CHAPTER 06 Antiderivatives

Basic Definitions and Formulae

Introduction: Integration is the inverse process of differentiation. The primary problem of differential calculus is: Given a function, to find it's differential coefficient. But the primary problem of integral Calculus is it's inverse, i–e "Given the differential coefficient of a function, to find the function itself".

$$f(x) \xrightarrow{\text{differentiation}} f'(x)$$

$$f'(x) \xrightarrow{\text{anti-differentiation}} f(x)$$

This means that integration is also known as anti-derivative.

Integral: If $\phi(x)$ is a differentiable function such that $\frac{d}{dx}\phi(x) = f(x)$, then $\phi(x)$ is called an integral or primitive or anti-derivative of $f(x)$ and is written as

$\int f(x)\,dx = \phi(x)$. The symbol dx indicates that integration is to be performed with respect to x.

Integration: The process of finding the integral of a function is called integration.

Integrand: The function to be integrated is called the integrand

$\int f(x)\,dx = \phi(x) + c$

↓ Integrand ↓ Integral

↓ Variable of Integration

Integral Sign: The symbol "$\int$" is used to denote the "sign of integration or integral sign" and is used to represent the process of integration. The mathematician leibnitz introduced this symbol.

Note: (1) The symbol $\int$ is an elongated S which is the first letter of sum, as the process of integration originated from summation of infinite series.

(2) The Symbols $\int$ and dx, Separately have no meanings. These two symbols may be regarded as something like a pair of brackets in which the function is to be integrated.

Constant of Integration and indefinite Integrals:

let $\frac{d}{dx}[\phi(x)] = f(x)$ then

$$\frac{d}{dx}[\phi(x) + c] = \frac{d}{dx}\phi(x) + \frac{d}{dx}(c)$$

Initial Condition: A condition is on the solution at one point.

Equation with variable Separable: A first order equation is called variable separable if it can be written $\frac{dy}{dx} = \frac{f(x)}{g(y)}$.

EXERCISE # 6.1

Evaluate the following indefinite integrals.

(1) $\int (x^2 + 4x + 13)\,dx.$

Solution: let $I = \int (x^2 + 4x + 13)\,dx.$

$I = \int x^2\,dx + \int 4x\,dx + \int 13dx \Rightarrow I = \int x^2\,dx + 4\int x\,dx + 13\int dx$

$I = \frac{x^3}{3} + \cancel{4}^{2}\frac{x^2}{\cancel{2}} + 13x + c \Rightarrow \boxed{I = \frac{x^3}{3} + 2x^2 + 13x + c}$ Ans

(2) $\int (3x^4 - 5x^3 - 4x^2 - 2)\,dx$

Solution: let $I = \int (3x^4 - 5x^3 - 4x^2 - 2)\,dx$

$I = \int 3x^4\,dx - \int 5x^3\,dx - \int 4x^2\,dx - \int 2x\,dx$

$I = 3\int x^4\,dx - 5\int x^3\,dx - 4\int x^2\,dx - 2\int dx$

$\boxed{I = \frac{3}{5}x^5 - \frac{5}{4}x^4 - \frac{4}{3}x^3 - 2x + c}$ Ans

(3) $\int (4x^3 - 12x^2 - 4x + 12)\,dx$

Solution: let $I = \int (4x^3 - 12x^2 - 4x + 12)\,dx$

$I = \int 4x^3\,dx - \int 12x^2\,dx - \int 4x\,dx + \int 12\,dx$

$I = 4\int x^3\,dx - 12\int x^2\,dx - 4\int x\,dx + 12\int dx$

$I = \cancel{4}\frac{x^4}{\cancel{4}} - {}^{4}\cancel{12}\frac{x^3}{\cancel{3}} - {}^{2}\cancel{4}\frac{x^2}{\cancel{2}} + 12x + c$

$\boxed{I = x^4 - 4x^3 - 2x^2 + 12x + c}$ Ans

(4) $\int x^2(x^2 - 4)\,dx$

Solution: let $I = \int x^2(x^2 - 4)\,dx$

$I = \int (x^4 - 4x^2)\,dx \Rightarrow I = \int x^4\,dx - \int 4x^2\,dx$

$I = \int x^4\,dx - 4\int x^2\,dx \Rightarrow \boxed{I = \frac{x^5}{5} - \frac{4}{3}x^3 + c}$ Ans

(5) $\int \sqrt{2x + 5}\,dx$

Solution: let $I = \int \sqrt{2x + 5}\,dx$

$I = \int (2x + 5)^{1/2}\,dx$

x & ÷ by 2

Initial Condition: A condition is on the solution at one point.

Equation with variable Separable: A first order equation is called variable separable if it can be written $\frac{dy}{dx} = \frac{f(x)}{g(y)}$.

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$I = \frac{x^3}{3} + \cancel{4}^2\frac{x^2}{\cancel{2}} + 13x + c \Rightarrow \boxed{I = \frac{x^3}{3} + 2x^2 + 13x + c}$ Ans

(2) $\int (3x^4 - 5x^3 - 4x^2 - 2)\,dx$

Solution: let $I = \int (3x^4 - 5x^3 - 4x^2 - 2)\,dx$

$I = \int 3x^4\,dx - \int 5x^3\,dx - \int 4x^2\,dx - \int 2x\,dx$

$I = 3\int x^4\,dx - 5\int x^3\,dx - 4\int x^2\,dx - 2\int dx$

$\boxed{I = \frac{3}{5}x^5 - \frac{5}{4}x^4 - \frac{4}{3}x^3 - 2x + c}$ Ans

(3) $\int (4x^3 - 12x^2 - 4x + 12)\,dx$

Solution: let $I = \int (4x^3 - 12x^2 - 4x + 12)\,dx$

$I = \int 4x^3\,dx - \int 12x^2\,dx - \int 4x\,dx + \int 12\,dx$

$I = 4\int x^3\,dx - 12\int x^2\,dx - 4\int x\,dx + 12\int dx$

$I = \cancel{4}\frac{x^4}{\cancel{4}} - {}^4\cancel{12}\frac{x^3}{\cancel{3}} - {}^2\cancel{4}\frac{x^2}{\cancel{2}} + 12x + c$

$\boxed{I = x^4 - 4x^3 - 2x^2 + 12x + c}$ Ans

(4) $\int x^2(x^2 - 4)\,dx$

Solution: let $I = \int x^2(x^2 - 4)\,dx$

$I = \int (x^4 - 4x^2)\,dx \Rightarrow I = \int x^4\,dx - \int 4x^2\,dx$

$I = \int x^4\,dx - 4\int x^2\,dx \Rightarrow \boxed{I = \frac{x^5}{5} - \frac{4}{3}x^3 + c}$ Ans

(5) $\int \sqrt{2x + 5}\,dx$

Solution: let $I = \int \sqrt{2x + 5}\,dx$

$I = \int (2x + 5)^{1/2}\,dx$

x & ÷ by 2

$I = \frac{1}{2}\int (2x + 5)^{1/2} . 2\,dx \Rightarrow I = \frac{1}{2}\frac{(2x + 5)^{3/2}}{\frac{3}{2}} + c$

$I = \frac{1}{\cancel{2}} \times \frac{\cancel{2}}{3}(2x + 5)^{3/2} + c \Rightarrow \boxed{I = \frac{1}{3}(2x + 5)^{3/2} + c}$ Ans

(6) $\int (2x + 3)^{2/3}\,dx$

Solution: let $I = \int (2x + 3)^{2/3}\,dx$

x & ÷ by 2.

$I = \frac{1}{2}\int (2x + 3)^{2/3} . 2\,dx \Rightarrow I = \frac{1}{2}\frac{(2x + 3)^{5/3}}{\frac{5}{3}} + c$

$\boxed{I = \frac{3}{10}(2x + 3)^{5/3} + c}$ Ans

(7) $\int (3x + 4)^{29}\,dx$

Solution: let $I = \int (3x + 4)^{29}\,dx$

x & ÷ by 3

$I = \frac{1}{3}\int (3x + 4)^{29} . 3\,dx \Rightarrow I = \frac{1}{3}\frac{(3x + 4)^{30}}{30} + c$

$\boxed{I = \frac{(3x + 4)^{30}}{90} + c}$ Ans

(8) $\int \frac{du}{u^2}$

Solution: let $I = \int \frac{du}{u^2}$

$I = \int u^{-2}\,du \Rightarrow I = \frac{u^{-1}}{-1} + c \Rightarrow \boxed{I = \frac{-1}{u} + c}$ Ans

(9) $\int \frac{6}{v^5}\,dv$

Solution: let $I = \int \frac{6}{v^5}\,dv.$

$I = \int 6v^{-5}\,du \Rightarrow I = 6\int v^{-5}\,du \Rightarrow I = {}^3\cancel{6}\frac{v^{-4}}{-\cancel{4}_2} + c$

$\boxed{I = \frac{-3}{2v^4} + c}$ Ans

(10) $\int \frac{dy}{\sqrt{ay+b}}$

Solution: let $I = \int \frac{dy}{\sqrt{ay+b}}$

$I = \int (ay+b)^{-1/2}\, dy$

x & ÷ by a

$I = \frac{1}{a}\int (ay+b)^{-1/2}.\, a\, dy \Rightarrow I = \frac{1}{a}\,\frac{(ay+b)^{1/2}}{1/2} + c$

$\boxed{I = \frac{2}{a}\sqrt{ay+b} + c}$ Ans

(11) $\int x(x^3+1)^2\, dx$

Solution: let $I = \int x(x^3+1)^2\, du$

$\boxed{\therefore (a+b)^2 = a^2 + 2ab + b^2}$

$I = \int x(x^6 + 2x^3 + 1)\, dx$

$I = \int (x^7 + 2x^4 + x)\, dx$

$I = \int x^7\, dx + 2\int x^4\, dx + \int x\, dx$

$\boxed{I = \frac{x^8}{8} + \frac{2}{5}x^5 + \frac{x^2}{2} + c}$ Ans

(12) $\int (x-1)(x-2)(x-3)\, dx$

Solution: let $I = \int (x-1)(x-2)(x-3)\, dx$

$I = \int (x-1)(x^2 - 3x - 2x + 6)\, dx$

$I = \int (x-1)(x^2 - 5x + 6)\, dx$

$I = \int (x^3 - 5x^2 + 6x - x^2 + 5x - 6)\, dx$

$I = \int (x^3 - 6x^2 + 11x - 6)\, dx$

$I = \int x^3\, dx - \int 6x^2\, dx + \int 11x\, dx - \int 6\, dx$

$I = \int x^3\, dx - 6\int x^2\, dx + 11\int x\, dx - 6\int dx$

$I = \frac{x^4}{4} - {}^{2}\cancel{6}\frac{x^3}{\cancel{3}} + 11\frac{x^2}{2} - 6x + c$

$\boxed{I = \frac{x^4}{4} - 2x^3 + \frac{11}{2}x^2 - 6x + c}$ Ans

(13) $\int (\sqrt{x} - 1)^2\, dx$

Solution: let $= \int (\sqrt{x} - 1)^2\, dx$

$\boxed{\therefore (a-b)^2 = a^2 - 2ab + b^2}$

$I = \int (x - 2\sqrt{x} + 1)\, dx \Rightarrow I = \int x\, dx - \int 2\sqrt{x}\, dx + \int dx$

$I = \int x\, dx - 2\int x^{1/2}\, dx + \int dx$

$I = \frac{x^2}{2} - 2\frac{x^{3/2}}{\frac{3}{2}} + x + c \Rightarrow I = \frac{x^2}{2} - 2\times\frac{2}{3}x^{3/2} + x + c$

$\boxed{I = \frac{x^2}{2} - \frac{4}{3}x^{3/2} + x + c}$ Ans

(14) $\int_0^2 \frac{dx}{\sqrt{1+x} + \sqrt{x}}$

Solution: let $I = \int_0^2 \frac{dx}{\sqrt{1+x} + \sqrt{x}} \times \frac{\sqrt{1+x} - \sqrt{x}}{\sqrt{1+x} - \sqrt{x}}$

$I = \int_0^2 \frac{(\sqrt{1+x} - \sqrt{x})\, dx}{(\sqrt{1+x} + \sqrt{x})(\sqrt{1+x} - \sqrt{x})}$

$\boxed{\therefore a^2 - b^2 = (a+b)(a-b)}$

$I = \int_0^2 \frac{(\sqrt{1+x} - \sqrt{x})\, dx}{(\sqrt{1+x})^2 - (\sqrt{x})^2} \Rightarrow I = \int_0^2 \frac{(\sqrt{1+x} - \sqrt{x})\, dx}{1 + \cancel{x} - \cancel{x}}$

$I = \int_0^2 (\sqrt{1+x} - \sqrt{x})\, dx \Rightarrow I = \int_0^2 (1+x)^{1/2}\, dx - \int_0^2 x^{1/2}\, dx$

$I = \left[\frac{(1+x)^{3/2}}{3/2}\right]_0^2 - \left[\frac{x^{3/2}}{3/2}\right]_0^2$

$I = \frac{2}{3}\left[(1+x)^{3/2}\right]_0^2 - \frac{2}{3}\left[x^{3/2}\right]_0^2$

$I = \frac{2}{3}\{(1+2)^{3/2} - (1+0)^{3/2}\} - \frac{2}{3}\{2^{3/2} - 0^{3/2}\}$

$I = \frac{2}{3}\{3^{3/2} - 1\} - \frac{2}{3}\{2^{3/2}\} \Rightarrow I = \frac{2}{3}\{3\sqrt{3} - 1\} - \frac{2}{3}\{2\sqrt{2}\}$

$\boxed{I = \frac{2}{3}(3\sqrt{3} - 2\sqrt{2} - 1)}$ Ans

(15) $\int \frac{x^2 + 3\sqrt{x} + 4}{3x^{\frac{3}{4}}}\, dx$

Solution: let $I = \int \left(\frac{x^2 + 3x^{1/2} + 4}{3x^{\frac{3}{4}}}\right) dx$

$I = \int \left\{\frac{x^2}{3x^{\frac{3}{4}}} + \frac{\cancel{3}x^{\frac{1}{2}}}{\cancel{3}x^{\frac{3}{4}}} + \frac{4}{3x^{\frac{3}{4}}}\right\} dx$

$$I = \int\left\{\frac{1}{3}x^2 \cdot x^{-3/4} + x^{\frac{1}{2}} \cdot x^{\frac{-3}{4}} + \frac{4}{3}\, x^{\frac{-3}{4}}\right\} dx$$

$$I = \int\left\{\frac{1}{3}x^{5/4} + x^{-1/4} + \frac{4}{3}x^{-3/4}\right\} dx$$

$$I = \frac{1}{3}\int x^{5/4}\, dx + \int x^{-1/4}\, dx + \frac{4}{3}\int x^{-3/4}\, dx$$

$$I = \frac{1}{3}\frac{x^{9/4}}{\frac{9}{4}} + \frac{x^{3/4}}{3/4} + \frac{4}{3}\,\frac{x^{1/4}}{1/4} + c$$

$$I = \frac{1}{3}\times\frac{4}{9}\, x^{9/4} + \frac{4}{3}x^{3/4} + \frac{4}{3}\times 4\, x^{1/4} + c$$

$$\boxed{I = \frac{4}{27}\, x^{9/4} + \frac{4}{3}x^{3/4} + \frac{16}{3}\, x^{1/4} + c} \quad \text{Ans}$$

(16) $\int \frac{x+8}{\sqrt{x}}\, dx$

Solution: let $I = \int \frac{x+8}{\sqrt{x}}\, dx$

$$I = \int\left(\frac{x+8}{x^{1/2}}\right)dx \Rightarrow I = \int\left(\frac{x}{x^{1/2}} + \frac{8}{x^{1/2}}\right)dx$$

$$I = \int(x\, x^{-1/2} + 8\, x^{-1/2})\, dx \Rightarrow I = \int(x^{1/2} + 8x^{-1/2})\, dx$$

$$I = \int x^{1/2}\, dx + 8\int x^{-1/2}\, dx \Rightarrow I = \frac{x^{3/2}}{3/2} + 8\,\frac{x^{1/2}}{1/2} + c$$

$$\boxed{I = \frac{2}{3}x^{3/2} + 16\sqrt{x} + c} \quad \text{Ans}$$

(17) $\int \frac{(\sqrt{\theta} - 1)^3}{\sqrt{\theta}}\, d\theta$

Solution: let $I = \int \frac{(\sqrt{\theta} - 1)^3}{\sqrt{\theta}}\, d\theta$

$$\boxed{\therefore (a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3}$$

$$I = \int\frac{(\theta^{3/2} - 3\theta + 3\theta^{1/2} - 1)}{\theta^{1/2}}\, d\theta$$

$$I = \int\left(\frac{\theta^{3/2}}{\theta^{1/2}} - \frac{3\theta}{\theta^{1/2}} + \frac{3\cancel{\theta^{1/2}}}{\cancel{\theta^{1/2}}} - \frac{1}{\theta^{1/2}}\right)d\theta$$

$$I = \int(\theta^{3/2}.\,\theta^{-1/2} - 3\theta.\theta^{-1/2} + 3 - \theta^{-1/2})\, d\theta$$

$$I = \int(\theta - 3\theta^{1/2} + 3 - \theta^{-1/2})\, d\theta$$

$$I = \int\theta\, d\theta - \int 3\theta^{1/2}\, d\theta + \int 3\, d\theta - \int\theta^{-1/2}\, d\theta$$

$$I = \int\theta\, d\theta - 3\int\theta^{1/2}\, d\theta + 3\int d\theta - \int\theta^{-1/2}\, d\theta$$

$$I = \frac{\theta^2}{2} - 3\,\frac{\theta^{3/2}}{3/2} + 3\theta - \frac{\theta^{1/2}}{1/2} + c$$

$$I = \frac{\theta^2}{2} - \cancel{3}\times\frac{2}{\cancel{3}}\,\theta^{3/2} + 3\theta - 2\sqrt{\theta} + c$$

$$\boxed{I = \frac{\theta^2}{2} - 2\theta^{3/2} + 3\theta - 2\sqrt{\theta} + c} \quad \text{Ans}$$

(18) $\int \frac{dx}{(2x+3)^{2/3}}$

Solution: let $I = \int\frac{dx}{(2x+3)^{2/3}}$

$$I = \int(2x+3)^{-2/3}\, dx$$

× & ÷ by 2

$$I = \frac{1}{2}\int(2x+3)^{-2/3}.\, 2dx \Rightarrow I = \frac{1}{2}\,\frac{(2x+3)^{1/3}}{1/3} + c$$

$$\boxed{I = \frac{3}{2}\,(2x+3)^{1/3} + c} \quad \text{Ans}$$

(19) $\int_{-1}^{1} (2x^2+4)^3\, (4x)\, dx$

Solution: let $I = \int_{-1}^{1}(2x^2+4)^3\,(4x)\, dx$

$$I = \left[\frac{(2x^2+4)^4}{4}\right]_{-1}^{1} \Rightarrow I = \frac{1}{4}\,\left[(2x^2+4)^4\right]_{-1}^{1}$$

$$I = \frac{1}{4}\left[\{2\,(1)^2 + 4\}^4 - \{2\,(-1)^2 + 4\}^4\right]$$

$$I = \frac{1}{4}\left[(6)^4 - (6)^4\right] \Rightarrow I = [\cancel{1296} - \cancel{1296}]$$

$$I = \frac{1}{4}\,(0) \Rightarrow \boxed{I = 0} \quad \text{Ans}$$

(20) $\int_{0}^{2} (x^2+bx+c)^{-2/3}\, (x + \frac{b}{2})\, dx$

Solution: let $I = \int_{0}^{2}(x^2+bx+c)^{-2/3}\,(x + \frac{b}{2})\, dx$

× & ÷ by 2

$$I = \frac{1}{2}\int_0^2 (x^2 + bx + c)^{-2/3}(2x + b)\,dx$$

$$I = \frac{1}{2}\left[\frac{(x^2 + bx + c)^{1/3}}{1/3}\right]_0^2 \Rightarrow I = \frac{3}{2}\left[(x^2 + bx + c)^{1/3}\right]_0^2$$

$$\boxed{I = \frac{3}{2}\left[(4 + 2b + c)^{1/3} - c^{1/3}\right]} \text{ Ans}$$

(21) $\displaystyle\int \frac{3x^2 + 2x + 1}{(x^3 + x^2 + x + 7)^{1/7}}\,dx$

Solution: let $I = \displaystyle\int \frac{3x^2 + 2x + 1}{(x^3 + x^2 + x + 7)^{1/7}}\,dx$

$$I = \int (x^3 + x^2 + x + 7)^{-1/7}(3x^2 + 2x + 1)\,dx$$

$$I = \frac{(x^3 + x^2 + x + 7)^{6/7}}{6/7} + c \Rightarrow \boxed{I = \frac{7}{6}(x^3 + x^2 + x + 7)^{6/7} + c} \text{ Ans}$$

Integrate the following with respect to their independent variable.

(22) $\displaystyle\int \left(\sqrt{\theta} - \frac{1}{\sqrt{\theta}}\right) d\theta$

Solution: let $I = \displaystyle\int \left(\sqrt{\theta} - \frac{1}{\sqrt{\theta}}\right) d\theta$

$$I = \int \sqrt{\theta}\,d\theta - \int \frac{1}{\sqrt{\theta}}\,d\theta \Rightarrow I = \int \theta^{1/2}\,d\theta - \int \theta^{-1/2}\,d\theta$$

$$I = \frac{\theta^{3/2}}{3/2} - \frac{\theta^{1/2}}{1/2} + c \Rightarrow \boxed{I = \frac{2}{3}\theta^{3/2} - 2\sqrt{\theta} + c} \text{ Ans}$$

(23) $\displaystyle\int (ax^2 + bx + c)^2\,dx$

Solution: let $I = \displaystyle\int (ax^2 + bx + c)^2\,dx$

$$\boxed{\therefore (a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ac}$$

$$I = \int (a^2x^4 + b^2x^2 + c^2 + 2abx^3 + 2bcx + 2acx^2)\,dx$$

$$I = \int \{a^2x^4 + 2abx^3 + (2ac + b^2)x^2 + 2bcx + c^2\}\,dx$$

$$I = a^2\int x^4\,dx + 2ab\int x^3\,dx + (2ac + b^2)\int x^2\,dx + 2bc\int x\,dx + c^2\int dx.$$

$$I = \frac{a^2x^5}{5} + \frac{\cancel{2}abx^4}{\cancel{4}_2} + (2ac + b^2)\frac{x^3}{3} + \cancel{2}bc\frac{x^2}{\cancel{2}} + c^2x + c$$

$$\boxed{I = \frac{1}{5}a^2x^5 + \frac{1}{2}abx^4 + \frac{1}{3}(b^2 + 2ac)x^3 + bcx^2 + c^2x + c} \text{ Ans}$$

(24) $\displaystyle\int \left(\sqrt{t} + \frac{1}{\sqrt{t}}\right)^2 dt$

Solution: let $I = \displaystyle\int \left(\sqrt{t} + \frac{1}{\sqrt{t}}\right)^2 dt$

$$\boxed{\therefore (a + b)^2 = a^2 + 2ab + b^2}$$

$$I = \int \left\{t + 2(\cancel{\sqrt{t}})\left(\frac{1}{\cancel{\sqrt{t}}}\right) + \frac{1}{t}\right\} dt$$

$$I = \int \left(t + 2 + \frac{1}{t}\right) dt \Rightarrow I = \int t\,dt + \int 2\,dt + \int \frac{1}{t}\,dt$$

$$I = \int t\,dt + 2\int dt + \int \frac{1}{t}\,dt \Rightarrow \boxed{I = \frac{t^2}{2} + 2t + \ell nt + c} \text{ Ans}$$

(25) $\displaystyle\int \left(\frac{1}{y^3} - y\right) dy$

Solution: let $I = \displaystyle\int \left(\frac{1}{y^3} - y\right) dy$

$$I = \int \frac{1}{y^3}\,dy - \int y\,dy \Rightarrow I = \int y^{-3}\,dy - \int y\,dy$$

$$I = \frac{y^{-2}}{-2} - \frac{y^2}{2} + c \Rightarrow \boxed{I = \frac{-1}{2y^2} - \frac{y^2}{2} + c} \text{ Ans}$$

(26) $\displaystyle\int x^3(x + 2)^3(x - 1)^2\,dx$

Solution: let $I = \displaystyle\int x^3(x + 2)^3(x - 1)^2\,dx$

$$\boxed{\begin{aligned}&\therefore (a - b)^2 = a^2 - 2ab + b^2\\ &\therefore (a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3\end{aligned}}$$

$$I = \int x^3(x^3 + 6x^2 + 12x + 8)(x^2 - 2x + 1)\,dx$$

$$I = \int (x^6 + 6x^5 + 12x^4 + 8x^3)(x^2 - 2x + 1)\,dx$$

$$I = \int (x^8 - 2x^7 + x^6 + 6x^7 - \cancel{12x^6} + 6x^5 + \cancel{12x^6} - 24x^5 + 12x^4 + 8x^5 - 16x^4 - 8x^3)\,dx$$

$$I = \int (x^8 + 4x^7 + x^6 - 10x^5 - 4x^4 + 8x^3)\,dx$$

$$I = \int x^8\,dx + 4\int x^7\,dx + \int x^6\,dx - 10\int x^5\,dx - 4\int x^4\,dx + 8\int x^3\,dx$$

$$I = \frac{x^9}{9} + \frac{\cancel{4}^{1}x^8}{\cancel{8}_2} + \frac{x^7}{7} - \frac{\cancel{10}^{5}x^6}{\cancel{6}_3} - 4\frac{x^5}{5} + \frac{\cancel{8}^{2}x^4}{\cancel{4}_1} + c$$

$$\boxed{I = \frac{x^9}{9} + \frac{x^8}{2} + \frac{x^7}{7} - \frac{5x^6}{3} - \frac{4x^5}{5} + 2x^4 + c} \text{ Ans}$$

(27) $\int_{-1}^{1}(x^4+2x^3+3x^2+4x+5)^{2/5}(4+6x+6x^2+4x^3)\,dx$

Solution: let $I=\int_{-1}^{1}(x^4+2x^3+3x^2+4x+5)^{2/5}(4x^3+6x^2+6x+4)\,dx$

$$I=\left[\frac{(x^4+2x^3+3x^2+4x+5)^{7/5}}{7/5}\right]_{-1}^{1}$$

$$I=\frac{5}{7}\left[(x^4+2x^3+3x^2+4x+5)^{7/5}\right]_{-1}^{1}$$

$$I=\frac{5}{7}\left[\{(1)^4+2(1)^3+3(1)^2+4(1)+5\}^{7/5}-\{(-1)^4+2(-1)^3+3(-1)^2+4(-1)+5\}^{7/5}\right]$$

$$I=\frac{5}{7}\{(1+2+3+4+5)^{7/5}-(1-2+3-4+5)^{7/5})\}$$

$\boxed{I=\frac{5}{7}\{(15)^{7/5}-(3)^{7/5}\}}$ Ans

(28) $\int_{-1}^{1}\frac{3x^2+1}{(x^3+x+6)^{1/2}}\,dx$

Solution: let $I=\int\int_{-1}^{1}\frac{3x^2+1}{(x^3+x+6)^{1/2}}\,dx$

$$I=\int_{-1}^{1}(x^3+x+6)^{-1/2}.(3x^2+1)\,dx$$

$$I=\left[\frac{(x^3+x+6)^{1/2}}{1/2}\right]_{-1}^{1}\Rightarrow I=2\left[(x^3+x+6)^{1/2}\right]_{-1}^{1}$$

$$I=2\left[\{(1)^3+(1)+6\}^{1/2}-\{(-1)^3+(-1)+6\}^{1/2}\right]$$

$$I=2[8^{1/2}-4^{1/2}]=2\{2\sqrt{2}-2\}$$

$\boxed{I=4(\sqrt{2}-1)}$ Ans

EXERCISE # 6.2

Evaluate the following integrals.

(1) $\int\frac{dx}{x^2-4}$

Solution: let $I=\int\frac{dx}{x^2-4}\,dx$

$$I=\int\frac{1}{(x)^2-(2)^2}\,dx$$

$\boxed{\text{Using formula } \therefore \int\frac{1}{x^2-a^2}\,dx=\frac{1}{2a}\ln\left(\frac{x-a}{x+a}\right)+c}$

$$I=\frac{1}{2(2)}\ln\left(\frac{x-2}{x+2}\right)+c$$

$\boxed{I=\frac{1}{4}\ln\left(\frac{x-2}{x+2}\right)+c}$ Ans

(2) $\int\frac{dy}{1-y^2}$

Solution: let $I=\int\frac{1}{1-y^2}\,dy$

$$I=\int\frac{1}{(1)^2-(y)^2}\,dy$$

$\boxed{\text{Using formula } \therefore \int\frac{1}{a^2-x^2}\,dx=\frac{1}{2a}\ln\left(\frac{a+x}{a-x}\right)+c}$

$I=\frac{1}{2(1)}\ln\left(\frac{1+y}{1-y}\right)+c\Rightarrow\boxed{I=\frac{1}{2}\ln\left(\frac{1+y}{1-y}\right)+c}$ Ans

(3) $\int\frac{dx}{4x^2-1}$

Solution: let $I=\int\frac{dx}{4x^2-1}\,dx$

$$I=\int\frac{1}{4(x^2-\frac{1}{4})}\,dx\Rightarrow I=\frac{1}{4}\int\frac{1}{(x)^2-\left(\frac{1}{2}\right)^2}\,dx$$

$\boxed{\text{Using formula } \therefore \int\frac{1}{x^2-a^2}\,dx=\frac{1}{2a}\ln\left(\frac{x-a}{x+a}\right)+c}$

$$I=\frac{1}{4}\left\{\frac{1}{\cancel{2}\left(\frac{1}{\cancel{2}}\right)}\ln\frac{1}{2}\left(\frac{x-\frac{1}{2}}{x+\frac{1}{2}}\right)\right\}+c$$

$I=\frac{1}{4}\ln\left(\frac{\frac{2x-1}{\cancel{2}}}{\frac{2x+1}{\cancel{2}}}\right)+c\Rightarrow\boxed{I=\frac{1}{4}\ln\left(\frac{2x-1}{2x+1}\right)+c}$ Ans

(4) $\int_0^2 \frac{dx}{9-x^2}$

Solution: let $I=\int_0^2 \frac{dx}{9-x^2}\,dx$

$$I=\int_0^2 \frac{1}{(3)^2-(x)^2}\,dx$$

$$\boxed{\text{Using formula } \therefore \int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\ln\left(\frac{a+x}{a-x}\right)+c}$$

$$I=\left[\frac{1}{2(3)}\ln\left(\frac{3+x}{3-x}\right)\right]_0^2$$

$$I=\frac{1}{6}\left\{\ln\left(\frac{3+2}{3-2}\right)-\ln\left(\frac{3+0}{3-0}\right)\right\}$$

$$I=\frac{1}{6}\left\{\ln\left(\frac{5}{1}\right)-\ln\left(\frac{\cancel{3}}{\cancel{3}}\right)\right\} \Rightarrow I=\frac{1}{6}\{\ln 5-\ln 1\}$$

$$I=\frac{1}{6}\{\ln 5-0\}$$

Note: $\boxed{\therefore \ln 1=0}$

$\boxed{I=\frac{1}{6}\ln 5}$ Ans

(5) $\int\frac{du}{\sqrt{u^2+9}}$

Solution: let $I=\int \frac{du}{\sqrt{u^2+9}}\,du.$

$$I=\int\frac{1}{\sqrt{(u)^2+(3)^2}}\,du.$$

$$\boxed{\text{Using formula } \therefore \frac{1}{\sqrt{x^2+a^2}}\,dx=\ln\left(x+\sqrt{x^2+a^2}\right)+c}$$

$\boxed{I=\ln\left(u+\sqrt{u^2+9}\right)+c}$ Ans

(6) $\int\frac{dy}{\sqrt{y^2-1}}$

Solution: let $I=\int\frac{dy}{\sqrt{y^2-1}}\,dy$

$$I=\int\frac{1}{\sqrt{(y)^2-(1)^2}}\,dy$$

$$\boxed{\text{Using formula } \therefore \int\frac{1}{\sqrt{x^2-a^2}}\,dx=\ln\left(x+\sqrt{x^2-a^2}\right)+c}$$

$\boxed{I=\ln\left(y+\sqrt{y^2-1}\right)+c}$ Ans

(7) $\int\sqrt{25-x^2}\,dx$

Solution: let $I=\int\sqrt{25-x^2}\,dx$

$$I=\int\sqrt{(5)^2-(x)^2}\,dx$$

$$\boxed{\text{Using formula } \therefore \int\sqrt{a^2-x^2}\,dx=\frac{1}{2}x\sqrt{a^2-x^2}+\frac{1}{2}a^2\sin^{-1}\frac{x}{a}+c}$$

$$I=\frac{1}{2}x\sqrt{25-x^2}+\frac{1}{2}(5)^2\sin^{-1}\frac{x}{5}+c$$

$\boxed{I=\frac{x}{2}\sqrt{25-x^2}+\frac{25}{2}\sin^{-1}\frac{x}{5}+c}$ Ans

(8) $\int\sqrt{v^2-36}\,dv.$

Solution: let $I=\int\sqrt{v^2-36}\,dv.$

$$I=\int\sqrt{(v)^2-(6)^2}\,dv$$

$$\boxed{\text{Using formula } \therefore \int\sqrt{x^2-a^2}\,dx=\frac{1}{2}x\sqrt{x^2-a^2}-\frac{1}{2}a^2\ln\left(x+\sqrt{x^2-a^2}\right)+c}$$

$$I=\frac{1}{2}v\sqrt{v^2-36}-\frac{1}{2}(6)^2\ln\left(v+\sqrt{v^2-36}\right)+c$$

$$I=\frac{v\sqrt{v^2-36}}{2}-\frac{\cancel{36}^{18}}{\cancel{2}}\ln\left(v+\sqrt{v^2-36}\right)+c$$

$\boxed{I=\frac{v\sqrt{v^2-36}}{2}-18\ln\left(v+\sqrt{v^2-36}\right)+c}$ Ans

(9) $\int\frac{dx}{x\sqrt{4x^2-9}}$

Solution: let $=\int\frac{dx}{x\sqrt{4x^2-9}}$

$$I=\int\frac{1}{x\sqrt{4\left(x^2-\frac{9}{4}\right)}}\,dx \Rightarrow I=\frac{1}{2}\int\frac{1}{x\sqrt{(x)^2-\left(\frac{3}{2}\right)^2}}\,dx$$

$$\boxed{\text{Using formula } \therefore \frac{1}{x\sqrt{x^2-a^2}}\,dx=\frac{1}{a}\sec^{-1}\frac{x}{a}+c}$$

$$I = \frac{1}{2}\left\{\frac{1}{\frac{3}{2}}\ \text{Sec}^{-1}\frac{x}{\frac{3}{2}}\right\} + c \Rightarrow I = \frac{1}{\cancel{2}}\ \frac{\cancel{2}}{3}\ \text{Sec}^{-1}\frac{2x}{3} + c$$

$$\boxed{I = \frac{1}{3}\text{Sec}^{-1}\left(\frac{2x}{3}\right) + c}\ \text{Ans}$$

(10) $\int \frac{dy}{25 - 16y^2}$

Solution: let $I = \int \frac{dy}{25 - 16y^2}\ dy$

$$I = \int \frac{1}{16\left(\frac{25}{16} - y^2\right)} dy. \Rightarrow I = \frac{1}{16}\int \frac{1}{\left(\frac{5}{4}\right)^2 - (y)^2} dy.$$

$$\boxed{\text{Using formula } \therefore \int \frac{1}{a^2 - x^2}\ dx = \frac{1}{2a}\ \ell n\left(\frac{a + x}{a - x}\right) + c}$$

$$I = \frac{1}{16}\left\{\frac{1}{\cancel{2}\left(\frac{5}{\cancel{4}_2}\right)}\ \ell n\left(\frac{\frac{5}{4} + y}{\frac{5}{4} - y}\right)\right\} + c$$

$$I = \frac{1}{16}\left\{\frac{2}{5}\ell n\left(\frac{\frac{5 + 4y}{\cancel{4}}}{\frac{5 - 4y}{\cancel{4}}}\right)\right\} + c$$

$$I = \frac{1}{\cancel{16}_8} \times \frac{\cancel{2}}{5}\ \ell n\left(\frac{5 + 4y}{5 - 4y}\right) + c \Rightarrow \boxed{I = \frac{1}{40}\ \ell n\left(\frac{5 + 4y}{5 - 4y}\right) + c}\ \text{Ans}$$

(11) $\int \frac{dx}{\sqrt{25 - 16x^2}}$

Solution: let $I = \int \frac{dx}{\sqrt{25 - 16x^2}}$

$$I = \int \frac{1}{\sqrt{16\left(\frac{25}{16} - x^2\right)}} dx \Rightarrow I = \frac{1}{4}\int \frac{1}{\sqrt{\left(\frac{5}{4}\right)^2 - (x)^2}}\ dx$$

$$\boxed{\text{Using formula } \therefore \int \frac{1}{\sqrt{a^2 - x^2}}\ dx = \text{Sin}^{-1}\frac{x}{a} + c}$$

$$I = \frac{1}{4}\left(\text{Sin}^{-1}\frac{x}{\frac{5}{4}}\right) + c \Rightarrow \boxed{I = \frac{1}{4}\ \text{Sin}^{-1}\left(\frac{4x}{5}\right) + c}\ \text{Ans}$$

(12) $\int \frac{dx}{4x^2 + 9}$

Solution: let $I = \int \frac{dx}{4x^2 + 9}$

$$I = \int = \frac{1}{4\left(x^2 + \frac{9}{4}\right)}\ dx \Rightarrow I = \frac{1}{4}\int \frac{1}{(x)^2 + \left(\frac{3}{2}\right)^2}\ dx$$

$$\boxed{\text{Using formula } \therefore \int \frac{1}{x^2 + a^2}\ dx = \frac{1}{a}\ \tan^{-1}\frac{x}{a} + c}$$

$$I = \frac{1}{4}\left\{\frac{1}{\frac{3}{2}}\ \tan^{-1}\frac{x}{\frac{3}{2}}\right\} + c \Rightarrow I = \frac{1}{4}\ \left\{\frac{2}{3}\tan^{-1}\frac{2x}{3}\right\} + c$$

$$I = \frac{1}{\cancel{4}_2} \times \frac{\cancel{2}}{3}\tan^{-1}\frac{2x}{3} + c \Rightarrow \boxed{I = \frac{1}{6}\ \tan^{-1}\left(\frac{2x}{3}\right) + c}\ \text{Ans}$$

(13) $\int \frac{dx}{\sqrt{4x^2 + 9}}$

Solution: let $I = \int \frac{dx}{\sqrt{4x^2 + 9}}$

$$I = \int \frac{1}{\sqrt{4\left(x^2 + \frac{9}{4}\right)}}\ dx \Rightarrow I = \frac{1}{2}\int \frac{1}{\sqrt{(x)^2 + \left(\frac{3}{2}\right)^2}}\ dx$$

$$\boxed{\text{Using formula } \therefore \int \frac{1}{\sqrt{x^2 + a^2}}\ dx = \ln\left(x + \sqrt{x^2 + a^2}\right) + c}$$

$$I = \frac{1}{2}\ell n\left(x + \sqrt{x^2 + \frac{9}{4}}\right) + c \Rightarrow I = \frac{1}{2}\ell n\left(x + \sqrt{\frac{4x^2 + 9}{4}}\right) + c$$

$$I = \frac{1}{2}\ell n\left(x + \frac{\sqrt{4x^2 + 9}}{2}\right) + c \Rightarrow I = \frac{1}{2}\ \ell n\left(\frac{2x + \sqrt{4x^2 + 9}}{2}\right) + c$$

$$I = \frac{1}{2}\ \left\{\ell n\left(2x + \sqrt{4x^2 + 9}\right) - \ell n 2\right\} + c$$

$$I = \frac{1}{2}\ell n\left(2x + \sqrt{4x^2 + 9}\right) + c - \frac{\ell n 2}{2}$$

$$\boxed{\text{Since } c - \frac{\ln 2}{2} = \text{constant} = c}$$

$50 - 0.5 = C$

$$\boxed{I = \frac{1}{2}\ \ell n\left(2x + \sqrt{4x^2 + 9}\right) + c}\ \text{Ans}$$

(14) $\int \frac{dz}{\sqrt{9z^2-25}}$

Solution: let $I = \int \frac{dz}{\sqrt{9z^2-25}}\, dz = \int \frac{1}{\sqrt{9\left(z^2-\frac{25}{9}\right)}}\, dz$

$$I = \frac{1}{3}\int \frac{1}{\sqrt{(z)^2-\left(\frac{5}{3}\right)^2}}\, dz$$

$\boxed{\text{Using formula } \therefore \int \frac{1}{\sqrt{x^2-a^2}}\, dx = \ln\left(x+\sqrt{x^2-a^2}\right)+c}$

$$I = \frac{1}{3}\ln\left(z+\sqrt{z^2-\frac{25}{9}}\right)+c$$

$$I = \frac{1}{3}\ln\left(z+\sqrt{\frac{9z^2-25}{9}}\right)+c$$

$$I = \frac{1}{3}\ln\left(z+\frac{\sqrt{9z^2-25}}{3}\right)+c$$

$$I = \frac{1}{3}\ln\left(\frac{3z+\sqrt{9z^2-25}}{3}\right)+c$$

$$I = \frac{1}{3}\left\{\ln\left(3z+\sqrt{9z^2-25}\right)-\ln 3\right\}+c$$

$$I = \frac{1}{3}\ln\left(3z+\left(3z+\sqrt{9z^2-25}\right)+c-\frac{\ln 3}{3}\right.$$

$\boxed{\text{Since } c-\frac{\ln 3}{3} = \text{constant} = c}$

$\boxed{I = \frac{1}{3}\ln\left(3z+\sqrt{9z^2-25}\right)+c}$ Ans

(15) $\int \sqrt{3-4x^2}\, dx.$

Solution: let $I = \int \sqrt{3-4x^2}\, dx$

$$I = \int \sqrt{4\left(\frac{3}{4}-x^2\right)}\, dx$$

$$I = 2\int \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2-(x)^2}\, dx$$

$\boxed{\text{Using formula } \therefore \int \sqrt{a^2-x^2}\, dx = \frac{1}{2}x\sqrt{a^2-x^2}+\frac{1}{2}a^2\,\text{Sin}^{-1}\frac{x}{a}+c}$

$$= 2\left\{\frac{1}{2}x\sqrt{\frac{3}{4}-x^2}+\frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)^2\text{Sin}^{-1}\left(\frac{x}{\frac{\sqrt{3}}{2}}\right)\right\}+c$$

$$I = 2\left\{\frac{x}{2}\sqrt{\frac{3-4x^2}{4}}+\frac{3}{8}\,\text{Sin}^{-1}\left(\frac{2x}{\sqrt{3}}\right)\right\}+c.$$

$$I = 2\left\{\frac{x}{4}\sqrt{3-4x^2}+\frac{3}{8}\,\text{Sin}^{-1}\left(\frac{2x}{\sqrt{3}}\right)\right\}+c$$

$$I = \cancel{2}\times\frac{x}{\cancel{4}_2}\sqrt{3-4x^2}+\cancel{2}\times\frac{3}{\cancel{8}_2}\,\text{Sin}^{-1}\left(\frac{2x}{\sqrt{3}}\right)+c$$

$\boxed{I = \frac{x}{2}\sqrt{3-4x^2}+\frac{3}{4}\,\text{Sin}^{-1}\left(\frac{2x}{\sqrt{3}}\right)+c}$ Ans.

(16) $\int \sqrt{16-9x^2}\, dx$

Solution: let $I = \int \sqrt{16-9x^2}\, dx$

$$I = \int \sqrt{9\left(\frac{16}{9}-x^2\right)}\, dx \Rightarrow I = 3\int \sqrt{\left(\frac{4}{3}\right)^2-(x)^2}\, dx$$

$\boxed{\text{Using formula } \therefore \int \sqrt{a^2-x^2}\, dx = \frac{1}{2}x\sqrt{a^2-x^2}+\frac{1}{2}a^2\,\text{Sin}^{-1}\frac{x}{a}+c}$

$$I = 3\left\{\frac{1}{2}x\sqrt{\frac{16}{9}-x^2}+\frac{1}{2}\left(\frac{4}{3}\right)^2\text{Sin}^{-1}\frac{x}{\frac{4}{3}}\right\}+c$$

$$I = 3\left\{\frac{x\sqrt{16-9x^2}}{6}+\frac{16}{18}\,\text{Sin}^{-1}\frac{3x}{4}\right\}+c$$

$$I = \cancel{3}\,\frac{x\sqrt{16-9x^2}}{\cancel{6}_2}+\cancel{3}\times\frac{\overset{8}{\cancel{16}}}{\underset{3}{\cancel{18}}}\,\text{Sin}^{-1}\frac{3x}{4}+c$$

$\boxed{I = \frac{x\sqrt{16-9x^2}}{2}+\frac{8}{3}\,\text{Sin}^{-1}\left(\frac{3x}{4}\right)+c}$

EXERCISE # 6.3

Evaluate

(i) $\int 4x^3(x^4+1)^{3/2}\, dx$

Solution: let $I = \int 4x^3(x^4+1)^{3/2}\, dx$

$$I = \frac{(x^4+1)^{5/2}}{5/2}+c \Rightarrow \boxed{I = \frac{2}{5}(x^4+1)^{5/2}+c}$$ Ans

(ii) $\int \frac{8x^2}{(x^3+2)^3}\,dx$

Solution: let $I = \int \frac{8x^2}{(x^3+2)^3}\,dx$

$I = \int (x^3+2)^{-3}\, 8x^2\,dx \Rightarrow I = 8\int (x^3+2)^{-3}\, x^2\,dx.$

$\times \,\&\, \div$ by 3

$I = \frac{8}{3}\int (x^3+2)^{-3}\, 3x^2\,dx \Rightarrow I = \frac{\cancel{8}^{\,4}}{3}\,\frac{(x^3+2)^{-2}}{-\cancel{2}} + c$

$\boxed{I = \frac{-4}{3}(x^3+2)^{-2} + c}$ Ans

(iii) $\int 3x\sqrt{1-2x^2}\,dx$

Solution: let $I = \int (1-2x^2)^{1/2}\, 3x\,dx$

$I = 3\int (1-2x^2)^{1/2}\, x\,dx.$

$\times \,\&\, \div$ by (-4)

$I = \frac{-3}{4}\int (1-2x^2)^{1/2}(-4x)\,dx \Rightarrow I = \frac{-3}{4}\,\frac{(1-2x^2)^{3/2}}{3/2} + c$

$I = -\frac{\cancel{3}}{\cancel{4}_2}\times\frac{\cancel{2}}{\cancel{3}}(1-2x^2)^{3/2} + c \Rightarrow \boxed{I = \frac{-1}{2}(1-2x^2)^{3/2} + c}$

(iv) $\int \frac{y+3}{(y^2+6y)^{1/2}}\,dy.$

Solution: let $I = \int \frac{y+3}{(y^2+6y)^{1/2}}\,dy.$

$I = \int (y^2+6y)^{-1/2}.(y+3)\,dy.$

$\times \,\&\, \div$ by 2.

$I = \frac{1}{2}\int (y^2+6y)^{-1/2}(2y+6)\,dy. \Rightarrow I = \frac{1}{2}\,\frac{(y^2+6y)^{1/2}}{1/2} + c$

$I = \frac{1}{\cancel{2}}\times\cancel{2}\sqrt{y^2+6y} + c \Rightarrow \boxed{I = \sqrt{y^2+6y} + c}$ Ans

(v) $\int \sqrt[3]{1-x^2}\; x\,dx$

Solution: let $I = \int \sqrt[3]{1-x^2}\; x\,dx$

$I = \int (1-x^2)^{1/3}\, x\,dx$

$\times \,\&\, \div$ by (-2)

$I = \frac{-1}{2}\int (1-x^2)^{1/3}.(-2x)\,dx \Rightarrow I = \frac{-1}{2}\,\frac{(1-x^2)^{4/3}}{4/3} + c$

$I = \frac{-1}{2}\times\frac{3}{4}(1-x^2)^{4/3} + c \Rightarrow \boxed{I = \frac{-3}{8}(1-x^2)^{4/3} + c}$

(vi) $\int \sqrt{x^2-24x^4}\,dx$

Solution: let $I = \int \sqrt{x^2-24x^4}\,dx$

$I = \int \sqrt{x^2(1-24x^2)}\,dx \Rightarrow I = \int \sqrt{1-24x^2}\; x\,dx$

$I = \int (1-24x^2)^{1/2}.\, x\,dx$

$\times \,\&\, \div$ by (-48)

$I = \frac{-1}{48}\int (1-24x^2)^{1/2}.(-48x)\,dx \Rightarrow I = \frac{-1}{48}\,\frac{(1-24x^2)^{3/2}}{3/2} + c$

$I = \frac{-1}{\cancel{48}}\times\frac{\cancel{2}}{3}(1-24x^2)^{3/2} + c \Rightarrow \boxed{I = \frac{-1}{72}(1-24x^2)^{3/2} + c}$ Ans

(vii) $\int \frac{x\,dx}{3x^2-4}$

Solution: let $I = \int \frac{x\,dx}{3x^2-4}$

$\times \,\&\, \div$ by 6

$I = \frac{1}{6}\int \frac{6x}{3x^2-4}\,dx \Rightarrow \boxed{I = \frac{1}{6}\,\ell n\,(3x^2-4) + c}$ Ans

(viii) $\int \frac{x^2\,dx}{(1-2x^3)^{2/3}}$

Solution: let $I = \int \frac{x^2\,dx}{(1-2x^3)^{2/3}}$

$I = \int (1-2x^3)^{-2/3}\, x^2\,dx$

$\times \,\&\, \div$ by (-6)

$I = \frac{-1}{6}\int (1-2x^3)^{-2/3}.(-6x^2)\,dx.$

$I = \frac{-1}{6}\,\frac{(1-2x^3)^{1/3}}{1/3} + c \Rightarrow I = \frac{-1}{\cancel{6}}\times\cancel{3}(1-2x^3)^{1/3} + c$

$\boxed{I = \frac{-1}{2}(1-2x^3)^{1/3} + c}$ Ans

(ix) $\int \frac{x+2}{x-3}\,dx.$

Solution: let $I = \int \frac{x+2}{x-3}\,dx.$

This question belongs to special case # 01 make N (x) as D (x).

$I = \int \frac{x-3+5}{x-3}\,dx. \Rightarrow I = \int\left(\frac{\cancel{(x-3)}}{\cancel{(x-3)}} + \frac{5}{x-3}\right)dx.$

$I = \int dx + 5\int \frac{1}{x-3}\,dx \Rightarrow \boxed{I = x + 5\,\ell n\,(x-3) + c}$ Ans

(x) $\int \frac{(1+x)(1-2x)}{1-2x^2}dx$

Solution: let $I = \int \frac{(1+x)(1-2x)}{1-2x^2}dx$

$I = \int \frac{1-2x+x-2x^2}{1-2x^2}dx \Rightarrow I = \int \frac{1-2x^2-x}{1-2x^2}dx$

$I = \int \left(\frac{\cancel{(1-2x^2)}}{\cancel{(1-2x^2)}} + \frac{x}{1-2x^2}\right)dx \Rightarrow I = \int\left(1 - \frac{x}{1-2x^2}\right)dx$

$I = \int dx - \int \frac{x}{1-2x^2}dx$

$\times$ & $\div$ by (-4) on 2^{nd} integral.

$I = \int dx + \frac{1}{4}\int \frac{-4x}{1-2x^2}dx \Rightarrow \boxed{I = x + \frac{1}{4}\ \ell n\,(1-2x^2)+c}$ Ans

(xi) $\int \frac{x}{\sqrt{1+x}}dx$

Solution: let $I = \int \frac{x}{\sqrt{1+x}}dx$

This question belongs to special case # 01 make N (x) as D (x).

$I = \int \frac{1+x-1}{(1+x)^{1/2}}dx$

$I = \int\left(\frac{1+x}{(1+x)^{1/2}} - \frac{1}{(1+x)^{1/2}}\right)dx$

$I = \int \{(1+x)(1+x)^{-1/2} - (1+x)^{-1/2}\}\,dx$

$I = \int \{(1+x)^{1/2} - (1+x)^{-1/2}\}\,dx$

$I = \int (1+x)^{1/2}dx - \int (1+x)^{-1/2}dx$

$I = \frac{(1+x)^{3/2}}{3/2} - \frac{(1+x)^{1/2}}{1/2} + c$

$\boxed{I = \frac{2}{3}(1+x)^{3/2} - 2\sqrt{1+x} + c}$ Ans

(xii) $\int x^2\sqrt{4+x}\ dx$

Solution: let $I = \int \sqrt{4+x}\ x^2\,dx$ ——(1)

This question belongs to special case # 02 break the derivative power.

let $t = 4+x \Rightarrow t-4 = x \Rightarrow \boxed{x^2 = (t-4)^2}$

differentiate w.r to x

$\frac{dt}{dx} = 1 \Rightarrow \boxed{dt = dx}$

$(1) \Rightarrow I = \int t^{1/2}\,(t-4)^2\,dt$

$\boxed{\therefore (a-b)^2 = a^2 - 2ab + b^2}$

$I = \int t^{1/2}(t^2 - 8t + 16)\,dt \Rightarrow I = \int (t^{5/2} - 8\,t^{3/2} + 16t^{1/2})\,dt$

$I = \int t^{5/2}\,dt - 8\int t^{3/2}\,dt + 16\int t^{1/2}\,dt$

$I = \frac{t^{7/2}}{7/2} - 8\frac{t^{5/2}}{5/2} + 16\frac{t^{3/2}}{3/2} + c$

$I = \frac{2}{7}t^{7/2} - 8\times\frac{2}{5}t^{5/2} + 16\times\frac{2}{3}t^{3/2} + c$ but $t = 4+x$

$\boxed{I = \frac{2}{7}(4+x)^{7/2} - \frac{16}{5}(4+x)^{5/2} + \frac{32}{3}(4+x)^{3/2} + c}$ Ans

(xiii) $\int \frac{3x+2}{\sqrt{x-1}}dx$

Solution: let $I = \int \frac{3x+2}{\sqrt{x-1}}dx$

This question belongs to special case # 01 make N (x) as D (x).

$I = \int \frac{3x-3+5}{(x-1)^{1/2}}dx.$

$I = \int\left\{\frac{3\cancel{(x-1)}}{\cancel{(x-1)}^{1/2}} + \frac{5}{(x-1)^{1/2}}\right\}dx$

$I = \int \{3\,(x-1)^{1/2} + 5\,(x-1)^{-1/2}\}\,dx$

$I = 3\int (x-1)^{1/2}dx + 5\int (x-1)^{-1/2}dx$

$I = 3\frac{(x-1)^{3/2}}{3/2} + 5\frac{(x-1)^{1/2}}{1/2} + c$

$I = \cancel{3}\times\frac{2}{\cancel{3}}(x-1)^{3/2} + 5\times 2\sqrt{x-1} + c$

$\boxed{I = 2\,(x-1)^{3/2} + 10\sqrt{x-1} + c}$ Ans

(xiv) $\int (2x^2-3)^{4/3}\,x^3\,dx$

Solution: let $I = \int (2x^2-3)^{4/3}\,x^2 \cdot x\,dx$ ——(1)

This question belongs to special case # 02 break derivative power.

let $t = 2x^2 - 3 \Rightarrow t+3 = 2x^2$

$\boxed{x^2 = \frac{t+3}{2}}$

differentiate w.r to x.

$\frac{dt}{dx} = 4x. \Rightarrow \boxed{\frac{dt}{4} = xdx.}$

$$(1) \Rightarrow I = \int t^{4/3}.\left(\frac{t+3}{2}\right)\frac{dt}{4} \Rightarrow I = \frac{1}{8}\int t^{4/3}(t+3)\,dt$$

$$I = \frac{1}{8}\int (t^{7/3} + 3t^{4/3})\,dt \Rightarrow I = \frac{1}{8}\int t^{7/3}\,dt + \frac{3}{8}\int t^{4/3}\,dx$$

$$I = \frac{1}{8}\frac{t^{10/3}}{10/3} + \frac{3}{8}\frac{t^{7/3}}{7/3} + c$$

but $t = 2x^2 - 3$

$$I = \frac{1}{8}\times\frac{3}{10}(2x^2-3)^{10/3} + \frac{3}{8}\times\frac{3}{7}(2x^2-3)^{7/3} + c$$

$$\boxed{I = \frac{3}{80}(2x^2-3)^{10/3} + \frac{9}{56}(2x^2-3)^{7/3} + c}$$

(xv) $\int (x^3+1)^{7/5}\cdot x^5\cdot dx$

Solution: let $I = \int (x^3+1)^{7/5}\cdot x^3\cdot x^2\,dx$ ———— (1)

This question belongs to special case # 02 break derivative power.

let $t = x^3 + 1 \Rightarrow \boxed{x^3 = t-1}$

differentiate w.r to x

$$\frac{dt}{dx} = 3x^2 \Rightarrow \boxed{\frac{dt}{3} = x^2\,dx}$$

$$(1) \Rightarrow I = \int t^{7/5}(t-1)\frac{dt}{3} \Rightarrow I = \frac{1}{3}\int t^{7/5}(t-1)\,dt$$

$$I = \frac{1}{3}\int (t^{12/5} - t^{7/5})\,dt \Rightarrow I = \frac{1}{3}\int t^{12/5}\,dt - \frac{1}{3}\int t^{7/5}\,dt$$

$$I = \frac{1}{3}\frac{t^{17/5}}{17/5} - \frac{1}{3}\frac{t^{12/5}}{12/5} + c$$

but $t = x^3 + 1$

$$I = \frac{1}{3}\times\frac{5}{17}(x^3+1)^{17/5} - \frac{1}{3}\times\frac{5}{12}(x^3+1)^{12/5} + c$$

$$\boxed{I = \frac{5}{51}(x^3+1)^{17/5} - \frac{5}{36}(x^3+1)^{12/5} + c}$$

(xvi) $\int (x^2 - 2x + 1)^{4/3}\,dx$

Solution: let $I = \int (x^2 - 2x + 1)^{4/3}\,dx$

$$I = \int \{(x-1)^2\}^{4/3}\,dx \Rightarrow I = \int (x-1)^{8/3}\,dx$$

$$I = \frac{(x-1)^{11/3}}{11/3} + c \Rightarrow \boxed{I = \frac{3}{11}(x-1)^{11/3} + c}$$ Ans

Q2. Obtain

(i) $\int e^{2-3x}\,dx$

Solution: let $I = \int e^{2-3x}\,dx$

× & ÷ by (−3)

$$I = \frac{-1}{3}\int e^{2-3x}(-3)\,dx \Rightarrow \boxed{I = \frac{-1}{3}e^{2-3x} + c}$$ Ans.

(ii) $\int x\,e^{3x^2+2}\,dx$

Solution: let $I = \int\int x\,e^{3x^2+2}\,x\,dx$

× & ÷ by 6

$$I = \frac{1}{6}\int e^{3x^2+2}\cdot(6x)\,dx \Rightarrow \boxed{I = \frac{1}{6}e^{3x^2+2} + c}$$ Ans

(iii) $\int a^{2y}\,dy.$

Solution: let $I = \int a^{2y}\,dy.$

$$\boxed{\text{Using formula} \therefore \int a^x\,dx = \frac{a^x}{\ln a} + c} \Rightarrow \boxed{I = \frac{a^{2y}}{2\ln 2y} + c}$$ Ans.

(iv) $\int \frac{e^{1/u}}{u^2}\,du.$

Solution: let $I = \int e^{1/u}\frac{du}{u^2}$ ———— (1)

× & ÷ by (−1)

$$I = -\int e^{1/u}\left(\frac{-1}{u^2}\right)du \Rightarrow \boxed{I = -e^{\frac{1}{u}} + c}$$ Ans

(v) $\int \frac{x^2\,dx}{e^{2x^3+3}}$

Solution: let $I = \int \frac{x^2\,dx}{e^{2x^3+3}}$ ———— (1)

let $t = 2x^3 + 3$

differentiate w.r to x

$$\frac{dt}{dx} = 6x^2 \Rightarrow \boxed{\frac{dt}{6} = x^2\,dx}$$

$$(1) \Rightarrow I = \int \frac{\frac{dt}{6}}{e^t} \Rightarrow I = \frac{1}{6}\int e^{-t}\,dt$$

$$I = \frac{1}{6}\left(\frac{e^{-t}}{-1}\right) + c \Rightarrow I = \frac{-1}{6e^t} + c$$

but $t = 2x^3 + 3$

$$\boxed{I = \frac{-1}{6^{2x^3+3}} + c}$$

(vi) $\int e^x (2e^{3x} - 5)^{2/5}\, dx.$

Solution: let $I = \int e^x (2e^{3x} - 5)^{2/5}\, dx$

$I = (2e^{3x} - 5)^{2/5} \int e^x\, dx \Rightarrow \boxed{I = (2e^{3x} - 5)^{2/5} e^x + c}$ Ans

Q3. Determine:

(i) $\int \text{Sin}(3x + 2)\, dx.$

Solution: let $I = \int \text{Sin}(3x + 2)\, dx$

$\times$ & $\div$ by (3)

$I = \frac{1}{3} \int \text{Sin}(3x + 2) \cdot 3dx \Rightarrow \boxed{I = \frac{-1}{3} \text{Cos}(3x + 2) + c}$ Ans

(ii) $\int \text{Cos}2y\, dy$

Solution: let $I = \int \text{Cos}2y\, dy$

$\times$ & $\div$ by 2

$I = \frac{1}{2} \int \text{Cos}2y \cdot 2dy \Rightarrow \boxed{I = \frac{1}{2} \text{Sin}(2y) + c}$ Ans

(iii) $\int \text{Sin}^2x\, \text{Cos}x\, dx.$

Solution: let $I = \int \text{Sin}^2x\, \text{Cos}x$

$\boxed{I = \frac{1}{3} \text{Sin}^3x + c}$ Ans

(iv) $\int \frac{\text{Sec}\sqrt{x}}{\sqrt{x}}\, dx.$

Solution: let $I = \int \text{Sec}\sqrt{x}\, \frac{dx}{\sqrt{x}}$

$\times$ & $\div$ by 2

$I = 2 \int \text{Sec}\sqrt{x} \cdot \frac{1}{2\sqrt{x}}\, dx \Rightarrow \boxed{I = 2\, \ell n \tan\left(\frac{\sqrt{x}}{2} + \frac{\pi}{4}\right) + c}$ Ans

(v) $\int x\, \text{Cot}x^2\, dx$

Solution: let $I = \int \text{Cot}x^2 . x dx.$

$\times$ & $\div$ by 2

$I = \frac{1}{2} \int \text{Cot}x^2 \cdot 2xdx \Rightarrow \boxed{I = \frac{1}{2}\, \ell n\, \text{Sin}x^2 + c}$ Ans

(vi) $\int \tan x\, dx.$

Solution: let $I = \int \tan x\, dx$

$\boxed{I = \ell n\, \text{Sec}x + c}$ Ans

(vii) $\int \text{Sec}^2 2ax\, dx.$

Solution: let $I = \int \text{Sec}^2 2ax\, dx$

$\times$ & $\div$ by 2a

$I = \frac{1}{2a} \int \text{Sec}^2 2ax \cdot 2a\, dx \Rightarrow \boxed{I = \frac{1}{2a} \tan 2ax + c}$ Ans

(viii) $\int \frac{\text{Sin}x + \text{Cos}x}{\text{Cos}x}\, dx.$

Solution: let $I = \int \frac{\text{Sin}x + \text{Cos}x}{\text{Cos}x}\, dx.$

$I = \int \left(\frac{\text{Sin}x}{\text{Cos}x} + \frac{\cancel{\text{Cos}x}}{\cancel{\text{Cos}x}}\right) dx \Rightarrow I = \int (\tan x + 1)\, dx$

$I = \int \tan x\, dx + \int dx \Rightarrow \boxed{I = \ell n\, \text{Sec}x + x + c}$ Ans

(ix) $\int (1 + \tan x)^2\, dx.$

Solution: let $I = \int (1 + \tan x)^2\, dx.$

$\boxed{\therefore (a + b)^2 = a^2 + 2ab + b^2}$

$I = \int (1 + \tan^2x + 2\tan x)\, dx \Rightarrow \boxed{\therefore 1 + \tan^2x = \text{Sec}^2x.}$

$I = \int (\text{Sec}^2x + 2\tan x)\, dx \Rightarrow I = \int \text{Sec}^2x\, dx + 2 \int \tan x\, dx$

$\boxed{I = \tan x + 2\, \ell n\, \text{Sec}x + c}$ Ans

(x) $\int e^x\, \text{Cos}\, e^x\, dx.$

Solution: let $I = \int \text{Cos}\, e^x . e^x\, dx.$

$\boxed{I = \text{Sin}e^x + c}$ Ans

(xi) $\int e^{3\cos 2x}\, \text{Sin}2x\, dx.$

Solution: let $I = \int e^{3\cos 2x} . \text{Sin}2x\, dx$

$\times$ & $\div$ by (−6)

$I = \frac{-1}{6} \int e^{3\text{Cos}2x} (-6\, \text{Sin}2x)\, dx \Rightarrow \boxed{I = \frac{-1}{6} e^{3\text{Cos}2x} + c}$ Ans

(xii) $\int (\tan 2x + \text{Sec}2x)^2\, dx.$

Solution: let $I = \int (\tan 2x + \text{Sec}2x)^2\, dx$

$\boxed{\therefore (a + b)^2 = a^2 + 2ab + b^2}$

$I = \int (\tan^2 2x + 2\tan 2x\, \text{Sec}2x + \text{Sec}^2 2x)\, dx$

$$\boxed{\therefore \tan^2 2x = \text{Sec}^2 2x - 1}$$

$$I = \int (\text{Sec}^2 2x - 1 + 2\tan 2x\,\text{Sec}2x + \text{Sec}^2 2x)\,dx$$

$$I = \int (2\text{Sec}^2 2x - 1 + 2\tan 2x\,\text{Sec}\,2x)\,dx$$

$$I = 2\int \text{Sec}^2 2x\,dx - \int dx + 2\int \tan 2x\,\text{Sec}2x\,dx$$

$$I = \cancel{2}\frac{\tan 2x}{\cancel{2}} - x + \cancel{2}\frac{\text{Sec}2x}{\cancel{2}} + c$$

$$\boxed{I = \tan 2x - x + \text{Sec}2x + c} \text{ Ans}$$

(xiii) $\int (\text{Sec}4x - 1)^2\,dx.$

Solution: let $I = \int (\text{Sec}4x - 1)^2\,dx$

$$\boxed{\therefore (a-b)^2 = a^2 - 2ab + b^2}$$

$$I = \int (\text{Sec}^2 4x - 2\text{Sec}4x + 1)\,dx$$

$$I = \int \text{Sec}^2 4x\,dx - 2\int \text{Sec}4x\,dx + \int dx.$$

$$I = \frac{\tan 4x}{4} - \frac{\cancel{2}^{1}}{\cancel{4}_{2}}\ln(\text{Sec}4x + \tan 4x) + x + c$$

$$I = \frac{1}{4}\tan 4x - \frac{1}{2}\ln(\text{Sec}4x + \tan 4x) + x + c$$

$$\boxed{I = \frac{1}{4}\tan 4x - \frac{1}{2}\ln\tan\left(2x + \frac{\pi}{4}\right) + c} \text{ Ans}$$

(xiv) $\int \frac{\textbf{Secx tanx}}{\textbf{a + bSecx}}\,dx$

Solution: let $I = \int \frac{\text{Secx tanx}}{a + b\text{Secx}}\,dx$

× & ÷ by b

$$I = \frac{1}{b}\int \frac{b\text{Secxtanxdx}}{a + b\text{Secx}} \Rightarrow \boxed{I = \frac{1}{b}\ln(a + b\,\text{Secx}) + c}$$

(xv) $\int \frac{dx}{\text{Cos}2x - \text{Cot}2x}$

Solution: let $I = \int \frac{1}{\text{Cos}2x - \text{Cot}2x}\,dx.$

$$I = \int \frac{1}{\frac{1}{\text{Sin}2x} - \frac{\text{Cos}2x}{\text{Sin}2x}} \Rightarrow I = \int \frac{1}{\frac{1 - \text{Cos}2x}{\text{Sin}2x}}\,dx$$

$$I = \int \frac{\text{Sin}2x}{1 - \text{Cos}2x}\,dx$$

× & ÷ by 2

$$I = \frac{1}{2}\int \frac{2\text{Sin}2x\,dx}{1 - \text{Cos}2x} \Rightarrow I = \frac{1}{2}\ln(1 - \text{Cos}2x) + c$$

$$\boxed{\therefore 1 - \text{Cos}\theta = 2\text{Sin}^2\frac{\theta}{2}}$$

$$\boxed{\therefore 1 - \text{Cos}2x = 2\text{Sin}^2 x}$$

$$I = \frac{1}{2}\ln\{2\text{Sin}^2 x\} + c \Rightarrow I = \frac{1}{2}\{\ln 2 + \ln\text{Sin}^2 x\} + c$$

$$I = \frac{1}{2}\{\ln 2 + 2\ln\text{Sinx}\} + c$$

$$I = \frac{\ln 2}{2} + \frac{\cancel{2}}{\cancel{2}}\ln\text{Sinx} + c \Rightarrow I = \ln\text{Sinx} + c + \frac{\ln 2}{2}$$

$$\boxed{\therefore c + \frac{\ln 2}{2} = \text{constant} = c} \Rightarrow \boxed{I = \ln\text{Sinx} + c} \text{ Ans.}$$

(xvi) $\int \frac{\textbf{Sinlnx dx}}{\textbf{x (3 - Coslnx)}^{1/2}}$

Solution: let $I = \int \frac{\text{Sinlnx dx}}{x(3 - \text{Coslnxc})^{1/2}}$ ——(1)

$$I = \int (3 - \text{Cos}\ln x)^{-1/2} \cdot \frac{\text{Sin}\ln x}{x}\,dx$$

$$\boxed{I = 2(3 - \text{Coslnx})^{1/2} + c} \quad \text{Ans}$$

Q4. Find

(i) $\int \frac{x^2\,dx}{\sqrt{1 - x^6}}$

Solution: let $I = \int \frac{x^2\,dx}{\sqrt{1 - x^6}}$

$$I = \int \frac{x^2\,dx}{\sqrt{1 - (x^3)^2}} \text{ ——(1)}$$

let $t = x^3$

differentiate w.r to x

$$\frac{dt}{dx} = 3x^2 \Rightarrow \boxed{\frac{dt}{3} = x^2\,dx.}$$

$$(1) \Rightarrow I = \int \frac{\frac{dt}{3}}{\sqrt{1 - t^2}} \Rightarrow I = \frac{1}{3}\int \frac{1}{\sqrt{1 - t^2}}\,dt$$

$$\boxed{\text{Using formula } \therefore \int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \text{Sin}^{-1}\frac{x}{a} + c}$$

$I = \frac{1}{3}\sin^{-1}\frac{t}{1} + c$

but $t = x^3$

$\boxed{I = \frac{1}{3}\sin^{-1}x^3 + c}$ Ans

(ii) $\int \frac{x\,dx}{x^4 + 3}$

Solution: let $I = \int \frac{x\,dx}{x^4+3}$

$I = \int \frac{x\,dx}{(x^2)^2 + (\sqrt{3})^2}$ ———— (1)

let $t = x^2$

differentiate w.r to x

$\frac{dt}{dx} = 2x \Rightarrow \boxed{\frac{dt}{2} = x\,dx}$

$(1) \Rightarrow I = \int \frac{\frac{dt}{2}}{(t)^2 + (\sqrt{3})^2}$

$\boxed{\text{Using formula } \therefore \int \frac{1}{a^2 + x^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c}$

$I = \frac{1}{2}\frac{1}{\sqrt{3}}\tan^{-1}\frac{t}{\sqrt{3}} + c$

but $t = x^2$

$\boxed{I = \frac{1}{2\sqrt{3}}\tan^{-1}\frac{x^2}{\sqrt{3}} + c}$ Ans

(iii) $\int \frac{dx}{x\sqrt{x^4 - 1}}$

Solution: let $I = \int \frac{dx}{x\sqrt{x^4-1}}$

$I = \int \frac{dx}{x\sqrt{(x^2)^2 - (1)^2}}$ ———— (1)

let $t = x^2$

differentiate w.r to x

$\frac{dt}{dx} = 2x \Rightarrow \boxed{\frac{dt}{2x} = dx.}$

$(1) \Rightarrow I = \int \frac{\frac{dt}{2x}}{x\sqrt{t^2 - 1}} \Rightarrow I = \int \frac{dt}{2x^2\sqrt{t^2-1}}$ but $t = x^2$

$I = \frac{1}{2}\int \frac{1}{t\sqrt{t^2 - 1}}dt$

$\boxed{\text{Using formula } \therefore \int \frac{1}{x\sqrt{x^2 - a^2}}dx = \frac{1}{a}\sec^{-1}\frac{x}{a} + c}$

$I = \frac{1}{2}\left(\frac{1}{1}\right)\sec^{-1}\left(\frac{t}{1}\right) + c$

but $t = x^2$

$\boxed{I = \frac{1}{2}\sec^{-1}(x^2) + c}$ Ans

(iv) $\int \frac{dx}{\sqrt{4 - (x+2)^2}}$

Solution: let $I = \int \frac{dx}{\sqrt{4 - (x+2)^2}}$ ———— (1)

let $t = x + 2$

differentiate w.r to x

$\frac{dt}{dx} = 1 \Rightarrow \boxed{dt = dx}$

$(1) \Rightarrow I = \int \frac{dt}{\sqrt{(2)^2 - (t)^2}}$

$\boxed{\text{using formula } \therefore \int \frac{1}{\sqrt{a^2 - x^2}}dx = \sin^{-1}\frac{x}{a} + c}$

$I = \sin^{-1}\frac{t}{2} + c$ but $t = x + 2$

$\boxed{I = \sin^{-1}\left(\frac{x+2}{2}\right) + c}$

(v) $\int \frac{\sec\theta\tan\theta\,d\theta}{9 + 4\sec^2\theta}$

Solution: let $I = \int \frac{\sec\theta\tan\theta\,d\theta}{9 + 4\sec^2\theta}$

$I = \int \frac{\sec\theta\tan\theta\,d\theta}{(3)^2 + (2\sec\theta)^2}$ ———— (1)

let $t = 2\sec\theta$

differentiate w.r to x

$\frac{dt}{d\theta} = 2\,\text{Sec}\theta\tan\theta \Rightarrow \boxed{\frac{dt}{2} = \text{Sec}\theta\tan\theta\, d\theta}$

$(1) \Rightarrow I = \int \frac{\frac{dt}{2}}{(3)^2 + (t)^2}$

$I = \frac{1}{2}\int \frac{1}{(3)^2 + (t)^2}\, dt.$

$\boxed{\text{Using formula } \therefore \int \frac{1}{a^2 + x^2}\, dx = \tan^{-1}\frac{x}{a} + c}$

$I = \frac{1}{2}\,\frac{1}{3}\tan^{-1}\frac{t}{3} + c$ but $t = 2\text{Sec}\theta$

$\boxed{I = \frac{1}{6}\tan^{-1}\frac{(2\,\text{Sec}\theta)}{3} + c}$ Ans

(vi) $\int \frac{(x+3)\,dx}{\sqrt{1-x^2}}$

Solution: let $I = \int \frac{(x+3)\,dx}{\sqrt{1-x^2}}$ breaking L.C.M

$I = \int \frac{x}{\sqrt{1-x^2}}\, dx + \int \frac{3}{\sqrt{1-x^2}}\, dx$

$I = \int (1-x^2)^{-1/2}\, x\, dx + 3\int \frac{1}{\sqrt{1-x^2}}\, dx$

$\times$ & $\div$ by (-2) to 1st integral.

and using formula $\therefore \int \frac{1}{\sqrt{a^2-x^2}}\, dx = \text{Sin}^{-1}\frac{x}{a} + c$ on Second integral.

$I = \frac{-1}{2}\int (1-x^2)^{-1/2}\cdot(-2x)\, dx + 3\int \frac{1}{\sqrt{1-x^2}}\, dx$

$= \frac{-1}{2}\,\frac{(1-x^2)^{1/2}}{1/2} + 3\,\text{Sin}^{-1}\frac{x}{1} + c$

$= \frac{-1}{2} \times 2\sqrt{1-x^2} + 3\,\text{Sin}^{-1} x + c$

$\boxed{I = -\sqrt{1-x^2} + 3\,\text{Sin}^{-1} x + c}$ Ans

(iii) $\int \frac{2u-7}{u^2+9}\, du.$

Solution: let $I = \int \frac{2u-7}{u^2+9}\, du.$

breaking L.C.M

$I = \int \frac{2u}{u^2+9}\, du - 7\int \frac{1}{u^2+9}\, du.$

$\boxed{\text{Using formula } \therefore \int \frac{1}{a^2 + x^2}\, dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c}$

on 2nd integral.

$I = \ln(u^2+9) - 7\left(\frac{1}{3}\tan^{-1}\frac{u}{3}\right) + c$

$\boxed{I = \ln(u^2+9) - \frac{7}{3}\tan{-1}\frac{u}{3} + c}$ Ans

(viii) $\int \frac{dt}{1 + (3t-4)^2}$

Solution: let $I = \int \frac{dt}{1 + (3t-4)^2}$ ——— (1)

let $u = 3t - 4$

differentiate w.r to x

$\frac{du}{dt} = 3 \Rightarrow \boxed{\frac{du}{3} = dt}$

$(1) \Rightarrow I = \int \frac{\frac{du}{3}}{(1)^2 + (u)^2}$

$I = \frac{1}{3}\int \frac{1}{(1)^2 + (u)^2}\, du.$

$\boxed{\text{Using formula } \therefore \int \frac{1}{a^2 + x^2}\, dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c}$

$I = \frac{1}{3}\,\frac{1}{1}\tan^{-1}\frac{u}{1} + c$

but $u = 3t - 4$

$\boxed{I = \frac{1}{3}\tan^{-1}(3t-4) + c}$ Ans

Q5. by using Trigonometric identities or otherwise, evalu following integrals.

(i) $\int \text{Sin}^2 x\, dx.$

Solution: let $I = \int \text{Sin}^2 x\, dx.$

$\boxed{\textbf{Using formula } \therefore \text{Sin}^2 x \frac{1-\text{Cos}2x}{2}}$

$I = \int\left(\frac{1-\cos 2x}{2}\right) dx \Rightarrow I = \frac{1}{2}\int dx - \frac{1}{2}\int \text{Cos}2x\, dx$

$\times$ & $\div$ by 2 on 2nd integral.

$$I = \frac{1}{2}\int dx - \frac{1}{2\times 2}\int \cos 2x \cdot 2\,dx$$

$$\boxed{I = \frac{x}{2} - \frac{1}{4}\sin 2x + c}\quad \text{Ans}$$

(ii) $\int \cos^2(3x+2)\,dx$

Solution: let $I = \int \cos^2(3x+2)\,dx$

$$\boxed{\text{Using formula } \therefore \cos^2 x = \frac{1+\cos 2x}{2}}$$

$$I = \int\left\{\frac{1+\cos 2(3x+2)}{2}\right\}dx$$

$$I = \int\left\{\frac{1+\cos(6x+4)}{2}\right\}dx$$

$$I = \frac{1}{2}\,dx + \frac{1}{2}\int \cos(6x+4)\,dx$$

$\times$ & $\div$ by 6 on 2nd integral.

$$I = \frac{1}{2}\int dx + \frac{1}{2\times 6}\int \cos(6x+4)\cdot 6\,dx$$

$$\boxed{I = \frac{x}{2} + \frac{1}{12}\sin(6x+4) + c}\quad \text{Ans}$$

(iii) $\int \sin^3 x\,dx$

Solution: let $I = \int \sin^3 x\,dx$

$$I = \int \sin^2 x.\sin x\,dx$$

$$\boxed{\therefore \sin^2 x = 1 - \cos^2 x}$$

$$I = \int(1-\cos^2 x)\sin x\,dx \Rightarrow I = \int \sin x\,dx - \int \cos^2 x \sin x\,dx.$$

$\times$ & $\div$ by (-1) on 2nd integral.

$$I = \int \sin x\,dx + \int \cos^2 x.(-\sin x)\,dx$$

$$\boxed{I = -\cos x + \frac{\cos^3 x}{3} + c}\quad \text{Ans}$$

$\int \cos^5 x\,dx$.

Solution: let $I = \int \cos^5 x\,dx$

$$I = \int(\cos^2 x)^2.\cos x\,dx$$

$$\boxed{\cos^2 x = 1 - \sin^2 x.}$$

$$I = \int(1-\sin^2 x)^2\cos x\,dx$$

breaking $\boxed{(a-b)^2 = a^2 - 2ab + b^2}$

$$I = \int(1 - 2\sin^2 x + \sin^4 x)\cos x\,dx.$$

$$I = \int \cos x\,dx - 2\int \sin^2 x \cos x\,dx + \int \sin^4 x \cos x\,dx$$

$$\boxed{I = \sin x - 2\frac{\sin^3 x}{3} + \frac{\sin^5 x}{5} + c}\quad \text{Ans}$$

(v) $\int \sin^2 x \cos^3 x\,dx$

Solution: let $I = \int \sin^2 x \cos^3 x\,dx$.

↓

break

$$I = \int \sin^2 x.\cos^2 x.\cos x\,dx$$

$$\boxed{\therefore \cos^2 x = 1 - \sin^2 x.}$$

$$I = \int \sin^2 x \cos x(1-\sin^2 x)\,dx.$$

$$I = \int \sin^2 x \cos x\,dx - \int \sin^4 x \cos x\,dx.$$

$$\boxed{I = \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + c}$$

(vi) $\int \cos^4 2x.\sin^3 2x\,dx$

Solution: let $I = \int \cos^4 2x.\sin^3 2x\,dx$

↓

break.

$$I = \int \cos^4 2x.\sin 2x.\sin^2 2x\,dx$$

$$\boxed{\therefore \sin^2 2x = 1 - \cos^2 2x}$$

$$I = \int \cos^4 2x.\sin 2x(1-\cos^2 2x)\,dx$$

$$I = \int \cos^4 2x \sin 2x\,dx - \int \cos^6 2x \sin 2x\,dx$$

$\times$ & $\div$ by (-2) to both intergrals.

$$I = \frac{-1}{2}\int \cos^4 2x.(-2\sin 2x)\,dx + \frac{1}{2}\int \cos^6 2x.(-2\sin 2x)\,dx$$

$$I = \frac{-1}{2}\frac{\cos^5 2x}{5} + \frac{1}{2}\frac{\cos^7 2x}{7} + c$$

$$\boxed{I = -\frac{\cos^5 2x}{10} + \frac{\cos^7 2x}{14} + c}\quad \text{Ans}$$

(vii) $\int \cos^3\frac{y}{3}\,dy$.

Solution: let $I = \int \cos^3\frac{y}{3}\,dy$.

$$I = \int \cos^2\frac{y}{3}.\cos\frac{y}{3}\,dy. \quad \boxed{\therefore \cos^2\frac{y}{3} = 1 - \sin^2\frac{y}{3}}$$

$$I = \int\left(1-\sin^2\frac{y}{3}\right)\cos\frac{y}{3}\,dy$$

$$I = \int \text{Cos}\frac{y}{3}\, dy - \int \text{Sin}^2\frac{y}{3}\,\text{Cos}\frac{y}{3}\, dy$$

× & ÷ by 3 on both integrals.

$$I = 3\int \text{Cos}\frac{y}{3}\bullet\frac{1}{3}dy - 3\int \text{Sin}^2\frac{y}{3}\bullet\left(\frac{1}{3}\,\text{Cos}\frac{y}{3}\right)dy$$

$$I = 3\,\text{Sin}\frac{y}{3} - \frac{\cancel{3}\,\text{Sin}^3\frac{y}{3}}{\cancel{3}} + c \Rightarrow \boxed{I = 3\,\text{Sin}\frac{y}{3} - \text{Sin}^3\frac{y}{3} + c} \text{ Ans}$$

(viii) $\int \text{Sin}^4x\, dx$.

Solution: let $I = \int \text{Sin}^4x\, dx$

$$I = \int(\text{Sin}^2x)^2\, dx$$

$$\boxed{\therefore \text{Sin}^2x = \frac{1 - \text{Cos}2x}{2}}$$

$$I = \int\left(\frac{1-\text{Cos}2x}{2}\right)^2 dx \Rightarrow I = \int\frac{(1-\text{Cos}2x)^2}{4}\, dx.$$

$$\boxed{\therefore (a-b)^2 = a^2 - 2ab + b^2}$$

$$I = \int\frac{(1 - 2\,\text{Cos}2x + \text{Cos}^2 2x)}{4}\, dx$$

$$I = \frac{1}{4}\int dx - \frac{\cancel{2}}{\cancel{4}_2}\int \text{Cos}2x\, dx + \frac{1}{4}\int \text{Cos}^2 2x\, dx$$

$$I = \frac{1}{4}\int dx - \frac{1}{2}\int \text{Cos}2x\, dx + \frac{1}{4}\int \text{Cos}^2 2x\, dx$$

$$\boxed{\therefore \text{Cos}^2x = \frac{1 + \text{Cos}2x}{2}}$$

$$I = \frac{1}{4}\int dx - \frac{1}{2}\int \text{Cos}2x\, dx + \frac{1}{4}\int\left(\frac{1 + \text{Cos}4x}{2}\right)dx$$

$$I = \frac{1}{4}\int dx - \frac{1}{2}\int \text{Cos}2x\, dx + \frac{1}{8}\int dx + \frac{1}{8}\int \text{Cos}4x\, dx.$$

× & ÷ by 2 on 2nd integral and
× & ÷ by 4 on 4th integral.

$$I = \frac{1}{4}\int dx - \frac{1}{2\times 2}\int \text{Cos}2x.2dx + \frac{1}{8}\int dx + \frac{1}{8\times 4}\int \text{Cos}4x.4dx$$

$$I = \frac{x}{4} - \frac{\text{Sin}2x}{4} + \frac{x}{8} + \frac{\text{Sin}4x}{32} + c$$

$$\boxed{I = \frac{3x}{8} - \frac{\text{Sin}2x}{4} + \frac{\text{Sin}4x}{32} + c} \text{ Ans}$$

(ix) $\int \text{Sin}^2x\,\text{Cos}^2x\, dx$

Solution: let $I = \int \text{Sin}^2x\,\text{Cos}^2x\, dx$.

$$\boxed{\therefore \text{Sin}^2x = \frac{1-\text{Cos}2x}{2}} \quad \boxed{\therefore \text{Cos}^2x = \frac{1+\text{Cos}2x}{2}}$$

$$I = \int\left(\frac{1-\text{Cos}2x}{2}\right)\left(\frac{1+\text{Cos}2x}{2}\right)dx$$

$$I = \frac{1}{4}\int(1-\text{Cos}2x)\,(1+\text{Cos}2x)\, dx$$

$$\boxed{\therefore a^2 - b^2 = (a-b)(a+b)}$$

$$I = \frac{1}{4}(1 - \text{Cos}^2 2x)\, dx \Rightarrow I = \frac{1}{4}\int dx - \frac{1}{4}\int \text{Cos}^2 2x\, dx$$

$$I = \frac{1}{4}\int dx - \frac{1}{4}\int\left(\frac{1+\text{Cos}4x}{2}\right)dx$$

$$I = \frac{1}{4}\int dx - \frac{1}{8}\int dx - \frac{1}{8}\int \text{Cos}4x\, dx$$

× & ÷ by 4 on 3rd integral.

$$I = \frac{1}{4}\int dx - \frac{1}{8}\int dx - \frac{1}{8\times 4}\int \text{Cos}4x.4dx$$

$$I = \frac{x}{4} - \frac{x}{8} - \frac{1}{32}\,\text{Sin}4x + c \Rightarrow \boxed{I = \frac{x}{8} - \frac{\text{Sin}4x}{32} + c} \text{ Ans}$$

(x) $\int \text{Sin}^4 3x\,\text{Cos}^3 3x\, dx$

Solution: let $I = \int \text{Sin}^4 3x\,\text{Cos}^3 3x\, dx$

↓

break

$$I = \int \text{Sin}^4 3x.\text{Cos}3x.\text{Cos}^2 3x\, dx$$

$$\boxed{\therefore \text{Cos}^2 3x = 1 - \text{Sin}^2 3x}$$

$$I = \int \text{Sin}^4 3x.\ \text{Cos}3x.\ (1 - \text{Sin}^2 3x)\, dx$$

$$I = \int \text{Sin}^4 3x\,\text{Cos}3x\, dx - \int \text{Sin}^6 3x\,\text{Cos}3x\, dx$$

× & ÷ by 3 to both integrals.

$$I = \frac{1}{3}\int \text{Sin}^4 3x.\ (3\,\text{Cos}3x)\, dx - \frac{1}{3}\int \text{Sin}^6 3x.\ (3\,\text{Cos}3x)\, dx.$$

$$I = \frac{1}{3}\frac{\text{Sin}^5 3x}{5} - \frac{1}{3}\frac{\text{Sin}^7 3x}{7} + c$$

$$\boxed{I = \frac{1}{15}\text{Sin}^5 3x - \frac{1}{7}\,\text{Sin}^7 3x + c} \text{ Ans}$$

(xi) $\int \text{Sin}2z\,\text{Sin}3z\, dz$.

Solution: let $I = \int \text{Sin}2z\,\text{Sin}3z\, dz$.

$\boxed{\text{Using formula } \sin\alpha\sin\beta = \frac{-1}{2}\left[\cos(\alpha+\beta) - \cos(\alpha-\beta)\right]}$

$$I = \int\left(\frac{-1}{2}\right)\left[\cos(2z+3z) - \cos(2z-3z)\right] dz$$

$$I = \frac{-1}{2}\int\left[\cos 5z - \cos(-z)\right] dz.$$

$\boxed{\therefore \cos(-\theta) = \cos\theta}$

$$I = \frac{-1}{2}\int\left[\cos 5z - \cos z\right] dz.$$

$$I = \frac{-1}{2}\int \cos 5z\, dz + \frac{1}{2}\int \cos z\, dz$$

$$I = \frac{-1}{2}\frac{\sin 5z}{5} + \frac{1}{2}\sin z + c$$

$$\boxed{I = -\frac{\sin 5z}{10} + \frac{\sin z}{2} + c} \quad \text{Ans}$$

(xii) $\int \sin 3y \cos 5y\, dy$

Solution: let $I = \int \sin 3y \cos 5y\, dy$

$\boxed{\text{Using formula } \therefore \sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right]}$

$$I = \int\frac{1}{2}\left[\sin(3y+5y) + \sin(3y-5y)\right] dy.$$

$$I = \frac{1}{2}\int\left[\sin 8y + \sin(-2y)\right] dy.$$

$\boxed{\therefore \sin(-\theta) = -\sin\theta}$

$$I = \frac{1}{2}\int\left[\sin 8y - \sin 2y\right] dy. \Rightarrow I = \frac{1}{2}\int \sin 8y\, dy - \frac{1}{2}\int \sin 2y\, dy.$$

$$I = \frac{1}{2}\left(\frac{-\cos 8y}{8}\right) - \frac{1}{2}\left(\frac{-\cos 2y}{2}\right) + c$$

$$\boxed{I = \frac{-\cos 8y}{16} + \frac{\cos 2y}{4} + c} \quad \text{Ans}$$

(xiii) $\int \cos 4x \cos 2x\, dx.$

Solution: let $I = \int \cos 4x \cos 2x\, dx$

$\boxed{\text{Using formula } \therefore \cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha+\beta) + \cos(\alpha-\beta)\right]}$

$$I = \int\frac{1}{2}\left[\cos(4x+2x) + \cos(4x-2x)\right] dx$$

$$I = \int\frac{1}{2}\left[\cos 6x + \cos 2x\right] dx \Rightarrow I = \frac{1}{2}\int \cos 6x\, dx + \frac{1}{2}\int \cos 2x\, dx$$

$$I = \frac{1}{2}\frac{\sin 6x}{6} + \frac{1}{2}\frac{\sin 2x}{2} + c \Rightarrow \boxed{I = \frac{\sin 6x}{12} + \frac{\sin 2x}{4} + c} \quad \text{Ans}$$

(xiv) $\int \sqrt{1 - \cos x}\, dx.$

Solution: let $I = \int \sqrt{1 - \cos x}\, dx$

$\boxed{\text{Using formula } \therefore 1 - \cos\theta = 2\sin^2\frac{\theta}{2}}$

$$I = \int\sqrt{2\sin^2\frac{x}{2}}\, dx \Rightarrow I = \sqrt{2}\int \sin\frac{x}{2}\, dx.$$

$$I = \sqrt{2}\left(\frac{-\cos\frac{x}{2}}{\frac{1}{2}}\right) + c \Rightarrow \boxed{I = -2\sqrt{2}\cos\frac{x}{2} + c} \quad \text{Ans}$$

(xv) $\int \sqrt{(1 + \cos 3x)^3}\, dx$

Solution: let $I = \int \sqrt{(1 + \cos 3x)^3}\, dx$

$\boxed{\therefore \text{Using formula } 1 + \cos\theta = 2\cos^2\frac{\theta}{2}}$

$$I = \int\sqrt{\left\{2\cos^2\frac{3x}{2}\right\}^3}\, dx \Rightarrow I = \int\sqrt{8\cos^6\frac{3x}{2}}\, dx$$

$$I = 2\sqrt{2}\int \cos^3\frac{3x}{2}\, dx \Rightarrow I = 2\sqrt{2}\int \cos^2\frac{3x}{2}.\cos\frac{3x}{2}\, dx$$

$\boxed{\therefore \cos^2\frac{3x}{2} = 1 - \sin^2\frac{3x}{2}}$

$$I = 2\sqrt{2}\int\left(1 - \sin^2\frac{3x}{2}\right)\cos\frac{3x}{2}\, dx$$

$$I = 2\sqrt{2}\int \cos\frac{3x}{2}\, dx - 2\sqrt{2}\int \sin^2\frac{3x}{2}\cos\frac{3x}{2}\, dx$$

$$I = 2\sqrt{2}\,\frac{\sin\frac{3x}{2}}{\frac{3}{2}} - 2\sqrt{2}\,\frac{\sin^3\frac{3x}{2}}{3\times\frac{3}{2}} + c$$

$$I = 2\sqrt{2}\times\frac{2}{3}\sin\frac{3x}{2} - 2\sqrt{2}\times\frac{2}{9}\sin^3\frac{3x}{2} + c$$

$$\boxed{I = \frac{4\sqrt{2}}{3}\sin\frac{3x}{2} - \frac{4\sqrt{2}}{9}\sin^3\frac{3x}{2} + c} \quad \text{Ans}$$

(xvi) $\int \frac{dx}{\sqrt{1 - \sin 2x}}$

Solution: let $I = \int \frac{dx}{\sqrt{1 - \sin 2x}}$ ——— (1)

$$\therefore \ \sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$$

$$\therefore \ \sin 2x = \cos\left(\frac{\pi}{2} - 2x\right)$$

$$1 - \sin 2x = 1 - \cos\left(\frac{\pi}{2} - 2x\right)$$

$$\boxed{\therefore 1 - \cos\theta = 2\sin^2\frac{\theta}{2}}$$

$$1 - \sin 2x = 2\sin^2\frac{\left(\frac{\pi - 4x}{2}\right)}{2}$$

$$1 - \sin 2x = 2\sin^2\left(\frac{\pi - 4x}{4}\right)$$

$$1 - \sin 2x = 2\sin^2\left(\frac{\pi}{4} - x\right)$$

$$(1) \Rightarrow I = \int \frac{dx}{\sqrt{2\sin^2\left(\frac{\pi}{4} - x\right)}} \Rightarrow I = \int \frac{dx}{\sqrt{2}\sin\left(\frac{\pi}{4} - x\right)}$$

$$I = \frac{1}{\sqrt{2}} \int \operatorname{Cosec}\left(\frac{\pi}{4} - x\right) dx. \Rightarrow I = \frac{-1}{\sqrt{2}} \int \operatorname{Cosec}\left(\frac{\pi}{4} - x\right)(-1)\, dx.$$

$$I = \left(\frac{-1}{\sqrt{2}}\right) \ln \tan\left(\frac{\frac{\pi}{4} - x}{2}\right) + c$$

$$\boxed{I = \frac{-1}{\sqrt{2}} \ln \tan\left(\frac{\pi}{8} - \frac{x}{2}\right) + c} \quad \text{Ans}$$

(xvii) $\int \tan^4 x\, dx$

Solution: let $I = \int \tan^4 x\, dx$

$$I = \int \tan^2 x . \tan^2 x\, dx$$

$$\boxed{\therefore \tan^2 x = \sec^2 x - 1}$$

$$I = \int (\sec^2 x - 1)\tan^2 x\, dx$$

$$I = \int \sec^2 x \tan^2 x\, dx - \int \tan^2 x\, dx$$

$$I = \int \sec^2 x \tan^2 x\, dx - \int (\sec^2 x - 1)\, dx$$

$$I = \int \sec^2 x \tan^2 x\, dx - \int \sec^2 x\, dx + \int dx$$

$$\boxed{I = \frac{\tan^3 x}{3} - \tan x + x + c} \quad \text{Ans}$$

(xviii) $\int \tan^5 x\, dx$

Solution: let $I = \int \tan^5 x\, dx$

$$I = \int \tan^3 x \cdot \tan^2 x\, dx.$$

$$\boxed{\therefore \tan^2 x = \sec^2 x - 1}$$

$$I = \int \tan^3 x.(\sec^2 x - 1)\, dx$$

$$I = \int \tan^3 x \sec^2 x\, dx - \int \tan^3 x\, dx$$

$$I = \int \tan^3 x \sec^2 x\, dx - \int \tan x \tan^2 x\, dx$$

$$I = \int \tan^3 x \sec^2 x\, dx - \int \tan x (\sec^2 x - 1)\, dx$$

$$I = \int \tan^3 x \sec^2 x\, dx - \int \tan x \sec^2 x\, dx + \int \tan x\, dx.$$

$$\boxed{I = \frac{\tan^4 x}{4} - \frac{\tan^2 x}{2} + \ln \sec x + c} \quad \text{Ans}$$

(xix) $\int \sec^4 2u\, du.$

Solution: let $I = \int \sec^4 2u\, du.$

$$I = \int \sec^2 2u . \sec^2 2u\, du.$$

$$\boxed{\therefore \sec^2 2u = 1 + \tan^2 2u}$$

$$I = \int (1 + \tan^2 2u)\sec^2 2u\, du.$$

$$I = \int \sec^2 2u\, du + \int \tan^2 2u \sec^2 2u\, du$$

$$I = \frac{\tan 2u}{2} + \frac{\tan^3 2u}{3\,(2)} + c$$

$$\boxed{I = \frac{\tan 2u}{2} + \frac{\tan^3 2u}{6} + c} \quad \text{Ans}$$

(xx) $\int \tan^3 3x \sec^4 3x\, dx$

Solution: let $I = \int \tan^3 3x \sec^4 3x\, dx$

$$I = \int \tan^3 3x . \sec^2 3x . \sec^2 3x\, dx$$

$$\boxed{\therefore \tan^2 3x = \sec^2 3x - 1}$$

$$I = \int \tan^3 3x \cdot \sec^2 3x (\tan^2 3x + 1)\, dx$$

$$I = \int \tan^5 3x \sec^2 x\, dx + \int \tan^3 3x \sec^2 3x\, dx$$

$$I = \frac{1}{3}\int \tan^5 3x . \sec^2 3x \cdot 3dx + \frac{1}{3}\int \tan^3 3x . \sec^2 3x . 3dx$$

$$I = \frac{1}{3}\frac{\tan^6 3x}{6} + \frac{1}{3}\frac{\tan^4 3x}{4} + c$$

$$\boxed{I = \frac{1}{18}\tan^6 3x + \frac{1}{12}\tan^4 3x + c} \quad \text{Ans}$$

(xxi) $\int \tan^4 \frac{x}{3} \operatorname{Sec}^4 \frac{x}{3} dx$

Solution: let $I = \int \tan^4 \frac{x}{3} \operatorname{Sec}^4 \frac{x}{3} dx$

$= \int \tan^4 \frac{x}{3} \cdot \operatorname{Sec}^2 \frac{x}{3} \cdot \operatorname{Sec}^2 \frac{x}{3} dx \Rightarrow = \int \tan^4 \frac{x}{3}\left(1 + \tan^2 \frac{x}{3}\right) \operatorname{Sec}^2 \frac{x}{3} dx$

$= \int \tan^4 \frac{x}{3} \operatorname{Sec}^2 \frac{x}{3} dx + \int \tan^6 \frac{x}{3} \operatorname{Sec}^2 \frac{x}{3} dx$

$= 3\int \tan^4 \frac{x}{3}\left(\operatorname{Sec}^2 \frac{x}{3} \cdot \frac{1}{3}\right) dx + 3\int \tan^6 \frac{x}{3} \cdot \left(\operatorname{Sec}^2 \frac{x}{3} \cdot \frac{1}{3}\right) dx$

$\boxed{I = \frac{3}{5}\tan^5 \frac{x}{3} + \frac{3}{7}\tan^7 \frac{x}{3} + c}$

(xxii) $\int \tan^2 x \operatorname{Sec} x\, dx$

Solution: let $I = \int \tan^2 x \cdot \operatorname{Sec} x\, dx$

$I = \int \tan x \cdot \tan x \operatorname{Sec} x\, dx$ ———— (1)

let $t = \operatorname{Sec} x$ $\therefore$ $1 + \tan^2 x = \operatorname{Sec}^2 x \Rightarrow \tan x = \sqrt{t^2 - 1}$

$\frac{dt}{dx} = \operatorname{Sec} x \tan x \Rightarrow \boxed{dt = \operatorname{Sec} x \tan x dx}$

(1)$\Rightarrow$ $I = \int \sqrt{t^2 - 1}\, dt$

$\boxed{\therefore \int \sqrt{x^2 - a^2}\, dx = \frac{1}{2}\sqrt{x^2 - a^2} - \frac{1}{2}a^2 \ln\left(x + \sqrt{x^2 - a^2}\right) + c}$

$= \frac{1}{2}\sqrt{t^2 - 1} - \frac{1}{2}(1)^2 \ln\left(t + \sqrt{t^2 - 1}\right) + c$

$= \frac{1}{2}\sqrt{\operatorname{Sec}^2 x - 1} - \frac{1}{2}\ln\left\{\operatorname{Sec} x + \sqrt{\operatorname{Sec}^2 x - 1}\right\} + c$

$= \frac{1}{2}\sqrt{\tan^2 x} - \frac{1}{2}\ln\left\{\operatorname{Sec} x + \sqrt{\tan^2 x}\right\} + c$

$\boxed{I = \frac{1}{2}\tan x - \frac{1}{2}\ln\{\operatorname{Sec} x + \tan x\} + c}$ Ans.

(xxiii) $\int \tan^3 2x \operatorname{Sec}^3 2x\, dx$

Solution: let $I = \int \tan^3 2x \cdot \operatorname{Sec}^3 2x\, dx$

$I = \int \tan^2 2x \cdot \operatorname{Sec}^3 2x \cdot \tan 2x\, dx$

$I = \int (\operatorname{Sec}^2 2x - 1)\operatorname{Sec}^3 2x \tan 2x\, dx$

$I = \int \operatorname{Sec}^5 2x \cdot \tan 2x\, dx - \int \operatorname{Sec}^3 2x \tan 2x\, dx$

$I = \frac{1}{2}\int \operatorname{Sec}^4 2x \cdot (\operatorname{Sec} 2x \tan 2x \cdot 2)\, dx - \frac{1}{2}\int \operatorname{Sec}^2 2x \cdot (\operatorname{Sec} 2x \cdot \tan 2x \cdot 2)\, dx$

$\boxed{I = \frac{1}{10}\operatorname{Sec}^5 2x - \frac{1}{6}\operatorname{Sec}^3 2x + c}$ Ans

(xxiv) $\int \operatorname{Cot}^3 2x\, dx$

Solution: let $I = \int \operatorname{Cot}^3 2x\, dx$

$I = \int \operatorname{Cot}^2 2x \cdot \operatorname{Cot} 2x\, dx \Rightarrow I = \int (\operatorname{Cosec}^2 2x - 1)\operatorname{Cot} 2x\, dx$

$I = \int \operatorname{Cosec}^2 2x \operatorname{Cot} 2x\, dx - \int \operatorname{Cot} 2x dx$

$I = -\frac{1}{2}\int \operatorname{Cot} 2x \cdot (-2\operatorname{Cosec}^2 2x)\, dx - \frac{1}{2}\int \operatorname{Cot} 2x \cdot 2dx$

$\boxed{I = \frac{-1}{4}\operatorname{Cot}^2 2x - \frac{1}{2}\ln \operatorname{Sin} 2x + c}$ Ans.

(xxv) $\int \operatorname{Cot}^4 3z\, dx$

Solution: let $I = \int \operatorname{Cot}^4 3z\, dx$

$I = \int \operatorname{Cot}^2 3z \cdot \operatorname{Cot}^2 3z\, dz$

$I = \int \operatorname{Cot}^2 3z\,(\operatorname{Cosec}^2 3z - 1)\, dz$

$I = \int \operatorname{Cot}^2 3z \operatorname{Cosec}^2 3z\, dx - \int \operatorname{Cot}^2 3z\, dz$

$I = -\frac{1}{3}\int \operatorname{Cot}^2 3z\,(-3\operatorname{Cosec}^2 3z)\, dx - \frac{1}{3}\int \operatorname{Cosec}^2 3z \cdot 3\, dz + \int dz$

$\boxed{I = \frac{-1}{9}\operatorname{Cot}^3 3z + \frac{1}{3}\operatorname{Cot} 3z + z + c}$

(xxvi) $\int \operatorname{Cosec}^6 x\, dx$

Solution: let $I = \int \operatorname{Cosec}^6 x\, dx$

$I = \int \operatorname{Cosec}^4 x \cdot \operatorname{Cosec}^2 x\, dx = \int (\operatorname{Cosec}^2 x)^2 \operatorname{Cosec}^2 x dx$

$I = \int (1 + \operatorname{Cot}^2 x)^2 \operatorname{Cosec}^2 x\, dx$

$I = \int (1 + 2\operatorname{Cot}^2 x + \operatorname{Cot}^4 x)\operatorname{Cosec}^2 x\, dx$

$I = \int \operatorname{Cosec}^2 x\, dx + 2\int \operatorname{Cot}^2 x \cdot \operatorname{Cosec}^2 x\, dx + \int \operatorname{Cot}^4 x \operatorname{Cosec}^2 x\, dx$

$I = \int \operatorname{Cosec}^2 x dx - 2\int \operatorname{Cot}^2 x \cdot (-\operatorname{Cosec}^2 x dx) - \int \operatorname{Cot}^4 x \cdot (-\operatorname{Cosec}^2 x)\, dx$

$\boxed{I = -\operatorname{Cot} x - \frac{2}{3}\operatorname{Cot}^3 x - \frac{1}{5}\operatorname{Cot}^5 x + c}$ Ans.

(xxvii) $\int \operatorname{Cot} 3x \operatorname{Cosec}^4 3x\, dx$

Solution: let $I = \int \operatorname{Cot} 3x \operatorname{Cosec}^4 3x\, dx$

$I = \int \operatorname{Cot} 3x \cdot \operatorname{Cosec}^2 3x \cdot \operatorname{Cosec}^2 3x\, dx$

$I = \int (1 + \operatorname{Cot}^2 3x)\operatorname{Cot} 3x \operatorname{Cosec}^2 3x\, dx$

$I = \int \operatorname{Cot} 3x \cdot \operatorname{Cosec}^2 3x \cdot dx + \int \operatorname{Cot}^3 3x \operatorname{Cosec}^2 3x dx$

$I = \frac{-1}{3}\int \operatorname{Cot} 3x \cdot (-3\operatorname{Cosec}^2 3x)\, dx - \frac{1}{3}\int \operatorname{Cot}^3 3x\,(-3\operatorname{Cosec}^2 3x)\, dx$

$\boxed{I = \frac{-1}{6}\operatorname{Cot}^2 3x - \frac{1}{12}\operatorname{Cot}^4 3x + c}$ Ans.

(xxviii) $\int Cot^5\frac{x}{2}\,Cosec^3\frac{x}{2}\,dx$

Solution: let $I = \int Cot^5\frac{x}{2}\,Cosec^3\frac{x}{2}\,dx$

$$I = \int \left(Cot^2\frac{x}{2}\right)^2 \cdot Cot\frac{x}{2} \cdot Cosec^3\frac{x}{2}\,dx$$

$$I = \int \left(Cosec^2\frac{x}{2} - 1\right)^2 Cot\frac{x}{2}\,Cosec^3\frac{x}{2}\,dx$$

$$I = \int \left(Cosec^4\frac{x}{2} - 2Cosec^2\frac{x}{2} + 1\right) Cot\frac{x}{2}\,Cosec^3\frac{x}{2}\,dx$$

$$I = \int Cosec^6\frac{x}{2} \cdot Cosec\frac{x}{2}\,Cot\frac{x}{2}\,dx - 2\int Cosec^4\frac{x}{2} \cdot Cosec\frac{x}{2} \cdot Cot\frac{x}{2}\,dx + \int Cosec^2\frac{x}{2} \cdot Cosec\frac{x}{2}\,Cot\frac{x}{2}\,dx$$

$$I = -2\int Cosec^6\frac{x}{2} \cdot \left(\frac{-1}{2}Cosec\frac{x}{2}\,Cot\frac{x}{2}\right)dx + 4\int Cosec^4\frac{x}{2} \cdot \left(\frac{-1}{2}Cosec\frac{x}{2}\,Cot\frac{x}{2}\right)dx - 2\int Cosec^2\frac{x}{2}\left(\frac{-1}{2}Cosec\frac{x}{2}\,Cot\frac{x}{2}\right)dx$$

$$\boxed{I = \frac{-2}{7}Cosec^7\frac{x}{2} + \frac{4}{5}Cosec^5\frac{x}{2} - \frac{2}{3}Cosec^3\frac{x}{2} + c}$$ Ans.

(xxix) $\int Cot^2 2x\,Cosec^4 2x\,dx$

Solution: let $I = \int Cot^2 2x \cdot Cosec^4 2x\,dx$

$$I = \int Cot^2 2x \cdot Cos^2 2x \cdot Cosec^2 2x\,dx$$

$$I = \int Cot^2 2x \cdot (1 + Cot^2 2x)\,Cosec^2 2x\,dx$$

$$I = \int Cot^2 2x \cdot Cosec^2 2x\,dx + \int Cot^4 2x\,Cosec^2 2x\,dx$$

$$I = \frac{-1}{2}\int Cot^2 2x \cdot (-2Cosec^2 2x\,dx) - \frac{1}{2}\int Cot^4 2x\,(-2Cosec^2 2x)dx$$

$$\boxed{I = \frac{-1}{6}Cot^3 2x - \frac{1}{10}Cot^5 3x + c}$$

(xxx) $\int Cot^2 x\,Cosecx\,dx$

Solution: let $I = \int Cot^2 x \cdot Cosecx\,dx$

$I = \int Cotx \cdot Cotx\,Cosecx\,dx$ ——— (1)

let $t = Cosecx$

$\frac{dt}{dx} = -Cosecx\,Cotx \Rightarrow \boxed{-dt = CosecxCotx\,dx}$

$\therefore\ 1 + Cot^2 x = Cosec^2 x \Rightarrow \boxed{Cotx = \sqrt{t^2 - 1}}$

$(1) \Rightarrow I = \int \sqrt{t^2 - 1}\,(-dt) = -\int \sqrt{t^2 - 1}\,dt$

$$\boxed{\therefore \int \sqrt{x^2 - a^2}\,dx = \frac{1}{2}\sqrt{x^2 - a^2} - \frac{1}{2}a^2\,\ell n\left(x + \sqrt{x^2 - a^2}\right) + c}$$

$$I = -\left\{\frac{1}{2}\sqrt{t^2 - 1} - \frac{1}{2}\ell n\left(t + \sqrt{t^2 - 1}\right)\right\} + c$$

$$I = \frac{-1}{2}\sqrt{Cosec^2 x - 1} + \frac{1}{2}\ell n\left\{Cosecx + \sqrt{Cosec^2 x - 1}\right\} + c$$

$$\boxed{I = \frac{1}{2}\ell n\{Cosecx + Cotx\} - \frac{1}{2}Cotx + c}$$ Ans.

EXERCISE # 6.4

Evaluate the following integrals.

Q1. $\int_0^3 \sqrt[3]{(3t-1)^2}\,dt$

Solution: let $I = \int_0^3 \sqrt[3]{(3t-1)^2}\,dt$

$$I = \int_0^3 \{(3t-1)^2\}^{1/3}\,dt \Rightarrow I = \int_0^3 (3t-1)^{2/3}\,dt$$

$\times$ & $\div$ by 3.

$$I = \frac{1}{3}\int_0^3 (3t-1)^{2/3} \cdot 3\,dt \Rightarrow I = \frac{1}{3}\left[\frac{(3t-1)^{5/3}}{5/3}\right]_0^3$$

$$I = \frac{1}{\cancel{3}} \times \frac{\cancel{3}}{5}\left[(3t-1)^{5/3}\right]_0^3$$

$$I = \frac{1}{5}\left[\{3(3)-1\}^{5/3} - \{3(0)-1\}^{5/3}\right]$$

$$I = \frac{1}{5}\{8^{5/3} - (-1)^{5/3}\} \Rightarrow I = \frac{1}{5}\{(2^3)^{5/3} + 1\}$$

$$I = \frac{1}{5}\{32 + 1\} \Rightarrow \boxed{I = \frac{33}{5}}$$ Ans

Q.2 $\int_{-2}^{1} \sqrt{2 - x}\,dx$

Solution: let $I = \int_{-2}^{1} (2 - x)^{1/2}\,dx$

$\times$ & $\div$ by (-1)

$$I = -\int_{-2}^{1} (2-x)^{1/2}.(-1)\,dx \Rightarrow I = -\left[\frac{(2-x)^{3/2}}{3/2}\right]_{-2}^{1}$$

$$I = \frac{-2}{3}\{(2-1)^{3/2} - (2+2)^{3/2}\} \Rightarrow I = \frac{-2}{3}\{(1)^{3/2} - 4^{3/2}\}$$

$$I = \frac{-2}{3}\{1 - (2^{2})^{3/2}\} \Rightarrow I = \frac{-2}{3}\{1-8\} \Rightarrow I = \frac{-2}{3}(-7)$$

$$\boxed{I = \frac{14}{3}}$$ Ans

Q3. $\int_{-3}^{-1} \frac{dx}{(x-1)^3}$

Solution: let $I = \int_{-3}^{-1} \frac{dx}{(x-1)^3}$

$$I = \int_{-3}^{-1} (x-1)^{-3}\,dx \Rightarrow I = \left[\frac{(x-1)^{-2}}{-2}\right]_{-3}^{-1}$$

$$I = \frac{-1}{2}\{(-1-1)^{-2} - (-3-1)^{-2}\}$$

$$I = \frac{-1}{2}\{(-)^{-2} - (-4)^{-2}\} \Rightarrow I = \frac{-1}{2}\left\{\frac{1}{(-2)^2} - \frac{1}{(-4)^2}\right\}$$

$$I = \frac{-1}{2}\left\{\frac{1}{4} - \frac{1}{16}\right\} \Rightarrow I = \frac{-1}{2}\left\{\frac{4-1}{16}\right\} \Rightarrow I = \frac{-1}{2}\left(\frac{3}{16}\right)$$

$$\boxed{I = \frac{-3}{32}}$$ Ans

Q4. $\int_{-2}^{1} x\sqrt{2x^2+3}\,dx$

Solution: let $I = \int_{-2}^{1} (2x^2+3)^{1/2} \cdot x\,dx$

× & ÷ by 4

$$I = \frac{1}{4}\int_{-2}^{1} (2x^2+3)^{1/2}\,4x\,dx \Rightarrow I = \frac{1}{4}\left[\frac{(2x^2+3)^{3/2}}{3/2}\right]_{-2}^{1}$$

$$I = \frac{1}{\cancel{4}_2} \times \frac{\cancel{2}}{3}\left[\{2(1)^2+3\}^{3/2} - \{2(-2)^2+3\}^{3/2}\right]$$

$$I = \frac{1}{6}\,[5^{3/2} - 11^{3/2}] \Rightarrow \boxed{I = \frac{1}{6}\{\sqrt{125} - \sqrt{1331}\}}$$ Ans

Q5. $\int_{0}^{5} \frac{2x+3}{\sqrt{x^2+3x+1}}\,dx$

Solution: let $I = \int_{0}^{5} \frac{2x+3}{\sqrt{x^2+3x+1}}\,dx$

$$I = \int_{0}^{5} (x^2+3x+1)^{-1/2}.(2x+3)\,dx \Rightarrow I = \left[\frac{(x^2+3x+1)^{1/2}}{1/2}\right]_{0}^{5}$$

$$I = 2\left[\{(5)^2+3(5)+1\}^{1/2} - \{(0)^2+3(0)+1\}^{1/2}\right]$$

$$I = 2\,[(25+15+1)^{1/2} - (1)^{1/2}]$$

$$\boxed{I = 2[\sqrt{41} - 1]}$$ Ans

Q6. $\int_{1}^{2} (2x+1)\sqrt[3]{x^2+x+1}\,dx$

Solution: let $I = \int_{1}^{2} (x^2+x+1)^{1/3}.(2x+1)\,dx$

$$I = \left[\frac{(x^2+x+1)^{4/3}}{4/3}\right]_{1}^{2}$$

$$I = \frac{3}{4}\left[\{(2)^2+2+1\}^{4/3} - \{(1)^2+1+1\}^{4/3}\right]$$

$$I = \frac{3}{4}\{(7)^{4/3} - (3)^{4/3}\}$$

$$\boxed{I = \frac{3}{4}\left[7\sqrt[3]{7} - 3\sqrt[3]{3}\right]}$$ Ans

Q7. $\int_{-2}^{1} x\sqrt{x^2+1}\,dx$

Solution: let $I = \int_{-2}^{1} (x^2+1)^{1/2}.\,x\,dx$

× & ÷ by 2

$$I = \frac{1}{2}\int_{-2}^{1} (x^2+1)^{1/2}.2x\,dx \Rightarrow I = \frac{1}{2}\left[\frac{(x^2+1)^{3/2}}{3/2}\right]_{-2}^{1}$$

$$I = \frac{1}{\cancel{2}} \times \frac{\cancel{2}}{3}\,[\{(1)^2+1\}^{3/2} - \{(-2)^2+1\}^{3/2}]$$

$$\boxed{I = \frac{1}{3}\,[2^{3/2} - 5^{3/2}]}$$ Ans

Q8. $\int_{\pi/3}^{\pi/6} \text{Sin}^2x \text{ Cos}x \, dx$

Solution: let $I = \int_{\pi/3}^{\pi/6} \text{Sin}^2x \text{ Cos}x \, dx$

$$I = \left[\frac{\text{Sin}^3x}{3}\right]_{\pi/3}^{\pi/6} \Rightarrow I = \frac{1}{3}\left[\left(\text{Sin}\frac{\pi}{6}\right)^3 - \left(\text{Sin}\frac{\pi}{3}\right)^3\right]$$

$$I = \frac{1}{3}\left[\left(\frac{1}{2}\right)^3 - \left(\frac{\sqrt{3}}{2}\right)^3\right] \Rightarrow I = \frac{1}{3}\left[\frac{1}{8} - \frac{3\sqrt{3}}{8}\right]$$

$$I = \frac{1}{3}\left(\frac{1 - 3\sqrt{3}}{8}\right) \Rightarrow \boxed{I = \frac{1}{24}\left(1 - 3\sqrt{3}\right)} \text{ Ans}$$

Q9. $\int_0^{\pi/2} \text{Cos}^4x \, dx.$

Solution: let $I = \int_0^{\pi/2} \text{Cos}^4x \, dx.$

$$I = \int_0^{\pi/2} (\text{Cos}^2x)^2 dx \Rightarrow \boxed{\therefore \text{Cos}^2x = \frac{1 + \text{Cos}2x}{2}}$$

$$I = \int_0^{\pi/2} \left(\frac{1 + \text{Cos}2x}{2}\right)^2 dx \Rightarrow I = \frac{1}{4} \int_0^{\pi/2} (1 + \text{Cos}2x)^2 dx.$$

$$\boxed{\therefore (a + b)^2 = a^2 + 2ab + b^2}$$

$$I = \frac{1}{4} \int_0^{\pi/2} (1 + 2\text{Cos}2x + \text{Cos}^2 2x) \, dx$$

$$I = \frac{1}{4} \int_0^{\pi/2} dx + \frac{\cancel{2}}{\cancel{4}_2} \int_0^{\pi/2} \text{Cos}2x \, dx + \frac{1}{4} \int_0^{\pi/2} \text{Cos}^2 2x \, dx$$

$$I = \frac{1}{4} \int_0^{\pi/2} dx + \frac{1}{2} \int_0^{\pi/2} \text{Cos}2x \, dx + \frac{1}{4} \int_0^{\pi/2} \left(\frac{1 + \text{Cos}4x}{2}\right) dx$$

$$I = \frac{1}{4} \int_0^{\pi/2} dx + \frac{1}{2} \int_0^{\pi/2} \text{Cos}2x \, dx + \frac{1}{8} \int_0^{\pi/2} dx + \frac{1}{8} \int_0^{\pi/2} \text{Cos}4x \, dx$$

$$I = \frac{1}{4} \int_0^{\pi/2} dx + \frac{1}{2 \times 2} \int_0^{\pi/2} \text{Cos}2x.2dx + \frac{1}{8} \int_0^{\pi/2} dx + \frac{1}{8 \times 4} \int_0^{\pi/2} \text{Cos}4x.4dx$$

$$I = \frac{1}{4}[x]_0^{\pi/2} + \frac{1}{4}[\text{Sin}2x]_0^{\pi/2} + \frac{1}{8}[x]_0^{\pi/2} + \frac{1}{32}[\text{Sin}4x]_0^{\pi/2}$$

$$I = \frac{1}{4}\left[\frac{\pi}{2} - 0\right] + \frac{1}{4}\left[\text{Sin}\cancel{2}\left(\frac{\pi}{\cancel{2}}\right) - \text{Sin}(0)\right] + \frac{1}{8}\left[\frac{\pi}{2} - 0\right] + \frac{1}{32}\left[\text{Sin}\cancel{4}^2\left(\frac{\pi}{\cancel{2}}\right) - \text{Sin}(0)\right]$$

$$I = \frac{1}{4}\left(\frac{\pi}{2}\right) + \frac{1}{4}[0 - 0] + \frac{1}{8}\left(\frac{\pi}{2}\right) + \frac{1}{32}(0 - 0)$$

$$I = \frac{\pi}{8} + \frac{\pi}{16} \Rightarrow I = \frac{2\pi + \pi}{16} \Rightarrow \boxed{I = \frac{3\pi}{16}} \text{ Ans}$$

Q10. $\int_{\pi/6}^{\pi/2} \frac{\text{Cos}^3x \, dx}{\sqrt{\text{Sin}x}}$

Solution: let $I = \int_{\pi/6}^{\pi/2} \frac{\text{Cos}^3x \, dx}{\sqrt{\text{Sin}x}}$

$$I = \int_{\pi/6}^{\pi/2} \frac{\text{Cos}^2x.\text{Cos}x \, dx}{\sqrt{\text{Sin}x}}$$

$$\boxed{\therefore \text{Cos}^2x = 1 - \text{Sin}^2x}$$

$$I = \int_{\pi/6}^{\pi/2} \frac{(1 - \text{Sin}^2x) \, \text{Cos}x \, dx}{\text{Sin}^{1/2}x} \text{ ———— (1)}$$

let $t = \text{Sin}x$.

diff w.r.t x

$\frac{dt}{dx} = \text{Cos}x. \Rightarrow \boxed{dt = \text{Cos}x \, dx}$

When $x = \frac{\pi}{2}$, $t = \text{Sin}\frac{\pi}{2}$, $\boxed{t = 1}$

When $x = \frac{\pi}{6}$, $t = \text{Sin}\frac{\pi}{6}$, $\boxed{t = \frac{1}{2}}$

$$(1) \Rightarrow I = \int_{1/2}^{1} \frac{(1 - t^2) \, dt}{t^{1/2}}$$

$$I = \int_{1/2}^{1} \left\{\frac{1}{t^{1/2}} - \frac{t^2}{t^{1/2}}\right\} dt \Rightarrow I = \int_{1/2}^{1} \left\{t^{-1/2} - t^{3/2}\right\} dt$$

$$I = \int_{1/2}^{1} t^{-1/2} \, dt - \int_{1/2}^{1} t^{3/2} \, dt$$

$$I = \left[\frac{t^{1/2}}{1/2}\right]_{1/2}^{1} - \left[\frac{t^{5/2}}{5/2}\right]_{1/2}^{1}$$

$$I = 2\left[t^{1/2}\right]_{1/2}^{1} - \frac{2}{5}\left[t^{5/2}\right]_{1/2}^{1}$$

$$I = 2\left\{(1)^{1/2} - \left(\frac{1}{2}\right)^{1/2}\right\} - \frac{2}{5}\left\{(1)^{5/2} - \left(\frac{1}{2}\right)^{5/2}\right\}$$

$$I = 2\left(1 - \frac{1}{\sqrt{2}}\right) - \frac{2}{5}\left(1 - \frac{1}{2^{5/2}}\right)$$

$$I = 2\left\{1 - \frac{1}{\sqrt{2}} - \frac{1}{5} + \frac{1}{5}\left(\frac{1}{2^{5/2}}\right)\right\}$$

$$I = 2\left\{\frac{5-1}{5} - \frac{1}{\sqrt{2}} + \frac{1}{5}\left(\frac{1}{4\sqrt{2}}\right)\right\}$$

$$I = 2\left\{\frac{4}{5} + \frac{1}{20\sqrt{2}} - \frac{1}{\sqrt{2}}\right\} \Rightarrow I = 2\left\{\frac{4}{5} + \frac{1-20}{20\sqrt{2}}\right\}$$

$$I = 2\left\{\frac{4}{5} - \frac{19}{20\sqrt{2}}\right\} = \frac{8}{5} - \frac{19}{10\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$

$$I = \frac{8}{5} - \frac{19\sqrt{2}}{20} \Rightarrow \boxed{I = \frac{32 - 19\sqrt{2}}{20}}$$

Q11. $\int_0^{\pi} \tan^3 x \,\text{Sec}x \, dx.$

Solution: let $I = \int_0^{\pi} \tan^3 x \,\text{Sec}x \, dx$

$$I = \int_0^{\pi} \tan x.\text{Sec}x.\tan^2 x \, dx \Rightarrow I = \int_0^{\pi} \tan x \,\text{Sec}x \,(\text{Sec}^2 x - 1)\, dx$$

$$I = \int_0^{\pi} \text{Sec}^3 x \tan x \, dx - \int_0^{\pi} \text{Sec}x \tan x \, dx$$

$$I = \int_0^{\pi} \text{Sec}^2 x.\ \sec x \tan x \, dx - \int_0^{\pi} \text{Sec}x \tan x \, dx$$

$$I = \left[\frac{\text{Sec}^3 x}{3}\right]_0^{\pi} - \left[\text{Sec}x\right]_0^{\pi}$$

$$I = \frac{1}{3}\left[(\text{Sec}\pi)^3 - (\text{Sec}0)^3\right] - \left[(\text{Sec}\pi) - (\text{Sec}0)\right]$$

$$I = \frac{1}{3}\left[-1 - 1\right] - \left[-1 - 1\right]$$

$$I = \frac{1}{3}(-2) - (-2) \Rightarrow I = \frac{-2}{3} + 2$$

$$I = \frac{-2+6}{3} \Rightarrow \boxed{I = \frac{4}{3}} \text{ Ans}$$

Q12. $\int_0^{\pi/4} \text{Sin}^2 2x.\ \text{Cos}^2 2x \, dx$

Solution: let $I = \int_0^{\pi/4} \text{Sin}^2 2x.\text{Cos}^2 2x \, dx$

$$I = \int_0^{\pi/4} \left(\frac{1 - \text{Cos}4x}{2}\right)\left(\frac{1 + \text{Cos}4x}{2}\right) dx$$

$$I = \frac{1}{4}\int_0^{\pi/4} (1 - \text{Cos}4x)(1 + \text{Cos}4x)\, dx$$

$$\boxed{\therefore a^2 - b^2 = (a-b)(a+b)}$$

$$I = \frac{1}{4}\int_0^{\pi/4} (1 - \text{Cos}^2 4x)\, dx$$

$$I = \frac{1}{4}\int_0^{\pi/4} \text{Sin}^2 4x \, dx. \Rightarrow I = \frac{1}{4}\int_0^{\pi/4} \left(\frac{1 - \text{Cos}8x}{2}\right) dx$$

$$I = \frac{1}{8}\int_0^{\pi/4} dx - \frac{1}{8}\int_0^{\pi/4} \text{Cos}8x \, dx$$

$$I = \frac{1}{8}\left[x\right]_0^{\pi/4} - \frac{1}{64}\left[\text{Sin}8x\right]_0^{\pi/4}$$

$$I = \frac{1}{8}\left(\frac{\pi}{4} - 0\right) - \frac{1}{64}\left\{\text{Sin}8\left(\frac{\pi}{4}\right) - \text{Sin}8(0)\right\}$$

$$I = \frac{1}{8}\left(\frac{\pi}{4}\right) - \frac{1}{64}(0 - 0) \Rightarrow \boxed{I = \frac{\pi}{32}} \text{ Ans}$$

EXERCISE # 6.5

Evaluate the following definite integrals.

(1) $\int_0^1 \frac{dx}{\sqrt{4 - x^2}}$

Solution: let $I = \int_0^1 \frac{dx}{\sqrt{4 - x^2}}$ ——— (1)

let $x = a\,\text{Sin}\theta$ $x = 2\,\text{Sin}\theta$ differentiate w.r to θ $\dfrac{dx}{d\theta} = 2\,\text{Cos}\theta$ $\boxed{dx = 2\,\text{Cos}\theta d\theta}$	when $\underline{x = 1}$ $x = 2\,\text{Sin}\theta$ $1 = 2\,\text{Sin}\theta$ $\text{Sin}\theta = \dfrac{1}{2}$ $\theta = \text{Sin}^{-1}\left(\dfrac{1}{2}\right) \Rightarrow \boxed{\theta = \dfrac{\pi}{6}}$ when $\underline{x = 0}$ $0 = 2\,\text{Sin}\theta$ $\text{Sin}\theta = 0$ $\theta = \text{Sin}^{-1}(0) \Rightarrow \boxed{\theta = 0}$

$$(1) \Rightarrow I = \int_0^{\pi/6} \frac{2\,\text{Cos}\theta\, d\theta}{\sqrt{4 - 4\text{Sin}^2\theta}}$$

$$I = \int_0^{\pi/6} \frac{2\,\text{Cos}\theta\, d\theta}{\sqrt{4\,(1 - \text{Sin}^2\theta)}} \Rightarrow I = \int_0^{\pi/6} \frac{\cancel{2}\text{Cos}\theta\, d\theta}{\cancel{2}\sqrt{1 - \text{Sin}^2\theta}}$$

$$I = \int_0^{\pi/6} \frac{\text{Cos}\theta\, d\theta}{\sqrt{\text{Cos}^2\theta}} \Rightarrow I = \int_0^{\pi/6} \frac{\cancel{\text{Cos}\theta}\, d\theta}{\cancel{\text{Cos}\theta}} \Rightarrow$$

$$I = \int_0^{\pi/6} d\theta \Rightarrow I = [\,\theta\,]_0^{\pi/6} \Rightarrow I = \frac{\pi}{6} - 0 \Rightarrow \boxed{I = \frac{\pi}{6}} \text{ Ans}$$

(2) $\displaystyle\int_{-6}^{-2\sqrt{3}} \frac{dx}{x\sqrt{x^2 - 9}}$

Solution: let $I = \displaystyle\int_{-6}^{-2\sqrt{3}} \frac{dx}{x\sqrt{x^2 - 9}}$ ——— (1)

let $x = a\,\text{Sec}\theta$ $x = 3\,\text{Sec}\theta$ differentiate w.r to θ $\dfrac{dx}{d\theta} = 3\,\text{Sec}\theta\tan\theta$ $\boxed{dx = 3\,\text{Sec}\theta\tan\theta\, d\theta}$	when $\underline{x = -2\sqrt{3}}$ $x = 3\,\text{Sec}\theta$ $\dfrac{1}{x} = \dfrac{1}{3\,\text{Sec}\theta}$ $\text{Cos}\theta = \dfrac{3}{x} \Rightarrow \text{Cos}\theta = \dfrac{3}{-2\sqrt{3}} \times \dfrac{\sqrt{3}}{\sqrt{3}}$ $\text{Cos}\theta = \dfrac{-\sqrt{3}}{2}$ $\theta = \text{Cos}^{-1}\left(\dfrac{-\sqrt{3}}{2}\right) \Rightarrow \boxed{\theta = \dfrac{5\pi}{6}}$ **when $\underline{x = 6}$** $x = 3\,\text{Sec}\theta$ $\text{Cos}\theta = \dfrac{3}{x} \Rightarrow \text{Cos}\theta = \dfrac{\cancel{3}}{-\cancel{6}_2} = \dfrac{-1}{2}$ $\theta = \text{Cos}^{-1}\left(\dfrac{-1}{2}\right) \Rightarrow \boxed{\theta = \dfrac{2\pi}{3}}$

$$(1) \Rightarrow = \int_{2\pi/3}^{5\pi/6} \frac{\cancel{3\text{Sec}\theta}\tan\theta\, d\theta}{\cancel{3\,\text{Sec}\theta}\sqrt{9\,\text{Sec}^2\theta - 9}}$$

$$I = \int_{2\pi/3}^{5\pi/6} \frac{\tan\theta\, d\theta}{\sqrt{9(\text{Sec}^2\theta - 1)}} \Rightarrow I = \frac{1}{3}\int_{2\pi/3}^{5\pi/6} \frac{\tan\theta\, d\theta}{\sqrt{\text{Sec}^2\theta - 1}}$$

$$I = \frac{1}{3}\int_{2\pi/3}^{5\pi/6} \frac{\tan\theta\, d\theta}{\sqrt{\tan^2\theta}} \Rightarrow I = \frac{1}{3}\int_{2\pi/3}^{5\pi/6} \frac{\cancel{\tan\theta}\, d\theta}{\cancel{\tan\theta}}$$

$$I = \frac{1}{3}\int_{2\pi/3}^{5\pi/6} d\theta \Rightarrow I = \frac{1}{3}[\,\theta\,]_{2\pi/3}^{5\pi/6}$$

$$I = \frac{1}{3}\left\{\frac{5\pi}{6} - \frac{2\pi}{3}\right\} \Rightarrow I = \frac{1}{3}\left\{\frac{5\pi - 4\pi}{6}\right\} \Rightarrow \boxed{I = \frac{\pi}{18}} \text{ Ans}$$

(3) $\displaystyle\int_0^2 \frac{x^2\, dx}{\sqrt{x^2 + 4}}$

Solution: let $I = \displaystyle\int_0^2 \frac{x^2\, dx}{\sqrt{x^2 + 4}}$ ——— (1)

let $x = a\tan\theta$ $x = 2\tan\theta$ differentiate w.r to θ $\dfrac{dx}{d\theta} = 2\text{Sec}\theta$ $\boxed{dx = 3\text{Sec}^2\theta\, d\theta}$	**When $\underline{x = 2}$** $x = 2\tan\theta$ $\cancel{2} = \cancel{2}\tan\theta$ $\tan\theta = 1 \Rightarrow \theta = \tan^{-1}(1)$ $\boxed{\theta = \dfrac{\pi}{4}}$ **When $\underline{x = 0}$** $x = 2\tan\theta$ $0 = 2\tan\theta$ $\tan\theta = 0$ $\theta = \tan^{-1}(0) \Rightarrow \boxed{\theta = 0}$

(1)⇒ $I=\int_0^{\pi/4}\frac{4\tan^2\theta\cdot 2\sec^2\theta\,d\theta}{\sqrt{4\tan^2\theta+4}} \Rightarrow I=\int_0^{\pi/4}\frac{4\tan^2\theta\cdot \cancel{2}\sec^2\theta\,d\theta}{\cancel{2}\sqrt{1+\tan^2\theta}}$

$I=4\int_0^{\pi/4}\frac{\tan^2\theta\cdot\sec^2\theta\,d\theta}{\sqrt{\sec^2\theta}} \Rightarrow I=4\int_0^{\pi/4}\frac{\tan^2\theta\cdot\sec^2\theta}{\sec\theta}\,d\theta$

$I=4\int_0^{\pi/4}\tan^2\theta\cdot\sec\theta\,d\theta$

$I=4\int_0^{\pi/4}\tan\theta\cdot\tan\theta\cdot\sec\theta\,d\theta$ ——— (2)

let $x=\sec\theta$

diff w.r to θ

$\frac{dx}{d\theta}=\sec\theta\tan\theta \Rightarrow \boxed{dx=\sec\theta\tan\theta\,d\theta}$

If $x=\sec\theta \Rightarrow x^2=\sec^2\theta \Rightarrow x^2=1+\tan^2\theta$

$\tan^2\theta=x^2-1 \Rightarrow \boxed{\tan\theta=\sqrt{x^2-1}}$

When $\theta = 0$

$x=\sec\theta$

$x=\sec 0$

$\boxed{x=1}$

When $\theta=\frac{\pi}{4}$

$x=\sec\theta \Rightarrow x=\sec\frac{\pi}{4}$

$\boxed{x=\sqrt{2}}$

(2)⇒ $I=4\int_1^{\sqrt{2}}\sqrt{x^2-1}\,dx$

$\boxed{\therefore \int\sqrt{x^2-a^2}\,dx=\frac{1}{2}x\sqrt{x^2-a^2}-\frac{1}{2}a^2\ln\left(x+\sqrt{x^2-a^2}+c\right)}$

$I=4\left[\frac{1}{2}x\sqrt{x^2-1}-\frac{1}{2}\ln\left(x+\sqrt{x^2-1}\right)\right]_1^{\sqrt{2}}$

$I=4\left[\frac{\sqrt{2}}{2}\sqrt{2-1}-\frac{1}{2}\ln\left(\sqrt{2}+\sqrt{2-1}\right)-\frac{1}{2}\sqrt{1-1}+\frac{1}{2}\ln\left(1+\sqrt{1-1}\right)\right]$

$I=4\left[\frac{\sqrt{2}}{2}-\frac{1}{2}\ln\left(\sqrt{2}+1\right)\right] \Rightarrow \boxed{I=2\sqrt{2}-2\ln\left(1+\sqrt{2}\right)}$ Ans.

(4) $\int_0^2\frac{y^3\,dy}{\sqrt{16-y^2}}$

Solution: let $I=\int_0^2\frac{y^3\,dy}{\sqrt{16-y^2}}$ ——— (1)

$\therefore \sqrt{a^2-x^2} \Rightarrow y=a\sin\theta$

$y=4\sin\theta$

differentiate w.r to θ

$\frac{dy}{d\theta}=4\cos\theta$

$\boxed{dy=4\cos\theta\,d\theta}$

when $y=2$

$y=4\sin\theta$

$2=4\sin\theta \Rightarrow \frac{\cancel{2}}{\cancel{4}_2}=\sin\theta$

$\sin\theta=\frac{1}{2} \Rightarrow \theta=\sin^{-1}\left(\frac{1}{2}\right)$

$\boxed{\theta=\frac{\pi}{6}}$

when $y=0$

$y=4\sin\theta$

$0=4\sin\theta$

$\sin\theta=0$

$\theta=\sin^{-1}(0)$

$\boxed{\theta=0}$

(1)⇒ $I=\int_0^{\pi/6}\frac{64\sin^3\theta.4\cos\theta\,d\theta}{\sqrt{16-16\sin^2\theta}} \Rightarrow I=\int_0^{\pi/6}\frac{64\sin^3\theta.4\cos\theta\,d\theta}{\sqrt{16(1-\sin^2\theta)}}$

$I=\int_0^{\pi/6}\frac{64\sin^3\theta.\cancel{4}\cos\theta\,d\theta}{\cancel{4}\sqrt{\cos^2\theta}} \Rightarrow I=\int_0^{\pi/6}\frac{64\sin^3\theta\,\cancel{\cos}\,\theta\,d\theta}{\cancel{\cos}\,\theta}$

$I=\int_0^{\pi/6}64\sin^3\theta\,d\theta \Rightarrow I=\int_0^{\pi/6}64.\sin\theta\sin^2\theta\,d\theta$

$I=\int_0^{\pi/6}64.\sin\theta.(1-\cos^2\theta)\,d\theta$

$I=64\int_0^{\pi/6}\sin\theta d\theta-64\int_0^{\pi/6}\cos^2\theta\sin\theta\,d\theta$

$I=64\left[-\cos\theta\right]_0^{\pi/6}+64\left[\frac{\cos^3\theta}{3}\right]_0^{\pi/6}$

$I=-64\left\{\cos\frac{\pi}{6}-\cos 0\right\}+\frac{64}{3}\left\{\left(\cos\frac{\pi}{6}\right)^3-(\cos 0)^3\right\}$

$I=-64\left\{\frac{\sqrt{3}}{2}-1\right\}+\frac{64}{3}\left\{\left(\frac{\sqrt{3}}{2}\right)^3-(1)^3\right\}$

$I=-32\sqrt{3}+64+\frac{64}{3}\left\{\frac{3\sqrt{3}}{8}-1\right\}$

$$I = -32\sqrt{3} + 64 + 8\sqrt{3} - \frac{64}{3} \Rightarrow I = \frac{192-94}{3} - 24\sqrt{3}$$

$$I = \frac{128}{3} - 24\sqrt{3} \Rightarrow \boxed{I = \frac{8}{3}\left(16 - 9\sqrt{3}\right)} \text{ Ans}$$

(5) $\displaystyle\int_0^{2\sqrt{3}} \frac{x^3\,dx}{\sqrt{x^2+4}}$

Solution: let $I = \displaystyle\int_0^{2\sqrt{3}} \frac{x^3\,dx}{\sqrt{x^2+4}}$ ——— (1)

$\therefore \sqrt{x^2+a^2} \qquad \Rightarrow x = a\tan\theta$

$x = 2\tan\theta$

differentiate w.r to θ

$$\frac{dx}{d\theta} = 2\,\mathrm{Sec}^2\theta$$

$$\boxed{dx = 2\,\mathrm{Sec}^2\theta\,d\theta}$$

when $x = 0$

$x = 2\tan\theta$

$0 = 2\tan\theta$

$\tan\theta = 0$

$\theta = \tan^{-1}(0)$

$\boxed{\theta = 0}$

when $x = 2\sqrt{3}$

$x = 2\,\mathrm{tam}\theta$

$\cancel{2}\sqrt{3} = \cancel{2}\tan\theta$

$\theta = \tan^{-1}(\sqrt{3}) \Rightarrow \boxed{\theta = \frac{\pi}{3}}$

$$(1) \Rightarrow I = \int_0^{\pi/3} \frac{8\tan^3\theta.\,2\,\mathrm{Sec}^2\theta\,d\theta}{\sqrt{4\tan^2\theta + 4}} \Rightarrow I = \int_0^{\pi/3} \frac{8\tan^3\theta.\,2\,\mathrm{Sec}^2\theta\,d\theta}{\sqrt{4(\tan^2\theta + 1)}}$$

$$I = \int_0^{\pi/3} \frac{8\tan^3\theta.\,\cancel{2}\,\mathrm{Sec}^2\theta\,d\theta}{\cancel{2}\sqrt{1+\tan^2\theta}} \Rightarrow I = \int_0^{\pi/3} \frac{8\tan^3\theta.\,\mathrm{Sec}^2\theta\,d\theta}{\sqrt{\mathrm{Sec}^2\theta}}$$

$$I = \int_0^{\pi/3} \frac{8\tan^3\theta.\,\cancel{\mathrm{Sec}^2\theta}\,d\theta}{\cancel{\mathrm{Sec}\theta}} \Rightarrow I = \int_0^{\pi/3} 8.\tan^3\theta.\,\mathrm{Sec}\theta\,d\theta$$

$$I = \int_0^{\pi/3} 8\tan^2\theta.\,\tan\theta\,\mathrm{Sec}\theta\,d\theta$$

$$I = \int_0^{\pi/3} 8(\mathrm{Sec}^2\theta - 1)\tan\theta\,\mathrm{Sec}\theta\,d\theta$$

$$I = 8\int_0^{\pi/3} \mathrm{Sec}^3\theta\tan\theta\,d\theta - 8\int_0^{\pi/3}\tan\theta\,\mathrm{Sec}\theta\,d\theta$$

$$I = 8\int_0^{\pi/3} \mathrm{Sec}^2\theta.\mathrm{Sec}\theta\tan\theta\,d\theta - 8\int_0^{\pi/3}\tan\theta\,\mathrm{Sec}\theta\,d\theta$$

$$I = 8\left[\frac{\mathrm{Sec}^3\theta}{3}\right]_0^{\pi/3} - 8\left[\mathrm{Sec}\theta\right]_0^{\pi/3}$$

$$I = \frac{8}{3}\left\{\left(\mathrm{Sec}\frac{\pi}{3}\right)^3 - (\mathrm{Sec}\,0)^3\right\} - 8\left\{\left(\mathrm{Sec}\frac{\pi}{3}\right) - (\mathrm{Sec}\,0)\right\}$$

$$I = \frac{8}{3}\{8-1\} - 8\{2-1\} \Rightarrow I = \frac{8}{3}(7) - 8(1)$$

$$I = \frac{56}{3} - 8 \Rightarrow I = \frac{56-24}{3} \Rightarrow \boxed{I = \frac{32}{3} = 10\frac{2}{3}} \text{ Ans}$$

(6) $\displaystyle\int_0^1 \frac{x^3\,dx}{\sqrt{4-x^2}}$

Solution: let $I = \displaystyle\int_0^1 \frac{x^3\,dx}{\sqrt{4-x^2}}$ ——— (1)

$\therefore \sqrt{a^2-x^2} \qquad \Rightarrow x = a\,\mathrm{Sin}\theta$

$x = 2\,\mathrm{Sin}\theta$

differentiate w.r to θ

$$\frac{dx}{d\theta} = 2\,\mathrm{Cos}\theta$$

$$\boxed{dx = 2\,\mathrm{Cos}\theta\,d\theta}$$

when $x = 0$

$x = 2\,\mathrm{Sin}\theta$

$0 = 2\,\mathrm{Sin}\theta$

$\mathrm{Sin}\theta = 0$

$\theta = \mathrm{Sin}^{-1}(0)$

$\boxed{\theta = 0}$

when $x = 1$

$x = 2\,\mathrm{Sin}\theta$

$1 = 2\,\mathrm{Sin}\theta \Rightarrow \frac{1}{2} = \mathrm{Sin}\theta$

$\theta = \mathrm{Sin}^{-1}\left(\frac{1}{2}\right) \Rightarrow \boxed{\theta = \frac{\pi}{6}}$

$$(1) \Rightarrow I = \int_0^{\pi/6} \frac{8\,\mathrm{Sin}^3\theta.\,2\,\mathrm{Cos}\theta\,d\theta}{\sqrt{4 - 4\,\mathrm{Sin}^2\theta}} \Rightarrow I = \int_0^{\pi/6} \frac{8\,\mathrm{Sin}^3\theta.\,2\,\mathrm{Cos}\theta\,d\theta}{\sqrt{4(1 - \mathrm{Sin}^2\theta)}}$$

$$I = \int_0^{\pi/6} \frac{8\,\mathrm{Sin}^3\theta.\,\cancel{2}\,\mathrm{Cos}\theta\,d\theta}{\cancel{2}\sqrt{1 - \mathrm{Sin}^2\theta}} \Rightarrow I = \int_0^{\pi/6} \frac{8\,\mathrm{Sin}^3\theta.\,\mathrm{Cos}\theta\,d\theta}{\sqrt{\mathrm{Cos}^2\theta}}$$

$$I = \int_0^{\pi/6} \frac{8\,\mathrm{Sin}^3\theta.\,\cancel{\mathrm{Cos}\theta}\,d\theta}{\cancel{\mathrm{Cos}\theta}} \Rightarrow I = \int_0^{\pi/6} 8.\,\mathrm{Sin}^3\theta.d\theta$$

$$I = \int_0^{\pi/6} 8\,Sin^2\theta.\,Sin\theta\,d\theta$$

$$I = \int_0^{\pi/6} 8\,(1 - Cos^2\theta)\,Sin\theta\,d\theta$$

$$I = 8\int_0^{\pi/6} Sin\theta\,d\theta - 8\int_0^{\pi/6} Cos^2\theta\,Sin\theta\,d\theta$$

$$I = 8\left[-Cos\theta\right]_0^{\pi/6} + 8\left[\frac{Cos^3\theta}{3}\right]_0^{\pi/6}$$

$$I = -8\left\{Cos\frac{\pi}{6} - Cos\,0\right\} + \frac{8}{3}\left\{\left(Cos\frac{\pi}{6}\right)^3 - (Cos\,0)^3\right\}$$

$$I = -8\left\{\frac{\sqrt{3}}{2} - 1\right\} + \frac{8}{3}\left\{\left(\frac{\sqrt{3}}{2}\right)^3 - (1)^3\right\}$$

$$I = -8\left\{\frac{\sqrt{3}}{2} - 1\right\} + \frac{8}{3}\left\{\frac{3\sqrt{3}}{8} - 1\right\}$$

$$I = -4\sqrt{3} + 8 + \sqrt{3} - \frac{8}{3} \Rightarrow I = -3\sqrt{3} + \frac{24-8}{3}$$

$$I = -3\sqrt{3} + \frac{16}{3} \Rightarrow \boxed{I = \frac{1}{3}\left(16 - 9\sqrt{3}\right)}\ \text{Ans}$$

(7) $\displaystyle\int_0^a \frac{dx}{(a^2 + x^2)^{3/2}}$

Solution: let $I = \displaystyle\int_0^a \frac{dx}{(a^2 + x^2)^{3/2}}$ ——— (1)

let $x = a\tan\theta$

differentiate w.r to θ

$\dfrac{dx}{d\theta} = a\,Sec^2\theta$

$\boxed{dx = a\,Sec^2\theta\,d\theta}$

when $x = a$

$x = a\tan\theta$

$\cancel{a} = \cancel{a}\tan\theta$

$\tan\theta = 1$

$\theta = \tan^{-1}(1) \Rightarrow \boxed{\theta = \frac{\pi}{4}}$

when $x = 0$

$x = a\tan\theta$

$0 = a\tan\theta$

$\tan\theta = 0$

$\theta = \tan^{-1}(0) \Rightarrow \boxed{\theta = 0}$

$$(1) \Rightarrow \int_0^{\pi/4} \frac{a\,Sec^2\theta\,d\theta}{(a^2 + a^2\tan^2\theta)^{3/2}} \Rightarrow I = \int_0^{\pi/4} \frac{a\,Sec^2\theta\,d\theta}{(a^2)^{3/2}(1 + \tan^2\theta)^{3/2}}$$

$$I = \int_0^{\pi/4} \frac{\cancel{a}\,Sec^2\theta\,d\theta}{a^{\cancel{3}}(Sec^2\theta)^{3/2}} \Rightarrow I = \int_0^{\pi/4} \frac{Sec^2\theta\,d\theta}{a^2\,Sec^3\theta}$$

$$I = \frac{1}{a^2}\int_0^{\pi/4} \frac{1}{Sec\theta}\,d\theta \Rightarrow I = \frac{1}{a^2}\int_0^{\pi/4} Cos\theta\,d\theta$$

$$I = \frac{1}{a^2}\left[Sin\theta\right]_0^{\pi/4} \Rightarrow I = \frac{1}{a^2}\left\{Sin\frac{\pi}{4} - Sin\,0\right\}$$

$$I = \frac{1}{a^2}\left\{\frac{1}{\sqrt{2}} - 0\right\} \Rightarrow \boxed{I = \frac{1}{a^2\sqrt{2}}}\ \text{Ans}$$

(8) $\displaystyle\int_0^1 \frac{x^2\,dx}{(4 - x^2)^{3/2}}$

Solution: let $I = \displaystyle\int_0^1 \frac{x^2\,dx}{(4 - x^2)^{3/2}}$ ——— (1)

let $x = a\,Sin\theta$

$x = 2\,Sin\theta$

differentiate w.r to θ

$\dfrac{dx}{d\theta} = 2\,Cos\theta$

$\boxed{dx = 2\,Cos\theta\,d\theta}$

when $x = 0$

$x = 2\,Sin\theta$

$0 = 2\,Sin\theta$

$Sin\theta = 0$

$\theta = Sin^{-1}(0) \Rightarrow \boxed{\theta = 0}$

when $x = 1$

$x = 2\,Sin\theta \Rightarrow 1 = 2\,Sin\theta$

$Sin\theta = \dfrac{1}{2}$

$\theta = Sin^{-1}\left(\dfrac{1}{2}\right) \Rightarrow \boxed{\theta = \frac{\pi}{6}}$

$$(1) \Rightarrow I = \int_0^{\pi/6} \frac{4\,Sin^2\theta.\,2\,Cos\theta\,d\theta}{(4 - 4\,Sin^2\theta)^{3/2}} \Rightarrow I = \int_0^{\pi/6} \frac{4\,Sin^2\theta.\,2\,Cos\theta\,d\theta}{4^{3/2}(1 - Sin^2\theta)^{3/2}}$$

$$I = \int_0^{\pi/6} \frac{\cancel{4}\,Sin^2\theta.\,\cancel{2}Cos\theta\,d\theta}{\cancel{8}\,(Cos^2\theta)^{3/2}} \Rightarrow I = \int_0^{\pi/6} \frac{Sin^2\theta.\cancel{Cos\theta}\,d\theta}{Cos^3\theta}$$

$$I = \int_0^{\pi/6} \frac{\sin^2\theta}{\cos^2\theta}\, d\theta \Rightarrow I = \int_0^{\pi/6} \tan^2\theta\, d\theta$$

$$I = \int_0^{\pi/6} (\sec^2\theta - 1)\, d\theta \Rightarrow I = \int_0^{\pi/6} \sec^2\theta\, d\theta - \int_0^{\pi/6} d\theta$$

$$I = [\tan\theta]_0^{\pi/6} - [\theta]_0^{\pi/6} \Rightarrow I = \left\{\tan\frac{\pi}{6} - \tan 0\right\} - \left\{\frac{\pi}{6} - 0\right\}$$

$$I = \frac{1}{\sqrt{3}} - 0 - \frac{\pi}{6} = \frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} - \frac{\pi}{6}$$

$$I = \frac{\sqrt{3}}{3} - \frac{\pi}{6} \Rightarrow \boxed{I = \frac{2\sqrt{3} - \pi}{6}} \text{ Ans}$$

(9) $\int_{\sqrt{3}}^{3\sqrt{3}} \frac{dx}{x^2\sqrt{x^2+9}}$

Solution: let $I = \int_{\sqrt{3}}^{3\sqrt{3}} \frac{dx}{x^2\sqrt{x^2+9}}$ ——— (1)

let $x = a\tan\theta$

$x = 3\tan\theta$

differentiate w • r to θ

$$\frac{dx}{d\theta} = 3\sec^2\theta$$

$$\boxed{dx = 3\sec^2\theta\, d\theta}$$

when $\underline{x = 3\sqrt{3}}$

$x = 3\tan\theta$

$\cancel{3}\sqrt{3} = \cancel{3}\tan\theta$

$\theta = \tan^{-1}(\sqrt{3})$

$\boxed{\theta = \frac{\pi}{3}}$

when $\underline{x = \sqrt{3}}$

$x = 3\tan\theta \Rightarrow \sqrt{3} = 3\tan\theta$

$\tan\theta = \frac{\sqrt{3}}{3} \Rightarrow \tan\theta = \frac{\cancel{\sqrt{3}}}{\cancel{\sqrt{3}} \times \sqrt{3}}$

$\theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) \Rightarrow \boxed{\theta = \frac{\pi}{6}}$

$$(1) \Rightarrow I = \int_{\pi/6}^{\pi/3} \frac{\cancel{3}\sec^2\theta\, d\theta}{{}_3\cancel{9}\tan^2\theta\sqrt{9\tan^2\theta + 9}}$$

$$I = \int_{\pi/6}^{\pi/3} \frac{\sec^2\theta\, d\theta}{3\tan^2\theta\sqrt{9(\tan^2\theta + 1)}} \Rightarrow I = \int_{\pi/6}^{\pi/3} \frac{\sec^2\theta\, d\theta}{9\tan^2\theta\sqrt{\sec^2\theta}}$$

$$I = \int_{\pi/6}^{\pi/3} \frac{\sec^{\cancel{2}}\theta\, d\theta}{9\tan^2\theta.\cancel{\sec\theta}} \Rightarrow I = \int_{\pi/6}^{\pi/3} \frac{\cot^2\theta}{9}\cdot\frac{1}{\cos\theta}\, d\theta$$

$$I = \int_{\pi/6}^{\pi/3} \frac{1}{9}\,\frac{\cos^{\cancel{2}}\theta}{\sin^2\theta} \times \frac{1}{\cancel{\cos\theta}}\, d\theta \Rightarrow I = \frac{1}{9}\int_{\pi/6}^{\pi/3} \operatorname{cosec}^2\theta.\cos\theta\, d\theta$$

$$I = \frac{1}{9}\int_{\pi/6}^{\pi/3} \operatorname{cosec}\theta.\cot\theta\, d\theta \Rightarrow I = \frac{1}{9}[-\operatorname{cosec}\theta]_{\pi/6}^{\pi/3}$$

$$I = \frac{1}{9}\left\{\operatorname{cosec}\frac{\pi}{3} - \operatorname{cosec}\frac{\pi}{6}\right\} \Rightarrow I = \frac{-1}{9}\left\{\frac{2}{\sqrt{3}} - 2\right\}$$

$$\boxed{I = \frac{-2}{9}\left(\frac{1}{\sqrt{3}} - 1\right)} \text{ Ans}$$

(10) $\int_0^{\sqrt{5}} x^2\sqrt{5 - x^2}\, dx$

Solution: let $I = \int_0^{\sqrt{5}} x^2\sqrt{(\sqrt{5})^2 - (x)^2}\, dx$ ——— (1)

let $x = a\sin\theta$

$x = \sqrt{5}\sin\theta$.

differentiate w•r to θ

$$\frac{dx}{d\theta} = \sqrt{5}\cos\theta$$

$$\boxed{dx = \sqrt{5}\cos\theta\, d\theta}$$

when $\underline{x = \sqrt{5}}$

$x = \sqrt{5}\sin\theta \Rightarrow \cancel{\sqrt{5}} = \cancel{\sqrt{5}}\sin\theta$

$\sin\theta = 1$

$\theta = \sin^{-1}(1) \Rightarrow \boxed{\theta = \frac{\pi}{2}}$

when $\underline{x = 0}$

$x = \sqrt{5}\sin\theta \Rightarrow 0 = \sqrt{5}\sin\theta$

$\sin\theta = 0 \Rightarrow$

$\theta = \sin^{-1}(0)$

$\boxed{\theta = 0}$

$$(1) \Rightarrow I = \int_0^{\pi/2} 5\sin^2\theta\sqrt{5 - 5\sin^2\theta}\,.\sqrt{5}\cos\theta\, d\theta$$

$$I = \int_0^{\pi/2} 5\sin^2\theta\sqrt{5(1 - \sin^2\theta)}\,.\sqrt{5}\cos\theta\, d\theta$$

$$I = \int_0^{\pi/2} 5\sin^2\theta\sqrt{5}\sqrt{1 - \sin^2\theta}\sqrt{5}\cos\theta\, d\theta$$

$$I = 25\int_0^{\pi/2} \sin^2\theta\sqrt{\cos^{\cancel{2}}\theta}\cos\theta\, d\theta$$

$$I = 25\int_0^{\pi/2} \sin^2\theta\cos^2\theta\, d\theta$$

$$I = 25\int_0^{\pi/2}\left(\frac{1-\text{Cos}2\theta}{2}\right)\left(\frac{1+\text{Cos}2\theta}{2}\right)d\theta$$

$$I = \frac{25}{4}\int_0^{\pi/2}(1-\text{Cos}^2 2\theta)\,d\theta$$

$$I = \frac{25}{4}\int_0^{\pi/2}d\theta - \frac{25}{4}\int_0^{\pi/2}\text{Cos}^2 2\theta\,d\theta$$

$$I = \frac{25}{4}\int_0^{\pi/2}d\theta - \frac{25}{8}\int_0^{\pi/2}\left(\frac{1+\text{Cos}4\theta}{2}\right)d\theta$$

$$I = \frac{25}{4}\int_0^{\pi/2}d\theta - \frac{25}{8}\int_0^{\pi/2}d\theta - \frac{25}{8}\int_0^{\pi/2}\text{Cos}4\theta\,d\theta$$

$$I = \frac{25}{4}\int_0^{\pi/2}d\theta - \frac{25}{8}\int_0^{\pi/2}d\theta - \frac{25}{8\times 4}\int_0^{\pi/2}\text{Cos}4\theta.4\,d\theta$$

$$I = \frac{25}{4}[\,\theta\,]_0^{\pi/2} - \frac{25}{8}[\,\theta\,]_0^{\pi/2} - \frac{25}{32}[\,\text{Sin}4\theta\,]_0^{\pi/2}$$

$$I = \frac{25}{4}\left\{\frac{\pi}{2}-0\right\} - \frac{25}{8}\left\{\frac{\pi}{2}-0\right\} - \frac{25}{32}\left\{\text{Sin}4\left(\frac{\pi}{2}\right) - \text{Sin}4\,(0)\right\}$$

$$I = \frac{25}{4}\left(\frac{\pi}{2}\right) - \frac{25}{8}\left(\frac{\pi}{2}\right) - \frac{25}{32}\{0-0\} \Rightarrow I = \frac{25\pi}{8} - \frac{25\pi}{16}$$

$$I = \frac{50\pi - 25\pi}{16} \Rightarrow \boxed{I = \frac{25\pi}{16}} \quad \text{Ans}$$

Determine the following integral.

Q11. $\int \frac{x^3\,dx}{\sqrt{a^2-x^2}}$

Solution: let $I = \int \frac{x^3\,dx}{\sqrt{a^2-x^2}}$ ———— (1)

let $x = a\,\text{Sin}\theta$

differentiate w • r to θ

$$\frac{dx}{d\theta} = a\,\text{Cos}\theta$$

$$\boxed{dx = a\,\text{Cos}\theta\,d\theta}$$

$$(1) \Rightarrow I = \int\frac{a^3\,\text{Sin}^3\theta.\ a\,\text{Cos}\theta\,d\theta}{\sqrt{a^2 - a^2\,\text{Sin}^2\theta}} \Rightarrow I = \int\frac{a^4\,\text{Sin}^3\theta\,\text{Cos}\theta\,d\theta}{\sqrt{a^2\,(1-\text{Sin}^2\theta)}}$$

$$I = \int\frac{a^4\,\text{Sin}^3\theta\,\text{Cos}\theta\,d\theta}{a\sqrt{1-\text{Sin}^2\theta}} \Rightarrow I = \int\frac{a^3\,\text{Sin}^3\theta\,\text{Cos}\theta\,d\theta}{\sqrt{\text{Cos}^2\theta}}$$

$$I = \int\frac{a^3\,\text{Sin}^3\theta\,\cancel{\text{Cos}\theta}\,d\theta}{\cancel{\text{Cos}\theta}} \Rightarrow I = a^3\int\text{Sin}^3\theta\,d\theta$$

$$I = a^3\int\text{Sin}^2\theta.\ \text{Sin}\theta\,d\theta \Rightarrow I = a^3\int(1-\text{Cos}^2\theta)\ \text{Sin}\theta\,d\theta$$

$$I = a^3\int\text{Sin}\theta\,d\theta - a^3\int\text{Cos}^2\theta\,\text{Sin}\theta\,d\theta$$

$$I = a^3[-\text{Cos}\theta] + a^3\left[\frac{\text{Cos}^3\theta}{3}\right] + c$$

$$I = a^3\left\{-\text{Cos}\theta + \frac{\text{Cos}^3\theta}{3}\right\} + c \text{ ———— (1)}$$

let $x = a\,\text{Sin}\theta$

Squaring on b • s

$$x^2 = a^2\,\text{Sin}^2\theta \Rightarrow \frac{x^2}{a^2} = \text{Sin}^2\theta \Rightarrow \frac{x^2}{a^2} = 1 - \text{Cos}^2\theta$$

$$\text{Cos}^2\theta = 1 - \frac{x^2}{a^2} \Rightarrow \text{Cos}^2\theta = \frac{a^2-x^2}{a^2} \Rightarrow \text{Cos}\theta = \frac{\sqrt{a^2-x^2}}{a}$$

$$(1) \Rightarrow I = a^3\left\{-\frac{\sqrt{a^2-x^2}}{a} + \frac{1}{3}\left(\frac{\sqrt{a^2-x^2}}{a}\right)^3\right\} + c$$

$$I = a^3\left\{-\frac{\sqrt{a^2-x^2}}{a} + \frac{1}{3a^3}(a^2-x^2)^{3/2}\right\} + c$$

$$\boxed{I = \frac{1}{3}(a^2-x^2)^{3/2} - a^2\sqrt{a^2-x^2} + c} \quad \text{Ans}$$

Q12. $\int \frac{x^3\,dx}{\sqrt{2x^2+7}}$

Solution: let $I = \int\frac{x^3\,dx}{\sqrt{2x^2+7}}$ ———— (1)

$$\therefore \sqrt{2x^2+7} = \sqrt{(\sqrt{2}\,x)^2 + (\sqrt{7})^2}$$

$$\therefore \sqrt{x^2+a^2} \Rightarrow x = a\tan\theta \Rightarrow \sqrt{2}\,x = \sqrt{7}\tan\theta \Rightarrow x = \frac{\sqrt{7}}{\sqrt{2}}\tan\theta$$

differentiate w • r to θ

$$\frac{dx}{d\theta} = \frac{\sqrt{7}}{\sqrt{2}}\text{Sec}^2\theta \Rightarrow \boxed{dx = \frac{\sqrt{7}}{\sqrt{2}}\text{Sec}^2\theta\,d\theta}$$

$$(1) \Rightarrow I = \frac{\int\frac{7\sqrt{7}}{2\sqrt{2}}\tan^3\theta \bullet \frac{\sqrt{7}}{\sqrt{2}}\text{Sec}^2\theta\,d\theta}{\sqrt{2\left(\frac{7}{2}\tan^2\theta\right)+7}}$$

$$I = \frac{49}{4}\int\frac{\tan^3\theta\,\text{Sec}^2\theta\,d\theta}{\sqrt{7\,(1+\tan^2\theta)}} \Rightarrow I = \frac{49}{4}\int\frac{\tan^3\theta.\text{Sec}^2\theta\,d\theta}{\sqrt{7}\,\sqrt{1+\tan^2\theta}}$$

$$I = \frac{49}{4}\int \frac{\tan^3\theta.\mathrm{Sec}^2\theta\, d\theta}{\sqrt{7}\sqrt{\mathrm{Sec}^{\cancel{2}}\theta}} \Rightarrow I = \frac{\frac{49}{4}}{\sqrt{7}}\int \frac{\tan^3\theta.\mathrm{Sec}^{\cancel{2}}\theta\, d\theta}{\cancel{\mathrm{Sec}\theta}}$$

$$I = \frac{49}{4\sqrt{7}} \times \frac{\sqrt{7}}{\sqrt{7}}\int \tan^2\theta \bullet \tan\theta.\mathrm{Sec}\theta\, d\theta$$

$$I = \frac{49\sqrt{7}}{4\times 7}\int (\mathrm{Sec}^2\theta - 1)\tan\theta\, \mathrm{Sec}\theta\, d\theta$$

$$I = \frac{7\sqrt{7}}{4}\int \mathrm{Sec}^2\theta.\mathrm{Sec}\theta\tan\theta\, d\theta - \frac{7\sqrt{7}}{4}\int \mathrm{Sec}\theta\tan\theta\, d\theta$$

$$I = \frac{7\sqrt{7}}{4}\int\left(\frac{\mathrm{Sec}^3\theta}{3}\right) - \frac{49}{4\sqrt{7}}(\mathrm{Sec}\theta) + c$$

$$I = \frac{7\sqrt{7}}{4}\left\{\frac{\mathrm{Sec}^3\theta}{3} - \mathrm{Sec}\theta\right\} + c \text{ ——— (1)}$$

$\therefore \sqrt{2}\,x = \sqrt{7}\tan\theta$

Squaring on b • S

$2x^2 = 7\tan^2\theta$

$$\frac{2x^2}{7} = \mathrm{Sec}^2\theta - 1 \Rightarrow \mathrm{Sec}^2\theta = \frac{2x^2}{7} + 1 \Rightarrow \mathrm{Sec}^2\theta = \frac{2x^2+7}{7}$$

$$\mathrm{Sec}\theta = \frac{\sqrt{2x^2+7}}{\sqrt{7}}$$

$$(1) \Rightarrow I = \frac{7\sqrt{7}}{4}\left\{\left[\frac{\sqrt{2x^2+7}}{\sqrt{7}}\right]^3 - \frac{\sqrt{2x^2+7}}{\sqrt{7}}\right\} + c$$

$$I = \frac{7\sqrt{7}}{4}\left\{\frac{1}{3}\left(\frac{2x^2+7}{7}\right)^{3/2} - \left(\frac{2x^2+7}{7}\right)^{1/2}\right\} + c \text{ Ans}$$

Q13. $\int u^3\sqrt{a^2u^2+b^2}\, du.$

Solution: let $I = \int u^3\sqrt{a^2u^2+b^2}\, du.$

$$I = \int u^3\sqrt{a^2\left(u^2 + \frac{b^2}{a^2}\right)}\, du.$$

$$I = a\int u^3\sqrt{(u)^2 + \left(\frac{b}{a}\right)^2}\, du. \text{ ——— (1)}$$

let $u = a\tan\theta$

$u = \frac{b}{a}\tan\theta$

differentiate w • r to θ

$$\frac{du}{d\theta} = \frac{b}{a}\mathrm{Sec}^2\theta \Rightarrow \boxed{du = \frac{b}{a}\mathrm{Sec}^2\theta\, d\theta}$$

$$(1) \Rightarrow I = \cancel{a}\int \frac{b^2}{a^{\cancel{3}}}\tan^3\theta\sqrt{\frac{b^2}{a^2}\tan^2\theta + \frac{b^2}{a^2}} \bullet \left(\frac{b}{a}\mathrm{Sec}^2\theta\, d\theta\right)$$

$$I = \frac{b^3}{a^2}\int \tan^3\theta\sqrt{\frac{b^2}{a^2}(\tan^2\theta + 1)}\;\frac{b}{a}\mathrm{Sec}^2\theta\, d\theta$$

$$I = \frac{b^3}{a^2}\int \tan^3\theta\,\frac{b}{a}\sqrt{1+\tan^2\theta}\;\frac{b}{a}\mathrm{Sec}^2\theta.d\theta$$

$$I = \frac{b^5}{a^4}\int \tan^3\theta\sqrt{\mathrm{Sec}^{\cancel{2}}\theta}\;\mathrm{Sec}^2\theta\, d\theta$$

$$I = \frac{b^5}{a^4}\int \tan^3\theta\,.\,\mathrm{Sec}\theta.\,\mathrm{Sec}^2\theta\, d\theta$$

$$I = \frac{b^5}{a^4}\int \tan^2\theta\,.\,\tan\theta.\,\mathrm{Sec}^2\theta.\,\mathrm{Sec}\theta\, d\theta$$

$$I = \frac{b^5}{a^4}\int (\mathrm{Sec}^2\theta - 1)\;\mathrm{Sec}^2\theta.\,\mathrm{Sec}\theta\tan\theta\, d\theta$$

$$I = \frac{b^5}{a^4}\int (\mathrm{Sec}^4\theta.\,\mathrm{Sec}\theta\tan\theta - \mathrm{Sec}^2\theta\,.\mathrm{Sec}\theta\tan\theta)\, d\theta$$

$$I = \frac{b^5}{a^4}\int \mathrm{Sec}^4\theta.\,\mathrm{Sec}\theta\tan\theta\, d\theta - \frac{b^5}{a^4}\int \mathrm{Sec}^2\theta.\,\mathrm{Sec}\theta\tan\theta\, d\theta$$

$$I = \frac{b^5}{a^4}\int\left(\frac{\mathrm{Sec}^5\theta}{5}\right) - \frac{b^5}{a^4}\left(\frac{\mathrm{Sec}^3\theta}{3}\right) + c$$

$$I = \frac{b^5}{a^4}\left\{\frac{\mathrm{Sec}^5\theta}{5} - \frac{\mathrm{Sec}^3\theta}{3}\right\} + c \text{ ——— (2)}$$

$\therefore u = \frac{b}{a}\tan\theta$

Squaring on b • s

$$u^2 = \frac{b^2}{a^2}\tan^2\theta \Rightarrow \frac{a^2u^2}{b^2} = \mathrm{Sec}^2\theta - 1 \Rightarrow \mathrm{Sec}^2\theta = \frac{a^2u^2}{b^2} + 1$$

$$\mathrm{Sec}^2\theta = \frac{a^2u^2+b^2}{b^2} \Rightarrow \boxed{\mathrm{Sec}\theta\sqrt{\frac{a^2u^2}{b^2}+1}}$$

$$(1) \Rightarrow I = \frac{b^5}{a^4}\left\{\frac{1}{5}\left(\sqrt{\frac{a^2u^2}{b^2}+1}\right)^5 - \frac{1}{3}\left(\sqrt{\frac{a^2u^2}{b^2}+1}\right)^3\right\} + c$$

$$I = \frac{b^5}{a^4}\left\{\frac{1}{5}\left(\frac{a^2u^2+b^2}{b^2}\right)^{5/2} - \frac{1}{3}\left(\frac{a^2u^2+b^2}{b^2}\right)^{3/2}\right\} + c$$

$$I = \frac{b^5}{a^4}\left\{\frac{1}{5}\frac{(a^2u^2+b^2)^{5/2}}{b^5} - \frac{1}{3}\frac{(a^2u^2+b^2)^{3/2}}{b^3}\right\} + c$$

$$\boxed{I = \frac{1}{5a^4}\left\{(a^2u^2+b^2)^{5/2} - \frac{1}{3a^4}(a^2u^2+b^2)^{3/2}\right\} + c} \text{ Ans}$$

(14) $\int \frac{\sqrt{x^2-a^2}}{x}\,dx$

Solution: let $I = \int \frac{\sqrt{x^2-a^2}}{x}\,dx$ ——— (1)

$\therefore \sqrt{x^2-a^2} \Rightarrow x = a\sec\theta$

differentiate w • r to θ

$\frac{dx}{d\theta} = a\sec\theta\tan\theta$

$\boxed{dx = a\sec\theta\tan\theta\, d\theta}$

(1)⇒ $I = \int \frac{\sqrt{a^2\sec^2\theta - a^2}}{\cancel{a}\,\cancel{\sec\theta}}\,\cancel{a}\,\cancel{\sec\theta}\tan\theta\, d\theta$

$I = \int\sqrt{a^2(\sec^2\theta-1)}\tan\theta\, d\theta \Rightarrow I = a\int\sqrt{\sec^2\theta - 1}\,\tan\theta\, d\theta$

$I = a\int\sqrt{\tan^2\theta}\,\tan\theta\, d\theta \Rightarrow I = a\int\tan^2\theta\, d\theta$

$I = a\int(\sec^2\theta-1)\,d\theta \Rightarrow I = a\int\sec^2\theta\, d\theta - a\int d\theta$

$I = a\tan\theta - a\theta + c$ ——— (2)

$\therefore \sqrt{x^2-a^2} = a\tan\theta$

$\boxed{\tan\theta = \sqrt{\frac{x^2-a^2}{a^2}}}$

$1+\tan^2\theta = \sec^2\theta \Rightarrow 1 + \left(\sqrt{\frac{x^2-a^2}{a^2}}\right)^2 = \sec^2\theta$

$1 + \frac{x^2-a^2}{a^2} = \sec^2\theta \Rightarrow \frac{\cancel{a^2}+x^2-\cancel{a^2}}{a^2} = \sec^2\theta$

$\sec^2\theta = \frac{x^2}{a^2} \Rightarrow \sec\theta = \frac{x}{a} \Rightarrow \boxed{\theta = \sec^{-1}\left(\frac{x}{a}\right)}$ Ans

(1)⇒ $I = a\sqrt{\frac{x^2-a^2}{a^2}} - a\sec^{-1}\left(\frac{x}{a}\right)+c$

$I = \cancel{a}\frac{\sqrt{x^2-a^2}}{\cancel{a}} - a\sec^{-1}\left(\frac{x}{a}\right)+c$

$\boxed{I = (x^2-a^2)^{1/2} - a\sec^{-1}\left(\frac{x}{a}\right)+c}$ Ans

15. $\int\frac{du}{u^2\sqrt{a^2-u^2}}$

Solution: let $I = \int \frac{du}{u^2\sqrt{a^2-u^2}}$ ——— (1)

let u = a Sinθ

differentiate w • r to θ

$\frac{du}{d\theta} = a\cos\theta \Rightarrow \boxed{du = a\cos\theta\, d\theta}$

(1)⇒ $I = \int \frac{\cancel{a}\cos\theta\, d\theta}{a^{\cancel{2}}\sin^2\theta\sqrt{a^2-a^2\sin^2\theta}} \Rightarrow I = \frac{1}{a}\int\frac{\cos\theta\, d\theta}{\sqrt{a^2(1-\sin^2\theta)}}$

$I = \frac{1}{a}\int\frac{\cos\theta\, d\theta}{\sin^2\theta\sqrt{1-\sin^2\theta}} \Rightarrow I = \frac{1}{a^2}\int\frac{\cos\theta\, d\theta}{\sin^2\theta\sqrt{\cos^{\cancel{2}}\theta}}$

$I = \frac{1}{a^2}\int\frac{\cancel{\cos\theta}\, d\theta}{\sin^2\theta\,\cancel{\cos\theta}} \Rightarrow I = \frac{1}{a^2}\int\csc^2\theta\, d\theta$

$I = \frac{1}{a^2}(-\cot\theta)+c$ ——— (2)

$\therefore u = a\sin\theta \Rightarrow \boxed{\sin\theta = \frac{u}{a}}$

$\therefore \sin^2\theta+\cos^2\theta = 1$

$\left(\frac{u}{a}\right)^2+\cos^2\theta = 1$

$\cos^2\theta = 1-\frac{u^2}{a^2} \Rightarrow \cos^2\theta = \frac{a^2-u^2}{a^2} \Rightarrow \boxed{\cos\theta = \frac{\sqrt{a^2-u^2}}{a}}$

$\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{\frac{\sqrt{a^2-u^2}}{\cancel{a}}}{\frac{u}{\cancel{a}}} = \frac{\sqrt{a^2-u^2}}{u}$

(2)⇒ $\boxed{I = \frac{-1}{a^2}\,\frac{\sqrt{a^2-u^2}}{u}+c}$ Ans

(16) $\int\frac{dy}{y^2\sqrt{y^2-a^2}}$

Solution: let $I = \int\frac{dy}{y^2\sqrt{y^2-a^2}}$ ——— (1)

let y = a Secθ

differentiate w • r to θ

$\frac{dy}{d\theta} = a\sec\theta\tan\theta$

$\boxed{dy = a\sec\theta\tan\theta\, d\theta}$

(1)⇒ $I = \int\frac{\cancel{a}\,\cancel{\sec\theta}\tan\theta\, d\theta}{a^{\cancel{2}}\sec^{\cancel{2}}\theta\sqrt{a^2\sec^2\theta-a^2}}$

$$I = \int \frac{\tan\theta\, d\theta}{a\sec\theta\sqrt{a^2(\sec^2\theta - 1)}} \Rightarrow I = \frac{1}{a^2}\int \frac{\tan\theta\, d\theta}{\sec\theta\sqrt{\sec^2\theta - 1}}$$

$$I = \frac{1}{a^2}\int \frac{\tan\theta\, d\theta}{\sec\theta\sqrt{\tan^2\theta}} \Rightarrow I = \frac{1}{a^2}\int \frac{\cancel{\tan\theta}\, d\theta}{\sec\theta\,\cancel{\tan\theta}}$$

$$I = \frac{1}{a^2}\int \cos\theta\, d\theta \Rightarrow I = \frac{1}{a^2}\sin\theta + c \quad \text{———} (2)$$

$\therefore\ y = a\sec\theta$

$$\frac{y}{a} = \sec\theta \Rightarrow \frac{a}{y} = \frac{1}{\sec\theta} \Rightarrow \boxed{\cos\theta = \frac{a}{y}}$$

$\therefore\ \cos^2\theta + \sin^2\theta = 1$

$$\frac{a^2}{y^2} + \sin^2\theta = 1$$

$$\sin^2\theta = 1 - \frac{a^2}{y^2} \Rightarrow \sin^2\theta = \frac{y^2 - a^2}{y^2} \Rightarrow \boxed{\sin\theta = \frac{\sqrt{y^2 - a^2}}{y}}$$

$$(2) \Rightarrow \boxed{I = \frac{1}{a^2}\left(\frac{\sqrt{y^2 - a^2}}{y}\right) + c} \quad \text{Ans}$$

(17) $\int \frac{dx}{(a^2 - x^2)^3}$

Solution: let $I = \int \frac{dx}{\sqrt{(a^2 - x^2)^3}}$ ——— (1)

let $x = a\sin\theta$

differentiate w • r to θ

$$\frac{dx}{d\theta} = a\cos\theta$$

$$\boxed{dx = a\cos\theta\, d\theta}$$

$$(1) \Rightarrow I = \int \frac{a\cos\theta\, d\theta}{(a^2 - a^2\sin^2\theta)^{3/2}} \Rightarrow I = \int \frac{\cancel{a}\cos\theta\, d\theta}{a^{\cancel{3}}(1 - \sin^2\theta)^{3/2}}$$

$$I = \frac{1}{a^2}\int \frac{\cos\theta\, d\theta}{(\cos^2\theta)^{3/2}} \Rightarrow I = \frac{1}{a^2}\int \frac{\cancel{\cos\theta}\, d\theta}{\cos^{\cancel{3}}\theta}$$

$$I = \frac{1}{a^2}\int \frac{1}{\cos^2\theta}\, d\theta \Rightarrow I = \frac{1}{a^2}\int \sec^2\theta\, d\theta$$

$$I = \frac{1}{a^2}\tan\theta + c \quad \text{———} (2)$$

$$\therefore\ x = a\sin\theta \Rightarrow \boxed{\sin\theta = \frac{x}{a}}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \Rightarrow \tan\theta = \frac{\frac{x}{\cancel{a}}}{\frac{\sqrt{a^2 - x^2}}{\cancel{a}}} \Rightarrow \tan\theta = \frac{x}{\sqrt{a^2 - x^2}}$$

Consider $\sin\theta = \frac{x}{a}$

Squaring on b • s

$$\sin^2\theta = \frac{x^2}{a^2} \Rightarrow 1 - \sin^2\theta = 1 - \frac{x^2}{a^2} \Rightarrow \cos^2\theta = \frac{a^2 - x^2}{a^2}$$

$$\boxed{\cos\theta = \frac{\sqrt{a^2 - x^2}}{a}}$$

$$(2) \Rightarrow \boxed{I = \frac{1}{a^2}\left(\frac{x}{\sqrt{a^2 - x^2}}\right) + c} \quad \text{Ans}$$

(18) $\int \frac{dx}{x^4\sqrt{a^2 - x^2}}$

Solution: let $I = \int \frac{dx}{x^4\sqrt{a^2 - x^2}}$ ——— (1)

let $x = a\sin\theta$

differentiate w • r to θ

$$\frac{dx}{d\theta} = a\cos\theta$$

$$\boxed{dx = a\cos\theta\, d\theta}$$

$$(1) \Rightarrow I = \int \frac{\cancel{a}\cos\theta\, d\theta}{a^4\sin^4\theta\sqrt{a^2 - a^2\sin^2\theta}} \Rightarrow I = \frac{1}{a^3}\int \frac{\cos\theta\, d\theta}{\sin^4\theta\sqrt{a^2(1 - \sin^2\theta)}}$$

$$I = \frac{1}{a^4}\int \frac{\cos\theta\, d\theta}{\sin^4\theta\sqrt{1 - \sin^2\theta}} \Rightarrow I = \frac{1}{a^4}\int \frac{\cos\theta\, d\theta}{\sin^4\theta\sqrt{\cos^2\theta}}$$

$$I = \frac{1}{a^4}\int \frac{\cancel{\cos\theta}\, d\theta}{\sin^4\theta.\cancel{\cos\theta}} \Rightarrow I = \frac{1}{a^4}\int \operatorname{cosec}^4\theta\, d\theta$$

$$I = \frac{1}{a^4}\int \operatorname{cosec}^2\theta.\ \operatorname{cosec}^2\theta\, d\theta$$

$$I = \frac{1}{a^4}\int (1 + \cot^2\theta)\operatorname{cosec}^2\theta\, d\theta$$

$$I = \frac{1}{a^4}\int \operatorname{cosec}^2\theta\, d\theta + \frac{1}{a^4}\int \cot^2\theta.\operatorname{cosec}^2\theta\, d\theta$$

$$I = \frac{1}{a^4}\int \operatorname{cosec}^2\theta\, d\theta - \frac{1}{a^4}\int \cot^2\theta.\ (-\operatorname{cosec}^2\theta)\, d\theta$$

$$I = \frac{1}{a^4}\int(-\mathrm{Cot}\theta) - \frac{1}{a^4}\left(\frac{\mathrm{Cot}^3\theta}{3}\right) + c \text{ ——— (2)}$$

$\therefore\ x = a\,\mathrm{Sin}\theta \Rightarrow \mathrm{Sin}\theta = \frac{x}{a}$

$$\mathrm{Cot}\theta = \frac{\mathrm{Cos}\theta}{\mathrm{Sin}\theta} \Rightarrow \mathrm{Cot}\theta = \frac{\frac{\sqrt{a^2-x^2}}{\cancel{a}}}{\frac{x}{\cancel{a}}} \Rightarrow \mathrm{Cot}\theta = \frac{\sqrt{a^2-x^2}}{x}$$

Consider $\mathrm{Sin}\theta = \frac{x}{a}$

Squaring on b • s

$\mathrm{Sin}^2\theta = \frac{x^2}{a^2}$

$$1 - \mathrm{Sin}^2\theta = 1 - \frac{x^2}{a^2} \Rightarrow \mathrm{Cos}^2\theta = \frac{a^2-x^2}{a^2} \Rightarrow \boxed{\mathrm{Cos}\theta = \frac{\sqrt{a^2-x^2}}{a}}$$

$$(2)\Rightarrow \boxed{I = \frac{-1}{a^4}\left\{\frac{1}{3}\frac{(a^2-x^2)^{3/2}}{x^3} + \frac{(a^2-x^2)^{1/2}}{x}\right\} + c} \text{ Ans}$$

19. $\int\frac{dx}{(a^2+x^2)^2}$

Solution: let $I = \int\frac{dx}{(a^2+x^2)^2}$ ——— (1)

let = $x = a\tan\theta$

differentiate w • r to θ

$\frac{dx}{d\theta} = a\,\mathrm{Sec}^2\theta$

$\boxed{dx = a\,\mathrm{Sec}^2\theta\, d\theta}$

$$(1)\Rightarrow I = \int\frac{a\,\mathrm{Sec}^2\theta\, d\theta}{(a^2+a^2\tan^2\theta)^2} \Rightarrow I = \int\frac{\cancel{a}\,\mathrm{Sec}^2\theta\, d\theta}{a^4(1+\tan^2\theta)^2}$$

$$I = \frac{1}{a^3}\int\frac{\mathrm{Sec}^2\theta\, d\theta}{(\mathrm{Sec}^2\theta)^2} \Rightarrow I = \frac{1}{a^3}\int\frac{\mathrm{Sec}^2\theta\, d\theta}{\mathrm{Sec}^4\theta} \Rightarrow I = \frac{1}{a^3}\int\frac{1}{\mathrm{Sec}^2\theta}\, d\theta$$

$$I = \frac{1}{a^3}\int\mathrm{Cos}^2\theta\, d\theta. \Rightarrow I = \frac{1}{a^3}\int\left(\frac{1+\mathrm{Cos}2\theta}{2}\right)d\theta$$

$$I = \frac{1}{2a^3}\int d\theta + \frac{1}{2a^3}\int\mathrm{Cos}2\theta.\, d\theta$$

$$I = \frac{1}{2a^3}\int d\theta + \frac{1}{4a^3}\int\mathrm{Cos}2\theta.\, 2\, d\theta$$

$$I = \frac{1}{2a^3}(\theta) + \frac{1}{4a^3}(\mathrm{Sin}2\theta) + c \text{ ——— (2)}$$

$\therefore\ x = a\tan\theta$

$$\tan\theta = \frac{x}{a} \Rightarrow \boxed{\theta = \tan^{-1}\left(\frac{x}{a}\right)}$$

$$1 + \tan^2\theta = \mathrm{Sec}^2\theta \Rightarrow 1 + \frac{x^2}{a^2} = \mathrm{Sec}^2\theta$$

$$\frac{a^2+x^2}{a^2} = \mathrm{Sec}^2\theta \Rightarrow \mathrm{Sec}\theta = \frac{\sqrt{a^2+x^2}}{a} \Rightarrow \boxed{\mathrm{Cos}\theta = \frac{a}{\sqrt{a^2+x^2}}}$$

$\therefore\ 1 + \mathrm{Cot}^2\theta = \mathrm{Cosec}^2\theta$

$1 + \frac{a^2}{x^2} = \mathrm{Cosec}^2\theta$

$$\frac{x^2+a^2}{x^2} = \mathrm{Cosec}^2\theta \Rightarrow \mathrm{Cosec}\theta = \frac{\sqrt{x^2+a^2}}{a}$$

$\boxed{\mathrm{Sin}\theta = \frac{x}{\sqrt{x^2+a^2}}}$

$$(2)\Rightarrow I = \frac{1}{2a^3}\theta + \frac{1}{{}_2\cancel{4}a^3}(\cancel{2}\mathrm{Sin}\theta\,\mathrm{Cos}\theta) + c$$

$$I = \frac{1}{2a^3}\tan^{-1}\left(\frac{x}{a}\right) + \frac{1}{2a^3}\left\{\frac{x}{\sqrt{x^2+a^2}} \times \frac{a}{\sqrt{x^2+a^2}}\right\} + c$$

$$\boxed{I = \frac{1}{2a^3}\left\{\frac{ax}{x^2+a^2} + \tan^{-1}\left(\frac{x}{a}\right)\right\} + c} \text{ Ans}$$

(20) $\int t^3\sqrt{a^2t^2-b^2}\, dt$

Solution: let = $I \int t^3\sqrt{a^2t^2-b^2}\, dt$

$$I = \int t^3\sqrt{a^2\left(t^2-\frac{b^2}{a^2}\right)}\, dt.$$

$$I = a\int\sqrt{(t)^2 - \left(\frac{b}{a}\right)^2}\; t^3\, dt. \text{ ——— (1)}$$

let $t = a\,\mathrm{Sec}\theta$

$t = \frac{b}{a}\mathrm{Sec}\theta$

differentiate w • r to θ

$$\frac{dt}{d\theta} = \frac{b}{a}\mathrm{Sec}\theta\tan\theta \Rightarrow \boxed{dt = \frac{b}{a}\mathrm{Sec}\theta\tan\theta\, d\theta}$$

$$(1) \Rightarrow I = \cancel{a}\int \sqrt{\frac{b^2}{a^2}Sec^2\theta - \frac{b^2}{a^2}}\ \frac{b^3}{a^3}\ Sec^3\theta\ \frac{b}{\cancel{a}}\ Sec\theta\ \tan\theta\ d\theta$$

$$I = \sqrt{\frac{b^2}{a^2}(Sec^2\theta - 1)}\ \frac{b^4}{a^3}\ Sec^4\theta.\ \tan\theta\ d\theta$$

$$I = \int \frac{b}{a}\sqrt{Sec^2\theta - 1}\ \frac{b^4}{a^3}\ Sec^4\theta\ \tan\theta\ d\theta$$

$$I = \frac{b^5}{a^4}\int \sqrt{\tan^2\theta}\ \ Sec^4\theta\ \tan\theta\ d\theta$$

$$I = \frac{b^5}{a^4}\int Sec^4\theta\ \tan^2\theta\ d\theta$$

$$I = \frac{b^5}{a^4}\int Sec^2\theta.\ Sec^2\theta.\ \tan^2\theta\ d\theta$$

$$I = \frac{b^5}{a^4}\int (1 + \tan^2\theta)\ Sec^2\theta\ \tan^2\theta\ d\theta$$

$$I = \frac{b^5}{a^4}\int Sec^2\theta\ \tan^2\theta\ d\theta + \frac{b^5}{a^4}\int \tan^4\theta\ Sec^2\theta\ d\theta$$

$$I = \frac{b^5}{a^4}\int\left(\frac{\tan^3\theta}{3}\right) + \frac{b^5}{a^4}\left(\frac{\tan^5\theta}{5}\right) + c \text{ ———— (2)}$$

$\therefore\ t = \dfrac{b}{a}\ Sec\theta$

$\dfrac{at}{b} = Sec\theta$

Squaring on b • S

$\dfrac{a^2t^2}{b^2} = Sec^2\theta$

$1 + \tan^2\theta = Sec^2\theta$

$\tan^2\theta = Sec^2\theta - 1$

$\tan^2\theta = \dfrac{a^2t^2}{b^2} - 1 \Rightarrow \tan^2\theta = \dfrac{a^2t^2 - b^2}{b^2}$

$$\boxed{\tan\theta = \sqrt{\frac{a^2t^2}{b^2} - 1}}$$

$$(1) \Rightarrow I = \frac{b^5}{a^4}\left\{\frac{1}{3}\left(\sqrt{\frac{a^2t^2}{b^2} - 1}\right)^3\right\} + \frac{b^5}{a^4}\left\{\frac{1}{5}\left(\sqrt{\frac{a^2t^2}{b^2} - 1}\right)^5\right\} + c$$

$$I = \frac{\cancel{b^5}}{3a^4}\ \frac{(a^2t^2 - b^2)^{3/2}}{\cancel{b^3}} + \frac{\cancel{b^5}}{5a^4}\ \frac{(a^2t^2 - b^2)^{5/2}}{\cancel{b^5}} + c$$

$$\boxed{I = \frac{b^2}{3a^4}(a^2t^2 - b^2)^{3/2} + \frac{1}{5a^4}(a^2t^2 - b^2)^{5/2} + c}\ \text{Ans}$$

EXERCISE # 6.6

Calculate the following integrals.

(1) $\displaystyle\int \frac{dx}{x^2 + 4x + 5}$

Solution: let $I = \displaystyle\int \frac{dx}{x^2 + 4x + 5}$ ———— (1)

$\therefore D(x) = x^2 + 4x + 5$

$D(x) = x^2 + 4x + 4 + 1 \Rightarrow D(x) = (x + 2)^2 + (1)^2$

$(1) \Rightarrow I = \displaystyle\int \frac{dx}{(x+2)^2 (1)^2}$

$$\boxed{\text{Using formula } \therefore \int \frac{1}{a^2 + x^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c}$$

$I = \dfrac{1}{1}.\tan^{-1}\dfrac{(x+2)}{1} + c$

$\boxed{I = \tan^{-1}(x + 2) + c}$ Ans

(2) $\displaystyle\int \frac{dx}{\sqrt{5 + 4x - x^2}}$

Solution: let $I = \displaystyle\int \frac{dx}{\sqrt{5 + 4x - x^2}}$ ———— (1)

$\therefore D(x) = 5 + 4x - x^2$

$D(x) = -(x^2 - 4x - 5) \Rightarrow D(x) = -\{x^2 - 4x + 4 - 4 - 5\}$

$D(x) = -\{(x-2)^2 - 9\} \Rightarrow D(x) = 9 - (x-2)^2$

$(1) \Rightarrow I = \displaystyle\int \frac{dx}{\sqrt{(3)^2 - (x-2)^2}}$

$$\boxed{\text{Using formula } \therefore \int \frac{1}{\sqrt{a^2 - x^2}}dx = Sin^{-1}\frac{x}{a} + c}$$

$\boxed{I = Sin^{-1}\dfrac{(x-2)}{3} + c}$ Ans

(3) $\displaystyle\int \frac{dx}{x^2 - x + 1}$

Solution: let $I = \displaystyle\int \frac{dx}{x^2 - x + 1}$ ———— (1)

$\therefore D(x) = x^2 - x + 1$

$D(x) = (x)^2 - 2(x)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 + 1 - \frac{1}{4}$

$$D(x) = \left(x - \frac{1}{2}\right)^2 + \frac{3}{4}$$

$$(1) \Rightarrow I = \int \frac{dx}{\left(x - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$\boxed{\text{Using formula } \therefore \int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c}$$

$$I = \frac{1}{\frac{\sqrt{3}}{2}} \tan^{-1} \frac{\left(x - \frac{1}{2}\right)}{\frac{\sqrt{3}}{2}} + c \Rightarrow \boxed{I = \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2x - 1}{\sqrt{3}}\right) + c} \text{ Ans}$$

(4) $\int \dfrac{ds}{\sqrt{4S - S^2}}$

Solution: let $I = \int \dfrac{ds}{\sqrt{4s - s^2}}$ ——— (1)

$\therefore D(S) = 4s - s^2$

$D(S) = -(s^2 - 4s)$

$D(S) = -\{(s)^2 - 2(s)(2) + (2)^2 - (2)^2\}$

$D(S) = -\{(s - 2)^2 - (2)^2\}$

$D(S) = (2)^2 - (s - 2)^2$

$$(1) \Rightarrow I = \int \frac{ds}{\sqrt{(2)^2 - (s - 2)^2}}$$

$$\boxed{\text{Using formula } \therefore \int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \text{Sin}^{-1}\frac{x}{a} + c}$$

$$\boxed{I = \text{Sin}^{-1}\left(\frac{s}{2} - 1\right) + c} \quad \text{Ans}$$

(5) $\int \dfrac{dx}{x^2 + 6x + 8}$

Solution: let $I = \int \dfrac{dx}{x^2 + 6x + 8}$ ——— (1)

$\therefore D(x) = x^2 + 6x + 8$

$D(x) = (x)^2 + 2(x)(3) + (3)^2 + 8 - 9$

$D(x) = (x + 3)^2 - (1)^2$

$$(1) \Rightarrow I = \int \frac{dx}{(x + 3)^2 - (1)^2}$$

$$\boxed{\text{Using formula } \int \frac{1}{x^2 - a^2}\,dx = \frac{1}{2a}\ln\left(\frac{x - a}{x + a}\right) + c}$$

$$I = \frac{1}{2(1)} \ln\left(\frac{x + 3 - 1}{x + 3 + 1}\right) + c \Rightarrow \boxed{I = \frac{1}{2}\ln\left(\frac{x + 2}{x + 4}\right) + c} \text{ Ans}$$

(6) $\int \dfrac{dx}{4x - x^2}$

Solution: let $I = \int \dfrac{dx}{4x - x^2}$ ——— (1)

$\therefore D(x) = 4x - x^2$

$D(x) = -(x^2 - 4x)$

$D(x) = -\{(x)^2 - 2(x)(2) + (2)^2 - (2)^2\}$

$D(x) = -\{(x - 2)^2 - (2)^2\}$

$D(x) = (2)^2 - (x - 2)^2$

$$(1) \Rightarrow I = \int \frac{dx}{(2)^2 - (x - 2)^2}$$

$$\boxed{\text{Using formula } \therefore \int \frac{1}{a^2 - x^2}\,dx = \frac{1}{2a}\ln\left(\frac{a + x}{a - x}\right) + c}$$

$$I = \frac{1}{2(2)} \ln\left(\frac{\cancel{2} + x - \cancel{2}}{2 - x + 2}\right) + c \Rightarrow \boxed{I = \frac{1}{4}\ln\left(\frac{x}{4 - x}\right) + c} \text{ Ans}$$

(7) $\int \dfrac{dx}{(x + 2)\sqrt{x^2 + 4x + 3}}$

Solution: let $I = \int \dfrac{dx}{(x + 2)\sqrt{x^2 + 4x + 3}}$ ——— (1)

$\therefore \quad D(x) = x^2 + 4x + 3$

$D(x) = (x)^2 + 2(x)(2) + (2)^2 + 3 - 4$

$D(x) = (x + 2)^2 - (1)^2$

$$(1) \Rightarrow I = \frac{dx}{(x + 2)\sqrt{(x + 2)^2 - (1)^2}}$$

$$\boxed{\text{Using formula } \int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\text{Sec}^{-1}\frac{x}{a} + c}$$

$$I = \frac{1}{1}\text{Sec}^{-1}\frac{(x + 2)}{1} + c \Rightarrow \boxed{I = \text{Sec}^{-1}(x + 2) + c} \text{ Ans}$$

(8) $\int \dfrac{dx}{(x - 1)\sqrt{x^2 - 2x - 3}}$

Solution: let $I = \int \dfrac{dx}{(x - 1)\sqrt{x^2 - 2x - 3}}$ ——— (1)

$\therefore D(x) = x^2 - 2x - 3$

$D(x) = (x)^2 - 2(x)(1) + (1)^2 - 3 - 1$

$D(x) = (x-1)^2 - (2)^2$

$$(1) \Rightarrow I = \int \frac{dx}{(x-1)\sqrt{(x-1)^2 - (2)^2}}$$

$$\boxed{\text{Using formula } \int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\,\text{Sec}^{-1}\frac{x}{a} + c}$$

$$\boxed{I = \frac{1}{2}\,\text{Sec}^{-1}\left(\frac{x-1}{2}\right) + c} \quad \text{Ans}$$

(9) $\displaystyle \int \frac{dx}{\sqrt{5 - 2x + x^2}}$

Solution: let $\displaystyle I = \int \frac{dx}{\sqrt{5 - 2x + x^2}}$ ——— (1)

$\therefore D(x) = 5 - 2x + x^2$

$D(x) = x^2 - 2x + 5 \Rightarrow D(x) = (x)^2 - 2(x)(1) + (1)^2 + 5 - 1$

$D(x) = (x-1)^2 - (2)^2$

$$(1) \Rightarrow I = \int \frac{dx}{\sqrt{(x-1)^2 + (2)^2}}$$

$$\boxed{\text{Using formula } \int \frac{1}{\sqrt{x^2 + a^2}} = \ell n\left(x + \sqrt{a^2 + x^2}\right) + c}$$

$$I = \ln\left\{(x-1) + \sqrt{(x-1)^2 + (2)^2}\right\} + c$$

$$\boxed{I = \ln\left\{(x-1) + \sqrt{x^2 - 2x + 5}\right\} + c} \quad \text{Ans}$$

(10) $\displaystyle \int \frac{(2x-5)\,dx}{\sqrt{4x - x^2}}$

Solution: let $\displaystyle I = \int \frac{(2x-5)\,dx}{\sqrt{4x - x^2}}$ ——— (1)

$\therefore \quad D(x) = 4x - x^2 \Rightarrow D(x) = -(x^2 - 4x)$

$D(x) = -\{(x)^2 - 2(x)(2) + (2)^2 - (2)^2\}$

$D(x) = -\{(x-2)^2 - 4\}$

$D(x) = 4 - (2-x)^2 \Rightarrow D(x) = 4 - (2-x)^2$

$$(1) \Rightarrow I = \int \frac{(2x-5)\,dx}{\sqrt{4 - (2-x)^2}} \text{ ——— (2)}$$

let $t = 2 - x$. $\Rightarrow \boxed{x = 2 - t}$

differentiate w.r to x

$\dfrac{dt}{dx} = -1 \quad \Rightarrow \boxed{-dt = dx}$

$$(1) \Rightarrow I = \int \frac{\{2(2-t) - 5\}(-dt)}{\sqrt{4 - t^2}} \Rightarrow I = \int \frac{(4 - 2t - 5)(-dt)}{\sqrt{4 - t^2}}$$

$$I = \int \frac{(-2t-1)(-dt)}{\sqrt{(2)^2 - (t)^2}} \Rightarrow I = \int \frac{(2t+1)\,dt}{\sqrt{(2)^2 - (t)^2}}$$

$$I = \int \frac{2t\,dt}{\sqrt{4 - t^2}} + \int \frac{dt}{\sqrt{(2)^2 - (t)^2}}$$

$$I = -\int (4 - t^2)^{-1/2}(-2t)\,dt + \int \frac{1}{\sqrt{(2)^2 - (t)^2}}\,dt.$$

$$\boxed{\therefore \int \frac{dx}{\sqrt{a^2 - x^2}} = \text{Sin}^{-1}\left(\frac{x}{a}\right) + c}$$

$$I = \frac{(4 - t^2)^{1/2}}{1/2} + \text{Sin}^{-1}\frac{t}{2} + c \qquad \text{but } t = 2 - x$$

$$I = 2\sqrt{4 - (2-x)^2} + \text{Sin}^{-1}\left(\frac{2-x}{2}\right) + c$$

$$I = 2\sqrt{\cancel{4} - \cancel{4} + 4x - x^2} + \text{Sin}^{-1}\left(1 - \frac{x}{2}\right) + c$$

$$\boxed{I = 2\sqrt{4x - x^2} + \text{Sin}^{-1}\left(1 - \frac{x}{2}\right) + c} \quad \text{Ans}$$

(11) $\displaystyle \int \frac{(x+3)\,dx}{x^2 + 2x + 5}$

Solution: let $\displaystyle I = \int \frac{(x+3)\,dx}{x^2 + 2x + 5}$

$\times$ & $\div$ by 2

$$I = \frac{1}{2}\int \frac{2x + 6\,dx}{x^2 + 2x + 5} \text{ ——— (1)}$$

$\therefore D(x) = x^2 + 2x + 5$

$D(x) = (x)^2 + 2(x)(1) + (1)^2 + 5 - 1$

$D(x) = (x+1)^2 + (2)^2$

$$(1) \Rightarrow I = \frac{1}{2}\int \frac{2x + 6\,dx}{(x+1)^2 + (2)^2} \Rightarrow I = \frac{1}{2}\int \frac{(2x + 2 + 4)\,dx}{(x+1)^2 + (2)^2}$$

$$I = \frac{1}{2}\int \frac{2x+2}{x^2 + 2x + 5}\,dx + \frac{1}{\cancel{2}}\int \frac{\overset{2}{\cancel{4}}\,dx}{(x+1)^2 + (2)^2}$$

$$I = \frac{1}{2}\int \frac{2x + 2\,dx}{x^2 + 2x + 5} + 2\int \frac{1\,dx}{(x+1)^2 + (2)^2}$$

$$I = \frac{1}{2}\ell n\,(x^2 + 2x + 5) + \cancel{2}\,\frac{1}{\cancel{2}}\tan^{-1}\frac{(x+1)}{2} + c$$

$$\boxed{I = \ell n\sqrt{x^2 + 2x + 5} + \tan^{-1}\frac{(x+1)}{2} + c}$$

EXERCISE # 6.7

Q1. Integrate by parts the following,

(i) $\int x\,\ell n\,x\,dx.$

Solution: let $I = \int x\,\ell n\,x\,dx.$

let $u = \ell n\,x$; $v = x$

differentiate w • r to x $\int v\,dx = \int x\,dx$

$\boxed{\dfrac{du}{dx} = \dfrac{1}{x}}$ $\boxed{\int v dx = \dfrac{x^2}{2}}$

$\boxed{\text{Using Formula } \int u \bullet v dx = u \int v dx - \int \left\{ \dfrac{du}{dx} \int v\,dx \right\} dx}$

$$I = \ell n\,x\,\frac{x^2}{2} - \int \frac{1}{\cancel{x}} \bullet \frac{\cancel{x^2}}{2}\,dx \Rightarrow I = \ell n\,x\,\frac{x^2}{2} - \frac{1}{2}\int x\,dx$$

$$I = \frac{x^2}{2}\ln x - \frac{1}{2}\frac{x^2}{2} + c \Rightarrow \boxed{I = \frac{x^2}{2}\ln x - \frac{x^2}{4} + c}$$ Ans

(ii) $\int x\,\text{Sin}^2x\,\text{Cos}x\,dx$

Solution: let $I = \int x\text{Sin}^2x\,\text{Cos}x\,dx$

let $u = \ell n\,x$; $v = \text{Sin}^2x\,\text{Cos}x$

differentiate w • r to x $\int v\,dx = \int \text{Sin}^2x\,\text{Cos}x\,dx$

$\boxed{\dfrac{du}{dx} = 1}$ $\boxed{\int v dx = \dfrac{\text{Sin}^3x}{3}}$

$\boxed{\text{Using Formula } \int u \bullet v du = u \int v dx - \int \left\{ \dfrac{du}{dx} \int v\,dx \right\} dx}$

$$I = \frac{\text{Sin}^3x}{3}.x - \int (1)\frac{\text{Sin}^3x}{3}.dx$$

$$I = \frac{\text{Sin}^3x}{3}.x - \frac{1}{3}\int \text{Sin}^2x.\,\text{Sin}x\,dx$$

$$I = \frac{\text{Sin}^3x}{3}.x - \frac{1}{3}\int (1 - \text{Cos}^2x)\,\text{Sin}x\,dx$$

$$I = x\frac{\text{Sin}^3x}{3} - \frac{1}{3}\int \text{Sin}x\,dx + \frac{1}{3}\int \text{Cos}^2x\,\text{Sin}x\,dx$$

$$I = x\frac{\text{Sin}^3x}{3} + \frac{\text{Cos}x}{3} - \frac{1}{3}\frac{\text{Cos}^3x}{3} + c$$

$$\boxed{I = \frac{1}{3}\left[x\,\text{Sin}^3x + \text{Cos}x - \frac{\text{Cos}^3x}{3}\right] + c}$$ Ans

(iii) $\int x^2\tan^{-1}x\,dx.$

Solution: let $I = \int x^2\tan^{-1}x\,dx$

let $u = \tan^{-1}x$; $v = x^2$

differentiate w • r to x $\int v\,dx = \int x^2\,dx$

$\boxed{\dfrac{du}{dx} = \dfrac{1}{1+x^2}}$ $\boxed{\int v dx = \dfrac{x^3}{3}}$

$\boxed{\text{Using Formula } \int u \bullet v dx = u \int v dx - \int \left\{ \dfrac{du}{dx} \int v\,dx \right\} dx}$

$$I = \frac{x^3}{3}\tan^{-1}x - \int \frac{1}{1+x^2}\frac{x^3}{3}dx$$

$$I = \frac{x^3}{3}\tan^{-1}x - \frac{1}{3}\int \frac{x^3}{1+x^2}dx.$$

$$\begin{array}{r|l} & x \\ \hline 1+x^2 & x^3 \\ & \underline{\pm x^3 \pm x} \\ & -x \end{array}$$

$$I = \frac{x^3}{3}\tan^{-1}x - \frac{1}{3}\int\left(x - \frac{x}{1+x^2}\right)dx$$

$$I = \frac{x^3}{3}\tan^{-1}x - \frac{1}{3}\int x\,dx + \frac{1}{3}\int\frac{x\,dx}{1+x^2}$$

$$I = \frac{x^3}{3}\tan^{-1}x - \frac{1}{3}\int x\,dx + \frac{1}{6}\int\frac{2x\,dx}{1+x^2}$$

$$I = \frac{x^3}{3}\tan^{-1}x - \frac{1}{3}\frac{x^2}{3} + \frac{1}{6}\,\ell n\,(1+x^2) + c$$

$$\boxed{I = \frac{x^3}{3}\tan^{-1}x - \frac{x^2}{6} + \frac{1}{6}\,\ell n\,(1+x^2) + c}$$ Ans

(iv) $\int x^3\tan^{-1}x\,dx.$

Solution: let $I = \int x^3\tan^{-1}x\,dx$

let $u = \tan^{-1}x$; $v = x^3$

differentiate w • r to x $\int v\,dx = \int x^3\,dx$

$\boxed{\dfrac{du}{dx} = \dfrac{1}{1+x^2}}$ $\boxed{\int v dx = \dfrac{x^4}{4}}$

$\boxed{\text{Using Formula } \int u \bullet v dx = u \int v dx - \int \left\{ \dfrac{du}{dx} \int v\,dx \right\} dx}$

$$I = \frac{x^4}{4}\tan^{-1}x - \int\frac{1}{1+x^2}\frac{x^4}{4}\,dx$$

$$I = \frac{x^4}{4}\tan^{-1}x - \frac{1}{4}\int \frac{x^4}{1+x^2}\,dx$$

$$\begin{array}{r|l} 1+x^2 & x^4 \\ & \cancel{x^4} + x^2 \\ \hline & -x^2 \\ & -x^2 - 1 \\ & +\quad + \\ \hline & 1 \end{array} \quad x^2 - 1$$

$$I = \frac{x^4}{4}\tan^{-1}x - \frac{1}{4}\int\left(x^2 - 1 + \frac{1}{1+x^2}\right)dx$$

$$I = \frac{x^4}{4}\tan^{-1}x - \frac{1}{4}\int x^2\,dx + \frac{1}{4}\int dx - \frac{1}{4}\int\frac{1}{1+x^2}\,dx$$

$$I = \frac{x^4}{4}\tan^{-1}x - \frac{1}{4}\frac{x^3}{3} + \frac{1}{4}x - \frac{1}{4}\tan^{-1}x + c$$

$$\boxed{I = \frac{x^4}{4}\tan^{-1}x - \frac{x^3}{12} + \frac{x}{4} - \frac{1}{4}\tan^{-1}x + c} \text{ Ans}$$

(v) $\int x^2 \ln x\, dx.$

Solution: let $I = \int x^2 \ln x\, dx.$

let $u = \ln x$; $v = x^2$

differentiate w • r to x $\int v\,dx = \int x^2\,dx$

$\boxed{\frac{du}{dx} = \frac{1}{x}}$ $\boxed{\int v dx = \frac{x^3}{3}}$

$$\boxed{\text{Using Formula } \int u \bullet v dx = u\int v dx - \int\left\{\frac{du}{dx}\int v\,dx\right\}dx}$$

$$I = \frac{x^3}{3}\ln x - \int\frac{1}{\cancel{x}}\cdot\frac{x^{\cancel{3}}}{3}dx \Rightarrow I = \frac{x^3}{3}\ln x - \frac{1}{3}\int x^2\,dx$$

$$I = \frac{x^3}{3}\ln x - \frac{1}{3}\frac{x^3}{3} + c \Rightarrow \boxed{I = \frac{x^3}{3}\ln x - \frac{x^3}{9} + c} \text{ Ans}$$

(vi) $\int \tan^{-1}x\, dx.$

Solution: let $I = \int \tan^{-1}x\, dx.$

let $u = \tan^{-1}x$; $v = dx$

differentiate w • r to x $\int v\,dx = \int dx$

$\boxed{\frac{du}{dx} = \frac{1}{1+x^2}}$ $\boxed{\int v dx = x}$

$$\boxed{\text{Using Formula } \int u \bullet v dx = u\int v dx - \int\left\{\frac{du}{dx}\int v\,dx\right\}dx}$$

$$I = x\tan^{-1}x - \int\frac{1}{1+x^2}\cdot x\,dx \Rightarrow I = x\tan^{-1}x - \int\frac{x}{1+x^2}\,dx$$

$$I = x\tan^{-1}x - \frac{1}{2}\int\frac{2x}{1+x^2}\cdot dx$$

$$\boxed{I = x\tan^{-1}x - \frac{1}{2}\ln(1+x^2) + c} \quad \text{Ans}$$

(vii) $\int x^4 \sin^2 x\, dx$

Solution: let $I = \int x^4 \sin^2 x\, dx$

$$I = \int x^4\left(\frac{1-\cos 2x}{2}\right)dx \Rightarrow I = \frac{1}{2}\int(x^4 - x^4\cos 2x)\,dx$$

$$I = \frac{1}{2}\int x^4\,dx - \frac{1}{2}\int x^4\cos 2x\,dx$$

$$I = \frac{x^5}{10} - \frac{1}{2}\int x^4\cos 2x\,dx \text{ ——— (1)}$$

Using by parts on 2^{nd} integral.

let $u = x^4$ $v = \cos 2x.$

differentiate w • r to x $\int v dx = \frac{1}{2}\int \cos 2x \bullet 2dx$

$\boxed{\frac{du}{dx} = 4x^3}$ $\boxed{\int v dx = \frac{\sin 2x}{2}}$

$$\boxed{\text{Using Formula } \int u \bullet v dx = u\int v dx - \int\left\{\frac{du}{dx}\int v\,dx\right\}dx}$$

$$I = \frac{x^4\sin 2x}{2} - \int {}^2\cancel{4}x^3.\frac{\sin 2x}{\cancel{2}}dx$$

$$I = \frac{x^4\sin 2x}{2} - 2\int x^3\sin 2x\,dx \text{ ——— (2)}$$

again using by parts

let $u = x^4$ $v = \sin 2x.$

$\boxed{\frac{du}{dx} = 3x^2}$ $\int v dx = \frac{1}{2}\int \sin 2x \bullet 2dx$

$\boxed{\int v dx = \frac{-\cos 2x}{2}}$

$$I = x^3\left(-\frac{\cos 2x}{2}\right) - \int 3x^2\left(\frac{-\cos 2x}{2}\right)dx$$

$$I = \frac{-x^3\cos 2x}{2} + \frac{3}{2}\int x^2\cos 2x\,dx \text{ ——— (3)}$$

again using by parts

let $u = x^2$ $\quad\quad$ $v = \text{Cos}2x.$

$$\boxed{\frac{du}{dx} = 2x} \qquad \int v dx = \frac{1}{2}\int \text{Cos}2x \bullet 2dx$$

$$\boxed{\int v dx = \frac{\text{Sin}2x}{2}}$$

$$I = \frac{x^2\,\text{Sin}2x}{2} - \int \cancel{2}x.\left(\frac{\text{Sin}2x}{\cancel{2}}\right)dx$$

$$I = \frac{x^2\,\text{Sin}2x}{2} - \int x\,\text{Sin}2x\,dx \text{ ———— (4)}$$

Using again by parts

let $u = x$ $\quad\quad$ $v = \text{Sin}2x.$

$$\boxed{\frac{du}{dx} = 1} \qquad \int v dx = \frac{1}{2}\int \text{Sin}2x \bullet 2dx$$

$$\boxed{\int v dx = \frac{-\text{Cos}2x}{2}}$$

$$I = x\left(\frac{-\text{Cos}2x}{2}\right) - \int(1)\left(\frac{-\text{Cos}2x}{2}\right)dx$$

$$I = \frac{-x\,\text{Cos}2x}{2} + \frac{1}{2.2}\int \text{Cos}2x \bullet 2dx$$

$$I = \frac{-x\,\text{Cos}2x}{2} + \frac{\text{Sin}2x}{4}$$

put this result in (4)

$$(4) \Rightarrow I = \frac{x^2\,\text{Sin}2x}{2} - \left\{\frac{-x\,\text{Cos}2x}{2} + \frac{\text{Sin}2x}{4}\right\}$$

$$I = \frac{x^2\,\text{Sin}2x}{2} + \frac{x\,\text{Cos}2x}{2} - \frac{\text{Sin}2x}{4}$$

put this result in (3)

$$(3) \Rightarrow I = \frac{-x^3\,\text{Cos}2x}{2} + \frac{3}{2}\left\{\frac{x^2\,\text{Sin}2x}{2} + \frac{x\,\text{Cos}2x}{2} - \frac{\text{Sin}2x}{4}\right\}$$

$$I = \frac{-x^3\,\text{Cos}2x}{2} + \frac{3}{4}x^2\,\text{Sin}2x + \frac{3}{4}x\,\text{Cos}2x - \frac{3}{8}\text{Sin}2x$$

put this result in (2)

$$(2) \Rightarrow I = \frac{x^4\,\text{Sin}2x}{2} - 2\left\{\frac{-x^3\,\text{Cos}2x}{2} + \frac{3}{4}x^2\,\text{Sin}2x + \frac{3}{4}x\,\text{Cos}2x - \frac{3}{8}\text{Sin}2x\right\}$$

$$I = \frac{x^4\,\text{Sin}2x}{2} + x^3\,\text{Cos}2x - \frac{3}{2}x^2\,\text{Sin}2x - \frac{3}{2}x\,\text{Cos}2x + \frac{3}{4}\text{Sin}2x.$$

put this result in (1)

$$I = \frac{x^5}{10} - \frac{1}{2}\left\{\frac{x^4\,\text{Sin}2x}{2} + x^3\,\text{Cos}2x - \frac{3}{2}x^2\,\text{Sin}2x - \frac{3}{2}x\,\text{Cos}2x + \frac{3}{4}\text{Sin}2x\right\} + c$$

$$I = \frac{x^5}{10} - \frac{1}{4}x^4\,\text{Sin}2x - \frac{1}{2}x^3\,\text{Cos}2x + \frac{3}{4}x^2\text{Sin}2x + \frac{3}{4}x\,\text{Cos}2x - \frac{3}{8}$$
$$\text{Sin}2x + c$$

$$I = \frac{x^5}{10} + \text{Sin}2x\left(\frac{3x^2}{4} - \frac{x^4}{4} - \frac{3}{8}\right) + \text{Cos}2x\left(\frac{3x}{4} - \frac{x^3}{2}\right) + c$$

$$I = \frac{x^5}{10} + \text{Sin}2x\left(\frac{6x^2 - 2x^4 - 3}{8}\right) + \text{Cos}2x\left(\frac{3x - 2x^3}{4}\right) + c$$

$$\boxed{I = \frac{x^5}{10} - \frac{1}{8}(2x^4 - 6x^2 + 3)\,\text{Sin}2x - \frac{1}{4}(2x^3 - 3x)\,\text{Cos}2x + c} \ \textbf{Ans}$$

(viii) $\int x^2\,\text{Sin}^2x\,dx.$

Solution: let $I = \int x^2\,\text{Sin}^2x\,dx.$

let $u = x^2$ $\quad ; \quad v = \text{Sin}^2x.$

differentiate w – r to x

$$\boxed{\frac{du}{dx} = 2x} \qquad \int v dx = \frac{1}{2}\int \text{Sin}^2x\,dx$$

$$\int v dx = \int\left(\frac{1 - \text{Cos}2x}{2}\right)dx$$

$$\int v dx = \frac{1}{2}\int dx - \frac{1}{2 \times 2}\int \text{Cos}2x.2dx$$

$$\boxed{\int v dx = \frac{1}{2}x - \frac{\text{Sin}2x}{4}}$$

$$I = u\int v dx - \int\left\{\frac{du}{dx}\int v\,dx\right\}dx$$

$$I = x^2\left(\frac{x}{2} - \frac{\text{Sin}2x}{4}\right) - \int 2x\left(\frac{x}{2} - \frac{\text{Sin}2x}{4}\right)dx$$

$$I = \frac{x^3}{2} - \frac{x^2\,\text{Sin}2x}{4} - \int x^2\,dx + \frac{1}{2}\int x\,\text{Sin}2x\,dx$$

$$I = \frac{x^3}{2} - \frac{x^2\,\text{Sin}2x}{4} - \frac{x^3}{3} + \frac{1}{2}\int x\,\text{Sin}2x\,dx \text{ ———— (1)}$$

Consider $I = \int x\,\text{Sin}2x\,dx.$

let $u = x$ $\quad ; \quad v = \text{Sin}2x.$

differentiate w • r to x $\quad\quad$ $\int v dx = \int \text{Sin}2x\,dx$

$$\boxed{\frac{du}{dx} = 1} \qquad \boxed{\int v dx = \frac{-\text{Cos}2x}{2}}$$

$$I = u\int v dx - \int\left\{\frac{du}{dx}\int v dx\right\}dx$$

$$I = \frac{-x\,Cos2x}{2} - \int (1)\left(\frac{-Cos2x}{2}\right)dx$$

$$I = \frac{-x\,Cos2x}{2} + \frac{1}{2\times 2}\int Cos2x \bullet 2x\,dx$$

$$I = \frac{-x\,Cos2x}{2} + \frac{Sin2x}{4}$$

put this result in (1)

$$(1) \Rightarrow I = \frac{x^3}{2} - \frac{x^2 Sin2x}{4} - \frac{x^3}{2} + \frac{1}{2}\left\{\frac{-x\,Cos2x}{2} + \frac{Sin2x}{4}\right\} + c$$

$$\boxed{I = \frac{x^3}{6} - \frac{x^2\,Sin2x}{4} - \frac{x\,Cos2x}{4} + \frac{Sin2x}{8} + c} \text{ Ans}$$

(ix) $\int (\ln x)^2\,dx.$

Solution: let $I = \int (\ln x)^2\,dx$

let $u = (\ln x)^2$; $v = dx$

differentiate w • r to x $\qquad \int vdx = \int dx$

$$\boxed{\frac{du}{dx} = 2(\ln x)\frac{1}{x}} \qquad \boxed{\int vdx = x}$$

$$I = uv - \int\left\{\frac{du}{dx}\int v\,dx\right\}dx \Rightarrow I = x(\ln x)^2 - \int\left\{\frac{2(\ln x)}{\cancel{x}}\right\}\cancel{x}\,dx$$

$$I = x(\ln x)^2 - 2\int \ln x\,dx. \text{ ——— } (1)$$

Consider $I = \int \ln x\,dx$

let $u = \ln x$; $v = dx$

differentiate w • r to x $\qquad \int vdx = \int dx$

$$\boxed{\frac{du}{dx} = \frac{1}{x}} \qquad \boxed{\int vdx = x}$$

$$I = u\int vdx - \int\left\{\frac{du}{dx}\int v\,dx\right\}dx \Rightarrow I = x\ln x - \int \frac{1}{\cancel{x}}.\cancel{x}\,dx$$

$$I = x\ln x - \int dx \Rightarrow I = x\ln x - x$$

put this result in (1)

$$(1) \Rightarrow \boxed{I = x(\ln x)^2 - 2x\ln x + 2x + c} \text{ Ans}$$

(x) $\int \ln\left(x + \sqrt{x^2+1}\right)dx$

Solution: let $I = \int \ln\left(x + \sqrt{x^2+1}\right)dx.$

$u = \ln\left(x + \sqrt{x^2+1}\right)$; $v = 1$

differentiate w • r to x $\qquad \int vdx = \int dx$

$$\boxed{\int vdx = x}$$

$$\frac{du}{dx} = \frac{1}{x+\sqrt{x^2+1}}\left\{1 + \frac{1}{\cancel{2}}(x^2+1)^{-1/2}\,\cancel{2}x\right\}$$

$$\frac{du}{dx} = \frac{1}{x+\sqrt{x^2+1}}\left\{1 + \frac{x}{\sqrt{x^2+1}}\right\}$$

$$\frac{du}{dx} = \frac{1}{\cancel{x+\sqrt{x^2+1}}}\left\{\frac{\cancel{\sqrt{x^2+1}+x}}{\sqrt{x^2+1}}\right\} \Rightarrow \boxed{\frac{du}{dx} = \frac{1}{\sqrt{x^2+1}}}$$

$$I = u\int vdx - \int\left\{\frac{du}{dx}\int v\,dx\right\}dx$$

$$I = x\ln\left(x+\sqrt{x^2+1}\right) - \int \frac{1}{\sqrt{x^2+1}}\,x\,dx$$

$$I = x\ln\left(x+\sqrt{x^2+1}\right) - \int (x^2+1)^{-1/2}.\,xdx$$

$$I = x\ln\left(x+\sqrt{x^2+1}\right) - \frac{1}{2}\int (x^2+1)^{-1/2}.2x\,dx$$

$$I = x\ln\left(x+\sqrt{x^2+1}\right) - \frac{1}{2}\frac{\sqrt{x^2+1}}{1/2} + c$$

$$I = x\ln\left(x+\sqrt{x^2+1}\right) - \frac{\cancel{2}}{\cancel{2}}\sqrt{x^2+1} + c$$

$$\boxed{I = x\ln\left(x+\sqrt{x^2+1}\right) - \sqrt{x^2+1} + c} \text{ Ans}$$

Q2. Use integration by parts to determine the following.

(i) $\int \sqrt{4-x^2}\,dx$

Solution: let $I = \int \sqrt{4-x^2}\,dx$ ——— (1)

let $u = (4-x^2)^{1/2}$; $v = dx$

differentiate w • r to x $\qquad \int vdx = \int dx$

$$\frac{du}{dx} = \frac{1}{\cancel{2}}(4-x^2)^{-1/2}.(-\cancel{2}x) \qquad \boxed{\int vdx = x}$$

$$\frac{du}{dx} = \frac{-x}{\sqrt{4-x^2}}$$

Using formula

$$I = u\int vdx - \int\left\{\frac{du}{dx}\int v\,dx\right\}dx$$

$$I = x\sqrt{4-x^2} - \int\left\{\frac{-x}{\sqrt{4-x^2}}\right\}x\,dx$$

$$I = x\sqrt{4-x^2} - \int \frac{-x^2\,dx}{\sqrt{4-x^2}}$$

$$I = x\sqrt{4-x^2} - \int\left\{\frac{4-x^2-4}{\sqrt{4-x^2}}\right\}dx.$$

$$I = x\sqrt{4-x^2} - \int \frac{(4-x^2)dx}{\sqrt{4-x^2}} + 4\int \frac{dx}{\sqrt{4-x^2}}$$

$$I = x\sqrt{4-x^2} - \int\sqrt{4-x^2}\,dx + 4\int\frac{1\,dx}{\sqrt{4-x^2}} \text{ by (1)}$$

$$I = x\sqrt{4-x^2} - I + 4\int\frac{1}{\sqrt{(2)^2-(x)^2}}\,dx$$

$$\boxed{\therefore \int\frac{1}{\sqrt{a^2-x^2}}\,dx = \text{Sin}^{-1}\frac{x}{a} + c}$$

$$I + I = x\sqrt{4-x^2} + 4\,\text{Sin}^{-1}\left(\frac{x}{2}\right) + c$$

$$2I = x\sqrt{4-x^2} + 4\,\text{Sin}^{-1}\left(\frac{x}{2}\right) + c.$$

$$\boxed{I = \frac{x\sqrt{4-x^2}}{2} + 2\,\text{Sin}^{-1}\left(\frac{x}{2}\right) + c} \text{ Ans}$$

(ii) $\int\sqrt{4-5x^2}\,dx.$

Solution: let $I = \int\sqrt{4-5x^2}\,dx.$ ——— (1)

let $u = (4-5x^2)^{1/2}$; $v = dx$

$\int vdx = \int dx$

$$\frac{du}{dx} = \frac{1}{\cancel{2}}(4-5x^2)^{-1/2}(-\cancel{10}^{5}x)$$

$\boxed{\int vdx = x}$

$$\boxed{\frac{du}{dx} = \frac{-5x}{\sqrt{4-5x^2}}}$$

$$I = u\int vdx - \int\left\{\frac{du}{dx}\int v\,dx\right\}dx$$

$$I = x\sqrt{4-5x^2} - \int\left\{\frac{-5x}{\sqrt{4-5x^2}}\right\}x\,dx$$

$$I = x\sqrt{4-5x^2} - \int\frac{-5x^2\,dx}{\sqrt{4-5x^2}}$$

$$I = x\sqrt{4-5x^2} - \int\frac{(4-5x^2-4)}{\sqrt{4-5x^2}}\,dx$$

$$I = x\sqrt{4-5x^2} - \int\frac{4-5x^2}{\sqrt{4-5x^2}}\,dx + 4\int\frac{1}{\sqrt{(2)^2-(\sqrt{5}\,x)^2}}\,dx.$$

$$I = x\sqrt{4-5x^2} - \int\sqrt{4-5x^2}\,dx + 4\int\frac{1\,dx}{\sqrt{5\left(\frac{4}{5}-x^2\right)}} \text{ by (1)}$$

$$I = x\sqrt{4-5x^2} - I + 4\int\frac{1\quad dx}{\sqrt{5}\sqrt{\left(\frac{2}{\sqrt{5}}\right)^2-(x)^2}}$$

$$I + I = x\sqrt{4-5x^2} + \frac{4}{\sqrt{5}}\int\frac{1\,dx}{\sqrt{\left(\frac{2}{\sqrt{5}}\right)^2-(x)^2}}$$

$$2I = x\sqrt{4-5x^2} + \frac{4}{\sqrt{5}}\,\text{Sin}^{-1}\frac{x}{\frac{2}{\sqrt{5}}} + c$$

$$\boxed{I = x\sqrt{4-5x^2} + \frac{2}{\sqrt{5}}\,\text{Sin}^{-1}\left(\frac{\sqrt{5}\,x}{2}\right) + c} \text{ Ans}$$

(iii) $\int\sqrt{a^2-x^2}\,dx.$

Solution: let $I = \int\sqrt{a^2-x^2}\,dx.$ ——— (1)

let $u = (a^2-x^2)^{1/2}$; $v = 1$

differentiate w.r to x ; $\int vdx = \int dx$

$$\frac{du}{dx} = \frac{1}{\cancel{2}}(a^2-x^2)^{-1/2}(-\cancel{2}x)$$

$\boxed{\int vdx = x}$

$$\boxed{\frac{du}{dx} = \frac{-x}{\sqrt{a^2-x^2}}}$$

$$I = u\int vdx - \int\left\{\frac{du}{dx}\int v\,dx\right\}dx$$

$$I = x\sqrt{a^2-x^2} - \int\left(\frac{-x}{\sqrt{a^2-x^2}}\right)x\,dx.$$

$$I = x\sqrt{a^2-x^2} - \int\frac{a^2-x^2-a^2}{\sqrt{a^2-x^2}}\,dx$$

$$I = x\sqrt{a^2-x^2} - \int\frac{a^2-x^2}{\sqrt{a^2-x^2}}\,dx + a^2\int\frac{1}{\sqrt{a^2-x^2}}\,dx.$$

$$I = x\sqrt{a^2-x^2} - \int\sqrt{a^2-x^2}\,dx + a^2\int\frac{1}{\sqrt{a^2-x^2}}\,dx.$$

$$I = x\sqrt{a^2-x^2} - I + a^2\int\frac{1}{\sqrt{a^2-x^2}}\,dx.$$

$$I + I = x\sqrt{a^2-x^2} + a^2\int\frac{1}{\sqrt{a^2-x^2}}\,dx.$$

$$2I = x\sqrt{a^2-x^2} + a^2\,\text{Sin}^{-1}\frac{x}{a} + c$$

÷ by 2

$$\boxed{I = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\,\text{Sin}^{-1}\frac{x}{a} + c} \text{ Ans}$$

Q3. Find the following indefinite integrals.

(i) $\int x^3 e^{2x}\,dx$

Solution: let $I = \int x^3 e^{2x}\,dx$

$u = x^3$; $v = e^{2x}$

differentiate w.r to x ; $\int v\,dx = \int e^{2x}\,dx$

$\boxed{\dfrac{du}{dx} = 3x^2}$ $\boxed{\int v\,dx = \dfrac{e^{2x}}{2}}$

$$I = u\int v\,dx - \int \left\{\frac{du}{dx}\int v\,dx\right\}dx$$

$$I = \frac{x^3 e^{2x}}{2} - \int 3x^2 \frac{e^{2x}}{2}\,dx$$

$$I = \frac{x^3 e^{2x}}{2} - \frac{3}{2}\int x^2 e^{2x}\,dx \text{ ——— (1)}$$

Consider $I_1 = \int x^2 e^{2x}\,dx$

$u = x^2$; $v = e^{2x}$

differentiate w.r to x ; $\int v\,dx = \int e^{2x}\,dx$

$\boxed{\dfrac{du}{dx} = 2x}$ $\boxed{\int v\,dx = \dfrac{e^{2x}}{2}}$

$$I = u\int v\,dx - \int \left\{\frac{du}{dx}\int v\,dx\right\}dx$$

$$I = \frac{x^2 e^{2x}}{2} - \int \cancel{2}x.\frac{e^{2x}}{\cancel{2}}\,dx$$

$$I = \frac{x^2 e^{2x}}{2} - \int x e^{2x}\,dx \text{ ——— (2)}$$

Condiser $I_2 = \int x e^{2x}\,dx$

$u = x$; $v = e^{2x}$

differentiate w.r to x ; $\int v\,dx = \int e^{2x}\,dx$

$\boxed{\dfrac{du}{dx} = 1}$ $\boxed{\int v\,dx = \dfrac{e^{2x}}{2}}$

$$I = u\int v\,dx - \int \left\{\frac{du}{dx}\int v\,dx\right\}dx \Rightarrow I = \frac{x e^{2x}}{2} - \int (1)\frac{e^{2x}}{2}\,dx$$

$$I = \frac{x e^{2x}}{2} - \frac{1}{2}\int e^{2x}\,dx \Rightarrow I = \frac{x e^{2x}}{2} - \frac{e^{2x}}{4}$$

put this result in (2)

$$(2) \Rightarrow I = \frac{x^2 e^{2x}}{2} - \frac{x e^{2x}}{2} + \frac{e^{2x}}{4}$$

put this result in (1)

$$(1) \Rightarrow I = \frac{x^3 e^{2x}}{2} - \frac{3}{2}\left\{\frac{x^2 e^{2x}}{2} - \frac{x e^{2x}}{2} + \frac{e^{2x}}{4}\right\} + c$$

$$I = \frac{x^3 e^{2x}}{2} - \frac{3}{4}x^2 e^{2x} + \frac{3}{4}x e^{2x} - \frac{3}{8}e^{2x} + c$$

$\boxed{I = \dfrac{1}{8}e^{2x}(4x^3 - 6x^2 + 6x - 3) + c}$ Ans

(ii) $\int \text{Sin}^{-1}x\,dx$

Solution: let $I = \int \text{Sin}^{-1}x\,dx$

$u = \text{Sin}^{-1}x$; $v = 1$

differentiate w.r to x ; $\int v\,dx = \int dx$

$\boxed{\dfrac{du}{dx} = \dfrac{1}{\sqrt{1-x^2}}}$ $\boxed{\int v\,dx = x}$

$$I = u\int v\,dx - \int \left\{\frac{du}{dx}\int v\,dx\right\}dx$$

$$I = x\,\text{Sin}^{-1}x - \int \frac{1}{\sqrt{1-x^2}}\,x\,dx$$

$$I = x\,\text{Sin}^{-1}x - \int (1-x^2)^{-1/2}\,x\,dx$$

$$I = x\,\text{Sin}^{-1}x + \frac{1}{2}\int (1-x^2)^{-1/2}(-2x)\,dx$$

$$I = x\,\text{Sin}^{-1}x + \frac{1}{2}\,\frac{(1-x^2)^{1/2}}{1/2} + c$$

$$I = x\,\text{Sin}^{-1}x + \frac{\cancel{2}}{\cancel{2}}\sqrt{1-x^2} + c$$

$\boxed{I = x\,\text{Sin}^{-1}x + \sqrt{1-x^2} + c}$ Ans

(iii) $\int x\tan^{-1}x\,dx.$

Solution: let $I = \int x\tan^{-1}x\,dx.$

$u = \tan^{-1}x$; $v = x$

diff w.r to x ; $\int v\,dx = \int x\,dx$

$\boxed{\dfrac{du}{dx} = \dfrac{1}{1+x^2}}$ $\boxed{\int v\,dx = \dfrac{x^2}{2}}$

$$I = u\int v\,dx - \int \left\{\frac{du}{dx}\int v\,dx\right\}dx$$

$$I = \frac{x^2}{2}\tan^{-1}x - \int \frac{1}{1+x^2}\,\frac{x^2}{2}\,dx$$

$$I = \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int \frac{x^2}{1+x^2}\,dx$$

$$I = \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int \frac{1+x^2-1}{1+x^2}\,dx$$

$$I = \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int \frac{\cancel{1+x^2}}{\cancel{1+x^2}}\,dx + \frac{1}{2}\int \frac{1}{1+x^2}\,dx$$

$$I = \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int dx + \frac{1}{2}\int \frac{1}{1+x^2}\,dx$$

$$I = \frac{x^2}{2}\tan^{-1}x - \frac{x}{2} + \frac{1}{2}\tan^{-1}\frac{x}{1} + c$$

$$\boxed{I = \frac{x^2}{2}\tan^{-1}x - \frac{x}{2} + \frac{1}{2}\tan^{-1}x + c} \quad \text{Ans}$$

(iv) $\int x\,\mathrm{Sec}^{-1}x\,dx.$

Solution: let $I = \int x\,\mathrm{Sec}^{-1}\,dx.$

$u = \mathrm{Sec}^{-1}x$; $v = x$

differentiate w • r to x ; $\int v\,dx = \int x\,dx$

$$\boxed{\frac{du}{dx} = \frac{1}{x\sqrt{x^2-1}}} \qquad \boxed{\int v\,dx = \frac{x^2}{2}}$$

$$I = u\int v\,dx - \int\left\{\frac{du}{dx}\int v\,dx\right\}dx$$

$$I = \frac{x^2}{2}\mathrm{Sec}^{-1}x - \int \frac{1}{\cancel{x}\sqrt{x^2-1}}\,\frac{x^{\cancel{2}}}{2}\,dx$$

$$I = \frac{x^2}{2}\mathrm{Sec}^{-1}x - \frac{1}{2\times 2}\int (x^2-1)^{-1/2}\,2x\,dx$$

$$I = \frac{x^2}{2}\mathrm{Sec}^{-1}x - \frac{1}{4}\,\frac{(x^2-1)^{1/2}}{1/2} + c$$

$$I = \frac{x^2}{2}\mathrm{Sec}^{-1}x - \frac{\cancel{2}}{\cancel{4}_2}\sqrt{x^2-1} + c$$

$$\boxed{I = \frac{x^2}{2}\mathrm{Sec}^{-1}x - \frac{1}{2}\sqrt{x^2-1} + c} \quad \text{Ans}$$

(v) $\int x\,\mathrm{Cosec}^2\frac{x}{2}\,dx.$

Solution: let $I = \int x\,\mathrm{Cosec}^2\frac{x}{2}\,dx.$

$u = x$; $v = \mathrm{Cosec}^2\frac{x}{2}$

differentiate w • r to x

$\int v\,dx = \int \mathrm{Cosec}^2\frac{x}{2}\,dx$

$$\boxed{\frac{du}{dx} = 1} \qquad \int v\,dx = \frac{-\mathrm{Cot}\frac{x}{2}}{\frac{1}{2}}$$

$$\boxed{\int v\,dx = -2\,\mathrm{Cot}\frac{x}{2}}$$

$$I = u\int v\,dx - \int\left\{\frac{du}{dx}\int v\,dx\right\}dx$$

$$I = -2x\,\mathrm{Cot}\frac{x}{2} - \int(1)\left(-2\,\mathrm{Cot}\frac{x}{2}\right)dx$$

$$I = -2x\,\mathrm{Cot}\frac{x}{2} + 2\int \mathrm{Cot}\frac{x}{2}\,dx$$

$$I = -2x\,\mathrm{Cot}\frac{x}{2} + 2\,\frac{\ln \mathrm{Sin}\frac{x}{2}}{\frac{1}{2}} + c$$

$$\boxed{I = -2x\,\mathrm{Cot}\frac{x}{2} + 4\ln \mathrm{Sin}\frac{x}{2} + c} \quad \text{Ans}$$

(vi) $\int \frac{2x\,dx}{\mathrm{Cos}^2 2x}$

Solution: let $I = \int \frac{2x\,dx}{\mathrm{Cos}^2 2x}$

$$I = 2\int x\,\mathrm{Sec}^2 2x\,dx \text{ ——— (1)}$$

$u = x$; $v = \mathrm{Sec}^2 2x$

differentiate w • r to x ; $\int v\,dx = \int \mathrm{Sec}^2 2x\,dx$

$$\boxed{\frac{du}{dx} = 1} \qquad \boxed{\int v\,dx = \frac{\tan 2x}{2}}$$

$$I = u\int v\,dx - \int\left\{\frac{du}{dx}\int v\,dx\right\}dx \Rightarrow I = \frac{x\tan 2x}{2} - \int(1)\frac{\tan 2x}{2}\,dx$$

$$I = \frac{x\tan 2x}{2} - \frac{1}{2}\int \tan 2x\,dx \Rightarrow I = \frac{x\tan 2x}{2} - \frac{1}{2\times 2}\ln \mathrm{Sec}2x$$

put this result in (1)

$$I = 2\left\{\frac{x\tan 2x}{2} - \frac{1}{4}\ln \mathrm{Sec}2x\right\} + c$$

$$\boxed{I = x\tan 2x - \frac{1}{2}\ln \mathrm{Sec}2x + c} \quad \text{Ans}$$

(vii) $\int 9x \tan^2 3x\, dx$.

Solution: let $I = \int 9x \tan^2 3x\, dx$.

$$I = \int 9x(\text{Sec}^2 3x - 1)\, dx$$

$$I = 9\int x\, \text{Sec}^2 3x - 9\int x\, dx$$

$$I = 9\int x\, \text{Sec}^2 3x - 9\frac{x^2}{2} + c \text{ ———— (1)}$$

Consider $I = \int x\text{Sec}^2 3x\, dx$

$u = x$; $v = \text{Sec}^2 3x$

differentiate w • r to x ; $\int v dx = \int \text{Sec}^2 3x\, dx$

$\boxed{\frac{du}{dx} = 1}$ $\boxed{\int v dx = \frac{\tan 3x}{3}}$

$$I = u\int v dx - \int \left\{\frac{du}{dx}\int v\, dx\right\} dx$$

$$I = \frac{x \tan 3x}{3} - \int (1)\frac{\tan 3x}{3}\, dx$$

$$I = \frac{x \tan 3x}{3} - \frac{1}{3}\int \tan 3x\, dx$$

$$I = \frac{x \tan 3x}{3} - \frac{1}{9}\ln \text{Sec}3x$$

put this result in (1)

$$(1) \Rightarrow I = 9\left\{\frac{x \tan 3x}{3} - \frac{1}{9}\ln \text{Sec}3x\right\} - \frac{9x^2}{2} + c$$

$$\boxed{I = 3x \tan 3x - \ln \text{Sec}3x - \frac{9x^2}{2} + c} \text{ Ans}$$

(viii) $\int \text{Sin}\sqrt{2x}\, dx$

Solution: let $I = \int \text{Sin}\sqrt{2x}\, dx$ ———— (1)

let $t = \sqrt{2x}$

Squaringon b • s

$t^2 = 2x$.

differentiate w • r to x

$$\cancel{2}t\frac{dt}{dx} = \cancel{2} \Rightarrow \boxed{dx = tdt}$$

$(1) \Rightarrow I = \int \text{Sin}t \cdot tdt$

$I = \int t\, \text{Sin}\, tdt$.

$u = t$; $v = \text{Sin}\, t$

differentiate w • r to t ; $\int v dt = \int \text{Sin}\, t\, dt$

$\boxed{\frac{du}{dx} = 1}$ $\boxed{\int v dt = -\text{Cos}\, t}$

$$I = u\int v\, dt - \int \left\{\frac{du}{dt}\int v\, dt\right\} dt$$

$$I = -t\,\text{Cos}\, t - \int (1)(-\text{Cos}\, t)\, dt \Rightarrow I = -t\,\text{Cos}t + \int \text{Cos}\, t\, dt$$

$$I = -t\,\text{Cos}\, t + \text{Sin}\, t + c$$

but $t = \sqrt{2x}$

$$\boxed{I = -\sqrt{2x}\,\text{Cos}\sqrt{2x} + \text{Sin}\sqrt{2x} + c} \text{ Ans}$$

(ix) $\int 6x^2 \text{Sin}^{-1} 2x\, dx$.

Solution: let $I = \int 6x^2 \text{Sin}^{-1} 2x\, dx$.

$I = 6\int x^2 \text{Sin}^{-1} 2x\, dx$. ———— (1)

Using integration by parts.

let $u = \text{Sin}^{-1}2x$; $v = x^2\, dx$

$\boxed{\frac{du}{dx} = \frac{2}{\sqrt{1-4x^2}}}$; $\int v dx = \int x^2\, dt$; $\boxed{\int v dx = \frac{x^3}{3}}$

$$I = u\int v\, dx - \int \left\{\frac{du}{dx}\int v\, dx\right\} dx$$

$$I = \frac{x^3}{3}\text{Sin}^{-1}2x - \int \frac{2}{\sqrt{1-4x^2}}\frac{x^3}{3}\, dx$$

$$I = \frac{x^3}{3}\text{Sin}^{-1}2x - \frac{2}{3}\int \frac{x^3}{\sqrt{1-4x^2}}\, dx$$

put this result in (1)

$$(1) \Rightarrow I = 2x^3\,\text{Sin}^{-1}2x - 4\int \frac{x^3}{\sqrt{1-4x^2}}\, dx \text{ ———— (2)}$$

$$\text{let } I = \int \frac{x^3\, dx}{\sqrt{1-4x^2}}$$

$$I = \int \frac{x^3\, dx}{\sqrt{4\left(\frac{1}{4} - x^2\right)}} = \frac{1}{2}\int \frac{x^3\, dx}{\sqrt{\left(\frac{1}{2}\right)^2 - (x)^2}}$$

Using Trigonometric Substitution method

let $x = a\,\text{Sin}\theta \Rightarrow x = \frac{1}{2}\text{Sin}\theta$

differentiate w • r to θ

$$\frac{dx}{d\theta} = \frac{\text{Cos}\theta}{2} \Rightarrow \boxed{dx = \frac{\text{Cos}\theta}{2}\, d\theta} \Rightarrow I = \frac{1}{2}\int \frac{\left(\frac{\text{Sin}\theta}{2}\right)^3 \frac{\text{Cos}\theta}{2}\, d\theta}{\sqrt{\frac{1}{4} - \frac{1}{4}\text{Sin}^2\theta}}$$

$$I = \frac{1}{2}\int \frac{\frac{Sin^3\theta}{8}\cdot\frac{Cos\theta}{2}\,d\theta}{\sqrt{\frac{1}{4}(1-Sin^2\theta)}} \Rightarrow I = \frac{1}{32}\int \frac{Sin^3\theta\,Cos\theta\,d\theta}{\frac{1}{2}\sqrt{1-Sin^2\theta}}$$

$$I = \frac{\cancel{2}}{\underset{16}{\cancel{32}}}\int \frac{Sin^3\theta\,Cos\theta\,d\theta}{\sqrt{Cos^2\theta}} \Rightarrow I = \frac{1}{16}\int \frac{Sin^3\theta\,\cancel{Cos\theta}\,d\theta}{\cancel{Cos\theta}}$$

$$I = \frac{1}{16}\int Sin^3\theta\,d\theta = \frac{1}{16}\int Sin^2\theta\,Sin\theta\,d\theta$$

$$I = \frac{1}{16}\int (1-Cos^2\theta)\,Sin\theta\,d\theta$$

$$I = \frac{1}{16}\int Sin\theta\,d\theta - \frac{1}{16}\int Cos^2\theta\,Sin\theta\,d\theta$$

$$I = \frac{1}{16}\int Sin\theta\,d\theta + \frac{1}{16}\int Cos^2\theta.\,(-Sin\theta)\,d\theta$$

$$I = \frac{1}{16}(-Cos\theta) + \frac{1}{16}\frac{Cos^3\theta}{3} \Rightarrow I = \frac{-Cos\theta}{16} + \frac{Cos^3\theta}{48}$$

put this result (2)

$$(2) \Rightarrow I = 2x^3\,Sin^{-1}2x - 4\left\{\frac{-Cos\theta}{16} + \frac{Cos^3\theta}{48}\right\} + c$$

$$I = 2x^3\,Sin^{-1}2x + \frac{Cos\theta}{4} - \frac{Cos^3\theta}{12} + c$$

$\therefore Sin\theta = 2x$

$$Sin^2\theta + Cos^2\theta = 1 \Rightarrow Cos\theta = \sqrt{1-4x^2}$$

$$I = 2x^3\,Sin^{-1}2x + \frac{\sqrt{1-4x^2}}{4} - \frac{(1-4x^2)^{3/2}}{12} + c$$

$$\boxed{I = 2x^3\,Sin^{-1}2x - \frac{1}{4}\left\{\frac{1}{3}(1-4x^2)^{3/2} - (1-4x^2)^{1/2}\right\} + c} \quad \text{Ans}$$

(x) $\int x^m \ln x\,dx$

Solution: let $I = \int x^m \ln x\,dx$

Using by parts.

let $u = \ln x$; $v = x^m$

$\boxed{\frac{du}{dx} = \frac{1}{x}}$ $\quad \int v\,dx = \int x^m\,dx$

$\boxed{\int v\,dx = \frac{x^{m+1}}{m+1}}$

$$I = u\int v\,dx - \int\left\{\frac{du}{dx}\int v\,dx\right\}dx$$

$$I = (\ln x)\left(\frac{x^{m+1}}{m+1}\right) - \int \frac{1}{x}\,\frac{x^{m+1}}{m+1}\,dx$$

$$I = (\ln x)\left(\frac{x^{m+1}}{m+1}\right) - \frac{1}{m+1}\int x^m\,dx$$

$$I = (\ln x)\left(\frac{x^{m+1}}{m+1}\right) - \frac{1}{m+1}\,\frac{x^{m+1}}{m+1} + c$$

$$\boxed{I = \frac{x^{m+1}}{m+1}\left\{\ln x - \frac{1}{m+1}\right\} + c} \quad \text{Ans}$$

(xi) $\int 2x^3 e^{x^2}\,dx$

Solution: let $I = \int 2x^3 e^{x^2}\,dx$ ——— (1)

let $t = x^2$

$$\frac{dt}{dx} = 2x. \Rightarrow \boxed{dx = \frac{dt}{2x}}$$

$$(1) \Rightarrow I = \int \cancel{2}\,\cancel{x}^3 e^t\left(\frac{dt}{\cancel{2x}}\right)$$

$$I = \int x^2 e^t\,dt. \Rightarrow I = \int t\,e^t\,dt.$$

using by parts.

let $u = t$; $v = e^t$

$\boxed{\frac{du}{dt} = 1}$ $\quad \int v\,dt = \int e^t\,dt$

$\boxed{\int v\,dt = e^t}$

$$I = u\int v\,dt - \int\left\{\frac{du}{dt}\int v\,dt\right\}dt$$

$$I = t\,e^t - \int (1)\cdot dt. \Rightarrow I = t.e^t - \int e^t\,dt.$$

$$I = t\,e^t - e^t + c \qquad \text{but } t = x^2$$

$$I = x^2 e^{x^2} - e^{x^2} + c \Rightarrow \boxed{I = e^{x^2}(x^2 - 1) + c} \quad \text{Ans}$$

(xii) $\int e^{3x}\,Sin2x\,dx$

Solution: let $I = \int e^{3x}\,Sin2x\,dx$

Using integration by parts.

let $u = Sin2x$; $v = e^{3x}\,dx$

$\boxed{\frac{du}{dx} = 2\,Cos2x}$ $\quad \int v\,dx = \int e^{3x}\,dx$

$\boxed{\int v\,dx = \frac{e^{3x}}{3}}$

$$I = u\int v\,dx - \int\left\{\frac{du}{dx}\int v\,dx\right\}dx$$

$$I = Sin2x.\,\frac{e^{3x}}{3} - \int 2\,Cos2x.\,\frac{e^{3x}}{3}\,dx$$

$$I = \frac{1}{3}\,Sin2x\, e^{3x} - \frac{2}{3}\int e^{3x}.\, Cos2x\, dx \text{ ——— (1)}$$

let $I = \int e^{3x}.\, Cos2x\, dx$

again using by parts.

$u = Cos2x$; $v = e^{3x}\, dx$

$\boxed{\frac{du}{dx} = -2\, Sin2x}$ $\int v\, dx = \int e^{3x}\, dx$

$\boxed{\int vdx = \frac{e^{3x}}{3}}$

$$I = Cos2x\,\frac{e^{3x}}{3} + \frac{2}{3}\int e^{3x}\, Sin2x\, dx$$

$$I = \frac{Cos2x\, e^{3x}}{3} + \frac{2}{3}\, I.$$

put this result in (1)

$$(1) \Rightarrow I = \frac{1}{3}\; Sin2x\, e^{3x} - \frac{2}{3}\left\{\frac{Cos2x\, e^{3x}}{3} + \frac{2}{3}\, I\right\} + c$$

$$I = \frac{1}{3}\; Sin2x\, e^{3x} - \frac{2}{9}\; Cos2x\, e^{3x} - \frac{4}{9}\; I + c$$

$$I + \frac{4I}{9} = \frac{1}{3}\, Sin2x\, e^{3x} - \frac{2}{9}\; Cos2x\, e^{3x} + c$$

$$\frac{13I}{9} = \frac{1}{3}\, Sin2x\, e^{3x} - \frac{2}{9}\; Cos2x\, e^{3x} + c$$

xplying throughout by $\frac{9}{13}$

$$I = \frac{3}{13}\; Sin2x\, e^{3x} - \frac{2}{13}\; Cos2x\, e^{3x} + c$$

$$\boxed{I = \frac{e^{3x}}{13}(3\; Sin2x - 2Cos2x) + c} \quad \text{Ans}$$

(xiii) $\int e^{-x}$ Cos3x dx.

Solution: let $I = \int e^{-x}\, Cos3x\, dx$.

Using integration by parts.

$u = Cos3x$; $v = e^{-x}$

$\boxed{\frac{du}{dx} = -3Sin3x}$ $\int v\, dx = \int e^{-x}\, dx$

$\boxed{\int v\, dx = -e^{-x}}$

$$I = u\int v\, dx - \int\left\{\frac{du}{dx}\int v\, dx\right\} dx$$

$$I = -3\; Cos3x\; e^{-x} - \int\{\, -3\; Sin3x\,\}(-e^{-x})\, dx$$

$$I = -3\; Cos3x\; e^{-x} - 3\int e^{-x}\; Sin3x\; dx \text{ ———— (1)}$$

let $I = \int e^{-x}\, Sin3x\, dx$

$u = Sin3x$; $v = e^{-x}$

$\boxed{\frac{du}{dx} = 3\; Cos3x}$ $\int v\, dx = \int e^{-x}\, dx$

$\boxed{\int v\, dx = -e^{-x}}$

$$I = -Sin3x\; e^{-x} - \int (3\; Cos3x)\,(-e^{-x})\, dx$$

$$I = -Sin3x\; e^{-x} + 3\int Cos3x\; e^{-x}\, dx$$

$$I = -Sin3x\; e^{-x} + 3\, I$$

put this result in (1)

$$(1) \Rightarrow I = -3Cos3x\; e^{-x} - 3\,(-Sin3x\; e^{-x} + 3\, I) + c$$

$$I = -3Cos3x\; e^{-x} + 3\; Sin3x\; e^{-x} - 9\, I + c$$

$$I + 9\, I = e^{-x}\{\,3\; Sin3x - Cos3x\,\} + c$$

$$10\, I = e^{-x}\{\,3\; Sin3x - Cos3x\,\} + c$$

$$\boxed{I = \frac{e^{-x}}{10}\{\,3\; Sin3x - Cos3x\,\} + c} \quad \text{Ans}$$

(xiv) $\int e^{ax}$ Cosbx dx

Solution: let $I = \int e^{ax}\, Cosbx\, dx$

let $u = Cosbx$; $v = e^{ax}$

$\boxed{\frac{du}{dx} = -b\; Sinbx}$ $\int v\, dx = \int e^{ax}\, dx$

$\boxed{\int vdx = \frac{e^{ax}}{3}}$

$$I = u\int v\, dx - \int\left\{\frac{du}{dx}\int v\, dx\right\} dx$$

$$I = Cosbx\,\frac{e^{ax}}{a} - \int(-\,b\; Sinbx)\,\frac{e^{ax}}{a}\; dx$$

$$I = \frac{Cosbx\; e^{ax}}{a} + \frac{b}{a}\int Sinbx\; e^{ax}\, dx \text{ ———— (2)}$$

let $I = \int Sinbx\; e^{ax}\, dx$

again using by parts

$u = Sinbx$; $v = e^{ax}$

$\boxed{\frac{du}{dx} = b\; Cosbx}$ $\int v\, dx = \int e^{ax}\, dx$

$\boxed{\int vdx = \frac{e^{ax}}{3}}$

$$I = Sinbx\,\frac{e^{ax}}{a} - \int b\; Cosbx.\,\frac{e^{ax}}{a}\; dx.$$

$$I = Sinbx\,\frac{e^{ax}}{a} - \frac{b}{a}\int Cosbx\; e^{ax}\, dx.$$

$$I = Sinbx\,\frac{e^{ax}}{a} - \frac{b}{a}\; I. \qquad \text{by (1)}$$

put this result in (2)

$$(2)\Rightarrow I = \frac{Cosbx}{a}e^{ax} + \frac{b}{a}\left\{ Sinbx\,\frac{e^{ax}}{a} - \frac{b}{a}\,I \right\} + c$$

$$I = \frac{Cosbx}{a}e^{ax} + \frac{b}{a^2}Sinbx\,e^{ax} - \frac{b^2}{a^2}\,I + c$$

$$I + \frac{b^2}{a^2}\,I = \frac{Cosbx}{a}e^{ax} + \frac{b}{a^2}Sinbx\,e^{ax} + c$$

$$I\left(\frac{a^2+b^2}{a^2}\right) = \frac{Cosbx}{a}e^{ax} + \frac{b}{a^2}Sinbx\,e^{ax} + c$$

xplying throughout by $\left(\frac{a^2}{a^2+b^2}\right)$

$$I = \frac{a}{a^2+b^2}\,Cosbx\,e^{ax} + \frac{a}{a^2+b^2}\,Sinbx\,e^{ax} + c$$

$$\boxed{I = \frac{e^{ax}}{a^2+b^2}(a\,Cosbx + b\,Sinbx) + c}\quad \text{Ans}$$

Q4. Calculate the following.

(i) $\int e^x(Sinx + Cosx)\,dx.$

Solution: let $I = \int e^x(Sinx + Cosx)\,dx.$

$$\boxed{\text{Using formula } \int e^x\{\, f(x) + f'(x) \,\}\,dx = e^x f(x) + c}$$

$$\boxed{I = e^x\,Sinx + c}\quad \text{Ans}$$

Second Method:

let $I = \int e^x(Sinx + Cosx)\,dx$

$I = \int e^x\,Sinx\,dx + \int e^x\,Cosx\,dx$ ——— (1)

Consider $I = \int e^x\,Sinx\,dx$

$u = Sinx$; $v = e^{ax}$

$\boxed{\frac{du}{dx} = Cosx}$; $\int v\,dx = \int e^{ax}\,dx$; $\boxed{\int vdx = e^x}$

$$I = u\int v\,dx - \int\left\{\frac{du}{dx}\int v\,dx\right\}dx$$

$$I = e^x\,Sinx - \int Cos\,x\,e^x\,dx$$

Put this result in (1)

$$(1)\Rightarrow I = e^x\,Sinx - \int Cosx\,e^x\,dx + \int \cancel{e^x\,Cos\theta}\,dx$$

$$\boxed{I = e^x\,Sinx + c}\quad \text{Ans}$$

(ii) $\int e^x\left\{ Sec^{-1}x + \frac{1}{x\sqrt{x^2-1}} \right\}dx$

Solution: $I = \int e^x\left\{ Sec^{-1}x + \frac{1}{x\sqrt{x^2-1}} \right\}dx$

$$\boxed{\text{Using formula } \int e^x\{\, f(x) + f'(x) \,\}\,dx = e^x f(x) + c}$$

$$\boxed{I = e^x\,Sec^{-1}x + c}\quad \text{Ans}$$

(iii) $\int \frac{x\,ex}{(1+x)^2}$

Solution: let $I = \int \frac{x\,ex}{(1+x)^2}$

$$I = \int\frac{(x+1-1)\,e^x\,dx}{(1+x)^2} \Rightarrow I = \int\frac{\cancel{(x+1)}\,e^x\,dx}{(1+x)^{\cancel{2}}} - \int\frac{e^x\,dx}{(1+x)^2}$$

$$I = \int\frac{e^x\,dx}{(1+x)} - \int\frac{e^x\,dx}{(1+x)^2} \text{ ——— (1)}$$

Consider $I = \int\frac{e^x\,dx}{(1+x)}$

Using by parts.

$u = (1+x)^{-1}$; $v = e^x$

differentiate w • r to x ; $\int v\,dx = \int e^x\,dx$

$\frac{du}{dx} = (-1)(1+x)^{-2}$; $\boxed{\int vdx = e^x}$

$$\boxed{\frac{du}{dx} = \frac{-1}{(1+x)^2}}$$

$$I = u\int v\,dx - \int\left\{\frac{du}{dx}\int v\,dx\right\}dx$$

$$I = \frac{e^x}{(1+x)} - \int\left\{\frac{-1}{(1+x)^2}\right\}e^x\,dx$$

$$I = \frac{e^x}{(1+x)} - \int\frac{e^x}{(1+x)^2}\,dx$$

put this result in (1)

$$(1)\Rightarrow I = \frac{e^x}{(1+x)} - \cancel{\int\frac{e^x\,dx}{(1+x)^2}} - \cancel{\int\frac{e^x\,dx}{(1+x)^2}} + c$$

$$\boxed{I = \frac{e^x}{(1+x)} + c}\quad \text{Ans}$$

(iv) $\int\frac{e^x(1+x)\,dx}{(2+x)^2}$

Solution: let $I = \int\frac{e^x(1+x)\,dx}{(2+x)^2}$

$$I = \int\frac{(2+x-1)\,e^x\,dx}{(2+x)^2} \Rightarrow I = \int\frac{\cancel{(2+x)}\,e^x\,dx}{(2+x)^{\cancel{2}}} - \int\frac{e^x\,dx}{(2+x)^2}$$

$$I = \int \frac{e^x\,dx}{(2+x)} - \int \frac{e^x\,dx}{(2+x)^2} \text{ ———— (1)}$$

Consider $I = \int \frac{e^x\,dx}{(2+x)}$

Using by parts.

$u = (2+x)^{-1}$; $v = e^x$

differentiate w • r to x $\quad \int v\,dx = \int e^x\,dx$

$\frac{du}{dx} = (-1)(2+x)^{-2}$ $\quad \boxed{\int v dx = e^x}$

$$\boxed{\frac{du}{dx} = \frac{-1}{(2+x)^2}}$$

$$I = u\int v\,dx - \int\left\{\frac{du}{dx}\int v\,dx\right\}dx$$

$$I = \frac{e^x}{(2+x)} - \int\left\{\frac{-1}{(2+x)^2}\right\}e^x\,dx$$

$$I = \frac{e^x}{(2+x)} - \int\frac{e^x}{(2+x)^2}\,dx$$

put this result in (1)

$$(1) \Rightarrow I = \frac{e^x}{(2+x)} - \cancel{\int\frac{e^x\,dx}{(2+x)^2}} - \cancel{\int\frac{e^x\,dx}{(2+x)^2}} + c$$

$$\boxed{I = \frac{e^x}{(2+x)} + c} \quad \text{Ans}$$

(v) $\int e^x \frac{1+\text{Sin}x}{1+\text{Cos}x}\,dx$

Solution: let $I = \int e^x \frac{1+\text{Sin}x}{1+\text{Cos}x}\,dx$

$$\boxed{\begin{array}{l} \therefore \text{Sin}x = 2\,\text{Sin}\frac{x}{2}\,\text{Cos}\frac{x}{2} \\ \therefore 1+\text{Cos}x = 2\,\text{Cos}^2\frac{x}{2} \end{array}}$$

$$I = \int e^x\left\{\frac{1 + 2\,\text{Sin}\frac{x}{2}\,\text{Cos}\frac{x}{2}}{2\,\text{Cos}^2\frac{x}{2}}\right\}dx$$

$$I = \int e^x\left\{\frac{1}{2\,\text{Cos}^2\frac{x}{2}} + \frac{\cancel{2}\text{Sin}\frac{x}{2}\,\cancel{\text{Cos}\frac{x}{2}}}{\cancel{2}\,\text{Cos}^{\cancel{2}}\frac{x}{2}}\right\}dx$$

$$I = \int e^x\left\{\frac{1}{2}\,\text{Sec}^2\frac{x}{2} + \tan\frac{x}{2}\right\}dx \Rightarrow I = \int e^x\left(\tan\frac{x}{2} + \frac{1}{2}\,\text{Sec}^2\frac{x}{2}\right)dx$$

Using formula

$\boxed{\text{Using formula } \int e^x\{f(x) + f'(x)\}\,dx = e^x f(x) + c}$

$\boxed{I = e^x\,\text{Sec}^{-1}x + c}$ Ans

(vi) $\int \text{Cos}\left(b\ln\frac{x}{a}\right) + c$

Solution: let $I = \int \text{Cos}\left(b\ln\frac{x}{a}\right)dx$ ———— (1)

let $y = \ln\frac{x}{a} \Rightarrow e^y = \frac{x}{a}$

differentiate w • r to x $\quad \boxed{x = ae^y}$

$$\frac{dy}{dx} = \frac{1}{\frac{x}{a}}\left(\frac{1}{a}\right) \Rightarrow \frac{dy}{dx} = \frac{\cancel{a}}{x}\cdot\frac{1}{\cancel{a}} \Rightarrow \boxed{xdy = dx}$$

$(1) \Rightarrow I = \int \text{Cos}\,(by)\,xdy. \Rightarrow I = \int \text{Cos}\,(by)\,ae^y)\,dy.$

$I = a\int e^y\,\text{Cos}\,by\,dy$ ———— (2)

let $I = \int e^y\,\text{Cos}\,by\,dy$

$u = \text{Cos}\,by$ $\quad v = e^y$

$\boxed{\frac{du}{dy} = b\,\text{Sin}\,by}$ $\quad \int v\,dy = \int e^y\,dx$ $\quad \boxed{\int v\,dy = e^y}$

$$I = u\int v\,dy - \int\left\{\frac{du}{dy}\int v\,dy\right\}dy$$

$I = e^y\,\text{Cos}\,by - \int(-\,b\,\text{Sin}\,by)\,e^y\,dy.$

$I = e^y\,\text{Cos}\,by + 6\int\text{Sin}\,by\,e^y\,dy.$ ———— (3)

let $I = \int\text{Sin}\,by\,e^y\,dy.$

$u = \text{Sin}\,by$ $\quad v = e^y$

$\boxed{\frac{du}{dy} = b\,\text{Cos}\,by}$ $\quad \int v\,dy = \int e^y\,dy$ $\quad \boxed{\int v\,dy = e^y}$

$I = \text{Sin}\,by\,e^y - \int b\,\text{Cos}\,by\,e^y\,dy.$

$I = \text{Sin}\,by\,e^y - b\int\text{Cos}\,by\,e^y\,dy.$

$I = \text{Sin}\,by\,e^y - b\,I$

put this result in (3)

$(3) \Rightarrow I = e^y\,\text{Cos}\,by + b\,\{\,\text{Sin}\,b^y\,ey - b\,I\,\} + c$

$I = e^y\,\text{Cos}\,by + b\,\text{Sin}\,by\,e^y - b^2 I + c$

$I = b^2 I = e^y\,\text{Cos}\,by + b\,\text{Sin}\,by\,e^y + c$

$I = (1 + b^2) = e^y \text{ Cos by} + b \sin by\, e^y + c.$

$I = \dfrac{e^y \text{ Cos by}}{1 + b^2} + \dfrac{b \text{ Sin by } e^y}{1 + b^2}$

$I = \dfrac{a\, e^y \text{ Cos by}}{1 + b^2} + \dfrac{ab \text{ Sin by } e^y}{1 + b^2} + c$

but $y = \ln \dfrac{x}{a}$ $\qquad ae^y = x$

$I = \dfrac{x \text{ Cos}\left(b \ln \frac{x}{a}\right)}{1 + b^2} + \dfrac{b \text{ Sin}\left(b\, \ell n \frac{x}{a}\right) x}{1 + b^2} + c$

$\boxed{I = \dfrac{x}{1 + b^2}\left\{\text{Cos}\left(b\, \ell n \dfrac{x}{a}\right) + b \text{ Sin}\left(b\, \ell n \dfrac{x}{a}\right)\right\} + c}$ Ans

EXERCISE # 6.8

Resolve into partial fractions.

(1) $\dfrac{7x - 25}{(x - 3)(x - 4)}$

Solution: let $\dfrac{7x - 25}{(x - 3)(x - 4)} = \dfrac{A}{x - 3} + \dfrac{B}{x - 4}$ ——— (1)

multiplying throughout eq^n (1) by $(x - 3)(x - 4)$

$7x - 25 = A(x - 4) + B(x - 3)$ ——— (2)

put $x - 3 = 0 \Rightarrow \boxed{x = 3}$ in eq^n (2) $\Rightarrow \boxed{A = 4}$

put $x - 4 = 0 \Rightarrow \boxed{x = 4}$ in eq^n (2) $\Rightarrow \boxed{B = 3}$

put the values of A & B in (1)

(1)$\Rightarrow$ $\boxed{\dfrac{7x - 25}{(x - 3)(x - 4)} = \dfrac{4}{x - 3} + \dfrac{3}{x - 4}}$ Ans.

(2) $\dfrac{6x^3 + 5x^2 - 7}{3x^2 - 2x - 1}$

Solution: $\dfrac{6x^3 + 5x^2 - 7}{3x^2 - 2x - 1}$

we first divide the N (x) by D (x).

$$\begin{array}{r|l} & 2x + 3 \\ \hline 3x^2 - 2x - 1 & \cancel{6x^3} + 5x^2 - 7 \\ & \underline{-\cancel{6x^3} \mp 4x^2 \mp 2x} \\ & \cancel{9x^2} + 2x - 7 \\ & \underline{-\cancel{9x^2} \mp 6x \mp 3} \\ & 8x - 4 \end{array}$$

$\therefore \quad \dfrac{N(x)}{D(x)} = Q(x) + \dfrac{R(x)}{D(x)}$

$\dfrac{6x^3 + 5x^2 - 7}{3x^2 - 2x - 1} = 2x + 3 + \dfrac{8x - 4}{3x^2 - 2x - 1}$

$\dfrac{8x - 4}{3x^2 - 2x - 1} = \dfrac{8x - 4}{(x - 1)(3x + 1)}$ (Now partial fraction)

$\dfrac{8x - 4}{(x - 1)(3x + 1)} = \dfrac{A}{x - 1} + \dfrac{B}{3x + 1}$ ——— (1)

Xplying throughout eq^n (1) by $(x - 1)(3x - 1)$

$8x - 4 = A(3x + 1) + B(x - 1)$ ——— (2)

put $x - 1 = 0 \Rightarrow \boxed{x = 1}$ in (2) $\Rightarrow \boxed{A = 1}$

put $3x + 1 = 0 \Rightarrow \boxed{x = \dfrac{-1}{3}}$ in (2) $\Rightarrow \boxed{B = 5}$

Now $\boxed{\dfrac{6x^3 + 5x^2 - 7}{3x^2 - 2x - 1} = 2x + 3 + \dfrac{1}{x - 1} + \dfrac{5}{3x + 1}}$ Ans.

(3) $\dfrac{2x^3 + x^2 - x - 3}{x(x - 1)(2x + 3)}$

Solution: $\dfrac{2x^3 + x^2 - x - 3}{x(x - 1)(2x + 3)} = \dfrac{2x^3 + x^2 - x - 3}{2x^3 + x^2 - 3x}$

first we divide N (x) by D (x)

$$\begin{array}{r|l} & 1 \\ \hline 2x^3 + x^2 - 3x & \cancel{2x^3} + \cancel{x^2} - x - 3 \\ & \underline{-\cancel{2x^3} \pm \cancel{x^2} \mp 3x} \\ & 2x - 3 \end{array}$$

$\dfrac{2x^3 + x^2 - x - 3}{x(x - 1)(2x + 3)} = 1 + \dfrac{2x - 3}{2x^3 + x^2 - 3x}$ ——— (1)

Consider $\dfrac{2x - 3}{2x^3 + x^2 - 3x} = \dfrac{A}{x} + \dfrac{B}{x - 1} + \dfrac{C}{2x + 3}$ ——— (2)

Xplying throughout eq^n (2) by $x(x - 1)(2x + 3)$

$2x - 3 = A(x - 1)(2x + 3) + Bx(2x + 3) + cx(x - 1)$ —— (3)

put $x = 0$ in (3) $\Rightarrow \boxed{A = 1}$

put $x - 1 = 0 \Rightarrow \boxed{x = 1}$ in (3) $\Rightarrow \boxed{B = \dfrac{-1}{5}}$

put $2x + 3 = 0 \Rightarrow \boxed{x = \frac{-3}{2}}$ in (3) $\Rightarrow \boxed{c = \frac{-8}{5}}$

(1)$\Rightarrow \boxed{\frac{2x^3 + x^2 - x - 3}{x(x-1)(2x+3)} = 1 + \frac{1}{x} - \frac{1}{5(x-1)} - \frac{-8}{5(2x+3)}}$ Ans.

(4) $\frac{1}{(1-ax)(1-bx)(1-cx)}$

Solution: $\frac{1}{(1-ax)(1-bx)(1-cx)} = \frac{A}{1-ax} + \frac{B}{1-bx} + \frac{C}{1-cx}$ —(1)

Xplying throughout eq^n (1) by $(1-ax)(1-bx)(1-cx)$

$1 = A(1-bx)(1-cx) + B(1-ax)(1-cx) + c(1-ax)(1-bx)$ —(2)

put $1 - ax = 0 \Rightarrow \boxed{x = \frac{1}{a}}$ in (2) $\Rightarrow \boxed{A = \frac{a^2}{(a-b)(a-c)}}$

put $1 - bx = 0 \Rightarrow \boxed{x = \frac{1}{b}}$ in (2) $\Rightarrow \boxed{B = \frac{b^2}{(b-a)(b-c)}}$

put $1 - cx = 0 \Rightarrow \boxed{x = \frac{1}{c}}$ in (2) $\Rightarrow \boxed{C = \frac{c^2}{(c-a)(c-b)}}$

$$\boxed{(1)\Rightarrow \frac{1}{(1-ax)(1-bx)(1-cx)} = \frac{a^2}{(a-b)(a-c)(1-ax)} + \frac{b^2}{(b-a)(b-c)(1-bx)} + \frac{c^2}{(c-a)(c-b)(1-cx)}}$$

(5) $\frac{x^2 + a^2}{(x^2+b^2)(x^2+c^2)(x^2+d^2)}$ **(Hint : put $x^2 = y$)**

Solution: $\frac{x^2+a^2}{(x^2+b^2)(x^2+c^2)(x^2+d^2)} = \frac{y+a^2}{(y+b^2)(y+c^2)(y+d^2)}$

$\frac{y+a^2}{(y+b^2)(y+c^2)(y+d^2)} = \frac{A}{y+b^2} + \frac{B}{y+c^2} + \frac{C}{y+d^2}$ —(1)

Xplying throughout eq^n (1) by $(y+b^2)(y+c^2)(y+d^2)$

$y + a^2 = A(y+c^2)(y+d^2) + B(y+b^2)(y+d^2) + c(y+c^2)(y+b^2)$ —(2)

put $y + b^2 = 0 \Rightarrow \boxed{y = -b^2}$ in (2)$\Rightarrow \boxed{A = \frac{a^2-b^2}{(c^2-b^2)(d^2-b^2)}}$

put $y + c^2 = 0 \Rightarrow \boxed{y = -c^2}$ in (2)$\Rightarrow \boxed{B = \frac{a^2-c^2}{(b^2-c^2)(d^2-c^2)}}$

put $y + d^2 = 0 \Rightarrow \boxed{y = -d^2}$ in (2)

(2)$\Rightarrow \boxed{C = \frac{d^2-a^2}{(c^2-d^2)(b^2-d^2)}}$

put the values of A, B & C in (1)

$$\frac{y+a^2}{(y+b^2)(y+c^2)(y+d^2)} = \frac{a^2-b^2}{(c^2-b^2)(d^2-b^2)(y+b^2)} + \frac{a^2-c^2}{(b^2-c^2)(d^2-c^2)(y+c^2)} + \frac{d^2-a^2}{(c^2-d^2)(b^2-d^2)(y+d^2)}$$

$$\boxed{\frac{x+a^2}{(x+b^2)(x+c^2)(x+d^2)} = \frac{a^2-b^2}{(c^2-b^2)(d^2-b^2)(x+b^2)} + \frac{a^2-c^2}{(b^2-c^2)(d^2-c^2)(x+c^2)} + \frac{d^2-a^2}{(c^2-d^2)(b^2-d^2)(x+d^2)}}$$

Ans.

(6) $\frac{1}{x^4(x+1)}$

Solution: $\frac{1}{x^4(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x^4} + \frac{E}{x+1}$ —(1)

Xplying throughout eq^n (1) by $x^4(x+1)$

$1 = Ax^3(x+1) + Bx^2(x+1) + Cx(x+1) + D(x+1) + Ex^4$ —(2)

put $x = 0$ in eq^n (2) $\Rightarrow \boxed{D = 1}$

eq^n (2) can be written as

$1 = A(x^4 + x^3) + B(x^3 + x^2) + C(x^2 + x) + D(x+1) + Ex^4$

equating the coefficients of x^4, x^3, x^2, x

$A + E = 0$ —(3)

$A + B = 0$ —(4)

$B + C = 0$ —(5)

$C + D = 0$ —(6)

put $D = 1$ in (6) $\Rightarrow \boxed{C = -1}$

put $C = -1$ in (5) $\Rightarrow \boxed{B = 1}$

put $B = 1$ in (4) $\Rightarrow \boxed{A = -1}$

put $A = -1$ in (3) $\Rightarrow \boxed{E = 1}$

put these values of A, B, C, D, E in (1)

(1)$\Rightarrow \boxed{\frac{1}{x^4(x+1)} = \frac{-1}{x} + \frac{1}{x^2} - \frac{1}{x^3} + \frac{1}{x^4} + \frac{1}{x+1}}$ Ans.

(7) $\dfrac{4x^3}{(x+1)^2(x^2-1)}$

Solution: $\dfrac{4x^3}{(x+1)^2(x^2-1)} = \dfrac{4x^3}{(x+1)^2(x+1)(x-1)} = \dfrac{4x^3}{(x+1)^3(x-1)}$

$$\frac{4x^3}{(x+1)^3(x-1)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3} + \frac{D}{x-1} \text{ —— (1)}$$

Xplying throughout eq^n (1) by $(x+1)^3(x-1)$

$4x^3 = A(x+1)^2(x-1) + B(x+1)(x-1) + C(x-1) + D(x+1)^3$ ——— (2)

put $x + 1 = 0 \Rightarrow \boxed{x=-1}$ in eq^n (2) $\Rightarrow \boxed{c=2}$

eqn (2) can be written as

$4x^3 = A(x^3+x^2-x-1) + B(x^2-1) + C(x-1) + D(x^3+3x^2+3x+1)$

equating the coefficient of x^3, x^2, x and constant terms.

$A + D = 4$ ——— (3)

$A + B + 3D = 0$ ——— (4)

$C - A + 3D = 0$ ——— (5)

$-A - B - C + D = 0$ ——— (6)

put $x - 1 = 0 \Rightarrow \boxed{x=1}$ in eq^n (2) $\Rightarrow \boxed{D=\frac{1}{2}}$

put the value of D in (3) $\Rightarrow \boxed{A=\frac{7}{2}}$

put the values of A & D in eq^n (4)

(4)$\Rightarrow \frac{7}{2} + B + \frac{3}{2} = 0 \Rightarrow \boxed{B=-5}$

put the values of A, B, C & D in (1)

$$(1)\Rightarrow \frac{4x^3}{(x+1)^3(x-1)} = \frac{\frac{7}{2}}{x+1} + \frac{-5}{(x+1)^2} + \frac{2}{(x+1)^3} + \frac{\frac{1}{2}}{x-1}$$

$$\boxed{\frac{4x^3}{(x+1)^3(x-1)} = \frac{7}{2(x+1)} - \frac{5}{(x+1)^2} + \frac{2}{(x+1)^3} + \frac{1}{2(x-1)}}$$

Ans.

(8) $\dfrac{x^4+1}{x^2(x-1)}$

Solution: $\dfrac{x^4+1}{x^2(x-1)} = \dfrac{x^4+1}{x^3-x^2}$

We first divide N(x) by D(x).

$$\begin{array}{r|l} & x+1 \\ x^3-x^2 & x^4+1 \\ & \underline{-x^4 \mp x^3} \\ & x^3+1 \\ & \underline{-x^3 \mp x^2} \\ & x^2+1 \end{array}$$

$$\frac{x^4+1}{x^2(x-1)} = x + 1 + \frac{x^2+1}{x^2(x-1)} \qquad \boxed{\therefore \frac{N(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)}}$$

let $\dfrac{x^2+1}{x^2(x-1)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x-1}$ ——— (1)

Xplying throughout eq^n (1) by $x^2(x-1)$

$x^2 + 1 = Ax(x-1) + B(x-1) + Cx^2$ ——— (2)

put $x = 0$ in eq^n (2) $\Rightarrow \boxed{B=-1}$

put $x - 1 = 0 \Rightarrow \boxed{x=1}$ in eq^n (2) $\Rightarrow \boxed{C=2}$

equating the coefficient of x^2

$A + C = 1 \Rightarrow A + 2 = 1 \Rightarrow \boxed{A=-1}$

$$(1)\Rightarrow \frac{x^2+1}{x^2(x-1)} = \frac{-1}{x} - \frac{1}{x^2} + \frac{2}{x-1}$$

$$\boxed{\therefore \frac{x^4+1}{x^2(x-1)} = x + 1 - \frac{1}{x} - \frac{1}{x^2} + \frac{2}{x-1}}$$ Ans.

(9) $\dfrac{2x+1}{(x+3)(x-1)(x+2)^2}$

Solution:

$$\frac{2x+1}{(x+3)(x-1)(x+2)^2} = \frac{A}{x+3} + \frac{B}{x-1} + \frac{C}{x+2} + \frac{D}{(x+2)^2} \text{ —— (1)}$$

Xplyng throughout eq^n (1) by $(x+3)(x-1)(x+2)^2$

$2x + 1 = A(x-1)(x+2)^2 + B(x+3)(x+2)^2 + C(x-1)(x+3)(x+2) + D(x+3)(x-1)$ ——— (2)

put $x + 3 = 0 \Rightarrow \boxed{x=-3}$ in eq^n (2) $\Rightarrow \boxed{A=\frac{5}{4}}$

put $x - 1 = 0 \Rightarrow \boxed{x=1}$ in eq^n (2) $\Rightarrow \boxed{B=\frac{1}{12}}$

put $x + 2 = 0 \Rightarrow \boxed{x=-2}$ in eq^n (2) $\Rightarrow \boxed{D=1}$

eq^n (2) can be written as

$2x + 1 = A(x^3 + 3x^2 - 4) + B(x^3 + 7x^2 + 16x + 12) + C(x^3 + 4x^2 + x - 6) + D(x^2 + 2x - 3)$

equating the coefficient of x^3

$$A + B + C = 0 \Rightarrow \frac{5}{4} + \frac{1}{12} + C = 0 \Rightarrow \boxed{C = \frac{-4}{3}}$$

put the values of A, B, C & D in (1)

$$(1) \Rightarrow \frac{2x+1}{(x+3)(x-1)(x+2)^2} = \frac{\frac{5}{4}}{x+3} + \frac{\frac{1}{12}}{x-1} - \frac{\frac{4}{3}}{x+2} + \frac{1}{(x+2)^2}$$

$$\boxed{\frac{2x+1}{(x+3)(x-1)(x+2)^2} = \frac{5}{4(x+3)} + \frac{1}{12(x-1)} - \frac{4}{3(x+2)} + \frac{1}{(x+2)^2}}$$

Ans

(10) $\dfrac{1}{(x+1)(x^2+1)}$

Solution: let $\dfrac{1}{(x+1)(x^2+1)} = \dfrac{A}{x+1} + \dfrac{Bx+C}{x^2+1}$ —— (1)

Xplying throughout eq^n (1) by $(x + 1)(x^2 + 1)$

$1 = A(x^2 + 1) + Bx(x + 1) + C(x + 1)$ —— (2)

put $x + 1 = 0 \Rightarrow \boxed{x = -1}$ in eq^n (2) $\Rightarrow \boxed{A = \frac{1}{2}}$

eq^n (2) can be written as

$1 = (A + B)x^2 + (B + C)x + (A + C)$

equating the coefficients of x^2 & x.

$$A + B = 0 \Rightarrow \frac{1}{2} + B = 0 \Rightarrow \boxed{B = \frac{-1}{2}}$$

$$B + C = 0 \Rightarrow \frac{-1}{2} + C = 0 \Rightarrow \boxed{C = \frac{1}{2}}$$

put the values of A, B & C in (1)

$$(1) \Rightarrow \frac{1}{(x+1)(x^2+1)} = \frac{\frac{1}{2}}{x+1} + \frac{\frac{-1}{2}x + \frac{1}{2}}{x^2+1}$$

$$\boxed{\frac{1}{(x+1)(x^2+1)} = \frac{1}{2(x+1)} + \frac{1-x}{2(x^2+1)}}$$

Ans.

(11) $\dfrac{3x+7}{(x^2+x+1)(x^2-4)}$

Solution: $\dfrac{3x+7}{(x^2+x+1)(x^2-4)} = \dfrac{3x+7}{(x^2+x+1)(x-2)(x+2)}$

let $\dfrac{3x+7}{(x^2+x+1)(x^2-4)} = \dfrac{Ax+B}{x^2+x+1} + \dfrac{C}{x-2} + \dfrac{D}{x+2}$ — (1)

Xplying throughout eq^n (1) by $(x^2 + x + 1)(x^2 - 4)$

$3x + 7 = Ax(x^2 - 4) + B(x^2 - 4) + C(x + 2)(x^2 + x + 1) + D(x - 2)(x^2 + x + 1)$ —— (2)

put $x - 2 = 0 \Rightarrow \boxed{x = 2}$ in eq^n (2) $\Rightarrow \boxed{C = \frac{13}{28}}$

put $x + 2 = 0 \Rightarrow \boxed{x = -2}$ in eq^n (2) $\Rightarrow \boxed{D = \frac{-1}{12}}$

eq^n (2) can be written as

$3x + 7 = (A + C + D)x^3 + (B + 3C - D)x^2 + (3C - 4A - D)x - 4B + 2C - 2D$

equating the coefficients of x^3, x^2 we have

$$A + C + D = 0 \Rightarrow A + \frac{13}{28} - \frac{1}{12} = 0 \Rightarrow A + \frac{39-7}{84} = 0$$

$$A + \frac{\cancel{32}^{\,8}}{\cancel{84}_{\,21}} = 0 \Rightarrow \boxed{A = \frac{-8}{21}}$$

$$B + 3C - D = 0 \Rightarrow B + \frac{39}{28} + \frac{1}{12} = 0$$

$$B + \frac{117+7}{84} = 0 \Rightarrow B + \frac{\cancel{124}^{\,31}}{\cancel{84}_{\,21}} = 0 \Rightarrow \boxed{B = \frac{-31}{21}}$$

put the values of A, B, C & D in (1)

$$(1) \Rightarrow \frac{3x+7}{(x^2+x+1)(x^2-4)} = \frac{\frac{-8}{21}x - \frac{31}{21}}{x^2+x+1} + \frac{\frac{13}{28}}{x-2} + \frac{\frac{-1}{12}}{x+2}$$

$$\boxed{\frac{3x+7}{(x^2+x+1)(x^2-4)} = \frac{-1}{21}\left(\frac{8x+31}{x^2+x+1}\right) + \frac{13}{28(x-2)} - \frac{1}{12(x+2)} \quad \text{Ans.}}$$

(12) $\dfrac{x^5}{x^4-1}$

Solution: first we ÷ N (x) by D (x)

$\dfrac{x^5}{x^4-1} = x + \dfrac{x}{x^4-1}$ —— (1)

$$\boxed{\therefore \frac{N(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)}}$$

$$\begin{array}{r|l} & x \\ x^4 - 1 & x^5 \\ & \underline{-x^5 \mp x} \\ & x \end{array}$$

let $\frac{x}{x^4-1} = \frac{x}{(x-1)(x+1)(x^2+1)}$

$\frac{x}{x^4-1} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1}$ ——— (2)

Xplying throughout eq^n (2) by $(x-1)(x+1)(x^2+1)$

$x = A(x+1)(x^2+1) + B(x-1)(x^2+1) + Cx(x^2-1) + D(x^2-1)$ ——— (3)

put $x - 1 = 0 \Rightarrow \boxed{x = 1}$ in eq^n (3) $\Rightarrow \boxed{A = \frac{1}{4}}$

put $x + 1 = 0 \Rightarrow \boxed{x = -1}$ in eq^n (3) $\Rightarrow \boxed{B = \frac{1}{4}}$

eq^n (2) can be written as

$x = (A+B+C)x^3 + (A-B+D)x^2 + (A+B-C)x + (A-B-D)$

equating the coefficient of x^3 & x^2

$A + B + C = 0 \Rightarrow \frac{1}{4} + \frac{1}{4} + C = 0 \Rightarrow \boxed{C = \frac{-1}{2}}$

$A - B + D = 0 \Rightarrow \frac{1}{4} - \frac{1}{4} + D = 0 \Rightarrow \boxed{D = 0}$

put the values of A, B, C & D in (1)

(1)$\Rightarrow \boxed{\frac{x^5}{x^4-1} = x + \frac{1}{4(x-1)} + \frac{1}{4(x+1)} - \frac{x}{2(x^2+1)}}$ Ans.

(13) $\frac{x+a}{x^2(x-a)(x^2+a^2)}$

Solution: $\frac{x+a}{x^2(x-a)(x^2+a^2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-a} + \frac{Dx+E}{x^2+a^2}$ ——— (1)

Xplying throughout eq^n (1) by $x^2(x-a)(x^2+a^2)$

$x + a = Ax(x-a)(x^2+a^2) + B(x-a)(x^2+a^2) + C(x^2+a^2)x^2 + Dx^3(x-a) + Ex^2(x-a)$ ——— (2)

put $x = 0$ in (2) $\Rightarrow \boxed{B = \frac{-1}{a^2}}$

put $x - a = 0 \Rightarrow \boxed{x = a}$ in (2) $\Rightarrow \boxed{C = \frac{1}{a^3}}$

eq^n (2) can be written as

$x + a = (A+C+D)x^4 + (B - aA - aD + E)x^3 + (a^2A - aB + a^2C - aE)x^2 + (a^2B - a^3A)x - a^3B$

equating the coefficient of x^4, x^3 & x

$a^2B - a^3A = 1 \Rightarrow \cancel{a^2}\left(\frac{-1}{\cancel{a^2}}\right) - a^3A = 1 \Rightarrow -1 - 1 = a^3A$

$\boxed{A = \frac{-2}{a^3}}$

$A + C + D = 0 \Rightarrow \frac{-2}{a^3} + \frac{1}{a^3} + D = 0 \Rightarrow \boxed{D = \frac{1}{a^3}}$

$B - aA - aD + E = 0 \Rightarrow \frac{-1}{a^2} - a\left(\frac{-2}{a^3}\right) - a\left(\frac{1}{a^3}\right) + E = 0$

$\frac{-1}{a^2} + \frac{2a}{a^3} - \frac{1}{a^2} + E = 0 \Rightarrow E + \frac{2-2}{a^2} = 0 \Rightarrow \boxed{E = 0}$

put the values of A, B, C, D & E in (1)

(1)$\Rightarrow \frac{x+a}{x^2(x-a)(x^2+a^2)} = \frac{\frac{-2}{a^3}}{x} + \frac{\frac{-1}{a^2}}{x^2} + \frac{\frac{1}{a^3}}{x-a} + \frac{\frac{1}{a^3}x}{x^2+a^2}$

$\boxed{\frac{x+a}{x^2(x-a)(x^2+a^2)} = \frac{-2}{a^3x} - \frac{1}{a^2x^2} + \frac{1}{a^3(x-a)} + \frac{x}{a^3(x^2+a^2)}}$ Ans.

(14) $\frac{4x^4 + 3x^3 + 6x^2 + 5x}{(x-1)(x^2+x+1)^2}$

Solution:

$\frac{4x^4 + 3x^3 + 6x^2 + 5x}{(x-1)(x^2+x+1)^2} = \frac{A}{x-1} + \frac{Bx+c}{x^2+x+1} + \frac{Dx+E}{(x^2+x+1)^2}$ —— (1)

Xplying throughout eq^n (1) by $(x-1)(x^2+x+1)$

$4x^4 + 3x^3 + 6x^2 + 5x = A(x^2+x+1)^2 + Bx(x-1)(x^2+x+1) + C(x-1)(x^2+x+1) + Dx(x-1) + E(x-1)$ —— (2)

put $x - 1 = 0 \Rightarrow \boxed{x = 1}$ in (2) $\boxed{A = 2}$

eq^n (2) can be written as

$4x^3 + 3x^3 + 6x^2 + 5x = (A+B)x^4 + (2A+C)x^3 + (3A+D)x^2 + (2A - B - D + E)x + (A - C - E)$

equating the Coefficient of x^4, x^3, x^2 and constant term

$A + B = 4 \Rightarrow 2 + B = 4 \Rightarrow \boxed{B = 2}$

$2A + C = 3 \Rightarrow 2(2) + C = 3 \Rightarrow \boxed{C = -1}$

$3A + D = 6 \Rightarrow 3(2) + D = 6 \Rightarrow \boxed{D = 0}$

$A - C - E = 0 \Rightarrow 2 + 1 - E = 0 \Rightarrow \boxed{E = 3}$

put the values of A, B, C, D & E in (1)

(1)$\Rightarrow \boxed{\frac{4x^4 + 3x^3 + 6x^2 + 5x}{(x-1)(x^2+x+1)^2} = \frac{2}{x-1} + \frac{2x-1}{x^2+x+1} + \frac{3}{(x^2+x+1)^2}}$ Ans.

(15) $\dfrac{x^3-15x^2-8x-7}{(1+x^2)^2(2x-5)}$

Solution: $\dfrac{x^3-15x^2-8x-7}{(1+x^2)^2(2x-5)}=\dfrac{Ax+B}{1+x^2}+\dfrac{Cx+D}{(1+x^2)^2}+\dfrac{E}{2x-5}$ — (1)

Xplying throughout eq^n (1) by $(1+x^2)^2(2x-5)$

$x^3-15x^2-8x-7=Ax(2x-5)(1+x^2)+B(2x-5)(1+x^2)+$
$Cx(2x-5)+D(2x-5)+E(1+x^2)^2$ —— (2)

put $2x-5=0\Rightarrow \boxed{x=\dfrac{5}{2}}$ in (2)

(2)$\Rightarrow \dfrac{125}{8}-\dfrac{375}{4}-\dfrac{40}{2}-7=E\left(\dfrac{29}{4}\right)^2$

$\dfrac{125-750-160-56}{8}=E\left(\dfrac{841}{16}\right)$

$-\dfrac{\cancel{841}}{\cancel{8}}=E\left(\dfrac{\cancel{841}}{\cancel{16}_{2}}\right)\Rightarrow \boxed{E=-2}$

eq^n (2) can be written as

$x^3-15x^2-8x-7=(2A+E)x^4+(2B-5A)x^3+(2A-5B+2C$
$+2E)x^2+(2B-5A-5C+2D)x-(5B+5D-E)$

equating the coefficients of x^4, x^3, x^2 and the constant terms.

$2A+E=0\Rightarrow 2A-2=0\Rightarrow \boxed{A=1}$

$2B-5A=1\Rightarrow 2B-5(1)=1\Rightarrow \boxed{B=3}$

$2A-5B+2C+2E=-15\Rightarrow 2(1)-5(3)+2C+2(-2)=-15$

$\Rightarrow \boxed{C=1}$ $\cancel{-}(5B+5D-E)=\cancel{-}7\Rightarrow 5(3)+5D+2=7$

$\Rightarrow \boxed{D=-2}$

put the values of A, B, C, D & E in (1)

$$\boxed{\dfrac{x^3-15x^2-8x-7}{(1+x^2)^2(2x-5)}=\dfrac{x+3}{1+x^2}+\dfrac{x-2}{(1+x^2)^2}-\dfrac{2}{2x-5}}$$ Ans.

(16) $\dfrac{8x^2}{(1-x^4)(1+x^2)}$

Solution: $\dfrac{8x^2}{(1-x^4)(1+x^2)}=\dfrac{8x^2}{(1-x)(1+x)(1+x^2)(1+x^2)}$

(or) $\dfrac{8x^2}{(1-x^4)(1+x^2)}=\dfrac{8x^2}{(1-x)(1+x)(1+x^2)^2}$

let $\dfrac{8x^2}{(1-x^4)(1+x^2)}=\dfrac{A}{1-x}+\dfrac{B}{1+x}+\dfrac{Cx+D}{1+x^2}+\dfrac{Ex+F}{(1+x^2)^2}$ — (1)

Xplying throughout eq^n (1) by $(1-x)(1+x)(1+x^2)^2$

$8x^2=A(1+x)(1+x^2)^2+B(1-x)(1+x^2)^2+Cx(1-x^2)(1+x^2)$
$+D(1-x^2)(1+x^2)+Ex(1-x^2)+F(1-x^2)$ — (2)

put $1-x=0\Rightarrow \boxed{x=1}$ in (2) $\Rightarrow \boxed{A=1}$

put $1+x=0\Rightarrow \boxed{x=-1}$ in (2) $\Rightarrow \boxed{B=1}$

eq^n (2) can be written as

$8x^2=(A-B-C)x^2+(A+B-D)x^4+(2A-2B-E)x^3+(2A$
$+2B-F)x^2+(A-B+C+E)x+A+B+D+F$

equating the coefficient of x^5, x^4, x^3 & x^2

$A-B-C=0\Rightarrow \cancel{1}-\cancel{1}-C=0\Rightarrow \boxed{C=0}$

$A+B-D=0\Rightarrow 1+1-D=0\Rightarrow \boxed{D=2}$

$2A-2B-E=0\Rightarrow 2(1)-2(1)-E=0\Rightarrow \boxed{E=0}$

$2A+2B-F=8\Rightarrow 2(1)+2(1)-F=8\Rightarrow \boxed{F=-4}$

put the values of A, B, C, D, E & F in (1)

(1)$\Rightarrow$ $$\boxed{\dfrac{8x^2}{(1-x^4)(1+x^2)}=\dfrac{1}{1-x}+\dfrac{1}{1+x}+\dfrac{2}{1+x^2}-\dfrac{4}{(1+x^2)^2}}$$ Ans.

(17) $\dfrac{x^3+1}{(x^2-1)^2}$

Solution: $\dfrac{x^3+1}{(x^2-1)^2}=\dfrac{\cancel{(x+1)}(x^2-x+1)}{(x-1)^2(x+1)^2}=\dfrac{x^2-x+1}{(x-1)^2(x+1)}$

let $\dfrac{x^2-x+1}{(x+1)(x-1)^2}=\dfrac{A}{x+1}+\dfrac{B}{x-1}+\dfrac{C}{(x-1)^2}$ —— (1)

Xplying throughout eq^n (1) by $(x+1)(x-1)^2$

$x^2-x+1=A(x-1)^2+B(x^2-1)+C(x+1)$ —— (2)

put $x+1=0\Rightarrow \boxed{x=-1}$ in (2) $\Rightarrow \boxed{A=\dfrac{3}{4}}$

put $x-1=0\Rightarrow \boxed{x=1}$ in (2) $\Rightarrow \boxed{C=\dfrac{1}{2}}$

eq^n (2) can be written as

$x^2-x+1=(A+B)x^2+(C-2A)x+A-B+C$

equating the Coefficient of x^2

$A+B=1\Rightarrow \dfrac{3}{4}+B=1\Rightarrow \boxed{B=\dfrac{1}{4}}$

put the values of A, B & C in (1)

$$\boxed{\frac{x^3+1}{(x^2-1)^2} = \frac{3}{4(x+1)} + \frac{1}{4(x-1)} + \frac{1}{2(x-1)^2}} \quad \text{Ans.}$$

(18) $\dfrac{x^2}{x^4-5x^2+4}$

Solution: $\dfrac{x^2}{x^4-5x^2+4} = \dfrac{x^2}{(x^2-4)(x^2-1)} = \dfrac{x^2}{(x-2)(x+2)(x-1)(x+1)}$

let $\dfrac{x^2}{(x-2)(x+2)(x-1)(x+1)} = \dfrac{A}{x-2} + \dfrac{B}{x+2} + \dfrac{C}{x-1} + \dfrac{D}{x+1}$ —(1)

Xplying throughout eq^n (1) by $(x-2)(x+2)(x-1)(x+1)$

$x^2 = A(x+2)(x-1)(x+1) + B(x-2)(x-1)(x+1) + C(x-2)(x+2)(x+1) + D(x-2)(x+2)(x-1)$ —— (2)

put $x-2=0 \Rightarrow \boxed{x=2}$ in (2) $\Rightarrow \boxed{A=\frac{1}{3}}$

put $x+2=0 \Rightarrow \boxed{x=-2}$ in (2) $\Rightarrow \boxed{B=\frac{-1}{3}}$

put $x-1=0 \Rightarrow \boxed{x=1}$ in (2) $\Rightarrow \boxed{C=\frac{-1}{6}}$

put $x+1=0 \Rightarrow \boxed{x=-1}$ in (2) $\Rightarrow \boxed{D=\frac{1}{6}}$

put the values of A, B, C & D in (1)

(1)$\Rightarrow \boxed{\dfrac{x^2}{x^4-5x^2+4} = \dfrac{1}{3(x-2)} - \dfrac{1}{3(x+2)} - \dfrac{1}{6(x-1)} + \dfrac{1}{6(x+1)}}$

Ans.

(19) $\dfrac{x^2+2x-1}{x^3-27}$

Solution: $\dfrac{x^2+2x-1}{x^3-27} = \dfrac{x^2+2x-1}{(x-3)(x^2+3x+9)}$

let $\dfrac{x^2+2x-1}{(x-3)(x^2+3x+9)} = \dfrac{A}{x-3} + \dfrac{Bx+C}{x^2+3x+9}$ —— (1)

Xplying throughout eq^n (1) by $(x-3)(x^2+3x+9)$

$x^2+2x-1 = A(x^2+3x+9) + Bx(x-3) + C(x-3)$ — (2)

put $x-3=0 \Rightarrow \boxed{x=3}$ in (2) $\Rightarrow \boxed{A=\frac{14}{27}}$

eq^n (2) can be written as

$x^2+2x-1 = (A+B)x^2 + (3A-3B+C)x + 9A-3C$

equating the coefficient of x^2 & constant terms

$A+B=1 \Rightarrow \frac{14}{27} + B = 1 \Rightarrow \boxed{B=\frac{13}{27}}$

$9A-3C=-1 \Rightarrow 9\left(\frac{14}{27}\right) - 3C = 1$

$\frac{126}{27} + 1 = 3C \Rightarrow \frac{\overset{51}{\cancel{153}}}{27\times\cancel{3}} = C \Rightarrow \boxed{C=\frac{51}{27}}$

put the values of A, B & C in (1)

(1)$\Rightarrow \dfrac{x^2+2x-1}{x^3-27} = \dfrac{14}{27(x-3)} + \dfrac{\frac{13}{27}x + \frac{51}{27}}{x^2+3x+9}$

$$\boxed{\frac{x^2+2x-1}{x^3-27} = \frac{14}{27(x-3)} + \frac{13x+51}{27(x^2+3x+9)}} \quad \text{Ans.}$$

(20) $\dfrac{x^2-3x+5}{x^4-8x^2+16}$

Solution: $\dfrac{x^2-3x+5}{x^4-8x^2+16} = \dfrac{x^2-3x+5}{(x^2-4)^2} = \dfrac{x^2-3x+5}{(x-2)^2(x+2)^2}$

let $\dfrac{x^2-3x+5}{(x-2)^2(x+2)^2} = \dfrac{A}{x-2} + \dfrac{B}{(x-2)^2} + \dfrac{C}{x+2} + \dfrac{D}{(x+2)^2}$ — (1)

Xplying throughout eq^n (1) by $(x-2)^2(x+2)^2$

$x^2-3x+5 = A(x-2)(x+2)^2 + B(x+2)^2 + C(x+2)(x-2)^2 + D(x-2)^2$ —— (2)

put $x-2=0 \Rightarrow \boxed{x=2}$ in (2) $\Rightarrow \boxed{B=\frac{3}{16}}$

put $x+2=0 \Rightarrow \boxed{x=-2}$ in (2) $\Rightarrow \boxed{D=\frac{15}{16}}$

eq^n (2) can be written as

$x^2-3x+5 = (A+C)x^3 + (2A+B-2C+D)x^2 + 4(B-A-C-D)x - 8A + 4B + 8C + 4D$

equating the coefficient of x^3 & x^2

$A+C=0 \Rightarrow A=-C$ —— (3)

$2A+B-2C+D=1 \Rightarrow 2(-c) + \frac{3}{16} - 2C + \frac{15}{16} = 1$

$-4C + \frac{18}{16} = 1 \Rightarrow \frac{18}{16} - 1 = 4c \Rightarrow \frac{\cancel{2}}{16\times\cancel{4}_2} = C$

$\boxed{C=\frac{1}{32}}$ put in (3) $\Rightarrow$ $\boxed{A=\frac{-1}{32}}$

put the values of A, B, C & D in (1)

$$(1)\Rightarrow \boxed{\frac{x^2-3x+5}{x^4-8x^2+16}=\frac{-1}{32\,(x-2)}+\frac{3}{16\,(x-2)^2}+\frac{1}{32\,(x+2)}+\frac{15}{16\,(x+2)^2}}$$

(21) $\dfrac{8x^2}{(1-x^4)(1+x^2)}$

Solution: $\dfrac{8x^2}{(1-x^4)(1+x^2)}=\dfrac{8x^2}{(1-x^2)(1+x^2)(1+x^2)}$

$$=\frac{8x^2}{(1-x)(1+x)(1+x^2)^2}$$

let $\dfrac{8x^2}{(1-x)(1+x)(1+x^2)^2}=\dfrac{A}{1-x}+\dfrac{B}{1+x}+\dfrac{Cx+D}{1+x^2}+\dfrac{Ex+F}{(1+x^2)^2}$ — (1)

Xplying throughout eq^n (1) by $(1-x)(1+x)(1+x^2)^2$

$8x^2=A(1+x)(1+x^2)^2+B(1-x)(1+x^2)^2+Cx(1-x^2)(1+x^2)+D(1-x^2)(1+x^2)+Ex(1-x^2)+F(1-x^2)$ — (2)

put $1-x=0\Rightarrow\boxed{x=1}$ in (2) $\Rightarrow\boxed{A=1}$

put $1+x=0\Rightarrow\boxed{x=-1}$ in (2) $\Rightarrow\boxed{B=1}$

eq^n (2) can be written as

$8x^2=(A-B-C)x^5+(-D+2A-B-E)x^3+(2A+2B-F)x^2+(A-B+C+E)x+(A+B+D+F)$

equating the coefficients of x^5, x^3, x^2, x

$A-B-C=0\Rightarrow\cancel{1}-\cancel{1}-C=0\Rightarrow\boxed{C=0}$

$2A+2B-F=8\Rightarrow 2(1)+2(1)-F=8\Rightarrow\boxed{F=-4}$

$A-B+C+E=O\Rightarrow\cancel{1}-\cancel{1}+O+E=O\Rightarrow\boxed{E=O}$

$A+B+D+F=O\Rightarrow 1+1+D-4=0\Rightarrow\boxed{D=2}$

put the values of A, B, C, D, E & F in (1)

$$(1)\Rightarrow\boxed{\frac{8x^2}{(1-x^4)(1+x^2)}=\frac{1}{1-x}+\frac{1}{1+x}+\frac{2}{1+x^2}-\frac{4}{(1+x^2)^2}}$$

(22) $\dfrac{x^2}{(x-1)^3(x+1)}$

Solution: $\dfrac{x^2}{(x+1)(x-1)^3}=\dfrac{A}{x+1}+\dfrac{B}{x-1}+\dfrac{C}{(x-1)^2}+\dfrac{D}{(x-1)^3}$ — (1)

Xplying throughout eq^n (1) by $(x+1)(x-1)^3$

$x^2=A(x-1)^3+B(x+1)(x-1)^2+C(x+1)(x-1)^2+D(x+1)$ — (2)

put $x+1=0\Rightarrow\boxed{x=-1}$ in (2) $\Rightarrow\boxed{A=\frac{-1}{8}}$

put $x-1=0\Rightarrow\boxed{x=1}$ in (2) $\Rightarrow\boxed{D=\frac{1}{2}}$

eq^n (2) can be written as

$x^2=(A+B)x^3+(C-3A-B)x^2+(3A-B+D)x-A+B-C+D$

equating the coefficient of x^3 & x^2

$A+B=O\Rightarrow\frac{-1}{8}+B=O\Rightarrow\boxed{B=\frac{1}{8}}$

$C-3A-B=1\Rightarrow C+\frac{3}{8}-\frac{1}{8}=1\Rightarrow C=1-\frac{2}{8}\Rightarrow\boxed{C=\frac{3}{4}}$

put the values of A, B, C & D in (1)

$$(1)\Rightarrow\boxed{\frac{x^2}{(x+1)(x-1)^3}=\frac{-1}{8\,(x+1)}+\frac{1}{8\,(x-1)}+\frac{3}{4(x-1)^2}+\frac{1}{2\,(x-1)^3}}$$

(23) $\dfrac{(x-1)(x-2)(x-3)}{(x-4)(x-5)(x-6)}$

Solution: $\dfrac{(x-1)(x-2)(x-3)}{(x-4)(x-5)(x-6)}=\dfrac{x^3-6x^2+11x-6}{x^3-15x^2+74x-120}$

first we ÷ N (x) by D (x)

$$\begin{array}{r|l}
 & \quad 1 \\
x^3-15x^2+74x-120 & x^3-6x^2+11x-6 \\
 & \underline{-x^3\mp 15x^2\pm 74x\mp 120} \\
 & 9x^2-63x+114
\end{array}$$

(or) $3\,(3x^2-21x+38)$

$$\frac{x^3-6x^2+11x-6}{x^3-15x^2+74x-120}=1+3\left\{\frac{3x^2-21x+38}{(x-4)(x-5)(x-6)}\right\}$$

Suppose $\dfrac{3x^2-21x+38}{(x-4)(x-5)(x-6)}=\dfrac{A}{x-4}+\dfrac{B}{x-5}+\dfrac{C}{x-6}$ — (1)

Xplying throughout (1) by $(x-4)(x-5)(x-6)$

$3x^2-21x+38=A(x-5)(x-6)+B(x-4)(x-6)+C(x-4)(x-5)$ — (2)

put $x-4=0\Rightarrow\boxed{x=4}$ in (2) $\Rightarrow\boxed{A=1}$

put $x-5=0\Rightarrow\boxed{x=5}$ in (2) $\Rightarrow\boxed{B=-8}$

put $x-6=0\Rightarrow\boxed{x=6}$ in (2) $\Rightarrow\boxed{C=10}$

put the values of A, B & C in (1)

$$\boxed{\frac{(x-1)(x-2)(x-3)}{(x-4)(x-5)(x-6)} = 1 + 3\left\{\frac{1}{x-4} - \frac{8}{x-5} + \frac{10}{x-6}\right\}} \text{ Ans.}$$

(24) $\dfrac{x^3+3x^2-2x+1}{x^4+5x^2+4}$

Solution: $\dfrac{x^3+3x^2-2x+1}{x^4+5x^2+4} = \dfrac{x^3+3x^2-2x+1}{(x^2+4)(x^2+1)}$

$$\frac{x^3+3x^2-2x+1}{x^4+5x^2+4} = \frac{Ax+B}{x^2+4} + \frac{Cx+D}{x^2+1} \quad \text{———— (1)}$$

Xplying throughout (1) by $(x^2+4)(x^2+1)$

$x^3+3x^2-2x+1 = Ax(x^2+1) + B(x^2+1) + Cx(x^2+4) + D(x^2+4)$ ———— (2)

$x^3+3x^2-2x+1 = A(x^3+x) + B(x^2+1) + C(x^3+4x) + D(x^2+4)$

$x^3+3x^2-2x+1 = (A+C)x^3 + (B+D)x^2 + (A+4C)x + B + 4D$

equating the Coefficient of x^3, x^2, x and constant terms

$A + C = 1$ ———— (3)

$B + D = 3$ ———— (4)

$A + 4C = -2$ ———— (5)

$B + 4D = 1$ ———— (6)

(3) – (5)

$$\begin{aligned} \cancel{A} + C &= 1 \\ -\cancel{A} \pm 4C &= \mp 2 \\ \hline -3C &= 3 \Rightarrow \boxed{C=-1} \end{aligned}$$

put the value of C in (3)

(3)⇒ $A - 1 = 1 \Rightarrow \boxed{A=2}$

(4) – (6)

$$\begin{aligned} \cancel{B} + D &= 3 \\ -\cancel{B} \pm 4D &= -1 \\ \hline -3D &= 2 \Rightarrow \boxed{D=\frac{-2}{3}} \end{aligned}$$

put the value of D in (4)

(4)⇒$B - \frac{2}{3} = 3 \Rightarrow \boxed{B=\frac{11}{3}}$

put the values of A, B, C & D in (1)

$$\frac{x^3+3x^2-2x+1}{x^4+5x^2+4} = \frac{2x+\frac{11}{3}}{x^2+4} + \frac{(-1)x-\frac{2}{3}}{x^2+1}$$

$$\frac{x^3+3x^2-2x+1}{x^4+5x^2+4} = \frac{6x+11}{3(x^2+4)} - \frac{(3x+2)}{3(x^2+1)}$$

$$\boxed{\frac{x^3+3x^2-2x+1}{x^4+5x^2+4} = \frac{1}{3}\left[\frac{6x+11}{x^2+4} - \frac{(3x+2)}{x^2+1}\right]} \text{ Ans.}$$

EXERCISE # 6.9

Calculate the following integrals.

(1) $\displaystyle\int \frac{dx}{x^2-25}$

Solution: let $I = \displaystyle\int \frac{dx}{x^2-25}$ let partial fraction

$$\frac{1}{x^2-25} = \frac{1}{(x-5)(x+5)} = \frac{A}{x-5} + \frac{B}{x+5} \quad \text{———— (1)}$$

$$\frac{1}{\cancel{(x-5)(x+5)}} = \frac{A(x+5)+B(x-5)}{\cancel{(x-5)(x+5)}}$$

$1 = A(x+5) + B(x-5)$ ———— (2)

put $x - 5 = 0 \Rightarrow \boxed{x=5}$ in (2)

(2)⇒ $1 = 10A \Rightarrow \boxed{A=\frac{1}{10}}$

put $x + 5 = 0 \Rightarrow \boxed{x=-5}$ in (2)

(2)⇒ $1 = -10B \Rightarrow \boxed{B=\frac{-1}{10}}$

put the values of A & B in (1)

(1)⇒ $\dfrac{1}{(x-5)(x+5)} = \dfrac{\frac{1}{10}}{x-5} - \dfrac{\frac{1}{10}}{x+5}$ on integration

$$\int \frac{1}{(x-5)(x+5)}dx = \frac{1}{10}\int \frac{1}{x-5}dx - \frac{1}{10}\int \frac{1}{x+5}dx$$

$$I = \frac{1}{10}\ell n(x-5) - \frac{1}{10}\ell n(x+5) + c$$

$$\boxed{I = \frac{1}{10}\ell n\frac{x-5}{x+5} + c} \quad \text{Ans.}$$

(2) $\displaystyle\int \frac{3x^2-12x+11}{(x-1)(x-2)(x-3)}dx$

Solution: let $I = \displaystyle\int \frac{3x^2-12x+11}{(x-1)(x-2)(x-3)}dx$

let partial fraction

$$\frac{3x^2-12x+11}{(x-1)(x-2)(x-3)}=\frac{A}{x-1}+\frac{B}{x-2}+\frac{C}{x-3} \text{ ——— (1)}$$

Xplying throughout (1) by $(x-1)(x-2)(x-3)$

$$3x^2-12x+11=A(x-2)(x-3)+B(x-1)(x-3)+C(x-1)(x-2) \text{ ——— (2)}$$

put $x-1=0 \Rightarrow \boxed{x=1}$ in (2) $\Rightarrow \boxed{A=1}$

put $x-2=0 \Rightarrow \boxed{x=2}$ in (2) $\Rightarrow \boxed{B=1}$

put $x-3=0 \Rightarrow \boxed{x=3}$ in (2) $\Rightarrow \boxed{C=1}$

put the values of A, B & C in (1)

$$(1)\Rightarrow \frac{3x^2-12x+11}{(x-1)(x-2)(x-3)}=\frac{1}{x-1}+\frac{1}{x-2}+\frac{1}{x-3} \text{ on integration}$$

$$\int\frac{3x^2-12x+11}{(x-1)(x-2)(x-3)}dx=\int\frac{1}{x-1}dx+\int\frac{1}{x-2}dx+\int\frac{1}{x-3}dx$$

$$I=\ell n(x-1)+\ell n(x-2)+\ell n(x-3)+\ell nc$$

$$\boxed{I=\ell n[c(x-1)(x-2)(x-3)]} \quad \text{Ans.}$$

(3) $\int\frac{(2x-1)\,dx}{x(x-1)(x-3)}$

Solution: let $I=\int\frac{(2x-1)\,dx}{x(x-1)(x-3)}$

We first resolve into partial fraction

$$\text{let } \frac{2x-1}{x(x-1)(x-3)}=\frac{A}{x}+\frac{B}{x-1}+\frac{C}{x-3} \text{ ——— (1)}$$

Xplying throughout eqⁿ (1) by $x(x-1)(x-3)$

$$2x-1=A(x-1)(x-3)+Bx(x-3)+Cx(x-1) \text{ —— (2)}$$

put $x=0$ in (2) $\Rightarrow -1=3A \Rightarrow \boxed{A=\frac{-1}{3}}$

put $x-1=0 \Rightarrow \boxed{x=1}$ in (2) $\Rightarrow \boxed{B=\frac{-1}{2}}$

put $x-3=0 \Rightarrow \boxed{x=3}$ in (2) $\Rightarrow \boxed{C=\frac{5}{6}}$

put the values of A, B & C in (1)

$$(1)\Rightarrow \frac{2x-1}{x(x-1)(x-3)}=\frac{-\frac{1}{3}}{x}-\frac{\frac{1}{2}}{x-1}+\frac{\frac{5}{6}}{x-3} \text{ on integration}$$

$$\int\frac{(2x-1)dx}{x(x-1)(x-3)}=\frac{-1}{3}\int\frac{1}{x}dx-\frac{1}{2}\int\frac{1}{x-1}dx+\frac{5}{6}\int\frac{1}{x-3}dx$$

$$I=\frac{-1}{3}\ell nx-\frac{1}{2}\ell n(x-1)+\frac{5}{6}\ell n(x-3)+\ell nc$$

$$I=-\ell nc^{1/3}-\ell n(x-1)^{1/2}+\ell n(x-3)^{5/6}+\ell nc$$

$$I=\ell nc(x-3)^{5/6}-\{\ell nx^{1/3}+\ell n(x-1)^{1/2}\}$$

$$I=\ell nc(x-3)^{5/6}-\ell nx^{1/3}\cdot(x-1)^{1/2}$$

$$\boxed{I=\ell n\left[\frac{c(x-3)^{5/6}}{x^{1/3}.(x-1)^{1/2}}\right]}$$

(4) $\int\frac{dx}{2x^2-5x+2}$

Solution: let $I=\int\frac{dx}{2x^2-5x+2}$

$$I=\int\frac{1}{2x^2-4x-x+2}dx=\int\frac{1}{2x(x-2)-(x-2)}dx$$

$$I=\int\frac{1}{(x-2)(2x-1)}dx \text{ ——— (1)}$$

We first resolve into partial fraction

$$\text{let } \frac{1}{(x-2)(2x-1)}=\frac{A}{x-2}+\frac{B}{2x-1} \text{ ——— (2)}$$

Xplying throughout eqⁿ (2) by $(x-2)(2x-1)$

$$1=A(2x-1)+B(x-2) \text{ ——— (3)}$$

put $x-2=0 \Rightarrow \boxed{x=2}$ in (3) $\Rightarrow \boxed{A=\frac{1}{3}}$

put $2x-1=0 \Rightarrow \boxed{x=\frac{1}{2}}$ in (3) $\Rightarrow \boxed{B=\frac{-2}{3}}$

put the values of A & B in (2)

$$(2)\Rightarrow \frac{1}{(x-2)(2x-1)}=\frac{\frac{1}{3}}{x-2}+\frac{\frac{-2}{3}}{2x-1} \quad \text{On integration}$$

$$\int\frac{1}{(x-2)(2x-1)}dx=\frac{1}{3}\int\frac{1}{x-2}dx-\frac{1}{3}\int\frac{2}{2x-1}dx$$

$$I=\frac{1}{3}\ell n(x-2)-\frac{1}{3}\ell n(2x-1)+\ell nc$$

$$I=\frac{1}{3}\ell n\frac{(x-2)}{(2x-1)}+\ell nc \Rightarrow \boxed{I=\frac{1}{3}\ell n\left\{\frac{c(x-2)}{(2x-1)}\right\}} \quad \text{Ans.}$$

(5) $\int \frac{\text{Cosx dx}}{\text{Sinx}(2+\text{Sinx})}$

Solution: let $I = \int \frac{\text{Cosx dx}}{\text{Sinx}(2+\text{Sinx})}$ ——— (1)

let $y = \text{Sinx}$ diff w.r to x

$\frac{dy}{dx} = \text{Cosx} \Rightarrow \boxed{dy = \text{Cosx dx}}$

$(1) \Rightarrow I = \int \frac{dy}{y(2+y)}$ ——— (2)

We first resolve into partial fraction

$\frac{1}{y(y+2)} = \frac{A}{y} + \frac{B}{y+2}$ ——— (3)

Xplying throughout eq^n (3) by $y(y+2)$

$1 = A(y+2) + By$ ——— (4)

put $y = 0$ in (4) $\Rightarrow \boxed{A = \frac{1}{2}}$

put $y + 2 = 0 \Rightarrow \boxed{y = -2}$ in (4) $\Rightarrow \boxed{B = \frac{-1}{2}}$

put the values of A & B in (3)

$(3) \Rightarrow \frac{1}{y(y+2)} = \frac{\frac{1}{2}}{y} + \frac{\frac{-1}{2}}{y+2}$ on integration

$\int \frac{1}{y(y+2)} dy = \frac{1}{2}\int \frac{1}{y} dy - \frac{1}{2}\int \frac{1}{y+2} dy$

$I = \frac{1}{2}\ell n y - \frac{1}{2}\ell n (y+2) + c$

$I = \frac{1}{2}\ell n \left\{\frac{y}{y+2}\right\} + c$ $\quad \therefore y = \text{Sinx}$

$\boxed{I = \frac{1}{2}\ell n \left\{\frac{\text{Sinx}}{2+\text{Sinx}}\right\} + c}$ Ans.

(6) $\int \frac{\text{Sinx dx}}{(1+\text{Cosx})(2+\text{Cosx})}$

Solution: let $I = \int \frac{\text{Sinx dx}}{(1+\text{Cosx})(2+\text{Cosx})}$ ——— (1)

let $y = \text{Cosx}$ diff w.r to x

$\frac{dy}{dx} = -\text{Sinx} \Rightarrow \boxed{-dy = \text{Sinx dx}}$

$(1) \Rightarrow I = \int \frac{-dy}{(1+y)(2+y)}$ ——— (2)

we first resolve into partial fraction

$\frac{-1}{(1+y)(2+y)} = \frac{A}{1+y} + \frac{B}{2+y}$ ——— (3)

Xplying throughout eq^n (3) by $(1+y)(2+y)$

$-1 = A(2+y) + B(1+y)$ ——— (4)

put $1 + y = 0 \Rightarrow \boxed{y = -1}$ in (4) $\Rightarrow \boxed{A = -1}$

put $2 + y = 0 \Rightarrow \boxed{y = -2}$ in (4) $\Rightarrow \boxed{B = 1}$

put the values of A & B in (3)

$(3) \Rightarrow \frac{-1}{(1+y)(2+y)} = \frac{-1}{1+y} + \frac{1}{2+y}$ on integration

$\int \frac{-1 \; dy}{(1+y)(2+y)} = -\int \frac{1}{1+y} dy + \int \frac{1}{y+2} dy$

$I = -\ell n (1+y) + \ell n (y+2) + c$

$I = \ell n \left\{\frac{2+y}{1+y}\right\} + c$ $\quad \therefore y = \text{Cosx}$

$\boxed{I = \ell n \left\{\frac{2+\text{Cosx}}{1+\text{Cosx}}\right\} + c}$ Ans.

(7) $\int \frac{x^2+3x+4}{x-2} dx$

Solution: let $I = \int \frac{x^2+3x+4}{x-2} dx$

We first divide the N (x) by D (x)

$$\begin{array}{r|l} & x+5 \\ \hline x-2 & x^2+3x+4 \\ & -x^2 \mp 2x \\ \hline & 5x+4 \\ & -5x \mp 10 \\ \hline & 14 \end{array}$$

$I = \int \left\{x + 5 + \frac{14}{x-2}\right\} dx$

$\boxed{\therefore \frac{N(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)}}$

$I = \int x dx + 5\int dx + 14\int \frac{1}{x-2} dx$

$\boxed{I = \frac{x^2}{2} + 5x + 14\ell n (x-2) + c}$ Ans.

(8) $\int \frac{x^3 - x^2 + 2x + 3}{x^2 + 3x + 2} dx$

Solution: let $I = \int \frac{x^3 - x^2 + 2x + 3}{x^2 + 3x + 2} dx$

We first divide the N (x) by D (x)

$$\begin{array}{r|l} & x - 4 \\ \hline x^2 + 3x + 2 & x^3 - x^2 + 2x + 3 \\ & -x^3 \pm 3x^2 \pm 2x \\ \hline & -4x^2 + 3 \\ & \mp 4x^2 \mp 12x \mp 8 \\ \hline & 12x + 11 \end{array}$$

$$\boxed{\therefore \frac{N(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)}}$$

$$I = \int \left\{ x - 4 + \frac{12x + 11}{x^2 + 3x + 2} \right\} dx$$

$$I = \int x\,dx - 4\int dx + \int \frac{12x + 11}{x^2 + 3x + 2}\,dx \text{ ——— (1)}$$

$$\text{let } I = \int \frac{12x + 11\,dx}{x^2 + 3x + 2}$$

resolve into partial fraction

$$\frac{12x + 11}{x^2 + 3x + 2} = \frac{12x + 11}{(x+2)(x+1)} = \frac{A}{x+1} + \frac{B}{x+2} \text{ ——— (2)}$$

Xplying throughout eq^n (2) by $(x + 1)(x + 2)$

$$12x + 11 = A(x + 2) + B(x + 1) \text{ ——— (3)}$$

put $x + 1 = 0 \Rightarrow \boxed{x = -1}$ in (3) $\Rightarrow \boxed{A = -1}$

put $x + 2 = 0 \Rightarrow \boxed{x = -2}$ in (3) $\Rightarrow \boxed{B = 13}$

put the values of A & B in (2)

$$(2) \Rightarrow \frac{12x + 11}{(x+2)(x+1)} = \frac{-1}{x+1} + \frac{13}{x+2} \text{ on integration}$$

$$\int \frac{12x + 11}{(x+2)(x+1)}\,dx = -\int \frac{1}{x+1}\,dx + 13\int \frac{1}{x+2}\,dx$$

put this in (1)

$$I = \int x dx - 4\int dx - \int \frac{1}{x+1}\,dx + 13\int \frac{1}{x+2}\,dx$$

$$\boxed{I = \frac{x^2}{2} - 4x - \ell n(x + 1) + 13\ell n(x + 2) + c}$$

(9) $\displaystyle\int \frac{2x\,dx}{(1 + x^2)(3 + x^2)}$

Solution: let $I = \displaystyle\int \frac{2x\,dx}{(1 + x^2)(3 + x^2)}$ ——— (1)

let $y = x^2$ diff w.r to x

$\dfrac{dy}{dx} = 2x \Rightarrow \boxed{2xdx = dy}$

$$(1) \Rightarrow I = \int \frac{dy}{(1 + y)(3 + y)}$$

Now resolve into partial fraction

$$\frac{1}{(1 + y)(3 + y)} = \frac{A}{1 + y} + \frac{B}{3 + y} \text{ ——— (2)}$$

Xplying throught eq^n (2) by $(1 + y)(3 + y)$

$$1 = A(3 + y) + B(1 + y) \text{ ——— (3)}$$

put $1 + y = 0 \Rightarrow \boxed{y = -1}$ in (3) $\Rightarrow \boxed{A = \frac{1}{2}}$

put $3 + y = 0 \Rightarrow \boxed{y = -3}$ in (3) $\Rightarrow \boxed{B = \frac{-1}{2}}$

put the values of A & B in (2)

$$(2) \Rightarrow \frac{1}{(1 + y)(3 + y)} = \frac{\frac{1}{2}}{1 + y} - \frac{\frac{1}{2}}{3 + y} \text{ on integration}$$

$$\int \frac{1}{(1 + y)(3 + y)}\,dy = \frac{1}{2}\int \frac{1}{1 + y}\,dy - \frac{1}{2}\int \frac{1}{3 + y}\,dy$$

$$I = \frac{1}{2}\ell n(1 + y) - \frac{1}{2}\ell n(3 + y) + c$$

$$I = \frac{1}{2}\ell n\left\{\frac{1 + y}{3 + y}\right\} + c \qquad \therefore y = x^2$$

$$\boxed{I = \frac{1}{2}\ell n\left\{\frac{1 + x^2}{3 + x^2}\right\} + c} \quad \text{Ans.}$$

(10) $\displaystyle\int \frac{x^2 + 2x + 3}{x^2 - 3x + 2}\,dx$

Solution: let $I = \displaystyle\int \frac{x^2 + 2x + 3}{x^2 - 3x + 2}\,dx$

first we divide the N (x) by D (x)

$$\begin{array}{r|l} & 1 \\ \hline x^2 - 3x + 2 & x^2 + 2x + 3 \\ & -x^2 \pm 3x^2 \pm 2 \\ \hline & 5x + 1 \end{array}$$

$$\boxed{\therefore \frac{N(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)}}$$

$$I = \int \left\{1 + \frac{5x + 1}{x^2 - 3x + 2}\right\} dx \Rightarrow I = \int \left\{1 + \frac{5x + 1}{x^2 - 2x - x + 2}\right\} dx$$

$$I = \int dx + \int \frac{(5x+1)}{(x-2)(x-1)}dx \text{ ——— (1)}$$

let $I = \int \frac{5x+1}{(x-2)(x-1)}dx$

resolve into partial fraction

$$\frac{5x+1}{(x-2)(x-1)} = \frac{A}{x-2} + \frac{B}{x-1} \text{ ——— (2)}$$

Xplying throught eq^n (2) by $(x-2)(x-1)$

$$5x + 1 = A(x-1) + B(x-2) \text{ ——— (3)}$$

put $x - 2 = 0 \Rightarrow \boxed{x = 2}$ in (3) $\Rightarrow \boxed{A = 11}$

put $x - 1 = 0 \Rightarrow \boxed{x = 1}$ in (3) $\Rightarrow \boxed{B = -6}$

put the values of A & B in (2)

$$(2) \Rightarrow \frac{5x+1}{(x-2)(x-1)} = \frac{11}{x-2} - \frac{6}{x-1} \text{ on integration}$$

$$\int \frac{(5x+1)\,dx}{(x-2)(x-1)} = 11\int \frac{1}{x-2}dx - 6\int \frac{1}{x-1}dx$$

put this result in (1)

$$(1) \Rightarrow I = \int dx + 11\int \frac{1}{x-2}dx - 6\int \frac{1}{x-1}dx$$

$$\boxed{I = x + 11\,\ell n\,(x-2) - 6\,\ell n\,(x-1) + c} \quad \text{Ans}$$

(11) $\int \frac{x^4+1}{x^3-x}dx$

Solution: let $I = \int \frac{x^4+1}{x^3-x}dx$

first we divide N (x) by D (x)

$$\begin{array}{r|l} & x \\ x^3 - x & x^4 + 1 \\ & -x^4 \mp x^2 \\ \hline & x^2 + 1 \end{array}$$

$$\boxed{\therefore \frac{N(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)}}$$

$$I = \int \left\{ x + \frac{x^2+1}{x^3-x} \right\} dx$$

$$I = \int x dx + \int \frac{x^2+1\ dx}{x(x-1)(x+1)} \text{ ——— (1)}$$

let $I = \int \frac{x^2+1\ dx}{x(x-1)(x+1)}$

resolve into partial fraction

$$\frac{x^2+1}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1} \text{ ——— (2)}$$

Xplying throught eq^n (2) by $x(x-1)(x+1)$

$$x^2 + 1 = A(x^2-1) + Bx(x+1) + Cx(x-1) \text{ ——— (3)}$$

put $x - 0$ in (3) $\Rightarrow \boxed{A = -1}$

put $x - 1 = 0 \Rightarrow \boxed{x = 1}$ in (3) $\Rightarrow \boxed{B = 1}$

put $x + 1 = 0 \Rightarrow \boxed{x = -1}$ in (3) $\Rightarrow \boxed{C = 1}$

put the values of A, B and C in (2)

$$(2) \Rightarrow \frac{x^2+1}{x(x-1)(x+1)} = \frac{-1}{x} + \frac{1}{x-1} + \frac{1}{x+1} \text{ on integration}$$

$$\int \frac{x^2+1\ dx}{x(x-1)(x+1)} = -\int \frac{1}{x}dx + \int \frac{1}{x-1}dx + \int \frac{1}{x+1}dx$$

put this result in (1)

$$(1) \Rightarrow I = \int x dx - \int \frac{1}{x}dx + \int \frac{1}{x-1}dx + \int \frac{1}{x+1}dx$$

$$I = \frac{x^2}{2} - \ell nx + \ell n\,(x-1) + \ell n\,(x+1) + c$$

$$I = \frac{x^2}{2} + \ell n\,(x^2-1) - \ell nx + c$$

$$\boxed{I = \frac{x^2}{2} + \ell n\left\{\frac{x^2-1}{x}\right\} + c} \quad \text{Ans.}$$

(12) $\int \frac{x^3+3}{x^2+4}dx$

Solution: let $I = \int \frac{x^3+3}{x^2+4}dx$

first we divide N (x) by D (x)

$$\begin{array}{r|l} & x \\ x^2 - 4 & x^3 + 3 \\ & -x^3 \pm 4x \\ \hline & -4x + 3 \end{array}$$

$$\boxed{\therefore \frac{N(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)}}$$

$$I = \int \left\{ x + \frac{-4x+3}{x^2+4} \right\} dx \Rightarrow I = \int x dx - \int \frac{4x dx}{x^2+4} + 3\int \frac{1}{x^2+4}dx$$

$$I = \int x dx - 2\int \frac{2x\,dx}{x^2+4} + 3\int \frac{1}{(x)^2+(2)^2}dx$$

$$\boxed{\therefore \int \frac{1}{x^2 + a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c}$$

$$\boxed{I = \frac{x^2}{2} - 2\,\ell n\,(x^2 + 4) + \frac{3}{2}\tan^{-1}\frac{x}{2} + c}$$ Ans.

(13) $\int \frac{(x^2 - 2)\,dx}{(x + 1)(x - 1)^2}$

Solution: let $I = \int \frac{(x^2 - 2)\,dx}{(x + 1)(x - 1)^2}$

first we resolve into partial fraction

$$\frac{x^2 - 2}{(x + 1)(x - 1)^2} = \frac{A}{x + 1} + \frac{B}{x - 1} + \frac{C}{(x - 1)^2} \text{ ——— (1)}$$

Xplying through out eq^n (1) by $(x + 1)(x - 1)^2$

$$x^2 - 2 = A(x - 1)^2 + B(x^2 - 1) + C(x + 1) \text{ ——— (2)}$$

put $x + 1 = 0 \Rightarrow \boxed{x = -1}$ in (3) $\Rightarrow \boxed{A = \frac{-1}{4}}$

put $x - 1 = 0 \Rightarrow \boxed{x = 1}$ in (3) $\Rightarrow \boxed{C = \frac{-1}{2}}$

eq^n (2) can be written as

$$x^2 - 2 = A(x^2 - 2x + 1) + B(x^2 - 1) + C(x + 1)$$

$$x^2 - 2 = (A + B)x^2 + (C - 2A)x + (A - B + C)$$

equating the coefficient of x^2

$A + B = 1 \Rightarrow \frac{-1}{4} + B = 1 \Rightarrow \boxed{B = \frac{5}{4}}$

put the values of A, B & C in (1)

$$(1) \Rightarrow \frac{x^2 - 2}{(x + 1)(x - 1)^2} = \frac{\frac{-1}{4}}{x + 1} + \frac{\frac{5}{4}}{x - 1} + \frac{\frac{-1}{2}}{(x - 1)^2}$$

on integration

$$\int \frac{(x^2 - 2)}{(x + 1)(x - 1)^2}\,dx = \frac{1}{4}\int \frac{1}{x + 1}\,dx + \frac{5}{4}\int \frac{1}{x - 1}\,dx$$

$$-\frac{1}{2}\int (x - 1)^{-2}\,dx$$

$$I = \frac{1}{4}\,\ell n\,(x + 1) + \frac{5}{4}\,\ell n\,(x - 1) + \frac{1}{2(x - 1)} + c$$

$$\boxed{I = \frac{1}{4}\,\ell n\left\{\frac{(x - 1)^5}{x + 1}\right\} + \frac{1}{2(x - 1)} + c}$$ Ans.

(14) $\int \frac{(x^2 + 3x + 3)\,dx}{(x + 1)(x^2 + 1)}$

Solution: let $I = \int \frac{(x^2 + 3x + 3)\,dx}{(x + 1)(x^2 + 1)}$

first resolve into partial fraction

$$\frac{x^2 + 3x + 3}{(x + 1)(x^2 + 1)} = \frac{A}{x + 1} + \frac{Bx + c}{x^2 + 1} \text{ ——— (1)}$$

Xplying throughout eq^n (1) by $(x + 1)(x^2 + 1)$

$$x^2 + 3x + 3 = A(x^2 + 1) + B(x^2 + x) + C(x + 1) \text{ ——— (2)}$$

put $x + 1 = 0 \Rightarrow \boxed{x = -1}$ in (2) $\Rightarrow \boxed{A = \frac{1}{2}}$

eqn (2) can be written as

$$x^2 + 3x + 3 = (A + B)x^2 + (B + C)x + (A + C)$$

equating the coefficient of x^2 & x

$1 = A + B \Rightarrow 1 = \frac{1}{2} + B \Rightarrow \boxed{B = \frac{1}{2}}$

$3 = B + C \Rightarrow 3 = \frac{1}{2} + C \Rightarrow \boxed{C = \frac{5}{2}}$

put the values of A, B and C in (1)

$$(1) \Rightarrow \frac{x^2 + 3x + 3}{(x + 1)(x^2 + 1)} = \frac{\frac{1}{2}}{x + 1} + \frac{\frac{1}{2}x + \frac{5}{2}}{x^2 + 1}$$ on integration

$$\int \frac{(x^2 + 3x + 3)\,dx}{(x + 1)(x^2 + 1)} = \frac{1}{2}\int \frac{1}{x + 1}\,dx + \frac{1}{2 \times 2}\int \frac{2x\,dx}{x^2 + 1}$$

$$+\frac{5}{2}\int \frac{1}{(x)^2 + (1)^2}\,dx$$

$$\boxed{\therefore \int \frac{1}{x^2 + a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c}$$

$$\boxed{I = \frac{1}{2}\,\ell n\,(x + 1) + \frac{1}{4}\,\ell n\,(x^2 + 1) + \frac{5}{2}\tan^{-1} x + c}$$

(15) $\int \frac{x^2 - 2x + 3}{(x - 1)(x^2 + 2x + 2)}\,dx$

Solution: let $I = \int \frac{x^2 - 2x + 3}{(x - 1)(x^2 + 2x + 2)}\,dx$

first we resolve into partial fraction

$$\frac{x^2 - 2x + 3}{(x - 1)(x^2 + 2x + 2)} = \frac{A}{x - 1} + \frac{Bx + c}{x^2 + 2x + 2} \text{ ———}$$

Xplying throughout eq^n (1) by $(x-1)(x^2+2x+2)$

$$x^2-2x+3 = A(x^2+2x+2) + B(x^2-x) + C(x-1) \text{ — (2)}$$

put $x-1=0 \Rightarrow \boxed{x=1}$ in (2) $\Rightarrow \boxed{A=\frac{2}{5}}$

eq^n (2) can be written as

$$x^2-2x+3 = (A+B)x^2 + (2A-B+C)x + (2A-C)$$

equating the Coefficient of x^2 & constant terms

$$1 = A+B \Rightarrow 1 = \frac{2}{5} + B \Rightarrow \boxed{B=\frac{3}{5}}$$

$$3 = 2A - C \Rightarrow 3 = 2\left(\frac{2}{5}\right) - C \Rightarrow \boxed{C=\frac{-11}{5}}$$

Put the values of A, B & C in (1)

$$(1) \Rightarrow \frac{x^2-2x+3}{(x-1)(x^2+2x+2)} = \frac{\frac{2}{5}}{x-1} + \frac{\frac{3}{5}x - \frac{11}{5}}{x^2+2x+2} \quad \text{on integration}$$

$$\int \frac{(x^2-2x+3)\,dx}{(x-1)(x^2+2x+2)} = \frac{2}{5}\int \frac{1}{x-1}\,dx + \frac{1}{5}\frac{(3x-11)}{x^2+2x+2}\,dx$$

$$I = \frac{2}{5}\int \frac{1}{x-1}\,dx + \frac{1}{5\times 2}\int \frac{(6x-22)}{x^2+2x+2}\,dx$$

$$I = \frac{2}{5}\int \frac{1}{x-1}\,dx + \frac{1}{10}\int \frac{(6x+6)\,dx}{x^2+2x+2} - \frac{1}{10}\int \frac{28}{(x^2+2x+1)+(1)^2}\,dx$$

$$I = \frac{2}{5}\int \frac{1}{x-1}\,dx + \frac{3}{10}\int \frac{(2x+2)\,dx}{x^2+2x+2} - \frac{28}{10}\int \frac{1}{(x+1)^2+(1)^2}\,dx$$

$$\boxed{\therefore \int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c}$$

$$I = \frac{2}{5}\ln(x-1) + \frac{3}{10}\ln(x^2+2x+2) - \frac{\overset{14}{\cancel{28}}}{\underset{5}{\cancel{10}}}\tan^{-1}(x+1) + c$$

$$\boxed{I = \frac{2}{5}\ln(x-1) + \frac{3}{10}\ln(x^2+2x+2) - \frac{14}{5}\tan^{-1}(x+1) + c}$$

Ans

(16) $\int \frac{(x-3)\,dx}{(x+1)^2(x-2)}$

Solution: let $I = \int \frac{(x-3)\,dx}{(x+1)^2(x-2)}$

We first resolve into partial fraction

$$\frac{(x-3)}{(x+1)^2(x-2)} = \frac{A}{x-2} + \frac{B}{x+1} + \frac{C}{(x+1)^2} \text{ — (1)}$$

Xplying through out eq^n (1) by $(x+1)^2(x-2)$

$$x-3 = A(x+1)^2 + B(x-2)(x+1) + C(x-2) \text{ — (2)}$$

put $x-2=0 \Rightarrow \boxed{x=2}$ in (3) $\Rightarrow \boxed{A=\frac{-1}{9}}$

put $x+1=0 \Rightarrow \boxed{x=-1}$ in (3) $\Rightarrow \boxed{C=\frac{4}{3}}$

The eq^n (3) can be written as

$$x-3 = A(x^2+2x+1) + B(x^2+x-2x-2) + C(x-2)$$

$$x-3 = (A+B)x^2 + (2A-B+C)x + (A-2B-2C)$$

equating the coefficient of x^2

$$A+B=0 \Rightarrow \frac{-1}{9} + B = 0 \Rightarrow \boxed{B=\frac{1}{9}}$$

put the values of A, B & C in (1)

$$(1) \Rightarrow \frac{x-3}{(x+1)^2(x-2)} = \frac{\frac{-1}{9}}{x-2} + \frac{\frac{1}{9}}{x+1} + \frac{\frac{4}{3}}{(x+1)^2} \quad \text{on integration}$$

$$\int \frac{(x-3)\,dx}{(x+1)^2(x-2)} = \frac{-1}{9}\int \frac{1}{x-2}\,dx + \frac{1}{9}\int \frac{1}{x+1}\,dx + \frac{4}{3}\int (x+1)^{-2}\,dx$$

$$I = \frac{-1}{9}\ln(x-2) + \frac{1}{9}\ln(x+1) - \frac{4}{3(x+1)} + C$$

$$\boxed{I = \frac{1}{9}\ln\left\{\frac{x+1}{x-2}\right\} - \frac{4}{3(x+1)} + C} \quad \text{Ans.}$$

(17) $\int \frac{(x^2+1)}{(x-1)^3}\,dx$

Solution: let $I = \int \frac{(x^2+1)\,dx}{(x-1)^3}$

We first resolve into partial fraction

$$\frac{(x^2+1)}{(x-1)^3} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} \text{ — (1)}$$

Xplying throughout eq^n (1) by $(x-1)^3$

$$x^2+1 = A(x-1)^2 + B(x-1) + C \text{ — (2)}$$

put $x-1=0 \Rightarrow \boxed{x=1}$ in (2) $\Rightarrow \boxed{C=2}$

eq^n (2) can be written as

$$x^2+1 = A(x^2-2x+1) + Bx - B + C$$

$$x^2+1 = Ax^2 + (B-2A)x + (A-B+C)$$

equating the coefficient of x^2 & x.

$\boxed{A=1}$ $B-2A=0 \Rightarrow B-2(1)=0 \Rightarrow \boxed{B=2}$

put the values of A , B & C in (1)

$$(1)\Rightarrow \frac{(x^2+1)}{(x-1)^3} = \frac{1}{x-1} + \frac{2}{(x-1)^2} + \frac{2}{(x-1)^3} \quad \text{on integration}$$

$$\int \frac{(x^2+1)\,dx}{(x-1)^3} = \int \frac{1}{x-1}\,dx + 2\int (x-1)^{-2}\,dx + 2\int (x-1)^{-3}\,dx$$

$$\boxed{I = \ell n\,(x-1) - \frac{2}{(x-1)} - \frac{1}{(x-1)^2} + C} \quad \text{Ans}$$

(18) $\int \frac{2x^2-1}{(x+1)^2\,(x-3)}\,dx$

Solution: let $I = \int \frac{2x^2-1}{(x+1)^2\,(x-3)}\,dx$

resolve into partial fraction

$$\frac{2x^2-1}{(x+1)^2\,(x-3)} = \frac{A}{x-3} + \frac{B}{x+1} + \frac{C}{(x+1)^2} \text{ —— (1)}$$

Xplying throughout eq^n (1) by $(x-3)(x+1)^2$

$$2x^2 - 1 = A\,(x+1)^2 + B\,(x+1)\,(x-3) + C\,(x-3) \text{ — (2)}$$

put $x - 3 = 0 \Rightarrow \boxed{x = 3}$ in (2) $\Rightarrow \boxed{A = \frac{17}{16}}$

put $x + 1 = 0 \Rightarrow \boxed{x = -1}$ in (2) $\Rightarrow \boxed{C = \frac{-1}{4}}$

eq^n (2) can be written as

$$2x^2 - 1 = A\,(x^2 + 2x + 1) + B\,(x^2 - 2x - 3) + C\,(x-3)$$

$$2x^2 - 1 = (A+B)\,x^2 + (2A - 2B + C)\,x + (A - 3B - 3C)$$

equating the coefficient of x^2

$2 = A + B \Rightarrow 2 = \frac{17}{16} + B \Rightarrow \boxed{B = \frac{15}{16}}$

put the values of A, B and C in (1)

$$(1)\Rightarrow \frac{2x^2-1}{(x+1)^2\,(x-3)} = \frac{\frac{17}{16}}{x-3} + \frac{\frac{15}{16}}{x+1} + \frac{\frac{-1}{4}}{(x+1)^2} \quad \text{on integration}$$

$$\int \frac{(2x^2-1)\,dx}{(x+1)^2\,(x-3)} = \frac{17}{16}\int \frac{1}{x-3}\,dx + \frac{15}{16}\int \frac{1}{x+1}\,dx - \frac{1}{4}\int (x+1)^{-2}\,dx$$

$$\boxed{I = \frac{17}{16}\,\ell n\,(x-3) + \frac{15}{16}\,\ell n\,(x+1) + \frac{1}{4\,(x+1)} + C}$$

(19) $\int \frac{(x^3+1)\,dx}{(x^2-1)^2}$

$\int (x^3+1)\,dx$

$$I = \int \frac{\cancel{(x+1)}\,(x^2 - x + 1)\,dx}{(x-1)^2\,\cancel{(x+1)^2}} \Rightarrow I = \int \frac{(x^2 - x + 1)\,dx}{(x+1)\,(x-1)^2}$$

resolve into partial fraction

$$\frac{x^2 - x + 1}{(x+1)(x-1)^2} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2} \text{ —— (1)}$$

Xplying throughout eq^n (1) by $(x+1)\,(x-1)^2$

$$x^2 - x + 1 = A\,(x-1)^2 + B\,(x^2 - 1) + C\,(x+1) \text{ —— (2)}$$

put $x + 1 = 0 \Rightarrow \boxed{x = -1}$ in (2) $\Rightarrow \boxed{A = \frac{3}{4}}$

put $x - 1 = 0 \Rightarrow \boxed{x = 1}$ in (2) $\Rightarrow \boxed{C = \frac{1}{2}}$

eq^n (2) can be written as

$$x^2 - x + 1 = A\,(x^2 - 2x + 1) + B\,(x^2 - 1) + C\,(x+1)$$

$$x^2 - x + 1 = (A+B)\,x^2 + (C - 2A)\,x + (A - B + C)$$

equating the coefficient of x^2

$1 = A + B \Rightarrow 1 = \frac{3}{4} + B \Rightarrow \boxed{B = \frac{1}{4}}$

put the values of A, B & C in (1)

$$(1)\Rightarrow \frac{x^2 - x + 1}{(x+1)(x-1)^2} = \frac{\frac{3}{4}}{x+1} + \frac{\frac{1}{4}}{x-1} + \frac{\frac{1}{2}}{(x-1)^2} \quad \text{on integration}$$

$$\int \frac{(x^2 - x + 1)\,dx}{(x+1)\,(x-1)^2} = \frac{3}{4}\int \frac{1}{x+1}\,dx + \frac{1}{4}\int \frac{1}{x-1}\,dx + \frac{1}{2}\int (x-2)^{-2}\,dx$$

$$I = \frac{3}{4}\,\ell n\,(x+1) + \frac{1}{4}\,\ell n\,(x-1) - \frac{1}{2\,(x-1)} + c$$

$$\boxed{I = \frac{1}{4}\,\ell n\,\{(x+1)^3\,(x-1)\} - \frac{1}{2\,(x-1)} + C}$$

(20) $\int \frac{(x^3 - 2x^2 + 3x - 4)\,dx}{(x-1)^2\,(x^2 + 2x + 2)}$

Solution: let $I = \int \frac{(x^3 - 2x^2 + 3x - 4)\,dx}{(x-1)^2\,(x^2 + 2x + 2)}$

Resolve into partial fraction

$$\frac{x^3 - 2x^2 + 3x - 4}{(x-1)^2(x^2 + 2x + 2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx + D}{x^2 + 2x + 2} \text{ — (1)}$$

Xplying throughout eq^n (1) by $(x-1)^2\,(x^2 + 2x + 2)$

$$x^3 - 2x^2 + 3x - 4 = A\,(x-1)(x^2 + 2x + 2)$$
$$+ B\,(x^2 + 2x + 2) + Cx\,(x-1)^2 + D\,(x-1)^2 \text{ —— (2)}$$

put $x - 1 = 0 \Rightarrow \boxed{x = 1}$ in eqⁿ (2) $\Rightarrow \boxed{B = \frac{-2}{5}}$

$x^3 - 2x^2 + 3x - 4 = A(x^3 + x^2 - 2) + B(x^2 + 2x + 2) + C(x^3 - 2x^2 + x) + D(x^2 - 2x + 1)$

$x^3 - 2x^2 + 3x - 4 = (A + C)x^3 + (A + B - 2C + D)x^2 + (2B + C - 2D)x + (2B - 2A + D)$

equating the coefficient of x^3, x^2, x & constants

$A + C = 1$ ——— (3)

$A + B - 2C + D = -2$ ——— (4)

$2B + C - 2D = 3$ ——— (5)

$2B - 2A + D = -4$ ——— (6)

put $B = \frac{-2}{5}$ & from (3) $A = 1 - C$ in (4)

$(4) \Rightarrow 1 - C - \frac{2}{5} - 2C + D = -2$

$-3C + D = -2 - \frac{3}{5} \Rightarrow -3C + D = \frac{-13}{5}$ ——— (7)

put $B = \frac{-2}{5}$ in (5)

$(5) \Rightarrow 2\left(\frac{-2}{5}\right) + C - 2D = 3 \Rightarrow C - 2D = 3 + \frac{4}{5}$

$C - 2D = \frac{19}{5}$ ——— (8)

Xplying throught by 3

$3C - 6D = \frac{57}{5}$ ——— (9)

eqn (7) + eqn (9)

$$
\begin{aligned}
-3C + D &= \frac{-13}{5}\\
3C - 6D &= \frac{57}{5}\\
\hline
-5D &= \frac{44}{5} \Rightarrow \boxed{D = \frac{-44}{25}}
\end{aligned}
$$

put the value of D in (8)

$(8) \Rightarrow C - 2\left(\frac{-44}{25}\right) = \frac{19}{5}$

$C = \frac{19}{5} - \frac{88}{25} \Rightarrow C = \frac{95 - 88}{25} \Rightarrow \boxed{C = \frac{7}{25}}$

put $\boxed{C = \frac{7}{25}}$ in (3)

$(3) \Rightarrow A = \frac{7}{25} = 1 \Rightarrow \boxed{A = \frac{18}{25}}$

put the values of A, B, C & D in (1)

$(1) \Rightarrow \frac{x^3 - 2x^2 + 3x - 4}{(x-1)^2(x^2 + 2x + 2)} = \frac{18}{25}\int \frac{1}{x-1}dx$

$-\frac{2}{5}\int (x-1)^{-2}dx + \frac{1}{25 \times 2}\int \frac{14x - 88}{x^2 + 2x + 2}dx$

$I = \frac{18}{25}\int \frac{1}{x-1}dx - \frac{2}{5}\int (x-1)^{-2}dx + \frac{1}{50}\int \frac{14x + 14 - 102}{x^2 + 2x + 2}dx$

$I = \frac{18}{25}\int \frac{1}{x-1}dx - \frac{2}{5}\int (x-1)^{-2}dx + \frac{1}{50}\int \frac{14x + 14}{x^2 + 2x + 2}dx$

$-\frac{1}{50}\int \frac{102}{x^2 + 2x + 2}dx$

$I = \frac{18}{25}\int \frac{1}{x-1}dx - \frac{2}{5}\int (x-1)^{-2}dx + \frac{7}{50}\int \frac{2x + 2}{x^2 + 2x + 2}dx$

$-\frac{1}{50}\int \frac{102\,dx}{(x+1)^2 + (1)^2}$

$I = \frac{18}{25}\ell n(x-1) + \frac{2}{5(x-1)} + \frac{7}{50}\ell n(x^2 + 2x + 2)$

$-\frac{\cancel{102}^{\,51}}{\cancel{50}_{\,25}}\tan^{-1}(x+1) + C$

$$\boxed{I = \frac{18}{25}\ell n(x-1) + \frac{2}{5(x-1)} + \frac{7}{50}\ell n(x^2 + 2x + 2) - \frac{51}{25}\tan^{-1}(x+1) + c}$$

(21) $\int \frac{x^2 - 3x + 5}{x^4 - 8x^2 + 16}dx$

Solution: let $I = \int \frac{x^2 - 3x + 5}{x^4 - 8x^2 + 16}dx$

$I = \int \frac{x^2 - 3x + 5}{(x^2 - 4)^2}dx = \int \frac{x^2 - 3x + 5}{(x-2)^2(x+2)^2}dx$

resolve into partial fraction

$\frac{x^2 - 3x + 5}{(x-2)^2(x+2)^2} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{x+2} + \frac{D}{(x+2)^2}$ ——— (1)

Xplying throughout eqⁿ (1) by $(x-2)^2(x+2)^2$

$x^2 - 3x + 5 = A(x-2)(x+2)^2 + B(x+2)^2 + C(x+2)(x-2)^2 + D(x-2)^2$ ——— (2)

put $x - 2 = 0 \Rightarrow \boxed{x = 2}$ in (2) $\Rightarrow \boxed{B = \frac{3}{16}}$

put $x + 2 = 0 \Rightarrow \boxed{x = -2}$ in (2) $\Rightarrow \boxed{D = \frac{15}{16}}$

eq^n (2) can be written as

$x^2 - 3x + 5 = A(x^3 + 2x^2 - 4x - 8) + B(x^2 + 4x + 4) + C(x^3 - 2x^2 - 4x + 8) + D(x^2 - 4x + 4)$

$x^2 - 3x + 5 = (A + C)x^3 + (2A + B - 2C + D)x + (4B - 4A - 4C - 4D)x + (4B - 8A + 8C + 4D)$

equating the coefficient of x^3, x^2 and constant term

$A + C = 0$ ——— (3)

$2A + B - 2C + D = 1$ ——— (4)

$4B - 8A + 8C + 4D = 5$ ——— (5)

put the values of B & D in (4)

$(4) \Rightarrow 2A + \frac{3}{16} - 2C + \frac{15}{16} = 1$

Xplying throughout by 16.

$32A + 3 - 32C + 15 = 16$

$32A - 32C = -2$ ——— (6)

Xply eq^n (3) by 32 + eq^n (6)

$$\begin{array}{l} 32A + \cancel{32C} = 0 \\ \underline{32A - \cancel{32C} = -2} \end{array}$$

$$\cancel{64}A = -\cancel{2} \Rightarrow \boxed{A = \frac{-1}{32}}$$

32

put the value of A in (3)

$\frac{-1}{32} + C = 0 \Rightarrow \boxed{C = \frac{1}{32}}$

put the values of A, B, C & D in (1)

$$(1) \Rightarrow \frac{x^2 - 3x + 5}{(x-2)^2(x+2)^2} = \frac{\frac{-1}{32}}{x-2} + \frac{\frac{3}{16}}{(x-2)^2} + \frac{\frac{1}{32}}{x+2} + \frac{\frac{15}{16}}{(x+2)^2}$$

on integration

$$\int \frac{(x^2 - 3x + 5)\,dx}{(x-2)^2(x+2)^2} = \frac{-1}{32}\int \frac{1}{x-2}dx + \frac{3}{16}\int (x-2)^{-2}dx + \frac{1}{32}\int \frac{1}{x+2}dx + \frac{15}{16}\int (x+2)^{-2}dx$$

$$I = \frac{-1}{32}\ell n(x-2) - \frac{3}{16(x-2)} + \frac{1}{32}\ell n(x+2) - \frac{15}{16(x+2)} + C$$

$$\boxed{I = \frac{-3}{16(x-2)} - \frac{1}{32}\ell n\left\{\frac{x-2}{x+2}\right\} - \frac{15}{16(x+2)} + C}$$

(22) $\int \frac{x^2 + 3x + 5}{x^3 + 8}dx$

Solution: let $I = \int \frac{x^2 + 3x + 5}{x^3 + 8}dx$

$I = \int \frac{x^2 + 3x + 5}{(x+2)(x^2 - 2x + 4)}dx$

resolve into partial fraction

$\frac{x^2 + 3x + 5}{(x+2)(x^2 - 2x + 4)} = \frac{A}{x+2} + \frac{Bx + c}{x^2 - 2x + 4}$ ——— (1)

Xplying throughout eq^n (1) by $(x + 2)(x^2 - 2x + 4)$

$x^2 + 3x + 5 = A(x^2 - 2x + 4) + Bx(x + 2) + C(x + 2)$ — (2)

put $x + 2 = 0 \Rightarrow \boxed{x = -2}$ in (2) $\Rightarrow \boxed{A = \frac{1}{4}}$

eq^n (2) can be written as.

$x^2 + 3x + 5 = (A + B)x^2 + (2B - 2A + C)x + 4A + 2C$

equating the coefficient of x^2 & x

$A + B = 1 \Rightarrow \frac{1}{4} + B = 1 \Rightarrow \boxed{B = \frac{3}{4}}$

$2B - 2A + C = 3$

$2\left(\frac{3}{4}\right) - 2\left(\frac{1}{4}\right) + C = 3 \Rightarrow \frac{6}{4} - \frac{2}{4} + C = 3$

$\frac{\cancel{4}}{\cancel{4}} + C = 3 \Rightarrow C + 1 = 3 \Rightarrow \boxed{C = 2}$

put the values of A, B & C in (1) x^2

$$(1) \Rightarrow \frac{x^2 + 3x + 5}{(x+2)(x^2 - 2x + 4)} = \frac{\frac{1}{4}}{x+2} + \frac{\frac{3}{4}x + 2}{x^2 - 2x + 4}$$ on integration

$$\int \frac{(x^2 + 3x + 5)\,dx}{(x+2)(x^2 - 2x + 4)} = \frac{1}{4}\int \frac{1}{x+2}dx + \frac{1}{4}\int \frac{3x + 8}{x^2 - 2x + 4}dx$$

$$I = \frac{1}{4}\int \frac{1}{x+2}dx + \frac{1}{8}\int \frac{(6x + 18)\,dx}{x^2 - 2x + 4}$$

$$I = \frac{1}{4}\int \frac{1}{x+2}dx + \frac{1}{8}\int \frac{(6x - 6 + 22)\,dx}{x^2 - 2x + 4}$$

$$I = \frac{1}{4}\int \frac{1}{x+2}dx + \frac{3}{8}\int \frac{(2x - 2)\,dx}{x^2 - 2x + 4} + \frac{22}{8}\int \frac{1}{(x - 1)^2 + (\sqrt{3})^2}dx$$

$$I=\frac{1}{4}\ln(x+2)+\frac{3}{8}\ln(x^2-2x+4)+\frac{\cancel{22}^{11}}{\cancel{8}_{4}}\times\frac{1}{\sqrt{3}}\tan^{-1}\frac{(x-1)}{\sqrt{3}}+C$$

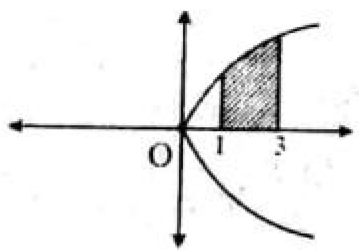
$$\boxed{I=\frac{1}{4}\ln(x+2)+\frac{3}{8}\ln(x^2-2x+4)+\frac{11\sqrt{3}}{12}\tan^{-1}\frac{(x-1)}{\sqrt{3}}+C}$$

EXERCISE # 6.10

Find the area, above the x–axis, under the following curves, between the given ordinates.

(1) $\mathbf{y^2 = x \;,\; x = 1 \;,\; x = 3}$

Solution: $y^2 = x \qquad a = 1, b = 3$



$\Rightarrow \quad y=\sqrt{x} \Rightarrow y = x^{1/2}$

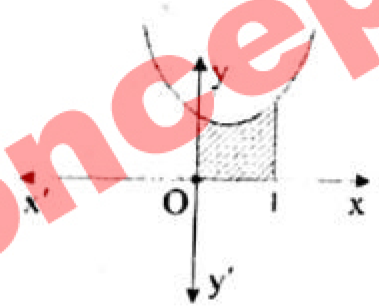
$$A=\int_a^b y\,dx$$

$$A=\int_1^3 x^{1/2}\,dx=\left[\frac{x^{3/2}}{3/2}\right]_1^3=\frac{2}{3}\left[x^{3/2}\right]_1^3$$

$$A=\frac{2}{3}\{3^{3/2}-1^{3/2}\}=\frac{2}{3}\{3\sqrt{3}-1\}\text{ Unit}^2$$

(2) $\mathbf{y = x^2 - 2x + 5 \;,\; x = 0 \;,\; x = 1}$

Solution: $y = x^2 - 2x + 5 \;,\; a = 0 \;,\; b = 1$



$$A=\int_a^b y\,dx$$

$$A=\int_0^1 (x^2-2x+5)\,dx$$

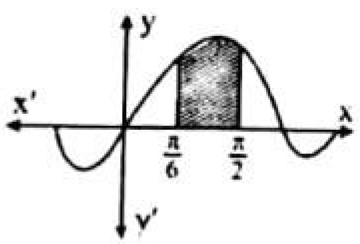
$$A=\left[\frac{x^3}{3}-\frac{\cancel{2}x^2}{\cancel{2}}+5x\right]_0^1=\left[\frac{x^3}{3}-x^2+5x\right]_0^1$$

$$A=\left[\frac{(1)^3}{3}-(1)^2+5(1)\right]-\left[\frac{(0)^3}{3}-(0)^2+5(0)\right]$$

$$A=\frac{1}{3}-1+5=\frac{1}{3}+4=\frac{13}{3}\Rightarrow\boxed{A=4\frac{1}{3}\text{ Unit}^2}$$

(3) $\mathbf{y = Sinx \;,\; x = \frac{\pi}{6}, x = \frac{\pi}{2}}$

Solution: $y = \text{Sin}x \;,\; a=\frac{\pi}{6} \;,\; b=\frac{\pi}{2}$

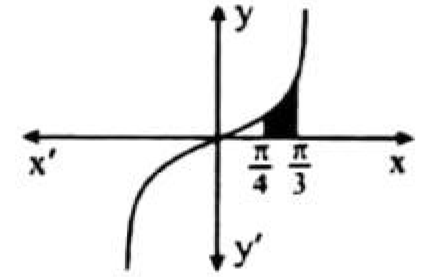


$$A=\int_a^b y\,dx \Rightarrow A=\int_{\pi/6}^{\pi/2}\text{Sin}x\,dx=[-\text{Cos}x]_{\pi/6}^{\pi/2}$$

$$A=-\left[\text{Cos}\frac{\pi}{2}-\text{Cos}\frac{\pi}{6}\right]=-\left[0-\frac{\sqrt{3}}{2}\right].$$

$$\boxed{A=\frac{\sqrt{3}}{2}\text{ Unit}^2}\qquad\text{Ans.}$$

(4) $\mathbf{y = tanx \;,\; x = \frac{\pi}{4} \;,\; x = \frac{\pi}{3}}$



Solution: $y = \tan x \;,\; a=\frac{\pi}{4} \;,\; b=\frac{\pi}{3}$

$$A=\int_a^b y\,dx \Rightarrow A=\int_{\pi/4}^{\pi/3}\tan x\,dx$$

$$A=[\ln \text{Sec}x]_{\pi/4}^{\pi/3}=\ln\text{Sec}\frac{\pi}{3}-\ln\text{Sec}\frac{\pi}{4}$$

$$A=\ln\left[\frac{\text{Sec}\frac{\pi}{3}}{\text{Sec}\frac{\pi}{4}}\right]=\ln\left[\frac{\text{Cos}\frac{\pi}{4}}{\text{Cos}\frac{\pi}{3}}\right]=\ln\left[\frac{\frac{1}{\sqrt{2}}}{\frac{1}{2}}\right]$$

$$A=\ln\left(\frac{2}{\sqrt{2}}\right)\Rightarrow A=\ln(1.4142)\Rightarrow\boxed{A=0.3465\text{ Unit}^2}$$

(5) $\mathbf{y = 3x^4 - 2x^3 + 1 \;,\; x = 1 \;,\; x = 2}$

Solution: $y = 3x^4 - 2x^3 + 1 \;,\; a = 1 \;,\; b = 2$

$$A=\int_a^b y\,dx=\int_1^2(3x^4-2x^3+1)\,dx\Rightarrow A=\left[\frac{3x^5}{5}-\frac{2x^4}{4}+x\right]_1^2$$

$$A=\left[\frac{3}{5}(2)^5-\frac{2}{5}(2)^4+2\right]-\left[\frac{3}{5}(1)^5-\frac{2}{5}(1)^4+1\right]$$

$$A=\frac{96}{5}-6-\frac{11}{10}=\frac{960-300-55}{50}$$

$$A=\frac{605}{50}\Rightarrow\boxed{A=12\frac{1}{10}\text{ unit}^2}\quad\text{Ans.}$$

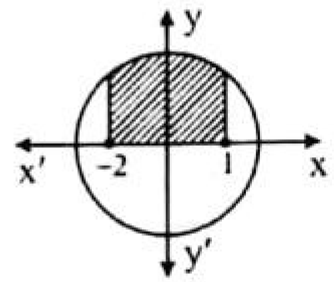
(6) $\mathbf{x^2 + y^2 = 9 \;,\; x = -2, \; x = 1}$

Solution: $x^2 + y^2 = 9 \;,\; a = -2 \;,\; b = 1$

$$y=\sqrt{9-x^2}\Rightarrow A=\int_a^b y\,dx \Rightarrow A\int_{-2}^{1}\sqrt{9-x^2}\,dx$$

$$\boxed{\therefore\int\sqrt{a^2-x^2}\,dx=\frac{1}{2}x\sqrt{a^2-x^2}+\frac{1}{2}a^2\,\text{Sin}^{-1}\left(\frac{x}{a}\right)+c}$$

$$A = \left[\frac{x}{2}\sqrt{9-x^2} + \frac{9}{2}\,\mathrm{Sin}^{-1}\left(\frac{x}{3}\right)\right]_{-2}^{1}$$



$$A = \left[\frac{(1)}{2}\sqrt{9-(1)^2} + \frac{9}{2}\,\mathrm{Sin}^{-1}\left(\frac{1}{3}\right)\right] - \left[\frac{(-\cancel{2})}{\cancel{2}}\sqrt{9-(-2)^2} + \frac{9}{2}\,\mathrm{Sin}^{-1}\left(\frac{-2}{3}\right)\right]$$

$$A = \left[\frac{\sqrt{8}}{2} + \frac{9}{2}\,\mathrm{Sin}^{-1}\left(\frac{1}{3}\right)\right] - \left[-\sqrt{5} - \frac{9}{2}\,\mathrm{Sin}^{-1}\left(\frac{2}{3}\right)\right]$$

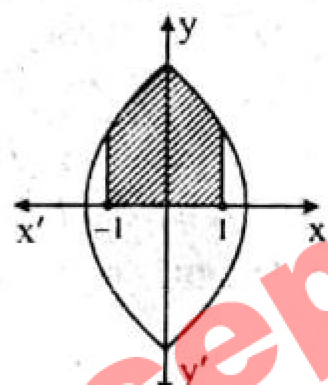
$$A = \frac{\sqrt{8}}{2} + \sqrt{5} + \frac{9}{2}\,\mathrm{Sin}^{-1}\left(\frac{1}{3}\right) + \frac{9}{2}\,\mathrm{Sin}^{-1}\left(\frac{2}{3}\right)$$

$$A = 1.4142 + 2.2360 + \frac{9}{2}\left\{\mathrm{Sin}^{-1}\left(\frac{1}{3}\right) + \mathrm{Sin}^{-1}\left(\frac{2}{3}\right)\right\}$$

$$A = 3.6502 + \frac{9}{2}\{\,0.3398 + 0.7297\,\}$$

$$A = 3.6502 + 4.5\,(1.0695) \Rightarrow \boxed{A = 8.46295\ \mathrm{Unit}^2}$$ Ans.

(7) $\frac{x^2}{4} + \frac{y^2}{9} = 1,\ x = -1,\ x = 1$



Solution: $\frac{x^2}{4} + \frac{y^2}{9} = 1,\ a = -1,\ b = 1$

$$\frac{y^2}{9} = 1 - \frac{x^2}{4} \Rightarrow \frac{y^2}{9} = \frac{4-x^2}{4}$$

$$y^2 = \frac{9\,(4-x^2)}{4} \Rightarrow y = \frac{3}{2}\sqrt{4-x^2}$$

$$A = \int_a^b y\,dx \Rightarrow A = \frac{3}{2}\int_{-1}^{1}\sqrt{4-x^2}\,dx$$

$$\boxed{\therefore \int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\,\mathrm{Sin}^{-1}\left(\frac{x}{a}\right) + c}$$

$$A = \frac{3}{2}\left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\,\mathrm{Sin}^{-1}\left(\frac{x}{2}\right)\right]_{-1}^{1}$$

$$A = \frac{3}{2}\left[\left\{\frac{1}{2}\sqrt{4-(1)^2} + 2\,\mathrm{Sin}^{-1}\left(\frac{1}{2}\right)\right\} - \left\{\frac{(-1)}{2}\sqrt{4-(-1)^2} + 2\,\mathrm{Sin}^{-1}\left(\frac{-1}{2}\right)\right\}\right]$$

$$A = \frac{3}{2}\left[\frac{\sqrt{3}}{2} + 2\mathrm{Sin}^{-1}\left(\frac{1}{2}\right) + \frac{\sqrt{3}}{2} + 2\mathrm{Sin}^{-1}\left(\frac{1}{2}\right)\right]$$

$$A = \frac{3}{2}\left[\frac{\cancel{2}(\sqrt{3})}{\cancel{2}} + 2\left\{\mathrm{Sin}^{-1}\left(\frac{1}{2}\right) + \mathrm{Sin}^{-1}\left(\frac{1}{2}\right)\right\}\right]$$

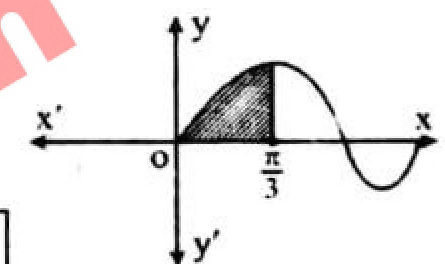
$$A = \frac{3}{2}[\,\sqrt{3} + 2\,(0.5235 + 0.5235)\,]$$

$\boxed{A = 5.7390\ \mathrm{Unit}^2}$ Ans.

Find the area under the curve given between the ordinates x = a , x = b.

(8) $y = 3\mathrm{Sin}x,\ a = 0,\ b = \frac{\pi}{3}$

Solution: $y = 3\mathrm{Sin}x,\ a = 0,\ b = \frac{\pi}{3}$



$$A = \int_a^b y\,dx = \int_0^{\pi/3} 3\mathrm{Sin}x\,dx$$

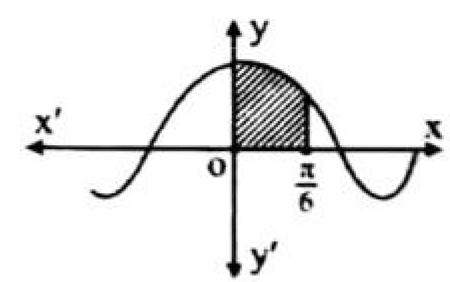
$$A = 3\int_0^{\pi/3}\mathrm{Sin}x\,dx = 3[-\mathrm{Cos}x]_0^{\pi/3}$$

$$A = -3\left[\mathrm{Cos}\frac{\pi}{3} - \mathrm{Cos}0\right] = -3\left[\frac{1}{2} - 1\right]$$

$\boxed{A = \frac{3}{2}\ \mathrm{Sq.\ Units}}$ Ans.

(9) $y = 2\mathrm{Cos}3x,\ a = 0,\ b = \frac{\pi}{6}$

Solution: $y = 2\mathrm{Cos}3x,\ a = 0,\ b = \frac{\pi}{6}$



$$A = \int_a^b y\,dx = \int_0^{\pi/6} 2\mathrm{Cos}3x\,dx$$

$$A = 2\int_0^{\pi/6}\mathrm{Cos}3x\,dx = 2\left[\frac{\mathrm{Sin}3x}{3}\right]_0^{\pi/6}$$

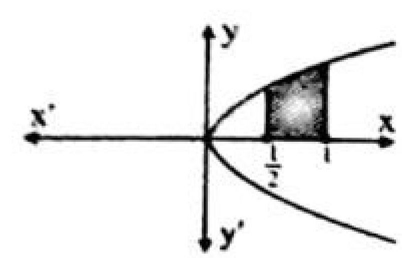
$$A = \frac{2}{3}\left[\mathrm{Sin}3\left(\frac{\pi}{6}\right) - \mathrm{Sin}3\,(0)\right]$$

$$A = \frac{2}{3}[\,1 - 0\,] \Rightarrow \boxed{A = \frac{2}{3}\ \mathrm{Sq.\ Units}}$$ Ans.

(10) $y^2 = 6x,\ a = \frac{1}{2},\ b = 1$

Solution: $y^2 = 6x,\ a = \frac{1}{2},\ b = 1$

$$y = \sqrt{6}\,x^{1/2}$$



$$A = \int_a^b y\,dx = \int_{1/2}^{1} \sqrt{6}\, x^{1/2}\,dx$$

$$A = \sqrt{6}\left[\frac{x^{3/2}}{3/2}\right]_{\frac{1}{2}}^{1} = \frac{2\sqrt{6}}{3}\left[(1)^{3/2} - \left(\frac{1}{2}\right)^{3/2}\right]$$

$$A = \frac{2\sqrt{6}}{3}\left[1 - \frac{1}{2\sqrt{2}}\right] = \frac{\cancel{2}\sqrt{6}}{3}\left[\frac{2\sqrt{2}-1}{\cancel{2}\sqrt{2}}\right]$$

$$A = \frac{\sqrt{3}}{3}(2\sqrt{2} - 1) \Rightarrow \boxed{A = 1.005 \text{ Sq. Units}} \quad \text{Ans.}$$

(portion of the curve in the first quadrant)

(11) $x^2 + y^2 = 25$, $a = 3$, $b = 4$

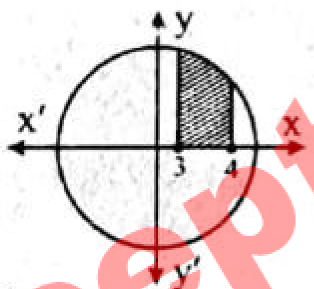
Solution: $x^2 + y^2 = 25,\ a = 3,\ b = 4,\ y = \sqrt{25 - x^2}$

$$A = \int_a^b y\,dx = \int_3^4 \sqrt{25 - x^2}\,dx$$

$$\boxed{\therefore \int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\text{Sin}^{-1}\left(\frac{x}{a}\right) + c}$$

$$A = \left[\frac{x}{2}\sqrt{25 - x^2} + \frac{25}{2}\text{Sin}^{-1}\left(\frac{x}{5}\right)\right]_3^4$$

$$A = \left[\frac{4}{2}\sqrt{25 - 16} + \frac{25}{2}\text{Sin}^{-1}\left(\frac{4}{5}\right)\right] - \left[\frac{3}{2}\sqrt{25 - 9} + \frac{25}{2}\text{Sin}^{-1}\left(\frac{3}{5}\right)\right]$$



$$A = 2\sqrt{9} + \frac{25}{2}\text{Sin}^{-1}\left(\frac{4}{5}\right) - \frac{3}{2}\sqrt{16} - \frac{25}{2}\text{Sin}^{-1}\left(\frac{3}{5}\right)$$

$$A = \cancel{6} - \cancel{6} + \frac{25}{2}\left[\text{Sin}^{-1}\left(\frac{4}{5}\right) - \text{Sin}^{-1}\left(\frac{3}{5}\right)\right]$$

$$A = \frac{25}{2}[\,0.9272 - 0.6435\,]$$

$$\boxed{A = 3.54625 \text{ Sq. Units}} \quad \text{Ans.}$$

(12) $y = \tan^2 x$, $a = \frac{\pi}{6}$, $b = \frac{\pi}{4}$

Solution: $y = \tan^2 x$, $a = \frac{\pi}{6}$, $b = \frac{\pi}{4}$

$$y = \text{Sec}^2 x - 1$$

$$A = \int_a^b y\,dx = \int_{\pi/6}^{\pi/4} (\text{Sec}^2 x - 1)\,dx = [\,\tan x - x\,]_{\pi/6}^{\pi/4}$$

$$A = \left\{\tan\frac{\pi}{4} - \frac{\pi}{4}\right\} - \left\{\tan\frac{\pi}{6} - \frac{\pi}{6}\right\}$$

$$A = \left(\tan\frac{\pi}{4} - \tan\frac{\pi}{6}\right) - \left(\frac{\pi}{4} - \frac{\pi}{6}\right)$$

$$A = \left(1 - \frac{1}{\sqrt{3}}\right) - \left(\frac{3\pi - 2\pi}{12}\right)$$

$$\boxed{A = 1 - \frac{1}{\sqrt{3}} - \frac{\pi}{12}\text{ Sq. Units}} \quad \text{Ans.}$$

(13) $y = 2e^{3x}$, $a = 2$, $b = 5$

Solution: $y = 2e^{3x}$, $a = 2$, $b = 5$

$$A = \int_a^b y\,dx = \int_2^5 2e^{3x}\,dx = \left[\frac{2e^{3x}}{3}\right]_2^5$$

$$A = \frac{2}{3}[\,e^{3(5)} - e^{3(2)}\,] = \frac{2}{3}\{\,e^{15} - e^{6}\,\}\text{ Sq. Units}$$

(14) $y = x - \frac{5}{x^2}$, $a = 2$, $b = 4$

Solution: $y = x - \frac{5}{x^2}$, $a = 2$, $b = 4$

$$A = \int_a^b y\,dx = \int_2^4 (x - 5x^{-2})\,dx$$

$$A = \left[\frac{x^2}{2} - \frac{5x^{-1}}{-1}\right]_2^4 = \left[\frac{x^2}{2} + \frac{5}{x}\right]_2^4$$

$$A = \left[\frac{(4)^2}{2} + \frac{5}{4}\right] - \left[\frac{(2)^2}{2} + \frac{5}{2}\right] = \left[8 + \frac{5}{4} - 2 - \frac{5}{2}\right]$$

$$A = \frac{32 + 5 - 8 - 10}{4} \Rightarrow A = \frac{19}{4} \Rightarrow \boxed{A = 4\frac{3}{4}\text{ Sq Units}} \quad \text{Ans.}$$