Phoenix Autocall & Barrier Reverse Pricing and Hedging

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Abstract

Structured products with so-called "performance enhancement" are exposed to a higher risk than products with capital protection, for example, as the capital invested here is not fully guaranteed. The investor's participation in the positive performance of the underlying is limited, but in case of a negative performance, the participation in the negative performance can be total. As this type of product is riskier, the expected return for the investor is higher: a higher risk generally results in a higher expected return. Among these products, we will study here the Phoenix Autocall and the Barrier Reverse Convertible. Firstly, we discuss basics, defining their features and explaining the payoff mechanism. Then, we present numerical studies for the price of the product by Monte Carlo, and analysis of the Greeks computed by finite differences based on Monte Carlo prices.

1 Phoenix Autocall

1.1 Mechanism

An autocall is as a structured product that pays a high coupon payment while exposing the buyer to the downside risk of a reference asset. An autocall can be "called" prior to its maturity due to a barrier condition. The barrier condition is checked on "call dates".

Autocollables are split into two categories: discrete Autocollables with discrete call dates and continous Autocollables with continous call dates. In this work we will only focus on discrete Autocollables.

A Phoenix Autocall is an Autocollable that is characterised by three barrier conditions that governs its payoff. The first barrier dictates whether the investor receives or not, the coupons on call dates. The second barrier is a threshold below which the capital is no longer protected and the investor would be exposed to the downside risk of the reference asset. The last barrier is a threshold above which the contract is called.

The Phoenix Autocall has the following payoff mechanism:

- As long as the price of the underlying is above the Barrier Coupon (BC) level at call dates, the investor receives a coupon.
- If the price of the Underlying rises above its Initial Fixing (IF) level at call dates, then the product is called (Up and Out).
- If the underlying moves below the Barrier (B) level, the investor is exposed to the negative performance of the underlying (Down and in Put).

Autocollables are designed for investors who anticipate a slight decline in the market. In the event of a stagnation or slight decline in the market, the coupon amounts are received and are fixed. This is an advantage for the investor, as he can receive coupons while the price of the underlying is stagnating in the market.

1.2 Payoff

Notations: C is the net invested capital, c is the coupon percentage, T is the maturity, B is the barrier, BC is the barrier coupon, IF is the Initial Fixing level and S_t is the price of the underlying at time t.

The payoff formula can be simplified by taking N as the number of coupons received by the investor:

$$N = \mathbb{1}_{S_1 > BC} + \mathbb{1}_{S_2 > BC}\mathbb{1}_{S_1 < IF} + \ldots + \mathbb{1}_{S_1 < IF, S_2 < IF, S_{T-1} < IF}\mathbb{1}_{S_T > BC}$$

The payoff formula at time T for this product is as follows:

$$\begin{aligned} Y_T = & \mathbb{1}_{\max_{1 \leqslant t \leqslant T}(S_t) > IF} C(1 + Nc) \\ & + \mathbb{1}_{\max_{1 \leqslant t \leqslant T}(S_t) < IF} \mathbb{1}_{S_T \geqslant B} C(1 + Nc) \\ & + \mathbb{1}_{\max_{1 \leqslant t \leqslant T}(S_t) < IF} \mathbb{1}_{S_T < B} C\left(Nc + 1 + \frac{S_T - IF}{IF}\right) \end{aligned}$$

The payoff of this product depends on the price of the underlying at each observation date and not only at maturity because of the possible early repayment of the capital in case the underlying exceeds the strike level. As far as the calculation of the discounted payoff is concerned, the coupons are not received on the same date, and are therefore discounted according to the date on which they are received by the investor.

To illustrate the return over time of the Phoenix autocall, here is a graph representing the probable scenarios for a product with a 3-year maturity:

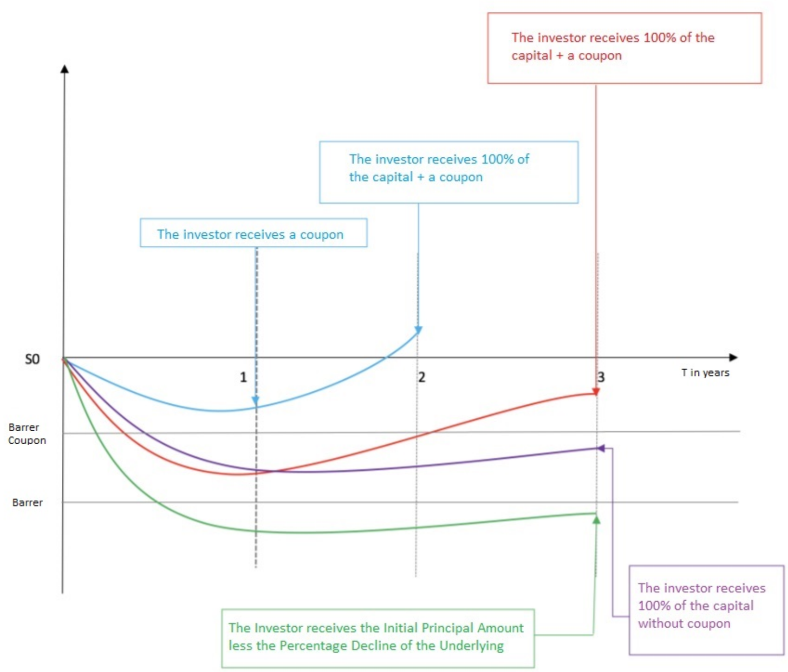

Figure 1: Scenarios for a 3-year Phoenix Autocall

The investor's gain is maximized when the underlying does not exceed its Initial Fixing level on the observation dates and does not fall below the Barrier coupon level to receive the coupons.

1.3 Pricing Monte Carlo

1.3.1 Framework

In the Black-Scholes model, the price of the underlying is modelled as a log-normal random variable. The stochastic differential equation (SDE) governing the dynamics of the price is given by

$$\mathrm{d}S(t) = rS(t)\mathrm{d}t + \sigma S(t)\mathrm{d}W_t$$

where r is the risk-free rate, and σ the volatility of the underlying. Like a typical SDE, this equation consists of a deterministic part and a random part. The part $\mathrm{d}S(t) = rS(t)\mathrm{d}t$ is a deterministic, ordinary differential equation, which can be written as $\frac{\mathrm{d}S(t)}{\mathrm{d}t} = rS(t)$.

The addition of the term $\sigma S(t)\mathrm{d}W_t$ introduces randomness into the equation, making it stochastic. The random part contains the term W_t, which is Brownian motion; it is a random process that is normally distributed with mean zero and variance t.

The assumption of a log-normal price implies that log prices are normally distributed. The log is another way of expressing returns, so in a different way this is saying that if the price is log-normally distributed, then the returns of the underlying are normally distributed.

1.3.2 Euler discretization

The idea of the Euler scheme is very simple: It reads

$$\begin{aligned} X_{t+h} &= X_t + \int_t^{t+h} b\left(X_s\right) ds + \int_t^{t+h} \sigma\left(X_s\right) dW_s \\ &\approx X_t + b\left(X_t\right) h + \sigma\left(X_t\right)\left(W_{t+h} - W_t\right). \end{aligned}$$

The Euler discretization $\bar{X}^m$ of X is thus constructed by setting $\bar{X}_0^m = x$ and then recursively for $0 \leqslant i \leqslant m-1$,

$$\bar{X}_{t_{i+1}}^m = \bar{X}_{t_i}^m + b\left(\bar{X}_{t_i}^m\right)\Delta t + \sigma\left(\bar{X}_{t_i}^m\right)\Delta W_i,$$

where $\Delta W_i := W_{t_{i+1}} - W_{t_i}$. In particular, the simulation of the Euler scheme reduces to generating the independent increments $\Delta W_i \sim \mathcal{N}\left(0, \sqrt{\Delta t} I_d\right)$.

Simulation We can simulate $\left(\bar{X}_{t_1}^m, \ldots, \bar{X}_{t_m}^m\right)$ as follows:

1. Generate $\left(Z_1, \ldots, Z_m\right)$ iid $\mathcal{N}\left(0, I_d\right)$
2. Set $\bar{X}_0^m = x$ and recursively for all $0 \leqslant i \leqslant m-1$,

$$\bar{X}_{t_{i+1}}^m = \bar{X}_{t_i}^m + b\left(\bar{X}_{t_i}^m\right)\Delta t + \sigma\left(\bar{X}_{t_i}^m\right)\sqrt{\Delta t} Z_{i+1}$$

The Euler discretization of the Black-Scholes model writes :

$$\bar{X}_{t_{i+1}}^m = \bar{X}_{t_i}^m + r\bar{X}_{t_i}^m \Delta t + \operatorname{diag}\left(\bar{X}_{t_i}^m\right)\sigma\Delta W_i.$$

1.4 Price Computation

In this subsection, we provide an example of the Autocall price with respect to the price of the underlying asset. The prices are illustrated in Figure 2. The following parameters were used:

- $\mathrm{S}_0 = 100$ Initial fixing spot
- $\mathrm{C} = 100$ Net invested capital
- $\mathrm{c} = 5\%$ Coupon Rate
- $\mathrm{T} = 3$ Maturity
- $\mathrm{B} = 80$ Barrier
- $\mathrm{BC} = 90$ Barrier Coupon
- $\sigma = 30\%$ Volatility
- $r = 1\%$ Risk Free Rate

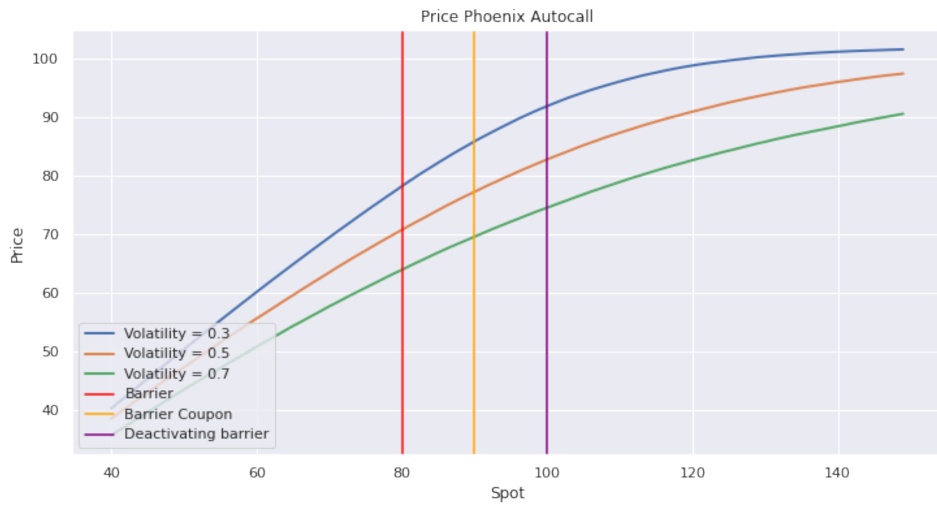

Figure 2: Price Phoenix Autocall v/s Spot

The price of the product is an increasing and concave function of the spot, limited by our net investment.Indeed, the higher the spot price, the higher the chance that the product will be recalled and therefore to have a 100% return on capital.

1.5 Greeks computation

In this section we will study the following Greeks : Delta, Gamma, Rho, Vega, Volga and Vanna. The Greeks where computed using finite difference method.

1.5.1 Delta

The delta measures the sensitivity of a derivative's price to a given change in the price of the underlying. Mathematically, it is the derivative of the price with respect to the underlying.

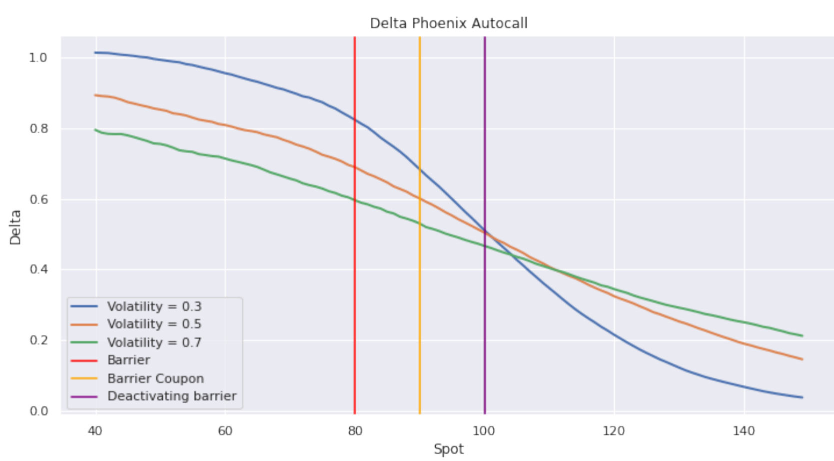

Figure 3: Delta Phoenix Autocall

The Delta is positive thus a positive variation in the underlying price results in a positive change in the derivative's price. Furthermore, the Delta is a decreasing function of the spot price: if the spot is high, the higher the spot price, the less the Autocall is sensitive to a change in the underlying

price and vice versa. When the spot price is far above the deactivating barrier the delta is 0: if the underlying rises by 1, the price of the product doesn't change. In this case the chances of being above the deactivating barrier at the time of observation and therefore of the product being recalled are high. Thus an increase in the underlying does not have much impact.
On the other hand, if the underlying is below the performance barrier and the coupon barrier, a variation in the underlying has a strong impact on the price: the delta is 1. Thus if the underlying rises by 1, the price of the product also rises by 1. If the spot price is low and the underlying rises, the product may end up above the knock-out barrier and also above the coupon barrier and therefore not suffer a performance penalty but also not receive a coupon.

The next Greek will guide us on how to maintain a Delta neutral position.

1.5.2 Gamma

The gamma measures the convexity of a derivative's price as a function of the underlying's price (second derivative of the price in relation to the underlying's price). This sensitivity indicates whether the derivative price moves faster or slower than the price of the underlying. In short, if the delta represents speed, the gamma represents acceleration. To compute gamma we use the following approximation of the second derivative:

$$Gamma \approx \frac{f\left(S+h_S\right)-2f\left(S\right)-f\left(S-h_S\right)}{h_S^2}$$

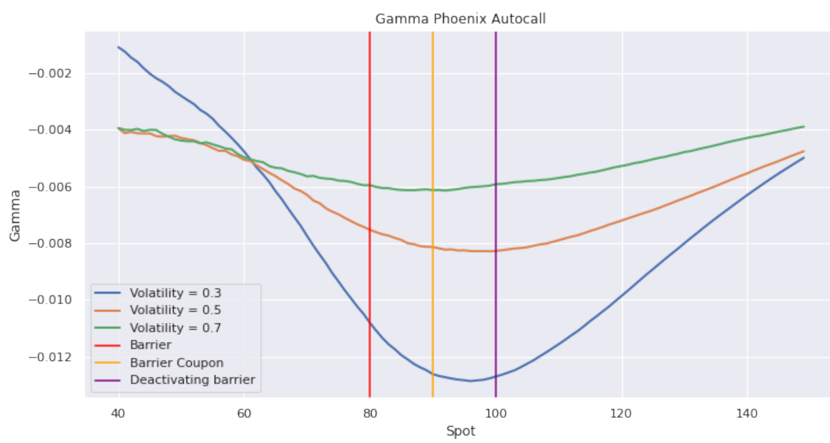

Figure 4: Gamma Phoenix Autocall

The Gamma is negative thus a positive variation in the underlying price results in a negative change in the derivative's price. In addition we observe that the Gamma peaks around the Coupon Barrier and the Deactivating barrier. Adjusting a Delta neutral position should be done frequently around this area.

1.5.3 Vega

Vega is the measurement of a derivative's price sensitivity to changes in the volatility of the underlying asset. Vega represents the amount of price change in reaction to a 1% change in the implied volatility of the underlying asset.

We approximate Vega using the first order derivative:

$$Vega \approx \frac{f\left(S,\sigma+h_\sigma\right)-f\left(S,\sigma-h_\sigma\right)}{2h_\sigma}$$

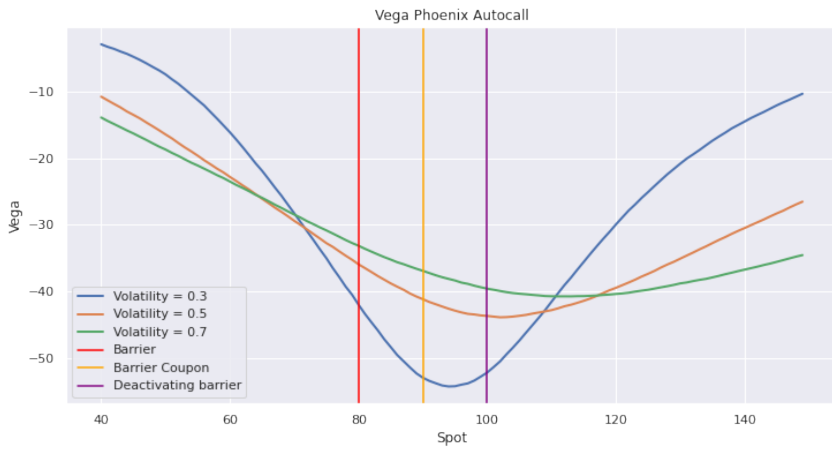

Figure 5: Vega Phoenix Autocall

The buyer is short Vega : it is negative thus a positive variation in volatility results in a negative change in the derivative's price. We observe that the Vega peaks between the coupon barrier and the deactivating barrier. As the underlying evolves around these two barriers, the Autocall has a high probability of being called out or pass below the coupon barrier thus not receiving a coupon. Thus the Vega is at it's peak in this interval.

1.5.4 Rho

Rho is the rate at which the price of a derivative changes relative to a change in the risk-free rate of interest. Rho measures the sensitivity of a derivative with respect to a change in interest rate.

We approximate Rho using the first order derivative:

$$Rho \approx \frac{f\left(S, r+h_r\right) - f\left(S, r-h_r\right)}{2h_r}$$

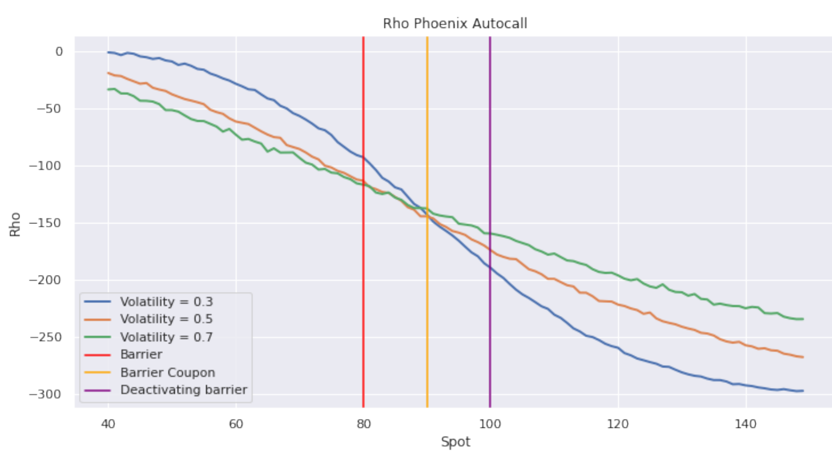

Figure 6: Rho Phoenix Autocall

The buyer is short Rho : a variation of the interest rate has a negative impact on the price of the product. As the price of the underlying increases, the effect of a variation of the interset rate grows and has a negative impact on the derivative. This is mainly due to the bond that is embedded in the Phoenix Autocall.

1.5.5 Volga

Volga measure the sensitivity of Vega with respect to the volatility of the underlying.

We approximate Volga using the second order derivative:

$$Volga \approx \frac{f(S, \sigma - h_v) - 2f(S, \sigma) - f(S, \sigma + h_v)}{h_v^2}$$

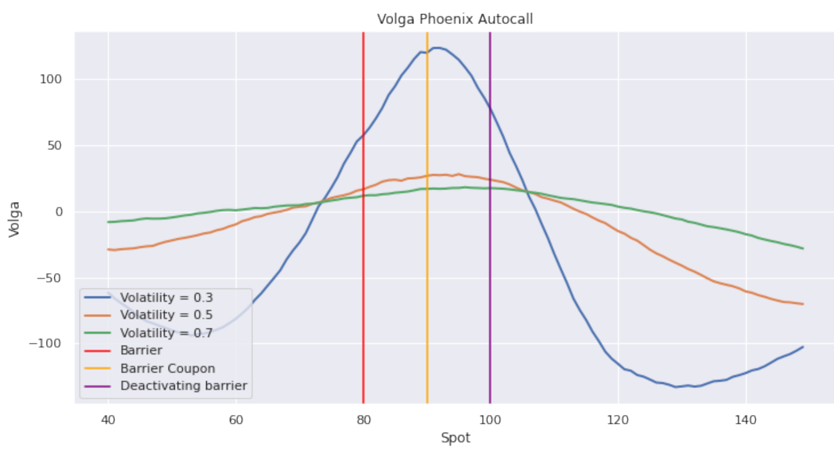

Figure 7: Volga Phoenix Autocall

The buyer is short-Volga when the spot is just before the Barrier or just after the Initial Fixing: at these levels, a change in volatility decreases the Vega.

Between the Barrier and the Initial Fixing, the buyer is long-Volga, a change in volatility will increase the Vega. The acceleration of the price as a function of volatility is therefore maximal when the spot is between the Barrier and the Initial Fixing at the level of the Barrier Coupon. Adjusting a Vega neutral position should done frequently in this area.

1.5.6 Vanna

Vanna measures the rate at which the delta of a derivative will change with respect to the volatility of the underlying market. It also measure the rate at which the Vega will change with respect to the underlying price. Vanna is used to assess changes in the relationship between the Greeks delta and Vega.

We approximate Vanna using the second order derivative:

$$Vanna \approx \frac{f(S + h_S, \sigma + h_v) - f(S - h_S, \sigma + h_v) - f(S + h_S, \sigma - h_v) + f(S - h_S, \sigma - h_v)}{4h_S h_v},$$

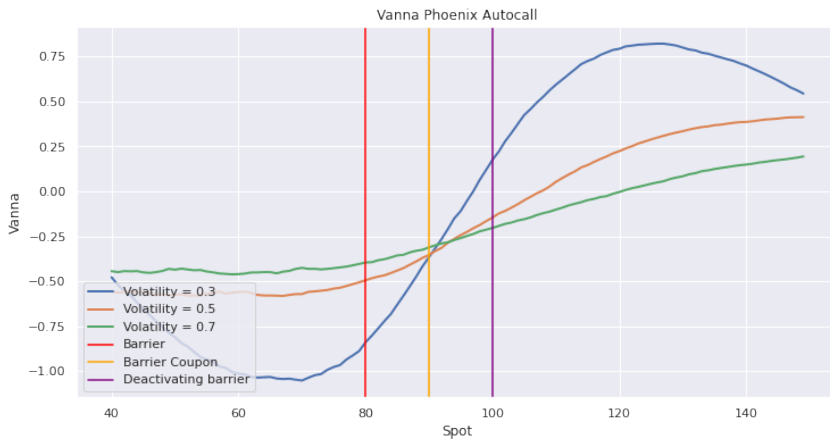

Figure 8: Vanna Phoenix Autocall

The buyer is short-Vanna before the Initial Fixing : a change in volatility decreases the Delta. In this case if volatility increases, there is a greater chance of approaching barriers and therefore an increase in the underlying will have less effect on the price. On the other hand, if we are above the Initial Fixing we are long-Vanna: an increase in volatility will increase the Delta. Indeed an increase in volatility increases our chances of being below the Initial Fixing and therefore decreases our chances of the product being recalled at the next observation time.

2 Barrier Reverse Convertible

2.1 Mechanism

The Barrier Reverse Convertible is a structured product consisting of a bond component and involves going short on a put option in exchange for a premium.

A coupon is received by the investor regardless of the performance of the underlying. The Barrier Reverse Convertible only includes a redemption at maturity.

If the price of the underlying is above the Barrier Level, the investor's invested capital remains intact and the investor receives the invested capital together with the Coupon Amounts determined in advance. Conversely, if the price of the underlying falls below the Barrier Level, the Investor receives the original principal Amount minus the fall in the Underlying, but still receives the Coupon Amount.

This type of product is designed for investors with a stable or slightly bullish view of the market. The investor receives the coupons regardless of the price development of the underlying. They are generally higher than the interest on a money market investment, as they are exposed to certain risks: the risk of default by the issuer, and the risk of a sharp decline in the market. This additional interest therefore remunerates a part of the risks encountered. However, if the market rises, the investor will not benefit from the positive performance of the underlying.

2.2 Payoff

Notations: C is the net invested capital, c is the coupon percentage, T is the maturity, B is the barrier and S_t is the price of the underlying at time t.

Here is the payoff X of this product at time T :

$$X_T = \mathbb{1}_{S_T \geqslant B} * C(1 + f * c) + \mathbb{1}_{S_T < B} * C\left(1 + f * c + \frac{S_T - S_0}{S_0}\right)$$

The graph represents probable scenarios for a 3-year Barrier Reverse Convertible:

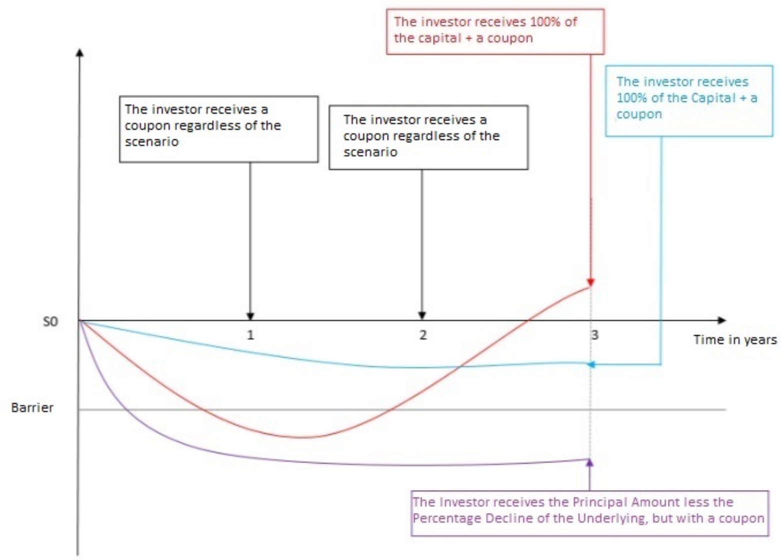

Figure 9: Scenarios for a 3-year Barrier reverse convertible

2.3 Price Computation

In this subsection, we provide an example of the Autocall price with respect to the price of the underlying asset. The prices are illustrated in Figure 2. The following parameters were used:

- $S_0 = 100$ Initial fixing spot
- $C = 100$ Net invested capital
- $T = 3$ maturity
- Barrier $= 90$
- $f = 3$ Frequency of coupons
- $\sigma = 30\%$ Volatility
- $r = 1\%$ Interest Rate

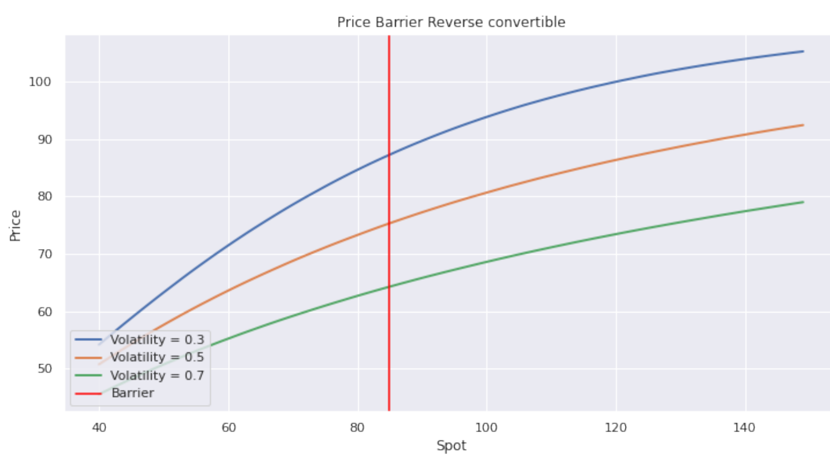

Figure 10: Price Reverse Convertible

2.4 Greeks computation

In this section we will study the following Greeks : Delta, Gamma, Rho, Vega, Volga and Vanna. The Greeks where computed using finite difference method.

2.4.1 Delta

The Delta is positive thus a positive variation in the underlying price results in a positive change in the derivative's price. Furthermore, the Delta is a decreasing function of the spot price: the higher the spot price, the less the Barrier Reverse is sensitive to a change in the underlying price and vice versa. When the spot price is far above the barrier the delta is 0: if the underlying rises by 1, the price of the product doesn't change. In this case the chances of being above the barrier at maturity and therefore recover 100% of the capital are high. Thus an increase in the underlying does not have much impact. On the other hand, if the underlying is below the barrier, a variation in the underlying has a strong impact on the price. If the spot price is low and the underlying rises, the product may end up above the barrier and therefore not suffer a performance penalty.

The next Greek will guide us on how to maintain a Delta neutral position.

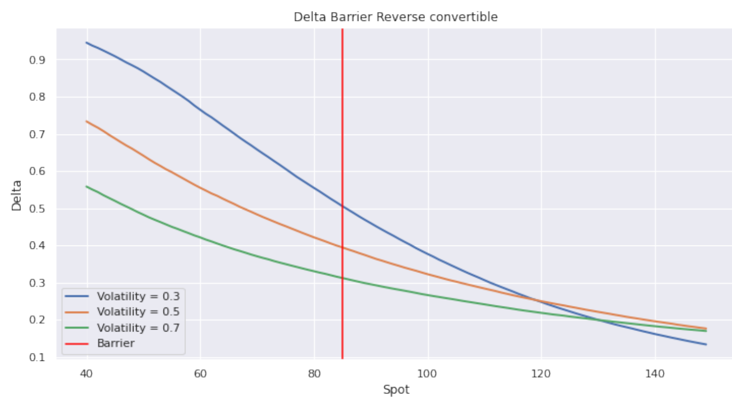

Figure 11: Delta Reverse Convertible

2.4.2 Gamma

The buyer is Short Gamma. Gamma decreases until it peaks just before the barrier and then increase. The increase in price acceleration relative to the underlying shows that when you are close to the barrier from below the change of the delta will be large so a movement in the underlying will greatly influence the price of the product.

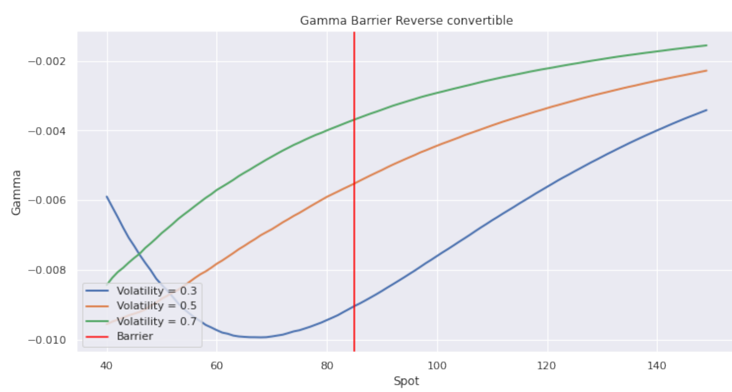

Figure 12: Gamma Reverse Convertible

2.4.3 Vega

The buyer is short Vega. The Vega decreases to a local minimum just after the barrier at the initial fixing level then rises again. The Vega before the barrier is higher than after the barrier, an increase in volatility when we are above the barrier increases the chances of being below the barrier at maturity and therefore of suffering an under-performance penalty. On the other hand, if we are below the barrier, an increase in volatility will increase our chances of going above the barrier and thus avoid penalties.

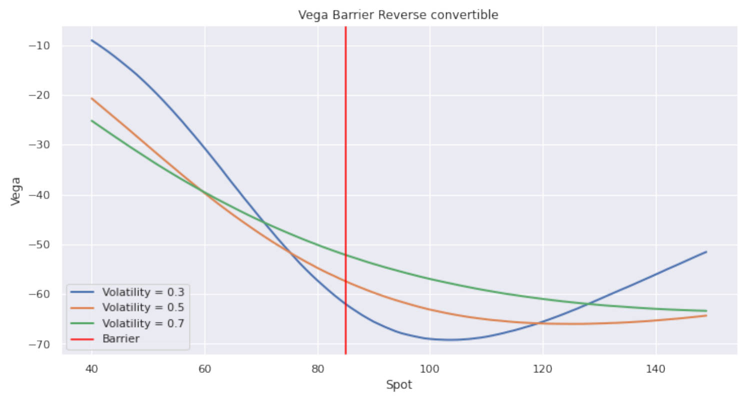

Figure 13: Vega Reverse Convertible

2.4.4 Rho

The buyer is short Rho : a variation of the interest rate has a negative impact on the price of the product. As the price of the underlying increases, the effect of a variation of the interset rate grows and has a negative impact on the derivative. This is mainly due to the bond that is embedded in the Barrier Convertible.

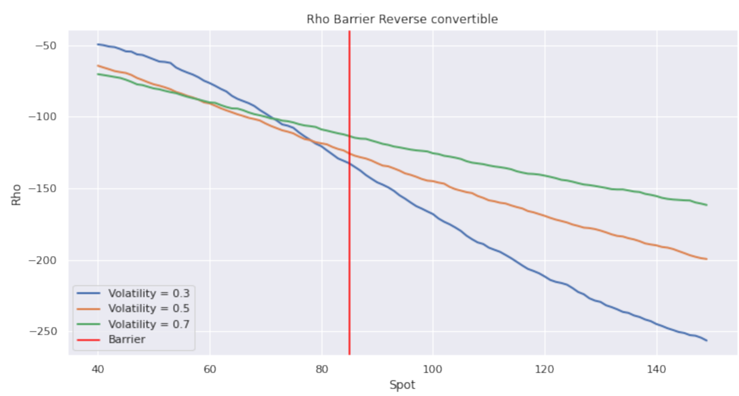

Figure 14: Rho Reverse Convertible

2.4.5 Volga

The buyer is long Volga when the spot is close to the barrier. At these levels, a change in volatility increases the Vega. When the spot is far away from the Barrier, the buyer is short Volga, a change in volatility will decrease the Vega. The acceleration of the price as a function of volatility is therefore maximal when the spot is close to the Barrier and the Initial Fixing at the level. Adjusting a Vega neutral position should done frequently in this area.

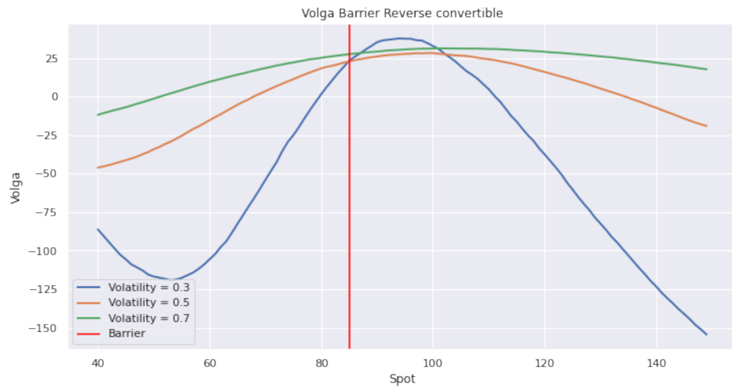

Figure 15: Volga Reverse Convertible

2.4.6 Vanna

The buyer is short Vanna beside the barrier level: a change in volatility decreases the Delta. In this case if volatility increases, there is a greater chance of approaching barrier and therefore an increase in the underlying will have less effect on the price . On the other hand, if we are far above the barrier we are long Vanna: an increase in volatility will increase the Delta. Indeed an increase in volatility increases our chances of being below the barrier and therefore increase the impact of an increase of the underlying price on the price of the product.

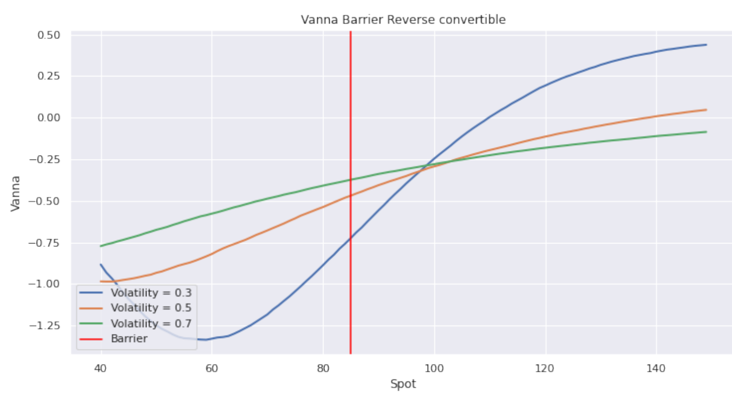

Figure 16: Vanna Reverse Convertible

3 Conclusion

This project was dedicated to the study of a Phoenix Autocall and a Barrier Reverse Convertible. We described the payoff mechanism of each structured product. We then proceeded with an in-depth study of the Greeks: Delta, Gamma, Vega, Rho, Volga and Vanna. These Greeks are risk measures that are utilised to hedge against uncertainty. They are related to a change in the underlying price, volatility and the interest rate. The Greeks were computed using Finite difference and Monte Carlo. The sensitivities of each product showed interesting behaviour around the discontinuity area of the payoff. To improve the calculations of sensitivities, one can use variance reduction methods e.g Antithetic control. The sensitivities of each structured product with respect to its barriers should also be studied in-depth.