Optimal Leapfrogging - A Complete Guide

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1 Introduction

Suppose we have some checkers placed in the lower left corner of a Go board, and we wish to move them to the upper right corner in as few moves as possible. There are no opponent pieces in the way, and the pieces move as they would in the game of Chinese Checkers, where a piece may either move one unit in any direction, or repeatedly leapfrog over other pieces.

We may consider the go board as a subset of the non-negative integer lattice $\mathbb{Z}^2$. For example, we have four pieces placed at the coordinates $(0,0)$, $(1,0)$, $(0,1)$, and $(1,1)$, and wish to move them to the squares $(9,9)$, $(10,9)$, $(9,10)$, and $(10,10)$. For the pieces to complete the task efficiently, the pieces must first be moved into a configuration such that they may jump over each other in an optimal way.

We may intuitively attempt lining the checkers up diagonally in what we will call a *snake configuration*, that is, moving the pieces to coordinates $(0,0)$, $(1,1)$, $(2,2)$, and $(3,3)$. By repeating the three-move process of shifting the backmost piece to the right $[(0,0) \to (1,0)]$, leapfrogging that piece to the front $[(1,0) \to (3,4)]$, then shifting it right again $[(3,4) \to (4,4)]$, we can reach our destination in $4+4+(3\times 7) = 29$ moves.

However a faster method exists. We first move the pieces into what we call a *serpent configuration*, with the pieces on coordinates $(0,0)$, $(1,0)$, $(1,1)$, and $(2,1)$. Then we repeat the two-move process of leapfrogging the backmost piece to the front $[(0,0) \to (2,2)]$ then leapfrogging the new backmost piece to the front again $[(1,0) \to (3,1)]$, we may reach our destination in $1+1+(2\times 8) = 18$ moves. This is indeed the fastest way of moving the checkers from the bottom left to the upper right.

2 Generalizing The Problem

We generalize the problem by supposing we have p indistinguishable pieces and wish to move them over the integer lattice $\mathbb{Z}^n$ under the taxicab metric. We define the movements of the pieces as we would in Chinese Checkers - the pieces occupy distinct coordinates of $\mathbb{Z}^n$, and each move allows for one piece to change locations. If a piece is located at coordinate $l \in \mathbb{Z}^n$, and some other coordinate $l+e_i$ is not occupied by a piece (for unit vector e_i), then the piece may *shift* there. Alternatively if $l+e_i$ is occupied but $l+2e_i$ is not, the piece may *hop* over the occupant of $l+e_i$ to land at $l+2e_i$, where it may remain or continue hopping over a different adjacent piece. One *move* consists of either a shift or a *jump* (a sequence of one or more hops by a single piece).

We may define a *placement* of pieces as a size p subset of $\mathbb{Z}^n$, denoted by $X = \{\mathbf{x_1}, ..., \mathbf{x_p}\}$. Define the *centroid* of placement X to be

$$c(X) = \frac{1}{p}\sum_{u=1}^{p} \mathbf{x_u}$$

a vector in $\frac{1}{p}\mathbb{Z}^n$. For placements X, Y, define their displacement as

$$d(X,Y) = \sum_{i=1}^{n} c_i(X) - c_i(Y)$$

Displacement can be considered a loose distance measurement between placements, though it can be negative, or zero for distinct placements. For $m \geq 1$, an *m-move trajectory* $X_0, X_1, ..., X_m$ is a sequence of placements where X_{u+1} is reachable from X_u in a single move. The *speed* of an m-move trajectory from X_0 to X_m is $d(X_0, X_m)/m$. We say that placements X, Y are *translates* if there exists $\mathbf{a} \in \mathbb{Z}^n$ such that

$X + \mathbf{a} = Y$. We say that X and Y are represented by the same *configuration* of pieces, and can define the *speed* of a configuration C to be the maximum speed attained by any trajectory between two translates represented by C.

Auslander, Benjamin, and Wilkerson proved in 1993 the following: the maximum speed of any configuration C is 1, and that only three configurations (called $speed-of-light$ configurations) attain this "speed of light" for $d \geq 1$. These placements are:

- The **atom** $\{x\}$ (if $p = 1$)
- The **frog** $\{x,\, x + e_i\}$, $1 \leq i \leq n$ (if $p = 2$)
- The **serpent** $\{x,\, x + e_i,\, x + e_i + e_j,\, x + 2e_i + e_j\}$ $1 \leq i \neq j \leq n$ (if $p = 4$ and $d > 1$)

It was conjectured that the maximum attainable speed for any configuration on $p \neq 1, 2, 4$ is $2/3$, which we may observe is attained by the snake configuration with any number of pieces. We will show that $2/3$ is indeed the maximum possible speed attainable for $p \neq 1, 2, 4$ in any dimension $d \geq 2$.

3 Some Useful Definitions and Lemmas

We may partition $\mathbb{Z}^n$ into subsets of dimension $n - 1$ in the following way: let $m \in \mathbb{Z}$. Then *border M* is defined by:

$$B_m = \{x \in \mathbb{Z}^n : ||x|| = m\}$$

In other words, B_m is the set of lattice points with in $\mathbb{Z}^n$ with coordinates that sum to m. For a placement X, we may define the *tail* and *head* borders of X, $t(X)$ and $h(X)$, as follows:

$$t(X) = \min_u\, ||x_u||,\; h(X) = \max_u\, ||x_u||$$

Let $T(X)$ and $H(X)$, the *tailset* and *headset* of X, be the complete sets of pieces that lay on $t(X)$ and $h(X)$, respectively. Define the *width* of a placement X $w(X) = h(X) - t(X) + 1$ (we give the placement consisting of no pieces width 0). Define the *back border* of X as $B_{t(X)}$ and the *front border* $B_{h(X)}$.

We now define an underlying placement structure which will motivate our proof. A *ladder* of length $k > 0$ is subset of a placement X: $L = \{l_0, l_1, ..., l_k\} \subseteq X$ such that l_0 is able to jump over l_1, l_2, ... , l_k in that order, with no other hops. If $\{l_0\} = T(X)$ and $l_k \in H(X)$, then we say L is a *true ladder* of X. We call the move consisting of l_0 jumping over the rest of the ladder pieces a *ladder climb*, and call l_0 the *base* of the ladder.

Proposition 3.0.1. If a configuration X contains a true ladder L, X has even width.

Proof. Observe that when a piece p hops over another piece p', p either increases or decreases what border it belongs to by 2. Therefore since l_0 jumps from $B_{t(X)}$ to $B_{h(X)+1}$, this implies $h(X) + 1 - t(X)$ is even. □

Proposition 3.0.2. A placement X with $n > 1$ pieces can perform a move that simultaneously increases $t(X)$ and $h(X)$ if and only if it has a true ladder.

Proof. First we observe that performing a ladder climb increases both $t(X)$ and $h(X)$, as l_0 moves from $T(X)$, leaving that border empty, and jumps in front of $l_k \in H(X)$, thus advancing the front border.

Now note that if a move on placement X exists that advances the front and back borders forward, since only one piece can move, the back border must only have one piece, and that it must be the piece that changes places for the move. Call this piece p, and its starting coordinate l_0. If X has width 1, this implies there are multiple pieces on the back border so $w(X) > 0$. Thus the tailset and headset do not intersect, and thus the lone piece in the tailset must hop over some piece in the headset to advance the front border. Therefore p must jump from the back border to the border directly in front of the front border. Take the sequence of coordinates hopped over by p and call these l_1, l_2, ... l_k, where l_k is the coordinate of the piece in the headset hopped over by p. $\{l_0, l_1, ..., l_k\}$ is by definition a ladder, and since l_0 must be the coordinates of the lone piece on the back border and l_k must lie on the front border, it is a true ladder of X.

□

We may classify possible types of moves a placement can make into seven distinct categories.

- *Ladder Climb*: A move that simultaneously increase the tail and head of X
- *Front Push*: A move that increases the head but not the tail of X
- *Back Push*: A move that increases the tail but not the head of X
- *Dead Move*: A move that changes neither the tail nor head of X
- *Front Retreat*: A move that decreases the head of X
- *Back Retreat*: A move that decreases the tail of X
- *Reverse Ladder Climb*: A move that decreases both the head and the tail of X.

Suppose we have some legal m-move trajectory $M = \{X_0, X_1, ..., X_m\}$, where X_0 is a translate of X_M. We shall represent the *moveset* of M as a collection of moves $m(M) = \{x_0 \to x'_0, \dots, x_{m-1} \to x'_{m-1}\}$, where x_i is the coordinates of the piece that moves in X_i, and x'_i is the location of that same piece in X_{i+1}.

For a move trajectory M, let $LC(M)$ represent the number of ladder climbs in $m(M)$, $FP(M)$ represent the number of front pushes, $BP(M)$ the number of back pushes, $DM(M)$ the number of dead moves, $FR(M)$ the number of front retreats, $BR(M)$ the number of back retreats, and $RLC(M)$ the number of reverse ladder climbs.

Now, define the *efficiency* of a trajectory M or it's corresponding moveset $m(M)$ as follows:

$$\omega(M) = \omega(m(M)) =$$
$$LC(M) - (1/2) \times (FP(M) + BP(M)) - 2DM(M) - (7/2) \times (FR(M) + BR(M)) - 5 \times RLC(M)$$

Note that if we partition a trajectory $M = M_1 \oplus M_2 \oplus ... \oplus M_k$, $\omega(M) = \omega(M_1) + ... + \omega(M_k)$.

Lemma 3.1. A m-move trajectory M of a configuration C has speed greater than $2/3$ if and only if $\omega(M) > 0$

Proof. Note that any move is classified by exactly one of our distinct categories, therefore $LC(M)+FP(M)+BP(M)+DM(M)+FR(M)+BR(M)+RLC(M) = m$.

Additionally, note that the displacement of m can be characterized by $LC(M) - RLC(M) + (1/2) \times (FP(M) - FR(M) + BP(M) - BR(M))$. Since the start and end placements of M are translates, they must have the same width, and their centroids must have the same distance as the amount that the front border was pushed forward. This is equivalent to $LC(M) - RLC(M) + FP(M) - FR(M)$. However since M is a trajectory between translates, the front and back borders must have been pushed equal amounts, so by performing the substitution $FP(M) - FR(M) = BP(M) - BR(M)$, we achieve the desired result.

Therefore the speed of M is $\frac{LC(M)-RLC(M)+\frac{FP(M)-FR(M)+BP(M)-BR(M)}{2}}{LC(M)+FP(M)+BP(M)+DM(M)+FR(M)+BR(M)+RLC(M)}$.

M has speed $> 2/3$ if and only if:

$$2/3 < \frac{LC(M)-RLC(M)+\frac{FP(M)-FR(M)+BP(M)-BR(M)}{2}}{LC(M)+FP(M)+BP(M)+DM(M)+FR(M)+BR(M)+RLC(M)}$$
$$\iff$$
$$0 < LC(M) - \frac{FP(M)+BP(M)}{2} - 2 \times DM(M) - 7 \times \frac{FR(M)+BR(M)}{2} - 5 \times RLC(M)$$
$$\iff$$
$$0 < \omega(M)$$

□

Thus to check if some translate has speed greater than 2/3, we can instead count the number of each type of move. We will demonstrate for $p > 4$ or $p = 3$, $\omega(M) > 0$ is impossible to achieve.

Lemma 3.2. For any translate M with no two ladder climb moves occurring in a row and not both beginning and ending with a ladder climb, $\omega(M) \leq 0$.

Proof. First note that after a ladder climb, if a front push, back push, front retreat, or back retreat occurs, this changes the width of the placement, and since prior to the first move the placement had even parity, for another ladder climb to occur, another one of the listed moves must occur.

Without loss of generality let us assume M begins with a ladder climb, therefore it cannot end with one. Let us split up the moveset $m(M)$ into separate blocks B_1, ..., B_k (with $m(M) = B_1 \oplus B_2 \oplus ... \oplus B_k$), where each block begins with a ladder climb and contains no other ladder climbs. Since no two ladder climbs can occur in a row, each block must consist of at least two moves. Additionally since each new block must begin with a ladder climb, a block must end with a placement with even width.

We may observe that $\omega(M) = \omega(B_1) + ... + \omega(B_k)$. Now for any B_i, it contains one ladder climb and at least one other move, call it a. If a is any type of move besides a front push or back push, $\omega(B_i)$ is negative, since there are no other ladder climbs in it. Otherwise, if a is a front push or back push the resulting placement has even width and another front push, back push, front retreat, or back retreat must occur. This implies $\omega(B_i) \leq 0$, with equality being maintained if and only if B_i is of the form $\{LC, FP/BP, FP/BP\}$. If $B_i \leq 0$ for all $1 \leq i \leq k$, $\omega(M) \leq 0$, as desired.

□

We add the condition that M cannot both begin and end with a ladder climb due to the cyclic nature of translations - if such a translation begins and ends with a ladder climb, and $M = \{X_0, ..., X_m\}$, then all other configurations represented in the orbit have translations with consecutive ladder climbs. For simplicity, when we say "in a row" from now on, we will act as if a moveset is cyclic and count the first move as adjacent to the last move .

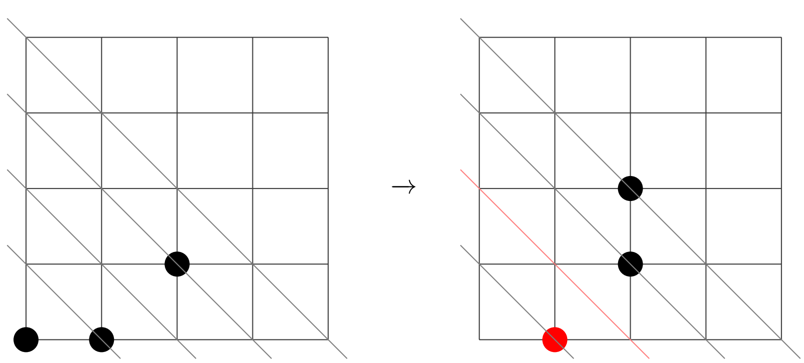
4 $p = 3$

Theorem 4.1. For any configuration C with 3 pieces, it is impossible to find a m-move trajectory M with $\omega(M) > 0$.

Proof. To show this, we will employ the previous lemma by demonstrating that no placement X exists for $p = 3$ such that two successive ladder climbs are possible, which implies that no translation can have two consecutive ladder climbs. Let us first assume that the first move is a ladder climb.

If X has two or three pieces occupying the same border, then the configuration can have pieces occupying at most two borders. Therefore either the tailset or headset has 2 members. If the tailset has 2 pieces, no ladder climb is possible, contradicting our assumption. If the headset has 2 pieces, then if a ladder climb is possible, the resulting headset has 1 piece and the former headset is the new tailset, so no ladder climb is possible after the first climb.

Otherwise assume X has pieces occupying all different borders. The only possible way for X to be able to perform a ladder climb is if it has width 4, since a piece jumping over two other pieces can travel distance at most 4. Let us consider the four borders passing through X: B_0, B_1, B_2, B_3. $B_{t(X)}$ and $B_{h(X)}$ must contain one piece each, implying the last piece, p can either lay on B_1 or B_2. If p lies on B_2, the backmost piece of X cannot jump as it is distance 2 away from all other pieces. Otherwise, p lays on $B_{t(X)+1}$. If the backmost piece can perform a ladder climb, the pieces now lay on B_1, B_3, and B_4, which implies p now is distance at least 2 away from all other pieces and cannot jump.



After a jump, p is isolated and cannot jump.

Therefore no placement X with $p = 3$ pieces exists such that two consecutive ladder climbs can be performed, as desired. No such configuration of $\mathbb{Z}^n$ exists with speed greater than $2/3$. $\square$

5 $p > 4$

Lemma 5.1. For $p > 4$, there does not exist a trajectory with moveset containing more than 4 consecutive ladder climbs.

Proof. The proof follows similarly to the proof of Theorem 3 in "Optimal Leapfrogging" $\square$

Suppose we have some configuration C with greater than 4 pieces with an m-move trajectory M. If $m(M)$ contains no two consecutive ladder climbs, we know M has speed less than or equal to $2/3$. Then let us suppose $m(M)$ at some point has two consecutive ladder climbs, and without loss of generality say M begins with consecutive ladder climbs. We first note that in addition to having even width, C must have a piece on every border from $B_{t(X)}$ to $B_{h(X)}$.

Lemma 5.2. If configuration C has a translation with moveset M such that M begins with two consecutive ladder climbs, M has even width and at least one piece occupying every border between the front and back borders of M.

Proof. Consider a placement X that allows two consecutive ladder climbs with initial ladder $L = \{l_0, l_1, ..., l_k\}$. Since after the first ladder climb the resulting placement X' has a true ladder L' as well, it must have a unique back piece which must be l_1, thus l_1 is the root of the newly formed ladder. Additionally after l_0 jumps past l_k, it becomes the unique front piece of X' and thus must be the last piece of ladder L'.

We note that the root of a ladder must occur on a border of opposite parity than all other ladder pieces. Additionally, if l_1 is on border B_i and l_k is on border B_j for $i \equiv j \mod 2$, then the ladder must occupy all other borders B_k for all k satisfying $k \equiv i \mod 2$ and $i \leq k \leq j$. Since X has at least two ladders (one true, one non-true) starting on adjacent borders of opposite parity, this implies that X has pieces occupying each border from $B_{t(X)}$ to $B_{h(X)}$. $\square$

We shall define a *isolating partition* of M. First, partition $m(M)$ sequentially into blocks $A_1, ...A_k$ such that each block begins with two or more consecutive ladder climbs, but does not have consecutive ladder climbs anywhere else, and additionally does not end with a ladder climb (this implies there are k sequences of 2 or more consecutive ladder climbs). Then, within each block A_i, sub-partition the blocks the in the same manner we did for Lemma 3.2, that is creating a new sub-partition $A_{(i,j)}$ at each ladder climb, so $A_i = A_{(i,1)} + ... + A_{(i,l_i)}$, where l_i is the number of ladder climbs in partition A_i.

Lemma 5.3. Let $L(A_i)$ be the number of ladder climbs A_i begins with. $\omega(A_i) < L(A_i)$.

Proof. Split A_i into two blocks B_1 and B_2, the first $L(A_i) - 1$ ladder climbs, and the rest of the moves. B_2 contains no consecutive ladder climbs by our definition of A_i, so $\omega(B_i) < 1$. $\omega(B_1) = L(A_i) - 1$, thus $\omega(A_i) = \omega(B_1 + B_2) < L(A_i)$. $\square$

We wish to show $\omega(A_i) < 0$, since $\omega(M) = \omega(\sum A_i) = \sum \omega(A_i)$. Since A_i can only begin with 2, 3, or 4 ladder climbs, we only have these two cases to consider.

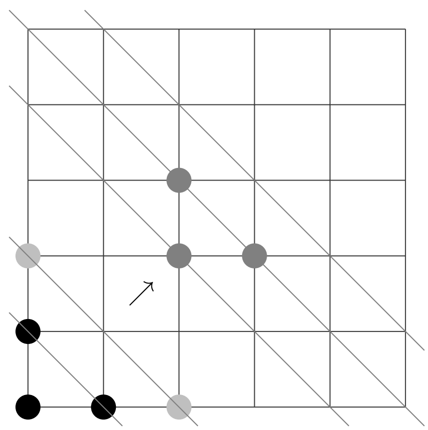
First, let us take into consideration the following - two translations with the same beginning and end placements, and with the same number of moves, must have the same speed. Therefore, in considering M, if we desire, we may instead consider some other arbitrary moveset M' with the same number of moves, and start and end placements, as M and any result proven about the speed of M will hold for M' and vice versa.

Let us make the following change to any M as a whole - if at any point, we have placement X with $T(X)$ containing exactly two elements, and proceed to first make a front push, then a back push, replace the ordering of the two moves *if we are able to do so*, that is, performing the moves in reverse order results in two legal moves. This changes the move types - we now first have a dead move, then a ladder climb. If we are not able to reverse the ordering, that is, the dead move prevents the ladder climb from occurring, then we leave that sequence as is. We note as a sanity check that this does not alter $\omega(M)$, as a front push followed by a back push has score -1, as does a dead move followed by a ladder climb.

Lemma 5.4. If some subpartition $A_{i,j}$ begins in configuration $C_{i,j}$ with the second border of $C_{i,j}$ containing 2 or more pieces (and the first move is a ladder climb), $\omega(A_{i,j}) \leq -1$. Additionally, if $\omega(A_{i,j}) = -1$, $A_{i,j}$ consisted of a ladder climb and a dead move, and the resulting $C_{i,j+1}$ has its second border containing 2 or more pieces, unless $C_{i,j}$ has a square subconfiguration as all of its backmost pieces (pieces on x, $x + e_u$, $x + e_v$, $x + e_u + e_v$).

Proof. The only way for any $\omega(A_{i,j}) = 0$ is if it begins with a ladder climb, then has two moves which are some combination of front and back pushes. However after the ladder climb in $A_{i.j}$, a back push is not possible. If we perform a dead move or do not move one of those two pieces, $\omega(A_{i,j}) \leq -1$. Let us then assume we move those pieces, p_1 and p_2. So we consider if one of the two backmost pieces performs the first front push, then the other performs a back push. WLOG let p_1 front push and p_2 back push. However by our earlier stipulation, we must reverse these moves if possible, which implies $\omega(A_{i,j}) = -1$ if the moves are reversible. If they are not reversible, p_2 must have jumped to block some part of p_1's ladder climb. This implies the pieces' coordinates have equivalent parity, so they are at least distance 4 apart (as they are on the same border), and therefore there must be two pieces on the third border (one for each piece's jump), and we have not yet left $A_{i,j}$, thus $\omega(A_{i,j}) \leq -1$.

Additionally if the moves were not reversible, $\omega(A_{i,j}) < -1$, as the only way for $\omega(A_{i,j}) = -1$ is for the next two moves to be front or back pushes, but we may use the same argument as above to demonstrate that we either have not left $A_{i,j}$ or were forced to perform a dead move instead. Thus if $\omega(A_{i,j}) = -1$, we know $A_{i,j}$ consists of a ladder climb and a dead move.



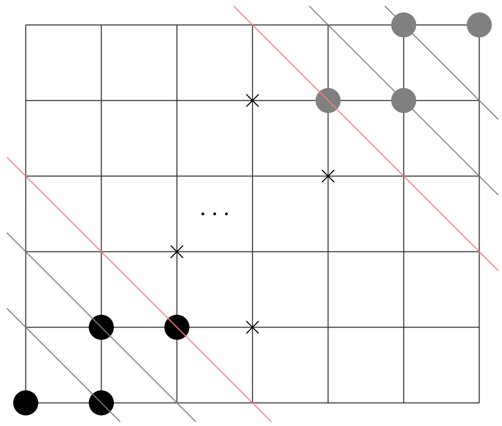
We must make 3 moves, after the ladder climb we optimally either perform a dead move or a front/back push, removing the possibility of a ladder climb after.

If $\omega(A_{i,j}) = -1$, we know the first move in $A_{i,j}$ was a ladder climb so we must have pieces on x and $x + e_u$, call these p_0, p_1. First, suppose the second piece p_2 is distance two away from p_1 on the second border. If p_2 is located on $x + e_v$, then both pieces can uniquely both jump over $x + e_u + e_v$ on the next border. Otherwise suppose there is no piece here, as this is the square configuration - since p_2 ladder climbs after p_1 moves, there must be a piece on the third border, p_3 that it jumped over, but p_1 could not have jumped over p_3 for its move so there either must be a second piece on the 3rd border or p_1 must have shifted to it. Now suppose p_2 is located at $x + 2e_u - e_v$ - the unique point both pieces can jump over is $x + 2e_u$, which cannot have a piece on it since it would block p_0. Thus either both pieces jumped over different pieces on the third border, or one piece shifts up. Therefore $C_{i,j+1}$ must have two pieces on its second border.

Otherwise, p_1 and p_2 are more than distance 2 apart and therefore must have either jumped over separate pieces, or one piece must have shifted up to the third border. Therefore $C_{i,j+1}$ must have two pieces on its second border. ☐

Lemma 5.5. If a partition A_i begins with exactly four consecutive ladder climbs, $\omega(A_i) \leq 0$.

Proof. First observe that we must have the width of the starting configuration greater than 4, since we have more than 4 pieces.



The forced start and end of the configuration. The x's mark where pieces cannot be, and the red borders are the open boundary of where any other pieces cannot be placed.

We note that climbing pieces, p_1, p_2, p_3, p_4, must be in a serpent configuration both before and after the ladder climbs, and after, they must be the four frontmost pieces. Additionally, we note that there must be two or more distinct pieces on the fifth border, the pieces that p_2 and p_4 jumped over, since if p_2 and p_4 jumped over the same piece at $x + 2e_i + 2e_j$, this would block the climb of p_1. A similar argument reveals the front border of our starting configuration must have 2 pieces. Therefore after the four jumps, we are now in $A_{i,4}$ and unable to back push. A quick observation lets us note that if we are leading with a serpent subconfiguration, the only piece that may front push are pieces in the serpent.

Consider the piece p_4 jumps over to begin its ladder climb, p_5 - if p_4 is located at $x + 2e_u + e_v$, we note as before that p_5 cannot be located at $x + 2e_u + 2e_v$. Additionally, p_5 cannot be located at $x + 3e_u + e_v$, as this would block the ladder climb of p_3. Therefore p_5 must be located at $x + 2e_u + e_v + e_w$, for $w \neq u, v$. This implies that p_5 is unable to at any point jump over p_1, p_2, p_3, because in $\mathbb{Z}_2^n$, it has distance greater than 1 from any of those pieces.

On the fifth border, the second piece hopped over in the ladder climb of p_2, p_5' must have form $x + e_u + 2e_v + e_w$ for some $1 \leq w \leq n$ and $w \neq u$. For p_5' to be able to jump, there must be a piece on the sixth border. However, we also note there must be a piece on the sixth border one away from $x + 3e_u + e_v$, a square p_3 hops to. These pieces must be distance 3 away, therefore if p_5' is able to jump there must be two pieces or more on the sixth border.

We will consider various cases considering where the next ladder climb occurs, and show that for each case, we must make moves that force $\omega(A_i) \leq 0$.

- Suppose the next ladder climb occurs from beyond the fifth border. This implies both p_5, p_5' must move forward before that occurs. We know p_5 is incapable of front pushing without considerable adjustment. Suppose p_5' cannot jump. p_5 may only ladder climb after a piece moves to the border p_2 is on, since p_2 cannot be jumped over by p_5. We may choose to move p_1 forward or p_3 backwards, this is the fastest we may do so but at best it is a dead move. Then if we suppose we front push with p_5 then back push with p_5', by our earlier stipulation we must reverse these moves (since p_5' is shifting, we know we can), which means we performed two dead moves in a row, and thus $\omega(A_{i,4}) \leq -3$ and $\omega(A_i) \leq 0$. So instead let us assume neither piece front pushes, and both move forward, a dead move and a back push. We know there are at least two pieces on the sixth border. We must fix the parity of the configuration, so we may perform a front push. However we still cannot ladder climb, meaning we are forced to make at least one more move, and thus $\omega(A_{i,4}) \leq -3$ and $\omega(A_i) \leq 0$.

- Otherwise let us assume p_5' can jump. The only new case we have is if we front push with p_5', as we are still forced to have two pieces on the sixth border. If p_5' desires to front push, since it must end by jumping over p_4 in our current configuration, either p_1 or p_3 must move. First consider if either dead moves as its move. Then if p_5' front pushes and p_5 back pushes, we are again forced to switch the ordering, and we perform two dead moves and thus $\omega(A_{i,4}) \leq -3$ and $\omega(A_i) \leq 0$. However if we cannot switch the ordering, we see after the moves that we still cannot ladder climb, as there are two pieces on the back border. Alternatively, consider if p_3 or p_1 front pushes instead of dead moving, by jumping over p_4, then front pushes again (as it must). Then p_5' can front push, so we have so far performed 3 front pushes. Now we must back push p_5 before we can ladder climb since we specified the next ladder climb happens beyond the fifth border. We must additionally push p_5' again, since it is currently blocking the only ladder reaching beyond p_4. However, we are now at an odd width, and have two pieces on the back border, and one of those pieces must move before the other can climb. Since we are at an odd width we cannot front push with one of those pieces, so if one moves, it is a dead move and $\omega(A_{i,4}) \leq -3$. Alternatively, even if front push again with p_5', it is distance 2 away from the nearest other piece and we are clearly unable to perform a ladder climb, and still must move one of the back pieces, so $\omega(A_{i,4})$ Therefore if the next ladder climb occurs beyond the fifth border, $\omega(A_i) \leq 0$.

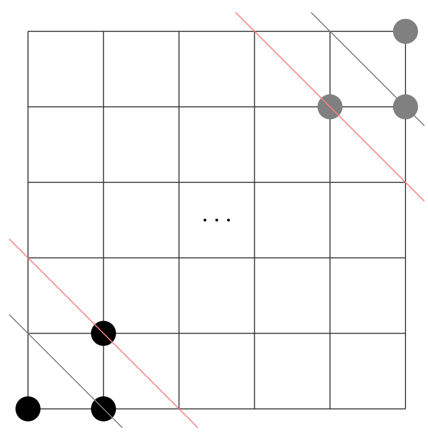
- So instead let us assume the next ladder climb begins on the fifth border. It is safe to assume p_5 or p_5' makes the climb. We have already covered the scenario where one of the pieces front pushes after the other dead moves forward in the previous cases. Thus the only case to consider is if one piece ladder climbs after the other performs a front push. We know for p_5 to front push optimally, p_1 or p_3 must move to the same border as p_2, and for p_5' to front push optimally, p_1 or p_3 must move, a dead move or two front pushes optimally. If these conditions are satisfied as fast as possible and p_5 or p_5' perform a ladder climb, we have $\omega(A_{i,1} + ... + A_{i,4}) = 1$. We cannot have left A_i, as there are two pieces on the new back border, and not all borders between the back and front border have a piece on them, since p_1 or p_3 moved. If these conditions were not satisfied as quickly, we have $\omega(A_i) \leq 0$. However we now see that it is impossible for $\omega(A_{i,5}) = 0$, since there are two pieces on the back border, and by our previous lemma $\omega(A_i) \leq 0$.

We have demonstrated that all possible movesets after performing four ladder climbs lead to a non-positive score.

□

Lemma 5.6. If a partition A_i begins with exactly three consecutive ladder climbs $\omega(A_i) \leq 0$.

Proof. From lemma 5.2, $\omega(A_i) < 3$, and we see $\omega(A_{i,1} + A_{i,2}) = 2$, as these subpartitions are only the first two ladder climbs, and $2 < i \leq l_i$, $\omega(A_{i,l}) \leq 0$. We wish to find two A_{i,j_1} and A_{i,j_2} with $\omega(A_{i,j_1}) < 0$ and $\omega(A_{i,j_2}) < 0$, or alternatively one $A_{i,j}$ such that $\omega(A_{i,j}) \leq -2$.

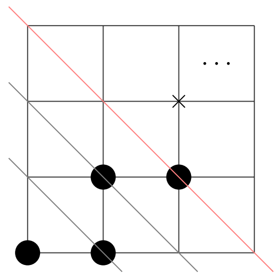


The forced start and end of the configuration. The x's mark where pieces cannot be, and the red borders are the open boundary of where any other pieces cannot be placed.

Here we have two initial cases: first, if the fourth border contains two pieces, second if the fourth border contains only one piece. WLOG let us assume p_1 began at x, p_2 began at $x+e_u$, and p_3 began at $x+e_u+e_v$ for $u \neq v$.

Suppose the fourth border contains two or more pieces. We can apply our lemma twice if we can demonstrate the third through fifth borders do not contain exactly a square configuration. p_1 must jump over one of these pieces however, and the only potential place to put a piece adjacent to p_3 for p_1 to jump over is at $x+2e_u+e_v$. p_2 must also jump over one of these pieces after its first hop, and the only place where both it after its first hop, $x + e_u + 2e_v$, and $x + 2e_u + e_v$ are adjacent on the fifth border is $x + 2e_u + 2e_v$, which would block p_1 from climbing, and thus we cannot have a square configuration.

Otherwise, the fourth border only contains one piece, which implies we must have started in a serpent subconfiguration, as shown:



We first note that p_4 must be located at $x + 2e_u + e_v$, completing the initial configuration's serpent sub-configuration. Before or during the next ladder climb, this piece must move.

- Suppose p_4 is unable to immediately perform a jump, or can do so, but chooses not to use its given opportunity. We know that for a ladder climb to occur, p_4 must move (whether it performs the ladder climb or not). If the next ladder climb to occur is p_4 moving from its current location, we know some other piece must move adjacent to it which must be a dead move. If the piece that moves adjacent to it moved from the same border, this is two dead moves, and $\omega(A_{i,3}) \leq -3$. Otherwise, it came from a different border implying there are two pieces on the next border, and applying our lemma, $\omega(A_{i,3}) \leq -2$. Additionally, if p_4 is to perform a ladder climb from that same border but somewhere else upon it, it must move at least twice to adjust its position which is equivalent to performing two dead moves, implying $\omega(A_{i,3}) \leq -3$, even worse. The last case to consider then is if p_4 shifts forward before a ladder climb occurs. WLOG let us assume that the first move is for p_4 translate forward. Then there must be two pieces on the new backmost border, as there must have been one present before (call it p_5). Before we can perform a climb we must fix the parity of the width of the configuration, so another back or front push must occur. Since there are two pieces on the back border a back push implies a dead move occurred first, which would set $\omega(A_{i,3}) \leq -2$, so let us instead assume a front push occurs. Now, no matter how the front push occurred, we still cannot true ladder climb since there are two pieces on the back border. If a dead move occurs next $\omega(A_{i,3}) \leq -2$, so we must front push then back push. However, given our earlier restriction, we must reverse these moves if possible, so instead they are a dead move then ladder climb, implying $\omega(A_{i,3}) \leq -2$. However, suppose then that the moves are NOT reversible, that is, the back push would have blocked the front push from occurring. This implies the pieces are on the same square modulo $\mathbb{Z}_2^n$, which then implies they are distance greater than 2 apart. Therefore, for both of these pieces' jumps, their first jump was over two separate pieces on the next border, and therefore after this sequence, we still cannot ladder climb and are forced to conclude $\omega(A_{i,3}) \leq -2$ in any scenario where p_4 does not jump as its move.

- Our next case then is to assume that p_4 can jump in the current position and does so. First consider if p_4s first move is not a ladder climb. We first note that this implies that there must be at least two pieces on the 5th border, the piece p_4 can jump over, and the piece p_2 jumps over second in its ladder climb (these cannot be the same piece, or else p_1's ladder climb is blocked. First note that if p_4 performs a back push, then a piece that is not p_1, p_2, p_3 performs a front push, we are reduced to the previous lemma as we may treat this as a less efficient ladder climb. Additionally if p_4 back pushes, then a dead move occurs, we are forced to make another non-ladder climb and thus $\omega(A_{i,3}) \leq -2$. So next suppose our next two moves are p_4 back pushing and one of p_2, p_3 front pushing. If p_2 front pushes, only p_2 or p_3 can front push again, and we cannot perform a back push since we have two pieces on the fifth border. However before ladder climbing, we must move one of those two pieces ahead. If we do so in the current configuration, it is a dead move and $\omega(A_{i,3}) \leq -2$. However if we adjust the front via two front pushes, we still cannot ladder climb and are forced to make more non-LC moves, and thus $\omega(A_{i,3}) \leq -2$. Otherwise suppose p_3 front pushes. If a dead move occurs next $\omega(A_{i,3}) \leq -2$. For the two pieces on the back border, neither can dead move without breaking the inequality, so the first to move must front push, and must front push again after to unblock the ladder and set the configuration to even width. If we cannot ladder climb here, $\omega(A_{i,3}) \le -2$. Otherwise if we can, $\omega(A_{i,3}) = -1$, but we have not left A_i since between the front and back borders, there are empty borders. Additionally, if both of the pieces on the previous back border were able to ladder climb, this implies they both hopped over p_3 and therefore they must be distance 4 or more apart, and hopped over separate pieces for their first hop. Thus by our earlier lemma $\omega(A_{i,4}) \le -1$ and thus $\omega(A_i) \le 0$.

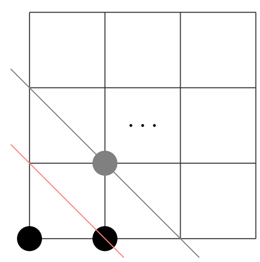
- Then let us suppose p_4s next move is a ladder climb, but not immediately. Observe that if there are two pieces on the fifth border, we may reduce this proof to the previous case, where we began with 4 ladder climbs. Otherwise, suppose there is only one piece on the next border. It cannot be the same piece that p_2 jumped over as we have stated, so it must move twice before p_4 can jump over it, two dead moves, and thus $\omega(A_{i,3}) \le -3$.

We have exhausted all cases and therefore $\omega(A_i) \le 0$ as desired.

□

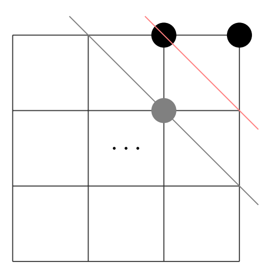
Lemma 5.7. If a partition A_i begins with exactly two consecutive ladder climbs, $\omega(A_i) \le 0$.

Proof. We see that we must start as depicted:



We note that $\omega(A_{(i,l)}) \le 0$ for $l > 1$ and $\omega(A_{(i,1)}) = 1$. Again WLOG let us assume p_1 begins on x, p_2 begins on $x + e_u$, and p_3 begins on $x + e_u + e_v$ with $1 \le u \ne v \le n$.

We know the first two moves are two ladder climbs, therefore $B_1 = \{LC\}$ and $B_2 = \{LC, ...\}$. Say performing the two ladder climbs gives us placement X'. Now consider $T(X')$. We must end up with the front of the configuration as such:



If $T(X')$ contains two or more pieces, we may simply apply our earlier lemma. If $T(X')$ contains exactly one piece (located at $x + e_i + e_j$), if the following move is not a front push or back push, we can conclude $\omega(A_{i,2}) \le 0$. Otherwise, we have three cases:

If our following two moves are two front pushes, we note there is only one allowable way to do so that could possibly set up a ladder climb - by having the piece that first climbed hop over the second piece that climbed, then having that piece push forward again in any direction. This implies that the next climb must at some point land where the first piece jumped to. However, $x \ne x + e_u + e_v$ in $(\mathbb{Z}/\mathbb{Z}^2)^n$, so that piece cannot perform the ladder climb, and thus $\omega(A_{i,2}) \le 0$.

If our following two moves are two back pushes, this implies that on $B_{t(X')+1}$ there is only one piece. Additionally, this implies that the piece on $T(X')$ must jump for its back push, and thus must jump over the piece on $B_{t(X')+1}$ for its push. We note that the only location for a piece on $B_{t(X')+1}$ that allows x to hop over it as well as $x + e_u + e_v$ is $x + 2e_u + e_v$. However we also note the only place that supports both the piece at $x+e_u$ and the piece at $x+2e_u+e_v$ to hop over it (since the piece at $x+e_u$ first hops to $x+e_u+2e_v$ is $x+2e_u+2e_v$, however having a piece here would block the piece at x from jumping, and therefore there must either be two pieces on one of the borders $B_{t(X')}$ or $B_{t(X')+1}$, or the piece at $x+2e_u+e_v$ must be given a new piece to jump over, the piece in $T(X')$ which performed the first back push. However, this then leaves us with two pieces on $B_{t(X')+2}$, leaving us unable to perform a ladder climb next. Therefore $\omega(A_{i,2}) \leq 0$.

If our following two moves are one back push and one front push, we first note that if the front push comes from a border before $H(X)$, we will not be able to perform a ladder climb after any back push occurs, which would imply $\omega(A_{i,2}) < 0$. Otherwise, we note that only front push move that allows us to perform any sort of ladder climb afterwards by having the frontmost piece move forward in any direction, so let us assume we have done so. Now if the backmost piece of X' shifts forward, we know there are two pieces on the new back border, so a ladder climb is not possible. If the piece jumps over a different piece than x jumped over, we have a the same situation. So therefore, the piece must have jumped over the same piece as x, which as we noted before must be at $x+2e_u+e_v$. However if we observe the board in $(\mathbb{Z}/\mathbb{Z}^2)^n$, we see that that piece cannot perform a ladder climb, as it differs by 1 at the uth and vth dimensions. Therefore a ladder climb is not possible in any case and therefore $\omega(A_{i,2}) \leq 0$.

□

Corollary 5.7.1. If C is a configuration with $p > 4$, C has speed less than or equal to 2/3.

Proof. Consider any m-move trajectory M of C. If M contains no consecutive ladder climbs $\omega(M) \leq 0$. Otherwise, adjust M as necessary to get M' and perform an isolating partition of M'. $\omega(M) = \omega(M') = \omega(\sum A_i) = \sum \omega(A_i) \leq 0$. Since in any case $\omega(M) \leq 0$, M has speed less than 2/3. □

We note that on an infinite board, any finite configuration is capable of coming arbitrarily close to 2/3 speed, by first arranging itself into the snake configuration, then moving arbitrarily far with the snake traverse, then moving back to the starting configuration.

We now pose the question of whether we replace the go board with a standard Chinese Checkers board (a tessellation of simplices), the same result holds.