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Interest Rate Derivatives

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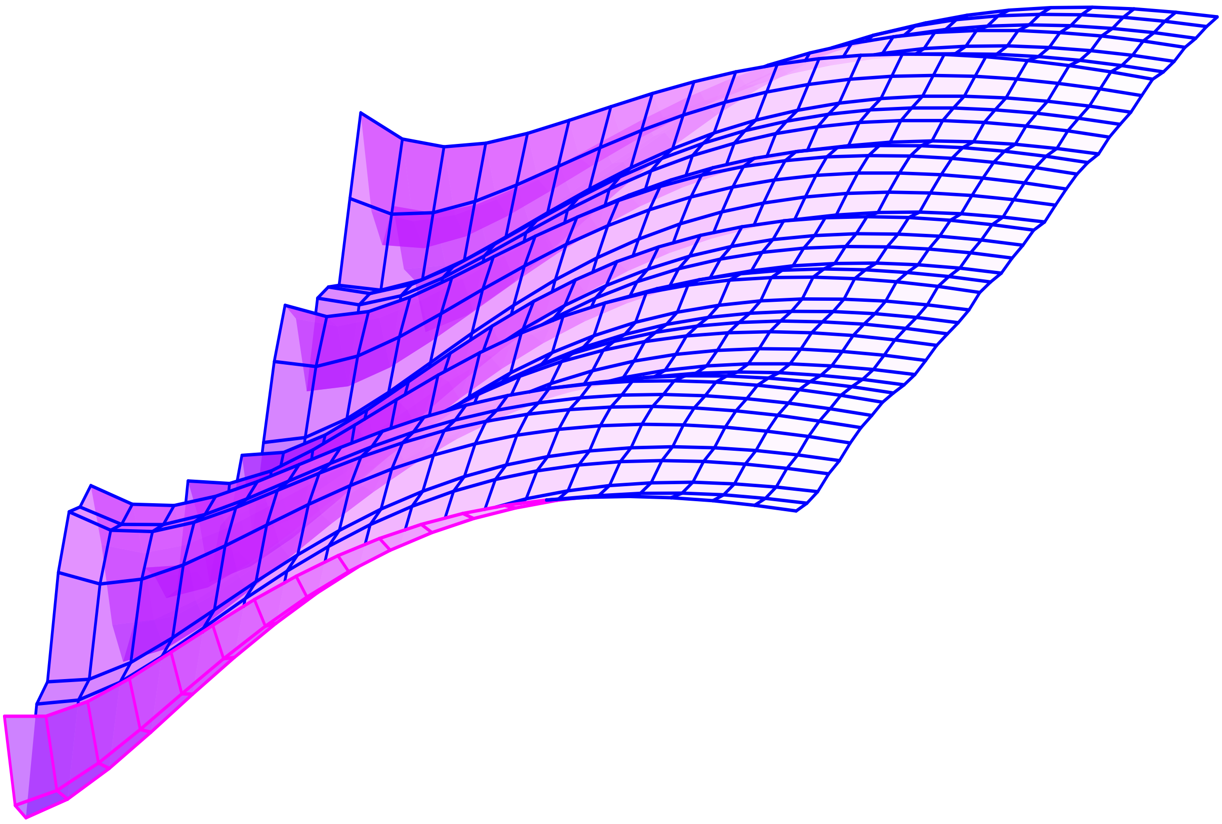

January 28, 2023 https://personal.ntu.edu.sg/nprivault/indext.html

This document deals the stochastic modeling of interest rates and the pricing of related derivative products. Bond pricing and the modeling of forward rates are introduced in Chapters 1 and 2. The general construction of forward measures by change of numéraire is given in Chapter 3 and is applied to the pricing of interest rate derivatives such as caplets, caps and swaptions in Chapter 4. Credit risk and stochastic default are considered in Chapter 5, with the pricing of defaultable bonds.

The pdf file contains internal and external links, and 43 figures, including 9 animated figures, *e.g.* Figures 1.16, 2.6, 2.9, 2.10, 2.17, that may require using Acrobat Reader for viewing on the complete pdf file. It also includes 2 Python code and 8 R codes, *e.g.* on pages 35 and 101. This text contains 22 exercises with solutions. Supplementary exercises, problems and solutions are available from the textbook **Stochastic Interest Rate Modeling with Fixed Income Derivative Pricing**, World Scientific, 2021.

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*Animated figures (work in Acrobat reader).

1. Short Rates and Bond Pricing

Short-term rates, typically daily rates, are the interest rates applied to short-term lending between financial institutions. The stochastic modeling of short-term interest rate processes is based on the mean reversion property, as in the Vasicek, CIR, CEV, and affine-type models studied in this chapter. The pricing of fixed income products, such as bonds, is considered in this framework using probabilistic and PDE arguments.

1.1 Short-Term Mean-Reverting Models

Money market accounts with price $(A_t)_{t\in\mathbb{R}_+}$ can be defined from a short-term interest rate process $(r_t)_{t\in\mathbb{R}_+}$ as

$$\frac{A_{t+dt}-A_t}{A_t} = r_t dt, \qquad \frac{dA_t}{A_t} = r_t dt, \qquad \frac{dA_t}{dt} = r_t A_t, \quad t \geqslant 0,$$

with

$$A_t = A_0 \exp\left(\int_0^t r_s ds\right), \qquad t \geqslant 0.$$

As short-term interest rates behave differently from stock prices, they require the development of specific models to account for properties such as positivity, boundedness, and return to equilibrium.

Vašíček, 1977 model

The first model to capture the mean reversion property of interest rates, a property not possessed by geometric Brownian motion, is the Vašíček, 1977 model, which is based on the Ornstein-Uhlenbeck process. Here, the short-term interest rate process $(r_t)_{t\in\mathbb{R}_+}$ solves the equation

$$dr_t = (a - br_t)dt + \sigma dB_t, \tag{1.1.1}$$

where $a, \sigma \in \mathbb{R}$, $b > 0$, and $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion, with solution

$$r_t = r_0\,\mathrm{e}^{-bt} + \frac{a}{b}\big(1 - \mathrm{e}^{-bt}\big) + \sigma\int_0^t \mathrm{e}^{-(t-s)b}dB_s, \quad t \geqslant 0, \tag{1.1.2}$$

see Exercise 1.1. The probability distribution of r_t is Gaussian at all times t, with mean

$$\mathbb{E}[r_t] = r_0\,\mathrm{e}^{-bt} + \frac{a}{b}\big(1 - \mathrm{e}^{-bt}\big),$$

and variance given from the Itô isometry as

$$\begin{aligned}
\mathrm{Var}[r_t] &= \mathrm{Var}\left[\sigma\int_0^t \mathrm{e}^{-(t-s)b}dB_s\right] \\
&= \sigma^2\int_0^t \big(\mathrm{e}^{-(t-s)b}\big)^2 ds \\
&= \sigma^2\int_0^t \mathrm{e}^{-2bs}ds \\
&= \frac{\sigma^2}{2b}\big(1 - \mathrm{e}^{-2bt}\big), \qquad t \geqslant 0,
\end{aligned}$$

i.e.

$$r_t \simeq \mathscr{N}\left(r_0\,\mathrm{e}^{-bt} + \frac{a}{b}\big(1 - \mathrm{e}^{-bt}\big), \frac{\sigma^2}{2b}\big(1 - \mathrm{e}^{-2bt}\big)\right), \quad t > 0.$$

In particular, the probability density function $f_t(x)$ of r_t at time $t > 0$ is given by

$$f_t(x) = \frac{\sqrt{b/\pi}}{\sigma\sqrt{1 - \mathrm{e}^{-2bt}}}\exp\left(-\frac{\big(r_0\,\mathrm{e}^{-bt} + a\big(1 - \mathrm{e}^{-bt}\big)/b - x\big)^2}{\sigma^2\big(1 - \mathrm{e}^{-2bt}\big)/b}\right), \quad x \in \mathbb{R}.$$

In the long run,* *i.e.* as time t becomes large we have, assuming $b > 0$,

$$\lim_{t\to\infty}\mathbb{E}[r_t] = \frac{a}{b} \quad \text{and} \quad \lim_{t\to\infty}\mathrm{Var}[r_t] = \frac{\sigma^2}{2b}, \tag{1.1.3}$$

and this distribution converges to the Gaussian $\mathscr{N}(a/b, \sigma^2/(2b))$ distribution, which is also the *invariant* (or stationary) distribution of $(r_t)_{t\in\mathbb{R}_+}$, see Exercise 1.1. In addition, the process tends to revert to its long term mean $a/b = \lim_{t\to\infty}\mathbb{E}[r_t]$ which makes the average drift vanish, *i.e.*:

$$\lim_{t\to\infty}\mathbb{E}[a - br_t] = a - b\lim_{t\to\infty}\mathbb{E}[r_t] = 0.$$

Figure 1.1 presents a random simulation of $t \mapsto r_t$ in the Vasicek model with $r_0 = 3\%$, and shows the mean-reverting property of the process with respect to $a/b = 2.5\%$.

*"But this *long run* is a misleading guide to current affairs. *In the long run* we are all dead." Keynes, 1924, Ch. 3, p. 80.

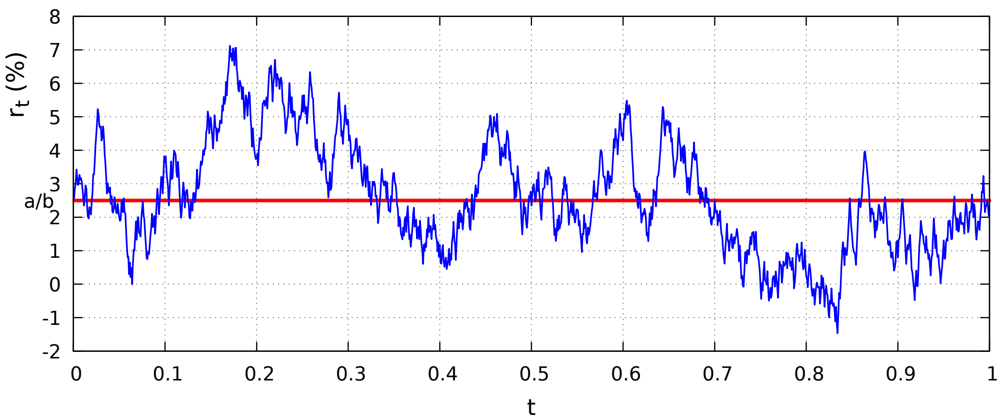


Figure 1.1: Graph of the Vasicek short rate $t \mapsto r_t$ with $a = 0.025$, $b = 1$, and $\sigma = 0.1$.

As can be checked from the simulation of Figure 1.1 the value of r_t in the Vasicek model may become negative due to its Gaussian distribution. Although real interest rates may sometimes fall below zero,[*] this can be regarded as a potential drawback of the Vasicek model. The next R code provides a numerical solution of the stochastic differential equation (1.1.1) using the Euler method, see Figure 1.1.

```r
N=10000;t<-0:(N-1);dt<-1.0/N;nsim<-2; a=0.025;b=1;sigma=0.1;
dB <- matrix(rnorm(nsim*N,mean=0,sd=sqrt(dt)), nsim, N)
R <- matrix(0,nsim,N);R[,1]=0.03
dev.new(width=10,height=7);
for (i in 1:nsim){for (j in 2:N){R[i,j]=R[i,j-1]+(a-b*R[i,j-1])*dt+sigma*dB[i,j]}}
par(mar=c(0,1,1,1));par(oma=c(0,1,1,1));par(mgp=c(-5,1,1))
plot(t,R[1,],xlab = "Time",ylab = "",type = "l",ylim = c(R[1,1]-0.2,R[1,1]+0.2),col = 0,axes=FALSE)
axis(2, pos=0,las=1);for (i in 1:nsim){lines(t, R[i, ], xlab = "time", type = "l", col = i+0)}
abline(h=a/b,col="blue",lwd=3);abline(h=0)
```

Example - TNX yield

We consider the yield of the 10 Year Treasury Note on the Chicago Board Options Exchange (CBOE), for later use in the calibration of the Vasicek model. Treasury notes usually have a maturity between one and 10 years, whereas treasury bonds have maturities beyond 10 years)

```r
library(quantmod)
getSymbols("^TNX",from="2012-01-01",to="2016-01-01",src="yahoo")
rate=Ad(`TNX`);rate<-rate[!is.na(rate)]
dev.new(width=10,height=7);chartSeries(rate,up.col="blue",theme="white")
n = length(!is.na(rate))
```

The next Figure 1.2 displays the yield of the 10 Year Treasury Note.

[*] Eurozone interest rates turned negative in 2014.

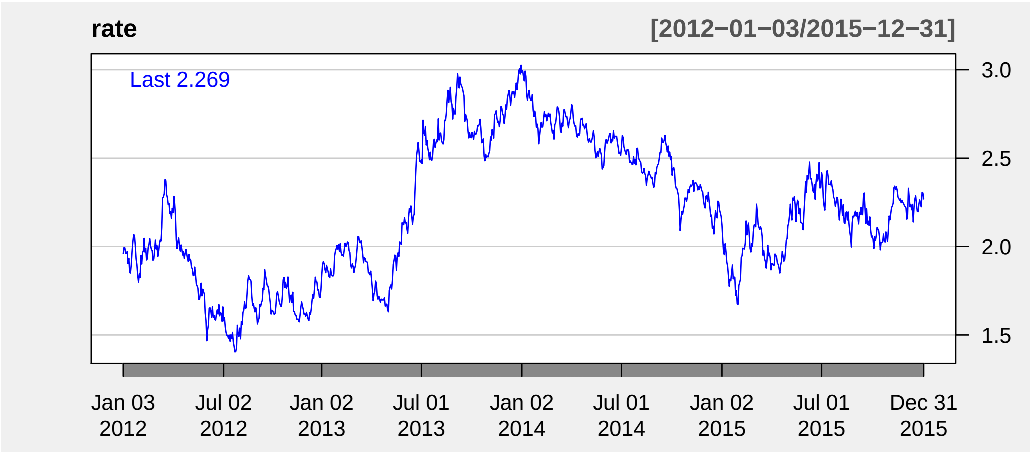


Figure 1.2: CBOE 10 Year Treasury Note (TNX) yield.

Cox-Ingersoll-Ross (CIR) model

The Cox, Ingersoll, and Ross, 1985 (CIR) model brings a solution to the positivity problem encountered with the Vasicek model, by the use the nonlinear stochastic differential equation

$$dr_t = \beta(\alpha - r_t)dt + \sigma\sqrt{r_t}dB_t, \tag{1.1.4}$$

with $\alpha > 0$, $\beta > 0$, $\sigma > 0$. The probability distribution of r_t at time $t > 0$ admits the noncentral Chi square probability density function given by

$$\begin{aligned} & f_t(x) \\ &= \frac{2\beta}{\sigma^2(1-\mathrm{e}^{-\beta t})} \exp\left(-\frac{2\beta(x + r_0\,\mathrm{e}^{-\beta t})}{\sigma^2(1-\mathrm{e}^{-\beta t})}\right)\left(\frac{x}{r_0\,\mathrm{e}^{-\beta t}}\right)^{\alpha\beta/\sigma^2-1/2} I_{2\alpha\beta/\sigma^2-1}\left(\frac{4\beta\sqrt{r_0 x\,\mathrm{e}^{-\beta t}}}{\sigma^2(1-\mathrm{e}^{-\beta t})}\right), \end{aligned} \tag{1.1.5}$$

$x > 0$, where

$$I_\lambda(z) := \left(\frac{z}{2}\right)^\lambda \sum_{k\geqslant 0} \frac{(z^2/4)^k}{k!\Gamma(\lambda + k + 1)}, \qquad z \in \mathbb{R},$$

is the modified Bessel function of the first kind, see Lemma 9 in Feller, 1951 and Corollary 24 in Albanese and Lawi, 2005. Note that $f_t(x)$ is not defined at $x = 0$ if $\alpha\beta/\sigma^2 - 1/2 < 0$, *i.e.* $\sigma^2 > 2\alpha\beta$, in which case the probability distribution of r_t admits a point mass at $x = 0$. On the other hand, r_t remains almost surely strictly positive under the Feller condition $2\alpha\beta \geqslant \sigma^2$, *cf.* the study of the associated probability density function in Lemma 4 of Feller, 1951 for $\alpha, \beta \in \mathbb{R}$.

Figure 1.3 presents a random simulation of $t \mapsto r_t$ in the Cox, Ingersoll, and Ross, 1985 (CIR) model in the case $\sigma^2 > 2\alpha\beta$, in which the process is mean reverting with respect to $\alpha = 2.5\%$ and has a nonzero probability of hitting 0.

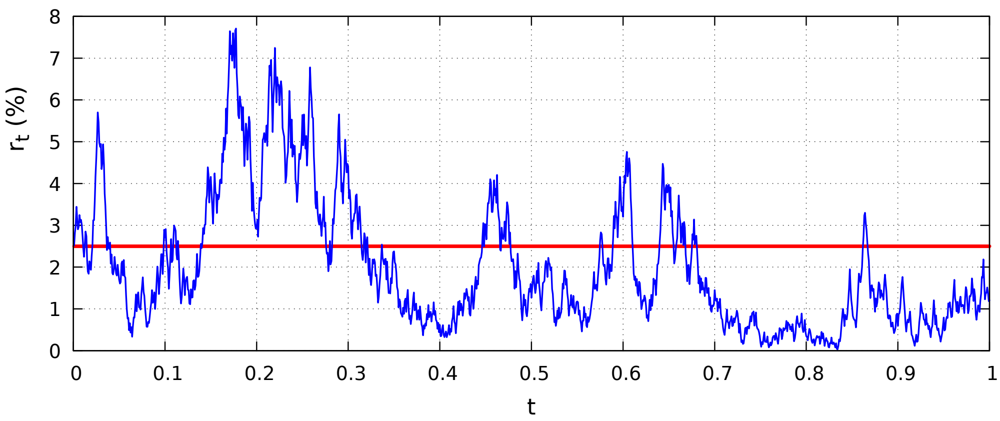


Figure 1.3: Graph of the CIR short rate $t \mapsto r_t$ with $\alpha = 2.5\%$, $\beta = 1$, and $\sigma = 1.3$.

The next R code provides a numerical solution of the stochastic differential equation (1.1.4) using the Euler method, see Figure 1.3.

```
N=10000; t <- 0:(N-1); dt <- 1.0/N; nsim <- 2
a=0.025; b=1; sigma=0.1; sd=sqrt(sigma^2/2/b)
X <- matrix(rnorm(nsim*N,mean=0,sd=sqrt(dt)), nsim, N)
R <- matrix(0,nsim,N);R[,1]=0.03
for (i in 1:nsim){for (j in 2:N){R[i,j]=max(0,R[i,j-1]+(a-b*R[i,j-1])*dt+sigma*sqrt(R[i,j-1])*X[i,j])}}
plot(t,R[1,],xlab="time",ylab="",type="l",ylim=c(0,R[1,1]+sd/5),col=0,axes=FALSE)
axis(2, pos=0)
for (i in 1:nsim){lines(t, R[i, ], xlab = "time", type = "l", col = i+8)}
abline(h=a/b,col="blue",lwd=3);abline(h=0)
```

In large time $t \to \infty$, using the asymptotics

$$I_\lambda(z) \simeq_{z\to 0} \frac{1}{\Gamma(\lambda+1)} \left(\frac{z}{2}\right)^\lambda,$$

the probability density function (1.1.5) becomes the gamma density function

$$f(x) = \lim_{t\to\infty} f_t(x) = \frac{1}{\Gamma(2\alpha\beta/\sigma^2)} \left(\frac{2\beta}{\sigma^2}\right)^{2\alpha\beta/\sigma^2} x^{-1+2\alpha\beta/\sigma^2} \mathrm{e}^{-2\beta x/\sigma^2}, \quad x > 0. \tag{1.1.6}$$

with shape parameter $2\alpha\beta/\sigma^2$ and scale parameter $\sigma^2/(2\beta)$, which is also the *invariant distribution* of r_t.

The family of classical mean-reverting models also includes the Courtadon, 1982 model

$$dr_t = \beta(\alpha - r_t)dt + \sigma r_t dB_t,$$

where α, β, σ are nonnegative, cf. Exercise 1.5, and the exponential Vasicek model

$$dr_t = r_t(\eta - a\log r_t)dt + \sigma r_t dB_t,$$

where $a, \eta, \sigma > 0$.

Constant Elasticity of Variance (CEV) model

Constant Elasticity of Variance models are designed to take into account nonconstant volatilities that can vary as a power of the underlying asset price. The Marsh and Rosenfeld, 1983 short-term interest rate model

$$dr_t = (\beta r_t^{\gamma-1} + \alpha r_t)dt + \sigma r_t^{\gamma/2} dB_t, \tag{1.1.7}$$

where $\alpha \in \mathbb{R}$, $\beta, \sigma > 0$ are constants and $\gamma > 0$ is the variance (or diffusion) elasticity coefficient, covers most of the CEV models. Here, the elasticity coefficient is defined as ratio

$$\frac{dv^2(r)/v^2(r)}{dr/r}$$

between the relative change $dv(r)/v(r)$ in the variance $v(r)$ and the relative change dr/r in r. Denoting by $v^2(r) := \sigma^2 r^\gamma$ the variance coefficient in (1.1.7), constant elasticity refers to the constant ratio

$$\frac{dv^2(r)/v^2(r)}{dr/r} = 2\frac{r}{v(r)}\frac{dv(r)}{dr} = 2\frac{d\log v(r)}{d\log r} = 2\frac{d\log r^{\gamma/2}}{d\log r} = \gamma.$$

For $\gamma = 1$, (1.1.7) yields the Cox, Ingersoll, and Ross, 1985 (CIR) equation

$$dr_t = (\beta + \alpha r_t)dt + \sigma\sqrt{r_t}dB_t.$$

For $\beta = 0$ we get the standard CEV model

$$dr_t = \alpha r_t dt + \sigma r_t^{\gamma/2} dB_t,$$

and for $\gamma = 2$ and $\beta = 0$ this yields the Dothan, 1978 model

$$dr_t = \alpha r_t dt + \sigma r_t dB_t,$$

which is a version of geometric Brownian motion used for short-term interest rate modeling.

Time-dependent affine models

The class of short rate interest rate models admits a number of generalizations (see the references quoted in the introduction of this chapter), including the class of affine models of the form

$$dr_t = (\eta(t) + \lambda(t)r_t)dt + \sqrt{\delta(t) + \gamma(t)r_t}dB_t. \tag{1.1.8}$$

Such models are called *affine* because the associated bonds can be priced using an *affine* PDE of the type (1.4.9) below with solution of the form (1.4.10), as will be seen after Proposition 1.2.

The family of affine models also includes:

i) the Ho and Lee, 1986 model

$$dr_t = \theta(t)dt + \sigma dB_t,$$

where $\theta(t)$ is a deterministic function of time, as an extension of the Merton model $dr_t = \theta dt + \sigma dB_t$,

ii) the Hull and White, 1990 model

$$dr_t = (\theta(t) - \alpha(t)r_t)dt + \sigma(t)dB_t$$

which is a time-dependent extension of the Vasicek model (1.1.1), with the explicit solution

$$r_t = r_0 \mathrm{e}^{-\int_0^t \alpha(\tau)d\tau} + \int_0^t \mathrm{e}^{-\int_u^t \alpha(\tau)d\tau}\theta(u)du + \int_0^t \sigma(u)\mathrm{e}^{-\int_u^t \alpha(\tau)d\tau}dB_u,$$

$t \geqslant 0$.

1.2 Calibration of the Vasicek Model

The Vasicek equation (1.1.1), *i.e.*

$$dr_t = (a - br_t)dt + \sigma dB_t,$$

can be discretized according to a discrete-time sequence $(t_k)_{k\geqslant 0} = (t_0, t_1, t_2, \ldots)$ of time instants, as

$$r_{t_{k+1}} - r_{t_k} = (a - br_{t_k})\Delta t + \sigma Z_k, \qquad k \geqslant 0,$$

where $\Delta t := t_{k+1} - t_k$ and $(Z_k)_{k\geqslant 0}$ is a Gaussian white noise with variance Δt, *i.e.* a sequence of independent, centered and identically distributed $\mathcal{N}(0, \Delta t)$ Gaussian random variables, which yields

$$r_{t_{k+1}} = r_{t_k} + (a - br_{t_k})\Delta t + \sigma Z_k = a\Delta t + (1 - b\Delta t)r_{t_k} + \sigma Z_k, \quad k \geqslant 0.$$

Based on a set $(\tilde{r}_{t_k})_{k=0,1,\ldots,n}$ of *market data*, we consider the quadratic residual

$$\sum_{k=0}^{n-1} \left(\tilde{r}_{t_{k+1}} - a\Delta t - (1 - b\Delta t)\tilde{r}_{t_k}\right)^2 \tag{1.2.1}$$

which represents the (squared) quadratic distance between the observed data sequence $(\tilde{r}_{t_k})_{k=1,2,\ldots,n}$ and its predictions $\big(a\Delta t + (1 - b\Delta t)\tilde{r}_{t_k}\big)_{k=0,1,\ldots,n-1}$.

In order to minimize the residual (1.2.1) over a and b we use Ordinary Least Square (OLS) regression, and equate the following derivatives to zero. Namely, we have

$$\begin{aligned}
&\frac{\partial}{\partial a} \sum_{l=0}^{n-1} \left(\tilde{r}_{t_{l+1}} - a\Delta t - (1 - b\Delta t)\tilde{r}_{t_l}\right)^2 \\
&= \ -2\Delta t \left(-an\Delta t + \sum_{l=0}^{n-1} \left(\tilde{r}_{t_{l+1}} - (1 - b\Delta t)\tilde{r}_{t_l}\right) \right) \\
&= \ 0,
\end{aligned}$$

hence

$$a\Delta t = \frac{1}{n} \sum_{l=0}^{n-1} \left(\tilde{r}_{t_{l+1}} - (1 - b\Delta t)\tilde{r}_{t_l}\right),$$

and

$$\begin{aligned}
&\frac{\partial}{\partial b} \sum_{k=0}^{n-1} \left(\tilde{r}_{t_{k+1}} - a\Delta t - (1 - b\Delta t)\tilde{r}_{t_k}\right)^2 \\
&= \ 2\Delta t \sum_{k=0}^{n-1} \tilde{r}_{t_k} \left(-a\Delta t + \tilde{r}_{t_{k+1}} - (1 - b\Delta t)\tilde{r}_{t_k}\right) \\
&= \ 2\Delta t \sum_{k=0}^{n-1} \tilde{r}_{t_k} \left(\tilde{r}_{t_{k+1}} - (1 - b\Delta t)\tilde{r}_{t_k} - \frac{1}{n} \sum_{l=0}^{n-1} \left(\tilde{r}_{t_{l+1}} - (1 - b\Delta t)\tilde{r}_{t_l}\right) \right) \\
&= \ 2\Delta t \sum_{k=0}^{n-1} \tilde{r}_{t_k}\tilde{r}_{t_{k+1}} - \frac{\Delta t}{n} \sum_{k,l=0}^{n-1} \tilde{r}_{t_k}\tilde{r}_{t_{l+1}} - \Delta t(1 - b\Delta t) \left(\sum_{k=0}^{n-1} (\tilde{r}_{t_k})^2 - \frac{1}{n} \sum_{k,l=0}^{n-1} \tilde{r}_{t_k}\tilde{r}_{t_l} \right) \\
&= \ 0.
\end{aligned}$$

This leads to estimators for the parameters a and b, respectively as the empirical mean and covariance of $(\tilde{r}_{t_k})_{k=0,1,\ldots,n}$, *i.e.*

$$
\begin{cases}
\widehat{a}\Delta t = \dfrac{1}{n}\displaystyle\sum_{k=0}^{n-1}\left(\tilde{r}_{t_{k+1}} - (1-\widehat{b}\Delta t)\tilde{r}_{t_k}\right), \\
\\
\text{and} \\
\\
1-\widehat{b}\Delta t = \dfrac{\displaystyle\sum_{k=0}^{n-1}\tilde{r}_{t_k}\tilde{r}_{t_{k+1}} - \frac{1}{n}\sum_{k,l=0}^{n-1}\tilde{r}_{t_k}\tilde{r}_{t_{l+1}}}{\displaystyle\sum_{k=0}^{n-1}(\tilde{r}_{t_k})^2 - \frac{1}{n}\sum_{k,l=0}^{n-1}\tilde{r}_{t_k}\tilde{r}_{t_l}} \\
\qquad = \dfrac{\displaystyle\sum_{k=0}^{n-1}\left(\tilde{r}_{t_k} - \frac{1}{n}\sum_{l=0}^{n-1}\tilde{r}_{t_l}\right)\left(\tilde{r}_{t_{k+1}} - \frac{1}{n}\sum_{l=0}^{n-1}\tilde{r}_{t_{l+1}}\right)}{\displaystyle\sum_{k=0}^{n-1}\left(\tilde{r}_{t_k} - \frac{1}{n}\sum_{k=0}^{n-1}\tilde{r}_{t_k}\right)^2}.
\end{cases}
\tag{1.2.2}
$$

This also yields

$$
\begin{array}{rcl}
\sigma^2\Delta t & = & \mathrm{Var}[\sigma Z_k] \\
& \simeq & \mathbb{E}\left[\left(\tilde{r}_{t_{k+1}} - (1-b\Delta t)\tilde{r}_{t_k} - a\Delta t\right)^2\right], \qquad k \geqslant 0,
\end{array}
$$

hence σ can be estimated as

$$
\widehat{\sigma}^2\Delta t = \frac{1}{n}\sum_{k=0}^{n-1}\left(\tilde{r}_{t_{k+1}} - \tilde{r}_{t_k}(1-\widehat{b}\Delta t) - \widehat{a}\Delta t\right)^2. \tag{1.2.3}
$$

Exercise. Show that (1.2.3) can be recovered by minimizing the residual

$$
\eta \mapsto \sum_{k=0}^{n-1}\left(\left(\tilde{r}_{t_{k+1}} - \tilde{r}_{t_k}(1-\widehat{b}\Delta t) - \widehat{a}\Delta t\right)^2 - \eta\Delta t\right)^2
$$

as a function of $\eta > 0$, see also Exercise 1.3.

Time series modeling

Defining $\widehat{r}_{t_n} := r_{t_n} - a/b$, $n \geqslant 0$, we have

$$
\begin{array}{rcl}
\widehat{r}_{t_{n+1}} & = & r_{t_{n+1}} - \dfrac{a}{b} \\
& = & r_{t_n} - \dfrac{a}{b} + (a - br_{t_n})\Delta t + \sigma Z_n \\
& = & r_{t_n} - \dfrac{a}{b} - b\left(r_{t_n} - \dfrac{a}{b}\right)\Delta t + \sigma Z_n \\
& = & \widehat{r}_{t_n} - b\widehat{r}_{t_n}\Delta t + \sigma Z_n \\
& = & (1-b\Delta t)\widehat{r}_{t_n} + \sigma Z_n, \qquad n \geqslant 0.
\end{array}
$$

In other words, the sequence $(\widehat{r}_{t_n})_{n\geqslant 0}$ is modeled according to an autoregressive AR(1) time series $(X_n)_{n\geqslant 0}$ with parameter $\alpha = 1 - b\Delta t$, in which the current state X_n of the system is expressed as the linear combination

$$
X_n := \sigma Z_n + \alpha X_{n-1}, \qquad n \geqslant 1, \tag{1.2.4}
$$

where $(Z_n)_{n\geqslant 1}$ another Gaussian white noise sequence with variance Δt. This equation can be solved recursively as the causal series

$$X_n = \sigma Z_n + \alpha(\sigma Z_{n-1} + \alpha X_{n-2}) = \cdots = \sigma \sum_{k\geqslant 0} \alpha^k Z_{n-k},$$

which converges when $|\alpha| < 1$, *i.e.* $|1 - b\Delta t| < 1$, in which case the time series $(X_n)_{n\geqslant 0}$ is weakly stationary, with

$$\begin{aligned}
\mathbb{E}[X_n] &= \sigma \sum_{k\geqslant 0} \alpha^k \mathbb{E}[Z_{n-k}] \\
&= \sigma \mathbb{E}[Z_0] \sum_{k\geqslant 0} \alpha^k \\
&= \frac{\sigma}{1-\alpha} \mathbb{E}[Z_0] \\
&= 0, \qquad n \geqslant 0.
\end{aligned}$$

The variance of X_n is given by

$$\begin{aligned}
\mathrm{Var}[X_n] &= \sigma^2 \, \mathrm{Var}\left[\sum_{k\geqslant 0} \alpha^k Z_{n-k}\right] \\
&= \sigma^2 \Delta t \sum_{k\geqslant 0} \alpha^{2k} \\
&= \sigma^2 \Delta t \sum_{k\geqslant 0} (1 - b\Delta t)^{2k} \\
&= \frac{\sigma^2 \Delta t}{1 - (1 - b\Delta t)^2} \\
&= \frac{\sigma^2 \Delta t}{2b\Delta t - b^2(\Delta t)^2} \\
&\simeq \frac{\sigma^2}{2b}, \qquad [\Delta t \simeq 0],
\end{aligned}$$

which coincides with the variance (1.1.3) of the Vasicek process in the stationary regime.

Example - TNX yield calibration

The next R code is estimating the parameters of the Vasicek model using the 10 Year Treasury Note yield data of Figure 1.2, by implementing the formulas (1.2.2).

```
ratek=as.vector(rate);ratekplus1 <- c(ratek[-1],0)
oneminusbdt <- (sum(ratek*ratekplus1) - sum(ratek)*sum(ratekplus1)/n)/(sum(ratek*ratek) -
    sum(ratek)*sum(ratek)/n)
adt <- sum(ratekplus1)/n-oneminusbdt*sum(ratek)/n;
sigmadt <- sqrt(sum((ratekplus1-oneminusbdt*ratek-adt)^2)/n)
```

Parameter estimation can also be implemented using the linear regression command

lm(c(diff(ratek)) ~ ratek[1:length(ratek)-1])

in R, which estimates the values of $a\Delta t \simeq 0.017110$ and $-b\Delta t \simeq -0.007648$ in the regression

$$r_{t_{k+1}} - r_{t_k} = (a - b r_{t_k})\Delta t + \sigma Z_k, \qquad k \geqslant 0,$$

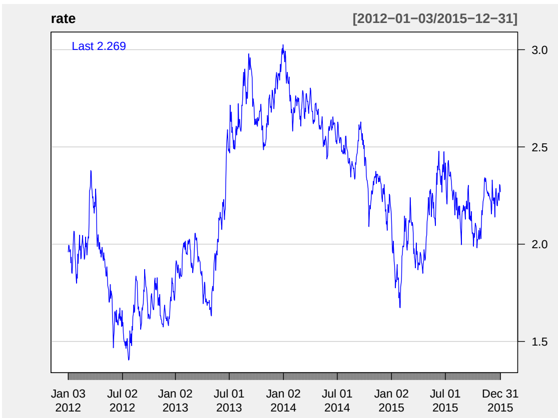
Coefficients:

(Intercept) ratek[1:length(ratek) - 1]

0.017110 -0.007648

```
dev.new(width=10,height=7);
for (i in 1:100) {ar.sim<-arima.sim(model=list(ar=c(oneminusbdt)),n.start=100,n)
y=adt/oneminusbdt+sigmadt*ar.sim;y=y+ratek[1]-y[1]
time <- as.POSIXct(time(rate), format = "%Y-%m-%d")
yield <- xts(x = y, order.by = time);
chartSeries(yield,up.col="blue",theme="white",yrange=c(0,max(ratek)))
Sys.sleep(0.5)}
```

The above R code is generating Vasicek random samples according to the AR(1) time series (1.2.4), see Figure 1.4.



(a) CBOE TNX market yield. (b) Calibrated Vasicek sample path.

Figure 1.4: Calibrated Vasicek simulation *vs* market data.

The R package Sim.DiffProc can also be used to estimate the coefficients $a\Delta t$ and $b\Delta t$.

```
install.packages("Sim.DiffProc");library(Sim.DiffProc)
fx <- expression( theta[1]-theta[2]*x ); gx <- expression( theta[3] )
fitsde(data = as.ts(ratek), drift = fx, diffusion = gx, start = list(theta1=0.1, theta2=0.1,
      theta3=0.1),pmle="euler")
```

1.3 Zero-Coupon and Coupon Bonds

A zero-coupon bond is a contract priced $P(t,T)$ at time $t < T$ to deliver the *face value* (or *par value*) $P(T,T) = \$1$ at time T. In addition to its value at maturity, a bond may yield a periodic *coupon* payment at regular time intervals until the maturity date.

Figure 1.5: Five-dollar 1875 Louisiana bond with 7.5% biannual coupons and maturity $T = 01/01/1886$.

The computation of the arbitrage-free price $P_0(t,T)$ of a zero-coupon bond based on an underlying short-term interest rate process $(r_t)_{t\in\mathbb{R}_+}$ is a basic and important issue in interest rate modeling.

Constant short rate

In case the short-term interest rate is a constant $r_t = r$, $t \geqslant 0$, a standard arbitrage argument shows that the price $P(t,T)$ of the bond is given by

$$P(t,T) = \mathrm{e}^{-r(T-t)}, \qquad 0 \leqslant t \leqslant T.$$

Indeed, if $P(t,T) > \mathrm{e}^{-r(T-t)}$ we could issue a bond at the price $P(t,T)$ and invest this amount at the compounded risk free rate r, which would yield $P(t,T)\,\mathrm{e}^{r(T-t)} > 1$ at time T.

On the other hand, if $P(t,T) < \mathrm{e}^{-r(T-t)}$ we could borrow $P(t,T)$ at the rate r to buy a bond priced $P(t,T)$. At maturity time T we would receive \$1 and refund only $P(t,T)\,\mathrm{e}^{r(T-t)} < 1$.

The price $P(t,T) = \mathrm{e}^{-r(T-t)}$ of the bond is the value of $P(t,T)$ that makes the potential profit $P(t,T)\,\mathrm{e}^{r(T-t)} - 1$ vanish for both traders.

Time-dependent deterministic short rates

Similarly to the above, when the short-term interest rate process $(r(t))_{t\in\mathbb{R}_+}$ is a deterministic function of time, a similar argument shows that

$$P(t,T) = \mathrm{e}^{-\int_t^T r(s)ds}, \qquad 0 \leqslant t \leqslant T. \tag{1.3.1}$$

Stochastic short rates

In case $(r_t)_{t\in\mathbb{R}_+}$ is an $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$-adapted random process the formula (1.3.1) is no longer valid as it relies on future information, and we replace it with the averaged discounted payoff

$$P(t,T) = \mathbb{E}^*\left[\mathrm{e}^{-\int_t^T r_s ds}\,\middle|\,\mathscr{F}_t\right], \qquad 0 \leqslant t \leqslant T, \tag{1.3.2}$$

under a risk-neutral probability measure $\mathbb{P}^*$. It is natural to write $P(t,T)$ as a conditional expectation under a martingale measure, as the use of conditional expectation helps to “filter out” the (random/unknown) future information past time t contained in $\displaystyle\int_t^T r_s ds$. The expression (1.3.2) makes sense as the “best possible estimate” of the future quantity $\mathrm{e}^{-\int_t^T r_s ds}$ in mean-square sense, given the information known up to time t.

Coupon bonds

Pricing bonds with nonzero coupon is not difficult since in general the amount and periodicity of coupons are deterministic.* In the case of a succession of coupon payments $c_1, c_2, \ldots, c_n$ at times $T_1, T_2, \ldots, T_n \in (t, T]$, another application of the above absence of arbitrage argument shows that the price $P_c(t,T)$ of the coupon bond with discounted (deterministic) coupon payments is given by the linear combination of zero-coupon bond prices

$$\begin{aligned} P_c(t,T) &:= \mathbb{E}^*\left[\sum_{k=1}^n c_k \mathrm{e}^{-\int_t^{T_k} r_s ds} \,\Big|\, \mathscr{F}_t\right] + \mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T} r_s ds} \,\Big|\, \mathscr{F}_t\right] \\ &= \sum_{k=1}^n c_k \mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_k} r_s ds} \,\Big|\, \mathscr{F}_t\right] + P_0(t,T) \\ &= P_0(t,T) + \sum_{k=1}^n c_k P_0(t,T_k), \qquad 0 \leqslant t \leqslant T_1, \end{aligned} \tag{1.3.3}$$

which represents the present value at time t of future $\$c_1, \$c_2, \ldots, \$c_n$ receipts respectively at times $T_1, T_2, \ldots, T_n$, in addition to a terminal \$1 principal payment.

In the case of a constant coupon rate c paid at regular time intervals $\tau = T_{k+1} - T_k$, $k = 0, 1, \ldots, n-1$, with $T_0 = t$, $T_n = T$, and a constant deterministic short rate r, we find

$$\begin{aligned} P_c(T_0, T_n) &= \mathrm{e}^{-rn\tau} + c\sum_{k=1}^n \mathrm{e}^{-(T_k - T_0)r} \\ &= \mathrm{e}^{-rn\tau} + c\sum_{k=1}^n \mathrm{e}^{-kr\tau} \\ &= \mathrm{e}^{-rn\tau} + c\frac{\mathrm{e}^{-r\tau} - \mathrm{e}^{-r(n+1)\tau}}{1 - \mathrm{e}^{-r\tau}}. \end{aligned}$$

In terms of the discrete-time interest rate $\tilde{r} := \mathrm{e}^{r\tau} - 1$, we also have

$$P_c(T_0, T_n) = \frac{c}{\tilde{r}} + \frac{\tilde{r} - c}{(1+\tilde{r})^n \tilde{r}}.$$

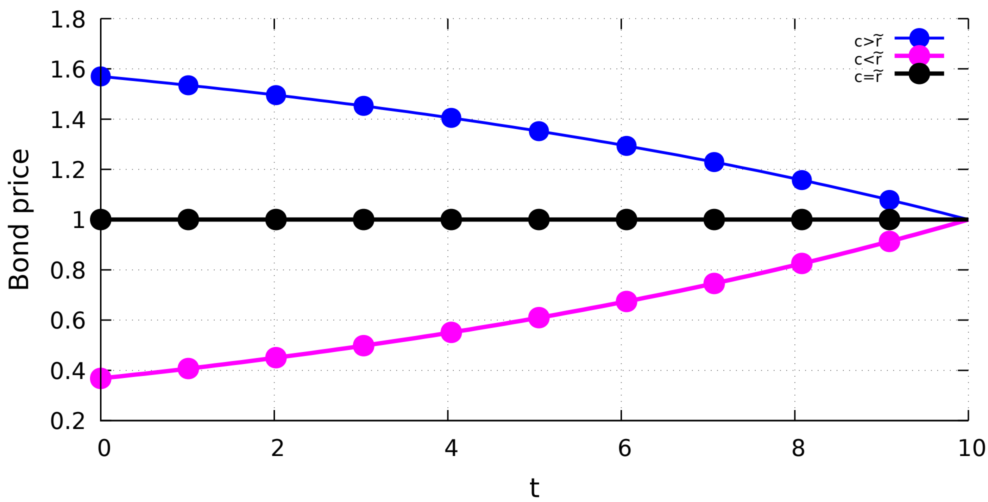


Figure 1.6: Discrete-time coupon bond pricing.

*However, coupon default cannot be excluded.

In the case of a continuous-time coupon rate $c > 0$, the above discrete-time calculation (1.3.3) can be reinterpreted as follows:

$$\begin{aligned}
P_c(t,T) &= P_0(t,T) + c\int_t^T P_0(t,u)du \qquad (1.3.4)\\
&= P_0(t,T) + c\int_t^T \mathrm{e}^{-(u-t)r}du\\
&= \mathrm{e}^{-(T-t)r} + c\int_0^{T-t} \mathrm{e}^{-ru}du\\
&= \mathrm{e}^{-(T-t)r} + \frac{c}{r}\big(1-\mathrm{e}^{-(T-t)r}\big),\\
&= \frac{c}{r} + \frac{r-c}{r}\mathrm{e}^{-(T-t)r}, \qquad 0 \leqslant t \leqslant T, \qquad (1.3.5)
\end{aligned}$$

where the coupon bond price $P_c(t,T)$ solves the ordinary differential equation

$$dP_c(t,T) = (r-c)\,\mathrm{e}^{-(T-t)r}dt = -cdt + rP_c(t,T)dt, \qquad 0 \leqslant t \leqslant T,$$

see also Figures 1.7 and 1.11 below.

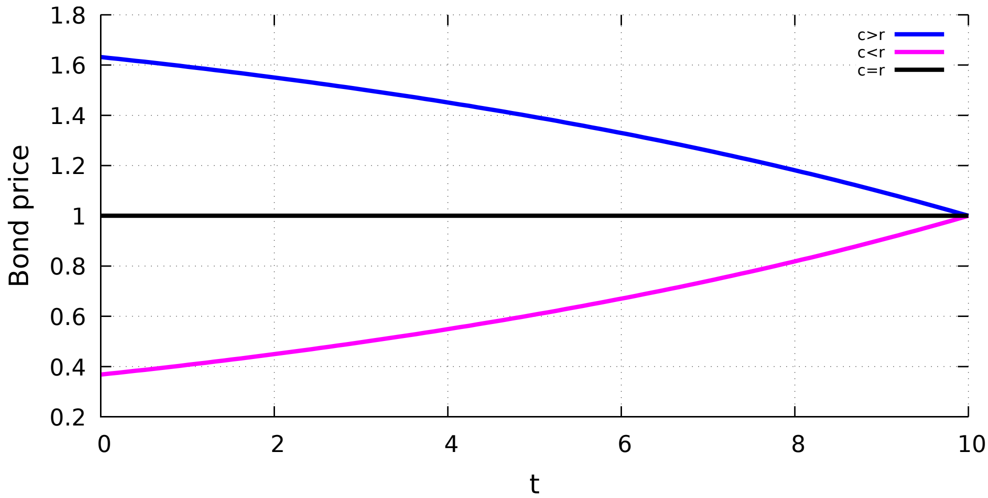


Figure 1.7: Continuous-time coupon bond pricing.

In what follows, we will mostly consider zero-coupon bonds priced as $P(t,T) = P_0(t,T)$, $0 \leqslant t \leqslant T$, in the setting of stochastic short rates.

Martingale property of discounted bond prices

The following proposition shows that Assumption 1 of Chapter 3 is satisfied, in other words, the bond price process $t \mapsto P(t,T)$ can be used as a numéraire.

Proposition 1.1 The discounted bond price process

$$t \mapsto \widetilde{P}(t,T) := \mathrm{e}^{-\int_0^t r_s ds}P(t,T)$$

is a martingale under $\mathbb{P}^*$.

Proof. By (1.3.2) we have

$$\begin{aligned}
\widetilde{P}(t,T) &= \mathrm{e}^{-\int_0^t r_s ds}P(t,T)\\
&= \mathrm{e}^{-\int_0^t r_s ds}\mathbb{E}^*\left[\mathrm{e}^{-\int_t^T r_s ds}\,\middle|\,\mathscr{F}_t\right]
\end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}^*\left[e^{-\int_0^t r_s ds} e^{-\int_t^T r_s ds} \,\Big|\, \mathscr{F}_t \right] \\
&= \mathbb{E}^*\left[e^{-\int_0^T r_s ds} \,\Big|\, \mathscr{F}_t \right], \qquad 0 \leqslant t \leqslant T,
\end{aligned}
$$

and this suffices in order to conclude, since by the tower property of conditional expectations, any process $(X_t)_{t\in\mathbb{R}_+}$ of the form $t \mapsto X_t := \mathbb{E}^*[F \mid \mathscr{F}_t]$, $F \in L^1(\Omega)$, is a martingale. In other words, we have

$$
\begin{aligned}
\mathbb{E}^*\left[\widetilde{P}(t,T) \mid \mathscr{F}_u\right] &= \mathbb{E}^*\left[\mathbb{E}^*\left[e^{-\int_0^T r_s ds} \,\Big|\, \mathscr{F}_t \right] \,\Big|\, \mathscr{F}_u \right] \\
&= \mathbb{E}^*\left[e^{-\int_0^T r_s ds} \,\Big|\, \mathscr{F}_u \right] \\
&= \widetilde{P}(u,T), \qquad 0 \leqslant u \leqslant t.
\end{aligned}
$$

□

1.4 Bond Pricing PDE

We assume from now on that the underlying short rate process solves the stochastic differential equation

$$
dr_t = \mu(t,r_t)dt + \sigma(t,r_t)dB_t \tag{1.4.1}
$$

where $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion under $\mathbb{P}^*$. Note that specifying the dynamics of $(r_t)_{t\in\mathbb{R}_+}$ under the historical probability measure $\mathbb{P}$ will also lead to a notion of market price of risk (MPoR) for the modeling of short rates.

As all solutions of stochastic differential equations such as (1.4.1) have the *Markov property*, cf. *e.g.* Theorem V-32 of Protter, 2004, the arbitrage-free price $P(t,T)$ can be rewritten as a function $F(t,r_t)$ of r_t, *i.e.*

$$
\begin{aligned}
P(t,T) &= \mathbb{E}^*\left[e^{-\int_t^T r_s ds} \,\Big|\, \mathscr{F}_t \right] \\
&= \mathbb{E}^*\left[e^{-\int_t^T r_s ds} \,\Big|\, r_t \right] \\
&= F(t,r_t),
\end{aligned} \tag{1.4.2}
$$

and depends on (t,r_t) only, instead of depending on the whole information available in $\mathscr{F}_t$ up to time t, meaning that the pricing problem can now be formulated as a search for the function $F(t,x)$.

Proposition 1.2 (Bond pricing PDE). Consider a short rate $(r_t)_{t\in\mathbb{R}_+}$ modeled by a diffusion equation of the form

$$
dr_t = \mu(t,r_t)dt + \sigma(t,r_t)dB_t.
$$

The bond pricing PDE for $P(t,T) = F(t,r_t)$ as in (1.4.2) is written as

$$
\boxed{xF(t,x) = \frac{\partial F}{\partial t}(t,x) + \mu(t,x)\frac{\partial F}{\partial x}(t,x) + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2 F}{\partial x^2}(t,x),} \tag{1.4.3}
$$

$t \geqslant 0$, $x \in \mathbb{R}$, subject to the terminal condition

$$F(T,x) = 1, \qquad x \in \mathbb{R}. \tag{1.4.4}$$

In addition, the bond price dynamics is given by

$$dP(t,T) = r_t P(t,T)dt + \sigma(t,r_t)\frac{\partial F}{\partial x}(t,r_t)dB_t. \tag{1.4.5}$$

Proof. By Itô's formula, we have

$$\begin{aligned}
& d\left(\mathrm{e}^{-\int_0^t r_s ds} P(t,T)\right) = -r_t\,\mathrm{e}^{-\int_0^t r_s ds} P(t,T)dt + \mathrm{e}^{-\int_0^t r_s ds} dP(t,T) \\
&= -r_t\,\mathrm{e}^{-\int_0^t r_s ds} F(t,r_t)dt + \mathrm{e}^{-\int_0^t r_s ds} dF(t,r_t) \\
&= -r_t\,\mathrm{e}^{-\int_0^t r_s ds} F(t,r_t)dt + \mathrm{e}^{-\int_0^t r_s ds} \frac{\partial F}{\partial x}(t,r_t)dr_t \\
&\quad + \frac{1}{2}\mathrm{e}^{-\int_0^t r_s ds}\frac{\partial^2 F}{\partial x^2}(t,r_t)(dr_t)^2 + \mathrm{e}^{-\int_0^t r_s ds}\frac{\partial F}{\partial t}(t,r_t)dt \\
&= -r_t\,\mathrm{e}^{-\int_0^t r_s ds} F(t,r_t)dt + \mathrm{e}^{-\int_0^t r_s ds} \frac{\partial F}{\partial x}(t,r_t)(\mu(t,r_t)dt + \sigma(t,r_t)dB_t) \\
&\quad + \mathrm{e}^{-\int_0^t r_s ds}\left(\frac{1}{2}\sigma^2(t,r_t)\frac{\partial^2 F}{\partial x^2}(t,r_t) + \frac{\partial F}{\partial t}(t,r_t)\right)dt \\
&= \mathrm{e}^{-\int_0^t r_s ds}\sigma(t,r_t)\frac{\partial F}{\partial x}(t,r_t)dB_t \\
&+ \mathrm{e}^{-\int_0^t r_s ds}\left(-r_t F(t,r_t) + \mu(t,r_t)\frac{\partial F}{\partial x}(t,r_t) + \frac{1}{2}\sigma^2(t,r_t)\frac{\partial^2 F}{\partial x^2}(t,r_t) + \frac{\partial F}{\partial t}(t,r_t)\right)dt.
\end{aligned} \tag{1.4.6}$$

Given that $t \mapsto \mathrm{e}^{-\int_0^t r_s ds}P(t,T)$ is a martingale, the above expression (1.4.6) should only contain terms in dB_t (cf. Corollary II-6-1, page 72 of Protter, 2004), and all terms in dt should vanish inside (1.4.6). This leads to the identities

$$\begin{cases}
r_t F(t,r_t) = \mu(t,r_t)\dfrac{\partial F}{\partial x}(t,r_t) + \dfrac{1}{2}\sigma^2(t,r_t)\dfrac{\partial^2 F}{\partial x^2}(t,r_t) + \dfrac{\partial F}{\partial t}(t,r_t) \\[2ex]
d\left(\mathrm{e}^{-\int_0^t r_s ds}P(t,T)\right) = \mathrm{e}^{-\int_0^t r_s ds}\sigma(t,r_t)\dfrac{\partial F}{\partial x}(t,r_t)dB_t,
\end{cases} \tag{1.4.7a}$$

which lead to (1.4.3) and (1.4.5). Condition (1.4.4) is due to the fact that $P(T,T) = \$1$. □

By (1.4.7a), the proof of Proposition 1.2 also shows that

$$\begin{aligned}
\frac{dP(t,T)}{P(t,T)} &= \frac{1}{P(t,T)}d\left(\mathrm{e}^{\int_0^t r_s ds}\mathrm{e}^{-\int_0^t r_s ds}P(t,T)\right) \\
&= \frac{1}{P(t,T)}\left(r_t P(t,T)dt + \mathrm{e}^{\int_0^t r_s ds} d\left(\mathrm{e}^{-\int_0^t r_s ds}P(t,T)\right)\right) \\
&= r_t dt + \frac{1}{P(t,T)}\mathrm{e}^{\int_0^t r_s ds} d\left(\mathrm{e}^{-\int_0^t r_s ds}P(t,T)\right) \\
&= r_t dt + \frac{1}{F(t,r_t)}\frac{\partial F}{\partial x}(t,r_t)\sigma(t,r_t)dB_t \\
&= r_t dt + \sigma(t,r_t)\frac{\partial}{\partial x}\log F(t,r_t)dB_t.
\end{aligned} \tag{1.4.8}$$

In the case of an interest rate process modeled by (1.1.8), we have

$$\mu(t,x) = \eta(t) + \lambda(t)x, \qquad \text{and} \qquad \sigma(t,x) = \sqrt{\delta(t) + \gamma(t)x},$$

hence (1.4.3) yields the *affine* PDE

$$xF(t,x) = \frac{\partial F}{\partial t}(t,x) + (\eta(t) + \lambda(t)x)\frac{\partial F}{\partial x}(t,x) + \frac{1}{2}(\delta(t) + \gamma(t)x)\frac{\partial^2 F}{\partial x^2}(t,x) \tag{1.4.9}$$

with time-dependent coefficients, $t \geqslant 0$, $x \in \mathbb{R}$.

Note that more generally, all affine short rate models as defined in Relation (1.1.8), including the Vasicek model, will yield a bond pricing formula of the form

$$P(t,T) = \mathrm{e}^{A(T-t)+r_t C(T-t)}, \tag{1.4.10}$$

cf. *e.g.* § 3.2.4. in Brigo and Mercurio, 2006.

Vašíček, 1977 model

In the Vasicek case

$$dr_t = (a - br_t)dt + \sigma dB_t,$$

the bond price takes the form

$$F(t,r_t) = P(t,T) = \mathrm{e}^{A(T-t)+r_t C(T-t)},$$

where $A(\cdot)$ and $C(\cdot)$ are deterministic functions of time, see (1.4.16)-(1.4.17) below, and (1.4.8) yields

$$\begin{aligned}\frac{dP(t,T)}{P(t,T)} &= r_t dt + \sigma C(T-t)dB_t \qquad (1.4.11)\\ &= r_t dt - \frac{\sigma}{b}\left(1 - \mathrm{e}^{-(T-t)b}\right)dB_t,\end{aligned}$$

since $F(t,x) = \mathrm{e}^{A(T-t)+xC(T-t)}$.

Probabilistic solution of the Vasicek PDE

Next, we solve the PDE (1.4.3), written with $\mu(t,x) = a - bx$ and $\sigma(t,x) = \sigma$ in the Vašíček, 1977 model

$$dr_t = (a - br_t)dt + \sigma dB_t \tag{1.4.12}$$

as

$$\begin{cases} xF(t,x) = \dfrac{\partial F}{\partial t}(t,x) + (a - bx)\dfrac{\partial F}{\partial x}(t,x) + \dfrac{\sigma^2}{2}\dfrac{\partial^2 F}{\partial x^2}(t,x), \\ F(T,x) = 1. \end{cases} \tag{1.4.13}$$

For this, Proposition 1.3 relies on a direct computation of the conditional expectation

$$F(t,r_t) = P(t,T) = \mathbb{E}^*\left[\mathrm{e}^{-\int_t^T r_s ds} \,\middle|\, \mathcal{F}_t\right]. \tag{1.4.14}$$

Proposition 1.3 The zero-coupon bond price in the Vasicek model (1.4.12) can be expressed as

$$P(t,T) = \mathrm{e}^{A(T-t)+r_tC(T-t)}, \qquad 0 \leqslant t \leqslant T, \tag{1.4.15}$$

where $A(x)$ and $C(x)$ are functions of time to maturity given by

$$C(x) := -\frac{1}{b}(1-\mathrm{e}^{-bx}), \tag{1.4.16}$$

and

$$\begin{aligned} A(x) &:= \frac{4ab-3\sigma^2}{4b^3} + \frac{\sigma^2-2ab}{2b^2}x + \frac{\sigma^2-ab}{b^3}\mathrm{e}^{-bx} - \frac{\sigma^2}{4b^3}\mathrm{e}^{-2bx} \\ &= -\left(\frac{a}{b}-\frac{\sigma^2}{2b^2}\right)(x+C(x)) - \frac{\sigma^2}{4b}C^2(x), \qquad x \geqslant 0. \end{aligned} \tag{1.4.17}$$

Proof. Recall that in the Vasicek model (1.4.12), the short rate process $(r_t)_{t\in\mathbb{R}_+}$ solution of (1.4.12) has the expression

$$r_t = g(t) + \int_0^t h(t,s)dB_s = r_0\mathrm{e}^{-bt} + \frac{a}{b}(1-\mathrm{e}^{-bt}) + \sigma\int_0^t \mathrm{e}^{-(t-s)b}dB_s,$$

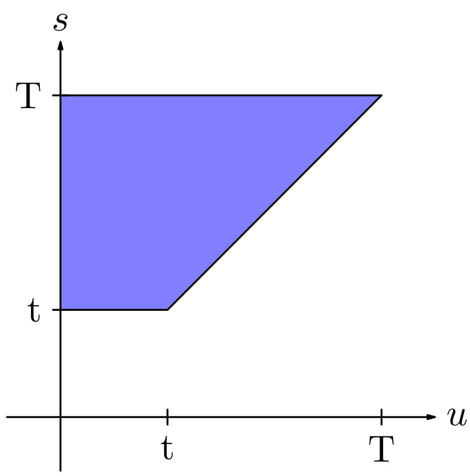
see Exercise 1.1, where g and h are the deterministic functions

$$g(t) := r_0\mathrm{e}^{-bt} + \frac{a}{b}(1-\mathrm{e}^{-bt}), \qquad t \geqslant 0,$$

and

$$h(t,s) := \sigma\mathrm{e}^{-(t-s)b}, \qquad 0 \leqslant s \leqslant t.$$

Using the fact that Wiener integrals are Gaussian random variables and the Gaussian moment generating function, and exchanging the order of integration between ds and du over $[t,T]$ according to the following picture,



we have

$$\begin{aligned} P(t,T) &= \mathbf{E}^*\left[\mathrm{e}^{-\int_t^T r_s ds} \,\middle|\, \mathscr{F}_t\right] \\ &= \mathbf{E}^*\left[\mathrm{e}^{-\int_t^T (g(s)+\int_0^s h(s,u)dB_u)ds} \,\middle|\, \mathscr{F}_t\right] \end{aligned}$$

$$
\begin{aligned}
&= \exp\left(-\int_t^T g(s)ds\right) \mathbb{E}^*\left[\mathrm{e}^{-\int_t^T \int_0^s h(s,u)dB_u ds} \,\Big|\, \mathscr{F}_t \right] \\
&= \exp\left(-\int_t^T g(s)ds\right) \mathbb{E}^*\left[\mathrm{e}^{-\int_0^T \int_{\mathrm{Max}(u,t)}^T h(s,u)dsdB_u} \,\Big|\, \mathscr{F}_t \right] \\
&= \exp\left(-\int_t^T g(s)ds - \int_0^t \int_{\mathrm{Max}(u,t)}^T h(s,u)dsdB_u\right) \mathbb{E}^*\left[\mathrm{e}^{-\int_t^T \int_{\mathrm{Max}(u,t)}^T h(s,u)dsdB_u} \,\Big|\, \mathscr{F}_t \right] \\
&= \exp\left(-\int_t^T g(s)ds - \int_0^t \int_t^T h(s,u)dsdB_u\right) \mathbb{E}^*\left[\mathrm{e}^{-\int_t^T \int_u^T h(s,u)dsdB_u} \,\Big|\, \mathscr{F}_t \right] \\
&= \exp\left(-\int_t^T g(s)ds - \int_0^t \int_t^T h(s,u)dsdB_u\right) \mathbb{E}^*\left[\mathrm{e}^{-\int_t^T \int_u^T h(s,u)dsdB_u} \right] \\
&= \exp\left(-\int_t^T g(s)ds - \int_0^t \int_t^T h(s,u)dsdB_u + \frac{1}{2}\int_t^T \left(\int_u^T h(s,u)ds\right)^2 du\right) \\
&= \exp\left(-\int_t^T \left(r_0 \mathrm{e}^{-bs} + \frac{a}{b}(1-\mathrm{e}^{-bs})\right)ds - \sigma \int_0^t \int_t^T \mathrm{e}^{-(s-u)b} dsdB_u\right) \\
&\quad \times \exp\left(\frac{\sigma^2}{2}\int_t^T \left(\int_u^T \mathrm{e}^{-(s-u)b}ds\right)^2 du\right) \\
&= \exp\left(-\int_t^T \left(r_0 \mathrm{e}^{-bs} + \frac{a}{b}(1-\mathrm{e}^{-bs})\right)ds - \frac{\sigma}{b}\left(1-\mathrm{e}^{-(T-t)b}\right)\int_0^t \mathrm{e}^{-(t-u)b}dB_u\right) \\
&\quad \times \exp\left(\frac{\sigma^2}{2}\int_t^T \mathrm{e}^{2bu}\left(\frac{\mathrm{e}^{-bu}-\mathrm{e}^{-bT}}{b}\right)^2 du\right) \\
&= \exp\left(-\frac{r_t}{b}\left(1-\mathrm{e}^{-(T-t)b}\right) + \frac{1}{b}\left(1-\mathrm{e}^{-(T-t)b}\right)\left(r_0 \mathrm{e}^{-bt} + \frac{a}{b}(1-\mathrm{e}^{-bt})\right)\right) \\
&\quad \times \exp\left(-\int_t^T \left(r_0 \mathrm{e}^{-bs} + \frac{a}{b}(1-\mathrm{e}^{-bs})\right)ds + \frac{\sigma^2}{2}\int_t^T \mathrm{e}^{2bu}\left(\frac{\mathrm{e}^{-bu}-\mathrm{e}^{-bT}}{b}\right)^2 du\right) \\
&= \mathrm{e}^{A(T-t)+r_tC(T-t)}, \qquad (1.4.18)
\end{aligned}
$$

where $A(x)$ and $C(x)$ are the functions given by (1.4.16) and (1.4.17). □

Analytical solution of the Vasicek PDE

In order to solve the PDE (1.4.13) analytically, we may start by looking for a solution of the form

$$
F(t,x) = \mathrm{e}^{A(T-t)+xC(T-t)}, \qquad (1.4.19)
$$

where $A(\cdot)$ and $C(\cdot)$ are functions to be determined under the conditions $A(0)=0$ and $C(0)=0$. Substituting (1.4.19) into the PDE (1.4.3) with the Vasicek coefficients $\mu(t,x)=(a-bx)$ and $\sigma(t,x)=\sigma$ shows that

$$
\begin{aligned}
x\mathrm{e}^{A(T-t)+xC(T-t)} &= -(A'(T-t)+xC'(T-t))\,\mathrm{e}^{A(T-t)+xC(T-t)} \\
&\quad +(a-bx)C(T-t)\,\mathrm{e}^{A(T-t)+xC(T-t)} \\
&\quad +\frac{1}{2}\sigma^2C^2(T-t)\,\mathrm{e}^{A(T-t)+xC(T-t)},
\end{aligned}
$$

i.e.

$$
x = -A'(T-t) - xC'(T-t) + (a-bx)C(T-t) + \frac{1}{2}\sigma^2C^2(T-t).
$$

By identification of terms for $x = 0$ and $x \neq 0$, this yields the system of Riccati and linear differential equations

$$\begin{cases} A'(s) = aC(s) + \dfrac{\sigma^2}{2} C^2(s) \\ \\ C'(s) = -1 - bC(s), \end{cases}$$

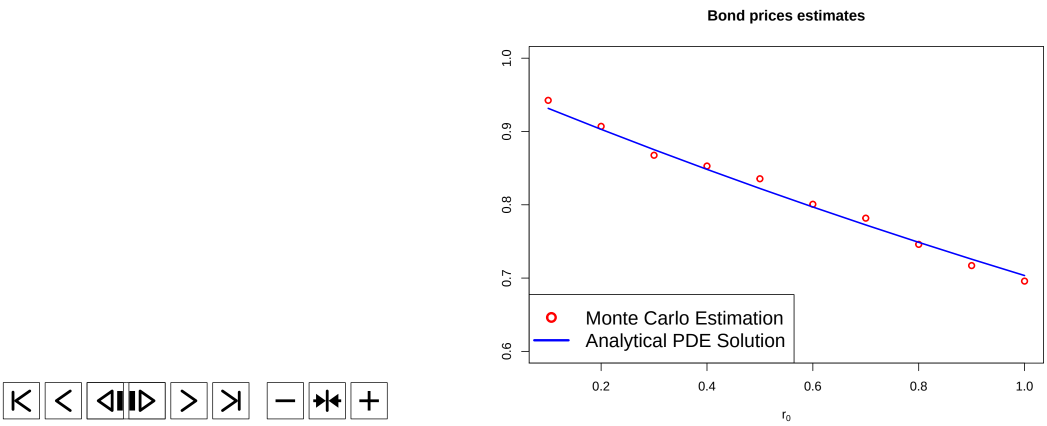
which can be solved to recover the above value of $P(t,T) = F(t,r_t)$ via

$$C(s) = -\frac{1}{b}(1 - \mathrm{e}^{-bs})$$

and

$$\begin{aligned} A(t) &= A(0) + \int_0^t A'(s)ds \\ &= \int_0^t \left(aC(s) + \frac{\sigma^2}{2} C^2(s) \right) ds \\ &= \int_0^t \left(\frac{a}{b}(1 - \mathrm{e}^{-bs}) + \frac{\sigma^2}{2b^2}(1 - \mathrm{e}^{-bs})^2 \right) ds \\ &= \frac{a}{b} \int_0^t (1 - \mathrm{e}^{-bs}) ds + \frac{\sigma^2}{2b^2} \int_0^t (1 - \mathrm{e}^{-bs})^2 ds \\ &= \frac{4ab - 3\sigma^2}{4b^3} + \frac{\sigma^2 - 2ab}{2b^2} t + \frac{\sigma^2 - ab}{b^3} \mathrm{e}^{-bt} - \frac{\sigma^2}{4b^3} \mathrm{e}^{-2bt}, \qquad t \geqslant 0. \end{aligned}$$

The next Figure 1.8 shows the output of the attached **R code** for the Monte Carlo and analytical estimation of Vasicek bond prices.



(a) Short rate paths. (b) Comparison with the PDE solution.

Figure 1.8: Comparison of Monte Carlo and analytical PDE solutions.

Vasicek bond price simulations

In this section we consider again the Vasicek model, in which the short rate $(r_t)_{t\in\mathbb{R}_+}$ is solution to (1.1.1). Figure 1.9 presents a random simulation of the zero-coupon bond price (1.4.15) in the Vasicek model with $\sigma = 10\%$, $r_0 = 2.96\%$, $b = 0.5$, and $a = 0.025$. The graph of the corresponding deterministic zero-coupon bond price with $r = r_0 = 2.96\%$ is also shown in Figure 1.9.

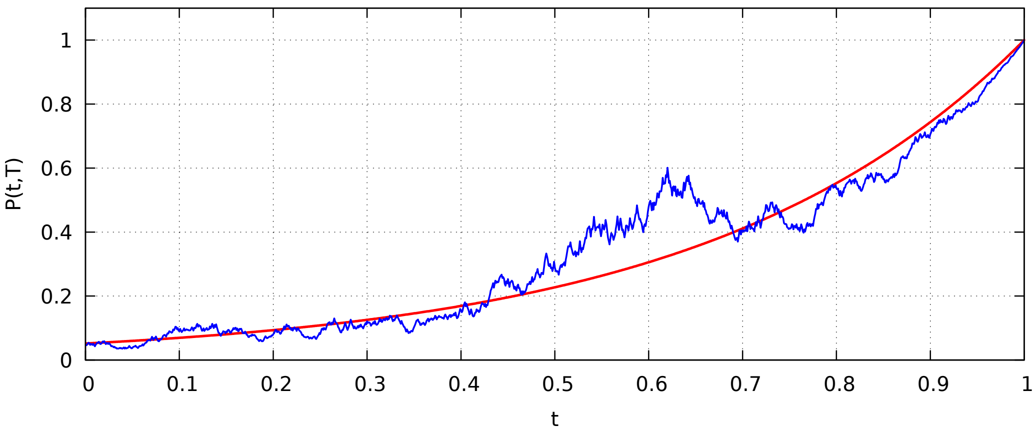


Figure 1.9: Graphs of $t \mapsto F(t,r_t) = P(t,T)$ *vs* $t \mapsto \mathrm{e}^{-r_0(T-t)}$.

Figure 1.10 presents a random simulation of the coupon bond price (1.3.4) in the Vasicek model with $\sigma = 2\%$, $r_0 = 3.5\%$, $b = 0.5$, $a = 0.025$, and coupon rate $c = 5\%$. The graph of the corresponding deterministic coupon bond price (1.3.5) with $r = r_0 = 3.5\%$ is also shown in Figure 1.10.

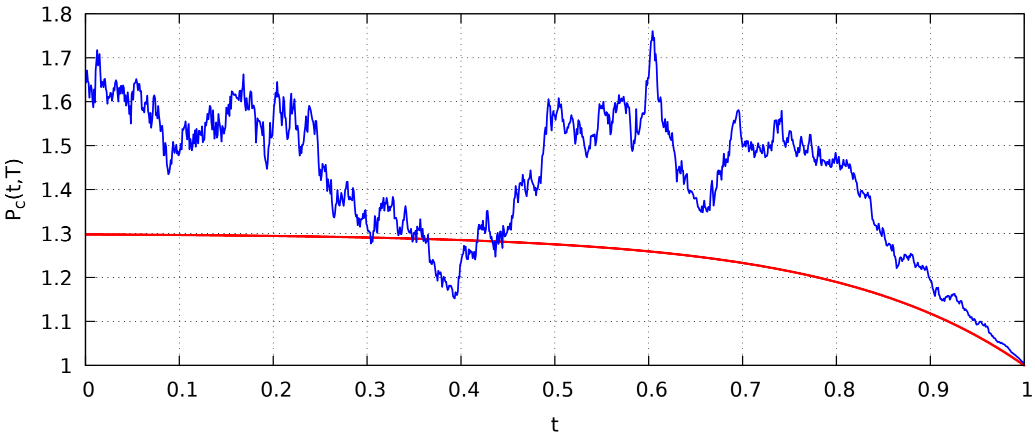


Figure 1.10: Graph of $t \mapsto P_c(t,T)$ for a bond with a 5% coupon rate.

Figure 1.11 presents market price data for a coupon bond with coupon rate $c = 6.25\%$.

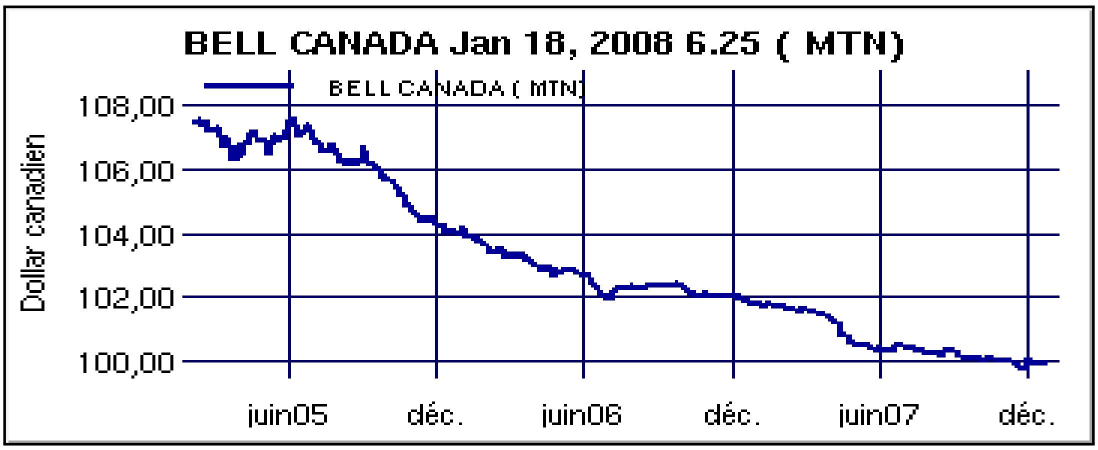


Figure 1.11: Bond price graph with maturity 01/18/08 and coupon rate 6.25%.

Zero-coupon bond price and yield data

The following zero-coupon public bond price data was downloaded from EMMA at the Municipal Securities Rulemaking Board.

ORANGE CNTY CALIF PENSION OBLIG CAP APPREC-TAXABLE-REF-SER A (CA)
CUSIP: 68428LBB9
Dated Date: 06/12/1996 (June 12, 1996)
Maturity Date: 09/01/2016 (September 1st, 2016)
Interest Rate: 0.0 %
Principal Amount at Issuance: $26,056,000
Initial Offering Price: 19.465

```
library(quantmod);getwd()
bondprice <- read.table("bond_data_R.txt",col.names =
     c("Date","HighPrice","LowPrice","HighYield","LowYield","Count","Amount"))
head(bondprice)
time <- as.POSIXct(bondprice$Date, format = "%Y-%m-%d")
price <- xts(x = bondprice$HighPrice, order.by = time)
yield <- xts(x = bondprice$HighYield, order.by = time)
dev.new(width=10,height=7);
chartSeries(price,up.col="blue",theme="white")
chartSeries(yield,up.col="blue",theme="white")
```

	Date	HighPrice	LowPrice	HighYield	LowYield	Count	Amount
1	2016-01-13	99.082	98.982	1.666	1.501	2	20000
2	2015-12-29	99.183	99.183	1.250	1.250	1	10000
3	2015-12-21	97.952	97.952	3.014	3.014	1	10000
4	2015-12-17	99.141	98.550	2.123	1.251	5	610000
5	2015-12-07	98.770	98.770	1.714	1.714	2	10000
6	2015-12-04	98.363	98.118	2.628	2.280	2	10000

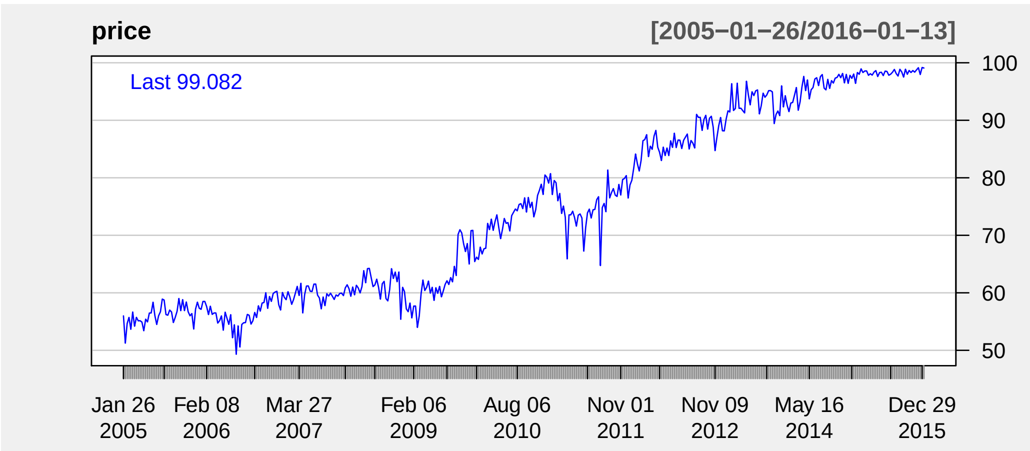


Figure 1.12: Orange Cnty Calif bond prices.

The next Figure 1.13 plots the *bond yield* $y(t,T)$ defined as

$$y(t,T) = -\frac{\log P(t,T)}{T-t}, \quad \text{or} \quad P(t,T) = \mathrm{e}^{-(T-t)y(t,T)}, \qquad 0 \leqslant t \leqslant T.$$

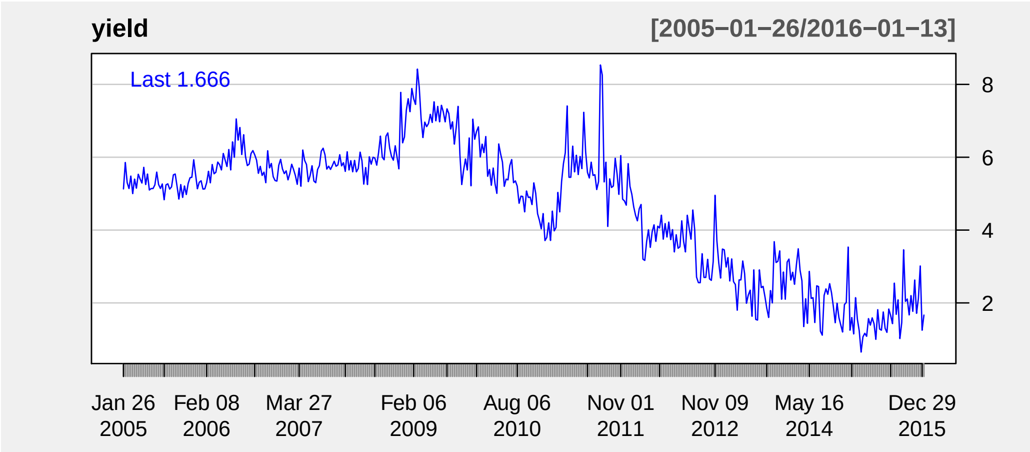


Figure 1.13: Orange Cnty Calif bond yields.

Bond pricing in the Dothan model

In the Dothan, 1978 model, the short-term interest rate process $(r_t)_{t\in\mathbb{R}_+}$ is modeled according to a geometric Brownian motion

$$dr_t = \mu r_t dt + \sigma r_t dB_t, \tag{1.4.20}$$

where the volatility $\sigma > 0$ and the drift $\mu \in \mathbb{R}$ are constant parameters and $(B_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion. In this model the short-term interest rate r_t remains always positive, while the proportional volatility term σr_t accounts for the sensitivity of the volatility of interest rate changes to the level of the rate r_t.

On the other hand, the Dothan model is the only lognormal short rate model that allows for an analytical formula for the zero-coupon bond price

$$P(t,T) = \mathbb{E}^*\left[\mathrm{e}^{-\int_t^T r_s ds} \,\middle|\, \mathscr{F}_t \right], \qquad 0 \leqslant t \leqslant T.$$

For convenience of notation we let $p = 1 - 2\mu/\sigma^2$ and rewrite (1.4.20) as

$$dr_t = (1-p)\frac{\sigma^2}{2} r_t dt + \sigma r_t dB_t,$$

with solution

$$r_t = r_0 \,\mathrm{e}^{\sigma B_t - p\sigma^2 t/2}, \qquad t \geqslant 0. \tag{1.4.21}$$

By the Markov property of $(r_t)_{t\in\mathbb{R}_+}$, the bond price $P(t,T)$ is a function $F(t,r_t)$ of r_t and time $t \in [0,T]$:

$$P(t,T) = F(t,r_t) = \mathbb{E}^*\left[\mathrm{e}^{-\int_t^T r_s ds} \,\middle|\, r_t \right], \qquad 0 \leqslant t \leqslant T. \tag{1.4.22}$$

By computation of the conditional expectation (1.4.22) we easily obtain the following result, cf. Proposition 1.2 of C. Pintoux and N. Privault, 2011, where the function $\theta(v,t)$ is defined using a double integral.

Proposition 1.4 The zero-coupon bond price $P(t,T) = F(t,r_t)$ is given for all $p \in \mathbb{R}$ by

$$F(t,x) \tag{1.4.23}$$
$$= e^{-\sigma^2 p^2 (T-t)/8} \int_0^\infty \int_0^\infty e^{-ux} \exp\left(-2\frac{(1+z^2)}{\sigma^2 u}\right) \theta\left(\frac{4z}{\sigma^2 u}, \frac{(T-t)\sigma^2}{4}\right) \frac{du}{u} \frac{dz}{z^{p+1}},$$

$x > 0$.

Proof. The probability distribution of the time integral $\int_0^{T-t} e^{\sigma B_s - p\sigma^2 s/2} ds$ is given by

$$\begin{aligned}
&\mathbb{P}\left(\int_0^{T-t} e^{\sigma B_s - p\sigma^2 s/2} ds \in dy\right) \\
&= \int_{-\infty}^\infty \mathbb{P}\left(\int_0^t e^{\sigma B_s - p\sigma^2 s/2} ds \in dy,\ B_t - p\sigma t/2 \in dz\right) \\
&= \frac{\sigma}{2} \int_{-\infty}^\infty e^{-p\sigma z/2 - p^2\sigma^2 t/8} \exp\left(-2\frac{1+e^{\sigma z}}{\sigma^2 y}\right) \theta\left(\frac{4 e^{\sigma z/2}}{\sigma^2 y}, \frac{\sigma^2 t}{4}\right) \frac{dy}{y} dz \\
&= e^{-(T-t)p^2\sigma^2/8} \int_0^\infty \exp\left(-2\frac{1+z^2}{\sigma^2 y}\right) \theta\left(\frac{4z}{\sigma^2 y}, \frac{(T-t)\sigma^2}{4}\right) \frac{dz}{z^{p+1}} \frac{dy}{y}, \quad y > 0,
\end{aligned}$$

where the exchange of integrals is justified by the Fubini theorem and the nonnegativity of integrands. Hence, by (1.4.21) we find

$$\begin{aligned}
&F(t,r_t) = P(t,T) \\
&= \mathbb{E}^*\left[\exp\left(-\int_t^T r_s ds\right) \Big| \mathscr{F}_t\right] \\
&= \mathbb{E}^*\left[\exp\left(-r_t \int_t^T e^{\sigma(B_s - B_t) - \sigma^2 p(s-t)/2} ds\right) \Big| \mathscr{F}_t\right] \\
&= \mathbb{E}^*\left[\exp\left(-x \int_t^T e^{\sigma(B_s - B_t) - \sigma^2 p(s-t)/2} ds\right)\right]_{x=r_t} \\
&= \mathbb{E}^*\left[\exp\left(-x \int_0^{T-t} e^{\sigma B_s - \sigma^2 ps/2} ds\right)\right]_{x=r_t} \\
&= \int_0^\infty e^{-r_t y}\, \mathbb{P}\left(\int_0^{T-t} e^{\sigma B_s - p\sigma^2 s/2} ds \in dy\right) \\
&= e^{-(T-t)p^2\sigma^2/8} \int_0^\infty e^{-r_t y} \int_0^\infty \exp\left(-2\frac{1+z^2}{\sigma^2 y}\right) \theta\left(\frac{4z}{\sigma^2 y}, \frac{(T-t)\sigma^2}{4}\right) \frac{dz}{z^{p+1}} \frac{dy}{y}.
\end{aligned}$$

□

The zero-coupon bond price $P(t,T) = F(t,r_t)$ in the Dothan model can also be written for all $p \in \mathbb{R}$ as

$$\begin{aligned}
F(t,x) &= \frac{(2x)^{p/2}}{2\pi^2\sigma^p} \int_0^\infty u\, e^{-(p^2+u^2)\sigma^2 t/8} \sinh(\pi u) \left|\Gamma\left(-\frac{p}{2} + i\frac{u}{2}\right)\right|^2 K_{iu}\left(\frac{\sqrt{8x}}{\sigma}\right) du \\
&+ \frac{(2x)^{p/2}}{\sigma^p} \sum_{k \geqslant 0} \frac{2(p-2k)^+}{k!(p-k)!} e^{\sigma^2 k(k-p)t/2} K_{p-2k}\left(\frac{\sqrt{8x}}{\sigma}\right), \quad x > 0,\ t > 0,
\end{aligned}$$

cf. Corollary 2.2 of C. Pintoux and N. Privault, 2010, see also Privault and Uy, 2013 for numerical computations. Zero-coupon bond prices in the Dothan model can also be computed by the conditional expression

$$\mathbb{E}\left[\exp\left(-\int_0^T r_t dt\right)\right] = \int_0^\infty \mathbb{E}\left[\exp\left(-\int_0^T r_t dt\right)\Big| r_T = z\right] d\mathbb{P}(r_T \leqslant z), \tag{1.4.24}$$

where r_T has the lognormal distribution

$$d\mathbb{P}(r_T \leqslant z) = d\mathbb{P}(r_0 \mathrm{e}^{\sigma B_T - p\sigma^2 T/2} \leqslant z) = \frac{1}{z\sqrt{2\pi\sigma^2 T}} \mathrm{e}^{-(p\sigma^2 T/2 + \log(z/r_0))^2/(2\sigma^2 T)}.$$

In Proposition 1.5 we note that the conditional Laplace transform

$$\mathbb{E}\left[\exp\left(-\int_0^T r_t dt\right)\Big| r_T = z\right]$$

cf. (1.4.28) above, can be computed by a closed-form integral expression based on the modified Bessel function of the second kind

$$K_\zeta(z) := \frac{z^\zeta}{2^{\zeta+1}} \int_0^\infty \exp\left(-u - \frac{z^2}{4u}\right) \frac{du}{u^{\zeta+1}}, \qquad \zeta \in \mathbb{R}, \quad z \in \mathbb{C}, \tag{1.4.25}$$

cf. *e.g.* Watson, 1995 page 183, provided that the real part $\mathscr{R}(z^2)$ of $z^2 \in \mathbb{C}$ is positive.

Proposition 1.5 (Privault and Yu, 2016, Proposition 4.1). Taking $r_0 = 1$, for all $\lambda, z > 0$ we have

$$\mathbb{E}\left[\exp\left(-\lambda \int_0^T r_s ds\right) \Big| \, r_T = z\right] = \frac{4\,\mathrm{e}^{-\sigma^2 T/8}}{\pi^{3/2}\sigma^2 p(z)} \sqrt{\frac{\lambda}{T}} \tag{1.4.26}$$
$$\times \int_0^\infty \mathrm{e}^{2(\pi^2-\xi^2)/(\sigma^2 T)} \sin\left(\frac{4\pi\xi}{\sigma^2 T}\right) \sinh(\xi) \frac{K_1\left(\sqrt{8\lambda}\sqrt{1+2\sqrt{z}\cosh\xi + z}/\sigma\right)}{\sqrt{1+2\sqrt{z}\cosh\xi + z}} d\xi.$$

Note however that the numerical evaluation of (1.4.26) can fail for small values of $T > 0$, and for this reason the integral can be estimated by a gamma approximation as in (1.4.27) below. Under the gamma approximation we can approximate the conditional bond price on the Dothan short rate r_t as

$$\mathbb{E}\left[\exp\left(-\lambda \int_0^T r_t dt\right) \Big| r_T = z\right] \simeq (1 + \lambda\theta(z))^{-\nu(z)},$$

where the parameters $\nu(z)$ and $\theta(z)$ are determined by conditional moment fitting to a gamma distribution, as

$$\theta(z) := \frac{\mathrm{Var}[\Lambda_T \mid S_T = z]}{\mathbb{E}[\Lambda_T \mid S_T = z]}, \quad \nu(z) := \frac{(\mathbb{E}[\Lambda_T \mid S_T = z])^2}{\mathrm{Var}[\Lambda_T \mid S_T = z]} = \frac{\mathbb{E}[\Lambda_T \mid S_T = z]}{\theta},$$

cf. Privault and Yu, 2016, which yields

$$\mathbb{E}\left[\exp\left(-\lambda \int_0^T r_s ds\right)\right] \simeq \int_0^\infty (1 + \lambda\theta(z))^{-\nu(z)} d\mathbb{P}(r_T \leqslant z). \tag{1.4.27}$$

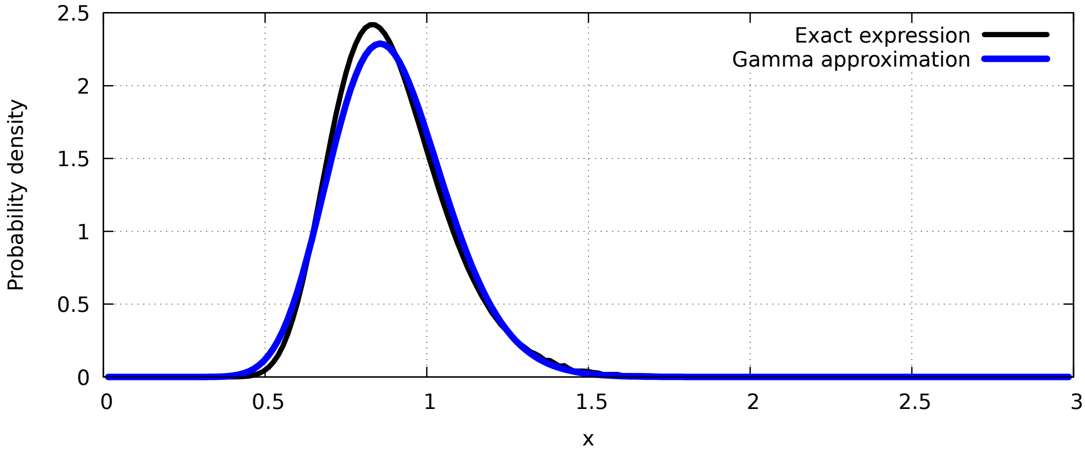


Figure 1.14: Fitting of a gamma probability density function.

The quantity $\theta(z)$ is also known in physics as the *Fano factor* or *dispersion index* that measures the dispersion of the probability distribution of Λ_T given that $S_T = z$. Figures 1.15 shows that the stratified gamma approximation (1.4.27) matches the Monte Carlo estimate, while the use of the integral expressions (1.4.24) and (1.4.26) leads to numerical instabilities.

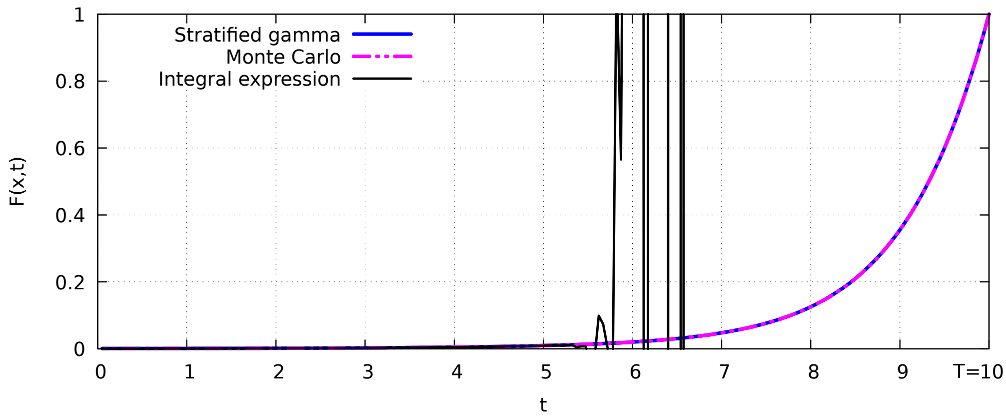


Figure 1.15: Approximation of Dothan bond prices $t \mapsto F(t,x)$ with $\sigma = 0.3$ and $T = 10$.

Related computations for yield options in the Cox, Ingersoll, and Ross, 1985 (CIR) model can also be found in Prayoga and Privault, 2017.

Path integrals in option pricing

Let $\hbar$ denote the Planck constant, and let $S(x(\cdot))$ denote the action functional given as

$$S(x(\cdot)) = \int_0^t L(x(s),\dot{x}(s),s)ds = \int_0^t \left(\frac{1}{2}m(\dot{x}(s))^2 - V(x(s))\right)ds,$$

where $L(x(s),\dot{x}(s),s)$ is the Lagrangian

$$L(x(s),\dot{x}(s),s) := \frac{1}{2}m(\dot{x}(s))^2 - V(x(s)).$$

In physics, the Feynman path integral

$$\psi(y,t) := \int_{x(0)=x,\ x(t)=y} \mathscr{D}x(\cdot)\exp\left(\frac{i}{\hbar}S(x(\cdot))\right)$$

solves the Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t}(x,t) = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}(x,t) + V(x(t))\psi(x,t).$$

After the Wick rotation $t \mapsto -it$, the function

$$\phi(y,t) := \int_{x(0)=x,\ x(t)=y} \mathscr{D}x(\cdot)\exp\left(-\frac{1}{\hbar}S(x(\cdot))\right)$$

solves the heat equation

$$\hbar\frac{\partial\phi}{\partial t}(x,t) = -\frac{\hbar^2}{2m}\frac{\partial^2\phi}{\partial x^2}(x,t) + V(x(t))\phi(x,t).$$

By reformulating the action functional $S(x(\cdot))$ as

$$\begin{aligned} S(x(\cdot)) &= \int_0^t \left(\frac{1}{2}m(\dot{x}(s))^2 + V(x(s))\right) ds \\ &\simeq \sum_{i=1}^N \left(\frac{(x(t_i)-x(t_{i-1}))^2}{2(t_i-t_{i-1})^2} + V(x(t_{i-1}))\right)\Delta t_i, \end{aligned}$$

we can rewrite the Euclidean path integral as

$$\begin{aligned} &\phi(y,t) = \int_{x(0)=x,\ x(t)=y} \mathscr{D}x(\cdot)\exp\left(-\frac{1}{\hbar}S(x(\cdot))\right) \\ &= \int_{x(0)=x,\ x(t)=y} \mathscr{D}x(\cdot)\exp\left(-\frac{1}{2\hbar}\sum_{i=1}^N \frac{(x(t_i)-x(t_{i-1}))^2}{2\Delta t_i} - \frac{1}{\hbar}\sum_{i=1}^N V(x(t_{i-1}))\right) \\ &= \mathbb{E}^*\left[\exp\left(-\frac{1}{\hbar}\int_0^t V(B_s)ds\right) \,\Big|\, B_0 = x, B_t = y\right]. \end{aligned}$$

This type of path integral computation

$$\phi(y,t) = \mathbb{E}^*\left[\exp\left(-\int_0^t V(B_s)ds\right) \,\Big|\, B_0 = x, B_t = y\right]. \tag{1.4.28}$$

is particularly useful for bond pricing, as (1.4.28) can be interpreted as the price of a bond with short-term interest rate process $(r_t)_{t\in\mathbb{R}_+} := (V(B_t))_{t\in\mathbb{R}_+}$ conditionally to the value of the endpoint $B_t = y$, cf. (1.4.26) below. The path integral (1.4.28) can be estimated either by closed-form expressions using Partial Differential Equations (PDEs) or probability densities, by approximations such as (conditional) Moment matching, or by Monte Carlo estimation, from the paths of a Brownian bridge as shown in Figure 1.16.

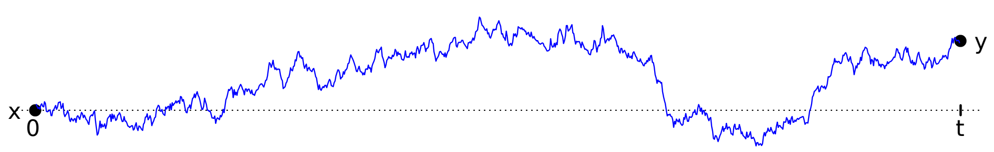


Figure 1.16: Brownian bridge.

Exercises

Exercise 1.1 We consider the stochastic differential equation

$$dr_t = (a - br_t)dt + \sigma dB_t, \tag{1.4.29}$$

where $a, \sigma \in \mathbb{R}$, $b > 0$.

a) Show that the solution of (1.4.29) is

$$r_t = r_0 \,\mathrm{e}^{-bt} + \frac{a}{b}\big(1 - \mathrm{e}^{-bt}\big) + \sigma \int_0^t \mathrm{e}^{-(t-s)b} dB_s, \quad t \geqslant 0. \tag{1.4.30}$$

b) Show that the Gaussian $\mathcal{N}(a/b, \sigma^2/(2b))$ distribution is the *invariant* (or stationary) distribution of $(r_t)_{t \in \mathbb{R}_+}$.

Exercise 1.2 (Brody, Hughston, and Meier, 2018) In the mean-reverting Vasicek model (1.4.30) with $b > 0$, compute:

i) The asymptotic bond yield, or exponential long rate of interest

$$r_\infty := -\lim_{T \to \infty} \frac{\log P(t,T)}{T-t}.$$

ii) The long-bond return

$$L_t := \lim_{T \to \infty} \frac{P(t,T)}{P(0,T)}.$$

Hint: Start from $\log(P(t,T)/P(0,T))$.

Exercise 1.3 Consider the Chan-Karolyi-Longstaff-Sanders (CKLS) interest rate model (Chan et al., 1992) parametrized as

$$dr_t = (a - br_t)dt + \sigma r_t^\gamma dB_t,$$

and time-discretized as

$$\begin{aligned} r_{t_{k+1}} &= r_{t_k} + (a - br_{t_k})\Delta t + \sigma r_{t_k}^\gamma Z_k \\ &= a\Delta t + (1 - b\Delta t) r_{t_k} + \sigma r_{t_k}^\gamma Z_k, \quad k \geqslant 0, \end{aligned}$$

where $\Delta t := t_{k+1} - t_k$ and $(Z_k)_{k \geqslant 0}$ is an *i.i.d.* sequence of $\mathcal{N}(0, \Delta t)$ random variables. Assuming that $a, b, \gamma > 0$ are known, find an unbiased estimator $\widehat{\sigma}^2$ for the variance coefficient σ^2, based on a market data set $(\tilde{r}_{t_k})_{k=0,1,\ldots,n}$.

Exercise 1.4 Let $(B_t)_{t \in \mathbb{R}_+}$ denote a standard Brownian motion started at 0 under the risk-neutral probability measure $\mathbb{P}^*$. We consider a short-term interest rate process $(r_t)_{t \in \mathbb{R}_+}$ in a Ho-Lee model with constant deterministic volatility, defined by

$$dr_t = adt + \sigma dB_t, \tag{1.4.31}$$

where $a \in \mathbb{R}$ and $\sigma > 0$. Let $P(t,T)$ denote the arbitrage-free price of a zero-coupon bond in this model:

$$P(t,T) = \mathbb{E}^*\left[\exp\left(-\int_t^T r_s ds\right) \middle| \mathcal{F}_t\right], \qquad 0 \leqslant t \leqslant T. \tag{1.4.32}$$

a) State the bond pricing PDE satisfied by the function $F(t,x)$ defined *via*

$$F(t,x) := \mathbb{E}^*\left[\exp\left(-\int_t^T r_s ds\right)\Bigg| r_t = x\right], \qquad 0 \leqslant t \leqslant T.$$

b) Compute the arbitrage-free price $F(t,r_t) = P(t,T)$ from its expression (1.4.32) as a conditional expectation.

Hint: One may use the *integration by parts* relation

$$\begin{aligned}\int_t^T B_s ds &= TB_T - tB_t - \int_t^T s dB_s \\ &= (T-t)B_t + T(B_T - B_t) - \int_t^T s dB_s \\ &= (T-t)B_t + \int_t^T (T-s) dB_s,\end{aligned}$$

and the Gaussian moment generating function $\mathbb{E}[e^{\lambda X}] = e^{\lambda^2\eta^2/2}$ for $X \simeq \mathcal{N}(0,\eta^2)$.

c) Check that the function $F(t,x)$ computed in Question (b)) does satisfy the PDE derived in Question (a)).

Exercise 1.5 Consider the Courtadon, 1982 model

$$dr_t = \beta(\alpha - r_t)dt + \sigma r_t dB_t, \tag{1.4.33}$$

where α, β, σ are nonnegative, which is a particular case of the Chan-Karolyi-Longstaff-Sanders (CKLS) model (Chan et al., 1992) with $\gamma = 1$. Show that the solution of (1.4.33) is given by

$$r_t = \alpha\beta\int_0^t \frac{S_t}{S_u} du + r_0 S_t, \qquad t \geqslant 0, \tag{1.4.34}$$

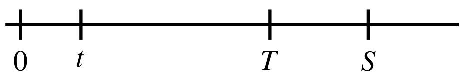
where $(S_t)_{t\in\mathbb{R}_+}$ is the geometric Brownian motion solution of $dS_t = -\beta S_t dt + \sigma S_t dB_t$ with $S_0 = 1$.

2. Forward Rate Modeling

Forward rates are interest rates used in Forward Rate Agreements (FRA) for financial transactions, such as loans, that can take place at a future date. This chapter deals with the modeling of forward rates and swap rates in the Heath-Jarrow-Morton (HJM) and Brace-Gatarek-Musiela (BGM) models. It also includes a presentation of the Nelson and Siegel, 1987 and Svensson, 1994 curve parametrizations for yield curve fitting, and an introduction to two-factor interest rate models.

2.1 Construction of Forward Rates

A forward interest rate contract (or Forward Rate Agreement, FRA) gives to its holder the possibility to lock an interest rate denoted by $f(t,T,S)$ at present time t for a loan to be delivered over a future period of time $[T,S]$, with $t \leqslant T \leqslant S$.



The rate $f(t,T,S)$ is called a forward interest rate. When $T=t$, the *spot* forward rate $f(t,t,T)$ coincides with the *yield*, see Relation (2.1.3) below.

Figure 2.1 presents a typical yield curve on the LIBOR (London Interbank Offered Rate) market with t =07 May 2003.

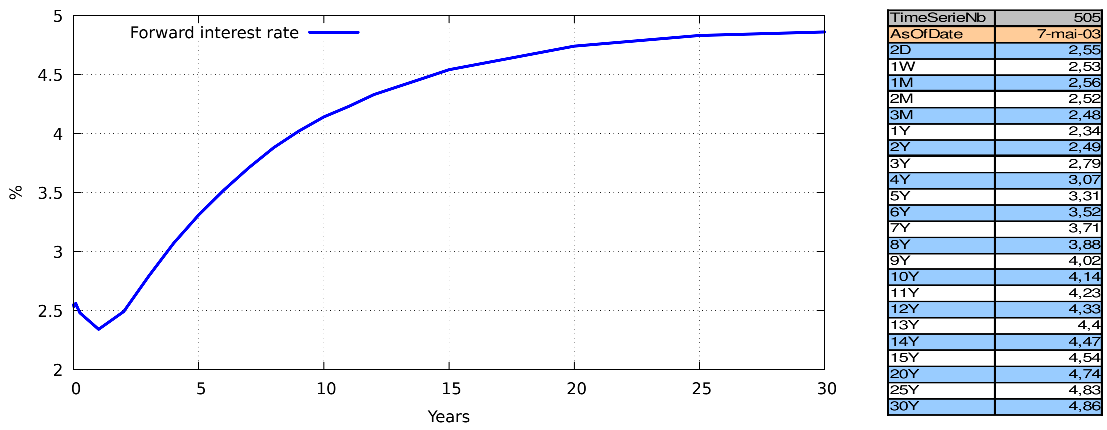


Figure 2.1: Graph of the spot forward rate $S \mapsto f(t,t,S)$.

Maturity transformation, *i.e.*, the ability to transform short-term borrowing (debt with short maturities, such as deposits) into long term lending (credits with very long maturities, such as loans), is among the roles of banks. Profitability is then dependent on the difference between long rates and short rates.

Another example of market data is given in the next Figure 2.2, in which the red and blue curves refer respectively to July 21 and 22 of year 2011.

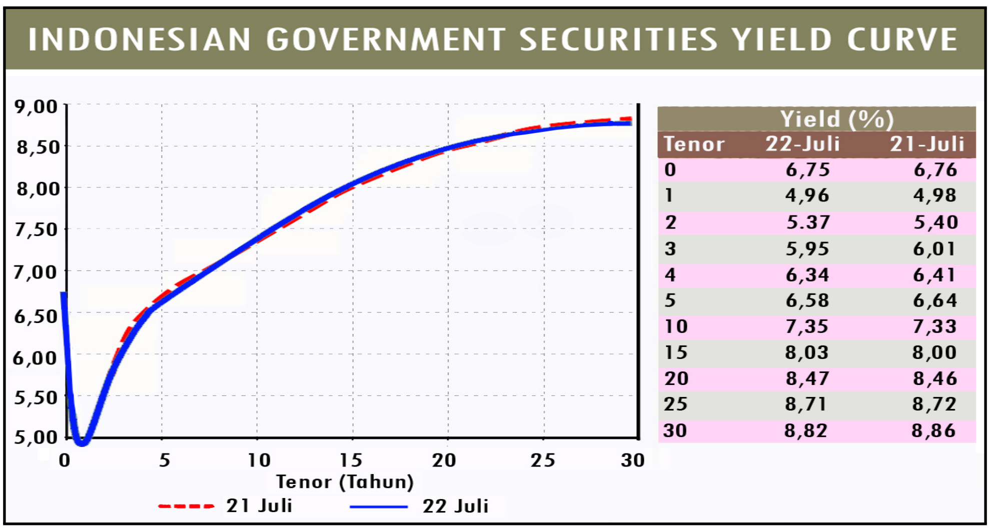


Figure 2.2: Market example of yield curves, cf. (2.1.3).

Long maturities usually correspond to higher rates as they carry an increased risk. The dip observed with short maturities can correspond to a lower motivation to lend/invest in the short-term.

Forward rates from bond prices

Let us determine the arbitrage or "fair" value of the forward interest rate $f(t,T,S)$ by implementing the Forward Rate Agreement using the instruments available in the market, which are bonds priced at $P(t,T)$ for various maturity dates $T > t$.

The loan can be realized using the available instruments (here, bonds) on the market, by proceeding in two steps:

1) At time t, borrow the amount $P(t,S)$ by issuing (or short selling) one bond with maturity S, which means refunding \$1 at time S.

2) Since the money is only needed at time T, the rational investor will invest the amount $P(t,S)$ over the period $[t,T]$ by buying a (possibly fractional) quantity $P(t,S)/P(t,T)$ of a bond with maturity T priced $P(t,T)$ at time t. This will yield the amount

$$\$1 \times \frac{P(t,S)}{P(t,T)}$$

at time $T > 0$.

As a consequence, the investor will actually receive $P(t,S)/P(t,T)$ at time T, to refund \$1 at time S.

The corresponding forward rate $f(t,T,S)$ is then given by the relation

$$\frac{P(t,S)}{P(t,T)} \exp\left((S-T)f(t,T,S)\right) = \$1, \qquad 0 \leqslant t \leqslant T \leqslant S, \tag{2.1.1}$$

where we used exponential compounding, which leads to the following definition (2.1.2).

Definition 2.1 The forward rate $f(t,T,S)$ at time t for a loan on $[T,S]$ is given by

$$\boxed{f(t,T,S) = \frac{\log P(t,T) - \log P(t,S)}{S-T}.} \tag{2.1.2}$$

The *spot* forward rate $f(t,t,S)$ coincides with the *yield* $y(t,S)$, with

$$\boxed{f(t,t,S) = y(t,S) = -\frac{\log P(t,S)}{T-t}, \quad \text{or} \quad P(t,S) = \mathrm{e}^{-(S-t)f(t,t,S)}, \quad 0 \leqslant t \leqslant S.} \tag{2.1.3}$$

Instantaneous forward rates

Proposition 2.2 The instantaneous forward rate $f(t,T) = f(t,T,T)$ is defined by taking the limit of $f(t,T,S)$ as $S \searrow T$, and satisfies

$$f(t,T) := \lim_{S \searrow T} f(t,T,S) = -\frac{1}{P(t,T)} \frac{\partial P}{\partial T}(t,T). \tag{2.1.4}$$

Proof. We have

$$\begin{aligned} f(t,T): &= \lim_{S \searrow T} f(t,T,S) \\ &= -\lim_{S \searrow T} \frac{\log P(t,S) - \log P(t,T)}{S-T} \\ &= -\lim_{\varepsilon \searrow 0} \frac{\log P(t,T+\varepsilon) - \log P(t,T)}{\varepsilon} \end{aligned}$$

$$
\begin{aligned}
&= -\frac{\partial}{\partial T}\log P(t,T) \\
&= -\frac{1}{P(t,T)}\frac{\partial P}{\partial T}(t,T).
\end{aligned}
$$

□

The above equation (2.1.4) can be viewed as a differential equation to be solved for $\log P(t,T)$ under the initial condition $P(T,T)=1$, which yields the following proposition.

Proposition 2.3 The bond price $P(t,T)$ can be recovered from the instantaneous forward rate $f(t,s)$ as

$$
P(t,T) = \exp\left(-\int_t^T f(t,s)ds\right), \qquad 0 \leqslant t \leqslant T. \tag{2.1.5}
$$

Proof. We check that

$$
\begin{aligned}
\log P(t,T) &= \log P(t,T) - \log P(t,t) \\
&= \int_t^T \frac{\partial}{\partial s}\log P(t,s)ds \\
&= -\int_t^T f(t,s)ds.
\end{aligned}
$$

□

Proposition 2.3 also shows that

$$
\begin{aligned}
f(t,t,t) &= f(t,t) \\
&= \frac{\partial}{\partial T}\int_t^T f(t,s)ds_{|T=t} \\
&= -\frac{\partial}{\partial T}\log P(t,T)_{|T=t} \\
&= -\frac{1}{P(t,T)_{|T=t}}\frac{\partial P}{\partial T}(t,T)_{|T=t} \\
&= -\frac{1}{P(T,T)}\frac{\partial}{\partial T}\mathbf{E}^*\left[\mathrm{e}^{-\int_t^T r_s ds} \,\Big|\, \mathscr{F}_t\right]_{|T=t} \\
&= \mathbf{E}^*\left[r_T\,\mathrm{e}^{-\int_t^T r_s ds} \,\Big|\, \mathscr{F}_t\right]_{|T=t} \\
&= \mathbf{E}^*[r_t \mid \mathscr{F}_t] \\
&= r_t,
\end{aligned} \tag{2.1.6}
$$

i.e. the short rate r_t can be recovered from the instantaneous forward rate as

$$
\boxed{r_t = f(t,t) = \lim_{T\searrow t} f(t,T).}
$$

As a consequence of (2.1.1) and (2.1.5) the forward rate $f(t,T,S)$ can be recovered from (2.1.2) and the instantaneous forward rate $f(t,s)$, as:

$$
\begin{aligned}
f(t,T,S) &= \frac{\log P(t,T) - \log P(t,S)}{S-T} \\
&= -\frac{1}{S-T}\left(\int_t^T f(t,s)ds - \int_t^S f(t,s)ds\right)
\end{aligned}
$$

$$= \frac{1}{S-T}\int_T^S f(t,s)ds, \qquad 0 \leqslant t \leqslant T < S. \tag{2.1.7}$$

Similarly, as a consequence of (2.1.3) and (2.1.5) we have the next proposition.

Proposition 2.4 The *spot* forward rate or *yield* $f(t,t,T)$ can be written in terms of bond prices as

$$\boxed{f(t,t,T) = -\frac{\log P(t,T)}{T-t} = \frac{1}{T-t}\int_t^T f(t,s)ds, \qquad 0 \leqslant t < T.} \tag{2.1.8}$$

Differentiation with respect to T of the above relation shows that the yield $f(t,t,T)$ and the instantaneous forward rate $f(t,s)$ are linked by the relation

$$\frac{\partial f}{\partial T}(t,t,T) = -\frac{1}{(T-t)^2}\int_t^T f(t,s)ds + \frac{1}{T-t}f(t,T), \quad 0 \leqslant t < T,$$

from which it follows that

$$\boxed{\begin{aligned} f(t,T) &= \frac{1}{T-t}\int_t^T f(t,s)ds + (T-t)\frac{\partial f}{\partial T}(t,t,T) \\ &= f(t,t,T) + (T-t)\frac{\partial f}{\partial T}(t,t,T), \qquad 0 \leqslant t < T. \end{aligned}}$$

Forward Vašíček, 1977 rates

In this section we consider the Vasicek model, in which the short rate process is the solution (1.1.2) of (1.1.1) as illustrated in Figure 1.1.

In the Vasicek model, the forward rate is given by

$$\begin{aligned} f(t,T,S) &= -\frac{\log P(t,S) - \log P(t,T)}{S-T} \\ &= -\frac{r_t(C(S-t) - C(T-t)) + A(S-t) - A(T-t))}{S-T} \\ &= -\frac{\sigma^2 - 2ab}{2b^2} \\ &\quad -\frac{1}{S-T}\left(\left(\frac{r_t}{b} + \frac{\sigma^2 - ab}{b^3}\right)\left(\mathrm{e}^{-(S-t)b} - \mathrm{e}^{-(T-t)b}\right) - \frac{\sigma^2}{4b^3}\left(\mathrm{e}^{-2(S-t)b} - \mathrm{e}^{-2(T-t)b}\right)\right), \end{aligned}$$

and the spot forward rate, or yield, satisfies

$$\begin{aligned} f(t,t,T) &= -\frac{\log P(t,T)}{T-t} = -\frac{r_t C(T-t) + A(T-t)}{T-t} \\ &= -\frac{\sigma^2 - 2ab}{2b^2} + \frac{1}{T-t}\left(\left(\frac{r_t}{b} + \frac{\sigma^2 - ab}{b^3}\right)\left(1 - \mathrm{e}^{-(T-t)b}\right) - \frac{\sigma^2}{4b^3}\left(1 - \mathrm{e}^{-2(T-t)b}\right)\right), \end{aligned}$$

with the mean

$$\begin{aligned} &\mathbb{E}[f(t,t,T)] \\ &= -\frac{\sigma^2 - 2ab}{2b^2} + \frac{1}{T-t}\left(\left(\frac{\mathbb{E}[r_t]}{b} + \frac{\sigma^2 - ab}{b^3}\right)\left(1 - \mathrm{e}^{-(T-t)b}\right) - \frac{\sigma^2}{4b^3}\left(1 - \mathrm{e}^{-2(T-t)b}\right)\right) \end{aligned}$$

$$= -\frac{\sigma^2 - 2ab}{2b^2} + \frac{1}{T-t}\left(\frac{r_0}{b}\,\mathrm{e}^{-bt} + \frac{a}{b^2}\left(1-\mathrm{e}^{-bt}\right) + \frac{\sigma^2 - ab}{b^3}\right)\left(1-\mathrm{e}^{-(T-t)b}\right)$$
$$-\frac{\sigma^2}{4b^3(T-t)}\left(1-\mathrm{e}^{-2(T-t)b}\right).$$

In this model, the forward rate $t \mapsto f(t,t,T)$ can be represented as in the following Figure 2.3, with $a = 0.06$, $b = 0.1$, $\sigma = 0.1$, $r_0 = \%1$ and $T = 50$.

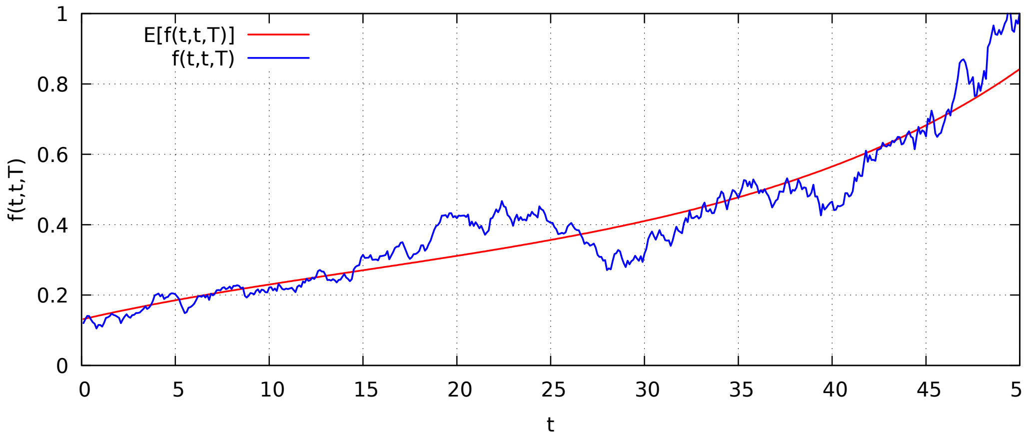


Figure 2.3: Forward rate process $t \mapsto f(t,t,T)$.

We note that the Vasicek forward rate curve $t \mapsto f(t,t,T)$ appears flat for small values of t, *i.e.* longer rates are more stable, while shorter rates show higher volatility or risk. Similar features can be observed in Figure 2.4 for the instantaneous short rate given by

$$\begin{aligned} f(t,T) : &= -\frac{\partial}{\partial T}\log P(t,T) \\ &= r_t\,\mathrm{e}^{-(T-t)b} + \frac{a}{b}\left(1-\mathrm{e}^{-(T-t)b}\right) - \frac{\sigma^2}{2b^2}\left(1-\mathrm{e}^{-(T-t)b}\right)^2, \end{aligned} \tag{2.1.9}$$

from which the relation $\lim_{T \searrow t} f(t,T) = r_t$ can be easily recovered. We can also evaluate the mean

$$\begin{aligned} \mathbb{E}[f(t,T)] &= \mathbb{E}[r_t]\,\mathrm{e}^{-(T-t)b} + \frac{a}{b}\left(1-\mathrm{e}^{-(T-t)b}\right) - \frac{\sigma^2}{2b^2}\left(1-\mathrm{e}^{-(T-t)b}\right)^2 \\ &= r_0\,\mathrm{e}^{-bT} + \frac{a}{b}\left(1-\mathrm{e}^{-bT}\right) - \frac{\sigma^2}{2b^2}\left(1-\mathrm{e}^{-(T-t)b}\right)^2. \end{aligned}$$

The instantaneous forward rate $t \mapsto f(t,T)$ can be represented as in the following Figure 2.4, with $a = 0.06$, $b = 0.1$, $\sigma = 0.1$ and $r_0 = \%1$.

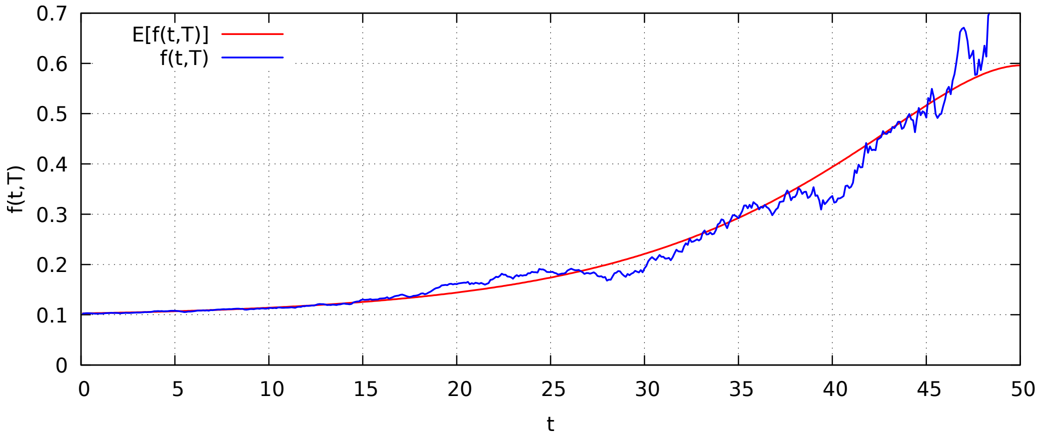


Figure 2.4: Instantaneous forward rate process $t \mapsto f(t,T)$.

Yield curve data

We refer to Chapter III-12 of Charpentier, 2014 on the R package "YieldCurve" Guirreri, 2015 for the following R code and further details on yield curve and interest rate modeling using R.

```
install.packages("YieldCurve");require(YieldCurve);data(FedYieldCurve)
first(FedYieldCurve,'3 month');last(FedYieldCurve,'3 month')
mat.Fed=c(0.25,0.5,1,2,3,5,7,10);n=50
plot(mat.Fed, FedYieldCurve[n,], type="o",xlab="Maturities structure in years", ylab="Interest rates
      values", col = "blue", lwd=3)
title(main=paste("Federal Reserve yield curve observed at",time(FedYieldCurve[n], sep=" ")))
grid()
```

The next Figure 2.5 is plotted using this code* which is adapted from
https://www.quantmod.com/examples/chartSeries3d/chartSeries3d.alpha.R

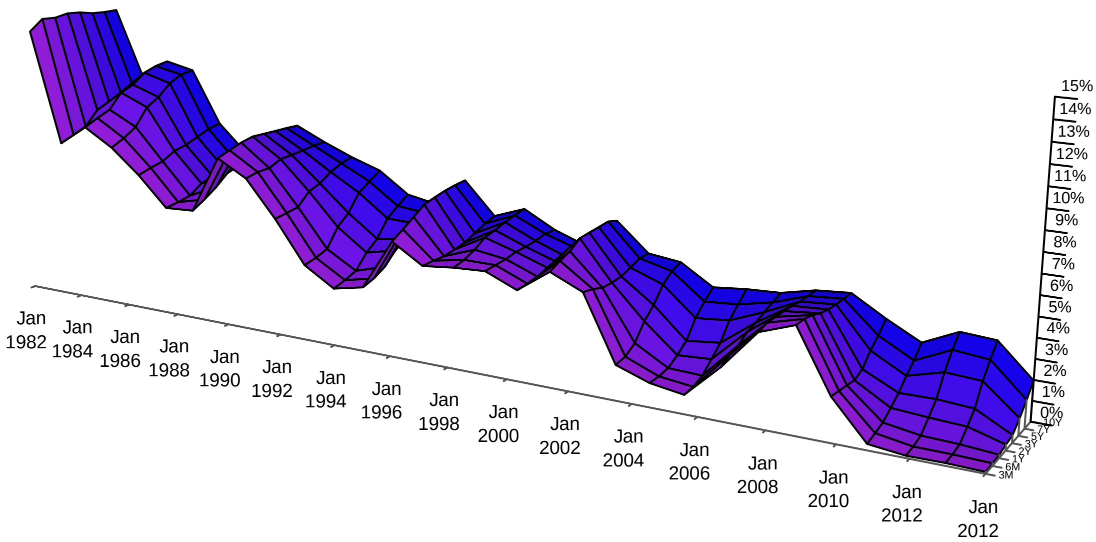


Figure 2.5: Federal Reserve yield curves from 1982 to 2012.

European Central Bank (ECB) data can be similarly obtained by the next R code.

*Click to open or download.

```r
data(ECBYieldCurve);first(ECBYieldCurve,'3 month');last(ECBYieldCurve,'3 month')
mat.ECB<-c(3/12,0.5,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23, 24,25,26,27,28, 29,30)
dev.new(width=16,height=7)
for (n in 200:400) {
plot(mat.ECB, ECBYieldCurve[n,], type="o",xlab="Maturity structure in years", ylab="Interest rates
    values",ylim=c(3.1,5.1),col="blue",lwd=2,cex.axis=1.5,cex.lab=1.5)
title(main=paste("European Central Bank yield curve observed at",time(ECBYieldCurve[n], sep=" ")))
grid();Sys.sleep(0.5)}
```

The next Figure 2.6 represents the output of the above script.

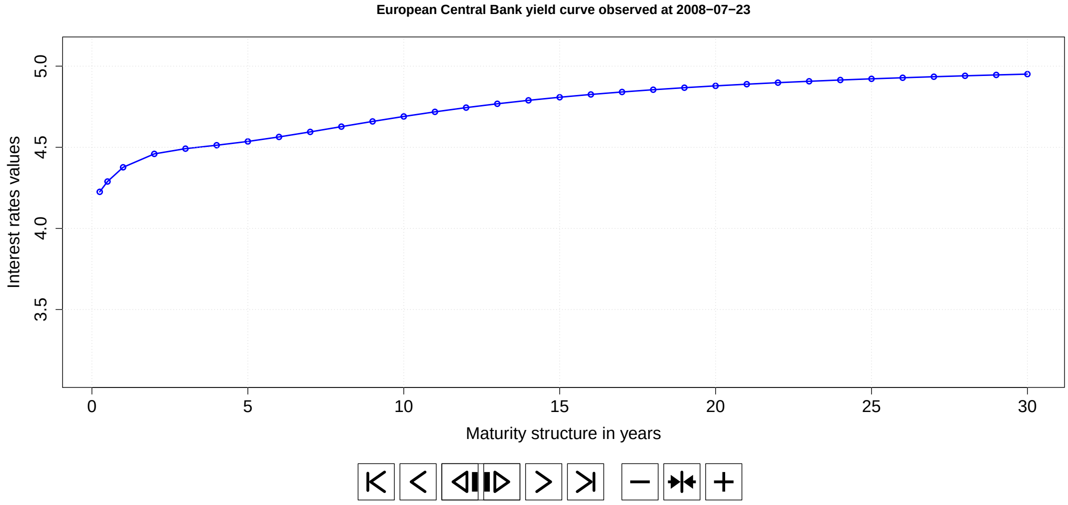


Figure 2.6: European Central Bank yield curves.*

Yield curve inversion

Increasing yield curves are typical of economic expansion phases. Decreasing yield curves can occur when central banks attempt to limit inflation by tightening interest rates, such as in the case of an economic recession. In this case, uncertainty triggers increased investment in long bonds whose rates tend to drop as a consequence, while reluctance to lend in the short term can lead to higher short rates.

*The animation works in Acrobat Reader on the entire pdf file.

Figure 2.7: August 2019 Federal Reserve yield curve inversion.*

The above Figure 2.7 illustrates a Federal Reserve (FED) yield curve inversions occurring in February and August 2019.

LIBOR (London Interbank Offered) Rates

Recall that the forward rate $f(t,T,S)$, $0 \leqslant t \leqslant T \leqslant S$, is defined using exponential compounding, from the relation

$$f(t,T,S) = -\frac{\log P(t,S) - \log P(t,T)}{S-T}. \tag{2.1.10}$$

In order to compute swaption prices one prefers to use forward rates as defined on the London InterBank Offered Rates (LIBOR) market instead of the standard forward rates given by (2.1.10). Other types of LIBOR rates include EURIBOR (European Interbank Offered Rates), HIBOR (Hong Kong Interbank Offered Rates), SHIBOR (Shanghai Interbank Offered Rates), SIBOR (Singapore Interbank Offered Rates), TIBOR (Tokyo Interbank Offered Rates), etc. Most LIBOR rates have been replaced by alternatives such as the Secured Overnight Financing Rate (SOFR) starting with the end of year 2021, see below, page 38.

The forward LIBOR rate $L(t,T,S)$ for a loan on $[T,S]$ is defined using linear compounding, *i.e.* by replacing (2.1.10) with the relation

$$1 + (S-T)L(t,T,S) = \frac{P(t,T)}{P(t,S)}, \qquad t \geqslant T,$$

which yields the following definition.

Definition 2.5 The forward LIBOR rate $L(t,T,S)$ at time t for a loan on $[T,S]$ is given by

$$\boxed{L(t,T,S) = \frac{1}{S-T}\left(\frac{P(t,T)}{P(t,S)} - 1\right), \qquad 0 \leqslant t \leqslant T < S.} \tag{2.1.11}$$

Note that (2.1.11) above yields the same formula for the (LIBOR) instantaneous forward rate

$$L(t,T) : \quad = \quad \lim_{S \searrow T} L(t,T,S)$$

*The animation works in Acrobat Reader on the entire pdf file.

$$
\begin{aligned}
&= \lim_{S\searrow T} \frac{P(t,T)-P(t,S)}{(S-T)P(t,S)} \\
&= \lim_{\varepsilon\searrow 0} \frac{P(t,T)-P(t,T+\varepsilon)}{\varepsilon P(t,T+\varepsilon)} \\
&= \frac{1}{P(t,T)} \lim_{\varepsilon\searrow 0} \frac{P(t,T)-P(t,T+\varepsilon)}{\varepsilon} \\
&= -\frac{1}{P(t,T)} \frac{\partial P}{\partial T}(t,T) \\
&= -\frac{\partial}{\partial T} \log P(t,T) \\
&= f(t,T),
\end{aligned}
$$

as in (2.1.4).

In addition, Relation (2.1.11) shows that the LIBOR rate can be viewed as a forward price $\widehat{X}_t = X_t/N_t$ with numéraire $N_t = (S-T)P(t,S)$ and $X_t = P(t,T) - P(t,S)$, according to Relation (3.2.4) of Chapter 3. As a consequence, from Proposition 3.4, the LIBOR rate $(L(t,T,S))_{t\in[T,S]}$ is a martingale under the forward measure $\widehat{\mathbb{P}}$ defined by

$$\frac{d\widehat{\mathbb{P}}}{d\mathbb{P}^*} = \frac{1}{P(0,S)} \mathrm{e}^{-\int_0^S r_t dt}.$$

SOFR (Secured Overnight Financing) Rates

The repurchase agreement ("repo") market is a market where government treasury securities can be borrowed on the short term. The SOFR rate is a measure of the cost of borrowing which is estimated using overnight activity on the repo market. In that sense, the SOFR, which is transaction-based, differs from LIBOR which is relied on a survey of a panel of banks and subject to manipulation. On the other hand, an important difference is that LIBOR rates are *forward-looking* using a term structure, whereas SOFR rates are *backward-looking*.

The next definition uses the integral convention $\int_a^b = -\int_b^a$, $a < b$.

Definition 2.6 The *backward-looking* bond price is defined for $t \geqslant T$ as

$$P(t,T) = \mathbb{E}\left[\mathrm{e}^{-\int_t^T r_u du} \,\middle|\, \mathcal{F}_t\right] = \mathbb{E}\left[\mathrm{e}^{\int_T^t r_u du} \,\middle|\, \mathcal{F}_t\right] = \mathrm{e}^{\int_T^t r_u du}, \qquad t \geqslant T.$$

The forward SOFR rate $R(t,T,S)$ for a loan on $[T,S]$ is defined using linear compounding with the relation

$$1 + (S-T)R(t,T,S) = \frac{P(t,T)}{P(t,S)}, \qquad 0 \leqslant T \leqslant t,$$

which yields the following definition.

Definition 2.7 The forward SOFR rate $R(t,T,S)$ at time t for a loan on $[T,S]$ is given by

$$\boxed{R(t,T,S) = \frac{1}{S-T}\left(\frac{P(t,T)}{P(t,S)} - 1\right), \qquad 0 \leqslant T \leqslant t \leqslant S.} \tag{2.1.12}$$

In particular, the spot Effective Federal Funds Rate (EFFR) is given for $t = S$ as

$$R(S,T,S) = \frac{1}{S-T}\left(\mathrm{e}^{\int_T^S r_u du} - 1\right).$$

The following proposition, see Rutkowski and Bickersteth, 2021, uses the forward S-measure $\mathbb{P}_S$ defined by its Randon-Nikodym

$$\frac{d\mathbb{P}_S}{d\mathbb{P}^*} := e^{-\int_0^S r_s ds} P(0,S),$$

with the numéraire process $N_t := P(t,S)$, $t \in [0,S]$, see Definition 3.1.

Proposition 2.8 The (simply compounded) SOFR forward rate $(R(t,T,S))_{t\in[T,S]}$ is a martingale under the forward S-measure $\mathbb{P}_S$, *i.e.* we have

$$R(t,T,S) = \mathbb{E}_S[R(S,T,S) \mid \mathscr{F}_t] = \mathbb{E}_S\left[\frac{1}{S-T}\left(e^{\int_T^S r_u du} - 1\right) \Big| \mathscr{F}_t\right],$$

$T \leqslant t \leqslant S$.

Proof. We have

$$\begin{aligned}
R(t,T,S) &= \frac{1}{S-T}\left(\frac{P(t,T)}{P(t,S)} - 1\right) \\
&= \frac{1}{S-T}\left(\frac{e^{\int_T^t r_u du}}{P(t,S)} - 1\right) \\
&= \frac{1}{S-T}\left(\frac{1}{P(t,S)}\mathbb{E}[P(t,T) \mid \mathscr{F}_t] - 1\right) \\
&= \frac{1}{S-T}\left(\frac{1}{P(t,S)}\mathbb{E}\left[e^{\int_T^t r_u du} \Big| \mathscr{F}_t\right] - 1\right) \\
&= \frac{1}{S-T}\left(\frac{1}{P(t,S)}\mathbb{E}\left[e^{-\int_t^S r_u du} e^{\int_T^S r_u du} \Big| \mathscr{F}_t\right] - 1\right) \\
&= \frac{1}{S-T}\left(\mathbb{E}_S\left[e^{\int_T^S r_u du} \Big| \mathscr{F}_t\right] - 1\right) \\
&= \frac{1}{S-T}(\mathbb{E}_S[P(S,T) \mid \mathscr{F}_t] - 1) \\
&= \mathbb{E}_S[R(S,T,S) \mid \mathscr{F}_t], \qquad T \leqslant t \leqslant S.
\end{aligned}$$

□

2.2 LIBOR and SOFR Swap Rates

The first interest rate swap occurred in 1981 between the World Bank, which was interested in borrowing German Marks and Swiss Francs, and IBM, which already had large amounts of those currencies but needed to borrow U.S. dollars.

The vanilla interest rate swap makes it possible to exchange a sequence of variable LIBOR rates $L(t,T_k,T_{k+1})$, $k = 1,2,\ldots,n-1$, against a fixed rate κ over a succession of time intervals $[T_i,T_{i+1}),\ldots,[T_{j-1},T_j]$ defining a *tenor structure*, see Section 4.1 for details.

Making the agreement fair results into an exchange of cashflows

$$\underbrace{(T_{k+1}-T_k)L(t,T_k,T_{k+1})}_{\text{floating leg}} - \underbrace{(T_{k+1}-T_k)\kappa}_{\text{fixed leg}},$$

at the dates $T_{i+1}, \ldots T_j$ between the two parties, therefore generating a cumulative discounted cash flow

$$\sum_{k=i}^{j-1} \mathrm{e}^{-\int_t^{T_{k+1}} r_s ds} (T_{k+1} - T_k)(L(t, T_k, T_{k+1}) - \kappa),$$

at time $t = T_0$, in which we used simple (or linear) interest rate compounding. This corresponds to a *payer swap* in which the swap holder receives the *floating leg* and pays the *fixed leg* κ, whereas the holder of a *seller swap* receives the *fixed leg* κ and pays the *floating leg*.

The above cash flow is used to make the contract fair, and it can be priced *at time t* as

$$\begin{aligned}
&\mathbb{E}^*\left[\sum_{k=i}^{j-1}(T_{k+1}-T_k)\,\mathrm{e}^{-\int_t^{T_{k+1}} r_s ds}(L(t,T_k,T_{k+1})-\kappa)\ \Big|\ \mathcal{F}_t\right] \\
&\quad = \sum_{k=i}^{j-1}(T_{k+1}-T_k)(L(t,T_k,T_{k+1})-\kappa)\,\mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_{k+1}} r_s ds}\ \Big|\ \mathcal{F}_t\right] \\
&\quad = \sum_{k=i}^{j-1}(T_{k+1}-T_k)P(t,T_{k+1})\big(L(t,T_k,T_{k+1})-\kappa\big). \qquad (2.2.1)
\end{aligned}$$

The swap rate $S(t, T_i, T_j)$ is by definition the value of the rate κ that makes the contract fair by making the above cash flow $\mathscr{C}(t)$ vanish.

Definition 2.9 The LIBOR swap rate $S(t, T_i, T_j)$ is the value of the break-even rate κ that makes the contract fair by making the cash flow (2.2.1) vanish, *i.e.*

$$\sum_{k=i}^{j-1}(T_{k+1}-T_k)P(t,T_{k+1})\big(L(t,T_k,T_{k+1})-\kappa\big) = 0. \qquad (2.2.2)$$

The next Proposition 2.10 makes use of the annuity numéraire

$$\begin{aligned}
P(t,T_i,T_j) &:= \mathbb{E}^*\left[\sum_{k=i}^{j-1}(T_{k+1}-T_k)\,\mathrm{e}^{-\int_t^{T_{k+1}} r_s ds}\ \Big|\ \mathcal{F}_t\right] \qquad (2.2.3)\\
&= \sum_{k=i}^{j-1}(T_{k+1}-T_k)\,\mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_{k+1}} r_s ds}\ \Big|\ \mathcal{F}_t\right] \\
&= \sum_{k=i}^{j-1}(T_{k+1}-T_k)P(t,T_{k+1}), \qquad 0 \leqslant t \leqslant T_2,
\end{aligned}$$

which represents the present value at time t of future \$1 receipts at times $T_i, \ldots, T_j$, weighted by the lengths $T_{k+1} - T_k$ of the time intervals $(T_k, T_{k+1}]$, $k = i, \ldots, j-1$.

The time intervals $(T_{k+1} - T_k)_{k=i,\ldots,j-1}$ in the definition (2.2.3) of the annuity numéraire can be replaced by coupon payments $(c_{k+1})_{k=i,\ldots,j-1}$ occurring at times $(T_{k+1})_{k=i,\ldots,j-1}$, in which case the annuity numéraire becomes

$$\begin{aligned}
P(t,T_i,T_j) &:= \mathbb{E}^*\left[\sum_{k=i}^{j-1} c_{k+1}\,\mathrm{e}^{-\int_t^{T_{k+1}} r_s ds}\ \Big|\ \mathcal{F}_t\right] \\
&= \sum_{k=i}^{j-1} c_{k+1}\,\mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_{k+1}} r_s ds}\ \Big|\ \mathcal{F}_t\right] \\
&= \sum_{k=i}^{j-1} c_{k+1}P(t,T_{k+1}), \qquad 0 \leqslant t \leqslant T_i, \qquad (2.2.4)
\end{aligned}$$

which represents the value at time t of the future coupon payments discounted according to the bond prices $(P(t,T_{k+1}))_{k=i,\dots,j-1}$. This expression can also be used to define *amortizing swaps* in which the value of the notional decreases over time, or *accreting swaps* in which the value of the notional increases over time.

LIBOR Swap rates

The LIBOR *swap rate* $S(t,T_i,T_j)$ is defined by solving Relation (2.2.2) for the forward rate $S(t,T_k,T_{k+1})$, *i.e.*

$$\sum_{k=i}^{j-1}(T_{k+1}-T_k)P(t,T_{k+1})\big(L(t,T_k,T_{k+1})-S(t,T_i,T_j)\big)=0. \tag{2.2.5}$$

Proposition 2.10 The LIBOR swap rate $S(t,T_i,T_j)$ is given by

$$S(t,T_i,T_j)=\frac{1}{P(t,T_i,T_j)}\sum_{k=i}^{j-1}(T_{k+1}-T_k)P(t,T_{k+1})L(t,T_k,T_{k+1}), \tag{2.2.6}$$

$0\leqslant t\leqslant T_i$.

Proof. By definition, $S(t,T_i,T_j)$ is the (fixed) break-even rate over $[T_i,T_j]$ that will be agreed in exchange for the family of forward rates $L(t,T_k,T_{k+1})$, $k=i,\dots,j-1$, and it solves (2.2.5), *i.e.* we have

$$\begin{aligned}
&\sum_{k=i}^{j-1}(T_{k+1}-T_k)P(t,T_{k+1})L(t,T_k,T_{k+1})-P(t,T_i,T_j)S(t,T_i,T_j)\\
&=\sum_{k=i}^{j-1}(T_{k+1}-T_k)P(t,T_{k+1})L(t,T_k,T_{k+1})\\
&\quad -S(t,T_i,T_j)\sum_{k=i}^{j-1}(T_{k+1}-T_k)P(t,T_{k+1})\\
&=\sum_{k=i}^{j-1}(T_{k+1}-T_k)P(t,T_{k+1})L(t,T_k,T_{k+1})-S(t,T_i,T_j)P(t,T_i,T_j)\\
&=0,
\end{aligned}$$

which shows (2.2.6) by solving the above equation for $S(t,T_i,T_j)$. □

The LIBOR swap rate $S(t,T_i,T_j)$ is defined by the same relation as (2.2.2), with the forward rate $L(t,T_k,T_{k+1})$ replaced with the LIBOR rate $L(t,T_k,T_{k+1})$. In this case, using the Definition 2.1.11 of LIBOR rates we obtain the next corollary.

Corollary 2.11 The LIBOR swap rate $S(t,T_i,T_j)$ is given by

$$S(t,T_i,T_j)=\frac{P(t,T_i)-P(t,T_j)}{P(t,T_i,T_j)},\qquad 0\leqslant t\leqslant T_i. \tag{2.2.7}$$

Proof. By (2.2.6), (2.1.11) and a telescoping summation argument we have

$$\begin{aligned} S(t,T_i,T_j) &= \frac{1}{P(t,T_i,T_j)}\sum_{k=i}^{j-1}(T_{k+1}-T_k)P(t,T_{k+1})L(t,T_k,T_{k+1}) \\ &= \frac{1}{P(t,T_i,T_j)}\sum_{k=i}^{j-1}P(t,T_{k+1})\left(\frac{P(t,T_k)}{P(t,T_{k+1})}-1\right) \\ &= \frac{1}{P(t,T_i,T_j)}\sum_{k=i}^{j-1}(P(t,T_k)-P(t,T_{k+1})) \\ &= \frac{P(t,T_i)-P(t,T_i)}{P(t,T_i,T_j)}. \end{aligned} \tag{2.2.8}$$

□

By (2.2.7), the bond prices $P(t,T_i)$ can be recovered from the values of the forward swap rates $S(t,T_i,T_j)$.

Clearly, a simple expression for the swap rate such as that of Corollary 2.11 cannot be obtained using the standard (*i.e.* non-LIBOR) rates defined in (2.1.10). Similarly, it will not be available for amortizing or accreting swaps because the telescoping summation argument does not apply to the expression (2.2.4) of the annuity numéraire.

When $n=2$, the LIBOR swap rate $S(t,T_1,T_2)$ coincides with the LIBOR rate $L(t,T_1,T_2)$, as from (2.2.4) we have

$$\begin{aligned} S(t,T_1,T_2) &= \frac{P(t,T_1)-P(t,T_2)}{P(t,T_1,T_2)} \\ &= \frac{P(t,T_1)-P(t,T_2)}{(T_2-T_1)P(t,T_2)} \\ &= L(t,T_1,T_2). \end{aligned} \tag{2.2.9}$$

Similarly to the case of LIBOR rates, Relation (2.2.7) shows that the LIBOR swap rate can be viewed as a forward price with (annuity) numéraire $N_t = P(t,T_i,T_j)$ and $X_t = P(t,T_i) - P(t,T_j)$. Consequently the LIBOR swap rate $(S(t,T_i,T_j))_{t\in[T,S]}$ is a martingale under the forward measure $\widehat{\mathbb{P}}$ defined from (3.2.1) by

$$\frac{d\widehat{\mathbb{P}}}{d\mathbb{P}^*} = \frac{P(T_i,T_i,T_j)}{P(0,T_i,T_j)}\mathrm{e}^{-\int_0^{T_i} r_t dt}.$$

SOFR Swap rate

The expressions

$$\boxed{S(t,T_i,T_j) = \frac{1}{P(t,T_i,T_j)}\sum_{k=i}^{j-1}(T_{k+1}-T_k)P(t,T_{k+1})R(t,T_k,T_{k+1})} \tag{2.2.10}$$

and

$$\boxed{S(t,T_i,T_j) = \frac{P(t,T_i)-P(t,T_j)}{P(t,T_i,T_j)}, \qquad T_i \leqslant t \leqslant T_j,} \tag{2.2.11}$$

defining the SOFR swap rate $S(t,T_i,T_j)$ are identical to the ones defining the LIBOR swap rate in (2.2.6) and (2.2.7) by taking $t \geqslant T_i$ in the case of the SOFR swap rate.

2.3 The HJM Model

In this section we turn to the modeling of instantaneous forward rate curves. From the beginning of this chapter we have started with the modeling of the short rate $(r_t)_{t\in\mathbb{R}_+}$, followed by its consequences on the pricing of bonds $P(t,T)$ and on the expressions of the forward rates $f(t,T,S)$ and $L(t,T,S)$.

In this section we choose a different starting point and consider the problem of directly modeling the instantaneous forward rate $f(t,T)$. The graph given in Figure 2.8 presents a possible random evolution of a forward interest rate curve using the Musiela convention, *i.e.* we will write

$$g(x) = f(t,t+x) = f(t,T), \tag{2.3.1}$$

under the substitution $x = T - t$, $x \geqslant 0$, and represent a sample of the instantaneous forward curve $x \mapsto f(t,t+x)$ for each $t \geqslant 0$.

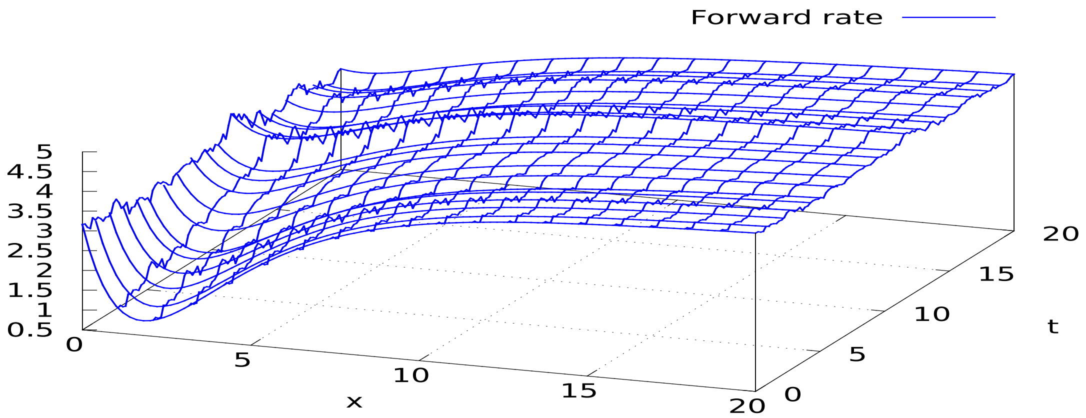


Figure 2.8: Stochastic process of forward curves.

Definition 2.12 In the Heath-Jarrow-Morton (HJM) model, the instantaneous forward rate $f(t,T)$ is modeled under $\mathbb{P}^*$ by a stochastic differential equation of the form

$$d_t f(t,T) = \alpha(t,T)dt + \sigma(t,T)dB_t, \qquad 0 \leqslant t \leqslant T, \tag{2.3.2}$$

where $t \mapsto \alpha(t,T)$ and $t \mapsto \sigma(t,T)$, $0 \leqslant t \leqslant T$, are allowed to be random, $(\mathscr{F}_t)_{t\in[0,T]}$-adapted, processes.

In the above equation, the date T is fixed and the differential d_t is with respect to the time variable t.

Under basic Markovianity assumptions, a HJM model with deterministic coefficients $\alpha(t,T)$ and $\sigma(t,T)$ will yield a short rate process $(r_t)_{t\in\mathbb{R}_+}$ of the form

$$dr_t = (a(t) - b(t)r_t)dt + \sigma(t)dB_t,$$

see § 7.4 in Privault, 2021, which is the Hull and White, 1990 model, with the explicit solution

$$r_t = r_s \mathrm{e}^{-\int_s^t b(\tau)d\tau} + \int_s^t \mathrm{e}^{-\int_u^t b(\tau)d\tau} a(u)du + \int_s^t \sigma(u)\,\mathrm{e}^{-\int_u^t b(\tau)d\tau} dB_u,$$

$0 \leqslant s \leqslant t$.

The HJM condition

How to "encode" absence of arbitrage in the defining HJM Equation (2.3.2) is an important question. Recall that under absence of arbitrage, the bond price $P(t,T)$ has been constructed as

$$P(t,T) = \mathbb{E}^*\left[\exp\left(-\int_t^T r_s ds\right) \Big| \mathscr{F}_t\right] = \exp\left(-\int_t^T f(t,s)ds\right), \tag{2.3.3}$$

cf. Proposition 2.3, hence the discounted bond price process is given by

$$t \mapsto \exp\left(-\int_0^t r_s ds\right) P(t,T) = \exp\left(-\int_0^t r_s ds - \int_t^T f(t,s)ds\right) \tag{2.3.4}$$

is a martingale under $\mathbb{P}^*$ by Proposition 1.1 and Relation (2.1.5) in Proposition 2.3. This shows that $\mathbb{P}^*$ is a risk-neutral probability measure, and by the first fundamental theorem of asset pricing we conclude that the market is without arbitrage opportunities.

Proposition 2.13 (HJM Condition Heath, Jarrow, and Morton, 1992). Under the condition

$$\alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,s)ds, \qquad 0 \leqslant t \leqslant T, \tag{2.3.5}$$

which is known as the *HJM absence of arbitrage condition*, the discounted bond price process (2.3.4) is a martingale, and the probability measure $\mathbb{P}^*$ is risk-neutral.

Proof. Using the process $(X_t)_{t\in[0,T]}$ defined as

$$X_t := \int_t^T f(t,s)ds = -\log P(t,T), \qquad 0 \leqslant t \leqslant T,$$

such that $P(t,T) = \mathrm{e}^{-X_t}$, we rewrite the spot forward rate, or yield

$$f(t,t,T) = \frac{1}{T-t}\int_t^T f(t,s)ds,$$

see (2.1.8), as

$$f(t,t,T) = \frac{1}{T-t}\int_t^T f(t,s)ds = \frac{X_t}{T-t}, \qquad 0 \leqslant t \leqslant T,$$

where the dynamics of $t \mapsto f(t,s)$ is given by (2.3.2). We also use the extended Leibniz integral rule

$$d_t \int_t^T f(t,s)ds = -f(t,t)dt + \int_t^T d_t f(t,s)ds = -r_t dt + \int_t^T d_t f(t,s)ds,$$

see (2.1.6). This identity can be checked in the particular case where $f(t,s) = g(t)h(s)$ is a smooth function that satisfies the separation of variables property, as

$$\begin{aligned} d_t\left(\int_t^T g(t)h(s)ds\right) &= d_t\left(g(t)\int_t^T h(s)ds\right) \\ &= \int_t^T h(s)dsdg(t) + g(t)d_t\int_t^T h(s)ds \\ &= g'(t)\left(\int_t^T h(s)ds\right)dt - g(t)h(t)dt. \end{aligned}$$

We have

$$\begin{aligned}
d_t X_t &= d_t \int_t^T f(t,s)ds \\
&= -f(t,t)dt + \int_t^T d_t f(t,s)ds \\
&= -f(t,t)dt + \int_t^T \alpha(t,s)dsdt + \int_t^T \sigma(t,s)dsdB_t \\
&= -r_t dt + \left(\int_t^T \alpha(t,s)ds\right)dt + \left(\int_t^T \sigma(t,s)ds\right)dB_t,
\end{aligned}$$

hence

$$|d_t X_t|^2 = \left(\int_t^T \sigma(t,s)ds\right)^2 dt.$$

By Itô's calculus, we find

$$\begin{aligned}
d_t P(t,T) &= d_t \, \mathrm{e}^{-X_t} \\
&= -\mathrm{e}^{-X_t} d_t X_t + \frac{1}{2}\mathrm{e}^{-X_t}(d_t X_t)^2 \\
&= -\mathrm{e}^{-X_t} d_t X_t + \frac{1}{2}\mathrm{e}^{-X_t}\left(\int_t^T \sigma(t,s)ds\right)^2 dt \\
&= -\mathrm{e}^{-X_t}\left(-r_t dt + \int_t^T \alpha(t,s)dsdt + \int_t^T \sigma(t,s)dsdB_t\right) \\
&\quad + \frac{1}{2}\mathrm{e}^{-X_t}\left(\int_t^T \sigma(t,s)ds\right)^2 dt,
\end{aligned}$$

and the discounted bond price satisfies

$$\begin{aligned}
& d_t\left(\exp\left(-\int_0^t r_s ds\right)P(t,T)\right) \\
&= -r_t \exp\left(-\int_0^t r_s ds - X_t\right)dt + \exp\left(-\int_0^t r_s ds\right)d_t P(t,T) \\
&= -r_t \exp\left(-\int_0^t r_s ds - X_t\right)dt - \exp\left(-\int_0^t r_s ds - X_t\right)d_t X_t \\
&\quad + \frac{1}{2}\exp\left(-\int_0^t r_s ds - X_t\right)\left(\int_t^T \sigma(t,s)ds\right)^2 dt \\
&= -r_t \exp\left(-\int_0^t r_s ds - X_t\right)dt \\
&\quad - \exp\left(-\int_0^t r_s ds - X_t\right)\left(-r_t dt + \int_t^T \alpha(t,s)dsdt + \int_t^T \sigma(t,s)dsdB_t\right) \\
&\quad + \frac{1}{2}\exp\left(-\int_0^t r_s ds - X_t\right)\left(\int_t^T \sigma(t,s)ds\right)^2 dt \\
&= -\exp\left(-\int_0^t r_s ds - X_t\right)\int_t^T \sigma(t,s)dsdB_t \\
&\quad - \exp\left(-\int_0^t r_s ds - X_t\right)\left(\int_t^T \alpha(t,s)ds - \frac{1}{2}\left(\int_t^T \sigma(t,s)ds\right)^2\right)dt.
\end{aligned}$$

Thus, the discounted bond price process

$$t \mapsto \exp\left(-\int_0^t r_s ds\right)P(t,T)$$

will be a martingale provided that

$$\int_t^T \alpha(t,s)ds - \frac{1}{2}\left(\int_t^T \sigma(t,s)ds\right)^2 = 0, \qquad 0 \leqslant t \leqslant T. \tag{2.3.6}$$

Differentiating the above relation with respect to T yields

$$\alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,s)ds,$$

which is in fact equivalent to (2.3.6). □

Forward Vasicek rates in the HJM model

The HJM coefficients in the Vasicek model are in fact deterministic, for example, taking $a = 0$, by (2.1.9) we have

$$d_t f(t,T) = \sigma^2 \mathrm{e}^{-(T-t)b}\int_t^T \mathrm{e}^{(t-s)b} ds dt + \sigma \mathrm{e}^{-(T-t)b} dB_t,$$

i.e.

$$\alpha(t,T) = \sigma^2 \mathrm{e}^{-(T-t)b}\int_t^T \mathrm{e}^{(t-s)b} ds = \sigma^2 \mathrm{e}^{-(T-t)b}\frac{1-\mathrm{e}^{-(T-t)b}}{b},$$

and $\sigma(t,T) = \sigma \mathrm{e}^{-(T-t)b}$, and the HJM condition reads

$$\alpha(t,T) = \sigma^2 \mathrm{e}^{-(T-t)b}\int_t^T \mathrm{e}^{(t-s)b} ds = \sigma(t,T)\int_t^T \sigma(t,s)ds. \tag{2.3.7}$$

Random simulations of the Vasicek instantaneous forward rates are provided in Figures 2.9 and 2.10 using the Musiela convention (2.3.1).

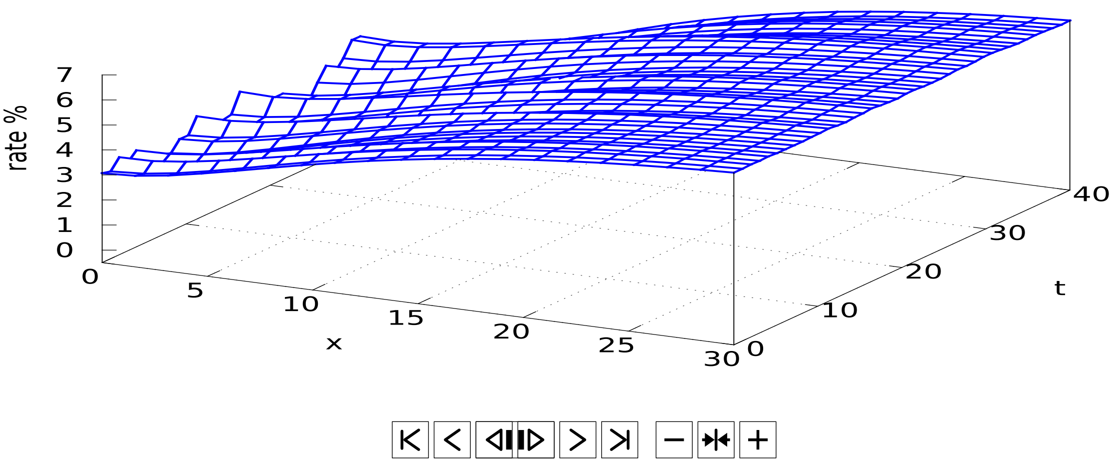


Figure 2.9: Forward instantaneous curve $(t,x) \mapsto f(t,t+x)$ in the Vasicek model.*

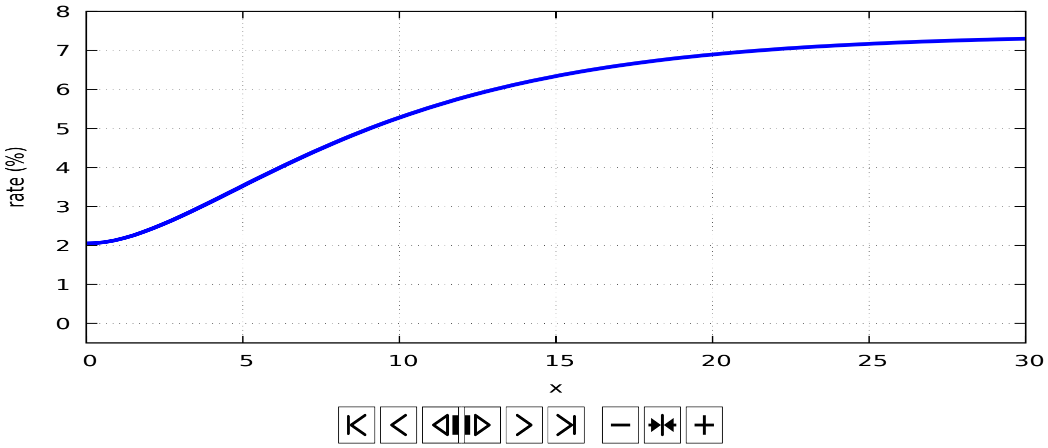


Figure 2.10: Forward instantaneous curve $x \mapsto f(0,x)$ in the Vasicek model.*

For $x = 0$ the first "slice" of this surface is actually the short rate Vasicek process $r_t = f(t,t) = f(t,t+0)$ which is represented in Figure 2.11 using another discretization.

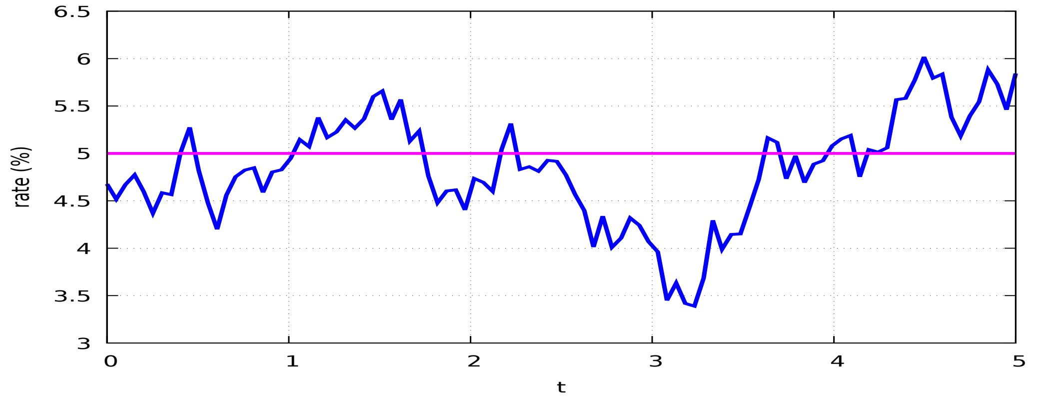


Figure 2.11: Short-term interest rate curve $t \mapsto r_t$ in the Vasicek model.

HJM-SOFR Model

In the HJM-SOFR model, the instantaneous forward rate $f(t,T)$ is extended to $t > T$ by taking

$$d_t f(t,T) = \mathbb{1}_{\{t \leqslant T\}} \alpha(t,T)dt + \mathbb{1}_{\{t \leqslant T\}} \sigma(t,T)dB_t, \qquad t \geqslant T,$$

i.e.

$$f(t,T) = f(T,T) = r_T, \qquad t \geqslant T,$$

see Lyashenko and Mercurio, 2020.

2.4 Yield Curve Modeling

Nelson-Siegel parametrization of instantaneous forward rates

In the Nelson and Siegel, 1987 parametrization the instantaneous forward rate curves are parametrized by 4 coefficients z_1, z_2, z_3, z_4, as

$$g(x) = z_1 + (z_2 + z_3 x)\,\mathrm{e}^{-xz_4}, \qquad x \geqslant 0.$$

*The animation works in Acrobat Reader on the entire pdf file.

An example of a graph obtained by the Nelson-Siegel parametrization is given in Figure 2.12, for $z_1 = 1$, $z_2 = -10$, $z_3 = 100$, $z_4 = 10$.

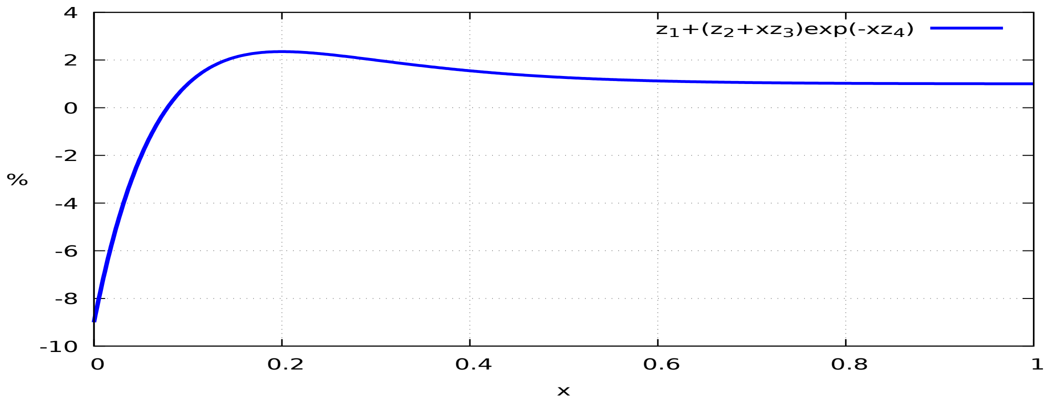


Figure 2.12: Graph of $x \mapsto g(x)$ in the Nelson-Siegel model.

Svensson parametrization of instantaneous forward rates

The Svensson, 1994 parametrization has the advantage to reproduce two humps instead of one, the location and height of which can be chosen *via* 6 parameters $z_1, z_2, z_3, z_4, z_5, z_6$ as

$$g(x) = z_1 + (z_2 + z_3 x)\,\mathrm{e}^{-xz_4} + z_5 x\,\mathrm{e}^{-xz_6}, \qquad x \geqslant 0.$$

A typical graph of a Svensson parametrization is given in Figure 2.13, for $z_1 = 6.6$, $z_2 = -5$, $z_3 = -100$, $z_4 = 10$, $z_5 = -1/2$, $z_6 = 1$.

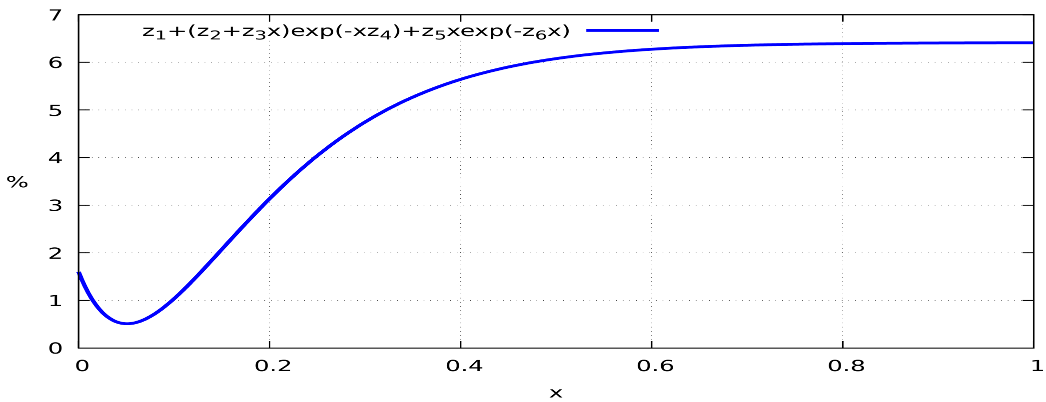


Figure 2.13: Graph of $x \mapsto g(x)$ in the Svensson model.

Figure 2.14 presents a fit of the market data of Figure 2.1 using a Svensson curve.

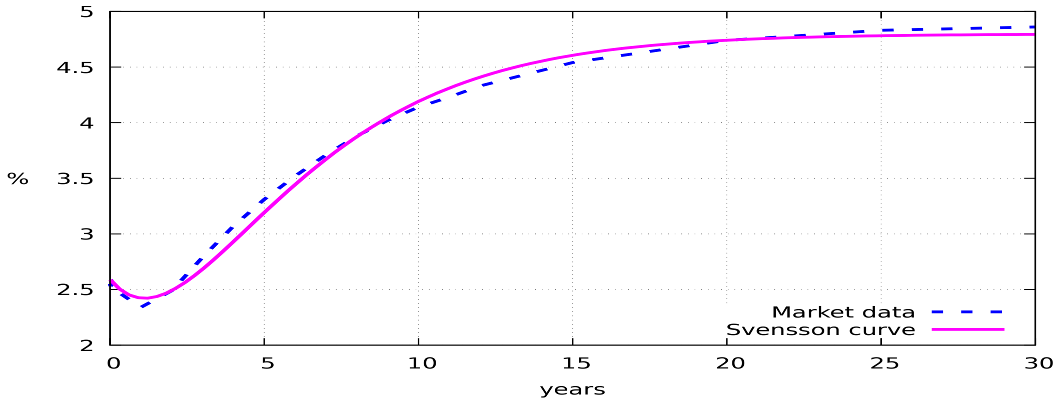


Figure 2.14: Fitting of a Svensson curve to market data.

The attached **IPython notebook** can be run **here** or **here** to fit a Svensson curve to market data.

Vasicek parametrization

In the Vasicek model, the instantaneous forward rate process is given from (2.1.9) and (2.3.1) as

$$f(t,T) = \frac{a}{b} - \frac{\sigma^2}{2b^2} + \left(r_t - \frac{a}{b} + \frac{\sigma^2}{b^2} \right) \mathrm{e}^{-bx} - \frac{\sigma^2}{2b^2} \mathrm{e}^{-2bx}, \tag{2.4.1}$$

in the Musiela notation ($x = T - t$), and we have

$$\frac{\partial f}{\partial T}(t,T) = \left(a - br_t - \frac{\sigma^2}{b}(1 - \mathrm{e}^{-(T-t)b}) \right) \mathrm{e}^{-(T-t)b}.$$

We check that the derivative $\partial f / \partial T$ vanishes when $a - br_t + a - \sigma^2 (1 - \mathrm{e}^{-bx})/b = 0$, *i.e.*

$$\mathrm{e}^{-bx} = 1 + \frac{b}{\sigma^2}(br_t - a),$$

which admits at most one solution, provided that $a > br_t$. As a consequence, the possible forward curves in the Vasicek model are limited to one change of "regime" per curve, as illustrated in Figure 2.15 for various values of r_t, and in Figure 2.16.

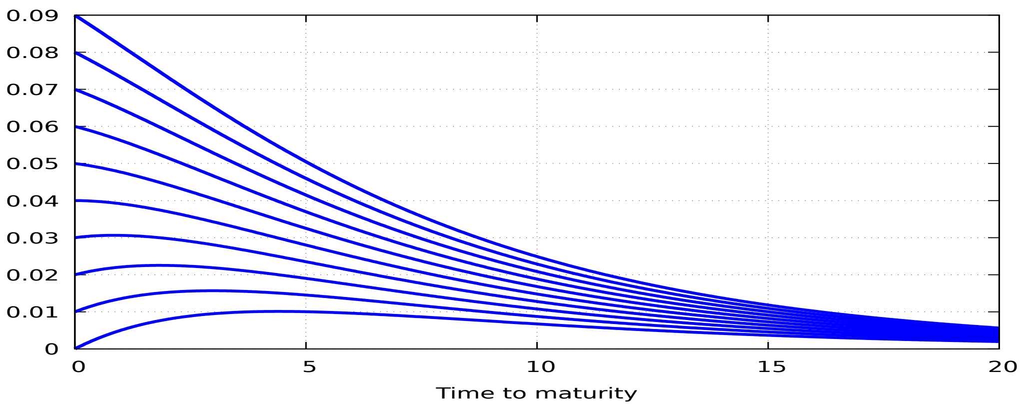


Figure 2.15: Graphs of forward rates with $b = 0.16$, $a/b = 0.04$, $r_0 = 2\%$, $\sigma = 4.5\%$.

The next Figure 2.16 is also using the parameters $b = 0.16$, $a/b = 0.04$, $r_0 = 2\%$, and $\sigma = 4.5\%$.

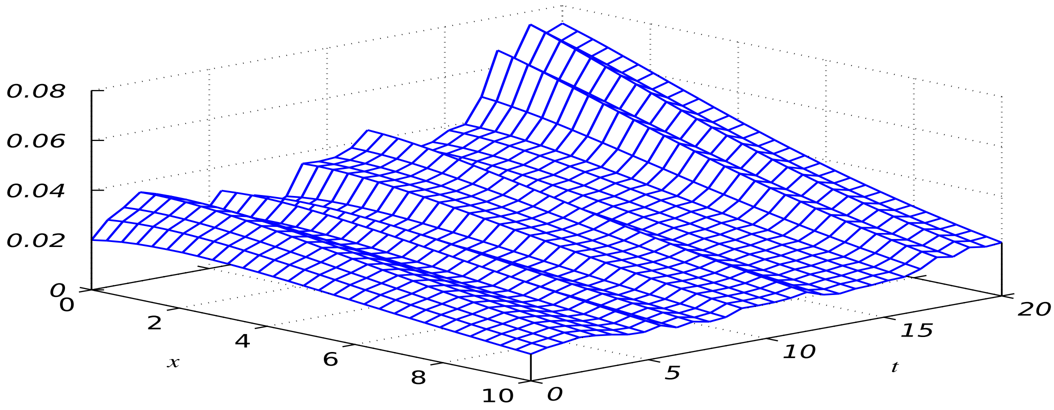


Figure 2.16: Forward instantaneous curve $(t,x) \mapsto f(t,t+x)$ in the Vasicek model.

One may think of constructing an instantaneous forward rate process taking values in the Svensson space, however this type of modeling is not consistent with absence of arbitrage, and it can be proved that the HJM curves cannot live in the Nelson-Siegel or Svensson spaces, see §3.5 of Björk, 2004. In other words, it can be shown that the forward yield curves produced by the Vasicek model are included neither in the Nelson-Siegel space, nor in the Svensson space. In addition, the Vasicek yield curves do not appear to correctly model the market forward curves cf. also Figure 2.1 above.

Another way to deal with the curve fitting problem is to use deterministic shifts for the fitting of one forward curve, such as the initial curve at $t = 0$, cf. *e.g.* § 6.3 in Privault, 2021.

Fitting the Nelson-Siegel and Svensson models to yield curve data

Recall that in the Nelson-Siegel parametrization the instantaneous forward rate curves are parametrized by four coefficients z_1, z_2, z_3, z_4, as

$$f(t,t+x) = z_1 + (z_2 + z_3 x)\,\mathrm{e}^{-xz_4}, \qquad x \geqslant 0. \tag{2.4.2}$$

Taking $x = T - t$, the yield $f(t,t,T)$ is given as

$$\begin{aligned} f(t,t,T) &= \frac{1}{T-t}\int_t^T f(t,s)ds \\ &= \frac{1}{x}\int_0^x f(t,t+y)dy \\ &= z_1 + \frac{z_2}{x}\int_0^x \mathrm{e}^{-yz_4}dy + \frac{z_3}{x}\int_0^x y\,\mathrm{e}^{-yz_4}dy \\ &= z_1 + z_2\frac{1-\mathrm{e}^{-xz_4}}{xz_4} + z_3\frac{1-\mathrm{e}^{-xz_4}+x\,\mathrm{e}^{-xz_4}}{xz_4}. \end{aligned}$$

The yield $f(t,t,T)$ can be reparametrized as

$$f(t,t+x) = z_1 + (z_2 + z_3 x)\,\mathrm{e}^{-xz_4} = \beta_0 + \beta_1\,\mathrm{e}^{-x/\lambda} + \frac{\beta_2}{\lambda}x\,\mathrm{e}^{-x/\lambda}, \qquad x \geqslant 0,$$

cf. Charpentier, 2014, with $\beta_0 = z_1$, $\beta_1 = z_2$, $\beta_2 = z_3/z_4$, $\lambda = 1/z_4$.

```r
require(YieldCurve);data(ECBYieldCurve)
mat.ECB<-c(3/12,0.5,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23, 24,25,26,27,28,29,30)
first(ECBYieldCurve, '1 month');Nelson.Siegel(first(ECBYieldCurve, '1 month'), mat.ECB)
```

```r
for (n in seq(from=70, to=290, by=10)) {
ECB.NS <- Nelson.Siegel(ECBYieldCurve[n,], mat.ECB)
ECB.S <- Svensson(ECBYieldCurve[n,], mat.ECB)
ECB.NS.yield.curve <- NSrates(ECB.NS, mat.ECB)
ECB.S.yield.curve <- Srates(ECB.S, mat.ECB,"Spot")
plot(mat.ECB, as.numeric(ECBYieldCurve[n,]), type="o", lty=1, col=1,ylab="Interest rates",
      xlab="Maturity in years", ylim=c(3.2,4.8),cex.lab=1.6,cex.axis=1.6)
lines(mat.ECB, as.numeric(ECB.NS.yield.curve), type="l", lty=3,col=2,lwd=2)
lines(mat.ECB, as.numeric(ECB.S.yield.curve), type="l", lty=2,col=6,lwd=2)
title(main=paste("ECB yield curve observed at",time(ECBYieldCurve[n], sep=" "),"vs fitted yield curve"))
legend('bottomright', legend=c("ECB data","Nelson-Siegel","Svensson"),col=c(1,2,6), lty=1, bg='gray90')
grid();}
```

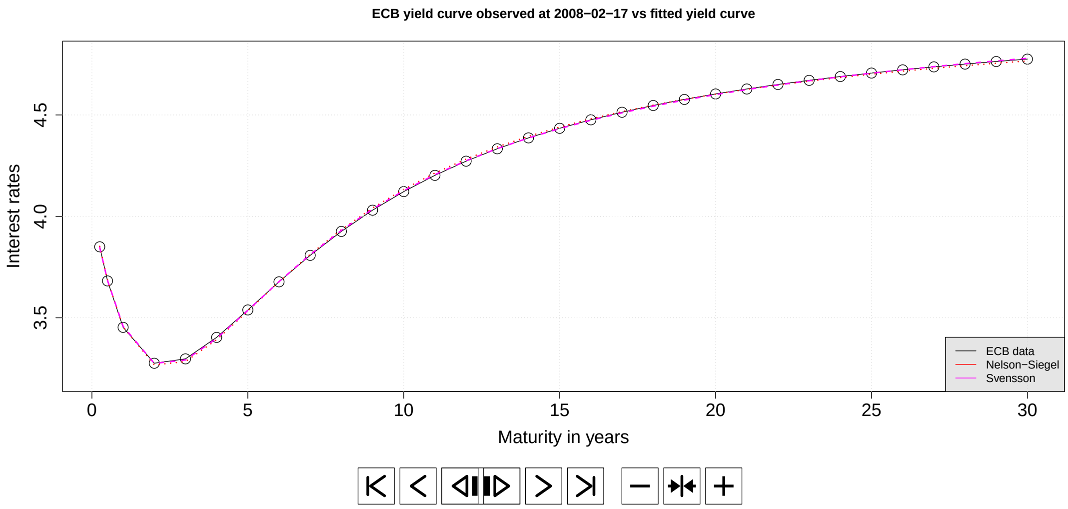


Figure 2.17: ECB data *vs* fitted yield curve.*

2.5 Two-Factor Model

The correlation problem is another issue of concern when using the affine models considered so far, see (1.1.8) and (1.4.10). Let us compare three bond price simulations with maturity $T_1 = 10$, $T_2 = 20$, and $T_3 = 30$ based on the same Brownian path, as given in Figure 2.18. Clearly, the bond prices

$$F(r_t, T_i) = P(t, T_i) = \mathrm{e}^{A(t,T_i) r_t C(t,T_i)}, \qquad 0 \leqslant t \leqslant T_i, \quad i = 1, 2,$$

with maturities T_1 and T_2 are linked by the relation

$$P(t, T_2) = P(t, T_1) \exp\big(A(t, T_2) - A(t, T_1) + r_t (C(t, T_2) - C(t, T_1))\big), \tag{2.5.1}$$

meaning that bond prices with different maturities could be deduced from each other, which is unrealistic.

*The animation works in Acrobat Reader on the entire pdf file.

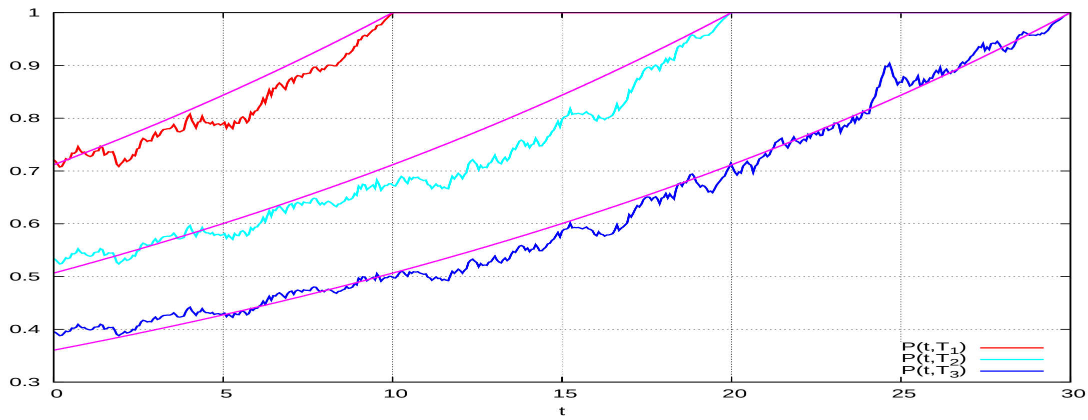


Figure 2.18: Graph of $t \mapsto P(t,T_1), P(t,T_2), P(t,T_3)$.

In affine short rate models, by (2.5.1), $\log P(t,T_1)$ and $\log P(t,T_2)$ are linked by the affine relationship

$$\begin{aligned}
\log P(t,T_2) &= \log P(t,T_1) + A(t,T_2) - A(t,T_1) + r_t(C(t,T_2) - C(t,T_1)) \\
&= \log P(t,T_1) + A(t,T_2) - A(t,T_1) + (C(t,T_2) - C(t,T_1)) \frac{\log P(t,T_1) - A(t,T_1)}{C(t,T_1)} \\
&= \left(1 + \frac{C(t,T_2) - C(t,T_1)}{A(t,T_1)}\right) \log P(t,T_1) + A(t,T_2) - A(t,T_1) \frac{C(t,T_2)}{C(t,T_1)}
\end{aligned}$$

with constant coefficients, which yields the perfect correlation or anticorrelation

$$\operatorname{Cor}(\log P(t,T_1), \log P(t,T_2)) = \pm 1,$$

depending on the sign of the coefficient $1 + (C(t,T_2) - C(t,T_1))/A(t,T_1)$, cf. § 6.4 in Privault, 2021,

A solution to the correlation problem is to consider a two-factor model based on two control processes $(X_t)_{t\in\mathbb{R}_+}$, $(Y_t)_{t\in\mathbb{R}_+}$ which are solution of

$$\begin{cases} dX_t = \mu_1(t,X_t)dt + \sigma_1(t,X_t)dB_t^{(1)}, \\ \\ dY_t = \mu_2(t,Y_t)dt + \sigma_2(t,Y_t)dB_t^{(2)}, \end{cases} \tag{2.5.2}$$

where $\big(B_t^{(1)}\big)_{t\in\mathbb{R}_+}$, $\big(B_t^{(2)}\big)_{t\in\mathbb{R}_+}$ are correlated Brownian motion, with

$$\operatorname{Cov}\big(B_s^{(1)}, B_t^{(2)}\big) = \rho \min(s,t), \qquad s,t \geqslant 0, \tag{2.5.3}$$

and

$$dB_t^{(1)} \bullet dB_t^{(2)} = \rho\, dt, \tag{2.5.4}$$

for some correlation parameter $\rho \in [-1,1]$. In practice, $\big(B^{(1)}\big)_{t\in\mathbb{R}_+}$ and $\big(B^{(2)}\big)_{t\in\mathbb{R}_+}$ can be constructed from two independent Brownian motions $\big(W^{(1)}\big)_{t\in\mathbb{R}_+}$ and $\big(W^{(2)}\big)_{t\in\mathbb{R}_+}$, by letting

$$\begin{cases} B_t^{(1)} = W_t^{(1)}, \\ \\ B_t^{(2)} = \rho W_t^{(1)} + \sqrt{1-\rho^2} W_t^{(2)}, \qquad t \geqslant 0, \end{cases}$$

and Relations (2.5.3) and (2.5.4) are easily satisfied from this construction.

In two-factor models one chooses to build the short-term interest rate r_t *via*

$$r_t := X_t + Y_t, \qquad t \geqslant 0.$$

By the previous standard arbitrage arguments we define the price of a bond with maturity T as

$$\begin{aligned}
P(t,T): &= \mathbb{E}^*\left[\exp\left(-\int_t^T r_s ds\right) \Big| \mathscr{F}_t\right] \\
&= \mathbb{E}^*\left[\exp\left(-\int_t^T r_s ds\right) \Big| X_t,\, Y_t\right] \\
&= \mathbb{E}^*\left[\exp\left(-\int_t^T (X_s+Y_s) ds\right) \Big| X_t,\, Y_t\right] \\
&= F(t,X_t,Y_t),
\end{aligned} \tag{2.5.5}$$

since the couple $(X_t,Y_t)_{t\in\mathbb{R}_+}$ is Markovian. Applying the Itô formula with two variables to

$$t \mapsto F(t,X_t,Y_t) = P(t,T) = \mathbb{E}^*\left[\exp\left(-\int_t^T r_s ds\right) \Big| \mathscr{F}_t\right],$$

and using the fact that the discounted process

$$t \mapsto \mathrm{e}^{-\int_0^t r_s ds} P(t,T) = \mathbb{E}^*\left[\exp\left(-\int_0^T r_s ds\right) \Big| \mathscr{F}_t\right]$$

is an $\mathscr{F}_t$-martingale under $\mathbb{P}^*$, we can derive the PDE

$$\begin{aligned}
&-(x+y)F(t,x,y) + \mu_1(t,x)\frac{\partial F}{\partial x}(t,x,y) + \mu_2(t,y)\frac{\partial F}{\partial y}(t,x,y) \\
&+\frac{1}{2}\sigma_1^2(t,x)\frac{\partial^2 F}{\partial x^2}(t,x,y) + \frac{1}{2}\sigma_2^2(t,y)\frac{\partial^2 F}{\partial y^2}(t,x,y) \\
&+\rho\sigma_1(t,x)\sigma_2(t,y)\frac{\partial^2 F}{\partial x\partial y}(t,x,y) + \frac{\partial F}{\partial t}(t,x,y) = 0,
\end{aligned} \tag{2.5.6}$$

on $\mathbb{R}^2$ for the bond price $P(t,T)$. In the Vasicek model

$$\begin{cases}
dX_t = -aX_t dt + \sigma dB_t^{(1)}, \\
dY_t = -bY_t dt + \eta dB_t^{(2)},
\end{cases}$$

this yields the solution $F(t,x,y)$ of (2.5.6) as

$$P(t,T) = F(t,X_t,Y_t) = F_1(t,X_t)F_2(t,Y_t)\exp\left(\rho U(t,T)\right), \tag{2.5.7}$$

where $F_1(t,X_t)$ and $F_2(t,Y_t)$ are the bond prices associated to X_t and Y_t in the Vasicek model, and

$$U(t,T) := \frac{\sigma\eta}{ab}\left(T-t+\frac{\mathrm{e}^{-(T-t)a}-1}{a}+\frac{\mathrm{e}^{-(T-t)b}-1}{b}-\frac{\mathrm{e}^{-(a+b)(T-t)}-1}{a+b}\right)$$

is a correlation term which vanishes when $\left(B_t^{(1)}\right)_{t\in\mathbb{R}_+}$ and $\left(B_t^{(2)}\right)_{t\in\mathbb{R}_+}$ are independent, *i.e.* when $\rho = 0$, cf. Ch. 4, Appendix A in Brigo and Mercurio, 2006, § 6.5 of Privault, 2021.

Partial differentiation of $\log P(t,T)$ with respect to T leads to the instantaneous forward rate

$$f(t,T) = f_1(t,T) + f_2(t,T) - \rho \frac{\sigma\eta}{ab}\left(1 - \mathrm{e}^{-(T-t)a}\right)\left(1 - \mathrm{e}^{-(T-t)b}\right), \tag{2.5.8}$$

where $f_1(t,T)$, $f_2(t,T)$ are the instantaneous forward rates corresponding to X_t and Y_t respectively, cf. § 6.5 of Privault, 2021.

An example of a forward rate curve obtained in this way is given in Figure 2.19.

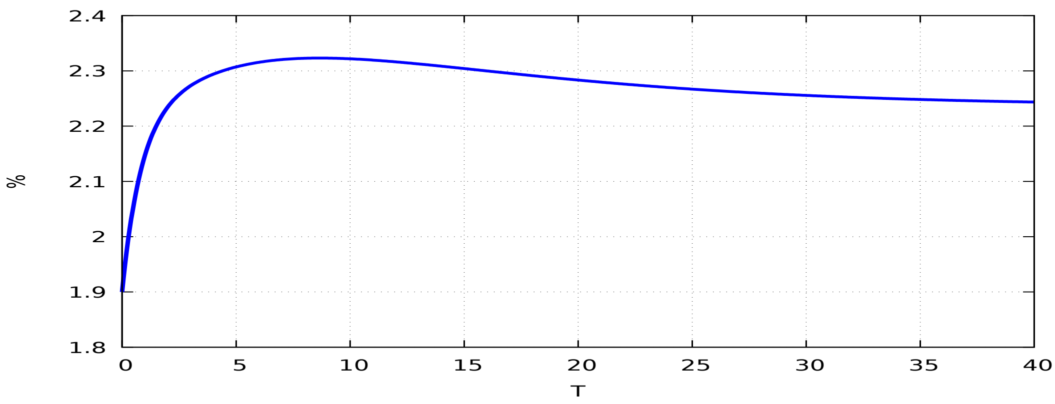


Figure 2.19: Graph of forward rates in a two-factor model.

Next, in Figure 2.20 we present a graph of the evolution of forward curves in a two-factor model.

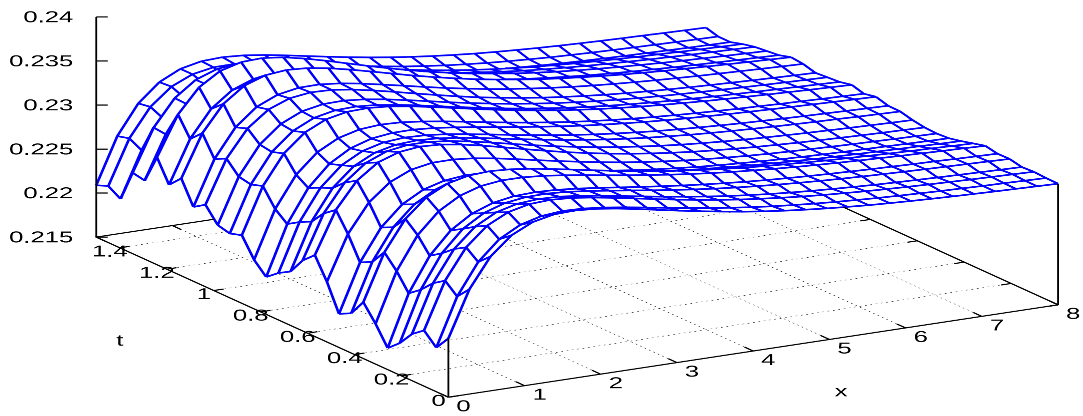


Figure 2.20: Random evolution of instantaneous forward rates in a two-factor model.

2.6 The BGM Model

The models (HJM, affine, etc.) considered in the previous chapter suffer from various drawbacks such as nonpositivity of interest rates in Vasicek model, and lack of closed-form solutions in more complex models. The Brace, Gatarek, and Musiela, 1997 (BGM) model has the advantage of yielding positive interest rates, and to permit to derive explicit formulas for the computation of prices for interest rate derivatives such as interest rate caps and swaptions on the LIBOR market.

In the BGM model we consider two bond prices $P(t,T_1)$, $P(t,T_2)$ with maturities T_1, T_2, and the forward probability measure $\mathbb{P}_2$ defined as

$$\frac{d\mathbb{P}_2}{d\mathbb{P}^*} = \frac{\mathrm{e}^{-\int_0^{T_2} r_s ds}}{P(0,T_2)},$$

with numéraire $P(t,T_2)$, cf. (3.2.10). The forward LIBOR rate $L(t,T_1,T_2)$ is modeled as a driftless geometric Brownian motion under $\mathbb{P}_2$, *i.e.*

$$\frac{dL(t,T_1,T_2)}{L(t,T_1,T_2)} = \gamma_1(t)dB_t, \tag{2.6.1}$$

$0 \leqslant t \leqslant T_1$, for some deterministic volatility function of time $\gamma_1(t)$, with solution

$$L(u,T_1,T_2) = L(t,T_1,T_2)\exp\left(\int_t^u \gamma_1(s)dB_s - \frac{1}{2}\int_t^u |\gamma_1|^2(s)ds\right),$$

i.e. for $u = T_1$,

$$L(T_1,T_1,T_2) = L(t,T_1,T_2)\exp\left(\int_t^{T_1} \gamma_1(s)dB_s - \frac{1}{2}\int_t^{T_1} |\gamma_1|^2(s)ds\right).$$

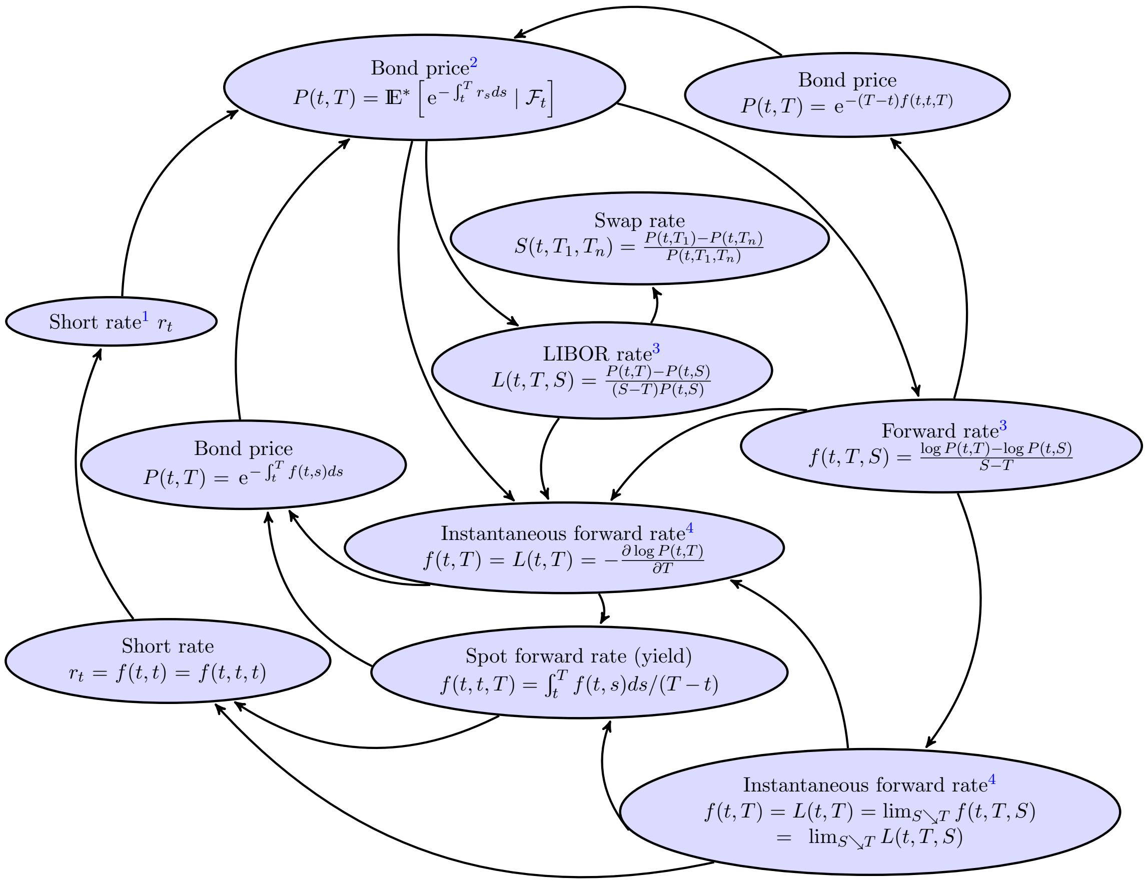
Since $L(t,T_1,T_2)$ is a geometric Brownian motion under $\mathbb{P}_2$, standard caplets can be priced at time $t \in [0,T_1]$ from the Black-Scholes formula.

In the next Table 2.1 we summarize some stochastic models used for interest rates.

	Model
Short rate r_t	Mean reverting SDEs
Instantaneous forward rate $f(t,s)$	HJM model
Forward rate $f(t,T,S)$	BGM model

Table 2.1: Stochastic interest rate models.

The following Graph 2.21 summarizes the notions introduced in this chapter.



[1]Can be modeled by Vasiçek and other short rate models
[2]Can be modeled from $dP(t,T)/P(t,T)$.
[3]Can be modeled in the BGM model
[4]Can be modeled in the HJM model

Figure 2.21: Roadmap of stochastic interest rate modeling.

Exercises

Exercise 2.1 We consider a bond with maturity T, priced $P(t,T) = \mathbb{E}^*\left[e^{-\int_t^T r_s ds} \,\Big|\, \mathscr{F}_t\right]$ at time $t \in [0,T]$.

a) Using the forward measure $\widehat{\mathbb{P}}$ with numéraire $N_t = P(t,T)$, apply the change of numéraire formula (3.2.9) to compute the derivative $\dfrac{\partial P}{\partial T}(t,T)$.
b) Using Relation (2.1.5), find an expression of the instantaneous forward rate $f(t,T)$ using the short rate r_T and the forward expectation $\widehat{\mathbb{E}}$.
c) Show that the instantaneous forward rate $(f(t,T))_{t\in[0,T]}$ is a martingale under the forward measure $\widehat{\mathbb{P}}$.

Exercise 2.2 Consider a tenor structure $\{T_1, T_2\}$ and a bond with maturity T_2 and price given at time $t \in [0, T_2]$ by

$$P(t,T_2) = \exp\left(-\int_t^{T_2} f(t,s)ds\right), \qquad t \in [0,T_2],$$

where the instantaneous yield curve $f(t,s)$ is parametrized as

$$f(t,s) = r_1 \mathbb{1}_{[0,T_1]}(s) + r_2 \mathbb{1}_{[T_1,T_2]}(s), \qquad t \leqslant s \leqslant T_2.$$

Find a formula to estimate the values of r_1 and r_2 from the data of $P(0,T_2)$ and $P(T_1,T_2)$.

Same question when $f(t,s)$ is parametrized as

$$f(t,s) = r_1 s \mathbb{1}_{[0,T_1]}(s) + (r_1 T_1 + (s-T_1) r_2) \mathbb{1}_{[T_1,T_2]}(s), \qquad t \leqslant s \leqslant T_2.$$

Exercise 2.3 Consider a short rate process $(r_t)_{t\in\mathbb{R}_+}$ of the form $r_t = h(t) + X_t$, where $h(t)$ is a deterministic function of time and $(X_t)_{\mathbb{R}_+}$ is a Vasicek process started at $X_0 = 0$.

a) Compute the price $P(0,T)$ at time $t = 0$ of a bond with maturity T, using $h(t)$ and the function $A(T)$ defined in (1.4.17) for the pricing of Vasicek bonds.

b) Show how the function $h(t)$ can be estimated from the market data of the initial instantaneous forward rate curve $f(0,t)$.

Exercise 2.4 Consider two assets whose prices $S_t^{(1)}$, $S_t^{(2)}$ at time $t \in [0,T]$ follow the Bachelier dynamics

$$dS_t^{(1)} = rS_t^{(1)} dt + \sigma_1 dW_t^{(1)} \quad dS_t^{(2)} = rS_t^{(2)} dt + \sigma_2 dW_t^{(2)} \quad t \in [0,T],$$

where $\big(W_t^{(1)}\big)_{t\in[0,T]}$, $\big(W_t^{(2)}\big)_{t\in[0,T]}$ are two standard Brownian motions with correlation $\rho \in [-1,1]$ under a risk-neutral probability measure $\mathbb{P}^*$.

Compute the price $\mathrm{e}^{-rT}\, \mathbb{E}^*[(S_T - K)^+]$ of the spread option on $S_T := S_T^{(2)} - S_T^{(1)}$ with maturity $T > 0$ and strike price $K > 0$.

Exercise 2.5

a) Given two LIBOR spot rates $L(t,t,T)$ and $L(t,t,S)$, compute the corresponding LIBOR forward rate $L(t,T,S)$.

b) Assuming that $L(t,t,T) = 2\%$, $L(t,t,S) = 2.5\%$ and $t = 0$, $T = 1$, $S = 2T = 2$, would you buy a LIBOR forward contract over $[T,2T]$ with rate $L(0,T,2T)$ if $L(T,T,2T)$ remained at the level $L(T,T,2T) = L(0,0,T) = 2\%$?

3. Change of Numéraire and Forward Measures

Change of numéraire is a powerful technique for the pricing of options under random discount factors by the use of forward measures. It has applications to the pricing of interest rate derivatives and other types of options, including exchange options (Margrabe formula) and foreign exchange options (Garman-Kohlagen formula). The computation of self-financing hedging strategies by change of numéraire is treated in Section 3.5, and the change of numéraire technique will be applied to the pricing of interest rate derivatives such as bond options and swaptions in Chapter 4.

3.1 Notion of Numéraire

A *numéraire* is any strictly positive $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$-adapted stochastic process $(N_t)_{t\in\mathbb{R}_+}$ that can be taken as a unit of reference when pricing an asset or a claim.

In general, the price S_t of an asset, when quoted in terms of the numéraire N_t, is given by

$$\widehat{S}_t := \frac{S_t}{N_t}, \qquad t \geqslant 0.$$

Deterministic numéraire transformations are easy to handle, as change of numéraire by a constant factor is a formal algebraic transformation that does not involve any risk. This can be the case for example when a currency is pegged to another currency,[*] for example the exchange rate of the French franc to the Euro was locked at €1 = FRF 6.55957 on 31 December 1998.

On the other hand, a random numéraire may involve new risks, and can allow for arbitrage opportunities.

Examples of numéraire processes $(N_t)_{t\in\mathbb{R}_+}$ include:

- *Money market account.*

Given $(r_t)_{t\in\mathbb{R}_+}$ a possibly random, time-dependent and $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$-adapted risk-free interest rate process, let[†]

$$N_t := \exp\left(\int_0^t r_s ds\right).$$

In this case,

$$\widehat{S}_t = \frac{S_t}{N_t} = \mathrm{e}^{-\int_0^t r_s ds} S_t, \qquad t \geqslant 0,$$

represents the discounted price of the asset at time 0.

- *Currency exchange rates*

In this case, $N_t := R_t$ denotes *e.g.* the EUR/SGD (EURSGD=X) exchange rate from a foreign currency (*e.g.* EUR) to a domestic currency (*e.g.* SGD), *i.e.* one unit of foreign currency (EUR) corresponds to R_t units of local currency (SGD). Let

$$\widehat{S}_t := \frac{S_t}{R_t}, \qquad t \geqslant 0,$$

denote the price of a local (SG) asset quoted in units of the foreign currency (EUR). For example, if $R_t = 1.63$ and $S_t = S\$1$, then

$$\widehat{S}_t = \frac{S_t}{R_t} = \frac{1}{1.63} \times \$1 \simeq €0.61,$$

and $1/R_t$ is the domestic SGD/EUR exchange rate. A question of interest is whether a local asset $\$S_t$, discounted according to a foreign risk-free rate r^f and priced in foreign currency as

$$\mathrm{e}^{-r^f t} \frac{S_t}{R_t} = \mathrm{e}^{-r^f t} \widehat{S}_t,$$

can be a martingale on the foreign market.

[*] Major currencies have started floating against each other since 1973, following the end of the system of fixed exchanged rates agreed upon at the Bretton Woods Conference, July 1-22, 1944.

[†] "Anyone who believes exponential growth can go on forever in a finite world is either a madman or an economist", K.E. Boulding, 1973, page 248.

My foreign currency account S_t grew by 5% this year.

Q: Did I achieve a positive return?

A:

(a) Scenario A.

My foreign currency account S_t grew by 5% this year.

The foreign exchange rate dropped by 10%.

Q: Did I achieve a positive return?

A:

(b) Scenario B.

Figure 3.1: Why change of numéraire?

- *Forward numéraire.*

  The price $P(t,T)$ of a bond paying $P(T,T) = \$1$ at maturity T can be taken as numéraire. In this case, we have

$$N_t := P(t,T) = \mathbb{E}^*\left[e^{-\int_t^T r_s ds} \,\Big|\, \mathscr{F}_t \right], \qquad 0 \leqslant t \leqslant T.$$

  Recall that

$$t \mapsto e^{-\int_0^t r_s ds} P(t,T) = \mathbb{E}^*\left[e^{-\int_0^T r_s ds} \,\Big|\, \mathscr{F}_t \right], \qquad 0 \leqslant t \leqslant T,$$

  is an $\mathscr{F}_t$- martingale.

- *Annuity numéraire.*

  Processes of the form

$$N_t = P(t,T_0,T_n) := \sum_{k=1}^{n} (T_k - T_{k-1}) P(t,T_k), \qquad 0 \leqslant t \leqslant T_0,$$

  where $P(t,T_1), P(t,T_2), \ldots, P(t,T_n)$ are bond prices with maturities $T_1 < T_2 < \cdots < T_n$ arranged according to a *tenor structure*.

- *Combinations of the above*: for example a foreign money market account $e^{\int_0^t r_s^{\mathrm{f}} ds} R_t$, expressed in local (or domestic) units of currency, where $\left(r_t^{\mathrm{f}}\right)_{t\in\mathbb{R}_+}$ represents a short-term interest rate on the foreign market.

When the numéraire is a random process, the pricing of a claim whose value has been transformed under change of numéraire, *e.g.* under a change of currency, has to take into account the risks existing on the foreign market.

In particular, in order to perform a fair pricing, one has to determine a probability measure under which the transformed (or forward, or deflated) process $\widehat{S}_t = S_t / N_t$ will be a martingale, see Section 3.3 for details.

For example in case $N_t := e^{\int_0^t r_s ds}$ is the money market account, the risk-neutral probability measure $\mathbb{P}^*$ is a measure under which the discounted price process

$$\widehat{S}_t = \frac{S_t}{N_t} = e^{-\int_0^t r_s ds} S_t, \qquad t \geqslant 0,$$

is a martingale. In the next section, we will see that this property can be extended to any type of numéraire.

See Exercise 3.5 for other examples of numéraires.

3.2 Change of Numéraire

In this section we review the pricing of options by a change of measure associated to a numéraire N_t, cf. *e.g.* Geman, El Karoui, and Rochet, 1995 and references therein.

Most of the results of this chapter rely on the following assumption, which expresses absence of arbitrage. In the foreign exchange setting where $N_t = R_t$, this condition states that the price of one unit of foreign currency is a martingale when quoted and discounted in the domestic currency.

Assumption 1. *The discounted numéraire*

$$t \mapsto M_t := \mathrm{e}^{-\int_0^t r_s ds} N_t$$

is an $\mathcal{F}_t$-martingale under the risk-neutral probability measure $\mathbb{P}^*$.

Definition 3.1 Given $(N_t)_{t\in[0,T]}$ a numéraire process, the associated *forward measure* $\widehat{\mathbb{P}}$ is defined by its Radon-Nikodym density

$$\frac{d\widehat{\mathbb{P}}}{d\mathbb{P}^*} := \frac{M_T}{M_0} = \mathrm{e}^{-\int_0^T r_s ds} \frac{N_T}{N_0}. \tag{3.2.1}$$

Recall that the above Relation (3.2.1) rewrites as

$$d\widehat{\mathbb{P}} = \frac{M_T}{M_0} d\mathbb{P}^* = \mathrm{e}^{-\int_0^T r_s ds} \frac{N_T}{N_0} d\mathbb{P}^*,$$

which is equivalent to stating that

$$\begin{aligned}
\widehat{\mathbb{E}}[F] &= \int_\Omega F(\omega) d\widehat{\mathbb{P}}(\omega) \\
&= \int_\Omega F(\omega)\, \mathrm{e}^{-\int_0^T r_s ds} \frac{N_T}{N_0} d\mathbb{P}^*(\omega) \\
&= \mathbb{E}^*\left[F\, \mathrm{e}^{-\int_0^T r_s ds} \frac{N_T}{N_0} \right],
\end{aligned}$$

for all integrable $\mathcal{F}_T$-measurable random variables F. More generally, by (3.2.1) and the fact that the process

$$t \mapsto M_t := \mathrm{e}^{-\int_0^t r_s ds} N_t$$

is an $\mathcal{F}_t$-martingale under $\mathbb{P}^*$ under Assumption 1, we find that

$$\begin{aligned}
\mathbb{E}^*\left[\frac{d\widehat{\mathbb{P}}}{d\mathbb{P}^*} \,\middle|\, \mathcal{F}_t \right] &= \mathbb{E}^*\left[\frac{N_T}{N_0} \mathrm{e}^{-\int_0^T r_s ds} \,\middle|\, \mathcal{F}_t \right] \\
&= \mathbb{E}^*\left[\frac{M_T}{M_0} \,\middle|\, \mathcal{F}_t \right] \\
&= \frac{M_t}{M_0} \\
&= \frac{N_t}{N_0} \mathrm{e}^{-\int_0^t r_s ds}, \qquad 0 \leqslant t \leqslant T.
\end{aligned} \tag{3.2.2}$$

In Proposition 3.5 we will show, as a consequence of next Lemma 3.2 below, that for any integrable random claim payoff C we have

$$\mathbf{E}^*\left[C\,\mathrm{e}^{-\int_t^T r_s ds} N_T \mid \mathscr{F}_t\right] = N_t \widehat{\mathbf{E}}[C \mid \mathscr{F}_t], \qquad 0 \leqslant t \leqslant T.$$

Similarly to the above, the Radon-Nikodym density $d\widehat{\mathbb{P}}_{|\mathscr{F}_t} / d\mathbb{P}^*_{|\mathscr{F}_t}$ of $\widehat{\mathbb{P}}_{|\mathscr{F}_t}$ with respect to $\mathbb{P}^*_{|\mathscr{F}_t}$ satisfies the relation

$$\begin{aligned}
\widehat{\mathbf{E}}[F \mid \mathscr{F}_t] &= \int_\Omega F(\omega) d\widehat{\mathbb{P}}_{|\mathscr{F}_t}(\omega) \\
&= \int_\Omega F(\omega)\,\mathrm{e}^{-\int_0^T r_s ds} \frac{d\widehat{\mathbb{P}}_{|\mathscr{F}_t}}{d\mathbb{P}^*_{|\mathscr{F}_t}} d\mathbb{P}^*_{|\mathscr{F}_t}(\omega) \\
&= \mathbf{E}^*\left[F \frac{d\widehat{\mathbb{P}}_{|\mathscr{F}_t}}{d\mathbb{P}^*_{|\mathscr{F}_t}} \,\middle|\, \mathscr{F}_t\right],
\end{aligned}$$

for all integrable $\mathscr{F}_T$-measurable random variables F, $0 \leqslant t \leqslant T$. Note that (3.2.2), which is $\mathscr{F}_t$-measurable, should not be confused with (3.2.3), which is $\mathscr{F}_T$-measurable.

Lemma 3.2 We have

$$\frac{d\widehat{\mathbb{P}}_{|\mathscr{F}_t}}{d\mathbb{P}^*_{|\mathscr{F}_t}} = \frac{M_T}{M_t} = \mathrm{e}^{-\int_t^T r_s ds} \frac{N_T}{N_t}, \qquad 0 \leqslant t \leqslant T. \tag{3.2.3}$$

Proof. The proof of (3.2.3) relies on an abstract version of the Bayes formula. For all bounded $\mathscr{F}_t$-measurable random variable G, by (3.2.2), the tower property of conditional expectations we have

$$\begin{aligned}
\widehat{\mathbf{E}}[G\widehat{X}] &= \mathbf{E}^*\left[G\widehat{X}\,\mathrm{e}^{-\int_0^T r_s ds} \frac{N_T}{N_0}\right] \\
&= \mathbf{E}^*\left[G \frac{N_t}{N_0}\,\mathrm{e}^{-\int_0^t r_s ds}\,\mathbf{E}^*\left[\widehat{X}\,\mathrm{e}^{-\int_t^T r_s ds} \frac{N_T}{N_t} \,\middle|\, \mathscr{F}_t\right]\right] \\
&= \mathbf{E}^*\left[G\,\mathbf{E}^*\left[\frac{d\widehat{\mathbb{P}}}{d\mathbb{P}^*} \,\middle|\, \mathscr{F}_t\right] \mathbf{E}^*\left[\widehat{X}\,\mathrm{e}^{-\int_t^T r_s ds} \frac{N_T}{N_t} \,\middle|\, \mathscr{F}_t\right]\right] \\
&= \mathbf{E}^*\left[G \frac{d\widehat{\mathbb{P}}}{d\mathbb{P}^*}\,\mathbf{E}^*\left[\widehat{X}\,\mathrm{e}^{-\int_t^T r_s ds} \frac{N_T}{N_t} \,\middle|\, \mathscr{F}_t\right]\right] \\
&= \widehat{\mathbf{E}}\left[G\,\mathbf{E}^*\left[\widehat{X}\,\mathrm{e}^{-\int_t^T r_s ds} \frac{N_T}{N_t} \,\middle|\, \mathscr{F}_t\right]\right],
\end{aligned}$$

for all bounded random variables $\widehat{X}$, which shows that

$$\widehat{\mathbf{E}}[\widehat{X} \mid \mathscr{F}_t] = \mathbf{E}^*\left[\widehat{X}\,\mathrm{e}^{-\int_t^T r_s ds} \frac{N_T}{N_t} \,\middle|\, \mathscr{F}_t\right],$$

i.e. (3.2.3) holds. □

We note that in case the numéraire $N_t = \mathrm{e}^{\int_0^t r_s ds}$ is equal to the money market account we simply have $\widehat{\mathbb{P}} = \mathbb{P}^*$.

Definition 3.3 Given $(X_t)_{t\in\mathbb{R}_+}$ an asset price process, we define the process of forward (or deflated) prices

$$\widehat{X}_t := \frac{X_t}{N_t}, \qquad 0 \leqslant t \leqslant T. \tag{3.2.4}$$

The process $(\widehat{X}_t)_{t\in\mathbb{R}_+}$ epresents the values at times t of X_t, expressed in units of the numéraire N_t. In the sequel, it will be useful to determine the dynamics of $(\widehat{X}_t)_{t\in\mathbb{R}_+}$ under the forward measure $\widehat{\mathbb{P}}$. The next proposition shows in particular that the process $\left(e^{\int_0^t r_s ds}/N_t\right)_{t\in\mathbb{R}_+}$ is an $\mathscr{F}_t$-martingale under $\widehat{\mathbb{P}}$.

Proposition 3.4 Let $(X_t)_{t\in\mathbb{R}_+}$ denote a continuous $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$-adapted asset price process such that

$$t \mapsto e^{-\int_0^t r_s ds} X_t, \qquad t \geqslant 0,$$

is a martingale under $\mathbb{P}^*$. Then, under change of numéraire,

the deflated process $(\widehat{X}_t)_{t\in[0,T]} = (X_t/N_t)_{t\in[0,T]}$ of forward prices is an $\mathscr{F}_t$-martingale under $\widehat{\mathbb{P}}$,

provided that $(\widehat{X}_t)_{t\in[0,T]}$ is integrable under $\widehat{\mathbb{P}}$.

Proof. We show that

$$\widehat{\mathbb{E}}\left[\frac{X_t}{N_t}\,\Big|\,\mathscr{F}_s\right] = \frac{X_s}{N_s}, \qquad 0 \leqslant s \leqslant t, \tag{3.2.5}$$

using the standard characterization of conditional expectation. Namely, for all bounded $\mathscr{F}_s$-measurable random variables G we note that using (3.2.2) under Assumption 1 we have

$$\begin{aligned}
\widehat{\mathbb{E}}\left[G\frac{X_t}{N_t}\right] &= \mathbb{E}^*\left[G\frac{X_t}{N_t}\frac{d\widehat{\mathbb{P}}}{d\mathbb{P}^*}\right] \\
&= \mathbb{E}^*\left[\mathbb{E}^*\left[G\frac{X_t}{N_t}\frac{d\widehat{\mathbb{P}}}{d\mathbb{P}^*}\,\Big|\,\mathscr{F}_t\right]\right] \\
&= \mathbb{E}^*\left[G\frac{X_t}{N_t}\mathbb{E}^*\left[\frac{d\widehat{\mathbb{P}}}{d\mathbb{P}^*}\,\Big|\,\mathscr{F}_t\right]\right] \\
&= \mathbb{E}^*\left[G e^{-\int_0^t r_u du}\frac{X_t}{N_0}\right] && (3.2.6) \\
&= \mathbb{E}^*\left[G e^{-\int_0^s r_u du}\frac{X_s}{N_0}\right] && (3.2.7) \\
&= \mathbb{E}^*\left[G\frac{X_s}{N_s}\mathbb{E}^*\left[\frac{d\widehat{\mathbb{P}}}{d\mathbb{P}^*}\,\Big|\,\mathscr{F}_s\right]\right] \\
&= \mathbb{E}^*\left[\mathbb{E}^*\left[G\frac{X_s}{N_s}\frac{d\widehat{\mathbb{P}}}{d\mathbb{P}^*}\,\Big|\,\mathscr{F}_s\right]\right] \\
&= \mathbb{E}^*\left[G\frac{X_s}{N_s}\frac{d\widehat{\mathbb{P}}}{d\mathbb{P}^*}\right]
\end{aligned}$$

$$= \widehat{\mathbb{E}}\left[G\frac{X_s}{N_s}\right], \qquad 0 \leqslant s \leqslant t,$$

where from (3.2.6) to (3.2.7)we used the fact that

$$t \mapsto \mathrm{e}^{-\int_0^t r_s ds} X_t$$

is an $\mathscr{F}_t$-martingale under $\mathbb{P}^*$. Finally, the identity

$$\widehat{\mathbb{E}}[G\widehat{X}_t] = \widehat{\mathbb{E}}\left[G\frac{X_t}{N_t}\right] = \widehat{\mathbb{E}}\left[G\frac{X_s}{N_s}\right] = \widehat{\mathbb{E}}[G\widehat{X}_s], \qquad 0 \leqslant s \leqslant t,$$

for all bounded $\mathscr{F}_s$-measurable G, implies (3.2.5). □

Pricing using change of numéraire

The change of numéraire technique is especially useful for pricing under random interest rates, in which case an expectation of the form

$$\mathbb{E}^*\left[\mathrm{e}^{-\int_t^T r_s ds} C \,\middle|\, \mathscr{F}_t\right]$$

becomes a *path integral*, see *e.g.* Dash, 2004 for an account of path integral methods in quantitative finance. The next proposition is the basic result of this section, it provides a way to price an option with arbitrary payoff C under a random discount factor $\mathrm{e}^{-\int_t^T r_s ds}$ by use of the forward measure. It will be applied in Chapter 4 to the pricing of bond options and caplets, cf. Propositions 4.1, 4.3 and 4.5 below.

Proposition 3.5 An option with integrable claim payoff $C \in L^1(\Omega, \mathbb{P}^*, \mathscr{F}_T)$ is priced at time t as

$$\mathbb{E}^*\left[\mathrm{e}^{-\int_t^T r_s ds} C \,\middle|\, \mathscr{F}_t\right] = N_t \widehat{\mathbb{E}}\left[\frac{C}{N_T} \,\middle|\, \mathscr{F}_t\right], \qquad 0 \leqslant t \leqslant T, \tag{3.2.8}$$

provided that $C/N_T \in L^1(\Omega, \widehat{\mathbb{P}}, \mathscr{F}_T)$.

Proof. By Relation (3.2.3) in Lemma 3.2 we have

$$\begin{aligned}
\mathbb{E}^*\left[\mathrm{e}^{-\int_t^T r_s ds} C \,\middle|\, \mathscr{F}_t\right] &= \mathbb{E}^*\left[\frac{d\widehat{\mathbb{P}}_{|\mathscr{F}_t}}{d\mathbb{P}^*_{|\mathscr{F}_t}} \frac{N_t}{N_T} C \,\middle|\, \mathscr{F}_t\right] \\
&= N_t \mathbb{E}^*\left[\frac{d\widehat{\mathbb{P}}_{|\mathscr{F}_t}}{d\mathbb{P}^*_{|\mathscr{F}_t}} \frac{C}{N_T} \,\middle|\, \mathscr{F}_t\right] \\
&= N_t \int_\Omega \frac{d\widehat{\mathbb{P}}_{|\mathscr{F}_t}}{d\mathbb{P}^*_{|\mathscr{F}_t}} \frac{C}{N_T} d\mathbb{P}^*_{|\mathscr{F}_t} \\
&= N_t \int_\Omega \frac{C}{N_T} d\widehat{\mathbb{P}}_{|\mathscr{F}_t} \\
&= N_t \widehat{\mathbb{E}}\left[\frac{C}{N_T} \,\middle|\, \mathscr{F}_t\right], \qquad 0 \leqslant t \leqslant T.
\end{aligned}$$

Equivalently, we can write

$$\begin{aligned}
N_t \widehat{\mathbb{E}}\left[\frac{C}{N_T} \,\middle|\, \mathscr{F}_t\right] &= N_t \mathbb{E}^*\left[\frac{C}{N_T} \frac{d\widehat{\mathbb{P}}_{|\mathscr{F}_t}}{d\mathbb{P}^*_{|\mathscr{F}_t}} \,\middle|\, \mathscr{F}_t\right] \\
&= \mathbb{E}^*\left[\mathrm{e}^{-\int_t^T r_s ds} C \,\middle|\, \mathscr{F}_t\right], \qquad 0 \leqslant t \leqslant T.
\end{aligned}$$

□

Each application of the change of numéraire formula (3.2.8) will require to:

a) pick a suitable numéraire $(N_t)_{t\in\mathbb{R}_+}$ satisfying Assumption 1,
b) make sure that the ratio C/N_T takes a sufficiently simple form,
c) use the Girsanov theorem in order to determine the dynamics of asset prices under the new probability measure $\widehat{\mathbb{P}}$,

so as to compute the expectation under $\widehat{\mathbb{P}}$ on the right-hand side of (3.2.8).

Next, we consider further examples of numéraires and associated examples of option prices.

Examples:

a) *Money market account.*

Take $N_t := \mathrm{e}^{\int_0^t r_s ds}$, where $(r_t)_{t\in\mathbb{R}_+}$ is a possibly random and time-dependent risk-free interest rate. In this case, Assumption 1 is clearly satisfied, we have $\widehat{\mathbb{P}} = \mathbb{P}^*$ and $d\mathbb{P}^*/d\widehat{\mathbb{P}}$, and (3.2.8) simply reads

$$\mathbb{E}^*\left[\mathrm{e}^{-\int_t^T r_s ds} C \,\Big|\, \mathscr{F}_t\right] = \mathrm{e}^{\int_0^t r_s ds}\,\mathbb{E}^*\left[\mathrm{e}^{-\int_0^T r_s ds} C \,\Big|\, \mathscr{F}_t\right], \qquad 0 \leqslant t \leqslant T,$$

which yields no particular information.

b) *Forward numéraire.*

Here, $N_t := P(t,T)$ is the price $P(t,T)$ of a bond maturing at time T, $0 \leqslant t \leqslant T$, and the discounted bond price process $\left(\mathrm{e}^{-\int_0^t r_s ds} P(t,T)\right)_{t\in[0,T]}$ is an $\mathscr{F}_t$-martingale under $\mathbb{P}^*$, *i.e.* Assumption 1 is satisfied and $N_t = P(t,T)$ can be taken as numéraire. In this case, (3.2.8) shows that a random claim payoff C can be priced as

$$\mathbb{E}^*\left[\mathrm{e}^{-\int_t^T r_s ds} C \,\Big|\, \mathscr{F}_t\right] = P(t,T)\widehat{\mathbb{E}}[C \mid \mathscr{F}_t], \qquad 0 \leqslant t \leqslant T, \tag{3.2.9}$$

since $N_T = P(T,T) = 1$, where the forward measure $\widehat{\mathbb{P}}$ satisfies

$$\frac{d\widehat{\mathbb{P}}}{d\mathbb{P}^*} = \mathrm{e}^{-\int_0^T r_s ds}\frac{P(T,T)}{P(0,T)} = \frac{\mathrm{e}^{-\int_0^T r_s ds}}{P(0,T)} \tag{3.2.10}$$

by (3.2.1).

c) *Annuity numéraires.*

We take

$$N_t := \sum_{k=1}^n (T_k - T_{k-1}) P(t,T_k)$$

where $P(t,T_1),\ldots,P(t,T_n)$ are bond prices with maturities $T_1 < T_2 < \cdots < T_n$. Here, (3.2.8) shows that a swaption on the cash flow $P(T,T_n) - P(T,T_1) - \kappa N_T$ can be priced as

$$\begin{aligned}
&\mathbb{E}^*\left[\mathrm{e}^{-\int_t^T r_s ds}(P(T,T_n) - P(T,T_1) - \kappa N_T)^+ \,\Big|\, \mathscr{F}_t\right] \\
&\quad = N_t \widehat{\mathbb{E}}\left[\left(\frac{P(T,T_n) - P(T,T_1)}{N_T} - \kappa\right)^+ \,\Bigg|\, \mathscr{F}_t\right],
\end{aligned}$$

$0 \leqslant t \leqslant T < T_1$, where $(P(T,T_n) - P(T,T_1))/N_T$ becomes a *swap rate*, cf. (2.2.7) in Proposition 2.11 and Section 4.5.

Girsanov theorem

We refer to *e.g.* Theorem III-35 page 132 of Protter, 2004 for the following version of the Girsanov Theorem.

Theorem 3.6 Assume that $\widehat{\mathbb{P}}$ is equivalent[a] to $\mathbb{P}^*$ with Radon-Nikodym density $d\widehat{\mathbb{P}}/d\mathbb{P}^*$, and let $(W_t)_{t\in[0,T]}$ be a standard Brownian motion under $\mathbb{P}^*$. Then, letting

$$\Phi_t := \mathbb{E}^*\left[\frac{d\widehat{\mathbb{P}}}{d\mathbb{P}^*}\,\Big|\,\mathscr{F}_t\right], \qquad 0 \leqslant t \leqslant T, \tag{3.2.11}$$

the process $(\widehat{W}_t)_{t\in[0,T]}$ defined by

$$d\widehat{W}_t := dW_t - \frac{1}{\Phi_t} d\Phi_t \bullet dW_t, \qquad 0 \leqslant t \leqslant T, \tag{3.2.12}$$

is a standard Brownian motion under $\widehat{\mathbb{P}}$.

[a] This means that the Radon-Nikodym densities $d\widehat{\mathbb{P}}/d\mathbb{P}^*$ and $d\mathbb{P}^*/d\widehat{\mathbb{P}}$ exist and are strictly positive with $\mathbb{P}^*$ and $\widehat{\mathbb{P}}$-probability one, respectively.

In case the martingale $(\Phi_t)_{t\in[0,T]}$ takes the form

$$\Phi_t = \exp\left(-\int_0^t \psi_s dW_s - \frac{1}{2}\int_0^t |\psi_s|^2 ds\right), \qquad 0 \leqslant t \leqslant T,$$

i.e.

$$d\Phi_t = -\psi_t \Phi_t dW_t, \qquad 0 \leqslant t \leqslant T,$$

the Itô multiplication table shows that Relation (3.2.12) reads

$$\begin{aligned} d\widehat{W}_t &= dW_t - \frac{1}{\Phi_t} d\Phi_t \bullet dW_t \\ &= dW_t - \frac{1}{\Phi_t}(-\psi_t \Phi_t dW_t) \bullet dW_t \\ &= dW_t + \psi_t dt, \qquad 0 \leqslant t \leqslant T, \end{aligned}$$

and shows that the shifted process $(\widehat{W}_t)_{t\in[0,T]} = \left(W_t + \int_0^t \psi_s ds\right)_{t\in[0,T]}$ is a standard Brownian motion under $\widehat{\mathbb{P}}$, which is consistent with the Girsanov Theorem. The next result is another application of the Girsanov Theorem.

Proposition 3.7 The process $(\widehat{W}_t)_{t\in[0,T]}$ defined by

$$d\widehat{W}_t := dW_t - \frac{1}{N_t} dN_t \bullet dW_t, \qquad 0 \leqslant t \leqslant T, \tag{3.2.13}$$

is a standard Brownian motion under $\widehat{\mathbb{P}}$.

Proof. Relation (3.2.2) shows that Φ_t defined in (3.2.11) satisfies

$$\begin{aligned} \Phi_t &= \mathbb{E}^*\left[\frac{d\widehat{\mathbb{P}}}{d\mathbb{P}^*}\,\Big|\,\mathscr{F}_t\right] \\ &= \mathbb{E}^*\left[\frac{N_T}{N_0} e^{-\int_0^T r_s ds}\,\Big|\,\mathscr{F}_t\right] \end{aligned}$$

$$= \frac{N_t}{N_0} \mathrm{e}^{-\int_0^t r_s ds}, \qquad 0 \leqslant t \leqslant T,$$

hence

$$\begin{aligned} d\Phi_t &= d\left(\frac{N_t}{N_0}\mathrm{e}^{-\int_0^t r_s ds}\right) \\ &= -\Phi_t r_t dt + \mathrm{e}^{-\int_0^t r_s ds} d\left(\frac{N_t}{N_0}\right) \\ &= -\Phi_t r_t dt + \frac{\Phi_t}{N_t} dN_t, \end{aligned}$$

which, by (3.2.12), yields

$$\begin{aligned} d\widehat{W}_t &= dW_t - \frac{1}{\Phi_t} d\Phi_t \bullet dW_t \\ &= dW_t - \frac{1}{\Phi_t}\left(-\Phi_t r_t dt + \frac{\Phi_t}{N_t} dN_t\right) \bullet dW_t \\ &= dW_t - \frac{1}{N_t} dN_t \bullet dW_t, \end{aligned}$$

which is (3.2.13), from Relation (3.2.12) and the Itô multiplication table. □

The next Proposition 3.8 is consistent with the statement of Proposition 3.4, and in addition it specifies the dynamics of $(\widehat{X}_t)_{t\in\mathbb{R}_+}$ under $\widehat{\mathbb{P}}$ using the Girsanov Theorem 3.7. As a consequence, we have the next proposition, see Exercise 3.1 for another calculation based on geometric Brownian motion.

Proposition 3.8 Assume that $(X_t)_{t\in\mathbb{R}_+}$ and $(N_t)_{t\in\mathbb{R}_+}$ satisfy the stochastic differential equations

$$dX_t = r_t X_t dt + \sigma_t^X X_t dW_t, \quad \text{and} \quad dN_t = r_t N_t dt + \sigma_t^N N_t dW_t, \tag{3.2.14}$$

where $(\sigma_t^X)_{t\in\mathbb{R}_+}$ and $(\sigma_t^N)_{t\in\mathbb{R}_+}$ are $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$-adapted volatility processes and $(W_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion under $\mathbb{P}^*$. Then we have

$$d\widehat{X}_t = (\sigma_t^X - \sigma_t^N)\widehat{X}_t d\widehat{W}_t, \tag{3.2.15}$$

hence $(\widehat{X}_t)_{t\in[0,T]}$ is given by the driftless geometric Brownian motion

$$\widehat{X}_t = \widehat{X}_0 \exp\left(\int_0^t (\sigma_s^X - \sigma_s^N) d\widehat{W}_s - \frac{1}{2}\int_0^t (\sigma_s^X - \sigma_s^N)^2 ds\right), \quad 0 \leqslant t \leqslant T.$$

Proof. First we note that by (3.2.13) and (3.2.14),

$$d\widehat{W}_t = dW_t - \frac{1}{N_t} dN_t \bullet dW_t = dW_t - \sigma_t^N dt, \qquad t \geqslant 0,$$

is a standard Brownian motion under $\widehat{\mathbb{P}}$. Next, by Itô's calculus and the Itô multiplication table and (3.2.14) we have

$$\begin{aligned} d\left(\frac{1}{N_t}\right) &= -\frac{1}{N_t^2} dN_t + \frac{1}{N_t^3} dN_t \bullet dN_t \\ &= -\frac{1}{N_t^2}(r_t N_t dt + \sigma_t^N N_t dW_t) + \frac{|\sigma_t^N|^2}{N_t} dt \end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{N_t^2}\left(r_t N_t dt + \sigma_t^N N_t \left(d\widehat{W}_t + \sigma_t^N dt\right)\right) + \frac{|\sigma_t^N|^2}{N_t} dt \\
&= -\frac{1}{N_t}\left(r_t dt + \sigma_t^N d\widehat{W}_t\right), \qquad (3.2.16)
\end{aligned}
$$

hence

$$
\begin{aligned}
d\widehat{X}_t &= d\left(\frac{X_t}{N_t}\right) \\
&= \frac{dX_t}{N_t} + X_t d\left(\frac{1}{N_t}\right) + dX_t \bullet d\left(\frac{1}{N_t}\right) \\
&= \frac{1}{N_t}\left(r_t X_t dt + \sigma_t^X X_t dW_t\right) - \frac{X_t}{N_t}\left(r_t dt + \sigma_t^N dW_t - |\sigma_t^N|^2 dt\right) \\
&\quad -\frac{1}{N_t}\left(r_t X_t dt + \sigma_t^X X_t dW_t\right) \cdot \left(r_t dt + \sigma_t^N dW_t - |\sigma_t^N|^2 dt\right) \\
&= \frac{1}{N_t}\left(r_t X_t dt + \sigma_t^X X_t dW_t\right) - \frac{X_t}{N_t}\left(r_t dt + \sigma_t^N dW_t\right) \\
&\quad + \frac{X_t}{N_t}|\sigma_t^N|^2 dt - \frac{X_t}{N_t}\sigma_t^X \sigma_t^N dt \\
&= \frac{X_t}{N_t}\sigma_t^X dW_t - \frac{X_t}{N_t}\sigma_t^N dW_t - \frac{X_t}{N_t}\sigma_t^X \sigma_t^N dt + X_t \frac{|\sigma_t^N|^2}{N_t} dt \\
&= \frac{X_t}{N_t}\left(\sigma_t^X dW_t - \sigma_t^N dW_t - \sigma_t^X \sigma_t^N dt + |\sigma_t^N|^2 dt\right) \\
&= \widehat{X}_t\left(\sigma_t^X - \sigma_t^N\right) dW_t - \widehat{X}_t\left(\sigma_t^X - \sigma_t^N\right)\sigma_t^N dt \\
&= \widehat{X}_t\left(\sigma_t^X - \sigma_t^N\right) d\widehat{W}_t,
\end{aligned}
$$

since $d\widehat{W}_t = dW_t - \sigma_t^N dt$, $0 \leqslant t \leqslant T$. □

We end this section with some comments on inverse changes of measure.

Inverse changes of measure

In the next proposition we compute the conditional inverse Radon-Nikodym density $d\mathbb{P}^*/d\widehat{\mathbb{P}}$, see also (3.2.2).

Proposition 3.9 We have

$$
\widehat{\mathbb{E}}\left[\frac{d\mathbb{P}^*}{d\widehat{\mathbb{P}}} \,\Big|\, \mathcal{F}_t\right] = \frac{N_0}{N_t}\exp\left(\int_0^t r_s ds\right), \qquad 0 \leqslant t \leqslant T, \qquad (3.2.17)
$$

and the process

$$
t \mapsto \frac{M_0}{M_t} = \frac{N_0}{N_t}\exp\left(\int_0^t r_s ds\right), \qquad 0 \leqslant t \leqslant T,
$$

is an $\mathcal{F}_t$-martingale under $\widehat{\mathbb{P}}$.

Proof. For all bounded and $\mathcal{F}_t$-measurable random variables F we have,

$$
\begin{aligned}
\widehat{\mathbb{E}}\left[F\frac{d\mathbb{P}^*}{d\widehat{\mathbb{P}}}\right] &= \mathbb{E}^*[F] \\
&= \mathbb{E}^*\left[F\frac{N_t}{N_t}\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \mathbb{E}^*\left[F\frac{N_T}{N_t}\exp\left(-\int_t^T r_s ds\right)\right] \\
&= \widehat{\mathbb{E}}\left[F\frac{N_0}{N_t}\exp\left(\int_0^t r_s ds\right)\right].
\end{aligned}
$$

□

By (3.2.16) we also have

$$
d\left(\frac{1}{N_t}\exp\left(\int_0^t r_s ds\right)\right) = -\frac{1}{N_t}\exp\left(\int_0^t r_s ds\right)\sigma_t^N d\widehat{W}_t,
$$

which recovers the second part of Proposition 3.9, *i.e.* the martingale property of

$$
t \mapsto \frac{M_0}{M_t} = \frac{1}{N_t}\exp\left(\int_0^t r_s ds\right)
$$

under $\widehat{\mathbb{P}}$.

3.3 Foreign Exchange

Currency exchange is a typical application of change of numéraire, that illustrates the absence of arbitrage principle.

Let R_t denote the foreign exchange rate, *i.e.* R_t is the (possibly fractional) quantity of local currency that correspond to one unit of foreign currency, while $1/R_t$ represents the quantity of foreign currency that correspond to a unit of local currency.

$$
\begin{tikzcd}
\text{Local} & \$1 \arrow[d] & \frac{R_t}{R_0}e^{t r^f} \arrow[r, "\text{Discount}"] & \frac{R_t}{R_0}e^{(r^f - r^l)t} \\
\text{Foreign} & \frac{1}{R_0} \arrow[r] & \frac{e^{t r^f}}{R_0} \arrow[u] &
\end{tikzcd}
$$

Consider an investor that intends to exploit an "overseas investment opportunity" by

a) at time 0, changing one unit of local currency into $1/R_0$ units of foreign currency,
b) investing $1/R_0$ on the foreign market at the rate r^f, which will yield the amount $e^{t r^f}/R_0$ at time $t > 0$,
c) changing back $e^{t r^f}/R_0$ into a quantity $e^{t r^f}R_t/R_0 = N_t/R_0$ of local currency.

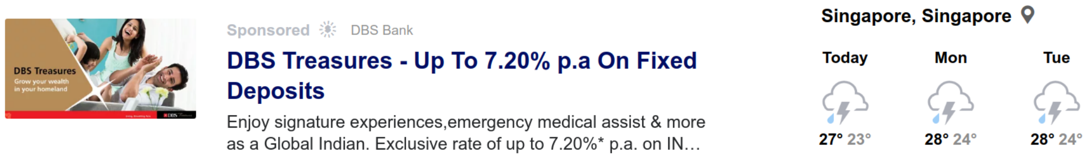


Figure 3.2: Overseas investment opportunity.[*]

In other words, the foreign money market account $e^{t r^f}$ is valued $e^{t r^f}R_t$ on the local (or domestic) market, and its discounted value on the local market is

$$
e^{-t r^l + t r^f}R_t, \qquad t > 0.
$$

[*] For illustration purposes only. *Not* an advertisement.

The outcome of this investment will be obtained by a martingale comparison of $\mathrm{e}^{t r^{\mathrm{f}}} R_t / R_0$ to the amount $\mathrm{e}^{t r^l}$ that could have been obtained by investing on the local market.

Taking

$$\boxed{N_t := \mathrm{e}^{t r^{\mathrm{f}}} R_t, \qquad t \geqslant 0,} \tag{3.3.1}$$

as *numéraire*, absence of arbitrage is expressed by Assumption 1, which states that the discounted numéraire process

$$t \mapsto \mathrm{e}^{-r^l t} N_t = \mathrm{e}^{-t(r^l - r^{\mathrm{f}})} R_t$$

is an $\mathcal{F}_t$-martingale under $\mathbb{P}^*$.

Next, we find a characterization of this arbitrage condition using the model parameters r, r^f, μ, by modeling the foreign exchange rates R_t according to a geometric Brownian motion (3.3.2).

Proposition 3.10 Assume that the foreign exchange rate R_t satisfies a stochastic differential equation of the form

$$dR_t = \mu R_t dt + \sigma R_t dW_t, \tag{3.3.2}$$

where $(W_t)_{t \in \mathbb{R}_+}$ is a standard Brownian motion under $\mathbb{P}^*$. Under the absence of arbitrage Assumption 1 for the numéraire (3.3.1), we have

$$\mu = r^l - r^{\mathrm{f}}, \tag{3.3.3}$$

hence the exchange rate process satisfies

$$dR_t = \left(r^l - r^{\mathrm{f}}\right) R_t dt + \sigma R_t dW_t. \tag{3.3.4}$$

under $\mathbb{P}^*$.

Proof. The equation (3.3.2) has solution

$$R_t = R_0 \, \mathrm{e}^{\mu t + \sigma W_t - \sigma^2 t / 2}, \qquad t \geqslant 0,$$

hence the discounted value of the foreign money market account $\mathrm{e}^{t r^{\mathrm{f}}}$ on the local market is

$$\mathrm{e}^{-t r^l} N_t = \mathrm{e}^{-t r^l + t r^{\mathrm{f}}} R_t = R_0 \, \mathrm{e}^{(r^{\mathrm{f}} - r^l + \mu) t + \sigma W_t - \sigma^2 t / 2}, \qquad t \geqslant 0.$$

Under the absence of arbitrage Assumption 1, the process $\mathrm{e}^{-(r^l - r^{\mathrm{f}}) t} R_t = \mathrm{e}^{-t r^l} N_t$ should be an $\mathcal{F}_t$-martingale under $\mathbb{P}^*$, and this holds provided that $r^{\mathrm{f}} - r + \mu = 0$, which yields (3.3.3) and (3.3.4). □

As a consequence of Proposition 3.10, under absence of arbitrage a local investor who buys a unit of foreign currency in the hope of a higher return $r^{\mathrm{f}} >> r$ will have to face a lower (or even more negative) drift

$$\mu = r^l - r^{\mathrm{f}} << 0$$

in his exchange rate R_t. The drift $\mu = r^l - r^{\mathrm{f}}$ is also called the *cost of carrying* the foreign currency.

The local money market account $X_t := \mathrm{e}^{t r^l}$ is valued $\mathrm{e}^{t r^l}/R_t$ on the foreign market, and its discounted value at time $t \geqslant 0$ on the foreign market is

$$\begin{aligned} \frac{\mathrm{e}^{(r^l - r^{\mathrm{f}})t}}{R_t} &= \frac{X_t}{N_t} = \widehat{X}_t \qquad (3.3.5) \\ &= \frac{1}{R_0}\mathrm{e}^{(r^l - r^{\mathrm{f}})t - \mu t - \sigma W_t + \sigma^2 t/2} \\ &= \frac{1}{R_0}\mathrm{e}^{(r^l - r^{\mathrm{f}})t - \mu t - \sigma \widehat{W}_t - \sigma^2 t/2}, \end{aligned}$$

where

$$\begin{aligned} d\widehat{W}_t &= dW_t - \frac{1}{N_t} dN_t \bullet dW_t \\ &= dW_t - \frac{1}{R_t} dR_t \bullet dW_t \\ &= dW_t - \sigma dt, \qquad t \geqslant 0, \end{aligned}$$

is a standard Brownian motion under $\widehat{\mathbb{P}}$ by (3.2.13). Under absence of arbitrage, the process $\mathrm{e}^{-(r^l - r^{\mathrm{f}})t} R_t$ is an $\mathscr{F}_t$-martingale under $\mathbb{P}^*$ and (3.3.5) is an $\mathscr{F}_t$-martingale under $\widehat{\mathbb{P}}$ by Proposition 3.4, which recovers (3.3.3).

```
library(quantmod)
getSymbols("EURTRY=X",src = "yahoo",from = "2018-01-01",to = "2021-12-31")
getSymbols("INTDSRTRM193N",src = "FRED")
Interestrate<-`INTDSRTRM193N`["2018-01-01::2021-31-12"]
EURTRY<-Ad(`EURTRY=X`); myPars <- chart_pars();myPars$cex<-1.2
Cumulative<-cumprod(1+Interestrate/100/12)
normalizedfxrate<-1+(as.numeric(last(Cumulative))-1)*(EURTRY-as.numeric(EURTRY[1]))/
    (as.numeric(last(EURTRY))-as.numeric(EURTRY[1]))
myTheme <- chart_theme();myTheme$col$line.col <- "blue"
dev.new(width=16,height=8)
chart_Series(Cumulative,pars=myPars, theme = myTheme)
add_TA(normalizedfxrate, col='black', lw =2, on = 1)
add_TA(Interestrate, col='purple', lw =2)
```

The above code plots an evolution of currency exchange rates compared with the evolution of interest rates, as shown in Figure 3.3.

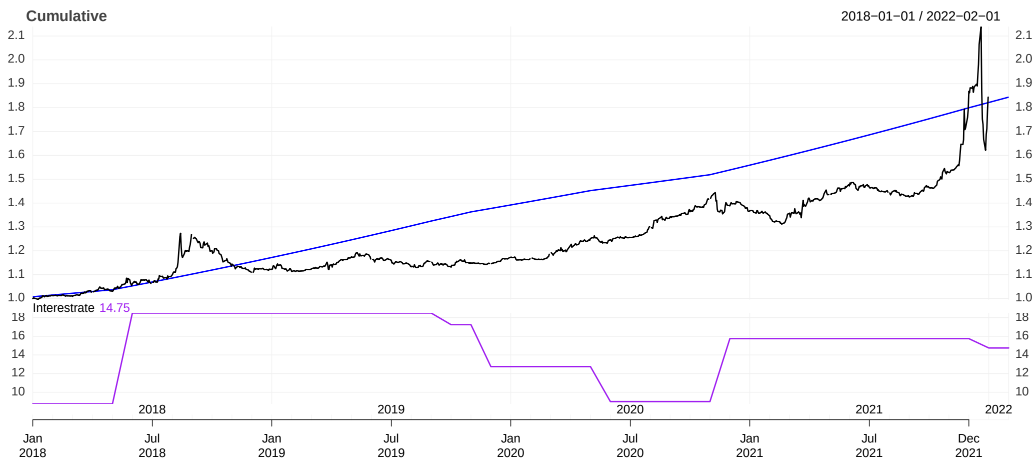


Figure 3.3: Evolution of exchange rate *vs* interest rate.*

Proposition 3.11 Under the absence of arbitrage condition (3.3.3), the inverse exchange rate $1/R_t$ satisfies

$$d\left(\frac{1}{R_t}\right) = \frac{r^{\mathrm{f}} - r^l}{R_t}dt - \frac{\sigma}{R_t}d\widehat{W}_t, \tag{3.3.6}$$

under $\widehat{\mathbb{P}}$, where $(R_t)_{t\in\mathbb{R}_+}$ is given by (3.3.4).

Proof. By (3.3.3), the exchange rate $1/R_t$ is written using Itô's calculus as

$$\begin{aligned} d\left(\frac{1}{R_t}\right) &= -\frac{1}{R_t^2}(\mu R_t dt + \sigma R_t dW_t) + \frac{1}{R_t^3}\sigma^2 R_t^2 dt \\ &= -\frac{\mu - \sigma^2}{R_t}dt - \frac{\sigma}{R_t}dW_t \\ &= -\frac{\mu}{R_t}dt - \frac{\sigma}{R_t}d\widehat{W}_t \\ &= \frac{r^{\mathrm{f}} - r^l}{R_t}dt - \frac{\sigma}{R_t}d\widehat{W}_t, \end{aligned}$$

where $(\widehat{W}_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion under $\widehat{\mathbb{P}}$. □

Consequently, under absence of arbitrage, a foreign investor who buys a unit of the local currency in the hope of a higher return $r >> r^{\mathrm{f}}$ will have to face a lower (or even more negative) drift $-\mu = r^{\mathrm{f}} - r$ in his exchange rate $1/R_t$ as written in (3.3.6) under $\widehat{\mathbb{P}}$.

Foreign exchange options

We now price a foreign exchange call option with payoff $(R_T - \kappa)^+$ under $\mathbb{P}^*$ on the exchange rate R_T by the Black-Scholes formula as in the next proposition, also known as the Garman and Kohlhagen, 1983 formula. The foreign exchange call option is designed for a local buyer of foreign currency.

Proposition 3.12 (Garman and Kohlhagen, 1983 formula for call options). Consider the exchange rate process $(R_t)_{t\in\mathbb{R}_+}$ given by (3.3.4). The price of the foreign exchange call option on R_T with maturity T and strike price $\kappa > 0$ is given in local currency units as

$$\boxed{\mathrm{e}^{-(T-t)r^l}\mathbb{E}^*\left[(R_T-\kappa)^+ \,\middle|\, \mathscr{F}_t\right] = \mathrm{e}^{-(T-t)r^{\mathrm{f}}}R_t\Phi_+(t,R_t) - \kappa\,\mathrm{e}^{-(T-t)r^l}\Phi_-(t,R_t),} \tag{3.3.7}$$

$0 \leqslant t \leqslant T$, where

$$\Phi_+(t,x) = \Phi\left(\frac{\log(x/\kappa) + (T-t)(r^l - r^{\mathrm{f}} + \sigma^2/2)}{\sigma\sqrt{T-t}}\right),$$

and

$$\Phi_-(t,x) = \Phi\left(\frac{\log(x/\kappa) + (T-t)(r^l - r^{\mathrm{f}} - \sigma^2/2)}{\sigma\sqrt{T-t}}\right).$$

Proof. As a consequence of (3.3.4), we find the numéraire dynamics

$$dN_t = d\left(\mathrm{e}^{tr^{\mathrm{f}}}R_t\right)$$

$$
\begin{aligned}
&= r^{\mathrm{f}} \mathrm{e}^{t r^{\mathrm{f}}} R_t dt + \mathrm{e}^{t r^{\mathrm{f}}} dR_t \\
&= r^{\mathrm{f}} \mathrm{e}^{t r^{\mathrm{f}}} R_t dt + \sigma \mathrm{e}^{t r^{\mathrm{f}}} R_t dW_t \\
&= r^{\mathrm{f}} N_t dt + \sigma N_t dW_t.
\end{aligned}
$$

Hence, a standard application of the Black-Scholes formula yields

$$
\begin{aligned}
& \mathrm{e}^{-(T-t) r^l} \mathbb{E}^* \left[(R_T - \kappa)^+ \mid \mathcal{F}_t \right] = \mathrm{e}^{-(T-t) r^l} \mathbb{E}^* \left[\left(\mathrm{e}^{-T r^{\mathrm{f}}} N_T - \kappa \right)^+ \mid \mathcal{F}_t \right] \\
&= \mathrm{e}^{-(T-t) r^l} \mathrm{e}^{-T r^{\mathrm{f}}} \mathbb{E}^* \left[\left(N_T - \kappa \mathrm{e}^{T r^{\mathrm{f}}} \right)^+ \mid \mathcal{F}_t \right] \\
&= \mathrm{e}^{-T r^{\mathrm{f}}} \left(N_t \Phi \left(\frac{\log(N_t \mathrm{e}^{-T r^{\mathrm{f}}} / \kappa) + (r + \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}} \right) \right. \\
&\quad \left. - \kappa \mathrm{e}^{T r^{\mathrm{f}} - (T-t) r^l} \Phi \left(\frac{\log(N_t \mathrm{e}^{-T r^{\mathrm{f}}} / \kappa) + (r^l - \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}} \right) \right) \\
&= \mathrm{e}^{-T r^{\mathrm{f}}} \left(N_t \Phi \left(\frac{\log(R_t / \kappa) + (T-t)(r^l - r^{\mathrm{f}} + \sigma^2/2)}{\sigma \sqrt{T-t}} \right) \right. \\
&\quad \left. - \kappa \mathrm{e}^{T r^{\mathrm{f}} - (T-t) r^l} \Phi \left(\frac{\log(R_t / \kappa) + (T-t)(r^l - r^{\mathrm{f}} - \sigma^2/2)}{\sigma \sqrt{T-t}} \right) \right) \\
&= \mathrm{e}^{-(T-t) r^{\mathrm{f}}} R_t \Phi_+(t, R_t) - \kappa \mathrm{e}^{-(T-t) r^l} \Phi_-(t, R_t).
\end{aligned}
$$

A similar conclusion can be reached by directly applying (3.3.4). □

Similarly, from (3.3.6) rewritten as

$$
d\left(\frac{\mathrm{e}^{t r^l}}{R_t} \right) = r^{\mathrm{f}} \frac{\mathrm{e}^{t r^l}}{R_t} dt - \sigma \frac{\mathrm{e}^{t r^l}}{R_t} d\widehat{W}_t,
$$

a foreign exchange put option with payoff $(1/\kappa - 1/R_T)^+$ can be priced under $\widehat{\mathbb{P}}$ in a Black-Scholes model by taking $\mathrm{e}^{t r^l}/R_t$ as underlying asset price, r^{f} as risk-free interest rate, and $-\sigma$ as volatility parameter. The foreign exchange put option is designed for the foreign seller of local currency, see for example the **buy back guarantee*** which is a typical example of a foreign exchange put option.

Proposition 3.13 (Garman and Kohlhagen, 1983 formula for put options). Consider the exchange rate process $(R_t)_{t \in \mathbb{R}_+}$ given by (3.3.4). The price of the foreign exchange put option on R_T with maturity T and strike price $1/\kappa > 0$ is given in foreign currency units as

$$
\boxed{
\begin{aligned}
& \mathrm{e}^{-(T-t) r^{\mathrm{f}}} \widehat{\mathbb{E}} \left[\left(\frac{1}{\kappa} - \frac{1}{R_T} \right)^+ \middle| \mathcal{F}_t \right] \\
&\quad = \frac{\mathrm{e}^{-(T-t) r^{\mathrm{f}}}}{\kappa} \Phi_- \left(t, \frac{1}{R_t} \right) - \frac{\mathrm{e}^{-(T-t) r^l}}{R_t} \Phi_+ \left(t, \frac{1}{R_t} \right),
\end{aligned}
} \tag{3.3.8}
$$

*Right-click to open or save the attachment.

$0 \leqslant t \leqslant T$, where

$$\Phi_+(t,x) := \Phi\left(-\frac{\log(\kappa x) + (T-t)(r^{\mathrm{f}} - r^l + \sigma^2/2)}{\sigma\sqrt{T-t}}\right),$$

and

$$\Phi_-(t,x) := \Phi\left(-\frac{\log(\kappa x) + (T-t)(r^{\mathrm{f}} - r^l - \sigma^2/2)}{\sigma\sqrt{T-t}}\right).$$

Proof. The Black-Scholes formula yields

$$\begin{aligned}
\mathrm{e}^{-(T-t)r^{\mathrm{f}}}\widehat{\mathbf{E}}\left[\left(\frac{1}{\kappa} - \frac{1}{R_T}\right)^+ \,\Big|\, \mathscr{F}_t\right] &= \mathrm{e}^{-(T-t)r^{\mathrm{f}}}\mathrm{e}^{-Tr^l}\widehat{\mathbf{E}}\left[\left(\frac{\mathrm{e}^{Tr^l}}{\kappa} - \frac{\mathrm{e}^{Tr^l}}{R_T}\right)^+ \,\Big|\, \mathscr{F}_t\right] \\
&= \frac{1}{\kappa}\mathrm{e}^{-(T-t)r^{\mathrm{f}}}\Phi_-\left(t, \frac{1}{R_t}\right) - \frac{\mathrm{e}^{-(T-t)r^l}}{R_t}\Phi_+\left(t, \frac{1}{R_t}\right),
\end{aligned}$$

which is the symmetric of (3.3.7) by exchanging R_t with $1/R_t$, and r with r^f. □

Call/put duality for foreign exchange options

Let $N_t = \mathrm{e}^{tr^{\mathrm{f}}}R_t$, where R_t is an exchange rate with respect to a foreign currency and r_f is the foreign market interest rate.

Proposition 3.14 The foreign exchange call and put options on the local and foreign markets are linked by the call/put duality relation

$$\mathrm{e}^{-(T-t)r^l}\mathbf{E}^*\left[(R_T - \kappa)^+ \,\big|\, \mathscr{F}_t\right] = \kappa R_t\,\mathrm{e}^{-(T-t)r^{\mathrm{f}}}\widehat{\mathbf{E}}\left[\left(\frac{1}{\kappa} - \frac{1}{R_T}\right)^+ \,\Big|\, \mathscr{F}_t\right], \tag{3.3.9}$$

between a put option with strike price $1/\kappa$ and a (possibly fractional) quantity $1/(\kappa R_t)$ of call option(s) with strike price κ.

Proof. By application of change of numéraire from Proposition 3.5 and (3.2.8) we have

$$\widehat{\mathbf{E}}\left[\frac{1}{\mathrm{e}^{Tr^{\mathrm{f}}}R_T}(R_T - \kappa)^+ \,\Big|\, \mathscr{F}_t\right] = \frac{1}{N_t}\mathrm{e}^{-(T-t)r^l}\mathbf{E}^*\left[(R_T - \kappa)^+ \,\big|\, \mathscr{F}_t\right],$$

hence

$$\begin{aligned}
\mathrm{e}^{-(T-t)r^{\mathrm{f}}}\widehat{\mathbf{E}}\left[\left(\frac{1}{\kappa} - \frac{1}{R_T}\right)^+ \,\Big|\, \mathscr{F}_t\right] &= \mathrm{e}^{-(T-t)r^{\mathrm{f}}}\widehat{\mathbf{E}}\left[\frac{1}{\kappa R_T}(R_T - \kappa)^+ \,\Big|\, \mathscr{F}_t\right] \\
&= \frac{1}{\kappa}\mathrm{e}^{tr^{\mathrm{f}}}\widehat{\mathbf{E}}\left[\frac{1}{\mathrm{e}^{Tr^{\mathrm{f}}}R_T}(R_T - \kappa)^+ \,\Big|\, \mathscr{F}_t\right] \\
&= \frac{1}{\kappa N_t}\mathrm{e}^{tr^{\mathrm{f}} - (T-t)r^l}\mathbf{E}^*\left[(R_T - \kappa)^+ \,\big|\, \mathscr{F}_t\right] \\
&= \frac{1}{\kappa R_t}\mathrm{e}^{-(T-t)r^l}\mathbf{E}^*\left[(R_T - \kappa)^+ \,\big|\, \mathscr{F}_t\right].
\end{aligned}$$

□

In the Black-Scholes case, the duality (3.3.9) can be directly checked by verifying that (3.3.8) coincides with

$$\frac{1}{\kappa R_t}\mathrm{e}^{-(T-t)r^l}\mathbf{E}^*\left[(R_T - \kappa)^+ \,\big|\, \mathscr{F}_t\right]$$

$$
\begin{aligned}
&= \frac{1}{\kappa R_t} \mathrm{e}^{-(T-t)r^l} \mathrm{e}^{-Tr^{\mathrm{f}}} \mathbb{E}^* \left[\left(\mathrm{e}^{Tr^{\mathrm{f}}} R_T - \kappa \mathrm{e}^{Tr^{\mathrm{f}}} \right)^+ \,\middle|\, \mathscr{F}_t \right] \\
&= \frac{1}{\kappa R_t} \mathrm{e}^{-(T-t)r^l} \mathrm{e}^{-Tr^{\mathrm{f}}} \mathbb{E}^* \left[\left(N_T - \kappa \mathrm{e}^{Tr^{\mathrm{f}}} \right)^+ \,\middle|\, \mathscr{F}_t \right] \\
&= \frac{1}{\kappa R_t} \left(\mathrm{e}^{-(T-t)r^{\mathrm{f}}} R_t \Phi_+^{\mathrm{c}}(t, R_t) - \kappa \mathrm{e}^{-(T-t)r^l} \Phi_-^{\mathrm{c}}(t, R_t) \right) \\
&= \frac{1}{\kappa} \mathrm{e}^{-(T-t)r^{\mathrm{f}}} \Phi_+^{\mathrm{c}}(t, R_t) - \frac{\mathrm{e}^{-(T-t)r^l}}{R_t} \Phi_-^{\mathrm{c}}(t, R_t) \\
&= \frac{1}{\kappa} \mathrm{e}^{-(T-t)r^{\mathrm{f}}} \Phi_- \left(t, \frac{1}{R_t} \right) - \frac{\mathrm{e}^{-(T-t)r^l}}{R_t} \Phi_+ \left(t, \frac{1}{R_t} \right),
\end{aligned}
$$

where

$$
\Phi_+^{\mathrm{c}}(t, x) := \Phi \left(\frac{\log(x/\kappa) + (T-t)(r^l - r^{\mathrm{f}} + \sigma^2/2)}{\sigma \sqrt{T-t}} \right),
$$

and

$$
\Phi_-^{\mathrm{c}}(t, x) := \Phi \left(\frac{\log(x/\kappa) + (T-t)(r^l - r^{\mathrm{f}} - \sigma^2/2)}{\sigma \sqrt{T-t}} \right).
$$

	Local market	Foreign market
Measure	$\mathbb{P}^*$	$\widehat{\mathbb{P}}$
Discount factor	$t \mapsto \mathrm{e}^{-r^l t}$	$t \mapsto \mathrm{e}^{-r^{\mathrm{f}} t}$
Martingale	$t \mapsto \mathrm{e}^{-tr^l} N_t = \mathrm{e}^{-t(r^l - r^{\mathrm{f}})} R_t$	$t \mapsto \dfrac{X_t}{N_t} = \widehat{X}_t = \dfrac{\mathrm{e}^{(r^l - r^{\mathrm{f}})t}}{R_t}$
Option	$\mathrm{e}^{-(T-t)r^l} \mathbb{E}^* \left[(R_T - \kappa)^+ \,\middle|\, \mathscr{F}_t \right]$	$\mathrm{e}^{-(T-t)r^{\mathrm{f}}} \widehat{\mathbb{E}} \left[\left(\dfrac{1}{\kappa} - \dfrac{1}{R_T} \right)^+ \,\middle|\, \mathscr{F}_t \right]$
Application	Local purchase of foreign currency	Foreign selling of local currency

Table 3.1: Local *vs* foreign exchange options.

Example - Buy back guarantee

The put option priced

$$
\begin{aligned}
&\mathrm{e}^{-(T-t)r^{\mathrm{f}}} \widehat{\mathbb{E}} \left[\left(\frac{1}{\kappa} - \frac{1}{R_T} \right)^+ \,\middle|\, \mathscr{F}_t \right] \\
&= \frac{1}{\kappa} \mathrm{e}^{-(T-t)r^{\mathrm{f}}} \Phi_- \left(t, \frac{1}{R_t} \right) - \frac{\mathrm{e}^{-(T-t)r^l}}{R_t} \Phi_+ \left(t, \frac{1}{R_t} \right)
\end{aligned}
$$

on the foreign market corresponds to a **buy back guarantee*** in currency exchange. In the case of an option "at the money" with $\kappa = R_t$ and $r^{\rm l} = r^{\rm f} \simeq 0$, we find

$$\begin{aligned}
\widehat{\mathbb{E}}\left[\left(\frac{1}{R_t}-\frac{1}{R_T}\right)^+ \Big| \mathscr{F}_t\right] &= \frac{1}{R_t}\times\left(\Phi\left(\frac{\sigma\sqrt{T-t}}{2}\right)-\Phi\left(-\frac{\sigma\sqrt{T-t}}{2}\right)\right)\\
&= \frac{1}{R_t}\times\left(2\Phi\left(\frac{\sigma\sqrt{T-t}}{2}\right)-1\right).
\end{aligned}$$

For example, let R_t denote the USD/EUR (USDEUR=X) exchange rate from a foreign currency (USD) to a local currency (EUR), *i.e.* one unit of the foreign currency (USD) corresponds to $R_t = 1/1.23$ units of local currency (EUR). Taking $T-t = 30$ days and $\sigma = 10\%$, we find that the foreign currency put option allowing the foreign sale of one EURO back into USDs is priced at the money in USD as

$$\begin{aligned}
\widehat{\mathbb{E}}\left[\left(\frac{1}{R_t}-\frac{1}{R_T}\right)^+ \Big| \mathscr{F}_t\right] &= 1.23\big(2\Phi\big(0.05\times\sqrt{31/365}\big)-1\big)\\
&= 1.23(2\times 0.505813-1)\\
&= \$0.01429998
\end{aligned}$$

per USD, or €0.011626 per exchanged unit of EURO. Based on a displayed option price of €4.5 and in order to make the contract fair, this would translate into an average amount of $4.5/0.011626 \simeq$ €387 exchanged at the counter by customers subscribing to the buy back guarantee.

3.4 Pricing Exchange Options

Based on Proposition 3.4, we model the process $\widehat{X}_t$ of forward prices as a continuous martingale under $\widehat{\mathbb{P}}$, written as

$$d\widehat{X}_t = \widehat{\sigma}_t d\widehat{W}_t, \qquad t \geqslant 0, \tag{3.4.1}$$

where $\big(\widehat{W}_t\big)_{t\in\mathbb{R}_+}$ is a standard Brownian motion under $\widehat{\mathbb{P}}$ and $(\widehat{\sigma}_t)_{t\in\mathbb{R}_+}$ is an $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$-adapted stochastic volatility process. More precisely, we assume that $\big(\widehat{X}_t\big)_{t\in\mathbb{R}_+}$ has the dynamics

$$d\widehat{X}_t = \widehat{\sigma}_t\big(\widehat{X}_t\big)d\widehat{W}_t, \tag{3.4.2}$$

where $x \mapsto \widehat{\sigma}_t(x)$ is a local volatility function which is Lipschitz in x, uniformly in $t \geqslant 0$. The Markov property of the diffusion process $\big(\widehat{X}_t\big)_{t\in\mathbb{R}_+}$, cf. Theorem V-6-32 of Protter, 2004, shows that when $\widehat{g}$ is a deterministic payoff function, the conditional expectation $\widehat{\mathbb{E}}\big[\widehat{g}\big(\widehat{X}_T\big) \mid \mathscr{F}_t\big]$ can be written using a (measurable) function $\widehat{C}(t,x)$ of t and $\widehat{X}_t$, as

$$\widehat{\mathbb{E}}\big[\widehat{g}\big(\widehat{X}_T\big) \mid \mathscr{F}_t\big] = \widehat{C}\big(t,\widehat{X}_t\big), \qquad 0 \leqslant t \leqslant T.$$

Consequently, a vanilla option with claim payoff $C := N_T\widehat{g}\big(\widehat{X}_T\big)$ can be priced from Proposition 3.5 as

$$\begin{aligned}
\mathbb{E}^*\left[e^{-\int_t^T r_s ds}N_T\widehat{g}\big(\widehat{X}_T\big) \Big| \mathscr{F}_t\right] &= N_t\widehat{\mathbb{E}}\big[\widehat{g}\big(\widehat{X}_T\big) \mid \mathscr{F}_t\big]\\
&= N_t\widehat{C}\big(t,\widehat{X}_t\big), \quad 0 \leqslant t \leqslant T.
\end{aligned} \tag{3.4.3}$$

In the next Proposition 3.15 we state the Margrabe, 1978 formula for the pricing of exchange options by the zero interest rate Black-Scholes formula. It will be applied in particular in Proposition 4.3 below for the pricing of bond options. Here, $(N_t)_{t\in\mathbb{R}_+}$ denotes any numéraire process satisfying Assumption 1.

*Right-click to open or save the attachment.

Proposition 3.15 (Margrabe, 1978 formula). Assume that $\widehat{\sigma}_t(\widehat{X}_t) = \widehat{\sigma}(t)\widehat{X}_t$, *i.e.* the martingale $(\widehat{X}_t)_{t\in[0,T]}$ is a (driftless) geometric Brownian motion under $\widehat{\mathbb{P}}$ with deterministic volatility $(\widehat{\sigma}(t))_{t\in[0,T]}$. Then we have

$$\mathbb{E}^*\left[\mathrm{e}^{-\int_t^T r_s ds}(X_T-\kappa N_T)^+ \,\Big|\, \mathcal{F}_t\right] = X_t\Phi_+^0(t,\widehat{X}_t) - \kappa N_t\Phi_-^0(t,\widehat{X}_t), \tag{3.4.4}$$

$t\in[0,T]$, where

$$\Phi_+^0(t,x) = \Phi\left(\frac{\log(x/\kappa)}{v(t,T)} + \frac{v(t,T)}{2}\right), \quad \Phi_-^0(t,x) = \Phi\left(\frac{\log(x/\kappa)}{v(t,T)} - \frac{v(t,T)}{2}\right), \tag{3.4.5}$$

and $v^2(t,T) = \int_t^T \widehat{\sigma}^2(s)ds$.

Proof. Taking $g(x) := (x-\kappa)^+$ in (3.4.3), the call option with payoff

$$\begin{aligned}(X_T-\kappa N_T)^+ &= N_T\left(\widehat{X}_T-\kappa\right)^+ \\ &= N_T\left(\widehat{X}_t\exp\left(\int_t^T\widehat{\sigma}(t)d\widehat{W}_t - \frac{1}{2}\int_t^T|\widehat{\sigma}(t)|^2dt\right)-\kappa\right)^+,\end{aligned}$$

and floating strike price κN_T is priced by (3.4.3) as

$$\begin{aligned}\mathbb{E}^*\left[\mathrm{e}^{-\int_t^T r_s ds}(X_T-\kappa N_T)^+ \,\Big|\, \mathcal{F}_t\right] &= \mathbb{E}^*\left[\mathrm{e}^{-\int_t^T r_s ds}N_T\left(\widehat{X}_T-\kappa\right)^+ \,\Big|\, \mathcal{F}_t\right] \\ &= N_t\widehat{\mathbb{E}}\left[\left(\widehat{X}_T-\kappa\right)^+ \,\big|\, \mathcal{F}_t\right] \\ &= N_t\widehat{C}(t,\widehat{X}_t),\end{aligned}$$

where the function $\widehat{C}(t,\widehat{X}_t)$ is given by the Black-Scholes formula

$$\widehat{C}(t,x) = x\Phi_+^0(t,x) - \kappa\Phi_-^0(t,x),$$

with zero interest rate, since $(\widehat{X}_t)_{t\in[0,T]}$ is a driftless geometric Brownian motion which is an $\mathcal{F}_t$-martingale under $\widehat{\mathbb{P}}$, and $\widehat{X}_T$ is a lognormal random variable with variance coefficient $v^2(t,T) = \int_t^T \widehat{\sigma}^2(s)ds$. Hence we have

$$\begin{aligned}\mathbb{E}^*\left[\mathrm{e}^{-\int_t^T r_s ds}(X_T-\kappa N_T)^+ \,\Big|\, \mathcal{F}_t\right] &= N_t\widehat{C}(t,\widehat{X}_t) \\ &= N_t\widehat{X}_t\Phi_+^0(t,\widehat{X}_t) - \kappa N_t\Phi_-^0(t,\widehat{X}_t),\end{aligned}$$

$t \geqslant 0$. □

In particular, from Proposition 3.8 and (3.2.15), we can take $\widehat{\sigma}(t) = \sigma_t^X - \sigma_t^N$ when $(\sigma_t^X)_{t\in\mathbb{R}_+}$ and $(\sigma_t^N)_{t\in\mathbb{R}_+}$ are deterministic.

Examples:

a) When the short rate process $(r(t))_{t\in[0,T]}$ is a *deterministic* function of time and $N_t = \mathrm{e}^{\int_t^T r(s)ds}$, $0 \leqslant t \leqslant T$, we have $\widehat{\mathbb{P}} = \mathbb{P}^*$ and Proposition 3.15 yields the Merton, 1973 "zero interest rate" version of the Black-Scholes formula

$$\begin{aligned} & \mathrm{e}^{-\int_t^T r(s)ds}\mathbb{E}^*\left[(X_T-\kappa)^+ \,\middle|\, \mathscr{F}_t\right] \\ & = X_t\Phi_+^0\left(t, \mathrm{e}^{\int_t^T r(s)ds}X_t\right) - \kappa\,\mathrm{e}^{-\int_t^T r(s)ds}\Phi_-^0\left(t, \mathrm{e}^{\int_t^T r(s)ds}X_t\right), \end{aligned}$$

where Φ_+^0 and Φ_-^0 are defined in (3.4.5) and $(X_t)_{t\in\mathbb{R}_+}$ satisfies the equation

$$\frac{dX_t}{X_t} = r(t)dt + \widehat{\sigma}(t)dW_t, \quad i.e. \quad \frac{d\widehat{X}_t}{\widehat{X}_t} = \widehat{\sigma}(t)dW_t, \quad 0 \leqslant t \leqslant T.$$

b) In the case of pricing under a *forward numéraire*, *i.e.* when $N_t = P(t,T)$, $t \in [0,T]$, we get

$$\mathbb{E}^*\left[\mathrm{e}^{-\int_t^T r_s ds}(X_T-\kappa)^+ \,\middle|\, \mathscr{F}_t\right] = X_t\Phi_+^0(t,\widehat{X}_t) - \kappa P(t,T)\Phi_-^0(t,\widehat{X}_t),$$

$0 \leqslant t \leqslant T$, since $N_T = P(T,T) = 1$. In particular, when $X_t = P(t,S)$ the above formula allows us to price a bond call option on $P(T,S)$ as

$$\mathbb{E}^*\left[\mathrm{e}^{-\int_t^T r_s ds}(P(T,S)-\kappa)^+ \,\middle|\, \mathscr{F}_t\right] = P(t,S)\Phi_+^0(t,\widehat{X}_t) - \kappa P(t,T)\Phi_-^0(t,\widehat{X}_t),$$

$0 \leqslant t \leqslant T$, provided that the martingale $\widehat{X}_t = P(t,S)/P(t,T)$ under $\widehat{\mathbb{P}}$ is given by a geometric Brownian motion, cf. Section 4.2.

3.5 Hedging by Change of Numéraire

In this section we reconsider and extend the standard Black-Scholes self-financing hedging strategies. For this, we use the stochastic integral representation of the forward claim payoffs and change of numéraire in order to compute self-financing portfolio strategies. Our hedging portfolios will be built on the assets (X_t, N_t), not on X_t and the money market account $B_t = \mathrm{e}^{\int_0^t r_s ds}$, extending the classical hedging portfolios that are available in from the Black-Scholes formula, using a technique from Jamshidian, 1996, cf. also Privault and Teng, 2012.

Consider a claim with random payoff C, typically an interest rate derivative, cf. Chapter 4. Assume that the forward claim payoff $C/N_T \in L^2(\Omega)$ has the stochastic integral representation

$$\widehat{C} := \frac{C}{N_T} = \widehat{\mathbb{E}}\left[\frac{C}{N_T}\right] + \int_0^T \widehat{\phi}_t d\widehat{X}_t, \tag{3.5.1}$$

where $(\widehat{X}_t)_{t\in[0,T]}$ is given by (3.4.1) and $(\widehat{\phi}_t)_{t\in[0,T]}$ is a square-integrable adapted process under $\widehat{\mathbb{P}}$, from which it follows that the forward claim price

$$\widehat{V}_t := \frac{V_t}{N_t} = \frac{1}{N_t}\mathbb{E}^*\left[\mathrm{e}^{-\int_t^T r_s ds}C \,\middle|\, \mathscr{F}_t\right] = \widehat{\mathbb{E}}\left[\frac{C}{N_T} \,\middle|\, \mathscr{F}_t\right], \quad 0 \leqslant t \leqslant T,$$

is an $\mathscr{F}_t$-martingale under $\widehat{\mathbb{P}}$, that can be decomposed as

$$\widehat{V}_t = \widehat{\mathbb{E}}[\widehat{C} \mid \mathscr{F}_t] = \widehat{\mathbb{E}}\left[\frac{C}{N_T}\right] + \int_0^t \widehat{\phi}_s d\widehat{X}_s, \qquad 0 \leqslant t \leqslant T. \tag{3.5.2}$$

The next Proposition 3.16 extends the argument of Jamshidian, 1996 to the general framework of pricing using change of numéraire. Note that this result differs from the standard formula that uses the money market account $B_t = \mathrm{e}^{\int_0^t r_s ds}$ for hedging instead of N_t, cf. *e.g.* Geman, El Karoui, and Rochet, 1995 pages 453-454.

Proposition 3.16 Letting $\widehat{\eta}_t := \widehat{V}_t - \widehat{X}_t\widehat{\phi}_t$, with $\widehat{\phi}_t$ defined in (3.5.2), $0 \leqslant t \leqslant T$, the portfolio allocation

$$\left(\widehat{\phi}_t, \widehat{\eta}_t\right)_{t\in[0,T]}$$

with value

$$V_t = \widehat{\phi}_t X_t + \widehat{\eta}_t N_t, \qquad 0 \leqslant t \leqslant T,$$

is self-financing in the sense that

$$dV_t = \widehat{\phi}_t dX_t + \widehat{\eta}_t dN_t,$$

and it hedges the claim payoff C, *i.e.*

$$V_t = \widehat{\phi}_t X_t + \widehat{\eta}_t N_t = \mathbb{E}^*\left[\mathrm{e}^{-\int_t^T r_s ds} C \,\middle|\, \mathcal{F}_t \right], \qquad 0 \leqslant t \leqslant T. \tag{3.5.3}$$

Proof. In order to check that the portfolio allocation $\left(\widehat{\phi}_t, \widehat{\eta}_t\right)_{t\in[0,T]}$ hedges the claim payoff C it suffices to check that (3.5.3) holds since by (3.2.8) the price V_t at time $t \in [0,T]$ of the hedging portfolio satisfies

$$V_t = N_t\widehat{V}_t = \mathbb{E}^*\left[\mathrm{e}^{-\int_t^T r_s ds} C \,\middle|\, \mathcal{F}_t \right], \qquad 0 \leqslant t \leqslant T.$$

Next, we show that the portfolio allocation $\left(\widehat{\phi}_t, \widehat{\eta}_t\right)_{t\in[0,T]}$ is self-financing. By *numéraire invariance*, cf. *e.g.* page 184 of Protter, 2001, we have, using the relation $d\widehat{V}_t = \widehat{\phi}_t d\widehat{X}_t$ from (3.5.2),

$$\begin{aligned}
dV_t &= d\left(N_t\widehat{V}_t\right) \\
&= \widehat{V}_t dN_t + N_t d\widehat{V}_t + dN_t \cdot d\widehat{V}_t \\
&= \widehat{V}_t dN_t + N_t\widehat{\phi}_t d\widehat{X}_t + \widehat{\phi}_t dN_t \cdot d\widehat{X}_t \\
&= \widehat{\phi}_t\widehat{X}_t dN_t + N_t\widehat{\phi}_t d\widehat{X}_t + \widehat{\phi}_t dN_t \cdot d\widehat{X}_t + \left(\widehat{V}_t - \widehat{\phi}_t\widehat{X}_t\right) dN_t \\
&= \widehat{\phi}_t d\left(N_t\widehat{X}_t\right) + \widehat{\eta}_t dN_t \\
&= \widehat{\phi}_t dX_t + \widehat{\eta}_t dN_t.
\end{aligned}$$

□

We now consider an application to the forward Delta hedging of European-type options with payoff $C = N_T\widehat{g}\left(\widehat{X}_T\right)$ where $\widehat{g} : \mathbb{R} \longrightarrow \mathbb{R}$ and $\left(\widehat{X}_t\right)_{t\in\mathbb{R}_+}$ has the Markov property as in (3.4.2), where $\widehat{\sigma} : \mathbb{R}_+ \times \mathbb{R}$ is a *deterministic* function. Assuming that the function $\widehat{C}(t,x)$ defined by

$$\widehat{V}_t := \widehat{\mathbb{E}}\left[\widehat{g}\left(\widehat{X}_T\right) \mid \mathcal{F}_t\right] = \widehat{C}\left(t, \widehat{X}_t\right)$$

is $\mathscr{C}^2$ on $\mathbb{R}_+$, we have the following corollary of Proposition 3.16, which extends the Black-Scholes Delta hedging technique to the general change of numéraire setup.

Corollary 3.17 Letting $\widehat{\eta}_t = \widehat{C}\left(t, \widehat{X}_t\right) - \widehat{X}_t \dfrac{\partial\widehat{C}}{\partial x}\left(t, \widehat{X}_t\right)$, $0 \leqslant t \leqslant T$, the portfolio allocation

$$\left(\frac{\partial\widehat{C}}{\partial x}\left(t, \widehat{X}_t\right), \widehat{\eta}_t\right)_{t\in[0,T]}$$

with value

$$V_t = \widehat{\eta}_t N_t + X_t \frac{\partial\widehat{C}}{\partial x}\left(t, \widehat{X}_t\right), \qquad t \geqslant 0,$$

is self-financing and hedges the claim payoff $C = N_T \widehat{g}(\widehat{X}_T)$.

Proof. This result follows directly from Proposition 3.16 by noting that by Itô's formula, and the martingale property of $\widehat{V}_t$ under $\widehat{\mathbb{P}}$ the stochastic integral representation (3.5.2) is given by

$$\begin{aligned}
\widehat{V}_T &= \widehat{C} \\
&= \widehat{g}(\widehat{X}_T) \\
&= \widehat{C}(0,\widehat{X}_0) + \int_0^T \frac{\partial \widehat{C}}{\partial t}(t,\widehat{X}_t)dt + \int_0^T \frac{\partial \widehat{C}}{\partial x}(t,\widehat{X}_t)d\widehat{X}_t \\
&\quad + \frac{1}{2}\int_0^T \frac{\partial^2 \widehat{C}}{\partial x^2}(t,\widehat{X}_t)|\widehat{\sigma}_t|^2 dt \\
&= \widehat{C}(0,\widehat{X}_0) + \int_0^T \frac{\partial \widehat{C}}{\partial x}(t,\widehat{X}_t)d\widehat{X}_t \\
&= \widehat{\mathbb{E}}\left[\frac{C}{N_T}\right] + \int_0^T \widehat{\phi}_t d\widehat{X}_t, \qquad 0 \leqslant t \leqslant T,
\end{aligned}$$

hence

$$\widehat{\phi}_t = \frac{\partial \widehat{C}}{\partial x}(t,\widehat{X}_t), \qquad 0 \leqslant t \leqslant T.$$

□

In the case of an exchange option with payoff function

$$C = (X_T - \kappa N_T)^+ = N_T\left(\widehat{X}_T - \kappa\right)^+$$

on the geometric Brownian motion $\left(\widehat{X}_t\right)_{t\in[0,T]}$ under $\widehat{\mathbb{P}}$ with

$$\widehat{\sigma}_t\left(\widehat{X}_t\right) = \widehat{\sigma}(t)\widehat{X}_t, \tag{3.5.4}$$

where $\left(\widehat{\sigma}(t)\right)_{t\in[0,T]}$ is a deterministic volatility function of time, we have the following corollary on the hedging of exchange options based on the Margrabe, 1978 formula (3.4.4).

Corollary 3.18 The decomposition

$$\mathbb{E}^*\left[\mathrm{e}^{-\int_t^T r_s ds}(X_T - \kappa N_T)^+ \,\middle|\, \mathcal{F}_t\right] = X_t \Phi_+^0\left(t,\widehat{X}_t\right) - \kappa N_t \Phi_-^0\left(t,\widehat{X}_t\right)$$

yields a self-financing portfolio allocation $\left(\Phi_+^0\left(t,\widehat{X}_t\right), -\kappa\Phi_-^0\left(t,\widehat{X}_t\right)\right)_{t\in[0,T]}$ in the assets (X_t, N_t), that hedges the claim payoff $C = (X_T - \kappa N_T)^+$.

Proof. We apply Corollary 3.17 and the relation

$$\frac{\partial \widehat{C}}{\partial x}(t,x) = \Phi_+^0(t,x), \qquad x \in \mathbb{R},$$

for the function $\widehat{C}(t,x) = x\Phi_+^0(t,x) - \kappa\Phi_-^0(t,x)$. □

Note that the Delta hedging method requires the computation of the function $\widehat{C}(t,x)$ and that of the associated finite differences, and may not apply to path-dependent claims.

Examples:

a) When the short rate process $(r(t))_{t\in[0,T]}$ is a *deterministic* function of time and $N_t = \mathrm{e}^{\int_t^T r(s)ds}$, Corollary 3.18 yields the usual Black-Scholes hedging strategy

$$\begin{aligned}&\left(\Phi_+(t,\widehat{X}_t), -\kappa\,\mathrm{e}^{\int_0^T r(s)ds}\Phi_-(t,X_t)\right)_{t\in[0,T]}\\ &= \left(\Phi_+^0\big(t,\mathrm{e}^{\int_t^T r(s)ds}\widehat{X}_t\big), -\kappa\,\mathrm{e}^{\int_0^T r(s)ds}\Phi_-^0\big(t,\mathrm{e}^{\int_t^T r(s)ds}X_t\big)\right)_{t\in[0,T]},\end{aligned}$$

in the assets $\left(X_t, \mathrm{e}^{\int_0^t r(s)ds}\right)$, that hedges the claim payoff $C = (X_T-\kappa)^+$, with

$$\Phi_+(t,x) := \Phi\left(\frac{\log(x/\kappa)+\int_t^T r(s)ds+(T-t)\sigma^2/2}{\sigma\sqrt{T-t}}\right),$$

and

$$\Phi_-(t,x) := \Phi\left(\frac{\log(x/\kappa)+\int_t^T r(s)ds-(T-t)\sigma^2/2}{\sigma\sqrt{T-t}}\right).$$

b) In case $N_t = P(t,T)$ and $X_t = P(t,S)$, $0 \leqslant t \leqslant T < S$, Corollary 3.18 shows that when $\big(\widehat{X}_t\big)_{t\in[0,T]}$ is modeled as the geometric Brownian motion (3.5.4) under $\widehat{\mathbb{P}}$, the bond call option with payoff $(P(T,S)-\kappa)^+$ can be hedged as

$$\mathbb{E}^*\left[\mathrm{e}^{-\int_t^T r_s ds}(P(T,S)-\kappa)^+ \,\Big|\, \mathscr{F}_t\right] = P(t,S)\Phi_+\big(t,\widehat{X}_t\big) - \kappa P(t,T)\Phi_-\big(t,\widehat{X}_t\big)$$

by the self-financing portfolio allocation

$$\left(\Phi_+\big(t,\widehat{X}_t\big), -\kappa\Phi_-\big(t,\widehat{X}_t\big)\right)_{t\in[0,T]}$$

in the assets $(P(t,S),P(t,T))$, *i.e.* one needs to hold the quantity $\Phi_+\big(t,\widehat{X}_t\big)$ of the bond maturing at time S, and to short a quantity $\kappa\Phi_-\big(t,\widehat{X}_t\big)$ of the bond maturing at time T.

Exercises

Exercise 3.1 Let $(B_t)_{t\in\mathbb{R}_+}$ be a standard Brownian motion started at 0 under the risk-neutral probability measure $\mathbb{P}^*$. Consider a numéraire $(N_t)_{t\in\mathbb{R}_+}$ given by

$$N_t := N_0\,\mathrm{e}^{\eta B_t-\eta^2 t/2}, \qquad t \geqslant 0,$$

and a risky asset $(X_t)_{t\in\mathbb{R}_+}$ given by

$$X_t := X_0\,\mathrm{e}^{\sigma B_t-\sigma^2 t/2}, \qquad t \geqslant 0,$$

in a market with risk-free interest rate $r=0$. Let $\widehat{\mathbb{P}}$ denote the forward measure relative to the numéraire $(N_t)_{t\in\mathbb{R}_+}$, under which the process $\widehat{X}_t := X_t/N_t$ of forward prices is known to be a martingale.

a) Using the Itô formula, compute

$$\begin{aligned}d\widehat{X}_t &= d\left(\frac{X_t}{N_t}\right)\\ &= \frac{X_0}{N_0}d\left(\mathrm{e}^{(\sigma-\eta)B_t-(\sigma^2-\eta^2)t/2}\right).\end{aligned}$$

b) Explain why the exchange option price $\mathbb{E}^*[(X_T-\lambda N_T)^+]$ at time 0 has the Black-Scholes form

$$\begin{aligned}&\mathbb{E}^*[(X_T-\lambda N_T)^+] \\ &= X_0\Phi\left(\frac{\log\big(\widehat{X}_0/\lambda\big)}{\widehat{\sigma}\sqrt{T}}+\frac{\widehat{\sigma}\sqrt{T}}{2}\right) - \lambda N_0\Phi\left(\frac{\log\big(\widehat{X}_0/\lambda\big)}{\widehat{\sigma}\sqrt{T}}-\frac{\widehat{\sigma}\sqrt{T}}{2}\right).\end{aligned} \tag{3.5.5}$$

Hints:

(i) Use the change of numéraire identity

$$\mathbb{E}^*[(X_T - \lambda N_T)^+] = N_0 \widehat{\mathbb{E}}\big[\big(\widehat{X}_T - \lambda\big)^+\big].$$

(ii) The forward price $\widehat{X}_t$ is a martingale under the forward measure $\widehat{\mathbb{P}}$ relative to the numéraire $(N_t)_{t\in\mathbb{R}_+}$.

c) Give the value of $\widehat{\sigma}$ in terms of σ and η.

Exercise 3.2 Let $\big(B_t^{(1)}\big)_{t\in\mathbb{R}_+}$ and $\big(B_t^{(2)}\big)_{t\in\mathbb{R}_+}$ be correlated standard Brownian motions started at 0 under the risk-neutral probability measure $\mathbb{P}^*$, with correlation $\mathrm{Corr}\big(B_s^{(1)}, B_t^{(2)}\big) = \rho \min(s,t)$, *i.e.* $dB_t^{(1)} \bullet dB_t^{(2)} = \rho dt$. Consider two asset prices $\big(S_t^{(1)}\big)_{t\in\mathbb{R}_+}$ and $\big(S_t^{(2)}\big)_{t\in\mathbb{R}_+}$ given by the geometric Brownian motions

$$S_t^{(1)} := S_0^{(1)} \mathrm{e}^{rt+\sigma B_t^{(1)} - \sigma^2 t/2}, \quad \text{and} \quad S_t^{(2)} := S_0^{(2)} \mathrm{e}^{rt+\eta B_t^{(2)} - \eta^2 t/2}, \qquad t \geqslant 0.$$

Let $\widehat{\mathbb{P}}_2$ denote the forward measure with numéraire $(N_t)_{t\in\mathbb{R}_+} := \big(S_t^{(2)}\big)_{t\in\mathbb{R}_+}$ and Radon-Nikodym density

$$\frac{\mathrm{d}\widehat{\mathbb{P}}_2}{\mathrm{d}\mathbb{P}^*} = \mathrm{e}^{-rT} \frac{S_T^{(2)}}{S_0^{(2)}} = \mathrm{e}^{\eta B_T^{(2)} - \eta^2 T/2}.$$

a) Using the Girsanov Theorem 3.7, determine the shifts $\big(\widehat{B}_t^{(1)}\big)_{t\in\mathbb{R}_+}$ and $\big(\widehat{B}_t^{(2)}\big)_{t\in\mathbb{R}_+}$ of $\big(B_t^{(1)}\big)_{t\in\mathbb{R}_+}$ and $\big(B_t^{(2)}\big)_{t\in\mathbb{R}_+}$ which are standard Brownian motions under $\widehat{\mathbb{P}}_2$.

b) Using the Itô formula, compute

$$\begin{aligned} d\widehat{S}_t^{(1)} &= d\left(\frac{S_t^{(1)}}{S_t^{(2)}}\right) \\ &= \frac{S_0^{(1)}}{S_0^{(2)}} d\big(\mathrm{e}^{\sigma B_t^{(1)} - \eta B_t^{(2)} - (\sigma^2 - \eta^2)t/2}\big), \end{aligned}$$

and write the answer in terms of the martingales $d\widehat{B}_t^{(1)}$ and $d\widehat{B}_t^{(2)}$.

c) Using change of numéraire, explain why the exchange option price

$$\mathrm{e}^{-rT}\, \mathbb{E}^*\left[\big(S_T^{(1)} - \lambda S_T^{(2)}\big)^+\right]$$

at time 0 has the Black-Scholes form

$$\begin{aligned} \mathrm{e}^{-rT}\, \mathbb{E}^*\left[\big(S_T^{(1)} - \lambda S_T^{(2)}\big)^+\right] &= S_0^{(1)} \Phi\left(\frac{\log\big(\widehat{S}_0^{(1)}/\lambda\big)}{\widehat{\sigma}\sqrt{T}} + \frac{\widehat{\sigma}\sqrt{T}}{2}\right) \\ &\quad -\lambda S_0^{(2)} \Phi\left(\frac{\log\big(\widehat{S}_0^{(1)}/\lambda\big)}{\widehat{\sigma}\sqrt{T}} - \frac{\widehat{\sigma}\sqrt{T}}{2}\right), \end{aligned}$$

where the value of $\widehat{\sigma}$ can be expressed in terms of σ and η.

Exercise 3.3 Consider two zero-coupon bond prices of the form $P(t,T) = F(t,r_t)$ and $P(t,S) = G(t,r_t)$, where $(r_t)_{t\in\mathbb{R}_+}$ is a short-term interest rate process. Taking $N_t := P(t,T)$ as a numéraire defining the forward measure $\widehat{\mathbb{P}}$, compute the dynamics of $(P(t,S))_{t\in[0,T]}$ under $\widehat{\mathbb{P}}$ using a standard Brownian motion $\big(\widehat{W}_t\big)_{t\in[0,T]}$ under $\widehat{\mathbb{P}}$.

Exercise 3.4 Forward contracts. Using a change of numéraire argument for the numéraire $N_t := P(t,T)$, $t \in [0,T]$, compute the price at time $t \in [0,T]$ of a forward (or future) contract with payoff $P(T,S)-K$ in a bond market with short-term interest rate $(r_t)_{t\in\mathbb{R}_+}$. How would you hedge this forward contract?

Exercise 3.5 (Question 2.7 page 17 of Downes, Joshi, and Denson, 2008). Consider a price process $(S_t)_{t\in\mathbb{R}_+}$ given by $dS_t = rS_t dt + \sigma S_t dB_t$ under the risk-neutral probability measure $\mathbb{P}^*$, where $r \in \mathbb{R}$ and $\sigma > 0$, and the option with payoff

$$S_T(S_T-K)^+ = \mathrm{Max}(S_T(S_T-K),0)$$

at maturity T.

a) Show that the option payoff can be rewritten as

$$(S_T(S_T-K))^+ = N_T(S_T-K)^+$$

for a suitable choice of numéraire process $(N_t)_{t\in[0,T]}$.

b) Rewrite the option price $\mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[(S_T(S_T-K))^+ \mid \mathscr{F}_t\right]$ using a forward measure $\widehat{\mathbb{P}}$ and a change of numéraire argument.

c) Find the dynamics of $(S_t)_{t\in\mathbb{R}_+}$ under the forward measure $\widehat{\mathbb{P}}$.

d) Price the option with payoff

$$S_T(S_T-K)^+ = \mathrm{Max}(S_T(S_T-K),0)$$

at time $t\in[0,T]$ using the Black-Scholes formula.

4. Pricing of Interest Rate Derivatives

Interest rate derivatives are option contracts whose payoffs can be based on fixed-income securities such as bonds, or on cash flows exchanged in *e.g.* interest rate swaps. In this chapter we consider the pricing and hedging of interest rate and fixed income derivatives such as bond options, caplets, caps and swaptions, using the change of numéraire technique and forward measures.

4.1 Forward Measures and Tenor Structure

The maturity dates are arranged according to a discrete *tenor structure*

$$\{0 = T_0 < T_1 < T_2 < \cdots < T_n\}.$$

A sample of forward interest rate curve data is given in Table 4.1, which contains the values of $(T_1, T_2, \ldots, T_{23})$ and of $\{f(t, t+T_i, t+T_i+\delta)\}_{i=1,2,\ldots,23}$, with $t = 07/05/2003$ and $\delta =$ six months.

Maturity	2D	1W	1M	2M	3M	1Y	2Y	3Y	4Y	5Y	6Y	7Y
Rate (%)	2.55	2.53	2.56	2.52	2.48	2.34	2.49	2.79	3.07	3.31	3.52	3.71
Maturity	8Y	9Y	10Y	11Y	12Y	13Y	14Y	15Y	20Y	25Y	30Y	
Rate (%)	3.88	4.02	4.14	4.23	4.33	4.40	4.47	4.54	4.74	4.83	4.86	

Table 4.1: Forward rates arranged according to a tenor structure.

Recall that by definition of $P(t,T_i)$ and absence of arbitrage the discounted bond price process

$$t \mapsto \mathrm{e}^{-\int_0^t r_s ds} P(t,T_i), \qquad 0 \leqslant t \leqslant T_i,$$

is an $\mathscr{F}_t$-martingale under the probability measure $\mathbb{P}^* = \mathbb{P}$, hence it satisfies the Assumption 1 on page 61 for $i = 1,2,\ldots,n$. As a consequence the bond price process can be taken as a numéraire

$$N_t^{(i)} := P(t,T_i), \qquad 0 \leqslant t \leqslant T_i,$$

in the definition

$$\frac{d\widehat{\mathbb{P}}_i}{d\mathbb{P}^*} = \frac{1}{P(0,T_i)} \mathrm{e}^{-\int_0^{T_i} r_s ds} \tag{4.1.1}$$

of the *forward measure* $\widehat{\mathbb{P}}_i$, see Definition 3.1. The following proposition will allow us to price contingent claims using the forward measure $\widehat{\mathbb{P}}_i$, it is a direct consequence of Proposition 3.5, noting that here we have $P(T_i,T_i) = 1$.

Proposition 4.1 For all sufficiently integrable random variables C we have

$$\mathbb{E}^*\left[C \mathrm{e}^{-\int_t^{T_i} r_s ds} \,\middle|\, \mathscr{F}_t \right] = P(t,T_i)\widehat{\mathbb{E}}_i[C \mid \mathscr{F}_t], \quad 0 \leqslant t \leqslant T_i, \quad i = 1,2,\ldots,n. \tag{4.1.2}$$

Recall that by Proposition 3.4, the deflated process

$$t \mapsto \frac{P(t,T_j)}{P(t,T_i)}, \qquad 0 \leqslant t \leqslant \min(T_i,T_j),$$

is an $\mathscr{F}_t$-martingale under $\widehat{\mathbb{P}}_i$ for all $T_i, T_j \geqslant 0$.

In the sequel we assume as in (1.4.8) that the dynamics of the bond price $P(t,T_i)$ is given by

$$\frac{dP(t,T_i)}{P(t,T_i)} = r_t dt + \zeta_i(t) dW_t, \tag{4.1.3}$$

for $i = 1,2,\ldots,n$, where $(W_t)_{t\in\mathbb{R}_+}$ is a standard Brownian motion under $\mathbb{P}^*$ and $(r_t)_{t\in\mathbb{R}_+}$ and $(\zeta_i(t))_{t\in\mathbb{R}_+}$ are adapted processes with respect to the filtration $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$ generated by $(W_t)_{t\in\mathbb{R}_+}$, *i.e.*

$$\boxed{P(t,T_i) = P(0,T_i) \exp\left(\int_0^t r_s ds + \int_0^t \zeta_i(s) dW_s - \frac{1}{2}\int_0^t |\zeta_i(s)|^2 ds \right),}$$

$0 \leqslant t \leqslant T_i$, $i = 1,2,\ldots,n$.

Forward Brownian motions

Proposition 4.2 For all $i = 1,2,\ldots,n$, the process

$$\widehat{W}_t^i := W_t - \int_0^t \zeta_i(s)ds, \qquad 0 \leqslant t \leqslant T_i, \tag{4.1.4}$$

is a standard Brownian motion under the forward measure $\widehat{\mathbb{P}}_i$.

Proof. The Girsanov Proposition 3.7 applied to the numéraire

$$N_t^{(i)} := P(t,T_i), \qquad 0 \leqslant t \leqslant T_i,$$

as in (3.2.13), shows that

$$\begin{aligned} d\widehat{W}_t^i &:= dW_t - \frac{1}{N_t^{(i)}} dN_t^{(i)} \bullet dW_t \\ &= dW_t - \frac{1}{P(t,T_i)} dP(t,T_i) \bullet dW_t \\ &= dW_t - \frac{1}{P(t,T_i)} (P(t,T_i) r_t dt + \zeta_i(t) P(t,T_i) dW_t) \bullet dW_t \\ &= dW_t - \zeta_i(t)dt, \end{aligned}$$

is a standard Brownian motion under the forward measure $\widehat{\mathbb{P}}_i$ for all $i = 1,2,\ldots,n$. □

We have

$$d\widehat{W}_t^i = dW_t - \zeta_i(t)dt, \qquad i = 1,2,\ldots,n, \tag{4.1.5}$$

and

$$d\widehat{W}_t^j = dW_t - \zeta_j(t)dt = d\widehat{W}_t^i + (\zeta_i(t) - \zeta_j(t))dt, \quad i,j = 1,2,\ldots,n,$$

which shows that $(\widehat{W}_t^j)_{t\in\mathbb{R}_+}$ has drift $(\zeta_i(t) - \zeta_j(t))_{t\in\mathbb{R}_+}$ under $\widehat{\mathbb{P}}_i$.

Bond price dynamics under the forward measure

In order to apply Proposition 4.1 and to compute the price

$$\mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_i} r_s ds} C \,\middle|\, \mathcal{F}_t \right] = P(t,T_i) \widehat{\mathbb{E}}_i[C \mid \mathcal{F}_t],$$

of a random claim payoff C, it can be useful to determine the dynamics of the underlying variables r_t, $f(t,T,S)$, and $P(t,T)$ via their stochastic differential equations written under the forward measure $\widehat{\mathbb{P}}_i$.

As a consequence of Proposition 4.2 and (4.1.3), the dynamics of $t \mapsto P(t,T_j)$ under $\widehat{\mathbb{P}}_i$ is given by

$$\frac{dP(t,T_j)}{P(t,T_j)} = r_t dt + \zeta_i(t)\zeta_j(t)dt + \zeta_j(t)d\widehat{W}_t^i, \quad i,j = 1,2,\ldots,n, \tag{4.1.6}$$

where $(\widehat{W}_t^i)_{t\in\mathbb{R}_+}$ is a standard Brownian motion under $\widehat{\mathbb{P}}_i$, and we have

$$\begin{aligned} &P(t,T_j) \\ &= P(0,T_j) \exp\left(\int_0^t r_s ds + \int_0^t \zeta_j(s) dW_s - \frac{1}{2} \int_0^t |\zeta_j(s)|^2 ds \right) \qquad \text{[under } \mathbb{P}^*\text{]} \end{aligned}$$

$$
\begin{aligned}
&= P(0,T_j)\exp\left(\int_0^t r_s ds + \int_0^t \zeta_j(s)d\widehat{W}_s^j + \frac{1}{2}\int_0^t |\zeta_j(s)|^2 ds\right) \qquad [\text{under } \widehat{\mathbb{P}}_j] \\
&= P(0,T_j)\exp\left(\int_0^t r_s ds + \int_0^t \zeta_j(s)d\widehat{W}_s^i + \int_0^t \zeta_j(s)\zeta_i(s)ds - \frac{1}{2}\int_0^t |\zeta_j(s)|^2 ds\right) \ [\text{under } \widehat{\mathbb{P}}_i] \\
&= P(0,T_j)\exp\left(\int_0^t r_s ds + \int_0^t \zeta_j(s)d\widehat{W}_s^i - \frac{1}{2}\int_0^t |\zeta_j(s)-\zeta_i(s)|^2 ds + \frac{1}{2}\int_0^t |\zeta_i(s)|^2 ds\right),
\end{aligned}
$$

$t \in [0,T_j]$, $i,j = 1,2,\ldots,n$. Consequently, the forward price $P(t,T_j)/P(t,T_i)$ can be written as

$$
\begin{aligned}
&\frac{P(t,T_j)}{P(t,T_i)} \\
&= \frac{P(0,T_j)}{P(0,T_i)}\exp\left(\int_0^t (\zeta_j(s)-\zeta_i(s))d\widehat{W}_s^j + \frac{1}{2}\int_0^t |\zeta_j(s)-\zeta_i(s)|^2 ds\right) \qquad [\text{under } \widehat{\mathbb{P}}_j] \\
&= \frac{P(0,T_j)}{P(0,T_i)}\exp\left(\int_0^t (\zeta_j(s)-\zeta_i(s))d\widehat{W}_s^i - \frac{1}{2}\int_0^t |\zeta_i(s)-\zeta_j(s)|^2 ds\right), \qquad [\text{under } \widehat{\mathbb{P}}_i]
\end{aligned}
\tag{4.1.7}
$$

$t \in [0,\min(T_i,T_j)]$, $i,j = 1,2,\ldots,n$, which also follows from Proposition 3.8.

Short rate dynamics under the forward measure

In case the short rate process $(r_t)_{t\in\mathbb{R}_+}$ is given as the (Markovian) solution to the stochastic differential equation

$$dr_t = \mu(t,r_t)dt + \sigma(t,r_t)dW_t,$$

by (4.1.5) its dynamics will be given under $\widehat{\mathbb{P}}_i$ by

$$
\begin{aligned}
dr_t &= \mu(t,r_t)dt + \sigma(t,r_t)\big(\zeta_i(t)dt + d\widehat{W}_t^i\big) \\
&= \mu(t,r_t)dt + \sigma(t,r_t)\zeta_i(t)dt + \sigma(t,r_t)d\widehat{W}_t^i.
\end{aligned}
\tag{4.1.8}
$$

In the case of the Vašíček, 1977 model, by (1.4.11) we have

$$dr_t = (a - br_t)dt + \sigma dW_t,$$

and

$$\zeta_i(t) = -\frac{\sigma}{b}\big(1 - \mathrm{e}^{-b(T_i-t)}\big), \qquad 0 \leqslant t \leqslant T_i,$$

hence from (4.1.8) we have

$$d\widehat{W}_t^i = dW_t - \zeta_i(t)dt = dW_t + \frac{\sigma}{b}\big(1 - \mathrm{e}^{-b(T_i-t)}\big)dt, \tag{4.1.9}$$

and

$$dr_t = (a - br_t)dt - \frac{\sigma^2}{b}\big(1 - \mathrm{e}^{-b(T_i-t)}\big)dt + \sigma d\widehat{W}_t^i \tag{4.1.10}$$

and we obtain

$$\frac{dP(t,T_i)}{P(t,T_i)} = r_t dt + \frac{\sigma^2}{b^2}\big(1 - \mathrm{e}^{-b(T_i-t)}\big)^2 dt - \frac{\sigma}{b}\big(1 - \mathrm{e}^{-b(T_i-t)}\big)d\widehat{W}_t^i,$$

from (1.4.11).

4.2 Bond Options

The next proposition can be obtained as an application of the Margrabe formula (3.4.4) of Proposition 3.15 by taking $X_t = P(t,T_j)$, $N_t^{(i)} = P(t,T_i)$, and $\widehat{X}_t = X_t/N_t^{(i)} = P(t,T_j)/P(t,T_i)$. In the Vasicek model, this formula has been first obtained in Jamshidian, 1989.

We work with a standard Brownian motion $(W_t)_{t\in\mathbb{R}_+}$ under $\mathbb{P}^*$, generating the filtration $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$, and an $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$-adapted short rate process $(r_t)_{t\in\mathbb{R}_+}$.

Proposition 4.3 Let $0 \leqslant T_i \leqslant T_j$ and assume as in (1.4.8) that the dynamics of the bond prices $P(t,T_i)$, $P(t,T_j)$ under $\mathbb{P}^*$ are given by

$$\frac{dP(t,T_i)}{P(t,T_i)} = r_t dt + \zeta_i(t) dW_t, \qquad \frac{dP(t,T_j)}{P(t,T_j)} = r_t dt + \zeta_j(t) dW_t,$$

where $(\zeta_i(t))_{t\in\mathbb{R}_+}$ and $(\zeta_j(t))_{t\in\mathbb{R}_+}$ are *deterministic* volatility functions. Then, the price of a bond call option on $P(T_i,T_j)$ with payoff

$$C := (P(T_i,T_j) - \kappa)^+$$

can be written as

$$\begin{aligned}
&\mathbb{E}^*\left[e^{-\int_t^{T_i} r_s ds} (P(T_i,T_j) - \kappa)^+ \,\middle|\, \mathscr{F}_t \right] \qquad (4.2.1)\\
&= P(t,T_j)\Phi\left(\frac{v(t,T_i)}{2} + \frac{1}{v(t,T_i)} \log \frac{P(t,T_j)}{\kappa P(t,T_i)} \right)\\
&\quad -\kappa P(t,T_i)\Phi\left(-\frac{v(t,T_i)}{2} + \frac{1}{v(t,T_i)} \log \frac{P(t,T_j)}{\kappa P(t,T_i)} \right),
\end{aligned}$$

where $v^2(t,T_i) := \int_t^{T_i} |\zeta_i(s) - \zeta_j(s)|^2 ds$ and

$$\Phi(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-y^2/2} dy, \qquad x \in \mathbb{R},$$

is the Gaussian cumulative distribution function.

Proof. First, we note that using $N_t^{(i)} := P(t,T_i)$ as a numéraire the price of a bond call option on $P(T_i,T_j)$ with payoff $F = (P(T_i,T_j) - \kappa)^+$ can be written from Proposition 3.5 using the forward measure $\widehat{\mathbb{P}}_i$, or directly by (3.2.9), as

$$\mathbb{E}^*\left[e^{-\int_t^{T_i} r_s ds} (P(T_i,T_j) - \kappa)^+ \,\middle|\, \mathscr{F}_t \right] = P(t,T_i)\widehat{\mathbb{E}}_i\big[(P(T_i,T_j) - \kappa)^+ \,\big|\, \mathscr{F}_t \big]. \qquad (4.2.2)$$

Next, by (4.1.7) or by solving (3.2.15) in Proposition 3.8 we can write $P(T_i,T_j)$ as the geometric Brownian motion

$$\begin{aligned}
P(T_i,T_j) &= \frac{P(T_i,T_j)}{P(T_i,T_i)}\\
&= \frac{P(t,T_j)}{P(t,T_i)} \exp\left(\int_t^{T_i} (\zeta_j(s) - \zeta_i(s)) d\widehat{W}_s^i - \frac{1}{2} \int_t^{T_i} |\zeta_i(s) - \zeta_j(s)|^2 ds \right),
\end{aligned}$$

under the forward measure $\widehat{\mathbb{P}}_i$, and rewrite (4.2.2) as

$$\begin{aligned}
&\mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_i} r_s ds}(P(T_i,T_j)-\kappa)^+ \,\Big|\, \mathscr{F}_t\right] \\
&= P(t,T_i)\widehat{\mathbb{E}}_i\left[\left(\frac{P(t,T_j)}{P(t,T_i)}\mathrm{e}^{\int_t^{T_i}(\zeta_j(s)-\zeta_i(s))d\widehat{W}_s^i-\frac{1}{2}\int_t^{T_i}|\zeta_i(s)-\zeta_j(s)|^2ds}-\kappa\right)^+ \,\Bigg|\, \mathscr{F}_t\right] \\
&= \widehat{\mathbb{E}}_i\left[\left(P(t,T_j)\,\mathrm{e}^{\int_t^{T_i}(\zeta_j(s)-\zeta_i(s))d\widehat{W}_s^i-\frac{1}{2}\int_t^{T_i}|\zeta_i(s)-\zeta_j(s)|^2ds}-\kappa P(t,T_i)\right)^+ \,\Bigg|\, \mathscr{F}_t\right].
\end{aligned}$$

Since $(\zeta_i(s))_{s\in[0,T_i]}$ and $(\zeta_j(s))_{s\in[0,T_j]}$ in (4.1.3) are deterministic volatility functions, $P(T_i,T_j)$ is a lognormal random variable given $\mathscr{F}_t$ under $\widehat{\mathbb{P}}_i$ and we can price the bond option by the zero-rate Black-Scholes formula

$$\mathrm{Bl}\big(P(t,T_j),\kappa P(t,T_i),v(t,T_i)/\sqrt{T_i-t},0,T_i-t\big)$$

with underlying asset price $P(t,T_j)$, strike level $\kappa P(t,T_i)$, volatility parameter

$$\frac{v(t,T_i)}{\sqrt{T_i-t}}=\sqrt{\frac{\int_t^{T_i}|\zeta_i(s)-\zeta_j(s)|^2ds}{T_i-t}},$$

time to maturity T_i-t, and zero interest rate, which yields (4.2.1). □

Note that from Corollary 3.17 the decomposition (4.2.1) gives the self-financing portfolio in the assets $P(t,T_i)$ and $P(t,T_j)$ for the claim with payoff $(P(T_i,T_j)-\kappa)^+$.

In the Vasicek case the above bond option price could also be computed from the joint distribution of $\left(r_T,\int_t^T r_s ds\right)$, which is Gaussian, or from the dynamics (4.1.6)-(4.1.10) of $P(t,T)$ and r_t under $\widehat{\mathbb{P}}_i$, see Kim, 2002 and § 8.3 of Privault, 2021.

4.3 Caplet Pricing

An interest rate caplet is an option contract that offers protection against the fluctuations of a variable (or floating) rate with respect to a fixed rate κ. The payoff of a LIBOR caplet on the yield (or spot forward rate) $L(T_i,T_i,T_{i+1})$ with strike level κ can be written as

$$(L(T_i,T_i,T_{i+1})-\kappa)^+,$$

and priced at time $t\in[0,T_i]$ from Proposition 3.5 using the forward measure $\widehat{\mathbb{P}}_{i+1}$ as

$$\begin{aligned}
&\mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_{i+1}} r_s ds}(L(T_i,T_i,T_{i+1})-\kappa)^+ \,\Big|\, \mathscr{F}_t\right] \\
&= P(t,T_{i+1})\widehat{\mathbb{E}}_{i+1}\left[(L(T_i,T_i,T_{i+1})-\kappa)^+ \mid \mathscr{F}_t\right],
\end{aligned} \tag{4.3.1}$$

by taking $N_t^{(i+1)}=P(t,T_{i+1})$ as a numéraire.

Proposition 4.4 The LIBOR rate

$$L(t,T_i,T_{i+1}):=\frac{1}{T_{i+1}-T_i}\left(\frac{P(t,T_i)}{P(t,T_{i+1})}-1\right),\quad 0\leqslant t\leqslant T_i<T_{i+1},$$

is a martingale under the forward measure $\widehat{\mathbb{P}}_{i+1}$ defined in (4.1.1).

Proof. The LIBOR rate $L(t,T_i,T_{i+1})$ is a deflated process according to the forward numéraire process $(P(t,T_{i+1}))_{t\in[0,T_{i+1}]}$. Therefore, by Proposition 3.4 it is a martingale under $\widehat{\mathbb{P}}_{i+1}$. □

The caplet on $L(T_i,T_i,T_{i+1})$ can be priced at time $t \in [0,T_i]$ as

$$\begin{aligned}
&\mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_{i+1}} r_s ds}(L(T_i,T_i,T_{i+1})-\kappa)^+ \,\Big|\, \mathcal{F}_t \right] \qquad (4.3.2)\\
&= \mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_{i+1}} r_s ds}\left(\frac{1}{T_{i+1}-T_i}\left(\frac{P(t,T_i)}{P(t,T_{i+1})} - 1 \right) - \kappa \right)^+ \,\Bigg|\, \mathcal{F}_t \right],
\end{aligned}$$

where the discount factor is counted from the settlement date T_{i+1}. The next pricing formula (4.3.4) allows us to price and hedge a caplet using a portfolio based on the bonds $P(t,T_i)$ and $P(t,T_{i+1})$, cf. (4.3.8) below, when $L(t,T_i,T_{i+1})$ is modeled in the BGM model of Section 2.6.

Proposition 4.5 (Black LIBOR caplet formula). Assume that $L(t,T_i,T_{i+1})$ is modeled in the BGM model as

$$\frac{dL(t,T_i,T_{i+1})}{L(t,T_i,T_{i+1})} = \gamma_i(t)d\widehat{W}_t^{i+1}, \qquad (4.3.3)$$

$0 \leqslant t \leqslant T_i$, $i = 1,2,\ldots,n-1$, where $\gamma_i(t)$ is a deterministic volatility function of time $t \in [0,T_i]$, $i = 1,2,\ldots,n-1$. The caplet on $L(T_i,T_i,T_{i+1})$ with strike level κ is priced at time $t \in [0,T_i]$ as

$$\begin{aligned}
&(T_{i+1}-T_i)\,\mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_{i+1}} r_s ds}(L(T_i,T_i,T_{i+1})-\kappa)^+ \,\Big|\, \mathcal{F}_t \right] \qquad (4.3.4)\\
&= (P(t,T_i)-P(t,T_{i+1}))\Phi(d_+(t,T_i)) - \kappa(T_{i+1}-T_i)P(t,T_{i+1})\Phi(d_-(t,T_i)),
\end{aligned}$$

$0 \leqslant t \leqslant T_i$, where

$$d_+(t,T_i) = \frac{\log(L(t,T_i,T_{i+1})/\kappa) + (T_i - t)\sigma_i^2(t,T_i)/2}{\sigma_i(t,T_i)\sqrt{T_i - t}}, \qquad (4.3.5)$$

and

$$d_-(t,T_i) = \frac{\log(L(t,T_i,T_{i+1})/\kappa) - (T_i - t)\sigma_i^2(t,T_i)/2}{\sigma_i(t,T_i)\sqrt{T_i - t}}, \qquad (4.3.6)$$

and

$$|\sigma_i(t,T_i)|^2 = \frac{1}{T_i - t}\int_t^{T_i} |\gamma_i|^2(s)ds. \qquad (4.3.7)$$

Proof. Taking $P(t,T_{i+1})$ as a numéraire, the forward price

$$\widehat{X}_t := \frac{P(t,T_i)}{P(t,T_{i+1})} = 1 + (T_{i+1}-T_i)L(T_i,T_i,T_{i+1})$$

and the forward LIBOR rate process $(L(t,T_i,T_{i+1}))_{t\in[0,T_i]}$ are martingales under $\widehat{\mathbb{P}}_{i+1}$ by Proposition 4.4, $i = 1,2,\ldots,n-1$. More precisely, by (4.3.3) we have

$$L(T_i,T_i,T_{i+1}) = L(t,T_i,T_{i+1})\exp\left(\int_t^{T_i} \gamma_i(s)d\widehat{W}_s^{i+1} - \frac{1}{2}\int_t^{T_i} |\gamma_i(s)|^2 ds \right),$$

$0 \leqslant t \leqslant T_i$, *i.e.* $t \mapsto L(t,T_i,T_{i+1})$ is a geometric Brownian motion with time-dependent volatility

$\gamma_i(t)$ under $\widehat{\mathbb{P}}_{i+1}$. Hence by (4.3.1), since $N^{(i+1)}_{T_{i+1}} = 1$, we have

$$\begin{aligned}
&\mathbb{E}^*\left[e^{-\int_t^{T_{i+1}} r_s ds}(L(T_i,T_i,T_{i+1})-\kappa)^+ \,\Big|\, \mathscr{F}_t \right] \\
&\quad = P(t,T_{i+1})\widehat{\mathbb{E}}_{i+1}\left[(L(T_i,T_i,T_{i+1})-\kappa)^+ \,\big|\, \mathscr{F}_t\right] \\
&\quad = P(t,T_{i+1})\left(L(t,T_i,T_{i+1})\Phi(d_+(t,T_i)) - \kappa\Phi(d_-(t,T_i))\right) \\
&\quad = P(t,T_{i+1})\mathrm{Bl}(L(t,T_i,T_{i+1}),\kappa,\sigma_i(t,T_i),0,T_i-t),
\end{aligned}$$

$t \in [0,T_i]$, where

$$\mathrm{Bl}(x,\kappa,\sigma,0,\tau) = x\Phi(d_+(t,T_i)) - \kappa\Phi(d_-(t,T_i))$$

is the zero-interest rate Black-Scholes function, with

$$|\sigma_i(t,T_i)|^2 = \frac{1}{T_i-t}\int_t^{T_i} |\gamma_i|^2(s)ds.$$

Therefore, we obtain

$$\begin{aligned}
&(T_{i+1}-T_i)\,\mathbb{E}^*\left[e^{-\int_t^{T_{i+1}} r_s ds}(L(T_i,T_i,T_{i+1})-\kappa)^+ \,\Big|\, \mathscr{F}_t \right] \\
&\quad = P(t,T_{i+1})L(t,T_i,T_{i+1})\Phi(d_+(t,T_i)) - \kappa P(t,T_{i+1})\Phi(d_-(t,T_i)) \\
&\quad = P(t,T_{i+1})\left(\frac{P(t,T_i)}{P(t,T_{i+1})} - 1\right)\Phi(d_+(t,T_i)) \\
&\qquad -\kappa(T_{i+1}-T_i)P(t,T_{i+1})\Phi(d_-(t,T_i)),
\end{aligned}$$

which yields (4.3.4). □

In addition, from Corollary 3.17 we obtain the self-financing portfolio strategy

$$(\Phi(d_+(t,T_i)), -\Phi(d_+(t,T_i)) - \kappa(T_{i+1}-T_i)\Phi(d_-(t,T_i))) \tag{4.3.8}$$

in the bonds priced $(P(t,T_i),P(t,T_{i+1}))$ with maturities T_i and T_{i+1}, cf. Corollary 3.18 and Privault and Teng, 2012.

The formula (4.3.4) can be applied to options on underlying futures or forward contracts on commodities whose prices are modeled according to (4.3.3), as in the next corollary.

Corollary 4.6 (Black, 1976 formula). Let $L(t,T_i,T_{i+1})$ be modeled as in (4.3.3) and let the bond price $P(t,T_{i+1})$ be given as $P(t,T_{i+1}) = e^{-(T_{i+1}-t)r}$. Then, (4.3.4) becomes

$$e^{-(T_{i+1}-t)r}L(t,T_i,T_{i+1})\Phi(d_+(t,T_i)) - \kappa\, e^{-(T_{i+1}-t)r}\Phi(d_-(t,T_i)),$$

$0 \leqslant t \leqslant T_i$.

Floorlet pricing

The floorlet on $L(T_i,T_i,T_{i+1})$ with strike level κ is a contract with payoff $(\kappa - L(T_i,T_i,T_{i+1}))^+$. Floorlets are analog to put options and can be similarly priced by the call/put parity in the Black-Scholes formula.

Proposition 4.7 Assume that $L(t,T_i,T_{i+1})$ is modeled in the BGM model as in (4.3.3). The floorlet on $L(T_i,T_i,T_{i+1})$ with strike level κ is priced at time $t \in [0,T_i]$ as

$$(T_{i+1}-T_i)\,\mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_{i+1}} r_s ds}(\kappa - L(T_i,T_i,T_{i+1}))^+ \,\Big|\, \mathcal{F}_t\right] \tag{4.3.9}$$
$$= \kappa(T_{i+1}-T_i)P(t,T_{i+1})\Phi\big(-d_-(t,T_i)\big) - (P(t,T_i)-P(t,T_{i+1}))\Phi\big(-d_+(t,T_i)\big),$$

$0 \leqslant t \leqslant T_i$, where $d_+(t,T_i)$, $d_-(t,T_i)$ and $|\sigma_i(t,T_i)|^2$ are defined in (4.3.5)-(4.3.7).

Proof. We have

$$\begin{aligned}
&(T_{i+1}-T_i)\,\mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_{i+1}} r_s ds}(\kappa - L(T_i,T_i,T_{i+1}))^+ \,\Big|\, \mathcal{F}_t\right]\\
&= (T_{i+1}-T_i)P(t,T_{i+1})\widehat{\mathbb{E}}_{i+1}\left[(\kappa - L(T_i,T_i,T_{i+1}))^+ \mid \mathcal{F}_t\right]\\
&= (T_{i+1}-T_i)P(t,T_{i+1})\left(\kappa\Phi\big(-d_-(t,T_i)\big) - (T_{i+1}-T_i)L(t,T_i,T_{i+1})\Phi\big(-d_+(t,T_i)\big)\right)\\
&= (T_{i+1}-T_i)P(t,T_{i+1})\kappa\Phi\big(-d_-(t,T_i)\big) - (P(t,T_i)-P(t,T_{i+1}))\Phi\big(-d_+(t,T_i)\big),
\end{aligned}$$

$0 \leqslant t \leqslant T_i$. □

Cap pricing

More generally, one can consider interest rate caps that are relative to a given tenor structure $\{T_1,T_2,\ldots,T_n\}$, with discounted payoff

$$\sum_{k=i}^{j-1}(T_{k+1}-T_k)\,\mathrm{e}^{-\int_t^{T_{k+1}} r_s ds}(L(T_k,T_k,T_{k+1})-\kappa)^+.$$

Pricing formulas for interest rate caps are easily deduced from analog formulas for caplets, since the payoff of a cap can be decomposed into a sum of caplet payoffs. Thus, the cap price at time $t \in [0,T_i]$ is given by

$$\begin{aligned}
&\mathbb{E}^*\left[\sum_{k=i}^{j-1}(T_{k+1}-T_k)\,\mathrm{e}^{-\int_t^{T_{k+1}} r_s ds}(L(T_k,T_k,T_{k+1})-\kappa)^+ \,\Bigg|\, \mathcal{F}_t\right]\\
&\quad= \sum_{k=i}^{j-1}(T_{k+1}-T_k)\,\mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_{k+1}} r_s ds}(L(T_k,T_k,T_{k+1})-\kappa)^+ \,\Big|\, \mathcal{F}_t\right]\\
&\quad= \sum_{k=i}^{j-1}(T_{k+1}-T_k)P(t,T_{k+1})\widehat{\mathbb{E}}_{k+1}\left[(L(T_k,T_k,T_{k+1})-\kappa)^+ \mid \mathcal{F}_t\right].
\end{aligned} \tag{4.3.10}$$

In the BGM model (4.3.3) the interest rate cap with payoff

$$\sum_{k=i}^{j-1}(T_{k+1}-T_k)(L(T_k,T_k,T_{k+1})-\kappa)^+$$

can be priced at time $t \in [0,T_1]$ by the Black formula

$$\sum_{k=i}^{j-1}(T_{k+1}-T_k)P(t,T_{k+1})\mathrm{Bl}(L(t,T_k,T_{k+1}),\kappa,\sigma_k(t,T_k),0,T_k-t),$$

where

$$|\sigma_k(t,T_k)|^2 = \frac{1}{T_k-t}\int_t^{T_k}|\gamma_k|^2(s)ds.$$

SOFR Caplets

The backward-looking SOFR caplet has payoff $(R(S,T,S)-K)^+$, which is known only at time S. By the Jensen, 1906 inequality we note the relation

$$\begin{aligned}\mathbb{E}_S[(R(S,T,S)-K)^+ \mid \mathcal{F}_t] &= \mathbb{E}_S[\mathbb{E}_S[(R(S,T,S)-K)^+ \mid \mathcal{F}_T] \mid \mathcal{F}_t] \\ &\geqslant \mathbb{E}_S[(\mathbb{E}_S[R(S,T,S) \mid \mathcal{F}_T]-K)^+ \mid \mathcal{F}_t] \\ &= \mathbb{E}_S[(R(T,T,S)-K)^+ \mid \mathcal{F}_t] \\ &= \mathbb{E}_S[(L(T,T,S)-K)^+ \mid \mathcal{F}_t],\end{aligned}$$

hence the backward-looking SOFR caplet is more expensive than the forward-looking LIBOR caplet. The caplet on the SOFR rate $R(T_{i+1},T_i,T_{i+1})$ with payoff $(R(T_{i+1},T_i,T_{i+1})-\kappa)^+$ and strike level κ can be priced at time $t \in [0,T_i]$ with a discount factor counted from the settlement date T_{i+1} from Proposition 3.5 as

$$\begin{aligned}&\mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_{i+1}} r_s ds}(R(T_{i+1},T_i,T_{i+1})-\kappa)^+ \,\Big|\, \mathcal{F}_t\right] \qquad (4.3.11)\\ &= \mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_{i+1}} r_s ds}\left(\frac{1}{T_{i+1}-T_i}\left(\frac{P(t,T_i)}{P(t,T_{i+1})}-1\right)-\kappa\right)^+ \,\Bigg|\, \mathcal{F}_t\right] \\ &= P(t,T_{i+1})\widehat{\mathbb{E}}_{i+1}\left[(R(T_{i+1},T_i,T_{i+1})-\kappa)^+ \mid \mathcal{F}_t\right],\end{aligned}$$

by taking $N_t^{(i+1)} := P(t,T_{i+1})$ as a numéraire and using the forward measure $\widehat{\mathbb{P}}_{i+1}$.

Proposition 4.8 The SOFR rate

$$R(t,T_i,T_{i+1}) := \frac{1}{T_{i+1}-T_i}\left(\frac{P(t,T_i)}{P(t,T_{i+1})}-1\right), \quad 0 \leqslant T_i \leqslant t \leqslant T_{i+1},$$

is a martingale under the forward measure $\widehat{\mathbb{P}}_{i+1}$.

Proof. The SOFR rate $R(t,T_i,T_{i+1})$ is a deflated process according to the forward numéraire process $(P(t,T_{i+1}))_{t\in[0,T_{i+1}]}$. Therefore, it is a martingale under $\widehat{\mathbb{P}}_{i+1}$ by Proposition 3.4. □

The next pricing formula (4.3.13) allows us to price and hedge a caplet using a portfolio based on the bonds $P(t,T_i)$ and $P(t,T_{i+1})$, cf. (4.3.14) below, when $R(t,T_i,T_{i+1})$ is modeled in the BGM model.

Proposition 4.9 (Black SOFR caplet formula). Assume that $R(t,T_i,T_{i+1})$ is modeled in the BGM model as

$$\frac{dR(t,T_i,T_{i+1})}{R(t,T_i,T_{i+1})} = \gamma_i(t)d\widehat{W}_t^{i+1}, \qquad (4.3.12)$$

$0 \leqslant t \leqslant T_{i+1}$, $i = 1,2,\ldots,n-1$, where $\gamma_i(t)$ is a deterministic volatility function of time $t \in [0,T_i]$, $i = 1,2,\ldots,n-1$. The caplet on $R(T_{i+1},T_i,T_{i+1})$ with strike level $\kappa > 0$ is priced at time $t \in [0,T_{i+1}]$ as

$$\boxed{\begin{aligned}&(T_{i+1}-T_i)\,\mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_{i+1}} r_s ds}(R(T_{i+1},T_i,T_{i+1})-\kappa)^+ \,\Big|\, \mathcal{F}_t\right] \qquad (4.3.13)\\ &= (P(t,T_i)-P(t,T_{i+1}))\Phi(d_+(t,T_{i+1})) - \kappa(T_{i+1}-T_i)P(t,T_{i+1})\Phi(d_-(t,T_{i+1})),\end{aligned}}$$

$0 \leqslant t \leqslant T_{i+1}$, where

$$d_+(t,T_{i+1}) = \frac{\log(R(t,T_i,T_{i+1})/\kappa) + (T_{i+1}-t)\sigma_i^2(t,T_{i+1})/2}{\sigma_i(t,T_{i+1})\sqrt{T_{i+1}-t}},$$

and

$$d_-(t,T_{i+1}) = \frac{\log(R(t,T_i,T_{i+1})/\kappa) - (T_{i+1}-t)\sigma_i^2(t,T_{i+1})/2}{\sigma_i(t,T_{i+1})\sqrt{T_{i+1}-t}},$$

and

$$|\sigma_i(t,T_{i+1})|^2 = \frac{1}{T_{i+1}-t}\int_t^{T_{i+1}} |\gamma_i|^2(s)ds.$$

Proof. The forward price

$$\widehat{X}_t := \frac{P(t,T_i)}{P(t,T_{i+1})} = 1 + (T_{i+1}-T_i)R(T_{i+1},T_i,T_{i+1})$$

and the SOFR rate process $(R(t,T_i,T_{i+1}))_{t\in[0,T_{i+1}]}$ are martingales under $\widehat{\mathbb{P}}_{i+1}$ by Proposition 4.8, $i = 1,2,\ldots,n-1$, and

$$R(T_{i+1},T_i,T_{i+1}) = R(t,T_i,T_{i+1})\exp\left(\int_t^{T_{i+1}} \gamma_i(s)d\widehat{W}_s^{i+1} - \frac{1}{2}\int_t^{T_{i+1}} |\gamma_i(s)|^2 ds\right),$$

$0 \leqslant t \leqslant T_{i+1}$, where $t \mapsto R(t,T_i,T_{i+1})$ is a geometric Brownian motion under $\widehat{\mathbb{P}}_{i+1}$ (4.3.12). Hence by (4.3.11) we have

$$\begin{aligned}
&\mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_{i+1}} r_s ds}(R(T_{i+1},T_i,T_{i+1})-\kappa)^+ \,\Big|\, \mathscr{F}_t\right] \\
&= P(t,T_{i+1})\widehat{\mathbb{E}}_{i+1}\left[(R(T_{i+1},T_i,T_{i+1})-\kappa)^+ \,\big|\, \mathscr{F}_t\right] \\
&= P(t,T_{i+1})\left(R(t,T_i,T_{i+1})\Phi(d_+(t,T_{i+1})) - \kappa\Phi(d_-(t,T_{i+1}))\right) \\
&= P(t,T_{i+1})\mathrm{Bl}(R(t,T_i,T_{i+1}),\kappa,\sigma_i(t,T_{i+1}),0,T_{i+1}-t),
\end{aligned}$$

$t \in [0,T_{i+1}]$, with

$$|\sigma_i(t,T_{i+1})|^2 = \frac{1}{T_{i+1}-t}\int_t^{T_{i+1}} |\gamma_i|^2(s)ds.$$

□

In addition, we obtain the self-financing portfolio strategy

$$\left(\Phi(d_+(t,T_{i+1})), -\Phi(d_+(t,T_{i+1})) - \kappa(T_{i+1}-T_i)\Phi(d_-(t,T_{i+1}))\right) \tag{4.3.14}$$

in the bonds priced $(P(t,T_i),P(t,T_{i+1}))$, $t \in [0,T_{i+1}]$, with maturities T_i and T_{i+1}.

4.4 Forward Swap Measures

In this section we introduce the forward swap (or annuity) measures, or annuity measures, to be used for the pricing of swaptions, and we study their properties. We start with the definition of the *annuity numéraire*

$$N_t^{(i,j)} := P(t,T_i,T_j) = \sum_{k=i}^{j-1}(T_{k+1}-T_k)P(t,T_{k+1}), \qquad 0 \leqslant t \leqslant T_i, \tag{4.4.1}$$

with in particular, when $j = i+1$,

$$P(t,T_i,T_{i+1}) = (T_{i+1}-T_i)P(t,T_{i+1}), \qquad 0 \leqslant t \leqslant T_i.$$

$1 \leqslant i < n$. The annuity numéraire can be also used to price a *bond ladder*. It satisfies the following martingale property, which can be proved by linearity and the fact that $t \mapsto \mathrm{e}^{-\int_0^t r_s ds} P(t,T_k)$ is a martingale for all $k = 1,2,\ldots,n$, under Assumption 1.

R The discounted annuity numéraire

$$t \mapsto \mathrm{e}^{-\int_0^t r_s ds} P(t,T_i,T_j) = \mathrm{e}^{-\int_0^t r_s ds} \sum_{k=i}^{j-1} (T_{k+1}-T_k) P(t,T_{k+1}), \quad 0 \leqslant t \leqslant T_i,$$

is a martingale under $\mathbb{P}^*$.

The forward swap measure $\widehat{\mathbb{P}}_{i,j}$ is defined, according to Definition 3.1, by

$$\frac{d\widehat{\mathbb{P}}_{i,j}}{d\mathbb{P}^*} := \mathrm{e}^{-\int_0^{T_i} r_s ds} \frac{N_{T_i}^{(i,j)}}{N_0^{(i,j)}} = \mathrm{e}^{-\int_0^{T_i} r_s ds} \frac{P(T_i,T_i,T_j)}{P(0,T_i,T_j)}, \tag{4.4.2}$$

$1 \leqslant i < j \leqslant n$.

R We have

$$\begin{aligned}
\mathbb{E}^*\left[\frac{d\widehat{\mathbb{P}}_{i,j}}{d\mathbb{P}^*} \,\middle|\, \mathscr{F}_t\right] &= \frac{1}{P(0,T_i,T_j)} \mathbb{E}^*\left[\mathrm{e}^{-\int_0^{T_i} r_s ds} P(T_i,T_i,T_j) \,\middle|\, \mathscr{F}_t\right] \\
&= \frac{1}{P(0,T_i,T_j)} \mathbb{E}^*\left[\mathrm{e}^{-\int_0^{T_i} r_s ds} \sum_{k=i}^{j-1} (T_{k+1}-T_k) P(T_i,T_{k+1}) \,\middle|\, \mathscr{F}_t\right] \\
&= \frac{1}{P(0,T_i,T_j)} \sum_{k=i}^{j-1} (T_{k+1}-T_k) \mathbb{E}^*\left[\mathrm{e}^{-\int_0^{T_i} r_s ds} P(T_i,T_{k+1}) \,\middle|\, \mathscr{F}_t\right] \\
&= \frac{1}{P(0,T_i,T_j)} \mathrm{e}^{-\int_0^t r_s ds} \sum_{k=i}^{j-1} (T_{k+1}-T_k) P(t,T_{k+1}) \\
&= \mathrm{e}^{-\int_0^t r_s ds} \frac{P(t,T_i,T_j)}{P(0,T_i,T_j)},
\end{aligned}$$

$0 \leqslant t \leqslant T_i$, by Remark 4.4, and

$$\frac{d\widehat{\mathbb{P}}_{i,j|\mathscr{F}_t}}{d\mathbb{P}^*_{|\mathscr{F}_t}} = \mathrm{e}^{-\int_t^{T_i} r_s ds} \frac{P(T_i,T_i,T_j)}{P(t,T_i,T_j)}, \qquad 0 \leqslant t \leqslant T_{i+1}, \tag{4.4.3}$$

by Relation (3.2.3) in Lemma 3.2.

Proposition 4.10 The LIBOR swap rate

$$S(t,T_i,T_j) = \frac{P(t,T_i)-P(t,T_j)}{P(t,T_i,T_j)}, \qquad 0 \leqslant t \leqslant T_i,$$

see Corollary 2.11, is a martingale under the forward swap measure $\widehat{\mathbb{P}}_{i,j}$.

Proof. We use the fact that the deflated process

$$t \mapsto \frac{P(t,T_k)}{P(t,T_i,T_j)}, \qquad i,j,k = 1,2,\ldots,n,$$

is an $\mathscr{F}_t$-martingale under $\widehat{\mathbb{P}}_{i,j}$ by Proposition 3.4. □

The following pricing formula is then stated for a given integrable claim with payoff of the form $P(T_i,T_i,T_j)F$, using the forward swap measure $\widehat{\mathbb{P}}_{i,j}$:

$$\begin{aligned} \mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_i} r_s ds} P(T_i,T_i,T_j)F \,\middle|\, \mathscr{F}_t \right] &= P(t,T_i,T_j)\,\mathbb{E}^*\left[F \frac{d\widehat{\mathbb{P}}_{i,j|\mathscr{F}_t}}{d\mathbb{P}^*_{|\mathscr{F}_t}} \,\middle|\, \mathscr{F}_t \right] \\ &= P(t,T_i,T_j)\widehat{\mathbb{E}}_{i,j}[F \mid \mathscr{F}_t], \end{aligned} \tag{4.4.4}$$

after applying (4.4.2) and (4.4.3) on the last line, or Proposition 3.5.

4.5 Swaption Pricing

Definition 4.11 A payer (or call) swaption gives the option, but not the obligation, to enter an interest rate swap as payer of a fixed rate κ and as receiver of floating LIBOR rates $L(T_i,T_k,T_{k+1})$ at time T_{k+1}, $k = i,\ldots,j-1$, and has the payoff

$$\begin{aligned} &\left(\sum_{k=i}^{j-1} (T_{k+1}-T_k)\,\mathbb{E}^*\left[\mathrm{e}^{-\int_{T_i}^{T_{k+1}} r_s ds} \,\middle|\, \mathscr{F}_{T_i} \right] (L(T_i,T_k,T_{k+1})-\kappa) \right)^+ \\ &\quad = \left(\sum_{k=i}^{j-1} (T_{k+1}-T_k)P(T_i,T_{k+1})(L(T_i,T_k,T_{k+1})-\kappa) \right)^+ \end{aligned} \tag{4.5.1}$$

at time T_i.

This swaption can be priced at time $t \in [0,T_i]$ under the risk-neutral probability measure $\mathbb{P}^*$ as

$$\mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_i} r_s ds} \left(\sum_{k=i}^{j-1} (T_{k+1}-T_k)P(T_i,T_{k+1})(L(T_i,T_k,T_{k+1})-\kappa) \right)^+ \,\middle|\, \mathscr{F}_t \right], \tag{4.5.2}$$

$t \in [0,T_i]$. When $j = i+1$, the swaption price (4.5.2) coincides with the price at time t of a caplet on $[T_i,T_{i+1}]$ up to a factor $\delta_i := T_{i+1}-T_i$, since

$$\begin{aligned} &\mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_i} r_s ds} ((T_{i+1}-T_i)P(T_i,T_{i+1})(L(T_i,T_i,T_{i+1})-\kappa))^+ \,\middle|\, \mathscr{F}_t \right] \\ &= (T_{i+1}-T_i)\,\mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_i} r_s ds} P(T_i,T_{i+1})\,(L(T_i,T_i,T_{i+1})-\kappa)^+ \,\middle|\, \mathscr{F}_t \right] \\ &= (T_{i+1}-T_i)\,\mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_i} r_s ds}\,\mathbb{E}^*\left[\mathrm{e}^{-\int_{T_i}^{T_{i+1}} r_s ds} \,\middle|\, \mathscr{F}_{T_i} \right] (L(T_i,T_i,T_{i+1})-\kappa)^+ \,\middle|\, \mathscr{F}_t \right] \\ &= (T_{i+1}-T_i)\,\mathbb{E}^*\left[\mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_i} r_s ds}\,\mathrm{e}^{-\int_{T_i}^{T_{i+1}} r_s ds} (L(T_i,T_i,T_{i+1})-\kappa)^+ \,\middle|\, \mathscr{F}_{T_i} \right] \,\middle|\, \mathscr{F}_t \right] \\ &= (T_{i+1}-T_i)\,\mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_{i+1}} r_s ds} (L(T_i,T_i,T_{i+1})-\kappa)^+ \,\middle|\, \mathscr{F}_t \right], \end{aligned} \tag{4.5.3}$$

$0 \leqslant t \leqslant T_i$, which coincides with the caplet price (4.3.1) up to the factor $T_{i+1}-T_i$. Unlike in the case of interest rate caps, the sum in (4.5.2) cannot be taken out of the positive part. Nevertheless, the price of the swaption can be bounded as in the next proposition.

Proposition 4.12 The payer swaption price (4.5.2) can be upper bounded by the interest rate cap price (4.3.10) as

$$\begin{aligned}
&\mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_i} r_s ds}\left(\sum_{k=i}^{j-1}(T_{k+1}-T_k)P(T_i,T_{k+1})(L(T_i,T_k,T_{k+1})-\kappa)\right)^+ \Bigg| \mathcal{F}_t\right]\\
&\quad\leqslant \sum_{k=i}^{j-1}(T_{k+1}-T_k)\,\mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_{k+1}} r_s ds}\left(L(T_i,T_k,T_{k+1})-\kappa\right)^+ \Big| \mathcal{F}_t\right],
\end{aligned}$$

$0 \leqslant t \leqslant T_i$.

Proof. Due to the inequality

$$(x_1+x_2+\cdots+x_m)^+ \leqslant x_1^+ + x_2^+ + \cdots + x_m^+, \quad x_1,x_2,\ldots,x_m \in \mathbb{R},$$

we have

$$\begin{aligned}
&\mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_i} r_s ds}\left(\sum_{k=i}^{j-1}(T_{k+1}-T_k)P(T_i,T_{k+1})(L(T_i,T_k,T_{k+1})-\kappa)\right)^+ \Bigg| \mathcal{F}_t\right]\\
&\leqslant \mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_i} r_s ds}\sum_{k=i}^{j-1}(T_{k+1}-T_k)P(T_i,T_{k+1})\left(L(T_i,T_k,T_{k+1})-\kappa\right)^+ \Big| \mathcal{F}_t\right]\\
&= \sum_{k=i}^{j-1}(T_{k+1}-T_k)\,\mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_i} r_s ds}P(T_i,T_{k+1})\left(L(T_i,T_k,T_{k+1})-\kappa\right)^+ \Big| \mathcal{F}_t\right]\\
&= \sum_{k=i}^{j-1}(T_{k+1}-T_k)\,\mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_i} r_s ds}\,\mathbb{E}^*\left[\mathrm{e}^{-\int_{T_i}^{T_{k+1}} r_s ds} \Big| \mathcal{F}_{T_i}\right]\left(L(T_i,T_k,T_{k+1})-\kappa\right)^+ \Big| \mathcal{F}_t\right]\\
&= \sum_{k=i}^{j-1}(T_{k+1}-T_k)\,\mathbb{E}^*\left[\mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_{k+1}} r_s ds}\left(L(T_i,T_k,T_{k+1})-\kappa\right)^+ \Big| \mathcal{F}_{T_i}\right] \Big| \mathcal{F}_t\right]\\
&= \sum_{k=i}^{j-1}(T_{k+1}-T_k)\,\mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_{k+1}} r_s ds}\left(L(T_i,T_k,T_{k+1})-\kappa\right)^+ \Big| \mathcal{F}_t\right]\\
&= \mathbb{E}^*\left[\sum_{k=i}^{j-1}(T_{k+1}-T_k)\,\mathrm{e}^{-\int_t^{T_{k+1}} r_s ds}\left(L(T_i,T_k,T_{k+1})-\kappa\right)^+ \Big| \mathcal{F}_t\right],
\end{aligned}$$

$0 \leqslant t \leqslant T_i$. □

The payoff of the payer swaption can be rewritten as in the following lemma which is a direct consequence of the definition of the swap rate $S(T_i,T_i,T_j)$, see Proposition 2.10 and Corollary 2.11.

Lemma 4.13 The payer swaption payoff (4.5.1) at time T_i with swap rate $\kappa = S(t,T_j,T_j)$ can be rewritten as

$$\begin{aligned}
&\left(\sum_{k=i}^{j-1}(T_{k+1}-T_k)P(T_i,T_{k+1})(L(T_i,T_k,T_{k+1})-\kappa)\right)^+\\
&\quad= \left(P(T_i,T_i)-P(T_i,T_j)-\kappa P(T_i,T_i,T_j)\right)^+ &(4.5.4)\\
&\quad= P(T_i,T_i,T_j)\left(S(T_i,T_i,T_j)-\kappa\right)^+. &(4.5.5)
\end{aligned}$$

Proof. The relation

$$\sum_{k=i}^{j-1}(T_{k+1}-T_k)P(t,T_{k+1})(L(t,T_k,T_{k+1})-S(t,T_i,T_j))=0$$

that defines the forward swap rate $S(t,T_i,T_j)$ shows that

$$\begin{aligned}
&\sum_{k=i}^{j-1}(T_{k+1}-T_k)P(t,T_{k+1})L(t,T_k,T_{k+1}) \\
&\qquad = S(t,T_i,T_j)\sum_{k=i}^{j-1}(T_{k+1}-T_k)P(t,T_{k+1}) \\
&\qquad = P(t,T_i,T_j)S(t,T_i,T_j) \\
&\qquad = P(t,T_i)-P(t,T_j)
\end{aligned}$$

as in the proof of Corollary 2.11, hence by the definition (4.4.1) of $P(t,T_i,T_j)$ we have

$$\begin{aligned}
&\sum_{k=i}^{j-1}(T_{k+1}-T_k)P(t,T_{k+1})(L(t,T_k,T_{k+1})-\kappa) \\
&\qquad = P(t,T_i)-P(t,T_j)-\kappa P(t,T_i,T_j) \\
&\qquad = P(t,T_i,T_j)\,(S(t,T_i,T_j)-\kappa),
\end{aligned}$$

and for $t=T_i$ we get

$$\begin{aligned}
&\left(\sum_{k=i}^{j-1}(T_{k+1}-T_k)P(T_i,T_{k+1})(L(T_i,T_k,T_{k+1})-\kappa)\right)^+ \\
&\qquad = P(T_i,T_i,T_j)\,(S(T_i,T_i,T_j)-\kappa)^+.
\end{aligned}$$

□

The next proposition simply states that a payer swaption on the LIBOR rate can be priced as a European call option on the swap rate $S(T_i,T_i,T_j)$ under the forward swap measure $\widehat{\mathbb{P}}_{i,j}$.

Proposition 4.14 The price (4.5.2) of the payer swaption with payoff

$$\left(\sum_{k=i}^{j-1}(T_{k+1}-T_k)P(T_i,T_{k+1})(L(T_i,T_k,T_{k+1})-\kappa)\right)^+ \tag{4.5.6}$$

on the LIBOR market can be written under the forward swap measure $\widehat{\mathbb{P}}_{i,j}$ as the European call price

$$P(t,T_i,T_j)\widehat{\mathbb{E}}_{i,j}\big[(S(T_i,T_i,T_j)-\kappa)^+\big|\mathcal{F}_t\big], \quad 0\leqslant t\leqslant T_i,$$

on the swap rate $S(T_i,T_i,T_j)$.

Proof. As a consequence of (4.4.4) and Lemma 4.13, we find

$$\begin{aligned}
&\mathbb{E}^*\left[e^{-\int_t^{T_i}r_sds}\left(\sum_{k=i}^{j-1}(T_{k+1}-T_k)P(T_i,T_{k+1})(L(T_i,T_k,T_{k+1})-\kappa)\right)^+\Bigg|\mathcal{F}_t\right] \\
&= \mathbb{E}^*\left[e^{-\int_t^{T_i}r_sds}(P(T_i,T_i)-P(T_i,T_j)-\kappa P(T_i,T_i,T_j))^+\Big|\mathcal{F}_t\right] \qquad (4.5.7)\\
&= \mathbb{E}^*\left[e^{-\int_t^{T_i}r_sds}P(T_i,T_i,T_j)\,(S(T_i,T_i,T_j)-\kappa)^+\Big|\mathcal{F}_t\right] \\
&= P(t,T_i,T_j)\,\mathbb{E}^*\left[\frac{d\widehat{\mathbb{P}}_{i,j|\mathcal{F}_t}}{d\mathbb{P}^*_{|\mathcal{F}_t}}\,(S(T_i,T_i,T_j)-\kappa)^+\Big|\mathcal{F}_t\right] \\
&= P(t,T_i,T_j)\widehat{\mathbb{E}}_{i,j}\big[(S(T_i,T_i,T_j)-\kappa)^+\big|\mathcal{F}_t\big]. \qquad (4.5.8)
\end{aligned}$$

□

In the next Proposition 4.15 we price the payer swaption with payoff (4.5.6) or equivalently (4.5.5), by modeling the swap rate $(S(t,T_i,T_j))_{0\leqslant t\leqslant T_i}$ using standard Brownian motion $(\widehat{W}_t^{i,j})_{0\leqslant t\leqslant T_i}$ under the swap forward measure $\widehat{\mathbb{P}}_{i,j}$.

Proposition 4.15 (Black swaption formula for payer swaptions). Assume that the LIBOR swap rate (2.2.7) is modeled as a geometric Brownian motion under $\widehat{P}_{i,j}$, *i.e.*

$$dS(t,T_i,T_j) = S(t,T_i,T_j)\widehat{\sigma}_{i,j}(t)d\widehat{W}_t^{i,j}, \tag{4.5.9}$$

where $(\widehat{\sigma}_{i,j}(t))_{t\in\mathbb{R}_+}$ is a deterministic volatility function of time. Then, the payer swaption with payoff

$$(P(T,T_i)-P(T,T_j)-\kappa P(T_i,T_i,T_j))^+ = P(T_i,T_i,T_j)\,(S(T_i,T_i,T_j)-\kappa)^+$$

can be priced using the Black-Scholes call formula as

$$\begin{aligned}
&\mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_i} r_s ds}P(T_i,T_i,T_j)\,(S(T_i,T_i,T_j)-\kappa)^+ \,\middle|\, \mathscr{F}_t\right] \\
&\quad = (P(t,T_i)-P(t,T_j))\Phi(d_+(t,T_i)) \\
&\qquad -\kappa\Phi(d_-(t,T_i))\sum_{k=i}^{j-1}(T_{k+1}-T_k)P(t,T_{k+1}),
\end{aligned}$$

$t\in[0,T_i]$, where

$$d_+(t,T_i) = \frac{\log(S(t,T_i,T_j)/\kappa)+\sigma_{i,j}^2(t,T_i)(T_i-t)/2}{\sigma_{i,j}(t,T_i)\sqrt{T_i-t}}, \tag{4.5.10}$$

and

$$d_-(t,T_i) = \frac{\log(S(t,T_i,T_j)/\kappa)-\sigma_{i,j}^2(t,T_i)(T_i-t)/2}{\sigma_{i,j}(t,T_i)\sqrt{T_i-t}}, \tag{4.5.11}$$

and

$$|\sigma_{i,j}(t,T_i)|^2 = \frac{1}{T_i-t}\int_t^{T_i}|\widehat{\sigma}_{i,j}(s)|^2ds, \qquad 0\leqslant t\leqslant T_i. \tag{4.5.12}$$

Proof. Since $S(t,T_i,T_j)$ is a geometric Brownian motion with volatility function $(\widehat{\sigma}(t))_{t\in\mathbb{R}_+}$ under $\widehat{\mathbb{P}}_{i,j}$, by (4.5.4)-(4.5.5) in Lemma 4.13 or (4.5.7)-(4.5.8) we have

$$\begin{aligned}
&\mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_i} r_s ds}P(T_i,T_i,T_j)\,(S(T_i,T_i,T_j)-\kappa)^+ \,\middle|\, \mathscr{F}_t\right] \\
&\quad = \mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T} r_s ds}(P(T,T_i)-P(T,T_j)-\kappa P(T_i,T_i,T_j))^+ \,\middle|\, \mathscr{F}_t\right] \\
&\quad = P(t,T_i,T_j)\widehat{\mathbb{E}}_{i,j}\left[(S(T_i,T_i,T_j)-\kappa)^+ \,\middle|\, \mathscr{F}_t\right] \\
&\quad = P(t,T_i,T_j)\mathrm{Bl}(S(t,T_i,T_j),\kappa,\sigma_{i,j}(t,T_i),0,T_i-t) \\
&\quad = P(t,T_i,T_j)\,(S(t,T_i,T_j)\Phi_+(t,S(t,T_i,T_j))-\kappa\Phi_-(t,S(t,T_i,T_j))) \\
&\quad = \big(P(t,T_i)-P(t,T_j)\big)\Phi_+(t,S(t,T_i,T_j))-\kappa P(t,T_i,T_j)\Phi_-(t,S(t,T_i,T_j)) \\
&\quad = \big(P(t,T_i)-P(t,T_j)\big)\Phi_+(t,S(t,T_i,T_j)) \\
&\qquad -\kappa\Phi_-(t,S(t,T_i,T_j))\sum_{k=i}^{j-1}(T_{k+1}-T_k)P(t,T_{k+1}).
\end{aligned}$$

□

In addition, the hedging strategy

$$
\begin{aligned}
&(\Phi_+(t, S(t, T_i, T_j)), -\kappa\Phi_-(t, S(t, T_i, T_j))(T_{i+1} - T_i), \ldots \\
&\qquad \ldots, -\kappa\Phi_-(t, S(t, T_i, T_j))(T_{j-1} - T_{j-2}), -\Phi_+(t, S(t, T_i, T_j)))
\end{aligned}
$$

based on the assets $(P(t, T_i), \ldots, P(t, T_j))$ is self-financing by Corollary 3.18, see also Privault and Teng, 2012. Similarly to the above, a receiver (or put) swaption gives the option, but not the obligation, to enter an interest rate swap as receiver of a fixed rate κ and as payer of floating LIBOR rates $L(T_i, T_k, T_{k+1})$ at times $T_{i+1}, \ldots, T_j$, and can be priced as in the next proposition.

Proposition 4.16 (Black swaption formula for receiver swaptions). Assume that the LIBOR swap rate (2.2.7) is modeled as the geometric Brownian motion (4.5.9) under the forward swap measure $\widehat{P}_{i,j}$. Then, the receiver swaption with payoff

$$
\left(\kappa P(T_i, T_i, T_j) - \left(P(T, T_i) - P(T, T_j)\right)\right)^+ = P(T_i, T_i, T_j)\left(\kappa - S(T_i, T_i, T_j)\right)^+
$$

can be priced using the Black-Scholes put formula as

$$
\begin{aligned}
&\mathbb{E}^*\left[e^{-\int_t^{T_i} r_s ds} P(T_i, T_i, T_j)\left(\kappa - S(T_i, T_i, T_j)\right)^+ \Big| \mathcal{F}_t \right] \\
&\quad = \kappa\Phi(-d_-(t, T_i)) \sum_{k=i}^{j-1} (T_{k+1} - T_k) P(t, T_{k+1}) \\
&\qquad -(P(t, T_i) - P(t, T_j))\Phi(-d_+(t, T_i)),
\end{aligned}
$$

where $d_+(t, T_i)$, and $d_-(t, T_i)$ and $|\sigma_{i,j}(t, T_i)|^2$ are defined in (4.5.10)-(4.5.12).

When the SOFR swap rate (2.2.11) is modeled as a geometric Brownian motion under $\widehat{\mathbb{P}}_{i,j}$ as in (4.5.9), SOFR swaptions are priced in the same way as LIBOR swaptions.

Swaption prices can also be computed by an approximation formula, from the exact dynamics of the swap rate $S(t, T_i, T_j)$ under the forward swap measure $\widehat{\mathbb{P}}_{i,j}$, based on the bond price dynamics of the form (4.1.3), cf. Schoenmakers, 2005, page 17.

Swaption volatilities can be estimated from swaption prices as implied volatilities from the Black pricing formula:

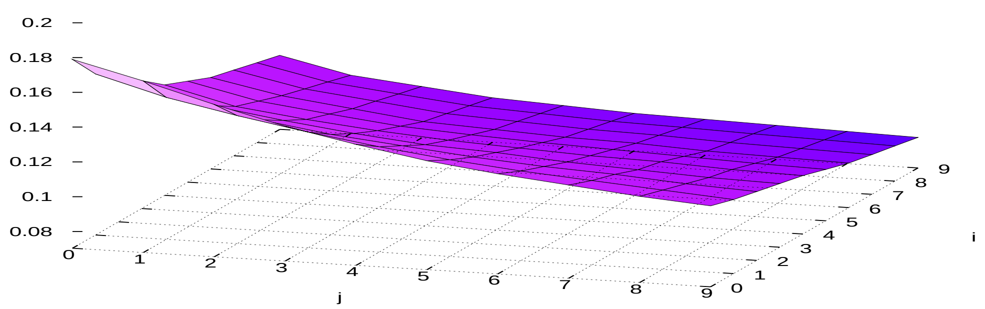


Figure 4.1: Implied swaption volatilities.

Implied swaption volatilities can then be used to calibrate the BGM model, cf. Schoenmakers, 2005, Privault and Wei, 2009, or § 9.5 of Privault, 2021.

LIBOR-SOFR Swaps

We consider the swap contract with payoff

$$\sum_{k=i}^{j-1}(T_{k+1}-T_k)(R(T_{k+1},T_k,T_{k+1})-L(T_k,T_k,T_{k+1})),$$

for the exchange of a backward-looking SOFR rate $R(T_{k+1},T_k,T_{k+1})$ with the forward-looking LIBOR rate $L(T_k,T_k,T_{k+1})$ over the time period $[T_k,T_{k+1}]$. The price of this interest rate swap vanishes at any time $t \in [0,T_1]$, as

$$\begin{aligned}
&(T_{k+1}-T_k)\,\mathbb{E}\left[\mathrm{e}^{-\int_t^{T_{k+1}} r_s ds}(R(T_{k+1},T_k,T_{k+1})-L(T_k,T_k,T_{k+1}))\,\Big|\,\mathscr{F}_t\right]\\
&\quad= (T_{k+1}-T_k)P(t,T_{k+1})\,\mathbb{E}_{k+1}[R(T_{k+1},T_k,T_{k+1})-L(T_k,T_k,T_{k+1})\mid\mathscr{F}_t]\\
&\quad= (T_{k+1}-T_k)P(t,T_{k+1})(R(t,T_k,T_{k+1})-L(t,T_k,T_{k+1})\\
&\quad= 0, \qquad 0\leqslant t\leqslant T_k.
\end{aligned}$$

see Mercurio, 2018. On the other hand, for any $i=1,\ldots,n$, we also have

$$\begin{aligned}
&(T_{k+1}-T_k)\,\mathbb{E}\left[\mathrm{e}^{-\int_t^{T_{k+1}} r_s ds}(R(T_{k+1},T_k,T_{k+1})-L(T_k,T_k,T_{k+1}))\,\Big|\,\mathscr{F}_{T_i}\right]\\
&\quad= (T_{k+1}-T_k)P(T_i,T_{k+1})\,\mathbb{E}_{k+1}[R(T_{k+1},T_k,T_{k+1})-L(T_k,T_k,T_{k+1})\mid\mathscr{F}_{T_k}]\\
&\quad= (T_{k+1}-T_k)P(T_i,T_{k+1})(R(T_i,T_k,T_{k+1})-L(T_i,T_k,T_{k+1})\\
&\quad= 0.
\end{aligned}$$

Bermudan swaption pricing in Quantlib

The Bermudan swaption on the tenor structure $\{T_i,\ldots,T_j\}$ is priced as the supremum

$$\begin{aligned}
&\operatorname*{Sup}_{l\in\{i,\ldots,j-1\}}\mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_l} r_s ds}\left(\sum_{k=l}^{j-1}\delta_k P(T_l,T_{k+1})(L(T_l,T_k,T_{k+1})-\kappa)\right)^+\,\Bigg|\,\mathscr{F}_t\right]\\
&\quad= \operatorname*{Sup}_{l\in\{i,\ldots,j-1\}}\mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_l} r_s ds}\left(P(T_l,T_l)-P(T_l,T_j)-\kappa P(T_l,T_l,T_j)\right)^+\,\Big|\,\mathscr{F}_t\right]\\
&\quad= \operatorname*{Sup}_{l\in\{i,\ldots,j-1\}}\mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_l} r_s ds}P(T_l,T_l,T_j)(S(T_l,T_l,T_j)-\kappa)^+\,\Big|\,\mathscr{F}_t\right],
\end{aligned}$$

where the supremum is over all stopping times taking values in $\{T_i,\ldots,T_j\}$.

Bermudan swaptions can be priced using this **Rcode**[*] in (R)quantlib, with the following output:

```
Summary of pricing results for Bermudan Swaption

Price (in bp) of Bermudan swaption is  24.92137
Strike is NULL (ATM strike is  0.05 )
Model used is: Hull-White using analytic formulas
Calibrated model parameters are:
a =  0.04641
sigma =  0.005869
```

This modified code[†] can be used in particular the pricing of ordinary swaptions, with the output:

[*] Click to open or download.
[†] Click to open or download.

```
Summary of pricing results for Bermudan Swaption

Price (in bp) of Bermudan swaption is  22.45436
Strike is NULL (ATM strike is  0.05 )
Model used is: Hull-White using analytic formulas
Calibrated model parameters are:
a =  0.07107
sigma =  0.006018
```

Table 4.2 summarizes some possible uses of change of numéraire in option pricing.

Application	Asset price	Numéraire process	Option payoff	Forward measure $\widehat{\mathbb{P}}$	Deflated process	Option price	Change of numéraire formula
Risk-neutral pricing	S_t	$N_t = \mathrm{e}^{\int_0^t r_s ds}$	C	$\frac{d\widehat{\mathbb{P}}}{d\mathbb{P}^*} = 1$	$\widetilde{S}_t = \mathrm{e}^{-\int_0^t r_s ds} S_t$	$\mathbb{E}^*\left[\mathrm{e}^{-\int_t^T r_s ds} C \,\middle\vert\, \mathcal{F}_t\right]$	$\mathrm{e}^{\int_0^t r_s ds}\mathbb{E}^*\left[\mathrm{e}^{-\int_0^T r_s ds} C \,\middle\vert\, \mathcal{F}_t\right]$
Exchange option	S_t	N_t	$(S_T - \kappa N_T)^+$	$\frac{d\widehat{\mathbb{P}}}{d\mathbb{P}^*} = \mathrm{e}^{-\int_0^T r_s ds}\frac{N_T}{N_0}$	$\widehat{X}_t = \frac{S_t}{N_t}$	$\mathbb{E}^*\left[\mathrm{e}^{-\int_t^T r_s ds}(S_T - \kappa N_T)^+ \,\middle\vert\, \mathcal{F}_t\right]$	$N_t \widehat{\mathbb{E}}\left[\left(\widehat{X}_T - \kappa\right)^+ \mid \mathcal{F}_t\right]$
Exotic	$S_t = S_0\,\mathrm{e}^{rt+\sigma W_t - \sigma^2 t/2}$	$N_t = S_t$	$S_T(S_T - K)^+$	$\frac{d\widehat{\mathbb{P}}}{d\mathbb{P}^*} = \mathrm{e}^{-(T-t)r}\frac{S_T}{S_0}$	$\widehat{X}_t = 1$	$\mathbb{E}^*\left[\mathrm{e}^{-(T-t)r} S_T \left(S_T - K\right)^+ \mid \mathcal{F}_t\right]$	$S_t \widehat{\mathbb{E}}[(S_T - K)^+ \mid \mathcal{F}_t]$
Foreign exchange	$\mathrm{e}^{r^f t} R_t$	$N_t = \mathrm{e}^{r^f t} R_t$	$(R_T - \kappa)^+$	$\frac{d\widehat{\mathbb{P}}}{d\mathbb{P}^*} = \mathrm{e}^{(r^f - r)T}\frac{R_T}{R_0}$	$\widehat{X}_t = 1$	$\mathrm{e}^{r^f T}\mathbb{E}^*\left[\mathrm{e}^{-(T-t)r}(R_T - \kappa) \mid \mathcal{F}_t\right]$	$\mathrm{e}^{r^f t} R_t \widehat{\mathbb{E}}\left[\left(1 - \frac{\kappa}{R_T}\right)^+ \,\middle\vert\, R_t\right]$
Bond option	$P(t,S)$	$N_t = P(t,T)$	$(P(T,S) - K)^+$	$\frac{d\widehat{\mathbb{P}}_T}{d\mathbb{P}^*} = \frac{\mathrm{e}^{-\int_0^T r_s ds}}{P(0,T)}$	$\widehat{X}_t = \frac{P(t,S)}{P(t,T)}$	$\mathbb{E}^*\left[\mathrm{e}^{-\int_t^T r_s ds}(P(T,S) - K)^+ \,\middle\vert\, \mathcal{F}_t\right]$	$P(t,T)\widehat{\mathbb{E}}_T[(P(T,S) - K)^+ \mid \mathcal{F}_t]$
Caplets and caps	$P(t,T)$	$N_t = P(t,S)$	$(S-T)(L(T,T,S) - \kappa)^+$	$\frac{d\widehat{\mathbb{P}}_S}{d\mathbb{P}^*} = \frac{\mathrm{e}^{-\int_0^S r_s ds}}{P(0,S)}$	$L(t,T,S) = \frac{1}{S-T}\left(\frac{P(t,T)}{P(t,S)} - 1\right)$	$(S-T)\,\mathbb{E}^*\left[\mathrm{e}^{-\int_t^S r_s ds}(L(T,T,S) - K)^+ \,\middle\vert\, \mathcal{F}_t\right]$	$(S-T)P(t,S)\widehat{\mathbb{E}}_S[(L(T,T,S) - K)^+ \mid \mathcal{F}_t]$
Swaption	$P(t,T_1), P(t,T_n)$	$N_t = P(t,T_1,T_n)$	$(P(T_1,T_1) - P(T_1,T_n) - \kappa P(T_1,T_1,T_n))^+$	$\frac{d\widehat{\mathbb{P}}_{1,n}}{d\mathbb{P}^*} = \mathrm{e}^{-\int_0^{T_1} r_s ds}\frac{P(T_1,T_1,T_n)}{P(0,T_1,T_n)}$	$S(t,T_1,T_n) = \frac{P(t,T_1) - P(t,T_n)}{P(t,T_1,T_n)}$	$\mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_1} r_s ds}(P(T_1,T_1) - P(T_1,T_n) - \kappa P(T_1,T_1,T_n))^+ \,\middle\vert\, \mathcal{F}_t\right]$	$P(t,T_1,T_n)\widehat{\mathbb{E}}_{1,n}[(S(T_1,T_1,T_n) - \kappa)^+ \vert \mathcal{F}_t]$
Black-Scholes	$S_t = S_0\,\mathrm{e}^{rt+\sigma W_t - \sigma^2 t/2}$	$N_t = S_t^n\,\mathrm{e}^{-(n-1)n\sigma^2 t/2 - (n-1)rt}$	$(S_T^n - K^n)^+$	$\frac{d\widehat{\mathbb{P}}}{d\mathbb{P}^*} = \mathrm{e}^{n\sigma W_T - (n\sigma)^2 T/2}$	$\widehat{X}_t = S_t^{n-1}\,\mathrm{e}^{(n-1)n\sigma^2 t/2 + (n-1)rt}$	$\mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[S_T^n \mathbb{1}_{\{S_T \geqslant K\}} \,\middle\vert\, \mathcal{F}_t\right] - K^n\,\mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[\mathbb{1}_{\{S_T \geqslant K\}} \,\middle\vert\, \mathcal{F}_t\right]$	$\mathrm{e}^{(n-1)n\sigma^2 T/2 + (n-1)rT} N_t \widehat{\mathbb{E}}\left[\mathbb{1}_{\{S_T \geqslant K\}} \,\middle\vert\, \mathcal{F}_t\right]$ $-K^n\,\mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[\mathbb{1}_{\{S_T \geqslant K\}} \,\middle\vert\, \mathcal{F}_t\right]$

Table 4.2: A list of numéraire processes and their applications.

Exercises

Exercise 4.1 Consider a floorlet on a three-month LIBOR rate in nine month's time, with a notional principal amount of $\$10,000$ per interest rate percentage point. The term structure is flat at 3.95% per year with discrete compounding, the volatility of the forward LIBOR rate in nine months is 10%, and the floor rate is 4.5%.

a) What are the key assumptions on the LIBOR rate in nine month in order to apply Black's formula to price this floorlet?
b) Compute the price of this floorlet using Black's formula as an application of Proposition 4.7 and (4.3.9), using the functions $\Phi(d_+)$ and $\Phi(d_-)$.

Exercise 4.2 Consider a payer swaption giving its holder the right, but not the obligation, to enter into a 3-year annual pay swap in four years, where a fixed rate of 5% will be paid and the LIBOR rate will be received. Assume that the yield curve is flat at 5% with continuous annual compounding and the volatility of the swap rate is 20%. The notional principal is $100,000 per interest rate percentage point.

a) What are the key assumptions in order to apply Black's formula to value this swaption?
b) Compute the price of this swaption using Black's formula as an application of Proposition 4.15.

Exercise 4.3 Consider a *receiver* swaption which is giving its holder the right, but not the obligation, to enter into a 2-year annual pay swap in three years, where a fixed rate of 5% will be received and the LIBOR rate will be paid. Assume that the yield curve is flat at 2% with continuous annual compounding and the volatility of the swap rate is 10%. The notional principal is $10,000 per percentage point, and the swaption price is quoted in basis points. Write down the expression of the price of this swaption using Black's formula.

Exercise 4.4 Consider two bonds with maturities T_1 and T_2, $T_1 < T_2$, which follow the stochastic differential equations

$$dP(t,T_1) = r_t P(t,T_1)dt + \zeta_1(t)P(t,T_1)dW_t$$

and

$$dP(t,T_2) = r_t P(t,T_2)dt + \zeta_2(t)P(t,T_2)dW_t.$$

a) Using Itô calculus, show that the forward process $P(t,T_2)/P(t,T_1)$ is a driftless geometric Brownian motion driven by $d\widehat{W}_t := dW_t - \zeta_1(t)dt$ under the T_1-forward measure $\widehat{\mathbb{P}}$.

b) Compute the price $\mathbb{E}^*\left[e^{-\int_t^{T_1} r_s ds}(K - P(T_1,T_2))^+ \,\middle|\, \mathscr{F}_t \right]$ of a bond put option at time $t \in [0,T_1]$ using change of numéraire and the Black-Scholes formula.

Hint: Given X a Gaussian random variable with mean m and variance v^2 given $\mathscr{F}_t$, we have:

$$\begin{aligned}\mathbb{E}\left[\left(\kappa - e^X\right)^+ \mid \mathscr{F}_t\right] &= \kappa\Phi\left(-\frac{1}{v}(m - \log\kappa)\right) \\ &\quad - e^{m+v^2/2}\Phi\left(-\frac{1}{v}(m + v^2 - \log\kappa)\right).\end{aligned} \tag{4.5.13}$$

5. Reduced-Form Approach to Credit Risk

The reduced-form approach to credit risk modeling focuses on modeling default probabilities as stochastic processes, in contrast to the structural approach in which bankruptcy is modeled from the firm's asset value. The modeling of default risk using failure rate processes and exogeneous random variables results into the use of enlarged filtration that can incorporate information on default events.

5.1 Survival Probabilities

Given $t > 0$, let $\mathbb{P}(\tau > t)$ denote the probability that a random system with lifetime τ survives at least t years. Assuming that survival probabilities $\mathbb{P}(\tau > t)$ are strictly positive for all $t > 0$, we can compute the conditional probability for that system to survive up to time T, given that it was still functioning at time $t \in [0, T]$, as

$$\mathbb{P}(\tau > T \mid \tau > t) = \frac{\mathbb{P}(\tau > T \text{ and } \tau > t)}{\mathbb{P}(\tau > t)} = \frac{\mathbb{P}(\tau > T)}{\mathbb{P}(\tau > t)}, \qquad 0 \leqslant t \leqslant T,$$

with

$$\begin{aligned} \mathbb{P}(\tau \leqslant T \mid \tau > t) &= 1 - \mathbb{P}(\tau > T \mid \tau > t) \\ &= \frac{\mathbb{P}(\tau > t) - \mathbb{P}(\tau > T)}{\mathbb{P}(\tau > t)} \end{aligned}$$

$$
\begin{aligned}
&= \frac{\mathbb{P}(\tau \leqslant T) - \mathbb{P}(\tau \leqslant t)}{\mathbb{P}(\tau > t)} \\
&= \frac{\mathbb{P}(t < \tau \leqslant T)}{\mathbb{P}(\tau > t)}, \qquad 0 \leqslant t \leqslant T.
\end{aligned} \tag{5.1.1}
$$

Such survival probabilities are typically found in life (or mortality) tables:

Age t	$\mathbb{P}(\tau \leqslant t+1 \mid \tau > t)$
20	0.0894%
30	0.1008%
40	0.2038%
50	0.4458%
60	0.9827%

Table 5.1: Mortality table.

The corresponding conditional survival probability distribution can be computed as follows:

$$
\begin{aligned}
\mathbb{P}(\tau \in dx \mid \tau > t) &= \mathbb{P}(x < \tau \leqslant x + dx \mid \tau > t) \\
&= \mathbb{P}(\tau \leqslant x + dx \mid \tau > t) - \mathbb{P}(\tau \leqslant x \mid \tau > t) \\
&= \frac{\mathbb{P}(\tau \leqslant x + dx) - \mathbb{P}(\tau \leqslant x)}{\mathbb{P}(\tau > t)} \\
&= \frac{1}{\mathbb{P}(\tau > t)} d\mathbb{P}(\tau \leqslant x) \\
&= -\frac{1}{\mathbb{P}(\tau > t)} d\mathbb{P}(\tau > x), \qquad x > t.
\end{aligned}
$$

Proposition 5.1 The *failure rate* function, defined as

$$
\lambda(t) := \frac{\mathbb{P}(\tau \leqslant t + dt \mid \tau > t)}{dt},
$$

satisfies

$$
\mathbb{P}(\tau > t) = \exp\left(-\int_0^t \lambda(u) du \right), \qquad t \geqslant 0. \tag{5.1.2}
$$

Proof. By (5.1.1), we have

$$
\begin{aligned}
\lambda(t) &:= \frac{\mathbb{P}(\tau \leqslant t + dt \mid \tau > t)}{dt} \\
&= \frac{1}{\mathbb{P}(\tau > t)} \frac{\mathbb{P}(t < \tau \leqslant t + dt)}{dt} \\
&= \frac{1}{\mathbb{P}(\tau > t)} \frac{\mathbb{P}(\tau > t) - \mathbb{P}(\tau > t + dt)}{dt} \\
&= -\frac{d}{dt} \log \mathbb{P}(\tau > t) \\
&= -\frac{1}{\mathbb{P}(\tau > t)} \frac{d}{dt} \mathbb{P}(\tau > t), \qquad t > 0,
\end{aligned}
$$

and the differential equation

$$
\frac{d}{dt} \mathbb{P}(\tau > t) = -\lambda(t) \mathbb{P}(\tau > t),
$$

which can be solved as in (5.1.2) under the initial condition $\mathbb{P}(\tau > 0) = 1$. $\square$

Proposition 5.1 allows us to rewrite the (conditional) survival probability as

$$\mathbb{P}(\tau > T \mid \tau > t) = \frac{\mathbb{P}(\tau > T)}{\mathbb{P}(\tau > t)} = \exp\left(-\int_t^T \lambda(u)du\right), \qquad 0 \leqslant t \leqslant T,$$

with

$$\mathbb{P}(\tau > t + h \mid \tau > t) = \mathrm{e}^{-\lambda(t)h} \simeq 1 - \lambda(t)h, \qquad [h \searrow 0],$$

and

$$\mathbb{P}(\tau \leqslant t + h \mid \tau > t) = 1 - \mathrm{e}^{-\lambda(t)h} \simeq \lambda(t)h, \qquad [h \searrow 0],$$

as h tends to 0. When the failure rate $\lambda(t) = \lambda > 0$ is a constant function of time, Relation (5.1.2) shows that

$$\mathbb{P}(\tau > T) = \mathrm{e}^{-\lambda T}, \qquad T \geqslant 0,$$

i.e. τ has the exponential distribution with parameter λ. Note that given $(\tau_n)_{n\geqslant 1}$ a sequence of *i.i.d.* exponentially distributed random variables, letting

$$T_n = \tau_1 + \tau_2 + \cdots + \tau_n, \qquad n \geqslant 1,$$

defines the sequence of jump times of a standard Poisson process with intensity $\lambda > 0$.

5.2 Stochastic Default

When the random time τ is a *stopping time* with respect to $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$ we have

$$\{\tau > t\} \in \mathscr{F}_t, \qquad t \geqslant 0,$$

i.e. the knowledge of whether default or bankruptcy has already occurred at time t is contained in $\mathscr{F}_t$, $t \in \mathbb{R}_+$. As a consequence, we can write

$$\mathbb{P}(\tau > t \mid \mathscr{F}_t) = \mathbb{E}\left[\mathbb{1}_{\{\tau>t\}} \mid \mathscr{F}_t\right] = \mathbb{1}_{\{\tau>t\}}, \qquad t \geqslant 0.$$

In what follows we will not assume that τ is an $\mathscr{F}_t$-stopping time, and by analogy with (5.1.2) we will write $\mathbb{P}(\tau > t \mid \mathscr{F}_t)$ as

$$\mathbb{P}(\tau > t \mid \mathscr{F}_t) = \exp\left(-\int_0^t \lambda_u du\right), \qquad t \geqslant 0, \tag{5.2.1}$$

where the failure rate function $(\lambda_t)_{t\in\mathbb{R}_+}$ is modeled as a random process adapted to a filtration $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$.

The process $(\lambda_t)_{t\in\mathbb{R}_+}$ can also be chosen among the classical mean-reverting diffusion processes, including jump-diffusion processes. In Lando, 1998, the process $(\lambda_t)_{t\in\mathbb{R}_+}$ is constructed as $\lambda_t := h(X_t)$, $t \in \mathbb{R}_+$, where h is a nonnegative function and $(X_t)_{t\in\mathbb{R}_+}$ is a stochastic process generating the filtration $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$.

The default time τ is then *defined* as

$$\tau := \inf\left\{t \in \mathbb{R}_+ \,:\, \int_0^t h(X_u)du \geqslant L\right\},$$

where L is an exponentially distributed random variable with parameter $\mu > 0$ and distribution function $\mathbb{P}(L > x) = \mathrm{e}^{-\mu x}$, $x \geqslant 0$, independent of $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$. In this case, as τ is not an $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$-stopping time, we have

$$\begin{aligned}
\mathbb{P}(\tau > t \mid \mathscr{F}_t) &= \mathbb{P}\left(\int_0^t h(X_u)du < L \,\Big|\, \mathscr{F}_t\right) \\
&= \exp\left(-\mu\int_0^t h(X_u)du\right) \\
&= \exp\left(-\mu\int_0^t \lambda_u du\right), \qquad t \geqslant 0.
\end{aligned}$$

Definition 5.2 Let $(\mathscr{G}_t)_{t\in\mathbb{R}_+}$ be the filtration defined by $\mathscr{G}_\infty := \mathscr{F}_\infty \vee \sigma(\tau)$ and

$$\mathscr{G}_t := \left\{B \in \mathscr{G}_\infty : \exists A \in \mathscr{F}_t \text{ such that } A\cap\{\tau > t\} = B\cap\{\tau > t\}\right\}, \tag{5.2.2}$$

with $\mathscr{F}_t \subset \mathscr{G}_t$, $t \geqslant 0$.

In other words, $\mathscr{G}_t$ contains insider information on whether default at time τ has occurred or not before time t, and τ is a $(\mathscr{G}_t)_{t\in\mathbb{R}_+}$-stopping time. Note that this information on τ may not be available to a generic user who has only access to the smaller filtration $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$. The next key Lemma 5.3, see Lando, 1998, Guo, Jarrow, and Menn, 2007, allows us to price a contingent claim given the information in the larger filtration $(\mathscr{G}_t)_{t\in\mathbb{R}_+}$, by only using information in $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$ and factoring in the default rate factor $\exp\left(-\int_t^T \lambda_u du\right)$.

Lemma 5.3 (Guo, Jarrow, and Menn, 2007, Theorem 1) For any $\mathscr{F}_T$-measurable integrable random variable F, we have

$$\mathbb{E}\left[F\mathbb{1}_{\{\tau>T\}} \mid \mathscr{G}_t\right] = \mathbb{1}_{\{\tau>t\}}\mathbb{E}\left[F\exp\left(-\int_t^T \lambda_u du\right) \,\Big|\, \mathscr{F}_t\right].$$

Proof. By (5.2.1) we have

$$\frac{\mathbb{P}(\tau > T \mid \mathscr{F}_T)}{\mathbb{P}(\tau > t \mid \mathscr{F}_t)} = \frac{\mathrm{e}^{-\int_0^T \lambda_u du}}{\mathrm{e}^{-\int_0^t \lambda_u du}} = \exp\left(-\int_t^T \lambda_u du\right),$$

hence, since F is $\mathscr{F}_T$-measurable,

$$\begin{aligned}
\mathbb{1}_{\{\tau>t\}}\mathbb{E}\left[F\exp\left(-\int_t^T \lambda_u du\right) \,\Big|\, \mathscr{F}_t\right] &= \mathbb{1}_{\{\tau>t\}}\mathbb{E}\left[F\frac{\mathbb{P}(\tau > T \mid \mathscr{F}_T)}{\mathbb{P}(\tau > t \mid \mathscr{F}_t)} \,\Big|\, \mathscr{F}_t\right] \\
&= \frac{\mathbb{1}_{\{\tau>t\}}}{\mathbb{P}(\tau > t \mid \mathscr{F}_t)}\mathbb{E}\left[F\,\mathbb{E}[\mathbb{1}_{\{\tau>T\}} \mid \mathscr{F}_T] \mid \mathscr{F}_t\right] \\
&= \frac{\mathbb{1}_{\{\tau>t\}}}{\mathbb{P}(\tau > t \mid \mathscr{F}_t)}\mathbb{E}\left[\mathbb{E}[F\mathbb{1}_{\{\tau>T\}} \mid \mathscr{F}_T] \mid \mathscr{F}_t\right] \\
&= \mathbb{1}_{\{\tau>t\}}\frac{\mathbb{E}\left[F\mathbb{1}_{\{\tau>T\}} \mid \mathscr{F}_t\right]}{\mathbb{P}(\tau > t \mid \mathscr{F}_t)} \\
&= \mathbb{1}_{\{\tau>t\}}\mathbb{E}\left[F\mathbb{1}_{\{\tau>T\}} \mid \mathscr{G}_t\right] \\
&= \mathbb{E}\left[F\mathbb{1}_{\{\tau>T\}} \mid \mathscr{G}_t\right], \qquad 0 \leqslant t \leqslant T.
\end{aligned}$$

In the last step of the above argument we used the key relation

$$\mathbb{1}_{\{\tau>t\}}\mathbb{E}\left[F\mathbb{1}_{\{\tau>T\}} \,\Big|\, \mathscr{G}_t\right] = \frac{\mathbb{1}_{\{\tau>t\}}}{\mathbb{P}(\tau > t \mid \mathscr{F}_t)}\mathbb{E}\left[F\mathbb{1}_{\{\tau>T\}} \mid \mathscr{F}_t\right],$$

cf. Relation (75.2) in § XX-75 page 186 of Dellacherie, Maisonneuve, and Meyer, 1992, Theorem VI-3-14 page 371 of Protter, 2004, and Lemma 3.1 of Elliott, Jeanblanc, and Yor, 2000, under the conditional probability measure $\mathbb{P}_{|\mathscr{F}_t}$, $0 \leqslant t \leqslant T$. Indeed, according to (5.2.2), for any $B \in \mathscr{G}_t$ we have, for some event $A \in \mathscr{F}_t$,

$$\begin{aligned}
\mathbb{E}\left[\mathbb{1}_B \mathbb{1}_{\{\tau>t\}} F \mathbb{1}_{\{\tau>T\}}\right] &= \mathbb{E}\left[\mathbb{1}_{B\cap\{\tau>t\}} F \mathbb{1}_{\{\tau>T\}}\right] \\
&= \mathbb{E}\left[\mathbb{1}_{A\cap\{\tau>t\}} F \mathbb{1}_{\{\tau>T\}}\right] \\
&= \mathbb{E}\left[\mathbb{1}_A \mathbb{1}_{\{\tau>t\}} F \mathbb{1}_{\{\tau>T\}}\right] \\
&= \mathbb{E}\left[\mathbb{1}_A \mathbb{1}_{\{\tau>t\}} \frac{\mathbb{E}[\mathbb{1}_{\{\tau>t\}} \mid \mathscr{F}_t]}{\mathbb{P}(\tau>t \mid \mathscr{F}_t)} F \mathbb{1}_{\{\tau>T\}}\right] \\
&= \mathbb{E}\left[\frac{\mathbb{E}[\mathbb{1}_A \mathbb{1}_{\{\tau>t\}} \mid \mathscr{F}_t]}{\mathbb{P}(\tau>t \mid \mathscr{F}_t)} F \mathbb{1}_{\{\tau>T\}}\right] \\
&= \mathbb{E}\left[\frac{\mathbb{E}[\mathbb{1}_A \mathbb{1}_{\{\tau>t\}} \mid \mathscr{F}_t]}{\mathbb{P}(\tau>t \mid \mathscr{F}_t)} \mathbb{E}\left[F \mathbb{1}_{\{\tau>T\}} \mid \mathscr{F}_t\right]\right] \\
&= \mathbb{E}\left[\frac{\mathbb{1}_A \mathbb{1}_{\{\tau>t\}}}{\mathbb{P}(\tau>t \mid \mathscr{F}_t)} \mathbb{E}\left[F \mathbb{1}_{\{\tau>T\}} \mid \mathscr{F}_t\right]\right] \\
&= \mathbb{E}\left[\frac{\mathbb{E}[\mathbb{1}_A \mathbb{1}_{\{\tau>t\}} \mid \mathscr{F}_t]}{\mathbb{P}(\tau>t \mid \mathscr{F}_t)} \mathbb{E}\left[F \mathbb{1}_{\{\tau>T\}} \mid \mathscr{F}_t\right]\right] \\
&= \mathbb{E}\left[\frac{\mathbb{1}_A \mathbb{1}_{\{\tau>t\}}}{\mathbb{P}(\tau>t \mid \mathscr{F}_t)} \mathbb{E}\left[F \mathbb{1}_{\{\tau>T\}} \mid \mathscr{F}_t\right]\right] \\
&= \mathbb{E}\left[\frac{\mathbb{1}_B \mathbb{1}_{\{\tau>t\}}}{\mathbb{P}(\tau>t \mid \mathscr{F}_t)} \mathbb{E}\left[F \mathbb{1}_{\{\tau>T\}} \mid \mathscr{F}_t\right]\right],
\end{aligned}$$

hence by a standard characterization of conditional expectations, we have

$$\mathbb{E}\left[\mathbb{1}_{\{\tau>t\}} F \mathbb{1}_{\{\tau>T\}} \mid \mathscr{G}_t\right] = \frac{\mathbb{1}_{\{\tau>t\}}}{\mathbb{P}(\tau>t \mid \mathscr{F}_t)} \mathbb{E}\left[F \mathbb{1}_{\{\tau>T\}} \mid \mathscr{F}_t\right]$$

□

Taking $F = 1$ in Lemma 5.3 allows one to write the survival probability up to time T, given the information known up to time t, as

$$\begin{aligned}
\mathbb{P}(\tau > T \mid \mathscr{G}_t) &= \mathbb{E}\left[\mathbb{1}_{\{\tau>T\}} \mid \mathscr{G}_t\right] \\
&= \mathbb{1}_{\{\tau>t\}} \mathbb{E}\left[\exp\left(-\int_t^T \lambda_u du\right) \Bigm| \mathscr{F}_t\right], \quad 0 \leqslant t \leqslant T.
\end{aligned} \tag{5.2.3}$$

In particular, applying Lemma 5.3 for $t = T$ and $F = 1$ shows that

$$\mathbb{E}\left[\mathbb{1}_{\{\tau>t\}} \mid \mathscr{G}_t\right] = \mathbb{1}_{\{\tau>t\}},$$

which shows that $\{\tau > t\} \in \mathscr{G}_t$ for all $t > 0$, and recovers the fact that τ is a $(\mathscr{G}_t)_{t\in\mathbb{R}_+}$-stopping time, while in general, τ is not $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$-stopping time.

The computation of $\mathbb{P}(\tau > T \mid \mathscr{G}_t)$ according to (5.2.3) is then similar to that of a bond price, by considering the failure rate $\lambda(t)$ as a "virtual" short-term interest rate. In particular the failure rate $\lambda(t,T)$ can be modeled in the HJM framework, and

$$\mathbb{P}(\tau > T \mid \mathscr{G}_t) = \mathbb{E}\left[\exp\left(-\int_t^T \lambda(t,u) du\right) \Bigm| \mathscr{F}_t\right]$$

can then be computed by applying HJM bond pricing techniques.

The computation of expectations given $\mathcal{G}_t$ as in Lemma 5.3 can be useful for pricing under insider trading, in which the insider has access to the augmented filtration $\mathcal{G}_t$ while the ordinary trader has only access to $\mathcal{F}_t$, therefore generating two different prices $\mathbb{E}^*[F \mid \mathcal{F}_t]$ and $\mathbb{E}^*[F \mid \mathcal{G}_t]$ for the same claim payoff F under the same risk-neutral probability measure $\mathbb{P}^*$. This leads to the issue of computing the dynamics of the underlying asset price by decomposing it using a $\mathcal{F}_t$-martingale *vs* a $\mathcal{G}_t$-martingale instead of using different forward measures as in *e.g.* § 19.1 of Privault, 2022. This can be obtained by the technique of enlargement of filtration, cf. Jeulin, 1980, Jacod, 1985, Yor, 1985, Elliott and Jeanblanc, 1999.

5.3 Defaultable Bonds

Bond pricing models are generally based on the terminal condition $P(T,T) = \$1$ according to which the bond payoff at maturity is always equal to \$1, and default does not occurs. In this chapter we allow for the possibility of default at a random time τ, in which case the terminal payoff of a bond is allowed to vanish at maturity.

The price $P_d(t,T)$ at time t of a default bond with maturity T, (random) default time τ and (possibly random) recovery rate $\xi \in [0,1]$ is given by

$$\begin{aligned} P_d(t,T) &= \mathbb{E}^*\left[\mathbb{1}_{\{\tau>T\}} \exp\left(-\int_t^T r_u du\right) \,\Big|\, \mathcal{G}_t\right] \\ &\quad + \mathbb{E}^*\left[\xi \mathbb{1}_{\{\tau\leqslant T\}} \exp\left(-\int_t^T r_u du\right) \,\Big|\, \mathcal{G}_t\right], \qquad 0 \leqslant t \leqslant T. \end{aligned}$$

Proposition 5.4 The default bond with maturity T and default time τ can be priced at time $t \in [0,T]$ as

$$\begin{aligned} P_d(t,T) &= \mathbb{1}_{\{\tau>t\}} \mathbb{E}^*\left[\exp\left(-\int_t^T (r_u+\lambda_u) du\right) \,\Big|\, \mathcal{F}_t\right] \\ &\quad + \mathbb{E}^*\left[\xi \mathbb{1}_{\{\tau\leqslant T\}} \exp\left(-\int_t^T r_u du\right) \,\Big|\, \mathcal{G}_t\right], \quad 0 \leqslant t \leqslant T. \end{aligned}$$

Proof. We take $F = \exp\left(-\int_t^T r_u du\right)$ in Lemma 5.3, which shows that

$$\mathbb{E}^*\left[\mathbb{1}_{\{\tau>T\}} \exp\left(-\int_t^T r_u du\right) \,\Big|\, \mathcal{G}_t\right] = \mathbb{1}_{\{\tau>t\}} \mathbb{E}^*\left[\exp\left(-\int_t^T (r_u+\lambda_u) du\right) \,\Big|\, \mathcal{F}_t\right],$$

cf. *e.g.* Lando, 1998, Duffie and Singleton, 2003, Guo, Jarrow, and Menn, 2007. □

In the case of complete default (zero-recovery), we have $\xi = 0$ and

$$P_d(t,T) = \mathbb{1}_{\{\tau>t\}} \mathbb{E}^*\left[\exp\left(-\int_t^T (r_s+\lambda_s) ds\right) \,\Big|\, \mathcal{F}_t\right], \quad 0 \leqslant t \leqslant T. \tag{5.3.1}$$

From the above expression (5.3.1) we note that the effect of the presence of a default time τ is to decrease the bond price, which can be viewed as an increase of the short rate by the amount λ_u. In a simple setting where the interest rate $r > 0$ and failure rate $\lambda > 0$ are constant, the default bond price becomes

$$P_d(t,T) = \mathbb{1}_{\{\tau>t\}} \mathrm{e}^{-(r+\lambda)(T-t)}, \qquad 0 \leqslant t \leqslant T.$$

In this case, the failure rate λ can be estimated at time $t \in [0,T]$ from a default bond price $P_d(t,T)$ and a non-default bond price $P(t,T) = \mathrm{e}^{-(T-t)r}$ as

$$\lambda = \frac{1}{T-t} \log \frac{P(t,T)}{P_d(t,T)}.$$

Finally, the bond price (5.3.1) can also be expressed under the forward measure $\widehat{\mathbb{P}}$ with maturity T, as

$$\begin{aligned} P_d(t,T) &= \mathbb{1}_{\{\tau>t\}} \mathbb{E}^* \left[\exp\left(- \int_t^T (r_s + \lambda_s) ds \right) \Big| \mathscr{F}_t \right] \\ &= \mathbb{1}_{\{\tau>t\}} \mathbb{E}^* \left[\exp\left(- \int_t^T r_s ds \right) \Big| \mathscr{F}_t \right] \widehat{\mathbb{E}} \left[\exp\left(- \int_t^T \lambda_s ds \right) \Big| \mathscr{F}_t \right] \\ &= \mathbb{1}_{\{\tau>t\}} N_t \widehat{\mathbb{P}}(\tau > T \mid \mathscr{G}_t), \end{aligned}$$

where $(N_t)_{t\in\mathbb{R}_+}$ is the numéraire process

$$N_t := P(t,T) = \mathbb{E}^* \left[\exp\left(- \int_t^T r_s ds \right) \Big| \mathscr{F}_t \right], \qquad 0 \leqslant t \leqslant T,$$

and by (5.2.3),

$$\widehat{\mathbb{P}}(\tau > T \mid \mathscr{G}_t) = \mathbb{1}_{\{\tau>t\}} \widehat{\mathbb{E}} \left[\exp\left(- \int_t^T \lambda_s ds \right) \Big| \mathscr{F}_t \right]$$

is the survival probability under the forward measure $\widehat{\mathbb{P}}$ defined as

$$\frac{d\widehat{\mathbb{P}}}{d\mathbb{P}} := \frac{N_T}{N_0} \mathrm{e}^{-\int_0^T r_t dt},$$

see Chen and Huang, 2001 and Chen, Cheng, et al., 2008.

Estimating the default rates

Recall that the price of a default bond with maturity T, (random) default time τ and (possibly random) recovery rate $\xi \in [0,1]$ is given by

$$\begin{aligned} P_d(t,T) &= \mathbb{1}_{\{\tau>t\}} \mathbb{E}^* \left[\exp\left(- \int_t^T (r_u + \lambda_u) du \right) \Big| \mathscr{F}_t \right] \\ &\quad + \mathbb{E}^* \left[\xi \mathbb{1}_{\{\tau\leqslant T\}} \exp\left(- \int_t^T r_u du \right) \Big| \mathscr{G}_t \right], \qquad 0 \leqslant t \leqslant T, \end{aligned}$$

where ξ denotes the recovery rate. We consider a simplified deterministic step function model with zero recovery rate and tenor structure

$$\{t = T_0 < T_1 < \cdots < T_n = T\},$$

where

$$r(t) = \sum_{l=0}^{n-1} r_l \mathbb{1}_{(T_l,T_{l+1}]}(t) \quad \text{and} \quad \lambda(t) = \sum_{l=0}^{n-1} \lambda_l \mathbb{1}_{(T_l,T_{l+1}]}(t), \quad t \geqslant 0. \tag{5.3.2}$$

i) Estimating the default rates from default bond prices.

From Proposition 5.4, we have

$$P_d(t,T_k) = \mathbb{1}_{\{\tau>t\}} \exp\left(- \int_t^{T_k} (r(u) + \lambda(u)) du \right)$$

$$= \mathbb{1}_{\{\tau>t\}} \exp\left(-\sum_{l=0}^{k-1}(r_l+\lambda_l)(T_{l+1}-T_l)\right),$$

$k=1,2,\ldots,n$, from which we can infer

$$\lambda_k = -r_k + \frac{1}{T_{k+1}-T_k}\log\frac{P_d(t,T_k)}{P_d(t,T_{k+1})} > 0, \quad k=0,1,\ldots,n-1.$$

ii) Estimating (implied) default probabilities $\mathbb{P}^*(\tau<T \mid \mathcal{G}_t)$ from default rates.

Based on the expression

$$\begin{aligned}\mathbb{P}^*(\tau>T \mid \mathcal{G}_t) &= \mathbb{E}^*\left[\mathbb{1}_{\{\tau>T\}} \mid \mathcal{G}_t\right] \\ &= \mathbb{1}_{\{\tau>t\}}\mathbb{E}^*\left[\exp\left(-\int_t^T \lambda_u du\right) \,\Big|\, \mathcal{F}_t\right], \qquad 0\leqslant t\leqslant T,\end{aligned} \tag{5.3.3}$$

of the survival probability up to time T, see (5.2.1), and given the information known up to time t, in terms of the hazard rate process $(\lambda_u)_{u\in\mathbb{R}_+}$ adapted to a filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$, we find

$$\begin{aligned}\mathbb{P}(\tau>T \mid \mathcal{G}_{T_k}) &= \mathbb{1}_{\{\tau>T_k\}}\exp\left(-\int_{T_k}^T \lambda_u du\right) \\ &= \mathbb{1}_{\{\tau>t\}}\exp\left(-\sum_{l=k}^{n-1}\lambda_l(T_{l+1}-T_l)\right), \quad k=0,1,\ldots,n-1,\end{aligned}$$

where

$$\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\{\tau\leqslant u\} : 0\leqslant u\leqslant t), \quad t\geqslant 0,$$

i.e. $\mathcal{G}_t$ contains the additional information on whether default at time τ has occurred or not before time t.

In Table 5.2, bond ratings are determined according to hazard (or failure) rate thresholds.

Bond Credit	Moody's		S & P	
Ratings	Municipal	Corporate	Municipal	Corporate
Aaa/AAAs	0.00	0.52	0.00	0.60
Aa/AA	0.06	0.52	0.00	1.50
A/A	0.03	1.29	0.23	2.91
Baa/BBB	0.13	4.64	0.32	10.29
Ba/BB	2.65	19.12	1.74	29.93
B/B	11.86	43.34	8.48	53.72
Caa-C/CCC-C	16.58	69.18	44.81	69.19
Investment Grade	0.07	2.09	0.20	4.14
Non-Invest. Grade	4.29	31.37	7.37	42.35
All	0.10	9.70	0.29	12.98

Table 5.2: Cumulative historic default rates (in percentage).*

Exercises

Exercise 5.1 Consider a standard zero-coupon bond with constant yield $r>0$ and a defaultable (risky) bond with constant yield r_d and default probability $\alpha\in(0,1)$. Find a relation between r, r_d, α and the bond maturity T.

*Source: Moody's, S&P.

Exercise 5.2 A standard zero-coupon bond with constant yield $r > 0$ and maturity T is priced $P(t,T) = \mathrm{e}^{-(T-t)r}$ at time $t \in [0,T]$. Assume that the company can get bankrupt at a random time $t+\tau$, and default on its final \$1 payment if $\tau < T-t$.

a) Explain why the defaultable bond price $P_d(t,T)$ can be expressed as

$$P_d(t,T) = \mathrm{e}^{-(T-t)r}\,\mathbb{E}^*\left[\mathbb{1}_{\{\tau > T-t\}}\right]. \tag{5.3.4}$$

b) Assuming that the default time τ is exponentially distributed with parameter $\lambda > 0$, compute the default bond price $P_d(t,T)$ using (5.3.4).

c) Find a formula that can estimate the parameter λ from the risk-free rate r and the market data $P_M(t,T)$ of the defaultable bond price at time $t \in [0,T]$.

Exercise 5.3 Consider a (random) default time τ with cumulative distribution function

$$\mathbb{P}(\tau > t \mid \mathscr{F}_t) = \exp\left(-\int_0^t \lambda_u du\right), \qquad t \geqslant 0,$$

where λ_t is a (random) default rate process which is adapted to the filtration $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$. Recall that the probability of survival up to time T, given the information known up to time t, is given by

$$\mathbb{P}(\tau > T \mid \mathscr{G}_t) = \mathbb{1}_{\{\tau > t\}}\,\mathbb{E}^*\left[\exp\left(-\int_t^T \lambda_u du\right) \,\Big|\, \mathscr{F}_t\right],$$

where $\mathscr{G}_t = \mathscr{F}_t \vee \sigma(\{\tau < u\} : 0 \leqslant u \leqslant t)$, $t \in \mathbb{R}_+$, is the filtration defined by adding the default time information to the history $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$. In this framework, the price $P(t,T)$ of defaultable bond with maturity T, short-term interest rate r_t and (random) default time τ is given by

$$\begin{aligned} P(t,T) &= \mathbb{E}^*\left[\mathbb{1}_{\{\tau > T\}} \exp\left(-\int_t^T r_u du\right) \,\Big|\, \mathscr{G}_t\right] \\ &= \mathbb{1}_{\{\tau > t\}}\,\mathbb{E}^*\left[\exp\left(-\int_t^T (r_u + \lambda_u) du\right) \,\Big|\, \mathscr{F}_t\right]. \end{aligned} \tag{5.3.5}$$

In what follows we assume that the processes $(r_t)_{t\in\mathbb{R}_+}$ and $(\lambda_t)_{t\in\mathbb{R}_+}$ are modeled according to the Vasicek processes

$$\begin{cases} dr_t = -ar_t dt + \sigma dB_t^{(1)}, \\ \\ d\lambda_t = -b\lambda_t dt + \eta dB_t^{(2)}, \end{cases}$$

where $\big(B_t^{(1)}\big)_{t\in\mathbb{R}_+}$ and $\big(B_t^{(2)}\big)_{t\in\mathbb{R}_+}$ are standard $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$-Brownian motions with correlation $\rho \in [-1,1]$, and $dB_t^{(1)} \bullet dB_t^{(2)} = \rho dt$.

a) Give a justification for the fact that

$$\mathbb{E}^*\left[\exp\left(-\int_t^T (r_u + \lambda_u) du\right) \,\Big|\, \mathscr{F}_t\right]$$

can be written as a function $F(t, r_t, \lambda_t)$ of t, r_t and λ_t, $t \in [0,T]$.

b) Show that

$$t \longmapsto \exp\left(-\int_0^t (r_s + \lambda_s) ds\right) \mathbb{E}^*\left[\exp\left(-\int_t^T (r_u + \lambda_u) du\right) \,\Big|\, \mathscr{F}_t\right]$$

is an $\mathscr{F}_t$-martingale under $\mathbb{P}$.

c) Use the Itô formula with two variables to derive a PDE on $\mathbb{R}^2$ for the function $F(t,x,y)$.

d) Taking $r_0 := 0$, show that we have

$$\int_t^T r_s ds = C(a,t,T)r_t + \sigma \int_t^T C(a,s,T) dB_s^{(1)},$$

and

$$\int_t^T \lambda_s ds = C(b,t,T)\lambda_t + \eta \int_t^T C(b,s,T) dB_s^{(2)},$$

where

$$C(a,t,T) = -\frac{1}{a}(\mathrm{e}^{-(T-t)a} - 1).$$

e) Show that the random variable

$$\int_t^T r_s ds + \int_t^T \lambda_s ds$$

is has a Gaussian distribution, and compute its conditional mean

$$\mathbb{E}^*\left[\int_t^T r_s ds + \int_t^T \lambda_s ds \;\middle|\; \mathscr{F}_t\right]$$

and variance

$$\mathrm{Var}\left[\int_t^T r_s ds + \int_t^T \lambda_s ds \;\middle|\; \mathscr{F}_t\right],$$

conditionally to $\mathscr{F}_t$.

f) Compute $P(t,T)$ from its expression (5.3.5) as a conditional expectation.

g) Show that the solution $F(t,x,y)$ to the 2-dimensional PDE of Question (c)) is

$$\begin{aligned} F(t,x,y) &= \exp(-C(a,t,T)x - C(b,t,T)y) \\ &\quad \times \exp\left(\frac{\sigma^2}{2}\int_t^T C^2(a,s,T)ds + \frac{\eta^2}{2}\int_t^T C^2(b,s,T)ds\right) \\ &\quad \times \exp\left(\rho\sigma\eta\int_t^T C(a,s,T)C(b,s,T)ds\right). \end{aligned}$$

h) Show that the defaultable bond price $P(t,T)$ can also be written as

$$P(t,T) = \mathrm{e}^{U(t,T)}\mathbb{P}(\tau > T \mid \mathscr{G}_t)\,\mathbb{E}^*\left[\exp\left(-\int_t^T r_s ds\right) \;\middle|\; \mathscr{F}_t\right],$$

where

$$U(t,T) = \rho\frac{\sigma\eta}{ab}(T - t - C(a,t,T) - C(b,t,T) + C(a+b,t,T)).$$

i) By partial differentiation of $\log P(t,T)$ with respect to T, compute the corresponding instantaneous short rate

$$f(t,T) = -\frac{\partial}{\partial T}\log P(t,T).$$

j) Show that $\mathbb{P}(\tau > T \mid \mathscr{G}_t)$ can be written using an HJM type default rate as

$$\mathbb{P}(\tau > T \mid \mathscr{G}_t) = \mathbb{1}_{\{\tau > t\}}\exp\left(-\int_t^T f_2(t,u)du\right),$$

where

$$f_2(t,u) = \lambda_t \mathrm{e}^{-(u-t)b} - \frac{\eta^2}{2}C^2(b,t,u).$$

k) Show how the result of Question (h)) can be simplified when $\big(B_t^{(1)}\big)_{t\in\mathbb{R}_+}$ and $\big(B_t^{(2)}\big)_{t\in\mathbb{R}_+}$ are independent.

Exercise Solutions

Chapter 1

Exercise 1.1

a) We have

$$\begin{aligned}
dr_t &= r_0 d\,\mathrm{e}^{-bt} + \frac{a}{b} d\left(1-\mathrm{e}^{-bt}\right) + \sigma d\left(\mathrm{e}^{-bt}\int_0^t \mathrm{e}^{bs} dB_s\right) \\
&= -br_0\,\mathrm{e}^{-bt}dt + a\,\mathrm{e}^{-bt}dt + \sigma\,\mathrm{e}^{-bt} d\int_0^t \mathrm{e}^{bs}dB_s + \sigma\int_0^t \mathrm{e}^{bs}dB_s d\,\mathrm{e}^{-bt} \\
&= -br_0\,\mathrm{e}^{-bt}dt + a\,\mathrm{e}^{-bt}dt + \sigma\,\mathrm{e}^{-bt}\,\mathrm{e}^{bt}dB_t - \sigma b\int_0^t \mathrm{e}^{bs}dB_s\,\mathrm{e}^{-bt}dt \\
&= -br_0\,\mathrm{e}^{-bt}dt + a\,\mathrm{e}^{-bt}dt + \sigma dB_t - \sigma b\int_0^t \mathrm{e}^{bs}dB_s\,\mathrm{e}^{-bt}dt \\
&= -br_0\,\mathrm{e}^{-bt}dt + a\,\mathrm{e}^{-bt}dt + \sigma dB_t - b\left(r_t - r_0\,\mathrm{e}^{-bt} - \frac{a}{b}\left(1-\mathrm{e}^{-bt}\right)\right)dt \\
&= (a-br_t)dt + \sigma dB_t,
\end{aligned}$$

which shows that r_t solves (1.4.29).

b) We note that

$$\begin{aligned}
r_t &= r_0\,\mathrm{e}^{-bt} + \frac{a}{b}\left(1-\mathrm{e}^{-bt}\right) + \sigma\int_0^t \mathrm{e}^{-(t-u)b}dB_u \\
&= r_0\,\mathrm{e}^{-bs}\,\mathrm{e}^{-(t-s)b} + \frac{a}{b}\mathrm{e}^{-(t-s)b}\left(1-\mathrm{e}^{-bs}\right) \\
&\quad + \frac{a}{b}\left(1-\mathrm{e}^{-(t-s)b}\right) + \sigma\,\mathrm{e}^{-(t-s)b}\int_0^s \mathrm{e}^{-(s-u)b}dB_u + \sigma\int_s^t \mathrm{e}^{-(t-u)b}dB_u \\
&= r_s\,\mathrm{e}^{-(t-s)b} + \frac{a}{b}\left(1-\mathrm{e}^{-(t-s)b}\right) + \sigma\int_s^t \mathrm{e}^{-(t-u)b}dB_u, \quad 0 \leqslant s \leqslant t.
\end{aligned}$$

Hence, assuming that r_s has the $\mathcal{N}(a/b, \sigma^2/(2b))$ distribution, the distribution of r_t is Gaussian with mean

$$\begin{aligned}
\mathbb{E}[r_t] &= \mathrm{e}^{-(t-s)b}\,\mathbb{E}[r_s] + \frac{a}{b}\left(1-\mathrm{e}^{(t-s)b}\right) \\
&= \frac{a}{b}\mathrm{e}^{-(t-s)b} + \frac{a}{b}\left(1-\mathrm{e}^{(t-s)b}\right) \\
&= \frac{a}{b},
\end{aligned}$$

and variance

$$\begin{aligned}
\mathrm{Var}[r_t] &= \mathrm{Var}\left[r_s\,\mathrm{e}^{-(t-s)b}+\frac{a}{b}\big(1-\mathrm{e}^{(t-s)b}\big)+\sigma\int_s^t \mathrm{e}^{-(t-u)b}dB_u\right]\\
&= \mathrm{Var}\left[r_s\,\mathrm{e}^{-(t-s)b}+\sigma\int_s^t \mathrm{e}^{-(t-u)b}dB_u\right]\\
&= \mathrm{Var}\left[r_s\,\mathrm{e}^{-(t-s)b}\right]+\mathrm{Var}\left[\sigma\int_s^t \mathrm{e}^{-(t-u)b}dB_u\right]\\
&= \mathrm{e}^{-2(t-s)b}\,\mathrm{Var}\left[r_s\right]+\sigma^2\,\mathrm{Var}\left[\int_s^t \mathrm{e}^{-(t-u)b}dB_u\right]\\
&= \mathrm{e}^{-2(t-s)b}\frac{\sigma^2}{2b}+\sigma^2\int_s^t \mathrm{e}^{-2(t-u)b}du\\
&= \mathrm{e}^{-2(t-s)b}\frac{\sigma^2}{2b}+\sigma^2\int_0^{t-s} \mathrm{e}^{-2bu}du\\
&= \frac{\sigma^2}{2b}, \qquad t\geqslant 0.
\end{aligned}$$

Exercise 1.2

a) The zero-coupon bond price $P(t,T)$ in the Vasicek model is given by

$$\log P(t,T)=A(T-t)+r_tC(T-t), \qquad 0\leqslant t\leqslant T,$$

where

$$C(T-t):=-\frac{1}{b}\big(1-\mathrm{e}^{-(T-t)b}\big),$$

and

$$\begin{aligned}
A(T-t) \;:=\; &\frac{4ab-3\sigma^2}{4b^3}+\frac{\sigma^2-2ab}{2b^2}(T-t)\\
&+\frac{\sigma^2-ab}{b^3}\,\mathrm{e}^{-(T-t)b}-\frac{\sigma^2}{4b^3}\,\mathrm{e}^{-2(T-t)b} \quad 0\leqslant t\leqslant T.
\end{aligned} \tag{A.1}$$

Since $\lim_{T\to\infty}C(T-t)/(T-t)=0$ and

$$\lim_{T\to\infty}A(T-t)/(T-t)=(\sigma^2-2ab)/(2b^2),$$

we find

$$r_\infty=-\lim_{T\to\infty}\frac{\log P(t,T)}{T-t}=-\frac{\sigma^2-2ab}{2b^2}=\frac{a}{b}-\frac{\sigma^2}{2b^2}.$$

b) We have

$$\begin{aligned}
\log\frac{P(t,T)}{P(0,T)} &= \log P(t,T)-\log P(0,T)\\
&= A(T-t)-A(T)+r_tC(T-t)-r_0C(T)\\
&= -t\frac{\sigma^2-2ab}{2b^2}-\mathrm{e}^{-(T-t)b}\left(\frac{\sigma^2-ab}{b^3}+\frac{\sigma^2}{4b^3}\,\mathrm{e}^{-(T-t)b}\right)\\
&\quad +\mathrm{e}^{-bT}\left(\frac{\sigma^2}{4b^3}-\frac{\sigma^2-ab}{b^3}\right)-\frac{r_t}{b}\big(1-\mathrm{e}^{-(T-t)b}\big)+\frac{r_0}{b}\big(1-\mathrm{e}^{-bT}\big),
\end{aligned}$$

hence

$$\lim_{T\to\infty}\log\frac{P(t,T)}{P(0,T)}=\left(\frac{a}{b}-\frac{\sigma^2}{2b^2}\right)t-\frac{r_t-r_0}{b}=-\frac{r_t-r_0}{b}+r_\infty t,$$

and[*]

$$\lim_{T\to\infty}\log\frac{P(t,T)}{P(0,T)}=\mathrm{e}^{-(r_t-r_0)/b+t(a/b-\sigma^2/(2b^2))}=\mathrm{e}^{-(r_t-r_0)/b+r_\infty t},$$

[*]The log function is continuous on $(0,\infty)$.

which shows that the yield of the long-bond return is the asymptotic bond yield r_∞.

Remark: By Relations (1.4.15)-(1.4.17), the Vasicek bond price $P(t,T)$ can be rewritten in terms of the asymptotic bond yield r_∞ as

$$P(t,T) = \mathrm{e}^{-(T-t)r_\infty+(r_t-r_\infty)C(T-t)-\sigma^2C^2(T-t)/(4b)}, \qquad 0 \leqslant t \leqslant T,$$

see *e.g.* Relation (3.12) in Brody, Hughston, and Meier, 2018.

Exercise 1.3 An estimator of σ can be obtained from the orthogonality relation

$$\begin{aligned}\sum_{l=0}^{n-1}\left(\left(\tilde{r}_{t_{l+1}} - a\Delta t - (1-b\Delta t)\tilde{r}_{t_l}\right)^2 - \sigma^2\Delta t(\tilde{r}_{t_l})^{2\gamma}\right) &= \sigma^2\sum_{l=0}^{n-1}(\tilde{r}_{t_l})^{2\gamma}\left((Z_l)^2-\Delta t\right)\\ &\simeq 0,\end{aligned}$$

which is due to the independence of t_{t_l} and Z_l, $l = 0,\ldots,n-1$, and yields

$$\widehat{\sigma}^2 = \frac{\displaystyle\sum_{l=0}^{n-1}\left(\tilde{r}_{t_{l+1}} - a\Delta t - (1-b\Delta t)\tilde{r}_{t_l}\right)^2}{\displaystyle\Delta t\sum_{l=0}^{n-1}(\tilde{r}_{t_l})^{2\gamma}}.$$

Regarding the estimation of γ, we can combine the above relation with the second orthogonality relation

$$\begin{aligned}&\sum_{l=0}^{n-1}\left(\left(\tilde{r}_{t_{l+1}} - a\Delta t - (1-b\Delta t)\tilde{r}_{t_l}\right)^2 - \sigma^2\Delta t(\tilde{r}_{t_l})^{2\gamma}\right)\tilde{r}_{t_l}\\ &= \sigma^2\sum_{l=0}^{n-1}(\tilde{r}_{t_l})^{2\gamma+1}\left((Z_l)^2-\Delta t\right)\\ &\simeq 0,\end{aligned}$$

cf. § 2.2 of Faff and Gray, 2006. One may also attempt to minimize the residual

$$\sum_{l=0}^{n-1}\left(\left(\tilde{r}_{t_{l+1}} - a\Delta t - (1-b\Delta t)\tilde{r}_{t_l}\right)^2 - \sigma^2\Delta t(\tilde{r}_{t_l})^{2\gamma}\right)^2$$

by equating the following derivatives to zero, as

$$\begin{aligned}&\frac{\partial}{\partial\sigma}\sum_{l=0}^{n-1}\left(\left(\tilde{r}_{t_{l+1}} - a\Delta t - (1-b\Delta t)\tilde{r}_{t_l}\right)^2 - \sigma^2\Delta t(\tilde{r}_{t_l})^{2\gamma}\right)^2\\ &= -4\sigma\sum_{l=0}^{n-1}(\tilde{r}_{t_l})^{2\gamma}\left(\left(\tilde{r}_{t_{l+1}} - a\Delta t - (1-b\Delta t)\tilde{r}_{t_l}\right)^2 - \sigma^2\Delta t(\tilde{r}_{t_l})^{2\gamma}\right)\\ &= 0,\end{aligned}$$

hence

$$\sum_{l=0}^{n-1}(\tilde{r}_{t_l})^{2\gamma}\left(\tilde{r}_{t_{l+1}} - a\Delta t - (1-b\Delta t)\tilde{r}_{t_l}\right)^2 - \sigma^2\Delta t\sum_{l=0}^{n-1}(\tilde{r}_{t_l})^{4\gamma} = 0,$$

which yields

$$\widehat{\sigma}^2 = \frac{\displaystyle\sum_{l=0}^{n-1}(\tilde{r}_{t_l})^{2\gamma}\left(\tilde{r}_{t_{l+1}} - a\Delta t - (1-b\Delta t)\tilde{r}_{t_l}\right)^2}{\displaystyle\Delta t\sum_{l=0}^{n-1}(\tilde{r}_{t_l})^{4\gamma}}. \tag{A.2}$$

We also have

$$\begin{aligned}
&\frac{\partial}{\partial\gamma}\sum_{l=0}^{n-1}\left(\left(\tilde{r}_{t_{l+1}}-a\Delta t-(1-b\Delta t)\tilde{r}_{t_l}\right)^2-\sigma^2\Delta t(\tilde{r}_{t_l})^{2\gamma}\right)^2\\
&= -4\sigma^2\Delta t\sum_{l=0}^{n-1}(\tilde{r}_{t_l})^{2\gamma}\left(\left(\tilde{r}_{t_{l+1}}-a\Delta t-(1-b\Delta t)\tilde{r}_{t_l}\right)^2-\sigma^2\Delta t(\tilde{r}_{t_l})^{2\gamma}\right)\log\tilde{r}_{t_l}\\
&= 0,
\end{aligned}$$

which yields

$$\widehat{\sigma}^2=\frac{\displaystyle\sum_{l=0}^{n-1}(\tilde{r}_{t_l})^{2\gamma}\left(\tilde{r}_{t_{l+1}}-a\Delta t-(1-b\Delta t)\tilde{r}_{t_l}\right)^2\log\tilde{r}_{t_l}}{\displaystyle\Delta t\sum_{l=0}^{n-1}(\tilde{r}_{t_l})^{4\gamma}\log\tilde{r}_{t_l}}, \tag{A.3}$$

and shows that γ can be estimated by matching Relations (A.2) and (A.3), *i.e.*

$$\begin{aligned}
&\frac{\displaystyle\sum_{l=0}^{n-1}(\tilde{r}_{t_l})^{2\gamma}\left(\tilde{r}_{t_{l+1}}-a\Delta t-(1-b\Delta t)\tilde{r}_{t_l}\right)^2}{\displaystyle\sum_{l=0}^{n-1}(\tilde{r}_{t_l})^{4\gamma}}\\
&= \frac{\displaystyle\sum_{l=0}^{n-1}(\tilde{r}_{t_l})^{2\gamma}\left(\tilde{r}_{t_{l+1}}-a\Delta t-(1-b\Delta t)\tilde{r}_{t_l}\right)^2\log\tilde{r}_{t_l}}{\displaystyle\sum_{l=0}^{n-1}(\tilde{r}_{t_l})^{4\gamma}\log\tilde{r}_{t_l}}.
\end{aligned}$$

Remarks.

i) Estimators of a and b can be obtained by minimizing the residual

$$\sum_{l=0}^{n-1}\left(\tilde{r}_{t_{l+1}}-a\Delta t-(1-b\Delta t)\tilde{r}_{t_l}\right)^2$$

as in the Vašíček, 1977 model, *i.e.* from the equations

$$\sum_{l=0}^{n-1}\left(\tilde{r}_{t_{l+1}}-a\Delta t-(1-b\Delta t)\tilde{r}_{t_l}\right)=0$$

and

$$\sum_{l=0}^{n-1}\left(\tilde{r}_{t_{l+1}}-a\Delta t-(1-b\Delta t)\tilde{r}_{t_l}\right)\tilde{r}_{t_l}=0.$$

ii) Instead of using the (generalised) method of moments, parameter estimation for stochastic differential equations can be achieved by maximum likelihood estimation, see *e.g.* Lindström, 2007 and references therein.

Exercise 1.4

a) We have $r_t=r_0+at+\sigma B_t$, and

$$F(t,r_t)=F(t,r_0+at+\sigma B_t),$$

hence by Proposition 1.2 the PDE satisfied by $F(t,x)$ is

$$-xF(t,x)+\frac{\partial F}{\partial t}(t,x)+a\frac{\partial F}{\partial x}(t,x)+\frac{1}{2}\sigma^2\frac{\partial^2 F}{\partial x^2}(t,x)=0, \tag{A.4}$$

with terminal condition $F(T,x)=1$.

b) Using the relation $r_t=r_0+at+\sigma B_t$ and the fact that the stochastic integral $\int_t^T (T-s)dB_s$ is independent of $\mathscr{F}_t$, we have

$$\begin{aligned}
F(t,r_t) &= \mathbb{E}^*\left[\exp\left(-\int_t^T r_s ds\right)\Big|\mathscr{F}_t\right]\\
&= \mathbb{E}^*\left[\exp\left(-r_0(T-t)-a\int_t^T sds-\sigma\int_t^T B_s ds\right)\Big|\mathscr{F}_t\right]\\
&= \mathbb{E}^*\left[\mathrm{e}^{-r_0(T-t)-a(T^2-t^2)/2}\exp\left(-(T-t)\sigma B_t-\sigma\int_t^T (T-s)dB_s\right)\Big|\mathscr{F}_t\right]\\
&= \mathrm{e}^{-r_0(T-t)-a(T^2-t^2)/2-(T-t)\sigma B_t}\,\mathbb{E}^*\left[\exp\left(-\sigma\int_t^T (T-s)dB_s\right)\Big|\mathscr{F}_t\right]\\
&= \mathrm{e}^{-r_0(T-t)-a(T-t)(T+t)/2-(T-t)\sigma B_t}\,\mathbb{E}^*\left[\exp\left(-\sigma\int_t^T (T-s)dB_s\right)\right]\\
&= \exp\left(-(T-t)r_t-a(T-t)^2/2+\frac{\sigma^2}{2}\int_t^T (T-s)^2 ds\right)\\
&= \exp\left(-(T-t)r_t-a(T-t)^2/2+(T-t)^3\sigma^2/6\right),
\end{aligned}$$

hence $F(t,x)=\exp\left(-(T-t)x-a(T-t)^2/2+(T-t)^3\sigma^2/6\right)$.

Note that the PDE (A.4) can also be solved by looking for a solution of the form $F(t,x)=\mathrm{e}^{A(T-t)+xC(T-t)}$, in which case one would find $A(s)=-as^2/2+\sigma^2 s^3/6$ and $C(s)=-s$.

c) We check that the function $F(t,x)$ of Question (b)) satisfies the PDE (A.4) of Question (a)), since $F(T,x)=1$ and

$$\begin{aligned}
-xF(t,x)+\left(x+a(T-t)-\frac{\sigma^2}{2}(T-t)^2\right)&F(t,x)-a(T-t)F(t,x)\\
&+\frac{\sigma^2}{2}(T-t)^2F(t,x)=0.
\end{aligned}$$

Exercise 1.5 We check from (1.4.34) and the differentiation rule $d\int_0^t f(u)du=f(t)dt$ that

$$\begin{aligned}
dr_t &= \alpha\beta d\left(S_t\int_0^t \frac{1}{S_u}du\right)+r_0 dS_t\\
&= \alpha\beta S_t d\int_0^t \frac{1}{S_u}du+\alpha\beta\int_0^t \frac{1}{S_u}du dS_t+r_0 dS_t\\
&= \alpha\beta\frac{S_t}{S_t}dt+\alpha\beta\int_0^t \frac{S_t}{S_u}du\frac{dS_t}{S_t}+r_0 dS_t\\
&= \alpha\beta dt+(r_t-r_0S_t)\frac{dS_t}{S_t}+r_0 dS_t\\
&= \alpha\beta dt+r_t\frac{dS_t}{S_t}\\
&= \alpha\beta dt+r_t(-\beta dt+\sigma dB_t)\\
&= \beta(\alpha-r_t)dt+\sigma dB_t, \qquad t\geqslant 0.
\end{aligned}$$

Chapter 2

Exercise 2.1

a) By partial differentiation with respect to T under the expectation $\widehat{\mathbb{E}}$, we have

$$\begin{aligned}\frac{\partial P}{\partial T}(t,T) &= \frac{\partial}{\partial T}\mathbb{E}^*\left[\mathrm{e}^{-\int_t^T r_s ds}\,\Big|\,\mathscr{F}_t\right]\\ &= \mathbb{E}^*\left[-r_T\,\mathrm{e}^{-\int_t^T r_s ds}\,\Big|\,\mathscr{F}_t\right]\\ &= -P(t,T)\widehat{\mathbb{E}}[r_T \mid \mathscr{F}_t].\end{aligned}$$

b) As a consequence of Question (a)), we find

$$f(t,T) = -\frac{1}{P(t,T)}\frac{\partial P}{\partial T}(t,T) = \widehat{\mathbb{E}}[r_T \mid \mathscr{F}_t], \qquad 0 \leqslant t \leqslant T, \tag{A.5}$$

see Relation (22) page 10 of Mamon, 2004.

c) The martingale property of $(f(t,T))_{t\in[0,T]}$ under the forward measure $\widehat{\mathbb{E}}$ follows from Relation (A.5) and the tower property of conditional expectations.

Remark. In the Vasicek model, by (1.1.2) and (4.1.9) we have

$$\begin{aligned} r_T &= r_0\,\mathrm{e}^{-bT} + \frac{a}{b}(1-\mathrm{e}^{-bT}) + \sigma\int_0^T \mathrm{e}^{-(T-s)b} dW_s\\ &= r_0\,\mathrm{e}^{-bT} + \frac{a}{b}(1-\mathrm{e}^{-bT}) + \sigma\int_0^T \mathrm{e}^{-(T-s)b} d\widehat{W}_s\\ &\quad -\frac{\sigma^2}{b}\int_0^T \mathrm{e}^{-(T-s)b}(1-\mathrm{e}^{-(T-s)b})ds\\ &= r_0\,\mathrm{e}^{-bT} + \frac{a}{b}(1-\mathrm{e}^{-bT}) + \sigma\int_0^T \mathrm{e}^{-(T-s)b} d\widehat{W}_s\\ &\quad -\frac{\sigma^2}{b}\int_0^T \mathrm{e}^{-(T-s)b}ds + \frac{\sigma^2}{b}\int_0^T \mathrm{e}^{-2(T-s)b}ds,\end{aligned}$$

hence

$$\begin{aligned}\widehat{\mathbb{E}}[r_T \mid \mathscr{F}_t] &= r_0\,\mathrm{e}^{-bT} + \frac{a}{b}(1-\mathrm{e}^{-bT}) + \sigma\int_0^t \mathrm{e}^{-(T-s)b} d\widehat{W}_s\\ &\quad -\frac{\sigma^2}{b}\int_0^T \mathrm{e}^{-(T-s)b}ds + \frac{\sigma^2}{b}\int_0^T \mathrm{e}^{-2(T-s)b}ds\\ &= r_0\,\mathrm{e}^{-bT} + \frac{a}{b}(1-\mathrm{e}^{-bT}) + \sigma\int_0^t \mathrm{e}^{-(T-s)b} dW_s\\ &\quad +\frac{\sigma^2}{b}\int_0^t \mathrm{e}^{-(T-s)b}(1-\mathrm{e}^{-(T-s)b})ds\\ &\quad -\frac{\sigma^2}{b}\int_0^T \mathrm{e}^{-(T-s)b}ds + \frac{\sigma^2}{b}\int_0^T \mathrm{e}^{-2(T-s)b}ds\\ &= r_0\,\mathrm{e}^{-bT} + \frac{a}{b}(1-\mathrm{e}^{-bT}) + \mathrm{e}^{-(T-t)b}\left(r_t - r_0\,\mathrm{e}^{-bt} - \frac{a}{b}(1-\mathrm{e}^{-bt})\right)\\ &\quad +\frac{\sigma^2}{b}\int_0^t \mathrm{e}^{-(T-s)b}ds - \frac{\sigma^2}{b}\int_0^t \mathrm{e}^{-2(T-s)b}ds\\ &\quad -\frac{\sigma^2}{b}\int_0^T \mathrm{e}^{-(T-s)b}ds + \frac{\sigma^2}{b}\int_0^T \mathrm{e}^{-2(T-s)b}ds\\ &= \frac{a}{b} + \mathrm{e}^{-(T-t)b}\left(r_t - \frac{a}{b}\right) - \frac{\sigma^2}{b}\int_t^T \mathrm{e}^{-(T-s)b}ds + \frac{\sigma^2}{b}\int_t^T \mathrm{e}^{-2(T-s)b}ds\\ &= \frac{a}{b} + \mathrm{e}^{-(T-t)b}\left(r_t - \frac{a}{b}\right) - \frac{\sigma^2}{b}\int_0^{T-t} \mathrm{e}^{-bs}ds + \frac{\sigma^2}{b}\int_0^{T-t} \mathrm{e}^{-2bs}ds\\ &= \frac{a}{b} + \mathrm{e}^{-(T-t)b}\left(r_t - \frac{a}{b}\right) - \frac{\sigma^2}{b^2}(1-\mathrm{e}^{-(T-t)b}) + \frac{\sigma^2}{b^2}(1-\mathrm{e}^{-2(T-t)b})\end{aligned}$$

$$= \frac{a}{b} - \frac{\sigma^2}{2b^2} + \mathrm{e}^{-(T-t)b}\left(r_t - \frac{a}{b} + \frac{\sigma^2}{b^2}\right) - \frac{\sigma^2}{b^2}\mathrm{e}^{-2(T-t)b},$$

which recovers (2.4.1).

Exercise 2.2 We have

$$P(0,T_2) = \exp\left(-\int_0^{T_2} f(t,s)ds\right) = \mathrm{e}^{-r_1T_1-r_2(T_2-T_1)}, \qquad t\in[0,T_2],$$

and

$$P(T_1,T_2) = \exp\left(-\int_{T_1}^{T_2} f(t,s)ds\right) = \mathrm{e}^{-r_2(T_2-T_1)}, \qquad t\in[0,T_2],$$

from which we deduce

$$r_2 = -\frac{1}{T_2-T_1}\log P(T_1,T_2),$$

and

$$\begin{aligned} r_1 &= -r_2\frac{T_2-T_1}{T_1} - \frac{1}{T_1}\log P(0,T_2) \\ &= \frac{1}{T_1}\log P(T_1,T_2) - \frac{1}{T_1}\log P(0,T_2) \\ &= -\frac{1}{T_1}\log\frac{P(0,T_2)}{P(T_1,T_2)}. \end{aligned}$$

Exercise 2.3

a) In the Vašíček, 1977 model, by (1.4.15) we have

$$\begin{aligned} P(t,T) &= \mathbb{E}\left[\exp\left(-\int_t^T r_s ds\right)\right] \\ &= \mathbb{E}\left[\exp\left(-\int_t^T h(s)ds - \int_t^T X_s ds\right)\right] \\ &= \exp\left(-\int_t^T h(s)ds\right)\mathbb{E}\left[\exp\left(-\int_t^T X_s ds\right)\right] \\ &= \exp\left(-\int_t^T h(s)ds + A(T-t) + X_tC(T-t)\right), \quad 0\leqslant t\leqslant T, \end{aligned}$$

hence, since $X_0 = 0$ we find $P(0,T) = \exp\left(-\int_0^T h(s)ds + A(T)\right)$.

b) By the identification

$$\begin{aligned} P(t,T) &= \exp\left(-\int_t^T h(s)ds + A(T-t) + X_tC(T-t)\right) \\ &= \exp\left(-\int_t^T f(t,s)ds\right), \end{aligned}$$

we find

$$\int_t^T h(s)ds = \int_t^T f(t,s)ds + A(T-t) + X_tC(T-t),$$

and by differentiation with respect ot T this yields

$$h(T) = f(t,T) + A'(T-t) + X_tC'(T-t), \qquad 0\leqslant t\leqslant T,$$

where

$$A(T-t) = \frac{4ab-3\sigma^2}{4b^3} + \frac{\sigma^2-2ab}{2b^2}(T-t) + \frac{\sigma^2-ab}{b^3}\,\mathrm{e}^{-b(T-t)} - \frac{\sigma^2}{4b^3}\,\mathrm{e}^{-2b(T-t)}.$$

Given an initial market data curve $f^M(0,T)$, the matching $f^M(0,T) = f(0,T)$ can be achieved at time $t=0$ by letting

$$h(T) := f^M(0,T) + A'(T) = f^M(0,T) + \frac{\sigma^2-2ab}{2b^2} - \frac{\sigma^2-ab}{b^2}\,\mathrm{e}^{-bT} + \frac{\sigma^2}{2b^2}\,\mathrm{e}^{-2bT},$$

$T > 0$. Note however that in general, at time $t \in (0,T]$ we will have

$$h(T) = f(t,T) + A'(T-t) + X_t C'(T-t) = f^M(0,T) + A'(T),$$

and the relation

$$f(t,T) = f^M(0,T) + A'(T) - A'(T-t) - X_t C'(T-t), \quad t \in [0,T],$$

will allow us to match market data at time $t=0$ only, *i.e.* for the initial curve. In any case, model calibration is to be done at time $t=0$.

Exercise 2.4 (See also Proposition 4.1 in Carmona and Durrleman, 2003). Letting $\sigma := \sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}$, we have

$$dS_t = rS_t dt + \sigma dW_t,$$

where $(W_t)_{t\in[0,T]}$, is a standard Brownian motions under $\mathbb{P}^*$, hence

$$S_t = S_0\,\mathrm{e}^{rt} + \sigma\int_0^t \mathrm{e}^{(t-s)r} dW_s, \quad t \geqslant 0,$$

The spread S_T has a Gaussian distribution with mean $\alpha := \mathbb{E}^*[S_T] = S_0\,\mathrm{e}^{rT}$ and variance

$$\begin{aligned}
\eta^2 &:= \mathrm{Var}^*[S_T] \\
&= \mathrm{Var}\left[\sigma\int_0^T \mathrm{e}^{(T-s)r} dB_s\right] \\
&= \sigma^2\int_0^T \left(\mathrm{e}^{(T-s)r}\right)^2 ds \\
&= \frac{\sigma^2}{2r}\left(\mathrm{e}^{2rT}-1\right),
\end{aligned}$$

and probability density function

$$\varphi(x) = \frac{\sqrt{r/\pi}}{\sigma\sqrt{\mathrm{e}^{2rT}-1}}\exp\left(-\frac{\left(S_0\,\mathrm{e}^{rT}-x\right)^2}{\sigma^2\left(\mathrm{e}^{2rT}-1\right)/r}\right), \quad x \in \mathbb{R}.$$

Hence, we have

$$\begin{aligned}
\mathrm{e}^{-rt}\,\mathbb{E}^*[(S_T-K)^+] &= \frac{\mathrm{e}^{-rt}}{\sqrt{2\pi\eta^2}}\int_{-\infty}^{\infty}(x-K)^+\,\mathrm{e}^{-(x-\alpha)^2/(2\eta^2)}dx \\
&= \frac{\mathrm{e}^{-rt}}{\sqrt{2\pi\eta^2}}\int_K^{\infty}(x-K)\,\mathrm{e}^{-(x-\alpha)^2/(2\eta^2)}dx \\
&= \frac{\mathrm{e}^{-rt}}{\sqrt{2\pi\eta^2}}\int_K^{\infty}x\,\mathrm{e}^{-(x-\alpha)^2/(2\eta^2)}dx - \frac{K\,\mathrm{e}^{-rt}}{\sqrt{2\pi\eta^2}}\int_K^{\infty}\mathrm{e}^{-(x-\alpha)^2/(2\eta^2)}dx
\end{aligned}$$

$$\begin{aligned}
&= \frac{\eta\,\mathrm{e}^{-rt}}{\sqrt{2\pi}}\int_{(K-\alpha)/\eta}^{\infty}(x+\alpha)\,\mathrm{e}^{-x^2/2}dx - \frac{K\,\mathrm{e}^{-rt}}{\sqrt{2\pi}}\int_{(K-\alpha)/\eta}^{\infty}\mathrm{e}^{-x^2/2}dx \\
&= -\frac{\eta\,\mathrm{e}^{-rt}}{\sqrt{2\pi}}\left[\mathrm{e}^{-x^2/2}\right]_{(K-\alpha)/\eta}^{\infty} - (K-\alpha)\,\mathrm{e}^{-rt}\Phi\left(-\frac{K-\alpha}{\eta}\right) \\
&= \frac{\eta\,\mathrm{e}^{-rt}}{\sqrt{2\pi}}\mathrm{e}^{-(K-\alpha)^2/(2\eta^2)} - (K-\alpha)\,\mathrm{e}^{-rt}\Phi\left(-\frac{K-\alpha}{\eta}\right).
\end{aligned}$$

Exercise 2.5 From the definition

$$L(t,t,T) = \frac{1}{T-t}\left(\frac{1}{P(t,T)}-1\right),$$

we have

$$P(t,T) = \frac{1}{1+(T-t)L(t,t,T)},$$

and similarly

$$P(t,S) = \frac{1}{1+(S-t)L(t,t,S)}.$$

Hence we get

$$\begin{aligned}
L(t,T,S) &= \frac{1}{S-T}\left(\frac{P(t,T)}{P(t,S)}-1\right) \\
&= \frac{1}{S-T}\left(\frac{1+(S-t)L(t,t,S)}{1+(T-t)L(t,t,T)}-1\right) \\
&= \frac{1}{S-T}\left(\frac{(S-t)L(t,t,S)-(T-t)L(t,t,T)}{1+(T-t)L(t,t,T)}\right).
\end{aligned}$$

When $T =$ one year and $L(0,0,T) = 2\%$, $L(0,0,2T) = 2.5\%$ we find

$$L(t,T,S) = \frac{1}{T}\left(\frac{2TL(0,0,2T)-TL(0,0,T)}{1+TL(0,0,T)}\right) = \frac{2\times 0.025-0.02}{1+0.02} = 2.94\%,$$

so we would not prefer a spot rate at $L(T,T,2T) = 2\%$ over a forward contract with rate $L(0,T,2T) = 2.94\%$.

Chapter 3

Exercise 3.1

a) We have

$$\begin{aligned}
d\widehat{X}_t &= d\left(\frac{X_t}{N_t}\right) \\
&= \frac{X_0}{N_0}d\left(\mathrm{e}^{(\sigma-\eta)B_t-(\sigma^2-\eta^2)t/2}\right) \\
&= \frac{X_0}{N_0}(\sigma-\eta)\,\mathrm{e}^{(\sigma-\eta)B_t-(\sigma^2-\eta^2)t/2}dB_t \\
&\quad + \frac{X_0}{2N_0}(\sigma-\eta)^2\,\mathrm{e}^{(\sigma-\eta)B_t-(\sigma^2-\eta^2)t/2}dt \\
&\quad - \frac{X_0}{2N_0}(\sigma^2-\eta^2)\,\mathrm{e}^{(\sigma-\eta)B_t-(\sigma^2-\eta^2)t/2}dt
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{X_t}{2N_t}(\sigma^2-\eta^2)dt + \frac{X_t}{N_t}(\sigma-\eta)dB_t + \frac{X_t}{2N_t}(\sigma-\eta)^2dt \\
&= -\frac{X_t}{N_t}\eta(\sigma-\eta)dt + \frac{X_t}{N_t}(\sigma-\eta)dB_t \\
&= \frac{X_t}{N_t}(\sigma-\eta)(dB_t-\eta dt) \\
&= (\sigma-\eta)\frac{X_t}{N_t}d\widehat{B}_t = (\sigma-\eta)\widehat{X}_t d\widehat{B}_t,
\end{aligned}
$$

where $d\widehat{B}_t = dB_t - \eta dt$ is a *standard Brownian motion* under $\widehat{\mathbb{P}}$.

b) By change of numéraire, we have

$$
\mathbb{E}[(X_T-\lambda N_T)^+] = \widehat{\mathbb{E}}\left[\frac{N_0}{N_T}(X_T-\lambda N_T)^+\right] = N_0\widehat{\mathbb{E}}\big[(\widehat{X}_T-\lambda)^+\big].
$$

Next, by the result of Question (a)), $\widehat{X}_t$ is a driftless geometric Brownian motion with volatility $\sigma-\eta$ under $\widehat{\mathbb{P}}$, hence we have

$$
\widehat{\mathbb{E}}[(\widehat{X}_T-\lambda)^+] = \widehat{X}_0\Phi\left(\frac{\log(\widehat{X}_0/\lambda)}{\widehat{\sigma}\sqrt{T}} + \frac{\widehat{\sigma}\sqrt{T}}{2}\right) - \lambda\Phi\left(\frac{\log(\widehat{X}_0/\lambda)}{\widehat{\sigma}\sqrt{T}} - \frac{\widehat{\sigma}\sqrt{T}}{2}\right),
$$

by the Black-Scholes formula with <u>zero interest rate</u> and volatility parameter <u>$\widehat{\sigma} = \sigma - \eta$</u>. By multiplication by N_0 and the relation $X_0 = N_0\widehat{X}_0$ we conclude to (3.5.5), *i.e.*

$$
\begin{aligned}
\mathbb{E}\left[(X_T-\lambda N_T)^+\right] &= N_0\widehat{\mathbb{E}}\big[(\widehat{X}_T-\lambda)^+\big] \\
&= N_0\widehat{X}_0\Phi(d_+) - \lambda N_0\Phi(d^-) \\
&= X_0\Phi(d_+) - \lambda N_0\Phi(d^-).
\end{aligned}
$$

c) We have $\widehat{\sigma} = \sigma - \eta$.

Exercise 3.2

a) By the Girsanov Theorem, the processes

$$
d\widehat{B}_t^{(1)} = dB_t^{(1)} - \frac{1}{N_t}dN_t \bullet dB_t^{(1)} = dB_t^{(1)} - \frac{1}{S_t^{(2)}}dS_t^{(2)} \bullet dB_t^{(1)} = dB_t^{(1)} - \eta\rho dt,
$$

and

$$
d\widehat{B}_t^{(2)} = dB_t^{(2)} - \frac{1}{N_t}dN_t \bullet dB_t^{(2)} = dB_t^{(2)} - \frac{1}{S_t^{(2)}}dS_t^{(2)} \bullet dB_t^{(2)} = dB_t^{(2)} - \eta dt
$$

are both standard Brownian motions (and martingales) under $\widehat{\mathbb{P}}_2$.

b) We have

$$
\begin{aligned}
d\widehat{S}_t^{(1)} &= d\left(\frac{S_t^{(1)}}{S_t^{(2)}}\right) \\
&= \frac{S_0^{(1)}}{S_0^{(2)}}d\left(\mathrm{e}^{\sigma B_t^{(1)}-\eta B_t^{(2)}-(\sigma^2-\eta^2)t/2}\right) \\
&= \frac{S_0^{(1)}}{S_0^{(2)}}\mathrm{e}^{\sigma B_t^{(1)}-\eta B_t^{(2)}-(\sigma^2-\eta^2)t/2} \\
&\quad \times\left(\sigma dB_t^{(1)} + \frac{\sigma^2}{2}dt - \eta dB_t^{(2)} + \frac{\eta^2}{2}dt - \frac{\sigma^2-\eta^2}{2}dt - \sigma\eta\rho dt\right) \\
&= \widehat{S}_t^{(1)}\big(\sigma d\widehat{B}_t^{(1)} - \eta d\widehat{B}_t^{(2)}\big).
\end{aligned}
$$

c) We note that the driftless geometric Brownian motion $(\widehat{S}_t^{(1)})_{t\in\mathbb{R}}$ can be written as

$$d\widehat{S}_t^{(1)} = \widehat{\sigma}\widehat{S}_t^{(1)} d\widehat{W}_t,$$

where $(\widehat{W}_t)_{t\in\mathbb{R}}$ is a standard Brownian motion under $\widehat{\mathbb{P}}_2$. In order to determine $\widehat{\sigma}$ we note that

$$\begin{aligned}
\widehat{\sigma}^2 dt &= \widehat{\sigma}^2 d\widehat{W}_t \bullet d\widehat{W}_t \\
&= \frac{d\widehat{S}_t^{(1)}}{\widehat{S}_t^{(1)}} \bullet \frac{d\widehat{S}_t^{(1)}}{\widehat{S}_t^{(1)}} \\
&= \left(\sigma dB_t^{(1)} - \eta d\widehat{B}_t^{(2)}\right) \cdot \left(\sigma dB_t^{(1)} - \eta d\widehat{B}_t^{(2)}\right) \\
&= (\sigma^2 + \eta^2 - 2\sigma\eta\rho)dt,
\end{aligned}$$

hence $\widehat{\sigma}^2 = \sigma^2 + \eta^2 - 2\sigma\eta\rho$. We conclude by applying the change of numéraire formula

$$\mathrm{e}^{-rT}\,\mathbb{E}^*\left[\left(S_T^{(1)} - \lambda S_T^{(2)}\right)^+\right] = S_0^{(2)}\widehat{\mathbb{E}}\left[\left(\widehat{S}_T^{(1)} - \lambda\right)^+\right]$$

and the Black-Scholes formula to the the driftless geometric Brownian motion $(\widehat{S}_t^{(1)})_{t\in\mathbb{R}}$.

Exercise 3.3 We have $N_t = P(t,T)$ and from (1.4.8) and the relations $P(t,T) = F(t,r_t)$ and $P(t,S) = G(t,r_t)$ we find

$$\begin{cases}
\dfrac{dP(t,S)}{P(t,S)} = r_t dt + \sigma(t,r_t)\dfrac{\partial}{\partial x}\log G(t,r_t)dW_t, \\[2ex]
\dfrac{dN_t}{N_t} = \dfrac{dP(t,T)}{P(t,T)} = r_t dt + \sigma(t,r_t)\dfrac{\partial}{\partial x}\log F(t,r_t)dW_t.
\end{cases}$$

By the Girsanov Theorem (3.2.12) we also have

$$d\widehat{W}_t = dW_t - \frac{dN_t}{N_t} \bullet dW_t = dW_t - \sigma(t,r_t)\frac{\partial}{\partial x}\log F(t,r_t)dt,$$

hence

$$\frac{dP(t,S)}{P(t,S)} = r_t dt + \sigma^2(t,r_t)\frac{\partial}{\partial x}\log F(t,r_t)\frac{\partial}{\partial x}\log G(t,r_t)dt + \sigma(t,r_t)\frac{\partial}{\partial x}\log G(t,r_t)d\widehat{W}_t.$$

Using the relation $P(t,S) = G(t,r_t)$ we can also write

$$dP(t,S) = r_t P(t,S)dt + \sigma^2(t,r_t)\frac{\partial}{\partial x}\log F(t,r_t)\frac{\partial}{\partial x}G(t,r_t)dt + \sigma(t,r_t)\frac{\partial}{\partial x}G(t,r_t)d\widehat{W}_t.$$

Exercise 3.4 Forward contract. Taking $N_t := P(t,T)$, $t \in [0,T]$, we have

$$\begin{aligned}
\mathbb{E}^*\left[\exp\left(-\int_t^T r_s ds\right)(P(T,S) - K)\,\Big|\,\mathscr{F}_t\right] &= N_t\widehat{\mathbb{E}}\left[\frac{(P(T,S) - K)}{N_T}\,\Big|\,\mathscr{F}_t\right] \\
&= P(t,T)\widehat{\mathbb{E}}\left[\frac{(P(T,S) - K)}{P(T,T)}\,\Big|\,\mathscr{F}_t\right] \\
&= P(t,T)\widehat{\mathbb{E}}[P(T,S) - K \mid \mathscr{F}_t] \\
&= P(t,T)\widehat{\mathbb{E}}[P(T,S) \mid \mathscr{F}_t] - KP(t,T) \\
&= P(t,T)\widehat{\mathbb{E}}\left[\frac{P(T,S)}{P(T,T)} \mid \mathscr{F}_t\right] - KP(t,T)
\end{aligned}$$

$$
\begin{aligned}
&= P(t,T)\frac{P(t,S)}{P(t,T)} - KP(t,T)\\
&= P(t,S) - KP(t,T),
\end{aligned}
$$

since

$$
t \longmapsto \frac{P(t,T)}{N_t} = \frac{P(t,S)}{P(t,T)}
$$

is a martingale under the forward measure $\widehat{\mathbb{P}}$. The corresponding (static) hedging strategy is given by buying one bond with maturity S and by short selling K units of the bond with maturity T.

Remark: The above result can also be obtained by a direct argument using the tower property of conditional expectations:

$$
\begin{aligned}
&\mathbb{E}^*\left[\exp\left(-\int_t^T r_s ds\right)(P(T,S)-K)\,\Big|\,\mathscr{F}_t\right]\\
&= \mathbb{E}^*\left[\exp\left(-\int_t^T r_s ds\right)\left(\mathbb{E}^*\left[\exp\left(-\int_T^S r_s ds\right)\,\Big|\,\mathscr{F}_T\right]-K\right)\,\Big|\,\mathscr{F}_t\right]\\
&= \mathbb{E}^*\left[\exp\left(-\int_t^T r_s ds\right)\mathbb{E}^*\left[\exp\left(-\int_T^S r_s ds\right)-K\,\Big|\,\mathscr{F}_T\right]\,\Big|\,\mathscr{F}_t\right]\\
&= \mathbb{E}^*\left[\mathbb{E}^*\left[\exp\left(-\int_t^S r_s ds\right)-K\exp\left(-\int_t^T r_s ds\right)\,\Big|\,\mathscr{F}_T\right]\,\Big|\,\mathscr{F}_t\right]\\
&= \mathbb{E}^*\left[\exp\left(-\int_t^S r_s ds\right)-K\exp\left(-\int_t^T r_s ds\right)\,\Big|\,\mathscr{F}_t\right]\\
&= P(t,S)-KP(t,T), \qquad t\in[0,T].
\end{aligned}
$$

Exercise 3.5

a) We choose $N_t := S_t$ as numéraire because this allows us to write the option payoff as $(S_T(S_T-K))^+ = N_T(S_T-K)^+$. In this case, the forward measure $\widehat{\mathbb{P}}$ satisfies

$$
\frac{d\widehat{\mathbb{P}}}{d\mathbb{P}} = \mathrm{e}^{-rT}\frac{N_T}{N_0} = \mathrm{e}^{-rT}\frac{S_T}{S_0},
$$

or

$$
\frac{d\widehat{\mathbb{P}}_{|\mathscr{F}_t}}{d\mathbb{P}_{|\mathscr{F}_t}} = \mathrm{e}^{-(T-t)r}\frac{N_T}{N_t} = \mathrm{e}^{-(T-t)r}\frac{S_T}{S_t}, \qquad 0\leqslant t\leqslant T.
$$

b) By the change of numéraire formula of Proposition 3.5, the option price becomes

$$
\begin{aligned}
\mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[(S_T(S_T-K))^+\,\big|\,\mathscr{F}_t\right] &= \mathbb{E}^*\left[\mathrm{e}^{-(T-t)r}N_T(S_T-K)^+\,\big|\,\mathscr{F}_t\right]\\
&= N_t\widehat{\mathbb{E}}\left[(S_T-K)^+\,\big|\,\mathscr{F}_t\right]\\
&= S_t\widehat{\mathbb{E}}\left[(S_T-K)^+\,\big|\,\mathscr{F}_t\right]. \qquad \text{(A.6)}
\end{aligned}
$$

c) In order to compute (A.6) it remains to determine the dynamics of $(S_t)_{t\in\mathbb{R}_+}$ under $\widehat{\mathbb{P}}$. Since $S_t = S_0\,\mathrm{e}^{\sigma B_t + rt - \sigma^2 t/2}$, we have

$$
\frac{d\widehat{\mathbb{P}}}{d\mathbb{P}} = \mathrm{e}^{-rT}\frac{S_T}{S_0} = \mathrm{e}^{\sigma B_T - \sigma^2 T/2},
$$

hence by the Girsanov Theorem, $\widehat{B}_t := B_t - \sigma t$ is a standard Brownian motion under $\widehat{\mathbb{P}}$, with

$$
\begin{aligned}
S_T &= S_0\,\mathrm{e}^{rT+\sigma B_T-\sigma^2 T/2}\\
&= S_0\,\mathrm{e}^{(r+\sigma^2)T+\sigma\widehat{B}_T-\sigma^2 T/2}\\
&= S_t\,\mathrm{e}^{(r+\sigma^2)(T-t)+(\widehat{B}_T-\widehat{B}_t)\sigma-(T-t)\sigma^2/2}, \qquad 0\leqslant t\leqslant T.
\end{aligned}
$$

d) According to the above, (A.6) becomes

$$\begin{aligned}
&\mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[\big(S_T(S_T-K)\big)^+\,\big|\,\mathcal{F}_t\right]=S_t\widehat{\mathbb{E}}\big[\big(S_T-K\big)^+\,\big|\,\mathcal{F}_t\big]\\
&= S_t\widehat{\mathbb{E}}\big[\big(S_0\,\mathrm{e}^{rT+\sigma^2T+\sigma\widehat{B}_T-\sigma^2T/2}-K\big)^+\,\big|\,\mathcal{F}_t\big]\\
&= S_t\widehat{\mathbb{E}}\big[\big(S_t\,\mathrm{e}^{(r+\sigma^2)(T-t)+(\widehat{B}_T-\widehat{B}_t)\sigma-(T-t)\sigma^2/2}-K\big)^+\,\big|\,\mathcal{F}_t\big]\\
&= S_t\,\mathrm{e}^{(T-t)(r+\sigma^2)}\mathrm{Bl}(S_t,K,r+\sigma^2,\sigma,T-t),\qquad 0\leqslant t\leqslant T,
\end{aligned}$$

since the Black-Scholes formula with interest rate $r+\sigma^2$ reads

$$\begin{aligned}
&\mathrm{e}^{-(T-t)(r+\sigma^2)}\widehat{\mathbb{E}}\big[\big(S_t\,\mathrm{e}^{(r+\sigma^2)(T-t)+(\widehat{B}_T-\widehat{B}_t)\sigma-(T-t)\sigma^2/2}-K\big)^+\,\big|\,\mathcal{F}_t\big]\\
&= \mathrm{Bl}(S_t,K,r+\sigma^2,\sigma,T-t),\qquad 0\leqslant t\leqslant T.
\end{aligned}$$

Remarks:

i) The option price can be rewritten using other Black-Scholes parametrizations, such as for example

$$S_t\mathrm{Bl}\big(S_t\,\mathrm{e}^{(T-t)(r+\sigma^2)},K,0,\sigma,T-t\big),$$

or

$$S_t\,\mathrm{e}^{(T-t)(r+\sigma^2)}\mathrm{Bl}\big(S_t,K\,\mathrm{e}^{-(T-t)(r+\sigma^2)},0,\sigma,T-t\big),$$

however we prefer to choose the simplest possibility.

ii) Deflated (or forward) processes such as $S_t/N_1=1$ or

$$\frac{\mathrm{e}^{-(T-t)r}}{N_t}\mathbb{E}^*\left[(S_T(S_T-K))^+\,\big|\,\mathcal{F}_t\right]=\widehat{\mathbb{E}}\big[(S_T-K)^+\,\big|\,\mathcal{F}_t\big],\quad 0\leqslant t\leqslant T,$$

are martingales under the forward measure $\widehat{\mathbb{P}}$.

iii) This option can also be priced via an integral calculation instead of using change of numéraire, as follows:

$$\begin{aligned}
&\mathrm{e}^{-(T-t)r}\mathbb{E}^*[S_T(S_T-K)^+\mid\mathcal{F}_t]\\
&= \mathrm{e}^{-(T-t)r}\mathbb{E}^*\left[S_t\,\mathrm{e}^{(T-t)r+(B_T-B_t)\sigma-(T-t)\sigma^2/2}\right.\\
&\qquad\left.\times(S_t\,\mathrm{e}^{(T-t)r+(B_T-B_t)\sigma-(T-t)\sigma^2/2}-K)^+\,\big|\,\mathcal{F}_t\right]\\
&= S_t\,\mathrm{e}^{-(T-t)\sigma^2/2}\mathbb{E}^*\left[(S_t\,\mathrm{e}^{(T-t)r+2(B_T-B_t)\sigma-(T-t)\sigma^2/2}-K\,\mathrm{e}^{(B_T-B_t)\sigma})^+\,\big|\,\mathcal{F}_t\right]\\
&= S_t\,\mathrm{e}^{-(T-t)\sigma^2/2}\mathbb{E}^*\left[(x\,\mathrm{e}^{(T-t)r+2(B_T-B_t)\sigma-(T-t)\sigma^2/2}-K\,\mathrm{e}^{(B_T-B_t)\sigma})^+\right]_{x=S_t}\\
&= S_t\,\mathrm{e}^{-(T-t)\sigma^2/2}\mathbb{E}^*\left[(\,\mathrm{e}^{m(x)+2X}-K\,\mathrm{e}^{X})^+\right]_{x=S_t},\qquad 0\leqslant t\leqslant T,
\end{aligned}$$

where $X\simeq\mathcal{N}(0,v^2)$ with $v^2=(T-t)\sigma^2$ and $m(x)=(T-t)r-(T-t)\sigma^2/2+\log x$. Next, we note that

$$\begin{aligned}
&\mathbb{E}\left[\big(\,\mathrm{e}^{m+2X}-K\,\mathrm{e}^{X}\big)^+\right]=\frac{1}{\sqrt{2\pi v^2}}\int_{-\infty}^{\infty}\big(\,\mathrm{e}^{m+2x}-K\,\mathrm{e}^{x}\big)^+\,\mathrm{e}^{-x^2/(2v^2)}dx\\
&=\frac{1}{\sqrt{2\pi v^2}}\int_{-m+\log K}^{\infty}(\,\mathrm{e}^{m+2x}-K\,\mathrm{e}^{x})\,\mathrm{e}^{-x^2/(2v^2)}dx\\
&=\frac{e^m}{\sqrt{2\pi v^2}}\int_{-m+\log K}^{\infty}\mathrm{e}^{2x-x^2/(2v^2)}dx-\frac{K}{\sqrt{2\pi v^2}}\int_{-m+\log K}^{\infty}\mathrm{e}^{x-x^2/(2v^2)}dx\\
&=\frac{\mathrm{e}^{m+2v^2}}{\sqrt{2\pi v^2}}\int_{-m+\log K}^{\infty}\mathrm{e}^{-(2v^2-x)^2/(2v^2)}dx-\frac{K\,\mathrm{e}^{v^2/2}}{\sqrt{2\pi}}\int_{-m+\log K}^{\infty}\mathrm{e}^{-(v^2-x)^2/(2v^2)}dx\\
&=\frac{\mathrm{e}^{m+2v^2}}{\sqrt{2\pi v^2}}\int_{-2v^2-m+\log K}^{\infty}\mathrm{e}^{-x^2/(2v^2)}dx-\frac{K\,\mathrm{e}^{v^2/2}}{\sqrt{2\pi v^2}}\int_{-v^2-m+\log K}^{\infty}\mathrm{e}^{-x^2/(2v^2)}dx\\
&=\frac{\mathrm{e}^{m+2v^2}}{\sqrt{2\pi}}\int_{(-2v^2-m+\log K)/v}^{\infty}\mathrm{e}^{-x^2/2}dx-\frac{K\,\mathrm{e}^{v^2/2}}{\sqrt{2\pi}}\int_{(-v^2-m+\log K)/v}^{\infty}\mathrm{e}^{-x^2/2}dx
\end{aligned}$$

$$= e^{m+2v^2}\Phi\left(2v+\frac{m-\log K}{v}\right)-K\,e^{v^2/2}\Phi\left(v+\frac{m-\log K}{v}\right),$$

hence

$$\begin{aligned}
& e^{-(T-t)r}\,\mathbb{E}^*\left[S_T\left(S_T-K\right)^+\,\middle|\,\mathcal{F}_t\right] \\
&= S_t\,e^{-(T-t)\sigma^2/2}\,\mathbb{E}^*\left[\left(e^{m(x)+2X}-K\,e^X\right)^+\right]_{x=S_t} \\
&= S_t^2\,e^{(T-t)(r+\sigma^2)}\Phi\left(\frac{(T-t)(r+\sigma^2)+(T-t)\sigma^2/2+\log(S_t/K)}{\sigma\sqrt{T-t}}\right) \\
&\quad -KS_t\Phi\left(\frac{(T-t)(r+\sigma^2)-(T-t)\sigma^2/2+\log(S_t/K)}{\sigma\sqrt{T-t}}\right) \\
&= S_t\,e^{(T-t)(r+\sigma^2)}\mathrm{Bl}(S_t,K,r+\sigma^2,\sigma,T-t), \qquad 0\leqslant t\leqslant T.
\end{aligned}$$

Chapter 4

Exercise 4.1

a) We price the floorlet at $t=0$, with $T_1=9$ months, $T_2=1$ year, $\kappa=4.5\%$. The LIBOR rate $(L(t,T_1,T_2))_{t\in[0,T_1]}$ is modeled as a driftless geometric Brownian motion with volatility coefficient $\widehat{\sigma}=\sigma_{1,2}(t)=0.1$ under the forward measure $\mathbb{P}_2$. The discount factors are given by

$$P(0,T_1)=e^{-9r/12}\simeq 0.970809519$$

and

$$P(0,T_2)=e^{-r}\simeq 0.961269954,$$

with $r=3.95\%$.

b) By (4.3.9), the price of the floorlet is

$$\begin{aligned}
&\mathbb{E}^*\left[e^{-\int_0^{T_2}r_s ds}(\kappa-L(T_1,T_1,T_2))^+\right] \\
&= P(0,T_2)\left(\kappa\Phi(-d_-(T_1))-L(0,T_1,T_2)\Phi(-d_+(T_1))\right), \qquad \text{(A.7)}
\end{aligned}$$

where

$$d_+(T_1)=\frac{\log(L(0,T_1,T_2)/\kappa)+\sigma^2T_1/2}{\sigma_1\sqrt{T_1}},$$

and

$$d_-(T_1)=\frac{\log(L(0,T_1,T_2)/\kappa)-\sigma^2T_1/2}{\sigma\sqrt{T_1}},$$

are given in Proposition 4.5, and the LIBOR rate $L(0,T_1,T_2)$ is given by

$$\begin{aligned}
L(0,T_1,T_2) &= \frac{P(0,T_1)-P(0,T_2)}{(T_2-T_1)P(0,T_2)} \\
&= \frac{e^{-3r/4}-e^{-r}}{0.25\,e^{-r}} \\
&= 4(e^{r/4}-1) \\
&\simeq 3.9695675\%.
\end{aligned}$$

Hence, we have

$$d_+(T_1)=\frac{\log(0.039695675/0.045)+(0.1)^2\times 0.75/2}{0.1\times\sqrt{0.75}}\simeq -1.404927033,$$

and

$$d_-(T_1)=\frac{\log(0.039695675/0.045)-(0.1)^2\times 0.75/2}{0.1\times\sqrt{0.75}}\simeq -1.491529573,$$

hence

$$\begin{aligned}
&\mathbb{E}^*\left[\mathrm{e}^{-\int_0^{T_2} r_s ds}(\kappa - L(T_1,T_1,T_2))^+\right] \\
&= 0.961269954 \times (\kappa\Phi(1.491529573) - L(0,T_1,T_2) \times \Phi(1.404927033)) \\
&= 0.961269954 \times (0.045 \times 0.932089 - 0.039695675 \times 0.919979) \\
&\simeq 0.52147141\%.
\end{aligned}$$

Finally, we need to multiply (A.7) by the notional principal amount of \$1 million per interest rate percentage point, *i.e.* \$10,000 per percentage point or \$100 per basis point, which yields \$5214.71.

Exercise 4.2

a) We price the swaption at $t = 0$, with $T_1 = 4$ years, $T_2 = 5$ years, $T_3 = 6$ years, $T_4 = 7$ years, $\kappa = 5\%$, and the swap rate $(S(t,T_1,T_4))_{t\in[0,T_1]}$ is modeled as a driftless geometric Brownian motion with volatility coefficient $\widehat{\sigma} = \sigma_{1,4}(t) = 0.2$ under the forward swap measure $\mathbb{P}_{1,4}$. The discount factors are given by $P(0,T_1) = \mathrm{e}^{-4r}$, $P(0,T_2) = \mathrm{e}^{-5r}$, $P(0,T_3) = \mathrm{e}^{-6r}$, $P(0,T_4) = \mathrm{e}^{-7r}$, where $r = 5\%$.

b) By Proposition 4.15 the price of the swaption is

$$\begin{aligned}
&(P(0,T_1) - P(0,T_4))\Phi(d_+(T_1 - t)) \\
&\quad -\kappa\Phi(d_-(T_1))(P(0,T_2) + P(0,T_3) + P(0,T_4)),
\end{aligned}$$

where $d_+(T_1)$ and $d_-(T_1)$ are given in Proposition 4.15, and the LIBOR swap rate $S(0,T_1,T_4)$ is given by

$$\begin{aligned}
S(0,T_1,T_4) &= \frac{P(0,T_1) - P(0,T_4)}{P(0,T_1,T_4)} \\
&= \frac{P(0,T_1) - P(0,T_4)}{P(0,T_2) + P(0,T_3) + P(0,T_4)} \\
&= \frac{\mathrm{e}^{-4r} - \mathrm{e}^{-7r}}{\mathrm{e}^{-5r} + \mathrm{e}^{-6r} + \mathrm{e}^{-7r}} \\
&= \frac{\mathrm{e}^{3r} - 1}{\mathrm{e}^{2r} + \mathrm{e}^{r} + 1} \\
&= \frac{\mathrm{e}^{0.15} - 1}{\mathrm{e}^{0.1} + \mathrm{e}^{0.05} + 1} \\
&= 0.051271096.
\end{aligned}$$

By Proposition 4.15 we also have

$$d_+(T_1) = \frac{\log(0.051271096/0.05) + (0.2)^2 \times 4/2}{0.2\sqrt{4}} = 0.526161682,$$

and

$$d_-(T_1) = \frac{\log(0.051271096/0.05) - (0.2)^2 2 \times 4/2}{0.2\sqrt{4}} = 0.005214714,$$

Hence, the price of the swaption is given by

$$\begin{aligned}
&(\mathrm{e}^{-4r} - \mathrm{e}^{-7r})\Phi(0.526161682) \\
&\quad -\kappa\Phi(0.005214714)(\mathrm{e}^{-5r} + \mathrm{e}^{-6r} + \mathrm{e}^{-7r}) \\
&= (0.818730753 - 0.70468809) \times 0.700612 \\
&\quad -0.05 \times 0.50208 \times (0.818730753 + 0.740818221 + 0.70468809) \\
&= 2.3058251\%.
\end{aligned} \tag{A.8}$$

Finally, we need to multiply (A.8) by the notional principal amount of \$10 million, *i.e.* \$100,000 by interest percentage point, or \$1,000 by basis point, which yields \$230,582.51.

Exercise 4.3 Taking $t=0$, we have $T_1=3$, $T_2=4$ and $T_3=5$. The LIBOR swap rate $S(t,T_1,T_3)$ is modeled as a driftless geometric Brownian motion with volatility $\sigma=0.1$ under the forward swap measure $\widehat{P}_{i,j}$. The receiver swaption is priced using the Black-Scholes formula as

$$\begin{aligned}
&\mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_1} r_s ds} P(T_1,T_1,T_3)\left(\kappa-S(T_1,T_1,T_3)\right)^+ \Big| \mathscr{F}_t\right] \\
&= \kappa\Phi(-d_-(T_1-t))\sum_{k=1}^{2}(T_{k+1}-T_k)P(t,T_{k+1}) \\
&\quad -(P(t,T_1)-P(t,T_3))\Phi(-d_+(T_1-t)),
\end{aligned}$$

where $\kappa=5\%$, $r=2\%$ and $P(t,T_1)=\mathrm{e}^{-3r}=0.9417$, $P(t,T_2)=\mathrm{e}^{-4r}=0.9231$, $P(t,T_3)=\mathrm{e}^{-5r}=0.9048$. Hence,

$$P(t,T_1,T_3)=P(t,T_2)+P(t,T_3)=0.92311+0.90483=1.82794$$

and

$$S(t,T_1,T_3)=\frac{P(t,T_1)-P(t,T_3)}{P(t,T_1,T_3)}=\frac{0.9417-0.9048}{1.82794}=0.02018.$$

We also have

$$\begin{aligned}
d_+(T_1-t) &= \frac{\log(S(t,T_1,T_3)/\kappa)+\sigma^2(T_1-t)/2}{\sigma\sqrt{T_1-t}} \\
&= \frac{\log(2.018/5)+0.1^2\times 3/2}{0.1\sqrt{3}}=-5.1518
\end{aligned}$$

and

$$d_-(T_1-t)=d_+(T_1-t)-\sigma\sqrt{T_1-t}=-5.3250,$$

hence

$$\begin{aligned}
&\mathbb{E}^*\left[\mathrm{e}^{-\int_t^{T_1} r_s ds} P(T_1,T_1,T_3)\left(\kappa-S(T_1,T_1,T_3)\right)^+ \Big| \mathscr{F}_t\right] \\
&= 0.05\times 1.82794\times\Phi(5.3250)-(0.9417-0.9048)\times\Phi(5.1518) \\
&= 0.05\times 1.82794\times 0.9999999-(0.9417-0.9048)\times 0.9999999 \\
&= 0.054496 \\
&= 5.4496\% \\
&= 544.96\ \text{bp},
\end{aligned}$$

which yields \$54,496 after multiplication by the \$10,000 notional principal.

Exercise 4.4

a) We have

$$\begin{aligned}
d\left(\frac{P(t,T_2)}{P(t,T_1)}\right) &= \frac{dP(t,T_2)}{P(t,T_1)}+P(t,T_2)d\left(\frac{1}{P(t,T_1)}\right)+dP(t,T_2)\bullet d\left(\frac{1}{P(t,T_1)}\right) \\
&= \frac{dP(t,T_2)}{P(t,T_1)}+P(t,T_2)\left(-\frac{dP(t,T_1)}{(P(t,T_1))^2}+\frac{dP(t,T_1)\bullet dP(t,T_1)}{(P(t,T_1))^3}\right) \\
&\quad -\frac{dP(t,T_1)\bullet dP(t,T_2)}{(P(t,T_1))^2} \\
&= \frac{1}{P(t,T_1)}\left(r_tP(t,T_2)dt+\zeta_2(t)P(t,T_2)dW_t\right)
\end{aligned}$$

$$
\begin{aligned}
& -\frac{P(t,T_2)}{(P(t,T_1))^2}(r_tP(t,T_1)dt+\zeta_1(t)P(t,T_1)dW_t) \\
& +\frac{P(t,T_2)}{(P(t,T_1))^3}((r_tP(t,T_1)dt+\zeta_1(t)P(t,T_1)dW_t)^2) \\
& -\frac{1}{(P(t,T_1))^2}((r_tP(t,T_1)dt+\zeta_1(t)P(t,T_1)dW_t)\bullet(r_tP(t,T_2)dt+\zeta_2(t)P(t,T_2)dW_t)) \\
&= \zeta_2(t)\frac{P(t,T_2)}{P(t,T_1)}dW_t-\zeta_1(t)\frac{P(t,T_2)}{P(t,T_1)}dW_t \\
& +(\zeta_1(t))^2\frac{P(t,T_2)}{P(t,T_1)}dt-\zeta_1(t)\zeta_2(t)\frac{P(t,T_2)}{P(t,T_1)}dt \\
&= -\frac{P(t,T_2)}{P(t,T_1)}\zeta_1(t)(\zeta_2(t)-\zeta_1(t))dt+\frac{P(t,T_2)}{P(t,T_1)}(\zeta_2(t)-\zeta_1(t))dW_t \\
&= \frac{P(t,T_2)}{P(t,T_1)}(\zeta_2(t)-\zeta_1(t))(dW_t-\zeta_1(t)dt) \\
&= (\zeta_2(t)-\zeta_1(t))\frac{P(t,T_2)}{P(t,T_1)}d\widehat{W}_t=(\zeta_2(t)-\zeta_1(t))\frac{P(t,T_2)}{P(t,T_1)}d\widehat{W}_t,
\end{aligned}
$$

where $d\widehat{W}_t=dW_t-\zeta_1(t)dt$ is a *standard Brownian motion* under the T_1-forward measure $\widehat{\mathbb{P}}$.

b) From Question (a)) or (4.1.7) we have

$$
\begin{aligned}
P(T_1,T_2) &= \frac{P(T_1,T_2)}{P(T_1,T_1)} \\
&= \frac{P(t,T_2)}{P(t,T_1)}\exp\left(\int_t^{T_1}(\zeta_2(s)-\zeta_1(s))d\widehat{W}_s-\frac{1}{2}\int_t^{T_1}(\zeta_2(s)-\zeta_1(s))^2ds\right) \\
&= \frac{P(t,T_2)}{P(t,T_1)}\mathrm{e}^{X-v^2/2},
\end{aligned}
$$

where X is a centered Gaussian random variable with variance $v^2=\int_t^{T_1}(\zeta_2(s)-\zeta_1(s))^2ds$, independent of $\mathscr{F}_t$ under $\widehat{\mathbb{P}}$. Hence by the hint (4.5.13) with $x:=P(t,T_2)/P(t,T_1)$ and $\kappa:=K/x$, we find

$$
\begin{aligned}
& \mathbb{E}^*\left[\mathrm{e}^{-\int_0^{T_1}r_sds}(K-P(T_1,T_2))^+\,\Big|\,\mathscr{F}_t\right]=P(t,T_1)\widehat{\mathbb{E}}\left[(K-P(T_1,T_2))^+\mid\mathscr{F}_t\right] \\
&= P(t,T_1)\left(K\Phi\left(\frac{v}{2}+\frac{1}{v}\log\frac{K}{x}\right)-\frac{P(t,T_2)}{P(t,T_1)}\Phi\left(-\frac{v}{2}+\frac{1}{v}\log\frac{K}{x}\right)\right) \\
&= KP(t,T_1)\Phi\left(\frac{v}{2}+\frac{1}{v}\log\frac{K}{x}\right)-P(t,T_2)\Phi\left(-\frac{v}{2}+\frac{1}{v}\log\frac{K}{x}\right).
\end{aligned}
$$

Chapter 5

Exercise 5.1 By absence of arbitrage we have $(1-\alpha)\mathrm{e}^{-r_dT}=\mathrm{e}^{-rT}$, hence $\alpha=1-\mathrm{e}^{-(r-r_d)T}$.

Exercise 5.2

a) The bond payoff $\mathbb{1}_{\{\tau>T-t\}}$ is discounted according to the risk-free rate, before taking expectation.
b) We have $\mathbb{E}\left[\mathbb{1}_{\{\tau>T-t\}}\right]=\mathrm{e}^{-\lambda(T-t)}$, hence $P_d(t,T)=\mathrm{e}^{-(\lambda+r)(T-t)}$.
c) We have $P_M(t,T)=\mathrm{e}^{-(\lambda+r)(T-t)}$, hence $\lambda=-r+\frac{1}{T-t}\log P_M(t,T)$.

Exercise 5.3

a) Use the fact that $(r_t,\lambda_t)_{t\in[0,T]}$ is a Markov process.

b) Use the tower property of the conditional expectation given $\mathscr{F}_t$.
c) Writing $F(t,r_t,\lambda_t) = P(t,T)$, we have

$$
\begin{aligned}
& d\left(\mathrm{e}^{-\int_0^t (r_s+\lambda_s)ds} P(t,T)\right) \\
& = -(r_t+\lambda_t)\,\mathrm{e}^{-\int_0^t (r_s+\lambda_s)ds} P(t,T)dt + \mathrm{e}^{-\int_0^t (r_s+\lambda_s)ds} dP(t,T) \\
& = -(r_t+\lambda_t)\,\mathrm{e}^{-\int_0^t (r_s+\lambda_s)ds} P(t,T)dt + \mathrm{e}^{-\int_0^t (r_s+\lambda_s)ds} dF(t,r_t,\lambda_t) \\
& = -(r_t+\lambda_t)\,\mathrm{e}^{-\int_0^t (r_s+\lambda_s)ds} P(t,T)dt + \mathrm{e}^{-\int_0^t (r_s+\lambda_s)ds} \frac{\partial F}{\partial x}(t,r_t,\lambda_t)dr_t \\
& + \mathrm{e}^{-\int_0^t (r_s+\lambda_s)ds} \frac{\partial F}{\partial y}(t,r_t,\lambda_t)d\lambda_t + \frac{1}{2}\mathrm{e}^{-\int_0^t (r_s+\lambda_s)ds} \frac{\partial^2 F}{\partial x^2}(t,r_t,\lambda_t)\sigma_1^2(t,r_t)dt \\
& + \frac{1}{2}\mathrm{e}^{-\int_0^t (r_s+\lambda_s)ds} \frac{\partial^2 F}{\partial y^2}(t,r_t,\lambda_t)\sigma_2^2(t,\lambda_t)dt \\
& + \mathrm{e}^{-\int_0^t (r_s+\lambda_s)ds} \rho \frac{\partial^2 F}{\partial x \partial y}(t,r_t,\lambda_t)\sigma_1(t,r_t)\sigma_2(t,\lambda_t)dt + \mathrm{e}^{-\int_0^t (r_s+\lambda_s)ds} \frac{\partial F}{\partial t}(t,r_t,\lambda_t)dt \\
& = \mathrm{e}^{-\int_0^t (r_s+\lambda_s)ds} \frac{\partial F}{\partial x}(t,r_t,\lambda_t)\sigma_1(t,r_t)dB_t^{(1)} + \mathrm{e}^{-\int_0^t (r_s+\lambda_s)ds} \frac{\partial F}{\partial y}(t,r_t,\lambda_t)\sigma_2(t,\lambda_t)dB_t^{(2)} \\
& + \mathrm{e}^{-\int_0^t (r_s+\lambda_s)ds} \left(-(r_t+\lambda_t)P(t,T) + \frac{\partial F}{\partial x}(t,r_t,\lambda_t)\mu_1(t,r_t) \right. \\
& + \frac{\partial F}{\partial y}(t,r_t,\lambda_t)\mu_2(t,\lambda_t) + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(t,r_t,\lambda_t)\sigma_1^2(t,r_t) + \frac{1}{2}\frac{\partial^2 F}{\partial y^2}(t,r_t,\lambda_t)\sigma_2^2(t,\lambda_t) \\
& \left. + \rho\frac{\partial^2 F}{\partial x \partial y}(t,r_t,\lambda_t)\sigma_1(t,r_t)\sigma_2(t,\lambda_t) + \frac{\partial F}{\partial t}(t,r_t,\lambda_t) \right) dt,
\end{aligned}
$$

hence the bond pricing PDE is

$$
\begin{aligned}
& -(x+y)F(t,x,y) + \mu_1(t,x)\frac{\partial F}{\partial x}(t,x,y) \\
& + \mu_2(t,y)\frac{\partial F}{\partial y}(t,x,y) + \frac{1}{2}\sigma_1^2(t,x)\frac{\partial^2 F}{\partial x^2}(t,x,y) \\
& + \frac{1}{2}\sigma_2^2(t,y)\frac{\partial^2 F}{\partial y^2}(t,x,y) + \rho\sigma_1(t,x)\sigma_2(t,y)\frac{\partial^2 F}{\partial x \partial y}(t,x,y) + \frac{\partial F}{\partial t}(t,r_t,\lambda_t) = 0.
\end{aligned}
$$

d) We have

$$
r_t = -a\int_0^t r_s ds + \sigma B_t^{(1)}, \quad t \geqslant 0,
$$

hence

$$
\begin{aligned}
\int_0^t r_s ds &= \frac{1}{a}\left(\sigma B_t^{(1)} - r_t\right) \\
&= \frac{\sigma}{a}\left(B_t^{(1)} - \int_0^t \mathrm{e}^{-(t-s)a} dB_s^{(1)}\right) \\
&= \frac{\sigma}{a}\int_0^t (1-\mathrm{e}^{-(t-s)a}) dB_s^{(1)},
\end{aligned}
$$

and

$$
\begin{aligned}
& \int_t^T r_s ds = \int_0^T r_s ds - \int_0^t r_s ds \\
& = \frac{\sigma}{a}\int_0^T (1-\mathrm{e}^{-(T-s)a}) dB_s^{(1)} - \frac{\sigma}{a}\int_0^t (1-\mathrm{e}^{-(t-s)a}) dB_s^{(1)} \\
& = -\frac{\sigma}{a}\left(\int_0^t (\mathrm{e}^{-(T-s)a} - \mathrm{e}^{-(t-s)a}) dB_s^{(1)} + \int_t^T (\mathrm{e}^{-(T-s)a} - 1) dB_s^{(1)}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{\sigma}{a}(\mathrm{e}^{-(T-t)a}-1)\int_0^t \mathrm{e}^{-(t-s)a} dB_s^{(1)} - \frac{\sigma}{a}\int_t^T (\mathrm{e}^{-(T-s)a}-1)dB_s^{(1)} \\
&= -\frac{1}{a}(\mathrm{e}^{-(T-t)a}-1)r_t - \frac{\sigma}{a}\int_t^T (\mathrm{e}^{-(T-s)a}-1)dB_s^{(1)}.
\end{aligned}
$$

The answer for λ_t is similar.

e) As a consequence of the previous question we have

$$
\mathbb{E}\left[\int_t^T r_s ds + \int_t^T \lambda_s ds \,\Big|\, \mathscr{F}_t\right] = C(a,t,T)r_t + C(b,t,T)\lambda_t,
$$

and

$$
\begin{aligned}
&\mathrm{Var}\left[\int_t^T r_s ds + \int_t^T \lambda_s ds \,\Big|\, \mathscr{F}_t\right] = \\
&= \mathrm{Var}\left[\int_t^T r_s ds \,\Big|\, \mathscr{F}_t\right] + \mathrm{Var}\left[\int_t^T \lambda_s ds \,\Big|\, \mathscr{F}_t\right] \\
&\quad +2\,\mathrm{Cov}\left(\int_t^T X_s ds, \int_t^T Y_s ds \,\Big|\, \mathscr{F}_t\right) \\
&= \frac{\sigma^2}{a^2}\int_t^T (\mathrm{e}^{-(T-s)a}-1)^2 ds \\
&\quad +2\rho\frac{\sigma\eta}{ab}\int_t^T (\mathrm{e}^{-(T-s)a}-1)(\mathrm{e}^{-(T-s)b}-1)ds \\
&\quad +\frac{\eta^2}{b^2}\int_t^T (\mathrm{e}^{-(T-s)b}-1)^2 ds \\
&= \sigma^2\int_t^T C^2(a,s,T)ds + 2\rho\sigma\eta\int_t^T C(a,s,T)C(b,s,T)ds \\
&\quad +\eta^2\int_t^T C^2(b,sT)ds,
\end{aligned}
$$

from the Itô isometry.

f) We have

$$
\begin{aligned}
P(t,T) &= \mathbb{1}_{\{\tau>t\}}\mathbb{E}\left[\exp\left(-\int_t^T r_s ds - \int_t^T \lambda_s ds\right) \,\Big|\, \mathscr{F}_t\right] \\
&= \mathbb{1}_{\{\tau>t\}}\exp\left(-\mathbb{E}\left[\int_t^T r_s ds \,\Big|\, \mathscr{F}_t\right] - \mathbb{E}\left[\int_t^T \lambda_s ds \,\Big|\, \mathscr{F}_t\right]\right) \\
&\quad \times\exp\left(\frac{1}{2}\mathrm{Var}\left[\int_t^T r_s ds + \int_t^T \lambda_s ds \,\Big|\, \mathscr{F}_t\right]\right) \\
&= \mathbb{1}_{\{\tau>t\}}\exp(-C(a,t,T)r_t - C(b,t,T)\lambda_t) \\
&\quad \times\exp\left(\frac{\sigma^2}{2}\int_t^T C^2(a,s,T)ds + \frac{\eta^2}{2}\int_t^T C^2(b,s,T)\,\mathrm{e}^{-(T-s)b}ds\right) \\
&\quad \times\exp\left(\rho\sigma\eta\int_t^T C(a,s,T)C(b,s,T)ds\right).
\end{aligned}
$$

g) This is a direct consequence of the answers to Questions (c)) and f).

h) The above analysis shows that

$$
\begin{aligned}
\mathbb{P}(\tau > T \mid \mathscr{G}_t) &= \mathbb{1}_{\{\tau>t\}}\mathbb{E}\left[\exp\left(-\int_t^T \lambda_s ds\right) \,\Big|\, \mathscr{F}_t\right] \\
&= \mathbb{1}_{\{\tau>t\}}\exp\left(-C(b,t,T)\lambda_t + \frac{\eta^2}{2}\int_t^T C^2(b,s,T)ds\right),
\end{aligned}
$$

for $a = 0$ and

$$
\mathbb{E}\left[\exp\left(-\int_t^T r_s ds\right) \,\Big|\, \mathscr{F}_t\right] = \exp\left(-C(a,t,T)r_t + \frac{\sigma^2}{2}\int_t^T C^2(a,s,T)ds\right),
$$

for $b = 0$, and this implies

$$\begin{aligned} U_\rho(t,T) &= \exp\left(\rho\sigma\eta\int_t^T C(a,s,T)C(b,s,T)ds\right) \\ &= \exp\left(\rho\frac{\sigma\eta}{ab}\left(T-t-C(a,t,T)-C(b,t,T)+C(a+b,t,T)\right)\right). \end{aligned}$$

i) We have

$$\begin{aligned} & f(t,T) = -\mathbb{1}_{\{\tau>t\}}\frac{\partial}{\partial T}\log P(t,T) \\ &= \mathbb{1}_{\{\tau>t\}}\left(r_t\,\mathrm{e}^{-(T-t)a}-\frac{\sigma^2}{2}C^2(a,t,T)+\lambda_t\,\mathrm{e}^{-(T-t)b}-\frac{\eta^2}{2}C^2(b,t,T)\right) \\ & \quad -\mathbb{1}_{\{\tau>t\}}\rho\sigma\eta C(a,t,T)C(b,t,T). \end{aligned}$$

j) We use the relation

$$\begin{aligned} \mathbb{P}(\tau>T\mid\mathscr{G}_t) &= \mathbb{1}_{\{\tau>t\}}\mathbb{E}\left[\exp\left(-\int_t^T\lambda_s ds\right)\Big|\,\mathscr{F}_t\right] \\ &= \mathbb{1}_{\{\tau>t\}}\exp\left(-C(b,t,T)\lambda_t+\frac{\eta^2}{2}\int_t^T C^2(b,s,T)ds\right) \\ &= \mathbb{1}_{\{\tau>t\}}\,\mathrm{e}^{-\int_t^T f_2(t,u)du}, \end{aligned}$$

where $f_2(t,T)$ is the Vasicek forward rate corresponding to λ_t, *i.e.*

$$f_2(t,u) = \lambda_t\,\mathrm{e}^{-(u-t)b}-\frac{\eta^2}{2}C^2(b,t,u).$$

k) In this case we have $\rho = 0$ and

$$P(t,T) = \mathbb{P}(\tau>T\mid\mathscr{G}_t)\,\mathbb{E}\left[\exp\left(-\int_t^T r_s ds\right)\Big|\,\mathscr{F}_t\right],$$

since $U_\rho(t,T) = 0$.

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