General Thesis Problem Notes

Sam Kim Miller

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1 C_2 Splendid Rickard Complex Construction

Overview of the problem

For simplicity, let us assume we have a p-group G and a p-modular system $(K, \mathcal{O}, k)$. Then in this case, $B_p(G) = \mathcal{O}G$, that is, G has only one block. Our goal is to construct a Splendid Rickard complex for G in the following manner: we have a sequence of maps:

$$\begin{tikzcd}
 & & \mathrm{Spl}(G,G) \arrow[d, squiggly] \\
B(G)^\times \arrow[r] \arrow[urr, dotted] & O(B(G,G)) \arrow[r] & O(T(G,G))
\end{tikzcd}$$

where $B(G)^\times$ is the multiplicative group of the Burnside ring $B(G)$ of G, $O(B(G,G))$ is the group of orthogonal units of the Burnside (G,G)-biset ring, $B(G,G)$, and $O(T(G,G))$ is the group of orthogonal units in the Burnside ring of trivial source (G,G)-bimodules, $T(G,G)$. One has that in $B(G)$, all unit elements have order 2, that is, $[U] = [U]^{-1}$. In $B(G,G)$, all elements u satisfy $[U] = [U^{\mathrm{op}}]$, that is, there is an adjoint operator (but its action is trivial). Finally in $T(G,G)$, we have an adjoint operator given by $[M^*] = \mathrm{Hom}_k(M,k)$ (or whatever ring M is defined over). It is known that if $[M]$ is invertible, that $[M]^{-1} = [N^*]$ for some trivial source module N.

The maps are as follows: for $B(G)^\times \to O(B(G,G))$, given $[U] \in B(G)^\times$, define the map by linear extension of the assignment $[U] \mapsto [\tilde{U}] \in B(G,G)$. This is a group homomorphism when restricted to the multiplicative groups of the respective rings. Note that since $[U] = [U]^{-1}$, $[\tilde{U}] = [\tilde{U}]^{-1} = [\tilde{U}^{\mathrm{op}}]$.

For $O(B(G,G)) \to O(T(G,G))$, given $[U] \in O(B(G,G))$, define the map by extension of the assignment $U \mapsto \mathcal{O}U$, the free module with basis elements given by the elements of U. The $\mathcal{O}G$ actions are given by the left and right actions on U, respectively. Again,this is a group homomorphism when restricted to the multiplicative groups. Note that $[\mathcal{O}U] = [\mathcal{O}U^*] =$

$[\mathcal{O}U]^{-1}$ by the transport of properties of $O(B(G,G))$.

Finally, given a splendid Rickard complex of (G,G) bimodules over p, we form an alternating sum of its components. That is, if $0 \to M_n \to M_{n-1} \to \cdots \to M_m \to 0$ is a splendid Rickard complex, send it to the element $\sum_{i=m}^{n}(-1)^i[M_i] \in O(T(G,G))$. It was proven by Boltje that this indeed forms a p-permutation equivalence.

Given some $U \in B(G)^\times$, we wish to construct a splendid Rickard complex which makes the above diagram commute. The image of U in $O(T(G,G))$ will immediately suggest the components of a splendid Rickard complex - however, the difficulty is first in the choice of transition maps, and second in verifying that the constructed chain complex is indeed a splendid Rickard complex.

Splendid Rickard Complexes

First recall that any $(\mathcal{O}G, \mathcal{O}H)$-bimodule M can equivalently be considered a left $\mathcal{O}(G \times H)$-module with action $(g,h)\cdot m = g\cdot m\cdot h^{-1}$. Let

$$\Gamma := \cdots \to 0 \to M_m \xrightarrow{d_m} M_{m-1} \xrightarrow{d_{m-1}} \cdots \xrightarrow{d_{m-(a-1)}} M_{m-a} \to 0 \to \cdots$$

be a bounded chain complex of modules on some $\mathcal{O}$-algebra. Then, the $\mathcal{O}$-dual of Γ is defined as the complex

$$\Gamma^* := \cdots \to 0 \to \mathrm{Hom}_{\mathcal{O}}(M_{m-a}, \mathcal{O}) \xrightarrow{d^*_{m-(a-1)}} \cdots \xrightarrow{d^*_m} \mathrm{Hom}_{\mathcal{O}}(M_m, \mathcal{O}) \to 0 \to \cdots$$

in other words, the chain complex applied by taking the contravariant functor $\mathrm{Hom}_{\mathcal{O}}(-,\mathcal{O})$. Let G b ea finite group whose Sylow p-subgroups are abelian. We denote by S_p one of those Sylow p-subgroups, and set $H := N_G(S_p)$.

Then Γ is a **Rickard complex** for the principal blocks $B_p(G)$ and $B_p(H)$ if it is a complex of $(B_p(G), B_p(H))$-bimodules, and satisfies the following properties:

1. Each M_n of Γ, viewed as a $\mathcal{O}(G \times H)$ module, is a p-permutation module with vertex contained in $\Delta_{G\times H^{\mathrm{op}}}(S_p)$, where $\Delta_{G\times H^{\mathrm{op}}}(S_p) = \{(x, x^{-1}) \in G \times H : x \in S_p\}$.

2. We have homotopy equivalences

$$\Gamma \otimes_{\mathcal{O}H} \Gamma^* \simeq B_p(G) \text{ as complexes of } (B_p(G), B_p(G))\text{-bimodules,}$$
$$\Gamma^* \otimes_{\mathcal{O}G} \Gamma \simeq B_p(H) \text{ as complexes of } (B_p(H), B_p(H))\text{-bimodules.}$$

Side Note: We may loosen these restrictions a bit - for the purposes of this problem, we will reformulate the definition as follows:
Let Γ be a bounded complex of (kG, kG)-bimodules with the following properties:

- Every indecomposable summand of Γ_n is a trivial source module, in other words, Γ_n is a p-permutation module.
- Every indecomposable summand has twisted diagonal vertices (condition 1 above). This is equivalent to Γ_n being projective as left- and as right-kG modules.
- $\Gamma \otimes_{kG} \Gamma^* \simeq kG$ and $\Gamma^* \otimes_{kG} \Gamma \simeq kG$.

Then Γ is a **splendid Rickard Equivalence** for kG and kG. It is conjectured that $kG - kG$ Rickard Equivalences lead to Rickard complexes of blocks via idempotents. For our purposes, we will mainly focus on $kG - kG$ Rickard equivalences for the time being.

The $G = C_2$ case

Fix $G = C_2$. As we have an injective homomorphism $B(G) \to \prod_{[s_G]} \mathbb{Z}$ where $[s_G]$ indexes conjugacy classes of subgroups of G, $|B(G)^\times| \leqslant 4$ (since there are only 2 subgroups of G). One may compute that in this case, $B(G)^\times$ has 4 elements, given by $[G/G], [G/G] - [G/1]$ and their negative counterparts.

We will take the element $g = [G/1] - [G/G]$. First let us compute its image in $O(B(G,G))$. Since the ring homomorphism is unital, $[\widetilde{G/G}] = [G]$. $\widetilde{G/1} = G \times G$ as a set, and with the group action it is not hard to compute that all elements have stabilizer 1×1, hence $[\widetilde{G/1}] = [G \times G]$. Hence the image in $O(B(G,G))$ is $[G \times G] - [G]$

Now, the image of g in $O(T(G,G))$ will be given by $[\mathcal{O}(G \times G)] - [\mathcal{O}G] = [\mathcal{O}G \otimes_O \mathcal{O}G] - [\mathcal{O}G]$. We wish to find a splendid Rickard complex send to that - the fairly clear choice is

$$\Gamma = \cdots \to 0 \to \mathcal{O}G \otimes_{\mathcal{O}} \mathcal{O}G \xrightarrow{d} \mathcal{O}G \to 0 \to \cdots,$$

where $\mathcal{O}G$ occurs at the 0 index, and the transition map $d : \mathcal{O}G \otimes_{\mathcal{O}} \mathcal{O}G \to \mathcal{O}G$ is given by $d(a \otimes b) = ab$. It is clear that this complex satisfies condition (1), as these modules are permutation modules which are free, and hence have trivial source. Moreover, one may verify in general that for any $h \leqslant G$, $\mathrm{Ind}_{\Delta H}^{G \times G}(\mathcal{O}) \cong \mathcal{O}G \otimes_{\mathcal{O}H} \mathcal{O}G$, so each module has diagonal vertices. The difficulty lies in verifying (2).

The Dual Complex

Let us compute what Γ^* is and attempt to simplify as much as possible. By definition, it is

$$\Gamma^* = \cdots \to 0 \to \mathcal{O}G^* \xrightarrow{d^*} (\mathcal{O}G \otimes_{\mathcal{O}} \mathcal{O}G)^* \to 0 \to \cdots.$$

Here, d^* is given by precomposition. It sends a map $f : \mathcal{O}G \to \mathcal{O}$ to a map $f' : \mathcal{O}G \otimes_{\mathcal{O}} \mathcal{O}G \to \mathcal{O}$ defined by $f'(a \otimes b) := f(ab)$.

Recall that for any $(\mathcal{O}G, \mathcal{O}H)$-bimodule M, $\mathrm{Hom}_{\mathcal{O}}(M, \mathcal{O})$ has $(\mathcal{O}H, \mathcal{O}G)$-bimodule structure given by

$$(h \cdot f \cdot g)(m) = f(g \cdot m \cdot h), \qquad \forall g \in \mathcal{O}G, h \in \mathcal{O}H, m \in M, f \in \mathrm{Hom}_{\mathcal{O}}(M, \mathcal{O})$$

With this, we have an isomorphism of $(\mathcal{O}G, \mathcal{O}G)$-bimodules $\mathcal{O}G \cong \mathcal{O}G^*$ given by the $\mathcal{O}$-linear extension of

$$\begin{aligned} \Phi_1 : \mathcal{O}G &\xrightarrow{\sim} \mathcal{O}G^* \\ g &\mapsto \delta_{g^{-1}} \\ \sum_{g \in G} f(g^{-1}) \cdot g &\leftarrow\!\shortmid f \end{aligned}$$

Call the map in the $\mathcal{O}G \to \mathcal{O}G^*$ direction Φ_1. We may consider Γ^* as a chain complex

$$0 \to \mathcal{O}G \xrightarrow{d^* \circ \Phi_1} (\mathcal{O}G \otimes \mathcal{O}G)^* \to 0$$

Note that by the Burnside ring structure of $B(G)^\times$ and $O(T(G,G))$ that we have an equality

$$[\mathcal{O}G \otimes_O \mathcal{O}G] - [\mathcal{O}G] = [(\mathcal{O}G \otimes_O \mathcal{O}G)^*] - [\mathcal{O}G^*] = [\mathcal{O}G \otimes_O \mathcal{O}G]^{-1} - [\mathcal{O}G]^{-1},$$

and we have demonstrated $[\mathcal{O}G] = [\mathcal{O}G]^*$, so we must have an isomorphism of $\mathcal{O}G$-modules $(\mathcal{O}G \otimes_{\mathcal{O}} \mathcal{O}G)^* \cong \mathcal{O}G \otimes_{\mathcal{O}} \mathcal{O}G$. In fact, we have a similar construction as before for the isomorphism. Set $\mathcal{B} = \{(g \otimes h) : g, h \in G\}$, an $\mathcal{O}$-basis of $\mathcal{O}G \otimes_O \mathcal{O}G$. Then, for any $g \otimes h \in \mathcal{B}$, we denote $(g \otimes h)^{-1} := h^{-1} \otimes g^{-1}$. One then may verify that the $\mathcal{O}$-linear extension of the map:

$$\begin{aligned} \Phi_2 : \mathcal{O}G \otimes_O \mathcal{O}G &\xrightarrow{\sim} (\mathcal{O}G \otimes_O \mathcal{O}G)^* \\ g \otimes h &\mapsto \delta_{h^{-1} \otimes g^{-1}} \\ \sum_{b \in \mathcal{B}} f(b^{-1}) \cdot b &\leftarrow\!\shortmid f \end{aligned}$$

is indeed an isomorphism of $(\mathcal{O}G, \mathcal{O}G)$-bimodules.

Now given these, we may try to find an isomorphism of complexes $\Gamma^* \cong \Gamma'$, where Γ' is of the form

$$0 \to \mathcal{O}G \to \mathcal{O}G \otimes_{\mathcal{O}} \mathcal{O}G \to 0.$$

More precisely, we wish to find the transition map d' that makes the following diagram commute:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}G \arrow[r, "d'"] \arrow[d, "\Phi_1"] & \mathcal{O}G \otimes_{\mathcal{O}} \mathcal{O}G \arrow[r] \arrow[d, "\Phi_2"] & 0 \\ 0 \arrow[r] & \mathcal{O}G^* \arrow[r, "d^*"] & (\mathcal{O}G \otimes_{\mathcal{O}} \mathcal{O}G)^* \arrow[r] & 0 \end{tikzcd}$$

We compute how d' acts on $\mathcal{O}$-basis elements to define it:

$$d'(1) = 1 \otimes 1 + c \otimes c, \qquad d'(c) = 1 \otimes c + c \otimes 1,$$

and check that it is indeed a $(\mathcal{O}G, \mathcal{O}G)$-bimodule homomorphism (it is). Thus we have an isomorphism of (kG, kG)-bimodules $\Gamma^* \cong \Gamma'$. We may identify the two interchangeably for computing the conditions of Rickard complexes.

Tensoring and simplifying

We now wish to take the tensor product $\Gamma \otimes_{\mathcal{O}G} \Gamma'$ - in this case we only need to perform one computation since $B_p(G) = B_p(H) = G$, and $\Gamma \otimes_{\mathcal{O}G} \Gamma' \cong \Gamma' \otimes_{\mathcal{O}G} \Gamma$.
Let's recall what the construction is for the tensor product of bounded chain complexes is. Given two chain complexes $C_\bullet$ and $D_\bullet$ of $\mathcal{O}G$-bimodules

$$(C \otimes_{\mathcal{O}G} D)_n = \bigoplus_{i+j=n} C_i \otimes_{\mathcal{O}G} D_j.$$

Transition maps are defined over the direct sum as follows: given $c_i \otimes d_j \in C_i \otimes_{\mathcal{O}G} D_j$, we set

$$d_n(c_i \otimes d_j) = d_i^C(c_i) \otimes d_j + (-1)^i c_i \otimes d_j^D(d_j),$$

then define d_n by linearizing. In our case, the tensor product $\Gamma \otimes_{\mathcal{O}G} \Gamma'$ will have three nonzero components. The modules are as follows:

$$\begin{gathered}
(\Gamma \otimes_{\mathcal{O}G} \Gamma')_1 = (\mathcal{O}G \otimes_{\mathcal{O}} \mathcal{O}G) \otimes_{\mathcal{O}G} \mathcal{O}G \\
(\Gamma \otimes_{\mathcal{O}G} \Gamma')_0 = \big(\mathcal{O}G \otimes_{\mathcal{O}G} \mathcal{O}G\big) \oplus \big((\mathcal{O}G \otimes_{\mathcal{O}} \mathcal{O}G) \otimes_{\mathcal{O}G} (\mathcal{O}G \otimes_{\mathcal{O}} \mathcal{O}G)\big) \\
(\Gamma \otimes_{\mathcal{O}G} \Gamma')_{-1} = \mathcal{O}G \otimes_{\mathcal{O}G} (\mathcal{O}G \otimes_{\mathcal{O}} \mathcal{O}G)
\end{gathered}$$

and the transition maps are as follows:

$$d_1 : (a \otimes b) \otimes c \mapsto \big(ab \otimes c, a \otimes b \otimes d^*(c)\big)$$

$$d_0 : (a \otimes b, c \otimes d \otimes e \otimes f) \mapsto -a \otimes d^*(b) + cd \otimes e \otimes f$$

We may simplify this chain complex by finding an isomorphism of chain complexes in what will follow. Denote for ease of notation $\Gamma \otimes_{\mathcal{O}G} \Gamma' = C_\bullet$ and $(\Gamma \otimes_{\mathcal{O}G} \Gamma')_i = C_i$. Then, we have obvious isomorphisms of $(\mathcal{O}G, \mathcal{O}G)$-bimodules as follows:

$$C_1 \cong \mathcal{O}G \otimes_{\mathcal{O}} \mathcal{O}G, \quad a \otimes b \otimes c \mapsto a \otimes bc$$

$$C_0 \cong \mathcal{O}G \oplus (\mathcal{O}G \otimes_{\mathcal{O}} \mathcal{O}G \otimes_{\mathcal{O}} \mathcal{O}G), \quad (a \otimes b, c \otimes d \otimes e \otimes f) \mapsto (ab, c \otimes de \otimes f)$$

$$C_{-1} \cong \mathcal{O}G \otimes_{\mathcal{O}} \mathcal{O}G, \quad a \otimes b \otimes c \mapsto ab \otimes c$$

Then, one may verify that the following squares both commute:

$$\begin{tikzcd}
C_1 \arrow[r, "d_1"] \arrow[d, "\cong"] & C_0 \arrow[r, "d_0"] \arrow[d, "\cong"] & C_{-1} \arrow[d, "\cong"] \\
\mathcal{O}G \otimes_{\mathcal{O}} \mathcal{O}G \arrow[r, "f"] & \mathcal{O}G \oplus (\mathcal{O}G \otimes_{\mathcal{O}} \mathcal{O}G \otimes_{\mathcal{O}} \mathcal{O}G) \arrow[r, "g"] & \mathcal{O}G \otimes_{\mathcal{O}} \mathcal{O}G
\end{tikzcd}$$

where $f(x \otimes y) = (xy, x \otimes d'(y))$ and $g(w, x \otimes y \otimes z) = xy \otimes z - d'(w)$. We conclude we have an isomorphism of chain complexes:

$$\Gamma \otimes_{\mathcal{O}G} \Gamma^* \cong \cdots \to 0 \to \mathcal{O}G \otimes_{\mathcal{O}} \mathcal{O}G \to \mathcal{O}G \oplus (\mathcal{O}G \otimes_{\mathcal{O}} \mathcal{O}G \otimes_{\mathcal{O}} \mathcal{O}G) \to \mathcal{O}G \otimes_{\mathcal{O}} \mathcal{O}G \to 0 \to \cdots$$

and will henceforth refer to this complex as $\Gamma \otimes_{\mathcal{O}} \Gamma^*$ instead, with differentials defined above.

Finding a homotopy equivalence

Before we do this, let's double check to make sure that this chain complex is the "right" one, that is, that it has the same homology as $\mathcal{O}G$ as a chain complex. Computing the full homology may be nontrivial but it's easier to at least show that f is injective and g is surjective, showing $H_1 = H_{-1} = 0$.

To see f is injective, first one may verify that d' as defined for Γ' is injective. Since $\mathcal{O}$ has no zero divisors, $x \otimes d'(y) = 0$ if and only if $x = 0$ or $d'(y) = 0$, which in turn is true only when $x = 0$ or $y = 0$, so $x \otimes y = 0$. Therefore $\ker f = 0$, and $\mathrm{im} f \cong \mathcal{O}G \otimes_{\mathcal{O}} \mathcal{O}G$.

On the other hand it is clear that g is surjective, as $a \otimes b \in \mathcal{O}G \otimes_{\mathcal{O}} \mathcal{O}G$ is mapped to by g via $(0, a \otimes 1 \otimes b)$, so all basis elements have preimage, and thus g is surjective. It follows that $H_1 = H_{-1} = 0$. We have everything we need to prove the homotopy equivalence.

Theorem. $\mathcal{O}G \simeq \Gamma \otimes_{\mathcal{O}G} \Gamma^*$

Proof. First, observe that $\mathcal{O}G \otimes_{\mathcal{O}} \mathcal{O}G$ is a free $(\mathcal{O}G, \mathcal{O}G)$-bimodule, hence projective and injective. Since f is injective, by injectivity of $\mathcal{O}G \otimes_{\mathcal{O}} \mathcal{O}G$, f splits with section f', and moreover, there is a decomposition

$$\mathcal{O}G \oplus \mathcal{O}G \otimes_{\mathcal{O}} \mathcal{O}G \otimes_{\mathcal{O}} \mathcal{O}G = \mathrm{im} f \oplus \ker f'.$$

Similarly, since g is surjective, by projectivity of $\mathcal{O}G \otimes_{\mathcal{O}} \mathcal{O}G$, g splits with section g', and moreover, there is a decomposition

$$\mathcal{O}G \oplus \mathcal{O}G \otimes_{\mathcal{O}} \mathcal{O}G \otimes_{\mathcal{O}} \mathcal{O}G = \ker g \oplus \mathrm{im}\ g'.$$

Since $\Gamma \otimes_{\mathcal{O}G} \Gamma^*$ is a complex, $\mathrm{im} f \subseteq \ker g$, and so we may write

$$\mathcal{O}G \oplus \mathcal{O}G \otimes_{\mathcal{O}} \mathcal{O}G \otimes_{\mathcal{O}} \mathcal{O}G = \mathrm{im} f \oplus \ker g' \oplus M, \quad \text{with } M \oplus \mathrm{im} f = \ker g$$

Therefore, $\mathrm{im} f \cap \ker g' = \{0\}$. Now, by injectivity of f and surjectivity of g, we have isomorphisms

$$f : \mathcal{O}G \otimes_{\mathcal{O}} \mathcal{O}G \xrightarrow{\sim} \mathrm{im} f$$

$$g : \ker g' \xrightarrow{\sim} \mathcal{O}G \otimes_{\mathcal{O}} \mathcal{O}G$$

from which we can form an acyclic, split chain complex:

$$A = \cdots \to 0 \to \mathcal{O}G \otimes_{\mathcal{O}} \mathcal{O}G \xrightarrow{f} \mathrm{im} f \oplus \ker g' \xrightarrow{g} \mathcal{O}G \otimes_{\mathcal{O}} \mathcal{O}G \to 0 \to \cdots .$$

(Linkelmann 1.18.15) implies that A is contractible. Moreover, we have that $A \oplus M = \Gamma \otimes_{\mathcal{O}G} \Gamma^*$, where M is the chain complex with $(\mathcal{O}G, \mathcal{O}G)$-bimodule M at degree 0, and 0s elsewhere, and $\mathcal{O}G \oplus \mathcal{O}G \otimes_{\mathcal{O}} \mathcal{O}G \otimes_{\mathcal{O}} \mathcal{O}G \cong \mathrm{im} f \oplus \ker g' \oplus M$. (Linkelmann 1.18.19) then implies that $M \simeq \Gamma \otimes_{\mathcal{O}G} \Gamma^*$, so it remains to show that $M \cong \mathcal{O}G$.

On the other hand, observe that

$$\mathcal{O}G \otimes_{\mathcal{O}} \mathcal{O}G \otimes_{\mathcal{O}} \mathcal{O}G \cong \mathcal{O}G \otimes_{\mathcal{O}} \big(\mathcal{O} \oplus \mathcal{O}\big) \otimes_{\mathcal{O}} \mathcal{O}G,$$

as $(\mathcal{O}G, \mathcal{O}G)$-bimodules, since the middle term in the triple tensor product can be restricted to be considered an $(\mathcal{O}, \mathcal{O})$-bimodule. Then, we have:

$$\begin{aligned}
OG \otimes_{\mathcal{O}} \mathcal{O}G \otimes_{\mathcal{O}} \mathcal{O}G &\cong \mathcal{O}G \otimes_{\mathcal{O}} \big(\mathcal{O} \oplus \mathcal{O}\big) \otimes_{\mathcal{O}} \mathcal{O}G \\
&\cong \big((\mathcal{O}G \otimes_{\mathcal{O}} \mathcal{O}) \oplus (\mathcal{O}G \otimes_{\mathcal{O}} \mathcal{O})\big) \otimes_{\mathcal{O}} \mathcal{O}G \\
&\cong (\mathcal{O}G \oplus \mathcal{O}G) \otimes_{\mathcal{O}} \mathcal{O}G \\
&\cong (\mathcal{O}G \otimes_{\mathcal{O}} \mathcal{O}G) \oplus (\mathcal{O}G \otimes_{\mathcal{O}} \mathcal{O}G)
\end{aligned}$$

Since we have a decomposition:

$$\mathcal{O}G \oplus (\mathcal{O}G \otimes_{\mathcal{O}} \mathcal{O}G \otimes_{\mathcal{O}} \mathcal{O}G) = \mathrm{im} f \oplus \mathrm{im}\ g' \oplus M \cong (\mathcal{O}G \otimes_{\mathcal{O}} \mathcal{O}G) \oplus (\mathcal{O}G \otimes_{\mathcal{O}} \mathcal{O}G) \oplus M,$$

it follows that $M \cong \mathcal{O}G$ by the Krull-Schmidt theorem, as desired. $\square$

Thus, we have constructed a Splendid Rickard complex for $G = C_2$!