Pricing Under Stochastic Volatility
Simulation of the Heston Model

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Abstract

Volatility in financial markets is not constant, in direct contradiction to the Black-Scholes paradigm. If we look at vanilla option prices with different strike prices and different maturities, we see that their implied volatility is different. Other one-factor models have been developed, for example, local volatility models where the volatility $\sigma(St, t)$ is a deterministic function of the underlying price and time, chosen to match observed European option prices exactly. With these models, the market remains complete and it is possible to find a smile structure. In parallel, multi-factor models have been developed, in particular stochastic volatility models where the volatility σ_t is modeled as a continuous Brownian semi-martingale always with the aim of reproducing the observed smile observed on the market. In this project we will study the simulation of the Heston model.

1 Heston Model

In this section we begin by presenting the dynamics of the Heston model. Then we proceed with the study of four different discretization schemes and compare their errors : Euler scheme, Milstein scheme, Anderson scheme and Almost exact scheme.

1.1 Mechanism

The Heston model is described by the bi-variate stochastic process for the stock price S_t and its variance v_t which follows the Cox-Ingersoll-Ross (CIR) model.

$$\begin{cases} dS_t = rS_t dt + \sqrt{v_t} S_t dW_{1,t} \\ dv_t = \kappa(\theta - v_t)\, dt + \sigma\sqrt{v_t} dW_{2,t} \end{cases}$$

where r is the risk-neutral interest rate, θ is the long run mean, κ is the rate at which the volatility converge to θ, σ is the volatility of the volatility and ρ is the correlation between the two brownian motion such that: $E[W_{1,t}W_{2,t}] = \rho$.
The volatility is always positive and cannot be zero or negative if the Feller condition $2\kappa\theta > \sigma^2$ is satisfied. The price of the option increases if θ increases. The parameters ρ and σ affect the Skewness and the Kurtosis respectively of the return distribution.

1.2 Discretization Schemes

1.2.1 Euler Scheme

Discretization of v_t :

The SDE for v_t in integral form is :

$$v_{t+dt} = v_t + \int_t^{t+dt} \kappa(\theta - v_u)\, du + \int_t^{t+dt} \sigma\sqrt{v_u} dW_{2,u}$$

The Euler discretization approximates the integrals using the left-point rule

$$\int_t^{t+dt} \kappa\left(\theta - v_u\right) du \approx \kappa\left(\theta - v_t\right) dt$$

$$\int_t^{t+dt} \sigma\sqrt{v_u} dW_{2,u} \approx \sigma\sqrt{v_t}\left(W_{2,t+dt} - W_{2,t}\right)$$

$$= \sigma\sqrt{v_t dt} Z_v$$

where Z_v is a standard normal random variable. The right hand side involves $(\theta - v_t)$ rather than $(\theta - v_{t+dt})$ since at time t we don't know the value of v_{t+dt}. This leaves us with

$$v_{t+dt} = v_t + \kappa\left(\theta - v_t\right) dt + \sigma\sqrt{v_t dt} Z_v$$

However, since this is a finite discretization of a continuous process, it is possible to introduce discretization errors where v_t may become negative. In order to properly handle negative values, we need to modify the above formula to include methods for eliminating negative values, we can replace v_t with $v_t^+ = \max\left(0, v_t\right)$: this is the full truncation scheme or to replace to replaces v_t with its absolute value $|v_t|$: this is the reflection scheme.

The literature generally proposes the full truncation method as the "best" and this is the one we will use in the following. The discretization equation of the scheme with full truncation for the volatility path will therefore be given by:

$$v_{t+dt} = v_t + \kappa\left(\theta - v_t^+\right) dt + \sigma\sqrt{v_t^+ dt} Z_v$$

Discretization of S_t :

In a similar fashion, the SDE for S_t is written in integral form as

$$S_{t+dt} = S_t + r\int_t^{t+dt} S_u du + \int_t^{t+dt} \sqrt{v_u} S_u dW_{1,u}$$

Euler discretization approximates the integrals with the left-point rule

$$\int_t^{t+dt} S_u du \approx S_t dt$$

$$\int_t^{t+dt} \sqrt{v_u} S_u dW_{1,u} \approx \sqrt{v_t} S_t\left(W_{1,t+dt} - W_{1,t}\right)$$

$$= \sqrt{v_t dt} S_t Z_s$$

where Z_s is a standard normal random variable that has correlation ρ with Z_v. We end up with

$$S_{t+dt} = S_t + r S_t dt + \sqrt{v_t dt} S_t Z_s$$

The Monte Carlo algorithm for a payoff $g(S_T)$ is written as follows:

Algorithm 1 Naive Monte Carlo Heston Model With Euler Discretization

Fix N and T
Fix dt
Fix v_0 and S_0
Draw W and $Z^s \sim \mathcal{N}(0,1)$

Set $Z^v = \rho Z^s + \sqrt{1-\rho^2} W$

Repeat

1. $S_{t+1} = S_t + S_t r dt + S_t\sqrt{v_t} Z_t^s$
2. $v_{t+1} = v_t + \kappa\left(\theta - v_t^+\right) dt + \sigma\sqrt{v_t^+ dt} Z_t^v$

Until $t = T$

Return $\frac{1}{N}\sum_{i=1}^N g(S_T)$

1.2.2 Milstein Scheme

The scheme works for SDEs for which the coefficients $\mu(S_t)$ and $\sigma(S_t)$ depend only on S, and do not depend on t directly. Hence we assume that the stock price S_t is driven by the SDE

$$dS_t = \mu(S_t)\,dt + \sigma(S_t)\,dW_t = \mu_t dt + \sigma_t dW_t$$

The general form of Milstein discretization is such :

$$S_{t+dt} = S_t + \mu_t dt + \sigma_t \sqrt{dt} Z + \frac{1}{2}\sigma_t' \sigma_t dt\left(Z^2 - 1\right)$$

with Z distributed distributed as standard and σ_t' refers to differentiation in S.

Discretization of S_t :

The coefficients of the stock price process are $\mu(S_t) = rS_t$ and $\sigma(S_t) = \sqrt{v_t}S_t$ so Milstein equation becomes

$$S_{t+dt} = S_t + rS_t dt + \sqrt{v_t dt} S_t Z_s + \frac{1}{2} v_t S_t dt\left(Z_s^2 - 1\right)$$

Discretization of v_t :

The coefficients of the variance process are $\mu(v_t) = \kappa(\theta - v_t)$ and $\sigma(v_t) = \sigma\sqrt{v_t}$ so an application of the Milstein equation for v_t produces

$$v_{t+dt} = v_t + \kappa(\theta - v_t)\,dt + \sigma\sqrt{v_t dt} Z_v + \frac{1}{4}\sigma^2 dt\left(Z_v^2 - 1\right)$$

Again, it is necessary to apply the full truncation scheme on the process v_t.

The Misltein Scheme becomes:

$$\begin{cases} S_{t+dt} = S_t + rS_t dt + \sqrt{v_t^+ dt} S_t Z_s + \frac{1}{2} v_t^+ S_t dt\left(Z_s^2 - 1\right) \\ v_{t+dt} = v_t + \kappa\left(\theta - v_t^+\right)dt + \sigma\sqrt{v_t^+ dt} Z_v + \frac{1}{4}\sigma^2 dt\left(Z_v^2 - 1\right) \end{cases}$$

The Monte Carlo algorithm for a payoff $g(S_T)$ is written as follows:

Algorithm 2 Naive Monte Carlo Heston Model With Milstein Discretization

Fix N and T
Fix dt
Fix v_0 and S_0
Draw W and $Z^s \sim \mathcal{N}(0,1)$
Set $Z^v = \rho * Z^s + \sqrt{1-\rho^2} * W$

Repeat

1. $S_t = S_t + rS_t dt + \sqrt{v^+ dt} * S_t Z_t^s + \frac{1}{2} v_t^+ S_t dt\left(Z_t^{s\,2} - 1\right)$
2. $v_{t+1} = v_t + \kappa\left(\theta - v_t^+\right)dt + \sigma\sqrt{v_t^+ dt} * Z_t^v + \frac{1}{4}\sigma^2 dt\left(Z_t^{v\,2} - 1\right)$

Until $t = T$

Return $\frac{1}{N}\sum_{i=1}^{N} g(S_T)$

1.3 Andersen's efficient simulation

The QE scheme for sampling from $v(t)$ consists of two different sampling algorithms where the switching between these algorithms dependends on the parameter values of the CIR process. Two different cases are considered: one for which the density of the process $v(t)$ is far from the zero region, and another where the distribution is close to the origin.

1.3.1 Discretization of $\mathbf{v}_t$

In the first case, the analysis is based on the observation that the noncentral chi-squared distribution can, for moderate or high levels of the noncentrality parameter $\bar{\kappa}$, be well approximated by a power function which is applied to a Gaussian variable Patnaik [1949] :

$$v(t) \approx v_1(t) = a\left(b + Z_v\right)^2,$$

with some constants a and b and $Z_v \sim \mathcal{N}(0,1)$.

To determine the constants a and b, the moment matching technique is applied, by which the first two moments from the two distributions are equated. The mean $\bar{m}$ and variance $\bar{s}^2$ of the CIR-type process $v(t)$ are known, and given by :

$$\begin{cases} \bar{m} := \mathbb{E}[v(t) \mid \mathcal{F}(0)] = \bar{c}(t,0)(\delta + \bar{\kappa}(t,0)) \\ \bar{s}^2 := \mathbb{V}\mathrm{ar}[v(t) \mid \mathcal{F}(0)] = \bar{c}^2(t,0)(2\delta + 4\bar{\kappa}(t,0)) \end{cases}$$

with the parameters $\delta, \bar{c}(t,0), \bar{\kappa}(t,0)$ as:

$$\bar{c}(t,s) = \frac{1}{4\kappa}\sigma^2\left(1 - \mathrm{e}^{-\kappa(t-s)}\right), \quad \delta = \frac{4\kappa\theta}{\sigma^2}, \quad \bar{\kappa}(t,s) = \frac{4\kappa v(s)\mathrm{e}^{-\kappa(t-s)}}{\sigma^2\left(1 - \mathrm{e}^{-\kappa(t-s)}\right)}$$

Saying that the $v(t)$ is distributed as a noncentral chi-squared distribution with one degree of freedom and with noncentrality parameter b^2 :

$$v_1(t) \sim a \cdot \chi^2\left(1, b^2\right)$$

implies the following expectation and variance:

$$\mathbb{E}\left[v_1(t) \mid \mathcal{F}(0)\right] = a\left(1 + b^2\right), \quad \mathrm{Var}\left[v_1(t) \mid \mathcal{F}(0)\right] = 2a^2\left(1 + 2b^2\right)$$

By equating these two sets of equations, $\mathbb{E}[v(t)] = \mathbb{E}\left[v_1(t)\right]$ and $\mathrm{Var}[v(t)] = \mathrm{Var}\left[v_1(t)\right]$, we find:

$$\bar{m} = a\left(1 + b^2\right), \quad \bar{s}^2 = 2a^2\left(1 + 2b^2\right)$$

It follows that $a = \bar{m}/\left(1 + b^2\right)$ and $b^4 - 2\frac{\bar{m}^2}{\bar{s}^2} + 1 + 2b^2\left(1 - 2\frac{\bar{m}^2}{\bar{s}^2}\right) = 0$

Setting $z := b^2$, we obtain a quadratic equation in z, as follows :

$$z^2 - 2\frac{\bar{m}^2}{\bar{s}^2} + 1 + 2z\left(1 - 2\frac{\bar{m}^2}{\bar{s}^2}\right) = 0$$

which has solutions if $\bar{s}^2/\bar{m}^2 \leqslant 2$. Under this condition, the solution for b^2 is given by :

$$b^2 = 2\frac{\bar{m}^2}{\bar{s}^2} - 1 + \sqrt{2\frac{\bar{m}^2}{\bar{s}^2}}\sqrt{2\frac{\bar{m}^2}{\bar{s}^2} - 1} \geqslant 0, \quad \text{and} \quad a = \frac{\bar{m}}{1 + b^2}$$

The condition $\bar{s}^2/\bar{m}^2 \leqslant 2$ corresponds to the case for which the density of the corresponding variance process $v(t)$ is far from zero (which can also be seen from the fact that the mean $\bar{m}$ is significantly larger than the variance s^2). As shown in Andersen [2007], the representation $v(t) \approx v_1(t) = a\left(b + Z_v\right)^2$ appears very accurate for large values of $v(t)$, however, when the probability mass of $v(t)$ accumulates around zero it may become inaccurate. In this case it is suggested to approximate the cumulative distribution function of $v(t)$ by the following exponential function :

$$F_{v(t)}(x) := P[v(t) \leqslant x] \approx F_{v_2(t)}(x) = c + (1 - c)\left(1 - \mathrm{e}^{-dx}\right), \quad \forall x \geqslant 0,$$

and the density by,

$$f_{v(t)}(x) \approx f_{v_2(t)}(x) = c\delta(0) + d(1 - c)\mathrm{e}^{-dx}, \quad \forall x \geqslant 0,$$

From the approximation above, it is again possible to derive the corresponding mean and variance :

$$\mathbb{E}\left[v_2(t) \mid \mathcal{F}\left(t_0\right)\right] = \frac{1 - c}{d}, \quad \mathrm{Var}\left[v_2(t) \mid \mathcal{F}\left(t_0\right)\right] = \frac{1 - c^2}{d^2}$$

To determine the parameters c and d, we apply once more the moment matching technique, which results in the following system of equations:

$$\bar{m} = \frac{1-c}{d}, \quad \bar{s}^2 = \frac{1-c^2}{\left(\frac{1-c}{m}\right)^2}$$

By solving this system for c and d and taking into account the condition $c \in [0,1]$, we find:

$$c = \frac{\frac{s^2}{m^2} - 1}{\frac{s^2}{m^2} + 1}, \quad \text{and} \quad d = \frac{1-c}{\bar{m}} = \frac{2}{\bar{m}\left(\frac{s^2}{m^2} + 1\right)}$$

It is clear that, in order to keep $c \in [0,1]$, we need to assume $\frac{\bar{s}^2}{\bar{m}^2} \geqslant 1$. The main advantage of such an exponential representation lies in the fact that $F_{v_2(t)}(\cdot)$ is invertible, so that sampling can take place directly from $v_2(t)$:

$$v_2(t) = F^{-1}_{v_2(t)}(u), \quad \text{for} \quad u \sim \mathcal{U}([0,1])$$

The inverse of $F_{v_2(t)}(x)$ is given by,

$$F^{-1}_{v_2(t)}(u) := F^{-1}_{v_2(t)}(u; c, d) = \begin{cases} 0, & 0 \leqslant u \leqslant c, \\ d^{-1} \log\left(\frac{1-c}{1-u}\right), & c < u \leqslant 1, \end{cases}$$

.

1.3.2 Discretization of S_t

As a final step, Andersen [2007] propose the propose the following discretization scheme for S_t :

$$\begin{aligned} S_{t+dt} &= S_t \exp\left(\frac{\rho}{\sigma}(v_{t+dt} - v_t - \kappa\theta dt) + dt\left(\frac{\kappa\rho}{\sigma} - \frac{1}{2}\right)(\gamma_1 v_t + \gamma_2 v_{t+dt}) + \sqrt{dt\,(1-\rho^2)\,(\gamma_1 v_t + \gamma_2 v_{t+dt})}\right) \\ &= S_t \exp\left(K_0 + K_1 v_t + K_2 v_{t+dt} + \sqrt{K_3 v_t + K_4 v_{t+dt}} Z\right) \end{aligned}$$

where Z is a standard Gaussian random variable, independent of $\hat{V}$, and $K_0, \ldots, K_4$ are given by

$$\begin{aligned} K_0 &= -\frac{\rho\kappa\theta}{\sigma} dt, \quad K_1 = \gamma_1 dt\left(\frac{\kappa\rho}{\sigma} - \frac{1}{2}\right) - \frac{\rho}{\sigma}, \\ K_2 &= \gamma_2 dt\left(\frac{\kappa\rho}{\sigma} - \frac{1}{2}\right) + \frac{\rho}{\sigma}, \quad K_3 = \gamma_1 dt\left(1-\rho^2\right), \quad K_4 = \gamma_2 dt\left(1-\rho^2\right). \end{aligned}$$

The final algorithm using Monte Carlo methods for a payoff $g(S_T)$ is written as follows:

Algorithm 3 Naive Monte Carlo with Andersen Efficient Simulation

Fix N and T
Fix dt
Fix v_0 and S_0
Fix γ_1 and γ_2
Repeat

1. $m := \mathbb{E}\left[v_{t+1} \mid v_t\right], s := \mathrm{Var}\left(v_{t+1} \mid v_i\right), \eta := \frac{s}{m^2}$
2. Draw $U \sim \mathcal{U}(0,1)$
3. $k_0 = -\frac{\rho\kappa\theta}{\sigma}dt$, $k_1 = \gamma_1 dt\left(\frac{\kappa\rho}{\sigma} - \frac{1}{2}\right) - \frac{\rho}{\sigma}$, $k_2 = \gamma_2 dt\left(\frac{\kappa\rho}{\sigma} - \frac{1}{2}\right) + \frac{\rho}{\sigma}$, $k_3 = \gamma_1 dt\left(1-\rho^2\right)$, $k_4 = \gamma_2 dt\left(1-\rho^2\right)$
4. **IF** $\eta < 1.5$
   (a) $b := \left(\frac{2}{\eta} - 1 + \sqrt{\frac{2}{\eta}}\sqrt{\frac{2}{\eta} - 1}\right)^{1/2}$, $a := \frac{m}{1+b^2}$
   (b) $Z_v := \Phi^{-1}\left(U_v\right) (\sim \mathcal{N}(0,1))$
   (c) $v_{t+1} := a\left(b + Z_v\right)^2$
5. **ELSE :**
   (a) $p := \frac{\eta-1}{\eta+1}$, $\beta := \frac{1-p}{m}$
   (b) $v_{t+1} = \begin{cases} 0 & \text{if } U_v \in [0,p] \\ \beta^{-1}\ln\left(\frac{1-p}{1-U_v}\right) & \text{if } U_v \in [p,1] \end{cases}$
6. $S_{t+1} = S_t \exp\left(k_0 + k_1 v_t + k_2 v_{t+1} + \sqrt{k_3 v_t + k_4 v_{t+1}}Z\right)$.

Until $t = T$

Return $\frac{1}{N}\sum_{i=1}^{N} g(S_T)$

1.4 Almost exact simulation of the Heston model

Under the log transformation, the Heston model becomes:

$$\begin{cases} \mathrm{d}X_t = \left(r - \frac{1}{2}v_t\right)\mathrm{d}t + \sqrt{v_t}dW_{1,t} \\ \mathrm{d}v_t = \kappa(\theta - v(t))\mathrm{d}t + \sigma\sqrt{v_t}dW_{2,t} \end{cases}$$

with $W_{1,t}$ and $W_{2,t}$ two ρ correlated Brownian Motions.

Which can be rewrite :

$$\begin{cases} \mathrm{d}X_t = \left(r - \frac{1}{2}v_t\right)\mathrm{d}t + \sqrt{v_t}\left(\rho dW_{v,t} + \sqrt{1-\rho^2}dW_{x,t}\right) \\ \mathrm{d}v_t = \kappa(\theta - v(t))\mathrm{d}t + \sigma\sqrt{v_t}dW_{v,t} \end{cases}$$

with $W_{x,t}$ and $W_{v,t}$ two intependant Brownian Motions.

It is crucial here that the variance process is driven by an independent Brownian motion, while the process for $X_t := \log(S_t)$ is now correlated to v_t. The motivation for this is that we are able to use the properties of the marginal distribution of v_t.

After integration of both processes in in a certain time interval $[t_i, t_{i+1}]$, the following discretization scheme is obtained:

$$X_{i+1} = X_i + \int_{t_i}^{t_{i+1}}\left(r - \frac{1}{2}v(t)\right)\mathrm{d}t + \rho\int_{t_i}^{t_{i+1}}\sqrt{v(t)}\mathrm{d}\widetilde{W}_v(t) + \sqrt{1-\rho^2}\int_{t_i}^{t_{i+1}}\sqrt{v(t)}\mathrm{d}\widetilde{W}_x(t)$$

and

$$v_{i+1} = v_i + \kappa\int_{t_i}^{t_{i+1}}(\theta - v(t))\mathrm{d}t + \sigma\int_{t_i}^{t_{i+1}}\sqrt{v(t)}\mathrm{d}\widetilde{W}_v(t).$$

Notice that the two integrals with $\widetilde{W}_v(t)$ in the SDEs above are the same, and in terms of the variance realizations they are given by:

$$\int_{t_i}^{t_{i+1}} \sqrt{v(t)} \mathrm{d}\widetilde{W}_v(t) = \frac{1}{\sigma}\left(v_{i+1} - v_i - \kappa \int_{t_i}^{t_{i+1}} (\theta - v(t)) \mathrm{d}t\right).$$

The variance v_{i+1} can then be simulated, for given value of v_i, by means of the CIR process, or, equivalently, by the QE scheme.

As a final step in the Heston model simulation, the discretization for x_{i+1} is given by:

$$x_{i+1} = x_i + \int_{t_i}^{t_{i+1}} \left(r - \frac{1}{2}v(t)\right) \mathrm{d}t + \frac{\rho}{\sigma}\left(v_{i+1} - v_i - \kappa \int_{t_i}^{t_{i+1}} (\theta - v(t)) \mathrm{d}t\right) + \sqrt{1-\rho^2} \int_{t_i}^{t_{i+1}} \sqrt{v(t)} \mathrm{d}\widetilde{W}_x(t).$$

We approximate all integrals appearing in the expression above by their left integration boundary values of the integrand, as in the Euler discretization scheme:

$$x_{i+1} \approx x_i + \int_{t_i}^{t_{i+1}} \left(r - \frac{1}{2}v_i\right) \mathrm{d}t + \frac{\rho}{\sigma}\left(v_{i+1} - v_i - \kappa \int_{t_i}^{t_{i+1}} (\theta - v_i) \, \mathrm{d}t\right) + \sqrt{1-\rho^2} \int_{t_i}^{t_{i+1}} \sqrt{v_i} \, \mathrm{d}\widetilde{W}_x(t)$$

The calculation of the integrals is now trivial and results in:

$$x_{i+1} \approx x_i + \left(r - \frac{1}{2}v_i\right)\Delta t + \frac{\rho}{\sigma}\left(v_{i+1} - v_i - \kappa\left(\theta - v_i\right)\Delta t\right) + \sqrt{1-\rho^2}\sqrt{v_i}\left(\widetilde{W}_x\left(t_{i+1}\right) - \widetilde{W}_x\left(t_i\right)\right)$$

After collection of all terms and using the well-known property of normally distributed random variables, $\widetilde{W}_x\left(t_{i+1}\right) - \widetilde{W}_x\left(t_i\right) \overset{\mathrm{d}}{=} \sqrt{\Delta t} Z_x$, with $Z_x \sim \mathcal{N}(0,1)$, we find:

$$\begin{cases} x_{i+1} \approx x_i + k_0 + k_1 v_i + k_2 v_{i+1} + \sqrt{k_3 v_i} Z_x \\ v_{i+1} = \bar{c}\left(t_{i+1}, t_i\right) \chi^2\left(\delta, \bar{\kappa}\left(t_{i+1}, t_i\right)\right) \end{cases}$$

with:

$$\bar{c}\left(t_{i+1}, t_i\right) = \frac{\sigma^2}{4\kappa}\left(1 - \mathrm{e}^{-\kappa\left(t_{i+1} - t_i\right)}\right), \quad \delta = \frac{4\kappa\bar{v}}{\sigma^2}, \quad \bar{\kappa}\left(t_{i+1}, t_i\right) = \frac{4\kappa \mathrm{e}^{-\kappa\Delta t}}{\sigma^2\left(1 - \mathrm{e}^{-\kappa\Delta t}\right)} v_i,$$

and $\chi^2(\delta, \bar{\kappa}(\cdot,\cdot))$ the noncentral chi-squared distribution with δ degrees of freedom and noncentrality parameter $\bar{\kappa}\left(t_{i+1}, t_i\right)$. The remaining constants are known as :

$$k_0 = \left(r - \frac{\rho}{\sigma}\kappa\theta\right) dt, \quad k_1 = \left(\frac{\rho\kappa}{\sigma} - \frac{1}{2}\right) dt - v\frac{\rho}{\sigma}, \quad k_2 = \frac{\rho}{\sigma}, \quad k_3 = \left(1 - \rho^2\right) dt$$

With the intention of having a true benchmark of the Almost exact scheme, we sampled directly the noncentral chi-squared distribution provided in the *numpy.random* library. The Monte Carlo algorithm for a payoff $g(S_T)$ is written as follows:

Algorithm 4 Naive Monte Carlo Heston Model With Almost Exact Scheme

Fix N and T
Fix dt
Fix v_0 and S_0
Fix $X_0 = log(S_0)$
Fix $\bar{c} = \frac{\sigma^2}{4\kappa}\left(1 - \mathrm{e}^{-\kappa dt}\right)$ and $\delta = \frac{4\kappa\theta}{\sigma^2}$
Fix $k_0 = \left(r - \frac{\rho}{\sigma}\kappa\theta\right)dt$, $k_1 = \left(\frac{\rho\kappa}{\sigma} - \frac{1}{2}\right)dt - \frac{\rho}{\sigma}$, $k_2 = \frac{\rho}{\sigma}$, $k_3 = \left(1 - \rho^2\right)dt$
Repeat

1. $\bar{\kappa} = \frac{4\kappa\mathrm{e}^{-\kappa dt}}{\sigma^2(1-\mathrm{e}^{-\kappa dt})}v_t$
2. Draw $Z \sim \mathcal{N}(0,1)$ and $Y \sim \chi^2(\delta, \bar{\kappa})$
3. $v_{t+1} = \bar{c}Y$
4. $X_{t+1} = X_t + k_0 + k_1 v_t + k_2 v_{t+1} + \sqrt{k_3 v_t}Z$

Until $t = T$

Return $\frac{1}{N}\sum_{i=1}^{N} g(e^{X_T})$

1.5 Pricing Under Heston Model

1.5.1 Heston PDE

For a European option with payoff $g(S_T)$ at maturity T, we have that the $\mathcal{C}^{1,2}$ function $C(t,s,v)$ of the pricing rule $C_t = C(t, S_t, V_t), t \in [0,T]$, for the option satisfies the following partial differential equation

$$\frac{\partial C}{\partial t} + rs\frac{\partial C}{\partial s} + \{\kappa\theta - v(\kappa + \sigma\lambda)\}\frac{\partial C}{\partial v} + \frac{1}{2}s^2 v\frac{\partial^2 C}{\partial s^2} + \rho\sigma vs\frac{\partial^2 C}{\partial v \partial s} + \frac{1}{2}\sigma^2 v\frac{\partial^2 C}{\partial v^2} = rC$$

with terminal condition $C(T,s,v) = g(s)$. Equivalently, by Feynman-Kac, this is to say that we have the usual risk-neutral pricing formula

$$C(t,s,v) = \mathbb{E}^Q\left[e^{-r(T-t)}g(S_T) \mid (S_t, V_t) = (s,v)\right]$$

where (S,V) follows the Q-dynamics with initial value $(S_t, V_t) = (s,v)$ at the initial time $t \in [0,T]$.

1.5.2 Closed Form Formula

The great advantage of the Heston model is the existence of a closed formula for the price of European Vanilla options. Indeed, the price C_t of a Call with underlying S and payoff $(S_T - K)_+$ under the Heston model is expressed in this way, by noting $\tau = T - t$ which is the tenor of the Call at time t:

$$C_t = C(t,S,v) = S_t P_1 - Ke^{-r\tau}P_2$$

where:

$$P_j = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{-i\Phi\ln(K)}f_j(t,x,v,\Phi)}{i\Phi}\right]d\Phi$$

with:

$x = \ln(S)$ and $f_j(t,x,v,\phi) = \exp\left(C_j(\tau,\phi) + D_j(\tau,\phi)v + i\phi x\right)$

and:

$$C_j(\tau,\Phi) = r\Phi i\tau + \frac{a}{\sigma^2}\left[(b_j - \rho\sigma\Phi + d)\tau - 2\ln\left(\frac{1-ge^{d\tau}}{1-g}\right)\right]$$

$$D_j(\tau,\Phi) = \frac{b_j - \rho\sigma\Phi i + d}{\sigma^2}\left[\frac{1-e^{d\tau}}{1-ge^{d\tau}}\right]$$

with:

- $g = \frac{b_j - \rho\sigma\Phi i + d}{b_j - \rho\sigma\Phi i - d}$
- $d = \sqrt{(\rho\sigma\Phi i - b_j)^2 - \sigma^2 (2u_j\Phi i - \Phi^2)}$
- $a = \kappa\theta, u_1 = \frac{1}{2}, u_2 = -\frac{1}{2}, b_1 = \kappa + \lambda - \rho\sigma, b_2 = \kappa + \lambda.$

The demonstration of this closed formula, inspired by the reasoning of the Black-Scholes formula, has been carried out and can be found in Heston [1993].

The solution is not so easy and immediate as the Black-Scholes one but this closed form precise solution is the benchmark by which we evaluate the performance our Monte Carlo estimator.

1.6 Numerical application on Vanilla options prices

This section is dedicated to a comparison of the four discretization schemes for pricing a European call option with the following parameters : $T = 1$, $S_0 = 100$, $r = 0.02$, $\kappa = 2$, $\theta = 0.3$, $v_0 = 0.3$, $\sigma = 0.9$, $\rho = -0.9$. These model parameters may be encountered in equity options markets.

Here we are interested in the Weak error which measures how far the distribution of the discretized process is from the original process :

$$\epsilon_{\text{weak}} = \left| \mathbb{E}\left[g\left(X_T\right)\right] - \mathbb{E}\left[g\left(\bar{X}_T^m\right)\right] \right|$$

The comparison is feasible thanks to the closed form formula provided in the previous section. The simulations were done with a number of simulations $n = 10^5$ and with a step $\Delta = T/40$.

Figure 1 describes the evolution of Weak error as a function of strike of the call option. We observe that the weak error decreases as the strike of the option increases. The figure 1 provides a clear rank of the accuracy of the schemes. The most accurate schemes is the Almost exact scheme and the Andersen scheme followed by the Euler and Milstein at the same level. The figure proves that the approximations done by Anderson in order to bypass the non-central chi-squared distribution are legitimate.

Figure 1: Weak error of the four schemes

The followings tables provides an in-depth study of the behavior of the weak error as both the strike and the step change. From table 1, we can observe that the Andersen and Almost exact schemes outperform both Euler and Milstein discretization for steps that are higher than $T/20$. We also notice that the error decreases at first to a minimum then starts to increase as the steps increases. Table 2 and Table 3 show the same behaviour. The confidence interval on the other hand increase as the step size increases. Finally, we remark that the optimal step depends on the strike of the call option.

	$K = 70$			
Δt	Euler	Milstein	Andersen	AE
$T/10$	0.72 (0.49)	0.52 (0.51)	0.46 (0.51)	0.85 (0.48)
$T/20$	1.13 (0.51)	0.83 (0.52)	0.91 (0.52)	0.32 (0.51)
$T/40$	1.37 (0.52)	1.16 (0.53)	1.16 (0.53)	0.63 (0.52)
$T/80$	1.25 (0.53)	1.26 (0.53)	1.20 (0.53)	1.33 (0.53)
$T/160$	1.69 (0.53)	1.42 (0.53)	1.42 (0.53)	1.40 (0.53)

Table 1: Test cases for the Heston schemes for Call with $S(0) = 100$

	$K = 100$			
Δt	Euler	Milstein	Andersen	AE
$T/10$	1.04 (0.38)	1.04 (0.40)	1.09 (0.40)	0.27 (0.37)
$T/20$	1.70 (0.40)	1.54 (0.41)	1.59 (0.41)	0.99 (0.39)
$T/40$	2.04 (0.41)	1.92 (0.42)	1.96 (0.41)	1.41 (0.40)
$T/80$	2.02 (0.41)	2.05 (0.42)	1.99 (0.41)	2.08 (0.42)
$T/160$	2.34 (0.42)	2.24 (0.42)	2.21 (0.42)	2.18 (0.30)

Table 2: Test cases for the Heston schemes for Call with $S(0) = 100$

	$K = 130$			
Δt	Euler	Milstein	Andersen	AE
$T/10$	0.88 (0.25)	1.20 (0.28)	1.27 (0.27)	0.12 (0.25)
$T/20$	1.60 (0.28)	1.69 (0.29)	1.73 (0.29)	1.20 (0.27)
$T/40$	2.00 (0.29)	2.03 (0.30)	2.07 (0.29)	1.59 (0.29)
$T/80$	2.06 (0.29)	2.14 (0.30)	2.06 (0.29)	2.14 (0.29)
$T/160$	2.32 (0.30)	2.30 (0.30)	2.26 (0.30)	2.24 (0.30)

Table 3: Test cases for the Heston schemes for Call with $S(0) = 100$

2 Conclusion

This work aimed to study the Heston model. The first part was dedicated to the implementation of four discretization schemes : Euler, Milstein, Anderson and Almost exact scheme. The closed form formula provided us with a benchmark from which we were able to evaluate the accuracy of each scheme. Our experiments showed that both Anderson and Almost exact scheme are superior to the Euler and Milstein schemes.

3 Annexe

3.1 Numerical Application on Vanilla options Greeks with Pathwise Differentiation

Heston Model :
Parameters: $T = 1$, $K = 100$, $r = 0.02$, $\kappa = 4$, $\theta = 0.3$, $v_0 = 0.3$, $\sigma = 0.9$.

Figure 2: Delta Call Heston Model

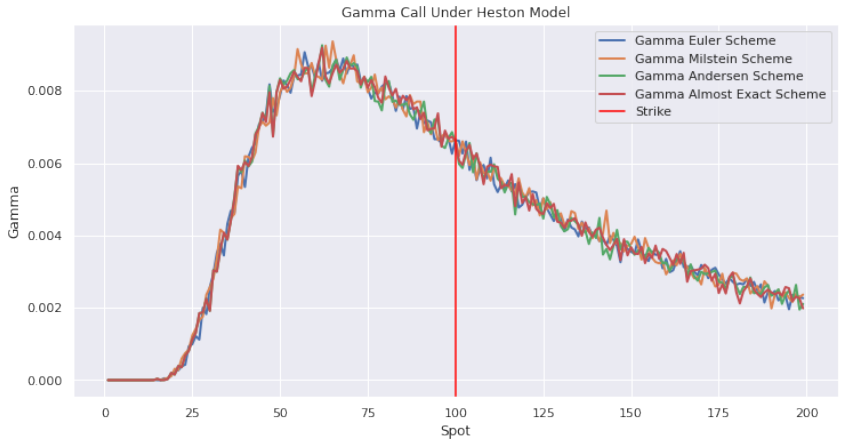

Figure 3: Gamma Call Heston Model

References

L. B. Andersen. Efficient simulation of the heston stochastic volatility model. *Available at SSRN 946405*, 2007.

Heston. A closed-form solution for. options with stochastic. volatility with applications to bond and currency. options. 1993.

Patnaik. The non-central χ^2 and f-distributions and their applications get access arrow. 1949.