^2 &= w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + w_3^2\sigma_3^2 + 2w_1w_2Cov\,(r_1, r_2)\\ &\quad +2w_1w_3Cov\,(r_1, r_3) + 2w_2w_3Cov\,(r_2, r_3)\\ &= 0.0283.\end{aligned}$$

**Exercise 2.4** Consider a portfolio $K$ with return $r_K$ and the market portfolio $K_M$ with return $r_M$, where $r_K$ and $r_M$ take the following values:

| Scenario | Probability | $r_K$ | $r_M$ |
|---|---|---|---|
| $\omega_1$ | 0.3 | 3% | 8% |
| $\omega_2$ | 0.2 | 2% | 7% |
| $\omega_3$ | 0.3 | 4% | 10% |
| $\omega_4$ | 0.2 | 1% | 9% |

Find the coefficients $\beta_K$ and $\alpha_K$ of the regression line.

**Solution** From the table above we obtain

$$\begin{aligned}&\mu_K = E(r_K) = 0.027; \ \mu_M = E(r_M) = 0.086;\\ &\sigma_M^2 = 0.000124; \qquad Cov(r_K, r_M) = 0.000058.\end{aligned}$$

By applying directly the formulas providing the coefficients $\beta_V$ and $\alpha_V$, we obtain

$$\begin{aligned}\beta_V &= \frac{Cov(r_K, r_M)}{\sigma_M^2} = \frac{0.000058}{0.000124} = 0.467\\ \alpha_K &= \mu_K - \beta_K\mu_M = 0.027 - 0.467 \cdot 0.086 = -0.01316.\end{aligned}$$

**Exercise 2.5**

1. Consider two assets $S_1$, $S_2$ with jointly normally distributed returns of expected value and standard deviation $E(r_1) = \mu_1 = 0.20$, $E(r_2) = \mu_2 = 0.16$, $\sigma_1 = \sqrt{Var(r_1)} = 0.30$, $\sigma_2 = \sqrt{Var(r_2)} = 0.36$, respectively, and with correlation coefficient $\rho_{12} = -0.5$.

Determine the portfolio maximizing the expected utility, assumed to be the exponential utility function:

$$U(x) = 1 - \exp(-\alpha x)$$

with parameter $\alpha = 0.04$.

2. Consider now a new portfolio $J$ made by the same asset $S_1$ of the previous item and another asset $B$ (riskless) with return $r_B = 0.06$. Find the optimal composition of the new portfolio with respect to the same utility function as in item 1. (exponential utility).

**Solution**

1. Denote by $K$ the portfolio with weights $w_1 = x$ (in the asset $S_1$) and $w_2 = 1 - x$ (in the asset $S_2$). The expectation and the variance of the return of $K$ are given by

$$\begin{aligned} E[r_K] &= E\left[w_1 r_1 + w_2 r_2\right] = 0.20 \cdot x + 0.16 \cdot (1 - x) = 0.16 - 0.04x \\ Var(r_K) &= Var\left(w_1 r_1 + w_2 r_2\right) \\ &= (0.3)^2 x^2 + (0.36)^2 (1 - x)^2 - 2 \cdot 0.5 \cdot 0.30 \cdot 0.36 \cdot x(1 - x) \\ &= 0.3276 \cdot x^2 - 0.828 \cdot x + 0.1296. \end{aligned}$$

If a random variable $Z$ is normally distributed, then $\exp(Z)$ is distributed according to a log-normal and the following relation holds:

$$E\left[\exp(Z)\right] = \exp\left\{E(Z) + \frac{1}{2} Var(Z)\right\}.$$

Since maximizing the expectation of the exponential utility is equivalent to maximizing the quantity:

$$E[U(Z)] = 1 - E\left[\exp(-\alpha Z)\right] = 1 - \exp\left\{-\alpha E(Z) + \frac{1}{2}\alpha^2 Var(Z)\right\},$$

under the assumption of jointly normally distributed returns of both assets, it is enough to maximize the following expression:

$$E[Z] - \frac{1}{2}\alpha Var(Z).$$

In the present case $Z = r_K$, so we need to find the value $x$ maximizing the second-order polynomial:

$$\begin{aligned} g(x) &= 0.16 - 0.04x - \tfrac{0.04}{2}(0.3276x^2 - 0.828x + 0.1296) \\ &= 0.1574 - 0.0234x - 0.0066x^2. \end{aligned}$$

It is easy to obtain that the optimal point is $x^* = -1.776$.

2. The expected return of the new portfolio $J$, identified via its weights $w_1 = y$ and $w_2 = 1 - y$, is then:

$$E[r_J] = E\,[w_1 r_1] + w_2 r_B = 0.20 \cdot y + (1-y) \cdot 0.06, \tag{2.13}$$

and its variance is:

$$Var(r_J) = (0.3)^2 \cdot y^2. \tag{2.14}$$

The new function to be maximized is now the following:

$$f(y) = 0.14 \cdot y + 0.06 - \frac{0.04}{2}\,(0.3 \cdot y)^2 = -0.0018 \cdot y^2 + 0.14 \cdot y + 0.06. \tag{2.15}$$

It is immediate to verify that the maximum point is $y^* = 38.89$. The corresponding weight for the riskless asset is $1 - y^* = -37.89$, that corresponds to a short position in the non-risky asset $B$.

**Exercise 2.6** Consider the market consisting of three assets $S_1, S_2, S_3$ with returns $r_1, r_2, r_3$, uncorrelated and all with unit variance, with expectations $E[r_1] = 1/4$, $E[r_2] = 1/2$, $E[r_3] = 3/4$, respectively.

Determine the efficient portfolio with expected return $\hat{r}$, under the constraint of no short selling (i.e. $w_i \geq 0$ for $i = 1, 2, 3$).

**Solution** Since unilateral constraints are imposed, the problem cannot be solved by means of a system of equations. It is a quadratic optimization problem with linear constraints. Since the number of assets involved is small, the problem can be solved by considering all different pairs of assets.

Denote by $K$ the portfolio with weight $w_i$ in the asset $S_i$ (for $i = 1, 2, 3$) and by $r_K$ its return. Since assets' returns are uncorrelated and all with unit variance, the expectation and the variance of $r_K$ are given by

$$\begin{aligned} E[r_K] &= w_1 E[r_1] + w_2 E[r_2] + w_3 E[r_3] \\ &= w_1 \cdot \frac{1}{4} + w_2 \cdot \frac{1}{2} + (1 - w_1 - w_2) \cdot \frac{3}{4} \\ &= \frac{3}{4} - \frac{w_1}{2} - \frac{w_2}{4}; \end{aligned}$$

$$
\begin{aligned}
Var(r_K) &= Var\ (w_1 r_1 + w_2 r_2 + w_3 r_3) \\
&= w_1^2 V(r_1) + w_2^2 V(r_2) + w_3^2 V(r_3) \\
&= w_1^2 + w_2^2 + (1 - w_1 - w_2)^2 \\
&= 2w_1^2 + 2w_2^2 + 1 - 2w_1 - 2w_2 + 2w_1 w_2.
\end{aligned}
$$

Set $x \triangleq w_1$ and $y \triangleq w_2$.

Our goal is to find a portfolio with return $\hat{r}$ and with minimum variance. Notice, however, that (because of the no-short-selling constraints) only target returns $\hat{r} \in [\frac{1}{4}; \frac{3}{4}]$ are admissible.

By the condition on the target return, we get

$$
\frac{3}{4} - \frac{x}{2} - \frac{y}{4} = \hat{r},
$$

so $y = 3 - 2x - 4\hat{r}$. We have thus to minimize (under the no-short-selling constraints) the following

$$
\begin{aligned}
Var(r_K) &= 2x^2 + 2y^2 + 1 - 2x - 2y + 2xy \\
&= x^2 + 2(3 - 2x - 4\hat{r})^2 + 1 - 2x - 2(3 - 2x - 4\hat{r}) + 2x(3 - 2x - 4\hat{r}) \\
&= 6x^2 - 16x + 24x\hat{r} + 12 + 32\hat{r}^2 - 40\hat{r} \triangleq f(x).
\end{aligned}
$$

Since $f'(x) = 12x - 16 + 24\hat{r}$, the weights

$$
\begin{cases}
x^* = \frac{4}{3} - 2\hat{r} \\
y^* = 3 - 2x^* - 4r = \frac{1}{3} \\
z^* = 1 - x^* - y^* = 2\hat{r} - \frac{2}{3}
\end{cases}
\tag{2.16}
$$

are optimal if admissible. It is easy to check that the optimal weights above satisfy all the constraints if and only if $\frac{1}{3} \le \hat{r} \le \frac{2}{3}$. In such a case, the efficient portfolio weights are the following:

$$
w_1 = \frac{4}{3} - 2\hat{r},\ w_2 = \frac{1}{3},\ w_3 = 2\hat{r} - \frac{2}{3},
$$

and the portfolio mean square deviation turns out to be $\sigma = \sqrt{31/9 - 8\hat{r} + 8\hat{r}^2}$.

Take $\frac{2}{3} < \hat{r} \le \frac{3}{4}$. Since $f'(x) = 12x - 16 + 24\hat{r} \ge 0$ for any $x \in [0, 1]$, it is easy to check that the efficient portfolio weights are the following:

$$
w_1 = 0,\ w_2 = 3 - 4r,\ w_3 = 4\hat{r} - 2
$$

and the portfolio mean square deviation turns out to be $\sigma = \sqrt{13 - 40\hat{r} + 32\hat{r}^2}$.

Take now $\frac{2}{3} < \hat{r} \leq \frac{3}{4}$. Since for any $x \in [0, 1]$, it is easy to check that the efficient portfolio weights are the following:

$$w_1 = 0, w_2 = 3 - 4\hat{r}, w_3 = 4\hat{r} - 2,$$

and the portfolio mean square deviation turns out to be $\sigma = \sqrt{13 - 40\hat{r} + 32\hat{r}^2}$.

Finally, take $\frac{1}{4} \leq \hat{r} < \frac{1}{3}$. In order to have admissible weights one should have

$$\begin{cases} x \in [0, 1] \\ y = 3 - 2x - 4r \in [0, 1] \\ z = 1 - x - y = x + 4\hat{r} - 2 \in [0, 1] \end{cases} \tag{2.17}$$

so $2 - 4\hat{r} \leq x \leq \frac{3}{2} - 2\hat{r}$. Since $\frac{4}{3} - 2\hat{r} \leq 2 - 4\hat{r}$ for $\hat{r} \leq \frac{1}{3}$, we deduce that the efficient portfolio weights are the following:

$$w_1 = 2 - 4\hat{r}, w_2 = 4\hat{r} - 1, w_3 = 0$$

and the portfolio mean square deviation turns out to be $\sigma = \sqrt{5 - 24\hat{r} + 32\hat{r}^2}$.

**Exercise 2.7** Consider a portfolio composed by five assets $S_1, S_2, S_3, S_4, S_5$ with expected returns $E[r_1] = 0.183$, $E[r_2] = 0.085$, $E[r_3] = 0.121$, $E[r_4] = 0.112$, $E[r_5] = 0.096$, respectively, and with return covariance matrix

$$\mathbf{C} = \begin{pmatrix} 0.0690 & 0.0279 & 0.0186 & 0.0222 & -0.0069 \\ 0.0279 & 0.042 & 0.0066 & 0.0168 & 0.0078 \\ 0.0186 & 0.0066 & 0.054 & 0.0234 & -0.0081 \\ 0.0222 & 0.0168 & 0.0234 & 0.102 & -0.168 \\ -0.0069 & 0.0078 & -0.0081 & -0.168 & 0.078 \end{pmatrix}. \tag{2.18}$$

Determine two funds whose linear combinations generate any efficient portfolio.

**Solution** The two funds we are looking for must belong to the set of portfolios with minimum variance. We briefly recall that the efficient portfolio (with expected return $\mu_K$) weights must satisfy the following linear system:

$$\begin{cases} \mathbf{C} \cdot \mathbf{w}^T - \lambda_1 \mathbf{m} - \lambda_2 = 0, \\ \mathbf{w} \cdot \mathbf{m}^T = \mu_K, \end{cases} \tag{2.19}$$

together with the normalization condition:

$$\sum_{j=1}^{n} w_j = 1. \tag{2.20}$$

Here $w_i$ are the portfolio weights, $\mathbf{w}$ in vector notation, and $\mu_i = E[r_i]$ the expected returns of assets $S_i$, $\mathbf{m}$ in vector notation. $\mathbf{C}$ is the returns' covariance matrix. The Two Funds theorem implies that, in order to compute an efficient portfolio for every value of $\mu_K$, it is sufficient to determine two solutions of (2.19) and (2.20), since every other solution is a linear combination of them. A simple method to determine two solutions is that of specifying two possible values for the Lagrange multipliers $\lambda_1$ and $\lambda_2$, for example we can pick the solution with $\lambda_1 = 0$ and $\lambda_2 = 1$, and the solution with $\lambda_1 = 1$ and $\lambda_2 = 0$. If we look for the solution with Lagrange multipliers $\lambda_1 = 0$, $\lambda_2 = 1$, we have to solve the following linear system:

$$\mathbf{C} \cdot \mathbf{v_1}^T = 1. \tag{2.21}$$

The solution can be easily found to be the following:

$$v_1 = (2.1191; 26.1615; 17.6417; -8.5490; -5.9895).$$

In order to compute the relative weights $w_{1,i}$ we must impose the normalization condition on the components of the vector $v_1$, i.e. we must divide each component by their sum:

$$w_{1,i} = \frac{v_i}{\sum_{j=1}^{n} v_j}.$$

We obtain then:

$$w_1 = (0.0721; 0.8233; 0.6004; -0.2909; -0.2038).$$

In order to determine the solution with $\lambda_1 = 1$ and $\lambda_2 = 0$ we must solve the second linear system:

$$\mathbf{C} \cdot \mathbf{v_2}^T = \mathbf{m}. \tag{2.22}$$

We obtain:

$$v_2 = (2.0736; 0.9003; 1.7540; -1.0116; -0.6725),$$

that, after normalization, provides the second fund expressed by its relative weights:

$$w_2 = (0.6813; 0.2958; 0.5763; -0.3324; -0.2210).$$

## 2.3 Proposed Exercises

**Exercise 2.8** Given two risky assets whose values are normally distributed with $S_1 \sim N(3; 0.5)$, $S_2 \sim N(4; 0.7)$ and with correlation $\rho_{12} = 0.5$, determine the efficient frontier and the optimal portfolio with respect to the exponential utility function with parameter $\alpha = 1.5$.

**Exercise 2.9** In the set of all admissible portfolios composed by three assets of expected returns $\mu_1 = 0.30$, $\mu_2 = 0.15$, $\mu_3 = 0.18$, with standard deviation $\sigma_1 = 0.22$, $\sigma_2 = 0.30$, $\sigma_3 = 0.26$, and correlations $\rho_{12} = 0.34$, $\rho_{23} = 0.02$, $\rho_{13} = 0.25$, find the minimum variance portfolio and compute its expected return and variance.

**Exercise 2.10** In the set of all admissible portfolios composed by the three assets of the previous exercise and with expected return $\mu_V = 20\%$, determine the minimum variance portfolio and compute its variance.

**Exercise 2.11** Assume that the risk-free interest rate is $r_f = 6\%$, the expectation and the variance of the market portfolio are 10% and 20%, respectively, and the correlation between a given asset and the market portfolio return is 0.5. Compute the expected return of the asset considered in a CAPM framework.

**Exercise 2.12** Consider a market with three assets $S_1$, $S_2$, $S_3$ with expected returns $E[r_1] = 1/8$, $E[r_2] = 1/4$, $E[r_3] = 3/8$, respectively. Suppose the returns are uncorrelated and all with unit variance. Find the efficient portfolio with expected return $\hat{r} = \frac{1}{3}$ when short-selling is allowed and also when it is not.

**Exercise 2.13** Consider a portfolio composed by four assets $S_1$, $S_2$, $S_3$, $S_4$ with expected returns $E[r_1] = 0.112$, $E[r_2] = 0.076$, $E[r_3] = 0.191$, $E[r_4] = 0.125$, and with return covariance matrix

$$\mathbf{C} = \begin{pmatrix} 0.1024 & 0.05376 & -0.0144 & 0.03456 \\ 0.05376 & 0.0441 & 0.0189 & 0.00756 \\ -0.0144 & 0.0189 & 0.2025 & -0.1296 \\ 0.03456 & 0.00756 & -0.1296 & 0.1296 \end{pmatrix}. \tag{2.23}$$

Determine two funds able to generate every efficient portfolio by linear combination.

# Chapter 3
# Binomial Model for Option Pricing

## 3.1 Review of Theory

In the following, we consider a market model where a non-risky asset (called *bond*) and a risky asset (called *stock*) are available. The bond price is denoted by $B$, while the stock price is denoted by $S$.

Let us focus, first, on a one-period model. This means that any asset on the market has to be evaluated just at the beginning and at the end of the given time interval. By convention, let $t = 0$ and $t = 1$ be the corresponding dates.

The initial prices of the (non-risky and risky) assets are known: $B_0$ for the bond, $S_0$ for the stock. The bond price is deterministic, equal to

$$B_1 = B_0(1 + r)$$

at $t = 1$, where $r$ is the risk-free interest rate and the rate is compounded once at the end of the period. The stock price at $t = 1$ is given by

$$S_1 = S_0 X,$$

where $X$ indicates a Bernoulli random variable that may assume $u$ and $d$, with probability $p$ and $(1 - p)$, respectively.

The market model described above (known as *one-period binomial* model) can be summarized as follows:

For $d < 1 + r < u$, the market above is *free of arbitrage*. Roughly speaking, it is not possible to have a profit for free (see Chap. 4 for details on arbitrage on more general market models). As the binomial model is arbitrage-free, the price of any financial derivative (or contingent claim) on the underlying $S$ is uniquely determined. This fact can be seen in two different ways. The first is based on the construction of a portfolio composed by one derivative and by a suitable number of stock shares rendering the portfolio riskless (hence its dynamics are deterministic).

 
E. Rosazza Gianin, C. Sgarra, *Mathematical Finance*,
La Matematica per il 3+2 149, https://doi.org/10.1007/978-3-031-28378-9_3

| | | |
|---|---|---|
| Bond | $B_0$ | $B_1 = B_0(1+r)$ |
| Stock | $S_0$ | $S_1 = \begin{cases} S_0u = S_1^u; & \text{with prob. } p \\ S_0d = S_1^d; & \text{with prob. } (1-p) \end{cases}$ |
| | $t=0$ | $t=1$ |

Such a strategy is called *Delta-Hedging* strategy. The second is based on the construction of a portfolio (composed by a suitable number of bonds and stock shares) "replicating" the value of the derivative, that is assuming at maturity $T$ the same value of the derivative in any state of the world. This last strategy is called *replicating strategy* (see Chap. 4 for more details).

Under the no-arbitrage assumption, the initial cost $F(S_0)$ of the derivative written on the stock $S$ is given by the following formula:

$$F(S_0) = \frac{1}{1+r}\left[q_u \cdot F(S_0u) + q_d \cdot F(S_0d)\right], \tag{3.1}$$

where $q_u = \frac{(1+r)-d}{u-d}$, $q_d = \frac{u-(1+r)}{u-d} = 1 - q_u$, and $F(S_0u)$ and $F(S_0d)$ denote the payoff of the derivative when the price of the underlying is equal to $S_0u$ and $S_0d$, respectively. In the following, we will write $\phi^u = F(S_0u)$ and $\phi^d = F(S_0d)$.

As we could expect, the initial price of the derivative depends on its payoff $F(S_1)$ at maturity.

Since $q_u, q_d$ verify $q_u, q_d \geq 0$ and $q_u + q_d = 1$, they can be seen as the new probabilities associated to the values $u, d$ assumed by the random variable $X$. The derivative price in formula (3.1) can be thus interpreted as the expected value of the discounted payoff of the derivative with respect to the new probability measure induced by $q_u, q_d$. In other words

$$F(S_0) = \frac{1}{1+r}E_Q\left[F(S_1)\right]. \tag{3.2}$$

The new probability measure $Q$ is called *risk-neutral measure* or *equivalent martingale measure*. By definition of $q_u$ and $q_d$, the expected value (with respect to $Q$) of the discounted final value of the stock is indeed equal to its current value:

$$\frac{1}{1+r}\left[q_u S_0 u + q_d S_0 d\right] = S_0, \tag{3.3}$$

which means that the discounted price process of the stock is a martingale under $Q$.

The above market model can be generalized to a multi-period setting as follows. Consider a discrete-time model where a bond and a stock are traded. In each period (from $t$ to $t+1$, for $t = 0, 1, \dots, T-1$), the bond price and the stock price are assumed to evolve as in a one-period binomial model. In other words:

$$
\begin{array}{ccccccc}
 & & & & & & S_n^{n\ up;0\ down} \\
 & & & & & \nearrow^{p} & \\
 & & & & & \searrow_{1-p} & \\
 & & & & \cdots & & S_n^{(n-1)\ up;1\ down} \\
 & & & \nearrow^{p} & & & \\
 & & S_0 u = S_1^u & & & \vdots & \vdots \\
 & \nearrow^{p} & & & & \vdots & \vdots \\
S_0 & & & & & & \\
 & \searrow_{1-p} & & & & \vdots & \vdots \\
 & & S_0 d = S_1^d & & & & \\
 & & & \searrow_{1-p} & & & S_n^{1\ up;(n-1)\ down} \\
 & & & & \cdots & \nearrow^{p} & \\
 & & & & & \searrow_{1-p} & \\
 & & & & & & S_n^{0\ up;n\ down}
\end{array}
$$

The market described above is known as *multi-period binomial model*.

By the Markov property fulfilled by the model, it is easy to check that under the risk-neutral measure $Q$

$$
S_k = \frac{1}{(1+r)^{l-k}} E_Q \left[ S_l | S_k \right] = \frac{1}{(1+r)^{l-k}} E_Q \left[ S_l | \mathscr{F}_k \right], \quad 0 \le k \le l \le n. \tag{3.4}
$$

Once discounted, the process $(S_k)_{k=0,1,\dots,n}$ is then a martingale under $Q$ and with respect to the filtration $(\mathscr{F}_k)_{k=0,1,\dots,n}$ generated by $(S_k)_{k=0,1,\dots,n}$.

The pricing method for the derivative $F(S)$ is based on the assumption that, in the binomial model, any derivative is *attainable*, namely there always exists a strategy consisting in a suitable number of bonds and of stocks replicating exactly the derivative value at any time and in any state of the world. Or, equivalently, there always exists a portfolio strategy composed by a suitable amount of the derivative and the underlying, so to "completely hedge" the risk. Such a property is called *completeness* of the market (see Chap. 4 for more details on completeness). More precisely, a market model is said to be complete if any derivative can be attained (or replicated).

Both the one-period and the multi-period binomial market models described above are complete. In the multi-period binomial model, the initial price $F(S_0)$ of the derivative is given by the expected value of the discounted payoff under the risk-neutral measure $Q$, i.e.

$$F(S_0) = \frac{1}{(1+r)^n} E_Q\left[F(S_n)\right] = \frac{1}{(1+r)^n} \sum_{k=0}^{n} \binom{n}{k} q_u^k q_d^{n-k} F(S_0 u^k d^{n-k}). \qquad (3.5)$$

Necessary and sufficient conditions for a (quite general) market model to be free of arbitrage and/or complete are given by the *Fundamental Theorems of the Asset Pricing*. What just recalled in a binomial model can be seen as a particular case of these conditions.

The *First Fundamental Theorem of the Asset Pricing* establishes that, in "suitable" market models, no-arbitrage and existence of an equivalent martingale measure are equivalent conditions.

For a market that is free of arbitrage, the *Second Fundamental Theorem of the Asset Pricing* guarantees the uniqueness of the equivalent martingale measure when the market model is complete.

As already underlined, the binomial model is complete. Nevertheless, if at any time the risky asset could assume not only two but three (or more) different values, such new market model would not any longer be complete. As we will see in Chap. 4, in such a case, indeed, an arbitrary derivative could not be attainable only by means of the bond $B$ and of the risky asset $S$.

European Put and Call options (sometimes called vanilla options) and American options, i.e. options that can be exercised at any time between the initial 0 and the maturity $T$, can be easily evaluated in a binomial model. Concerning European options, it is sufficient to apply the arguments above to the payoff of a European Call (respectively, Put) option with strike $K$, maturity $T$ and written on the stock $S$:

$$F_{Call}(S_T) = (S_T - K)^+, \qquad \text{resp. } F_{Put}(S_T) = (K - S_T)^+.$$

It is also worth to mention that the price (at time $t = 0, 1, \dots, T$) of a European Call option ($C_t$) and of a European Put option ($P_t$) with the same characteristics are related by the so called *Put-Call Parity*:

$$C_t - P_t = S_t - \frac{K}{(1+r)^{T-t}}. \qquad (3.6)$$

The pricing of some other options whose value depends not only on the underlying value at maturity, but also on its value at intermediate times between 0 and $T$, will be illustrated in some exercises of the present chapter and of the next ones.

Based on the arguments above, it is also possible to obtain the strategy able to make the portfolio riskless between two consecutive times: it would be composed by a long position in the derivative and by $\Delta = (F_u - F_d)/(u - d)$ short positions in the underlying.

For a detailed treatment and further details on the subject, we refer to Björk [6], Hull [25], Pliska [37] and Ross [39], among others. A rich collection of examples and exercises on multi-period models is provided in the textbook [36].

## 3.2 Solved Exercises

**Exercise 3.1** Different options having the same stock as underlying are available on the market. Suppose that the risk-free interest rate is 4% per year, that the current stock price is 20 euros and that such price may go up or down by 25% in each of the next 2 semesters.

1. Assume that the stock price may go up or down in each semester with (objective) probability of 50%. Verify whether the exercise of a European Call option with strike of 18 euros and with maturity of 6 months is more likely than the exercise of a Call option as the one above, but with maturity of 1 year.
2. Compare the prices of the Call options of item 1.
3. Discuss whether the options prices of items 1. and 2. would change if:
   (a) the objective probability of an increase in the stock price was of 80% (consequently, 20% of a decrease);
   (b) if the strike was of 20 euros. If yes, compute the price of a European Call option with maturity 6 of months and with strike of 20 euros.

**Solution** We deduce that the stock price evolves as follows:

$$
\begin{array}{ccccc}
 & & & & 31.25 = S_0u^2 \\
 & & & \nearrow & \\
 & & S_0u = 25 & & \\
 & \nearrow & & \searrow & \\
S_0 = 20 & & & & 18.75 = S_0ud \\
 & \searrow & & \nearrow & \\
 & & S_0d = 15 & & \\
 & & & \searrow & \\
 & & & & 11.25 = S_0d^2 \\
 & & & & \\
---- & & ------- & & ------- \\
0 & & 6\,\text{months} & & T = 1\,\text{year}
\end{array}
$$

where $u = 1.25$ (growth factor) and $d = 0.75$.

1. Remember that a European Call option is exercised when the underlying price is greater (or equal) than the strike. With this fact in mind, it is easy to compute the probability to exercise the options above.

   Before doing that, it is worth to underline that the behavior of the stock price under the "real" probability is

$$
\begin{array}{ccc|c}
 & & 25 & p \\
 & \nearrow & & \\
20 & & & \\
 & \searrow & & \\
 & & 15 & 1-p \\
\hline
0 & & T' = 6\,\text{months} & \text{with probability}
\end{array}
$$

$$
\begin{array}{ccc|c}
 & & 31.25 & p^2 \\
 & \nearrow & & \\
20 & \rightarrow & 18.75 & 2p(1-p) \\
 & \searrow & & \\
 & & 11.25 & (1-p)^2 \\
\hline
0 & & T = 1\,\text{year} & \text{with probability}
\end{array}
$$

   where $p = 1 - p = 0.5$.

   It follows that the option with maturity ($T'$) of 6 months and with strike of 18 euros is exercised with probability

$$P\left(S_{T'} \geq K\right) = P\left(S_{T'} \geq 18\right) = P\left(S_{T'} = 25\right) = 0.5,$$

   while the option with maturity ($T$) of 1 year and with strike of 18 euros is exercised with probability

$$
\begin{aligned}
P\left(S_T \geq K\right) &= P\left(S_T \geq 18\right) \\
&= P\left(\{S_T = 18.75\} \cup \{S_T = 31.25\}\right) \\
&= p^2 + 2p\left(1-p\right) = (0.5)^2 + 2 \cdot 0.5 \cdot 0.5 = 0.75.
\end{aligned}
$$

   The exercise of a European Call option with strike of 18 euros and with maturity of 1 year is then more likely than the exercise of a Call option as the one above but with maturity of 6 months.

2. In order to compare the prices of the options taken into account, we look for the "risk-neutral" probability $Q$ that will be useful to evaluate the options. Such a probability $Q$ corresponds to:

$$\begin{aligned} q_u &= \frac{(1+r)^{1/2}-d}{u-d} = \frac{\sqrt{1.04}-0.75}{1.25-0.75} = 0.54 \\ q_d &= 1-q_u = 0.46, \end{aligned}$$

because $r = 0.04$ is the annual interest rate and the time intervals are of 6 months.

For the European Call (A) with maturity $T' = 6$ months we get:

$$\begin{array}{ccc|c} & & S_0u = 25 & \phi_A^u = 7 \\ & \nearrow^{q_u} & & \\ S_0 = 20 & & & \\ & \searrow^{q_d} & & \\ & & S_0d = 15 & \phi_A^d = 0 \\ ---- & & ------- & --------- \\ 0 & & T' = 6\,\text{months} & \text{Payoff of option A} \end{array}$$

since the payoff of a European Call is given by

$$(S_{T'}-K)^+ = \begin{cases} S_{T'}-K; & \text{if } S_{T'} \geq K \\ 0; & \text{if } S_{T'} < K \end{cases} .$$

It follows that the initial price of the European Call option A is given by

$$C_0^A = \frac{1}{(1+r)^{1/2}}\left[q_u \cdot \phi_A^u + q_d \cdot \phi_A^d\right] = \frac{1}{\sqrt{1.04}}\,[0.54 \cdot 7 + 0] = 3.71 \text{ euros.}$$

For the European Call (B) with maturity $T = 1$ year, we have

$$\begin{array}{ccccc|c} & & & & S_0u^2 = 31.25 & \phi_B^{uu} = 13.25 \\ & & & \nearrow^{q_u} & & \\ & & S_0u = 25 & & & \\ & \nearrow^{q_u} & & \searrow^{q_d} & & \\ S_0 = 20 & & & & S_0ud = 18.75 & \phi_B^{ud} = 0.75 \\ & \searrow^{q_d} & & \nearrow^{q_u} & & \\ & & S_0d = 15 & & & \\ & & & \searrow^{q_d} & & \\ & & & & S_0d^2 = 11.25 & \phi_B^{dd} = 0 \\ ---- & & ------- & & --------- & ----------- \\ 0 & & 6\,\text{months} & & 1\,\text{year} & \text{Payoff of option B} \end{array}$$

To compute the initial cost of such an option we can proceed in two different ways:

- translate the problem in a one-period setting by considering only the initial date (0) and the maturity $T$. Consequently,

$$
\begin{array}{ccccc|c}
0 & & T = 1\,\text{year} & & \text{with probability} & \text{payoff} \\
\hline
 & & 31.25 & & (q_u)^2 & 13.25 = \phi_B^{uu} \\
 & \nearrow & & & & \\
20 & \longrightarrow & 18.75 & & 2q_u q_d & 0.75 = \phi_B^{ud} \\
 & \searrow & & & & \\
 & & 11.25 & & (q_d)^2 & 0 = \phi_B^{dd}
\end{array}
$$

It follows that the initial price of the European Call B is given by

$$
\begin{aligned}
C_0^B &= \frac{1}{1+r}\left[q_u^2 \phi_B^{uu} + 2q_u q_d \phi_B^{ud} + q_d^2 \phi_B^{dd}\right] \\
&= \frac{1}{1.04}\left[(0.54)^2 \cdot 13.25 + 2 \cdot 0.54 \cdot 0.46 \cdot 0.75 + 0\right] = 4.07 \text{ euros;}
\end{aligned}
$$

- proceed backwards in time as illustrated below.

  We first consider the sub-tree in bold and compute the "price" $C_{6m}^u$ at the node "$S_{6m}^u$" at time $t = 6$ months.

$$
\begin{array}{ccccc|c}
 & & & & \mathbf{S_0 u^2 = 31.25} & \phi_B^{uu} = 13.25 \\
 & & & \nearrow & & \\
 & & \mathbf{S_0 u = 25} & & & \\
 & \nearrow & & \searrow & & \\
S_0 = 20 & & & & \mathbf{S_0 ud = 18.75} & \phi_B^{ud} = 0.75 \\
 & \searrow & & \nearrow & & \\
 & & S_0 d = 15 & & & \\
 & & & \searrow & & \\
 & & & & S_0 d^2 = 11.25 & \phi_B^{dd} = 0 \\
\hline
0 & & 6\,\text{months} & & 1\,\text{year} & \text{Payoff of option B}
\end{array}
$$

This reduces to option pricing in a one-period model. Hence

$$
\begin{aligned}
C_{6m}^u &= \frac{1}{(1+r)^{1/2}}\left[q_u \cdot \phi_B^{uu} + q_d \cdot \phi_B^{ud}\right] \\
&= \frac{1}{\sqrt{1.04}}[0.54 \cdot 13.25 + 0.46 \cdot 0.75] = 7.3545.
\end{aligned}
$$

The same procedure can be applied to the sub-tree below:

$$
\begin{array}{ccccc|c}
 & & & & S_0u^2 = 31.25 & \phi_B^{uu} = 13.25 \\
 & & & \nearrow & & \\
 & & S_0u = 25 & & & \\
 & \nearrow & & \searrow & & \\
S_0 = 20 & & & & \mathbf{S_0ud = 18.75} & \phi_B^{ud} = 0.75 \\
 & \searrow & & \nearrow & & \\
 & & \mathbf{S_0d = 15} & & & \\
 & & & \searrow & & \\
 & & & & \mathbf{S_0d^2 = 11.25} & \phi_B^{dd} = 0 \\
---- & & ------- & & --------- & ------- \\
0 & & 6\,\text{months} & & 1\,\text{year} & \text{Payoff of option B}
\end{array}
$$

It follows that:

$$
\begin{aligned}
C_{6m}^{d} &= \frac{1}{(1+r)^{1/2}} \left[ q_u \cdot \phi_B^{ud} + q_d \cdot \phi_B^{dd} \right] \\
&= \frac{1}{\sqrt{1.04}} [0.54 \cdot 0.75 + 0] = 0.397.
\end{aligned}
$$

Finally, there remains to deduce the initial price of the Call from the prices $C_{6m}^{u}$ and $C_{6m}^{d}$ just computed. In other words,

$$
\begin{array}{ccc|c}
 & & C_{\text{6m}}^{u} = 7.3545 & q_u \\
 & \swarrow & & \\
C_0^B = ??? & & & \\
 & \nwarrow & & \\
 & & C_{\text{6m}}^{d} = 0.397 & q_d \\
------- & & --------- & --------- \\
0 & & t = 6\,\text{months} & \text{with prob.}
\end{array}
$$

Also in the present case the option pricing reduces to the pricing in a one-period setting, hence:

$$
\begin{aligned}
C_0^B &= \frac{1}{(1+r)^{1/2}} \left[ q_u \cdot C_{\text{6m}}^{u} + q_d \cdot C_{6m}^{d} \right] \\
&= \frac{1}{\sqrt{1.04}} [0.54 \cdot 7.3545 + 0.46 \cdot 0.397] = 4.07 \text{ euros.}
\end{aligned}
$$

3.

(a) The prices of options A and B studied in items 1.–2. do not change on varying the real probability. Such a probability, indeed, does not influence the option pricing.

(b) The price of the European Call option with maturity of 6 months decreases as the strike increases.

Recall that the initial price of the European Call option A with maturity $T' = 6$ months and with strike of $K = 18$ euro is $C_0^A = 3.85$ euros.

Consider now the same option as before except for the strike, now of $\tilde{K} = 20$ euros. We get

$$
\begin{array}{ccc|c}
 & & S_0 u = 25 & \tilde{\phi}^u = 5 \\
 & \nearrow^{q_u} & & \\
S_0 = 20 & & & \\
 & \searrow_{q_d} & & \\
 & & S_0 d = 15 & \tilde{\phi}^d = 0 \\
\hline
0 & & T' = 6\,\text{months} & \text{Payoff of option with strike } \tilde{K} = 20
\end{array}
$$

Consequently, the price of the European Call (as A) but with strike $\tilde{K}$ is given by

$$\tilde{C}_0 = \frac{1}{(1+r)^{1/2}} \left[ q_u \cdot \tilde{\phi}^u + q_d \cdot \tilde{\phi}^d \right] = \frac{1}{\sqrt{1.04}} \left[ 0.54 \cdot 5 + 0 \right] = 2.65 < C_0^A.$$

**Exercise 3.2** Different options having the same stock as underlying are available on the market. Suppose that the risk-free interest rate is 4% per year, that the current stock price is 20 euros and that such price may go up or down by 25% in each of the next 3 semesters.

1. Evaluate a European Call option with strike of 18 euros and with maturity of 18 months.
2. Evaluate the corresponding Put option.
3. Consider the Call and Put options above.

   (a) How many shares of the stock would you be able to buy, and how much money could you invest in a bank account at the initial time, if you sold eight Call options and bought one Put option (supposing it is not possible to buy fractions of shares)?

   (b) Compute the profit (or the loss) of the investment above if the stock price grows in the next 3 semesters. What about the profit (or the loss) if the stock price grows in 2 semesters and decreases in one semester?

4. Evaluate the Call option of item 1. when the length of the periods is 1 year instead one semester.

**Solution**

1. Recall that, when the model consists of $n$ periods (months, years, ...) and $r_p$ denotes for the interest rate per period[1] (monthly, annual, ...), we have:

$$\begin{aligned} F(S_0) &= \text{initial option price} \\ &= \frac{1}{(1+r_p)^n} \sum_{k=0}^{n} \binom{n}{k} q_u^k \cdot q_d^{n-k} \cdot \phi^{(k)\ up;(n-k)\ down}, \end{aligned}$$

where $\phi^{(k)\ up;(n-k)\ down}$ stands for the payoff of the option when the price of the underlying has moved $k$ times up and $(n-k)$ times down.

In the present case, we have that time periods are of 6 months and that the price of the underlying evolves as follows

$$\begin{array}{ccccccc|c} & & & & & & 39.0625 & \phi^{uuu} = 21.0625 \\ & & & & & \nearrow & & \\ & & & & 31.25 & & & \\ & & & \nearrow & & \searrow & & \\ & & 25 & & & & 23.4375 & \phi^{uud} = 5.4375 \\ & \nearrow & & \searrow & & \nearrow & & \\ 20 & & & & 18.75 & & & \\ & \searrow & & \nearrow & & \searrow & & \\ & & 15 & & & & 14.0625 & \phi^{udd} = 0 \\ & & & \searrow & & \nearrow & & \\ & & & & 11.25 & & & \\ & & & & & \searrow & & \\ & & & & & & 8.4375 & \phi^{ddd} = 0 \\ -- & & ---- & & ---- & & ------- & --------- \\ 0 & & 6\,\text{months} & & 1\,\text{year} & & 18\,\text{months} & \text{Payoff of the Call} \end{array}$$

[1] If time and interest rate $r_p$ refer to a different unit of time, it is enough to transform the interest rate according to compounding. For instance, if $r_{year}$ is the annual interest rate and time intervals are months, then the monthly interest rate (equivalent to $r_{year}$) is given by

$$r_{month} = (1 + r_{year})^{1/12} - 1.$$

The monthly interest rate (equivalent to $r = 0.04$ per year) is $r_m = (1+r)^{1/12} - 1 = (1.04)^{1/12} - 1$, so $(1+r_{sem})^3 = (1+r)^{3/2}$. From the arguments above it follows that

$$\begin{aligned}
&C_0 = \text{price of the European Call with maturity } T\,(18\text{ months })\\
&= \tfrac{1}{(1+r)^{3/2}}\left[(q_u)^3 \cdot \phi^{uuu} + 3\,(q_u)^2\,q_d \cdot \phi^{uud} + 3q_u\,(q_d)^2 \cdot \phi^{udd} + (q_d)^3 \cdot \phi^{ddd}\right]\\
&= \tfrac{1}{(1.04)^{3/2}}\left[(0.54)^3 \cdot 21.0625 + 3 \cdot (0.54)^2 \cdot 0.46 \cdot 5.4375 + 0 + 0\right] = 5.19,
\end{aligned}$$

where $q_u = 0.54$ and $q_d = 0.46$ are the same as in Exercise 3.1.

2. We can evaluate the European Put option in two different ways.

First, we can proceed as above. Since the payoff of the European Put option is

$$(K - S_T)^+ = \max\{K - S_T; 0\} = \begin{cases} K - S_T; & \text{if } S_T \le K \\ 0; & \text{if } S_T > K \end{cases},$$

it follows that

| | | | | |
|---|---|---|---|---|
| | | 39.0625 | $(q_u)^3$ | $\phi^{uuu}_{Put} = 0$ |
| | $\nearrow$ | | | |
| | $\nearrow$ | 23.4375 | $3\,(q_u)^2\,q_d$ | $\phi^{uud}_{Put} = 0$ |
| 20 | | | | |
| | $\searrow$ | 14.0625 | $3q_u\,(q_d)^2$ | $\phi^{udd}_{Put} = 3.9375$ |
| | $\searrow$ | | | |
| | | 8.4375 | $(q_d)^3$ | $\phi^{ddd}_{Put} = 9.5625$ |
| 0 | | $T = 18$ months | with probability | Payoff of the Put |

Hence

$$\begin{aligned}
P_0 &= \text{price of the European Put with maturity } T\\
&= \frac{1}{(1+r)^{3/2}}\Big[(q_u)^3 \cdot \phi^{uuu}_{Put} + 3\,(q_u)^2\,q_d \cdot \phi^{uud}_{Put}\\
&\quad + 3q_u\,(q_d)^2 \cdot \phi^{udd}_{Put} + (q_d)^3 \cdot \phi^{ddd}_{Put}\Big]\\
&= \frac{1}{(1.04)^{3/2}}\left[0 + 0 + 3 \cdot 0.54 \cdot (0.46)^2 \cdot 3.9375 + (0.46)^3 \cdot 9.5625\right]\\
&= 2.15 \text{ euros.}
\end{aligned}$$

An alternative way to evaluate the Put option above is to apply the Put-Call Parity true for European options with the same maturity and strike and written on the same underlying, that is

$$C_0 - P_0 = S_0 - \frac{K}{(1+r)^T}.$$

Due to the equality above, we obtain immediately

$$P_0 = C_0 - S_0 + \frac{K}{(1+r)^T} \cong 2.15 \text{ euros.}$$

3. By the previous items, $C_0 = 5.19$ and $P_0 = 2.15$ euros.
   (a) If we sell eight Call options and buy one Put option as above, we dispose of

   $$\pi = 8 \cdot C_0 - P_0 = 8 \cdot 5.19 - 2.15 = 39.37 \text{ euros.}$$

   Under the restrictions imposed, we can buy only one share of the underlying (its initial price is indeed $S_0 = 20$ euros) and invest the remaining budget of 19.37 euros at risk-free interest rate.
   (b) If the stock price grows in each of the next three semesters (hence, $S_T = S_0 u^3 = 39.0625$ euros), the profit (or loss) of our strategy is

| | |
|---|---|
| 8 call sold | $-8\,(S_T - K)^+ = -8 \cdot \phi^{uuu} = -168.5$ |
| 1 put bought | $(K - S_T)^+ = 0$ |
| Profit/loss by the stock | $(S_T - S_0) = 19.0625$ |
| Cash invested in a bank account | $19.37 \cdot (1.04)^{3/2} = 20.54375$ |
| $Total\ profit\ /\ loss$ | $-128.894$ euro |

   If the stock price grows in two of the next three semesters and decreases in one (hence $S_T = S_0 u^2 d = 23.4375$ euros), the profit (or loss) of our strategy is

| | |
|---|---|
| 8 calls sold | $-8\,(S_T - K)^+ = -8 \cdot \phi^{uud} = -43.5$ |
| 1 put bought | $(K - S_T)^+ = 0$ |
| Profit/loss by the stock | $(S_T - S_0) = 3.4375$ |
| Cash invested in a bank account | $19.37 \cdot (1.04)^{3/2} = 20.54375$ |
| $Total\ profit\ /\ loss$ | $-19.5188$ euros |

4. With annual periods instead of semestral ones, the stock price would evolve as before:

$$
\begin{array}{ccccccc|c}
 & & & & & & 39.0625 & \phi^{uuu} = 21.0625 \\
 & & & & & \nearrow & & \\
 & & & & 31.25 & & & \\
 & & & \nearrow & & \searrow & & \\
 & & 25 & & & & 23.4375 & \phi^{uud} = 5.4375 \\
 & \nearrow & & \searrow & & \nearrow & & \\
20 & & & & 18.75 & & & \\
 & \searrow & & \nearrow & & \searrow & & \\
 & & 15 & & & & 14.0625 & \phi^{udd} = 0 \\
 & & & \searrow & & \nearrow & & \\
 & & & & 11.25 & & & \\
 & & & & & \searrow & & \\
 & & & & & & 8.4375 & \phi^{ddd} = 0 \\
--- & & ---- & & ---- & & ------ & --------- \\
0 & & \text{1 year} & & \text{2 years} & & \text{3 years} & \text{Payoff of the Call}
\end{array}
$$

while the maturity and the equivalent martingale measure $Q$ would change. For annual periods, indeed, $Q$ would correspond to

$$
\begin{aligned}
q_u^* &= \frac{(1+r)-d}{u-d} = \frac{1.04-0.75}{1.25-0.75} = 0.58 \\
q_d^* &= 1 - q_u^* = 0.42,
\end{aligned}
$$

hence the price of the European Call with maturity 3 years would be given by

$$
\begin{aligned}
C_0^{(3y)} &= \frac{1}{(1+r)^3}\Big[\left(q_u^*\right)^3 \cdot \phi^{uuu} + 3\left(q_u^*\right)^2 q_d^* \cdot \phi^{uud} \\
&\quad +3q_u^*\left(q_d^*\right)^2 \cdot \phi^{udd} + \left(q_d^*\right)^3 \cdot \phi^{ddd}\Big] \\
&= \frac{1}{(1.04)^3}\Big[(0.58)^3 \cdot 21.0625 + 3 \cdot (0.58)^2 \cdot 0.42 \cdot 5.4375 \\
&\quad +0+0] = 5.70.
\end{aligned}
$$

**Exercise 3.3** Consider a stock whose current price is $S_0 = 40$ euros. In each of the next two 4-month periods, the stock price may move up by 4% or down by 2%. The risk-free rate available on the market is of 4% per year.

Compute the initial price of an option with maturity of $T = 8$ months and with payoff

$$\phi = \left(S_T - \frac{(K-S_0)^2}{10}\right)^+ = \begin{cases} 0; & S_T < \frac{(K-S_0)^2}{10} \\ S_T - \frac{(K-S_0)^2}{10}; & S_T \geq \frac{(K-S_0)^2}{10} \end{cases}$$

when the strike $K$ is of 60 euros.

**Solution** We know that on each of the next 4-month periods the stock price may increase by a growth factor $u = 1.04$ or decrease by a factor $d = 0.98$. Accordingly, the stock price evolves as below:

$$\begin{array}{ccccc} & & & & 43.264 = S_0u^2 \\ & & & \nearrow & \\ & & 41.6 & & \\ & \nearrow & & \searrow & \\ 40 & & & & 40.768 = S_0ud \\ & \searrow & & \nearrow & \\ & & 39.2 & & \\ & & & \searrow & \\ & & & & 38.416 = S_0d^2 \\ --- & & ----- & & --------- \\ 0 & & 4\text{ months} & & 8\text{ months} \end{array}$$

The payoff of the option may thus take the following values:

$$\phi^{uu} = \left(S_T^{uu} - \frac{(60-40)^2}{10}\right)^+ = \left(43.264 - \frac{20^2}{10}\right)^+ = 3.264$$

$$\phi^{ud} = \left(S_T^{ud} - \frac{(60-40)^2}{10}\right)^+ = \left(40.768 - \frac{20^2}{10}\right)^+ = 0.768$$

$$\phi^{dd} = \left(S_T^{dd} - \frac{(60-40)^2}{10}\right)^+ = \left(38.416 - \frac{20^2}{10}\right)^+ = 0.$$

Consequently,

$$
\begin{array}{ccccc|c}
 & & & & 43.264 = S_0 u^2 & 3.264 = \phi^{uu} \\
 & & & \nearrow & & \\
 & & 41.6 & & & \\
 & \nearrow & & \searrow & & \\
40 & & & & 40.768 = S_0 ud & 0.768 = \phi^{ud} \\
 & \searrow & & \nearrow & & \\
 & & 39.2 & & & \\
 & & & \searrow & & \\
 & & & & 38.416 = S_0 d^2 & 0 = \phi^{dd} \\
--- & & ---- & & --------- & -------- \\
0 & & 4\text{ months} & & 8\text{ months} & \text{Payoff of the option}
\end{array}
$$

The risk-neutral probability $Q$ to be used for option pricing corresponds to:

$$
\begin{aligned}
q_u &= \frac{(1+r)^{4/12} - d}{u - d} = 0.55 \\
q_d &= 1 - q_u = 0.45,
\end{aligned}
$$

since periods are of 4 months while $r$ is the annual interest rate.

From the arguments above, the option with payoff $\phi$ now costs

$$
\begin{aligned}
F(S_0) &= \frac{1}{(1+r)^T}\left[(q_u)^2 \phi^{uu} + 2 q_u q_d \phi^{ud} + (q_d)^2 \phi^{dd}\right] \\
&= \frac{1}{(1+r)^{8/12}}\left[(0.55)^2 \cdot 3.264 + 2 \cdot 0.55 \cdot 0.45 \cdot 0.768 + 0\right] = 1.33 \text{ euros.}
\end{aligned}
$$

**Exercise 3.4** We take a short position in a European Call option with maturity 4 months and with strike of 20 euros, having a stock with current price of 20 euros as underlying. In the next 4 months, the stock price may increase by a growth factor $u = 1.2$ or decrease by a factor $d = 0.8$. The risk-free interest rate available on the market is of 4% per year.

1. Find the replicating strategy of the option.
2. Compute the initial price of the option.
3. Suppose now that the stock price does not follow a binomial model any more, but that in 4 months it may either increase by a growth factor $u = 1.2$, decrease by a factor $d = 0.8$ or remain unchanged. Discuss whether it is still possible to replicate the Call option above only with the underlying and with cash (to be deposited or borrowed).

**Solution**

1. The stock price and the option payoff are the following:

$$
\begin{array}{ccc|c}
 & & S_0u = 24 & \phi^u = 4 \\
 & \nearrow & & \\
S_0 = 20 & & & \\
 & \searrow & & \\
 & & S_0d = 16 & \phi^d = 0 \\
---- & & -------- & ------- \\
0 & & T = 4\,\text{months} & \text{Payoff}
\end{array}
$$

In order to find the replicating strategy of the option we need to find a number $\Delta$ of shares of the underlying and an amount $x$ of cash so that the seller of the option holding such a portfolio can be "covered" against losses (due to the option) at maturity and in any state of the world. We are then looking for $\Delta$ and $x$ such that

$$
\begin{cases}
\Delta S_0u + x\,(1+r)^T = \phi^u \\
\Delta S_0d + x\,(1+r)^T = \phi^d
\end{cases}.
$$

The previous system of linear equations is equivalent to

$$
\begin{cases}
\Delta S_0u - \Delta S_0d = \phi^u - \phi^d \\
\Delta S_0d + x\,(1+r)^T = \phi^d
\end{cases},
$$

whose solution is given by:

$$
\begin{cases}
\Delta = \frac{\phi^u - \phi^d}{S_0(u-d)} \\
x = \frac{1}{(1+r)^T}\left[\phi^d - \Delta S_0d\right]
\end{cases}.
$$

The replicating strategy is then composed as below:

$$
\begin{cases}
\Delta = \frac{\phi^u - \phi^d}{S_0(u-d)} = \frac{4-0}{20(1.2-0.8)} = 0.5 \\
x = \frac{1}{(1.04)^{4/12}}\left[0 - 0.5 \cdot 20 \cdot 0.8\right] = -7.9
\end{cases}
$$

To be covered against losses deriving from the exercise of the option, the seller has to buy 0.5 shares of the stock and to borrow 7.9 euros.

Since the cost of such a strategy is equal to

$$
\Delta S_0 + x = 0.5 \cdot 20 - 7.9 = 2.1,
$$

the No Arbitrage principle implies that the Call price should be equal to the cost of the replicating strategy, i.e. $C_0 = \Delta S_0 + x = 2.1$.

2. We have already computed the option price above by means of the replicating strategy.

   An alternative way to compute $C_0$ is to proceed as in Exercises 3.1 and 3.2. By the no arbitrage principle, the result has to coincide with the one above.

   Since the equivalent martingale measure corresponds to $q_u = \frac{(1+r)^{4/12}-d}{u-d} = 0.53$ and $q_d = 1 - q_u = 0.47$, we obtain indeed

$$C_0 = \frac{1}{(1+r)^T}\left[q_u\phi^u + q_d\phi^d\right]$$
$$= \frac{1}{(1.04)^{4/12}}\left[0.53 \cdot 4 + 0\right] = 2.10 \text{ euros.}$$

3. The stock price is now assumed to evolve as follows:

$$\begin{array}{ccc|c}
 & & S_0u = 24 & \phi^{\mathrm{u}} = 4 \\
 & \nearrow & & \\
S_0 = 20 & \rightarrow & S_0 \cdot 1 = 20 & \phi^{\mathrm{m}} = 0 \\
 & \searrow & & \\
 & & S_0d = 16 & \phi^{\mathrm{d}} = 0 \\
---- & & --------- & ------- \\
0 & & T = 4\,\text{months} & \text{Payoff}
\end{array}$$

Verifying the existence of a replicating strategy for the option is equivalent to verifying the existence of a number $\Delta$ of shares of the underlying and of an amount $x$ of cash such that

$$\begin{cases} \Delta S_0u + x(1+r)^T = \phi^u \\ \Delta S_0 + x(1+r)^T = \phi^m \\ \Delta S_0d + x(1+r)^T = \phi^d \end{cases}.$$

Since the previous system (of 3 linear equations in 2 variables) is equivalent to the following ones

$$\begin{cases} \Delta S_0u + x(1+r)^T = \phi^u \\ \Delta S_0 + x(1+r)^T = \phi^m \\ \Delta S_0(u-d) = \phi^u - \phi^d \end{cases} \qquad \begin{cases} \Delta S_0u + x(1+r)^T = \phi^u \\ \Delta S_0 + x(1+r)^T = \phi^m \\ \Delta = \frac{\phi^u-\phi^d}{S_0(u-d)} \end{cases}$$

$$\begin{cases} x(1+r)^T = \phi^u - \Delta S_0u \\ x(1+r)^T = \phi^m - \Delta S_0 \\ \Delta = \frac{1}{2} \end{cases} \qquad \begin{cases} x(1+r)^T = -8 \\ x(1+r)^T = -10 \\ \Delta = \frac{1}{2} \end{cases},$$

it is evident that there is no solution. Consequently, it is not possible to replicate the option only by means of the underlying and of cash invested at risk-free rate.

From the arguments above it follows that the market model is incomplete (i.e. not any option is attainable) and that the option price is not uniquely determined, in general.

**Exercise 3.5** We take a short position in a European Call option with maturity 2 years and with strike of 25 euros, having a stock with current price of 40 euros as underlying. In each of the next 2 years, the stock price may increase by a growth factor $u = 1.5$ or decrease by a factor $d = 0.5$, while the risk-free interest rate is of 4% per year.

1. Find the replicating trading strategy of the option above and deduce the option price.
2. Find the replicating strategy of a portfolio formed by two short positions in a Call with maturity of 1 year, in one long position in a Put with maturity of 1 year (where both Call and Put options have strike of 25 euros and are written on the underlying described above) and in one long position in the underlying.
3. Discuss whether the replicating strategy and the option price change if everything (underlying dynamics, strike, maturity, ...) stays the same except for the risk-free rate that increases to 6% per year.

**Solution**

1. The stock price evolves as follows:

$$\begin{array}{ccccc}
 & & & & 90 = S_0u^2 \\
 & & & \nearrow & \\
 & & S_0u = 60 & & \\
 & \nearrow & & \searrow & \\
S_0 = 40 & & & & 30 = S_0ud \\
 & \searrow & & \nearrow & \\
 & & S_0d = 20 & & \\
 & & & \searrow & \\
 & & & & 10 = S_0d^2 \\
---- & & ------ & & -------- \\
0 & & 1\text{ year} & & 2\text{ years}
\end{array}$$

where $u = 1.5$ (growth factor) and $d = 0.5$.

First of all, let us find the risk-neutral measure to be used for option pricing. Such a measure $Q$ corresponds to:

$$q_u = \frac{(1+r)-d}{u-d} = \frac{1.04-0.5}{1.5-0.5} = 0.54$$
$$q_d = 1 - q_u = 0.46,$$

because both the risk-free rate $r$ and the periods are annual.

We will find simultaneously the option price and the replicating strategy. For a European Call option with strike of 25 euros:

$$
\begin{array}{ccccc|c}
 & & & & 90 = S_0u^2 & \phi^{uu} = 65 \\
 & & & \nearrow & & \\
 & & S_0u = 60 & & & \\
 & \nearrow & & \searrow & & \\
S_0 = 40 & & & & 30 = S_0ud & \phi^{ud} = 5 \\
 & \searrow & & \nearrow & & \\
 & & S_0d = 20 & & & \\
 & & & \searrow & & \\
 & & & & 10 = S_0d^2 & \phi^{dd} = 0 \\
---- & & ------ & & -------- & --------- \\
0 & & 1\text{ year} & & 2\text{ years} & \text{Payoff of the option}
\end{array}
$$

To find the replicating strategy of the option, we consider separately any period and we proceed backwards in time.

We first consider the sub-tree in bold so to compute the option "price" $C_1^u$ at the node "$S_1^u$" in $t = 1$ year.

$$
\begin{array}{ccccc|c}
 & & & & \mathbf{90 = S_0u^2} & \phi^{uu} = 65 \\
 & & & \nearrow & & \\
 & & \mathbf{S_0u = 60} & & & \\
 & \nearrow & & \searrow & & \\
S_0 = 40 & & & & \mathbf{30 = S_0ud} & \phi^{ud} = 5 \\
 & \searrow & & \nearrow & & \\
 & & S_0d = 30 & & & \\
 & & & \searrow & & \\
 & & & & 10 = S_0d^2 & \phi^{dd} = 0 \\
---- & & ------- & & ------- & ------- \\
0 & & 1\text{ year} & & 2\text{ years} & \text{Payoff of the option}
\end{array}
$$

We have reduced to evaluate an option and/or find its replicating trading strategy in a one-period model. We obtain that:

$$
C_1^u = \frac{1}{1+r}\left[q_u \cdot \phi^{uu} + q_d \cdot \phi^{ud}\right] = \frac{1}{1.04}\,[0.54 \cdot 65 + 0.46 \cdot 5] = 35.96
$$

and that the replicating strategy $\left(\Delta_1^u, x_1^u\right)$ at time $t = 1$ and at the node "$S_1^u$" is given by:

$$
\begin{cases}
\Delta_1^u = \frac{\phi^{uu}-\phi^{ud}}{S_1^u(u-d)} = \frac{65-5}{60(1.5-0.5)} = 1 \\
x_1^u = \frac{1}{1+r}\left[\phi^{ud} - \Delta_1^u S_1^u d\right] = \frac{1}{1.04}\,[5-30] = -24.04
\end{cases}
$$

As expected,

$$C_1^u = \Delta_1^u S_1^u + x_1^u = \Delta_1^u S_0 u + x_1^u = 35.96.$$

Proceeding analogously for the sub-tree in bold below:

$$\begin{array}{cccc|c}
 & & & 90 = S_0 u^2 & \phi^{uu} = 65 \\
 & & \nearrow & & \\
 & S_0 u = 60 & & & \\
 \nearrow & & \searrow & & \\
S_0 = 40 & & & \mathbf{30 = S_0 ud} & \phi^{ud} = 5 \\
 \searrow & & \nearrow & & \\
 & \mathbf{S_0 d = 20} & & & \\
 & & \searrow & & \\
 & & & \mathbf{10 = S_0 d^2} & \phi^{dd} = 0 \\
---- & ------- & & ------- & ------- \\
0 & 1\ \text{year} & & 2\ \text{years} & \text{Payoff of the option}
\end{array}$$

we get

$$C_1^d = \frac{1}{1+r}\left[q_u \cdot \phi^{ud} + q_d \cdot \phi^{dd}\right] = \frac{1}{1.04}\left[0.54 \cdot 5 + 0\right] = 2.6$$

and

$$\begin{cases} \Delta_1^d = \frac{\phi^{ud} - \phi^{dd}}{S_1^d(u-d)} = \frac{5-0}{20(1.5-0.5)} = 0.25 \\ x_1^d = \frac{1}{1+r}\left[\phi^{dd} - \Delta_1^d S_1^d d\right] = -2.4 \end{cases}$$

Also in the present case, we obtain the same result as previously, that is

$$C_1^d = \Delta_1^d S_1^d + x_1^d = \Delta_1^d S_0 d + x_1^d = 2.6.$$

Finally, we consider the first period:

$$\begin{array}{cccc}
 & & C_1^u = 35.96 & q_u \\
 & \swarrow & & \\
C_0 = ??? & & & \\
 & \nwarrow & & \\
 & & C_1^d = 2.6 & q_d \\
---- & & ------- & \\
0 & & t = 1\ \text{year} &
\end{array}$$

It is a one-period case, hence the option price can be computed immediately as

$$\begin{aligned} C_0 &= \frac{1}{1+r}\left[q_u \cdot C_1^u + q_d \cdot C_1^d\right] \\ &= \frac{1}{1.04}\,[0.54 \cdot 35.96 + 0.46 \cdot 2.6] = 19.8 \text{ euros,} \end{aligned}$$

while the replicating strategy is given by

$$\begin{cases} \Delta_0 = \frac{C_1^u - C_1^d}{S_0(u-d)} = \frac{35.96-2.6}{40(1.5-0.5)} = 0.834 \\ x_0 = (1+r)^{-1}\left[C_1^d - \Delta_0 S_0 d\right] = -13.54 \end{cases}$$

Consequently, we find again $C_0 = \Delta_0 S_0 + x_0 = 19.8$ euros.

Summing up: the initial price of the option is 19.8 euros and its replicating trading strategy consists in the following. The seller of the option buys 0.834 stocks and borrows 13.54 euros at $t = 0$. Then, at $t = 1$: (a) if the stock price is increased to $S_0 u = 60$, he buys $(1 - 0.834) = 0.166$ more stocks and borrows $(24.04 - 13.54) = 10.50$ euros more. So, his portfolio consists in 1 stock and in 24.04 euros borrowed. (b) If the stock price is decreased to $S_0 d = 20$, he sells $(0.834 - 0.25) = 0.584$ stocks and gives back $(13.54 - 2.4) = 11.14$ euros of what he had borrowed. Thus, his portfolio consists in 0.25 stocks and in 2.4 euros borrowed.

2. We consider now a portfolio formed by two short positions in a Call with maturity of 1 year, by one long position in a Put with maturity of 1 year (where both Call and Put options have strike of 25 euros and are written on the underlying described above) and by one long position in the underlying.

   Since the underlying and the Call and Put payoffs read:

$$\begin{array}{cccc} & S_0 u = 60 & \phi^u_{Call} = 35 & \phi^u_{Put} = 0 \\ & \nearrow & & \\ S_0 = 40 & & & \\ & \searrow & & \\ & S_0 d = 20 & \phi^d_{Call} = 0 & \phi^d_{Put} = 5 \\ ---- & ------- & ------- & ------- \\ 0 & T^* = 1 \text{ year} & \text{Call payoff} & \text{Put payoff} \end{array}$$

the portfolio $\pi$ to be replicated is given by

$$\begin{aligned}
\phi_\pi &= -2\phi_{Call} + \phi_{Put} + S_{T^*} \\
&= \begin{cases} \phi_\pi^u;\ \text{if } S_1 = S_1^u \\ \\ \phi_\pi^d;\ \text{if } S_1 = S_1^d \end{cases} \\
&= \begin{cases} -2\phi_{Call}^u + \phi_{Put}^u + S_1^u = -10;\ \text{if } S_1 = S_1^u \\ \\ \quad -2\phi_{Call}^d + \phi_{Put}^d + S_1^d = 25;\ \text{if } S_1 = S_1^d \end{cases}
\end{aligned}$$

In order to find a replicating strategy of the above portfolio of options and stocks, we have to look for a number $\Delta$ of shares of the underlying and for an amount $x$ of cash such that

$$\begin{cases} \Delta S_0 u + x\,(1+r)^{T^*} = \phi_\pi^u \\ \Delta S_0 d + x\,(1+r)^{T^*} = \phi_\pi^d \end{cases}.$$

In the present case, we obtain

$$\begin{cases} \Delta S_0 u - \Delta S_0 d = \phi_\pi^u - \phi_\pi^d \\ \Delta S_0 d = \phi_\pi^d - x\,(1+r)^{T^*} \end{cases} \quad \begin{cases} \Delta = \frac{\phi_\pi^u - \phi_\pi^d}{S_0(u-d)} = \frac{-10-25}{40(1.5-0.5)} = -0.875 \\ x = \frac{1}{1+r}\left[\phi_\pi^d - \Delta S_0 d\right] = 40.865 \end{cases}$$

Consequently, the replicating strategy we are looking for consists in selling 0.875 shares of the underlying and in investing 40.865 euros in a bank account. Such a strategy costs then $\Delta S_0 + x = -35 + 40.865 = 5.865$ euros.

3. To find the replicating trading strategy of the option (that exists for sure!), we recall that the dynamics of the stock, as well as the option payoff, do not depend on the risk-free rate and are the following:

$$\begin{array}{cccccc|c}
 & & & & & 90 = S_0u^2 & \phi^{uu} = 65 \\
 & & & & \nearrow & & \\
 & & S_0u = 60 & & & & \\
 & \nearrow & & & \searrow & & \\
S_0 = 40 & & & & & 30 = S_0ud & \phi^{ud} = 5 \\
 & \searrow & & & \nearrow & & \\
 & & S_0d = 20 & & & & \\
 & & & & \searrow & & \\
 & & & & & 10 = S_0d^2 & \phi^{dd} = 0 \\
---- & & ------- & & & ------- & ------- \\
0 & & 1\ \text{year} & & & 2\ \text{years} & \text{Payoff of the option}
\end{array}$$

In order to find the replicating strategy we analyze separately the two periods and we proceed backwards in time as in the previous item.

First, let us consider the sub-tree in bold so to compute the option price $C_1^u$ at the node "$S_1^u$" and at time $t = 1$:

$$\begin{array}{ccccc|c}
 & & & & \mathbf{90 = S_0u^2} & \phi^{uu} = 65 \\
 & & & \nearrow & & \\
 & & \mathbf{S_0u = 60} & & & \\
 & \nearrow & & \searrow & & \\
S_0 = 40 & & & & \mathbf{30 = S_0ud} & \phi^{ud} = 5 \\
 & \searrow & & \nearrow & & \\
 & & S_0d = 20 & & & \\
 & & & \searrow & & \\
 & & & & 10 = S_0d^2 & \phi^{dd} = 0 \\
- - - - & & - - - - - - & & - - - - - - & - - - - - - \\
0 & & \text{1 year} & & \text{2 years} & \text{Payoff of the option}
\end{array}$$

Under the new risk-free rate $\tilde{r}$, the risk-neutral measure corresponds now to

$$\tilde{q}_u = \frac{(1+\tilde{r}) - d}{u - d} = \frac{1.06 - 0.5}{1.5 - 0.5} = 0.56$$
$$\tilde{q}_d = 1 - \tilde{q}_u = 0.44.$$

Accordingly, the replicating strategy $\left(\Delta_1^u, x_1^u\right)$ to be held at $t = 1$ and at the node "$S_1^u$" is now

$$\begin{cases} \Delta_1^u = \frac{\phi^{uu} - \phi^{ud}}{S_1^u(u-d)} = \frac{65-5}{60(1.5-0.5)} = 1 \\ x_1^u = (1+\tilde{r})^{-1}\left[\phi^{ud} - \Delta_1^u S_1^u d\right] = (1+0.06)^{-1}\,[5 - 30] = -23.58 \end{cases}$$

Therefore, $C_1^u = \Delta_1^u S_1^u + x_1^u = \Delta_1^u S_0 u + x_1^u = 36.42$ euros.

Analogously for the sub-tree in bold below:

$$\begin{array}{ccccc|c}
 & & & & 90 = S_0u^2 & \phi^{uu} = 65 \\
 & & & \nearrow & & \\
 & & S_0u = 60 & & & \\
 & \nearrow & & \searrow & & \\
S_0 = 40 & & & & \mathbf{30 = S_0ud} & \phi^{ud} = 5 \\
 & \searrow & & \nearrow & & \\
 & & \mathbf{S_0d = 20} & & & \\
 & & & \searrow & & \\
 & & & & \mathbf{10 = S_0d^2} & \phi^{dd} = 0 \\
- - - - & & - - - - - - & & - - - - - - & - - - - - - \\
0 & & \text{1 year} & & \text{2 years} & \text{Payoff of the option}
\end{array}$$

In the present case, we get

$$\begin{cases} \Delta_1^d = \frac{\phi^{ud} - \phi^{dd}}{S_1^d (u-d)} = \frac{5-0}{20(1.5-0.5)} = 0.25 \\ x_1^d = (1+\tilde{r})^{-1} \left[ \phi^{dd} - \Delta_1^d S_1^d d \right] = -2.36 \end{cases}$$

and, consequently, $C_1^d = \Delta_1^d S_1^d + x_1^d = \Delta_1^d S_0 d + x_1^d = 2.64$ euros.

Finally, we take into account the first period:

$$\begin{array}{ccccc}
 & & C_1^u = 36.42 & & \tilde{q}_u \\
 & \swarrow & & & \\
C_0 = ??? & & & & \\
 & \nwarrow & & & \\
 & & C_1^d = 2.64 & & \tilde{q}_d \\
- - -- & & - - - - -- & & \\
0 & & 1 \text{ year} & &
\end{array}$$

The replicating trading strategy is given by

$$\begin{cases} \Delta_0 = \frac{C_1^u - C_1^d}{S_0 (u-d)} = \frac{36.42-2.64}{40(1.5-0.5)} = 0.8445 \\ x_0 = \frac{1}{1+\tilde{r}} \left[ C_1^d - \Delta_0 S_0 d \right] = -13.44 \end{cases},$$

so we deduce that $C_0 = \Delta_0 S_0 + x_0 = 20.34$ euros.

**Exercise 3.6** Consider a market where a bond (corresponding to a risk-free rate of 4% per year) and a stock are traded. The current price of the stock is $S_0^1 = 8$ euros that, in 1 year, may move up to 16 euros or to 12 euros, or down to 2 euros.

1. Consider a European Call option on the stock above, with strike of 8 euros and with maturity of 1 year. Verify if it is possible to replicate such an option only by means of the underlying and of cash invested (or borrowed) at risk-free rate.
2. Consider another stock of current price $S_0^2 = 10$ euros and that, in 1 year, may cost 20, 8 or 6 euros whenever the other stock costs, respectively, 16, 12 or 2 euros.

   A new option is also available, with maturity $T$ of 1 year, with the same strike $K$ as the Call above and with payoff

$$\phi = \left( \tilde{S}_T - K \right)^+,$$

where $\tilde{S}_T = \frac{S_T^1 + S_T^2}{2} - \left| \frac{S_T^1 - S_T^2}{4} \right|$.

Verify if it is possible to replicate the option above by means of the two stocks and of cash invested (or borrowed). If yes, find the cost of the replicating strategy.

**Solution**

1. The price of the first stock and the payoff of the European Call on the stock, with maturity of 1 year and strike of 8 euros are the following:

$$
\begin{array}{ccc|c}
 & & S_1^{1,u} = 16 & \phi_{Call}^u = 8 \\
 & \nearrow & & \\
S_0^1 = 8 & \rightarrow & S_1^{1,m} = 12 & \phi_{Call}^m = 4 \\
 & \searrow & & \\
 & & S_1^{1,d} = 2 & \phi_{Call}^d = 0 \\
---- & & ------- & --------- \\
0 & & 1 \text{ year} & \text{Payoff of the Call}
\end{array}
$$

In order to verify if the Call option above is attainable by means of the underlying stock and of cash, we need to check if the following system has solutions or not:

$$
\begin{cases}
\Delta S_1^{1,u} + x\,(1+r)^T = \phi_{Call}^u \\
\Delta S_1^{1,m} + x\,(1+r)^T = \phi_{Call}^m \\
\Delta S_1^{1,d} + x\,(1+r)^T = \phi_{Call}^d
\end{cases},
$$

where $\Delta$ represents the number of shares of the underlying to buy or sell and $x$ the amount of cash to be invested or borrowed.

Replacing $T$ with 1 (year), the previous system (of 3 equations in 2 variables) is equivalent to the following ones:

$$
\begin{cases}
\Delta S_1^{1,u} + x\,(1+r) = \phi_{Call}^u \\
\Delta \left(S_1^{1,u} - S_1^{1,m}\right) = \phi_{Call}^u - \phi_{Call}^m \\
\Delta S_1^{1,d} + x\,(1+r) = \phi_{Call}^d
\end{cases}
\qquad
\begin{cases}
\Delta S_1^{1,u} + x\,(1+r) = \phi_{Call}^u \\
\Delta = \frac{\phi_{Call}^u - \phi_{Call}^m}{S_1^{1,u} - S_1^{1,m}} \\
\Delta S_1^{1,d} + x\,(1+r) = \phi_{Call}^d
\end{cases}
$$

$$
\begin{cases}
x\,(1+r) = \phi_{Call}^u - \Delta S_1^{1,u} \\
\Delta = 1 \\
x\,(1+r) = \phi_{Call}^d - \Delta S_1^{1,d}
\end{cases}
\qquad
\begin{cases}
x\,(1+r) = -8 \\
\Delta = 1 \\
x\,(1+r) = -2
\end{cases}
$$

Since the last system has no solution, the Call option cannot be replicated only by means of the underlying and of cash.

2. We display below the dynamics of the two stocks and the payoff of the option written on them:

$$
\begin{array}{ccc|c}
 & & S_1^{1,u} = 16; \quad S_1^{2,u} = 20 & \phi^u = 9 \\
 & \nearrow & & \\
S_0^1 = 8; \quad S_0^2 = 10 & \rightarrow & S_1^{1,m} = 12; \quad S_1^{2,m} = 8 & \phi^m = 1 \\
 & \searrow & & \\
 & & S_1^{1,d} = 2; \quad S_1^{2,d} = 6 & \phi^d = 0 \\
\hline
0 & & 1 \text{ year} & \text{Payoff the option}
\end{array}
$$

Verifying the existence of a replicating strategy for the option is equivalent to verifying the existence of a number $\Delta^1$ of shares of one stock, of a number $\Delta^2$ of shares of the other stock and of an amount $x$ of cash such that

$$
\begin{cases}
\Delta^1 S_1^{1,u} + \Delta^2 S_1^{2,u} + x\,(1+r) = \phi^u \\
\Delta^1 S_1^{1,m} + \Delta^2 S_1^{2,m} + x\,(1+r) = \phi^m \\
\Delta^1 S_1^{1,d} + \Delta^2 S_1^{2,d} + x\,(1+r) = \phi^d
\end{cases}.
$$

In the present case, the previous system (of 3 equations in 3 variables) is equivalent to

$$
\begin{cases}
16\Delta^1 + 20\Delta^2 + x\,(1+r) = 9 \\
12\Delta^1 + 8\Delta^2 + x\,(1+r) = 1 \\
2\Delta^1 + 6\Delta^2 + x\,(1+r) = 0
\end{cases}
\qquad
\begin{cases}
16\Delta^1 + 20\Delta^2 + x\,(1+r) = 9 \\
10\Delta^1 + 2\Delta^2 = 1 \\
2\Delta^1 + 6\Delta^2 + x\,(1+r) = 0
\end{cases}
$$

$$
\begin{cases}
16\Delta^1 + 20\Delta^2 + x\,(1+r) = 9 \\
\Delta^2 = 1/2 - 5\Delta^1 \\
14\Delta^1 + 14\Delta^2 = 9
\end{cases}
\qquad
\begin{cases}
16\Delta^1 + 20\Delta^2 + x\,(1+r) = 9 \\
\Delta^2 = 1/2 - 5\Delta^1 \\
-56\Delta^1 = 2
\end{cases}
$$

$$
\begin{cases}
x = \frac{9-16\Delta^1-20\Delta^2}{1+r} \\
\Delta^2 = \frac{19}{28} \\
\Delta^1 = -\frac{1}{28}
\end{cases}
\qquad
\begin{cases}
x = -3.846 \\
\Delta^2 = \frac{19}{28} \\
\Delta^1 = -\frac{1}{28}
\end{cases}
$$

It is then possible to replicate the option by selling $-\Delta^1 = \frac{1}{28}$ shares of the first stock, buying $\Delta^2 = \frac{19}{28}$ shares of the second stock and borrowing 3.846 euros. So, the cost of the replicating trading strategy is equal to

$$
\Delta^1 S_0^1 + \Delta^2 S_0^2 + x = -\frac{1}{28} \cdot 8 + \frac{19}{28} \cdot 10 - 3.846 = 2.654 \text{ euros.}
$$

**Exercise 3.7** A stock (indexed by A) is available on the market at the current price $S_0^A = 8$ euros. In 1 year, the price may increase by 25% or decrease by 25%.

Another stock (indexed by B) is also available. Its current price is $S_0^B = 12$ euros that, in 1 year, may increase by 25% (when also the price of stock A is increased) or decreased by 25% (when also the price of stock A is decreased). The risk-free interest rate on the market is 4% per year.

1. Consider a European Call option with maturity of 1 year, with strike of 8 euros and written on stock A. Verify if it is possible to replicate such an option only by means of stock A and of cash invested or borrowed at risk-free rate.
2. Verify if it is possible to replicate the European Call option above by taking long positions in stock A, short positions in stock B and cash. If yes, compute the cost of the replicating strategy.
3. Verify if it is possible to replicate the European Call option above only by means of stock A, of cash and of one (and only one) short position in stock B. If yes, compute the cost of the replicating strategy.

**Solution**

1. The prices of stocks A and B may increase by a factor $u = 1.25$ (growth factor) or decrease by a factor $d = 0.75$. These prices and the payoff of the European Call option on stock A, with maturity of 1 year and strike of 8 euros, can be summarized as follows:

$$
\begin{array}{ccc|c}
 & & \begin{array}{c} S_1^{A,u} = S_0^A u = 10 \\ S_1^{B,u} = S_0^B u = 15 \end{array} & \phi_{Call}^u = 2 \\
 & \nearrow & & \\
\begin{array}{c} S_0^A = 8 \\ S_0^B = 12 \end{array} & & & \\
 & \searrow & & \\
 & & \begin{array}{c} S_1^{A,d} = S_0^A d = 6 \\ S_1^{B,d} = S_0^B u = 9 \end{array} & \phi_{Call}^d = 0 \\
\hline
0 & & 1 \text{ year} & \text{Payoff of Call on A}
\end{array}
$$

In order to verify if the Call option above is attainable by means of stock A and of cash, we need to check if the following system has solutions or not:

$$
\begin{cases}
\Delta^A S_1^{A,u} + x\,(1+r)^T = \phi_{Call}^u \\
\Delta^A S_1^{A,d} + x\,(1+r)^T = \phi_{Call}^d
\end{cases},
$$

where $\Delta^A$ represents the number of shares of stock A to buy or sell and $x$ the amount of cash to be invested or borrowed.

Substituting the price of A, the previous system (of 2 equations in 2 variables) becomes

$$\begin{cases} 10\Delta^A + x\,(1+r) = 2 \\ 6\Delta^A + x\,(1+r) = 0 \end{cases} \quad \begin{cases} 10\Delta^A + x\,(1+r) = 2 \\ 4\Delta^A = 2 \end{cases}$$

$$\begin{cases} x = -\frac{3}{1.04} \\ \Delta^A = \frac{1}{2} \end{cases} \quad \begin{cases} x = -2.88 \\ \Delta^A = \frac{1}{2} \end{cases}$$

Consequently, the Call option can be replicated by buying $\Delta^A = \frac{1}{2}$ shares of stock A and by borrowing 2.88 euros at risk-free rate. The replicating strategy then costs

$$\Delta^A S_0^A + x = 4 - 2.88 = 1.12 \text{ euros.}$$

2. Verifying the existence of a replicating strategy for the Call option above by means of long positions in stock A, of short positions in stock B and of cash is equivalent to verifying if there exist a number $\Delta^A \geq 0$ of shares of stock A, a number $\Delta^B \leq 0$ of shares of stock B and an amount $x$ of cash such that

$$\begin{cases} \Delta^A S_1^{A,u} + \Delta^B S_1^{B,u} + x\,(1+r) = \phi^u \\ \Delta^A S_1^{A,d} + \Delta^B S_1^{B,d} + x\,(1+r) = \phi^d \\ \qquad \Delta^A \geq 0; \quad \Delta^B \leq 0 \end{cases} .$$

Substituting our data, the previous system becomes

$$\begin{cases} 10\Delta^A + 15\Delta^B + x\,(1+r) = 2 \\ 6\Delta^A + 9\Delta^B + x\,(1+r) = 0 \\ \Delta^A \geq 0; \quad \Delta^B \leq 0 \end{cases} \quad \begin{cases} 10\Delta^A + 15\Delta^B + x\,(1+r) = 2 \\ 4\Delta^A + 6\Delta^B = 2 \\ \Delta^A \geq 0; \quad \Delta^B \leq 0 \end{cases}$$

$$\begin{cases} 10\Delta^A + 15\Delta^B + x\,(1+r) = 2 \\ \Delta^A = \frac{1}{2} - \frac{3}{2}\Delta^B \\ \Delta^A \geq 0; \quad \Delta^B \leq 0 \end{cases} \quad \begin{cases} 5 - 15\Delta^B + 15\Delta^B + x\,(1+r) = 2 \\ \Delta^A = \frac{1}{2} - \frac{3}{2}\Delta^B \\ \Delta^A \geq 0; \quad \Delta^B \leq 0 \end{cases}$$

$$\begin{cases} x = -\frac{3}{1.04} = -2.88 \\ \Delta^A = \frac{1}{2} - \frac{3}{2}\Delta^B \\ \Delta^B \leq 0 \end{cases}$$

Consequently, the Call option can be replicated with $\Delta^B \leq 0$ stocks B, $\Delta^A = \frac{1}{2} - \frac{3}{2}\Delta^B$ stocks A and by borrowing 2.88 euros. The replication cost is then equal to

$$\Delta^A S_0^A + \Delta^B S_0^B + x = \left(\frac{1}{2} - \frac{3}{2}\Delta^B\right) \cdot 8 + 12\Delta^B - 2.88 = 1.12 \text{ euros.}$$

3. In the present case, we should verify if it is possible to replicate the Call option only by means of stock A, of cash and by selling one (and only one) short position in stock B. We just need to impose $\Delta^B = -1$ and apply the result obtained in the previous item.

   The required replicating strategy of the Call reduces to

$$\begin{cases} x = -\frac{3}{1.04} = -2.88 \\ \Delta^A = \frac{1}{2} - \frac{3}{2}(-1) \\ \Delta^B = -1 \end{cases} \qquad \begin{cases} x = -\frac{3}{1.04} = -2.88 \\ \Delta^A = 2 \\ \Delta^B = -1 \end{cases}$$

   In the present case, then, the Call option can be replicated by buying 2 shares of stock A, selling one share of stock B and borrowing 2.88 euros. It is immediate to check that such a replicating strategy, as well, costs 1.12 euros.

**Exercise 3.8** The current price of a stock is $S_0 = 8$ euros. In 1 year, such price may move up to 16 euros or to 12 euros, or move down to 8.32 euros. Each event may happen with probability $1/3$.

On the market, the risk-free interest rate is 4% per year and it is possible to sell (at the price of one euro) a European Call option written on the stock above, with maturity of 1 year and with strike of 11 euros.

Consider the following strategy:

- buy one share of the stock;
- sell the Call;
- borrow 7 euros at the risk-free rate.

Verify that such a strategy represents an arbitrage opportunity.

**Solution** Consider strategy (A):

- buy one share of the stock;
- sell the Call;
- borrow 7 euros at the risk-free rate.

The initial value $V_0(A)$ and the final value $V_1(A)$ of strategy A are the following:

| $t = 0$ | $t = 1$ year |
|---|---|
| buy one stock $\Rightarrow \quad -S_0 = -8$ | stock price $S_T$ |
| sell the Call $\Rightarrow \quad +C_0 = 1$ | Call payoff $-(S_T - K)^+$ |
| borrow 7 euros $\Rightarrow \quad +c = 7$ | give back the loan $-c(1+r)^1 = -7 \cdot 1.04 = -7.28$ |
| $V_0(A) = C_0 - S_0 + c = 0$ | $V_1(A) = S_T - (S_T - K)^+ - c(1+r)$ |

Recall that $S_T - (S_T - K)^+ = \min\{S_T; K\}$.

Since $V_0(A) = 0$, in order to verify that the strategy above is an arbitrage opportunity we just need to check that $V_1(A) = S_T - (S_T - K)^+ - c(1+r) = \min\{S_T; K\} - c(1+r) \geq 0$ with $P(V_1(A) > 0) > 0$.

Since $P(S_T = 16) = P(S_T = 12) = P(S_T = 8.32) = \frac{1}{3}$, the final value $V_1(A)$ of the strategy A is given by

$$V_1(A) = \begin{cases} \min\{16; 11\} - 7.28 = 3.72; & \text{with prob. } 1/3 \\ \min\{12; 11\} - 7.28 = 3.72; & \text{with prob. } 1/3 \\ \min\{8.32; 11\} - 7.28 = 1.04; & \text{with prob. } 1/3 \end{cases}$$

Therefore, $V_1(A) \geq 0$ and $P(V_1(A) > 0) = 1 > 0$.

As $V_0(A) = 0$, and because of the previous arguments, strategy A is an arbitrage opportunity.

## 3.3 Proposed Exercises

**Exercise 3.9** A stock A is available on the market at the current price of 12 euros. In 4 months, such price may move up to 16 or down to 10 euros. In each of the next 4-month periods, the stock price may increase or decrease by the same percentage as in the previous period. The risk-free interest rate on the market is 4% per year.

Consider also a stock B whose price is the square of the price of stock A at any time and at any node of the tree.

1. Denote by $S_t$ and $\tilde{S}_t$ the prices (at time $t$) of stocks A and B, and by $u$ and $d$ (respectively, $\tilde{u}$ and $\tilde{d}$) the increase factor and the decrease factor on each period.

Verify that the price of stock B follows a binomial model with $\tilde{S}_0 = S_0^2$, $\tilde{u} = u^2$ and $\tilde{d} = d^2$.

2. Compute the current price of a European Put option on stock B, with maturity of 8 months and strike of 144 euros.
3. Find the replicating strategy of the European Put option. Verify if the result found in item 2. is in line with the replicating strategy cost.
4. Compare the above Put with a similar Put option on stock A with strike of 12 euros. Which one is more expensive?
5. Using the Put-Call Parity, deduce the prices of the corresponding European Call options.

**Exercise 3.10** The current price of a stock is $S_0 = 8$ euros. In 1 year, such price ($S_1$) may move up to 16 euros, to 12 euros or down to 8.32 euros, each with probability $1/3$.

The risk-free interest rate available on the market is 4% per year.

1. Consider the following strategy: buy one share of the stock and borrow 8 euro at the risk-free rate.

   Verify if the strategy is an arbitrage opportunity or not.
2. Discuss what happens if the stock price (in 1 year) assumes the values 16, 12 and 8.32 euros with probability $1/2$, $1/4$ and $1/4$, respectively.
3. Discuss whether the interest rate can be chosen so that the corresponding market model is free of arbitrage.
4. Verify if the strategy in item 1. remains an arbitrage opportunity even if $\tilde{S}_1 = S_1 - 4$ represents the value of another stock in 1 year and the interest rate is 4% per year.

# Chapter 4
# Absence of Arbitrage and Completeness of Market Models

## 4.1 Review of Theory

In the following, we recall the notions of arbitrage, completeness and option pricing in quite general one-period (or multi-period) market models, but always based on a finite sample space.

For a detailed treatment and further details on the subject, we refer to Pliska [37], among others.

Consider a market model consisting in a non-risky asset, called *bond*, whose value will be denoted by $B$, and in $n$ risky assets, called *stocks*, whose values will be denoted by $S^1, S^2, \ldots, S^n$.

Let us focus, first, on a *one-period model*. This means that any asset on the market can be exchanged just at the beginning and at the end of the given interval of time. By convention, let $t = 0$ and $t = 1$ be the corresponding dates.

The initial prices of the (non-risky and risky) assets are known: $B_0$ for the bond, $S_0^1, S_0^2, \ldots, S_0^n$ for the stocks. The bond price is deterministic, equal to

$$B_1 = B_0(1 + r)$$

at $t = 1$, where $r$ is the risk-free interest rate on the period. On the other hand, the prices of the stocks at $t = 1$ will be random, namely

$$S_1^i = S_1^i(\omega), \text{ for } i = 1, 2, \ldots, n,$$

with $\omega \in \Omega = \{\omega_1, \omega_2, \ldots, \omega_m\}$ and $m \in \mathbb{N}$ (finite sample space). Any state of the world will be assumed to be possible, that is $P(\omega_k) > 0$ for any $k = 1, 2, \ldots, m$ (under the real probability $P$ on $\Omega$).

 
E. Rosazza Gianin, C. Sgarra, *Mathematical Finance*,
La Matematica per il 3+2 149, https://doi.org/10.1007/978-3-031-28378-9_4

It is then possible to form different trading portfolios by means of the risky and non-risky assets above. A *trading strategy* or *portfolio* is characterized by a vector

$$\alpha = \left(\alpha^0, \alpha^1, \ldots, \alpha^n\right) \in \mathbb{R}^{n+1},$$

where $\alpha^0$ and $\alpha^i$ $(i = 1, 2, \ldots, n)$ stand, respectively, for the number of non-risky assets and the number of shares of stock $S^i$ in the portfolio. In particular, negative values of $\alpha^j$ $(j = 0, 1, 2, \ldots, n)$ correspond to short positions, while positive values correspond to long positions in asset $j$.

Given a strategy $\alpha$, its value at time $t \in \{0, 1\}$ is given by

$$V_t(\alpha) = \alpha^0 B_t + \sum_{i=1}^{n} \alpha^i S_t^i. \tag{4.1}$$

A strategy $\alpha$ is said to be an *arbitrage opportunity* if $V_0(\alpha) = 0$ and $V_1(\alpha) \geq 0$ with $P(V_1(\alpha) > 0) > 0$. In other words, an arbitrage opportunity is a way to make a profit at zero cost (hence the name of *free lunch*).

It is therefore reasonable to exclude arbitrage opportunities or, better, to focus on market models with no arbitrage opportunities. As already recalled in Chap. 3, the *First Fundamental Theorem of Asset Pricing* guarantees that the one-period market model above has no arbitrage opportunities if and only if there exists (at least) one equivalent martingale measure (EMM, for short), where with *equivalent martingale measure* (or *risk-neutral measure*) we mean a probability measure $Q$ such that $Q(\omega_k) > 0$ for any $k = 1, 2, \ldots, m$ (hence, equivalent) and such that

$$E_Q\left[\frac{S_1^i}{B_1}\right] = S_0^i \tag{4.2}$$

for any $i = 1, 2, \ldots, n$ (hence, martingale measure). The set of all equivalent martingale measures will be denoted by $\mathscr{M}$.

There remains to establish how to evaluate a derivative. First of all, we will distinguish between attainable derivatives and non-attainable derivatives. We recall that a derivative with payoff $\Phi$ is said to be *attainable* if there exists a trading strategy $\alpha$ such that $V_1(\alpha) = \Phi$. If all the derivatives available on the market are attainable (that is, they can be replicated), then the market is said to be *complete*; otherwise, it is said to be *incomplete*.

The following result provides a necessary and sufficient condition for a one-period model to be complete.

**Theorem 4.1** *Suppose that the market model considered is free of arbitrage. It is also complete if and only if the matrix*

$$\begin{bmatrix} B_1(\omega_1) & S_1^1(\omega_1) & \cdots & S_1^n(\omega_1) \\ B_1(\omega_2) & S_1^1(\omega_2) & \cdots & S_1^n(\omega_2) \\ \vdots & \vdots & \vdots & \vdots \\ B_1(\omega_m) & S_1^1(\omega_m) & \cdots & S_1^n(\omega_m) \end{bmatrix}$$

*has rank equal to the cardinality of $\Omega$, that is $m$.*

The *Second Fundamental Theorem of Asset Pricing* guarantees that if a one-period model as above is Free of Arbitrage, then it is complete if and only if there exists a unique equivalent martingale measure (i.e. $\text{card}(\mathscr{M}) = 1$).

If the market model is free of arbitrage, it is well known that:

- a derivative (with payoff $\Phi$) is attainable if and only if $E_Q\left[\frac{\Phi}{B_1}\right]$ is constant for any $Q \in \mathscr{M}$;
- if a derivative (with payoff $\Phi$) is attainable, then its initial price coincides with the cost of its replicating strategy:

$$E_Q\left[\frac{\Phi}{B_1}\right], \qquad \text{with } Q \in \mathscr{M}; \tag{4.3}$$

- if a derivative (with payoff $\Phi$) is not attainable, then its initial price belongs to the *no-arbitrage interval* $(V_-(\Phi), V_+(\Phi))$, where

$$V_-(\Phi) = \inf_{Q\in\mathscr{M}} E_Q\left[\frac{\Phi}{B_1}\right]; \quad V_+(\Phi) = \sup_{Q\in\mathscr{M}} E_Q\left[\frac{\Phi}{B_1}\right]. \tag{4.4}$$

  If the infimum and supremum $V_-(\Phi)$ and $V_+(\Phi)$ are attained, then the no-arbitrage interval is closed.

Consider now a *multi-period model* where transactions may occur at times

$$0, 1, 2, \ldots, T$$

and consisting in a non-risky asset (called *bond*), whose value will be denoted by $B$, and in $n$ risky assets (called *stocks*), whose values will be denoted by $S^1, S^2, \ldots, S^n$. Assume (as in the one-period model) that the sample space is finite

$$\Omega = \{\omega_1, \omega_2, \ldots, \omega_m\}$$

and that any elementary event is possible with respect to a probability $P$ given a priori, that is $P(\omega_k) > 0$ for any $k = 1, \ldots, m$. The information available in time is represented by a filtration $(\mathscr{F}_t)_{t=0,1,\dots,T}$ with $\mathscr{F}_0 = \{\emptyset, \Omega\}$ and $\mathscr{F}_T = \mathscr{P}(\Omega)$, where $\mathscr{P}(\Omega)$ denotes the power set of $\Omega$.

The bond price at $t \in \{1, 2, \dots, T\}$ is assumed to be equal to

$$B_t = B_0(1+r)^t, \tag{4.5}$$

where $B_0 = 1$ and $r$ stands as usual for the risk-free interest rate. The stocks prices $\left(S_t^i\right)_{t=0,1,\dots,T}$ (for $i = 1, 2, \dots, n$) are assumed to be stochastic processes, adapted to $(\mathscr{F}_t)_{t=0,1,\dots,T}$ and such that $S_0^i > 0$ for any $i = 1, 2, \dots, n$.

In the multi-period model, a *trading strategy* is a vector $\alpha = \left(\alpha_t^0, \alpha_t^1, \dots, \alpha_t^n\right)_{t=1,\dots,T}$ of processes, where $\alpha_t^0$ and $\alpha_t^i$ $(i = 1, 2, \dots, n)$ represent, respectively, the number of bonds and stock $S^i$ in the portfolio from time $(t-1)$ to $t$.

The value of a trading strategy $\alpha$ is then given by

$$V_0(\alpha) = \alpha_1^0 B_0 + \sum_{i=1}^{n} \alpha_1^i S_0^i$$

$$V_t(\alpha) = \alpha_t^0 B_t + \sum_{i=1}^{n} \alpha_t^i S_t^i, \quad \text{for } t = 1, \dots, T.$$

A strategy $\alpha$ is said to be *self-financing* if $V_t(\alpha) = \alpha_{t+1}^0 B_t + \sum_{i=1}^n \alpha_{t+1}^i S_t^i$ for any $t = 1, \dots, T$. In other words, a strategy is self-financing when at any time $t$ what one obtains by selling the strategy $\alpha_t$ is enough (and not too much) to finance the purchase of the strategy $\alpha_{t+1}$; hence nothing has to be added and nothing can be withdrawn.

Furthermore, a strategy $\alpha$ is called an *arbitrage opportunity* if it is self-financing and if it satisfies $V_0(\alpha) = 0$ and $V_T(\alpha) \geq 0$, with $P(V_T(\alpha) > 0) > 0$.

Similarly to the one-period case, the *First Fundamental Theorem of Asset Pricing* guarantees that the multi-period model described above is free of arbitrage opportunities if and only if there exists (at least) one equivalent martingale measure, that is, a probability measure $Q$ such that $Q(\omega_k) > 0$ for any $k = 1, 2, \dots, m$ (equivalent) and such that

$$E_Q\left[\left.\frac{S_t^i}{B_t}\right| \mathscr{F}_s\right] = \frac{S_s^i}{B_s}, \tag{4.6}$$

for any $s \leq t$ and $i = 1, 2, \dots, n$ (a martingale measure). The set of all equivalent martingale measures will be denoted by $\mathscr{M}$.

It is also well known that if the multi-period model is free of arbitrage, then all one-period sub-markets are free of arbitrage.

Furthermore, a derivative with payoff $\Phi$ is said to be *attainable* if there exists a self-financing trading strategy $\alpha$ such that $V_T(\alpha) = \Phi$. If all the derivatives available on the market are attainable (that is, they can be replicated), then the market is said to be *complete*; otherwise, it is called *incomplete*.

The *Second Fundamental Theorem of Asset Pricing* guarantees that if a multi-period model as above is free of arbitrage, then it is complete if and only if there exists a unique equivalent martingale measure (i.e. card$(\mathscr{M}) = 1$).

Moreover, if the multi-period model is free of arbitrage, then it is complete if and only if all its one-period sub-markets are complete.

As in the one-period case, if the market model is free of arbitrage it is well known that:

- a derivative (with payoff $\Phi$) is attainable if and only if $E_Q\left[\frac{\Phi}{B_T}\right]$ is constant for any $Q \in \mathscr{M}$;
- if a derivative (with payoff $\Phi$) is attainable, then its price at $t \in \{0, 1, \ldots, T\}$ coincides with the cost of its replicating strategy, which equals

$$V_t(\Phi) = B_t \cdot E_Q\left[\frac{\Phi}{B_T}\middle|\mathscr{F}_t\right], \text{ with } Q \in \mathscr{M}. \tag{4.7}$$

## 4.2 Solved Exercises

**Exercise 4.2** Consider a one-period (annual) market model consisting in a non-risky asset (paying a risk-free rate of 5% per year) and in two stocks with prices $S^1$ and $S^2$:

$$S_0^1 = 10;\ S_1^1(\omega) = \begin{cases} 12; & \text{if } \omega = \omega_1 \\ 8; & \text{if } \omega = \omega_2 \\ 6; & \text{if } \omega = \omega_3 \end{cases}$$

$$S_0^2 = 5;\quad S_1^2(\omega) = \begin{cases} 10; & \text{if } \omega = \omega_1 \\ 4; & \text{if } \omega = \omega_2 \\ 5; & \text{if } \omega = \omega_3 \end{cases},$$

where $P(\omega_1) = P(\omega_2) = P(\omega_3) = \frac{1}{3}$.

Establish if there exist arbitrage opportunities on the market. If yes, give an example.

**Solution** In the present example a one-period market model is considered, where time is measured in years. The bond price evolves as follows:

$$B_0 = 1$$

$$B_1 = 1 + r$$

with $r = 0.05$ per year, while the prices of the stocks can be summarized as

$$
\begin{array}{cccl}
 & \nearrow & & \left(S_1^{1,u}; S_1^{2,u}\right) = (12; 10) \quad (\text{on } \omega_1) \\
\left(S_0^1; S_0^2\right) = (10; 5) & \longrightarrow & & \left(S_1^{1,m}; S_1^{2,m}\right) = (8; 4) \quad (\text{on } \omega_2) \\
 & \searrow & & \left(S_1^{1,d}; S_1^{2,d}\right) = (6; 5) \quad (\text{on } \omega_3) \\
\hline
t = 0 & & & T = 1 \text{ year}
\end{array}
$$

By the First Fundamental Theorem of Asset Pricing, a market is free of arbitrage if and only if there exists an equivalent martingale measure. So, we only need to establish if there exists a probability measure $Q$ satisfying $q_k = Q(\omega_k) > 0$ for any $k = 1, 2, 3$ and

$$
\begin{cases}
S_0^1 = E_Q\left[\frac{S_1^1}{B_1}\right] \\
S_0^2 = E_Q\left[\frac{S_1^2}{B_1}\right]
\end{cases}.
$$

The previous system can be rewritten as

$$
\begin{cases}
\frac{12}{1+r} q_1 + \frac{8}{1+r} q_2 + \frac{6}{1+r} q_3 = 10 \\
\frac{10}{1+r} q_1 + \frac{4}{1+r} q_2 + \frac{5}{1+r} q_3 = 5 \\
q_1 + q_2 + q_3 = 1
\end{cases}
$$

$$
\begin{cases}
12 q_1 + 8 q_2 + 6 q_3 = \frac{21}{2} \\
10 q_1 + 4 q_2 + 5 q_3 = \frac{21}{4} \\
q_1 + q_2 + q_3 = 1
\end{cases},
$$

so its unique solution is

$$
\begin{cases}
q_1 = \frac{5}{16} \\
q_2 = \frac{21}{16} > 1 \\
q_3 = -\frac{5}{8} < 0
\end{cases}
$$

Accordingly, there do not exist any equivalent martingale measures. Hence, the market above is not free of arbitrage.

An example of arbitrage opportunity is given by the strategy $\alpha = \left(\alpha^0, \alpha^1, \alpha^2\right) = (0; -1; 2)$, consisting in zero positions in the bond, a short position in stock $S^1$ and two long positions in stock $S^2$. For such a strategy, indeed,

$$V_0(\alpha) = 0 \cdot B_0 - 1 \cdot S_0^1 + 2 \cdot S_0^2 = -10 + 10 = 0,$$

while

$$\begin{aligned} V_1(\alpha) &= -1 \cdot S_1^1 + 2 \cdot S_1^2 \\ &= \begin{cases} 8; & \text{if } \omega = \omega_1 \\ 0; & \text{if } \omega = \omega_2 \\ 4; & \text{if } \omega = \omega_3 \end{cases} . \end{aligned}$$

Consequently, $V_0(\alpha) = 0$ and $V_1(\alpha) \geq 0$ with $P(V_1(\alpha) > 0) = \frac{2}{3}$.

**Exercise 4.3** Consider a one-period (annual) model formed by a bond (paying a risk-free rate of 5% per year) and by two stocks with prices $S^1$ and $S^2$ evolving as follows:

$$S_0^1 = 10; \qquad S_1^1(\omega) = \begin{cases} 12; \text{ if } \omega = \omega_1 \\ 10; \text{ if } \omega = \omega_2 \\ \ \ 6; \text{ if } \omega = \omega_3 \end{cases}$$

$$S_0^2 = 10; \qquad S_1^2(\omega) = \begin{cases} 15; \text{ if } \omega = \omega_1 \\ \ \ 8; \text{ if } \omega \in \{\omega_2, \omega_3\} \end{cases},$$

with $P(\omega_1), P(\omega_2), P(\omega_3) > 0$ and $P(\omega_1) + P(\omega_2) + P(\omega_3) = 1$.

1. Establish if the market is free of arbitrage and complete.
2. Consider two derivatives (A and B) with maturity $T$ of 1 year and with payoffs $\Phi^A = \left(\frac{S_T^1 + S_T^2}{2} - 8\right)^+$ and $\Phi^B = \left(S_T^1 - S_T^2\right)^+$, respectively. Find their initial prices and their replicating strategies.
3. Discuss whether the market formed only by the bond and by stock $S^1$ would remain free of arbitrage and complete.
4. In the market formed by the bond and by the stock $S^1$, find the no-arbitrage intervals for the prices of derivatives A and B.

**Solution**

1. The market model we are dealing with is a one-period model with initial time 0 and final time 1 year. The bond price evolves as

$$B_0 = 1$$

$$B_1 = 1 + r$$

with $r = 0.05$ per year, while the prices of the stocks can be summarized as follows:

$$
\begin{array}{ccccc}
 & & \left(S_1^{1,u};\, S_1^{2,u}\right) = (12;\, 15) & & (\text{on } \omega_1) \\
 & \nearrow & & & \\
\left(S_0^1;\, S_0^2\right) = (10;\, 10) & \longrightarrow & \left(S_1^{1,m};\, S_1^{2,m}\right) = (10;\, 8) & & (\text{on } \omega_2) \\
 & \searrow & & & \\
 & & \left(S_1^{1,d};\, S_1^{2,d}\right) = (6;\, 8) & & (\text{on } \omega_3) \\
 & & & & \\
t = 0 & & T = 1 \text{ year} & &
\end{array}
$$

In order to establish if the market above is free of arbitrage (or not), we need to verify if there exists (at least) one equivalent martingale measure, that is a probability measure $Q$ satisfying $q_k = Q(\omega_k) > 0$ for any $k = 1, 2, 3$ and

$$
\begin{cases}
S_0^1 = E_Q\left[\frac{S_1^1}{B_1}\right] \\
S_0^2 = E_Q\left[\frac{S_1^2}{B_1}\right]
\end{cases}.
$$

The previous system is equivalent to

$$
\begin{cases}
\frac{12}{1+r}q_1 + \frac{10}{1+r}q_2 + \frac{6}{1+r}q_3 = 10 \\
\frac{15}{1+r}q_1 + \frac{8}{1+r}q_2 + \frac{8}{1+r}q_3 = 10 \\
q_1 + q_2 + q_3 = 1
\end{cases}
$$

$$
\begin{cases}
12q_1 + 10q_2 + 6q_3 = \frac{21}{2} \\
15q_1 + 8q_2 + 8q_3 = \frac{21}{2} \\
q_1 + q_2 + q_3 = 1
\end{cases}.
$$

The systems above have a unique solution $(q_1, q_2, q_3)$ satisfying $q_1, q_2, q_3 \in (0, 1)$, namely $(q_1, q_2, q_3) = \left(\frac{5}{14}, \frac{33}{56}, \frac{3}{56}\right)$. Hence, the market is free of arbitrage and, by the Second Fundamental Theorem of Asset Pricing, also complete. The unique equivalent martingale measure $Q$ is given by $Q(\omega_1) = \frac{5}{14}$, $Q(\omega_2) = \frac{33}{56}$ and $Q(\omega_3) = \frac{3}{56}$.

An alternative way to deduce the completeness of the market consists in observing

$$rk \begin{bmatrix} B_1(\omega_1) & S_1^1(\omega_1) & S_1^2(\omega_1) \\ B_1(\omega_2) & S_1^1(\omega_2) & S_1^2(\omega_2) \\ B_1(\omega_3) & S_1^1(\omega_3) & S_1^2(\omega_3) \end{bmatrix} = rk \begin{bmatrix} 1.05 & 12 & 15 \\ 1.05 & 10 & 8 \\ 1.05 & 6 & 8 \end{bmatrix} = 3,$$

i.e. the rank is equal to the cardinality of the sample space.

2. Since the market formed by the bond and stocks $S^1$ and $S^2$ is free of arbitrage and complete, derivatives A and B (with maturity of 1 year and with payoffs $\Phi^A$ and $\Phi^B$) are attainable. Their initial prices are given, respectively, by

$$V_0\left(\Phi^A\right) = \frac{1}{1+r} E_Q\left[\Phi^A\right]; \qquad V_0\left(\Phi^B\right) = \frac{1}{1+r} E_Q\left[\Phi^B\right]. \tag{4.8}$$

It is immediate to check that

$$\Phi^A(\omega) = \begin{cases} 5.5; \text{ if } \omega = \omega_1 \\ 1; \text{ if } \omega = \omega_2 \\ 0; \text{ if } \omega = \omega_3 \end{cases}; \qquad \Phi^B(\omega) = \begin{cases} 0; \text{ if } \omega = \omega_1 \\ 2; \text{ if } \omega = \omega_2 \\ 0; \text{ if } \omega = \omega_3 \end{cases},$$

hence, by (4.8), we deduce that

$$V_0\left(\Phi^A\right) = \frac{1}{1+r} E_Q\left[\Phi^A\right] = \frac{1}{1.05}\left[5.5 \cdot Q(\omega_1) + 1 \cdot Q(\omega_2) + 0\right] = 2.432$$

$$V_0\left(\Phi^B\right) = \frac{1}{1+r} E_Q\left[\Phi^B\right] = \frac{1}{1.05}\left[0 + 2 \cdot Q(\omega_2) + 0\right] = 1.1224.$$

There remains to find the replicating strategies of the two derivatives above.

A replicating trading strategy for the derivative A with payoff $\Phi^A$ is a vector $\alpha_A = \left(\alpha_A^0, \alpha_A^1, \alpha_A^2\right)$ satisfying $V_1(\alpha) = \Phi^A$; equivalently, a vector solving the following system of equations

$$\begin{cases} \alpha_A^0 \cdot B_1 + \alpha_A^1 \cdot S_1^{1,u} + \alpha_A^2 \cdot S_1^{2,u} = 5.5 \\ \alpha_A^0 \cdot B_1 + \alpha_A^1 \cdot S_1^{1,m} + \alpha_A^2 \cdot S_1^{2,m} = 1 \\ \alpha_A^0 \cdot B_1 + \alpha_A^1 \cdot S_1^{1,d} + \alpha_A^2 \cdot S_1^{2,d} = 0 \end{cases}$$

$$\begin{cases} \alpha_A^0 \cdot 1.05 + \alpha_A^1 \cdot 12 + \alpha_A^2 \cdot 15 = 5.5 \\ \alpha_A^0 \cdot 1.05 + \alpha_A^1 \cdot 10 + \alpha_A^2 \cdot 8 = 1 \\ \alpha_A^0 \cdot 1.05 + \alpha_A^1 \cdot 6 + \alpha_A^2 \cdot 8 = 0 \end{cases}$$

The unique solution of the systems is $\alpha_A = (-5.7823, 0.25, 0.5714)$, giving the required replicating strategy of A. Its cost $V_0(\alpha_A) = -5.7823 + 0.25 \cdot 10 + 0.5714 \cdot 10 = 2.432$ coincides with the initial price of the derivative A (as we should expect, and indeed happens).

Similarly, one obtains that the replicating strategy of option B is $\alpha_B = (\alpha_B^0, \alpha_B^1, \alpha_B^2) = (0.40816, 0.5, -0.42857)$. Consequently, its cost $V_0(\alpha_B) = 0.40816 + 5 - 4.2857 = 1.1225$ coincides with the initial price of derivative B (again, as it should be).

3. Proceeding as above, it is easy to check that the one-period model formed only by the bond and by stock $S^1$ is free of arbitrage but incomplete. There exist, indeed, infinitely many equivalent martingale measures, i.e. probability measures $Q$ satisfying $q_k = Q(\omega_k) > 0$ (for any $k = 1, 2, 3$) and $S_0^1 = E_Q\left[\frac{S_1^1}{B_1}\right]$. In the present case, the set of equivalent martingale measures is given by

$$\mathscr{M} = \left\{ Q \,\middle|\, Q(\omega_1) = q \in \left(\frac{1}{4}; \frac{3}{4}\right); Q(\omega_2) = \frac{9}{8} - \frac{3}{2}q; Q(\omega_3) = \frac{1}{2}q - \frac{1}{8} \right\}.$$

4. According to the previous item, the no-arbitrage interval for the price of the option with payoff $\Phi^A$ is given by

$$\left( \frac{1}{1+r} \inf_{Q \in \mathscr{M}} E_Q\left[\Phi^A\right]; \frac{1}{1+r} \sup_{Q \in \mathscr{M}} E_Q\left[\Phi^A\right] \right).$$

Since $E_Q\left[\Phi^A\right] = 5.5 \cdot Q(\omega_1) + 1 \cdot Q(\omega_2) + 0 = 5.5q + \frac{9}{8} - 1.5q = 4q + \frac{9}{8}$ for $Q \in \mathscr{M}$, we get

$$\frac{1}{1+r} \inf_{Q \in \mathscr{M}} E_Q\left[\Phi^A\right] = \frac{1}{1.05} \inf_{\frac{1}{4} < q < \frac{3}{4}} \left(4q + \frac{9}{8}\right) = \frac{1}{1.05}\left(1 + \frac{9}{8}\right) = 2.0238$$

$$\frac{1}{1+r} \sup_{Q \in \mathscr{M}} E_Q\left[\Phi^A\right] = \frac{1}{1.05} \sup_{\frac{1}{4} < q < \frac{3}{4}} \left(4q + \frac{9}{8}\right) = \frac{1}{1.05}\left(3 + \frac{9}{8}\right) = 3.9286.$$

It follows that the no-arbitrage interval for the price of option A is $(2.0238; 3.9286)$.

Similarly, one can obtain that the no-arbitrage price of option B belongs to the interval $(0; 1.4286)$.

**Exercise 4.4** Consider a bond (paying a risk-free rate of 3% per year) and two stocks with prices $S^1$ and $S^2$ evolving as follows:

$$
\begin{array}{ccccccc}
 & & & & \nearrow & S_2^{1,C} = 14; & S_2^{2,C} = 8 \\
 & & & & \nearrow & S_2^{1,D} = 12; & S_2^{2,D} = 4 \\
 & & S_1^{1,A} = 12; & S_1^{2,A} = 6 & \rightarrow & S_2^{1,E} = 9; & S_2^{2,E} = 2 \\
 & \nearrow & & & \searrow & S_2^{1,F} = 7; & S_2^{2,F} = 9 \\
S_0^1 = 10; \quad S_0^2 = 5 & & & & & & \\
 & \searrow & & & \nearrow & S_2^{1,G} = 8; & S_2^{2,G} = 4 \\
 & & S_1^{1,B} = 6; & S_1^{2,B} = 3 & & & \\
 & & & & \searrow & & \\
 & & & & & S_2^{1,H} = 4; & S_2^{2,H} = 5 \\
\hline
0 & & \text{1 year} & & & \text{2 years} &
\end{array}
$$

Establish if the two-period market model formed by the bond and two stocks above is free of arbitrage or not.

**Solution** Denote by $\omega_1$ the state of the world corresponding to node C where at $T = 2$ the price of the first stock has reached 14 and the price of the second stock has reached 8, by $\omega_2$ the state of the world corresponding to node D, and so on.

By the two-period model described above, we deduce that

$$\Omega = \{\omega_1, \omega_2, \ldots, \omega_6\}$$

and the filtration corresponding to the information available in time is the following one:

$$
\begin{aligned}
\mathscr{F}_0 &= \{\emptyset, \Omega\} \\
\mathscr{F}_1 &= \{\emptyset, \{\omega_1, \omega_2, \omega_3, \omega_4\}, \{\omega_5, \omega_6\}, \Omega\} \\
\mathscr{F}_2 &= \mathscr{P}(\Omega) .
\end{aligned}
$$

We shall write $U = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ and $D = \{\omega_5, \omega_6\}$.

Recall that if a multi-period model is free of arbitrage then all its one-period sub-markets are free of arbitrage.

Let us consider the first one-period sub-market, written in bold:

$$
\begin{array}{ccccccc}
 & & & & \nearrow & S_2^{1,C} = 14; & S_2^{2,C} = 8 \\
 & & & & \nearrow & S_2^{1,D} = 12; & S_2^{2,D} = 4 \\
 & & \mathbf{S_1^{1,A} = 12}; \quad \mathbf{S_1^{2,A} = 6} & & \rightarrow & S_2^{1,E} = 9; & S_2^{2,E} = 2 \\
 & \nearrow & & & \searrow & S_2^{1,F} = 7; & S_2^{2,F} = 9 \\
\mathbf{S_0^1 = 10}; \quad \mathbf{S_0^2 = 5} & & & & & & \\
 & \searrow & & & \nearrow & S_2^{1,G} = 8; & S_2^{2,G} = 4 \\
 & & \mathbf{S_1^{1,B} = 6}; \quad \mathbf{S_1^{2,B} = 3} & & & & \\
 & & & & \searrow & & \\
 & & & & & S_2^{1,H} = 4; & S_2^{2,H} = 5 \\
\hline
0 & & 1\,\text{year} & & & 2\,\text{years} &
\end{array}
$$

The sub-market above is free of arbitrage if and only if there exists at least an equivalent martingale measure, i.e. a probability measure $Q$ such that $Q(U), Q(D) \in (0, 1)$ and

$$
\begin{cases} E_Q\left[\frac{S_1^1}{B_1}\right] = S_0^1 \\ E_Q\left[\frac{S_1^2}{B_1}\right] = S_0^2 \end{cases}.
$$

Calling $q = Q(U)$ and observing that $Q(D) = 1 - Q(U)$, the previous condition becomes

$$
\begin{cases} \frac{S_1^{1,A}}{1+r} \cdot Q(U) + \frac{S_1^{1,B}}{1+r} \cdot Q(D) = S_0^1 \\ \frac{S_1^{2,A}}{1+r} \cdot Q(U) + \frac{S_1^{2,B}}{1+r} \cdot Q(D) = S_0^2 \end{cases}
$$

$$
\begin{cases} \frac{12}{1+r} \cdot q + \frac{6}{1+r} \cdot (1-q) = 10 \\ \frac{6}{1+r} \cdot q + \frac{3}{1+r} \cdot (1-q) = 5 \end{cases}.
$$

Trivially, the systems above reduce to $12q + 6(1-q) = 10(1+r)$. Hence

$$
q = \frac{2}{3} + \frac{5}{3} r \cong 0.72
$$

and

$$
Q(U) = q = 0.72; \quad Q(D) = 1 - q = 0.28.
$$

It follows that the first sub-market is free of arbitrage and, because of the uniqueness of the equivalent martingale measure, complete.

We consider now the one-period sub-market in bold:

$$
\begin{array}{ccccc}
 & & & \nearrow & \mathbf{S_2^{1,C}= 14;\quad S_2^{2,C}= 8} \\
 & & & \nearrow & \mathbf{S_2^{1,D}= 12;\quad S_2^{2,D}= 4} \\
 & & \mathbf{S_1^{1,A}= 12;\quad S_1^{2,A}= 6} & \rightarrow & \mathbf{S_2^{1,E}= 9;\quad S_2^{2,E}= 2} \\
 & \nearrow & & \searrow & \mathbf{S_2^{1,F}= 7;\quad S_2^{2,F}= 9} \\
S_0^1 = 10;\quad S_0^2 = 5 & & & & \\
 & \searrow & & \nearrow & S_2^{1,G} = 8;\quad S_2^{2,G} = 4 \\
 & & S_1^{1,B} = 6;\quad S_1^{2,B} = 3 & & \\
 & & & \searrow & \\
 & & & & S_2^{1,H} = 4;\quad S_2^{2,H} = 5 \\
\text{-----------} & & \text{-------------} & & \text{-------------} \\
0 & & 1\text{ year} & & 2\text{ years}
\end{array}
$$

As previously, the sub-market is free of arbitrage if and only if there exists at least one probability measure $Q\left(\cdot|\,U\right)$ satisfying $q_k = Q\left(\omega_k|\,U\right) > 0$ (for $k = 1, 2, 3, 4$) and

$$
\begin{cases}
E_Q\left[\left.\frac{S_2^1}{B_2}\right| U\right] = \frac{S_1^{1,A}}{B_1} \\
E_Q\left[\left.\frac{S_2^2}{B_2}\right| U\right] = \frac{S_1^{2,A}}{B_1}
\end{cases}.
$$

The condition above becomes

$$
\begin{cases}
\frac{14}{(1+r)^2}\cdot q_1 + \frac{12}{(1+r)^2}\cdot q_2 + \frac{9}{(1+r)^2}\cdot q_3 + \frac{7}{(1+r)^2}\cdot q_4 = \frac{12}{1+r} \\
\frac{8}{(1+r)^2}\cdot q_1 + \frac{4}{(1+r)^2}\cdot q_2 + \frac{2}{(1+r)^2}\cdot q_3 + \frac{9}{(1+r)^2}\cdot q_4 = \frac{6}{1+r} \\
q_1 + q_2 + q_3 + q_4 = 1
\end{cases}
$$

$$
\begin{cases}
7q_1 + 5q_2 + 2q_3 = 5 + 12r \\
-q_1 - 5q_2 - 7q_3 = -3 + 6r \\
q_4 = 1 - q_1 - q_2 - q_3
\end{cases}
$$

Consequently,

$$
\begin{cases}
q_1 = \frac{5}{6}q_3 + \frac{1}{3} + 3r \\
q_2 = \frac{8-27r}{15} - \frac{47}{30}q_3 \\
q_3 \in (0, 1) \\
q_4 = \frac{2}{15} - \frac{6}{5}r - \frac{4}{15}q_3
\end{cases}.
$$

By imposing $q_k > 0$, we obtain

$$\begin{cases} q_1 = \frac{5}{6}q_3 + \frac{1}{3} + 3r \\ q_2 = \frac{8-27r}{15} - \frac{47}{30}q_3 \\ 0 < q_3 < \min\{\frac{2-18r}{4}; \frac{4-18r}{5}; \frac{16-54r}{47}\} = 0.306 \\ q_4 = \frac{2}{15} - \frac{6}{5}r - \frac{4}{15}q_3 \end{cases}.$$

It follows that also the second sub-market is free of arbitrage (but incomplete). Furthermore,

$$Q(\omega_k) = Q(\omega_k | U) \cdot Q(U), \quad \text{for } k = 1, 2, 3, 4.$$

Finally, we consider the last one-period sub-market:

$$\begin{array}{ccccc}
 & & & \nearrow & S_2^{1,C} = 14; \quad S_2^{2,C} = 8 \\
 & & & \nearrow & S_2^{1,D} = 12; \quad S_2^{2,D} = 4 \\
 & & S_1^{1,A} = 12; \quad S_1^{2,A} = 6 & \rightarrow & S_2^{1,E} = 9; \quad S_2^{2,E} = 2 \\
 & \nearrow & & \searrow & S_2^{1,F} = 7; \quad S_2^{2,F} = 9 \\
S_0^1 = 10; \quad S_0^2 = 5 & & & & \\
 & \searrow & & \nearrow & \mathbf{S_2^{1,G} = 8; \quad S_2^{2,G} = 4} \\
 & & \mathbf{S_1^{1,B} = 6; \quad S_1^{2,B} = 3} & & \\
 & & & \searrow & \\
 & & & & \mathbf{S_2^{1,H} = 4; \quad S_2^{2,H} = 5} \\
\text{-----------} & & \text{-------------} & & \text{-------------} \\
0 & & 1\,\text{year} & & 2\,\text{years}
\end{array}$$

As previously, such a sub-market is free of arbitrage if and only if there exists at least one probability measure $Q(\cdot| D)$ satisfying $q_k = Q(\omega_k | D) > 0$ (for $k = 5, 6$) and

$$\begin{cases} E_Q\left[ \frac{S_2^1}{B_2} \middle| D \right] = \frac{S_1^{1,B}}{B_1} \\ E_Q\left[ \frac{S_2^2}{B_2} \middle| D \right] = \frac{S_1^{2,B}}{B_1} \end{cases}.$$

The condition above can be rewritten as

$$\begin{cases} \frac{8}{(1+r)^2} \cdot q_5 + \frac{4}{(1+r)^2} \cdot q_6 = \frac{6}{1+r} \\ \frac{4}{(1+r)^2} \cdot q_5 + \frac{5}{(1+r)^2} \cdot q_6 = \frac{3}{1+r} \\ q_5 + q_6 = 1 \end{cases}$$

$$\begin{cases} 8q_5 + 4q_6 = 6 + 6r \\ 4q_5 + 5q_6 = 3 + 3r \\ q_5 + q_6 = 1 \end{cases} .$$

It is easy to check that the system above is incompatible, hence there does not exist any equivalent martingale measure for the sub-market taken into account. This implies that such sub-market is not free of arbitrage and, consequently, the same holds for the overall two-period market model.

**Exercise 4.5** Consider a two-period model (with annual periods) formed by a non-risky asset ($B$, paying a risk-free rate of 2% per year) and by a stock ($S$) whose price evolves as follows:

$$\begin{array}{ccccc}
 & & & \nearrow & S_2^C = 18 \\
 & & S_1^A = 15 & \rightarrow & S_2^D = 15 \\
 & \nearrow & & \searrow & S_2^E = 10 \\
S_0 = 10 & & & & \\
 & \searrow & & \nearrow & S_2^F = 10 \\
 & & S_1^B = 5 & & \\
 & & & \searrow & \\
 & & & & S_2^G = 4 \\
------ & & -------- & & ---------- \\
0 & & 1\text{ year} & & T = 2\text{ years}
\end{array}$$

Denote by $\omega_1$ (respectively, $\omega_2, \ldots, \omega_5$) the state of the world corresponding to the node C (respectively, D, E, F, G). The probability measure $P$ given a priori satisfies $P(\omega_i) = \frac{1}{5}$ for any $i = 1, 2, \ldots, 5$.

1. Check whether the two-period model above is free of arbitrage and complete.
2. Consider a European Call option with maturity of 2 years and with strike of 10, written on stock $S$. Establish if such option is attainable and compute its price of no arbitrage.
3. Find the no-arbitrage interval for the price of a derivative with payoff

$$\Phi^C = \max\{(S_T - 10)^+ ; C_T\},$$

where $C_T(\omega_1) = C_T(\omega_3) = 4$ and $C_T(\omega_2) = C_T(\omega_4) = C_T(\omega_5) = 0$.

4. Compute the price of the derivative with payoff $\Phi^C$ by means of the martingale measure $Q^* \in \mathscr{M}_1 = \{Q \in \mathscr{M} \,|\, 0.1 \leq Q(\omega_1) \leq 0.2\}$ minimizing the relative entropy between all martingale measures in $\mathscr{M}_1$ and the probability measure $P$.

**Solution** By the two-period model specified above, we deduce that

$$\Omega = \{\omega_1, \omega_2, \ldots, \omega_5\}$$

and the filtration corresponding to the information available is

$$\begin{aligned}
\mathscr{F}_0 &= \{\emptyset, \Omega\} \\
\mathscr{F}_1 &= \{\emptyset, \{\omega_1, \omega_2, \omega_3\}, \{\omega_4, \omega_5\}, \Omega\} \\
\mathscr{F}_2 &= \mathscr{P}(\Omega).
\end{aligned}$$

Denote by $U = \{\omega_1, \omega_2, \omega_3\}$ and by $D = \{\omega_4, \omega_5\}$.

1. Recall that if a multi-period model is free of arbitrage then all its one-period sub-markets are free of arbitrage. Furthermore, it is complete if and only if all its one-period sub-markets are complete.

   Consider now the first one-period sub-market model in bold:

$$\begin{array}{ccccccc}
 & & & & \nearrow & & S_2^C = 18 \\
 & & \mathbf{S_1^A = 15} & & \rightarrow & & S_2^D = 15 \\
 & \nearrow & & & \searrow & & S_2^E = 10 \\
\mathbf{S_0 = 10} & & & & & & \\
 & \searrow & & & \nearrow & & S_2^F = 10 \\
 & & \mathbf{S_1^B = 5} & & & & \\
 & & & & \searrow & & \\
 & & & & & & S_2^G = 4 \\
-\,-\,-\,-\,- & & -\,-\,-\,-\,-\,-\,-\,- & & & & -\,-\,-\,-\,-\,-\,-\,-\,-- \\
0 & & 1\,\text{year} & & & & T = 2\,\text{years}
\end{array}$$

   Such a sub-market is free of arbitrage if and only if there exists at least one probability measure $Q$ satisfying $Q(U), Q(D) \in (0, 1)$ and

$$E_Q\left[\frac{S_1}{B_1}\right] = S_0.$$

   Denoting by $q = Q(U)$ and observing that $Q(D) = 1 - Q(U)$, the condition above becomes

$$\frac{15}{1+r} q + \frac{5}{1+r}(1-q) = 10.$$

Hence

$$Q(U) = q = \frac{1}{2} + r = 0.52$$

$$Q(D) = 1 - q = \frac{1}{2} - r = 0.48.$$

By the arguments above, we conclude that the first one-period sub-market is free of arbitrage and complete (because of the existence and uniqueness of the martingale measure).

We focus now on the one-period sub-market in bold:

$$\begin{array}{ccccc}
 & & & \nearrow & \mathbf{S_2^C = 18} \\
 & & \mathbf{S_1^A = 15} & \rightarrow & \mathbf{S_2^D = 15} \\
 & \nearrow & & \searrow & \mathbf{S_2^E = 10} \\
S_0 = 10 & & & & \\
 & \searrow & & \nearrow & S_2^F = 10 \\
 & & S_1^B = 5 & & \\
 & & & \searrow & \\
 & & & & S_2^G = 4 \\
----- & & -------- & & ---------- \\
0 & & 1\,\text{year} & & T = 2\,\text{years}
\end{array}$$

As previously, such a sub-market is free of arbitrage if and only if there exists at least one probability measure $Q(\cdot|\,U)$ (conditional probability, given the event $U$) satisfying $q_k = Q(\omega_k|\,U) > 0$ (for $k = 1, 2, 3$), $q_1 + q_2 + q_3 = 1$ and

$$E_Q\left[\left.\frac{S_2}{B_2}\right| U\right] = \frac{S_1^A}{B_1}.$$

The condition above becomes

$$\begin{cases} \frac{18}{(1+r)^2} \cdot q_1 + \frac{15}{(1+r)^2} \cdot q_2 + \frac{10}{(1+r)^2} \cdot q_3 = \frac{15}{1+r} \\ q_1 + q_2 + q_3 = 1 \end{cases}$$

$$\begin{cases} 8q_1 + 5q_2 = 5 + 15r \\ q_3 = 1 - q_1 - q_2 \end{cases}$$

Hence

$$\begin{cases} q_1 \in (0, 1) \\ q_2 = 1 + 3r - \frac{8}{5}q_1 \\ q_3 = \frac{3}{5}q_1 - 3r \end{cases}.$$

Nevertheless, we have to restrict our attention only to those solutions that correspond to equivalent martingale measures. We need, therefore, to impose $q_k > 0$ for any $k = 1, 2, 3$, obtaining that

$$\begin{cases} 0.1 = 5r < q_1 < \frac{5}{8}(1+3r) \cong 0.66 \\ q_2 = 1.06 - \frac{8}{5}q_1 \\ q_3 = \frac{3}{5}q_1 - 0.06 \end{cases}.$$

By the arguments above, the sub-market taken into account is free of arbitrage but incomplete.

Furthermore, $Q(\omega_k) = Q(\omega_k | U) \cdot Q(U)$ for $k = 1, 2, 3$, so

$$\begin{aligned} Q(\omega_1) &= \left(\frac{1}{2} + r\right) q_1 \\ Q(\omega_2) &= \left(\frac{1}{2} + r\right)\left(1.06 - \frac{8}{5}q_1\right) \\ Q(\omega_3) &= \left(\frac{1}{2} + r\right)\left(\frac{3}{5}q_1 - 0.06\right). \end{aligned}$$

We finally focus on the last one-period sub-market in bold:

$$\begin{array}{ccccccc}
 & & & & \nearrow & & S_2^C = 18 \\
 & & S_1^A = 15 & & \rightarrow & & S_2^D = 15 \\
 & \nearrow & & & \searrow & & S_2^E = 10 \\
S_0 = 10 & & & & & & \\
 & \searrow & & & \nearrow & & \mathbf{S_2^F = 10} \\
 & & \mathbf{S_1^B = 5} & & & & \\
 & & & & \searrow & & \\
 & & & & & & \mathbf{S_2^G = 4} \\
\text{-----} & & \text{--------} & & & & \text{----------} \\
0 & & 1\text{ year} & & & & T = 2\text{ years}
\end{array}$$

As previously, such a sub-market is free of arbitrage if and only if there exists at least one probability measure $Q(\cdot | D)$ (conditional probability, given the event $D$) satisfying $q_k = Q(\omega_k | D) > 0$ (for $k = 4, 5$), $q_4 + q_5 = 1$ and

$$E_Q\left[\frac{S_2}{B_2}\bigg| D\right] = \frac{S_1^B}{B_1}.$$

The previous condition becomes

$$\begin{cases} \frac{10}{(1+r)^2} q_4 + \frac{4}{(1+r)^2} q_5 = \frac{5}{1+r} \\ q_4 + q_5 = 1 \end{cases},$$

that admits as unique solution

$$\begin{cases} q_4 = \frac{1}{6}(1+5r) \in (0,1) \\ q_5 = \frac{5}{6}(1-r) \in (0,1) \end{cases}.$$

Accordingly, such a sub-market is free of arbitrage and complete. Furthermore, $Q(\omega_k) = Q(\omega_k | D) \cdot Q(D)$ for $k = 4, 5$, so

$$Q(\omega_4) = \frac{1}{6}\left(\frac{1}{2} - r\right)(1+5r)$$

$$Q(\omega_5) = \frac{5}{6}\left(\frac{1}{2} - r\right)(1-r).$$

To conclude, the two-period model taken into account is free of arbitrage and incomplete. The set of all equivalent martingale measures for the two-period model is

$$\mathscr{M} = \left\{ Q \;\middle|\; \begin{array}{l} Q(\omega_1) = \left(\frac{1}{2} + r\right) q_1 \\ Q(\omega_2) = \left(\frac{1}{2} + r\right)\left(1 + 3r - \frac{8}{5} q_1\right) \\ Q(\omega_3) = \left(\frac{1}{2} + r\right)\left(\frac{3}{5} q_1 - 3r\right) \\ Q(\omega_4) = \frac{1}{6}\left(\frac{1}{2} - r\right)(1+5r) \\ Q(\omega_5) = \frac{5}{6}\left(\frac{1}{2} - r\right)(1-r) \end{array} \quad ;\ \text{for } 5r < q_1 < \frac{5}{8}(1+3r) \right\}. \tag{4.9}$$

2. Due to the incompleteness of the above two-period model, there is no guarantee that the European Call option with maturity of 2 years, with strike of 10 and written on $S$ is attainable. The payoff of the Call option is given by

$$\Phi(\omega) = (S_T(\omega) - 10)^+ = \begin{cases} 8; & \omega = \omega_1 \\ 5; & \omega = \omega_2 \\ 0; & \omega \in \{\omega_3, \omega_4, \omega_5\} \end{cases}.$$

Looking for a self-financing replicating strategy for the option with payoff $\Phi$ is the same as looking for a self-financing replicating strategy $(\alpha_t)_{t=1,2}$ with $\alpha_t = \left(\alpha_t^B, \alpha_t^S\right)$ (where $\alpha^B$, resp. $\alpha^S$, represents the number of bonds, resp. of stocks, to be held in portfolio).

At maturity ($T = 2$), such a strategy (if it exists) has to satisfy $V_2(\alpha) = \Phi$, i.e.

$$\begin{cases} \alpha_2^{B,u} \cdot B_2 + \alpha_2^{S,u} \cdot 18 = 8 \\ \alpha_2^{B,u} \cdot B_2 + \alpha_2^{S,u} \cdot 15 = 5 \\ \alpha_2^{B,u} \cdot B_2 + \alpha_2^{S,u} \cdot 10 = 0 \\ \alpha_2^{B,d} \cdot B_2 + \alpha_2^{S,d} \cdot 10 = 0 \\ \alpha_2^{B,d} \cdot B_2 + \alpha_2^{S,d} \cdot 4 = 0 \end{cases}$$

where $\alpha_2^{\cdot,u}$ and $\alpha_2^{\cdot,d}$ denote, respectively, the number of shares to be held in portfolio from time 1 to time 2 in the node "up" or "down" at $t = 1$. The unique $\alpha_2$ satisfying the system above is given by

$$\begin{cases} \alpha_2^{B,u} = -\frac{10}{(1+r)^2} \\ \alpha_2^{S,u} = 1 \\ \alpha_2^{B,d} = 0 \\ \alpha_2^{S,d} = 0 \end{cases}.$$

It follows that $V_1^u(\alpha) = \alpha_2^{B,u} \cdot B_1 + \alpha_2^{S,u} \cdot S_1^A = \frac{5+15r}{1+r}$ and $V_1^d(\alpha) = \alpha_2^{B,d} \cdot B_1 + \alpha_2^{S,d} \cdot S_1^B = 0$.

At time $t = 1$, a replicating self-financing strategy has to satisfy

$$\begin{cases} \alpha_1^B \cdot B_1 + \alpha_1^S \cdot S_1^A = V_1^u(\alpha) \\ \alpha_1^B \cdot B_1 + \alpha_1^S \cdot S_1^B = V_1^d(\alpha) \end{cases},$$

hence

$$\begin{cases} \alpha_1^B(1+r) + \alpha_1^S \cdot 15 = \frac{5+15r}{1+r} \\ \alpha_1^B(1+r) + \alpha_1^S \cdot 5 = 0 \end{cases}$$

$$\begin{cases} \alpha_1^B = -\frac{5(1+3r)}{2(1+r)^2} \\ \alpha_1^S = \frac{1+3r}{2(1+r)} \end{cases}.$$

The European Call option of concern is then attainable, so the price of no arbitrage has to coincide with the cost of the self-financing replicating strategy, that is

$$V_0 = \alpha_1^B \cdot B_0 + \alpha_1^S \cdot S_0 = -\frac{5(1+3r)}{2(1+r)^2} + \frac{1+3r}{2(1+r)} \cdot 10 = 2.6490. \tag{4.10}$$

An alternative way to compute the option price is to find the no-arbitrage interval and observe that it reduces to a point, namely the value $V_0$ in (4.10).

3. The payoff of the claim C is equal to

$$\Phi^C(\omega) = \max\{(S_T(\omega) - 10)^+; C_T(\omega)\} = \begin{cases} 8; & \omega = \omega_1 \\ 5; & \omega = \omega_2 \\ 4; & \omega = \omega_3 \\ 0; & \omega \in \{\omega_4, \omega_5\} \end{cases}.$$

It is easy to check that such a claim is not attainable in the market we are dealing with. We will then look for the no-arbitrage interval for its price.

Let $Q$ be an equivalent martingale measure, i.e. $Q \in \mathscr{M}$. From (4.9) it follows that

$$\begin{aligned} &E_Q\left[\frac{\Phi^C}{B_2}\right] \\ &= \frac{1}{(1+r)^2}\left[8 \cdot Q(\omega_1) + 5 \cdot Q(\omega_2) + 4 \cdot Q(\omega_3)\right] \\ &= \frac{1}{(1+r)^2}\left[8\left(\frac{1}{2}+r\right)q_1 + 5\left(\frac{1}{2}+r\right)\left(1+3r-\frac{8}{5}q_1\right) + 4\left(\frac{1}{2}+r\right)\left(\frac{3}{5}q_1 - 3r\right)\right] \\ &= \frac{1+2r}{2(1+r)^2}\left[5+3r+\frac{12}{5}q_1\right]. \end{aligned}$$

Hence

$$\begin{aligned} \inf_{Q \in \mathscr{M}} E_Q\left[\frac{\Phi^C}{B_2}\right] &= \inf_{5r < q_1 < \frac{5}{8}(1+3r)} \left\{\frac{1+2r}{2(1+r)^2}\left[5+3r+\frac{12}{5}q_1\right]\right\} \\ &= \frac{5(1+2r)(1+3r)}{2(1+r)^2} = 2.65 \end{aligned}$$

and

$$\begin{aligned} \sup_{Q \in \mathscr{M}} E_Q\left[\frac{\Phi^C}{B_2}\right] &= \sup_{5r < q_1 < \frac{5}{8}(1+3r)} \left\{\frac{1+2r}{2(1+r)^2}\left[5+3r+\frac{12}{5}q_1\right]\right\} \\ &= \frac{(13+15r)(1+2r)}{4(1+r)^2} = 3.32. \end{aligned}$$

The price of option C belongs then to the no-arbitrage interval (2.65; 3.32).

4. As verified in the previous items, the market considered is free of arbitrage and incomplete (there exist indeed infinitely many equivalent martingale measures) and the derivative with payoff $\Phi^C$ is not attainable. A priori, therefore, we are only able to find the no-arbitrage interval for the price of such a claim.

   A criterion to "choose" one among the infinitely many equivalent martingale measures (using which we can evaluate the derivative C) is the *Minimal Relative Entropy Criterion*. More precisely, such a criterion suggests to select the equivalent martingale measure $Q$ that minimizes the relative entropy of $Q$ with respect to $P$:

$$H(Q, P) \triangleq E_P\left[\frac{dQ}{dP}\ln\left(\frac{dQ}{dP}\right)\right].$$

In the present case, $H(Q, P)$ (with $Q \in \mathscr{M}_1$) becomes

$$\begin{aligned}
&H(Q, P) = E_P\left[\tfrac{dQ}{dP}\ln\left(\tfrac{dQ}{dP}\right)\right] \\
&= \textstyle\sum_{i=1}^5 \left[\frac{Q(\omega_i)}{P(\omega_i)}\ln\left(\frac{Q(\omega_i)}{P(\omega_i)}\right)\right] P(\omega_i) \\
&= \textstyle\sum_{i=1}^5 Q(\omega_i)\left[\ln(Q(\omega_i)) - \ln(P(\omega_i))\right] \\
&= \textstyle\sum_{i=1}^5 Q(\omega_i)\ln(Q(\omega_i)) + \ln 5 \\
&= \left(\tfrac{1}{2}+r\right)\cdot\left[q_1\ln(q_1) + \left(1+3r-\tfrac{8}{5}q_1\right)\ln\left(1+3r-\tfrac{8}{5}q_1\right)\right. \\
&\quad \left.+\left(\tfrac{3}{5}q_1 - 3r\right)\ln\left(\tfrac{3}{5}q_1-3r\right)\right] \\
&\quad +\left(\tfrac{1}{2}-r\right)\left[\tfrac{1}{6}(1+5r)\ln\left(\tfrac{1}{6}(1+5r)\right)+\tfrac{5}{6}(1-r)\ln\left(\tfrac{5}{6}(1-r)\right)\right] \\
&\quad +\left(\tfrac{1}{2}+r\right)\ln\left(\tfrac{1}{2}+r\right)+\left(\tfrac{1}{2}-r\right)\ln\left(\left(\tfrac{1}{2}-r\right)\right)+\ln 5.
\end{aligned}$$

We deduce that

$$\begin{aligned}
&\arg\min_{Q\in\mathscr{M}_1} H(Q, P) \\
&= \arg\min_{Q\in\mathscr{M}_1}\left[\begin{array}{l} q_1\ln(q_1) + \left(1+3r-\frac{8}{5}q_1\right)\ln\left(1+3r-\frac{8}{5}q_1\right) \\ +\left(\frac{3}{5}q_1-3r\right)\ln\left(\frac{3}{5}q_1-3r\right)\end{array}\right].
\end{aligned} \tag{4.11}$$

Since $Q \in \mathscr{M}_1$ if and only if $Q \in \mathscr{M}$ and $0.1 \leq Q(\omega_1) \leq 0.2$ (or, equivalently, $0.1 \leq \left(\frac{1}{2}+r\right)q_1 \leq 0.2$), we deduce that $Q \in \mathscr{M}_1$ if and only if

$$\begin{cases} 0.1 \leq \left(\frac{1}{2}+r\right)q_1 \leq 0.2 \\ 5r < q_1 < \frac{5}{8}(1+3r)\end{cases} \quad \begin{cases} \frac{1}{5+10r} \leq q_1 \leq \frac{2}{5+10r} \\ 5r < q_1 < \frac{5}{8}(1+3r)\end{cases}$$

$$\begin{cases} 0.199 \leq q_1 \leq 0.398 \\ 0.1 < q_1 < 0.6625\end{cases},$$

hence $q_1 \in [0.199; 0.398]$. It follows that (4.11) can be rewritten as

$$\arg\min_{Q\in\mathscr{M}_1} H(Q, P) = \arg\min_{q_1\in[0.199;0.398]} g(q_1),$$

where $g(q_1) = q_1 \ln(q_1) + \left(1 + 3r - \frac{8}{5}q_1\right) \ln\left(1 + 3r - \frac{8}{5}q_1\right) + \left(\frac{3}{5}q_1 - 3r\right) \cdot$ $\cdot \ln\left(\frac{3}{5}q_1 - 3r\right)$. Since $g'(q_1) = \ln(q_1) - \frac{8}{5}\ln\left(1 + 3r - \frac{8}{5}q_1\right) + \frac{3}{5}\ln\left(\frac{3}{5}q_1 - 3r\right)$ $\leq 0$ on the interval $[0.199; 0.398]$, $\min_{Q \in \mathscr{M}_1} H(Q, P)$ is attained at $q_1^* = \frac{2}{5+10r} = 0.398$. Accordingly, the equivalent martingale measure chosen by the criterion is given by

$$
\begin{aligned}
Q^*(\omega_1) &= \left(\tfrac{1}{2} + r\right) q_1^* = 0.2 \\
Q^*(\omega_2) &= \left(\tfrac{1}{2} + r\right)\left(1 + 3r - \tfrac{8}{5}q_1^*\right) = 0.22 \\
Q^*(\omega_3) &= \left(\tfrac{1}{2} + r\right)\left(\tfrac{3}{5}q_1^* - 3r\right) = 0.09 \\
Q^*(\omega_4) &= \tfrac{1}{6}\left(\tfrac{1}{2} - r\right)(1 + 5r) = 0.09 \\
Q^*(\omega_5) &= \tfrac{5}{6}\left(\tfrac{1}{2} - r\right)(1 - r) = 0.4
\end{aligned}
.
$$

It follows that the price of the derivative C (evaluated by means of $Q^*$) is equal to

$$
\begin{aligned}
E_{Q^*}\left[\frac{\Phi^C}{B_2}\right] &= \frac{1}{(1+r)^2}\left[8 \cdot Q^*(\omega_1) + 5 \cdot Q^*(\omega_2) + 4 \cdot Q^*(\omega_3)\right] \\
&= \frac{1}{(1.02)^2}\left[8 \cdot 0.2 + 5 \cdot 0.22 + 4 \cdot 0.09\right] = 2.94
\end{aligned}
$$

which, as expected, belongs to the no-arbitrage interval of the price of C found previously.

## 4.3 Proposed Exercises

**Exercise 4.6** Consider a one-period (annual) model formed by a non-risky asset ($B$, paying a risk-free interest rate of 4% per year) and by a stock ($S$) whose price is

$$
S_0 = 5; \quad S_1(\omega) = \begin{cases} 8; & \text{if } \omega = \omega_1 \\ 5; & \text{if } \omega = \omega_2 \\ 3; & \text{if } \omega = \omega_3 \end{cases},
$$

with $P(\omega_1), P(\omega_2), P(\omega_3) > 0$.

1. Establish if the market above is free of arbitrage and complete.
2. Find the no-arbitrage price (or the no-arbitrage interval for the price) of a European Put option on the underlying $S$, with strike of 5 euros and with maturity of 1 year.
3. Suppose now that an additional asset is introduced on the market. Such an asset is a derivative (denoted by $C$) with initial price of 1 euro and with payoff

$$\Phi^C(\omega) = \begin{cases} 0; \text{ if } \omega = \omega_1 \\ 2; \text{ if } \omega = \omega_2 \\ 1; \text{ if } \omega = \omega_3 \end{cases}.$$

   Discuss if the new market (formed by $B, S$ and the derivative $C$) is free of arbitrage and complete.

**Exercise 4.7** Consider a two-period model (with annual periods) formed by a non-risky asset ($B$, paying a risk-free interest rate of 4% per year) and by a stock ($S$) whose price evolves as follows

$$\begin{array}{ccccc}
 & & & \nearrow & S_2^D = S_1^A \cdot u \\
 & & S_1^A = 8 & \rightarrow & S_2^E = S_1^A \cdot d \\
 & \nearrow & & & \\
S_0 = 5 & \longrightarrow & S_1^B = 5 & \longrightarrow & S_2^F = S_1^B \cdot u \\
 & \searrow & & \searrow & S_2^G = S_1^B \cdot d \\
 & & S_1^C = 3 & \longrightarrow & S_2^H = S_1^C \cdot u \\
 & & & \searrow & S_2^L = S_1^C \cdot d \\
 & & & & \\
---- & & ------- & & --------- \\
0 & & 1\text{ year} & & T = 2\text{ years}
\end{array}$$

with $u = 1.5$ and $d = 0.5$.

1. Establish if the two-period market above is free of arbitrage and complete.
2. Find the no-arbitrage price (or the no-arbitrage interval for the price) of a European Call option on the underlying $S$, with strike of 7 euros and with maturity of 2 years.

**Exercise 4.8** Consider a one-period (annual) model formed by a non-risky asset ($B$, paying a risk-free interest rate of 4% per year) and by two stocks ($S^1$ and $S^2$) whose prices evolve as follows

$$
\begin{array}{lll}
 & \nearrow & S_1^{1,A} = 12; \quad S_1^{2,A} = 15 \\
 & \nearrow & S_1^{1,B} = 10; \quad S_1^{2,B} = 8 \\
S_0^1 = 10; \quad S_0^2 = 10 & & \\
 & \searrow & S_1^{1,C} = 8; \quad S_1^{2,C} = 4 \\
 & \searrow & S_1^{1,D} = 4; \quad S_1^{2,D} = 8 \\
\hline
0 & & 1 \text{ year}
\end{array}
$$

Establish if the one-period market above is free of arbitrage and complete.

# Chapter 5
# Itô's Formula and Stochastic Differential Equations

## 5.1 Review of Theory

Given a stochastic process $(V_t)_{t\geq 0}$ having trajectories with bounded variation and a sufficiently regular function $f$, it is possible to define the integral of $Z_t = f(V_t)$ with respect to $dV_t$ as follows

$$\int_0^t Z_s dV_s = \int_0^t f(V_s)dV_s, \tag{5.1}$$

i.e. in the sense of a Riemann-Stieltjes integral.

On the other hand, if the trajectories of the process $(V_t)_{t\geq 0}$ are not of bounded variation and if the function $f$ is not regular enough, the integral above may not make sense as a Riemann-Stieltjes integral. This is the case of the Brownian motion, for which (in general) one cannot define the integral

$$\int f(W_s)dW_s$$

as a Riemann-Stieltjes integral.

A way out is to introduce *Itô's stochastic integral* $\int_0^t H_s dW_s$ for processes $(H_t)_{t\geq 0}$ that satisfy a "suitable measurability assumption" (for instance, progressive measurability) together with $P\left(\int_0^t H_s^2 ds < +\infty\right) = 1$. Under these hypotheses, one can write the following equation

$$X_t = X_0 + \int_0^t g(X_s, s)ds + \int_0^t \sigma(X_s, s)dW_s, \tag{5.2}$$

where (under some suitable regularity hypothesis on $g$ and $\sigma$) the first integral can be understood as a Riemann-Stieltjes integral, while the second as an Itô integral.



E. Rosazza Gianin, C. Sgarra, *Mathematical Finance*,
La Matematica per il 3+2 149, https://doi.org/10.1007/978-3-031-28378-9_5

The previous expression can also be written as:

$$\begin{cases} dX_t = g(X_t, t)dt + \sigma(X_t, t)dW_t \\ X_0 = x \end{cases},$$

which is referred to as a *Stochastic Differential Equation* (SDE). Note that the two formulations are equivalent: the latter is just a different way to write the former.

A fundamental result in stochastic analysis is the so-called Itô's Lemma.

In the following, we will denote with $f(x,t) \in C^{2,1}(\mathbb{R}, \mathbb{R}^+)$ a function that is continuously differentiable twice in $x$ and once in $t$. Similarly for $f(x_1, \dots, x_n, t) \in C^{2,1}(\mathbb{R}^n, \mathbb{R}^+)$.

**Lemma 5.1 (Itô's Lemma for Functions of One Variable Plus Time)** *Let $f(x,t) \in C^{2,1}(\mathbb{R}, \mathbb{R}^+)$ be a given function.*

*If the process $(X_t)_{t \geq 0}$ satisfies the following equation:*

$$dX_t = \mu(t)\, dt + \sigma(t)\, dW_t, \tag{5.3}$$

*then the process $(Z_t)_{t \geq 0}$, defined by $Z_t \triangleq f(X_t, t)$, satisfies*

$$\begin{aligned} dZ_t &= \frac{\partial f}{\partial t}(X_t, t)\, dt + \frac{\partial f}{\partial x}(X_t, t)\, dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(X_t, t)\, \sigma^2(t)\, dt \\ &= \left(\frac{\partial f}{\partial t} + \mu \frac{\partial f}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial x^2}\right)(X_t, t)\, dt + \sigma(t)\frac{\partial f}{\partial x}(X_t, t)\, dW_t. \end{aligned} \tag{5.4}$$

For functions of two or more variables plus time, Itô's Lemma can be formulated as follows.

**Lemma 5.2 (Itô's Lemma for Functions of Two or More Variables Plus Time)** *Let $f(x_1, x_2, \dots, x_n, t)$ be a function in $C^{2,1}(\mathbb{R}^n, \mathbb{R}^+)$ depending on $n$ variables $x_1, \dots, x_n$ and on $t$.*

*If, for any $i = 1, 2, \dots, n$, $X^i = \left(X_t^i\right)_{t \geq 0}$ satisfies the following equation:*

$$dX_t^i = \mu^i(t)\, dt + \sigma^i(t)\, dW_t^i, \tag{5.5}$$

*then the process $(Z_t)_{t \geq 0}$, defined by $Z_t \triangleq f\left(X_t^1, X_t^2, \dots, X_t^n, t\right)$, satisfies*

$$\begin{aligned} dZ_t = &\frac{\partial f}{\partial t}\left(X_t^1, \dots, X_t^n, t\right) dt + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\left(X_t^1, \dots, X_t^n, t\right) dX_t^i \\ &+ \frac{1}{2}\sum_{i,j=1}^{n} \sigma^i \sigma^j \rho_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j}\left(X_t^1, \dots, X_t^n, t\right) dt \end{aligned}$$

$$= \left[ \frac{\partial f}{\partial t} + \sum_{i=1}^{n} \mu^i \frac{\partial f}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{n} \sigma^i \sigma^j \rho_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j} \right] \left( X_t^1, \ldots, X_t^n, t \right) dt$$

$$+ \sum_{i=1}^{n} \sigma^i \frac{\partial f}{\partial x_i} \left( X_t^1, \ldots, X_t^n, t \right) dW_t^i, \tag{5.6}$$

*where $\rho_{ij} = E\left[dW^i dW^j\right]/dt$ is the correlation between the (standard) Brownian motions $W^i$ and $W^j$.*

When $f$ does not depend on time $t$, Eq. (5.4) simplifies to

$$dZ_t = \left( \mu f'(X_t) + \frac{\sigma^2}{2} f''(X_t) \right) dt + \sigma f'(X_t) \, dW_t, \tag{5.7}$$

while Eq. (5.6) reduces to

$$dZ_t = \left[ \sum_{i=1}^{n} \mu^i \frac{\partial f}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{n} \sigma^i \sigma^j \rho_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j} \right] \left( X_t^1, \ldots, X_t^n \right) dt$$

$$+ \sum_{i=1}^{n} \sigma^i \frac{\partial f}{\partial x_i} \left( X_t^1, \ldots, X_t^n \right) dW_t^i. \tag{5.8}$$

We finally recall the notion of *Brownian local time* and the Tanaka formula. The *occupation time* of a set $A$ by time $t$ of a Brownian motion is defined by $Y_t^A \triangleq \int_0^t 1_A(W_u)\, du$, where $1_A(x)$ is the indicator function of the set $A$ taking value 1 if $x \in A$, taking value 0 if $x \notin A$. The Brownian local time $L_t$ is the Radon-Nikodym derivative of $Y_t$ with respect to the Lebesgue measure on the real line (they can be proved to be equivalent), i.e.

$$Y_t^A = \int_A L_t(u)\, du. \tag{5.9}$$

The Tanaka formula provides a representation of the function of a Brownian motion $f(x) = (x - K)^+ = \max\{x - K; 0\}$ in terms of the Brownian local time of $K$. Namely,

$$(W_t - K)^+ = (W_0 - K)^+ + \int_0^t \mathbf{1}_{(K,+\infty)}(W_u) dW_u + \frac{1}{2} L_t(K), \tag{5.10}$$

where $L_t(K)$ can be proved to be given by

$$L_t(K) = \int_0^t \delta_K(W_u) du \tag{5.11}$$

and $\delta_K(x)$ is the *Dirac Delta distribution* defined by:

$$\int_0^t \delta_K(W_u)du = \lim_{n\to+\infty}\left(n\int_0^t \mathbf{1}_{[K-\frac{1}{2n},K+\frac{1}{2n})}(W_u)du\right). \tag{5.12}$$

For an exhaustive treatment we recommend the texts by Björk [6], Mikosch [32], Øksendal [34] and Pascucci [35] among many others.

## 5.2 Solved Exercises

**Exercise 5.3** Let $(W_t)_{t\geq 0}$ be a standard Brownian motion, and assume $\left(S_t^1\right)_{t\geq 0}$ and $\left(S_t^2\right)_{t\geq 0}$ satisfy the following SDEs:

$$\begin{aligned} dS_t^1 &= \mu_1 S_t^1 dt + \sigma_1 S_t^1 dW_t; \quad S_0^1 = s_0^1 > 0 \\ dS_t^2 &= \mu_2 S_t^2 dt + \sigma_2 S_t^2 dW_t; \quad S_0^2 = s_0^2 > 0 \end{aligned}$$

with $\mu_1, \mu_2 \in \mathbb{R}$ and $\sigma_2 > \sigma_1 > 0$.

1. For $f(x) = \ln x$, find the stochastic differential equation satisfied by the process $\left(f\left(S_t^1\right)\right)_{t\geq 0}$ and the SDE satisfied by $\left(f\left(S_t^2\right)\right)_{t\geq 0}$.
2. Find the stochastic differential equation satisfied by $Y_t = g\left(S_t^1, S_t^2\right) = \ln\left(\frac{S_t^1}{S_t^2}\right)$ when $\mu = \mu_1 = \mu_2$. Establish whether $(Y_t)_{t\geq 0}$ can be a geometric Brownian motion and/or a Brownian motion and/or a Brownian motion with drift.
3. Do the same as above but this time for $Z_t = h\left(S_t^1, S_t^2\right) = \ln\left(S_t^1 \cdot S_t^2\right)$.

**Solution**

1. Since $f'(x) = \frac{1}{x}$ and $f''(x) = -\frac{1}{x^2}$, Itô's formula implies that

$$\begin{aligned} df\left(S_t^1\right) = d\left(\ln S_t^1\right) &= f'\left(S_t^1\right)dS_t^1 + \frac{1}{2}f''\left(S_t^1\right)\left[\sigma_1 S_t^1\right]^2 dt \\ &= \frac{1}{S_t^1}dS_t^1 - \frac{1}{2}\frac{1}{\left(S_t^1\right)^2}\left[\sigma_1 S_t^1\right]^2 dt \\ &= \frac{1}{S_t^1}\left[\mu_1 S_t^1 dt + \sigma_1 S_t^1 dW_t\right] - \frac{1}{2}\sigma_1^2 dt \\ &= \left(\mu_1 - \frac{1}{2}\sigma_1^2\right)dt + \sigma_1 dW_t. \end{aligned}$$

It follows that $\left(f\left(S_t^1\right)\right)_{t\geq 0}$ satisfies the following stochastic differential equation:

$$\begin{cases} df\left(S_t^1\right) = \left(\mu_1 - \frac{1}{2}\sigma_1^2\right)dt + \sigma_1 dW_t \\ f\left(S_0^1\right) = \ln\left(S_0^1\right) \end{cases}.$$

Note that the previous SDE implies

$$\ln\left(S_t^1\right) = \ln\left(S_0^1\right) + \left(\mu_1 - \frac{1}{2}\sigma_1^2\right)t + \sigma_1 W_t.$$

Hence

$$S_t^1 = S_0^1 \cdot \exp\left(\left(\mu_1 - \frac{1}{2}\sigma_1^2\right)t + \sigma_1 W_t\right). \tag{5.13}$$

Proceeding as above, we get

$$\begin{cases} df\left(S_t^2\right) = \left(\mu_2 - \frac{1}{2}\sigma_2^2\right)dt + \sigma_2 dW_t \\ f\left(S_0^2\right) = \ln\left(S_0^2\right) \end{cases}.$$

2. In order to find the stochastic differential equation satisfied by $g\left(S_t^1, S_t^2\right)$, we can proceed (at least) in two ways.

   One (longer) is to apply Itô's formula to the function $g$.

   Another is based on the fact that $Y_t = g\left(S_t^1, S_t^2\right) = \ln\left(\frac{S_t^1}{S_t^2}\right) = \ln\left(S_t^1\right) - \ln\left(S_t^2\right)$ (using the properties of logarithms, as $S_0^1, S_0^2 > 0$). From item 1. and from $\mu = \mu_1 = \mu_2$ it follows that

$$\begin{aligned} dY_t &= d\left(\ln S_t^1\right) - d\left(\ln S_t^2\right) \\ &= df\left(S_t^1\right) - df\left(S_t^2\right) \\ &= \frac{1}{2}\left(\sigma_2^2 - \sigma_1^2\right)dt + \left(\sigma_2 - \sigma_1\right)dW_t. \end{aligned}$$

Hence $(Y_t)_{t\geq 0}$ satisfies the following SDE:

$$\begin{cases} dY_t = \frac{1}{2}\left(\sigma_2^2 - \sigma_1^2\right)dt + \left(\sigma_2 - \sigma_1\right)dW_t \\ Y_0 = \ln\left(\frac{S_0^1}{S_0^2}\right) \end{cases}.$$

We deduce that $(Y_t)_{t\geq 0}$ is neither a Brownian motion, nor a Brownian motion with drift, nor a geometric Brownian motion, yet it is a Brownian motion with drift $\mu^* = \frac{1}{2}\left(\sigma_2^2 - \sigma_1^2\right)$ and diffusion $\sigma^* = \sigma_2 - \sigma_1$.

3. Proceeding as above, we obtain

$$\begin{aligned}
dZ_t &= d\left(\ln S_t^1\right) + d\left(\ln S_t^2\right) \\
&= df\left(S_t^1\right) + df\left(S_t^2\right) \\
&= \left(2\mu - \frac{1}{2}\left(\sigma_1^2 + \sigma_2^2\right)\right) dt + (\sigma_1 + \sigma_2)\, dW_t.
\end{aligned}$$

Hence, $(Z_t)_{t\geq 0}$ is a Brownian motion with drift $\hat{\mu} = 2\mu - \frac{1}{2}\left(\sigma_1^2 + \sigma_2^2\right)$ and diffusion $\hat{\sigma} = \sigma_1 + \sigma_2$.

**Exercise 5.4** Assume that the price dynamics of two stocks is represented by the processes $\left(S_t^1\right)_{t\geq 0}$ and $\left(S_t^2\right)_{t\geq 0}$, solutions of the following SDEs:

$$\begin{aligned}
dS_t^1 &= \mu^1 S_t^1 dt + \sigma^1 S_t^1 dW_t^1, \quad S_0^1 = s_1, \\
dS_t^2 &= \mu^2 S_t^2 dt + \sigma^2 S_t^2 dW_t^2, \quad S_0^2 = s_2,
\end{aligned}$$

where $\left(W_t^1\right)_{t\geq 0}$, $\left(W_t^2\right)_{t\geq 0}$ are independent standard Brownian motions, $\mu^1, \mu^2 \in \mathbb{R}$ and $\sigma^1, \sigma^2 > 0$.

1. Find the SDE satisfied by

$$f\left(S_t^1, S_t^2\right) = \left(S_t^1\right)^2 - S_t^1 S_t^2 - K,$$

with $K > 0$, representing the value of a derivative on $S^1$ and $S^2$.

2. Find the SDE satisfied by

$$g\left(t, S_t^1, S_t^2\right) = \left(S_t^1\right)^2 - S_t^1 S_t^2 - Kt.$$

**Solution**

1. By Itô's formula in several variables (see (5.8)),

$$\begin{aligned}
df\left(S_t^1, S_t^2\right) &= \tfrac{\partial f}{\partial S^1}\left(S_t^1, S_t^2\right) dS_t^1 + \tfrac{\partial f}{\partial S^2}\left(S_t^1, S_t^2\right) dS_t^2 \\
&\quad + \tfrac{1}{2}\left[\frac{\partial^2 f}{\partial \left(S^1\right)^2}\left(S_t^1, S_t^2\right)\left(\sigma_t^1 S_t^1\right)^2 dt + \frac{\partial^2 f}{\partial \left(S^2\right)^2}\left(S_t^1, S_t^2\right)\left(\sigma_t^2 S_t^2\right)^2 dt\right] \\
&\quad + \tfrac{1}{2}\left[2\frac{\partial^2 f}{\partial S^1 \partial S^2}\left(S_t^1, S_t^2\right)\sigma^1 S_t^1 \sigma^2 S_t^2 \rho_{12} dt\right] \\
&= \left(2S_t^1 - S_t^2\right) dS_t^1 - S_t^1 dS_t^2 \\
&\quad + \tfrac{1}{2}\left[2\left(\sigma_t^1 S_t^1\right)^2 dt + 0 - 2\sigma^1 S_t^1 \sigma^2 S_t^2 \rho_{12} dt\right] \\
&= \left(2S_t^1 - S_t^2\right) dS_t^1 - S_t^1 dS_t^2 + \left(\sigma_t^1 S_t^1\right)^2 dt,
\end{aligned}$$

because, in our case, $\frac{\partial f}{\partial x_1}(x_1, x_2) = 2x_1 - x_2$, $\frac{\partial f}{\partial x_2}(x_1, x_2) = -x_1$, $\frac{\partial^2 f}{\partial x_1^2}(x_1, x_2) = 2$, $\frac{\partial^2 f}{\partial x_2^2}(x_1, x_2) = 0$, $\frac{\partial^2 f}{\partial x_1 \partial x_2}(x_1, x_2) = -1$ and $\rho_{12} = 0$ (by the independence of the two Brownian motions).

Furthermore, $f\left(S_0^1, S_0^2\right) = \left(S_0^1\right)^2 - S_0^1 S_0^2 - K$.

2. Using Itô's formula in several variables (see (5.6)) and proceeding as above, we obtain

$$
\begin{aligned}
dg\left(t, S_t^1, S_t^2\right) = {} & \tfrac{\partial g}{\partial t}\left(t, S_t^1, S_t^2\right) dt + \tfrac{\partial g}{\partial S^1}\left(t, S_t^1, S_t^2\right) dS_t^1 + \tfrac{\partial g}{\partial S^2}\left(t, S_t^1, S_t^2\right) dS_t^2 \\
& + \tfrac{1}{2}\left[\tfrac{\partial^2 g}{\partial (S^1)^2}\left(t, S_t^1, S_t^2\right)\left(\sigma_t^1 S_t^1\right)^2 dt \right. \\
& \left. + \tfrac{\partial^2 g}{\partial (S^2)^2}\left(t, S_t^1, S_t^2\right)\left(\sigma_t^2 S_t^2\right)^2 dt\right] \\
& + \tfrac{1}{2}\left[2\tfrac{\partial^2 g}{\partial S^1 \partial S^2}\left(t, S_t^1, S_t^2\right)\sigma^1 S_t^1 \sigma^2 S_t^2 \rho_{12} dt\right] \\
= {} & -K dt + \left(2S_t^1 - S_t^2\right) dS_t^1 - S_t^1 dS_t^2 + \left(\sigma_t^1 S_t^1\right)^2 dt,
\end{aligned}
$$

with $g\left(0, S_0^1, S_0^2\right) = \left(S_0^1\right)^2 - S_0^1 S_0^2$.

**Exercise 5.5** Consider a stock price evolving as the following geometric Brownian motion

$$
dS_t = \mu S_t dt + \sigma S_t dW_t, \tag{5.14}
$$

with current stock price $S_0 = 40$ euros, drift $\mu = 0.1$ and volatility $\sigma = 0.4$ (per year).

1. Establish whether the probability to exercise a European Call option is greater than the probability to exercise a European Put option, both with strike of 24 euros, maturity of 2 years and written on the stock above.
2. Suppose now that the price $\tilde{S}_T$ of another stock at time $T = 2$ (years) is a multiple of the price of the first stock, i.e. $\tilde{S}_T = cS_T$ for some $c > 0$. Establish if there exist $c > 0$ such that the probability to exercise the European Put option having the second stock as underlying is at least twice the probability to exercise the corresponding Call option.
3. Find the greatest strike $K^*$ making the Call payoff $\phi = (S_T - K^*)^+$ greater than 6 euros (or equal) with probability at least of 20%.

**Solution** As already pointed out in Exercise 5.3 (see Eq. (5.13)), the stock price is given by:

$$
S_t = S_0 \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t\right\}. \tag{5.15}
$$

1. In order to exercise the European Call option, the stock price at maturity should be greater than or equal to the strike. Hence,

$$\begin{aligned}
P\left(\{\text{ Call is exercised}\}\right) &= P\left(S_T \geq K\right)\\
&= P\left(\ln(S_T) \geq \ln(K)\right)\\
&= P\left(\ln\left(S_0\right) + \left(\mu - \frac{1}{2}\sigma^2\right)T + \sigma W_T \geq \ln\left(K\right)\right)\\
&= P\left(W_T \geq \frac{\ln(K) - \ln(S_0) - \left(\mu - \frac{1}{2}\sigma^2\right)T}{\sigma}\right).
\end{aligned}$$

Since $W_T \sim N\left(0; T\right)$ and, consequently, $\frac{W_T}{\sqrt{T}} \sim N\left(0; 1\right)$, we get

$$\begin{aligned}
P\left(\{\text{Call is exercised}\}\right) &= P\left(\frac{W_T}{\sqrt{T}} \geq \frac{\ln(K) - \ln(S_0) - \left(\mu - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right)\\
&= 1 - N\left(\frac{\ln(K) - \ln(S_0) - \left(\mu - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right)\\
&= 1 - N\left(\frac{\ln(24) - \ln(40) - \left(0.1 - \frac{1}{2}\left(0.4\right)^2\right)\cdot 2}{0.4\cdot\sqrt{2}}\right)\\
&= 1 - N\left(-0.974\right) = N\left(0.974\right) = 0.835,
\end{aligned}$$

where, as usual, $N(\cdot)$ denotes the cumulative distribution function of a standard normal.

In order to exercise the European Put option, at maturity the stock price should not exceed the strike. Since $S_t$ is a continuous random variable for any fixed $t > 0$ (by Eq. (5.15)), we deduce that $P\left(S_T \leq K\right) = P\left(S_T < K\right)$, so

$$\begin{aligned}
P\left(\{\text{Put is exercised}\}\right) &= P\left(S_T \leq K\right) = P\left(S_T < K\right)\\
&= 1 - P\left(S_T \geq K\right)\\
&= 1 - P\left(\{\text{Call is exercised}\}\right) = 0.165.
\end{aligned}$$

The probability of exercising the Call option is therefore greater than the one of exercising the Put option.

2. Suppose now that $\tilde{S}_T = cS_T$ for some $c > 0$. We have to establish if there exist some suitable $c > 0$ so that

$$P\left(\{\text{new Put is exercised}\}\right) \geq 2P\left(\{\text{new Call is exercised}\}\right). \tag{5.16}$$

In order to solve this problem, we start with computing the probabilities above. On the one hand,

$$\begin{aligned} P\left(\{\text{new Call is exercised}\}\right) &= P\left(\tilde{S}_T \geq K\right) \\ &= P\left(cS_T \geq K\right) = P\left(S_T \geq \frac{K}{c}\right), \end{aligned}$$

which corresponds to the probability to exercise a Call with strike $K/c$ and with the first stock as underlying. By the arguments of item 1., we get

$$\begin{aligned} P\left(\{\text{new Call is exercised}\}\right) &= P\left(S_T \geq \tfrac{K}{c}\right) \\ &= 1 - N\left(\frac{\ln(K/c) - \ln(S_0) - \left(\mu - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right) \\ &= 1 - N\left(\frac{\ln(K) - \ln(c) - \ln(S_0) - \left(\mu - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right). \end{aligned}$$

On the other hand,

$$\begin{aligned} P\left(\{\text{new Put is exercised}\}\right) &= 1 - P\left(\tilde{S}_T \geq K\right) \\ &= N\left(\frac{\ln(K) - \ln(c) - \ln(S_0) - \left(\mu - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right). \end{aligned}$$

Equation (5.16) becomes, therefore,

$$3N\left(\frac{\ln(K) - \ln(c) - \ln(S_0) - \left(\mu - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right) \geq 2$$

$$\ln(K) - \ln(S_0) - \left(\mu - \frac{1}{2}\sigma^2\right)T - q_{2/3} \cdot \sigma\sqrt{T} \geq \ln(c),$$

where $q_{2/3}$ denotes the quantile at level $\alpha = 2/3$ of the standard Normal, defined as the solution of $N\left(q_{2/3}\right) = 2/3$. Since $q_{2/3} = 0.4307$, it follows $0 < c \leq 0.452$.

3. First, notice that

$$\begin{aligned} P\left((S_T-K)^+\geq 6\right) &= P\left(S_T\geq K+6\right)\\ &= P\left(\ln S_T\geq \ln\left(K+6\right)\right)\\ &= P\left(W_T\geq \frac{\ln\left(\frac{K+6}{S_0}\right)-\left(\mu-\frac{1}{2}\sigma^2\right)T}{\sigma}\right)\\ &= P\left(\frac{W_T}{\sqrt{T}}\geq \frac{\ln\left(\frac{K+6}{S_0}\right)-\left(\mu-\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right)\\ &= 1-N\left(\frac{\ln\left(\frac{K+6}{S_0}\right)-\left(0.1-\frac{1}{2}(0.4)^2\right)\cdot 2}{0.4\sqrt{2}}\right). \end{aligned}$$

Since such a probability has to be greater than (or equal to) 20%, we deduce that

$$\begin{aligned} &1-N\left(\frac{\ln\left(\frac{K+6}{S_0}\right)-\left(0.1-\frac{1}{2}(0.4)^2\right)\cdot 2}{0.4\cdot\sqrt{2}}\right)\geq 0.2\\ &N\left(\frac{\ln\left(\frac{K+6}{S_0}\right)-\left(0.1-\frac{1}{2}(0.4)^2\right)\cdot 2}{0.4\cdot\sqrt{2}}\right)\leq 0.8\\ &\frac{\ln\left(\frac{K+6}{S_0}\right)-\left(0.1-\frac{1}{2}(0.4)^2\right)\cdot 2}{0.4\cdot\sqrt{2}}\leq 0.8416\\ &\ln\left(\frac{K+6}{40}\right)\leq 0.516\\ &K\leq 61.01. \end{aligned}$$

The largest 'admissible' strike is, then, $K^* = 61.01$ euros.

**Exercise 5.6** Consider a standard Brownian motion $(W_t)_{t\geq 0}$ and a stochastic process $(S_t)_{t\geq 0}$ satisfying the following stochastic differential equation:

$$dS_t = \mu_t dt + \sigma_t dW_t, \tag{5.17}$$

with $\mu_t$ and $\sigma_t$ not constant everywhere but time-dependent.

1. When $\sigma_t = 0.1$ for $t \in [0, 4]$ and

$$\mu_t = \begin{cases} 0.04t; & \text{for } 0\leq t\leq 2\\ 0.02\,(10-t)\,; & \text{for } 2< t\leq 4 \end{cases},$$

compute: (a) the probability of having (at time $t = 4$) a profit greater than (or equal to) 0.4; (b) the average profit at time $t = 4$.

2. When $\mu_t$ is the same as above, while

$$\sigma_t = \begin{cases} 0.3; & \text{for } 0 \leq t \leq 2 \\ 0.4; & \text{for } 2 < t \leq 4 \end{cases},$$

compute: (a) the probability of having a profit at least equal to 0.4 at time $t = 4$; (b) the average profit at time $t = 4$.

3. Set $X_t \triangleq S_t - S_0$ for any $t \geq 0$. Discuss whether there exist $(\mu_t)_{t\geq 0}$ and $(\sigma_t)_{t\geq 0}$ such that $(X_t)_{t\geq 0}$ is a martingale (with respect to the filtered space considered at the beginning).

**Solution**

1. By integrating Eq. (5.17), we obtain

$$S_t = S_0 + \int_0^t \mu_s ds + \int_0^t \sigma_s dW_s.$$

Hence

$$\begin{aligned} S_4 - S_0 &= \int_0^4 \mu_s ds + \int_0^4 \sigma_s dW_s \\ &= \int_0^2 0.04\, s\, ds + \int_2^4 0.02\,(10 - s)\, ds + \int_0^4 0.1\, dW_s \\ &= \left[0.02\, s^2\right]\Big|_0^2 + \left[0.02(10s - \frac{s^2}{2})\right]\Big|_2^4 + 0.1 \cdot (W_4 - W_0) \\ &= 0.36 + 0.1 \cdot (W_4 - W_0) \\ &= 0.36 + 0.1\, W_4. \end{aligned}$$

(b) It is straightforward to obtain that the average profit at time $t = 4$ is equal to $E[S_4 - S_0] = 0.36 + 0.1 \cdot E[W_4] = 0.36$, where the last equality is due to a well-known property of standard Brownian motions.

For item (a), we deduce that

$$\begin{aligned} &P\,(\{\text{profit at time } t = 4 \text{ is at least equal to } 0.4\}) \\ &\quad = P\,(0.36 + 0.1\, W_4 \geq 0.4) \\ &\quad = P\,(0.1\, W_4 \geq 0.04) = P\left(\tfrac{W_4}{\sqrt{4}} \geq \tfrac{0.04}{0.1\sqrt{4}}\right) \\ &\quad = 1 - N\,(0.2) = 0.42. \end{aligned}$$

2. Under the hypothesis on $\mu_t$ and on $\sigma_t$, we get

$$\begin{aligned}S_4 - S_0 &= \int_0^4 \mu_s ds + \int_0^4 \sigma_s dW_s \\ &= \int_0^2 0.04\, s\, ds + \int_2^4 0.02\,(10-s)\, ds + \int_0^2 0.3\, dW_s + \int_2^4 0.4\, dW_s \\ &= 0.36 + 0.3\,(W_2 - W_0) + 0.4\,(W_4 - W_2) \\ &= 0.36 + 0.3\; W_2 + 0.4\,(W_4 - W_2)\,.\end{aligned}$$

By the properties of standard Brownian motions we know, indeed, that $W_2$ and $(W_4 - W_2)$ are independent Normal random variables. We remind also that:
$X \sim N\left(m_X; s_X^2\right)$ and $Y \sim N\left(m_Y; s_Y^2\right)$ are independent
$\Rightarrow aX + bY \sim N\left(am_X + bm_Y; a^2 s_X^2 + b^2 s_Y^2\right)$ for real numbers $a$ and $b$.
Consequently, we deduce that $S_4 - S_0 \sim N\,(0.36; 0.09 \cdot 2 + 0.16 \cdot 2)$, so $S_4 - S_0 \sim N\,(0.36; 0.5)$. The average profit at time $t = 4$ is then equal to $E\,[S_4 - S_0] = 0.36$, while

$$\begin{aligned}&P\;(\text{profit at time } t = 4 \text{ is at least equal to } 0.4) \\ &= P\,(S_4 - S_0 \geq 0.4) \\ &= P\left(\tfrac{S_4 - S_0 - E[S_4 - S_0]}{\sqrt{V(S_4 - S_0)}} \geq \tfrac{0.4 - E[S_4 - S_0]}{\sqrt{V(S_4 - S_0)}}\right) \\ &= 1 - N\left(\tfrac{0.04}{\sqrt{0.5}}\right) = 0.48.\end{aligned}$$

3. Taking $\mu_t = 0$ and $\sigma_t = \sigma$ (with $\sigma \in \mathbb{R}$) for any $t \geq 0$, we get

$$X_t = S_t - S_0 = \int_0^t \sigma dW_s = \sigma W_t$$

for any $t \geq 0$. By the properties of Brownian motions and by the arguments above, it follows that $(X_t)_{t \geq 0}$ is a martingale (when $\mu_t \equiv 0$, $\sigma_t \equiv \sigma$).

**Exercise 5.7** By applying the Tanaka formula and the definition of Brownian local time, prove that the payoff of a European Call option, on a stock whose dynamics is described by a geometric Brownian motion with null drift and diffusion coefficient $\sigma$, admits the following representation:

$$(S_T - K)^+ = (S_0 - K)^+ + \int_0^T \mathbf{1}_{(K,+\infty)} dS_t + \frac{1}{2}\int_0^T \sigma^2 S_t^2 \delta_K(S_t) dt.$$

**Solution** A naive application of Itô's formula would provide:

$$f(S_T) = f(S_0) + \int_0^T f'(S_t)dS_t + \frac{1}{2}\sigma^2 \int_0^t f''(S_t)dt,$$

but in order to apply correctly the Itô's formula we must verify the condition that both the first and the second derivative of the function $f$ exist and are continuous. Actually, Itô's formula holds also if there is a finite number of points where $f''$ is not defined, provided that $f'$ is continuous on the whole domain of $f$ (and $|f''|$ is bounded wherever it is defined). This is not the case of the function under consideration. We can nevertheless apply an alternative procedure which could provide the desired result. Define the following sequence of functions:

$$f_n(x) = \begin{cases} 0; & x \le K - \frac{1}{2n} \\ \frac{n}{2}(x-K)^2 + \frac{1}{2}(x-K) + \frac{1}{8n}; & K - \frac{1}{2n} < x < K + \frac{1}{2n} \\ x - K; & x \ge K + \frac{1}{2n} \end{cases}.$$

Hence

$$f_n'(x) = \begin{cases} 0; & x \le K - \frac{1}{2n} \\ n(x-K) + \frac{1}{2}; & K - \frac{1}{2n} < x < K + \frac{1}{2n} \\ 1; & x \ge K + \frac{1}{2n} \end{cases}$$

and it can be immediately verified that all the functions $f_n$ are continuous on the whole domain of $f$. Moreover, their second derivatives

$$f_n''(x) = \begin{cases} 0; & x < K - \frac{1}{2n}, \\ n; & K - \frac{1}{2n} < x < K + \frac{1}{2n} \\ 0; & x > K + \frac{1}{2n} \end{cases}$$

are not defined at the points $x = K \pm \frac{1}{2n}$ (even if their absolute values are bounded everywhere else). Itô's formula can therefore be applied to the functions $f_n$, so that

$$f_n(S_T) = f_n(S_0) + \int_0^T f_n'(S_t)dS_t + \frac{1}{2}\sigma^2 \int_0^T f_n''(S_t)\, S_t^2 dt \tag{5.18}$$

since the stock price $S_t$ is assumed to solve $dS_t = \sigma S_t dW_t$.

We can also remark that the sequence $(f_n)_{n\in\mathbb{N}}$ converges to the function $f$, and the sequence of the first derivatives converges to

$$f'(x) = \lim_{n\to\infty} f_n'(x) = \begin{cases} 0; & x < K \\ \frac{1}{2}; & x = K \\ 1; & x > K \end{cases}.$$

By passing to the limit in the first integral appearing in Eq. (5.18), we notice that the value of $f'$ at the single point $x = K$ will not affect its value, hence that integral can be expressed as follows:

$$\int_0^T \mathbf{1}_{(K,+\infty)}(S_t)dS_t. \tag{5.19}$$

By collecting the results and passing to the limit $n \to +\infty$ (this is possible by the dominated convergence theorem), we finally obtain:

$$f(S_T) = f(S_0) + \int_0^T \mathbf{1}_{(K,+\infty)}(S_t)dS_t + \frac{1}{2}\sigma^2 \lim_{n\to\infty} n \int_0^T S_t^2 \mathbf{1}_{\left(K-\frac{1}{2n},K+\frac{1}{2n}\right)}(S_t)dt$$

which, by (5.12), can be written as:

$$(S_T - K)^+ = (S_0 - K)^+ + \int_0^T \mathbf{1}_{(K,\infty)}(S_t)dS_t + \frac{1}{2}\int_0^T \sigma^2 S_t^2 \delta_K(S_t)dt.$$

## 5.3 Proposed Exercises

**Exercise 5.8** Consider a stock price $(S_t)_{t\geq 0}$ evolving as the following geometric Brownian motion

$$dS_t = \mu S_t dt + \sigma S_t dW_t.$$

Find the stochastic differential equation satisfied by $f(S_t, t) = e^{rt} \cdot \ln\left(S_t^4 - K\right)$ and compare it with the one satisfied by $g(S_t) = \ln\left(S_t^4 - K\right)$ (where $K$ is a given positive real number).

**Exercise 5.9** The financial position $X_t$ (at time $t$) evolves as

$$dX_t = \mu dt + \sigma_t dW_t,$$

with $X_0 = 80{,}000$ euros, $\mu = 400$ (per year) and with $\sigma_t$ (per year) given by

$$\sigma_t = \begin{cases} \sum_{k=0}^{8} 100\,(k+1)\,\mathbf{1}_{[k;k+1)}(t); & 0 \leq t < 9 \\ \sigma^*; & t \geq 9 \end{cases} \tag{5.20}$$

where $\sigma^* > 0$. (We remind that the indicator function $\mathbf{1}_A(t)$ equals 1 when $t \in A$, and 0 when $t \notin A$.)

1. Find the distribution of $X_t$ (as a function of $t$).
2. Make $X_t$ explicit as function of $X_0$, $\mu$, $\sigma_t$, $t$ and $W_t$.

3. Find $\sigma^*$ such that $P\,(X_{10} < 10000) < \frac{1}{100}$. Is such a $\sigma^*$ reasonable from a financial point of view?
4. Compute $E\,[X_{100} - X_{10}]$, $V\,(X_{100} - X_{10})$, $E\,[X_{200} - X_{10}]$ and $V\,(X_{200} - X_{10})$ for $(\sigma_t)_{t\geq 0}$ as in (5.20).

   Due to the properties of Brownian motions, the quantities above are easy to compute. Is the same true for $E\,[X_{200}/X_{10}]$ and $V\,(X_{200}/X_{10})$?

**Exercise 5.10** Let $S_t^1$, $S_t^2$ and $S_t^3$ be the prices of three different stocks at time $t$. Suppose that $\left(S_t^1\right)_{t\geq 0}$, $\left(S_t^2\right)_{t\geq 0}$, $\left(S_t^3\right)_{t\geq 0}$ satisfy

$$\begin{aligned} dS_t^1 &= \mu^1 S_t^1 dt + \sigma^1 S_t^1 dW_t^1 \\ dS_t^2 &= \mu^2 S_t^2 dt + \sigma^2 S_t^2 dW_t^2 \\ dS_t^3 &= \mu^3 S_t^3 dt + \sigma^3 S_t^3 dW_t^3, \end{aligned}$$

respectively, where $W^1$, $W^2$ and $W^3$ are standard Brownian motions.

1. Find the dynamics of the value $f\left(S_t^1, S_t^2, S_t^3\right)$ of the portfolio formed by four shares of the first stock, two shares of the second and one share of the third.
2. Recall that cash invested (or borrowed) in a bank account at a risk-free rate $r$ evolves as follows:

$$X_t = X_0 e^{rt}.$$

   Applying the fact above, find the dynamics of the value $g\left(S_t^1, S_t^2, S_t^3, t\right)$ of the portfolio formed by four shares of the first stock, two shares of the second and one of the third and with a bank loan equal to the quantity of cash necessary (at the initial time) to buy the stocks.

# Chapter 6
# Partial Differential Equations in Finance

## 6.1 Review of Theory

Let $u$ be a function of several real variables $(t, x_1, x_2, \ldots, x_n)$:

$$\begin{aligned} u : \mathbb{R}^+ \times \mathbb{R}^n &\to \mathbb{R} \\ (t, x_1, x_2, \ldots, x_n) &\mapsto u\,(t, x_1, x_2, \ldots, x_n) \end{aligned}$$

An equation involving one or more partial derivatives of the function $u$ with respect to its variables (sometimes also $u$ itself) is called a *Partial Differential Equation* (PDE). Every function $u$ satisfying the equation is called a *solution*.

The notion of PDE generalizes in a natural way the notion of *Ordinary Differential Equation* (ODE) to the case where the unknown map depends on more than one variable. While the most general solution of an ODE depends in general on some arbitrary constants, the set of solutions satisfying a PDE is parameterized by arbitrary functions. Sometimes it is necessary to determine a solution satisfying extra conditions, for example some initial and/or boundary conditions. These conditions are also necessary in order to have existence, uniqueness and stability results for the corresponding solutions.

PDEs are one of the richest and most fruitful subjects of Mathematical Analysis. A detailed discussion, even of the most basic results of the theory, would require a much deeper treatment than that allowed in a summary like the present one. Below we shall just recall in a very concise way the concepts necessary to understand and solve the exercises we are going to propose, and we prompt the interested reader to consult one of the several textbooks written on the subject. We recommend, for instance, Salsa [40] and, concerning the most important applications of derivatives valuation, Wilmott et al. [43].

In our applications we shall focus on PDEs for functions of two real variables *of semilinear type*, where the partial derivatives of highest order appear always with power 1.

 
E. Rosazza Gianin, C. Sgarra, *Mathematical Finance*,
La Matematica per il 3+2 149, https://doi.org/10.1007/978-3-031-28378-9_6

Semilinear PDEs of second order can be classified according to some basic properties. In the particular framework we have chosen to work, i.e. real functions of two real variables, the classification can be described in the following way.

Once mixed partial derivatives of second order have been eliminated (possibly by a linear transformation of the independent variables), one faces the following possibilities:

- if both second-order derivatives appear with the same sign, the equation is said to be *of elliptic type*;
- if both second-order derivatives appear with opposite sign, the equation is said to be *of hyperbolic type*;
- if the equation contains only one second-order derivative, and the first-order derivative in the other variable, the equation is *of parabolic type*.

For parabolic equations a further distinction is introduced: if the two derivatives with highest order with respect to the two variables have the same sign, the equation is called *backward parabolic*, otherwise *forward parabolic*.

The classification of a PDE provides some useful information about the kind of conditions to assign in order to ensure existence of solutions. It is possible to prove, for example, that the initial-value problem for elliptic equations is ill-posed: in this case in fact, the existence of a solution is not guaranteed and, if a solution exists, it can be not unique and/or it can depend on the initial data in a discontinuous manner.

The Black-Scholes equation

$$\frac{\partial u}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 u}{\partial S^2} + rS\frac{\partial u}{\partial S} - ru = 0 \tag{6.1}$$

(see Chap. 7) is the most popular PDE in financial applications. The unknown function depends on two independent variables $S$, $t$ and the equation involves first derivatives in both, but only the second derivative in $S$. Moreover, the highest derivatives in $S$ and $t$ (the first derivative with respect to time $t$ and the second with respect to the underlying $S$) appear with the same sign. The Black-Scholes equation is thus a *backward parabolic* PDE.

For a backward parabolic equation it is possible to prove that, under quite general regularity conditions on both data and coefficients, given a final condition (in $t$) and (possibly) two boundary conditions (in $S$), a solution exists, is unique and depends continuously on the data.

For forward parabolic equations existence and uniqueness of a solution is guaranteed once an initial (instead of final) datum is assigned.

The interest in parabolic equations, in particular in the backward parabolic ones, for financial applications is motivated by a deep relationship between this type of equations and diffusion stochastic processes, i.e. processes described by stochastic differential equations driven by a standard Brownian motion. This relationship is clarified by the following very important result, known in the literature as the *Feynman-Kac Representation Theorem*.

**Theorem 6.1** *Let $u(x,t)$ be a solution of the PDE:*

$$\frac{\partial u}{\partial t} + \frac{1}{2}\sigma^2(x,t)\frac{\partial^2 u}{\partial x^2} + \mu(x,t)\frac{\partial u}{\partial x} - ru = 0 \tag{6.2}$$

*with final condition:*

$$u(x,T) = \Phi(x). \tag{6.3}$$

*Then this solution can be represented as follows:*

$$u(x,t) = e^{-r(T-t)}E_{t,x}[\Phi(X_T)], \tag{6.4}$$

*where $(X_s)_{s\geq t}$ satisfies the stochastic differential equation:*

$$\begin{cases} dX_s = \mu(X_s, s)ds + \sigma(X_s, s)dW_s \\ X_t = x \end{cases} \tag{6.5}$$

*and $E_{t,x}$ denotes the dependence on the variables $t$ and $x$. This dependence is inherited via the condition (6.5).*

The result holds under the further hypothesis that the process $\sigma(s, X_s)\frac{\partial u}{\partial x}(s, X_s)$ is square-integrable, i.e. $\int_0^t \left[\sigma(s, X_s)\frac{\partial u}{\partial x}(s, X_s)\right]^2 ds < +\infty$, $P$-almost surely (a.s.).

The relevance of this result is primary: it allows to represent the solution of a backward parabolic PDE as the expectation of its final condition with respect to a probability measure determined by a suitable stochastic differential equation. The reader eager to study in depth the relationship existing between backward parabolic PDEs and stochastic differential equations of diffusion type can find a more detailed treatment in the textbook by Björk [6].

Since only for a very limited number of PDEs explicit solutions are available, numerical procedures are needed to obtain approximate solutions. Several fast and accurate numerical methods can be used to compute solutions when these are not available in explicit form.

Now let us concentrate on one specific class of methods allowing to get solutions of PDEs in explicit form: the so-called *similarity methods*, sometimes successful in providing parabolic PDE solutions, in particular for *diffusion equations*:

$$\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t) \tag{6.6}$$

and some variations thereof.

The similarity method is based on the invariance properties of the PDE and its initial (final) and boundary data under a class of transformations of the independent variables, and it can be applied if this invariance property holds and can be identified. It is easy to see, for example, that the classical diffusion Eq. (6.6) does not change if $t$ is replaced by $\lambda^2 t$ and at the same time $x$ is replaced by $\lambda x$. This invariance property, known as *scaling invariance* (in this particular case the scaling is called parabolic for obvious reasons), suggests to look for a solution of a special form, where the two independent variables appear always in the same combination, depending on the ratio $x/\sqrt{t}$. It is necessary that also the data exhibit the same invariance property and, as we shall see immediately in the applications, sometimes the data themselves suggest the particular form to look for.

Once this special combination has been detected, both the unknown and the data of the PDE can be expressed as functions of this new variable, which becomes the only independent variable for the problem, and the PDE is reduced to an ordinary differential equation.

Here is a rule of thumb to find similarity solutions for second order parabolic PDEs with two independent variables. The rule consists in finding, by trials, a solution of the form $u = t^\alpha f(x/t^\beta)$ by looking for $\alpha$ and $\beta$ such that the equation admits a similarity reduction, i.e. it becomes an ordinary differential equation for the unknown $U(y)$ in the new independent variable $y = x/t^\beta$. The applications we are going to provide should explain in a more satisfactory way the concepts just exposed.

## 6.2 Solved Exercises

**Exercise 6.2** Show that, for functions $a, b : \mathbb{R} \to \mathbb{R}$ of class $C^1$,

$$V(S_t, t) = a(t)S_t + b(t)$$

satisfies the Black-Scholes equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2} + r S_t \frac{\partial V}{\partial S} - rV = 0 \tag{6.7}$$

if and only if $a(t) = a$ and $b(t) = be^{rt}$ with $a$ and $b$ real constants.

**Solution** First of all, we should remark that in the problem considered no *boundary condition* has been assigned. If (6.7) is furnished with the final condition $V(S_T, T) =$ *option payoff*, then the problem admits one and only one solution: the one provided by the Black-Scholes formula for Call or Put options.

Let us consider $V(S_t, t) = a(t)S_t + b(t)$ and verify that if $V$ satisfies Eq. (6.7) then $a$ and $b$ are constant. We get

$$\frac{\partial V}{\partial t} = a'(t)S_t + b'(t)$$
$$\frac{\partial V}{\partial S} = a(t)$$
$$\frac{\partial^2 V}{\partial S^2} = 0,$$

and, by substituting these partial derivatives in (6.7), we can verify that, if $V$ satisfies Eq. (6.7), then

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2} + rS_t \frac{\partial V}{\partial S} - rV = a'(t)S_t + b'(t) + rS_t a(t) - ra(t)S_t - rb(t) = 0$$

or, equivalently,

$$a'(t)\, S_t + b'(t) - rb(t) = 0.$$

From the previous equation and by imposing the coefficients of both the first- and the second-degree terms in $S$ to be equal to zero, we get

$$a'(t) = 0$$
$$b'(t) - rb(t) = 0.$$

Consequently, if $V$ satisfies (6.7) then $a(t) = a$ and $b(t) = be^{rt}$ with $a, b$ real constants.

The inverse implication is immediate to prove: if $a$ and $b$ are constant, then $V(S_t, t) = aS_t + b$ satisfies Eq. (6.7).

**Exercise 6.3** Suppose that the real function $u(x, t)$ satisfies the following problem on the real, positive half-line:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad \text{for } x > 0,\ t > 0$$

with

$$\begin{aligned} &u(x, 0) = u_0(x), && \text{for } x > 0,\\ &u(0, t) = 0, && \text{for } t > 0. \end{aligned}$$

Find a function $h(x, s, t)$ allowing to write the solution of the problem in the form:

$$u(x,t) = \int_0^{+\infty} u_0(s)\, h(x,s,t)ds. \tag{6.8}$$

**Solution** Let us start by pointing out that the Green function for the problem considered allows to write the solution of the problem in the following form:

$$u(x,t) = \int_{-\infty}^{+\infty} u_0(s)\, g(x-s,t)ds, \tag{6.9}$$

i.e. as a convolution integral extended to the whole real line. The function $g$ playing the role of the integral kernel in the previous expression is the fundamental solution of the PDE under examination, i.e. the solution with Dirac's function $\delta(x)$ as initial datum. It is well known that the Green function $g(x, s, t)$ for the present problem is the following:

$$g(x,s,t) = \frac{1}{2\sqrt{\pi t}} \exp\left\{-\frac{(x-s)^2}{4t}\right\}. \tag{6.10}$$

Notice that the function we are looking for is not the Green function, which is already known, since the integral providing the solution is not necessarily of convolution type and the integration domain is just the positive real half-line.

With this in mind, let us consider the following auxiliary problem (AP). Let the function $v(x, t)$ be defined by reflection with respect to $x = 0$, i.e.

$$v(x,t) = \begin{cases} -u(-x,t); & \text{for } x < 0 \\ \phantom{-}u(x,t); & \text{for } x > 0 \end{cases}$$

with $v_0(x) = v(x, 0)$ for $x \in \mathbb{R}$.

We will verify that $v(0, t) = 0$ and prove that the solution of the problem:

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2}, \qquad x \in \mathbb{R}, t > 0$$

with

$$\begin{array}{ll} v(x,0) = v_0(x), & \text{for } x > 0 \\ v(0,t) = 0, & \text{for } t > 0 \end{array}$$

is

$$v(x,t) = \frac{1}{2\sqrt{\pi t}} \int_0^{+\infty} u_0(s) \left[ e^{-\frac{(x-s)^2}{4t}} - e^{-\frac{(x+s)^2}{4t}} \right] ds.$$

By definition of $v$ and by the simple remark that $u(0,t) = 0$, one obtains simultaneously the following conditions:

$$v\left(0^+,t\right) = \lim_{x\to 0^+} u(x,t) = 0$$
$$v\left(0^-,t\right) = \lim_{x\to 0^-} \left[-u(-x,t)\right] = 0.$$

Since $v$ must be continuous, then $v(0,t) = 0$.

From the preliminary remark, if $v(x,t)$ is regular enough and if $\lim_{x\to\pm\infty} v(x,t) = 0$, we know that the following equality holds:

$$v(x,t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{+\infty} v_0(s)\, e^{-\frac{(x-s)^2}{4t}}\, ds. \tag{6.11}$$

By definition, however,

$$v_0(x) = v(x,0) = \begin{cases} -u(-x,0) = -u_0(-x)\,; & \text{for } x < 0 \\ u(x,0) = u_0(x)\,; & \text{for } x > 0 \end{cases} \tag{6.12}$$

By plugging (6.12) into (6.11), we obtain:

$$\begin{aligned} v(x,t) =& \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{+\infty} v_0(s)\, e^{-\frac{(x-s)^2}{4t}}\, ds \\ =& \frac{1}{2\sqrt{\pi t}} \left[ \int_{-\infty}^{0} -u_0(-s)\, e^{-\frac{(x-s)^2}{4t}}\, ds + \int_{0}^{+\infty} u_0(s)\, e^{-\frac{(x-s)^2}{4t}}\, ds \right] \end{aligned} \tag{6.13}$$

$$\begin{aligned} =& \frac{1}{2\sqrt{\pi t}} \left[ \int_{+\infty}^{0} u_0(y)\, e^{-\frac{(x+y)^2}{4t}}\, dy + \int_{0}^{+\infty} u_0(s)\, e^{-\frac{(x-s)^2}{4t}}\, ds \right] \qquad (6.14) \\ =& \frac{1}{2\sqrt{\pi t}} \left[ -\int_{0}^{+\infty} u_0(y)\, e^{-\frac{(x+y)^2}{4t}}\, dy + \int_{0}^{+\infty} u_0(s)\, e^{-\frac{(x-s)^2}{4t}}\, ds \right] \\ =& \frac{1}{2\sqrt{\pi t}} \left[ -\int_{0}^{+\infty} u_0(z)\, e^{-\frac{(x+z)^2}{4t}}\, dz + \int_{0}^{+\infty} u_0(z)\, e^{-\frac{(x-z)^2}{4t}}\, dz \right] \\ =& \frac{1}{2\sqrt{\pi t}} \int_{0}^{+\infty} u_0(z) \left[ e^{-\frac{(x-z)^2}{4t}} - e^{-\frac{(x+z)^2}{4t}} \right] dz, \end{aligned}$$

where equality (6.14) follows from the change of variable $y = -s$ ($dy = -ds$ ).

From the last equality and the relationship between the initial problem and problem (AP), one obtains that the function $h(x,s,t)$ we are looking for is $h(x,s,t) = \frac{1}{2\sqrt{\pi t}} \left[ e^{-\frac{(x-s)^2}{4t}} - e^{-\frac{(x+s)^2}{4t}} \right]$.

*Remark* The result just obtained can be applied to barrier option valuation and consists in a slightly modified version of the "image method".

**Exercise 6.4** Find the similarity solution of the following mixed problem:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 3x^2, \qquad \text{for } x > 0,\ t > 0 \tag{6.15}$$

with

$$u(0,t) = 0, \qquad \text{for } t > 0 \tag{6.16}$$
$$\lim_{x\to\infty} \frac{u(x,t)}{x^4} = 0, \qquad \text{for } t > 0 \tag{6.17}$$
$$u(x,0) = 0, \qquad \text{for } x > 0. \tag{6.18}$$

**Solution** Our goal is to find a transformation of $x$ and $t$ of the kind:

$$\xi = \frac{x}{t^\alpha},$$

generating a solution $u(x,t)$ of the following type:

$$u(x,t) = t^\beta U\left(\frac{x}{t^\alpha}\right) = t^\beta U(\xi),$$

and to write (6.15) in terms of $\xi$ and $U$. The resulting second-order ODE in $U$ will be much simpler to solve.

Let us begin by determining suitable coefficients $\alpha$ and $\beta$. First of all, in terms of $U$ and $\xi$, we get

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \beta t^{\beta-1} U\left(\frac{x}{t^\alpha}\right) + t^\beta U'\left(\frac{x}{t^\alpha}\right)\left(-\alpha \frac{x}{t^{\alpha+1}}\right) \\
&= \beta t^{\beta-1} U(\xi) - \alpha x t^{\beta-\alpha-1} U'(\xi) = \beta t^{\beta-1} U(\xi) - \alpha \xi t^{\beta-1} U'(\xi), \\
\frac{\partial u}{\partial x} &= t^\beta U'\left(\frac{x}{t^\alpha}\right)\frac{1}{t^\alpha} = t^{\beta-\alpha} U'\left(\frac{x}{t^\alpha}\right) = t^{\beta-\alpha} U'(\xi), \\
\frac{\partial^2 u}{\partial x^2} &= t^{\beta-\alpha} U''\left(\frac{x}{t^\alpha}\right)\frac{1}{t^\alpha} = t^{\beta-2\alpha} U''\left(\frac{x}{t^\alpha}\right) = t^{\beta-2\alpha} U''(\xi).
\end{aligned}$$

Equation (6.15) becomes then:

$$\beta t^{\beta-1} U(\xi) - \alpha \xi t^{\beta-1} U'(\xi) = t^{\beta-2\alpha} U''(\xi) + 3x^2$$
$$\beta t^{\beta-1} U(\xi) - \alpha \xi t^{\beta-1} U'(\xi) = t^{\beta-2\alpha} U''(\xi) + 3\xi^2 t^{2\alpha}$$

and, dividing both sides by $t^{2\alpha}$,

$$t^{\beta-2\alpha-1}\left[\beta U(\xi)-\alpha\xi U'(\xi)\right]=t^{\beta-4\alpha}U''(\xi)+3\xi^2. \tag{6.19}$$

In order to eliminate the explicit dependence on $t$ in the previous equation, let us choose $\alpha$ and $\beta$ such that:

$$\begin{cases}\beta-2\alpha-1=0\\ \beta-4\alpha=0\end{cases}.$$

This implies $\alpha=1/2$ and $\beta=2$.

The transformation induced on $x$ and $t$ is then given by

$$\xi=\frac{x}{\sqrt{t}}$$

and we are looking for a solution $u(x,t)$ of the following type:

$$u(x,t)=t^2U(\xi).$$

By writing again (6.15) in terms of $\xi$ and $U$ or, in other words, Eq. (6.19) with $\alpha=1/2$ and $\beta=2$, one obtains

$$\begin{aligned}&\beta U(\xi)-\alpha\xi U'(\xi)=U''(\xi)+3\xi^2\\&U''(\xi)+\frac{1}{2}\xi U'(\xi)-2U(\xi)=-3\xi^2.\end{aligned} \tag{6.20}$$

Condition (6.16) on $u$ becomes $U(0)=0$, while condition (6.17) becomes $\lim_{\xi\to+\infty}\frac{U(\xi)}{\xi^4}=0$. That is: as $t\to0$ we have $\xi\to+\infty$ and

$$0\leftarrow\frac{u(x,t)}{x^4}=\frac{t^2U(\xi)}{x^4}=\frac{U(\xi)}{\xi^4}.$$

We must then find the general solution of Eq. (6.20). Since this is a non-homogeneous ordinary differential equation, its general solution will be provided by the general solution of the associated homogeneous equation:

$$U''(\xi)+\frac{1}{2}\xi U'(\xi)-2U(\xi)=0 \tag{6.21}$$

plus a particular solution of (6.20).

Let us start from the particular solution. We can try with a polynomial, and we immediately observe that $U_p(\xi)=-\frac{1}{4}\xi^4$ is a particular solution of (6.20).

Now we look for the general solution of Eq. (6.21). If we knew two “independent” solutions $U_1^H$ and $U_2^H$ of Eq. (6.21), the general solution would be provided by a linear combination of $U_1^H$ and $U_2^H$.

By looking again for a polynomial solution, we find that $U_1^H(\xi) = \xi^4 + 12\xi^2 + 12$ satisfies Eq. (6.21).

Another solution of Eq. (6.21) can be obtained using the method of *variation of constants*:

$$U_2^H(\xi) = a(\xi)\, U_1^H(\xi),$$

with $a$ being a suitable function, to be determined. To this aim we require that $U_2^H$ satisfies Eq. (6.21).

Since

$$\left(U_2^H\right)'(\xi) = a'(\xi)\, U_1^H(\xi) + a(\xi)\left(U_1^H\right)'(\xi)$$
$$\left(U_2^H\right)''(\xi) = a''(\xi)\, U_1^H(\xi) + 2a'(\xi)\left(U_1^H\right)'(\xi) + a(\xi)\left(U_1^H\right)''(\xi),$$

the function $a$ must satisfy

$$a''U_1 + 2a'U_1' + aU_1'' + \frac{1}{2}\xi a U_1' + \frac{1}{2}\xi a' U_1 - 2aU_1 = 0, \tag{6.22}$$

where for notational simplicity the dependence on $\xi$ has been omitted and $U_1$ denotes $U_1^H$. Since $U_1^H$ satisfies (6.21), Eq. (6.22) can be written as:

$$a''U_1 + 2a'U_1' + \frac{1}{2}\xi a' U_1 = 0,$$

i.e.

$$\frac{a''}{a'} = -\frac{\xi}{2} - 2\frac{U_1'}{U_1}. \tag{6.23}$$

By denoting by $h$ the function $a'$, the differential Eq. (6.23) can be written as a first-order ordinary differential equation whose solution can be obtained via separation of variables.

Excluding $h(\xi) \equiv 0$, we obtain:

$$\ln(h(\xi)) = \ln(h(0)) - \frac{\xi^2}{4} - 2\ln(\xi^4 + 12\xi^2 + 12), \tag{6.24}$$

hence

$$h(\xi) = a'(\xi) = a'(0)\frac{1}{\left(\xi^4 + 12\xi^2 + 12\right)^2} e^{-\xi^2/4}.$$

Now, in order to get $U_2^H$ we have two possibilities. First, we can integrate $a'$ to get $a$ and consequently $U_2^H$. Second (more simply), looking at $a'$ we notice that a good candidate for $U_2^H$ is

$$f(\xi)\, e^{-\xi^2/4} + g(\xi) \int_{-\infty}^{\xi} e^{-\frac{1}{4}s^2} ds,$$

where $f(\xi)$ and $g(\xi)$ are polynomials in $\xi$, the former of degree not higher than 3 and the latter of degree not higher than 4. The coefficients of these polynomials can be easily computed by substituting $f(\xi)$ and $g(\xi)$ in the equation that $U_2^H$ must satisfy.

Proceeding as described above we obtain another solution of Eq. (6.21), independent of $U_1^H$:

$$U_2^H(\xi) = 2\xi\left(\xi^2 + 10\right) e^{-\frac{1}{4}\xi^2} + \left(\xi^4 + 12\xi^2 + 12\right) \int_0^{\xi} e^{-\frac{1}{4}s^2} ds.$$

The general solution of the homogeneous Eq. (6.21) is then

$$U_G^H(\xi) = aU_1^H(\xi) + bU_2^H(\xi)$$

and, consequently, the general solution of the non-homogeneous (6.20) is

$$U_G(\xi) = U_p(\xi) + aU_1^H(\xi) + bU_2^H(\xi).$$

By imposing $U(0) = 0$ and $\lim_{\xi\to+\infty} \frac{U(\xi)}{\xi^4} = 0$, it is possible to determine the constants $a$ and $b$. By the first condition we get $a = 0$. Since $\lim_{\xi\to\infty} \frac{U(\xi)}{\xi^4} = -\frac{1}{4} + b\sqrt{\pi}$, from the second condition we get $b = \frac{1}{4\sqrt{\pi}}$.

The similarity solution required is then:

$$U(\xi) = -\frac{1}{4}\xi^4 + \frac{1}{2\sqrt{\pi}}(\xi^3 + 10\xi)e^{-\frac{1}{4}\xi^2} + \\ + \frac{1}{4\sqrt{\pi}}(\xi^4 + 12\xi^2 + 12\xi) \int_0^{\xi} e^{-\frac{s^2}{4}} ds.$$

Finally, the solution of the initial problem (6.15)–(6.18) is

$$\begin{aligned} u(x,t) &= t^2 U\left(\frac{x}{\sqrt{t}}\right) \\ &= -\frac{1}{4}x^4 + \frac{1}{2\sqrt{\pi}}\left[\sqrt{t}x^3 + 10xt^{3/2}\right]e^{-\frac{1}{4}\xi^2} + \\ &\quad + \frac{1}{4\sqrt{\pi}}\left(x^4 + 12x^2 t + 6xt^{3/2}\right)\int_0^{x/\sqrt{t}} e^{-\frac{s^2}{4}}\, ds. \end{aligned}$$

It is easy to check that this expression solves our problem by direct substitution into (6.15).

**Exercise 6.5** By applying the Feynman-Kac representation formula, find the solution of the following problem on the domain $[0, T] \times \mathbb{R}$:

$$\begin{aligned} &\frac{\partial V}{\partial t} + \mu x \frac{\partial V}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 V}{\partial x^2} = 0 \\ &V(T, x) = \ln\left(x^4\right) + k \end{aligned} \tag{6.25}$$

with given constants $\mu, \sigma$ and $k$.

**Solution** By the Feynman-Kac representation formula, we know that the solution of the problem:

$$\begin{aligned} &\frac{\partial V}{\partial t} + m(t,x)\frac{\partial V}{\partial x} + \frac{1}{2}s^2(t,x)\frac{\partial^2 V}{\partial x^2} = 0 \\ &V(T,x) = \Phi(x) \end{aligned}$$

is

$$V(t,x) = E\left[\Phi(X_T)\right],$$

where, for $u \geq t$

$$\begin{cases} dX_u = m(u, X_u)\,du + s(u, X_u)\,dW_u \\ X_t = x \end{cases}.$$

In the present case, $m(t,x) = \mu x$, $s(t,x) = \sigma x$ and $\Phi(x) = \ln\left(x^4\right) + k$. The solution of the problem proposed is then given by:

$$V(t,x) = E\left[\Phi(X_T)\right],$$

where for $u \geq t$

$$\begin{cases} dX_u = \mu X_u du + \sigma X_u dW_u \\ X_t = x \end{cases} \tag{6.26}$$

Since the solution of (6.26) is given by

$$\begin{aligned} X_u &= X_t \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)(u-t) + \sigma\left(W_u - W_t\right)\right\} \\ &= x \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)(u-t) + \sigma\left(W_u - W_t\right)\right\}, \end{aligned}$$

we obtain that

$$X_T = x \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)(T-t) + \sigma\left(W_T - W_t\right)\right\}$$

and

$$\begin{aligned} \ln\left(X_T^4\right) + k &= \ln\left(x^4 \exp\left\{4\left(\mu - \frac{1}{2}\sigma^2\right)(T-t) + 4\sigma\left(W_T - W_t\right)\right\}\right) + k \\ &= \ln\left(x^4\right) + 4\left(\mu - \frac{1}{2}\sigma^2\right)(T-t) + 4\sigma\left(W_T - W_t\right) + k. \end{aligned}$$

We finally get the solution of the problem proposed:

$$\begin{aligned} V(t,x) &= E\left[\Phi\left(X_T\right)\right] \\ &= E\left[\ln\left(x^4\right) + 4\left(\mu - \frac{1}{2}\sigma^2\right)(T-t) + 4\sigma\left(W_T - W_t\right) + k\right] \\ &= \ln\left(x^4\right) + 4\left(\mu - \frac{1}{2}\sigma^2\right)(T-t) + k, \end{aligned}$$

once we observe $E\left[W_T - W_t\right] = 0$ ($W$ is a standard Brownian motion).

**Exercise 6.6** In the Black-Scholes setting, consider an option written on an underlying with the following dynamics:

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

with $S_0 = 20$ euro, $\mu = 0.16$ and $\sigma = 0.36$ (per year). The risk-free interest rate is 0.04 per year.

By applying the Feynman-Kac formula, compute the initial value of the option with underlying $S$ and payoff

$$\Phi\left(S_T\right)=\left(\ln\left(S_T^2\right)-K\right)^+$$

with strike $K=6$ euros and with maturity $T=1$ year.

**Solution** In the Black-Scholes model the value of the option above at time $t$ is assumed to be a function of $S_t$ and $t$ (denoted by $F\left(S_t,t\right)$) satisfying the following PDE:

$$\begin{aligned}
&\frac{\partial F}{\partial t}+rx\frac{\partial F}{\partial x}+\frac{1}{2}\sigma^2x^2\frac{\partial^2 F}{\partial x^2}-rF=0\\
&F\left(T,S_T\right)=\left(\ln\left(S_T^2\right)-K\right)^+=\Phi\left(S_T\right).
\end{aligned}$$

By the Feynman-Kac formula, we know that the solution of

$$\begin{aligned}
&\frac{\partial F}{\partial t}+m\left(t,x\right)\frac{\partial F}{\partial x}+\frac{1}{2}s^2\left(t,x\right)\frac{\partial^2 F}{\partial x^2}-rF=0\\
&F\left(T,x\right)=\Phi\left(x\right)
\end{aligned}$$

is given by

$$F\left(t,x\right)=e^{-r(T-t)}E\left[\Phi\left(X_T\right)\right],$$

where, for $u\geq t$

$$\begin{cases}dX_u=m\left(u,X_u\right)du+s\left(u,X_u\right)dW_u\\ \quad X_t=x\end{cases}.$$

Since, in our case, $m\left(t,x\right)=rx$, $s\left(t,x\right)=\sigma x$ and $\Phi\left(x\right)=\left(\ln\left(x^2\right)-K\right)^+$, the solution of the initial problem is given by

$$F\left(t,x\right)=e^{-r(T-t)}E\left[\Phi\left(X_T\right)\right],$$

where, for $u\geq t$

$$\begin{cases}dX_u=rX_udu+\sigma X_udW_u\\ \quad X_t=x\end{cases}. \tag{6.27}$$

Since the solution of Eq. (6.27) is

$$X_u = x \exp\left\{\left(r - \frac{1}{2}\sigma^2\right)(u-t) + \sigma\left(W_u - W_t\right)\right\},$$

we obtain that

$$X_T = x \exp\left\{\left(r - \frac{1}{2}\sigma^2\right)(T-t) + \sigma\left(W_T - W_t\right)\right\}$$

and

$$\begin{aligned}\ln\left(X_T^2\right) &= \ln\left(x^2 \exp\left\{2\left(r - \frac{1}{2}\sigma^2\right)(T-t) + 2\sigma\left(W_T - W_t\right)\right\}\right)\\ &= \ln\left(x^2\right) + 2\left(r - \frac{1}{2}\sigma^2\right)(T-t) + 2\sigma\left(W_T - W_t\right).\end{aligned}$$

We finally get the initial value of the option considered:

$$\begin{aligned}F\left(S_0, 0\right) &= e^{-rT} E\left[\Phi\left(X_T\right)\right]\\ &= e^{-rT} E\left[\left(\ln\left(X_0^2\right) + 2\left(r - \frac{1}{2}\sigma^2\right)T + 2\sigma W_T - K\right)^+\right]\\ &= e^{-rT} E\left[\left(\ln\left(S_0^2\right) + \left(2r - \sigma^2\right)T + 2\sigma W_T - K\right)^+\right].\end{aligned}$$

There remains then to compute

$$E\left[\left(\ln\left(S_0^2\right) + \left(2r - \sigma^2\right)T + 2\sigma W_T - K\right)^+\right].$$

Since $(W_t)_{t \geq 0}$ is a standard Brownian motion and $T = 1$,

$$W_T = W_1 \sim N(0; 1).$$

Accordingly, if we denote by $Z = W_1 \sim N(0; 1)$, we get:

$$\begin{aligned}F\left(S_0, 0\right) &= e^{-r} E\left[\left(2\ln\left(S_0\right) + 2r - \sigma^2 + 2\sigma Z - K\right)^+\right]\\ &= e^{-r} \int_{\mathbb{R}} \left(2\ln\left(S_0\right) + 2r - \sigma^2 + 2\sigma z - K\right)^+ f_Z(z)\, dz\\ &= e^{-r} \int_{\mathbb{R}} \left(2\ln\left(S_0\right) + 2r - \sigma^2 + 2\sigma z - K\right)^+ \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\, dz\end{aligned}$$

$$
\begin{aligned}
&= e^{-r}\int_{\frac{K-2\ln(S_0)-2r+\sigma^2}{2\sigma}}^{+\infty}\left(2\ln(S_0)+2r-\sigma^2+2\sigma z-K\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz\\
&= e^{-r}\int_{\frac{K-2\ln(S_0)-2r+\sigma^2}{2\sigma}}^{+\infty}\left(2\ln(S_0)+2r-\sigma^2-K\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz\\
&\quad +e^{-r}\int_{\frac{K-2\ln(S_0)-2r+\sigma^2}{2\sigma}}^{+\infty}2\sigma z\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz\\
&= e^{-r}\left(2\ln(S_0)+2r-\sigma^2-K\right)\left[1-N\left(\frac{K-2\ln(S_0)-2r+\sigma^2}{2\sigma}\right)\right]\\
&\quad +e^{-r}\left[-2\sigma\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}\right]\Bigg|_{\frac{K-2\ln(S_0)-2r+\sigma^2}{2\sigma}}^{+\infty}\\
&= e^{-r}\left(2\ln(S_0)+2r-\sigma^2-K\right)\left[1-N\left(\frac{K-2\ln(S_0)-2r+\sigma^2}{2\sigma}\right)\right]\\
&\quad +e^{-r}2\sigma\frac{1}{\sqrt{2\pi}}e^{-\frac{\left[K-2\ln(S_0)-2r+\sigma^2\right]^2}{8\sigma^2}}\\
&= 0.25,
\end{aligned}
$$

where $N(\cdot)$ denotes the cumulative function of a standard normal random variable. The initial value of the option is then 0.25 euros.

**Exercise 6.7** Compute the solution $\phi(x,t)$ of the diffusion equation:

$$\frac{\partial\phi}{\partial t}=\frac{\partial^2\phi}{\partial x^2},\qquad x\in\mathbb{R},\ t>0, \tag{6.28}$$

satisfying the following conditions:

$$
\begin{array}{l}
\phi(x,0)=\delta(x),\quad \lim_{x\to\pm\infty}\phi(x,t)=0,\\
\lim_{x\to\pm\infty}\frac{\partial\phi}{\partial x}=0,\ \int_{-\infty}^{+\infty}\phi(x,t)dx=1.
\end{array}
$$

**Solution** The solution to be determined is called the *fundamental solution* of the diffusion equation; it can be used to find solutions of the diffusion equation satisfying more general data, as we shall see later.

A first method to solve this initial/boundary-value problem is based on the Fourier transform. The Fourier transform of a function $f(x,t)$ is defined as

$$\varphi(k,t)\triangleq\int_{-\infty}^{+\infty}f(x,t)e^{ikx}dx. \tag{6.29}$$

Notice that, by integrating twice by parts, the following relations hold for the Fourier transforms of the derivatives of $f$ (under the hypothesis that both $f$ and its first derivative in $x$ vanish at infinity):

$$\int_{-\infty}^{+\infty} \frac{\partial f}{\partial x} e^{ikx} dx = \left[ f(x,t) e^{ikx} \right]_{-\infty}^{+\infty} - ik \int_{-\infty}^{+\infty} f(x,t) e^{ikx} dx = -ik\varphi,$$

$$\int_{-\infty}^{+\infty} \frac{\partial^2 f}{\partial x^2} e^{ikx} dx = \left[ \frac{\partial f}{\partial x} e^{ikx} \right]_{-\infty}^{+\infty} - ik \int_{-\infty}^{+\infty} \frac{\partial f}{\partial x} e^{ikx} dx = -k^2\varphi.$$

It is then immediate to verify that the function $\phi$ must satisfy

$$\frac{\partial \varphi}{\partial t} = -k^2 \varphi, \tag{6.30}$$

with the initial condition

$$\varphi(k, 0) = 1. \tag{6.31}$$

The solution satisfying the required initial condition of the previous ODE can be found immediately

$$\varphi(k, t) = e^{-k^2 t}. \tag{6.32}$$

Now, in order to find the solution $\phi(x, t)$ of the original problem we have to invert the Fourier transform:

$$\phi(x, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \varphi(k, t) e^{-ikx} dx. \tag{6.33}$$

The explicit computation of the integral (easily performed by completing the square appearing in the argument of the exponential function) provides the following result

$$\phi(x, t) = \frac{1}{2\sqrt{\pi t}} \exp\left( -\frac{x^2}{4t} \right). \tag{6.34}$$

Another method for solving the problem is based on a similarity technique. We look for a solution $f(x, t)$ of the following form: $f(x, t) = t^\alpha U(\xi)$, with $\xi = x/t^\beta$. The form of the PDE suggests for $\beta$ the value $\beta = 1/2$, while the condition on the integral of the unknown extended to the entire real line suggests for $\alpha$ the value $\alpha = 1/2$. The similarity solution we are looking for is then of the following kind:

$$f(x, t) = \frac{1}{\sqrt{t}} U(\xi). \tag{6.35}$$

The similarity reduction provides the following ODE for the new unknown $U(\xi)$:

$$U''(\xi) + \frac{\xi}{2}U'(\xi) + \frac{1}{2}U(\xi) = 0, \tag{6.36}$$

together with boundary conditions:

$$\lim_{\xi \to \pm\infty} U(\xi) = 0$$
$$\int_{-\infty}^{+\infty} U(\xi) d\xi = 1.$$

A straightforward computation provides the solution $U(\xi)$:

$$U(\xi) = Ae^{-\frac{\xi^2}{4}} + B. \tag{6.37}$$

While the first boundary condition implies $B = 0$, the second implies $A = 1/(2\sqrt{\pi})$. Hence the solution of our original problem $\phi(x,t)$ becomes:

$$\phi(x,t) = \frac{1}{2\sqrt{\pi t}} \exp\left(-\frac{x^2}{4t}\right). \tag{6.38}$$

**Exercise 6.8** Let $\phi$ be the fundamental solution of the diffusion equation obtained in the previous exercise. Verify that the function

$$f(x,t) \triangleq \int_{-\infty}^{+\infty} \phi(\xi - x, t) u(\xi) d\xi \tag{6.39}$$

is a solution of the diffusion equation

$$\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2}, \quad x \in \mathbb{R},\ t > 0, \tag{6.40}$$

satisfying the initial condition $f(x,0) = u(x)$.

**Solution** We already know that the function $\phi(x,t)$ is a solution of the diffusion equation satisfying the initial condition $\phi(x,0) = \delta(x)$. It is immediate to verify that the diffusion equation is invariant under the change of variable $x \to \xi - x$, i.e. if $\phi(x,t)$ is a solution, $\phi(\xi - x, t)$ is also a solution, but satisfying the initial datum $\phi(\xi - x, 0) = \delta(\xi - x)$. Moreover the diffusion equation is linear, so that multiplying the equation:

$$\frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial x^2} \tag{6.41}$$

by $u(\xi)$ and integrating in $\xi$, the function

$$f(x,t) \triangleq \int_{-\infty}^{+\infty} \phi(x-\xi,t)u(\xi)d\xi \tag{6.42}$$

satisfies the diffusion equation with initial datum

$$f(x,0) = \int_{-\infty}^{+\infty} \phi(x-\xi,0)u(\xi)d\xi = \int_{-\infty}^{+\infty} \delta(x-\xi)u(\xi)d\xi = u(x). \tag{6.43}$$

**Exercise 6.9** Find a suitable change of variables turning the Black-Scholes equation into a PDE with constant coefficients.

**Solution** The Black-Scholes PDE, holding for the value $F(S_t,t)$ of every derivative, written on an underlying asset whose price dynamics is described by a geometric Brownian motion, and consistent with the no-arbitrage requirement, is the following:

$$\frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} + rS\frac{\partial F}{\partial S} - rF = 0. \tag{6.44}$$

This PDE exhibits an explicit dependence on the variable $S$, denoting the underlying value, in both the coefficients of the terms involving the first and the second derivatives in $S$.

As a geometric Brownian motion is the exponential of a Brownian motion (with drift), an "educated guess" suggests the following change of variable:

$$\begin{aligned} x &= \ln(S), \\ S &= e^x, \\ F(S,t) &= h(x(S),t), \\ h(x,t) &= F(S(x),t). \end{aligned}$$

By computing the partial derivatives of the new unknown with respect to the new independent variables, we get

$$\begin{aligned} \frac{\partial F}{\partial S} &= \frac{\partial h}{\partial x}\frac{\partial x}{\partial S} = \frac{1}{S}\frac{\partial h}{\partial x}, \\ \frac{\partial^2 F}{\partial S^2} &= \frac{\partial^2 h}{\partial x^2}\left(\frac{\partial x}{\partial S}\right)^2 + \frac{\partial h}{\partial x}\frac{\partial^2 x}{\partial S^2} = \frac{1}{S^2}\frac{\partial^2 h}{\partial x^2} - \frac{1}{S^2}\frac{\partial h}{\partial x}. \end{aligned}$$

By plugging these expressions into the original equation, one obtains

$$\frac{\partial h}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 h}{\partial x^2} + \left(r - \frac{\sigma^2}{2}\right)\frac{\partial h}{\partial x} - rh = 0, \tag{6.45}$$

where the coefficients' dependence on the independent variables has been dropped.

**Exercise 6.10** Find a suitable change of variables turning the Black-Scholes equation for a European Call option into the diffusion equation with proper initial-boundary conditions.

**Solution** Solving the problem requires two steps. In the first, we shall remove the dependence on the independent variable $S$ in the PDE coefficients. To this end, inspired both by Exercise 6.9 and by the functional form of the European Call option payoff, we exploit a change of variables of the following kind:

$$\begin{aligned}
x &= \ln(S/K),\\
S &= Ke^x,\\
\tau &= \frac{\sigma^2}{2}(T - t),\\
t &= T - \frac{2}{\sigma^2}\tau\\
F(S, t) &= Kh(x(S), \tau(t)),\\
h(x, t) &= K^{-1}F(S(x), t(\tau)),
\end{aligned}$$

where $K$ is the strike of the European option considered. We introduce the new time variable $\tau$, which is the dimensionless version of the "time to maturity" variable $(T - t)$; this change of sign in the time variable allows to turn the parabolic type of the Black-Scholes equation (parabolic backward) into the parabolic type of the diffusion equation (parabolic forward). Moreover, we introduce the new parameter $q = 2r/\sigma^2$ to simplify the notation. After these changes, the Black-Scholes equation becomes

$$\frac{\partial h}{\partial \tau} - \frac{\partial^2 h}{\partial x^2} - (q - 1)\frac{\partial h}{\partial x} + qh = 0. \tag{6.46}$$

The final condition of the Black-Scholes equation for the European Call option is its payoff $F(S, T) = \max(S - K; 0)$; expressed in the new variables, it can be written as $h(x, 0) = \max(e^x - 1; 0)$.

As far as the second step is concerned, we try to remove the terms involving $h$ and its first derivative in $x$. To this end we introduce a further change of variables of the following kind:

$$h(x, \tau) = \exp\{\alpha x + \beta\tau\}g(x, \tau), \tag{6.47}$$

where $\alpha$, $\beta$ have to be determined so that the required terms can be removed, and we look for the PDE that the new function $g$ must satisfy. After a direct computation of all the derivatives of the old function in terms of the new variables, we obtain

$$\beta g + \frac{\partial g}{\partial \tau} = \frac{\partial^2 g}{\partial x^2} + (q-1)\left[\alpha g + \frac{\partial g}{\partial x}\right] + 2\alpha \frac{\partial g}{\partial x} + (\alpha^2 - q)g. \tag{6.48}$$

To remove the terms involving $g$ and its first derivative in $x$, the coefficient $\alpha$, $\beta$ must fulfill the following conditions:

$$\begin{cases} 2\alpha + (q-1) = 0 \\ \beta = \alpha^2 + (q-1)\alpha - q \end{cases}.$$

The required values are: $\alpha = -(q-1)/2$, $\beta = -(q+1)^2/4$. By choosing then $\alpha$ and $\beta$ as above, the function $g$ appearing in

$$h(x,\tau) = \exp\left\{-\frac{q-1}{2}x - \frac{(q+1)^2}{4}\tau\right\} g(x,\tau), \tag{6.49}$$

must satisfy the diffusion equation for $x \in (-\infty, +\infty)$ and $\tau > 0$:

$$\frac{\partial g}{\partial \tau} = \frac{\partial^2 g}{\partial x^2}, \tag{6.50}$$

with

$$g(x,0) = \max\left\{e^{\frac{q+1}{2}x} - e^{\frac{q-1}{2}x}; 0\right\}. \tag{6.51}$$

**Exercise 6.11** Find the explicit solution of the Black-Scholes equation for a European Call option by using the results obtained in the previous exercises.

**Solution** By the previous exercise we know that the Black-Scholes equation for a European Call option can be reduced, by a suitable change of variables (which we performed in two separate steps), to the diffusion equation:

$$\frac{\partial g}{\partial \tau} = \frac{\partial^2 g,}{\partial x^2} \tag{6.52}$$

with initial datum:

$$g(x,0) = \max\left\{e^{\frac{q+1}{2}x} - e^{\frac{q-1}{2}x}; 0\right\}. \tag{6.53}$$

Moreover, we know from Exercise 6.8 that the solution of the diffusion equation satisfying a general initial datum can be expressed, via an integral representation formula, as a convolution of the fundamental solution:

$$g(x,\tau) \triangleq \int_{-\infty}^{+\infty} \phi(\xi - x, \tau) g(\xi, 0) d\xi, \tag{6.54}$$

where

$$\phi(\xi - x, \tau) = \frac{1}{2\sqrt{\pi\tau}} \exp\left\{-\frac{(\xi - x)^2}{4\tau}\right\}. \tag{6.55}$$

Hence, in order to find explicitly this solution, we must compute the following integral:

$$g(x,\tau) \triangleq \int_{-\infty}^{+\infty} \frac{1}{2\sqrt{\pi\tau}} \exp\left\{-\frac{(\xi - x)^2}{4\tau}\right\} \max\left(e^{\frac{q+1}{2}\xi} - e^{\frac{q-1}{2}\xi}; 0\right) d\xi. \tag{6.56}$$

We remark that the support of the integrand is simply $\xi > 0$. Due to monotonicity of the exponential function ( $q > 1$), we have indeed

$$e^{\frac{q+1}{2}\xi} - e^{\frac{q-1}{2}\xi} > 0 \Longleftrightarrow \xi > 0. \tag{6.57}$$

So, by changing the integration domain, the integral can be written in the following way:

$$g(x,\tau) \triangleq \int_{0}^{+\infty} \frac{1}{2\sqrt{\pi\tau}} \exp\left\{-\frac{(\xi - x)^2}{4\tau}\right\} \left[e^{\frac{q+1}{2}\xi} - e^{\frac{q-1}{2}\xi}\right] d\xi. \tag{6.58}$$

Let us consider separately the two contributions:

$$J_1 = \int_{0}^{+\infty} \frac{1}{2\sqrt{\pi\tau}} \exp\left\{-\frac{(\xi - x)^2}{4\tau}\right\} e^{\frac{q+1}{2}\xi} d\xi$$

$$J_2 = \int_{0}^{+\infty} \frac{1}{2\sqrt{\pi\tau}} \exp\left\{-\frac{(\xi - x)^2}{4\tau}\right\} e^{\frac{q-1}{2}\xi} d\xi.$$

By introducing the new variable $z = (\xi - x)/\sqrt{2\tau}$ ($dz = d\xi/\sqrt{2\tau}$), the integral $J_1$ can be written as follows:

$$J_1 = \frac{1}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{+\infty} \exp\left(-\frac{z^2}{2}\right) e^{\frac{q+1}{2}\left(x + z\sqrt{2\tau}\right)} dz, \tag{6.59}$$

by completing the square in the argument of the exponential function, it can be computed in the following way:

$$J_1 = \frac{1}{\sqrt{2\pi}} e^{\frac{q+1}{2}x} \int_{-x/\sqrt{2\tau}}^{+\infty} e^{\frac{(q+1)^2}{4}\tau} \exp\left\{ -\frac{\left(z - \frac{q+1}{2}\sqrt{2\tau}\right)^2}{2} \right\} dz$$
$$= \frac{1}{\sqrt{2\pi}} \left[ e^{\frac{q+1}{2}x} e^{\frac{(q+1)^2}{4}\tau} \right] \int_{-\frac{x}{\sqrt{2\tau}} - \frac{q+1}{2}\sqrt{2\tau}}^{+\infty} \exp\left\{ -\frac{y^2}{2} \right\} dy,$$

where we introduced a further integration variable $y = z - (q+1)\sqrt{2\tau}/2$ ($dy = dz$). The last integral can be immediately expressed in terms of the standard normal distribution function, simply observing that:

$$\int_{-x}^{+\infty} \exp\left(-\frac{u^2}{2}\right) du = \int_{-\infty}^{x} \exp\left(-\frac{u^2}{2}\right) du = N(x). \tag{6.60}$$

Hence we finally get:

$$J_1 = \exp\left\{ \frac{q+1}{2}x + \frac{(q+1)^2}{4}\tau \right\} N\left( \frac{x}{\sqrt{2\tau}} + \frac{q+1}{2}\sqrt{2\tau} \right). \tag{6.61}$$

The integral $J_2$ can be computed similarly:

$$J_2 = \exp\left\{ \frac{q-1}{2}x + \frac{(q-1)^2}{4}\tau \right\} N\left( \frac{x}{\sqrt{2\tau}} + \frac{q-1}{2}\sqrt{2\tau} \right). \tag{6.62}$$

In order to find the solution to our original problem (the Black-Scholes equation for a European Call option) we need to write our solution in terms of the original variables. From

$$x = \ln(S/K), \qquad \tau = \frac{\sigma^2}{2}(T-t),$$
$$C(S,t) = K \cdot h(x(S), \tau(t)),$$
$$h(x,\tau) = \exp\left\{ -\frac{q-1}{2}x - \frac{(q+1)^2}{4}\tau \right\} g(x,\tau),$$

we obtain the well-known formula:

$$C(S,t) = S \cdot N(d_1) - Ke^{-r(T-t)} N(d_2), \tag{6.63}$$

where $d_1, d_2$ are defined, as usual, by

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sqrt{T-t}},$$

$$d_2 = \frac{\ln\left(\frac{S}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sqrt{T-t}}.$$

**Exercise 6.12** By using a similarity reduction method, derive an explicit valuation formula for Exchange options, i.e. for options written on two different underlying assets with values $S_1(t), S_2(t)$ and with payoff given by $C_{EX}(S_1(T), S_2(T)) :=$ max $S_1(T) - S_2(T)$; 0, by assuming that the dynamics of the two underlying assets is described by two geometric Brownian motions driven by two Wiener processes $W_1(t), W_2(t)$ correlated with correlation coefficient $\rho$, i.e.

$$dS_1(t) = \mu_1 S_1(t)dt + \sigma_1 dW_1(t)$$
$$dS_2(t) = \mu_2 S_2(t)dt + \sigma_2 dW_1(t),$$

with $dW_1(t)dW_2(t) = \rho dt$.

**Solution** We must solve the following two-dimensional Black-Scholes equation:

$$\frac{\partial f}{\partial t} + rS_1\frac{\partial f}{\partial S_1} + rS_2\frac{\partial f}{\partial S_2} + \sigma_1^2 S_1^2\frac{\partial^2 f,}{\partial S_1^2} + \sigma_2^2 S_2^2\frac{\partial^2 f,}{\partial S_2^2} + \rho\sigma_1\sigma_2\frac{\partial^2 f,}{\partial S_1 \partial S_2} - rf = 0. \tag{6.64}$$

with final datum:

$$f(S_1, S_2, T) = \max\{S_1 - S_2; 0\}. \tag{6.65}$$

By introducing the new independent variable $z := S_1/S_2$ and the new unknown $f(S_1, S_2, t) := S_2 g(S_1/S_2, t)$, and observing that $\frac{\partial f}{\partial t} = S_2\frac{\partial g}{\partial t}$, $\frac{\partial f}{\partial S_1} = S_2\frac{1}{S_2}\frac{\partial g}{\partial z} = \frac{\partial g}{\partial z}$, $\frac{\partial f}{\partial S_2} = g - S_2\frac{S_1}{S_2^2}\frac{\partial g}{\partial z} = g - z\frac{\partial g}{\partial z}$, $\frac{\partial^2 f,}{\partial S_1^2} = \frac{1}{S_2}\frac{\partial^2 g,}{\partial z^2}$, $\frac{\partial^2 f,}{\partial S_2^2} = z\frac{z}{S_2}\frac{\partial^2 g,}{\partial z^2}$, $\frac{\partial^2 f,}{\partial S_1^2} = -\frac{z}{S_2}\frac{\partial^2 g,}{\partial z^2}$, the previous PDE for $g$ becomes:

$$\frac{\partial g}{\partial t} + \frac{1}{2}\sigma^2 z^2\frac{\partial^2 g}{\partial z^2} = 0, \tag{6.66}$$

where

$$\sigma^2 = \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2, \tag{6.67}$$

and the final datum:

$$g(z, T) = \max\left(z(T) - 1; 0\right). \tag{6.68}$$

The PDE just written is of Black-Scholes type, with $r = 0$, $K = 1$, so we can write immediately its solution as follows:

$$g(z, t) = zN(d_1) - N(d_2), \tag{6.69}$$

where:

$$d_1 = \frac{1}{\sigma\sqrt{T-t}}[\ln z + \sigma^2 T/2]$$
$$d_2 = \frac{1}{\sigma\sqrt{T-t}}[\ln z - \sigma^2 T/2].$$

By introducing the old variables $f, S_1, S_2$) the previous formula can be written as follows:

$$\begin{aligned} f(S_1, S_2, t) &= S_1 N(d_1) - S_2 N(d_2) \\ d_1 &= \frac{1}{\sigma\sqrt{T-t}}[\ln(\frac{S_1}{S_2}) + \sigma^2 T/2] \\ d_2 &= \frac{1}{\sigma\sqrt{T-t}}[\ln(\frac{S_1}{S_2}) - \sigma^2 T/2], \end{aligned}$$

and this is the solution to our problem.

## 6.3 Proposed Exercises

**Exercise 6.13** In a Black-Scholes setting, consider an option with underlying evolving according to the following dynamics:

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

with $S_0 = 16$ euros, $\mu = 0.12$ and $\sigma = 0.3$. The risk-free interest rate $r$ available on the market is 0.04. The time unit is 1 year.

1. If the payoff of the option considered is

$$\Phi(S_T) = \left(\ln(S_T/K)^3\right)^+ = \left(\ln\left(S_T^3\right) - 3\ln K\right)^+$$

with $K = 20$ euros and maturity $T = 1$, compute the price of the option today ($t = 0$) by using the Feynman-Kac representation formula.
2. What is the event with higher probability: to have a strictly positive payoff for the option of item 1. or for a European Call option with the same $K$ and $T$? Or are their probabilities the same?
3. Is the probability to get a payoff at least of 10 euros for the option of item 1. higher than for a European Call option with the same $K$ and $T$? Or are their probabilities the same?

**Exercise 6.14** By solving the Black-Scholes equation, equipped with the proper final condition, find the formula for the European Put option price.

# Chapter 7
# Black-Scholes Model for Option Pricing and Hedging Strategies

## 7.1 Review of Theory

As in the previous chapters, we consider a market model consisting in two assets: one non-risky (bond), the other risky (stock). While before we focused on *discrete-time* market models, here we introduce the so-called *Black-Scholes model*: a well-known example of *continuous-time* market model.

The key assumption behind the Black-Scholes model is that the stock price $S$ evolves as a geometric Brownian motion, i.e.

$$S_t = S_0 \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right\}, \tag{7.1}$$

where $S_0$ is the initial stock price, $\mu$ the drift, $\sigma$ the volatility (or diffusion parameter) and $(W_t)_{t\geq 0}$ is a standard Brownian motion. By Itô's Lemma, the price of $S$ is described by the following Stochastic Differential Equation:

$$\begin{cases} dS_t = \mu S_t dt + \sigma S_t dW_t \\ S_0 = s_0 \end{cases}.$$

The bond price $B$ is evaluated by continuous compounding with a risk-free rate $r$. Namely, if $B_0$ is the initial price of the bond:

$$B_t = B_0 \exp(rt)\,, \tag{7.2}$$

or, equivalently,

$$\begin{cases} dB_t = rB_t dt \\ B_0 = b_0 \end{cases}.$$

 
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La Matematica per il 3+2 149, https://doi.org/10.1007/978-3-031-28378-9_7

Furthermore, the model is based on the following assumptions: perfect liquidity (one can buy or sell any real number of an asset), short selling is allowed, transaction costs are never applied, bid and ask prices of any asset coincide. Moreover, the market model is also assumed to be free of arbitrage.

In the Black-Scholes model, it has been proved (see [6, 21, 33] for details) that there exists an equivalent martingale measure useful for the pricing of a derivative and consistent with the no-arbitrage assumption. The equivalent martingale measure is also called "risk-neutral"; roughly speaking, in fact, under such a measure the underlying evolves (more or less) as it would with a non-risky asset. Under the risk-neutral measure, the underlying price is described by the following Stochastic Differential Equation:

$$\begin{cases} dS_t = rS_t dt + \sigma S_t dW_t \\ S_0 = s_0 \end{cases} .$$

It was also shown that the Black-Scholes model, as well as the binomial model, is complete. The Second Fundamental Theorem of Asset Pricing implies therefore the uniqueness of the risk-neutral measure for such a market model. For complete market models, the price of any derivative is uniquely determined by the no-arbitrage assumption.

Denote by $F(S_t, t)$ the price (at time $t$) of a derivative having $S$ as underlying. Such a price can be obtained by solving the following Partial Differential Equation (called *Black-Scholes equation*):

$$\frac{\partial F}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 F}{\partial S^2} + rS\frac{\partial F}{\partial S} - rF = 0, \tag{7.3}$$

endowed with some suitable final and boundary conditions. Equivalently, $F(S_t, t)$ can be obtained by the discounted expected value of the payoff of the derivative, under the equivalent martingale measure and conditioned by $\mathscr{F}_t$ (where $(\mathscr{F}_t)_{t\geq 0}$ is the filtration generated by $(S_t)_{t\geq 0}$). In other words,

$$F(S_t, t) = e^{-r(T-t)} E_Q[F(S_T, T)|\mathscr{F}_t]. \tag{7.4}$$

**Black-Scholes Formula for European Call/Put Options on Stocks Without Dividends**

Consider a European Call option, whose payoff is given by

$$C(S_T, T) = \max\{S_T - K; 0\},$$

where $K$ stands for the *strike* of the option and $S_T$ for the underlying price at maturity. The solution of the Black-Scholes PDE (as well as the discounted expected value of the payoff under the equivalent martingale measure) gives the following price:

$$C(S_t, t) = S_t N(d_1) - K e^{-r(T-t)} N(d_2), \quad 0 \le t < T \tag{7.5}$$

where $N(\cdot)$ denotes the cumulative distribution function of a standard normal and $d_1, d_2$ are defined as:

$$d_1 \triangleq \frac{\ln(S_t/K) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \tag{7.6}$$

$$d_2 \triangleq \frac{\ln(S_t/K) + (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t} \tag{7.7}$$

For the pricing of a Put option one can proceed similarly by taking into account that the final value (equal to the payoff) of a Put is given by

$$P(S_T, T) = \max\{K - S_T; 0\}.$$

An alternative way to evaluate the Put option is based on the relation between the prices of Call and a Put options written on the same underlying, with same strike and maturity. The prices of Call and Put options as above, indeed, satisfy the so-called *Put-Call parity*:

$$C(S_t, t) - P(S_t, t) = S_t - K e^{-r(T-t)}. \tag{7.8}$$

Proceeding as explained above, the price (at time $t$) of a European Put option with maturity $T$, with strike $K$ and written on the underlying $S$ is given by

$$P(S_t, t) = K e^{-r(T-t)} N(-d_2) - S_t N(-d_1), \quad 0 \le t < T. \tag{7.9}$$

**Black-Scholes Formula for European Call/Put Options on Stocks Paying "Continuous" Dividends**

The stock $S$ is said to pay "continuous" dividends when the dividend rate $D_0$ is constant in time, that is, the sum $D_0 S dt$ is distributed in the infinitesimal interval $dt$ to the owners of the stock.

The Black-Scholes formula for a European Call option written on an underlying paying continuous dividends can be obtained by (7.5), by replacing $r$ with the risk-free rate $(r - D_0)$ in the expressions of $d_1$ and $d_2$ ($d_1'$, $d_2'$ will denote the new values) and by replacing $S_t$ with $S_t e^{-D_0(T-t)}$ in the first term of the formula. This gives

$$C(S_t, t) = S_t e^{-D_0(T-t)} N(d_1') - K e^{-r(T-t)} N(d_2'), \tag{7.10}$$

where

$$d_1' \triangleq \frac{\ln(S_t/K) + (r - D_0 + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \tag{7.11}$$

$$d_2' \triangleq \frac{\ln(S_t/K) + (r - D_0 - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \tag{7.12}$$

A similar formula holds for European Put options.

**Black-Scholes Formula for European Call/Put Options on Stocks Paying "Discrete" Dividends**

When dividends are not paid "continuously" (as in the previous case where a dividend rate was applied) but just at fixed times (known a priori), dividends are said to be "discrete".

The Black-Scholes formula for a European Call option written on a stock paying discrete dividends is the following:

$$C\left(S_0, 0\right) = \left(S_0 - D\right) N\left(\hat{d}_1\right) - Ke^{-rT} N\left(\hat{d}_2\right), \tag{7.13}$$

where $D$ denotes the discounted value of all dividends paid until the option maturity $T$ and

$$\hat{d}_1 \triangleq \frac{\ln\left((S_0 - D)/K\right) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

$$\hat{d}_2 \triangleq \frac{\ln\left((S_0 - D)/K\right) + \left(r - \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} = \hat{d}_1 - \sigma\sqrt{T}$$

A similar formula holds for European Put options written on stocks paying discrete dividends.

As for options on stocks without dividends, a *Put-Call Parity* holds for European Call/Put options with the same maturity and strike and written on the same stock paying discrete dividends:

$$C\left(S_0, 0\right) - P\left(S_0, 0\right) = S_0 - D - Ke^{-rT}.$$

In order to simplify notations, in the following we will often write $C_t$ instead of $C\left(S_t, t\right)$ and $P_t$ instead of $P\left(S_t, t\right)$.

The Black-Scholes model allows not only to compute explicitly the price of a European Call/Put option on a given underlying $S$, but also to find the so-called *hedging strategy* that the seller of a derivative should have to be hedged against the risk assumed. An hedging strategy can be *static* (sometimes called "buy and hold" strategy) or *dynamic*. In the first case, the portfolio composition, once built, will be unchanged for the whole duration of the strategy; in the dynamic case, continuous (or periodic) changes in the composition of the portfolio would be needed.

The simplest dynamic hedging strategy is the one called *Delta-hedging*, where an "instantaneously riskless" portfolio is composed by one derivative and by $-\Delta$ (Delta) shares of the underlying (the negative sign denotes a "short" position). The 'Delta' of a derivative of value $F(S_t, t)$ is defined as:

$$\Delta_F \triangleq \frac{\partial F}{\partial S}, \tag{7.14}$$

and, obviously, it changes with time. By the Black-Scholes formula, one can deduce that the Delta of a European Call option on a stock without dividends is given by:

$$\Delta_{Call} = N(d_1),$$

while the Delta of the corresponding Put option is given by

$$\Delta_{Put} = N(d_1) - 1.$$

Notice that $\Delta_{Call} > 0$, while $\Delta_{Put} < 0$.

From a theoretical point of view, in order to keep the portfolio riskless one should be able to modify its composition at any instant of time. Obviously, this is unreasonable and too expensive in practice. Nevertheless, the portfolio should be re-balanced at intervals of time close enough to keep the riskiness under control. When the value of the derivative varies so quickly that it becomes difficult to control the risk associated to such a variation, *Gamma-hedging* strategies are used. The 'Gamma' $\Gamma$ of a derivative is defined as follows:

$$\Gamma_F \triangleq \frac{\partial^2 F}{\partial S^2}. \tag{7.15}$$

In the Black-Scholes model, the Gamma of a European Call/Put option written on a stock without dividends is given by:

$$\Gamma_{Call} = \Gamma_{Put} = \frac{N'(d_1)}{\sigma S_t \sqrt{T-t}}, \tag{7.16}$$

where $N'(\cdot)$ denotes the density function of a standard normal.

The Gamma-hedging strategy consists in composing a portfolio whose Gamma is equal to zero. In order to cancel both Delta and Gamma of a portfolio (in such a case the portfolio is said to be Delta and Gamma-neutral), the portfolio is required to be composed by more than two assets (differently from the Delta-neutral case). We will explain how to apply this idea in the exercises.

Besides Delta and Gamma, that quantify the riskiness of a derivative associated to changes in the underlying price, the Greek letters $\rho$ (Rho), $\nu$ (Vega) and $\theta$ (Theta) are used to index the risk associated to changes in the interest rate $r$, in the volatility $\sigma$ and in time $t$, respectively. Because of their symbols, these quantities are collectively known as "Greeks". They are used to introduce portfolios that are either Rho-neutral, Vega-neutral or Theta-neutral.

For a European Call option on a stock without dividends, $\rho$, $\nu$ and $\theta$ are given by

$$\rho \triangleq \frac{\partial F}{\partial r} = K(T-t)e^{-r(T-t)}N(d_2) \tag{7.17}$$

$$\nu \triangleq \frac{\partial F}{\partial \sigma} = S_t N'(d_1)\sqrt{T-t} \tag{7.18}$$

$$\theta \triangleq \frac{\partial F}{\partial t} = -\frac{\sigma S_t N'(d_1)}{2\sqrt{T-t}} - rKe^{-r(T-t)}N(d_2). \tag{7.19}$$

For hedging strategies we will adopt the convention (and approximation) not to discount their future cash-flows. This approximation is supported by the short duration of such strategies.

By composing portfolios formed by suitable shares of the underlying assets and of the derivatives one can achieve different risk profiles, which fit different goals and risk attitudes of investors. Some of these tools will be considered in the exercises below.

The most popular combinations of European options in use in order to obtain some of the risk profiles mentioned above are the following:

- Spreads: Bull Spread (two calls written on the same underlying, with the same maturity, but $K_1 < K_2$, long position on $C_1$, short position on $C_2$), Bear Spread (two calls written on the same underlying, with the same maturity, but $K_1 < K_2$, short position on $C_1$, long position on $C_2$), Butterfly Spread (4 call options written on the same underlying, with the same maturity, but $K_1 < K_2 < K_3$, $K_2 = (K_1 + K_3)/2$, long position on $C_1$ and $C_3$, two short position on $C_2$).
- Straddles: a long position on one call and one put written on the same underlying, with the same maturity and the same strike.
- Strangles: a long position on one call and one put written on the same underlying, with the same maturity, but different strikes, the strike of the call greater than the strike of the put.
- Strips (long position on one call and two puts written on the same underlying, with the same maturity and the same strike) and Straps (long position on two calls and one put written on the same underlying, with the same maturity and the same strike).

For a detailed treatment and further details on the subject, we send the interested reader to Björk [6], Hull [25] and Wilmott et al. [43, 44], among many others.

Options can be written also on exchange rates. If the exchange rate dynamics is described by a geometric Brownian motion with diffusion coefficient $\sigma_X$ , it is possible to prove that a European call option written on an exchange rate $X(t) :=$ units of domestic currency/units of foreign currency, can be evaluated like a usual call option written on a dividend distributing asset, where the (constant) dividend rate is given by the foreign risk-free interest rate:

$$C(X(t), t) = xe^{-r_f(T-t)}N(d_1) - Ke^{-r_d(T-t)}N(d_2), \tag{7.20}$$

$$d_1 = \frac{\ln(x/K) + (r_d - r_f + \sigma_X^2/2)}{\sigma\sqrt{T-t}} \tag{7.21}$$

$$d_2 = \frac{\ln(x/K) + (r_d - r_f - \sigma_X^2/2)}{\sigma\sqrt{T-t}}, \tag{7.22}$$

and $r_d, r_f$ are the risk-free interest rates for the domestic and the foreign currency respectively.

## 7.2 Solved Exercises

**Exercise 7.1** It is well known that, in market models that are free of arbitrage, the prices of a European Call and a European Put option written on the same stock (without dividends) satisfy

$$C_0 \le S_0 \qquad \text{and} \qquad P_0 \le Ke^{-rT}, \tag{7.23}$$

where $K$ stands for the strike of the options, $T$ for the maturity of the options and $r$ for the risk-free rate per year.

1. Verify that

$$-Ke^{-rT} \le C_0 - P_0 \le S_0$$

and find an arbitrage opportunity in the following cases:

(a) $C_0 - P_0 > S_0$;
(b) $C_0 - P_0 < -Ke^{-rT}$.

2. It is well known that, in market models that are free of arbitrage, the following Put-Call Parity holds for European options written on the same underlying paying discrete dividends:

$$C_0 - P_0 = S_0 - D - Ke^{-rT},$$

where $D$ stands for the discounted value of all the dividends paid until the maturity $T$ of the options.

(a) Find an arbitrage opportunity when $C_0 = P_0 + S_0 - Ke^{-rT}$ and the discounted value of dividends is strictly positive ($D > 0$).
(b) Show that, under the no-arbitrage assumption, $C_0 > S_0 - D$ can never occur.

**Solution**

1. From inequalities (7.23) it follows immediately that

$$C_0 - Ke^{-rT} \leq C_0 - P_0 \leq S_0 - P_0.$$

Since $C_0, P_0 \geq 0$ (see below), we deduce

$$-Ke^{-rT} \leq C_0 - Ke^{-rT} \leq C_0 - P_0 \leq S_0 - P_0 \leq S_0,$$

that is, the claim.

Let us verify that $C_0, P_0 \geq 0$. Suppose by contradiction that $C_0 < 0$ and consider the following strategy:

| $t = 0$ | $t = T$ |
|---|---|
| Buy the call $\Rightarrow \quad -C_0 > 0$ | $(S_T - K)^+$ |
| Invest $(-C_0)$ at rate $r$ $\Rightarrow \quad +C_0$ | $-C_0 e^{rT} > 0$ |
| $-C_0 + C_0 = 0$ | $(S_T - K)^+ - C_0 e^{rT} > 0$ |

Consequently, if $C_0$ were strictly negative then there would exist arbitrage opportunities (for instance, the one built above). This would contradict the no-arbitrage assumption, hence it follows that $C_0 \geq 0$.

Similarly, one can check $P_0 \geq 0$.

(a) Assume now that $C_0 > P_0 + S_0$.
As shown below, the following strategy represents an arbitrage opportunity.

| $t = 0$ | $t = T$ |
|---|---|
| Sell the call<br>$\Rightarrow \quad +C_0$ | $-(S_T - K)^+$ |
| Buy one stock and one put<br>$\Rightarrow \quad -S_0 - P_0$ | $S_T + (K - S_T)^+$ |
| Invest $(C_0 - S_0 - P_0)$<br>$\Rightarrow \quad -(C_0 - S_0 - P_0)$ | $(C_0 - S_0 - P_0)\, e^{rT}$ |
| $C_0 - S_0 - P_0 - (C_0 - S_0 - P_0) = 0$ | $K + (C_0 - S_0 - P_0)\, e^{rT} > 0$ |

(b) Suppose now that $C_0 - P_0 < -Ke^{-rT}$, from which $C_0 - P_0 < 0$.
As checked below, the following strategy represents an arbitrage opportunity.

| $t = 0$ | $t = T$ |
|---|---|
| Sell the put<br>$\Rightarrow \quad +P_0$ | $-(K - S_T)^+$ |
| Buy the call<br>$\Rightarrow \quad -C_0$ | $(S_T - K)^+$ |
| Invest $(P_0 - C_0)$<br>$\Rightarrow \quad -(P_0 - C_0)$ | $(P_0 - C_0)\, e^{rT}$ |
| $P_0 - C_0 - (P_0 - C_0) = 0$ | $S_T - K + (P_0 - C_0)\, e^{rT} > S_T > 0$ |

2. Note, first, that the Put-Call Parity (in the presence of dividends) can be rewritten as

$$C_0 + D = S_0 + P_0 - Ke^{-rT}.$$

(a) It is clear that if we had $C_0 = P_0 + S_0 - Ke^{-rT}$ for $D > 0$, then there would exist arbitrage opportunities.

Suppose indeed that $C_0 = P_0 + S_0 - Ke^{-rT}$ and consider the following portfolio:

- one long position in the stock;
- one long position in the Put;
- one short position in the Call;
- bank loan of $Ke^{-rT}$.

The value $V_0$ of such a portfolio at $t = 0$ is then equal to

$$V_0 = -S_0 - P_0 + C_0 + Ke^{-rT} = 0,$$

while its value $V_T$ at maturity $T$ of the two options is equal to

$$V_T = S_T + De^{rT} + (K - S_T)^+ - (S_T - K)^+ - K = De^{rT} > 0.$$

The portfolio above represents then an arbitrage opportunity.

(b) Suppose now that $C_0 > S_0 - D$.

The following strategy is an arbitrage opportunity.

| $t = 0$ | $t = T$ |
|---|---|
| Sell the call<br>$\Rightarrow \quad +C_0$ | $-(S_T - K)^+$ |
| Buy one share of the underlying<br>$\Rightarrow \quad -S_0$ | $S_T + De^{rT}$ |
| Invest/borrow $(C_0 - S_0)$<br>$\Rightarrow \quad -(C_0 - S_0)$ | $(C_0 - S_0)\, e^{rT}$ |
| $C_0 - S_0 - (C_0 - S_0) = 0$ | $-(S_T - K)^+ + S_T$<br>$+De^{rT} + (C_0 - S_0)\, e^{rT}$ |

In fact, the initial value of such a strategy is zero, while its final value equals

$$\begin{aligned} V_T &= -(S_T - K)^+ + S_T + De^{rT} + (C_0 - S_0)\,e^{rT} \\ &= \min\{S_T;\,K\} + (C_0 - S_0 + D)\,e^{rT} \\ &\geq (C_0 - S_0 + D)\,e^{rT} > 0, \end{aligned}$$

where the last inequality is due to the initial assumption $C_0 > S_0 - D$.

**Exercise 7.2** Consider a stock (without dividends) whose price evolves as in the Black-Scholes model with drift $\mu$ of 10% per year, volatility $\sigma$ of 40% per year and current price of $S_0 = 16$ euros. The risk-free interest rate $r$ on the market is 4% per year.

1.
   (a) Compute the initial price of a European Call option on the stock above, with strike of 18 euros and maturity $T$ of 1 year.
   (b) Deduce the price of the corresponding Put option.

2. Suppose now that in 6 months' time the stock will cost 16.4 euros. Discuss whether it is convenient to wait 6 months before buying the Call option above and investing (at risk-free rate) what we would have paid for buying the Call at the initial time.
   What if the stock costed 19.2 euros in 6 months from now?
3. Discuss whether it is possible to establish a priori some bounds on the initial prices of the Call and the Put option. If yes, verify that such bounds hold in the present case.
4. Consider the same Call option as above but written on a stock with $S_0 = 16$ euros, paying a dividend of 2 euros in 4 months and of 4 euros in 8 months.
   (a) Compute the initial price of such an option and compare it with the price of the Call option (of item 1.) on the stock without dividends. Determine whether the option price on the stock paying dividends is greater than the one not paying dividends.
   (b) Discuss whether, in general, the price of a Call option on a stock with dividends could be greater than the same without dividends. Justify the result by means of no-arbitrage arguments.

**Solution**

1.
   (a) Recall that the Black-Scholes formula for European Call options on stock without dividends is the following:

$$C_0 = S_0 N(d_1) - Ke^{-rT} N(d_2),$$

where $N$ denotes the cumulative distribution function of a standard normal and

$$d_1 = \frac{\ln (S_0/K) + \left(r + \frac{1}{2}\sigma^2\right) T}{\sigma \sqrt{T}};$$
$$d_2 = d_1 - \sigma \sqrt{T}.$$

In the present case, we get

$$d_1 = \frac{\ln (16/18) + \left(0.04 + \frac{1}{2}(0.4)^2\right)}{0.4} = 0.0055$$
$$d_2 = 0.0055 - 0.4 = -0.3945,$$

since $T = 1$ year.

The initial price of the European Call option above is then equal to

$$C_0 = 16 \cdot N(0.0055) - 18 \cdot e^{-0.04} \cdot N(-0.3945) = 2.04 \text{ euros.}$$

(b) Applying the Put-Call Parity (for European options with same maturity and strike, written on the same stock without dividends), we deduce that

$$P_0 = C_0 - S_0 + Ke^{-rT} = 2.04 - 16 + 18 \cdot e^{-0.04} = 3.34 \text{ euros.}$$

We would obtain the same result by applying again the Black-Scholes formula in the case of European Put options on stocks without dividends. In such a case, indeed,

$$P_0 = Ke^{-rT} N(-d_2) - S_0 N(-d_1).$$

2. We compute now the price of the Call option at $t = 6$ months when we assume that $S_{6m}^{(1)} = 16.4$ euros. Under this assumption,

$$d_1^{(1)} = \frac{\ln\left(S_{6m}^{(1)}/K\right) + \left(r + \frac{1}{2}\sigma^2\right)(T - t)}{\sigma\sqrt{T - t}}$$
$$= \frac{\ln (16.4/18) + \left(0.04 + \frac{1}{2}(0.4)^2\right)\frac{1}{2}}{0.4 \cdot \sqrt{\frac{1}{2}}} = -0.117$$
$$d_2^{(1)} = d_1^{(1)} - 0.4 \cdot \sqrt{\frac{1}{2}} = -0.40,$$

hence

$$C_{6m}^{(1)} = 16.4 \cdot N\left(d_1^{(1)}\right) - 18 \cdot e^{-0.04 \cdot 0.5} N\left(d_2^{(1)}\right) = 1.356 \text{ euros.}$$

It would be convenient to wait 6 months before buying the Call option of item 1. and investing (at risk-free rate) what we would have paid for buying the Call at the initial time depending on whether

$$C_0 \cdot e^{r/2} > C_{6m}^{(1)}.$$

Since $C_0 \cdot e^{r/2} = 2.04 \cdot e^{0.04 \cdot 0.5} = 2.08 > C_{6m}^{(1)} = 1.356$, we conclude that it would be convenient to wait before buying the Call option.

Proceeding as above, we obtain that for $S_{6m}^{(2)} = 19.2$ euros

$$d_1^{(2)} = \frac{\ln(19.2/18) + \left(0.04 + \frac{1}{2}(0.4)^2\right)\frac{1}{2}}{0.4 \cdot \sqrt{\frac{1}{2}}} = 0.440$$

$$d_2^{(2)} = d_1^{(2)} - 0.4 \cdot \sqrt{\frac{1}{2}} = 0.157$$

and

$$C_{6m}^{(2)} = 19.2 \cdot N\left(d_1^{(2)}\right) - 18 \cdot e^{-0.04 \cdot 0.5} N\left(d_2^{(2)}\right) = 2.94 \text{ euros.}$$

Since $C_0 \cdot e^{r/2} = 2.04 \cdot e^{0.04 \cdot 0.5} = 2.08 < C_{6m}^{(2)} = 2.94$, we conclude that in such a case it would not be convenient to wait 6 months to buy the option.

3. As well known, prices of European options written on stocks without dividends fulfill the following inequalities:

$$S_0 - Ke^{-rT} \le C_0 \le S_0$$
$$Ke^{-rT} - S_0 \le P_0 \le Ke^{-rT}$$

In the present case, we deduce immediately that

$$0 = \max\{-1.29; 0\} = \max\{S_0 - Ke^{-rT}; 0\} \le C_0 \le S_0 = 16$$
$$1.29 = \max\{1.29; 0\} = \max\{Ke^{-rT} - S_0; 0\} \le P_0 \le Ke^{-rT} = 17.29.$$

The previous bounds are then fulfilled by the initial price of the Call and the Put.

4. Consider now a stock paying a dividend of 2 euros in 4 months and of 4 euros in 8 months.

   (a) We recall that the Black-Scholes formula for European options written on stocks paying discrete dividends is the following:

$$\hat{C}_0 = (S_0 - D)\, N\left(\hat{d}_1\right) - K e^{-rT} N\left(\hat{d}_2\right),$$

   where $N$ denotes as usual the cumulative distribution function of a standard normal and

$$\hat{d}_1 = \frac{\ln\left((S_0 - D)/K\right) + \left(r + \frac{1}{2}\sigma^2\right) T}{\sigma\sqrt{T}}$$
$$\hat{d}_2 = \hat{d}_1 - \sigma\sqrt{T}.$$

   Since the discounted dividends amount to

$$D = 2 \cdot e^{-0.04 \cdot \frac{4}{12}} + 4 \cdot e^{-0.04 \cdot \frac{8}{12}} = 5.87$$

   and

$$\hat{d}_1 = \frac{\ln\left((16 - 5.87)/18\right) + \left(0.04 + \frac{1}{2}\,(0.4)^2\right)}{0.4} = -1.137$$
$$\hat{d}_2 = -1.137 - 0.4 = -1.537,$$

   it follows that the initial price of the new Call option equals

$$\hat{C}_0 = (16 - 5.87) \cdot N\,(-1.137) - 18 \cdot e^{-0.04} N\,(-1.537) = 0.22 \text{ euros}.$$

   Consequently, the price of the Call "without dividends" is greater than the other one. More precisely, the difference between the two prices amounts to $C_0 - \hat{C}_0 = 2.04 - 0.22 = 1.82$ euros.

(b) In general, we cannot have $\hat{C}_0 > C_0$. Suppose by contradiction that $\hat{C}_0 > C_0$ and consider the following strategy, where $S_T$ stands for the price (at time $T$) of the stock that does not pay any dividend:

| $t = 0$ | $t = T$ |
|---|---|
| Buy the call without dividends $\Rightarrow \quad -C_0$ | $(S_T - K)^+$ |
| Sell the call paying dividends $\Rightarrow \quad +\hat{C}_0$ | $-\left(S_T - De^{rT} - K\right)^+$ |
| Invest $\left(\hat{C}_0 - C_0\right)$ $\Rightarrow \quad -\left(\hat{C}_0 - C_0\right)$ | $\left(\hat{C}_0 - C_0\right) e^{rT}$ |
| $\hat{C}_0 - C_0 - \left(\hat{C}_0 - C_0\right) = 0$ | $V_T$ |

Since $\left(S_T - De^{rT} - K\right)^+ \leq (S_T - K)^+$, we deduce that the final value $V_T$ of the strategy above is

$$\begin{aligned} V_T &= -\left(S_T - De^{rT} - K\right)^+ + (S_T - K)^+ + \left(\hat{C}_0 - C_0\right) e^{rT} \\ &\geq \left(\hat{C}_0 - C_0\right) e^{rT} > 0. \end{aligned}$$

If $\hat{C}_0 > C_0$ at any time, there would exist an arbitrage opportunity (e.g. the one just built). Under the no-arbitrage assumption, then, necessarily $\hat{C}_0 \leq C_0$.

**Exercise 7.3** Consider the same stock A—paying discrete dividends—and the same European Call option of Exercise 7.2, item 4.

The risk-free interest rate $r$ available on the market is of 4% per year.

1. Consider a Call option with strike of 18 euros, maturity of 1 year and written on a stock B with the same features as stock A but paying continuous dividends (with dividend rate $d$).

   Find the dividend rate $d$ such that the price of the Call option written on the stock A paying discrete dividends coincides with the price of the Call option above written on stock B.
2. Compute the discounted value of the dividends paid by stock B.

**Solution**

1. From Exercise 7.2 we already know that the price of the Call option written on stock A (paying discrete dividends), with strike of 18 euros and maturity of 1 year is equal to $\hat{C}_0 = 0.22$ euros.

   Furthermore, we remind that the price of a Call with strike of 18 euros, maturity of 1 year and written on stock B (paying continuous dividends) is given by

$$C_0^d = S_0 e^{-dT} N(d_1') - K e^{-rT} N(d_2'),$$

   where

$$d_1' = \frac{\ln(S_0/K) + (r - d + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$$
$$d_2' = \frac{\ln(S_0/K) + (r - d - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}.$$

   Hence

$$C_0^d = 16 \cdot e^{-d} \cdot N\left(\frac{\ln(16/18) + 0.12 - d}{0.4}\right) - 18 \cdot e^{-0.04} \cdot N\left(\frac{\ln(16/18) - 0.04 - d}{0.4}\right).$$

   We look then for a dividend rate $d$ so that

$$C_0^d = \hat{C}_0 = 0.22 \text{ euros}.$$

   Numerically, we get

| | $C_0^d$ |
|---|---|
| $d = 0.1$ | 1.3505 |
| $d = 0.2$ | 0.8586 |
| $d = 0.3$ | 0.5232 |
| $d = 0.4$ | 0.3050 |
| $d = 0.45$ | 0.2289 |
| $d = 0.454$ | 0.2236 |
| **d= 0.4567** | **0.2200** |
| $d = 0.457$ | 0.2196 |
| $d = 0.46$ | 0.2158 |
| $d = 0.5$ | 0.1697 |

We can conclude that the "implicit" dividend rate we are looking for is $d \cong 0.4567$.

2. Since the price of stock B evolves as

$$S_t^B = S_0^B \cdot \exp\left\{\left(r - d - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right\}$$

with $S_0^B = S_0 = 16$, we deduce that the discounted value of all dividends paid by stock B is equal to

$$D^* = S_0\left(1 - e^{-dT}\right) = 16 \cdot \left(1 - e^{-0.4567}\right) = 5.866 \text{ euros.}$$

As expected, $D^*$ is approximately equal to the discounted value $D$ of all dividends paid by stock A (where $D = 5.87$, see Exercise 7.2).

**Exercise 7.4** Consider a stock whose price evolves as in the Black-Scholes model, with volatility $\sigma = 0.32$ (per year) and initial price $S_0 = 30$ euros. Such a stock pays a dividend of one euro in 3 months and of one euro in 9 months. The risk-free interest rate available on the market is of 4% per year.

1. Compute the price of a European Call option with maturity of 1 year, strike of 25 euros and written on the stock above.
2. What is the price of the corresponding Put option?
3. Discuss whether the price would change if the risk-free interest rate decreased from 4% to 2% per year. What would such a change affect?
4. Suppose now that the risk-free interest rate is equal to 4% per year. Compute the new price of the stock of item 1. "adjusted" by the discounted value of dividends, i.e. obtained by detracting the discounted value of dividends.
5. Compare the price of the Call of item 1. to the price of the Call with strike of 25 euros, maturity of 1 year and written on the stock without dividends, with volatility $\sigma = 0.32$ (per year) and with the current price found in item 4.

**Solution**

1. The discounted value $D$ of all dividends of the stock amounts to

$$D = 1 \cdot e^{-0.04 \cdot \frac{3}{12}} + 1 \cdot e^{-0.04 \cdot \frac{9}{12}} = 1.96 \text{ euros.}$$

Recall that, by the Black-Scholes formula, the price of a European Call option written on a stock paying discrete dividends is given by

$$C_0 = \hat{S}_0 \cdot N\left(\hat{d}_1\right) - Ke^{-rT} \cdot N\left(\hat{d}_2\right),$$

where $\hat{S}_0 = S_0 - D$ and

$$\hat{d}_1 = \frac{\ln\left(\hat{S}_0/K\right) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$
$$\hat{d}_2 = \hat{d}_1 - \sigma\sqrt{T}.$$

In the present case, we get

$$\hat{d}_1 = \frac{0.206}{0.32} = 0.644$$
$$\hat{d}_2 = \hat{d}_1 - 0.32 = 0.324$$

and, accordingly,

$$\begin{aligned} C_0 &= 28.04 \cdot N\,(0.644) - 25 \cdot e^{-0.04} \cdot N\,(0.324) \\ &= 28.04 \cdot 0.74 - 24.02 \cdot 0.627 = 5.69 \text{ euros.} \end{aligned}$$

2. By the Put-Call parity on stocks paying dividends, we obtain immediately the Put price:

$$\begin{aligned} P_0 &= C_0 - S_0 + D + Ke^{-rT} \\ &= 5.69 - 30 + 1.96 + 24.02 = 1.67 \text{ euros.} \end{aligned}$$

3. Changes in the interest rate affect the discounted value of the dividends paid and the strike, as well as $d_1$ and $d_2$ in the Black-Scholes formula.

   If the risk-free interest rate was $r^* = 0.02$, the new discounted value $D^*$ of the dividends paid by the stock would amount to

$$D^* = 1 \cdot e^{-0.02 \cdot \frac{3}{12}} + 1 \cdot e^{-0.02 \cdot \frac{9}{12}} = 1.98.$$

Hence, we would get

$$d_1^* = \frac{\ln\left(\left(S_0 - D^*\right)/K\right) + \left(r^* + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} = 0.579$$
$$d_2^* = d_1^* - \sigma\sqrt{T} = 0.259$$

and, accordingly,

$$\begin{aligned} C_0^* &= \left(S_0 - D^*\right) N\left(d_1^*\right) - K e^{-r^* T} N\left(d_2^*\right) \\ &= 28.02 \cdot N(0.579) - 25 \cdot e^{-0.02} \cdot N(0.259) \\ &= 5.38 \text{ euros.} \end{aligned}$$

4. In case $r^* = 0.04$, the current stock price adjusted by the discounted value of the dividends would be

$$\hat{S}_0 = S_0 - D = 28.04 \text{ euros.}$$

5. The price of the Call written on the stock paying the dividends of item 1. is equal to 5.69 euros.

   If we computed the price of a Call option with maturity of 1 year, strike of 25 euros and written on a stock with volatility $\sigma = 0.32$ (per year) and current price $\hat{S}_0 = 28.04$ euros, we would find exactly the same price as the one of the Call of item 1.

**Exercise 7.5** Consider a market model where it is possible to trade on a stock (with current price $S_0 = 20$ euros) and on different European Call and Put options written on such a stock.

The price of the stock follows a geometric Brownian motion, namely $S_t = S_0 \exp\{\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t\}$ with $S_0 = 20$, $\mu \in \mathbb{R}$ and $\sigma > 0$.

Our goal is an investment with 1 year as horizon of time.

1. Consider two European Call options (both with maturity $T = 1$ year and written on the stock above), with strikes of

$$K_1 = 20 \text{ euros}$$
$$K_2 = 40 \text{ euros}$$

   and prices

$$C_0^1 = 6 \text{ euros}$$
$$C_0^2 = 2 \text{ euros,}$$

   respectively.

   Establish under which conditions on $\mu$ and $\sigma > 0$ the profit (in 1 year) due to a bear spread[1] formed by the previous options is non-negative with probability at least of 50%.

[1] A *spread* is a trading strategy composed by two or more options of the same kind, that is all options are either European Calls or European Puts written on the same underlying.

2. Compare the profit due to the bear spread with profit arising from the following strategy:
   - short position in a Call with strike $K_1$ (the lower strike)
   - short selling a share of the underlying at the beginning and giving it back at maturity.

   Establish which strategy is most convenient as the price of the underlying decreases to 16 euros. What if the price increases to 32 euros?
3. Suppose that on the market it is also possible to buy/sell Call options on the same underlying as before but with different strikes (in particular, we could think that options of any strike between $K_1$ and $K_2$ are available). Assume that on this market (where arbitrage opportunities might exist) the prices of the options above decrease linearly from the price of the option with strike $K_1 = 20$ to the price $C^* = 4$ euros for the option with strike $K^* = 28$. Furthermore, the prices decrease linearly from the price of the option with strike $K^*$ to that of the option with strike $K_2$.
   (a) Build a "butterfly spread"[2] using the options above, which necessarily relies on the Call options with strikes $K_1$ and $K_2$.
   What is the profit when we use a butterfly spread instead of the previous strategy (see item 2.)?
   (b) Suppose now that the stock price in 1 year is distributed as in item 1. with $\mu = 0.24$ and $\sigma = 0.48$. Compute the probability with which it is more convenient to use the butterfly spread instead of the bear spread.

**Solution**

1. By assumption, the price (at time $T$) of the underlying is given by

$$S_T = S_0 \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)T + \sigma W_T\right\}$$

with $S_0 = 20$ and $T = 1$ year.

We need to find under which conditions on $\mu$ and $\sigma$ the profit (in 1 year) due to the bear spread is non-negative with probability at least of 50%. In other words, $\mu$ and $\sigma$ have to satisfy the following inequality

$$P\left(\{\text{profit of the bear spread } \geq 0\}\right) \geq 0.5. \tag{7.24}$$

Among the different spreads, the so-called "bear spread" is used when a decrease in the stock price is expected. For Call options, a bear spread is obtained by selling the Call option with lower strike and by buying the Call with higher strike.

[2] The so-called "butterfly spread" is used when we expect that the stock price remains more or less stable. A butterfly spread for Call options can be obtained by buying one Call option with lower strike, buying one Call option with higher strike and selling two Call options with intermediate strike.

Then we have to analyze the random variable representing the profit due to the bear spread.

Recall that a bear spread can be built—by means of Call options—as follows:

(A) sell the Call option with lower strike ($K_1$, in the present case)
(B) buy the Call options with higher strike ($K_2$)

The payoff of a bear spread is then given by:

| | (A) | (B) | Total payoff |
|---|---|---|---|
| if $S_T \le K_1$ | 0 | 0 | 0 |
| if $K_1 < S_T \le K_2$ | $-(S_T - K_1)$ | 0 | $K_1 - S_T$ |
| if $S_T > K_2$ | $-(S_T - K_1)$ | $S_T - K_2$ | $K_1 - K_2$ |

since the payoff of a Call option is $(S_T - K)^+$ for the buyer and $-(S_T - K)^+$ for the seller. Consequently, the profit (due to the payoff and to the Call prices) is given by:

| | Total profit |
|---|---|
| if $S_T \le 20$ | $C_0^1 - C_0^2 = 4$ |
| if $20 < S_T \le 40$ | $K_1 - S_T + C_0^1 - C_0^2 = 24 - S_T$ |
| if $S_T > 40$ | $K_1 - K_2 + C_0^1 - C_0^2 = -16$ |

Figure 7.1 shows the profit due to the bear spread on varying the underlying price.

Condition (7.24) reduces then to

$$0.5 \le P\left(\{\text{profit } \ge 0\}\right) = P\left(\{0 \le S_T \le 20\} \cup \{20 < S_T \le 40; 24 - S_T \ge 0\}\right).$$

By the arguments above and by the assumptions on $S_T$, we deduce that

$$\begin{aligned} P\left(\{\text{profit } \ge 0\}\right) &= P\left(0 \le S_T \le 24\right) \\ &= P\left(20e^{\left(\mu - \frac{1}{2}\sigma^2\right)T + \sigma W_T} \le 24\right) \\ &= P\left(\left(\mu - \frac{1}{2}\sigma^2\right) + \sigma W_1 \le \ln\left(\frac{24}{20}\right)\right) \end{aligned}$$

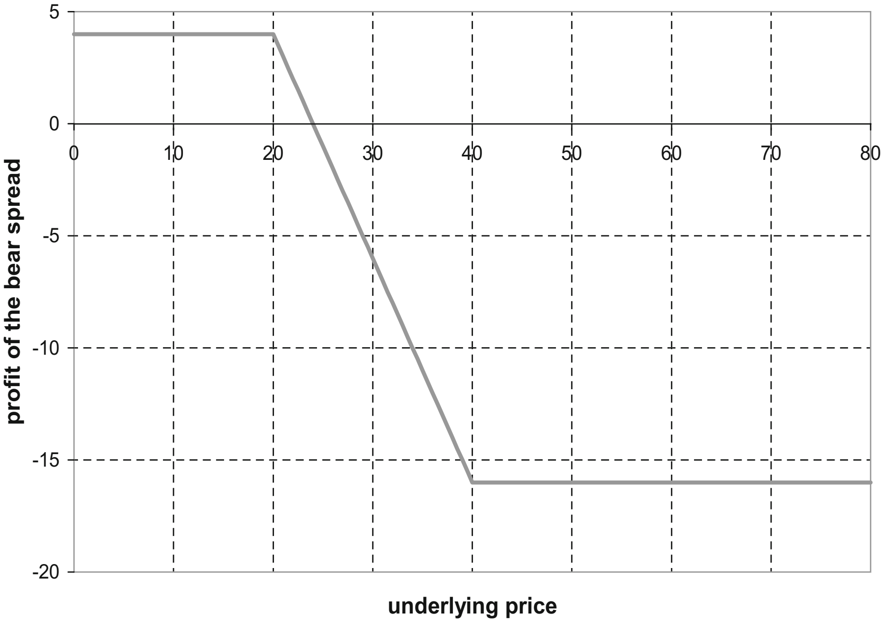


**Fig. 7.1** Profit of the bear spread

$$= P\left(W_1 \leq \frac{\ln(1.2) - \left(\mu - \frac{1}{2}\sigma^2\right)}{\sigma}\right)$$

$$= N\left(\frac{\ln(1.2) - \left(\mu - \frac{1}{2}\sigma^2\right)}{\sigma}\right),$$

because $W_1$ is distributed as a standard normal.

Since $\mu$ and $\sigma$ have to verify $N\left(\frac{\ln(1.2)-\left(\mu-\frac{1}{2}\sigma^2\right)}{\sigma}\right) \geq 0.5$, we obtain

$$\frac{\ln(1.2) - \left(\mu - \frac{1}{2}\sigma^2\right)}{\sigma} \geq 0.$$

The condition on $\mu$ and $\sigma$ $(> 0)$ becomes then

$$\mu \leq \frac{1}{2}\sigma^2 + \ln(1.2)\,.$$

Let us compare the profit due to the bear spread with the profit due to the following strategy:

– short position in the Call with strike $K_1$ (the lower strike)
– short selling a share of the underlying at the beginning and giving it back at maturity.

The profit (at maturity) of the strategy above is then given by:

| | Profit due to the call | Profit due to the stock | Total profit |
|---|---|---|---|
| if $S_T \le K_1$ | $C_0^1$ | $S_0 - S_T$ | $26 - S_T$ |
| if $S_T > K_1$ | $C_0^1 - (S_T - K_1)$ | $S_0 - S_T$ | $46 - 2S_T$ |

The comparison between the strategy just considered and the bear spread is manifest from Fig. 7.2.

Notice that "our strategy" is very convenient when the stock price decreases, while it is very risky and could cause big losses when the stock price increases steeply.

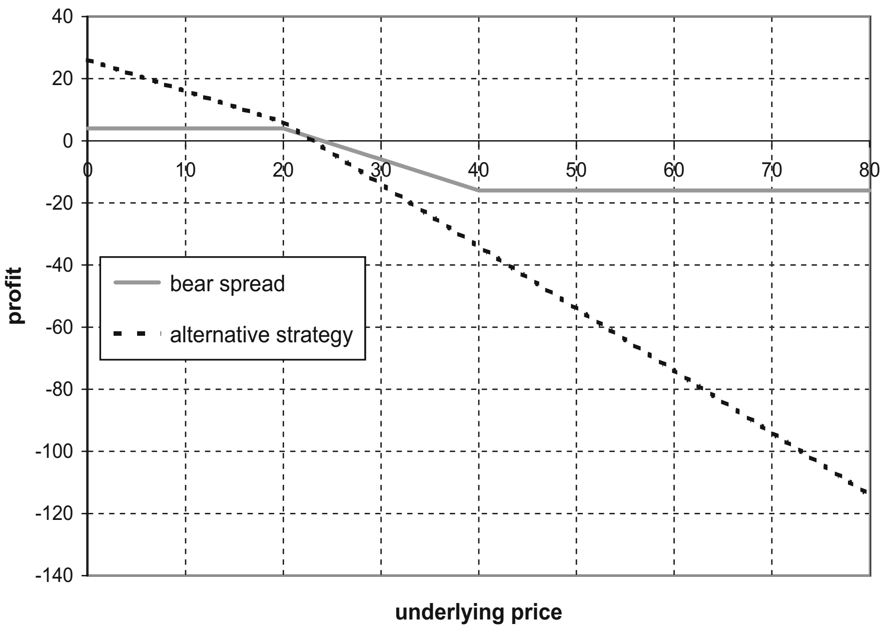


**Fig. 7.2** Comparison between the profit of a bear spread and the profit of our strategy

If the stock price decreased to 16 euros, we would have a total profit of 10 euros with our strategy, while a profit of 4 euros with the bear spread, making our strategy more convenient.

If the stock price increased to 32 euros, we would have a total loss of 18 euros with our strategy, while a loss of 8 euros with the bear spread. The bear spread would be thus more favorable.

2. First, remind that a butterfly spread can be built—by means of Call options—as follows:

(A) buy the Call with lower strike ($K_1$);
(B) buy the Call with higher strike ($K_2$);
(C) sell 2 Calls with intermediate strike (namely, with strike $K_3 = \frac{K_1+K_2}{2}$).

(a) By the initial assumptions, we obtain that the prices of the Call options with strike between $K_1$ and $K_2$ are given by

$$C_0^i = \begin{cases} 6 - \frac{K_i-20}{4}; \text{ if } 20 \le K_i \le 28 \\ \\ 4 - \frac{K_i-28}{6}; \text{ if } 28 \le K_i \le 40 \end{cases}.$$

In the present case, then, the butterfly spread would be built as explained before where the Call options in (C) have strike $K_3 = \frac{K_1+K_2}{2} = 30$ and price $C_0^3 = 4 - \frac{30-28}{6} = \frac{11}{3}$.

The payoff of the butterfly spread is then given by

| | (A) | (B) | (C) | Total payoff |
|---|---|---|---|---|
| if $S_T \le K_1$ | 0 | 0 | 0 | 0 |
| if $K_1 < S_T \le K_3$ | $S_T - K_1$ | 0 | 0 | $S_T - K_1$ |
| if $K_3 < S_T \le K_2$ | $S_T - K_1$ | 0 | $-2\,(S_T - K_3)$ | $K_2 - S_T$ |
| if $S_T > K_2$ | $S_T - K_1$ | $S_T - K_2$ | $-2\,(S_T - K_3)$ | 0 |

Consequently, the profit (due to the payoff and the Call prices) is given by:

| | Total Profit |
|---|---|
| if $S_T \leq 20$ | $2C_0^3 - C_0^1 - C_0^2 = -\frac{2}{3}$ |
| if $20 < S_T \leq 30$ | $S_T - K_1 + 2C_0^3 - C_0^1 - C_0^2 = S_T - \frac{62}{3}$ |
| if $30 < S_T \leq 40$ | $K_2 - S_T + 2C_0^3 - C_0^1 - C_0^2 = \frac{118}{3} - S_T$ |
| if $S_T > 40$ | $2C_0^3 - C_0^1 - C_0^2 = -\frac{2}{3}$ |

By comparing the profit of the strategy of item 2. to the profit of the butterfly spread, we deduce that the latter is more advantageous when the stock price is close to the intermediate strike, while it is less advantageous when the stock price goes under 20 euros. When the underlying price increases, however, the butterfly spread becomes more advantageous than "our strategy", since loss is bounded with the butterfly spread.

(b) In order to compute the probability for which the butterfly spread is more convenient than the bear spread, we need to compare the butterfly profit (see Fig. 7.3) with the bear profit.

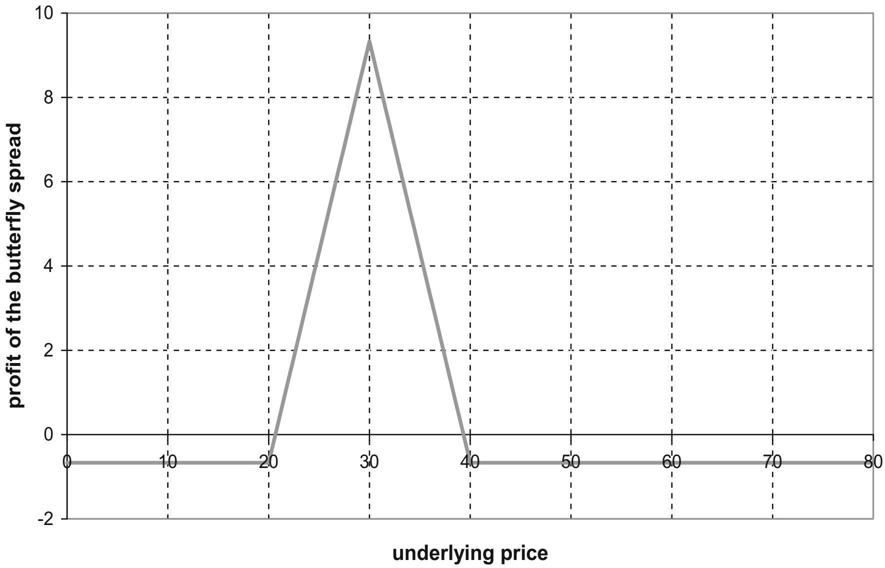


**Fig. 7.3** Profit of the butterfly spread

It is evident that for $S_T \leq 20$ the profit due to the bear is greater than the profit of the butterfly; vice versa for $S_T \geq 40$. For $20 < S_T < 40$ the profit of the bear is equal to $24 - S_T$ while the profit of the butterfly is equal to $S_T - \frac{62}{3}$ for $20 < S_T < 30$ and $\frac{118}{3} - S_T$ for $30 \leq S_T < 40$.

So, we deduce that

| | |
|---|---|
| for $S_T \leq 20$: | The bear is the more advantageous |
| for $20 < S_T < \frac{67}{3}$: | The bear is the more advantageous |
| for $S_T = \frac{67}{3}$: | Indifferent |
| for $\frac{67}{3} < S_T < 40$: | The butterfly is the more advantageous |
| for $S_T \geq 40$: | The butterfly is the more advantageous |

Consequently, the probability for which it is more advantageous to use a butterfly spread than a bear spread is equal to

$$\begin{aligned} P\left(S_T > \frac{67}{3}\right) &= P\left(S_0 \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)T + \sigma W_T\right\} > \frac{67}{3}\right) \\ &= P\left(\left(\mu - \frac{1}{2}\sigma^2\right)T + \sigma W_T > \ln\left(\frac{67}{3S_0}\right)\right) \\ &= P\left(W_T > \frac{\ln(67/60) - \left(0.24 - \frac{1}{2}(0.48)^2\right)}{0.48}\right) \\ &= 1 - N\left(\frac{\ln(67/60) - \left(0.24 - \frac{1}{2}(0.48)^2\right)}{0.48}\right) \\ &= 0.512. \end{aligned}$$

Notice that $\mu = 0.24$ and $\sigma = 0.48$ fulfill the condition found in item 1., hence the profit due to the bear spread is non-negative with probability at least of 50%.

**Exercise 7.6** We take a short position in a European Call option on a stock "Smart-and-Fast" with strike $K_1 = 8$ euros and maturity $T = 1$ year. The price of the underlying follows a geometric Brownian motion with $S_0 = 8$ euros, $\mu = 20\%$ and $\sigma = 40\%$ per year. The risk-free interest rate available on the market is $r = 4\%$ per year.

1.

(a) Compute the Delta ($\Delta_1$) of the Call option and, accordingly, establish how many shares of stock "Smart-and-Fast" we need to buy/sell in order to make our short position Delta-neutral.

(b) What if our short position was in a Put option (instead of a Call option)?

(c) Find the probability for which the seller of the Call has a (net) loss by the Delta-neutral portfolio built in item (a), given that the Call is exercised.

2. Discuss whether we can make a Delta-neutral short position in a Call option (as the one above) but with maturity $T^* = 2$ years instead of $T = 1$ year.
3. Suppose that another Call option on the same underlying "Smart-and-Fast" but with strike $K_2 = 12$ euros is available on the market.

   (a) Compute Delta and Gamma ($\varDelta_2$ and $\varGamma_2$) of the new option.
   (b) Establish how one could make Delta- and Gamma-neutral the portfolio with a short position in the first Call option. Is it possible for the Delta- and Gamma-neutral portfolio to contain a short position in the second Call?
   (c) We take a short position in the Call option of item 1. Explain how to make such a portfolio Gamma-neutral and with a total Delta smaller or equal (in absolute value) to 1. How many shares of the underlying could we buy at most?

4. Consider the portfolio composed by a short position in the Call (of item 1.) and by two long positions in the Put (of item 1.). Verify whether it is possible to make it Delta-, Gamma- and Vega-neutral if we dispose of a third Call option on the same underlying but with strike $K_3 = 6$ euros. If yes, explain how.

**Solution** Remind that:

- once a time $t$ is fixed, the Delta of an option is defined by $\varDelta = \varDelta_t \triangleq \frac{\partial F}{\partial S}(S_t, t)$, where $F(S_t, t)$ stands for the price at time $t$ of the option with underlying $S$;
- the Delta of the underlying ($\varDelta_S$) is equal to 1;
- a portfolio is said to be Delta-neutral if the Delta of the portfolio is zero;
- for European options written on a stock without dividends:

$$\varDelta_{Call} = N(d_1) > 0$$
$$\varDelta_{Put} = N(d_1) - 1 < 0.$$

1.

(a) By the arguments above, we obtain immediately that

$$\varDelta_1 = N(d_1) = N\left(\frac{\ln(S_0/K_1) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right)$$
$$= N\left(\frac{\ln(1) + \left(0.04 + \frac{1}{2}(0.4)^2\right)}{0.4}\right) = N(0.3) = 0.618.$$

Since our goal is to make a short position in the Call option Delta-neutral, we need to find the number $x$ of shares of the underlying to be bought (if $x > 0$) or sold (if $x < 0$) in order to make the portfolio of stocks-option Delta-neutral. We are then looking for $x$ verifying

$$-1 \cdot \Delta_1 + x \cdot \Delta_S = 0,$$

where the sign "-" is due to the short position in the option.

Since $\Delta_S = 1$, $x$ has to solve

$$-0.618 + x = 0.$$

So, $x = 0.618$. We conclude that to make a short position in the Call option Delta-neutral we need to buy 0.618 shares of the underlying.

(b) Suppose now that the short position to be "hedged" is in a Put option instead of a Call option. In order to make it Delta-neutral we look for a number $x$ of shares of the underlying to be bought (if $x > 0$) or sold (if $x < 0$) satisfying

$$\begin{aligned} -1 \cdot \Delta_{1,Put} + x \cdot \Delta_S &= 0 \\ -1 \cdot \Delta_{1,Put} + x &= 0. \end{aligned}$$

Since $\Delta_{1,Put} = \Delta_{1,Call} - 1 = 0.618 - 1 = -0.382$, to make the portfolio above Delta-neutral we need to sell 0.382 shares of the underlying.

(c) We have to compute the probability that the seller of the Call has a net loss by the Delta-neutral portfolio built in item (a), given that the Call option is exercised. This reduces to compute the following conditional probability

$$P\left(C_0 - (S_T - K_1)^+ + x\,(S_T - S_0) \le 0 \middle|\, S_T \ge K_1\right).$$

First, let us compute the initial price $C_0$ of the Call option. By item (a), we already know that $d_1 = 0.3$, hence $d_2 = d_1 - \sigma\sqrt{T} = -0.1$ and

$$\begin{aligned} C_0 &= S_0 \cdot N(d_1) - K_1 e^{-rT} \cdot N(d_2) \\ &= 8 \cdot N(0.3) - 8 \cdot e^{-0.04} \cdot N(-0.1) = 1.407 \text{ euros.} \end{aligned}$$

It follows that the probability we are looking for equals

$$P\left(C_0 - (S_T - K_1)^+ + x\,(S_T - S_0) \le 0 \middle|\, S_T \ge K_1\right)$$

$$= \frac{P\left(S_T \ge K_1; S_T \ge \frac{C_0+K_1-xS_0}{1-x}\right)}{P(S_T \ge K_1)}$$

$$= \frac{P\left(S_T \ge \frac{C_0+K_1-xS_0}{1-x}\right)}{P(S_T \ge K_1)}$$

$$= \frac{P\left(W_T \ge \frac{1}{\sigma}\left[\ln\left(\frac{C_0+K_1-xS_0}{(1-x)S_0}\right) - \left(\mu - \frac{1}{2}\sigma^2\right)T\right]\right)}{P\left(W_T \ge \frac{1}{\sigma}\left[\ln\left(\frac{K_1}{S_0}\right) - \left(\mu - \frac{1}{2}\sigma^2\right)T\right]\right)}$$

$$= \frac{1 - N\left(\frac{1}{\sigma}\left[\ln\left(\frac{C_0+K_1-xS_0}{(1-x)S_0}\right) - \left(\mu - \frac{1}{2}\sigma^2\right)\right]\right)}{1 - N\left(\frac{1}{\sigma}\left[\ln\left(\frac{K_1}{S_0}\right) - \left(\mu - \frac{1}{2}\sigma^2\right)\right]\right)}$$

$$= 0.419.$$

2. For a Call option as in item 1.(a) but with maturity $T^* = 2$ years (instead of $T = 1$) the Delta would be equal to

$$\Delta_1^{(T^*=2)} = N\left(\frac{\ln(S_0/K_1) + \left(r + \frac{\sigma^2}{2}\right)T^*}{\sigma\sqrt{T^*}}\right)$$

$$= N\left(\frac{\ln(1) + \left(0.04 + \frac{1}{2}(0.4)^2\right)\cdot 2}{0.4\sqrt{2}}\right) = 0.664.$$

   It is then easy to check that for Delta-neutrality we need to buy 0.664 shares of the underlying.
3. Remember that:

   - the Gamma of the underlying ($\Gamma_S$) is zero;
   - a portfolio is said to be Gamma-neutral if the Gamma of the portfolio is zero.

   Since the Gamma of the underlying is zero, to make the initial portfolio Gamma-neutral we should buy or sell other options.

   (a) We start computing $\Delta_2$ and $\Gamma_2$ (Delta and Gamma of the new option) as well as $\Gamma_1$ (Gamma of the Call option we are selling).

Proceeding as in item 1., we get

$$\Delta_2 = N(d_1) = N\left(\frac{\ln(S_0/K_2) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right)$$
$$= N\left(\frac{\ln(8/12) + \left(0.04 + \frac{1}{2}(0.4)^2\right)}{0.4}\right) = N(-0.71) = 0.238.$$

To compute Gamma, we recall that for European Call or Put options in the Black-Scholes model we have

$$\Gamma_{Call} = \Gamma_{Put} = \frac{N'(d_1)}{S_0\sigma\sqrt{T}} = \frac{1}{\sqrt{2\pi}S_0\sigma\sqrt{T}}e^{-\frac{d_1^2}{2}}.$$

Hence

$$\Gamma_1 == \frac{1}{\sqrt{2\pi}\cdot 8\cdot 0.4}e^{-\frac{(0.3)^2}{2}} = 0.12$$
$$\Gamma_2 = \frac{1}{\sqrt{2\pi}\cdot 8\cdot 0.4}e^{-\frac{(-0.71)^2}{2}} = 0.097$$

(b) First of all, let us make our portfolio Gamma-neutral. We will subsequently make it Delta-neutral by buying or selling a suitable number of shares of the underlying. Notice that it is necessary to respect this order. Indeed, making Gamma-neutral a portfolio that is already Delta-neutral might generate a portfolio that, in the end, is no more Delta-neutral.

The idea is then to find a number $y$ of options of the second kind to be bought/sold, that makes the portfolio Gamma-neutral. Namely, we are looking for $y$ solving

$$-1\cdot\Gamma_1 + y\cdot\Gamma_2 = 0, \tag{7.25}$$

where the sign "–" is due to the short position in the Call option. The equation above becomes then

$$-0.12 + y\cdot 0.097 = 0.$$

Hence $y = 1.237$. In order to make the portfolio Gamma-neutral we have to buy 1.237 options with strike $K_2$.

Since $\Gamma_1, \Gamma_2 > 0$ and $y$ has to solve (7.25), it is evident that to make our portfolio Gamma-neutral it is necessary to take a long position in the second Call.

In order to make the portfolio Delta-neutral as well, we need to find a number $x$ of shares of the underlying to be bought or sold that makes the new stocks-options portfolio Delta-neutral, i.e. such that

$$-1 \cdot \Delta_1 + 1.237 \cdot \Delta_2 + x = 0$$
$$-0.618 + 1.237 \cdot 0.238 + x = 0.$$

It follows that $x = 0.324$. To summarize: in order to make our portfolio Delta- and Gamma-neutral we have to buy 1.237 Call options with strike $K_2$ and 0.324 shares of the underlying.

(c) Our aim is to make our portfolio Gamma-neutral and with Delta smaller or equal (in absolute value) to 1. We have therefore to look for a number $y$ of Call options of the second kind and for a number $x$ of shares of the underlying satisfying the following conditions

$$\begin{cases} -1 \cdot \Gamma_1 + y \cdot \Gamma_2 = 0 \\ -1 \le -1 \cdot \Delta_1 + y \cdot \Delta_2 + x \le 1 \end{cases}$$

Consequently,

$$\begin{cases} y = 1.237 \\ -0.676 \le x \le 1.324 \end{cases}$$

As we would expect, the number of shares of the underlying to be bought to make the portfolio Delta-neutral (equal to $x = 0.618$, see item 1.(a)) belongs to the interval above. Furthermore, the number of shares that we are allowed to buy so to respect the constraints above is 1 (and no more).

4. In the following we will make our portfolio Delta-, Gamma- and Vega-neutral. Remember that:

- the Vega of the underlying is zero;
- a portfolio is said to be Vega-neutral if the Vega of the portfolio is null.

First of all, let us make our portfolio Gamma- and Vega-neutral. Only afterwards we will make it Delta-neutral by buying or selling a suitable number of shares of the underlying.

The idea is then to find a number $y$ of options of the second kind and $z$ of the third kind to be bought or sold to make the portfolio Gamma- and Vega-neutral.

Let us compute $\Delta$, $\Gamma$ and $\nu$ of the three options available:

$$\begin{aligned}\Delta_3 &= N(d_1) = N\left(\frac{\ln(S_0/K_3)+\left(r+\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right)\\ &= N\left(\frac{\ln(8/6)+\left(0.04+\frac{1}{2}(0.4)^2\right)}{0.4}\right) = N(1.02) = 0.846\end{aligned}$$

$$\Gamma_3 = \frac{1}{\sqrt{2\pi}\cdot 8\cdot 0.4}e^{-\frac{(1.02)^2}{2}} = 0.07$$

It is well known that the Vega of European Call or Put options in the Black-Scholes model is given by

$$\nu_{Call} = \nu_{Put} = S_0\sqrt{T}N'(d_1) = S_0^2\sigma\Gamma T.$$

Hence, in our setting we get

$$\begin{aligned}\nu_1 &= 8^2\cdot 0.4\cdot\Gamma_1 = 3.072\\ \nu_2 &= 8^2\cdot 0.4\cdot\Gamma_2 = 2.483\\ \nu_3 &= 8^2\cdot 0.4\cdot\Gamma_3 = 1.792\end{aligned}$$

In order to make our portfolio Gamma-neutral and Vega-neutral $y$ and $z$ have to solve the following system of linear equations

$$\begin{cases}-\Gamma_{1,Call} + 2\Gamma_{1,Put} + y\cdot\Gamma_2 + z\cdot\Gamma_3 = 0\\ -\nu_{1,Call} + 2\nu_{1,Put} + y\cdot\nu_2 + z\cdot\nu_3 = 0\end{cases}$$

$$\begin{cases}\Gamma_1 + y\cdot\Gamma_2 + z\cdot\Gamma_3 = 0\\ \nu_1 + y\cdot\nu_2 + z\cdot\nu_3 = 0\end{cases}$$

$$\begin{cases}y = -1.24\\ z = -0.000069\end{cases},$$

because $\Gamma_{1,Call} = \Gamma_{1,Put} = \Gamma_1$ and $\nu_{1,Call} = \nu_{1,Put} = \nu_1$.

Finally, to make such a portfolio Delta-neutral we have to find a number $x$ of shares of the underlying to be bought or sold so that the new portfolio of stocks-options becomes Delta-neutral. This means that $x$, $y$ and $z$ have to solve

$$-\Delta_{1,Call} + 2\Delta_{1,Put} + y\cdot\Delta_2 + z\cdot\Delta_3 + x = 0.$$

It follows that $x = 1.68$.

To conclude: in order to make our position Delta-, Gamma- and Vega-neutral we have to: sell 1.24 options with strike $K_2$; sell 0.000069 options with strike $K_3$; and buy 1.68 shares of the underlying.

**Exercise 7.7** There are three European call options traded on the market. They are written on the same underlying, with initial value $S(0) = 30$ euros, and the dynamics of its price is described by a geometric Brownian motion with volatility $\sigma = 0.6$. The risk-free interest rate is $r = 0.06$. The options have the same strike $K = 25$ and maturity $T = 1$ year. Two of them are call options, one of them is a put option. Define a portfolio of *strap* type composed by the European options above. Compute the value at time $t = 0$ and the Delta, the Vega and the Gamma of the *strap*.
By using the same European options above (two calls and one put, written on the same underlying following the dynamics specified before, with the same maturity and strike) and one unit of the underlying asset, compose a new portfolio which is both $\Delta$ and $\Gamma$-neutral.

**Solution** Let's compute the values of the three options included in the portfolio.

$$C_1 = C_2 = 30N(d_1) - 25e^{-0.06\times 1}N(d_2) = 30N(0.7038)$$
$$-25 \times 0.9418 \times N(0.1038) = 23.094 - 13.065 = 10.09 euros,$$

$$P_1 = C_1 - S(0) + Ke^{-0.06\times 1} = 10.09 - 30 + 25 \times 0.9418 = 3.635 euros,$$

where we computed the put option value by using the Put-Call parity.
The value at time $t = 0$ of the *strap* is then the following:

$$2C_1 + P_1 = 10.09 + 3.635 = 13.725 euros.$$

Let's compute now the Delta, the Gamma and the Vega of the options considered.

$$\Delta_C = N(d_1) = 0.7698$$

$$\Gamma_C = \frac{N'(d_1)}{\sigma S(0)\sqrt{T}} = \frac{0.3114}{18} = 0.0173$$

$$\nu_C = S(0) \times \sqrt{T} N'(d_1) = 30 \times 0.3114 = 9.342$$

$$\Delta_P = N(d_1) - 1 = -0.2302$$

$$\Gamma_P = \Gamma_C = 0.0173$$

$$\nu_P = \nu_C = 9.342.$$

By linearity, the $\Delta$, $\Gamma$, $\nu$ of the *strap* $\pi$ are then given by:

$$\Delta_\pi = 2\Delta_C + \Delta_P = 1.309$$

$$\Gamma_\pi = 2\Gamma_C + \Gamma_P = 0.0519$$

$$\nu_\pi = 2\nu_C + \nu_P = 28.026.$$

Now, we forget about the *strap* and compose a new portfolio, which is both $\Delta$ and $\Gamma$-neutral, by using the same options introduced before and one unit (short position) of the underlying asset. The conditions that must be satisfied are then:

$$\begin{aligned} 2\alpha\Delta_C + \beta\Delta_P - 1 = 0 \\ 2\alpha\Gamma_C + \beta\Gamma_P = 0. \end{aligned}$$

By remembering that $\Gamma_C = \Gamma_P$, and that $\Delta_P = \Delta_C - 1$, we get $\beta = -2\alpha$, $\alpha = 1/2$.

**Exercise 7.8** Compute the value of a European call option written on the exchange rate $X$ between between euros and Swedish Crowns. Suppose euro is the domestic currency and Swedish Crown is the foreign currency, and the domestic and foreign risk-free rates are $r_d = 0.08$ and $r_f = 0.04$ respectively. Assume moreover that the exchange rate dynamics is described by a geometric Brownian motion with diffusion coefficient $\sigma_X = 0.8$ and the initial value $X(0) = 9.8$. The option strike is $K = 9$ and the maturity $T = 1$ year.

**Solution** We simply need to apply the Black-Scholes formula for a European option written on an exchange rate:

$$C(X(t), t) = xe^{-r_f(T-t)}N(d_1) - Ke^{-r_d(T-t)}N(d_2), \tag{7.26}$$

$$d_1 = \frac{\ln(x/K) + (r_d - r_f + \sigma_X^2/2)}{\sigma\sqrt{T-t}} \tag{7.27}$$

$$d_2 = \frac{\ln(x/K) + (r_d - r_f - \sigma_X^2/2)}{\sigma\sqrt{T-t}}, \tag{7.28}$$

and $r_d$, $r_f$ are the risk-free interest rates for the domestic and the foreign currency respectively. In the present case we have:

$$d_1 = \frac{\ln(9.8/9) + (0.08 - 0.04 + 0.64/2)}{0.8} = 0.5565 \tag{7.29}$$

$$d_2 = \frac{\ln(9.8/9) + (0.08 - 0.04 - 0.64/2)}{0.8} = -0.2435, \tag{7.30}$$

$$C(X(0), 0) = 9.8e^{-0.04}N(d_1) - 9e^{-0.08}N(d_2) = 3.3414 euros. \tag{7.31}$$

**Exercise 7.9** Suppose we want to hedge a European Call option written on an underlying stock with dynamics described by a geometric Brownian motion with null drift by means of the following strategy (known as *stop-loss strategy*): whenever the stock price is greater than the strike $K$ we buy one unit of the stock, and we sell it whenever the stock price becomes smaller than the strike. By applying the Tanaka formula verify that this strategy cannot replicate the European Call option payoff.

**Solution** In order to replicate the Call option, we should determine an adapted process $(h_t)_{t\in[0,T]}$ such that:

$$(S_T - K)^+ = (S_0 - K)^+ + \int_0^T h_t dS_t. \tag{7.32}$$

The stop-loss strategy can be expressed as follows:

$$h_t \triangleq \mathbf{1}_{(K,+\infty)}(S_t).$$

By applying the Tanaka formula to the Call option payoff, as in Chap. 5, we get:

$$(S_T - K)^+ = (S_0 - K)^+ + \int_0^T \mathbf{1}_{(K,+\infty)}(S_t)dS_t + \frac{1}{2}\int_0^T \sigma^2 S_t^2 \delta_K(S_t)dt.$$

The last term in this expression does not vanish, unless the stock never assumes the value $K$ (in which case the strategy becomes trivial) and, although the process $(h_t)_{t\in[0,T]}$ is adapted to the natural filtration generated by the Brownian motion, it does not allow to write the option payoff in the desired form (7.32). We can conclude that the stop-loss strategy cannot be a replicating strategy for a European Call option.

## 7.3 Proposed Exercises

**Exercise 7.10** Consider a stock paying a dividend of 2 euros in 3 months and of 3 euros in 4 months. The stock price follows a geometric Brownian motion with current price of 36 euros and volatility $\sigma = 0.28$ (per year).

1. Compute the discounted value of all dividends paid when the risk-free interest rate is $r = 6\%$ per year.
2. Find the initial price of a Call option with strike of 32 euros and maturity of 1 year, written on the stock above. Using the Put-Call Parity, deduce the initial price of the corresponding Put option.
3. Discuss whether the probability that the Call of item 1. is exercised is greater, smaller than or equal to the probability to exercise the Call in item 2. written on the same stock but without dividends.

4. Establish if it is possible to choose suitable drift, volatility and current price of the stock (without dividends and distributed as a log-normal) so that the price of the Call option with strike of 32 euros, maturity of 1 year and written on such a stock can equal the Call price found in item 1.
5. Discuss what would happen to the prices of the Call and Put of item 1. if the dividends were paid in 4 and 8 months and the maturity of the options were unchanged.
6. What if the amount of dividends stayed unchanged while their dates of payment were swapped?

**Exercise 7.11** A spread is a strategy consisting in investing in two or more options of the same kind. Whereas a butterfly spread is used when a small change in the underlying price is expected, a bull spread is considered when an increase in the underlying price is expected.

1. Build a butterfly spread based on the following three Call options with the same maturity and with strikes and prices as below:

   Call A: strike of 20 euros; price of 4 euros;
   Call B: strike of 24 euros; price of 2 euros;
   Call C: strike of 28 euros; price of 1 euro.

2. Build a bull spread using only two of the above options. [Hint: you should buy the Call with lower strike (between the two options chosen) and sell the Call of greater strike (same two).]
3. Suppose now that at maturity of the options nothing of what is expected actually happens and the stock is, instead, $S_T = 16$ euros.
   (a) Compare the Profit and Loss due to the butterfly spread (respectively, bull spread) with the profit of a strategy consisting in a long position in a Put option with strike of 20 euros and price of 2 euros.
   (b) Now compare with the Profit and Loss due to a bear spread built with the options of item 1.
4. Suppose that the price (at $T = 1$ year) of the underlying is given by $S_T = S_0 \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right) T + \sigma W_T\right\}$, with $\mu = 0.1$ (per year), $\sigma = 0.4$ (per year) and that the risk-free interest rate available on the market is $r = 0.02$ (per year).
   (c) Compute the probability to lose at most one euro with the bear, with the bull and with the butterfly spread, respectively.
   (d) Compute the probability that the profit due to the bear spread is not smaller than the profit of the butterfly, and the probability that the loss due to the bear spread is not greater than the loss of the butterfly.
   Is there any relationship between the probabilities just computed?
5. Say if one can obtain a more satisfactory result by means of long/short positions in the stock and in the options.

**Exercise 7.12** We take a short position in a European Call option with strike $K_1 = 8$ euros and maturity $T = 1$ year, written on a stock "Smart-and-Fast" whose price is distributed as a log-normal with $S_0 = 8$ euros, $\sigma = 40\%$ per year. We also take a long position in two European Put options with strike $K_2 = 10$ euros and maturity of 1 year, written on the same stock.

The risk-free interest rate available on the market is $r = 4\%$ per year.

1. After having computed the Delta of the Call option (denoted by $\Delta_1$) and of the Put option ($\Delta_2$), find the number of shares of the stock "Smart-and-Fast" to be bought/sold to make Delta-neutral a portfolio PORT1 consisting in four short positions in the Call option and in a long position in the Put option.
2. Suppose that a Call option with strike $K_3 = 12$ euros written on the same stock as before is also available on the market.
   (a) Compute Delta and Gamma ($\Delta_3$ and $\Gamma_3$) of such a new option.
   (b) Explain how to make Delta- and Gamma-neutral a short position in the first Call option.
   (c) Discuss what would change in items (a) and (b) if we had a Put option with strike $K_3$ (instead of the Call option with strike $K_3$).
3. Establish whether it is possible to make the portfolio (PORT1) Delta-, Gamma- and Vega-neutral when a third Put option with strike $K_4 = 6$ euro and written on stock "Smart-and-Fast" is also available. If yes, explain how.
   (d) Would this be still possible if we were forced to buy/sell at least one option with strike $K_3$ and at least one option with strike $K_4$?
   (e) And if we had to buy/sell at most one option with strike $K_3$ and at most one option with strike $K_4$?

# Chapter 8
# American Options

## 8.1 Review of Theory

An *American Option* is a contract giving the buyer the right to buy (Call) or sell (Put) a financial underlying asset for a strike price $K$ at every instant between the agreement date and the maturity. The main difference between American and European options consists thus in the *early exercise* feature.

If the underlying does not pay any dividend, it is possible to prove that the early exercise of an American Call option is never optimal. The American Call option value—when written on dividend-free underlying—is then the same as that of the European Call option with the same parameters. The same conclusion does not hold for an American Call option on an underlying paying dividends, nor for an American Put option. This means that Put-Call parity cannot be formulated in the same way as for European options.

Since American options provide the holder with "at least" the same rights of their European counterpart (actually some more), their value cannot be less than the corresponding European options value. The initial values of both kind of options are therefore related by the following inequalities:

$$C_0^{Am} \geq C_0^{Eur}$$
$$P_0^{Am} \geq P_0^{Eur}.$$

During the options' lifetime it may happen that early exercise becomes more remunerative than exercise at maturity, and the option holder will exploit this opportunity, which is available for American options. The exercise time will be then chosen so to optimize the income. A problem of this type is well known in Stochastic Analysis as *Optimal Stopping* problem.

 
E. Rosazza Gianin, C. Sgarra, *Mathematical Finance*,
La Matematica per il 3+2 149, https://doi.org/10.1007/978-3-031-28378-9_8

The American options valuation problem in a Black-Scholes setting can be formulated as a *free boundary problem* for the Black-Scholes PDE, and this formulation can be proved to be equivalent to the *optimal stopping problem* for the stochastic process describing the underlying dynamics. The American option fair value (i.e. the option price consistent with the no-arbitrage requirement) can be equivalently characterized as the free boundary problem solution for the Black-Scholes PDE, or as the supremum over all possible risk-neutral expectations of the option payoff with respect to all exercise times. Both formulations do not enjoy a closed-form solution and the methods to solve them require some numerical techniques that give approximate solutions. A systematic presentation of these techniques is beyond the purpose of this textbook.

We invite the reader interested in a deeper understanding of both the analytical and the stochastic aspects of free boundary problems, and in particular American options valuation problems, to consult some classical textbooks on the subject, like Wilmott et al. [43], and Musiela and Rutkowski [33]. We are going to propose a number of exercises on American options valuation in the simple context where the detailed knowledge of more complex techniques is not required, and we mainly focus on the binomial model setting. We will show how to evaluate an American option by assigning to each node of the binomial tree an option value that takes the early exercise feature into account, and by proceeding with a backward recursion in time starting from maturity. It is useful to remember that a geometric Brownian motion with diffusion coefficient $\sigma$ can be approximated by a binomial process with $u = e^{\sigma\sqrt{\Delta t}}$ and $d = 1/u = e^{-\sigma\sqrt{\Delta t}}$, where $\Delta t$ is the time step size. This approximation can always be used whenever an exact valuation formula is not available and the quality of this approximation improves as the time step is smaller. Finally we shall present a few approximate methods proposed for American options valuation.

## 8.2 Solved Exercises

**Exercise 8.1** A Put option (with strike of 8 euros) is written on an underlying stock with current price of 8 euros and without dividends. At each ensuing two semesters the stock price can move up by 40% with probability $p = 0.8$ or down by 60% with probability $1 - p = 0.2$. Denote by $(S_t)_{t=0,1,2}$ the stochastic process representing the underlying stock price, where $t = 1$ stands for one semester and $t = 2$ for two semesters.

The risk-free interest rate available on the market is 4% (per year).

1.

   (a) Compute the discounted expectation of the random variable $S_2$ at the present date.
   Is the probability measure $P$ obtained by the previous data an equivalent martingale measure?
   Is the stochastic process $(S_t)_{t=0,1,2}$ describing the underlying stock dynamics a martingale with respect to $P$?
   (b) By letting $\tilde{S}_t$ denote the underlying value at time $t$ discounted at the present date, find a probability measure $Q$ with respect to which $\left(\tilde{S}_t\right)_{t=0,1,2}$ is a martingale.

2. What is the fair initial value of a European Call option written on the underlying considered before with maturity $T = 1$ year? And what is the fair value of the Put option on the same underlying and with the same parameters?
3. Establish if it is optimal to exercise the American Call on the same underlying and with the same parameters of the options of the previous item before maturity. What would then be its fair value?
4. Verify if there exist arbitrage opportunities in the market where only the (risky) stock and the (risk-free) bond are traded if, for each one of the next two semesters, the stock price can rise by 4% with probability 80% or fall by 4% with probability of 20% and the risk-free interest rate is 15% (per year).
5. In the market model of item 4., establish if it is possible to evaluate both the European and the American options as in items 2. and in 3..

**Solution** From the data we see that we are working in a binomial model with factors $u = 1.4$ and $d = 0.4$ in each time period (in the present case one semester). The dynamics of the stock price can then be described as follows:

$$
\begin{array}{ccccc}
 & & & & S_2^{uu} = 15.68 \\
 & & & p \nearrow & \\
 & & S_1^u = 11.2 & & \\
 & p{=}0.8 \nearrow & & 1-p \searrow & \\
S_0 = 8 & & & & S_2^{ud} = 4.48 \\
 & 1-p{=}0.2 \searrow & & p \nearrow & \\
 & & S_1^d = 3.2 & & \\
 & & & 1-p \searrow & \\
 & & & & S_2^{dd} = 1.28 \\
---- & & -------- & & -------- \\
t = 0 & & t = 1 \text{ (semester)} & & T = 2 \text{ (semesters)}
\end{array}
$$

1.

(a) By the binomial tree just described, we immediately get:

$$P\left(S_2 = S_2^{uu}\right) = p^2 = 0.64$$
$$P\left(S_2 = S_2^{ud}\right) = 2p\,(1-p) = 0.32$$
$$P\left(S_2 = S_2^{dd}\right) = (1-p)^2 = 0.04$$

and

$$E_P\,[S_2] = 15.68 \cdot 0.64 + 4.48 \cdot 0.32 + 1.28 \cdot 0.04 = 11.52.$$

Since $E_P\,[S_2] = 11.52 \neq S_0$, the process $(S_t)_{t=0,1,2}$ is not a martingale with respect to the probability measure $P$.
Let us check if the stochastic process $\left(\tilde{S}_t\right)_{t=0,1,2}$, describing the discounted (at the present date) underlying value, is a martingale with respect to $P$, i.e. if $P$ is a martingale measure.
We immediately verify that $P$ is not a martingale measure because:

$$E_P\left[\tilde{S}_2\right] = \frac{E_P\,[S_2]}{1+r} = \frac{11.52}{1.04} = 11.076 \neq 8 = S_0 = \tilde{S}_0.$$

(b) In order to determine the martingale measure $Q$, it is necessary to check if there exists a probability measure $Q$ such that

$$\begin{cases} \frac{E_Q[S_2]}{1+r} = S_0 \\ \frac{E_Q[S_1]}{(1+r)^{1/2}} = S_0 \\ \frac{E_Q\left[\,S_2|S_1=S_1^u\right]}{(1+r)^{1/2}} = S_1^u \\ \frac{E_Q\left[\,S_2|S_1=S_1^d\right]}{(1+r)^{1/2}} = S_1^d \end{cases}$$

or, equivalently, if there exists $q_u \in [0, 1]$ (the “new” growth probability of the underlying for each time period) solving the following system of equations:

$$\begin{cases} S_2^{uu} \cdot q_u^2 + S_2^{ud} \cdot 2q_u\,(1-q_u) + S_2^{dd} \cdot (1-q_u)^2 = S_0\,(1+r) \\ S_1^u \cdot q_u + S_1^d \cdot (1-q_u) = S_0\,(1+r)^{1/2} \\ S_2^{uu} \cdot q_u + S_2^{ud} \cdot (1-q_u) = S_1^u\,(1+r)^{1/2} \\ S_2^{ud} \cdot q_u + S_2^{dd} \cdot (1-q_u) = S_1^d\,(1+r)^{1/2} \end{cases}.$$

Since $S_2^{uu} = S_1^u \cdot u$, $S_2^{ud} = S_1^u \cdot d = S_1^d \cdot u$, $S_2^{dd} = S_1^d \cdot d$, $S_1^u = S_0 \cdot u$ and $S_1^d = S_0 \cdot d$, the system can be reduced to the following form:

$$\begin{cases} S_0 u^2 q_u^2 + S_0 ud \cdot 2q_u (1-q_u) + S_0 d^2 (1-q_u)^2 = S_0 (1+r) \\ S_0 u q_u + S_0 d (1-q_u) = S_0 (1+r)^{1/2} \end{cases}$$

$$\begin{cases} u^2 q_u^2 + ud \cdot 2q_u (1-q_u) + d^2 (1-q_u)^2 = 1+r \\ u q_u + d (1-q_u) = (1+r)^{1/2} \end{cases}$$

Since the first equation is essentially the second one squared, we immediately recognize that the solution of the previous system is the following:

$$q_u = \frac{(1+r)^{1/2} - d}{u-d} = \frac{\sqrt{1.04} - 0.4}{1.4 - 0.4} = 0.62.$$

Moreover, we conclude that the martingale measure (required in order to compute the options prices) corresponds to the following rise/fall probabilities for each time step:

$$q_u = 0.62$$
$$1 - q_u = 0.38.$$

2. We remark that the European Call option payoff is 7.68, 0 and 0 when $S_T = 15.68$, $S_T = 4.48$ and $S_T = 1.28$, respectively. The initial value of the European Call is then:

$$C_0^{Eur} = \frac{1}{1+r}\left[q_u^2 \cdot 7.68 + 2q_u (1-q_u) \cdot 0 + (1-q_u)^2 \cdot 0\right] = 2.84 \text{ euros.}$$

By the Put-Call parity we immediately get the European Put price:

$$P_0^{Eur} = C_0^{Eur} - S_0 + \frac{K}{1+r} = 2.84 - 8 + \frac{8}{1.04} = 2.53 \text{ euros.}$$

3. We know that for an American Call option, written on an underlying without dividends, early exercise is never optimal; so the initial value of such an option coincides with that of a European Call option on the same underlying and with the same parameters:

$$C_0^{Am} = C_0^{Eur} = 2.84 \text{ euros.}$$

4. In the present case, the framework for valuation is a binomial model with factors $u = 1.04$ and $d = 0.96$ for each time step. The stock price dynamics is then the following:

$$
\begin{array}{ccccc}
 & & & & S_2^{uu} = 8.6528 \\
 & & & p \nearrow & \\
 & & S_1^{u} = 8.32 & & \\
 & p{=}0.8 \nearrow & & 1{-}p \searrow & \\
S_0 = 8 & & & & S_2^{ud} = 7.9872 \\
 & 1{-}p{=}0.2 \searrow & & p \nearrow & \\
 & & S_1^{d} = 7.68 & & \\
 & & & 1{-}p \searrow & \\
 & & & & S_2^{dd} = 7.3728 \\
 & & & & \\
---- & & --------- & & --------- \\
t = 0 & & t = 1 \text{ (semester)} & & T = 2 \text{ (semesters)}
\end{array}
$$

Since the risk-free interest rate for each semester is equivalent to the yearly rate $r_{year}$ of 15%, we obtain $r_{sem} = \left(1 + r_{year}\right)^{1/2} - 1 = 0.072$, and $1 + r_{sem} = 1.072 > u = 1.04$. So in the market model considered, where a stock and a bond are traded, there exist arbitrage opportunities. We are going to present one of them:

| $t = 0$ | $t = 2 = T$ |
|---|---|
| short-sell the stock<br>$\Rightarrow \quad +S_0$ | give back the stock<br>$\Rightarrow \quad -S_T$ |
| invest $S_0$ at the risk-free rate $r$<br>$\Rightarrow \quad -S_0$ | $S_0\,(1 + r_{sem})^2$ |
| $S_0 - S_0 = 0$ | $S_0\,(1 + r_{sem})^2 - S_T > S_0 u^2 - S_T \geq 0$ |

Moreover,

$$
\begin{aligned}
P\left(S_0 u^2 - S_T > 0\right) &= P\left(\{S_2 = S_0 ud\} \cup \left\{S_2 = S_0 d^2\right\}\right) \\
&= 2p\,(1 - p) + (1 - p)^2 > 0.
\end{aligned}
$$

This shows that the strategy illustrated is an arbitrage opportunity.

5. By the previous item and from the First Fundamental Theorem of Asset Pricing we can immediately conclude that an equivalent martingale measure for the market model considered does not exist. This implies that the options previously mentioned cannot be evaluated as in items 2. and 3.
   The non-existence of an equivalent martingale measure can be proved also directly. If such a measure (let us denote it by $Q$) existed, the following condition should be satisfied:

$$\frac{E_Q[S_1]}{1 + r_{sem}} = S_0,$$

   i.e. there should exist $q_u \in [0, 1]$ such that

$$S_0 u q_u + S_0 d (1 - q_u) = S_0 (1 + r_{sem}).$$

   Since the last equation is equivalent to

$$\begin{aligned} u q_u + d (1 - q_u) &= 1 + r_{sem} \\ q_u &= \frac{1 + r_{sem} - d}{u - d} > \frac{u - d}{u - d} = 1, \end{aligned}$$

   it follows immediately that an equivalent martingale measure does not exist.

**Exercise 8.2** An option is written on a stock without dividends and with current price of 10 euros. At the end of each one of next two semesters the stock price can rise by 25% or fall by 20%, the risk-free interest rate is 4% (per year) and the strike of the option is 11 euros.

1. Find the current price of a European Put option written on this underlying and with maturity $T = 1$ year.
2. Is it optimal to exercise the corresponding (i.e. with the same parameters) American option before maturity? What is its fair value?
3. Assume now that the underlying stock pays a dividend of 1.5 euros in 6 months.
   (a) Compute the fair value of both the American and European Put options with strike of 11 euros.
   (b) Find the fair value of the American Call option with the same parameters of the previous one.

4. Can the difference between the initial values of the American Put and the American Call (both with strike $K = 11$ euros) be more than (or equal to) 0.5 and less than (or equal to) 3 euros?

**Solution** The market model described above corresponds to a binomial model with factors $u = 1.25$ and $d = 0.8$ for each time step (in the present case one semester). The stock price dynamics is then the following:

$$
\begin{array}{ccccc}
 & & & & S_{1y}^{uu} = 15.625 \\
 & & & \nearrow & \\
 & & S_{6m}^{u} = 12.5 & & \\
 & \nearrow & & \searrow & \\
S_0 = 10 & & & & S_{1y}^{ud} = 10 \\
 & \searrow & & \nearrow & \\
 & & S_{6m}^{d} = 8 & & \\
 & & & \searrow & \\
 & & & & S_{1y}^{dd} = 6.4 \\
---- & & --------- & & --------- \\
t = 0 & & t = 6\,\text{months} & & T = 1\,\text{year}
\end{array}
$$

The risk-neutral probability measure, necessary to compute the options' value, can be easily obtained and is defined by the following rise/fall probabilities for each time step:

$$
\begin{aligned}
q_u &= \frac{(1+r)^{1/2} - d}{u - d} = \frac{\sqrt{1.04} - 0.8}{1.25 - 0.8} = 0.488 \\
1 - q_u &= 0.512
\end{aligned}
$$

1. Since the European Put option payoff is 0, 1 and 4.6 (when $S_T = 15.625$, $S_T = 10$ and $S_T = 6.4$, respectively), by the previous remark we can conclude that the initial fair value of the European Put option is given by:

$$
P_0^{Eur} = \frac{1}{1+r}\left[q_u^2 \cdot 0 + 2q_u\,(1-q_u) \cdot 1 + (1-q_u)^2 \cdot 4.6\right] = 1.64 \text{ euros.}
$$

2. We want to verify now if the early exercise of the American Put is optimal or not. To check this we are going to proceed backwards along the tree, starting from maturity and considering all subtrees corresponding to one time-step separately. We shall denote by a capital letter each node of the tree:

$$\begin{tikzcd}[row sep=small]
 & & 15.625\ (D) \\
 & 12.5\ (B) \arrow[ur] \arrow[dr] & \\
10\ (A) \arrow[ur] \arrow[dr] & & 10\ (E) \\
 & 8\ (C) \arrow[ur] \arrow[dr] & \\
 & & 6.4\ (F) \\
---- & ------- & ------- \\
t = 0 & t = 6\,\text{months} & T = 1\,\text{year}
\end{tikzcd}$$

Moreover, for each node we write the payoff of the American Put considered:

$$\begin{tikzcd}[row sep=small]
 & & \begin{array}{c}15.625\ (D)\\ \text{payoff} = 0\end{array} \\
 & \begin{array}{c}12.5\ (B)\\ \text{payoff} = 0\end{array} \arrow[ur] \arrow[dr] & \\
\begin{array}{c}10\ (A)\\ \text{payoff} = 1\end{array} \arrow[ur] \arrow[dr] & & \begin{array}{c}10\ (E)\\ \text{payoff} = 1\end{array} \\
 & \begin{array}{c}8\ (C)\\ \text{payoff} = 3\end{array} \arrow[ur] \arrow[dr] & \\
 & & \begin{array}{c}6.4\ (F)\\ \text{payoff} = 4.6\end{array}
\end{tikzcd}$$

Let us start by considering the subtree including nodes B, D and E. The American Put, when restricted to this subtree is "European-like". So we can calculate the corresponding (i.e. with the same parameters) European Put value at the nodes B, D and E. To avoid confusion between the European Put of item 1. (we mean the European Put written at $t = 0$ and maturing in 1 year) and the "artificial" put options just introduced, we shall call the latter "European"(*).

At the nodes D and E the Put price coincides with its payoff, so

$$P_{T,D}^{Eur^*} = \text{payoff at node D} = 0$$
$$P_{T,E}^{Eur^*} = \text{payoff at node E} = 1$$
$$P_{6m,B}^{Eur^*} = \frac{1}{(1+r)^{1/2}}\left[q_u \cdot 0 + (1-q_u)\cdot 1\right] = 0.50.$$

Since the payoff of the American Put in B is 0, therefore less than the European* Put price in $t = 6$ months with maturity 1 year ($P_{6m,B}^{Eur^*} = 0.50$), early exercise at node B is *not* optimal.
The American Put value at each node is the maximum between the payoff and the value of the corresponding European* Put. In other words, it coincides with the European* counterpart when the early exercise is not optimal, while it equals the payoff at the node considered when the early exercise is optimal.
At the nodes B, D and E we have then:

$$P_{T,D}^{Am} = P_{T,D}^{Eur} = \text{payoff in D} = 0$$
$$P_{T,E}^{Am} = P_{T,E}^{Eur} = \text{payoff in E} = 1$$
$$P_{6m,B}^{Am} = \max\{P_{6m,B}^{Eur}; \text{payoff in B}\} = \max\{0.50; 0\} = 0.50.$$

Let us now consider the subtree with nodes C, E and F. The American Put option, restricted to this subtree is again "European-like". We can then compute the corresponding European* Put value at nodes C, E and F.
At nodes E and F, the option value coincides with the payoff. Hence,

$$P_{T,E}^{Eur^*} = \text{payoff in E} = 1$$
$$P_{T,F}^{Eur^*} = \text{payoff in F} = 4.6$$
$$P_{6m,C}^{Eur^*} = \frac{1}{(1+r)^{1/2}}\left[q_u \cdot 1 + (1-q_u)\cdot 4.6\right] = 2.79.$$

Since the American Put payoff at node C is equal to 3 (hence greater than the European* Put value at $t = 6$ months and with maturity $T = 1$ year ($P_{6m,C}^{Eur^*} = 2.79$)), the early exercise at node C is optimal.
The American Put price at nodes C, E, F is then given by:

$$P_{T,E}^{Am} = P_{T,E}^{Eur^*} = \text{payoff at node E} = 1$$
$$P_{T,F}^{Am} = P_{T,F}^{Eur^*} = \text{payoff at node F} = 4.6$$
$$P_{6m,C}^{Am} = \max\{P_{6m,C}^{Eur^*}; \text{payoff at node C}\} = \max\{2.79; 3\} = 3.$$

Let us finally consider the subtree corresponding to nodes A, B and C. The American Put option restricted to this subtree is again "European-like". We can compute the corresponding European* Put value at nodes A, B and C.
In B and C the value coincides with that of the American Put computed before, so

$$\begin{aligned}
P^{Eur^*}_{6m,B} &= P^{Am}_{6m,B} = 0.50 \\
P^{Eur^*}_{6m,C} &< P^{Am}_{6m,C} = 3 \\
P^{Eur^*}_{0,A} &= \frac{1}{(1+r)^{1/2}}\left[q_u \cdot 0.50 + (1-q_u)\cdot 3\right] = 1.75.
\end{aligned}$$

Since the payoff of the American Put at node A is 1, so less than the value of the European* Put in $t = 0$ with maturity $t = 6$ months ($P^{Eur^*}_{0,A} = 1.75$), the early exercise in A *is not* optimal.
Eventually, we can conclude that the American Put value in A is given by:

$$P^{Am}_{0,A} = \max\{P^{Eur^*}_{0,A};\ \text{payoff at node A}\} = \max\{1.75;\ 1\} = 1.75 \text{ euros.}$$

We remark that, as one could expect, the initial value of the American Put is greater than that of the corresponding European (as computed in 1. and equal to 1.64 euros).

3. If the underlying stock pays a dividend of 1.5 euros in 6 months, its dynamics can be described as follows:

$$\begin{array}{ccccc}
 & & & & S^{Div,uu}_{1\,\text{year}} = S^{Div,u}_{6m}\cdot u = 13.75 \\
 & & & \nearrow & \\
 & & S^{Div,u}_{6m} = S_0 u - 1.5 = 11 & & \\
 & \nearrow & & \searrow & S^{Div,ud}_{1y} = S^{Div,u}_{6m}\cdot d = 8.8 \\
S^{Div}_0 = 10 & & & & \\
 & \searrow & & \nearrow & S^{Div,du}_{1y} = S^{Div,d}_{6m}\cdot u = 8.125 \\
 & & S^{Div,d}_{6m} = S_0 d - 1.5 = 6.5 & & \\
 & & & \searrow & \\
 & & & & S^{Div,dd}_{1y} = S^{Div,d}_{6m}\cdot d = 5.2 \\
---- & & ------------- & & ------------- \\
t = 0 & & t = 6\,\text{months} & & T = 1\,\text{year}
\end{array}$$

(a) We describe, with the help of the following diagram, the American Put and the European Put payoffs, both with strike $K = 11$ euros:

$$
\begin{array}{ccccc}
 & & & & 13.75 \text{ (node D)} \\
 & & & & \text{payoff}_{Am} = \text{payoff}_{Eur} = 0 \\
 & & & \nearrow & \\
 & & 11 \text{ (node B)} & & \\
 & & \text{payoff}_{Am} = 0 & & \\
 & \nearrow & & \searrow & 8.8 \text{ (node E)} \\
 & & & & \text{payoff}_{Am} = \text{payoff}_{Eur} = 2.2 \\
10 \text{ (node A)} & & & & \\
\text{payoff}_{Am} = 1 & & & & \\
 & & & & 8.125 \text{ (node F)} \\
 & \searrow & & \nearrow & \text{payoff}_{Am} = \text{payoff}_{Eur} = 2.875 \\
 & & 6.5 \text{ (node C)} & & \\
 & & \text{payoff}_{Am} = 4.5 & & \\
 & & & \searrow & \\
 & & & & 5.2 \text{ (node G)} \\
 & & & & \text{payoff}_{Am} = \text{payoff}_{Eur} = 5.8
\end{array}
$$

Since

$$q_u = \frac{(1.04)^{1/2} - d}{u - d} = 0.488,$$

we have that

$$P^{Eur}_{6m,\mathrm{B}} = \frac{1}{\sqrt{1.04}}\,[0 + (1 - q_u) \cdot 2.2] = 1.105$$

$$P^{Eur}_{6m,\mathrm{C}} = \frac{1}{\sqrt{1.04}}\,[q_u \cdot 2.875 + (1 - q_u) \cdot 5.8] = 4.288.$$

Consequently,

$$P^{Am}_{6m,\mathrm{B}} = \max\{P^{Eur}_{6m,\mathrm{B}};\ \text{payoff}_{Am;\ \text{node B}}\} = \max\{1.105; 0\} = 1.105$$

$$P^{Am}_{6m,\mathrm{C}} = \max\{P^{Eur}_{6m,\mathrm{C}};\ \text{payoff}_{Am;\ \text{node C}}\} = \max\{4.288; 4.5\} = 4.5.$$

Finally we obtain:

$$
\begin{aligned}
P_0^{Eur} &= \frac{1}{\sqrt{1.04}}\left[q_u \cdot 1.105 + (1-q_u)\cdot 4.288\right] = 2.682 \text{ euro} \\
P_0^{Am} &= \max\left\{\frac{1}{\sqrt{1.04}}\left[q_u \cdot 1.105 + (1-q_u)\cdot 4.5\right];\, \text{payoff}_{Am;\,\text{node A}}\right\} \\
&= \max\{2.788;\, 1\} = 2.788 \text{ euros.}
\end{aligned}
$$

(b) If the American option considered in the previous point was a Call instead of a Put, the procedure would change as follows:

$$
\begin{array}{ccccc}
 & & & & \begin{array}{c}13.75 \text{ (node D)} \\ \text{payoff}_{\text{Call Am}} = 2.75\end{array} \\
 & & & \nearrow & \\
 & & \begin{array}{c}11 \text{ (node B)} \\ \text{payoff}_{\text{Call Am}} = 0\end{array} & & \\
 & \nearrow & & \searrow & \begin{array}{c}8.8 \text{ (node E)} \\ \text{payoff}_{\text{Call Am}} = 0\end{array} \\
\begin{array}{c}10 \text{ (node A)} \\ \text{payoff}_{\text{Call Am}} = 0\end{array} & & & & \\
 & \searrow & & \nearrow & \begin{array}{c}8.125 \text{ (node F)} \\ \text{payoff}_{\text{Call Am}} = 0\end{array} \\
 & & \begin{array}{c}6.5 \text{ (node C)} \\ \text{payoff}_{\text{Call Am}} = 0\end{array} & & \\
 & & & \searrow & \\
 & & & & \begin{array}{c}5.2 \text{ (node G)} \\ \text{payoff}_{\text{Call Am}} = 0\end{array}
\end{array}
$$

By proceeding in the same way outlined before, we have that:

$$
\begin{aligned}
C_{6m,\text{B}}^{Eur} &= \frac{1}{\sqrt{1.04}}\left[q_u \cdot 2.75 + 0\right] = 1.316 \\
C_{6m,\text{C}}^{Eur} &= 0
\end{aligned}
$$

and, consequently,

$$\begin{aligned} C_{6m,\mathrm{B}}^{Am} &= \max\{C_{6m,\mathrm{B}}^{Eur}; \mathrm{payoff}_{\text{Call Am; node B}}\} \\ &= \max\{1.316; 0\} = 1.316 \\ C_{6m,\mathrm{C}}^{Am} &= \max\{C_{6m,\mathrm{C}}^{Eur}; \mathrm{payoff}_{\text{Call Am; node C}}\} = 0. \end{aligned}$$

Finally, we obtain

$$\begin{aligned} C_0^{Am} &= \max\left\{\frac{1}{\sqrt{1.04}}\left[q_u \cdot 1.316 + 0\right]; \mathrm{payoff}_{Am;\,\text{node A}}\right\} \\ &= \max\{0.629; 0\} = 0.629. \end{aligned}$$

4. From the previous items, we immediately get

$$P_0^{Am} - C_0^{Am} = 2.788 - 0.629 = 2.159 \in [0.5; 3]\,.$$

Consequently, the difference between the initial American Put and Call values (both with strike $K = 11$ euros) is included in the interval $[a, b]$ where $a = 0.5$ and $b = 3$ euros (endpoints included).
We could obtain the same conclusion without having to compute the two options prices directly. We know, indeed, that

$$\frac{K}{1+r} - S_0 < P_0^{Am} - C_0^{Am} < D + K - S_0,$$

where $D$ is the discounted value of the dividends distributed between the initial date 0 and the maturity $T$. In the present case we have $D = \frac{1.5}{\sqrt{1.04}} = 1.471$ and

$$0.577 = \frac{K}{1+r} - S_0 < P_0^{Am} - C_0^{Am} < D + K - S_0 = 2.471.$$

**Exercise 8.3** Consider the American Put option of item 2. in Exercise 8.2. This option expires in 1 year, its strike is $K = 11$ euros and it is written on an underlying stock without dividends with current price of 10 euros. During each one of next two semesters the stock price can rise by 25% or fall by 20%. The risk-free interest rate is 4% (per year).

Find the hedging strategy of the American option considered.

**Solution** From Exercise 8.2, we recall that the dynamics of the underlying stock and the dynamics of the American Put can be described as follows:

$$
\begin{tikzcd}
 & & \begin{array}{c} S_{1y}^{uu} = 15.625 \text{ (node D)} \\ \text{value}_{\text{Put Am}} = 0 \end{array} \\
 & \begin{array}{c} S_{6m}^{u} = 12.5 \text{ (node B)} \\ \text{value}_{\text{Put Am}} = 0.50 \end{array} \arrow[ur] \arrow[dr] & \\
\begin{array}{c} S_0 = 10 \text{ (node A)} \\ \text{value}_{\text{Put Am}} = 1.75 \end{array} \arrow[ur] \arrow[dr] & & \begin{array}{c} S_{1y}^{ud} = 10 \text{ (node E)} \\ \text{value}_{\text{Put Am}} = 1 \end{array} \\
 & \begin{array}{c} S_{6m}^{d} = 8 \text{ (node C)} \\ \text{value}_{\text{Put Am}} = 3 \end{array} \arrow[ur] \arrow[dr] & \\
 & & \begin{array}{c} S_{1y}^{dd} = 6.4 \text{ (node F)} \\ \text{value}_{\text{Put Am}} = 4.6 \end{array}
\end{tikzcd}
$$

and that at node C early exercise is optimal.

In order to compute the hedging strategy, it is necessary to determine the strategy at time $t = 0$ and at node B, but not in C since at this node early exercise is viable.

To find the hedging strategy at time $t = 0$ means to find a quantity $\Delta_0$ of the underlying stock and an amount of money $x_0$ such that

$$
\begin{cases} \Delta_0 S_0 u + x_0 \left(1 + r\right)^{1/2} = P_{6m,\text{B}}^{Am} \\ \Delta_0 S_0 d + x_0 \left(1 + r\right)^{1/2} = P_{6m,\text{C}}^{Am} \end{cases},
$$

i.e.

$$
\begin{cases} \Delta_0 = \frac{P_{6m,\text{B}}^{Am} - P_{6m,\text{C}}^{Am}}{S_0(u-d)} = -0.556 \\ x_0 = \frac{1}{(1+r)^{1/2}} \left[ P_{6m,\text{B}}^{Am} - \Delta_0 S_0 u \right] = 7.31 \end{cases}.
$$

This means that the strategy consists in selling 0.556 stock units and investing 7.3 euros. As one could expect, its initial cost is given by:

$$
\Delta_0 S_0 + x_0 = -0.556 \cdot 10 + 7.31 = 1.75 = P_0^{Am}.
$$

The replicating strategy $\left(\Delta_{6m}^{B}, x_{6m}^{B}\right)$ to be performed at $t = 6$ months in the node B must solve the following equations:

$$
\begin{cases} \Delta_{6m}^{B} S_{6m}^{u} u + x_{6m}^{B} \left(1 + r\right)^{1/2} = P_{6\,\text{months, D}}^{Am} \\ \Delta_{6m}^{B} S_{6m}^{u} d + x_{6m}^{B} \left(1 + r\right)^{1/2} = P_{6m,\text{E}}^{Am} \end{cases}.
$$

Then

$$\begin{cases} \Delta^B_{6m} = \frac{P^{Am}_{6m,\mathrm{D}} - P^{Am}_{6m,\mathrm{E}}}{S^u_{6m}(u-d)} = -0.1778 \\ x^B_{6m} = \frac{1}{(1+r)^{1/2}} \left[ P^{Am}_{6m,\mathrm{D}} - \Delta^B_{6m} S^u_{6m} u \right] = 2.724 \end{cases}.$$

As expected, then, at node B the cost of the strategy just outlined is given by:

$$\Delta^B_{6m} S^u_{6m} + x^B_{6m} = \Delta^B_{6m} S_0 u + x^B_{6m} = 0.50 = P^{Am}_{6m,\mathrm{B}}.$$

**Exercise 8.4** Compute the price at time $t = 0$ of a perpetual American Put option with strike $K$ and written on an underlying whose price dynamics is described by a geometric Brownian motion with diffusion coefficient $\sigma$. The risk-free interest rate available on the market is $r$.

**Solution** Since the option is perpetual, i.e. its maturity is $T = +\infty$, the valuation problem does not depend explicitly on $t$. By removing the $t$-dependence in the PDE providing the American Put value, we obtain the following ODE:

$$\frac{\sigma^2}{2} S^2 \frac{d^2 P}{dS^2} + rS \frac{dP}{dS} - rP = 0. \tag{8.1}$$

This ordinary differential equation must be solved taking into account the proper boundary conditions that, for the American Put option, turn out to be the following

$$P(+\infty) = \lim_{S \to +\infty} P(S) = 0,$$
$$P(S_f) = K - S_f,$$
$$P'(S_f) = \left. \frac{dP}{dS} \right|_{S=S_f} = -1.$$

We should remark that we need three conditions in order to determine the solution of the ODE, since the free-boundary value $S_f$ is unknown, i.e. we need one more condition than for a "fixed" boundary. Actually, the free-boundary value $S_f$ has to be computed as part of the solution of the problem considered, and becomes an extra unknown. The second and third conditions are called the "value matching" condition and the "smooth pasting" condition, respectively. Together with the condition at infinity they can be proved to be both necessary and sufficient in order to guarantee the existence and uniqueness of the solution for the problem under examination.

To compute the general solution of the ODE above, we can perform a simple variable change:

$$S = e^x. \tag{8.2}$$

Hence, $x = \ln S$ and the ODE for the new unknown $P(x)$ becomes

$$\frac{\sigma^2}{2}\frac{d^2P}{dx^2} + \left(r - \frac{\sigma^2}{2}\right)\frac{dP}{dx} - rP = 0. \tag{8.3}$$

This is a linear homogeneous ordinary differential equation, whose solution can be immediately written as

$$P(x) = Ae^{\lambda_1 x} + Be^{\lambda_2 x}, \tag{8.4}$$

where $A, B$ are constants determined by imposing the boundary conditions and $\lambda_1, \lambda_2$ $(\lambda_1 < \lambda_2)$ are the two distinct solutions of the quadratic equation:

$$\frac{\sigma^2}{2}\lambda^2 + \left(r - \frac{\sigma^2}{2}\right)\lambda - r = 0. \tag{8.5}$$

Hence,

$$\begin{aligned}\lambda_{1,2} &= \sigma^{-2}\left[-(r - \frac{\sigma^2}{2}) \pm \sqrt{(r - \frac{\sigma^2}{2})^2 + 2r\sigma^2}\right] \\ &= \sigma^{-2}\left[-(r - \frac{\sigma^2}{2}) \pm (r + \frac{\sigma^2}{2})\right],\end{aligned}$$

$$\lambda_1 = -\frac{2r}{\sigma^2}, \qquad \lambda_2 = 1. \tag{8.6}$$

Written in terms of the old independent variable $S$ , the solution is then

$$P(S) = AS^{\lambda_1} + BS^{\lambda_2} = AS^{-\frac{2r}{\sigma^2}} + BS. \tag{8.7}$$

The first boundary condition, i.e. $P(\infty) = 0$, implies that the constant $B$ must vanish, while the second and third conditions allow to determine simultaneously the constant $A$ and the free-boundary value $S_f$:

$$\begin{aligned}P(S_f) &= AS_f^{-\frac{2r}{\sigma^2}} = K - S_f, \\ P'(S_f) &= -\frac{2r}{\sigma^2}AS_f^{-\frac{2r}{\sigma^2}-1} = -1.\end{aligned}$$

From these conditions, we immediately obtain

$$A = \left(K - S_f\right) S_f^{\frac{2r}{\sigma^2}}$$

$$S_f = \frac{K}{1 + \frac{\sigma^2}{2r}}.$$

**Exercise 8.5** The *early exercise premium* of an American option $p(S,t)$ is defined as $p(S,t) \triangleq P^{Am}(S,t) - P^{Eur}(S,t)$, i.e. as the difference at time $t$ between the values of an American and a European Put option on the same underlying and with the same parameters. Assume as ansatz that the dependence of $P$ on the time to maturity $\tau = T - t$ can be factorized through the function $H(\tau) = 1 - \exp(-r\tau)$ as follows:

$$p(S,\tau) = H(\tau) f(S, H(\tau)), \tag{8.8}$$

for a suitable function $f$.

1. Find the PDE satisfied by the function $f$ in the Black-Scholes framework (i.e. by assuming that the underlying price dynamics is described by a geometric Brownian motion).
2. Assume that the nonlinear terms appearing in the PDE determined in the previous step are much smaller than the remaining terms. Solve the ODE obtained by neglecting those terms by taking into account the proper boundary conditions.
3. By using the solution obtained in the previous step, provide an approximate formula for the American Put option value.

**Solution** We know already that the function $p(S,\tau)$ is non-negative. Since $p$ is the difference between two functions satisfying the Black-Scholes equation, it must also satisfy the same equation (due to the linearity of the Black-Scholes equation):

$$-\frac{\partial p}{\partial \tau} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 p}{\partial S^2} + rS\frac{\partial p}{\partial S} - rp = 0. \tag{8.9}$$

The initial condition in the variable $\tau$ for the function $p$ must be $p(S,0) = 0$, since the American and the European options satisfy the same initial condition (they have the same payoff).

1. By assuming that the dependence of $p$ on $\tau$ can be factorized in the suggested form, it is easy to compute the partial derivatives with respect to the new independent variables:

$$\frac{\partial p}{\partial S} = H\frac{\partial f}{\partial S},$$

$$\frac{\partial^2 p}{\partial S^2} = H\frac{\partial^2 f}{\partial S^2},$$

$$\frac{\partial p}{\partial \tau} = H\frac{\partial f}{\partial H}\frac{dH}{d\tau} + f\frac{dH}{d\tau},$$
$$\frac{dH}{d\tau} = r(1-H).$$

Hence, we get the following PDE for $f$:

$$S^2\frac{\partial^2 f}{\partial S^2} + kS\frac{\partial f}{\partial S} - \frac{k}{H}f\left[1 + H(1-H)\frac{1}{f}\frac{\partial f}{\partial H}\right] = 0, \tag{8.10}$$

where we introduced the parameter $k = 2r/\sigma^2$.

2. By neglecting the quadratic term in $H$, and thus eliminating the explicit dependence of $f$ on $H$, we finally obtain the approximate ODE for $f$:

$$S^2\frac{\partial^2 f}{\partial S^2} + kS\frac{\partial f}{\partial S} - \frac{k}{H}f = 0. \tag{8.11}$$

This ODE, in which $H$ plays the role of a parameter, can be solved explicitly. By solving the ODE exactly as in the previous exercise, we obtain the solution:

$$f(S) = AS^\lambda, \tag{8.12}$$

where $\lambda$ is defined as follows:

$$\lambda \triangleq -\frac{1}{2}\left[(k-1) + \sqrt{(k-1)^2 + \frac{4k}{H}}\right]. \tag{8.13}$$

The constant $A$ must be determined by imposing the proper conditions.

3. Combining the results obtained, we can write an approximation for the American Put option value for $S > S_f$ :

$$P^{Am}(S,\tau) \approx P^{Eur}(S,\tau) + AH(\tau)S^\lambda, \tag{8.14}$$

where the constant $A$ must be such that the option value matches its payoff for $S = S_f$:

$$P^{Eur}(S,\tau) + AH(\tau)S_f^\lambda = K - S_f. \tag{8.15}$$

There is still the unknown $S_f$ to be determined, and this can be done by imposing the "smooth pasting" condition

$$\begin{aligned}\frac{\partial P^{Am}}{\partial S}(S_f,\tau) &= \frac{\partial P^{Eur}}{\partial S}(S_f,\tau) + A\lambda H S_f^{\lambda-1}\\ &= N(d_1(S_f)) - 1 + A\lambda H S_f^{\lambda-1} = -1,\end{aligned} \tag{8.16}$$

where $N(\cdot)$ stands, as usual, for the cumulative distribution function of a standard normal. Solving the last equation, we get an expression for $A$ in terms of $S_f$:

$$A = -\frac{N(d_1(S_f))}{\lambda H S_f^{\lambda-1}}. \tag{8.17}$$

By substituting into the Eq. (8.15), we obtain an equation for the remaining unknown $S_f$:

$$P^{Eur}(S_f, \tau) - N(d_1(S_f))\frac{S_f}{\lambda} = K - S_f, \tag{8.18}$$

which, taking into account the Put-Call parity and the trivial identity $N(-d) = 1 - N(d)$, can be written as

$$S_f N(d_1(S_f)) - K e^{-r\tau} N(d_2(S_f)) - S_f + K e^{-r\tau} - N(d_1(S_f))\frac{S_f}{\lambda} - K + S_f = 0.$$

Collecting terms, we write it as

$$S_f N(d_1(S_f))(1 - \frac{1}{\lambda}) + K e^{-r\tau}\left[1 - N(d_2(S_f))\right] - K = 0. \tag{8.19}$$

Since $d_1$ and $d_2$ depend on $S_f$, this is an implicit equation in the unknown $S_f$ and must be solved by an approximate iterative method, like Newton's method, where the initial seed $S_0 = K$ may be used for starting the iterative procedure.

**Exercise 8.6** F. Black [7] suggested the following approximate procedure for evaluating American call options written on distributing dividends assets. He proposed to evaluate both the European call maturing at time $T$ and the European call maturing just before the last dividend distribution date, then to approximate the corresponding American option value by the maximum between the values of the two European options considered. As an example, consider an American call option written on the asset with initial value $S(0) = 30$ euros, maturity 1 year ($T = 1$),strike $K = 25$, and assume that its dynamics is described by a geometric Brownian motion with diffusion coefficient $\sigma = 0.4$, and the risk-free interest rate $r = 0.06$. Suppose moreover that, during the lifetime of the option, the asset distributes two dividends, one after 4 months and one after 8 months, of the same amount $d_i = 1$ euro ($i = 1, 2$). Evaluate the American call written on the asset considered by applying the procedure proposed by F. Black.

**Solution** We first have to evaluate the two European options written on the same asset, the first one with the same maturity of the American option, and the second one just before the last dividend distribution date, by applying the Black-Scholes formula for options written on asset distributing dividends at discrete time:

$$C_1 = [S(0) - d_1 e^{-0.333\times 0.06} - d_2 e^{-0.667\times 0.06}]N(\hat{d}_1^1) - Ke^{-0.06}N(\hat{d}_2^1) =$$
$$= 28.059N(2.8) - 25e^{-0.06}N(2.4) = 27.986 - 23.351 = 4.635 euro.$$

$$C_2 = [S(0) - d_1 e^{-0.333\times 0.06}]N(\hat{d}_1^2) - Ke^{-0.667\times 0.06}N(\hat{d}_2^2) =$$
$$= 29.02N(3.511) - 25e^{-0.667\times 0.06}N(3.184) = 5.006 euro.$$

Since the maximum between the two values is the second, we approximate the American call option value with this: $C_A = C_2 = 5.006$ euro.

*Remark 8.7* Notice that in this case the approximation proposed by Black is very rough. By computing $K[1 - e^{-r(T-t_2)}] = Ke^{-0.333\times 0.06} = 24.5 > 1$, it is immediate to conclude that it is not optimal to exercise the option before the date of the second dividend distribution, but it is optimal to exercise the option before the first dividend distribution. This typically happens since in this case, when the asset price approaches the strike value, the dividend yield of the asset is bigger than the risk free rate. When this is the case, it is optimal to exercise the option as late as possible. This implies that the true value of the option will be closer to that of the corresponding European option. Intuitively speaking, the dividends are not big enough to compensate the potential remuneration due to late exercise.

## 8.3 Proposed Exercises

**Exercise 8.8** The current price of a stock is 40 euros. At the end of this year the stock price will be either 42 or 36 euros. The growth and fall factors of the stock in the following years will remain the same and the risk-free interest rate is 4% per year.

1. Compute the price of the American Put with maturity $T = 2$ years and with strike of 40 euros. Is it optimal to exercise this option before maturity?
2. Suppose a first dividend of 1 euro is paid after 1 year and a second one of 3 euros after 2 years: what changes in the valuation procedure? Which option is more expensive: the American Put or the American Call?
3. For which American option is the early exercise more likely to take place: a Call or a Put option?
4. Compute the price of the American Put and Call options with the same parameters as in the previous point, but with maturity delayed by 2 years.
5. Is it possible to find a different value of the growth factor in a binomial model, so that the value of a Call option with maturity $T = 2$ years and strike $K = 40$ euros written on a stock without dividends coincides with the value of an American Put

option with the same parameters written on a stock with the same dividends as item 2.?

**Exercise 8.9** Compute the price (at time $t = 0$) of a perpetual American Call option, written on an underlying paying dividends at a constant rate $q$, and whose price dynamics is described by a geometric Brownian motion with diffusion coefficient $\sigma$. The risk-free interest rate is $r$, the strike $K$.

**Exercise 8.10** By applying the approximation procedure proposed by F. Black, compute the price (at time $t = 0$) of an American Call option, written on an underlying asset with initial value $S(0) = 50$ euros, maturity $T = 16$ months, paying two dividends during the option lifetime, one after 6 months, one after 1 year, both of the same amount $d_i = 1.5$ $(i = 1, 2)$ euros. Assume the asset price dynamics is described by a geometric Brownian motion with diffusion coefficient $\sigma = 0.2$. The risk-free interest rate is $r = 0.04$, the strike $K 40$ euros.

# Chapter 9
# Exotic Options

## 9.1 Review of Theory

An *Exotic option* is an option that is neither a European Call or Put option nor an American Call or Put. A *path-dependent option* is an option whose payoff does not depend only on the underlying value at maturity, but also on one or more values that it can assume during its lifetime. Although the two option classes do not coincide, many exotic options exhibit path-dependence features. Among the most popular Exotic options that are not path-dependent we just recall *binary (or digital) options*, *compound options* and *chooser options*.

*Binary Options* The most traded kind of binary option is the so called "cash or nothing" Call. This is a contract guaranteeing the owner a fixed amount of money $B$ if the underlying value at maturity exceeds a threshold $L$, and nothing if the underlying value is less than $L$.

The payoff of this option can be expressed in the following form:

$$B\mathbf{1}_{\{S_T \geq L\}} = B \cdot H(S_T - L),$$

where the function $H$ is the so-called Heaviside step-function:

$$H(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}. \tag{9.1}$$

Another kind of binary option is the "asset or nothing" Call. This option expires out-of-the-money if the underlying value at maturity is less than the threshold $L$, while it guarantees a unit of the underlying asset if the threshold value is exceeded. The expression of this second kind of binary option payoff is given by:

$$S_T\mathbf{1}_{\{S_T \geq L\}} = S_T \cdot H(S_T - L). \tag{9.2}$$



E. Rosazza Gianin, C. Sgarra, *Mathematical Finance*,
La Matematica per il 3+2 149, https://doi.org/10.1007/978-3-031-28378-9_9

*"Chooser" Options* are options with other options as underlying assets, giving the owner the right to decide, at maturity $T_1 < T_2$, whether the underlying option must be a Call or a Put. The chosen option will be (possibly) exercised at maturity $T_2$ and its strike will be $K_2$.

The valuation of these options can be performed in a way that is strictly similar to the European options, but with the following final condition:

$$F(S_{T_2}, T_2) = \max\left\{C(S_{T_2}, T_2) - K_2;\, P(S_{T_2}, T_2) - K_2;\, 0\right\}.$$

As far as path-dependent options are concerned, we are going to propose a few exercises on Asian options, Barrier options and Lookback options.

*Asian Options* are options depending on the average of the values assumed by the underlying during the option lifetime.

Asian options can be of different kinds according to the average considered: this can be the *arithmetic mean* or the *geometric mean*, and the underlying values contributing to the average can be "observed" in continuous or discrete time.

A further distinction among Asian options arises according to the functional dependence of the payoff on the average considered; this can in fact substitute the underlying final value or the strike in the payoff involved. In the former case we shall refer to *average rate* options, in the latter to *average strike* options.

The valuation problem for these kind of options can be cast into the Black-Scholes framework and, for some particular cases, closed-form solutions providing options prices exist. Except for the very simple case of the geometric Asian options (with continuous time monitoring), these solutions do not admit a representation that is easy to handle for applications. We shall therefore focus our attention mainly on examples of valuation problems in a binomial setting.

*Barrier Options* are options with the same payoff at maturity of the European options, but with one extra feature: they expire valueless (or not) if, at any instant during the option lifetime, the underlying reaches some threshold value from above or from below. In the first case they will be called *up-and-out options*, in the second *down-and-out* (or *up-and-in* and *down-and-in* respectively).

For Barrier options (with continuous monitoring), explicit valuation formulas are available in a Black-Scholes setting. We recall these formulas below.

*Down-and-Out Call Pricing Formula*

$$C_{DO}(S_t, t) = C(S_t, t) - \left(\frac{L}{S_t}\right)^{\frac{2r}{\sigma^2}-1} C\left(\frac{L^2}{S_t}, t\right), \tag{9.3}$$

where $C(S_t, t)$ is the value of the European Call option written on the same underlying and with the same parameters (strike, maturity, volatility and risk-free interest rate) of the Barrier option, and $L$ is the barrier value. The previous formula holds when $K > L$. When $K < L$, the down-and-out Call pricing formula reduces to the following:

$$\begin{aligned} C_{DO}(S_t, t) = S_t\left[N(d_3) - b(1 - N(d_6))\right] \\ -Ke^{-r(T-t)}\left[N(d_4) - a(1 - N(d_5))\right], \end{aligned} \tag{9.4}$$

where $a$, $b$, $d_3$, $d_4$, $d_5$, $d_8$ are defined as follows:

$$a = \left(\frac{L}{S_t}\right)^{\frac{2r}{\sigma^2}-1}, \quad b = \left(\frac{L}{S_t}\right)^{\frac{2r}{\sigma^2}+1}, \tag{9.5}$$

$$d_3 = \frac{\ln\left(\frac{S_t}{L}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}, \tag{9.6}$$

$$d_4 = \frac{\ln(\frac{S_t}{L}) + (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}, \tag{9.7}$$

$$d_5 = \frac{\ln\left(\frac{S_t}{L}\right) - \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}, \tag{9.8}$$

$$d_6 = \frac{\ln\left(\frac{S_t}{L}\right) - \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}. \tag{9.9}$$

*Up-and-Out Call Pricing Formula*

For up-and-out Call, in case $U > K$, the expression providing the price at time $t$ is the following:

$$\begin{aligned} C_{UO}(S_t, t) &= C(S_t, t) - S_t\left[N(d_3) + b(N(d_6) - N(d_8))\right] \\ &\quad + Ke^{-r(T-t)}\left[N(d_4) + a(N(d_5) - N(d_7))\right] \qquad (9.10) \\ &= S_t\left[N(d_1) - N(d_3) - b\left(N(d_6) - N(d_8)\right)\right] \\ &\quad - Ke^{-r(T-t)}\left[N(d_2) - N(d_4) - a(N(d_5) - N(d_7)\right], \qquad (9.11) \end{aligned}$$

where

$$d_7 = \frac{\ln\left(\frac{S_t K}{U^2}\right) - \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}, \tag{9.12}$$

$$d_8 = \frac{\ln\left(\frac{S_t K}{U^2}\right) - \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \tag{9.13}$$

and the quantities $d_3, d_4, d_5, d_6$ are defined as before, with $U$ instead of $L$. It is easy to observe that the value of an up-and-out Call option with $U < K$ is zero.

For Barrier options of "in" type it is possible to obtain immediately the pricing formulas from those just presented, by observing that a portfolio composed by two Barrier options of in and out type respectively, written on the same underlying, with the same parameters and with the same threshold values, is equivalent to a European option, still written on the same underlying and with the same parameters. As an example, we provide the following pricing formula for a down-and-in Call option, holding if $K > L$:

$$C_{DI}(S_t, t) = \left(\frac{L}{S_t}\right)^{2r/\sigma^2 - 1} C\left(\frac{L^2}{S_t}, t\right). \tag{9.14}$$

*Lookback Options* are options with payoff assuming one of the following forms:

$$F_{PM}(S_T, S_{\max}, T) = \max\{S_{\max} - S_T; 0\} = \max\{M - S_T; 0\} \tag{9.15}$$

for the Put,

$$F_{Cm}(S_T, S_{\min}, T) = \max\{S_T - S_{\min}; 0\} = \max\{S_T - m; 0\} \tag{9.16}$$

for the Call, where $m = S_{\min}$, $M = S_{\max}$ are, respectively, the minimum and the maximum values assumed by the underlying during the option lifetime. We remark that $M$ and $m$ "substitute the strike" in the payoff of the European Put and Call options respectively.

Explicit analytical expressions are available for the Lookback options just mentioned in the Black-Scholes setting; they can be obtained both by solving the relative PDE with similarity methods or by computing the discounted expectation of their payoff, since the joint probability distributions of the running maximum and minimum of a Brownian motion and of the Brownian motion itself is known. We briefly recall these pricing formulas for the reader's convenience:

$$\begin{aligned} F_{Cm}(S_t, m, t) = {} & S_t N(d_1) - m e^{-r(T-t)} N(d_2) \\ & + S_t e^{-r(T-t)} \tfrac{\sigma^2}{2r} \cdot \Big[ (\tfrac{S_t}{m})^{-2r/\sigma^2} N(-d_1 + \tfrac{2r}{\sigma}\sqrt{T-t}) \\ & - e^{r(T-t)} N(-d_1) \Big], \end{aligned}$$

$$\begin{aligned} F_{PM}(S_t, M, t) = {} & M e^{-r(T-t)} N(-d_2) - S_t N(-d_1) \\ & + S_t e^{-r(T-t)} \tfrac{\sigma^2}{2r} \cdot \Big[ -(\tfrac{S_t}{M})^{-2r/\sigma^2} N(d_1 - \tfrac{2r}{\sigma}\sqrt{T-t}) \\ & + e^{r(T-t)} N(d_1) \Big]. \end{aligned}$$

It is useful to remember that a geometric Brownian motion with diffusion coefficient $\sigma$ can be approximated by a binomial process with $u = e^{\sigma\sqrt{\Delta t}}$ and $d = 1/u = e^{-\sigma\sqrt{\Delta t}}$, where $\Delta t$ is the time step size. This approximation can always be used whenever an exact valuation formula is not available and the quality of this approximation improves as the time step is smaller. It is worth mentioning a few issues related to Exotic options hedging. While some of these options are easy to hedge—like Asian options (since the average becomes more and more stable as time goes on, and the number of observations grows), other options—like barrier options—can be extremely difficult to hedge: a small variation of the underlying price, close to the barrier value, can give rise to a huge variation of the option value, i.e. the Delta can be discontinuous across the barrier. In these cases, a static replication approach can be used. The static replication methodology is based on a standard result of functional analysis that can be roughly resumed as follows: every continuous real function with compact support can be approximated (with the desired accuracy) by piecewise linear functions; this means, in practice, that every payoff (with compact support) can be approximated by the payoff of a portfolio consisting of a suitable number of units of the underlying and a suitable number of Put and/or Call European options. We shall illustrate this procedure by an example in the exercises.

The reader interested in a more detailed presentation of pricing methods for Exotic options can find many results in El Karoui [18], Hull [25], Musiela and Rutkowski [33] and Wilmott et al. [43].

## 9.2 Solved Exercises

**Exercise 9.1** Consider a stock with value at time $t$ denoted by $(S_t)_{t\geq 0}$ and with current value $S_0 = 8$ euros. Compute the price at time $t = 0$ of a "cash or nothing" Call option, written on the stock considered, with maturity $T = 3$ months, strike $L = 10$ euros and rebate $B = 20$ euros. The stock price dynamics is assumed to be described by a geometric Brownian motion with volatility $\sigma = 0.25$ (per year). The risk-free interest rate on the market is $r = 0.04$ (per year).

We recall that, for a "cash or nothing" Call the strike is the threshold value $L$ such that, if the underlying at maturity is greater than or equal to $L$, the option buyer gets the cash amount $B$ (the rebate).

**Solution** In order to compute the value at time $t = 0$ of a binary option of "cash or nothing" Call type, we can apply the usual risk-neutral valuation procedure. Hence, we need to determine the discounted expectation of the option payoff with respect to the equivalent martingale measure:

$$C_D(S_t, t) = e^{-r(T-t)} E_Q\left[\Phi(S_T)\right], \tag{9.17}$$

where we denote by $C_D$ the binary option's value at time $t$, by $\Phi$ its payoff and by $E_Q$ the expectation with respect to $Q$ (the risk-neutral measure). We recall that the payoff of a binary option of "cash or nothing" Call type is given by the following expression: $\Phi(S_T) = B \cdot H(S_T - L)$, where $L$ is the "threshold" value and $B$ the rebate.

In the present case, since we are working in a Black-Scholes framework, we can write:

$$\begin{aligned} C_D(S_t, t) &= e^{-r(T-t)} \int_{-\infty}^{+\infty} B \cdot H\left(S_t e^{\left(r-\sigma^2/2\right)(T-t)+\sigma x} - L\right) \\ &\quad \times \frac{1}{\sqrt{2\pi(T-t)}} e^{-\frac{x^2}{2(T-t)}}\, dx \\ &= e^{-r(T-t)} \int_{-\infty}^{+\infty} \frac{B \cdot H(S_t e^z - L)}{\sigma\sqrt{2\pi(T-t)}} \\ &\quad \times \exp\left\{-\frac{\left(z - (r-\sigma^2/2)(T-t)\right)^2}{2\sigma^2(T-t)}\right\} dz. \end{aligned}$$

The integral appearing in the last formula can be explicitly computed:

$$\begin{aligned} &\int_{-\infty}^{+\infty} \frac{B \cdot H(S_t e^z - L)}{\sigma\sqrt{2\pi(T-t)}} \exp\left\{-\frac{\left(z - (r-\sigma^2/2)(T-t)\right)^2}{2\sigma^2(T-t)}\right\} dz \\ &= B \int_{\ln L - \ln S_t}^{+\infty} \exp\left\{-\frac{\left(z - (r-\sigma^2/2)(T-t)\right)^2}{2\sigma^2(T-t)}\right\} \frac{dz}{\sigma\sqrt{2\pi(T-t)}} \\ &= B \int_{\frac{\ln L - \ln S_t - \left(r-\sigma^2/2\right)(T-t)}{\sigma\sqrt{T-t}}}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du \\ &= B\left[1 - N\left(\frac{\ln(L/S_t) - \left(r-\sigma^2/2\right)(T-t)}{\sigma\sqrt{T-t}}\right)\right] \\ &= B \cdot N(d_2). \end{aligned}$$

In the last expression we have denoted by $N$ the cumulative distribution function of a standard normal and set

$$d_2 = \frac{\ln(S_t/L) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

(in strict analogy with the term appearing in the Black-Scholes formula for the European Call option price).

The "cash or nothing" Call option value is then given by

$$C_D(S_t, t) = Be^{-r(T-t)} N(d_2).$$

In our case, we have $t = 0$, $T = 1/4$, $S_0 = 8$, $B = 20$, $L = 10$, $r = 0.04$, $\sigma^2 = 0.0625$, so

$$d_2 = \frac{\ln(8/10) + (0.04 - 0.0625/2)\frac{1}{4}}{0.25 \cdot \sqrt{1/4}} = -1.767$$

and

$$C_D(0, 8) = 20 \cdot e^{-0.04 \cdot \frac{1}{4}} N(-1.767) = 0.764 \text{ euros.}$$

**Exercise 9.2** Consider a chooser option with maturity $T_2$ and strike $K$, i.e. an option written at time $t = 0$ with maturity $T_2$ for which at time $T_1$ (with $T_1 < T_2$) the option buyer can choose if the option will be exercised as a Call or a Put on the same underlying asset.

Compute the initial price of the chooser option with maturity $T_1 = 3$ months, $T_2 = 6$ months and strike $K = 30$ euros, if the underlying dynamics is described by a geometric Brownian motion with initial value $S_0 = 20$ euros, volatility $\sigma = 0.25$ (per year), and the risk-free interest rate is $r = 0.06$ (per year).

**Solution** A chooser option can be easily valuated by showing that it is equivalent to a portfolio composed by a European Call with maturity $T_1$ and strike $Ke^{-r(T_2-T_1)}$ and by a European Put with maturity $T_2$ and strike $K$ (see Hull [25] for a detailed proof). We denote by $C_0\left(Ke^{-r(T_2-T_1)}; T_1\right)$ and $P_0(K; T_2)$ the prices at $t = 0$ of these options.

The initial price $Ch_0$ of the chooser option is then given by:

$$Ch_0 = P_0(K; T_2) + C_0\left(Ke^{-r(T_2-T_1)}; T_1\right), \tag{9.18}$$

i.e. the initial prices of the Put and Call options mentioned before. By substituting the data assigned, we get:

$$Ch_0 = P_0(30, 1/2) + C_0(30e^{-0.06\frac{1}{4}}; 1/4).$$

By applying the Black-Scholes formulas for European options directly, one obtains:

$$C_0(30e^{-0.06\frac{1}{4}}; 1/4) = 0.038$$
$$P_0(30; 1/2) = 9.172,$$

by which we can compute the initial value of the chooser option:

$$Ch_0 = 0.038 + 9.172 = 9.21 \text{ euros.}$$

**Exercise 9.3** Consider a stock with initial value $S_0 = 8$ euros, with dynamics described by a geometric Brownian motion with volatility $\sigma = 0.36$ (per year) in a market model with risk-free interest rate $r = 0.04$ (per year).

1. Compute the value at time $t = 0$ of an up-and-in and of an up-and-out Call option with barrier $U = 12$ euros, strike $K = 9$ euros and maturity $T = 4$ months.
2. What is the (risk-neutral) probability that, at $t = 2$ months, the underlying value will reach the barrier?

**Solution**

1. First of all, we recall the valuation formula for a Barrier Call option of up-and-out type. By denoting by $C_{UO}$ the value of this Barrier option and by $C_0$ the value the corresponding vanilla option (both at time $t = 0$), from (9.10) we get:

$$C_{UO,0} = C_0 - S_0\left[N(d_3) + b(N(d_6) - N(d_8))\right] \tag{9.19}$$
$$+Ke^{-rT}\left[N(d_4) + a(N(d_5) - N(d_7))\right] \tag{9.20}$$

where $a$, $b$, $d_3$, $d_4$, $d_5$, $d_6$, $d_7$ and $d_8$ are defined by (9.5)–(9.13).

By the formulas just mentioned, with the assigned data, we have that:

$$a = \left(\frac{12}{8}\right)^{\frac{0.08}{(0.36)^2}-1} = 0.856; \quad b = \left(\frac{12}{8}\right)^{\frac{0.08}{(0.36)^2}+1} = 1.927$$

$$d_3 = \frac{\ln\left(\frac{8}{12}\right) + \left(0.04 + \frac{(0.36)^2}{2}\right)\frac{4}{12}}{0.36 \cdot \sqrt{\frac{4}{12}}} = -1.783$$

$$d_4 = \frac{\ln\left(\frac{8}{12}\right) + \left(0.04 - \frac{(0.36)^2}{2}\right)\frac{4}{12}}{0.36 \cdot \sqrt{\frac{4}{12}}} = -1.991$$

$$d_5 = \frac{\ln\left(\frac{8}{12}\right) - \left(0.04 - \frac{(0.36)^2}{2}\right)\frac{4}{12}}{0.36 \cdot \sqrt{\frac{4}{12}}} = -1.911$$

$$d_6 = \frac{\ln\left(\frac{8}{12}\right) - \left(0.04 + \frac{(0.36)^2}{2}\right)\frac{4}{12}}{0.36 \cdot \sqrt{\frac{4}{12}}} = -2.119$$

$$d_7 = \frac{\ln\left(\frac{8\cdot 9}{(12)^2}\right) - \left(0.04 - \frac{(0.36)^2}{2}\right)\frac{4}{12}}{0.36 \cdot \sqrt{\frac{4}{12}}} = -3.295$$

$$d_8 = \frac{\ln\left(\frac{8\cdot 9}{(12)^2}\right) - \left(0.04 + \frac{(0.36)^2}{2}\right)\frac{4}{12}}{0.36 \cdot \sqrt{\frac{4}{12}}} = -3.50$$

Since

$$d_1 = \frac{\ln(S_0/K) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} = -0.399$$
$$d_2 = d_1 - \sigma\sqrt{T} = -0.606,$$

we conclude that the initial (at time $t = 0$) value of the European Call option is

$$\begin{aligned} C_0 &= S_0 \cdot N(d_1) - Ke^{-rT} \cdot N(d_2) \\ &= 8 \cdot N(-0.399) - 9 \cdot e^{-0.04 \cdot 4/12} \cdot N(-0.606) \\ &= 0.342 \text{ euros.} \end{aligned}$$

Finally, we obtain the initial value of the up-and-out Call:

$$C_{UO,0} = 0.20 \text{ euros.}$$

By the relation holding between up-and-out, up-and-in and vanilla option values (strictly analogous to that between down-and-out, down-and-in and vanilla), we
have that:

$$C_0 = C_{UI,0} + C_{UO,0}, \tag{9.21}$$

where $C_{UI,0}$ denotes the initial value of the up-and-in Call.
From (9.21) we finally obtain:

$$C_{UI,0} = C_0 - C_{UO,0} = 0.342 - 0.20 = 0.142 \text{ euros.}$$

We remark that both the up-and-in Call and the up-and-out Call values are smaller that the value of the corresponding European Call. That is because, in general, there is a non-vanishing probability that the barrier will be reached. In general, the following relations hold:

$$C_0 \geq C_{DI,0}$$
$$C_0 \geq C_{DO,0}.$$

2. We now want to compute the probability that, at $t = 2$ months, the underlying value will reach the barrier, i.e. we want to compute $P(S_t \geq U)$ at $t = 2/12$ (2 months).

   Since the risk-neutral dynamics of the underlying's value is described by a geometric Brownian motion,

$$S_t = S_0 \exp\left\{\left(r - \sigma^2/2\right) t + \sigma W_t\right\}$$

and the probability that its value at $t = 2/12$ will be greater than $U$ is the following:

$$\begin{aligned} P(S_t \geq U) &= P(\ln(S_t) \geq \ln(U)) \\ &= P\left(\ln(S_0) + \left(r - \frac{\sigma^2}{2}\right) t + \sigma W_t \geq \ln(U)\right) \\ &= P\left(\frac{W_t}{\sqrt{t}} \geq \frac{\ln(U) - \ln(S_0) - \left(r - \frac{\sigma^2}{2}\right) t}{\sigma\sqrt{t}}\right) \\ &= 1 - N\left(\frac{\ln(U) - \ln(S_0) - \left(\mu - \frac{\sigma^2}{2}\right) t}{\sigma\sqrt{t}}\right). \end{aligned}$$

With the assigned data we obtain:

$$\begin{aligned} P(S_{2\,months} \geq U) &= 1 - N\left(\frac{\ln(12) - \ln(8) - \left(0.04 - \frac{(0.36)^2}{2}\right)\frac{2}{12}}{0.36\sqrt{2/12}}\right) \\ &= 1 - N(2.480) = 0.0066. \end{aligned}$$

**Exercise 9.4** In a binomial setting, consider a Lookback Call option with maturity $T = 8$ months, written on an underlying stock (without dividends) whose price can rise or fall by 50% during each 2-months period of the next 8 months. The underlying's initial value is $S_0 = 10$ euros, the risk-free interest rate is 8% (per year).

Compute the initial value of the Lookback option considered. What is the probability (risk-neutral) that its payoff will be strictly positive?

**Solution** The data provide $u = 1.5$ and $d = 0.5$ as parameters of the binomial model and $S_0 = 10$ as initial stock price. In the present setting, the underlying's dynamics can be described as follows:

$$
\begin{array}{ccccccccc}
 & & & & & & & & 50.625 \\
 & & & & & & & \nearrow & \\
 & & & & & & 33.75 & & \\
 & & & & & \nearrow & & \searrow & \\
 & & & & 22.5 & & & & 16.875 \\
 & & & \nearrow & & \searrow & & \nearrow & \\
 & & 15 & & & & 11.25 & & \\
 & \nearrow & & \searrow & & \nearrow & & \searrow & \\
S_0 = 10 & & & & 7.5 & & & & 5.625 \\
 & \searrow & & \nearrow & & \searrow & & \nearrow & \\
 & & 5 & & & & 3.75 & & \\
 & & & \searrow & & \nearrow & & \searrow & \\
 & & & & 2.5 & & & & 1.875 \\
 & & & & & \searrow & & \nearrow & \\
 & & & & & & 1.25 & & \\
 & & & & & & & \searrow & \\
 & & & & & & & & 0.625 \\
 & & & & & & & & \\
---- & & ---- & & ---- & & ---- & & ---- \\
t = 0 & & t = 1 & & t = 2 & & t = 3 & & T = 4 \\
 & & \text{(2 months)} & & \text{(4 months)} & & \text{(6 months)} & & \text{(8 months)}
\end{array}
$$

In order to simplify the notation, we shall denote as follows the nodes of the tree just described:

$$
\begin{array}{ccccccccc}
 & & & & & & & & M \\
 & & & & & & & \nearrow & \\
 & & & & & & G & & \\
 & & & & & \nearrow & & \searrow & \\
 & & & & D & & & & N \\
 & & & \nearrow & & \searrow & & \nearrow & \\
 & & B & & & & H & & \\
 & \nearrow & & \searrow & & \nearrow & & \searrow & \\
A & & & & E & & & & O \\
 & \searrow & & \nearrow & & \searrow & & \nearrow & \\
 & & C & & & & I & & \\
 & & & \searrow & & \nearrow & & \searrow & \\
 & & & & F & & & & P \\
 & & & & & \searrow & & \nearrow & \\
 & & & & & & L & & \\
 & & & & & & & \searrow & \\
 & & & & & & & & Q
\end{array}
$$

We recall that the payoff of a Lookback Call option is $(S_T - S_{\min})$ , where $S_{\min}$ denotes the minimum value assumed by the underlying during the option lifetime. The initial value of this option (written on an underlying with a binomial dynamics) is then given by:

$$F_{Cm}(0) = \frac{1}{(1+r)^T} E_Q [S_T - S_{\min}] , \tag{9.22}$$

where $Q$ is the risk-neutral probability measure.

Moreover, since $S_{\min}$ depends on the "trajectory" of the underlying value process, we must examine the contribution of each single path followed by the underlying along the binomial tree.

Before considering all possible final nodes, let us compute the risk-neutral probability measure with respect to which the expectation must be calculated. As usual, in the present binomial setting, $q_u$ is given by:

$$q_u = \frac{(1+r)^{2/12} - d}{u - d} = \frac{(1.08)^{2/12} - 0.5}{1.5 - 0.5} = 0.513$$

and $q_d = 1 - q_u = 0.487$.

Now we must consider all possible paths arriving at the final nodes M, N, O, P and Q.

**final node: M** If the underlying value at maturity was 50.625 (node M), then the only possible path would be that described by a growth of the underlying value at each time step. In this case we should have

$$\left.\begin{array}{r} S_{\min} = 10 \\ S_T = 50.625 \end{array}\right\} \quad \Longrightarrow \quad S_T - S_{\min} = 40.625.$$

The probability that this occurs is $q_u^4 = 0.0693$.

**final node: Q** If the underlying value at maturity was 0.625 (node Q), then the only possible path would be that described by a fall of the underlying value at each time step. In this case we should have

$$\left.\begin{array}{r} S_{\min} = 0.625 \\ S_T = 0.625 \end{array}\right\} \quad \Longrightarrow \quad S_T - S_{\min} = 0.$$

The probability that this occurs is $q_d^4 = (1 - q_u)^4 = 0.0562$.

**final node: N** If the underlying value at maturity was 16.875 (node N), then there would be 4 possible paths to consider.

*First Path* The price dynamics is the following:

$$\begin{array}{ccccccccc}
 & & & & & & G & & \\
 & & & & & \nearrow & & \searrow & \\
 & & & & D & & & & N \\
 & & & \nearrow & & & & & \\
 & & B & & & & & & \\
 & \nearrow & & & & & & & \\
A & & & & & & & &
\end{array}$$

and the corresponding probability is $q_u^3(1 - q_u) = 0.0654$. In this case we have:

$$\left.\begin{array}{r} S_{\min} = 10 \\ S_T = 16.875 \end{array}\right\} \quad \Longrightarrow \quad S_T - S_{\min} = 6.875.$$

*Second Path* The price dynamics is the following:

$$\begin{array}{ccccccccc}
 & & & & D & & & & N \\
 & & & \nearrow & & \searrow & & \nearrow & \\
 & & B & & & & H & & \\
 & \nearrow & & & & & & & \\
A & & & & & & & &
\end{array}$$

with probability $q_u^3\,(1-q_u) = 0.0654$. In this case we have:

$$\left.\begin{aligned} S_{\min} &= 10 \\ S_T &= 16.875 \end{aligned}\right\} \quad\Longrightarrow\quad S_T - S_{\min} = 6.875.$$

*Third Path* The price dynamics is the following:

$$\begin{array}{ccccccccc}
 & & & & & & & & N \\
 & & & & & & & \nearrow & \\
 & & B & & & & H & & \\
 & \nearrow & & \searrow & & \nearrow & & & \\
A & & & & E & & & &
\end{array}$$

and the probability of this path is $q_u^3\,(1-q_u) = 0.0654$. In this case we have:

$$\left.\begin{aligned} S_{\min} &= 7.5 \\ S_T &= 16.875 \end{aligned}\right\} \quad\Longrightarrow\quad S_T - S_{\min} = 9.375.$$

*Fourth Path* The price dynamics is the following:

$$\begin{array}{ccccccccc}
 & & & & & & & & N \\
 & & & & & & & \nearrow & \\
 & & & & & & H & & \\
 & & & & & \nearrow & & & \\
A & & & & E & & & & \\
 & \searrow & & \nearrow & & & & & \\
 & & C & & & & & &
\end{array}$$

and the probability is $q_u^3\,(1-q_u) = 0.0654$. In this case we have:

$$\left.\begin{aligned} S_{\min} &= 5 \\ S_T &= 16.875 \end{aligned}\right\} \quad\Longrightarrow\quad S_T - S_{\min} = 11.875.$$

**final node: O** If the underlying value at maturity was 5.625 (node O), there would be 6 possible paths to consider.

*First Path* The price dynamics is the following:

$$\begin{array}{ccccccccc}
 & & & & D & & & & \\
 & & & \nearrow & & \searrow & & & \\
 & & B & & & & H & & \\
 & \nearrow & & & & & & \searrow & \\
A & & & & & & & & O
\end{array}$$

and the probability is $q_u^2 (1 - q_u)^2 = 0.0560$. In this case we have:

$$\left.\begin{array}{r}S_{\min} = 5.625\\ S_T = 5.625\end{array}\right\} \implies S_T - S_{\min} = 0.$$

*Second Path* The price dynamics is the following:

$$\begin{array}{ccccccc} & & B & & H & & \\ & \nearrow & & \searrow \; \nearrow & & \searrow & \\ A & & & E & & & O \end{array}$$

and the probability is $q_u^2 (1 - q_u)^2 = 0.0560$. In this case we have:

$$\left.\begin{array}{r}S_{\min} = 5.625\\ S_T = 5.625\end{array}\right\} \implies S_T - S_{\min} = 0.$$

*Third Path* The price dynamics is the following:

$$\begin{array}{ccccccc} A & & & E & & & O \\ & \searrow & & \nearrow \; \searrow & & \nearrow & \\ & & C & & I & & \end{array}$$

and the probability is $q_u^2 (1 - q_u)^2 = 0.0560$. In this case we have:

$$\left.\begin{array}{r}S_{\min} = 3.75\\ S_T = 5.625\end{array}\right\} \implies S_T - S_{\min} = 1.875.$$

*Fourth Path* The price dynamics is the following:

$$\begin{array}{ccccccc} A & & & & & & O \\ & \searrow & & & & \nearrow & \\ & & C & & I & & \\ & & & \searrow \; \nearrow & & & \\ & & & F & & & \end{array}$$

and the probability is $q_u^2 (1 - q_u)^2 = 0.0560$. In this case we have:

$$\left.\begin{array}{r}S_{\min} = 2.5\\ S_T = 5.625\end{array}\right\} \implies S_T - S_{\min} = 3.125.$$

*Fifth Path* The price dynamics is the following:

$$
\begin{tikzcd}
 & B \arrow[dr] & & & \\
A \arrow[ur] & & E \arrow[dr] & & O \\
 & & & I \arrow[ur] &
\end{tikzcd}
$$

and the probability is $q_u^2\,(1-q_u)^2 = 0.0560$. In this case we have:

$$
\left.\begin{array}{r} S_{\min} = 3.75 \\ S_T = 5.625 \end{array}\right\} \implies S_T - S_{\min} = 1.875.
$$

*Sixth Path* The price dynamics is the following:

$$
\begin{tikzcd}
 & & & H \arrow[dr] & \\
A \arrow[dr] & & E \arrow[ur] & & O \\
 & C \arrow[ur] & & &
\end{tikzcd}
$$

and the probability that such occurs is $q_u^2\,(1-q_u)^2 = 0.0560$. In this case we have:

$$
\left.\begin{array}{r} S_{\min} = 5 \\ S_T = 5.625 \end{array}\right\} \implies S_T - S_{\min} = 0.625.
$$

**final node: P** If the underlying value at maturity was 1.875 (node P), there would be 4 possible paths to consider.

*First Path* The price dynamics is the following:

$$
\begin{tikzcd}
 & B \arrow[dr] & & & \\
A \arrow[ur] & & E \arrow[dr] & & \\
 & & & I \arrow[dr] & \\
 & & & & P
\end{tikzcd}
$$

and the probability is $q_u\,(1-q_u)^3 = 0.0593$. In this case we have:

$$\left.\begin{array}{r} S_{\min} = 1.875 \\ S_T = 1.875 \end{array}\right\} \quad \Longrightarrow \quad S_T - S_{\min} = 0.$$

*Second Path* The price dynamics is the following:

$$\begin{array}{ccccccccc} A & & & & & & & & \\ & \searrow & & & & & & & \\ & & C & & & & & & \\ & & & \searrow & & & & & \\ & & & & F & & & & P \\ & & & & & \searrow & & \nearrow & \\ & & & & & & L & & \end{array}$$

and the probability is $q_u\,(1-q_u)^3 = 0.0593$. In this case we have:

$$\left.\begin{array}{r} S_{\min} = 1.25 \\ S_T = 1.875 \end{array}\right\} \quad \Longrightarrow \quad S_T - S_{\min} = 0.625.$$

*Third Path* The price dynamics is the following:

$$\begin{array}{ccccccccc} A & & & & & & & & \\ & \searrow & & & & & & & \\ & & C & & & & I & & \\ & & & \searrow & & \nearrow & & \searrow & \\ & & & & F & & & & P \end{array}$$

and the probability is $q_u\,(1-q_u)^3 = 0.0593$. In this case we have:

$$\left.\begin{array}{r} S_{\min} = 1.875 \\ S_T = 1.875 \end{array}\right\} \quad \Longrightarrow \quad S_T - S_{\min} = 0.$$

*Fourth Path* The price dynamics is the following:

$$\begin{array}{ccccccccc} A & & & & E & & & & \\ & \searrow & & \nearrow & & \searrow & & & \\ & & C & & & & I & & \\ & & & & & & & \searrow & \\ & & & & & & & & P \end{array}$$

and the probability is $q_u\,(1-q_u)^3 = 0.0593$. In this case we have:

$$\left.\begin{array}{r} S_{\min} = 1.875 \\ S_T = 1.875 \end{array}\right\} \quad \Longrightarrow \quad S_T - S_{\min} = 0.$$

By summing up the contributions of all cases, we conclude that the Lookback Call option value at time $t = 0$ is given by:

$$\begin{aligned} F_{Cm}\,(0) = {} & \frac{1}{(1+r)^{8/12}}[q_u^4 \cdot 40.625 \\ & + q_u^3\,(1-q_u)\,(6.875 + 6.875 + 9.375 + 11.875) \\ & + q_u^2\,(1-q_u)^2\,(0 + 0 + 1.875 + 3.125 + 1.875 + 0.625) \\ & + q_u\,(1-q_u)^3\,(0 + 0.625 + 0 + 0) + (1-q_u)^4 \cdot 0] \\ = {} & 5.561 \text{ euros.} \end{aligned}$$

Moreover, we can easily find that the (risk-neutral) probability that the payoff of the Lookback Call will be strictly positive is given by:

$$Q\,(\{\text{payoff } > 0\}) = q_u^4 + 4q_u^3\,(1-q_u) + 4q_u^2\,(1-q_u)^2 + q_u\,(1-q_u)^3 = 0.6142.$$

**Exercise 9.5** With the same data of the previous exercise and again in a binomial setting, compute the value at time $t = 0$ of an average strike Call and the (risk-neutral) probability that its payoff will be strictly positive.

**Solution** An "average strike" option is an Asian option with payoff at maturity explicitly depending on the average of the values assumed by the underlying asset during the option lifetime, according to the following expression:

$$F_{AS}(S_T, S_{med}, T) = (S_T - S_{med})^+ = \max\{S_T - S_{med}; 0\}.$$

We shall assume that the average considered is the arithmetic mean, which in our binomial setting with 4 time steps is:

$$S_{med} = \frac{\sum_{t=1}^{4} S_t}{4}. \tag{9.23}$$

By recalling the scheme provided in the previous exercise we have to consider the following possible paths.

**final node: M** If the underlying value at maturity was 50.625 (node M), then the only possible path for the underlying price would be that of a growth at each time step. In this case we have:

$$\left.\begin{array}{r} S_{med} = 30.469 \\ S_T = 50.625 \end{array}\right\} \implies (S_T - S_{med})^+ = 20.156.$$

The probability that this occurs is $q_u^4 = 0.06935$.

**final node: Q** If the underlying value at maturity was 0.625 (node Q), then the only possible path for the underlying price would be that of a fall at each time step. In this case we have:

$$\left.\begin{array}{r} S_{med} = 2.344 \\ S_T = 0.625 \end{array}\right\} \implies (S_T - S_{med})^+ = 0.$$

The probability that this occurs is $q_d^4 = (1 - q_u)^4 = 0.0562$.

**final node: N** If the underlying value at maturity was 16.875 (node N), there would be 4 possible paths for the underlying price.

*First Path* The price dynamics is the following:

$$\begin{array}{ccccccccc}
 & & & & & & G & & \\
 & & & & & \nearrow & & \searrow & \\
 & & & & D & & & & N \\
 & & & \nearrow & & & & & \\
 & & B & & & & & & \\
 & \nearrow & & & & & & & \\
A & & & & & & & &
\end{array}$$

and the probability of this is $q_u^3\,(1 - q_u) = 0.0654$. In this case we have:

$$\left.\begin{array}{r} S_{med} = 22.031 \\ S_T = 16.875 \end{array}\right\} \implies (S_T - S_{med})^+ = 0.$$

*Second Path* The price dynamics is the following:

$$\begin{array}{ccccccccc}
 & & & & D & & & & N \\
 & & & \nearrow & & \searrow & & \nearrow & \\
 & & B & & & & H & & \\
 & \nearrow & & & & & & & \\
A & & & & & & & &
\end{array}$$

and the probability is $q_u^3(1-q_u) = 0.0654$. In this case we have:

$$\left.\begin{array}{r} S_{med} = 16.406 \\ S_T = 16.875 \end{array}\right\} \Longrightarrow \quad (S_T - S_{med})^+ = 0.469.$$

*Third Path* The price dynamics is the following:

$$\begin{array}{ccccccccc}
 & & & & & & & & N \\
 & & & & & & & \nearrow & \\
 & & B & & & & H & & \\
 & \nearrow & & \searrow & & \nearrow & & & \\
A & & & & E & & & &
\end{array}$$

and the probability is $q_u^3(1-q_u) = 0.0654$. In this case we have:

$$\left.\begin{array}{r} S_{med} = 12.656 \\ S_T = 16.875 \end{array}\right\} \Longrightarrow \quad (S_T - S_{med})^+ = 4.219.$$

*Fourth Path* The price dynamics is the following:

$$\begin{array}{ccccccccc}
 & & & & & & & & N \\
 & & & & & & & \nearrow & \\
 & & & & & & H & & \\
 & & & & & \nearrow & & & \\
A & & & & E & & & & \\
 & \searrow & & \nearrow & & & & & \\
 & & C & & & & & &
\end{array}$$

and the probability is $q_u^3(1-q_u) = 0.0654$. In this case we have:

$$\left.\begin{array}{r} S_{med} = 10.156 \\ S_T = 16.875 \end{array}\right\} \Longrightarrow \quad (S_T - S_{med})^+ = 6.719.$$

**final node: O** If the underlying value at maturity was 5.625 (node O), then there would be 6 possible paths for the underlying price.

*First Path* The price dynamics is the following:

$$\begin{array}{ccccccccc}
 & & & & D & & & & \\
 & & & \nearrow & & \searrow & & & \\
 & & B & & & & H & & \\
 & \nearrow & & & & & & \searrow & \\
A & & & & & & & & O
\end{array}$$

and the probability is $q_u^2\,(1-q_u)^2 = 0.0560$. In this case we have:

$$\left.\begin{array}{r} S_{med} = 13.594 \\ S_T = 5.625 \end{array}\right\} \implies (S_T - S_{med})^+ = 0.$$

*Second Path* The price dynamics is the following:

$$\begin{array}{ccccccccc} & & B & & & & H & & \\ & \nearrow & & \searrow & & \nearrow & & \searrow & \\ A & & & & E & & & & O \end{array}$$

and the probability is $q_u^2\,(1-q_u)^2 = 0.0560$. In this case we have:

$$\left.\begin{array}{r} S_{med} = 9.844 \\ S_T = 5.625 \end{array}\right\} \implies (S_T - S_{med})^+ = 0.$$

*Third Path* The price dynamics is the following:

$$\begin{array}{ccccccccc} A & & & & E & & & & O \\ & \searrow & & \nearrow & & \searrow & & \nearrow & \\ & & C & & & & I & & \end{array}$$

and the probability is $q_u^2\,(1-q_u)^2 = 0.0560$. In this case we have:

$$\left.\begin{array}{r} S_{med} = 5.469 \\ S_T = 5.625 \end{array}\right\} \implies (S_T - S_{med})^+ = 0.156.$$

*Fourth Path* The price dynamics is the following:

$$\begin{array}{ccccccccc} A & & & & & & & & O \\ & \searrow & & & & & & \nearrow & \\ & & C & & & & I & & \\ & & & \searrow & & \nearrow & & & \\ & & & & F & & & & \end{array}$$

and the probability is $q_u^2\,(1-q_u)^2 = 0.0560$. In this case we have:

$$\left.\begin{array}{r} S_{med} = 4.219 \\ S_T = 5.625 \end{array}\right\} \implies (S_T - S_{med})^+ = 1.406.$$

*Fifth Path* The price dynamics is the following:

$$\begin{tikzcd}[row sep=small, column sep=small]
& B \arrow[dr] & & & \\
A \arrow[ur] & & E \arrow[dr] & & O \\
& & & I \arrow[ur] &
\end{tikzcd}$$

and the probability is $q_u^2 (1 - q_u)^2 = 0.0560$. In this case we have:

$$\left. \begin{array}{r} S_{med} = 7.969 \\ S_T = 5.625 \end{array} \right\} \implies (S_T - S_{med})^+ = 0.$$

*Sixth Path* The price dynamics is the following:

$$\begin{tikzcd}[row sep=small, column sep=small]
& & & H \arrow[dr] & \\
A \arrow[dr] & & E \arrow[ur] & & O \\
& C \arrow[ur] & & &
\end{tikzcd}$$

and the probability is $q_u^2 (1 - q_u)^2 = 0.0560$. In this case we have:

$$\left. \begin{array}{r} S_{med} = 7.344 \\ S_T = 5.625 \end{array} \right\} \implies (S_T - S_{med})^+ = 0.$$

**final node: P** If the underlying value at maturity was 1.875 (node P), there would be 4 possible paths for the underlying price.

*First Path* The price dynamics is the following:

$$\begin{tikzcd}[row sep=small, column sep=small]
& B \arrow[dr] & & & \\
A \arrow[ur] & & E \arrow[dr] & & \\
& & & I \arrow[dr] & \\
& & & & P
\end{tikzcd}$$

and the probability is $q_u (1 - q_u)^3 = 0.0593$. In this case we have:

$$\left. \begin{array}{r} S_{med} = 7.031 \\ S_T = 1.875 \end{array} \right\} \implies (S_T - S_{med})^+ = 0.$$

*Second Path* The price dynamics is the following:

$$\begin{array}{ccccccccc}
A &&&&&&&& \\
& \searrow &&&&&&& \\
&& C &&&&&& \\
&&& \searrow &&&&& \\
&&&& F &&&& P \\
&&&&& \searrow && \nearrow & \\
&&&&&& L && 
\end{array}$$

and the probability is $q_u\,(1-q_u)^3 = 0.0593$. In this case we have:

$$\left.\begin{array}{r} S_{med} = 2.656 \\ S_T = 1.875 \end{array}\right\} \quad \Longrightarrow \quad (S_T - S_{med})^+ = 0.$$

*Third Path* The price dynamics is the following:

$$\begin{array}{ccccccccc}
A &&&&&&&& \\
& \searrow &&&&&&& \\
&& C &&&& I && \\
&&& \searrow && \nearrow && \searrow & \\
&&&& F &&&& P
\end{array}$$

and the probability is $q_u\,(1-q_u)^3 = 0.0593$. In this case we have:

$$\left.\begin{array}{r} S_{med} = 3.281 \\ S_T = 1.875 \end{array}\right\} \quad \Longrightarrow \quad (S_T - S_{med})^+ = 0.$$

*Fourth Path* The price dynamics is the following:

$$\begin{array}{ccccccccc}
A &&&& E &&&& \\
& \searrow && \nearrow && \searrow &&& \\
&& C &&&& I && \\
&&&&&&& \searrow & \\
&&&&&&&& P
\end{array}$$

and the probability is $q_u\,(1-q_u)^3 = 0.0593$. In this case we have:

$$\left.\begin{array}{r} S_{med} = 4.531 \\ S_T = 1.875 \end{array}\right\} \quad \Longrightarrow \quad (S_T - S_{med})^+ = 0.$$

The initial value of the option considered is then given by:

$$\begin{aligned}
&F_{AS}\ (S_T; S_{med}; 0)\\
=\ &\tfrac{1}{(1.08)^{8/12}}[0.0693 \cdot 20.156 + 0.0654 \cdot (0.469 + 4.219 + 6.719)\\
&+0.0560 \cdot (0.156 + 1.406) + 0.0593 \cdot 0 + 0.0562 \cdot 0]\\
=\ &2.26.
\end{aligned}$$

Moreover, we can easily find that the (risk-neutral) probability that the payoff of the average strike Call will be strictly positive is given by:

$$\begin{aligned}
Q\left(\{\text{payoff } > 0\}\right) &= q_u^4 + 3q_u^3\left(1 - q_u\right) + 2q_u^2\left(1 - q_u\right)^2\\
&= 0.0693 + 3 \cdot 0.0654 + 2 \cdot 0.0560 = 0.3775.
\end{aligned}$$

**Exercise 9.6** Compute the value at time $t = 0$ of a geometric Asian option, of the average-rate kind, written on an underlying stock whose price dynamics is described by a geometric Brownian motion with diffusion coefficient $\sigma$. The risk-free interest rate is $r$, the strike $K$.

**Solution** We first remark that the payoff of a geometric Asian option of average-rate type can be written as follows:

$$C_{AR}(T, \hat{S}_T) = \max\{\hat{S}_T - K; 0\} = \max\{S_0 \exp(\bar{X}_T) - K; 0\}, \tag{9.24}$$

where $\hat{S}_T$ denotes the geometric average of the values assumed by the underlying during the time interval $[0, T]$, while $\bar{X}_T$ denotes the arithmetic mean of the values assumed by the log-returns during the same time interval. The arithmetic average of the log-returns can be easily derived when the underlying price dynamics is described by a geometric Brownian motion. If the underlying price evolves as follows:

$$dS_t = rS_t dt + \sigma S_t dW_t,$$

i.e. $S_t = S_0 \exp(X_t)$ where

$$dX_t = (r - \sigma^2/2)dt + \sigma dW_t,$$

then, by integrating by parts, we can write immediately:

$$\int_0^t X_u du = tX_t - \int_0^t u dX(u). \tag{9.25}$$

By inserting the dynamics of $X$ and dividing by $t$, (9.25) becomes:

$$\bar{X}_t = \left(r - \frac{\sigma^2}{2}\right)\int_0^t \left(1 - \frac{u}{t}\right) du + \int_0^t \left(1 - \frac{u}{t}\right)\sigma dW_u.$$

The last equation implies that $\bar{X}_T$ is distributed as a normal random variable with mean:

$$E[\bar{X}_T] = \left(r - \frac{\sigma^2}{2}\right)\int_0^T \left(1 - \frac{u}{T}\right) du = \left(r - \frac{\sigma^2}{2}\right)\frac{T}{2}, \tag{9.26}$$

and variance:

$$V(\bar{X}_T) = \int_0^T \left(1 - \frac{u}{T}\right)^2 \sigma^2 du = \frac{\sigma^2}{3}T. \tag{9.27}$$

Now, since the value at time $t = 0$ of an average rate option is just the discounted value of its expected payoff (computed with respect to the risk-neutral measure):

$$C_{AR}(0, S_0) = e^{-rT} E\left[\max\{S_0 \exp(\bar{X}_T) - K; 0\}\right], \tag{9.28}$$

we can obtain this value by applying the Black-Scholes formula for European Call options, simply by comparing the payoffs and substituting the new values of the mean and the variance in the arguments. From the previous results, by remembering that $\hat{S}_t = e^{\bar{X}_t}$, we can write the stochastic differential equation satisfied by $\hat{S}_t$:

$$d\hat{S}_t = [(r - \frac{\sigma^2}{2})\frac{1}{2} + \frac{\sigma^2}{6}]\hat{S}_t dt + \frac{\sigma}{\sqrt{3}}\hat{S}_t dW_t. \tag{9.29}$$

In order to simplify the computation, we can consider (just as a computational trick) the extra term subtracted by $r$ in the previous equation like a constant continuous dividend rate:

$$(r - \frac{\sigma^2}{2})\frac{1}{2} + \frac{\sigma^2}{6} = r - q, \tag{9.30}$$

where:

$$q = \frac{r}{2} + \frac{\sigma^2}{2} - \frac{\sigma^2}{6} = \frac{r}{2} + \frac{\sigma^2}{12} = \frac{1}{2}[r + \frac{\sigma^2}{6}] \tag{9.31}$$

A direct substitution provides the following formula for the (geometric) average rate Call option:

$$C_{AR}(0, S_0) = S_0 e^{-\left(r+\frac{\sigma^2}{6}\right)\frac{T}{2}} N\left(d_* + \sigma\sqrt{\frac{T}{3}}\right) - Ke^{-rT} N(d_*), \tag{9.32}$$

where $d_*$ is defined as follows:

$$d_* \triangleq \frac{\ln(\frac{S_0}{K}) + \left(r - \frac{\sigma^2}{2}\right)\frac{T}{2}}{\sigma\sqrt{\frac{T}{3}}}. \tag{9.33}$$

**Exercise 9.7** Consider the following down-and-out Call option with maturity $T = 1$ year, strike $K = 30$ euros and barrier threshold $L = 25$ euros. The underlying price dynamics is described by a geometric Brownian motion with volatility $\sigma = 0.40$ (per year). The current price of the underlying is $S_0 = 30$ euros, while the risk-free interest rate is $r = 0.05$ (per year).

Construct a portfolio consisting of European options (approximately and statically) replicating the Barrier option considered.

**Solution** There is a large amount of arbitrariness in constructing a portfolio replicating the Barrier option. We start the process with a simple remark: if two portfolios have the same value on the contour of a region in the $(t, S)$-space, then they must have the same value inside the region. The idea is then to approximate the Barrier option on the contour of such region by means of European options with suitable maturities and strikes.

In the present case, since the barrier option is of down-and-out type, the domain is the strip delimited by vertical half-line $t = 0$ on the left, the interval $t \in [0, 1]$ on the horizontal line $S = 25$ below, and by the vertical half-line $t = T = 1$ on the right, and this is the contour on which we want to replicate the barrier option. The down-and-out Call takes the following values on these lines:

$$C(S, 1) = \max\{S - 30; 0\} \quad (\text{for } S > 25) \tag{9.34}$$

and

$$C(25, t) = 0 \quad (\text{for } 0 \leq t \leq 1). \tag{9.35}$$

The simplest way to replicate the option on the right contour ($t = 1$) is to buy a European Call option with strike $K_A = 30$ and maturity $T = 1$. We denote by A such an option and by $C_A$ its value.

To replicate the option on lower contour we can divide the time interval [0, 1] on the line $S = 25$ in three sub-intervals of 4 months (i.e. 1/3 of a year) each, and choose three different European options satisfying condition 9.35 for $t = 0, \frac{1}{3}, \frac{2}{3}$, respectively.

So, a second option (denoted by B) to be considered for replication on the lower contour can be a Put with strike $K_B = 25$ and maturity $T_B = 1$. Please, note that the value of this option vanishes on the previous contour considered, as it has to do, while a Call option with the same strike and maturity does not. By the Black-Scholes formula for European options (see (7.5)), the values of options A and B at $t = 2/3$ are $C_A(2/3, 25, 30, 1) = 0.86$ and $P_B(2/3, 25, 25, 1) = 2.07$ respectively, where the first and the second arguments denote the valuation point coordinates $(t, S)$ and the third and the fourth denote the strike and the maturity $(K, T)$.

In order to satisfy the condition (9.35) for $t = 1/3$ we need another option (vanishing at all the points previously considered): this third option (denoted by C) can be a Put option with maturity $T_C = 2/3$ and strike $K_C = 25$ and its value at $t = 1/3$, $S = 25$ is $P_C(1/3, 25, 25, 2/3) = 2.07$.

Finally, we need to satisfy the condition (9.35) at time $t = 0$. To this end we choose a fourth European option of type Call (denoted by D) with maturity $T_D = 1/3$ and strike $K_D = 25$ and its value at $t = 0$, $S = 25$ is $C_D(0, 25, 25, 1/3) = 2.48$.

The number of options of each kind to be held in the replicating portfolio is determined by the condition that the replicating portfolio assumes value zero at each of the points considered on the contour. So, by assuming that one unit of the option A is held in the replicating portfolio, the number $\beta$ of options B to buy (or sell) is obtained by imposing that at point $(t = 2/3, S = 25)$

$$C_A(2/3, 25, 30, 1) + \beta P_B(2/3, 25, 25, 1) = 0.$$

Since the values of options A and B at $(t = 2/3, S = 25)$ are 0.86 and 2.07, respectively, this condition provides for $\beta$ the value $\beta = -0.42$.

As far as the number of options C is concerned, we can repeat the same argument and impose the condition at $(t = 1/3, S = 25)$:

$$C_A(1/3, 25, 30, 1) + \beta P_B(1/3, 25, 25, 1) + \gamma P_C(1/3, 25, 25, 2/3) = 0.$$

Since A, B, and C at $(S = 25, t = 1/3)$ are valued 1.83, 2.79 and 2.07, respectively, this gives $\gamma = -0.32$.

Finally, we can determine the number of options of type D by imposing the analogous condition at $(t = 0, S = 25)$:

$$C_A(0, 25, 30, 1) + \beta P_B(0, 25, 25, 1) + \gamma P_C(0, 25, 25, 2/3) + \delta C_D(0, 25, 25, 1/3) = 0.$$

Since the options' values are 2.70, 3.29, 2.79 and 2.48 for A, B, C, and D, respectively, the value of $\delta = -0.35$ follows.

The replicating portfolio has then the following composition:

$$(1, -0.42, -0.32, -0.35)$$

in options A, B, C, D, respectively.

## 9.3 Proposed Exercises

**Exercise 9.8** Compute the value at time $t = 0$ of a Chooser option written on an underlying stock whose price dynamics is described by a geometric Brownian motion, with strike $K = 20$ euros and maturities $T_1 = 6$ months and $T_2 = 1$ year. The initial value of the underlying is $S_0 = 20$ euros, the risk-free interest rate $r = 10\%$ (per year) and the volatility $\sigma = 20\%$ (per year).

**Exercise 9.9** Compute the value at time $t = 0$ of a Barrier option of "up-and-in" Call type, written on an underlying stock whose price dynamics is described by a geometric Brownian motion, with initial value $S_0 = 10$ euros, $\sigma^2 = 0.016$, $r = 0.04$, $K = 6$ euros and barrier value $U = 8$ euros. By using the relationship between Barrier and European Call options, find the initial price of the corresponding "up-and-out" Call.

**Exercise 9.10** Compute the value at time $t = 0$ of a Binary option of "asset or nothing" Call type, with the same parameters (except the strike $K$) of Exercise 9.1.

**Exercise 9.11** With the same data of Exercises 9.4 and 9.5, and again in a binomial setting, compute the initial price of a Lookback Call option (on the minimum) with maturity $T = 10$ months. Find, moreover, the initial value of the corresponding Put option.

**Exercise 9.12** With the same data of Exercises 9.4 and 9.5, and again in a binomial setting, compute the initial price of an average strike Call option, with a payoff at maturity expressed by $(S_T - S_{MG})^+ = \max\{S_T - S_{MG}; 0\}$, where $S_{MG}$ is the geometric mean of the values assumed by the underlying during the option lifetime, i.e. $S_{MG} = \sqrt[n]{\Pi_{i=1}^n S_i}$.

**Exercise 9.13** With the same data of Exercises 9.4 and 9.5, and again in a binomial setting, compute the price at time $t = 0$ of an average rate Call option, with a payoff at maturity expressed by $(S_{med} - K)^+ = \max\{S_{med} - K; 0\}$, where $S_{med}$ denotes the arithmetic mean of the values assumed by the underlying during the option lifetime and $K = 5$ euros.

**Exercise 9.14** Determine the payoff at maturity of a portfolio composed by the Lookback Call of Exercises 9.4 and by the corresponding Lookback Put option. Compute the value of this portfolio at time $t = 0$.

**Exercise 9.15** Compute the initial value of a geometric Asian option, of the average-rate type, with maturity $T = 1$ year, strike $K = 30$ euros, written on an underlying stock with dynamics described by a geometric Brownian motion and with initial value $S_0 = 36$ euros. The risk-free interest rate is $r = 0.04$ (per year) and the volatility $\sigma = 0.4$ (per year).

**Exercise 9.16** Compute the value at $t = 0$ of a (floating strike) Lookback Call option on the minimum, with maturity $T = 1$ year, written on an underlying stock with dynamics described by a geometric Brownian motion and with initial value $S_0 = 36$ euros. The risk-free interest rate is $r = 0.04$ (per year) and the volatility $\sigma = 0.4$ (per year).

Consider the binomial model with four time steps (each of 3 months) approximating the geometric Brownian motion above. After computing the Lookback option value at time $t = 0$ in this binomial setting, evaluate the binomial approximation's quality by comparing the two results obtained.

**Exercise 9.17** Compute the initial value of a down-and-out Call option with the same data of Exercise 9.7, except the barrier value (which is now a lower threshold), here assumed to be $L = 40$ euros, and construct a static replicating portfolio composed by 4 European options.

Do Call options give a better or worse replication than Put options?

**Exercise 9.18** Compute the initial value of a financial derivative with payoff $\{S_{\max} - S_{\min}\}$, i.e. with payoff represented by the difference between the maximum and the minimum of the values assumed by $S(t)$, $t \in [0, T]$. Hint: the financial derivative above can be replicated by a portfolio composed by a lookback put on the maximum (long position) and a lookback call on the minimum (short position).

# Chapter 10
# Interest Rate Models

## 10.1 Review of Theory

In the valuation problems presented in the previous chapters, the interest rate was assumed to be a deterministic parameter, constant in most cases. This is of course a crude approximation of the real situation where interest rates vary over time in an unpredictable way. This assumption can be reasonable only when dealing with financial contracts with short maturities (like options), while for financial derivatives with longer lifetime (like bonds) or any other fixed-income products, it can be quite misleading. It is necessary, then, to adopt stochastic models for interest-rate dynamics.

Let us start by defining the relevant quantities that we wish to describe.

A *zero-coupon bond* (ZCB) is a contract that guarantees the buyer one unit (by convention) of the reference currency at maturity $T$. Since the value of this contract depends both on the date at which it is written and on its maturity $T$, we will denote by $Z(t, T)$ its value at time $t$. If the zero-coupon bond guarantees the buyer the cash amount $N$ at maturity $T$, we will specify that the *zero-coupon bond's face value* is $N$. In this case, its value at time $t$ will be expressed by $N \cdot Z(t, T)$.

A *coupon bond* with face value $N$, maturity $T$ and periodically paid coupons (at the end of each semester, for example) at the annual rate $r_c$ is a contract that guarantees the buyer the cash amount $N$ at time $T$ and the coupons with value $c$ at the end of each period (in the case of one semester: $c = \frac{r_c}{2} \cdot N$) until maturity $T$.

We define *instantaneous forward rate with respect to maturity* $T$ the following quantity:

$$f(t, T) \triangleq -\frac{\partial \ln Z(t, T)}{\partial T} \tag{10.1}$$

and *short rate* the quantity $r_t \triangleq f(t, t)$.

 
E. Rosazza Gianin, C. Sgarra, *Mathematical Finance*,
La Matematica per il 3+2 149, https://doi.org/10.1007/978-3-031-28378-9_10

The bond valuation is then strictly related to the interest-rate dynamics. Without any further specification, in the following we shall refer to short rates. When dealing with the valuation problem for zero-coupon bonds the first difficulty is the incompleteness of a short-rate stochastic model, so the no-arbitrage requirement does not provide a unique price for a generic derivative (in our case the zero-coupon bond). Nevertheless, no arbitrage implies some consistency condition on the zero-coupon bonds' values (with different maturities) which allows to reduce the valuation problem to the final-value problem for a partial differential equation; the coefficient of this PDE must be determined according to some "exogenous" criterion.

If the short-rate dynamics is described by a model of the following kind:

$$dr_t = u(r_t, t)dt + w(r_t, t)dW_t, \tag{10.2}$$

then the zero-coupon value $Z(t, T)$ must satisfy the following PDE:

$$\frac{\partial Z}{\partial t}(t, T) + \frac{1}{2}w^2(t, T)\frac{\partial^2 Z}{\partial r^2}(t, T) + (u - \lambda w)(t, T)\frac{\partial Z}{\partial r}(t, T) - (rZ)(t, T) = 0$$

together with the final condition $Z(T, T) = 1$. The term $\lambda$ in the previous equation is called the *Market Price of Risk* and, by the consistency relation implied by the no-arbitrage condition mentioned above, it can be shown to be the same for all bonds available in the market considered. The computation of the market price of risk turns out to be equivalent to the determination of the risk-neutral measure, which is not unique any more due to market incompleteness, chosen by the market in order to assign prices to traded securities. Several different approaches can be followed in order to determine the coefficient $\lambda$: among these we just mention the general equilibrium framework. The most popular in practice, anyway, is the one based on a *"calibration"* procedure. In the following, we will assume that the coefficient $\lambda$ is known.

An explicit solution of the bond pricing PDE for short-rate models is available only in a few relevant cases. Among these the most relevant are those assuming the following general form:

$$Z(t, T) = \exp\{A(t, T) - r_t B(t, T)\}, \tag{10.3}$$

where $A$ and $B$ are deterministic functions of $t$ and $T$. These solutions are called *affine term structures*. It is possible to prove that a sufficient condition for the zero-coupon bond equation to admit a solution of the form (10.3) is that the coefficients $u, w$ of the stochastic differential equation describing the short-rate dynamics in (10.2) have the following form:

$$u(r, t) = \alpha(t)r + \beta(t) \tag{10.4}$$

$$w(r, t) = \sqrt{\gamma(t)r + \delta(t)}. \tag{10.5}$$

Moreover, the functions $A, B$ are related to the coefficients $\alpha, \beta, \gamma, \delta$ by the following ordinary differential equations (of Riccati type):

$$\begin{aligned} &\frac{\partial B(t,T)}{\partial t} + \alpha(t)B(t,T) - \frac{1}{2}\gamma(t)B^2(t,T) = -1, \; B(T,T) = 0 \\ &\frac{\partial A(t,T)}{\partial t} - \beta(t)B(t,T) + \frac{1}{2}\delta(t)B^2(t,T) = 0, \quad A(T,T) = 0. \end{aligned} \tag{10.6}$$

Among these models, we briefly recall the most popular (and easiest to handle) on which we will focus in view of their applications. In particular, we shall deal with the models proposed by Vasiček, Ho and Lee, Hull and White.

*Vasiček Model*

$$dr_t = a\,(b - r_t)\,dt + \sigma dW_t, \tag{10.7}$$

i.e. $\alpha(t) = -a$, $\beta(t) = ab$, $\gamma(t) = 0$ and $\delta(t) = \sigma^2$.

In this case, we have:

$$B(t,T) = \frac{1 - e^{-a(T-t)}}{a} \tag{10.8}$$

$$A(t,T) = \frac{[B(t,T) - (T-t)]\,(a^2 b - \sigma^2/2)}{a^2} - \frac{\sigma^2 B^2(t,T)}{4a}. \tag{10.9}$$

The explicit solution of the stochastic differential Eq. (10.7) describing the short-rate dynamics is the following:

$$r_t = r_0 e^{-at} + b\left(1 - e^{-at}\right) + \sigma \int_0^t e^{-a(t-s)} dW_s. \tag{10.10}$$

Moreover:

$$E\,[r_t] = r_0 e^{-at} + b\left(1 - e^{-at}\right). \tag{10.11}$$

*Ho-Lee Model*

$$dr_t = \vartheta_t dt + \sigma dW_t, \tag{10.12}$$

i.e. $\alpha(t) = 0$, $\beta(t) = \vartheta_t$, $\gamma(t) = 0$ and $\delta(t) = \sigma^2$.

In this case, we have that:

$$B(t,T) = T - t \tag{10.13}$$

$$A(t,T) = \int_t^T \vartheta_s (s - T) ds + \frac{\sigma^2 (T-t)^3}{6}. \tag{10.14}$$

*Hull-White Model*

$$dr_t = (\vartheta_t - ar_t)\,dt + \sigma dW_t, \tag{10.15}$$

i.e. $\alpha(t) = -a$, $\beta(t) = \vartheta_t$, $\gamma(t) = 0$ and $\delta(t) = \sigma^2$.

In this case, we have that:

$$B(t,T) = \frac{1 - e^{-a(T-t)}}{a} \tag{10.16}$$

$$A(t,T) = \int_t^T \left[\frac{1}{2}\sigma^2 B^2(s,T) - \vartheta_s B(s,T)\right] ds. \tag{10.17}$$

By a suitable calibration procedure it is possible to obtain the functions $\vartheta_t$ appearing in the Ho-Lee and Hull-White valuation formulas:

$$\vartheta_t^{HL} = \frac{\partial f(0,t)}{\partial t} + \sigma^2 t \tag{10.18}$$

$$\vartheta_t^{HW} = \frac{\partial f(0,t)}{\partial t} + af(0,t) + \frac{\sigma^2}{2a}(1 - e^{-2at}). \tag{10.19}$$

In both expressions $f(0,t)$ denotes the instantaneous forward rate at time $t = 0$.

When the above models are taken as pricing framework, the zero-coupon bonds can be easily valuated, and so are the coupon bonds and the bond options. We shall examine in a more detailed way these valuation problems in the exercises.

The formulas providing the value of a Call option written on a zero-coupon bond with face value $N$ are, for the three models presented, the following.

*Price of a Call Option on a ZCB in the Vasiček Model*

$$C(t,T,K,T^*) = N \cdot Z(t,T^*) \cdot N(d) - K \cdot Z(t,T) \cdot N(d - \sigma_p), \tag{10.20}$$

where $K$, $T$ denote the strike and the maturity of the Call option, respectively, while $T^*$ is the bond maturity, $N(\cdot)$ the cumulative distribution function of a standard normal random variable and the quantities $d$ and $\sigma_p$ are defined as follows:

$$d = \frac{1}{\sigma_p} \ln\left[\frac{N \cdot Z(t,T^*)}{K \cdot Z(t,T)}\right] + \frac{1}{2}\sigma_p \tag{10.21}$$

$$\sigma_p = \frac{1 - e^{-a(T^*-T)}}{a}\sqrt{\frac{\sigma^2}{2a}\left[1 - e^{-2a(T-t)}\right]}. \tag{10.22}$$

*Price of a Call Option on a ZCB in the Ho-Lee Model*

$$C(t,T,K,T^*) = N \cdot Z(t,T^*) \cdot N(d) - K \cdot Z(t,T) \cdot N(d - \sigma_p), \tag{10.23}$$

where

$$d = \frac{1}{\sigma_p} \ln \left[ \frac{N \cdot Z(t, T^*)}{K \cdot Z(t, T)} \right] + \frac{1}{2}\sigma_p \tag{10.24}$$

$$\sigma_p = \sigma(T^* - T)\sqrt{T}. \tag{10.25}$$

*Price of a Call Option on a ZCB in the Hull-White Model*

$$C(t, T, K, T^*) = N \cdot Z(t, T^*) \cdot N(d) - K \cdot Z(t, T) \cdot N(d - \sigma_p), \tag{10.26}$$

where

$$d = \frac{1}{\sigma_p} \ln \left[ \frac{N \cdot Z(t, T^*)}{K \cdot Z(t, T)} \right] + \frac{1}{2}\sigma_p \tag{10.27}$$

$$\sigma_p = \frac{1 - e^{-a(T^* - T)}}{a} \sqrt{\frac{\sigma^2}{2a} \left[1 - e^{-2a(T-t)}\right]}. \tag{10.28}$$

Strictly analogous formulas exist for European Put options. These formulas can be proved quite easily with the help of a useful technique, called the *Change of Numéraire*. The latter provides a solution to several valuation problems arising in several, different contexts, like interest-rate models, currency derivatives, and some exotic options. We briefly recall the main ideas underlying this powerful method. Roughly speaking, a *Numéraire* is the unit in which all assets in a market model are expressed. This can be a currency unit, or a money market account, which is simply given by $B_t = \exp\left(\int_0^t r_u du\right)$ in market models with a deterministic short rate. Changing numéraire means that a new unit is assumed to express financial values, and the new numéraire does not have to be constant, nor deterministic, but it can be described by a stochastic dynamics; the only requirement it must fulfill is to be strictly positive for all $t$. The fundamental theorems of asset pricing are usually expressed with respect to the numéraire $B_t$ (bond) in such a way that, in order to avoid arbitrage opportunities, all asset price processes in the market models considered (including derivatives), divided by $B_t$, must be local martingales with respect to the risk-neutral measure. The same results can be expressed in terms of a different numéraire, but the risk-neutral measure for the market model will be different as well. It can be proved that, under suitable integrability conditions, the relationship between the two risk-neutral measures is provided by the following formula:

$$L_t = \left. \frac{dQ^1}{dQ^0} \right|_{\mathscr{F}_t} = \frac{S_t^1}{S_t^0} \frac{S_0^0}{S_0^1}, \tag{10.29}$$

where $L_t$ denotes the Radon-Nikodym derivative of the risk-neutral measure $Q^1$ (with $S^1$ as numéraire) with respect to the risk-neutral measure $Q^0$ (with $S^0$ as numéraire) at time $t$.

A special role in interest rate models is played by the *T-forward measure*, that is the risk-neutral measure adopting the value $p(t, T)$ $(= Z(t, T))$ of a zero-coupon bond with maturity $T$ as numéraire. We shall illustrate in the exercises how this method can be used to obtain pricing formulas.

Up to now we have focused our attention on short-rate models. In market practice, however, other relevant models play a role; among these, a very important one is the LIBOR (London InterBank Offer Rate) market model describing the dynamics of rates. The simple forward LIBOR rate at time $t$ for the time interval $[S, T]$ is defined by:

$$L(t, S, T) \triangleq \frac{p(t, S) - p(t, T)}{(T - S)p(t, T)}, \tag{10.30}$$

where $0 \leq S \leq T$ and $p(t, T)$ $(= Z(t, T))$ denotes the value at time $t$ of a zero-coupon bond with maturity $T$ (supposed to be quoted on the market). The LIBOR spot rate for the time interval $[S, T]$ is defined by:

$$L(S, T) \triangleq \frac{1 - p(S, T)}{(T - S)p(S, T)}, \tag{10.31}$$

and it is simply the forward rate for $t = S$.

If the dynamics of the LIBOR rate for each interval $[T_{i-1}, T_i]$ (where $T_i$ are the revision dates for the interest rate) is assumed to be described by a geometric Brownian motion, the well-known *Black Formula* provides an explicit value for the *Caplets*. These are financial contracts with the following payoff:

$$X_i \triangleq \alpha_i \cdot \max\left\{L(T_{i-1}, T_i) - R; 0\right\},$$

where $\alpha_i = T_i - T_{i-1}$ is the tenor and $R$ is the Cap rate. The Black Formula for Caplets is the following:

$$Capl_i(t) = \alpha_i \cdot p(t, T_i)\left[L(t, T_{i-1}, T_i) \cdot N(d_1) - R \cdot N(d_2)\right], \quad i = 1, ..., N, \tag{10.32}$$

where:

$$d_1 = \frac{1}{\sigma_i\sqrt{T_{i-1} - t}}\left[\ln\left(\frac{L(t, T_{i-1}, T_i)}{R}\right) + \frac{\sigma_i^2}{2}(T_{i-1} - t)\right]$$

$$d_2 = \frac{1}{\sigma_i\sqrt{T_{i-1} - t}}\left[\ln\left(\frac{L(t, T_{i-1}, T_i)}{R}\right) - \frac{\sigma_i^2}{2}(T_{i-1} - t)\right]$$

and $\sigma_i$ denotes the Black volatility for the $i$-th Caplet. The Black volatilities must be calibrated on the market values and, in order to simplify the treatment, in the exercises we shall assume that $\sigma_i$ take a constant value (we shall assume a flat structure for the Black volatilities).

The reader interested in a rigorous and systematic treatment of the valuation problems for interest rate derivatives should consult the books by Björk [6], Brigo and Mercurio [9], El Karoui [18] and Mikosch [32].

## 10.2 Solved Exercises

**Exercise 10.1** Consider a short rate model of Vasiček type, that is $(r_t)_{t\geq 0}$ satisfying

$$dr_t = a\,(b - r_t)\,dt + \sigma dW_t,$$

with $a = 0.4$, $b = 0.01$, $\sigma = 0.2$ and initial value $r_0 = 4\%$ on an annual basis.

1.
   (a) Compute the value at time $t = 0$ of a zero-coupon bond with face value 40 euros and maturity $T^* = 2.5$ years.
   (b) Discuss whether there exists a constant interest rate value $r^*$ such that the value of a zero-coupon bond with face value 40 euros and maturity $T^* = 2.5$ coincides with the bond value considered in item 1.(a).

2. Consider a European Call option with maturity $T$ of 1 year and 2 months, strike $K = 36$ euros and written on a zero-coupon bond with face value 40 euros.
   (a) Compute the initial value of the European Call considered.
   (b) Compute the value of the same Call option of the previous item under the assumption that the short rate has constant value $r_e = E\,[r_T]$, where $T$ denotes the maturity of the Call option.

3. Consider now a coupon bond with face value $N^*$, maturity $T^* = 2.5$ years, with coupon paid at the end of each semester with annual rate of 8%.
   (a) Find the value of $N^*$ such that the zero-coupon bond of item 1.(a) coincides with the present value of the coupon bond, if this value is computed under the assumption of a constant short rate with value $r_0$.
   (b) Compute the value (at time $t = 0$) of a European Call option with maturity $T$ of 1 year and 2 months, strike $K = 36$ euros, written on the coupon bond with face value $N^*$ and maturity $T^* = 2.5$.
   (c) Compute the value (at time $t = 0$) of a European Call option with maturity $T$ of 1 year and 2 months, strike $K = 36$ euros and written on a zero-coupon bond with face value $N^*$.

4. Compute the value of the European Put options with the same parameters of the Call options considered in items 2.(a) and 3.(b).

**Solution**

1.

(a) The price of the zero-coupon bond (with unit face value and maturity $T^*$) at time $t$ in the Vasiček model is given by:

$$Z\left(t, T^*\right) = e^{A(t,T^*) - r_t B(t,T^*)},$$

where

$$B\left(t, T^*\right) = \frac{1 - e^{-a(T^*-t)}}{a}$$

and

$$A\left(t; T^*\right) = \frac{\left[B\left(t, T^*\right) - \left(T^* - t\right)\right]\left(a^2 b - \frac{\sigma^2}{2}\right)}{a^2} - \frac{\sigma^2 B^2\left(t, T^*\right)}{4a}.$$

To find the initial value of a zero-coupon bond with face value 40 euros and maturity $T^* = 2.5$ years, we need to compute the quantities $A\,(0; 2.5)$ and $B\,(0; 2.5)$.
Since

$$B\,(0; 2.5) = \frac{1 - e^{-0.4 \cdot 2.5}}{0.4} = 1.58$$

$$\begin{aligned} A\,(0; 2.5) &= \frac{\left[B\,(0; 2.5) - 2.5\right]\left((0.4)^2 \cdot 0.01 - \frac{(0.2)^2}{2}\right)}{(0.4)^2} - \frac{(0.2)^2\, B^2\,(0; 2.5)}{4 \cdot 0.4} \\ &= 0.043, \end{aligned}$$

the initial value of the zero-coupon bond with face value 40 euros is then given by:

$$N \cdot Z\,(0; 2.5) = 40 \cdot e^{A(0;2.5) - r_0 \cdot B(0;2.5)} = 40 \cdot e^{0.043 - 0.04 \cdot 1.58} = 39.2.$$

This means that we have to invest 39.2 euros today in order to get 40 euros in two-and-a-half years.

By the previous results, it follows that the initial value of a zero-coupon bond (with unit face value) is $Z(0; 2.5) = 0.980$ euros.

(b) We have to find a constant short rate value $r^*$ such that the price of a zero-coupon bond with face value 40 euros and maturity $T^* = 2.5$ years coincides with the value found in item (a).
This short rate $r^*$ must then satisfy the following equation:

$$N \cdot Z\left(0; T^*\right) = N \cdot e^{-r^* T^*}$$

or, equivalently, $Z(0; 2.5) = e^{-r^* \cdot 2.5}$. We get then:

$$r^* = -\frac{\ln Z(0; 2.5)}{2.5} = 0.008.$$

2. We focus now on a European Call option with maturity $T$ of 1 year and 2 months (that is $T = 7/6$ years) and strike $K = 36$, written on a zero-coupon bond with face value $N = 40$ euros and maturity $T^* = 2.5$ years.

(a) We recall that the initial value of a European Call option written on a zero-coupon bond with face value $N$ is given by:

$$C_0 = N \cdot Z\left(0; T^*\right) \cdot N(d) - K \cdot Z(0; T) \cdot N\left(d - \sigma_p\right), \qquad (10.33)$$

where $T^*$ and $T$ stand, respectively, for the maturity of the ZCB and of the option, and

$$d = \frac{1}{\sigma_p} \ln\left[\frac{N \cdot Z(0, T^*)}{K \cdot Z(0, T)}\right] + \frac{1}{2}\sigma_p \qquad (10.34)$$

$$\sigma_p = \frac{1 - e^{-a(T^* - T)}}{a} \sqrt{\frac{\sigma^2}{2a}\left[1 - e^{-2aT}\right]}. \qquad (10.35)$$

First of all, we remark that the previous formula extends the "classical" Black-Scholes formula and that the factor $N \cdot Z(0; T^*)$ simply represents the current value of the underlying security, while $K \cdot Z(0; T)$ represents the current value (at $t = 0$) of the strike. This happens just because $Z(0; s)$ represents the present value of a unit of currency paid at time $s$.
Now, we turn to the Call option valuation problem. Looking at formula (10.33), we can immediately observe that only $Z(0; T)$ must be computed, since the value of $Z(0; T^*)$ has been already computed in 1.(a).

By proceeding in the same way as in 1.(a), we obtain that, for maturity $T = 7/6$ years (i.e. 1 year and 2 months),

$$B(0; 7/6) = \frac{1 - e^{-0.4 \cdot \frac{7}{6}}}{0.4} = 0.9323$$

$$A(0; 7/6) = \frac{\left[B\left(0; \frac{7}{6}\right) - \frac{7}{6}\right]\left((0.4)^2 \cdot 0.01 - \frac{(0.2)^2}{2}\right)}{(0.4)^2} - \frac{(0.2)^2 B^2\left(0; \frac{7}{6}\right)}{4 \cdot 0.4}$$
$$= 0.0052.$$

The zero-coupon bond (with unit face value) price is then given by:

$$Z(0; 7/6) = e^{A(0;7/6) - r_0 \cdot B(0;7/6)} = e^{0.0052 - 0.04 \cdot 0.9323} = 0.9684.$$

Hence

$$\sigma_p = \frac{1 - e^{-0.4(2.5 - 7/6)}}{0.4} \sqrt{\frac{(0.2)^2}{2 \cdot 0.4}\left[1 - e^{-2 \cdot 0.4 \cdot 7/6}\right]} = 0.18$$

$$d = \frac{\ln\left[\frac{40 \cdot Z(0;2.5)}{36 \cdot Z(0;7/6)}\right]}{0.18} + \frac{0.18}{2} = 0.741.$$

As a direct consequence, the initial value of the European Call with maturity $T = 7/6$, strike $K = 36$, written on the zero-coupon bond with face value $N = 40$ and maturity $T^* = 2.5$ years is

$$C_0 = 40 \cdot 0.980 \cdot N(0.741) - 36 \cdot 0.9684 \cdot N(0.741 - 0.18) = 5.365 \text{ euros}.$$

(b) Assume that the short rate is constant with value $r_e = E[r_T]$ and consider a European Call option with maturity $T = 7/6$ years, strike $K = 36$ euros, written on a zero-coupon bond with face value $N = 40$ euros and maturity $T^* = 2.5$ years.
We recall that for the Vasiček model the explicit solution of the stochastic differential equation describing the short-rate dynamics is the following:

$$r_t = r_0 e^{-at} + b\left(1 - e^{-at}\right) + \sigma \int_0^t e^{-a(t-s)} dW_s.$$

Moreover, $E[r_t] = r_0 e^{-at} + b\left(1 - e^{-at}\right) = r_0 + (b - r_0)\left(1 - e^{-aT}\right)$. This implies that:

$$\begin{aligned} r_e &= E[r_T] = r_0 + (b - r_0)\left(1 - e^{-aT}\right) \\ &= 0.04 + (0.01 - 0.04)\left(1 - e^{-0.4\cdot\frac{7}{6}}\right) = 0.029. \end{aligned}$$

At maturity, the Call option payoff is then the following:

$$\begin{aligned} \left(40 \cdot Z^{r_e}\left(T; T^*\right) - 36\right)^+ &= \left(40 \cdot e^{-r_e(T^* - T)} - 36\right)^+ \\ &= \left(40 \cdot e^{-0.029\cdot(2.5-7/6)} - 36\right)^+ = 2.483 \end{aligned}$$

and the value of the Call option with this payoff is its present value (computed by using the short rate $r_e$), i.e.

$$\begin{aligned} C_0^{r_e} &= Z^{r_e}(0; T) \cdot \left(40 \cdot Z^{r_e}\left(T; T^*\right) - 36\right)^+ \\ &= e^{-r_e T} \cdot \left(40 \cdot Z^{r_e}\left(T; T^*\right) - 36\right)^+ = 2.40 \text{ euros.} \end{aligned}$$

3. We focus now on a coupon bond with face value $N^*$, maturity $T = 2.5$ and coupons paid at the end of each semester with annual rate 8%.

   (a) We need to compute the value $N^*$ that makes the price of the zero-coupon bond of item 1.(a) equal to the present value of the coupon bond, when the present value is computed assuming a constant short rate $r_0$. Since this value is:

   $$\frac{4N^*}{100}\left[e^{-r_0/2} + e^{-r_0} + e^{-3r_0/2} + e^{-2r_0}\right] + \frac{104N^*}{100} e^{-5r_0/2},$$

   we have to find $N^*$ as a solution of the following equation:

   $$\frac{4N^*}{100}\left[e^{-r_0/2} + e^{-r_0} + e^{-3r_0/2} + e^{-2r_0}\right] + \frac{104N^*}{100} e^{-5r_0/2} = 40 \cdot Z(0; 2.5).$$

   We obtain then:

   $$N^* = 100 \cdot \frac{40 \cdot Z(0; 2.5)}{4\left[e^{-r_0/2} + e^{-r_0} + e^{-3r_0/2} + e^{-2r_0}\right] + 104 e^{-5r_0/2}} = 35.86 \text{ euros.}$$

   (b) Let us compute now the initial value of a European Call option with maturity $T = 7/6$, strike 36 euros, written on the coupon bond with face value $N^* = 35.86$, with maturity $T^* = 2.5$ and coupons paid at the end of each semester at annual rate 8%.

A few preliminary remarks are in order. After the option maturity (which does not coincide with any coupon payment date), the payment flow of the coupons will take place as follows: the cash amount $c = N^* \cdot 0.04 = 1.43$ euros) will be paid at times (always expressed in yearly units) $T_1^* = 1.5$, $T_2^* = 2$ and $T_3^* = T^* = 2.5$ (this is the bond maturity). Since at this date the bond cash value is paid back, the total cash amount received by the bond buyer is $N^* + c = 37.29$ euros.

The basic idea in computing the price of a European option written on a coupon bond is the following:

– "decompose" the underlying coupon bond as a linear combination of $n$ zero-coupon bonds of suitable face value, where $n$ is the number of coupons paid after the option maturity;
– "decompose" the option considered as a sum of $n$ options, each one written on the zero-coupon bonds of the previous step, and with suitable strike;
– compute the value of the relevant option as the sum of the options constructed in the previous step.

According to the previous remarks, let us try to decompose the Call option strike into three contributions ($K_1$, $K_2$ and $K_3$) representing: the strike ($K_1$) of the "virtual" option with maturity $T$ and written on the zero-coupon bond ($ZCB_1$), with face value $N_1 = c = 1.43$ and maturity $T_1^* = 1.5$ years; the strike ($K_2$) of a further "virtual" option with maturity $T$ and written the zero-coupon bond ($ZCB_2$) with face value $N_2 = c = 1.43$ and maturity $T_2^* = 2$ years; and the strike ($K_3$) of another "virtual" option with maturity $T$ and written on the zero-coupon bond ($ZCB_3$) with face value $N_3 = N^* + c = 37.29$ and maturity $T_3^* = T^* = 2.5$ years.

We impose the following requirement:

$$N_1 \cdot Z\left(T; T_1^*\right) + N_2 \cdot Z\left(T; T_2^*\right) + N_3 \cdot Z\left(T; T_3^*\right) = K \tag{10.36}$$

and define

$$\begin{aligned} K_1 &\triangleq N_1 \cdot Z\left(T; T_1^*\right) \\ K_2 &\triangleq N_2 \cdot Z\left(T; T_2^*\right) \\ K_3 &\triangleq N_3 \cdot Z\left(T; T_3^*\right). \end{aligned}$$

We remark that condition (10.36) requires just the decomposition of $K$ into the three strikes $K_1$, $K_2$ and $K_3$ taking into account the respective face values and the times to maturity of the three zero-coupon bonds.

In order to compute $K_1$, $K_2$ and $K_3$, we just need to calculate $Z\left(T; T_1^*\right)$, $Z\left(T; T_2^*\right)$ and $Z\left(T; T_3^*\right)$.

By proceeding in a strictly similar way to that outlined in item 1.(a), we get:

$$\begin{aligned}
B\left(T;T_1^*\right) &= \frac{1-e^{-a\left(T_1^*-T\right)}}{a} = \frac{1-e^{-0.4\cdot(1.5-7/6)}}{0.4} = 0.312 \\
A\left(T;T_1^*\right) &= 0.00001 \\
B\left(T;T_2^*\right) &= \frac{1-e^{-a\left(T_2^*-T\right)}}{a} = \frac{1-e^{-0.4\cdot(2-7/6)}}{0.4} = 0.709 \\
A\left(T;T_2^*\right) &= 0.0018 \\
B\left(T;T_3^*\right) &= \frac{1-e^{-a\left(T_3^*-T\right)}}{a} = \frac{1-e^{-0.4\cdot(2.5-7/6)}}{0.4} = 1.033 \\
A\left(T;T_3^*\right) &= 0.0078
\end{aligned}$$

and

$$\begin{aligned}
Z\left(T;T_1^*\right) &= e^{A\left(T;T_1^*\right)-r_T^*\cdot B\left(T;T_1^*\right)} = e^{0.00001-0.312\cdot r_T^*} \\
Z\left(T;T_2^*\right) &= e^{A\left(T;T_2^*\right)-r_T^*\cdot B\left(T;T_2^*\right)} = e^{0.0018-0.709\cdot r_T^*} \\
Z\left(T;T_3^*\right) &= e^{A\left(T;T_3^*\right)-r_T^*\cdot B\left(T;T_3^*\right)} = e^{0.0078-1.033\cdot r_T^*},
\end{aligned}$$

where $r_T^*$ is unknown and can be determined via (10.36). By the previous remarks and using $N_1 = N_2 = 1.43$ and $N_3 = 37.29$, indeed, equality (10.36) can be written as follows:

$$1.43\cdot e^{0.00001-0.312\cdot r_T^*} + 1.43\cdot e^{0.0018-0.709\cdot r_T^*} + 37.29\cdot e^{0.0078-1.033\cdot r_T^*} = 36.$$

The solution of this equation is $r_T^* = 0.1168$. The required decomposition is then given by:

$$\begin{aligned}
K_1 &= N_1\cdot Z\left(T;T_1^*\right) = 1.38 \\
K_2 &= N_2\cdot Z\left(T;T_2^*\right) = 1.32 \\
K_3 &= N_3\cdot Z\left(T;T_3^*\right) = 33.31.
\end{aligned}$$

By collecting the results available at this stage:

- the underlying coupon bond has been written as a linear combination of three zero-coupon bonds with face values $N_1 = 1.43$, $N_2 = 1.43$ and $N_3 = 37.29$, respectively;
- the relevant option has been decomposed into three options with maturity $T = 7/6$ years in the following way:
  one option (Call 1) with strike $K_1$ and written on $ZCB_1$ with face value $N_1$ and maturity $T_1^* = 1.5$ years;

a second option (Call 2) with strike $K_2$ and written on $ZCB_2$ with face value $N_2$ and maturity $T_2^* = 2$ years;
a third option (Call 3) with strike $K_3$ and written on $ZCB_3$ with face value $N_3$ and maturity $T_3^* = 2.5$ years;

– the initial value of the option we want to evaluate $C_0^{CB}$ is then:

$$C_0^{CB} = C_0^{(1)} + C_0^{(2)} + C_0^{(3)},$$

where $C_0^{(i)}$, $i = 1, 2, 3$, are the initial values of Call $i$.

So, we need to compute $C_0^{(1)}$, $C_0^{(2)}$ and $C_0^{(3)}$. Since Call 1, Call 2 and Call 3 are options written on zero-coupon bonds, the following relation holds:

$$C_0^{(i)} = N_i \cdot Z\left(0; T_i^*\right) \cdot N\left(d^{(i)}\right) - K_i \cdot Z\left(0; T\right) \cdot N\left(d^{(i)} - \sigma_p^{(i)}\right),$$

where, for $i = 1, 2, 3$,

$$d^{(i)} = \frac{\ln\left[\frac{N_i \cdot Z(0,T_i^*)}{K_i \cdot Z(0,T)}\right]}{\sigma_p^{(i)}} + \frac{1}{2}\sigma_p^{(i)}$$

$$\sigma_p^{(i)} = \frac{1 - e^{-a\left(T_i^* - T\right)}}{a}\sqrt{\frac{\sigma^2}{2a}\left[1 - e^{-2aT}\right]}.$$

In our case, we have:

$$\sigma_p^{(1)} = \frac{1 - e^{-a\left(T_1^* - T\right)}}{a}\sqrt{\frac{\sigma^2}{2a}\left[1 - e^{-2aT}\right]} = 0.054$$

$$d^{(1)} = \frac{\ln\left[\frac{N_1 \cdot Z(0,T_1^*)}{K_1 \cdot Z(0,T)}\right]}{\sigma_p^{(1)}} + \frac{1}{2}\sigma_p^{(1)} = 0.6446.$$

Moreover, it is immediate to verify that $Z\left(0; T_1^*\right) = 0.9664$ and $Z\left(0; T\right) = 0.9684$. Consequently,

$$C_0^{(1)} = 1.43 \cdot 0.9664 \cdot N\left(0.664\right) - 1.38 \cdot 0.9684 \cdot N\left(0.664 - 0.054\right) = 0.058.$$

In the same way we can obtain:

$$C_0^{(2)} = 0.1335$$

$$C_0^{(3)} = 5.1871.$$

We can then conclude that the price of the European Call option considered is:

$$C_0^{CB} = C_0^{(1)} + C_0^{(2)} + C_0^{(3)} = 0.058 + 0.1335 + 5.1871 = 5.3786.$$

(c) If we have a European Call written on a zero-coupon bond with face value $N^* = 35.86$ euros, by the same procedure outlined before we obtain:

$$\sigma_p = \frac{1 - e^{-0.4(2.5-7/6)}}{0.4}\sqrt{\frac{(0.2)^2}{2\cdot 0.4}\left[1 - e^{-2\cdot 0.4\cdot 7/6}\right]} = 0.18$$

$$d = \frac{\ln\left[\frac{35.86\cdot Z(0;2.5)}{36\cdot Z(0;7/6)}\right]}{0.18} + \frac{0.18}{2} = 0.135.$$

Finally,

$$\begin{aligned}C_0^{ZCB} &= N^* \cdot Z\left(0; T^*\right) \cdot N(d) - K \cdot Z(0; T) \cdot N\left(d - \sigma_p\right)\\ &= 35.86 \cdot 0.980 \cdot N(0.135) - 36 \cdot 0.9684 \cdot N(0.135 - 0.18)\\ &= 2.65 \text{ euros.}\end{aligned}$$

4. In order to compute the prices of the Put options with the same parameters of the Call examined in items 2.(a) and 3.(b) we can adopt two different approaches. The first consists in applying the European Put valuation formula, the second in applying the Put-Call Parity relation for options written on zero-coupon bonds.
Let us compute the price of the Put option with the same parameters of the Call in item 2.(a).
We recall that the initial value of a Put with maturity $T$ and strike $K$, written on a zero-coupon bond with face value $N$ and maturity $T^*$, is given by:

$$P_0 = K \cdot Z(0; T) \cdot N\left(\sigma_p - d\right) - N \cdot Z\left(0; T^*\right) \cdot N(-d), \tag{10.37}$$

where $d$ and $\sigma_p$ are defined as in (10.34) and in (10.35).
Since all the quantities appearing in (10.37) have been already computed in item 2.(a), we get immediately that:

$$P_0 = 36 \cdot 0.9684 \cdot N(0.18 - 0.741) - 40 \cdot 0.98025 \cdot N(-0.741) = 1.029 \text{ euros.}$$

Alternatively, and by directly applying the Put-Call parity holding for a European Call and European Put with maturity $T$ and strike $K$, written on the same zero-coupon bond with face value $N$ and maturity $T^*$:

$$C_0 - P_0 = N \cdot Z\left(0; T^*\right) - K \cdot Z(0; T),$$

we obtain the same result as before, i.e. $P_0 = 1.0274 \cong 1.029$ euros.

Let us compute now the price of the Put option with the same parameters of the Call considered in item 3.(b). As in the previous case, we have two possibilities: one consists in repeating step-by-step the computations performed for the Call option, the other consists in applying the Put-Call Parity to each of the Call options appearing in the decomposition proposed (Call 1, Call 2 and Call 3, in the present case) and finally sum up the values of the Put options in the "decomposition".

We are going to compute the Put value relying on the second approach, while the calculations with the former strategy can be performed as an exercise by the motivated student.

We have that:

$$P_0^{CB} = P_0^{(1)} + P_0^{(2)} + P_0^{(3)},$$

where Put $i$ is a European Put with maturity $T_i^*$, strike $K_i$ and is written on a zero-coupon bond with face value $N_i$. By the Put-Call Parity we obtain that

$$\begin{aligned}
P_0^{(1)} &= C_0^{(1)} - N_1 \cdot Z\left(0; T_1^*\right) + K_1 \cdot Z\left(0; T\right) = 0.012 \\
P_0^{(2)} &= C_0^{(2)} - N_2 \cdot Z\left(0; T_2^*\right) + K_2 \cdot Z\left(0; T\right) = 0.025 \\
P_0^{(3)} &= C_0^{(3)} - N_3 \cdot Z\left(0; T_3^*\right) + K_3 \cdot Z\left(0; T\right) = 0.889,
\end{aligned}$$

since $Z\left(0; T_1^*\right) = 0.9664$, $Z\left(0; T_2^*\right) = 0.9697$, $Z\left(0; T_3^*\right) = 0.9803$ and $Z\left(0; T\right) = 0.9684$. Finally,

$$P_0^{CB} = P_0^{(1)} + P_0^{(2)} + P_0^{(3)} = 0.926 \text{ euros}.$$

**Exercise 10.2** Consider the instantaneous forward rate with the following affine dynamics:

$$f\left(0, t\right) = c + mt$$

for $t \geq 0$.

1. Determine $\vartheta_t$ and the price of a zero-coupon bond with maturity $t$, as a function of the parameters $c$ and $m$ when the short rate dynamics is described by one of the following models: (a) the Ho-Lee model; (b) the Hull-White model.
2. Assume that the short rate $(r_t)_{t\geq 0}$ dynamics is described by the Ho-Lee model:

$$dr_t = \vartheta_t dt + \sigma dW_t,$$

with $r_0 = c = 0.04$, $\sigma = 0.2$ and with the $\vartheta_t$ obtained in the previous item.

(a) Find $m$ such that the current value of a zero-coupon bond with maturity $T = 7/6$ and face value 40 euros is 36 euros.

(b) By using the parameters $r_0$ and $\sigma$ above and the value of $m$ found in the previous item, compute the value of a European Call option with maturity $T = 7/6$, strike 36 euros and written on a zero-coupon bond with face value 40 euros and maturity of two years and a half.

3. Suppose now that the short rate $(r_t)_{t\geq 0}$ is described by the Hull-White model:

$$dr_t = (\vartheta_t - ar_t)\,dt + \sigma dW_t,$$

with $r_0 = c = 0.04$, $\sigma = 0.2$ and with the value of $m$ found in item 2.(a).

(a) Since the parameter $a$ value is not known beforehand, but we have $\vartheta_{1/(2a)} - \vartheta_0 = 0.084$, determine $a$.
(b) Compute the value of a European Call option with maturity $T = 7/6$, strike 36 euros and written on a zero-coupon bond with face value 40 euros and maturity of two years and a half.

4. Determine $\vartheta_t$ and the value of a zero-coupon bond with maturity $t$, as a function of $c$ and $m$, when the instantaneous forward rate is of quadratic type, i.e.

$$f(0,t) = c + mt + \frac{m}{2}t^2,$$

and when the short rate dynamics is described by the Ho-Lee model with the parameters of item 2.(a).
Compute the value of a zero-coupon bond with maturity one year and a half and face value 40 euros.

**Solution**

1. Let us start by considering case (a) of a short rate following a dynamics of Ho-Lee type. In this case we have:

$$\vartheta_t^{HL} = \vartheta_t = \frac{\partial f}{\partial t}(0,t) + \sigma^2 t = m + \sigma^2 t. \tag{10.38}$$

As far as the price $Z^{HL}(0,t)$ of a zero-coupon bond with maturity $t$ is concerned, we need to find the quantities $A(0,t)$ and $B(0,t)$. Recall that for the Ho-Lee model they are given by:

$$B(t,T) = T - t$$
$$A(t,T) = \int_t^T \vartheta_s (s - T) ds + \frac{\sigma^2 (T-t)^3}{6},$$

so we immediately get $B(0,t) = t$. It is then enough to determine $A(0,t)$ in order to find $Z^{HL}(0,t)$.

Since $r_0 = f(0,0) = c$ and

$$\begin{aligned} A(0,t) &= \int_0^t \vartheta_s (s-t) ds + \frac{\sigma^2 t^3}{6} \\ &= \int_0^t \left(m + \sigma^2 s\right)(s-t) ds + \frac{\sigma^2 t^3}{6} \\ &= \left[\frac{ms^2}{2} + \frac{\sigma^2 s^3}{3} - mst - \frac{\sigma^2 s^2 t}{2}\right]\Bigg|_0^t + \frac{\sigma^2 t^3}{6} \\ &= \frac{mt^2}{2} + \frac{\sigma^2 t^3}{3} - mt^2 - \frac{\sigma^2 t^3}{2} + \frac{\sigma^2 t^3}{6} \\ &= -\frac{mt^2}{2}, \end{aligned}$$

we obtain that, in a Ho-Lee model with the assigned parameters, the initial value of a zero-coupon bond with maturity $t$ is given by:

$$Z^{HL}(0,t) = e^{A(0,t) - r_0 B(0,t)} = e^{-\frac{mt^2}{2} - r_0 t} = e^{-\frac{mt^2}{2} - ct}.$$

In case (b) of a Hull-White type dynamics, we have that:

$$\begin{aligned} \vartheta_t^{HW} = \vartheta_t &= \frac{\partial f}{\partial t}(0,t) + a \cdot f(0,t) + \frac{\sigma^2}{2a}\left(1 - e^{-2at}\right) \\ &= m + a(c + mt) + \frac{\sigma^2}{2a}\left(1 - e^{-2at}\right) \\ &= m + ac + amt + \frac{\sigma^2}{2a}\left(1 - e^{-2at}\right). \end{aligned}$$

As far as the price of a zero-coupon bond with maturity $t$ is concerned, we just recall that $Z^{HW}(0,t) = e^{A(0,t) - r_0 B(0,t)}$ with

$$\begin{aligned} B(0,t) &= \frac{1 - e^{-at}}{a} \\ A(0,t) &= \int_0^t \left[\frac{1}{2}\sigma^2 B^2(s,t) - \vartheta_s B(s,t)\right] ds. \end{aligned}$$

Hence

$$A(0,t) = \int_0^t \left[\frac{1}{2}\sigma^2 \left(\frac{1 - e^{-a(t-s)}}{a}\right)^2 - \vartheta_s \frac{1 - e^{-a(t-s)}}{a}\right] ds.$$

In order to simplify the notation, we denote by $C_s$ the integrand:

$$C_s \triangleq \frac{1}{2}\sigma^2\left(\frac{1-e^{-a(t-s)}}{a}\right)^2 - \vartheta_s\frac{1-e^{-a(t-s)}}{a}.$$

Since

$$\begin{aligned}
C_s &= \tfrac{\sigma^2}{2a^2}\left[1+e^{-2a(t-s)}-2e^{-a(t-s)}\right]-\tfrac{m}{a}-c-ms-\tfrac{\sigma^2}{2a^2}\left(1-e^{-2as}\right)\\
&\quad+\tfrac{m}{a}e^{-a(t-s)}+c\cdot e^{-a(t-s)}+ms\cdot e^{-a(t-s)}+\tfrac{\sigma^2}{2a^2}e^{-a(t-s)}-\tfrac{\sigma^2}{2a^2}e^{-at-as}\\
&= \tfrac{\sigma^2}{2a^2}e^{-2a(t-s)}-\left(\tfrac{m}{a}+c\right)-ms+\tfrac{\sigma^2}{2a^2}e^{-2as}+\left(\tfrac{m}{a}+c-\tfrac{\sigma^2}{2a^2}\right)e^{-a(t-s)}\\
&\quad+ms\cdot e^{-a(t-s)}-\tfrac{\sigma^2}{2a^2}e^{-at-as},
\end{aligned}$$

we get:

$$\begin{aligned}
A(0,t) &= \left[\tfrac{\sigma^2}{4a^3}e^{-2a(t-s)}-\left(\tfrac{m}{a}+c\right)s-\tfrac{ms^2}{2}-\tfrac{\sigma^2}{4a^3}e^{-2as}\right]\Big|_0^t\\
&\quad+\left[\left(\tfrac{m}{a^2}+\tfrac{c}{a}-\tfrac{\sigma^2}{2a^3}\right)e^{-a(t-s)}+\tfrac{\sigma^2}{2a^3}e^{-at-as}\right]\Big|_0^t+\textstyle\int_0^t ms\cdot e^{-a(t-s)}ds\\
&= \tfrac{\sigma^2}{4a^3}-\tfrac{\sigma^2}{4a^3}e^{-2at}-\left(\tfrac{m}{a}+c\right)t-\tfrac{mt^2}{2}-\tfrac{\sigma^2}{4a^3}e^{-2at}+\tfrac{\sigma^2}{4a^3}+\tfrac{m}{a^2}+\tfrac{c}{a}-\tfrac{\sigma^2}{2a^3}\\
&\quad-\left(\tfrac{m}{a^2}+\tfrac{c}{a}-\tfrac{\sigma^2}{2a^3}\right)e^{-at}+\tfrac{\sigma^2}{2a^3}e^{-2at}-\tfrac{\sigma^2}{2a^3}e^{-at}\\
&\quad+\left[\tfrac{ms}{a}e^{-a(t-s)}-\tfrac{m}{a^2}e^{-a(t-s)}\right]\Big|_0^t\\
&= -\left(\tfrac{m}{a}+c\right)t-\tfrac{mt^2}{2}+\tfrac{m}{a^2}+\tfrac{c}{a}-\left(\tfrac{m}{a^2}+\tfrac{c}{a}\right)e^{-at}+\tfrac{mt}{a}-\tfrac{m}{a^2}+\tfrac{m}{a^2}e^{-at}\\
&= -ct-\tfrac{mt^2}{2}+\tfrac{c}{a}-\tfrac{c}{a}e^{-at}\\
&= \tfrac{c}{a}\left(1-e^{-at}\right)-ct-\tfrac{mt^2}{2}.
\end{aligned}$$

Since $r_0 = f(0,0) = c$, in the Hull-White model considered the initial value of a zero-coupon with maturity $t$ is given by:

$$\begin{aligned}
Z^{HW}(0,t) &= e^{A(0,t)-r_0B(0,t)}\\
&= \exp\left\{\frac{r_0}{a}\left(1-e^{-at}\right)-r_0t-\frac{mt^2}{2}-r_0\frac{1-e^{-at}}{a}\right\}\\
&= \exp\left\{-r_0t-\frac{mt^2}{2}\right\}.
\end{aligned}$$

2. Assume now that the short rate $(r_t)_{t\geq 0}$ is described by the following Ho-Lee model:

$$dr_t = \vartheta_t dt + \sigma dW_t,$$

with $r_0 = c = 0.04$, $\sigma = 0.2$ and $\vartheta_t$ as obtained in (10.38), i.e.

$$\vartheta_t^{HL} = \vartheta_t = m + \sigma^2 t.$$

(a) Let us start by determining the value $m$ such that the current price of a zero-coupon bond with face value 40 euros and maturity $T = 7/6$ years is equal to 36 euros.
We have already obtained (see item 1.) the current price of a zero-coupon bond as a function of $m$, that is:

$$Z^{HL}(0,t) = e^{A(0,t)-r_0 B(0,t)} = e^{-\frac{mt^2}{2}-r_0 t}.$$

We need then to find the value $m$ such that $40 \cdot Z(0;T) = 36$ or, equivalently,

$$e^{-\frac{mT^2}{2}-r_0 T} = \frac{36}{40}.$$

It follows that:

$$m = -2\frac{\ln(36/40) + r_0 T}{T^2} = -2\frac{\ln(36/40) + 0.04 \cdot 7/6}{(7/6)^2} = 0.0862.$$

The value $m$ required is then $m = 0.0862$. Consequently, $\vartheta_t^{HL} = 0.0862 + 0.04 \cdot t$ and $f(0,t) = 0.04 + 0.0862 \cdot t$.

(b) We have to find the price of a European Call with maturity of 1 year and 2 months, strike 36 euros and written on a zero-coupon bond with face value 40 euros and maturity of two years and a half years.
Since, in a Ho-Lee model, the initial value of such a Call option is given by

$$C_0 = 40 \cdot Z(0,T^*) \cdot N(d) - 36 \cdot Z(0,T) \cdot N(d - \sigma_p),$$

with

$$d = \sigma_p^{-1} \ln\left[\frac{40 \cdot Z(0,T^*)}{36 \cdot Z(0,T)}\right] + \frac{1}{2}\sigma_p$$
$$\sigma_p = \sigma(T^* - T)\sqrt{T},$$

it is necessary to compute $Z(0,T^*)$, $Z(0,T)$, $d$ and $\sigma_p$.
The value $Z(0;T)$ is immediate to obtain: $m$ has been actually chosen in order to get $Z(0;T) = 36/40 = 0.9$ euros.

Moreover: since $T = 7/6$, $T^* = 2.5$ and $\sigma = 0.2$, we obtain also

$$Z\left(0, T^*\right) = e^{-\frac{m(T^*)^2}{2} - r_0 T^*} = e^{-\frac{0.0862\cdot(2.5)^2}{2} - 0.04\cdot 2.5} = 0.691$$

$$\sigma_p = 0.2\,(2.5 - 7/6)\,\sqrt{7/6} = 0.288$$

$$d = \frac{\ln\left[\frac{40\cdot 0.691}{36\cdot 0.9}\right]}{0.288} + \frac{0.288}{2} = -0.408.$$

Finally, the initial value of a European Call with maturity $T = 7/6$ (i.e. 1 year and 2 months), strike 36 euros and written on a zero-coupon bond with face value 40 euros and maturity of two years and a half, is given by:

$$C_0 = 40 \cdot 0.691 \cdot N(-0.408) - 36 \cdot 0.9 \cdot N(-0.408 - 0.288) = 1.563 \text{ euros.}$$

3. Let us consider now a short rate $(r_t)_{t\geq 0}$ with dynamics described by the Hull-White model with $\vartheta_t$ as in item 1., $a$ to be determined and with parameters $r_0 = 0.04$ and $\sigma = 0.2$.

   (a) We need to find the value of $a$ first. We know that $a$ must satisfy the following condition:

$$\vartheta_{1/(2a)} - \vartheta_0 = \alpha, \tag{10.39}$$

with $\alpha = 0.084$.
By condition (10.39) we can obtain $a$. More precisely, (10.39) is equivalent to:

$$am \cdot \tfrac{1}{2a} + \tfrac{\sigma^2}{2a}\left(1 - e^{-2a\cdot\frac{1}{2a}}\right) = \alpha$$
$$\tfrac{m}{2} + \tfrac{\sigma^2}{2a}\left(1 - e^{-1}\right) = \alpha,$$

because (see item 1.)

$$\vartheta_t = m + ac + amt + \frac{\sigma^2}{2a}\left(1 - e^{-2at}\right).$$

Then:

$$a = \frac{\sigma^2\left(1 - e^{-1}\right)}{2\left(\alpha - \frac{m}{2}\right)} = \frac{\sigma^2\,(e-1)}{e\,(2\alpha - m)}$$
$$= \frac{(0.2)^2\,(e-1)}{e\,(2\cdot 0.084 - 0.0862)} = 0.309.$$

(b) We compute now the current price of a European Call with maturity of 1 year and 2 months, strike 36 euros and written on a zero-coupon bond with face value 40 euros and maturity of two years and a half.
Since, in the Hull-White model, the initial value of such a Call option is given by:

$$C_0 = 40 \cdot Z(0, T^*) \cdot N(d) - 36 \cdot Z(0, T) \cdot N(d - \sigma_p),$$

with

$$d = \sigma_p^{-1} \ln \left[ \frac{40 \cdot Z(0, T^*)}{36 \cdot Z(0, T)} \right] + \frac{1}{2}\sigma_p$$

$$\sigma_p = \frac{1 - e^{-a(T^*-T)}}{a} \sqrt{\frac{\sigma^2}{2a} \left[1 - e^{-2aT}\right]},$$

we need then to compute $Z(0, T^*)$, $Z(0, T)$, $d$ and $\sigma_p$.
Since $T = 7/6$, $T^* = 2.5$, $\sigma = 0.2$, $m = 0.0862$ and $a = 0.309$, we obtain:

$$Z\left(0, T^*\right) = e^{-\frac{m(T^*)^2}{2} - r_0 T^*} = e^{-\frac{0.0862 \cdot (2.5)^2}{2} - 0.04 \cdot 2.5} = 0.691$$

$$Z(0; T) = e^{-\frac{mT^2}{2} - r_0 T} = e^{-\frac{0.0862 \cdot (7/6)^2}{2} - 0.04 \cdot 7/6} = 0.9$$

$$\sigma_p = \frac{1 - e^{-0.309 \cdot (2.5 - 7/6)}}{0.309} \sqrt{\frac{(0.2)^2}{2 \cdot 0.309} \left[1 - e^{-2 \cdot 0.309 \cdot 7/6}\right]} = 0.199$$

$$d = \frac{\ln\left[\frac{40 \cdot 0.691}{36 \cdot 0.9}\right]}{0.199} + \frac{0.199}{2} = -0.699.$$

Finally, the initial price of the European Call option (with maturity of 1 year and 2 months, strike of 36 euros and written on a zero-coupon bond with face value 40 euros and maturity of two years and a half) equals

$$C_0 = 40 \cdot 0.691 \cdot N(-0.699) - 36 \cdot 0.9 \cdot N(-0.699 - 0.199) = 0.716 \text{ euros}.$$

4. If the short rate dynamics is described by the Ho-Lee model and the instantaneous forward rate is $f(0, t) = c + mt + \frac{m}{2}t^2$, we have that:

$$\vartheta_t = \frac{\partial f}{\partial t}(0, t) + \sigma^2 t = m + mt + \sigma^2 t = m + \left(m + \sigma^2\right) t.$$

As far as the price $Z(0, t)$ of a zero-coupon bond with maturity $t$ is concerned, the unique quantity we still have to compute is $A(0, t)$, since $B(0, t) = t$.

As $r_0 = f(0,0) = c$ and

$$\begin{aligned}
A(0,t) &= \int_0^t \vartheta_s (s-t) ds + \frac{\sigma^2 t^3}{6} \\
&= \int_0^t \left[ m + \left( m + \sigma^2 \right) s \right] (s-t) ds + \frac{\sigma^2 t^3}{6} \\
&= \left[ \frac{ms^2}{2} + \frac{\left(m+\sigma^2\right) s^3}{3} - mst - \frac{\left(m+\sigma^2\right) s^2 t}{2} \right] \Bigg|_0^t + \frac{\sigma^2 t^3}{6} \\
&= \frac{mt^2}{2} + \frac{mt^3}{3} + \frac{\sigma^2 t^3}{3} - mt^2 - \frac{mt^3}{2} - \frac{\sigma^2 t^3}{2} + \frac{\sigma^2 t^3}{6} \\
&= -\frac{mt^2}{2} - \frac{mt^3}{6},
\end{aligned}$$

we obtain that

$$Z(0,t) = e^{A(0,t) - r_0 B(0,t)} = e^{-\frac{mt^2}{2} - \frac{mt^3}{6} - r_0 t} = e^{-\frac{mt^2}{2} - \frac{mt^3}{6} - ct}.$$

The value (at time $t = 0$) of a zero-coupon bond with maturity of 1 year and 2 months and face value 40 euros is then equal to:

$$40 \cdot Z(0; 7/6) = 40 \cdot e^{-\frac{0.0862 \cdot (7/6)^2}{2} - \frac{0.0862 \cdot (7/6)^3}{6} - 0.04 \cdot 7/6} = 35.189 \text{ euros}.$$

**Exercise 10.3** Assume the short rate $(r_t)_{t \geq 0}$ dynamics is described by the Hull-White model:

$$dr_t = (\vartheta_t - ar_t)\, dt + \sigma dW_t, \tag{10.40}$$

with $r_0 = 0.04$, $\sigma = 0.2$ and $\vartheta_t$ associated to an instantaneous forward rate with the following affine dependence on $t$:

$$f(0,t) = c + mt,$$

with $m = 0.0862$ and $a = 0.309$—as in Exercise 10.2, items 2.(a), 3.(a).

1. Compute the current price of a zero-coupon bond with maturity $T$ of 1 year and 2 months and face value 32 euros, in the case of a short rate dynamics described by the Vasiček model (considered as a particular case of Hull-White model with $\vartheta_t = ab$) with parameter $b$ satisfying the following condition: $ab = \vartheta_T^4 / \vartheta_0$.
2. Consider now the Hull-White dynamics for the short rate as in (10.40), with *flat term structure*, i.e. with constant instantaneous forward rate $f(0,t) = r_0$ for any $t \geq 0$.

Compute the value (at time $t = 0$) of a European Call option with maturity of 1 year and 8 months, strike of 38 euros and written on a coupon bond with face value 40 euros, maturity of two years and a half and coupons paid at the end of each semester at the annual coupon rate of 8%.

**Solution**

1. We already know from Exercise 10.2 that

$$\begin{aligned}\vartheta_t &= \frac{\partial f}{\partial t}(0,t) + a \cdot f(0,t) + \frac{\sigma^2}{2a}\left(1 - e^{-2at}\right) \\ &= m + a(c + mt) + \frac{\sigma^2}{2a}\left(1 - e^{-2at}\right) \\ &= m + ar_0 + amt + \frac{\sigma^2}{2a}\left(1 - e^{-2at}\right).\end{aligned}$$

Let us now consider the Vasiček model corresponding to the Hull-White model considered, i.e.

$$dr_t = a(b - r_t)\,dt + \sigma dW_t, \tag{10.41}$$

where the parameter $b$ satisfies the condition $ab = \vartheta_T^4/\vartheta_0$. The value of $b$ is then:

$$b = \frac{\vartheta_T^4}{a\vartheta_0} = \frac{\left(m + ar_0 + amT + \frac{\sigma^2}{2a}\left(1 - e^{-2aT}\right)\right)^4}{a(m + ar_0)} = 0.228,$$

because $a = 0.309$, $m = 0.0862$, $r_0 = 0.04$, $\sigma = 0.2$ and $T = 20/12$ (1 year and 8 months).
We need to compute the current value of a zero-coupon bond with face value 32 euros and maturity of 1 year and 8 months. In a Vasiček model, we obtain the following results:

$$\begin{aligned}B(0,T) &= \frac{1 - e^{-aT}}{a} = \frac{1 - e^{-0.309\cdot\frac{20}{12}}}{0.309} = 1.303 \\ A(0,T) &= \frac{[B(0,T) - T]\left(a^2 b - \sigma^2/2\right)}{a^2} - \frac{\sigma^2 B^2(0,T)}{4a} = -0.06166 \\ Z(0,T) &= e^{A(0,T) - r_0 B(0,T)} = e^{-0.06166 - 0.04\cdot 1.303} = 0.8925.\end{aligned}$$

The value (at $t = 0$) of a zero-coupon bond with face value 32 euros and maturity of 1 year and 8 months is then equal to:

$$32 \cdot Z(0; 20/12) = 28.56 \text{ euros.}$$

2. Let us determine first $\vartheta_t$ in the case $f(0,t) = r_0$ for any $t \geq 0$. Since we are working now with the Hull-White model, we can write

$$\vartheta_t = \frac{\partial f}{\partial t}(0,t) + a \cdot f(0,t) + \frac{\sigma^2}{2a}\left(1 - e^{-2at}\right) = ar_0 + \frac{\sigma^2}{2a}\left(1 - e^{-2at}\right).$$

We just recall that for any affine model the price of zero-coupon bond with maturity $t$ can be written as $Z(0,t) = e^{A(0,t) - r_0 B(0,t)}$ with

$$B(0,t) = \frac{1 - e^{-at}}{a}$$
$$A(0,t) = \int_0^t \left[\frac{1}{2}\sigma^2 B^2(s,t) - \vartheta_s B(s,t)\right] ds.$$

We get then $A(0,t)$:

$$A(0,t) = \int_0^t \left[\frac{1}{2}\sigma^2\left(\frac{1 - e^{-a(t-s)}}{a}\right)^2 - \vartheta_s \frac{1 - e^{-a(t-s)}}{a}\right] ds.$$

In order to simplify the notation, denote by $C_s$ the integrand, that is

$$C_s \triangleq \frac{1}{2}\sigma^2\left(\frac{1 - e^{-a(t-s)}}{a}\right)^2 - \vartheta_s \frac{1 - e^{-a(t-s)}}{a}.$$

Since

$$\begin{aligned} C_s = & \tfrac{\sigma^2}{2a^2}\left[1 + e^{-2a(t-s)} - 2e^{-a(t-s)}\right] - r_0\left(1 - e^{-a(t-s)}\right) - \tfrac{\sigma^2}{2a^2}\left(1 - e^{-2as}\right) \\ & + \tfrac{\sigma^2}{2a^2} e^{-a(t-s)} - \tfrac{\sigma^2}{2a^2} e^{-at-as} \\ = & \tfrac{\sigma^2}{2a^2} e^{-2a(t-s)} - \tfrac{\sigma^2}{2a^2} e^{-a(t-s)} - r_0\left(1 - e^{-a(t-s)}\right) + \tfrac{\sigma^2}{2a^2} e^{-2as} - \tfrac{\sigma^2}{2a^2} e^{-at-as}, \end{aligned}$$

the following relation holds:

$$\begin{aligned} A(0,t) = & \left[\frac{\sigma^2}{4a^3} e^{-2a(t-s)} - \frac{\sigma^2}{2a^3} e^{-a(t-s)} - r_0 s + \frac{r_0}{a} e^{-a(t-s)}\right]\Bigg|_0^t \\ & + \left[-\frac{\sigma^2}{4a^3} e^{-2as} + \frac{\sigma^2}{2a^3} e^{-at-as}\right]\Bigg|_0^t \\ = & -r_0 t + \frac{r_0}{a} - \frac{r_0}{a} e^{-at}. \end{aligned}$$

Consequently,

$$Z(0,t) = e^{A(0,t) - r_0 B(0,t)} = e^{-r_0 t},$$

as we could actually have guessed from the beginning.
In a strictly analogous way we can prove that:

$$\begin{aligned} A(t,T) = &-r_0 (T-t) + \frac{r_0}{a}[1 - e^{-a(T-t)}] \\ &+ \frac{\sigma^2}{4a^3}[2e^{-a(T-t)} - 1 - e^{-2a(T-t)} \\ &+ e^{-2aT} - 2e^{-a(T+t)} + e^{-2at}] \end{aligned} \tag{10.42}$$

$$\begin{aligned} Z(t,T) = &\exp\left\{-r_0(T-t) + \frac{r_0}{a}(1 - e^{-a(T-t)}) - r_t \frac{1 - e^{-a(T-t)}}{a}\right\} \\ &\cdot \exp\left\{\frac{\sigma^2}{4a^3}\left[2e^{-a(T-t)} - 1 - e^{-2a(T-t)}\right.\right. \\ &\left.\left. + e^{-2aT} - 2e^{-a(T+t)} + e^{-2at}\right]\right\}. \end{aligned} \tag{10.43}$$

Let us now turn our attention to the initial problem, that is compute the initial value of a European Call option with maturity of 1 year and 8 months, strike of 38 euros and written on a coupon bond with face value 40 euros, maturity of two years and a half and with coupons paid at the end of each semester with coupon rate of 8%.
We remark that, after the option maturity (which does not coincide with any of the coupon payment dates), the coupons are paid at dates $T_1^* = 2$ years and $T_2^* = T^* = 2.5$ years (the bond maturity). At this last date, moreover, the face value is paid back, the total amount paid then is $N + c = 41.6$ euros since the coupons (paid in each semester) are equal to $c = N \cdot 0.04 = 1.6$ euros.
In strict analogy with Exercise 10.2, we try to decompose the Call option strike in two components ($K_1$ and $K_2$) representing, respectively, the strike of the "virtual" option with maturity $T$ and written on the zero-coupon bond ($ZCB_1$) with face value $N_1 = c = 1.6$ and maturity $T_1^* = 2$ years; the strike of the "virtual" option with maturity $T$ and written on the zero-coupon bond ($ZCB_2$) with face value $N_2 = N + c = 41.6$ and maturity $T_2^* = T^* = 2.5$ years.
We then impose the condition:

$$N_1 \cdot Z\left(T; T_1^*\right) + N_2 \cdot Z\left(T; T_2^*\right) = K \tag{10.44}$$

and define:

$$\begin{aligned} K_1 &\triangleq N_1 \cdot Z\left(T; T_1^*\right) \\ K_2 &\triangleq N_2 \cdot Z\left(T; T_2^*\right). \end{aligned}$$

We remark that condition (10.44) only requires to decompose $K$ into two strikes $K_1$ and $K_2$ as a function of the face values and the time to maturity of the two zero-coupon bonds.
In order to determine $K_1$ and $K_2$ as mentioned above, we need $Z\left(T; T_1^*\right)$ and $Z\left(T; T_2^*\right)$.
From (10.43) we deduce

$$\begin{aligned}
Z\left(T; T_1^*\right) &= \exp\left\{-r_0\left(T_1^* - T\right) + \frac{r_0}{a}\left(1 - e^{-a(T_1^*-T)}\right) - r_T^* \frac{1 - e^{-a(T_1^*-T)}}{a}\right\} \\
&\quad + \frac{\sigma^2}{4a^3}[2e^{-a(T_1^*-T)} - 1 - e^{-2a(T_1^*-T)} \\
&\quad + e^{-2aT_1^*} - 2e^{-a(T_1^*+T)} + e^{-2aT}] \\
&= e^{-0.0021-0.3167\cdot r_T^*} \\
Z\left(T; T_2^*\right) &= e^{-0.0112-0.7347\cdot r_T^*},
\end{aligned}$$

where $r_T^*$ is unknown but can be obtained by (10.44). In fact, as a consequence of our previous remarks and since $N_1 = 1.6$ and $N_2 = 41.6$, equality (10.44) can be written as follows:

$$1.6 \cdot e^{-0.0021-0.3167\cdot r_T^*} + 41.6 \cdot e^{-0.0112-0.7347\cdot r_T^*} = 38.$$

By solving explicitly this equation, we find the solution $r_T^* \cong 0.16337$. We get then:

$$\begin{aligned}
K_1 &\triangleq N_1 \cdot Z\left(T; T_1^*\right) = 1.516 \\
K_2 &\triangleq N_2 \cdot Z\left(T; T_2^*\right) = 36.484.
\end{aligned}$$

At this stage we have:

- the decomposition of the underlying coupon bond as a linear combination of two zero-coupon bonds with face values $N_1 = 1.6$ and $N_2 = 41.6$ respectively;
- the option considered has been then decomposed into two options (both with maturity $T = 20/12$): the first one (Call 1) with strike $K_1$, written on $ZCB_1$ with face value $N_1$ and maturity $T_1^* = 2$ years; the second one (Call 2) with strike $K_2$, written on $ZCB_2$ with face value $N_2$ and maturity $T_2^* = 2.5$ years;
- the initial price of our option $C_0^{CB}$ is then given by:

$$C_0^{CB} = C_0^{(1)} + C_0^{(2)},$$

where $C_0^{(i)}$, $i = 1, 2$ denotes the initial price of the European Call $i$.

The remaining quantities to compute are then $C_0^{(1)}$ and $C_0^{(2)}$. Since Call 1 and Call 2 are options written on zero-coupon bonds, in the Hull-White model, the following relation holds:

$$C_0^{(i)} = N_i \cdot Z\left(0; T_i^*\right) \cdot N\left(d^{(i)}\right) - K_i \cdot Z\left(0; T\right) \cdot N\left(d^{(i)} - \sigma_p^{(i)}\right),$$

where, for $i = 1, 2$,

$$d^{(i)} = \frac{\ln\left[\frac{N_i \cdot Z(0,T_i^*)}{K_i \cdot Z(0,T)}\right]}{\sigma_p^{(i)}} + \frac{1}{2}\sigma_p^{(i)}$$

$$\sigma_p^{(i)} = \frac{1 - e^{-a(T_i^* - T)}}{a}\sqrt{\frac{\sigma^2}{2a}\left[1 - e^{-2aT}\right]}.$$

In this case we have:

$$\sigma_p^{(1)} = \frac{1 - e^{-a(T_1^* - T)}}{a}\sqrt{\frac{\sigma^2}{2a}\left[1 - e^{-2aT}\right]} = 0.0646$$

$$d^{(1)} = \frac{\ln\left[\frac{N_1 \cdot Z(0,T_1^*)}{K_1 \cdot Z(0,T)}\right]}{\sigma_p^{(1)}} + \frac{1}{2}\sigma_p^{(1)} = 0.6606.$$

It is immediate to verify that $Z\left(0; T_1^*\right) = 0.9231$ and $Z\left(0; T\right) = 0.9355$. Consequently:

$$\begin{aligned} C_0^{(1)} &= 1.6 \cdot 0.9231 \cdot N\left(0.6606\right) - 1.516 \cdot 0.9355 \cdot N\left(0.6606 - 0.0646\right) \\ &= 0.0739 \text{ euros.} \end{aligned}$$

By proceeding in a strictly similar way we obtain

$$C_0^{(2)} = 4.3384.$$

We conclude then that the value at $t = 0$ of the European Call option considered at the beginning is the following:

$$C_0^{CB} = C_0^{(1)} + C_0^{(2)} = 0.0739 + 4.3384 = 4.4123 \text{ euros.}$$

**Exercise 10.4** By using the Change of Numéraire technique, prove that the pricing formula for a European Call option with strike $K$, maturity $T$, written on a zero-coupon bond (with unit face value and maturity $T^*$) when the short rate dynamics is described by the Ho-Lee model, is the following:

$$C(t, T, K, T^*) = p(t, T^*)N(d) - Kp(t, T)N(d - \sigma_p), \tag{10.45}$$

where $N$ denotes the cumulative density of a standard normal and $d$ and $\sigma_p$ are defined as follows:

$$d = \frac{\left[\ln\left(\frac{p(t,T^*)}{Kp(t,T)}\right) + \frac{\sigma_p^2}{2}\right]}{\sigma_p}$$

$$\sigma_p = \sigma(T^* - T)\sqrt{T - t}.$$

**Solution** The payoff of a Call option is the following: $\Phi = \max\{p(T, T^*) - K; 0\}$. A risk-neutral valuation argument provides the following expression for the Call option price at time $t$:

$$C(t, T, K, T^*) = E_Q\left[\exp\left(-\int_t^T r_s ds\right)\max\left\{p(T, T^*) - K; 0\right\} \mid \mathscr{F}_t\right]. \tag{10.46}$$

Using the zero-coupon bond with maturity $T$ as a numéraire, this formula can be rewritten as:

$$C(t, T, K, T^*) = p(t, T) \cdot E_{Q^T}\left[\max\left\{\frac{p(T, T^*)}{p(T, T)} - K; 0\right\}\middle| \mathscr{F}_t\right], \tag{10.47}$$

where $E_{Q^T}$ denotes expectation computed with respect to the T-forward measure $Q^T$, defined as the probability measure with respect to which the value of every asset divided by $p(t, T)$ is a martingale.

Set now

$$Z_t \triangleq \frac{p(t, T^*)}{p(t, T)}. \tag{10.48}$$

$Z_t$ is a $Q^T$-martingale and its $Q^T$-dynamics can be easily obtained by that of the bond prices. By the zero-coupon bond pricing formula in the Ho-Lee model, the bond dynamics with respect to $Q$ is in fact:

$$dp(t, T) = r_t p(t, T)dt - \sigma B(t, T)p(t, T)dW_t. \tag{10.49}$$

By applying Itô's formula, we obtain the dynamics of $Z$:

$$\begin{aligned} dZ_t &= \sigma^2 Z_t B(t, T)(B(t, T) - B(t, T^*))dt + \sigma Z_t[B(t, T) - B(t, T^*)]dW_t \\ &= \sigma Z_t(B(t, T) - B(t, T^*))[\sigma B(t, T)dt + dW_t] \end{aligned}$$

$$
\begin{aligned}
&= Z_t \sigma \left[ B(t,T) - B(t,T^*) \right] dW_t^T \\
&= Z_t \sigma \left[ (T-t) - (T^*-t) \right] dW_t^T \\
&= Z_t \sigma \left( T - T^* \right) dW_t^T ,
\end{aligned}
$$

where $W_t^T = W_t + \sigma \int_0^t B(s,T)ds$ and, by Girsanov's Theorem, $Q^T$ is the new probability measure for which $(W_t^T)_{t \geq 0}$ is a Brownian motion.

By comparing the present case with the usual one in the Black-Scholes model, we deduce that

$$
\begin{aligned}
C(t,T,K,T^*) &= p(t,T) E_{Q^T} \left[ \max\{Z(T) - K; 0\} \mid \mathscr{F}_t \right] \\
&= p(t,T^*) N(d) - K p(t,T) N(d - \sigma_p).
\end{aligned}
$$

**Exercise 10.5** By applying the Black Formula, find the value (at time $t = 0$) of a Caplet with maturity of 2 years and tenor of one semester in a LIBOR market model. Assume a flat term structure for the Black (forward) implied volatility $\sigma_i = 0.6$ (on annual basis) and for the spot LIBOR at level 2% and a CAP rate $R = 0.06$ (i.e. the forward implied volatility and the LIBOR rate are assumed to be constant over the period considered). The face value is $M = 100.000$ euros.

**Solution** We recall the Black formula for Caplets:

$$
Capl(0) = M \cdot \alpha_i \cdot p(0,T_i) \left[ L(0,T_{i-1},T_i) N(d_1) - R \cdot N(d_2) \right], \tag{10.50}
$$

where $\alpha_i = T_i - T_{i-1}$ and

$$
d_1 \triangleq \frac{\left[ \ln\left( \frac{L(0,T_{i-1},T_i)}{R} \right) + \frac{1}{2} \Sigma_i^2(0,T_{i-1}) \right]}{\Sigma_i(0,T_{i-1})}
$$

$$
d_2 \triangleq \frac{\left[ \ln\left( \frac{L(0,T_{i-1},T_i)}{R} \right) - \frac{1}{2} \Sigma_i^2(0,T_{i-1}) \right]}{\Sigma_i(0,T_{i-1})}
$$

with

$$
\Sigma_i(0,T_{i-1}) \triangleq \sqrt{\int_0^{T_i - 1} \sigma_i^2(u) du}. \tag{10.51}
$$

In the present case, $\alpha_i = T_i - T_{i-1} = 0.5$ years, $\sigma_i = 0.6$ , $T_i = 2$ years and $\Sigma_i(0,T_{i-1}) \triangleq \sqrt{\int_0^{T_i - 1} \sigma_i^2(u) du} = \sqrt{\sigma_i^2 T_{i-1}} = \sigma_i \sqrt{T_{i-1}} = 0.7348$.

From the definition of LIBOR spot rate we get:

$$
1 = p(0,T) \left[ 1 + L(0,T) T \right], \tag{10.52}
$$

hence

$$p(0,T) = \frac{1}{1 + L(0,T)T} = \frac{1}{1 + 0.02 \cdot T}. \tag{10.53}$$

In particular,

$$p(0,T_i) = \frac{1}{1 + 0.02 \cdot T_i} = \frac{1}{1 + 0.02 \cdot 2} = 0.96$$
$$p(0,T_{i-1}) = \frac{1}{1 + 0.02 \cdot T_{i-1}} = \frac{1}{1 + 0.02 \cdot 1.5} = 0.97.$$

It follows that the forward LIBOR rate at time $t = 0$ and for the interval $[T_{i-1}, T_i]$ equals

$$L(0, T_{i-1}, T_i) = \frac{p(0,T_{i-1}) - p(0,T_i)}{\alpha_i \cdot p(0,T_i)} = 0.0208. \tag{10.54}$$

Consequently,

$$\begin{aligned} N(d_1) &= N\left(\frac{\left[\ln\left(\frac{0.0208}{0.06}\right) + \frac{1}{2}(0.7348)^2\right]}{0.7348}\right) \\ &= N(-1.07) = 1 - N(1.07) = 1 - 0.8577 = 0.1423 \\ N(d_2) &= N\left(\frac{\left[\ln\left(\frac{0.0208}{0.06}\right) - \frac{1}{2}(0.7348)^2\right]}{0.7348}\right) \\ &= N(-1.81) = 1 - N(1.81) = 1 - 0.9649 = 0.0351. \end{aligned}$$

Finally, $Capl(0) = 40.98$ euros.

## 10.3 Proposed Exercises

### Exercise 10.6

1. Compute the current value of a zero-coupon bond with maturity of 1 year and 2 months and face value of 24 euros, by assuming that the short rate dynamics is described by the Hull-White model with the same value of $a$ as in item 3.(a) of Exercise 10.2.
2. Compare the price obtained in the previous item with the one obtained by assuming for the short rate dynamics of Vasiček type with parameter $b$ satisfying the condition $ab = \vartheta_0$, where $a$ and $\vartheta_0$ are the same as for the Hull-White model of item 1.

**Exercise 10.7** Assume that the short rate dynamics is described by the Vašiček model:

$$dr_t = a\,(b - r_t)\,dt + \sigma dW_t,$$

where the values of the parameters $a$, $b$ and $\sigma$ are: $a = 0.2$, $b = 0.1$ and $\sigma = 0.2$. Assume that, at time $t = 0$, the short rate is $r_0 = 4\%$ (per year).

1. Compute the value (at time $t = 0$) of a zero-coupon bond with maturity $T^* = 4$ years.
2. Keeping the results of item 2. in mind, compute the price (at time $t = 0$) of a zero-coupon bond with maturity $T^* = 4$ years and face value of 36 euros.
3. Establish if there exists a meaningful value of $r_0$ (i.e. $r_0 \in (0, 1)$) such that the current value of the bond of item 2. coincides with the current value of the same bond under the hypothesis of a constant short rate equal to the expected value of $r_{T^*}$.
4. Consider a European Call option with maturity $T = 11/3$ years and strike $K = 32$, written on a coupon bond with maturity $T^* = 4$ years, face value of 36 euros and coupon rate of 8% per year (coupon rate paid at the end of each semester).
   (a) Compute the value (at $t = 0$) of this European Call option.
   (b) How would the price of the option considered change if its maturity was 3 years and 2 months?
   (c) Compute the price of the Put option with the same parameters of the Call of item 4.(b).

**Exercise 10.8** Consider three short rates $\left(r_t^{(1)}\right)_{t\geq 0}$, $\left(r_t^{(2)}\right)_{t\geq 0}$ and $\left(r_t^{(3)}\right)_{t\geq 0}$, with the following dynamics:

$$\begin{aligned} dr_t^{(1)} &= a_1\left(b - r_t^{(1)}\right)dt + \sigma dW_t \\ dr_t^{(2)} &= \vartheta_t dt + \sigma dW_t \\ dr_t^{(3)} &= \left(\vartheta_t - a_2 r_t^{(3)}\right)dt + \sigma dW_t, \end{aligned}$$

with real parameters $a_1, a_2, b \in \mathbb{R}$, $\sigma > 0$ and $a_2 \neq 0$.

1. Establish what is the model describing the dynamics of the short rates defined as follows:

$$\begin{aligned} \hat{r}_t &\triangleq r_t^{(2)} - r_t^{(3)} - \frac{a_1}{a_2} r_t^{(1)} \\ r_t^* &\triangleq r_t^{(2)} - r_t^{(3)} + r_t^{(1)}. \end{aligned}$$

2. For $a_1 = a_2 = a$, find the probability distribution of $r_T^{(2)}$ with $T = 1$ year.

Discuss whether $P\left(r_T^{(2)} \geq 0\right) = 0.9$ holds for some value of the parameters $b$ and $\sigma$.

**Exercise 10.9** Assume that the short rate dynamics is described by the Hull-White model:

$$dr_t = (\vartheta_t - ar_t)\,dt + \sigma dW_t,$$

where $\vartheta_t$ is related to $f(0,t)$ of Exercise 10.2, item 4., $r_0 = 0.04$, $\sigma = 0.2$ and with the same parameters $a$ and $m$ of Exercise 10.2.

1. Compute the initial value of a European Put option of strike 36 euros, maturity of 1 year and 2 months and written on a zero-coupon bond with face value of 40 euros and maturity of two years and a half.
2. In the corresponding Ho-Lee model (obtained by assigning the values $a = 0$ and $\sigma$), compute the price of the European Put of item 1.
3. Compute the value of the Put option of items 1.–2. in a Vasiček model with parameters $a, \sigma$ and $r_0$ as before, $\vartheta_t = c$ for every $t$ and with $c$ constant and arbitrary (but reasonable).

**Exercise 10.10** By applying the Change of Numéraire technique, prove that the initial value of a European Call option with maturity $T$ and strike $K$, written on a zero-coupon bond (with unit face value) with maturity $S$, when the short rate dynamics is described by the Vasiček model, is given by the following formula:

$$C(0,T,K,S) = p(0,S)\cdot N(d) - p(0,T)\cdot K \cdot N(d-\sigma_p),$$

where:

$$\sigma_p = \sigma(S-T)\sqrt{T}$$
$$d = \frac{1}{\sigma_p}\ln\left(\frac{p(0,S)}{p(0,T)K}\right) + \frac{1}{2}\sigma_p.$$

**Exercise 10.11** By the Black Formula, compute the value (at time $t = 0$) of a Caplet with maturity of 2 years and tenor of one semester in a LIBOR market model, by assuming a flat term structure for the Black (forward) implied volatility $\sigma_i = 0.4$ (on annual basis), and for the initial spot LIBOR at level 4% and a CAP rate $R = 0.04$ (i.e. the forward implied volatility and the LIBOR rate are assumed to be constant over the period considered). The face value is $M = 400.000$ euros.

**Exercise 10.12** Solve the previous exercise proposed with the same data, but now assume that the Black (forward) implied volatility is not constant in time, and it grows linearly through different time intervals: $\sigma_0 = 0.4$, $\sigma_i = 1.5\sigma_{i-1}$, $i = 2, 3, 4$ (it is still constant inside each time interval), then solve again the previous exercise proposed with the same data, but now assume that the LIBOR initial structure is not flat any more and $L_1(0) = 0.04$, $L_i(0) = 1.5L_{i-1}(0)$, $i = 2, 3, 4$.

# Chapter 11
# Pricing Models Beyond Black-Scholes

## 11.1 Review of Theory

In the previous chapters we presented several pricing and hedging problems both in a discrete- and in a continuous-time setting. The basic model assumed in the first case was the binomial model, while for the continuous-time case the Black-Scholes model was assumed to be the framework, and in this last case the dynamics of the risky assets was described by a geometric Brownian motion.

Empirical evidence suggests that these models provide a very rough description of the financial markets' behaviour, and they can be used just as a first approximation of real markets' modelling.

These models are nevertheless still somehow popular since they give several explicit results that are not so easily available in more sophisticated models, and they provide a simple and consistent approach to valuation and hedging problems. In spite of this remark, some empirical features exhibited by asset prices that cannot be explained by the Black-Scholes model are impossible to ignore; we briefly mention the so-called *Aggregational Gaussianity*, *Volatility Clustering*, *Fat Tails* and *Leverage Effects*. The first indicates that prices tend to be more normally distributed when observed on a bigger time-scale, the second that volatility exhibits some kind of persistence behaviour; moreover, the log-returns seem to assume extreme values more likely than expected by a Gaussian distribution (fat tails), and prices look negatively correlated with volatility (when volatility grows, prices go down on average). Finally, a well-known effect exhibited by option prices is completely outside the reach of the Black-Scholes description, and this is the so-called *Volatility Smile*. This can be shortly summarized as follows. The coefficient $\sigma$ entering in the Black-Scholes formula for European Call options can, in principle, be estimated by the asset prices from historical data, but in practice it can be obtained by inverting the same formula if some of these options are actively traded on the market. The volatility value obtained by this second procedure is called Black-Scholes *implied volatility*. Empirical evidence shows that implied volatility exhibits

 
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a strong dependence both on the strike price and on time to maturity, while, if its behaviour is consistent with the Black-Scholes description, it should be independent from both.

In order to obtain a description of financial markets incorporating the previous features, more complex models must be introduced. Models proposed in the literature in the attempt to capture at least some of the previous features can be collected into two main groups: the so-called Stochastic Volatility models and models with jumps.

The *Stochastic Volatility models* propose to model the dynamics of volatility (which is assumed simply to be constant in the Black-Scholes model) by a stochastic differential equation driven by a Brownian motion correlated with the one driving the asset's dynamics. The most popular models of this kind are the following:

$$\begin{cases} dS_t = \mu S_t dt + \sqrt{Y_t} S_t dW_t^{(1)} \\ dY_t = \xi(\eta - Y_t)dt + \beta dW_t^{(2)} \end{cases}, \tag{11.1}$$

$$\begin{cases} dS_t = \mu S_t dt + \sqrt{Y_t} S_t dW_t^{(1)} \\ dY_t = \alpha Y_t dt + \beta Y_t dW_t^{(2)} \end{cases}, \tag{11.2}$$

$$\begin{cases} dS_t = \mu S_t dt + \sqrt{Y_t} S_t dW_t^{(1)} \\ dY_t = \xi(\eta - Y_t)dt + \theta\sqrt{Y_t} dW_t^{(2)} \end{cases}, \tag{11.3}$$

where $Y$ is the square of the volatility process and $\left(W_t^{(1)}\right)_{t\geq 0}$ and $\left(W_t^{(2)}\right)_{t\geq 0}$ are two correlated Brownian motions with $dW_t^{(1)}dW_t^{(2)} = \rho dt$. The first of the models just written was proposed by Stein and Stein [41], the second by Hull and White [26], and the third by Heston [24]. All three models allow to obtain, under suitable simplifying assumptions, some kind of semi-closed form expression for European option prices. Typically, it is possible to obtain an explicit expression for the characteristic function of the log-return distribution, and hence—through a technique pioneered by Carr and Madan [11] and now available in several versions—get an explicit formula for the Fourier transform of the option prices. Once this is available, a standard algorithm for inverting the Fourier transform can provide the option prices for different strikes and maturities. Only the last step of this procedure requires a numerical treatment, and this is the reason why these pricing methods are said to provide solutions in semi-closed form. We shall illustrate by an example how the log-return characteristic function can be computed.

Another direction explored in the attempt to explain some of the relevant features of the market behaviour is the one introducing jumps in the stochastic process describing the asset price dynamics.

The first model of this kind was proposed by Merton [31], and is called the *Jump-Diffusion model* since it combines a geometric Brownian motion dynamics with a Compound Poisson process.

The asset price dynamics is then assumed to be:

$$S_t = S_0 \exp\left\{\mu t + \sigma W_t + \sum_{i=1}^{N_t} Z_i\right\}, \tag{11.4}$$

where the term $\mu t$ represents the usual deterministic drift part, $\sigma W_t$ represents the diffusion term and the term $\sum_{i=1}^{N_t} Z_i$ consists of a Compound Poisson process, i.e. it is the sum on $N_t$ independent and identically distributed random variables $Z_i$, where $N_t$ is a random variable (independent of $W_t$ and all the $Y_i$) distributed as a Poisson with intensity $\lambda t$ (that is, $P(N_t = k) = e^{-\lambda t}(\lambda t)^k/k!$). The original choice proposed by Merton for the jump-size distribution was Gaussian with mean $\gamma$ and variance $\delta^2$, i.e. $Z_i \sim N(\gamma, \delta^2)$.

Also in the jump-diffusion model described above, under suitable simplifying assumptions, some explicit formulas for option prices are available. Merton [31] proved, assuming that the risk-neutral dynamics of the underlying price differs from the same dynamics under the historical measure only in the drift coefficient, that the initial price of a European Call option with strike $K$ and maturity $T$ is provided by the following formula:

$$C_{JD}(S_0, T, K, r, \sigma, \lambda, \gamma, \delta^2) = e^{-\lambda k T} \sum_{n=0}^{+\infty} \frac{(\lambda k T)^n}{n!} C_{BS}(S_0, T, K, \sigma(n), r(n)), \tag{11.5}$$

where $C_{BS}(S_0, T, K, \sigma(n), r(n))$ denotes the value expressed by the Black-Scholes formula of a European Call option with volatility parameter $\sigma(n)$ and risk-free interest rate parameter $r(n)$ with

$$k = \exp\left(\gamma + \frac{\delta^2}{2}\right),$$

$$\sigma(n) = \sigma^2 + n\frac{\delta^2}{T},$$

$$r(n) = r + \lambda(1-k) + \frac{n}{T}\left(\gamma + \frac{\delta^2}{2}\right).$$

Still in the attempt to capture some of the features exhibited by asset price behaviour, some stochastic volatility models in discrete-time have been proposed. Among them we just mention the class of GARCH (Generalized Auto-Regressive Conditional Heteroskedasticity) models introduced by Bollerslev [8], generalizing the class of ARCH models previously proposed by Engle [19]. The asset price dynamics described by the simplest version of the GARCH model, called GARCH(1,1), is the following:

$$\begin{aligned} S_n &= S_{n-1} \exp(X_n) = S_0 \exp\{X_1 + \ldots + X_n\}, \\ X_n &= \sigma_n \varepsilon_n, \\ \sigma_n^2 &= \alpha_0 + \alpha_1 X_{n-1}^2 + \beta\sigma_{n-1}^2 = \alpha_0 + (\alpha_1 \varepsilon_{n-1}^2 + \beta)\sigma_{n-1}^2, \end{aligned}$$

where $\alpha_0 > 0$, $\alpha_1 \geq 0$ and $\beta \geq 0$ are real parameters, $S_0$ and $\sigma_0$ are the initial values of the asset price and the volatility, respectively, and $(\varepsilon_i)_i$ are i.i.d. (independent and identically distributed) random variables, often assumed to be standard normal. This model can explain the volatility persistence mentioned before and can provide fatter tails for the log-returns distributions than those predicted by the log-normal model. The option pricing issues arising in this model have been addressed by some authors [14, 17], but their application goes far beyond the purpose of the present textbook.

Although both the stochastic volatility models and the models with jumps improve the description of financial markets with respect to the Black-Scholes model, they still provide a description of the volatility term structure that is not completely realistic: while stochastic volatility models usually perform quite well in describing volatility smiles for long maturities, models with jumps perform better in capturing volatility smiles for short maturities. Some models have been proposed that incorporate both features, stochastic volatility and jumps, in order to remove the inconsistencies exhibited by both classes of the previous models. Some of them allow to obtain an explicit expression for the characteristic function of log-returns distribution, in such a way that semi-explicit valuation formulas for European options are available. We shall provide a single example of how to compute the log-returns characteristic function for a model of this class. An exhaustive list of examples related to these topics is again far beyond the purpose of this textbook.

## 11.2 Solved Exercises

**Exercise 11.1** Compute the value (at time $t = 0$) of a European Call option with maturity $T = 1$ year, strike $K = 30$ euros, written on an underlying asset whose dynamics is of jump-diffusion type with $\sigma = 0.4$, $r = 0.05$, and $S_0 = 30$ euros; the jump process is a compound Poisson with intensity $\lambda = 1$ (this means an average of one jump each year) and jump-size distributed as a Gaussian with mean $\gamma = 1$ and variance $\delta^2 = 0.49$. Assume that the change of measure only affects the drift coefficient in the SDE describing the asset price dynamics.

**Solution** Assume (as suggested by Merton) that, when the historical probability is replaced by the risk-neutral one, only the drift part of the dynamics is affected. The explicit valuation formula for European Call options obtained by Merton can therefore be applied. Hence we need to compute:

$$C_{JD}(S_0, T, K, r, \sigma, \lambda, \gamma, \delta^2) = e^{-\lambda k T} \sum_{n=0}^{+\infty} \frac{(\lambda k T)^n}{n!} C_{BS}(S_0, T, K, \sigma(n), r(n)), \tag{11.6}$$

where

$$k = \exp\left(\gamma + \frac{\delta^2}{2}\right),$$

$$\sigma^2(n) = \sigma^2 + n\frac{\delta^2}{T},$$

$$r(n) = r + \lambda(1-k) + \frac{n}{T}\left(\gamma + \frac{\delta^2}{2}\right),$$

and $C_{BS}$ denotes the Black-Scholes value of a European Call option computed by assigning the values $\sigma(n)$, $r(n)$ to the volatility and the risk-free interest rate, respectively.

The valuation formula for the Call option in a jump-diffusion setting is actually provided by a series, and only an approximate value can be computed by truncating the series.

We remark that the general term contains a negative exponential-like factor (with $n!$ in the denominator) which makes the convergence rate of this series fast enough; for this reason we decide to consider only the first 5 terms.

By adopting the following notation, $C_{BS}^{(n)} \triangleq C_{BS}(S_0, T, K, \sigma(n), r(n))$, we obtain:

$$C_{BS}^{(0)} \simeq 0, \quad C_{BS}^{(1)} = 16.41, \quad C_{BS}^{(2)} = 12.90, \quad C_{BS}^{(3)} = 23.37, \quad C_{BS}^{(4)} = 25.59.$$

We remark that the first term is similar to the value of a European Call option valuated according to the Black-Scholes formula, since for $n = 0$ no jumps occur in the underlying dynamics. Note that with more than two jumps the probability that the Call option will expire out of the money is negligible. On the other hand, for $n \geq 4$, the factor $1/n!$ makes the contribution very small, allowing us to consider only the contribution of the first 5 terms of the series relevant for the result.

By summing up all these contributions, we finally obtain:

$$\begin{aligned} C_{JD} &\simeq e^{-\lambda k T}\left[C_{BS}^{(0)} + \lambda k T C_{BS}^{(1)} + \frac{(\lambda k T)^2}{2} C_{BS}^{(2)} + \frac{(\lambda k T)^3}{6} C_{BS}^{(3)} + \frac{(\lambda k T)^4}{24} C_{BS}^{(4)}\right] \\ &= 14.05. \end{aligned}$$

**Exercise 11.2** Compute the conditional characteristic function of the log-returns for the stochastic volatility model proposed by Heston:

$$\begin{cases} dS_t = \mu S_t dt + \sqrt{Y_t} S_t dW_t^{(1)} \\ dY_t = \xi(\eta - Y_t)dt + \theta\sqrt{Y_t} dW_t^{(2)} \end{cases}.$$

**Solution** The market model proposed by Heston [24] is incomplete. There is one Brownian motion appearing in the dynamics of the asset price and a second Brownian motion appearing in the square of the diffusion coefficient's dynamics: these are two different (although correlated) sources of randomness, and only one risky asset is traded on the market. We must specify then which one, among the infinitely many equivalent measures turning the discounted asset price into a martingale, we are going to assume describes the market behaviour. For simplicity reasons we are going to assume that the market price of risk associated to the volatility vanishes, i.e. the risk-neutral volatility dynamics is the same under both the historical and the risk-neutral measures.

Applying Itô's formula and a change of measure (from historical to risk-neutral), it is easy to check that the dynamics of the log-return $X_t \triangleq \ln(S_t/S_0)$ is described by the following SDE:

$$\begin{cases} dX_t = \left(r - \frac{Y_t}{2}\right) dt + \sqrt{Y_t} dW_t^{(1)} \\ X_0 = 0 \end{cases}. \tag{11.7}$$

The characteristic function (conditionally to the log-return and volatility values at time $t$) is defined as follows:

$$f(x, y, t) \triangleq E\left[\exp(iuX_T) \,|X_t = x, Y_t = y\right], \quad u \in \mathbb{R}. \tag{11.8}$$

The function $f$, depending on $t, x, y$, must satisfy a stochastic differential equation which can be easily obtained by a straightforward application of Itô's Lemma:

$$\begin{aligned} df = &\left[\frac{1}{2}Y_t\frac{\partial^2 f}{\partial x^2} + \rho\theta Y_t\frac{\partial^2 f}{\partial x \partial y}\right. \\ &\left.+\frac{1}{2}\theta^2 Y_t\frac{\partial^2 f}{\partial y^2} + \left(r - \frac{Y_t}{2}\right)\frac{\partial f}{\partial x} + \xi(\eta - Y_t)\frac{\partial f}{\partial y} + \frac{\partial f}{\partial t}\right] dt \\ &+\sqrt{Y_t}\frac{\partial f}{\partial x} dW_t^{(1)} + \theta\sqrt{Y_t}\frac{\partial f}{\partial y} dW_t^{(2)}. \end{aligned} \tag{11.9}$$

By looking at the definition of the function $f$ we should remark that this is a stochastic process obtained by conditioning an $\mathscr{F}_T$-adapted random variable with respect to the information available at time $t$ (i.e. with respect to the filtration $\mathscr{F}_t$). This means that it must be an $\mathscr{F}_t$-martingale. In a diffusion framework, like the present one, this means that the drift coefficient appearing in the SDE describing the dynamics of $f$ must vanish identically, so that:

$$\frac{1}{2}Y_t\frac{\partial^2 f}{\partial x^2}+\rho\theta Y_t\frac{\partial^2 f}{\partial x\partial y}+\frac{1}{2}\theta^2 Y_t\frac{\partial^2 f}{\partial y^2}+\left(r-\frac{Y_t}{2}\right)\frac{\partial f}{\partial x}+\xi(\eta-Y_t)\frac{\partial f}{\partial y}+\frac{\partial f}{\partial t}=0. \tag{11.10}$$

In order to compute the solution of this PDE we need to impose a final condition (this is a backward parabolic PDE). This turns out to be:

$$f(x,y,T)=e^{iux}. \tag{11.11}$$

Inspired by PDEs of similar kind encountered in the previous chapters, we make an educated guess of the following type:

$$f(x,y,t)=\exp\left\{A(\tau)+yB(\tau)+iux\right\}, \tag{11.12}$$

where $\tau=T-t$ is the time to maturity. By plugging this expression for $f$ in PDE (11.10), we get

$$\begin{aligned}&fY_t\left[-\frac{1}{2}u^2+\rho\theta iuB(\tau)+\frac{1}{2}B^2(\tau)-\frac{iu}{2}-\xi B(\tau)\right]\\&+f\left[riu+\xi\eta B(\tau)-A'(\tau)-yB'(\tau)\right]=0.\end{aligned}$$

By imposing that both the sum of terms multiplying $y$ and the sum of the terms in which $y$ does not appear vanish separately, we obtain the following system of ordinary differential equations together with their initial conditions:

$$\begin{cases}A'(\tau)=\xi\eta B(\tau)+iur, & A(0)=0\\ B'(\tau)=(iu\rho\theta-\xi)\,B(\tau)+\frac{\theta^2}{2}B^2(\tau)-\frac{u^2+iu}{2}, & B(0)=0\end{cases}.$$

Although non-linear, the previous system exhibits some nice features: once the differential equations satisfied by $B$ is solved, the first equation can be immediately reduced to quadratures and the unknown function $A$ can be computed as an integral. The second equation does not contain the unknown $A$ and is a differential equation which admits a closed-form solution; it is actually a Riccati equation with constant coefficients, and the explicit solution can be found by separating the variables as follows:

$$\frac{dB}{aB^2+bB+c}=d\tau, \tag{11.13}$$

where $a = \theta^2/2$, $b = iu\rho\theta - \xi$ , $c = -(u^2 + iu)/2$. The right-hand side of the previous equation can be written as follows:

$$\frac{[a\,(y_2 - y_1)]^{-1}\,dB}{B - y_2} - \frac{[a\,(y_2 - y_1)]^{-1}\,dB}{B - y_1}, \tag{11.14}$$

where $y_{1,2}$ are the roots of the second-order algebraic equation $aB^2 + bB + c = 0$:

$$y_1 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} = \frac{1}{\theta^2}\left[\xi - iu\rho\theta - \sqrt{(iu\rho\theta - \xi)^2 + \theta^2\left(u^2 + iu\right)}\right]$$
$$y_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} = \frac{1}{\theta^2}\left[\xi - iu\rho\theta + \sqrt{(iu\rho\theta - \xi)^2 + \theta^2\left(u^2 + iu\right)}\right].$$

By integrating the two rational functions and inverting the solution obtained, we finally get:

$$\begin{cases} A(\tau) = iru\tau + \frac{\xi\eta\tau(\xi - i\rho\theta u)}{\theta^2} - \frac{2\xi\eta}{\theta^2}\ln\left[\cosh\left(\frac{\beta\tau}{2}\right) + \frac{\xi - i\rho\theta u}{\beta}\sinh\left(\frac{\beta\tau}{2}\right)\right] \\ B(\tau) = -\frac{u^2 + iu}{\beta\coth\left(\frac{\beta\tau}{2}\right) + \xi - i\rho\theta u}, \end{cases}, \tag{11.15}$$

where $\beta = \sqrt{(\xi - i\rho\theta u)^2 + \theta^2\left(u^2 + iu\right)}$. By substituting $A(\tau)$ and $B(\tau)$ in (11.12), we obtain the characteristic function we were looking for.

**Exercise 11.3** Compute the characteristic function of the log-returns in the jump-diffusion model proposed by Merton:

$$S_t = S_0 \exp\left\{\mu t + \sigma W_t + \sum_{i=1}^{N_t} Y_i\right\}. \tag{11.16}$$

**Solution** The log-return process in the jump-diffusion model proposed by Merton is the following:

$$X_t = \ln\left(\frac{S_t}{S_0}\right)\mu t + \sigma W_t + \sum_{i=1}^{N_t} Y_i. \tag{11.17}$$

It is the sum of a linear deterministic component ($\mu t$), a diffusion component (a Brownian motion with diffusion coefficient $\sigma$), and a jump part represented by a compound Poisson process with intensity $\lambda$ and normal jump-size distribution.

The characteristic function of $X_t$ is given by:

$$E\left[\exp\left(iuX_t\right)\right] = E\left[\exp\left\{iu\left(\mu t + \sigma W_t + \sum_{i=1}^{N_t} Z_i\right)\right\}\right], \quad u \in \mathbb{R}. \tag{11.18}$$

We remark that in the Merton model the jump and the diffusion processes are assumed to be independent, so that the expectation above can be computed as the product of the expectations of each contribution. The first contribution is the characteristic function of a Brownian motion with drift, i.e. it is the characteristic function of a normal random variable with mean $\mu t$ and variance $\sigma^2 t$:

$$E\left[\exp\left\{iu\left(\mu t + \sigma W_t\right)\right\}\right] = \exp\left\{iu\mu t - iu^2\frac{\sigma^2}{2}t\right\}. \tag{11.19}$$

The characteristic function of the compound Poisson process can be computed as follows, by conditioning with respect to the variable $N_t$:

$$\begin{aligned} E\left[\exp\left(iu\sum_{i=1}^{N_t} Z_i\right)\right] &= E\left[E\left[\exp\left(iu\sum_{i=1}^{N_t} Z_i\right)\middle| N_t\right]\right] \\ &= \sum_{n=0}^{+\infty} E\left[\exp\left(iu\sum_{i=1}^{N_t} Z_i\right)\middle| N_t = n\right] P(N_t = n). \end{aligned}$$

Since the $Z_i$ are independent and identically distributed, the last conditional expectation in the previous expression becomes:

$$\begin{aligned} E\left[\exp\left(iu\sum_{i=1}^{N_t} Z_i\right)\middle| N_t = n\right] &= E\left[\prod_{i=0}^{n} \exp(iuZ_i)\right] \\ &= \prod_{i=0}^{n} E[\exp(iuZ_i)] \\ &= [\phi(u)]^n, \end{aligned}$$

where $\phi(u)$ is the characteristic function of the random variable $Z_i$. For a normal jump-size distribution with mean $\gamma$ and variance $\delta^2$, we have:

$$\phi(u) = \exp\left\{i\gamma u - \frac{\delta^2}{2}u^2\right\}. \tag{11.20}$$

Finally, we obtain:

$$
\begin{aligned}
E\left[\exp\left(iu\sum_{i=1}^{N_t} Z_i\right)\right] &= \sum_{n=0}^{+\infty} E\left[\exp\left(iu\sum_{i=1}^{N_t} Z_i\right)\middle| N_t = n\right] P(N_t = n) \\
&= \sum_{n=0}^{+\infty} [\phi(u)]^n \frac{e^{-\lambda t}(\lambda t)^n}{n!} \\
&= \exp\left\{\lambda t\left(\phi(u) - 1\right)\right\}.
\end{aligned}
$$

Putting together the results in (11.19) and (11.21), we conclude that

$$
E\left[\exp(iuX_t)\right] = \exp\left\{iu\mu t - iu^2\frac{\sigma^2}{2}t\right\}\cdot\exp\left\{\lambda t(\phi(u)-1)\right\}. \tag{11.21}
$$

**Exercise 11.4** By combining the results obtained in Exercises 11.2–11.3, compute the characteristic function of the log-returns for the following model proposed by Bates [5]:

$$
\begin{cases}
S_t = S_0 \exp\left\{\mu t + \sqrt{Y_t} W_t^{(1)} + \sum_{i=1}^{N_t} Z_i\right\} \\
dY_t = \xi(\eta - Y_t)dt + \theta\sqrt{Y_t}dW_t^{(2)}
\end{cases}.
$$

Specify by which equivalent martingale measure you are going to describe the risk-neutral dynamics of the asset price.

**Solution** As far as the jump-diffusion dynamics of the asset price is concerned, we shall work under the usual assumption that the change of measure from the historical to the risk-neutral probability affects only the drift component, in such a way that in order to get a local martingale for the discounted asset price dynamics we can write:

$$
S_t = S_0\exp\left\{\left(r - \frac{Y_t}{2} - \lambda\kappa\right)t + \sqrt{Y_t}W_t^{(1)} + \sum_{i=1}^{N_t} Z_i\right\}, \tag{11.22}
$$

where $\kappa$ is defined by:

$$
\kappa = \lambda\left\{\exp\left(\gamma + \frac{\delta^2}{2}\right) - 1\right\}. \tag{11.23}
$$

Moreover, we shall assume a vanishing risk-premium for the volatility component, so that the variance dynamics under the risk-neutral measure remains unchanged.

Under these assumptions, the characteristic function can be computed by observing that in the model under consideration the jump process (the compound Poisson) appearing in the asset price dynamics is independent of the diffusion part.

Consequently, the characteristic function of the log-returns is simply the product of the two characteristic functions of the diffusion and the jump components, which can be computed separately. The diffusion component is just the same as the one appearing in the Heston model (computed explicitly in Exercise 11.2), while the characteristic function of the compound Poisson has been computed in Exercise 11.3. The characteristic function of the log-return $X_t = \ln(S_t/S_0)$ is the given by:

$$\begin{aligned} E\left[\exp\left(iuX_t\right)\right] = \exp\left\{i\left(r-\lambda\kappa\right)ut + \frac{\xi\eta t\left(\xi - i\rho\theta u\right)}{\theta^2}\right\}\cdot \\ \cdot\left[\cosh\left(\frac{\beta t}{2}\right) + \frac{\xi - i\rho\theta u}{\beta}\sinh\left(\frac{\beta t}{2}\right)\right]^{-\frac{2\xi\eta}{\theta^2}}\cdot \\ \cdot\exp\left\{-\frac{u^2+iu}{\beta\coth\left(\frac{\beta t}{2}\right) + \xi - i\rho\theta u}Y_0\right\}\cdot\exp\left\{\lambda t\left(\phi(u)-1\right)\right\}, \end{aligned}$$

where $\phi(u)$ is the characteristic function of a normal random variable with mean $\gamma$ and variance $\delta^2$, that is,

$$\phi(u) = \exp\left\{i\gamma u - \frac{\delta^2}{2}u^2\right\}. \tag{11.24}$$

**Exercise 11.5** Prove that the solution of the recurrence equation describing the GARCH(1,1) model is the following:

$$\begin{cases} \sigma_n^2 = \alpha_0\left[1 + \sum_{i=1}^n \prod_{j=1}^i (\alpha_1\varepsilon_{n-j}^2 + \beta)\right] \\ X_n = \varepsilon_n\sqrt{\alpha_0\left[1 + \sum_{i=1}^n \prod_{j=1}^i (\alpha_1\varepsilon_{n-j}^2 + \beta)\right]} \end{cases}. \tag{11.25}$$

**Solution** The equation characterizing the GARCH(1,1) model is a particular case of a class of recurrence equations of the following form:

$$M_n = A_n M_{n-1} + B_n, \tag{11.26}$$

where $(A_n)_{n\in\mathbb{N}}$ and $(B_n)_{n\in\mathbb{N}}$ are sequences of independent and identically distributed (i.i.d.) random variables. The solution of this equation can be easily obtained by induction

$$\begin{aligned} M_n &= A_n\left(A_{n-1}M_{n-2} + B_{n-1}\right) + B_n \\ &= B_n + \sum_{i=1}^{k}\left(B_{n-i}\prod_{j=0}^{i-1}A_{n-j}\right) + M_{n-k-1}\prod_{i=0}^{k}A_{n-i}. \end{aligned}$$

In order to ensure the convergence of this expression, as $k \to +\infty$ the third term must vanish and the second must converge. By the Strong Law of Large Numbers we know that:

$$\frac{1}{k+1}\sum_{i=0}^{k} \ln|A_{n-i}| \to E[\ln|A_n|], \quad P\text{-almost surely.} \tag{11.27}$$

So, under the assumption $E[\ln|A_n|] < 0$, we have:

$$\begin{aligned}\prod_{i=0}^{k}|A_{n-i}| &= \exp\left\{\sum_{i=0}^{k}\ln|A_{n-i}|\right\}\\ &= \left(\exp\left\{\frac{1}{k+1}\sum_{i=0}^{k}\ln|A_{n-i}|\right\}\right)^{k+1} \to 0, \quad P\text{-a.s.}\end{aligned}$$

By proceeding in a similar way we can prove that the condition $E[\ln|A_n|] < 0$ ensures that the term:

$$\sum_{i=1}^{k}\left(B_{n-i}\prod_{j=0}^{i-1}A_{n-j}\right) \tag{11.28}$$

converges as $k \to \infty$, in such a way that the unique solution is

$$M_n = B_n + \sum_{i=1}^{+\infty}\left(B_{n-i}\prod_{j=0}^{i-1}A_{n-j}\right). \tag{11.29}$$

In the particular case of the GARCH(1,1) models, $M_n = \sigma_n^2$, $A_n = \alpha_1\varepsilon_{n-1}^2 + \beta$ and $B_n = \alpha_0$ (with $B_{n-i} = 0$ for $i > n$). Equation (11.29) implies:

$$\sigma_n^2 = \alpha_0\left[1 + \sum_{i=1}^{n}\prod_{j=1}^{i}(\alpha_1\varepsilon_{n-j}^2 + \beta)\right]. \tag{11.30}$$

Since $X_n = \sigma_n\varepsilon_n$, (11.25) follows.

**Exercise 11.6** Recall that a time series $(X_n)_{n\in\mathbb{Z}}$ is called stationary (in strict sense) if the random vector $(X_{t_1}, ..., X_{t_n})$ has the same distribution of $(X_{t_1+k}, ..., X_{t_n+k})$ for all $t_1, ..., t_n, k \in \mathbb{Z}$ and for all $n \in \mathbb{N}$.

Prove that for the GARCH(1,1) model the condition $\alpha_1 + \beta < 1$ is sufficient to ensure that the sequence $(\sigma_n)_{n\in\mathbb{N}}$ is stationary.

**Solution** The unique solution of the recurrence relation $M_n = A_n M_{n-1} + B_n$ ($n \in \mathbb{Z}$), under the assumption $E[\ln |A_n|] < 0$, has been shown (see Exercise 11.5) to be the following:

$$M_n = B_n + \sum_{i=1}^{+\infty} \left( B_{n-i} \prod_{j=0}^{i-1} A_{n-j} \right).$$

For a GARCH(1,1) model, $A_n = \alpha_1 \varepsilon_{n-1}^2 + \beta$ and $B_n = \alpha_0$ (with $B_{n-i} = 0$ for $i > n$). The condition $E[\ln |A_n|] < 0$, ensuring $(\sigma_n)_{n \in \mathbb{Z}}$ is stationary, becomes $E\left[\ln\left(\alpha_1 \varepsilon_n^2 + \beta\right)\right] < 0$, which in turn translates into the condition $\alpha_1 + \beta < 1$, since the innovations $(\varepsilon_i)_i$ are i.i.d. standard normal random variables. The solution obtained is a function of the i.i.d. random variables $A_n$, $B_n$, and this implies that the process $(M_n)_{n \in \mathbb{Z}}$ is strictly stationary.

## 11.3 Proposed Exercises

**Exercise 11.7** Compute the value (at time $t = 0$) of a European Call option with maturity $T = 1$ year, strike $K = 50$ euros, written on an underlying asset whose dynamics is of jump-diffusion type with $\sigma = 0.8$, $r = 0.04$, and $S_0 = 45$ euros; the jump process is a compound Poisson with intensity $\lambda = 0.5$ and jump-size distributed as a normal with mean $\gamma = 2$ and variance $\delta^2 = 0.64$. Assume that the change of measure only affects the drift coefficient in the SDE describing the asset price dynamics.

**Exercise 11.8** Compute the conditional characteristic function of the log-returns for the stochastic volatility model proposed by Stein and Stein and under the same assumptions of Exercise 11.2:

$$\begin{cases} dS_t = \mu S_t dt + \sqrt{Y_t} S_t dW_t^{(1)} \\ dY_t = \xi(\eta - Y_t)dt + \beta dW_t^{(2)} \end{cases}.$$

**Exercise 11.9** Consider a GARCH(1,1) model with parameters $\alpha_0 = 0.7$, $\alpha_1 = 0.5$ and $\beta = 0.4$. Verify that the sequence $\sigma_n^2$ is stationary and write explicitly $\sigma_{10}^2$.

**Exercise 11.10** An ARCH(1) model describes the log-returns (squared) $X_n^2$ of a financial asset through the following recurrence equation:

$$X_n^2 = \alpha_0 \varepsilon_n^2 + \alpha_1 \varepsilon_n^2 X_{n-1}^2.$$

Assume that $(\varepsilon_n)_n$ is a sequence of i.i.d. standard normal random variables, and prove that the condition $\alpha_1 < 3.562$ is sufficient to ensure $\left(X_n^2\right)_n$ is strictly stationary, and also that the solution of the recurrence equation is the following:

$$X_n^2 = \alpha_0 \sum_{i=0}^{+\infty} \left( \alpha_1^i \prod_{j=0}^{i} \varepsilon_{n-j}^2 \right).$$

# Chapter 12
# Risk Measures: Value at Risk and Beyond

## 12.1 Review of Theory

In the late 90s an increasing interest has been developing towards risk measures, in particular the Value at Risk (VaR) and the Conditional Value at Risk (CVaR). The use of such risk measures is due, on the one hand, to the rules imposed by the Basel Accord on the deposit of margins by banks and financial institutions because of the financial risks they are exposed to. On the other hand, these tools are important to quantify the riskiness assumed by an investor or an intermediary because of his financial transactions.

We recall briefly some notions on risk measures. For a detailed treatment and further details, we refer to Föllmer and Schied [21], Hull [25], Jorion [27] and Barucci et al. [4], among many others.

Consider the Profit & Loss (P&L) or the return of a financial position at a future date $T$, and denote with $\mathscr{X}$ the family of random variables representing P&Ls or returns of the financial positions taken into account. The space $\mathscr{X} = L^{\infty}(\Omega, \mathscr{F}, P)$ is often considered, where $L^{\infty}(\Omega, \mathscr{F}, P)$ (or, simply, $L^{\infty}$) is formed by all the essentially bounded random variables, i.e. by random variables $X$ satisfying $P(|X| \leq K) = 1$ for some $K > 0$.

Given any random variable $X \in \mathscr{X}$, the *Value at Risk* at the level $\alpha \in (0, 1)$ of $X$ is defined as

$$VaR_{\alpha}(X) \triangleq -q_{\alpha}^{+}(X) = -\inf\{x \in \mathbb{R} : F_X(x) > \alpha\}, \tag{12.1}$$

i.e. as the opposite of the greatest $\alpha$-quantile of $X$. Notice that, for continuous random variables $X$, $VaR_{\alpha}(X)$ is nothing but the opposite of the quantile $q_{\alpha}(X)$ solution of $F_X(x) = \alpha$.

The financial interpretation of the $VaR$ is the following. If $VaR_{\alpha}(X) > 0$, then $VaR_{\alpha}(X)$ represents the minimal amount to be deposited as a margin for $X$.

 
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La Matematica per il 3+2 149, https://doi.org/10.1007/978-3-031-28378-9_12

We list now some results on the $VaR$ for stock returns of the following form:

$$\frac{\Delta S}{S_0} \cong \mu \cdot \Delta t + \sigma \cdot \varepsilon \cdot \sqrt{\Delta t},$$

where $\Delta t = T$ is the length of the period of time considered, $S_0 > 0$ is the initial stock price, $\mu \in \mathbb{R}$ its drift, $\sigma > 0$ its volatility and $\varepsilon \sim N(0; 1)$.

The $VaR$ of the Profit & Loss of a stock of current price $S_0$ and daily return distributed as a normal with drift $\mu \in \mathbb{R}$ and volatility $\sigma > 0$, is:

$$VaR_\alpha(\Delta S) = S_0 \left[\sigma\sqrt{\Delta t} \cdot N^{-1}(1-\alpha) - \mu \cdot \Delta t\right].$$

The $VaR$ of the Profit & Loss of a portfolio formed by $x_i$ (for $i = 1, 2, \ldots, n$) stocks $i$ with current prices $S_0^i$, where the daily returns are jointly normal distributed with drift $\mu_i \in \mathbb{R}$, volatility $\sigma_i > 0$ and correlation $\rho_{ij}$ $(i, j = 1, 2, \ldots n)$, is:

$$VaR_\alpha(\Delta P) = N^{-1}(1-\alpha) \cdot \sqrt{\Delta t} \sqrt{\sum_{i=1}^n \sum_{j=1}^n x_i x_j S_0^i S_0^j \sigma_i \sigma_j \rho_{ij}} - \sum_{j=1}^n x_j S_0^j \mu_j (\Delta t). \tag{12.2}$$

In particular: if all drifts are zero, then

$$VaR_\alpha(\Delta P) = N^{-1}(1-\alpha) \cdot \sqrt{\Delta t} \sqrt{\sum_{i=1}^n \sum_{j=1}^n x_i x_j S_0^i S_0^j \sigma_i \sigma_j \rho_{ij}}. \tag{12.3}$$

The first-order approximation (Delta approximation) of the $VaR$ for the Profit & Loss of a portfolio formed by $x_i$ $(i = 1, 2, \ldots, n)$ options having Delta $\Delta_i$, underlying stocks of current prices $S_0^i$ and daily returns that are jointly normal with null drifts, volatility $\sigma_i > 0$ and correlation $\rho_{ij}$ $(i, j = 1, 2, \ldots n)$, is given by:

$$VaR_\alpha^{\Delta\, approx}(\Delta P) \cong N^{-1}(1-\alpha) \cdot \sqrt{\Delta t} \sqrt{\sum_{i=1}^n \sum_{j=1}^n x_i x_j \Delta_0^i \Delta_0^j S_0^i S_0^j \sigma_i \sigma_j \rho_{ij}}. \tag{12.4}$$

Although the $VaR$ satisfies monotonicity (and, more in general, consistency with respect to first-order stochastic dominance), cash-invariance and positive homogeneity (recalled below), many authors (see, among others, Artzner et al. [1]) have underlined its several drawbacks. First of all, the $VaR$ does not encourage diversification of risk (namely, it does not satisfies subadditivity); furthermore, it does not distinguish tails that are even very different at the left of $-VaR$ (i.e. for losses greater than $VaR$). The need to overcome the latter problem led to the introduction of the so-called Tail Conditional Expectation. More precisely, if $X$ is a random variable having finite expected value $E[X]$, the *Tail Conditional Expectation* (TCE) of $X$ at level $\alpha \in (0, 1)$ is defined as:

$$TCE_\alpha(X) \triangleq E_P[-X | X \leq -VaR_\alpha(X)] = \frac{E_P\left[-X\mathbf{1}_{\{X \leq -VaR_\alpha(X)\}}\right]}{P(X \leq -VaR_\alpha(X))}. \tag{12.5}$$

Unfortunately, neither the $TCE$ encourages diversification in general. For further details, see Artzner et al. [1] and Delbaen [15].

In order to amend the drawbacks of $VaR$ and $TCE$ but at the same time preserve the main properties, Artzner et al. [1] and Delbaen [15] introduced the so-called *coherent* risk measures. More details on coherent risk measures and, in particular, on the $CVaR$ can be found in Artzner et al. [1], Delbaen [15], Föllmer and Schied [21], Rockafellar and Uryasev [38], Barucci et al. [4] and Fiori et al. [20], among many others.

A *coherent* risk measure $\rho$ is defined (see [1], [15]) as a risk measure $\rho : \mathscr{X} \to \mathbb{R}$ satisfying the following properties:

- monotonicity: if $X \leq Y$, $P$-a.s., with $X, Y \in \mathscr{X}$, then $\rho(X) \geq \rho(Y)$;
- cash-invariance: $\rho(X + c) = \rho(X) - c$ for any $X \in \mathscr{X}$, $c \in \mathbb{R}$;
- subadditivity: $\rho(X + Y) \leq \rho(X) + \rho(Y)$ for any $X, Y \in \mathscr{X}$;
- positive homogeneity: $\rho(\lambda X) = \lambda \cdot \rho(X)$ for any $X \in \mathscr{X}$, $\lambda \geq 0$.

When $\mathscr{X} = L^\infty$, the most relevant examples of coherent risk measures are the Conditional Value at Risk and the risk measures generated by a set of probability measures. The *Conditional Value at Risk* at level $\alpha \in (0, 1)$ is defined by:

$$CVaR_\alpha(X) \triangleq \inf_{x \in \mathbb{R}} \left\{ \frac{E[(x - X)^+]}{\alpha} - x \right\}; \tag{12.6}$$

while a risk measure associated to a set $\mathscr{Q}$ of probability measures (known as set of generalized scenarios) is given by:

$$\rho_{\mathscr{Q}}(X) \triangleq \sup_{Q \in \mathscr{Q}} E_Q[-X]. \tag{12.7}$$

Any coherent risk measure defined on a finite sample space or satisfying a further hypothesis on continuity can be represented as in (12.7) (see Artzner et al. [1]), and conversely.

It is also well known (see Föllmer and Schied [21]) that

$$CVaR_\alpha(X) = \frac{E[(q_\alpha - X)^+]}{\alpha} - q_\alpha, \tag{12.8}$$

where $q_\alpha$ is an arbitrary $\alpha$-quantile of $X$. Moreover, if $X$ is a continuous random variable, then $CVaR_\alpha(X) = TCE_\alpha(X)$.

Since $\rho(X) > 0$ can be understood as the amount to deposit as a margin for $X$, the acceptance set $\mathscr{A}$ of $\rho$ is defined as the set

$$\mathscr{A}_\rho \triangleq \{X \in \mathscr{X} : \rho(X) \leq 0\}.$$

Coherent risk measures and acceptance sets are in one-to-one correspondence. Consider, for simplicity, a finite sample space. Then

- If $\mathscr{A}$ satisfies the following properties $(A)$:
  $\mathscr{A} \supseteq \{X \in \mathscr{X} : X \geq 0\}$
  $\mathscr{A} \cap \{X \in \mathscr{X} : X \leq 0 \text{ with } X(\omega) < 0 \text{ for some } \omega \in \Omega\} = \emptyset$
  $\mathscr{A}$ is a convex cone,
  then $\rho_{\mathscr{A}}(X) \triangleq \inf\{m \in \mathbb{R} : m + X \in \mathscr{A}\}$ is a coherent risk measure.
- If $\rho$ is a coherent risk measure, then $\mathscr{A}_\rho$ satisfies properties $(A)$ and $\rho_{\mathscr{A}_\rho} = \rho$.

Several extensions of coherent risk measures are studied in the literature both in a static and in a dynamic setting. Problems concerning the choice of portfolios that minimize risk, capital allocations, and dynamic risk measures and their time-consistency will be addressed in some of the exercises below. For portfolio optimization we refer to Gaivoronski and Pflug [23], for the capital allocation problem to Kalkbrener [28], Centrone and Rosazza Gianin [12], Dhaene et al. [16], Tasche [42], while for (dynamic time-consistent) risk measures see Artzner et al. [2], Cheridito and Stadje [13], Föllmer and Schied [21], Frittelli and Rosazza Gianin [22], among many others.

## 12.2 Solved Exercises

**Exercise 12.1** Consider two stocks whose Profit & Loss (per year) are represented, respectively, by the following random variables:

$$X = \begin{cases} -20; & \text{on } \omega_1 \\ -8; & \text{on } \omega_2 \\ 0; & \text{on } \omega_3 \\ 12; & \text{on } \omega_4 \end{cases}; \qquad Y = \begin{cases} 6; & \text{on } \omega_1 \\ 0; & \text{on } \{\omega_2, \omega_3\} \\ -2; & \text{on } \omega_4 \end{cases}$$

with $P(\omega_1) = 0.01$, $P(\omega_2) = 0.09$, $P(\omega_3) = 0.8$ and $P(\omega_4) = 0.1$.

1. Consider the portfolio composed by 3 shares of the first stock (corresponding to $X$) and by 4 shares of the other (corresponding to $Y$). Establish whether risk diversification is encouraged or not by $VaR$ at 1%.
2. Consider the portfolio composed by one share of the first stock and by 8 shares of the second. Establish whether risk diversification is encouraged or not by $VaR$ at 10%.

3. Consider the same portfolio as in item 2. Establish whether diversification of risk is encouraged by $CVaR$ at 10% and compute the profit due to diversification.

**Solution**

1. By the Profit & Loss of the two stocks we get the cumulative distribution functions of $X$ and $Y$:

$$F_X(x) = \begin{cases} 0; & \text{if } x < -20 \\ 0.01; & \text{if } -20 \le x < -8 \\ 0.1; & \text{if } -8 \le x < 0 \\ 0.9; & \text{if } 0 \le x < 12 \\ 1; & \text{if } x \ge 12 \end{cases} \qquad F_Y(x) = \begin{cases} 0; & \text{if } x < -2 \\ 0.1; & \text{if } -2 \le x < 0 \\ 0.99; & \text{if } 0 \le x < 6 \\ 1; & \text{if } x \ge 6 \end{cases}$$

As a consequence, we deduce that

$$VaR_{0.01}(X) = -q^{+}_{0.01}(X) = -\inf\{x \in \mathbb{R} : F_X(x) > 0.01\} = 8$$
$$VaR_{0.01}(Y) = -q^{+}_{0.01}(Y) = -\inf\{x \in \mathbb{R} : F_Y(x) > 0.01\} = 2.$$

Let us consider a portfolio composed by 3 shares of $X$ and by 4 shares of $Y$. The Profit & Loss of such a portfolio is then given by

$$3X + 4Y = \begin{cases} -60 + 24; & \text{on } \omega_1 \\ -24; & \text{on } \omega_2 \\ 0; & \text{on } \omega_3 \\ 36 - 8; & \text{on } \omega_4 \end{cases} = \begin{cases} -36; & \text{on } \omega_1 \\ -24; & \text{on } \omega_2 \\ 0; & \text{on } \omega_3 \\ 28; & \text{on } \omega_4 \end{cases}$$

whose cumulative distribution function is

$$F_{3X+4Y}(x) = \begin{cases} 0; & \text{if } x < -36 \\ 0.01; & \text{if } -36 \le x < -24 \\ 0.1; & \text{if } -24 \le x < 0 \\ 0.9; & \text{if } 0 \le x < 28 \\ 1; & \text{if } x \ge 28 \end{cases}.$$

Since

$$VaR_{0.01}(3X + 4Y) = 24 < 3 \cdot VaR_{0.01}(X) + 4 \cdot VaR_{0.01}(Y) = 32,$$

we can conclude that, for such a portfolio, $VaR$ at 1% encourages diversification of risk. The profit due to diversification (in terms of cash-saving on the margin to be deposited) is then 8 euros.

2. By the cumulative distribution functions of $X$ and of $Y$ found above, we deduce that

$$VaR_{0.1}(X) = -q^+_{0.1}(X) = -\inf\{x \in \mathbb{R} : F_X(x) > 0.1\} = 0$$
$$VaR_{0.1}(Y) = -q^+_{0.1}(Y) = -\inf\{x \in \mathbb{R} : F_Y(x) > 0.1\} = 0.$$

Consider now a portfolio composed by one share of $X$ and by 8 shares of $Y$. The Profit & Loss of such a portfolio is then given by

$$X + 8Y = \begin{cases} 28; & \text{on } \omega_1 \\ -8; & \text{on } \omega_2 \\ 0; & \text{on } \omega_3 \\ -4; & \text{on } \omega_4 \end{cases}.$$

The cumulative distribution function of the Profit & Loss of the portfolio above is then

$$F_{X+8Y}(x) = \begin{cases} 0; & \text{if } x < -8 \\ 0.09; & \text{if } -8 \le x < -4 \\ 0.19; & \text{if } -4 \le x < 0 \\ 0.99; & \text{if } 0 \le x < 28 \\ 1; & \text{if } x \ge 28 \end{cases},$$

hence we can conclude that risk diversification is not encouraged by $VaR$ at 10%. In fact,

$$VaR_{0.1}(X + 8Y) = 4 > VaR_{0.1}(X) + 8VaR_{0.1}(Y) = 0.$$

3. By formula (12.8) for the computation of $CVaR$ and by item 2., we obtain

$$CVaR_{0.1}(X) = \frac{E[(q_{0.1}(X)-X)^+]}{\alpha} - q_{0.1}(X) = \frac{20\cdot 0.01+8\cdot 0.09}{0.1} = 9.2$$

$$CVaR_{0.1}(Y) = \frac{2\cdot 0.1}{0.1} = 2$$

$$CVaR_{0.1}(X + 8Y) = \frac{4\cdot 0.09}{0.1} + 4 = 7.6,$$

by considering the quantiles $q^+_{0.1}(X) = q^+_{0.1}(Y) = 0$ and $q^+_{0.1}(X + 8Y) = -4 = q^-_{0.1}(X + 8Y)$. Since

$$CVaR_{0.1}(X + 8Y) = 7.6 < CVaR_{0.1}(X) + 8 \cdot CVaR_{0.1}(Y) = 25.2,$$

we conclude that there is diversification of risk with respect to $CVaR$ for the portfolio considered above (as well as for any other portfolio, because of the properties of the $CVaR$) with a profit (in terms of cash-saving on the margin to be deposited) of 17.6 euros.

**Exercise 12.2** Consider a portfolio whose Profit & Loss is represented by the following random variable:

$$X = \begin{cases} -400; & \text{on } \omega_1 \\ -160; & \text{on } \omega_2 \\ -40; & \text{on } \omega_3 \\ 0; & \text{on } \omega_4 \\ 80; & \text{on } \omega_5 \\ 1200; & \text{on } \omega_6 \end{cases}$$

with $P(\omega_1) = 0.01$, $P(\omega_2) = p$, $P(\omega_3) = q$, $P(\omega_4) = 0.8$, $P(\omega_5) = 0.05$ and $P(\omega_6) = 0.05$, and $p, q > 0$ so that $P$ is a probability measure.

1. Compute the Value at Risk at 1% and at 10% of $X$. Would we find a different result with a maximum loss of 4000 (instead of 400)?
2. Find (if possible) $p$ and $q$ such that the $VaR$ at 5% of $X$ is equal to 40 and the Tail Conditional Expectation at 5% of $X$ does not exceed 100.

**Solution** The condition $p + q = 0.09$ has to be satisfied in order for $P$ to be a probability measure. Moreover, the cumulative distribution function of $X$ (depending on $p, q$) is

$$F_X(x) = \begin{cases} 0; & \text{if } x < -400 \\ 0.01; & \text{if } -400 \le x < -160 \\ 0.01 + p; & \text{if } -160 \le x < -40 \\ 0.1; & \text{if } -40 \le x < 0 \\ 0.9; & \text{if } 0 \le x < 80 \\ 0.95; & \text{if } 80 \le x < 1200 \\ 1 & \text{if } x \ge 1200 \end{cases},$$

because $0.01 + p + q = 0.1$ (by the arguments above).

1. By the results above, it is straightforward to deduce that $VaR_{0.01}(X) = 160$ and $VaR_{0.1}(X) = 0$. It is easy to check (also using $F_X$) that such results would remain unchanged even if the maximum loss was 4000 instead of 400.
2. The conditions to be imposed on $p$ and $q$ (so that the $VaR$ at 5% of $X$ is equal to 40 and the Tail Conditional Expectation at 5% of $X$ does not exceed 100) are the following:

$$\begin{cases} p + q = 0.09 & (P \text{ prob. measure}) \\ 0.01 + p \le 0.05 & (\text{condition on } VaR) \\ 40 + 1600p + 400q \le 100 & (\text{condition on } TCE) \end{cases} \tag{12.9}$$

where the last inequality is due to

$$TCE_{0.05}(X) = E_P[-X|X \le -VaR_{0.05}(X)] = \frac{E_P[-X\mathbf{1}_{\{X\le -40\}}]}{P(X\le -40)} \\ = \frac{400\cdot 0.01+160p+40q}{0.1} = 40+1600p+400q.$$

Solving system (12.9) we obtain

$$\begin{cases} q = 0.09 - p \\ p \le 0.04 \\ p \le 0.02 \end{cases} \qquad \begin{cases} 0 < p \le 0.02 \\ q = 0.09 - p \end{cases}.$$

A pair of admissible $p$ and $q$ is, for instance, $p = 0.02$ and $q = 0.07$.

**Exercise 12.3** Consider two stocks (A and B) whose daily returns are jointly normally distributed. The correlation of the stock returns is 0.25, the current price of stock A is 4.2 euros, its daily drift 0.0002 and its daily volatility 1.2%, while the current price of the stock B is 3.6 euros, its daily drift 0.0008 and its daily volatility 2.8%.

1. Find the $VaR$ (at 1% and on a time interval of 10 days) of the portfolio (PORT1) composed by 20 shares of A and by 15 shares of B.
2. Consider two other stocks (C and D) having the same features as stocks A and B, respectively, but both having daily drift equal to zero.
   Compute the $VaR$ (at 1% and on a time interval of 10 days) of the portfolio (PORT2) composed by 20 shares of C and 15 shares of D.
3. Consider now a portfolio (PORT3) composed by 20 options (with Delta $\Delta_C = 0.4$) on the underlying C and by 15 options (with Delta $\Delta_D = -0.1$) on the underlying D. Give the Delta-approximation of the $VaR$ (at 1% and on a time interval of 10 days) of portfolio PORT3.

**Solution**

1. Applying formula (12.2) for the computation of the $VaR$ of a portfolio of stocks whose returns are jointly normally distributed, we get immediately that the $VaR$ (at 1% and on a time interval of 10 days) of portfolio PORT1 is

$$\begin{aligned} VaR_{0.01}^{10gg}&\left(\Delta P^{port1}\right) \\ &= N^{-1}(1-\alpha)\cdot\sqrt{\Delta t}\cdot\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j S_0^i S_0^j \sigma_i\sigma_j\rho_{ij}} \\ &\quad -\sum_{j=1}^{n} x_j S_0^j \mu_j(\Delta t) \\ &= N^{-1}(0.99)\cdot\sqrt{10}\cdot\left[(20\cdot 4.2\cdot 0.012)^2+(15\cdot 3.6\cdot 0.028)^2\right. \\ &\quad \left.+2\cdot 0.25\cdot(20\cdot 4.2\cdot 0.012\cdot 15\cdot 3.6\cdot 0.028)\right]^{1/2} \\ &\quad -[20\cdot 4.2\cdot 0.0002+15\cdot 3.6\cdot 0.0008]\cdot 10 \\ &= 14.831-0.6=14.231. \end{aligned}$$

In other words, the amount of cash to be deposited as a margin for PORT1 is equal to 14.231 euros.

2. Since the features of stocks C and D are the same as those of A and B, except for the drifts that are zero for C and D, the $VaR$ of portfolio PORT2 can be obtained by taking only the first term in the computation above for the $VaR$ of PORT1, i.e.

$$VaR_{0.01}^{10gg}\left(\Delta P^{port2}\right) = N^{-1}(1-\alpha)\cdot\sqrt{\Delta t}\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j S_0^i S_0^j \sigma_i \sigma_j \rho_{ij}} = 14.831.$$

In other words, the amount of money to be deposited as a margin for PORT2 is 14.831 euros.

3. Applying formula (12.4) for the Delta approximation of the $VaR$ of a portfolio of options having underlyings with jointly normal returns, we obtain immediately that the $VaR$ (at 1% and on a time interval of 10 days) of portfolio PORT3 is

$$\begin{aligned} VaR_{0.01}^{10gg}\left(\Delta P^{port3}\right) \\ &= N^{-1}(1-\alpha)\cdot\sqrt{\Delta t}\cdot\sqrt{\textstyle\sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j \Delta_0^i \Delta_0^j S_0^i S_0^j \sigma_i \sigma_j \rho_{ij}} \\ &= \ldots = 2.896. \end{aligned}$$

In other words, the amount of cash to be deposited as a margin for PORT3 is 2.896 euros. Note that this margin is much smaller than the one of PORT2. This ca be explained by the fact that some options in PORT3 have positive $\Delta$, others negative $\Delta$. Hence, PORT3 is more diversified than PORT2.

**Exercise 12.4** Consider the daily log-return of a stock represented by the random variable $X$ of mean 0.0002 and standard deviation 0.02.

1. Compute the $VaR$ of $X$ at 5% and on a period of 1 day, once $X$ is assumed to be normally distributed.
2. Applying the Cornish-Fisher expansion, compute the $VaR$ of $X$ at 5% and on a period of 1 day when the skewness of $X$ is supposed to be equal to $(-0.3)$.

**Solution** Denote by $m$, $s$ and $\xi$ the mean, the standard deviation and the skewness of $X$, respectively.

1. If $X$ is distributed as a normal with mean $m = 0.0002$ and standard deviation $s = 0.02$, then the $VaR$ of $X$ at 5% and on a period of 1 day is given by

$$VaR_{0.05}(X) = s\cdot N^{-1}(0.95) - m = 0.0327.$$

2. Applying the Cornish-Fisher expansion, the $VaR$ of $X$ at $\alpha = 0.05$ and on a period of 1 day can be approximated as follows:

$$VaR_{0.05}(X) \cong -m - s \cdot \left[ N^{-1}(0.05) + \frac{1}{6}\xi\left( \left( N^{-1}(0.05) \right)^2 - 1 \right) \right] = 0.0344.$$

**Exercise 12.5** Consider a financial position that is distributed as a Pareto with parameters $x_0 = 100$ and $a = 2$.

1. Compute the $VaR$ at 10% of the position. Is a margin of 400 euros sufficient as a deposit?
2. What about the $VaR$ at 1%?
3. Compute the $CVaR$ at 10% of the position. Is the margin of 400 euros also sufficient in this case?

**Solution** Denote by $X$ the random variable representing the P&L of the initial position. The cumulative distribution function and the density function of $X$ are, respectively,

$$F_X(x) = \begin{cases} 0; & \text{if } x < x_0 \\ 1 - \left(\frac{x_0}{x}\right)^a; & \text{if } x \geq x_0 \end{cases}$$

$$f_Y(y) = \begin{cases} 0; & \text{if } x < x_0 \\ a x_0^a x^{-a-1}; & \text{if } x \geq x_0 \end{cases}$$

First of all, note that $E[X] = \frac{a}{a-1}x_0 = 200$ and that $X \notin L^\infty$ since $X$ may take any value greater or equal to $x_0$.

1. Because $X$ is a continuous random variable, the $\alpha_0$-quantile of $X$ solves

$$F_X(x) = \alpha_0.$$

It follows, therefore, that

$$\begin{aligned} \left(\frac{x_0}{x}\right)^a &= 1 - \alpha_0 \\ x &= \frac{x_0}{(1-\alpha_0)^{1/a}}. \end{aligned}$$

For $\alpha_0 = 0.10$, we obtain that $q_{0.10}(X) = \frac{100}{\sqrt{0.9}} = 105.41$ and that $VaR_{0.10}(X) = -105.41$, hence that $X$ is acceptable with respect to $VaR_{0.10}$ as it is and it would not require any extra margin to be deposited. Furthermore, $VaR_{0.10}(X + 400) = -505.41$, so the margin of 400 euros would be (more than) enough for $X$.

2. The $VaR$ at $\alpha_1 = 0.01$ of $X$ can be computed in (at least) two ways. The first way consists in proceeding as before, so to obtain

$$VaR_{0.01}(X) = -\frac{x_0}{(1-\alpha_1)^{1/a}} = -100.50.$$

The second consists in recalling (see Barucci et al. [4], Fiori et al. [20]) that for Pareto distributed random variables one has

$$VaR_{\alpha_1}(X) = VaR_{\alpha_0}(X) \cdot \left(\frac{1-\alpha_0}{1-\alpha_1}\right)^{1/a},$$

hence

$$VaR_{\alpha_1}(X) = -105.41 \cdot \left(\frac{0.9}{0.99}\right)^{1/2} = -100.50.$$

As from the item above, $X$ would not require any extra margin to be deposited. The margin of 400 euros would be therefore sufficient also when the riskiness of $X$ is evaluated by means of $VaR$ at the level 1%.

3. Since $X$ does not belong to $L^\infty$ but to $L^1$, it is easy to check that $CVaR_{\alpha_0}(X) = TCE_{\alpha_0}(X) = E_P\left[-X | X \leq -VaR_{\alpha_0}(X)\right]$ is still true. Hence:

$$CVaR_{\alpha_0}(X) = E_P\left[-X | X \leq -VaR_{\alpha_0}(X)\right] = -\frac{E_P\left[X\mathbf{1}_{\{X\leq -VaR_{\alpha_0}(X)\}}\right]}{P\left(X \leq -VaR_{\alpha_0}(Y)\right)}. \tag{12.10}$$

By item 1., we deduce that $VaR_{\alpha_0}(X) = -\frac{x_0}{(1-\alpha_0)^{1/a}}$ and that $P\left(X \leq -VaR_{\alpha_0}(X)\right) = \alpha_0$. Set now $x^* = \frac{x_0}{(1-\alpha_0)^{1/a}} = -VaR_{\alpha_0}(X)$. We obtain that

$$\begin{aligned} E_P\left[X\mathbf{1}_{\{X\leq -VaR_{\alpha_0}(X)\}}\right] &= \int_{x_0}^{x^*} a x_0^a x^{-a} dx = \left[\frac{a}{1-a} x_0^a x^{1-a}\right]\Bigg|_{x_0}^{x^*} \\ &= \frac{a}{a-1} x_0 \left[1 - \frac{1}{(1-\alpha_0)^{1/a-1}}\right] \\ &= E[X] + \frac{a}{a-1}(1-\alpha_0)\, VaR_{\alpha_0}(X). \end{aligned}$$

From the argument above and from (12.10) it follows that

$$
\begin{aligned}
CVaR_{\alpha_0}(X) &= E_P\left[-X|X \le -VaR_{\alpha_0}(X)\right] \\
&= -\frac{E[X] + \frac{a}{a-1}(1-\alpha_0)\,VaR_{\alpha_0}(X)}{\alpha_0} \\
&= -\frac{200 + 2\cdot 0.9\cdot(-105.41)}{0.1} = -102.62.
\end{aligned}
$$

We can then conclude that a margin of 400 euros would be (more than) enough also in this case.

**Exercise 12.6** Consider a portfolio whose Profit & Loss is represented by the following random variable:

$$
X = \begin{cases} -1200; & \text{on } \omega_1 \\ -400; & \text{on } \omega_2 \\ -80; & \text{on } \omega_3 \\ 0; & \text{on } \omega_4 \\ 16; & \text{on } \omega_5 \\ x; & \text{on } \omega_6 \end{cases},
$$

where $16 < x < 4000$, $P(\omega_1) = 0.01$, $P(\omega_2) = 0.04$, $P(\omega_3) = 0.05$, $P(\omega_4) = 0.7$, $P(\omega_5) = 0.1$ and $P(\omega_6) = 0.1$.

Find the smallest $x$ such that, according to the coherent risk measure generated by the set $\mathcal{Q} = \{P, Q_1\}$ (with $Q_1(\omega_1) = Q_1(\omega_2) = Q_1(\omega_3) = Q_1(\omega_6) = 1/8$ and $Q_1(\omega_4) = Q_1(\omega_5) = 1/4$), the margin to be deposited for $X$ is smaller than or equal to 200.

**Solution** First of all, recall that the coherent risk measure generated by the set of generalized scenarios $\mathcal{Q} = \{P, Q_1\}$ is given by $\rho_{\mathcal{Q}}(X) \triangleq \sup_{Q\in\mathcal{Q}} E_Q[-X]$. Let us start by computing $E_P[-X]$ and $E_{Q_1}[-X]$:

$$
\begin{aligned}
E_P[-X] &= 1200\cdot 0.01 + 400\cdot 0.04 + 80\cdot 0.05 \\
&\quad +0 - 16\cdot 0.1 - x\cdot 0.1 = 30.4 - \frac{x}{10} \\
E_{Q_1}[-X] &= \frac{1200}{8} + \frac{400}{8} + \frac{80}{8} + 0 - \frac{16}{4} - \frac{x}{8} = 206 - \frac{x}{8}.
\end{aligned}
$$

Since $30.4 - \frac{x}{10} \ge 206 - \frac{x}{8}$ holds if and only if $x \ge 7024$, in the present case $(16 < x < 4000)$ we deduce that

$$
\rho_{\mathcal{Q}}(X) = \sup_{Q\in\mathcal{Q}} E_Q[-X] = \sup\left\{E_P[-X]; E_{Q_1}[-X]\right\} = 206 - \frac{x}{8}.
$$

In order to fulfill the constraint on the margin for $X$ (according to $\rho_{\mathscr{Q}}$), one should have $206 - \frac{x}{8} \leq 200$, hence $x \geq 48$.

Consequently, 48 is the smallest $x \in (16, 4000)$ verifying the conditions required in terms of margin for $X$.

**Exercise 12.7** Consider a sample space with only two elementary events, i.e. $\Omega = \{\omega_1, \omega_2\}$.

Find the coherent risk measure $\rho_{\mathscr{A}}$ associated to the acceptance set

$$\mathscr{A} = \left\{(x, y) \in \mathbb{R}^2 : y \geq -2x; \quad y \geq -\frac{x}{2}\right\}.$$

Establish which of the following positions is preferred according to the risk measure $\rho_{\mathscr{A}}$:

$$X_1 = \begin{cases} -30; & \text{on } \omega_1 \\ 150; & \text{on } \omega_2 \end{cases}; \qquad X_2 = 0.$$

**Solution** It is easy to check that the acceptance set $\mathscr{A}$ (see Fig. 12.1) satisfies all the properties $(A)$, hence the risk measure

$$\rho_{\mathscr{A}}(X) = \inf\{m \in \mathbb{R} : m + X \in \mathscr{A}\}$$

associated to $\mathscr{A}$ is coherent.

Since $\Omega$ has only two elements, any random variable $X$ on $\Omega$ can be identified with $(X(\omega_1), X(\omega_2)) \in \mathbb{R}^2$. Note that $m + X \in \mathscr{A}$ if and only if

$$\begin{cases} m + X(\omega_2) \geq -2(m + X(\omega_1)) \\ m + X(\omega_2) \geq -\frac{m+X(\omega_1)}{2} \end{cases} \begin{cases} m \geq -\frac{2}{3}X(\omega_1) - \frac{1}{3}X(\omega_2) \\ m \geq -\frac{1}{3}X(\omega_1) - \frac{2}{3}X(\omega_2) \end{cases}.$$

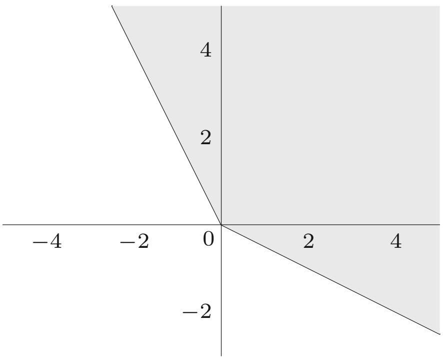


**Fig. 12.1** Acceptance set $\mathscr{A}$

It follows that

$$m \geq \sup \left\{ -\frac{2}{3} X(\omega_1) - \frac{1}{3} X(\omega_2); -\frac{1}{3} X(\omega_1) - \frac{2}{3} X(\omega_2) \right\},$$

hence

$$\rho_{\mathscr{A}}(X) = \sup \left\{ -\frac{2}{3} X(\omega_1) - \frac{1}{3} X(\omega_2); -\frac{1}{3} X(\omega_1) - \frac{2}{3} X(\omega_2) \right\}.$$

In terms of sets of generalized scenarios, $\rho_{\mathscr{A}}$ can also be written as

$$\rho_{\mathscr{A}}(X) = \sup_{Q \in \{Q_1, Q_2\}} E_Q[-X]$$

where $Q_1(\omega_1) = \frac{2}{3}$, $Q_1(\omega_2) = \frac{1}{3}$ and $Q_2(\omega_1) = \frac{1}{3}$, $Q_2(\omega_2) = \frac{2}{3}$.

For what concerns positions $X_1$ and $X_2$, we obtain that

$$\rho_{\mathscr{A}}(X_1) = \sup \left\{ -\frac{2}{3} \cdot (-30) - \frac{1}{3} \cdot 150; -\frac{1}{3} \cdot (-30) - \frac{2}{3} \cdot 150 \right\} = -30$$
$$\rho_{\mathscr{A}}(X_2) = 0,$$

hence both positions are acceptable even if $X_1$ is preferable over $X_2$ (with respect to $\rho_{\mathscr{A}}$).

**Exercise 12.8** Consider a sample space with only two elementary events, i.e. $\Omega = \{\omega_1, \omega_2\}$.

Find the risk measure $\rho_{\mathscr{A}}$ associated to the acceptance set

$$\mathscr{A} = \left\{ (x, y) \in \mathbb{R}^2 : y \geq \frac{x}{2}; \quad y \leq 4x \right\}$$

and verify that $\rho_{\mathscr{A}}$ does not satisfy monotonicity.

**Solution** It is easy to check that the acceptance set $\mathscr{A}$ (see Fig. 12.2) does not satisfy properties $(A)$ (in particular $\mathscr{A} \not\supseteq \mathbb{R}^2_+$). Consequently, the risk measure $\rho_{\mathscr{A}}$ associated to such a set is not coherent.

Since $\Omega$ has only two elements, any random variable $X$ on $\Omega$ can be identified with $(X(\omega_1), X(\omega_2)) \in \mathbb{R}^2$. Note that $m + X \in \mathscr{A}$ if and only if

$$\begin{cases} m + X(\omega_2) \geq \frac{m + X(\omega_1)}{2} \\ m + X(\omega_2) \leq 4(m + X(\omega_1)) \end{cases} \begin{cases} m \geq X(\omega_1) - 2X(\omega_2) \\ m \geq -\frac{4}{3} X(\omega_1) + \frac{1}{3} X(\omega_2) \end{cases}.$$

It follows that

$$m \geq \sup \left\{ X(\omega_1) - 2X(\omega_2); -\frac{4}{3} X(\omega_1) + \frac{1}{3} X(\omega_2) \right\},$$

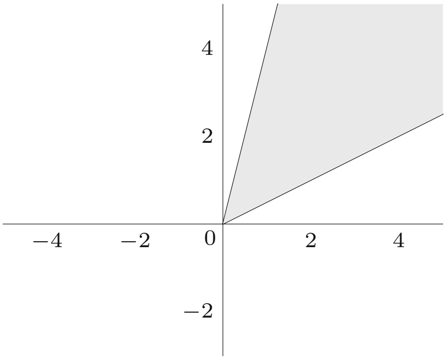


**Fig. 12.2** Acceptance set $\mathscr{A}$

hence

$$\rho_{\mathscr{A}}(X) = \sup\left\{X(\omega_1) - 2X(\omega_2); -\frac{4}{3}X(\omega_1) + \frac{1}{3}X(\omega_2)\right\}.$$

It is immediate to verify that $\rho_{\mathscr{A}}$ satisfies subadditivity, positive homogeneity and cash-invariance.

The following counterexamples shows that monotonicity is not guaranteed by $\rho_{\mathscr{A}}$.

Taking, for instance,

$$X_1 = \begin{cases} 0; & \text{on } \omega_1 \\ 30; & \text{on } \omega_2 \end{cases},$$

we obtain that $X_1 \geq 0$ but $\rho_{\mathscr{A}}(X_1) = 10 > 0 = \rho_{\mathscr{A}}(0)$, so $X_1 \notin \mathscr{A}$. In other words, even if $X_1$ will never give a loss, an amount of 10 euros has to be deposited as a margin.

Furthermore, take the two positions:

$$Y_1 = \begin{cases} -30, & \text{on } \omega_1 \\ 60, & \text{on } \omega_2 \end{cases}; \qquad Y_2 = \begin{cases} -6, & \text{on } \omega_1 \\ 180, & \text{on } \omega_2 \end{cases}.$$

We obtain that $\rho_{\mathscr{A}}(Y_2) > \rho_{\mathscr{A}}(X_1)$ even if $Y_2(\omega) \geq Y_1(\omega)$ for any $\omega \in \Omega$. In fact,

$$\rho_{\mathscr{A}}(Y_1) = \sup\left\{-30 - 120; \frac{4}{3} \cdot 30 + \frac{1}{3} \cdot 60\right\} = 60$$

$$\rho_{\mathscr{A}}(Y_2) = \sup\left\{-6 - 360; \frac{4}{3} \cdot 6 + \frac{1}{3} \cdot 180\right\} = 68.$$

It is also straightforward to check that

$$Y_2 - Y_1 = \begin{cases} 24, & \text{on } \omega_1 \\ 120, & \text{on } \omega_2 \end{cases} \geq 0$$

and $\rho_{\mathscr{A}}(Y_2 - Y_1) = 8 > 0$. Consequently, $Y_2 - Y_1 \notin \mathscr{A}$.

**Exercise 12.9** Consider a portfolio formed by two stocks whose returns are jointly distributed as bivariate normal. Suppose that the mean returns of the stocks are, respectively, 10% and 6% per year, their standard deviations 40% and 30% per year and their correlation amounts to 0.5.

1. Supposing short-selling is not allowed, find the optimal portfolio so to minimize the Conditional Value at Risk of the portfolio return [minus its mean] at level $\alpha = 0.01$ and so to have an average return of 6.5% at least. What about the optimal portfolio for an average return of at least 8%?
2. Show that the optimal portfolio of the previous item is the same when the riskiness is measured by Value at Risk, Conditional Value at Risk or standard deviation.

**Solution** Our goal is to construct a portfolio by investing a percentage $x_1$ in one stock (with return $R_1$) and a percentage $x_2$ in the other stock (with return $R_2$). Since short-selling is not allowed, $x_1$ and $x_2$ have to satisfy:

$$\begin{cases} x_1, x_2 \geq 0, \\ x_1 + x_2 = 1, \end{cases}$$

hence $x_1 = x \in [0, 1]$ and $x_2 = 1 - x$.

Let $\mu_i = E[R_i]$, $\sigma_i = \sqrt{V(R_i)}$ (for $i = 1, 2$) and denote with $\rho$ the correlation between $R_1$ and $R_2$. Since $R_1$ and $R_2$ are jointly normal, then also the portfolio return

$$R_p = x_1 R_1 + x_2 R_2 = x R_1 + (1 - x) R_2$$

is distributed as a normal with mean $E[R_p] = xE[R_1] + (1 - x)E[R_2] = x(\mu_1 - \mu_2) + \mu_2$ and with variance

$$V(R_p) = V(xR_1 + (1-x)R_2) = x^2\sigma_1^2 + (1-x)^2\sigma_2^2 + 2\rho\sigma_1\sigma_2 x(1-x). \tag{12.11}$$

1. When Conditional Value at Risk is used as a criterion for risk minimization, the optimization problem can be written as

$$\min_{\substack{x \in [0,1]; \\ E[R_p] \geq l}} \quad CVaR_\alpha\left(R_p - E[R_p]\right), \tag{12.12}$$

where $l$ stands for the target on the portfolio return.

For $\mu_1 > \mu_2$, the constraint in the minimization becomes $x \in [0, 1]$ such that $x \geq \frac{l-\mu_2}{\mu_1-\mu_2}$. Hence, the minimization problem above reduces to

$$\min_{\substack{x \in [0,1]; \\ x \geq \frac{l-\mu_2}{\mu_1-\mu_2}}} \quad CVaR_\alpha\left(R_p - E[R_p]\right). \tag{12.13}$$

Since $R_p - E[R_p] \sim N(0; V(R_p))$ and $CVaR_\alpha(Y) = -m + s\frac{N'(N^{-1}(\alpha))}{\alpha}$ for $Y \sim N(m; s^2)$ (see, for instance, Barucci et al. [3]) with $N'(\cdot)$ denoting the density function of a standard normal, problem (12.13) becomes

$$\min_{\substack{x \in [0,1]; \\ x \geq \frac{l-\mu_2}{\mu_1-\mu_2}}} \left\{\frac{N'(N^{-1}(\alpha))}{\alpha}\sqrt{x^2(\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2)-2x(\sigma_2^2-\rho\sigma_1\sigma_2)+\sigma_2^2}\right\}. \tag{12.14}$$

It is then easy to check that the unconstrained minimum is attained at $\tilde{x} = \frac{\sigma_2^2-\rho\sigma_1\sigma_2}{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}$. In our case, $\tilde{x} = \frac{3}{13}$.

Consider now $l_1 = 0.065$. Since the constraints on $x$ become $x \in [0, 1]$ and $x \geq \frac{l_1-\mu_2}{\mu_1-\mu_2} = \frac{1}{8}$, the constrained minimization is obtained for $x^* = \frac{3}{13}$. The optimal portfolio consists thus in investing in $\frac{3}{13}$ shares of the first stock and in $\frac{10}{13}$ shares of the second stock. The corresponding value of the variance of the portfolio is $V(R_p) = 0.127$ and, consequently, the Conditional Value at Risk is $CVaR_{0.01}(R_p - E[R_p]) = 0.950$.

When $l_2 = 0.08$, instead, the constraints on $x$ become $x \in [0, 1]$ and $x \geq \frac{l_2-\mu_2}{\mu_1-\mu_2} = 0.5$ and the constrained minimization is obtained for $x^* = 0.5$. The optimal portfolio consists thus in investing in 0.5 shares of the first stock and in 0.5 shares of the second stock. The corresponding value of the variance of the portfolio is $V(R_p) = 0.155$ and, consequently, the Conditional Value at Risk is $CVaR_{0.01}(R_p - E[R_p]) = 1.049$. As one could expect, the minimal riskiness of this last portfolio is greater than the other (we have indeed required an average portfolio return greater than before).

2. Consider now the optimization problem studied above, where the criterion of risk minimization of Conditional Value at Risk is replaced by that of Value at Risk or of standard deviation.

   Because of the normality assumption on the joint distribution of $(R_1, R_2)$ (hence, normality of $R_p$), the $VaR$ minimization reduces to

$$\min_{\substack{x \in [0,1];\\ x \geq \frac{l-\mu_2}{\mu_1-\mu_2}}} \left\{ N^{-1}(1-\alpha)\sqrt{x^2(\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2)-2x(\sigma_2^2-\rho\sigma_1\sigma_2)+\sigma_2^2} \right\}, \tag{12.15}$$

   while the minimization with respect to the standard deviation becomes

$$\min_{\substack{x \in [0,1];\\ x \geq \frac{l-\mu_2}{\mu_1-\mu_2}}} \sqrt{x^2(\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2)-2x(\sigma_2^2-\rho\sigma_1\sigma_2)+\sigma_2^2}. \tag{12.16}$$

   By the arguments above and since $N^{-1}(1-\alpha) > 0$ for $\alpha = 0.01$, in the present case the optimal composition of the portfolio is the same with respect to the $CVaR$ criterion, to the $VaR$ criterion and to the standard deviation criterion. Indeed, the minimum in problems (12.13), (12.15) and (12.16) is always attained at the same $x^*$. The minimal riskiness of the portfolio is, respectively, equal to

$$\begin{aligned} CVaR_{0.01}(R_p) &= 0.950 \\ VaR_{0.01}(R_p) &= 0.828 \\ \sigma(R_p) &= 0.356 \end{aligned}$$

   for $l_1 = 0.065$, while for $l_2 = 0.08$

$$\begin{aligned} CVaR_{0.01}(R_p) &= 1.049 \\ VaR_{0.01}(R_p) &= 0.916 \\ \sigma(R_p) &= 0.394. \end{aligned}$$

**Exercise 12.10** Consider a financial investment whose Profit and Loss in 2 years is represented by the random variable:

$$X = \begin{cases} 100, & \text{on } \omega_1 \\ 20, & \text{on } \omega_2 \\ 0, & \text{on } \omega_3 \\ -10, & \text{on } \omega_4 \\ -40, & \text{on } \omega_5 \end{cases},$$

where the filtrated probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t=0,1,2}, P)$ is given by

$$\begin{aligned} \Omega &= \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\} \\ U &= \{\omega_1, \omega_2\} \\ D &= \{\omega_3, \omega_4, \omega_5\} \\ \mathscr{F}_0 &= \{\emptyset, \Omega\} \\ \mathscr{F}_1 &= \{\emptyset, U, D, \Omega\} \\ \mathscr{F}_2 &= \mathscr{P}(\Omega) \end{aligned}$$

and

$$\begin{array}{lll} P(U) = 0.8; & P(D) = 0.2 & \\ P(\omega_1|U) = 0.4; & P(\omega_2|U) = 0.6 & \\ P(\omega_3|D) = 0.6; & P(\omega_4|D) = 0.3; & P(\omega_5|D) = 0.1 \end{array}$$

1. Verify whether $VaR^{(0)}_{0.05}(X) = VaR^{(0)}_{0.05}(-VaR^{(1)}_{0.05}(X))$, where $VaR^{(t)}_{\alpha}(X)$ stands for the Value at Risk of $X$ at time $t$.[1]
2. Verify whether $CVaR^{(0)}_{0.05}(X) = CVaR^{(0)}_{0.05}(-CVaR^{(1)}_{0.05}(X))$, where $CVaR^{(t)}_{\alpha}(X)$ stands for the Conditional Value at Risk of $X$ at time $t$.
3. Consider now the dynamic risk measure $(\rho_t)_{t=0,1,2}$ defined as

$$\rho_t(X) \triangleq \operatorname*{ess\,sup}_{Q \in \mathscr{Q}} E_Q[-X|\mathscr{F}_t], \tag{12.17}$$

[1] We recall (see, among many others, [2]) that a dynamic risk measure is a family $(\rho_t)_{t=0,1,2}$ defined on a space $\mathscr{X}$ of random variables such that $\rho_t(X)$ is $\mathscr{F}_t$-measurable, i.e. it takes into account all the information available until time $t$. In particular, $\rho_0(X) \in \mathbb{R}$. A risk measure is then said to be time-consistent if $\rho_0(-\rho_t(X)) = \rho_0(X)$ for any $X \in \mathscr{X}$ and $t \in [0, T]$.

where $\mathscr{Q} = \{P, Q_1, Q_2, Q_3\}$ and

$$
\begin{aligned}
&Q_1(U) = Q_2(U) = Q_3(U) = 0.8;\ Q_1(D) = Q_2(D) = Q_3(D) = 0.2\\
&Q_1(\omega_1|U) = Q_2(\omega_1|U) = 0.5;\ Q_1(\omega_2|U) = Q_2(\omega_2|U) = 0.5\\
&Q_3(\omega_1|U) = 0.4;\ Q_3(\omega_2|U) = 0.6;\ Q_1(\omega_3|D) = Q_3(\omega_3|D) = 0.4\\
&Q_1(\omega_4|D) = Q_3(\omega_4|D) = 0.4;\ Q_1(\omega_5|D) = Q_3(\omega_5|D) = 0.2\\
&Q_2(\omega_3|D) = 0.6;\ Q_2(\omega_4|D) = 0.3;\ Q_2(\omega_5|D) = 0.1
\end{aligned}
$$

Verify whether $\rho_0(X) = \rho_0(-\rho_1(X))$ or not.

**Solution** By the evolution of information in time, we obtain the following tree:

$$
\begin{array}{ccccccc}
 & & & & & & \omega_1 \\
 & & & & \nearrow & & \\
 & & U = \{\omega_1, \omega_2\} & & & & \\
 & \nearrow & & & \searrow & & \\
 & & & & & & \omega_2 \\
\cdot & & & & & & \\
 & & & & & & \omega_3 \\
 & \searrow & & & \nearrow & & \\
 & & D = \{\omega_3, \omega_4, \omega_5\} & & \rightarrow & & \omega_4 \\
 & & & & \searrow & & \\
 & & & & & & \omega_5 \\
--- & & --------- & & & & ---- \\
0 & & 1\text{ year} & & & & 2\text{ years}
\end{array}
$$

Furthermore, it also follows that

$$
\begin{aligned}
&P(\omega_1) = 0.32; \quad && P(\omega_2) = 0.48;\\
&P(\omega_3) = 0.12; \quad && P(\omega_4) = 0.06;\\
&P(\omega_5) = 0.02.
\end{aligned}
$$

1. In order to compute the Value at Risk of $X$ at the initial time 0 we can just consider the last nodes of the tree and the corresponding probabilities (evaluated by means of $P$). We deduce that

$$
VaR_{0.05}^{(0)}(X) = 10. \tag{12.18}
$$

Let us compute now $VaR_{0.05}^{(1)}(X)$, a random variable taking two values: one at the node U (denoted by $VaR_{0.05}^{(1,U)}(X)$), the other at the node D (denoted by $VaR_{0.05}^{(1,D)}(X)$).

Note that $VaR_{0.05}^{(1,U)}(X)$ is the $VaR$ of the random variable $X$ on the sub-tree starting from time $t = 1$ and node $U$. More precisely:

$$
\begin{array}{ccc}
 & & \omega_1 \\
 & \nearrow & \\
U = \{\omega_1, \omega_2\} & & \\
 & \searrow & \\
 & & \omega_2 \\
------- & & ---- \\
\text{1 year} & & \text{2 years}
\end{array}
$$

or, in terms of $X$ and $P$,

$$
\begin{array}{ccc}
 & & X(\omega_1) = 100 \\
 & \nearrow^{prob. 0.4} & \\
U & & \\
 & \searrow^{prob. 0.6} & \\
 & & X(\omega_2) = 20 \\
---- & & --------- \\
\text{1 year} & & \text{2 years}
\end{array}
$$

It follows that $VaR_{0.05}^{(1,U)}(X) = -20$.

In order to compute $VaR_{0.05}^{(1,D)}(X)$ we can proceed as previously by considering the sub-tree starting from time $t = 1$ and node $D$. More precisely:

$$
\begin{array}{ccc}
 & & \omega_3 \\
 & \nearrow & \\
D = \{\omega_3, \omega_4, \omega_5\} & \rightarrow & \omega_4 \\
 & \searrow & \\
 & & \omega_5 \\
--------- & & ---- \\
\text{1 year} & & \text{2 years}
\end{array}
$$

or, in terms of $X$ and $P$,

$$
\begin{array}{ccc}
 & & X(\omega_3) = 0 \\
 & \nearrow^{prob.0.6} & \\
D & \rightarrow^{prob.0.3} & X(\omega_4) = -10 \\
 & \searrow^{prob.0.1} & \\
 & & X(\omega_5) = -40 \\
----- & & ----------- \\
1 \text{ year} & & 2 \text{ years}
\end{array}
$$

It follows that $VaR_{0.05}^{(1,D)}(X) = 40$.

Finally, to find $VaR_{0.05}^{(0)}(-VaR_{0.05}^{(1)}(X))$ we proceed backwards once again by considering the sub-tree starting from 0 and ending at nodes $U$, $D$:

$$
\begin{array}{ccc}
 & & U = \{\omega_1, \omega_2\} \\
 & \nearrow & \\
\cdot & & \\
 & \searrow & \\
 & & D = \{\omega_3, \omega_4, \omega_5\} \\
--- & & ---------- \\
0 & & 1 \text{ year}
\end{array}
$$

or, in terms of $VaR_{0.05}^{(1)}(X)$ and $P$,

$$
\begin{array}{ccc}
 & & -VaR_{0.05}^{(1,U)}(X) = 20 \quad \text{(with prob. 0.8)} \\
 & \nearrow & \\
\cdot & & \\
 & \searrow & \\
 & & -VaR_{0.05}^{(1,D)}(X) = -40 \ \text{(with prob. 0.2)} \\
--- & & ------------- \\
0 & & 1 \text{ year}
\end{array}
$$

Hence

$$VaR_{0.05}^{(0)}(-VaR_{0.05}^{(1)}(X)) = 40 \neq 10 = VaR_{0.05}^{(0)}(X),$$

i.e. the dynamic version of $VaR$ is not time-consistent.

2. Proceeding as above, we deduce that

$$CVaR_{0.05}^{(0)}(X) = \frac{E[(-10 - X)^+]}{0.05} + 10 = 22. \tag{12.19}$$

$CVaR^{(1)}_{0.05}(X)$ is then a random variable taking two values: $CVaR^{(1,U)}_{0.05}(X)$ at node U, $CVaR^{(1,D)}_{0.05}(X)$ at node D.

Note that $CVaR^{(1,U)}_{0.05}(X)$ is the $CVaR$ of the random variable $X$ on the subtree starting from time $t = 1$ and node $U$. More precisely:

$$
\begin{array}{ccc}
 & & \omega_1 \\
 & \nearrow & \\
U = \{\omega_1, \omega_2\} & & \\
 & \searrow & \\
 & & \omega_2 \\
------- & & ---- \\
1 \text{ year} & & 2 \text{ years}
\end{array}
$$

or, in terms of $X$ and $P$,

$$
\begin{array}{ccc}
 & & X(\omega_1) = 100 \\
 & \nearrow prob.0.4 & \\
U & & \\
 & \searrow prob.0.6 & \\
 & & X(\omega_2) = 20 \\
---- & & ------- \\
1 \text{ year} & & 2 \text{ years}
\end{array}
$$

It follows that $CVaR^{(1,U)}_{0.05}(X) = \frac{(20-100)^+ \cdot 0.4 + (20-20)^+ \cdot 0.6}{0.05} - 20 = -20$. Similarly, we obtain that $CVaR^{(1,D)}_{0.05}(X) = 40$ and

$$CVaR^{(0)}_{0.05}(-CVaR^{(1)}_{0.05}(X)) = 40 \neq 22 = CVaR^{(0)}_{0.05}(X), \tag{12.20}$$

i.e. the dynamic version of $CVaR$ is not time-consistent.

3. We compute now $\rho_0(X) = \sup_{Q \in \{P, Q_1, Q_2, Q_3\}} E_Q[-X]$. It is easy to check what follows:

| $\omega_i$ | $P$ | $Q_1$ | $Q_2$ | $Q_3$ |
|---|---|---|---|---|
| $\omega_1$ | 0.32 | 0.4 | 0.4 | 0.32 |
| $\omega_2$ | 0.48 | 0.4 | 0.4 | 0.48 |
| $\omega_3$ | 0.12 | 0.08 | 0.12 | 0.08 |
| $\omega_4$ | 0.06 | 0.08 | 0.06 | 0.08 |
| $\omega_5$ | 0.02 | 0.04 | 0.02 | 0.04 |

Hence, we obtain

$$\begin{aligned}
E_P[-X] &= -100 \cdot 0.32 - 20 \cdot 0.48 + 0 + 10 \cdot 0.06 + 40 \cdot 0.02 = -40.2 \\
E_{Q_1}[-X] &= -100 \cdot 0.4 - 20 \cdot 0.4 + 0 + 10 \cdot 0.08 + 40 \cdot 0.04 = -45.6 \\
E_{Q_2}[-X] &= -100 \cdot 0.4 - 20 \cdot 0.4 + 0 + 10 \cdot 0.06 + 40 \cdot 0.02 = -46.6 \\
E_{Q_3}[-X] &= -100 \cdot 0.32 - 20 \cdot 0.48 + 0 + 10 \cdot 0.08 + 40 \cdot 0.04 = -39.2,
\end{aligned}$$

so

$$\rho_0(X) = \sup_{Q \in \{P, Q_1, Q_2, Q_3\}} E_Q[-X] = -39.2. \tag{12.21}$$

In order to find $\rho_1(X)$, we proceed as in items 1.–2. so to obtain that $\rho_1(X)$ takes the following values at U and D, respectively,

$$\begin{aligned}
\rho_1^U(X) &= \sup\{-100 \cdot 0.4 - 20 \cdot 0.6; \\
&\quad -100 \cdot 0.5 - 20 \cdot 0.5; -100 \cdot 0.5 - 20 \cdot 0.5; -100 \cdot 0.4 - 20 \cdot 0.6\} \\
&= -52; \\
\rho_1^D(X) &= \sup\{0 + 10 \cdot 0.3 + 40 \cdot 0.1; 0 + 10 \cdot 0.4 \\
&\quad +40 \cdot 0.2; 0 + 10 \cdot 0.3 + 40 \cdot 0.1; 0 + 10 \cdot 0.4 + 40 \cdot 0.2\} \\
&= 12.
\end{aligned}$$

Proceeding backwards once more, we deduce that

$$\rho_0(-\rho_1(X)) = -39.2. \tag{12.22}$$

By the arguments above and by (12.21), it follows that

$$\rho_0(-\rho_1(X)) = \rho_0(X) = -39.2.$$

Note that one can also prove that the dynamic risk measure $(\rho_t)_{t=0,1,2}$ defined in (12.17) is time-consistent. In such a case, indeed, the set $\mathcal{Q}$ contains any probability measure obtained by "pasting" other probability measures of $\mathcal{Q}$. Note that stability under pasting is not always satisfied by the set of generalized scenarios in the representation of $CVaR$.

**Exercise 12.11** Consider a portfolio whose Profit & Loss is represented by a random variable $X$ that is distributed as a Uniform on the interval $[-200, 300]$.

Compute the the $VaR$ at 10% of the position. Furthermore, if there exists, find the level $\alpha \in (0, 1)$ (and the corresponding scaling factor $c$ with $\alpha = c \cdot 0.1$) for which $VaR$ at 10% and $CVaR$ at level $\alpha$ of $X$ coincide.

Note the scaling factor $c$ is known as PELVE (Probability Equivalent Level of VaR and ES) and studied in Li and Wang [29].

**Solution** It is well known that a Uniform random variable on the interval $[-200, 300]$ has the following cumulative distribution function

$$F_X(x) = \begin{cases} 0; & \text{if } x < -200 \\ \frac{1}{500}(x+200); & \text{if } -200 \leq x < 300 \\ 1 & \text{if } x \geq 300 \end{cases}.$$

The quantile at 10% of $X$ then solves

$$\begin{aligned} F_X(q_{0.1}) &= 0.1 \\ \frac{1}{500}(q_{0.1}+200) &= 0.1, \end{aligned}$$

implying that $VaR_{0.1}(X) = -q_{0.1}(X) = 150$ and, more in general, that $VaR_\alpha(X) = 200 - 500\alpha$.

It remains to deduce the level $\alpha$ such that $CVaR_\alpha(X) = VaR_{0.1}(X)$. Since $X$ is a continuous random variable, $CVaR_\alpha(X)$ coincides with $TCE_\alpha(X)$ and, consequently,

$$\begin{aligned} CVaR_\alpha(X) = TCE_\alpha(X) &= \frac{E_P\left[-X\mathbf{1}_{\{X \leq -VaR_\alpha(X)\}}\right]}{P\left(X \leq -VaR_\alpha(X)\right)} \\ &= \frac{1}{\alpha}\int_{-200}^{-VaR_\alpha(X)} \left(-\frac{x}{500}\right) dx \\ &= \frac{200^2 - (-VaR_\alpha(X))^2}{1000\alpha}. \end{aligned}$$

Assuming that $CVaR_\alpha(X) = VaR_{0.1}(X)$, $\alpha \in (0,1)$ should then solve the following equation

$$\begin{aligned} \frac{200^2 - (-VaR_\alpha(X))^2}{1000\alpha} &= 150 \\ 40{,}000 - (500\alpha - 200)^2 &= 150{,}000\alpha \\ 250{,}000\alpha^2 - 50{,}000\alpha &= 0 \\ \alpha &= 0.2 \end{aligned}$$

The scaling factor $c$ (or, better, the PELVE at 10% for $X$) for which $VaR$ at 10% and $CVaR$ at level $\alpha = c \cdot 0.1$ of $X$ coincide is then given by $c = \frac{\alpha}{0.1} = 2$.[2]

[2] This result can be found in Li and Wang [29] who proved that any Uniform random variable has PELVE equal to 2.

**Exercise 12.12** Consider three stocks (or sub-portfolios) whose Profit & Loss (per year) are represented, respectively, by the following random variables:

$$X_1 = \begin{cases} -200; & \text{on } \omega_1 \\ -400; & \text{on } \omega_2 \\ 0; & \text{on } \omega_3 \\ 800; & \text{on } \omega_4 \\ -400; & \text{on } \omega_5 \end{cases}; \qquad X_2 = \begin{cases} 400; & \text{on } \omega_1 \\ 0; & \text{on } \{\omega_2; \omega_3\} \\ -200; & \text{on } \omega_4 \\ -400; & \text{on } \omega_5 \end{cases};$$

$$X_3 = \begin{cases} 0; & \text{on } \{\omega_1; \omega_2\} \\ -200; & \text{on } \omega_3 \\ 2000; & \text{on } \omega_4 \\ -800; & \text{on } \omega_5 \end{cases}$$

with $P(\omega_1) = 0.01$, $P(\omega_2) = 0.09$, $P(\omega_3) = 0.8$ and $P(\omega_4) = P(\omega_5) = 0.05$.

Consider now the whole portfolio $X = X_1 + X_2 + X_3$.

1. Compute the $VaR$ at 5% of $X$.
2. Assume now we need to share the margin given by $VaR_{0.05}(X)$ among the different sub-portfolios. Compute the capital to be allocated to each sub-portfolio by means of the marginal capital allocation, given by $\rho(X) - \rho(X - X_i)$ for sub-portfolio $X_i$.
3. Compute the capital to be allocated to each sub-portfolio by means of the haircut capital allocation at the level $p = 5\%$, given by

$$K \cdot \frac{VaR_p(X_i)}{\sum_{i=1}^{3} VaR_p(X_i)}$$

   for sub-portfolio $X_i$, where $K$ denotes the total margin of the position ($VaR_{0.05}(X)$ in this case). Note that the haircut allocation is a particular case of "proportional" capital allocations.
4. Compute the capital to be allocated to each sub-portfolio by means of the haircut capital allocation at the level $p = 1\%$.

**Solution** First of all, the Profit & Loss of the whole portfolio $X$ is given by

$$X = X_1 + X_2 + X_3 = \begin{cases} -1600; & \text{on } \omega_5 \\ -400; & \text{on } \omega_2 \\ -200; & \text{on } \omega_3 \\ 200; & \text{on } \omega_1 \\ 2600; & \text{on } \omega_4 \end{cases}$$

1. It easily follows that $VaR_{0.05}(X) = 400$.

2. In order to compute the marginal allocations we need to evaluate the impact (better, marginal contribution) of any sub-portfolio $X_i$ on the whole portfolio $X$, given by $\rho(X) - \rho(X - X_i)$.

   Since

$$X - X_1 = X_2 + X_3 = \begin{cases} 400; & \text{on } \omega_1 \\ 0; & \text{on } \omega_2 \\ -200; & \text{on } \omega_3 \\ 1800; & \text{on } \omega_4 \\ -1200; & \text{on } \omega_5 \end{cases};$$

$$X - X_2 = X_1 + X_3 = \begin{cases} -200; & \text{on } \omega_1 \\ -400; & \text{on } \omega_2 \\ -200; & \text{on } \omega_3 \\ 2800; & \text{on } \omega_4 \\ -1200; & \text{on } \omega_5 \end{cases};$$

$$X - X_3 = X_1 + X_2 = \begin{cases} 200; & \text{on } \omega_1 \\ -400; & \text{on } \omega_2 \\ 0; & \text{on } \omega_3 \\ 600; & \text{on } \omega_4 \\ -800; & \text{on } \omega_5 \end{cases},$$

   it follows that the capital to be allocated to each sub-portfolio with respect to the marginal allocation is given, respectively, by

$$\bar{K}_1 = VaR_{0.05}(X) - VaR_{0.05}(X - X_1) = 400 - 200 = 200 \quad \text{for } X_1$$
$$\bar{K}_2 = VaR_{0.05}(X) - VaR_{0.05}(X - X_2) = 400 - 400 = 0 \quad \text{for } X_2$$
$$\bar{K}_3 = VaR_{0.05}(X) - VaR_{0.05}(X - X_3) = 400 - 400 = 0 \quad \text{for } X_3.$$

3. Since $VaR_{0.05}(X_1) = 400$, $VaR_{0.05}(X_2) = 200$, $VaR_{0.05}(X_3) = 200$ and the total margin $K = VaR_{0.05}(X) = 400$, we deduce that the capital to be allocated to each sub-portfolio with respect to the haircut allocation with $p = 5\%$ is given, respectively, by

$$K_1 = K \cdot \frac{VaR_{0.05}(X_1)}{\sum_{i=1}^{3} VaR_{0.05}(X_i)} = 400 \cdot \frac{400}{800} = 200 \quad \text{for } X_1$$
$$K_2 = K \cdot \frac{VaR_{0.05}(X_1)}{\sum_{i=1}^{3} VaR_{0.05}(X_i)} = 400 \cdot \frac{200}{800} = 100 \quad \text{for } X_2$$
$$K_3 = K \cdot \frac{VaR_{0.05}(X_1)}{\sum_{i=1}^{3} VaR_{0.05}(X_i)} = 400 \cdot \frac{200}{800} = 100 \quad \text{for } X_3.$$

Note that the sum of $K_i$'s as above is equal to the total margin $VaR_{0.05}(X) = 400$ to be deposited for $X$. This property is known as full allocation.

4. Proceeding as above, we obtain that $VaR_{0.01}(X_1) = 400$, $VaR_{0.01}(X_2) = 400$, $VaR_{0.01}(X_3) = 800$. Since the total margin to be allocated is $K = VaR_{0.05}(X) = 400$, the capital to be allocated to each sub-portfolio with respect to the haircut allocation with $p = 1\%$ is then given, respectively, by

$$K_1^* = K \cdot \frac{VaR_{0.01}(X_1)}{\sum_{i=1}^3 VaR_{0.01}(X_i)} = 400 \cdot \frac{400}{1600} = 100 \quad \text{for } X_1$$

$$K_2^* = K \cdot \frac{VaR_{0.01}(X_2)}{\sum_{i=1}^3 VaR_{0.01}(X_i)} = 400 \cdot \frac{400}{1600} = 100 \quad \text{for } X_2$$

$$K_3^* = K \cdot \frac{VaR_{0.01}(X_3)}{\sum_{i=1}^3 VaR_{0.01}(X_i)} = 400 \cdot \frac{800}{1600} = 200 \quad \text{for } X_3.$$

## 12.3 Proposed Exercises

**Exercise 12.13** Consider two stocks (A and B) whose daily returns are jointly normal. Assume that the correlation between their returns is $-0.8$, that stock A has current price 2 euros, negligible daily drift and daily volatility of 1.2%, and that stock B has current price 1 euro, negligible daily drift and daily volatility of 1.6%.

Rank the following financial positions, based on the riskiness evaluated with respect to $VaR$ at 2% and on a period of 1 day:

- 40 shares of stock A;
- 10 shares of stock B;
- a portfolio composed by 40 shares of A and by 10 shares of B.

Is diversification of risk encouraged in the present setting?

**Exercise 12.14** Consider a portfolio whose Profit & Loss is represented by the following random variable:

$$X = \begin{cases} -4000; & \text{on } \omega_1 \\ -48; & \text{on } \omega_2 \\ 0; & \text{on } \omega_3 \\ 640; & \text{on } \omega_4 \end{cases}$$

where $P(\omega_1) = 0.02$, $P(\omega_2) = 0.08$, $P(\omega_3) = 0.8$ and $P(\omega_4) = 0.1$.

1. Compute the Value at Risk at 5% and the Tail Conditional Expectation at 5% of $X$.
2. In $X$, replace the value $-4000$ with $x$. Find the largest possible loss $x$ for which the margin to be deposited according to $TCE$ at 5% does not exceed twice the corresponding $VaR$.

**Exercise 12.15** Consider two portfolios whose Profit & Loss are represented by the following random variables:

$$X = \begin{cases} -160; & \text{on } \omega_1 \\ -20; & \text{on } \omega_2 \\ 0; & \text{on } \omega_3 \\ 120; & \text{on } \omega_4 \end{cases} \qquad Y = \begin{cases} -240; & \text{on } \omega_1 \\ 0; & \text{on } \omega_2 \\ 160; & \text{on } \omega_3 \\ -280; & \text{on } \omega_4 \end{cases}$$

where $P(\omega_1) = 0.01$, $P(\omega_2) = 0.02$, $P(\omega_3) = 0.57$ and $P(\omega_4) = 0.40$.

1. Compute the Value at Risk at 2% of $X$, of $Y$ and of $X + Y$. Is diversification of risk encouraged or not?
2. Compute the Conditional Value at Risk at 2% of $X$, of $Y$ and of $X + Y$. Is diversification of risk encouraged or not?

**Exercise 12.16** Consider a portfolio whose Profit & Loss is represented by the following random variable:

$$X = \begin{cases} -800; & \text{on } \omega_1 \\ -200; & \text{on } \omega_2 \\ 0; & \text{on } \omega_3 \\ 120; & \text{on } \omega_4 \\ 400; & \text{on } \omega_5 \\ 1200; & \text{on } \omega_6 \end{cases}$$

with $P(\omega_1) = 0.02$, $P(\omega_2) = 0.08$, $P(\omega_3) = 0.64$, $P(\omega_4) = 0.16$, $P(\omega_5) = 0.06$ and $P(\omega_6) = 0.04$.

Let $\rho_{\mathscr{Q}}$ be the coherent risk measure generated by the set $\mathscr{Q} = \{P, Q_1, Q_2\}$ of generalized scenarios, where

$$Q_1(\omega_1) = Q_1(\omega_2) = Q_1(\omega_3) = Q_1(\omega_4) = Q_1(\omega_5) = Q_1(\omega_6) = 1/6$$

$$Q_2(\omega_1) = Q_2(\omega_2) = 0.2; \quad Q_2(\omega_3) = Q_2(\omega_4) = Q_2(\omega_5) = Q_2(\omega_6) = 0.15$$

Compute $\rho_{\mathscr{Q}}(X)$ and establish which between $\rho_{\mathscr{Q}}$ and the expected value of losses is the stronger (or more conservative) risk measure.

**Exercise 12.17** Consider a portfolio whose Profit & Loss is represented by the following random variable:

$$X = \begin{cases} -480; & \text{with prob. } 0.02 \\ -20; & \text{with prob. } 0.04 \\ 0; & \text{with prob. } 0.64 \\ 40; & \text{with prob. } 0.2 \\ 800; & \text{with prob. } 0.1 \end{cases}$$

1. Compute the Value at Risk at 10% and the Conditional Value at Risk at 10% of $X$.
2. What is the sign of $CVaR$ and $VaR$ of a position having as P & L the one above increased by an amount of 80 euros? Is it possible to obtain $CVaR$ and $VaR$ of such a new position with one direct computation only?

**Exercise 12.18** Consider the filtrated probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t=0,1,2}, P)$ given by

$$\begin{aligned} \Omega &= \{\omega_1, \omega_2, \omega_3, \omega_4\} \\ U &= \{\omega_1, \omega_2\} \\ D &= \{\omega_3, \omega_4\} \\ \mathscr{F}_0 &= \{\emptyset, \Omega\} \\ \mathscr{F}_1 &= \{\emptyset, U, D, \Omega\} \\ \mathscr{F}_2 &= \mathscr{P}(\Omega) \end{aligned}$$

and a financial investment whose Profit and Loss in 2 years is represented by the random variable

$$X = \begin{cases} 20, & \text{on } \omega_1 \\ 10, & \text{on } \omega_2 \\ 0, & \text{on } \omega_3 \\ -30, & \text{on } \omega_4 \end{cases},$$

defined on the space above. Consider now the dynamic risk measures $(\rho_t^{\mathscr{R}})_{t=0,1,2}$ and $(\rho_t^{\mathscr{S}})_{t=0,1,2}$ defined as

$$\rho_t^{\mathscr{R}}(X) \triangleq \operatorname{ess\,sup}_{Q\in\mathscr{R}} E_Q[-X|\,\mathscr{F}_t] \tag{12.23}$$

$$\rho_t^{\mathscr{S}}(X) \triangleq \operatorname{ess\,sup}_{Q\in\mathscr{S}} E_Q[-X|\,\mathscr{F}_t], \tag{12.24}$$

where $\mathscr{R} = \{P, Q_1, Q_2\}$, $\mathscr{S} = \{P, Q_1, Q_2, Q_3\}$ and the probability measures are as follows:

| event | $P$ | $Q_1$ | $Q_2$ | $Q_3$ |
|---|---|---|---|---|
| $\omega_1 \vert U$ | 0.5 | 0.5 | 0.5 | 0.5 |
| $\omega_2 \vert U$ | 0.5 | 0.5 | 0.5 | 0.5 |
| $\omega_3 \vert D$ | 0.5 | 0.8 | 0.8 | 0.5 |
| $\omega_4 \vert D$ | 0.5 | 0.2 | 0.2 | 0.5 |
| $U$ | 0.5 | 0.4 | 0.5 | 0.4 |
| $D$ | 0.5 | 0.6 | 0.5 | 0.6 |

Verify if $\rho_0^{\mathscr{R}}(X) = \rho_0^{\mathscr{R}}(-\rho_1^{\mathscr{R}}(X))$ and if $\rho_0^{\mathscr{S}}(X) = \rho_0^{\mathscr{S}}(-\rho_1^{\mathscr{S}}(X))$. Explain and discuss the results obtained.

**Exercise 12.19** Consider the P & L of a stock represented by the random variable $X$ distributed as a Uniform on the interval $[-100, 400]$.

1. Compute the $VaR$ of $X$ at 10%.
2. Applying the Cornish-Fisher expansion, approximate the $VaR$ of $X$ at 10% by using the skewness of $X$ of a Uniform.

**Exercise 12.20** Consider four stocks (or sub-portfolios) whose Profit & Loss (per year) are represented, respectively, by the following random variables:

$$X_1 = \begin{cases} -200; & \text{on } \omega_1 \\ -400; & \text{on } \omega_2 \\ 0; & \text{on } \omega_3 \\ 800; & \text{on } \omega_4 \\ 400; & \text{on } \omega_5 \end{cases}; \qquad X_2 = \begin{cases} 400; & \text{on } \omega_1 \\ 0; & \text{on } \{\omega_2; \omega_3\} \\ 200; & \text{on } \omega_4 \\ -400; & \text{on } \omega_5 \end{cases};$$

$$X_3 = \begin{cases} 0; & \text{on } \{\omega_1; \omega_2\} \\ -200; & \text{on } \omega_3 \\ 1000; & \text{on } \omega_4 \\ 200; & \text{on } \omega_5 \end{cases}; \qquad X_4 = \begin{cases} 200; & \text{on } \{\omega_1, \omega_3\} \\ 0; & \text{on } \{\omega_2; \omega_4\} \\ -400; & \text{on } \omega_5 \end{cases}$$

with $P(\omega_1) = 0.05$, $P(\omega_2) = 0.15$, $P(\omega_3) = 0.7$ and $P(\omega_4) = P(\omega_5) = 0.05$.
Consider now the whole portfolio $X = X_1 + X_2 + X_3 + X_4$.

1. Compute the $VaR$ at 5% of $X$.
2. Assume now we need to share the margin given by $VaR_{0.05}(X)$ among the different sub-portfolios. Compute the capital to be allocated to each sub-portfolio by means of the marginal capital allocation, given by $\rho(X) - \rho(X - X_i)$ for sub-portfolio $X_i$.
3. Compute the capital to be allocated to each sub-portfolio by means of the haircut capital allocation at the level $p = 5\%$.

# References


1. Artzner, P., Delbaen, F., Eber, J.M., Heath, D.: Coherent measures of risk. Math. Financ. **4**, 203–228 (1999)
2. Artzner, P., Delbaen, F., Eber, J.M., Heath, D., Ku, H.: Coherent multiperiod risk adjusted values and Bellman s principle. Ann. Oper. Res. **152**, 5–22 (2007)
3. Barucci, E. : Financial Markets Theory. Springer-Verlag, London (2003)
4. Barucci, E., Marsala, C., Nencini, M., Sgarra, C.: Ingegneria finanziaria. Un'introduzione quantitativa. Egea, Milano (2009)
5. Bates, D.: Jumps and stochastic volatility: the exchange rate processes implicit in Deutschemark options. Rev. Financ. Stud. **9**, 69–107 (1996)
6. Björk, T. : Arbitrage Theory in Continuous Time, 3rd edn. Oxford University Press, Oxford (2009)
7. Black, F.: Fact and fantasy in the use of options. Financ. Anal. J. **31**, 36–41, (1975)
8. Bollerslev, T.: General autoregressive conditional heteroskedasticity. J. Econ. **31**, 307–327 (1986)
9. Brigo, D., Mercurio, F.: Interest rate models. Theory and Practice, 2nd edn. Springer-Verlag, Berlin Heidelberg (2006)
10. Capiński, M., Zastawniak, T.: Mathematics for Finance, 2nd edn. Springer-Verlag, London (2011)
11. Carr, P., Madan, D.: Option valuation using the fast Fourier transform. J. Comput. Financ. **2**, 61–73 (1998)
12. Centrone, F., Rosazza Gianin, E.: Capital allocation à la Aumann–Shapley for non-differentiable risk measures. Eur. J. Oper. Res. **267**, 667–675 (2018)
13. Cheridito, P., Stadje, M.: Time-inconsistency of VaR and time-consistent alternatives. Financ. Res. Lett. **6**, 40–46 (2009)
14. Christoffersen, P., Elkamhi, R., Feunou, B., Jacobs, K.: Option valuation with conditional heteroskedasticity and non-normality. Rev. Financ. Stud. **23**, 2139–2183 (2010)
15. Delbaen, F. : Coherent risk measures on general probability spaces. In: Sandmann, K., Schönbucher, P.J. (eds.) Advances in Finance and Stochastics, pp. 1–37. Springer-Verlag, Berlin/Heidelberg (2002)
16. Dhaene, J., Tsanakas, A., Valdez, E.A., Vanduffel, S.: Optimal capital allocation principles. J. Risk Insur. **79**(1), 1–28 (2012)
17. Duan, J.C.: The GARCH option pricing model. Math. Financ. **5**, 13–32 (1995)
18. El Karoui, N.: Dynamic Pricing and Hedging of Derivatives: Stochastic and Numerical Methods, 1st edn. Chapman and Hall/CRC, London/New York (2013)




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19. Engle, R.: Autoregressive conditional heteroskedasticity with estimates of the variance of the UK inflation. Econometrica **50**, 987–1008 (1982)
20. Fiori, A.M., Rosazza Gianin, E., Spasova, A.: Risk measures and Pareto style tails. In: Perna, C., Sibillo, M. (eds.) Mathematical and Statistical Methods for Actuarial Sciences and Finance, pp. 183–191. Springer-Verlag, Milano (2012)
21. Föllmer, H., Schied, A.: Stochastic finance. An Introduction in Discrete Time, 2nd edn. De Gruyter, Berlin/New York (2004)
22. Frittelli, M., Rosazza Gianin, E.: Dynamic convex risk measures. In: Szegö, G. (ed.) Risk Measures for the 21st Century, pp. 227–248. Wiley, New York (2004)
23. Gaivoronski, A., Pflug, G.C.: Value-at-risk in portfolio optimization: properties and computational approach. J. Risk **7**, 1–31 (2005)
24. Heston, S.L.: A closed-form solution for options with stochastic volatility with applications to bond and currency options. Rev. Financ. Stud. **6**, 327–343 (1993)
25. Hull, J.: Options, Futures and other Derivatives, 8th edn. Pearson, London (2011)
26. Hull, J., White, A.: The pricing of options on assets with stochastic volatility. J. Financ. **42**, 281–300 (1987)
27. Jorion, P.: Value at Risk. Mc Graw Hill, New York (2000)
28. Kalkbrener, M.: An axiomatic approach to capital allocation. Math. Financ. **15**(3), 425–437 (2005)
29. Li, H., Wang, R.: PELVE: Probability equivalent level of VaR and ES. J. Econ. **234**(1), 353–370 (2023).
30. Luenberger, D.G.: Investment Science. Oxford University Press, New York (1997)
31. Merton, R.: Option pricing when underlying stock returns are discontinuous. J. Financ. Econ. **3**, 125–144 (1976)
32. Mikosch, T.: Elementary Stochastic Calculus with Finance in View. World Scientific, Singapore (1998)
33. Musiela, M., Rutkowski, M.: Martingale Methods in Financial Modelling. Springer-Verlag, Berlin (1997)
34. Øksendal, B.: Stochastic Differential Equations, 6th edn. Springer-Verlag, Berlin Heidelberg (2003)
35. Pascucci, A.: PDE and Martingale Methods in Option Pricing. Springer, Bocconi University Press, Milano (2011)
36. Pascucci, A. Runggaldier, W.J.: Financial mathematics. Theory and Problems for Multi-Period Models. Springer, Milano (2011)
37. Pliska, S.R.: Introduction to Mathematical Finance: Discrete Time Models. Blackwell Publishers, Oxford (1997)
38. Rockafellar, R.T., Uryasev, S.: Conditional value-at-risk for general loss distributions. J. Bank. Financ. **26**, 1443–1471 (2002)
39. Ross, S.M.: An Elementary Introduction to Mathematical Finance, 3rd edn. Cambridge University Press, New York (2011)
40. Salsa, S.: Equazioni alle Derivate Parziali. Springer-Verlag, Milano (2004)
41. Stein, E., Stein, J.: Stock price distributions with stochastic volatility: an analytic approach. Rev. Financ. Stud. **4**, 727–752 (1991)
42. Tasche, D.: Allocating portfolio economic capital to sub-portfolios. In: Dev, A. (ed.) Economic Capital: A Practitioner Guide, pp.275–302. Risk Books, London (2004)
43. Wilmott, P., Howison, S., Dewynne, J.: Option Pricing: Mathematical Models and Computation. Oxford Financial Press, Oxford (1993)
44. Wilmott, P., Howison, S., Dewynne, J.: The Mathematics of Financial Derivatives: A Student Introduction. Cambridge University Press, Cambridge (1995)

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