# CUBIC FOURFOLDS FIBERED IN SEXTIC DEL PEZZO SURFACES

NICOLAS ADDINGTON, BRENDAN HASSETT, YURI TSCHINKEL, AND ANTHONY VÁRILLY-ALVARADO

Abstract. We exhibit new examples of rational cubic fourfolds. Our examples admit fibrations in sextic del Pezzo surfaces over the projective plane, which are rational whenever they have a section. They are parametrized by a countably infinite union of codimension-two subvarieties in the moduli space.

The rationality problem for complex cubic fourfolds has been studied by many authors; see [Has16, §1] for background and references to the extensive literature on this subject. The moduli space $\mathcal{C}$ of smooth cubic fourfolds has dimension 20. Since the 1990's, the only cubic fourfolds *known* to be rational are:

- Pfaffian cubic fourfolds and their limits. These form a divisor $\mathcal{C}_{14} \subset \mathcal{C}$.
- Cubic fourfolds containing a plane $P \subset X$ such that the induced quadric surface fibration $\mathrm{Bl}_P(X) \to \mathbb{P}^2$ has an odd-degree multisection. These form a countably infinite union of codimension-two loci $\bigcup \mathcal{C}_K \subset \mathcal{C}$, dense in the divisor $\mathcal{C}_8$ parametrizing cubic fourfolds containing a plane.

Our main result is:

**Theorem 1.** *Let $\mathcal{C}_{18} \subset \mathcal{C}$ denote the divisor of cubic fourfolds of discriminant* 18. *There is a Zariski open subset of $U \subset \mathcal{C}_{18}$ and a countably infinite union of codimension-two loci $\bigcup \mathcal{C}_K \subset \mathcal{C}$, dense in $\mathcal{C}_{18}$, such that $\bigcup \mathcal{C}_K \cap U$ parametrizes rational cubic fourfolds.*

We show that the generic element $X$ of $\mathcal{C}_{18}$ is birational to a fibration in sextic del Pezzo surfaces over $\mathbb{P}^2$. Its generic fiber and thus $X$ are rational if the fibration admits a section. In fact, a multisection of of degree prime to three suffices to establish rationality of $X$. This condition can be expressed in Hodge-theoretic terms, and is satisfied along a countably infinite union of divisors in $\mathcal{C}_{18}$.

In Section 1 we show that a generic cubic fourfold in $\mathcal{C}_{18}$ contains an elliptic ruled surface. In Section 2 we construct fibrations in sextic del Pezzo surfaces from elliptic ruled surfaces. After reviewing rationality of sextic del Pezzo surfaces in Section 3, we prove Theorem 1 in Section 4. In Section 5 we analyze the degenerate fibers of fourfolds fibered in sextic del Pezzo surfaces. Our approach is grounded in assumptions on the behavior of 'generic' cases; Section 6 validates these in a concrete example.

**Acknowledgments:** We are grateful to Asher Auel and Alexander Perry for stimulating discussions on related topics. The second author is partially supported by NSF grant 1551514; the fourth author is supported by NSF grant 1352291.

## 1. Sextic elliptic ruled surfaces and cubic fourfolds

**Theorem 2.** *A generic cubic $X \in \mathcal{C}_{18}$ contains an elliptic ruled surface $T$ of degree 6, with the following property: the linear system of quadrics in $\mathbb{P}^5$ containing $T$ is 2-dimensional, and its base locus is a complete intersection $\Pi_1 \cup T \cup \Pi_2$, where $\Pi_1$ and $\Pi_2$ are disjoint planes, which meet $T$ along curves $E_i$ that give sections of the $\mathbb{P}^1$-fibration $T \to E$.*

Before proving this we prove two preliminary results. We take the planes $\Pi_1, \Pi_2$ as our starting point.

**Proposition 3.** *Let $\Pi_1, \Pi_2 \subset \mathbb{P}^5$ be two disjoint planes. Then for a generic choice of three quadrics $Q_1, Q_2, Q_3$ containing $\Pi_1, \Pi_2$, we have*

$$Q_1 \cap Q_2 \cap Q_3 = \Pi_1 \cup T \cup \Pi_2,$$

*where $T$ is an elliptic ruled surface of degree 6. Moreover, the curves $E_i := T \cap \Pi_i$ are sections of the $\mathbb{P}^1$-fibration $T \to E$.*

*Proof.* Observe that a point in $\mathbb{P}^5 \setminus (\Pi_1 \cup \Pi_2)$ is contained in a unique line that meets $\Pi_1$ and $\Pi_2$, giving a rational map $\mathbb{P}^5 \dashrightarrow \Pi_1 \times \Pi_2$. To resolve this rational map, let $W$ be the blow-up of $\mathbb{P}^5$ along $\Pi_1$ and $\Pi_2$, and let $p$ and $q$ be as shown:

$$\begin{tikzcd} W \arrow[r, "p"] \arrow[d, "q"] & \Pi_1 \times \Pi_2 \\ \mathbb{P}^5. & \end{tikzcd}$$

Let $F_1, F_2 \subset W$ be the exceptional divisors, let $H$ be the hyperplane class on $\mathbb{P}^5$, and let $H_1, H_2$ denote the pullbacks of the hyperplane classes on $\Pi_1 \times \Pi_2$. We find that $p^*H_i = q^*H - F_i$, so $q^*(H_1 + H_2) = q^*(2H) - F_1 - F_2$; that is, divisors of type $(1, 1)$ on $\Pi_1 \times \Pi_2$ correspond to quadrics in $\mathbb{P}^5$ containing $\Pi_1$ and $\Pi_2$.

By the Bertini theorem and the adjunction formula, a generic intersection of three divisors of type $(1,1)$ is an elliptic curve $E \subset \Pi_1 \times \Pi_2$. The projection onto either factor maps $E$ isomorphically onto a smooth cubic curve $E_i \subset \Pi_i$, as follows. The three divisors of type $(1,1)$ determine a $3 \times 3$ matrix of linear forms on $\Pi_i$, and the image of $E$ is the locus where the determinant vanishes. The fibers of $E$ over $E_1$ are linear spaces in $\Pi_2$ but $E$ contains no lines so $E \to E_i$ is one-to-one. Thus $p_g(E_i) = 1 = p_a(E_i)$, so $E_i$ is smooth and $E \to E_i$ is an isomorphism.

Now $\tilde{T} := p^{-1}(E)$ is an elliptic ruled surface, and $F_1, F_2 \subset W$ give sections of $p$, hence sections of $\tilde{T} \to E$. Let $T = q(\tilde{T}) \subset \mathbb{P}^5$. From our analysis of the projections $E \to E_i \subset \Pi_i$, it follows that $\tilde{T} \to T$ is an ismorphism and $T \cap \Pi_i = E_i$. □

**Proposition 4.** *Let $T \subset \mathbb{P}^5$ be an elliptic ruled surface as in Proposition 3. Then the homogeneous ideal of $T$ is generated by quadrics and cubics. Moreover,*

$$\begin{aligned} h^0(\mathcal{I}_T(1)) &= 0 \\ h^0(\mathcal{I}_T(2)) &= 3 \\ h^0(\mathcal{I}_T(3)) &= 20. \end{aligned}$$

*Proof.* The first statement follows from [Hom80, Thm. 3.3]. The second can be found in [Hom80, §2], or we can prove it as follows. From the inclusion $T \subset (\Pi_1 \cup T \cup \Pi_2) \subset \mathbb{P}^5$ we get an exact sequence

$$0 \to \mathcal{I}_{(\Pi_1 \cup T \cup \Pi_2)/\mathbb{P}^5} \to \mathcal{I}_{T/\mathbb{P}^5} \to \mathcal{I}_{T/(\Pi_1 \cup T \cup \Pi_2)} \to 0.$$

Since $\Pi_1 \cup T \cup \Pi_2$ is a complete intersection of three quadrics, we can compute cohomology of the first term using a Koszul complex. The third term is isomorphic to

$$\mathcal{I}_{E_1/\Pi_1} \oplus \mathcal{I}_{E_2/\Pi_2} = \mathcal{O}_{\Pi_1}(-3) \oplus \mathcal{O}_{\Pi_2}(-3).$$

Then the calculation is straightforward. □

*Proof of Theorem 2.* First we claim that the Hilbert scheme of surfaces $T$ of the form appearing in Proposition 3 is smooth and irreducible of dimension 36. Indeed, it is isomorphic to an open subset of a $\mathrm{Gr}(3,9)$-bundle over $\mathrm{Gr}(3,6) \times \mathrm{Gr}(3,6)$. To see this, observe that each $T$ arises from a *unique* pair of planes $\Pi_1, \Pi_2$: by intersecting the three quadrics containing $T$ we recover $\Pi_1$ and $\Pi_2$.

Next, the Hilbert scheme of pairs $T \subset X$, where $X$ is a smooth cubic fourfold, is smooth and irreducible of dimension 55. It is an open subset of a $\mathbb{P}^{19}$-bundle over the previous Hilbert scheme.

Next, we claim that if $X$ is a smooth cubic fourfold containing $T$ then $X \in \mathcal{C}_{18}$. The intersection form on a smooth cubic fourfold $X$ containing $T$ is as follows:

| | $h^2$ | $T$ |
|---|---|---|
| $h^2$ | 3 | 6 |
| $T$ | 6 | 18 |

Only $T^2 = 18$ needs to be justified. By [Has00, §4.1], we have

$$T^2 = 6h^2 + 3h.K_T + K_T^2 - \chi_T.$$

We have $K_T^2 = 0 = \chi_T$ because $T$ is an elliptic ruled surface. Moreover we have $K_T = -E_1 - E_2$, so $h.K_T = -6$.

Modding out the action of PGL(6), we get a map from an irreducible, 20-dimensional space of pairs $(T, X)$ to an irreducible 19-dimensional space $\mathcal{C}_{18}$. To show that this is dominant, it is enough to show that for a generic pair $(T, X)$, the deformation space $H^0(\mathcal{N}_{T/X})$ is at most 1-dimensional. By semi-continuity it is enough to verify this for one pair. We do this with Macaulay2 in §6. □

*Remark* 5. The ruled surface $T \subset X$ determines an elliptic curve $E$ in the variety of lines $F_1(X)$. By [Ran95, Cor. 5.1], such a curve moves in a family of dimension at least 1.

## 2. Constructing the del Pezzo fibration

**Theorem 6.** *Let $X$ be a cubic fourfold containing an elliptic ruled surface $T$ as in Theorem 2, and let*

$$\pi \colon X' := \mathrm{Bl}_T(X) \to \mathbb{P}^2$$

*be the map induced by the linear system of quadrics containing $T$. For generic $X$, the generic fiber of $\pi$ is a del Pezzo surface of degree 6, and the preimages of the curves $E_1, E_2 \subset T$ induce trisections of $\pi$.*

*Proof.* Let $\Pi_1, \Pi_2$ be the two planes appearing in Theorem 2. We will show that for a generic choice of two quadrics $Q_1, Q_2 \supset \Pi_1 \cup \Pi_2$ and a cubic $X \supset T$, we have $Q_1 \cap Q_2 \cap X = T \cup S$, where $S$ is a del Pezzo surface of degree 6.

By Proposition 4 we know that $T$ is cut out by cubics, so a generic cubic containing $T$ does not contain $\Pi_1$ or $\Pi_2$. Thus by the Bertini theorem, we can say that $Q_1 \cap Q_2 \cap X = T \cup S$, where $S$ is irreducible and smooth away from $T$. First we argue that the surface $S$ is smooth everywhere, and that $S$ and $T$ meet transversely in a smooth curve $C$.

Let $Y = Q_1 \cap Q_2$. We claim that $Y$ has ordinary double points

$$s_{11}, s_{12}, s_{13} \in \Pi_1 \cap T, \quad s_{21}, s_{22}, s_{23} \in \Pi_2 \cap T.$$

(Compare [Kap09, 2.1].) To see this, let

$$\begin{tikzcd} W \arrow[r, "p"] \arrow[d, "q"'] & \Pi_1 \times \Pi_2 \\ \mathbb{P}^5 & \end{tikzcd}$$

be as in the proof of Proposition 3, and let $Z \subset \Pi_1 \times \Pi_2$ be the intersection of two divisors of type $(1,1)$ corresponding to $Q_1, Q_2$. (By coincidence, this is a sextic del Pezzo surface.) The projections $Z \to \Pi_i$ each contract three lines; thus $Y = q(p^{-1}(Z))$ is smooth away from six ordinary double points as claimed.

Now let

$$r\colon V \to \mathbb{P}^5$$

be the blow-up of $\mathbb{P}^5$ along $T$, let $F \subset V$ be the exceptional divisor, and let $Y' \subset V$ be the proper transform of $Y$. This is a small resolution of $Y$; indeed, it is the flop of $p^{-1}(Z)$ along the six exceptional lines. Let $T' = Y' \cap F$, which is the blow-up of $T$ at the six points $s_{ij}$.

Let $X'$ be a generic divisor in the linear system $|r^*(3H) - F|$. Because $T$ is cut out by cubics, this linear system is basepoint-free, so $X'$ is smooth, and moreover the surface $S' := X' \cap Y'$ and the curve $C' := X' \cap T'$ are smooth. We claim that $r$ maps $S'$ isomorphically onto $S = r(S')$, and $C'$ isomorphically onto $C = r(C')$. For this it is enough to observe that $X'$ meets the exceptional lines of $Y' \to Y$ in one point each; otherwise one of the lines would be contained in the base locus of $|r^*(3H) - F|$, which is impossible. (As $Q_1$ and $Q_2$ vary, these six points give rise to the two trisections of $\pi$.)

Finally, we argue that $S$ is a del Pezzo surface of degree 6. By adjunction we have $K_{S'} = r^*H - F$, so $K_S = H - C$. But $C = S \cap T = S \cap Q_3$, so $K_S = H - 2H = -H$, so $S$ is a del Pezzo surface. And $S \cup T = Q_1 \cap Q_2 \cap X$ has degree 12, and $T$ has degree 6, so $S$ has degree 6. □

*Remark* 7. For a generic pencil of quadrics generated by $Q_1$ and $Q_2$, the intersection $D := T \cap S$ is a smooth curve of degree 12 and genus 7. This is an application of the adjunction formula on $T$: we have $K_T = -E_1 - E_2$ and $H = K_T + D$, thus

$$H.D = 12, \quad D^2 + K_T.D = 12.$$

## 3. Rationality of sextic del Pezzo surfaces

To determine when our fibration in sextic del Pezzo surfaces $\pi\colon X' \to \mathbb{P}^2$ is rational, we review rationality properties of sextic del Pezzos in general.

Let $S$ be a del Pezzo surface of degree six over a perfect field $F$. (In our application $F$ is the function field of $\mathbb{P}^2$.) Over an algebraic closure $\bar{F}$, $\bar{S} = S_{\bar{F}}$ is isomorphic to $\mathbb{P}^2_{\bar{F}}$ blown up at three non-collinear points. The lines ($(-1)$-curves) on $\bar{S}$ consist of the exceptional divisors and the proper transforms of lines joining pairs of the points; these form a hexagon with dihedral symmetry $\mathfrak{D}_{12} \simeq \mathfrak{S}_2 \times \mathfrak{S}_3$.

The Galois action on the Picard group gives a representation

$$\rho_S : \mathrm{Gal}(\bar{F}/F) \to \mathfrak{S}_2 \times \mathfrak{S}_3,$$

where the first factor indexes the geometric realizations of $S$ as a blow-up of $\mathbb{P}^2$ and the second factor corresponds to the conic bundle structures. Let $K/F$ denote the quadratic étale algebra associated with first factor and $L/F$ a cubic étale algebra associated with the second factor. Blunk [Blu10] has studied Azumaya algebras $B/K$ and $Q/L$ with the following properties:

- The Brauer-Severi variety $\mathrm{BS}(B)$ has dimension two over $K$, and there is a birational morphism
$$S_K \to \mathrm{BS}(B)$$
realizing $S_K$ as the blow-up over a cycle of three points;
- $\mathrm{BS}(Q)$ has dimension one over $L$, and there is a morphism
$$S_L \to \mathrm{BS}(Q)$$
realizing $S_L$ as a conic fibration with two degenerate fibers;
- the corestrictions $\mathrm{cor}_{K/F}(B)$ and $\mathrm{cor}_{L/F}(Q)$ are split over $F$;
- $B$ and $Q$ both contain a copy of the compositum $KL$, and thus are split over this field.

**Proposition 8.** *The following are equivalent:*

1. *$S$ is rational over $F$;*
2. *$S$ admits an $F$-rational point;*
3. *$S$ admits a zero-cycle of degree prime to six;*
4. *the Brauer classes $B$ and $Q$ are trivial.*

*Proof.* The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are clear. The implication (2) $\Rightarrow$ (1) is [Man66, Cor. 1 to Thm. 3.10]. The implication (3) $\Rightarrow$ (4) is straightforward: note that $B$ is trivial if and only if $\mathrm{BS}(B)$ admits a zero-cycle of degree prime to 3, that $Q$ is trivial if and only if $\mathrm{BS}(Q)$ admits a zero-cycle of odd degree, and that $S_K$ maps to $\mathrm{BS}(B)$ and $S_L$ maps to $\mathrm{BS}(Q)$. The implication (4) $\Rightarrow$ (2) is [Blu10, Cor. 3.5]. □

*Remark* 9. It may happen that a del Pezzo surface $S$ has maximal Galois representation $\rho_S$, but $B$ and $Q$ are nonetheless trivial. For example, let $F = \mathbb{C}(t)$, and choose a surjective representation

$$\rho : \mathrm{Gal}(\bar{F}/F) \to \mathfrak{S}_2 \times \mathfrak{S}_3$$

corresponding to a branched cover $C \to \mathbb{P}^1$. The geometric automorphism group admits a split exact sequence [Blu10, §2]

$$1 \to T \to \mathrm{Aut}(\bar{S}) \to \mathfrak{S}_2 \times \mathfrak{S}_3 \to 1.$$

Here $T$ is a torus with a natural $\mathfrak{S}_2 \times \mathfrak{S}_3$ action via conjugation. Composing $\rho$ with the splitting, we obtain a del Pezzo surface $S/\mathbb{C}(t)$ of degree six with $\rho_S = \rho$. However, the Brauer group of any complex curve is trivial, so the Brauer classes above necessarily vanish.

## 4. Rationality of our cubic fourfolds

*Proof of Theorem 1.* Let $U \subset \mathcal{C}_{18}$ be the Zariski open set parametrizing cubics fourfolds as in Theorem 6. Given $X \in U$, we have the blow-up $r \colon X' \to X$ along $T$ and the sextic del Pezzo fibration $\pi \colon X' \to \mathbb{P}^2$. Let $S' \subset X'$ be a smooth fiber of $\pi$, and let $S = r(S')$. We claim that if there is a cohomology class $\Sigma \in H^4(X, \mathbb{Z}) \cap H^{2,2}(X)$ with $\Sigma.S = 1$, then $X$ is rational. The integral Hodge conjecture holds for cubic fourfolds [Voi13, Th. 1.4], so we can promote $\Sigma$ to an algebraic cycle. Then $\Sigma' = r^*\Sigma$ satisfies $\Sigma'.S' = 1$ by the projection formula. Proposition 8 implies that $X'$ is rational over the function field $\mathbb{C}(\mathbb{P}^2)$, hence over $\mathbb{C}$, so $X$ is rational over $\mathbb{C}$.

Thus our goal is to produce a countable dense set of divisors

$$\mathcal{C}_K \subset \mathcal{C}_{18}$$

such that if $X$ is in some $\mathcal{C}_K$ then there is a Hodge class $\Sigma$ with $\Sigma.S = 1$.

Let $h \in H^2(X, \mathbb{Z})$ be the hyperplane class, and let $L$ be the lattice underlying $H^4(X, \mathbb{Z})$. Nikulin's results on embeddings of lattices [Nik79, §1.14] imply that, for each positive definite rank-three overlattice

$$\begin{array}{c|cc} & h^2 & S \\ \hline & 3 & 6 \\ & 6 & 18 \end{array} \subset K_{a,b} := \begin{array}{c|ccc} & h^2 & S & \Sigma \\ \hline h^2 & 3 & 6 & a \\ S & 6 & 18 & 1 \\ \Sigma & a & 1 & b \end{array},$$

the embedding

$$\langle h^2, S \rangle \hookrightarrow L$$

extends to an embedding of $K_{a,b}$ in $L$; compare [Has99, §4].

The lattice $K_{a,b}$ has discriminant

$$\Delta = -3 + 12a - 18a^2 + 18b.$$

Replacing $\Sigma$ with $\Sigma + m(3h^2 - S)$ for a suitable $m \in \mathbb{Z}$, we may assume that $a = -1, 0, 1$. Thus positive integers $\Delta \equiv 3 \pmod 6$ arise as discriminants, each for precisely one lattice, denoted $K_\Delta$ from now on.

Excluding finitely many small $\Delta$, the lattice $K_\Delta$ defines a divisor

$$\mathcal{C}_{K_\Delta} \subset \mathcal{C}_{18}.$$

See [Has16, §2.3] for details on which $\Delta$ must be excluded, and [AT14, §4] for details of a similar calculation in $\mathcal{C}_8$.

The divisors $\mathcal{C}_{K_\Delta}$ are the ones we want. The density in the Euclidean topology follows from the Torelli theorem [Voi86] and from [Voi07, 5.3.4]. □

Let $X$ be a cubic fourfold containing an elliptic ruled surface $T$ as in Theorem 2, and let $\pi : X' \to \mathbb{P}^2$ be the associated fibration into del Pezzo surfaces $S$ of degree 6. A labelling of such a cubic fourfold is a choice of lattice

$$\left\langle h^2, T \right\rangle = \left\langle h^2, S \right\rangle \subset A(X) := H^4(X, \mathbb{Z}) \cap H^2(X, \Omega^2)$$

associated with such a fibration.

**Proposition 10** (Hodge-theoretic interpretation)**.** *Let $X$ be a labelled cubic fourfold of discriminant 18, and let $\Lambda$ be the Hodge structure on the orthogonal complement of the labelling lattice. Then there exists an embedding of polarized Hodge structures*

$$\Lambda(-1) \hookrightarrow H^2_{prim}(Y', \mathbb{Z}),$$

*where $(Y', f')$ is a polarized K3 surface of degree two, and $\Lambda(-1)$ is an index-three sublattice, expressible as*

$$\Lambda(-1) \cong \langle \eta' \rangle^\perp,$$

*for some $\eta' \in H^2(Y', \mathbb{Z})$ whose image in $H^2(Y', \mathbb{Z}/3\mathbb{Z})/\langle f' \rangle$ is isotropic under the intersection form modulo 3. If $A(X)$ has rank 2, then the class $\eta'$ gives rise to a nontrivial element in $\mathrm{Br}(Y')[3]$.*

*Proof.* The discriminant group of the lattice $\Lambda(-1)$ is isomorphic to $\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}$ [Has00, Proposition 3.2.5]. On the other hand, Theorem 9 in [MSTVA14] shows that there exist polarized K3 surfaces $(Y', f')$ of degree two such that $H^2_{prim}(Y', \mathbb{Z})$ contains a lattice $\Gamma$ of index three having the same rank and discriminant group as $\Lambda(-1)$. The lattices $\Lambda(-1)$ and $\Gamma$ are even, indefinite and have few generators for their discriminant groups relative to their rank (2 vs. 21). Results of Nikulin [Nik79] now imply they are isometric. The isomorphism

$$\begin{aligned} H^2(Y', \mathbb{Z})/\langle f' \rangle \otimes \mathbb{Z}/3\mathbb{Z} &\xrightarrow{\sim} \mathrm{Hom}(\langle f' \rangle^\perp, \mathbb{Z}/3\mathbb{Z}) \\ v \otimes 1 &\mapsto [t \mapsto v \cdot t \bmod 3] \end{aligned}$$

identifies $\Gamma$ with the kernel of a map $\langle f'\rangle^\perp \to \mathbb{Z}/3\mathbb{Z}$ corresponding to some $\eta'\otimes 1$, so $\Gamma = \{t \in \langle f'\rangle^\perp : t\cdot\eta' \in 3\mathbb{Z}\}$; see [vG05, §2.1] for details.

If $A(X)$ has rank 2, then $\mathrm{NS}(Y') = \langle f'\rangle$. The exponential sequence then gives rise to the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & H^2(Y',\mathbb{Z})/\langle f'\rangle \arrow[r] \arrow[d, "\times 3"] & H^2(Y',\mathscr{O}_{Y'}) \arrow[r] \arrow[d, "\times 3"] & \mathrm{Br}(Y') \arrow[r] \arrow[d, "\times 3"] & 0 \\
0 \arrow[r] & H^2(Y',\mathbb{Z})/\langle f'\rangle \arrow[r] & H^2(Y',\mathscr{O}_{Y'}) \arrow[r] & \mathrm{Br}(Y') \arrow[r] & 0
\end{tikzcd}$$

The snake lemma then shows that to $[\eta'] \in H^2(Y',\mathbb{Z}/3\mathbb{Z})/\langle f'\rangle$ there corresponds a class in $\mathrm{Br}(Y')[3]$. □

## 5. Singular fibers and discriminant curves

In this section we study the locus where fibers of $X' \to \mathbb{P}^2$ degenerate, and its relationship to the Azumaya algebras $B/K$ and $Q/L$ of Section 3.

A *singular del Pezzo surface* is a surface $S$ with ADE singularities and ample anticanonical class. Those of degree six ($K_S^2 = 6$), are classified as follows [CT88, Prop. 8.3]:

- type I: $S$ has one $A_1$ singularity and is obtained by blowing up $\mathbb{P}^2$ in three collinear points, and blowing down the proper transform of the line containing them;
- type II: $S$ also has one $A_1$ singularity and is obtained by blowing up $\mathbb{P}^2$ in two infinitely near points and a third point not on the line associated with the infinitely near points, then blowing down the proper transform of the first exceptional divisor over the infinitely near points;
- type III: $S$ has two $A_1$ singularities and is obtained by blowing up two infinitely near points and a third point all contained in a line, and blowing down the proper transforms of the first exceptional divisor over the infinitely near point and the line;
- type IV: $S$ has an $A_2$ singularity and is obtained by blowing up a curvilinear subscheme of length three not contained in a line and blowing down the first two exceptional divisors;
- type V: $S$ has an $A_1$ and and $A_2$ singularity and is obtained by blowing up a curvilinear subscheme contained in a line, and blowing down the proper transforms of the first two exceptional divisors and the line.

Types I and II occur in codimension one and correspond to conjugacy classes of involutions associated with the factors of $\mathfrak{S}_2 \times \mathfrak{S}_3$. Type

III occurs in codimension two and corresponds to conjugacy classes of Klein four-groups. Type IV also occurs in codimension two and corresponds to three cycles. Type V occurs in codimension three and corresponds to the full group.

**Definition 11.** Let $P$ be a smooth complex projective surface. A *good del Pezzo fibration* consists of a smooth fourfold $\mathcal{S}$ and a flat projective morphism $\pi\colon \mathcal{S} \to P$ with the following properties:

- The fibers of $\pi$ are either smooth or singular del sextic del Pezzo surfaces. Let $B_I, \ldots, B_V$ denote the closure of the corresponding loci in $P$.
- $B_I$ is a nonsingular curve.
- $B_{II}$ is a curve, non-singular away from $B_{IV}$.
- $B_{III}$ is finite and coincides with the intersection of $B_I$ and $B_{II}$, which is transverse.
- $B_{IV}$ is finite, and $B_{II}$ has cusps at $B_{IV}$.
- $B_V$ is empty.

*Remark* 12. The point is that the classifying map from $P$ is transverse to each singular stratum of the moduli stack of singular del Pezzo surfaces. For example, the discriminant locus is cuspidal at points with an $A_2$ singularity. In particular, good del Pezzo fibrations are Zariski open in the moduli space of all del Pezzo fibrations with fixed invariants.

**Proposition 13.** *Let $\pi\colon \mathcal{S} \to P$ be a good del Pezzo fibration. Then Blunk's construction yields:*

- *a non-singular double cover $Y \to P$ branched along $B_I$;*
- *an element $\eta \in \mathrm{Br}(Y)[3]$;*
- *a non-singular degree-three cover $Z \to P$ branched along $B_{II}$;*
- *an element $\zeta \in \mathrm{Br}(Z)[2]$.*

*Proof.* As before, let $K$ and $L$ be the quadratic and cubic extensions of $\mathbb{C}(P)$ introduced in Section 3. Let $Y$ and $Z$ denote the normalizations of $P$ in the fields (or étale algebras) $K$ and $L$ respectively.

We first address the double cover. Since $B_I$ is smooth, the double cover branched along $B_I$ is also smooth. Consider the base change $\pi_Y\colon \mathcal{S} \times_P Y \to Y$, a sextic del Pezzo fibration with singular fibers of types I, II, III, IV and geometric generic fiber $\overline{S}$. Let $G$ be the Galois group of $KL$ over $K$. Let $H_1, H_2 \in \mathrm{Pic}(\overline{S})^G$ be the classes corresponding to the birational morphisms

$$\beta_1, \beta_2\colon \overline{S} \to \mathbb{P}^2_K.$$

These specialize to Weil divisor classes in each geometric fiber of $\pi_Y$. These are Cartier for fibers of types II and IV; $H_1$ and $H_2$ are disjoint from the vanishing cycles, reflecting that the resulting divisors are disjoint from the $A_1$ and $A_2$ singularities. For fibers of type I, the specializations of $H_1$ and $H_2$ coincide and yield smooth Weil divisors containing the $A_1$ singularity; their order in the local class group is two. For fibers of type III, they contain the $A_2$ singularity and have order three in the local class group. In each case the resulting curves are parametrized by $\mathbb{P}^2$.

We relativize this as follows: Let $\mathcal{H} \to P$ denote the relative Hilbert scheme parametrizing connected genus zero curves of anticanonical degree three. The analysis above shows that its Stein factorization takes the form

$$\mathcal{H} \to Y \to P,$$

where the first morphism is an *étale* $\mathbb{P}^2$-bundle. Thus we obtain the desired class $\eta \in \mathrm{Br}(Y)[3]$.

Next we turn to the triple cover. Let $\mathcal{H}' \to P$ denote the relative Hilbert scheme parametrizing connected genus zero curves of anticanonical degree two. These are fibers of the conic bundle fibrations

$$\gamma_1, \gamma_2, \gamma_3 \colon \overline{S} \to \mathbb{P}^1_L.$$

A case-by-case analysis shows the conics in type I-IV fibers are still parametrized by $\mathbb{P}^1$'s. Repeating the analysis above, the Stein factorization

$$\mathcal{H}' \to Z \to P$$

consists of an *étale* $\mathbb{P}^1$-bundle followed by a triple cover ramified along $B_{II}$. The fact that $B_{II}$ has cusps at the points of threefold ramification shows that $Z$ is nonsingular. Indeed, étale locally such covers take the form

$$\mathbb{A}^2 = \{(r_1, r_2, r_3) : r_1 + r_2 + r_3 = 0\} \to \mathbb{A}^2/\{(12)\} \to \mathbb{A}^2/\mathfrak{S}_3$$

where $Z \to P$ corresponds to the second morphism, branched over the discriminant divisor which is cuspidal at the origin. The étale $\mathbb{P}^1$ bundle yields $\zeta \in \mathrm{Br}(Z)[2]$, the desired Brauer class. $\square$

**Proposition 14.** *Let $\pi \colon \mathcal{S} \to P$ be a good del Pezzo fibration and fix*

$$\begin{aligned} b_{IV} &= \chi(B_{IV}) = |B_{IV}| \\ b_{III} &= \chi(B_{III}) = |B_{III}| = |B_I \cap B_{II}| \\ b_{II} &= \chi(B_{II} \setminus (B_{III} \cup B_{IV})) \\ b_I &= \chi(B_I \setminus B_{III}), \end{aligned}$$

*where $\chi$ is the topological Euler characteristic. Then we have*

$$\chi(\mathcal{S}) = 6\chi(P) - b_I - b_{II} - 2b_{III} - 2b_{IV}.$$

*Proof.* This follows from the stratification of the fibration by singularity type. A smooth sextic del Pezzo surface has $\chi = 6$. For types I and II we have $\chi = 5$; for types III and IV we have $\chi = 4$. □

We specialize Proposition 14 to the case where the base is $\mathbb{P}^2$, using Bezout's Theorem and the genus formula:

**Corollary 15.** *Let $\pi\colon \mathcal{S} \to \mathbb{P}^2$ be a good del Pezzo fibration; write $d_I = \deg(B_I)$ and $d_{II} = \deg(B_{II})$. Then we have*

$$\chi(\mathcal{S}) = 14 + (d_I - 1)(d_I - 2) + (d_{II} - 1)(d_{II} - 2) - 3b_{IV}.$$

We specialize further to the case where $\pi\colon X' \to \mathbb{P}^2$ is as in Theorem 6. Recall the notation of Proposition 13: the double cover $Y \to \mathbb{P}^2$ and $\eta \in \mathrm{Br}(Y)[3]$, as well as the triple cover $Z \to \mathbb{P}^2$ and $\zeta \in \mathrm{Br}(Z)[2]$.

Let $T \to E$ be the elliptic ruled surface; assume that $T = \mathbb{P}(\mathcal{V}^\vee)$, where $\mathcal{V} = \mathcal{L}_1 \oplus \mathcal{L}_2$ is a direct sum of nonisomorphic degree-three line bundles on $E$. Let $r\colon X' \to X$ be the blow-up along $T \subset X$, and let $E_1, E_2 \subset T$ be as in Theorem 2. We have seen that $R_i := r^{-1}(E_i)$ is a trisection of $\pi$ for $i = 1, 2$; thus in particular $\zeta \in \mathrm{Br}(Z)$ is always zero.

**Proposition 16.** *The map $\pi\colon R_i \to \mathbb{P}^2$ is branched over the dual curve to the image of*

$$\phi : E \hookrightarrow \mathbb{P}^2, \quad \phi^*\mathcal{O}_{\mathbb{P}^2}(1) = \mathcal{L}_1^2 \otimes \mathcal{L}_2^{-1},$$

*a sextic curve $B_{II}$ with nine cusps.*

*Proof.* Let $L \subset \Gamma(\mathcal{I}_T(2))$ be a pencil of quadrics. We saw in the proof of Theorem 6 and in Remark 7 that for generic $L$, we have $\cap_{\ell\in L} Q_\ell \cap X = T \cup S_L$, where $S_L$ is a del Pezzo surface of degree six and $D_L := T \cap S_L$ is a smooth curve of degree 12 and genus 7.

As $L$ varies in $\mathbb{P}(\Gamma(\mathcal{I}_T(2))^\vee)$, the curve $D_L$ moves in a linear system on $T$. In Remark 7 we saw that $D_L = H + E_1 + E_2$, which meets $E_i$ in degree three, precisely

$$D_L|E_1 = 2[\mathcal{L}_1] - [\mathcal{L}_2].$$

The branch locus of $\pi\colon R_i \to \mathbb{P}^2$ corresponds to elements of this linear series tangent to $\phi(E)$, which is just the dual curve. □

**Proposition 17.** *If the fibration $\pi\colon X' \to \mathbb{P}^2$ is good in the sense of Definition 11, then the curve $B_I \subset \mathbb{P}^2$ is a sextic, and $Y$ is a K3 surface of degree 2.*

*Proof.* Since $X' = \mathrm{Bl}_T(X)$ and $\chi(T) = 0$, we have

$$\chi(X') = \chi(X) = 1 + 1 + 23 + 1 + 1 = 27.$$

In the previous proposition we have seen that $d_{II} = 6$ and $b_{IV} = 9$. Thus from Corollary 15 we find that $d_I = 6$. For the last statement, recall that $Y$ is the double cover of $\mathbb{P}^2$ branched over $B_I$. □

*Remark* 18. Let $f$ denote the degree-two polarization on $Y$ associated with the double cover. We expect that $(Y, f, \eta)$ coincides (up to sign) with the triple $(Y', f', \eta')$ obtained via Hodge theory in Proposition 10.

In the next section we will see that there are cubics $X \in \mathcal{C}_{18}$ for which $\pi\colon X' \to \mathbb{P}^2$ is good.

## 6. An explicit example

The computations below were verified symbolically with MAGMA [BCP97] and Macaulay 2 [GS].

Let $\mathbb{P}^5 = \mathrm{Proj}(\mathbb{F}_5[x_0, \ldots, x_5])$. Define quadrics

$$\begin{aligned}
Q_1 &= 3x_0x_3 + 2x_0x_5 + 4x_1x_3 + 2x_1x_4 + x_2x_3 + x_2x_4 + 2x_2x_5;\\
Q_2 &= x_0x_3 + 2x_0x_5 + x_1x_3 + 3x_1x_5 + 2x_2x_4 + 3x_2x_5;\\
Q_3 &= 2x_0x_4 + x_0x_5 + x_1x_3 + 2x_1x_5 + 4x_2x_3 + 3x_2x_5.
\end{aligned}$$

The net of quadrics $xQ_1 + yQ_2 + zQ_3$ contains the disjoint planes

$$\{x_0 = x_1 = x_2 = 0\} \qquad \text{and} \qquad \{x_3 = x_4 = x_5 = 0\}.$$

The sextic elliptic ruled surface $T$, obtained by saturating the ideal of the net of quadrics by the defining ideals of the planes, is given by

$$\begin{aligned}
x_0x_4 + 4x_1x_4 + 3x_1x_5 + 4x_2x_3 &= 0,\\
x_0x_4 + x_0x_5 + x_1x_3 + x_1x_4 + 4x_1x_5 + 3x_2x_5 &= 0,\\
x_0x_3 + 4x_0x_4 + x_0x_5 + 4x_1x_4 + 4x_1x_5 + 2x_2x_4 &= 0,\\
x_3^3 + 2x_3x_4^2 + x_3x_4x_5 + 4x_3x_5^2 + 4x_4^3 + 4x_5^3 &= 0,\\
x_0^3 + 4x_0^2x_1 + x_0^2x_2 + 2x_0x_1^2 + 2x_0x_1x_2 + 4x_0x_2^2 + x_1^3 + 3x_1^2x_2 + x_2^3 &= 0.
\end{aligned}$$

Note that $\mathcal{I}_T(2) = \langle Q_1, Q_2, Q_3\rangle$.

This surface is contained in the cubic fourfold $X = \{f = 0\}$, where

$$
\begin{aligned}
f := {} & x_0^3 + 4x_0^2x_1 + x_0^2x_2 + x_0^2x_3 + 3x_0^2x_4 + 3x_0^2x_5 + 2x_0x_1^2 + 2x_0x_1x_2 \\
& + 4x_0x_1x_3 + 3x_0x_1x_4 + 4x_0x_1x_5 + 4x_0x_2^2 + x_0x_2x_3 + 3x_0x_2x_4 \\
& + 2x_0x_2x_5 + 3x_0x_3^2 + 4x_0x_3x_5 + 4x_0x_4^2 + 2x_0x_4x_5 + x_0x_5^2 + x_1^3 \\
& + 3x_1^2x_2 + 4x_1^2x_3 + x_1x_2x_3 + 3x_1x_2x_4 + 4x_1x_2x_5 + 3x_1x_3^2 + x_1x_3x_4 \\
& + 2x_1x_4^2 + x_1x_4x_5 + 2x_1x_5^2 + x_2^3 + 4x_2^2x_3 + x_2^2x_4 + 4x_2^2x_5 \\
& + 4x_2x_3^2 + 3x_2x_3x_5 + 3x_2x_4^2 + 2x_2x_4x_5 + 4x_2x_5^2 + 4x_3^3 + 3x_3x_4^2 \\
& + 4x_3x_4x_5 + x_3x_5^2 + x_4^3 + x_5^3
\end{aligned}
$$

A direct computation of the partial derivatives of $f$ show that $X$ is smooth. The first order deformations of $T$ as a subscheme of $X$ are given by

$$\Gamma(T, \mathcal{N}_{T/X}) = \mathrm{Hom}(\mathcal{I}_T, \mathcal{O}_T);$$

a direct computation (e.g., in Macaulay 2) gives that this is one-dimensional, as was required at the end of the proof of Theorem 2. (To pass from $\mathbb{F}_5$ to $\mathbb{C}$ we use the semi-continuity theorem and flat base change.)

The discriminant locus of the map $\pi\colon X' := \mathrm{Bl}_T(X) \to \mathbb{P}^2 = \mathbb{P}(\mathcal{I}_T(2)^\vee)$ is a reducible curve of degree 12, with two irreducible components:

$$
\begin{aligned}
B_I : {} & x^6 + 2x^4y^2 + x^3y^3 + 4x^3y^2z + 2x^3z^3 + 4x^2y^4 + 4x^2y^2z^2 \\
& + 4x^2yz^3 + 4xy^5 + xy^4z + xy^2z^3 + xyz^4 + 2xz^5 + 4y^6 \\
& + 3y^5z + y^3z^3 + y^2z^4 + 4yz^5 = 0, \\
B_{II} : {} & x^6 + 2x^5y + 2x^4y^2 + x^4yz + 4x^3y^3 + 3x^3y^2z + 4x^3yz^2 + x^3z^3 \\
& + 3x^2y^4 + 4x^2y^2z^2 + x^2yz^3 + 3x^2z^4 + 3xy^5 + 2xy^4z \\
& + 3xy^3z^2 + 3xyz^4 + xz^5 + y^5z + 4y^4z^2 + 3y^3z^3 \\
& + 2y^2z^4 + 4yz^5 = 0
\end{aligned}
$$

The curve $B_I$ is smooth, and $B_{II}$ has 9 cusps. Their intersection is a reduced 0-dimensional scheme of degree 36 and is thus transverse. Thus $\pi$ is good in the sense of Definition 11.

## References


[AT14] Nicolas Addington and Richard Thomas. Hodge theory and derived categories of cubic fourfolds. *Duke Math. J.*, 163(10):1885–1927, 2014.

[BCP97] Wieb Bosma, John Cannon, and Catherine Playoust. The Magma algebra system. I. The user language. *J. Symbolic Comput.*, 24(3-4):235–265, 1997. Computational algebra and number theory (London, 1993).

[Blu10] Mark Blunk. Del Pezzo surfaces of degree 6 over an arbitrary field. *J. Algebra*, 323(1):42–58, 2010.

[CT88] D. F. Coray and M. A. Tsfasman. Arithmetic on singular Del Pezzo surfaces. *Proc. London Math. Soc. (3)*, 57(1):25–87, 1988.

[GS] Daniel R. Grayson and Michael E. Stillman. Macaulay2, a software system for research in algebraic geometry. Available at `http://www.math.uiuc.edu/Macaulay2/`.

[Has99] Brendan Hassett. Some rational cubic fourfolds. *J. Algebraic Geom.*, 8(1):103–114, 1999.

[Has00] Brendan Hassett. Special cubic fourfolds. *Compositio Math.*, 120(1):1–23, 2000.

[Has16] Brendan Hassett. Cubic fourfolds, K3 surfaces, and rationality questions, 2016. `arXiv:1601.05501`.

[Hom80] Yuko Homma. Projective normality and the defining equations of ample invertible sheaves on elliptic ruled surfaces with $e \geq 0$. *Natur. Sci. Rep. Ochanomizu Univ.*, 31(2):61–73, 1980.

[Kap09] Grzegorz Kapustka. Primitive contractions of Calabi-Yau threefolds. II. *J. Lond. Math. Soc. (2)*, 79(1):259–271, 2009.

[Man66] Ju. I. Manin. Rational surfaces over perfect fields. *Inst. Hautes Études Sci. Publ. Math.*, (30):55–113, 1966. English translation: American Mathematical Society Translations, Ser. 2, Vol. 84 (1969): Twelve papers on algebra, algebraic geometry and topology.

[MSTVA14] Kelly McKinnie, Justin Sawon, Sho Tanimoto, and Anthony Várilly-Alvarado. Braur groups on K3 surfaces and arithmetic applications, 2014. `arXiv:1404.5460v2`.

[Nik79] V. V. Nikulin. Integer symmetric bilinear forms and some of their geometric applications. *Izv. Akad. Nauk SSSR Ser. Mat.*, 43(1):111–177, 238, 1979.

[Ran95] Ziv Ran. Hodge theory and deformations of maps. *Compositio Math.*, 97(3):309–328, 1995.

[vG05] Bert van Geemen. Some remarks on Brauer groups of $K3$ surfaces. *Adv. Math.*, 197(1):222–247, 2005.

[Voi86] Claire Voisin. Théorème de Torelli pour les cubiques de $\mathbf{P}^5$. *Invent. Math.*, 86(3):577–601, 1986.

[Voi07] Claire Voisin. *Hodge theory and complex algebraic geometry. II*, volume 77 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, english edition, 2007. Translated from the French by Leila Schneps.

[Voi13] Claire Voisin. Abel-Jacobi map, integral Hodge classes and decomposition of the diagonal. *J. Algebraic Geom.*, 22(1):141–174, 2013.

Department of Mathematics, University of Oregon, Eugene, OR 97403-1222, USA
*E-mail address*: adding@uoregon.edu

Department of Mathematics, Brown University, Box 1917, 151 Thayer Street, Providence, RI 02912, USA
*E-mail address*: bhassett@math.brown.edu

Courant Institute, New York University, New York, NY 10012, USA

Simons Foundation, 160 Fifth Avenue, New York, NY 10010, USA
*E-mail address*: tschinkel@cims.nyu.edu

Department of Mathematics, Rice University MS 136, Houston, Texas 77251-1892, USA
*E-mail address*: av15@rice.edu