Pricing Under Stochastic Volatility
Simulation of the Heston Model

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Abstract

Volatility in financial markets is not constant, in direct contradiction to the BlackScholes paradigm. If we look at vanilla option prices with different strike prices and different maturities, we see that their implied volatility is different. Other one-factor models have been developed, for example, local volatility models where the volatility $\sigma(St, t)$ is a deterministic function of the underlying price and time, chosen to match observed European option prices exactly. With these models, the market remains complete and it is possible to find a smile structure. In parallel, multi-factor models have been developed, in particular stochastic volatility models where the volatility σ_t is modeled as a continuous Brownian semi-martingale always with the aim of reproducing the observed smile observed on the market. In this project we will study the simulaton of the Heston model.

1 Heston Model

In this section we begin by presenting the dynamics of the Heston model. Then we proceed with the study of four different discretization scheme and compare their errors : Euler scheme, Milstein scheme, Anderson scheme and Almost exact scheme.

1.1 Mechanism

The Heston model is described by the bivariate stochastic process for the stock price S_t and its variance v_t which follows the Cox-Ingersoll-Ross (CIR) model.

$$dS_t = rS_t dt + \sqrt{v_t} S_t dW_{1,t}$$
$$dv_t = \kappa(\theta - v_t)\, dt + \sigma\sqrt{v_t} dW_{2,t}$$

where r is the risk-neutral interest rate, θ is the long run mean, κ is the rate at which the volatility converge to θ, σ is the volatility of the volatility and ρ is the corellation between the two brownian motion such that: $E\left[dW_{1,t} dW_{2,t}\right] = \rho dt$.
The volatility is always positive and cannot be zero or negative if the Feller condition $2\kappa\theta > \sigma^2$ is satisfied. The price of the option increases if θ increases. The parameters ρ and σ affect the Skewness and the Kurtosis respectively of the return distribution.

1.2 Discretization Schemes

1.2.1 Euler Scheme

Discretization of v_t :

The SDE for v_t in integral form is :

$$v_{t+dt} = v_t + \int_t^{t+dt} \kappa(\theta - v_u)\, du + \int_t^{t+dt} \sigma\sqrt{v_u} dW_{2,u}$$

The Euler discretization approximates the integrals using the left-point rule

$$\int_t^{t+dt} \kappa\left(\theta - v_u\right) du \approx \kappa\left(\theta - v_t\right) dt$$

$$\begin{aligned} \int_t^{t+dt} \sigma\sqrt{v_u} dW_{2,u} &\approx \sigma\sqrt{v_t}\left(W_{2,t+dt} - W_{2,t}\right) \\ &= \sigma\sqrt{v_t dt} Z_v \end{aligned}$$

where Z_v is a standard normal random variable. The right hand side involves $(\theta - v_t)$ rather than $(\theta - v_{t+dt})$ since at time t we don't know the value of v_{t+dt}. This leaves us with

$$v_{t+dt} = v_t + \kappa\left(\theta - v_t\right) dt + \sigma\sqrt{v_t dt} Z_v$$

However, since this is a finite discretization of a continuous process, it is possible to introduce discretization errors where v_t may become negative. In order to properly handle negative values, we need to modify the above formula to include methods for eliminating negative values, we can replace v_t with $v_t^+ = \max\left(0, v_t\right)$: this is the full truncation scheme or to replace to replaces v_t with its absolute value $|v_t|$: this is the reflection scheme.

The literature generally proposes the full truncation method as the "best" and this is the one we will use in the following. The discretization equation of the scheme with full truncation for the volatility path will therefore be given by:

$$v_{t+dt} = v_t + \kappa\left(\theta - v_t^+\right) dt + \sigma\sqrt{v_t^+ dt} Z_v$$

Discretization of S_t :

In a similar fashion, the SDE for S_t is written in integral form as

$$S_{t+dt} = S_t + r\int_t^{t+dt} S_u du + \int_t^{t+dt} \sqrt{v_u} S_u dW_{1,u}$$

Euler discretization approximates the integrals with the left-point rule

$$\int_t^{t+dt} S_u du \approx S_t dt$$

$$\begin{aligned} \int_t^{t+dt} \sqrt{v_u} S_u dW_{1,u} &\approx \sqrt{v_t} S_t\left(W_{1,t+dt} - W_{1,t}\right) \\ &= \sqrt{v_t dt} S_t Z_s \end{aligned}$$

where Z_s is a standard normal random variable that has correlation ρ with Z_v. We end up with

$$S_{t+dt} = S_t + rS_t dt + \sqrt{v_t dt} S_t Z_s$$

The Monte Carlo algorithm for a payoff $g(S_T)$ is written as follows:

Algorithm 1 Naive Monte Carlo Heston Model With Euler Discretization

Fix N and T
Fix dt
Fix v_0 and S_0
Draw W and $Z^s \sim \mathcal{N}(0, 1)$

Set $Z^v = \rho Z^s + \sqrt{1-\rho^2} W$

Repeat

1. $S_{t+1} = S_t + S_t r dt + S_t\sqrt{v_t} Z_t^s$
2. $v_{t+1} = v_t + \kappa\left(\theta - v_t^+\right) dt + \sigma\sqrt{v_t^+ dt} Z_t^v$

Until $t = T$

Return $\frac{1}{N}\sum_{i=1}^N g(S_T)$

1.2.2 Milstein Scheme

The scheme works for SDEs for which the coefficients $\mu\left(S_t\right)$ and $\sigma\left(S_t\right)$ depend only on S, and do not depend on t directly. Hence we assume that the stock price S_t is driven by the SDE

$$\begin{aligned} dS_t &= \mu\left(S_t\right)dt + \sigma\left(S_t\right)dW_t \\ &= \mu_t dt + \sigma_t dW_t \end{aligned}$$

The general form of Milstein discretization is such :

$$S_{t+dt} = S_t + \mu_t dt + \sigma_t \sqrt{dt} Z + \frac{1}{2}\sigma_t' \sigma_t dt\left(Z^2 - 1\right)$$

with Z distributed distributed as standard and σ_t' refers to differentiation in S.

Discretization of S_t :

The coefficients of the stock price process are $\mu\left(S_t\right) = rS_t$ and $\sigma\left(S_t\right) = \sqrt{v_t}S_t$ so Milstein equation becomes

$$S_{t+dt} = S_t + rS_t dt + \sqrt{v_t dt} S_t Z_s + \frac{1}{2} v_t S_t dt\left(Z_s^2 - 1\right)$$

Discretization of v_t :

The coefficients of the variance process are $\mu\left(v_t\right) = \kappa\left(\theta - v_t\right)$ and $\sigma\left(v_t\right) = \sigma\sqrt{v_t}$ so an application of the Milstein equation for v_t produces

$$v_{t+dt} = v_t + \kappa\left(\theta - v_t\right)dt + \sigma\sqrt{v_t dt} Z_v + \frac{1}{4}\sigma^2 dt\left(Z_v^2 - 1\right)$$

Again, it is necessary to apply the full truncation scheme on the process v_t.

The Misltein Scheme becomes:

$$\begin{aligned} S_{t+dt} &= S_t + rS_t dt + \sqrt{v_t^+ dt} S_t Z_s + \frac{1}{2} v_t^+ S_t dt\left(Z_s^2 - 1\right) \\ v_{t+dt} &= v_t + \kappa\left(\theta - v_t^+\right)dt + \sigma\sqrt{v_t^+ dt} Z_v + \frac{1}{4}\sigma^2 dt\left(Z_v^2 - 1\right) \end{aligned}$$

The Monte Carlo algorithm for a payoff $g(S_T)$ is written as follows:

Algorithm 2 Naive Monte Carlo Heston Model With Milstein Discretization

---

Fix N and T
Fix dt
Fix v_0 and S_0
Draw W and $Z^s \sim \mathcal{N}(0,1)$
Set $Z^v = \rho * Z^s + \sqrt{1-\rho^2} * W$

Repeat

1. $S_t = S_t + rS_t dt + \sqrt{v^+ dt} * S_t Z_t^s + \frac{1}{2} v_t^+ S_t dt\left({Z_t^s}^2 - 1\right)$
2. $v_{t+1} = v_t + \kappa\left(\theta - v_t^+\right)dt + \sigma\sqrt{v_t^+ dt} * Z_t^v + \frac{1}{4}\sigma^2 dt\left({Z_t^v}^2 - 1\right)$

Until $t = T$

Return $\frac{1}{N}\sum_{i=1}^N g(S_T)$

1.3 Andersen's efficient simulation

The proposed scheme by Anderson in Andersen [2007] is based on the observation that v_{t+dt} conditionally on v_t follows a non-central chi-square distribution:

$$\Pr(v_{t+dt} < x \mid v_t) = F_{\chi^2}\left(\frac{x \cdot n(t, t+dt)}{e^{-\kappa(dt)}}; d, v_t \cdot n(t, t+dt)\right)$$

with :

$$d=4\kappa\theta/\sigma^2; \quad n(t,t+dt)=\frac{4\kappa e^{-\kappa(dt)}}{\sigma^2\left(1-e^{-\kappa(dt)}\right)}$$

and the following moments :

$$m:=\mathrm{E}(v_{t+dt}\mid v_t)=\theta+(v_t-\theta)e^{-\kappa(dt)}$$

$$s^2:=\mathrm{Var}(v_{t+dt}\mid v_t)=\frac{v_t\sigma^2e^{-\kappa(dt)}}{\kappa}\left(1-e^{-\kappa(dt)}\right)+\frac{\theta\sigma^2}{2\kappa}\left(1-e^{-\kappa(dt)}\right)^2$$

Discretization of v_t :

Anderson then proposed two discretisation schemes for V using the following facts : the non-central chi-square distribution approaches a Gaussian distribution as the non-centrality parameter approaches infinity and it can also be estimated by an ordinary chi-square distribution as the non-centrality parameter approaches zero.

The proposed schemes are the Truncated Gaussian (TG) and the Quadratic Exponential scheme (QE). In the following we will only focus on the QE approximations as it is more accurate than the TG approximations.

In the QE scheme, Anderson approximates the noncentral chi-square with moderate or high non-centrality parameter by a power-function applied to a Gaussian variable. He chose the quadratic representation to ensure that v remains positive even though a cubic transformation is more accurate. v is thus expressed as follows:

$$\hat{v}(t+dt)=a\left(b+Z_v\right)^2$$

with a and b constants that should be matched with the moments expressed earlier.
For small values of the non-centrality parameter, Andreson proposed to estimate v with the asymptotic approximation of the centered chi-square distribution:

$$\Psi(x)=\Pr(\hat{v}(t+dt)\leqslant x)=p+(1-p)\left(1-e^{-\beta x}\right), \quad x\geqslant 0$$

where p and β should be matched with the m and s^2.

The moment matching is conditioned by the value $\psi:=s^2/m^2$ as follows:
If $\psi\leqslant 2$ then set

$$b^2=2\psi^{-1}-1+\sqrt{2\psi^{-1}}\sqrt{2\psi^{-1}-1}\geqslant 0 \text{ and } a=\frac{m}{1+b^2}$$

If $\psi\geqslant 1$ then set

$$p=\frac{\psi-1}{\psi+1}\in[0,1) \text{ and } \beta=\frac{1-p}{m}=\frac{2}{m(\psi+1)}>0$$

Fortunately, the two condition overlap and we can thus set a threshold ψ_c in $[1,2]$ to select which approximation to apply.

Discretization of S_t :

A naive approach would be to use a Euler discretization scheme with ρ as the correlation between the two standard normal distributions that drives the scheme for S and v. However the non linearity used in the QE scheme would result in an effective correlation that is closer to zero rather than ρ between the estimated volatility and the price of the underlying. Anderson thus uses a discretization that explicitly relates v and S with the correlation coefficient ρ.

The scheme is obtained as follows:
Under the log transformation $X_t=log(S_t)$ we have by the Ito formula :

$$dX_t=\left(r-\frac{1}{2}v_t\right)dt+\sqrt{v_t}dW_{1,t}$$

$$dv_t=\kappa\left(\theta-v_t\right)dt+\sigma\sqrt{v_t}dW_{2,t}$$

Which can also be written using the Cholesky decomposition as:

$$dX_t=\left(r-\frac{1}{2}v_t\right)\mathrm{d}t+\sqrt{v_t}\left[\rho\,\mathrm{d}\widetilde{W}_v(s)+\sqrt{1-\rho^2}\,\mathrm{d}\widetilde{W}_x(s)\right]$$

$$dv_t=\kappa(\theta-v_t)\mathrm{d}t+\sigma\sqrt{v_t}\mathrm{d}\widetilde{W}_v(t)$$

with $\widetilde{W}_v = W_2$ and $\widetilde{W}_x$ independent of $\widetilde{W}_v$.

After integration of both processes on $[t, t+dt]$, the following discretization scheme is obtained:

$$
\begin{aligned}
X_{t+dt} =& X_t + \int_t^{t+dt} \left(r - \frac{1}{2}v_s\right) ds + \rho \int_t^{t+dt} \sqrt{v_s} d\widetilde{W}_v(s) + \sqrt{1-\rho^2} \int_t^{t+dt} \sqrt{v_s} d\widetilde{W}_x(s) \\
v_{t+dt} =& v_t + \kappa \int_t^{t+dt} (\theta - v_s) ds + \sigma \int_t^{t+dt} \sqrt{v_s} d\widetilde{W}_v(s).
\end{aligned}
$$

Notice that the two integrals with $\widetilde{W}_v(t)$ in the SDEs above are the same, and in terms of the variance realizations they are given by:

$$
\int_t^{t+dt} \sqrt{v_s} d\widetilde{W}_v(s) = \frac{1}{\sigma} \left(v_{t+dt} - v_t - \kappa \int_t^{t+dt} (\theta - v_s) ds \right).
$$

Injecting the previous integral in the stock scheme gives:

$$
\begin{aligned}
X_{t+dt} = X_t + \int_{t_i}^{t+dt} \left(r - \frac{1}{2}v_s\right) ds + \frac{\rho}{\sigma} \left(v_{t+dt} - v_t - \kappa \int_t^{t+dt} (\theta - v_s) ds \right) \\
+ \sqrt{1-\rho^2} \int_t^{t+dt} \sqrt{v_s} \mathrm{d}\widetilde{W}_x(s).
\end{aligned}
$$

We approximate all integrals appearing in the expression above by their left integration boundary values of the integrand, as in the Euler discretization scheme:

$$
\begin{aligned}
X_{t+dt} \approx X_t + \left(r - \frac{1}{2}v_t\right) dt + \frac{\rho}{\sigma} \left(v_{t+dt} - v_t - \kappa \left(\theta - v_t\right) dt \right) \\
+ \sqrt{1-\rho^2} \sqrt{v_t} \left(\widetilde{W}_x\left(t+dt\right) - \widetilde{W}_x\left(t\right) \right).
\end{aligned}
$$

Finally we obtain:

$$
X_{t+dt} \approx X_t + k_0 + k_1 v_t + k_2 v_{t+dt} + \sqrt{k_3 v_t} Z_x
$$

with

$$
\begin{aligned}
& k_0 = \left(r - \frac{\rho}{\sigma}\kappa\theta\right) dt, \quad k_1 = \left(\frac{\rho\kappa}{\sigma} - \frac{1}{2}\right) dt - \frac{\rho}{\sigma} \\
& k_2 = \frac{\rho}{\sigma}, \quad k_3 = \left(1 - \rho^2\right) dt
\end{aligned}
$$

The final algorithm for the stock and the volatility schemes use direct inversion method to sample accordingly to the mentioned distributions. The Monte Carlo algorithm for a payoff $g(S_T)$ is written as follows:

Algorithm 3 Naive Monte Carlo with Andersen Efficient Simulation

Fix N and T
Fix dt
Fix v_0 and S_0
Fix γ_1 and γ_2
Repeat

1. $m := \mathbb{E}\left[v_{t+1} \mid v_t\right], s := \operatorname{Var}\left(v_{t+1} \mid v_i\right), \eta := \frac{s}{m^2}$
2. Draw $U \sim \mathcal{U}(0,1)$
3. $k_0 = -\frac{\rho\kappa\theta}{\sigma}dt$, $k_1 = \gamma_1 dt\left(\frac{\kappa\rho}{\sigma} - \frac{1}{2}\right) - \frac{\rho}{\sigma}$, $k_2 = \gamma_2 dt\left(\frac{\kappa\rho}{\sigma} - \frac{1}{2}\right) + \frac{\rho}{\sigma}$, $k_3 = \gamma_1 dt\left(1-\rho^2\right)$, $k_4 = \gamma_2 dt\left(1-\rho^2\right)$
4. **IF** $\eta < 1.5$
   (a) $b := \left(\frac{2}{\eta} - 1 + \sqrt{\frac{2}{\eta}}\sqrt{\frac{2}{\eta}-1}\right)^{1/2}$, $a := \frac{m}{1+b^2}$
   (b) $Z_v := \Phi^{-1}\left(U_v\right)\left(\sim \mathcal{N}(0,1)\right)$
   (c) $v_{t+1} := a\left(b+Z_v\right)^2$
5. **ELSE :**
   (a) $p := \frac{\eta-1}{\eta+1}$, $\beta := \frac{1-p}{m}$
   (b) $v_{t+1} = \begin{cases} 0 & \text{if } U_v \in [0,p] \\ \beta^{-1}\ln\left(\frac{1-p}{1-U_v}\right) & \text{if } U_v \in [p,1] \end{cases}$
6. $S_{t+1} = S_t \exp\left(k_0 + k_1 v_t + k_2 v_{t+1} + \sqrt{k_3 v_t + k_4 v_{t+1}} Z\right)$.

Until $t = T$

Return $\frac{1}{N}\sum_{i=1}^{N} g(S_T)$

1.4 Almost exact simulation of the Heston model

The almost exact method is based on the previous modeling of the noncentral chi-squared distribution. We recover the schemes for the stock price and the volatility :

$$
\begin{aligned}
X_{t+dt} &\approx X_t + k_0 + k_1 v_t + k_2 v_{t+dt} + \sqrt{k_3 v_t} Z_x \\
v_{t+dt} &= e^{-\kappa(dt)}/n(t,t+dt)\chi^2(d, v_t n(t,t+dt))
\end{aligned}
$$

with

$$
d = 4\kappa\theta/\sigma^2; \quad n(t,t+dt) = \frac{4\kappa e^{-\kappa(dt)}}{\sigma^2\left(1-e^{-\kappa(dt)}\right)}
$$

In Broadie and Kaya [2006], the authors suggest to use an acceptance and rejection method to generate directly the noncentral Chi-squared distribution. In Van Haastrecht and Pelsser [2010], the authors proposed a Non-central Chi-squared Inversion (NCI) scheme that uses the following representation with a Poisson distribution:

$$
v(t+dt) \mid v(t) = e^{-\kappa(dt)}/n(t,t+dt)\chi^2_{d+2N} \quad \text{for } d > 0
$$

with N a Poisson distribution with mean $\mu = \frac{1}{2}v_t n(t,t+dt)$.

Thus sampling from a noncentral chi-squared distribution is equivalent to sampling from a Poisson conditioned Chi-squared distribution.

With the intention of having a true benchmark of the Almost exact scheme, we sampled directly the noncentral chi-squared distribution provided in the *numpy.random* library. The Monte Carlo algorithm for a payoff $g(S_T)$ is written as follows:

Algorithm 4 Naive Monte Carlo Heston Model With Almost Exact Scheme

Fix N and T
Fix dt
Fix v_0 and S_0
Fix $X_0 = log(S_0)$
Repeat

1. $\bar{c} = \frac{\sigma^2}{4\kappa}\left(1 - \mathrm{e}^{-\kappa dt}\right)$, $\delta = \frac{4\kappa\theta}{\sigma^2}$, $\bar{\kappa} = \frac{4\kappa \mathrm{e}^{-\kappa dt}}{\sigma^2(1-\mathrm{e}^{-\kappa dt})}v_t$
2. $k_0 = \left(r - \frac{\rho}{\sigma}\kappa\theta\right)dt$, $k_1 = \left(\frac{\rho\kappa}{\sigma} - \frac{1}{2}\right)dt - \frac{\rho}{\sigma}$, $k_2 = \frac{\rho}{\sigma}$, $k_3 = \left(1 - \rho^2\right)dt$
3. Draw $Z \sim \mathcal{N}(0,1)$, Draw $Y \sim \chi^2(\delta, \bar{\kappa})$
4. $v_{t+1} = \bar{c}Y$
5. $X_{t+1} = X_t + k_0 + k_1 v_t + k_2 v_{t+1} + \sqrt{k_3 v_t}Z$

Until $t = T$

Return $\frac{1}{N}\sum_{i=1}^{N} g(e^{X_T})$

1.5 Pricing Under Heston Model

1.5.1 Heston PDE

For a European option with payoff $g\left(S_T\right)$ at maturity T, we have that the $\mathcal{C}^{1,2}$ function $C(t,s,v)$ of the pricing rule $C_t = C\left(t, S_t, V_t\right), t \in [0,T]$, for the option satisfies the following partial differential equation

$$\frac{\partial C}{\partial t} + rs\frac{\partial C}{\partial s} + \{\kappa\theta - v(\kappa + \sigma\lambda)\}\frac{\partial C}{\partial v} + \frac{1}{2}s^2 v\frac{\partial^2 C}{\partial s^2} + \rho\sigma v s\frac{\partial^2 C}{\partial v \partial s} + \frac{1}{2}\sigma^2 v\frac{\partial^2 C}{\partial v^2} = rC$$

with terminal condition $C(T,s,v) = g(s)$. Equivalently, by Feynman-Kac, this is to say that we have the usual risk-neutral pricing formula

$$C(t,s,v) = \mathbb{E}^Q\left[e^{-r(T-t)}g\left(S_T\right) \mid \left(S_t, V_t\right) = (s,v)\right]$$

where (S,V) follows the Q-dynamics with initial value $(S_t, V_t) = (s,v)$ at the initial time $t \in [0,T]$.

1.5.2 Closed Form Formula

The great advantage of the Heston model is the existence of a closed formula for the price of European Vanilla options. Indeed, the price C_t of a Call with underlying S and payoff $\left(S_T - K\right)_+$ under the Heston model is expressed in this way, by noting $\tau = T - t$ which is the tenor of the Call at time t:

$$C_t = C(t,S,v) = S_t P_1 - Ke^{-r\tau}P_2$$

where:

$$P_j = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{-i\Phi\ln(K)}f_j(t,x,v,\Phi)}{i\Phi}\right]d\Phi$$

with:

$x = \ln(S)$ and $f_j(t,x,v,\phi) = \exp\left(C_j(\tau,\phi) + D_j(\tau,\phi)v + i\phi x\right)$

with:

$$C_j(\tau,\Phi) = r\Phi i\tau + \frac{a}{\sigma^2}\left[\left(b_j - \rho\sigma\Phi + d\right)\tau - 2\ln\left(\frac{1-ge^{d\tau}}{1-g}\right)\right]$$
$$D_j(\tau,\Phi) = \frac{b_j - \rho\sigma\Phi i + d}{\sigma^2}\left[\frac{1-e^{d\tau}}{1-ge^{d\tau}}\right]$$

with:

- $g = \frac{b_j - \rho\sigma\Phi i + d}{b_j - \rho\sigma\Phi i - d}$

- $d = \sqrt{\left(\rho\sigma\Phi i - b_j\right)^2 - \sigma^2\left(2u_j\Phi i - \Phi^2\right)}$

- $a = \kappa\theta, u_1 = \frac{1}{2}, u_2 = -\frac{1}{2}, b_1 = \kappa + \lambda - \rho\sigma, b_2 = \kappa + \lambda.$

The demonstration of this closed formula, inspired by the reasoning of the Black Scholes formula, has been carried out and can be found in Heston [1993].

The solution is not so easy and immediate as the Black-Scholes one but this closed form precise solution is the benchmark by which we evaluate the performance our Monte Carlo estimator.

1.6 Numerical application on Vanilla options prices

This section is dedicated to a comparison of the four discretization schemes for pricing a European call option with the following parameters : $T = 3$, $K = 100$, $r = 0.02$, $\kappa = 2$, $\theta = 0.3$, $v_0 = 0.3$, $\sigma = 0.9$. These model parameters may be encountered in equity options markets.

Here we are interested in the Weak error which measures how far the distribution of the discretized process is from the original process :

$$\epsilon_{\text{weak}} = \left|\mathbb{E}\left[g\left(X_T\right)\right] - \mathbb{E}\left[g\left(\bar{X}_T^m\right)\right]\right|$$

The comparison is feasible thanks to the closed form formula provided in the previous section. The simulations were done with a number of simulations $n = 10^6$ and with a step $\Delta = T/40$.

Figure 1 describes the evolution of Weak error as a function of strike of the call option. We observe that the weak error decreases as the strike of the option increases. The figure 1 provides a clear rank of the accuracy of the schemes. The most accurate schemes is the Almost exact scheme and the Andersen scheme followed by the Euler and Milstein at the same level. The figure proves that the approximations done by Anderson in order to bypass the non-central chi-squared distribution are legitimate.

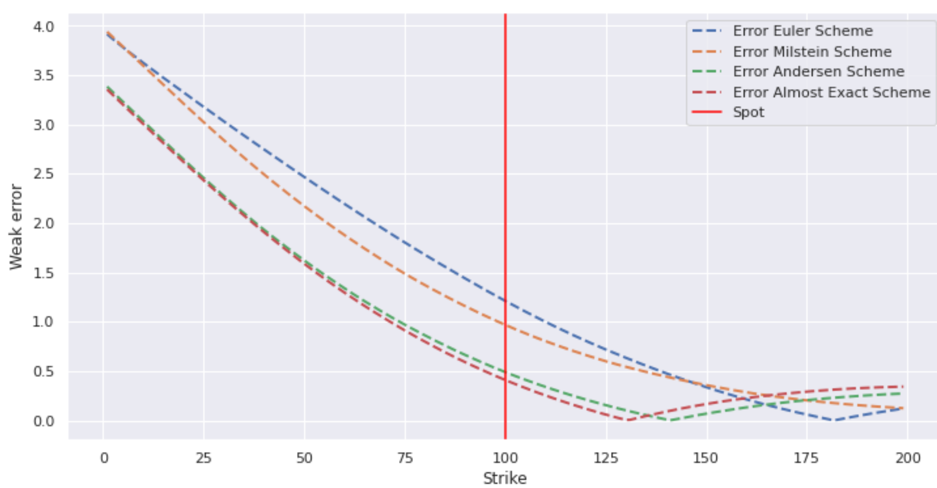

Figure 1: Weak error of the four schemes

The figure 4 provides the corresponding confidence interval. We observe that the confidence interval are the same for the four schemes and that they are decreasing with respect to the strike. It is highest for the Milstein scheme.

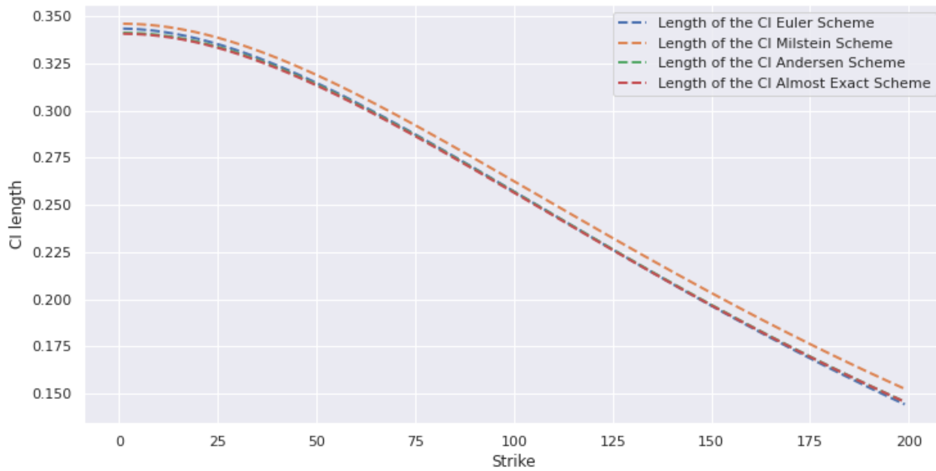

Figure 2: Confidence interval of the prices for the four schemes

The followings tables provides an in-depth study of the behavior of the weak error as both the strike and the step change. From table 1, we can observe that the Andersen and Almost exact schemes outperform both Euler and Milstein discretization for steps that are higher than $T/20$. We also notice that the error decreases at first to a minimum then starts to increase as the steps increases. Table 2 and Table 3 show the same behaviour. The confidence interval on the other hand increase as the step size increases. Finally, we remark that the optimal step depends on the strike of the call option.

	$K = 70$			
Δt	Euler	Milstein	Andersen	AE
T/10	1.86(0.27)	1.78(0.29)	2.03(0.27)	2.25(0.26)
$T/20$	1.90(0.28)	1.04(0.29)	0.22(0.28)	0.39(0.28)
$T/40$	1.91(0.29)	1.47(0.30)	0.84(0.29)	0.58(0.29)
$T/80$	1.87(0.30)	1.72(0.30)	1.43(0.30)	1.39(0.30)
$T/160$	1.96(0.30)	1.88(0.30)	1.71(0.30)	1.62(0.30)

Table 1: Test cases for the Heston schemes for Call with $K = 70$, $S(0) = 100$ and $r = 0.02$

	$K = 100$			
Δt	Euler	Milstein	Andersen	AE
T/10	0.44(0.23)	0.65(0.26)	2.83(0.23)	3.20(0.23)
$T/20$	0.89(0.25)	0.24(0.26)	0.82(0.25)	0.99(0.24)
$T/40$	1.18(0.26)	0.85(0.26)	0.05(0.25)	0.01(0.25)
$T/80$	1.33(0.26)	1.02(0.26)	0.71(0.26)	0.86(0.26)
$T/160$	1.33(0.26)	1.20(0.26)	1.20(0.26)	1.08(0.26)

Table 2: Test cases for the Heston schemes for Call with $K = 100$, $S(0) = 100$ and $r = 0.02$

	$K = 130$			
Δt	Euler	Milstein	Andersen	AE
T/10	1.79(0.27)	1.81(0.29)	1.92(0.27)	2.34(0.26)
$T/20$	1.82(0.28)	0.89(0.29)	0.36(0.28)	0.41(0.28)
$T/40$	1.86(0.29)	1.46(0.30)	0.76(0.29)	0.94(0.29)
$T/80$	1.99(0.30)	1.68(0.30)	1.33(0.29)	1.46(0.30)
$T/160$	2.03(0.30)	1.96(0.30)	1.65(0.30)	1.64(0.30)

Table 3: Test cases for the Heston schemes for Call with $K = 130$, $S(0) = 100$ and $r = 0.02$

2 Conclusion

This work aimed to study the Heston model. The first part was dedicated to the implementation of four discretization schemes : Euler, Milstein, Anderson and Almost exact scheme. The closed form formula provided us with a benchmark from which we were able to evaluate the accuracy of each scheme. Our experiments showed that both Anderson and Almost exact scheme are superior to the Euler and Milstein schemes.

3 Annexe

3.1 Numerical Application on Vanilla options Greeks with Pathwise Differentiation

Heston Model :
Parameters: $T = 1$, $K = 100$, $r = 0.02$, $\kappa = 4$, $\theta = 0.3$, $v_0 = 0.3$, $\sigma = 0.9$.

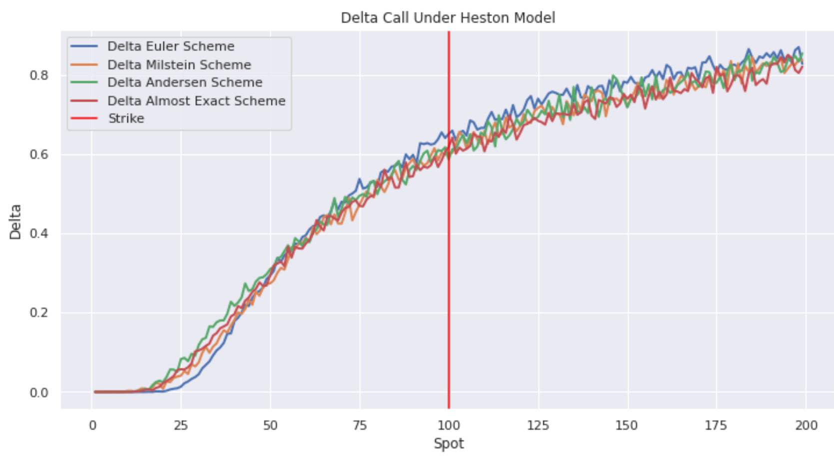

Figure 3: Delta Call Heston Model

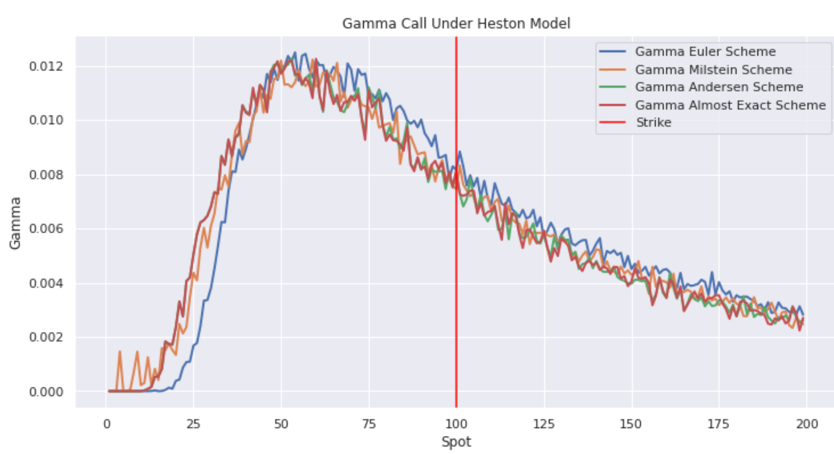

Figure 4: Gamma Call Heston Model

References

L. B. Andersen. Efficient simulation of the heston stochastic volatility model. *Available at SSRN 946405*, 2007.

M. Broadie and Ö. Kaya. Exact simulation of stochastic volatility and other affine jump diffusion processes. *Operations research*, 54(2):217–231, 2006.

Heston. A closed-form solution for. options with stochastic. volatility with applications to bond and currency. options. 1993.

A. Van Haastrecht and A. Pelsser. Efficient, almost exact simulation of the heston stochastic volatility model. *International Journal of Theoretical and Applied Finance*, 13(01):1–43, 2010.