Modèles avancés de la courbe des taux (cours VII)

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Swaption pricing in the LMM model and the swap market model

Swaption pricing

Consider a swap defined on the set of equidistant dates $T_1 < \cdots < T_n$, with $\delta = T_{k+1} - T_k$ and a related swaption with exercise date $T = T_1$ and strike rate R
Recall that the payoff of the swaption is given by

$$\begin{aligned}\left(\pi_{T_1}^{swap}(T_1, T_n, R)\right)^+ &= \left(1 - B_{T_1}(T_n) - R\delta \sum_{k=2}^{n} B_{T_1}(T_k)\right)^+ \\ &= \delta \sum_{k=2}^{n} B_{T_1}(T_k)(S(T_1, T_1, T_n) - R)^+,\end{aligned}$$

where $S(T_1, T_1, T_n)$ is the swap rate at time T_1 defined by

$$S(T_1, T_1, T_n) = \frac{1 - B_{T_1}(T_n)}{\delta \sum_{k=2}^{n} B_{T_1}(T_k)}$$

Swaption pricing

The time-t price of the swaption $\pi_t^{swaption}(T_1, T_n, R)$ is thus given under the forward measure $\mathbb{Q}_{T_1}$ by

$$\pi_t^{swaption}(T_1, T_n, R) = \delta B_t(T_1)\mathbb{E}^{\mathbb{Q}_{T_1}}\left[\sum_{k=2}^{n} B_{T_1}(T_k)(S(T_1, T_1, T_n) - R)^+ | \mathcal{F}_t\right]$$

To evaluate this expectation we need the joint law of $B_{T_1}(T_k), k = 2, \ldots, n$, under the measure $\mathbb{Q}_{T_1}$, or equivalently the joint law of the Libor rates $L(T_1, T_k), k = 1, \ldots, n-1$ under $\mathbb{Q}_{T_1}$.

$\Rightarrow$ We will try to simplify the calculation by changing the numéraire

Swaption pricing

Define the process

$$A_t := \sum_{k=2}^{n} B_t(T_k), \quad t \leq T_1$$

This is a sum of prices of traded assets (ZC bonds) and thus a valid numéraire.
Note that

$$\left(\frac{A_t}{B_t(T_1)}\right)_{t \leq T_1}$$

is a $\mathbb{Q}_{T_1}$-martingale (simply by definition of the forward measure).

Then we can introduce a new measure $\mathbb{Q}^{swap}$ on $\mathcal{F}_{T_1}$, equivalent to $\mathbb{Q}_{T_1}$, by setting the Radon-Nikodym derivative

$$\left.\frac{\mathrm{d}\mathbb{Q}^{swap}}{\mathrm{d}\mathbb{Q}_{T_1}}\right|_{\mathcal{F}_t} = \frac{A_t}{B_t(T_1)} \frac{B_0(T_1)}{A_0}$$

Swaption pricing

It is easy to show using Bayes formula that the swap rate

$$\left(S(t, T_1, T_n) = \frac{B_t(T_1) - B_t(T_n)}{\delta \sum_{k=2}^{n} B_t(T_k)} \right)_{t \leq T_1}$$

is a $\mathbb{Q}^{swap}$-martingale.

The measure $\mathbb{Q}^{swap}$ is called the forward swap measure associated with the numéraire A_t.

Once again by Bayes formula we get

$$\begin{aligned} \pi_t^{swaption}(T_1, T_n, R) &= \delta B_t(T_1) \mathbb{E}^{\mathbb{Q}_{T_1}} \left[A_{T_1} (S(T_1, T_1, T_n) - R)^+ | \mathcal{F}_t \right] \\ &= \delta A_t \mathbb{E}^{\mathbb{Q}^{swap}} \left[(S(T_1, T_1, T_n) - R)^+ | \mathcal{F}_t \right] \end{aligned}$$

$\Rightarrow$ we reduced the problem to pricing a call option with strike R and underlying $S(t, T_1, T_n)$

Swaption pricing

It can be shown that the swap rate $S(\cdot, T_1, T_n)$ being a positive, $\mathbb{Q}^{swap}$-martingale has the following representation

$$dS(t, T_1, T_n) = S(t, T_1, T_n)\varrho^{swap}(t)dW_t^{swap},$$

where W^{swap} is a Brownian motion with respect to the measure $\mathbb{Q}^{swap}$ and ϱ^{swap} is the swap volatility process.

$\Rightarrow$ If the volatility ϱ^{swap} *would be deterministic*, then the swaption price would be given by the Black formula

Swaption pricing

$\Rightarrow$ In the Libor market model ϱ^{swap} *cannot be deterministic* and thus we have the following choices:

1. Stick to the Libor market model, have the Black formula for caplets/floorlets and price swaptions using Monte Carlo methods or via analytic approximations ("Rebonato formula")

   (*see Filipovic(2009) Section 11.5.2 for approximate swaption pricing using Rebonato formula and Section 11.6 for some hints on Monte Carlo simulation*)

2. Model the swap rate $S(\cdot, T_1, T_n)$ directly and postulate that it is log-normal under the forward swap measure $\mathbb{Q}^{swap}$ $\Rightarrow$ swap market model, developed by Jamshidian (1997)

   Note that in the swap market model, the swaption prices are given by the Black formula, but the forward Libor rate volatilities cannot be deterministic in this model and we do not have the Black formula for caplets/floorlets. There is no model where the Black formula is available both for caplets and swaptions

Swaption pricing

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⇒ voir Brigo et Mercurio (2005), p. 203, Section 6.2 et p. 244, Section 6.8

Extensions of the Libor Market Model

Producing a caplet volatility smile

In order to produce a caplet volatility smile, the LMM has been extended in several ways. Recalling the dynamics of the forward Libor rate $L(\cdot, T_k)$ under the forward measure $\mathbb{Q}_{T_{k+1}}$

$$dL(t, T_k) = L(t, T_k)\lambda(t, T_k)\mathrm{d}W_t^{T_{k+1}}$$

- Replace the Brownian motion $W^{T_{k+1}}$ by a Lévy process (or another semimartingale with jumps) $X^{T_{k+1}}$ (as in e.g. the Lévy Libor model)
- Replace the deterministic volatility $\lambda(t, T_k)$ by a stochastic volatility, as in e.g. the SABR forward rate model given by

$$\begin{aligned} dL(t, T_k) &= L(t, T_k)^{\beta_k}\lambda(t, T_k)\mathrm{d}W_t^{T_{k+1}} \\ d\lambda(t, T_k) &= \alpha_k\lambda(t, T_k)\mathrm{d}Z_t^{T_{k+1}} \end{aligned}$$

  where $Z^{T_{k+1}}$ is a $\mathbb{Q}_{T_{k+1}}$-Brownian motion correlated with $W_t^{T_{k+1}}$ such that $d\langle W^{T_{k+1}}, Z^{T_{k+1}}\rangle_t = \rho_k dt$, and β_k, α_k are constants such that $0 \leq \beta_k \leq 1$ and $\alpha_k \geq 0$.

Producing a caplet volatility smile

$\Rightarrow$ A detailed overview of the various modeling approaches to produce a volatility smile in the LMM framework is given in Brigo and Mercurio (2005), Sections 9-11

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Let $[T_j, T_{j+1}]$ denote a generic time interval and F^j denote a generic forward rate for that interval.

Under the corresponding forward measure $\mathbb{Q}^{T_{j+1}}$ (recall that $W^{T_{j+1}}$ is a Brownian motion under $\mathbb{Q}^{T_{j+1}}$, the dynamics of F^j is given by

$$dF_t^j = \nu(t, F_t^j) dW_t^{T_{j+1}}$$

such that F^j is a martingale under $\mathbb{Q}^{T_{j+1}}$ and $\nu(t, F)$ can be either a deterministic or a stochastic function of F.

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$\Rightarrow$ In order to produce a non-flat volatility smile (see Brigo and Mercurio (2005), Section 9) and be able to calibrate well the model to the market data, the function ν has to be carefully chosen

Producing a caplet volatility smile

- If $\nu(t, F) = \lambda_t F$, where λ is a deterministic function of time, then we obtain the classical LMM
- If ν is a deterministic function of F, we obtain so-called local volatility models (e.g. $\nu(t, F) = \sigma(t)F^\gamma$, with $0 \leq \gamma \leq 1$ and σ some deterministic function of time, as in the CEV model)
- If ν is a stochastic function of F, we obtain stochastic volatility models (e.g. $\nu(t, F) = \alpha_t F$, where α satisfies another SDE, as in the SABR model or Wu and Zhang model)

Some popular local volatility models

- The shifted log-normal model (shifted LMM): $\nu(t, F) = F - \alpha$, where α is a real constant
- Constant Elasticity of Variance (CEV) Model (LMM based on CEV was introduced by Andersen and Andreasen (2000)): $\nu(t, F) = \sigma(t)F^{\gamma}$, with $0 < \gamma < 1$ and σ some deterministic function of time
- A Lognormal-Mixture (LM) Model by Brigo and Mercurio (2000): based on a mixture of log-normal densities
- A GMB mixture model à la Dupire (1994): local volatility function implied by the market data for caplets

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$\Rightarrow$ See Brigo and Mercurio (2005), Section 10

Some popular stochastic volatility models

- The Andersen and Brotherton-Ratcliffe Model (2001)
- The Wu and Zhang (2002) Model: the model given under a common measure $\mathbb{Q}$, with a common stochastic volatility given by a CIR process (cf. Heston model for asset prices)
- The Piterbarg (2003) Model: similarly to Wu and Zhang, the volatility is a CIR process, but the correlation between the forward rate and the volatility is assumed to be zero; however the parameters can be time-dependent. Efficient pricing of swaptions
- The SABR model (see next page)

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- The SABR model (see next page)

$\Rightarrow$ See Brigo and Mercurio (2005), Section 11

SABR model

- The model was proposed in the paper by Hagan, Kumar, Lesniewski and Woodward (2002)
- The name SABR is an acronym for stochastic, alpha, beta and rho (three of the four model parameters)
- Widely used in practice thanks to its simplicity and tractability (analytical approximations for the implied volatility)
- Intuitive meaning of the parameters which play specific roles in the generation of smiles and skews

SABR model

Let again $[T_j, T_{j+1}]$ denote a generic time interval and F^j denote a generic forward rate for that interval. The SABR model postulates the following dynamics of the forward rate under the associated forward measure $\mathbb{Q}^{T_{j+1}}$:

$$\begin{aligned} dF_t^j &= \alpha_t \, (F_t^j)^\beta \, dW_t^{T_{j+1}} \\ d\alpha_t &= \nu \, \alpha_t dZ_t^{T_{j+1}} \qquad \alpha_0 = \alpha \end{aligned}$$

where $Z^{T_{j+1}}$ is a $\mathbb{Q}_{T_{j+1}}$-Brownian motion correlated with $W_t^{T_{j+1}}$ such that $d\langle W^{T_{j+1}}, Z^{T_{j+1}} \rangle_t = \rho dt$, for $\rho \in [-1, 1]$, and β, α are constants such that $0 \le \beta \le 1$ and $\alpha, \nu \ge 0$.

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$\Rightarrow$ It can be shown that F^j is a martingale for $\beta < 1$. When $\beta = 1$, F^j is a martingale if and only if $\rho \leq 0$ (constraint on the admissible parameters.

SABR model

- The SABR model allows for a closed-form expression of a European option price whose underlying asset is F^j.
- Using singular perturbation techniques, Hagan, Kumar, Lesniewski and Woodward obtain an analytical approximation of the option implied volatility, which is then inserted into the Black-Scholes formula to obtain a closed-form expression of the European option price

SABR model

Proposition:

In the SABR the price $\pi^{caplet}(0, T_j, T_{j+1}, K)$ at time $t \leq T_j$ of a caplet with maturity T_{j+1} and strike K with pay-off at T_{k+1}

$$\delta(F^j_{T_j} - K)^+$$

is given by the following Black formula

$$\pi^{caplet}(0, T_j, T_{j+1}, K) = \delta B_0(T_{j+1})(F^j_0 \mathcal{N}(d_1) - K\mathcal{N}(d_2)),$$

where $\mathcal{N}(\cdot)$ denotes the cumulative distribution function of the standard normal random variable $N(0, 1)$,

$$d_{1,2} = \frac{\ln\left(\frac{F^j_0}{K}\right) \pm \frac{1}{2}(\sigma^{imp}(K, F^j_0))^2 T_j}{\sigma^{imp}(K, F^j_0)\sqrt{T_j}}$$

SABR model

and $\sigma^{imp}(K,F)$ denotes the SABR implied volatility given by the following analytical approximation

$$\sigma^{imp}(K,F) = \frac{\alpha}{(FK)^{\frac{1-\beta}{2}}\left[1+\frac{(1-\beta)^2}{24}\left(\ln\left(\frac{F}{K}\right)\right)^2+\frac{(1-\beta)^4}{1920}\left(\ln\left(\frac{F}{K}\right)\right)^4+\cdots\right]}\frac{z}{x(z)}$$
$$\cdot\left\{1+\left[\frac{(1-\beta)^2\alpha^2}{24(FK)^{1-\beta}}+\frac{\rho\beta\nu\alpha}{4(FK)^{\frac{1-\beta}{2}}}+\nu^2\frac{2-3\rho^2}{24}\right]T_j+\cdots\right\}$$

with

$$z := \frac{\nu}{\alpha}(FK)^{\frac{1-\beta}{2}}\ln\left(\frac{F}{K}\right)$$

and

$$x(z) := \ln\left(\frac{\sqrt{1-2\rho z+z^2}+z-\rho}{1-\rho}\right)$$

SABR model

The expression simplifies considerably in case of the ATM caplet implied volatility, i.e. for $K = F_0^j$:

$$\begin{aligned}\sigma^{ATM} &= \sigma^{imp}(F_0^j, F_0^j) \\ &= \frac{\alpha}{(F_0^j)^{1-\beta}} \left\{ 1 + \left[\frac{(1-\beta)^2\alpha^2}{24(F_0^j)^{2-2\beta}} + \frac{\rho\beta\nu\alpha}{4(F_0^j)^{1-\beta}} + \nu^2 \frac{2-3\rho^2}{24} \right] T_j + \cdots \right\}\end{aligned}$$

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\end{aligned}
$$

$\Rightarrow$ The curve obtained as a graph of the ATM implied volatility, as a function of F_0^j, is called a backbone

$\Rightarrow$ Note that α and β present in the leading term impact both the level and the slope of ATM implied volatilities

SABR model

A different approximation, based on an expansion of the expression for $\sigma^{imp}(K, F_0^j)$ in powers of $\ln\left(\frac{K}{F_0^j}\right)$, is given by:

$$\sigma^{imp}(K, F_0^j) = \frac{\alpha}{(F_0^j)^{1-\beta}} \left\{ 1 - \frac{1}{2}(1 - \beta - \rho\lambda) \ln\left(\frac{K}{F_0^j}\right) + \frac{1}{12}\left[(1-\beta)^2 + (2 - 3\rho^2)\lambda^2\right] \ln\left(\frac{K}{F_0^j}\right)^2 + \cdots \right\},$$

with

$$\lambda := \nu \frac{(F_0^j)^{1-\beta}}{\alpha}$$

SABR model

Remark: Note that the original SABR model postulates the evolution of a forward price of a single asset under a corresponding forward measure and produces an implied volatility smile for a single maturity.

$\Rightarrow$ The SABR model is often applied for the modeling of forward interest rates as presented above. Note that this model is not an extension of the LMM. In order to produce a stochastic volatility LMM one has to specify the joint evolution of all forward Libor rates in the tenor structure and the relation among the volatility dynamics of each forward rate

(see P. Henry-Labordère (2007). *Combining the SABR and LMM models*. Risk, October 2007, 102-107)

$\Rightarrow$The SABR model can be used for modeling of a swap rate under the swap measure and for analytical pricing of swaptions based on the approximation of the implied Black swaption volatility.
For a given swaption, we simply replace F by the swap rate S_t

Multiple curve LMM

- Similarly to other interest rate models, after the financial crisis and the appearance of the multiple interest rate curves in the market, the LMM was extended to take this phenomenon into account (*see Mercurio (2009, 2010)*).

  The OIS forward rates are defined and modeled as in the classical LMM framework

  $$dF^{OIS}(t, T_k) = F^{OIS}(t, T_k)\lambda^{OIS}(t, T_k)\mathrm{d}W_t^{T_{k+1}}$$

  under the OIS-forward measure $\mathbb{Q}_{T_{k+1}}$ and the risky Libor forward rates are modeled using the spreads $S(\cdot, T_k)$ above the OIS forward curve

  $$L(t, T_k) = F^{OIS}(t, T_k) + S(t, T_k)$$

  The model for $S(\cdot, T_k)$ is chosen such that it ensures positivity (market data shows that the LIBOR-OIS spreads are positive) and analytical tractability of the model (easy manipulation of the sum $F^{OIS}(t, T_k) + S(t, T_k)$ appearing in the option price calculations)

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$\Rightarrow$ see the paper by Mercurio (2010) and Grbac and Runggaldier (2016), Chapter 4

Shifted LMM for negative rates

- In order to take into account the negative rates observed in the markets, the LMM is modified by shifting the rates by a deterministic shift γ (the shifted LMM). Under the forward measure $\mathbb{Q}_{T_{k+1}}$ the dynamics of the Libor rate is given by

$$dL(t, T_k) = (L(t, T_k) + \gamma)\lambda(t, T_k)\mathrm{d}W_t^{T_{k+1}}$$

  The parameter $\gamma > 0$ is obtained from calibration to market data. The lower boundary for the values of the Libor rates is given by $-\gamma$.

- The shifted model allows moreover for better calibration in the presence of very low interest rates

- A very popular model in practice is a shifted SABR model for forward rates

$$\begin{aligned} dL(t, T_k) &= (L(t, T_k) + \gamma)^{\beta_k}\lambda(t, T_k)\mathrm{d}W_t^{T_{k+1}} \\ d\lambda(t, T_k) &= \alpha_k \lambda(t, T_k)\mathrm{d}Z_t^{T_{k+1}} \end{aligned}$$

Replacement of the Libor/Euribor rates

⇒ This work on this topic is still very much on-going, with a number of papers already published, since the reform is currently still not finished

⇒ For the latest contributions concerning this problem, see the reference list given at the first lecture (page 4, Réforme de Libor: Articles scientifiques)