Optimal Leapfrogging, A Complete Guide

Sam K. Miller

IN PROGRESS FOR SUBMISSION (revision: 12/28/2021)

1 Introduction

Suppose we have some checkers placed in the lower left corner of a Go board, and we wish to move them to the upper right corner in as few moves as possible. There are no opponent pieces present, and the pieces move as they would in the game of Chinese Checkers, where for one move, a piece may either shift one unit in any direction, or repeatedly leapfrog over other pieces.

Let us consider the Go board as a subset of the non-negative integer lattice $\mathbb{Z}^2$. As an example, suppose we have four pieces placed at the coordinates $(0,0)$, $(1,0)$, $(0,1)$, and $(1,1)$, and wish to move them to the squares $(9,9)$, $(10,9)$, $(9,10)$, and $(10,10)$. For the pieces to complete the task in as few moves as possible, the pieces must first be moved into a configuration such that they may jump over each other in an optimal way.

We may intuitively attempt lining the checkers up diagonally in what we will call a *snake configuration*, that is, moving the pieces to coordinates $(0,0)$, $(1,1)$, $(2,2)$, and $(3,3)$. By repeating the three-move process of shifting the backmost piece to the right $[(0,0) \to (1,0)]$, leapfrogging that piece to the front $[(1,0) \to (3,4)]$, then shifting it right again $[(3,4) \to (4,4)]$, we can reach our destination in $4+4+(3\times 7) = 29$ moves.

However a faster method exists. We first move the pieces into what we call a *serpent configuration*, with the pieces on coordinates $(0,0)$, $(1,0)$, $(1,1)$, and $(2,1)$. Then we repeat the two-move process of leapfrogging the backmost piece to the front $[(0,0) \to (2,2)]$ then leapfrogging the new backmost piece to the front again $[(1,0) \to (3,1)]$, we may reach our destination in $1+1+(2\times 8) = 18$ moves. This is indeed the fastest way of moving the checkers from the bottom left to the upper right.

We define a measure of the movement efficiency of a placement of pieces, and it may be shown that under this measure, the serpent is the most efficient configuration possible. In fact, it was shown by Auslander, Benjamin, and Wilkerson that the serpent configuration is maximally efficient, with only three configurations attaining this efficiency. For any non-maximal configurations, their efficiency was conjectured to have a strict upper bound, which we prove. [1]

2 Abstracting the game

Suppose we have p indistinguishable pieces and wish to move them in the positive direction over the integer lattice $\mathbb{Z}^n$. If a piece is located at coordinate $l \in \mathbb{Z}^n$, and some other coordinate $l + e_i$ is not occupied by a piece (for unit vector e_i), then the piece may *shift* there. Alternatively if $l + e_i$ is occupied but $l + 2e_i$ is not, the piece may *hop* over the occupant of $l + e_i$ to land at $l + 2e_i$, where it may remain or continue hopping over other adjacent pieces. One legal *move* consists of either a shift or a *jump*, a sequence of one or more hops by a single piece.

Define a **placement** of size p as a finite subset of $\mathbb{Z}^n$, denoted by $X = \{\vec{x}_1, \ldots, \vec{x}_p\}$. Define the *centroid* of placement X to be

$$c(X) = \frac{1}{p}\sum_{u=1}^{p} \vec{x}_u$$

For placements X, Y, define their *displacement* as

$$d(X,Y) = \sum_{i=1}^{n} |c_i(X) - c_i(Y)|$$

For $m \geq 1$, an *m-move trajectory* $X_0, X_1, ..., X_m$ is a sequence of placements where X_{u+1} is reachable from X_u in a single legal move. The *speed* of an m-move trajectory from X_0 to X_m is

$$s = \frac{d(X_0, X_m)}{m}.$$

We say that placements X, Y are *translates* if there exists $\vec{a} \in \mathbb{Z}^n$ such that $X + \vec{a} = Y$. X and Y are represented by the same *configuration* of pieces, and we define the *speed* of a configuration C to be the maximum speed attained by any trajectory between two translates represented by C.

Auslander, Benjamin, and Wilkerson proved in 1993 the following: the maximum speed of any configuration C is 1, and that only three configurations (called *speed-of-light* configurations) attain this "speed of light" for $d \geq 1$. [1] These configurations are:

- The **atom** $\{x\}$ (if $p = 1$)
- The **frog** $\{x,\ x + e_i\}$, $1 \leq i \leq n$ (if $p = 2$)
- The **serpent** $\{x,\ x + e_i,\ x + e_i + e_j,\ x + 2e_i + e_j\}$ $1 \leq i \neq j \leq n$ (if $p = 4$ and $d > 1$)

It was conjectured that the maximum attainable speed for any configuration on $p \neq 1, 2, 4$ is $2/3$, which we may observe is attained by the snake configuration with any number of pieces. [1] We will show that $2/3$ is indeed the maximum possible speed attainable for any non-speed-of-light configuration in any dimension $n \geq 2$.

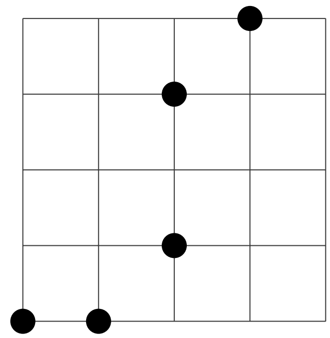
3 Definitions and Properties

Let $m \in \mathbb{Z}$ and placement $X \in \mathbb{Z}^n$. Then *border* l_m is defined by:

$$l_m = \{x \in X : ||x|| = m\}$$

For a placement X, we may define the *tail* (respectively, *head*) of X, by $t(X) = \min_u |l_u| > 0$ (respectively, $h(X) = \max_u |l_u| > 0$). Define the *width* of a placement X $w(X) = h(X) - t(X) + 1$. Define the *back border* (respectively, *front border* of X as $T(X) := l_{t(X)}$ (respectively, $H(X) := l_{h(X)}$).

We now define an underlying configuration which reoccurs in optimal play. A *ladder* of length $k > 0$ is subset of a placement X: $L = \{p_0, p_1, ..., p_k\} \subseteq X$ such that p_0 is able to hop over $p_1, \ldots, p_k$ successively. If $\{p_0\} = T(X)$ and $p_k \in H(X)$, then we say L is a *true ladder* of X. We call the move consisting of p_0 jumping over the rest of the ladder pieces an *climb*, call p_0 the *base* of the ladder, and the other pieces the *rungs*.



An example of a ladder.

Proposition 3.0.1. If a configuration X contains a true ladder L, X has even width.

Proof. Observe that when a piece p hops over another piece p', p either increases or decreases what border it belongs to by 2. Therefore since l_0 jumps from $B_{t(X)}$ to $B_{h(X)+1}$, this implies $h(X) + 1 - t(X)$ is even. □

Proposition 3.0.2. A placement X with $n > 1$ pieces can perform a move that simultaneously increases $t(X)$ and $h(X)$ if and only if it has a true ladder.

Proof. ($\Longleftarrow$)Performing a ladder climb increases both $t(X)$ and $h(X)$, as l_0 moves from $T(X)$, leaving that border empty, and jumps in front of $l_k \in H(X)$, thus advancing the front border.

($\Longrightarrow$) If a move on X exists that advances the front and back borders forward, since only one piece can change positions, the back border must only have one piece, and it must be the piece which moves. Call this piece p. Since $n > 1$ $w(X) > 1$, so p must jump from $T(X)$ to in front of $H(X)$. Denote the sequence of pieces hopped over by p by p_1, p_2, ... p_k. Since $p_k \in H(X)$, $\{p_0, p_1, ..., p_k\}$. □

We may classify possible moves into seven categories.

- *Ascent*: A move that increases $h(X)$ and $t(X)$. If $p > 1$, this is necessarily a ladder climb.
- *Front Push*: A move that increases $h(X)$ but not $t(X)$.
- *Back Push*: A move that increases $t(X)$ but not $h(X)$.
- *Dead Move*: A move that changes neither the tail nor head of X
- *Front Retreat*: A move that decreases the head of X
- *Back Retreat*: A move that decreases the tail of X

- *Reverse Ladder Climb*: A move that decreases both the head and the tail of X.

An ascent is necessarily a ladder climb for nontrivial placements. For a legal m-move trajectory $M = \{X_0, X_1, ..., X_m\}$, where X_0 is a translate of X_M, define the *moveset* of M as a collection of moves $m(M) = \{x_0 \to x'_0, \, ... \, , x_{m-1} \to x'_{m-1}\}$, where x_i is the location of the piece that moves in X_i, and x'_i is the location of the moved piece in X_{i+1}.

For a move trajectory M, let $A(M)$ represent the number of ascents in $m(M)$, $FP(M)$ represent the number of front pushes, $BP(M)$ the number of back pushes, $DM(M)$ the number of dead moves, $FR(M)$ the number of front retreats, $BR(M)$ the number of back retreats, and $RA(M)$ the number of reverse ascents.

Now, define the *efficiency* $\omega(M)$ of a trajectory M or it's corresponding moveset $m(M)$ as follows:

$$\omega(M) := A(M) - (1/2) \cdot (FP(M) + BP(M)) - 2 \cdot DM(M) - (7/2) \cdot (FR(M) + BR(M)) - 5 \cdot RA(M)$$

Additionally, call the coefficient corresponding to each move type the *move weight*. If a sequence of moves are performed, then the weight of the sequence is the sum of all the move weights.

Note that if we partition a trajectory $M = M_1 \oplus M_2 \oplus ... \oplus M_k$, then $\omega(M) = \omega(M_1) + ... + \omega(M_k)$.

Lemma 3.1. A m-move trajectory M of a configuration C has speed greater than $2/3$ if and only if $\omega(M) > 0$

Proof. Since the move types are mutually exclusive, $A(M) + FP(M) + BP(M) + DM(M) + FR(M) + BR(M) + RA(M) = m$. Additionally, the displacement of M can be characterized by $A(M) - RA(M) + (1/2) \times (FP(M) - FR(M) + BP(M) - BR(M))$. Therefore the speed of M is

$$\frac{A(M) - RA(M) + FP(M)/2 - FR(M)/2 + BP(M)/2 - BR(M)/2}{A(M) + FP(M) + BP(M) + DM(M) + FR(M) + BR(M) + RA(M)}.$$

Therefore,

$$2/3 < \frac{A(M) - RA(M) + FP(M)/2 - FR(M)/2 + BP(M)/2 - BR(M)/2}{A(M) + FP(M) + BP(M) + DM(M) + FR(M) + BR(M) + RA(M)} \iff 0 < \omega(M)$$

□

Theorem 3.2. For any trajectory M with no two ascents occurring in a row and not both beginning and ending with an ascent, $\omega(M) \leq 0$.

Proof. After an ascent, if a front push, back push, front retreat, or back retreat occurs, this changes the width of the placement, and since prior to the first move the placement had even parity, for another ascent to occur, another one of the listed moves must occur.

Without loss of generality let us assume M begins with an ascent, therefore it cannot end with one. Let us partition the moveset $m(M)$ into separate blocks $B_1, \ldots, B_k$ where each block begins with an ascent and contains no other ascents. Since no two ascents can occur in a row, each block must consist of at least two moves. Additionally since each new block must begin with an ascent, a block must end with a placement with even width.

B_i contains one ascent and at least one other move a. If a is any type of move besides a front or back push, $\omega(B_i)$ is negative. Otherwise, if a is a front push or back push the resulting placement has odd width and another front push, back push, front retreat, or back retreat must occur. This implies $\omega(B_i) \leq 0$, with equality holding only when B_i is of the form $\{A, FP/BP, FP/BP\}$. Since $B_i \leq 0$ for all $1 \leq i \leq k$, $\omega(M) \leq 0$, as desired.

□

We add the condition that M cannot both begin and end with an ascent due to the cyclic nature of translations - if such a translation begins and ends with an ascent, and $M = \{X_0, ..., X_m\}$, then all other configurations represented in the orbit have translations with consecutive ladder climbs. For simplicity, when we say "in a row" from now on, we will act as if a moveset for translates is cyclic and count the first move as adjacent to the last move.

4 $p = 1, 2, 4$

For a configuration which is not speed of light, observe that it may reach speed arbitrarily close to speed 1 by first shifting into a speed-of-light configuration, repeating its set of moves sufficiently long, then moving back to the original configuration. We will consider only trajectories which do not use this strategy. Rigorously, if a trajectory between translates does not perform any moves corresponding to any "speed-of-light" configuration's optimal moveset, call the corresponding configuration *non-speed-of-light*.

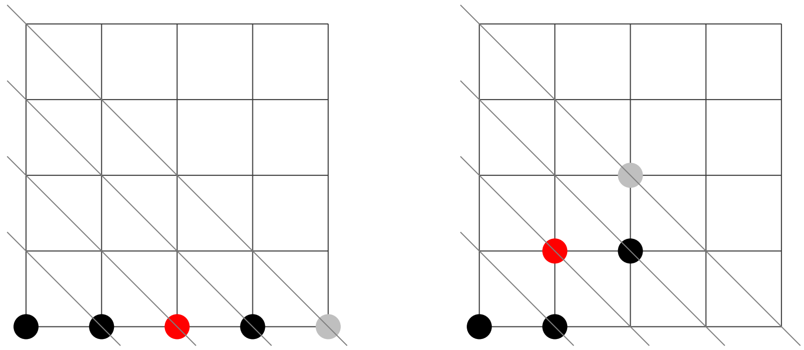
Theorem 4.1. A non-speed-of-light configuration C with 1, 2, or 4 pieces cannot have speed greater than 2/3.

Proof. There is only one configuration with 1 piece which is speed of light, so $p = 1$ is true.

For $p = 2$, any trajectory containing a jump (which must be an ascent) must contain the frog's optimal trajectory (a jump), therefore any non-speed-of-light configuration is limited to only shifts. Therefore, the optimal trajectory of C cannot contain any ascents, thus C cannot have speed greater than 2/3.

For $p = 4$, if C has optimal trajectory without two ascents in a row, C has speed at most 2/3. Suppose then that C has two ascents in a row, without loss of generality let us assume its trajectory begins with two ascents. C must have even width, and every border must contain a piece due to the parity of the ends of each successive ladder. Since C can perform two ascents in a row, it has width 4. $C = \{p_1, p_2, p_3, p_4\}$ with p_i on l_i. Say p_1 has jump $p_1 :\xrightarrow{p_2} a_1 \xrightarrow{p_4} a_2$ for open locations a_1, a_2. $d(p_1, p_2) = 1$, $d(a_1, p_2) = 1$, and $d(a_1, p_4) = 1$. Similarly, write the next move by p_2 as $p_2 :\xrightarrow{p_3} b_1 \xrightarrow{p_1 = a_2} b_2$. $d(p_2, p_3) = 1$ $d(p_3, b_1) = 1$, and $d(a_2, b_1) = 1$.

Suppose p_1, p_2, and p_4 are colinear (p_1 on x, p_2 on $x + u_i$, p_4 on $x + 3u_i$), $a_2 = x + 4u_i$ and $a_1 = x + 2u_i$. However for p_2 to jump over p_3 and p_1, it must start on a_1, a contradiction. Say p_1 starts on x, p_2 on $x + u_i$, then a_1 is $x + 2u_i$, p_4 starts on $x + 2u_i + u_j$, and a_2 is $x + 2u_i + 2u_j$. There are only two locations both adjacent to p_2 and 2 away from a_2, a_1 and $x + u_i + u_j$. However if p_3 is on $x + u_i + u_j$ and we perform the two ascents, we have performed the serpent configuration's trajectory, a contradiction. Thus it is impossible for a non-speed-of-light configuration C to have two ascents in a row, so C must have speed at most 2/3. □

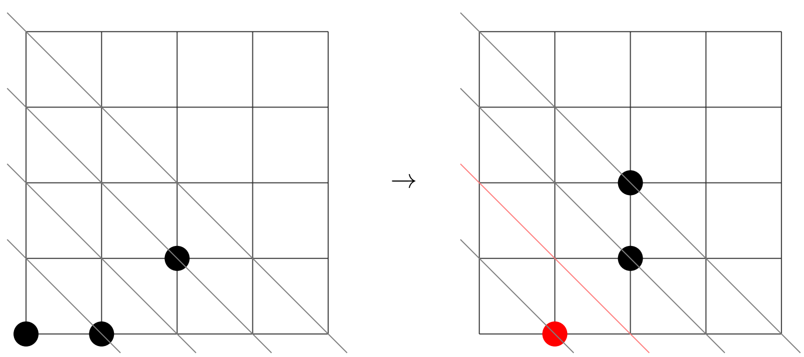


We cannot place p_3 without a contradiction.

5 $p = 3$

Theorem 5.1. No configuration of 3 pieces C exists with speed greater than 2/3.

Proof. To show this, we will demonstrate that no placement X exists for $p = 3$ such that two successive ascents are possible. If X has two or three pieces occupying the same border, then the back or front border has two pieces, rendering successive ascents impossible. Otherwise assume X has pieces occupying all different borders. The only possible way for X to be able to perform a ladder climb is if it has width 4, since a piece jumping over two other pieces can travel distance at most 4. Let us consider the four borders passing through X, without loss of generality say $l_1 - l_4$. l_1 and l_4 must contain one piece each, p_1 and p_3 respectively, implying the last piece, p_2 can either lay on l_2 or l_3. If p_2 lies on l_3, p_1 cannot jump. Otherwise, p_2 lays on l_2. If p_1 can perform an ascent, the pieces now lay on l_2, l_4, and l_5, which implies p_2 cannot jump.



After a jump, p is isolated and cannot jump.

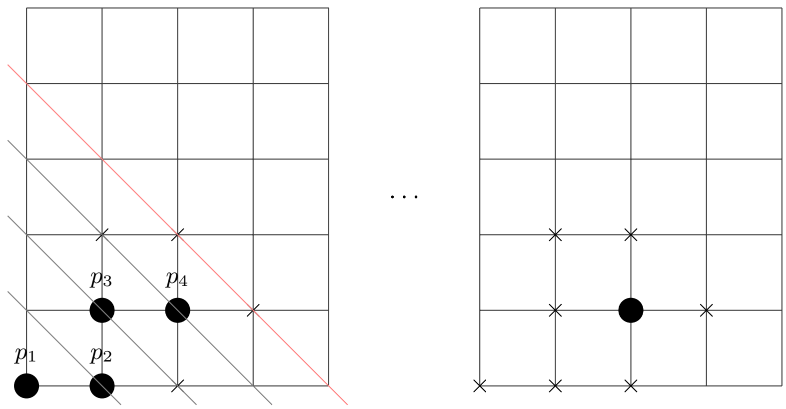
Therefore no placement X with $p = 3$ pieces exists such that two consecutive ascents can be performed, as desired. No such configuration of $\mathbb{Z}^n$ exists with speed greater than 2/3. $\square$

6 $p > 4$ for $\mathbb{Z}^2$

We provide a proof that no configuration of greater than 4 pieces has a speed greater than 2/3 in the 2-dimensional case. The details are less cumbersome than in the general case, but the basic idea of the proof in the general case remains the same. For this section, we will work only in $\mathbb{Z}^2$.

Lemma 6.1. For $p > 4$, there does not exist a configuration with a moveset containing more than 3 consecutive ascents.

Proof. Suppose for contradiction that there exists a configuration C with moveset containing more than 3 consecutive ascents. Then the width of the C prior to moving must be at least 6. Additionally, there can only be one piece on each of the four backmost borders. It follows that without loss of generality, we can assume the first four pieces which ascend are located at $(0, 0)$, $(1, 0)$, $(1, 1)$, & $(2, 1)$.



However, after the first 3 ascents, the piece at $(2, 1)$ is not adjacent to any pieces and therefore cannot ascend, a contradiction. $\square$

Define a piece's *measure* by taking its location modulo 2, $\overline{(x_1, x_2, ...x_n)}_2$. Note that when a piece jumps, its measure stays constant. This restricts the number of locations in $\mathbb{Z}^n$ a piece can jump to, given its starting position.

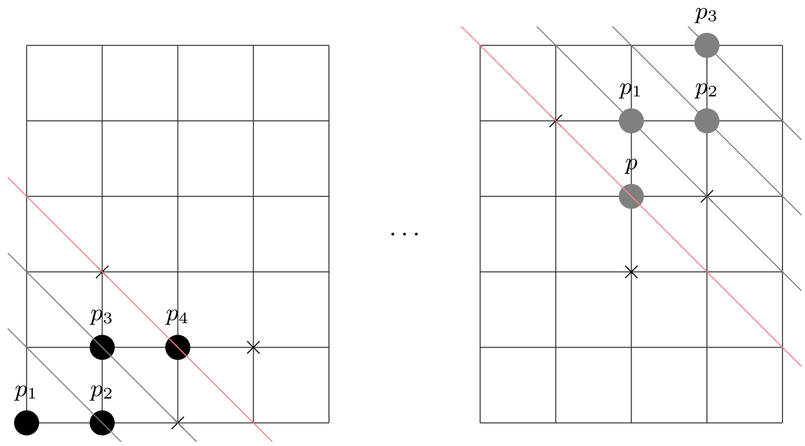
Define a *isolating partition* of M as follows. First, partition $m(M)$ sequentially into blocks $A_1, ...A_k$ such that each block begins with two or more consecutive ascents, but does not have consecutive ascents anywhere else and does not end with an ascent. Note that this partition of M is unique. Then, within each block A_i, sub-partition the blocks the in the same manner we did previously, that is creating a new sub-partition $A_{(i,j)}$ at each ladder climb, so $A_i = A_{(i,1)} + ... + A_{(i,j_i)}$, where j_i is the number of ladder climbs in partition A_i. Since A_i begins with consecutive ladder climbs, $|A_{(i,1)}| = 1$.

Let $L(A_i)$ be the number of ladder climbs A_i begins with. It is clear $\omega(A_i) < L(A_i)$. We wish to show $\omega(A_i) \leq 0$ for all i, since $\omega(M) = \omega(\sum A_i) = \sum \omega(A_i)$. Since A_i can only begin with 2 or 3 ladder climbs, we only have these two cases to consider.

Call a subpartition A_i for which $\omega(A_i) \leq 0$ *suboptimal*. If a consecutive sequence of moves $S \subset A_i$ satisfies $\omega(S) + L(A_i) \leq 0$, then necessarily, $\omega(A_i) \leq 0$, and thus A_i is suboptimal. Call S a *suboptimal sequence of moves*, and call any consecutive sequence of moves S that is not suboptimal *optimal*. Our strategy is to show that any subpartition A_i must contain a suboptimal sequence of moves, and we do so by trying to build an optimal sequence of moves, and show that it is eventually forced to become suboptimal.

Lemma 6.2. If a partition A_i begins with exactly three ascents, $\omega(A_i) \leq 0$.

Proof. Observe that if A_i begins with exactly three consecutive ladder climbs, the initial placement is forced to have a serpent configuration at the back, and after the three ascents, the resulting configuration is forced to have a serpent configuration in the front, as demonstrated below.



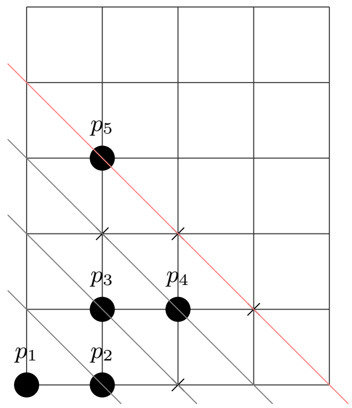
The borders before/after the ones indicated by red in the above diagrams are forced placements. A $\times$ indicates a location a piece cannot be.

By considering move weights, it suffices to demonstrate that a consecutive sequence of moves in A_i occurs with weight -3, that is, a suboptimal sequence of moves must occur. Note that if five non-ascents occur in a row, this condition is forced to be satisfied. Alternatively, if two dead moves occur or one dead move and one front/back push occurs, the condition is also satisfied. We presume that two dead moves cannot simultaneously occur. Call the border containing the backmost piece prior to moving l_1. We first consider two cases, if l_4 has exactly 1 piece or if l_4 has two or more pieces.

If l_4 has two or more pieces, first suppose the next ascent occurs from l_4. Observe that a ladder climb to the front from l_4 cannot be performed unless p moves, and if p ladder climbs to perform a front push, a ladder climb from l_4 still cannot be performed. Thus, for a piece from l_4 to perform a front push in the following move, p must perform a dead move. However, since another front or back push must occur again before an ascent is possible, this forces a sequence of moves of weight -3.

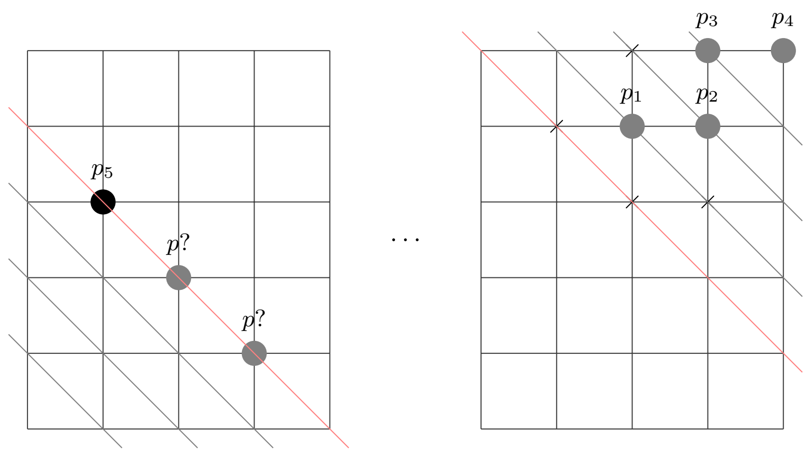
Otherwise, suppose the next ascent happens beyond l_4. In this case l_4 must be cleared, but since neither piece can front push, at least two moves must occur with at least one being a dead move, again forcing a sequence of moves with weight -3. Thus if l_4 has two pieces, a sequence of moves with weight -3 is forced to occur.

Now, suppose l_4 has one piece. The back of the initial configuration must necessarily must be in the following situation:



The left diagram is the initial configuration, and the right diagram is the configuration after the three consecutive ascents.

If the next ascent occurs on l_4, then again, p is forced to dead move, and since p_4 is isolated, another piece must move to be adjacent to it. If these are separate moves, then two dead moves have been performed, a sequence of moves with weight -4. Otherwise, suppose these are the same move, so p_4 ascends right after. Then there is a new serpent configuration at the front, and two pieces on l_5, p_5 and the piece p_4 hopped over (which is necessarily p).



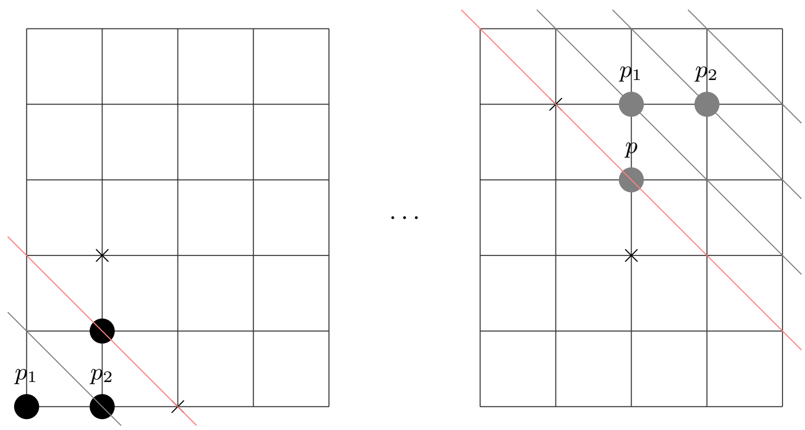
The configuration after p_4 ascends. There are necessarily two pieces on l_5 and a serpent configuration in the front preventing an ascent from occurring.

Since the previous two moves have weight -1, it suffices to show the next sequence of moves must have weight -2. Thus it suffices to assume no dead move can occur. Note that the parity is correct for an ascent to occur, but given the front of the configuration, no ascent can occur. Thus, at least two front pushes (or a dead move) must occur before one of the pieces on l_5 can climb to the front. However, this is three front pushes, implying a fourth must occur before an ascent can occur, so there must be a sequence of moves with weight -2 if the next ascent is to occur from l_5.

Otherwise, if the next ascent happens beyond l_5, it is clear that a sequence of moves must first occur with weight -3, and we conclude that in any case, $\omega(A_i) \leq 0$. $\square$

Lemma 6.3. If a partition A_i begins with exactly two ascents, $\omega(A_i) \leq 0$.

Proof. By considering move weights, it suffices to demonstrate that a sequence of moves in A_i after the two ascents occurs with weight -2, that is, a suboptimal sequence of moves must occur. In particular, it suffices to show either a dead move occurs or 3 front or back pushes occur between ascents. First observe that the starting and ending configurations after the two ascents must be as follows at the front and back borders respectively:

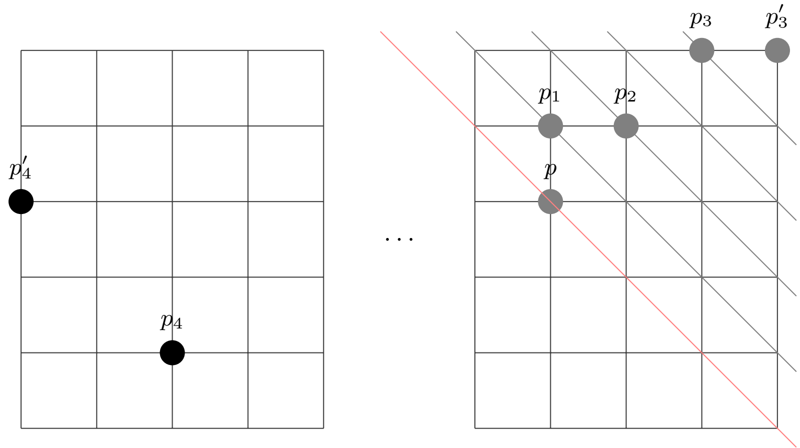


The borders before/after the ones indicated by red in the above diagrams are forced placements. A $\times$ indicates a location a piece cannot be.

Call the border containing the backmost piece prior to moving l_1. We consider two cases, if l_3 has exactly 1 piece, p_3, or if it has multiple pieces p_3 and p_3'. Then, we consider sub-cases and determine that any trajectory must eventually become suboptimal.

First, suppose l_3 has at least 2 pieces initially. Let us consider the next two moves following the ascents. If either is a dead move or worse, the trajectory is suboptimal, so suppose the next two moves are front or back pushes. Since l_3 has at least 2 pieces, the first move must be a front push, a ladder climb from l_3 to the front. Without loss of generality suppose p_3 makes the climb. If the second move is a back push, since there is a serpent configuration consisting of p, p_1, p_2, p_3, a ladder climb is not possible afterwards, and thus the trajectory is suboptimal.

Instead, suppose the second move is a front push. The only move which allows for a ladder climb (necessarily by p_3') is p_3 shifting forward. Then, a ladder climb by p_3' must finish by hopping to where p_3 was prior to shifting, then hopping over p_3. This implies p_3 and p_3' originally had the same measure, so they were distance at least 4 away. Therefore, there must be at least 2 pieces on l_4 which p_3 and p_3' hopped over, p_4 and p_4'.

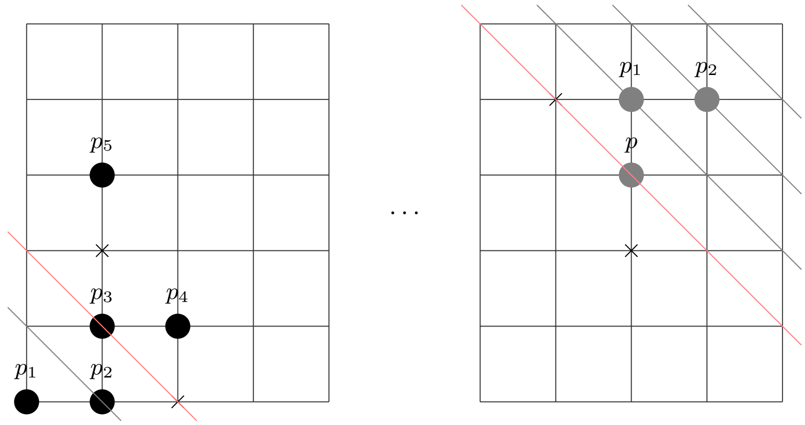


Right: the only possible optimal trajectory if 2 pieces started on l_3, five moves in.
Left: a corresponding possible placement on l_4 five moves in.

By a parity argument, p_4 and p_4' cannot hop over p, p_2, or p_3, as they are on an even-numbered border. If the next ladder climb occurs from beyond l_4, then p_4, p_4', and p_3' all must move first, which is necessarily a sequence of moves with weight at maximum -2. Otherwise, if the next ladder climb occurs on l_4, the pieces in front of p which can be used in the ladder are p_1 and p_3'. However, a piece needs to move to the border between the ones containing p_2 and p_3, and by parity, this piece cannot come from l_4. Hence, this must be a dead move, and we conclude that if l_3 began with 2 or more pieces, then $\omega(A_i) \leq 0$.

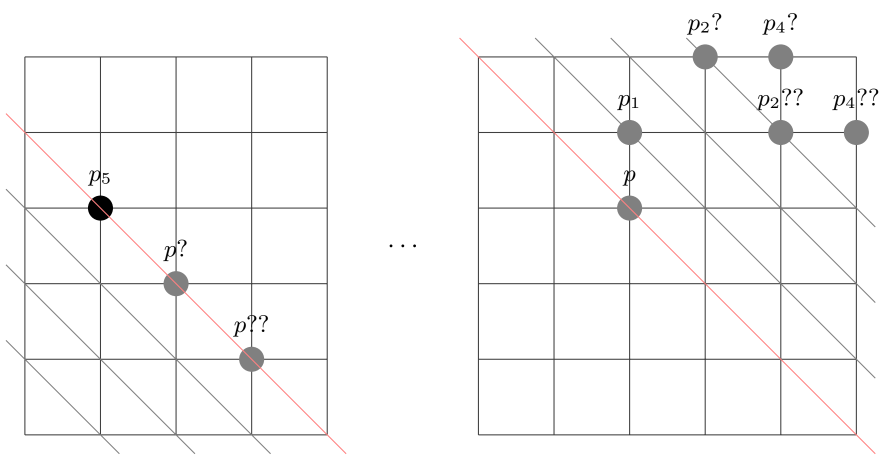
Now, suppose l_3 began with exactly 1 piece on it. We consider subcases based on where the next ascent in the the trajectory occurs from and the number of pieces on the following borders. First, suppose the next ascent comes from l_4 or beyond. If l_4 contains more than one piece, then in order for the trajectory to be optimal, the next two moves must be a back push from l_3, then a front push from l_4. However, this is impossible, since a back push from l_4 must be a ladder climb, but by a parity argument of where the frontmost border is, a ladder climb from l_4 cannot advance the front border.

Otherwise, suppose the next ascent comes from l_4 or beyond and that l_4 contains exactly one piece. Then for an optimal trajectory, the next two moves following the two ascents must be a back push from l_3 and a front push elsewhere (by assumption an ascent is not allowed). Assume the same piece does not perform the two moves, since otherwise, an ascent would be a faster trajectory, so that case would reduce to the previous lemma. Note that the front push cannot occur from l_4 by parity. Moreover, the back push from l_3 must hop over the sole piece on l_4, because otherwise, after the two moves there would be 2 pieces on l_4.



On the left, part of the necessary configuration if l_4 contains one piece, prior to the first two ascents. On the right, the front borders after the first two ascents.

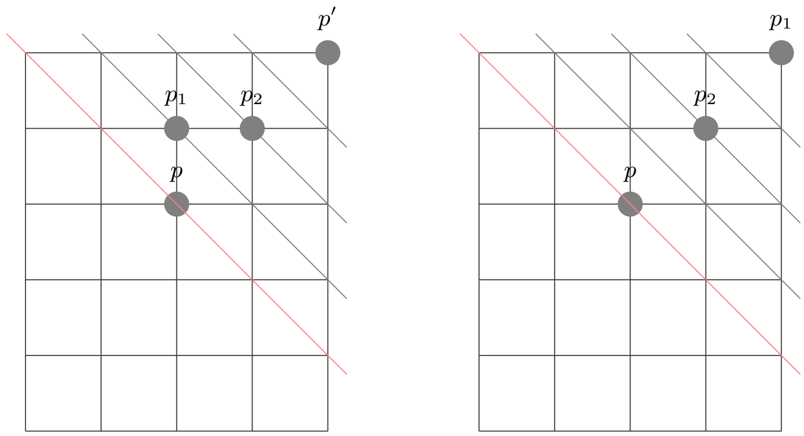
Finally, note that the front push must necessarily come from p_2 if an ascent is possible afterwards, since it is clear that a front push from p_1 prevent ascents, and front pushes from any other piece result in a serpent configuration at the front which prevents an ascent. Hence, after the front and back push, if an ascent is possible, there must be two pieces on l_5: the piece which p_2 hopped over in its ascent, and the piece l_4 is to hop over first in its ascent (this may be p_3). After p_4 ascends, we now are in the following configuration:



The configuration after p_4 ascends, question marks denote where piece locations are not unique.

Now, for the trajectory to stay optimal, the next ascent must occur from l_5, since otherwise it would take two moves to clear l_5 and at least one to alter the front borders so that an ascent is possible. Pieces from l_5 may only hop over p and p_4 in the front borders, so they must be in the next ladder. This implies that in the next two moves before the ascent, a piece must move to the border between p_1 and p_2. However, this must be a dead move or worse, since l_5 contains at least two pieces and thus cannot initiate a back push. Thus, all trajectories for which the next ascent comes from l_4 or beyond are suboptimal.

Finally, we consider the case where the next ascent comes from l_3, and l_3 contains only one piece. Since an ascent cannot immediately occur, the next two moves if they are to be optimal must be front pushes. It is quick to see that the only possibilities are either p_1 pushing twice or another unlabeled piece coming from an odd-numbered border pushing twice, as below:



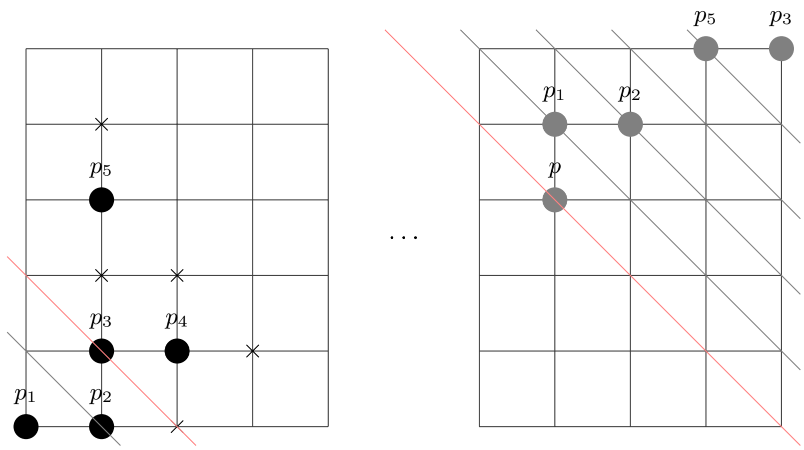
Note that the location of p' on the left or p_1 on the right are not uniquely determined.

If p_3 cannot ascend, the sequence of moves is suboptimal, so assume p_3 can ascend, the third ascent in the trajectory. Now, if the fourth ascent occurs from l_4 or any even numbered border, these pieces cannot hop over p, p_2, or $p'(p_1)$ in the left-hand (resp. right-hand) cases.

In the left-hand case, before the next ascent, a piece must move to the border between p_2 and p', p_2 must move for p_1 to be free to be hopped over, and p' must move so the frontmost piece can be hopped over. These must necessarily be performed as front or back pushes for optimality, however if these are performed as front or back pushes, this is at least 3 separate moves, making the sequence suboptimal.

In the right-hand case, pieces must move to the border between p and p_2 and the border between p_2 and p_1, and p_1 must move so the frontmost piece can be hopped over. Again, these must be performed as front or back pushes for optimality, but as front or back pushes, this is at least 3 separate moves, making the sequence suboptimal.

Finally, suppose the fourth ascent occurs from l_5 or beyond. There is necessarily at least one piece on l_4 which must front push, and p_3 must move to open the ladder it climbed. If there are two pieces on l_4, then there are 3 moves which must occur which is suboptimal, so suppose there is only one piece on l_4, p_4, which both p_3 and p_1 hopped over. By similar arguments to earlier, p_4 cannot hop forward, it must shift to l_5. Now, if there are 2 or more pieces on l_5, then suboptimality is forced, as one must move before the ascent, which totals 3 moves before the ascent. However, it is possible that l_5 contains only one piece, p_4, as p_5 as shown below could have been the piece to front push prior to the ascent of p_3.



On the left, the necessary setup if l_4 and l_5 only have one piece, and on the right, in the lone scenario where l_5 front pushed before p_3 ascended.

In this case, then p_4 can ascend and optimality is still preserved. However, there must be two different pieces on l_6, the piece p_5 hops over when it front pushes, and the piece p_3 hops over after hopping over p_4 when it ascends. By a similar argument as earlier, the next ascent occurring from l_6 forces suboptimality, and it is straightforward to see that at least 3 unique moves must occur for the next ascent to occur from l_7 or beyond. Thus, if after the initial 2 ascents, the next ascent occurs from l_3, and l_3 only contained one piece, suboptimality is forced. We have exhausted all cases, and conclude that $\omega(A_i) \leq 0$.

$\square$

Corollary 6.3.1. If C is a configuration with $p > 4$, C has speed less than or equal to $2/3$.

Proof. Consider any m-move trajectory M of C. Perform an isolating partition of $M = \sum_{i=1}^{n} A_i$. We compute $\omega(M) = \omega(\sum A_i) = \sum \omega(A_i) \leq 0$, as desired. $\square$

References

[1] Arthur T. Benjamin Joel Auslander and Daniel S. Wilkerson. Optimal leapfrogging. *Mathematics Magazine*, 66(1):14–19, 1993.