CS1112
Sets and Collections II

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Sets and Collections (continued)

Set complement

Cardinality

Sets containing other sets

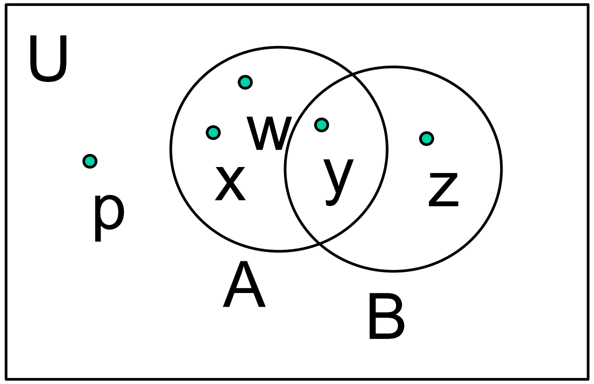
Power set
Partition

Laws of set operations

Cartesian product

The set complement of a set is a new set consisting of every element of the universal set that is not an element of the first set

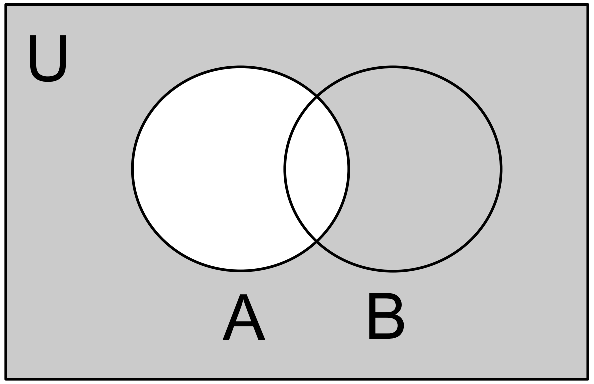
We write this as A’



A = {x,y,w}   B = {y,z}

A’ = {p,z}

In general:



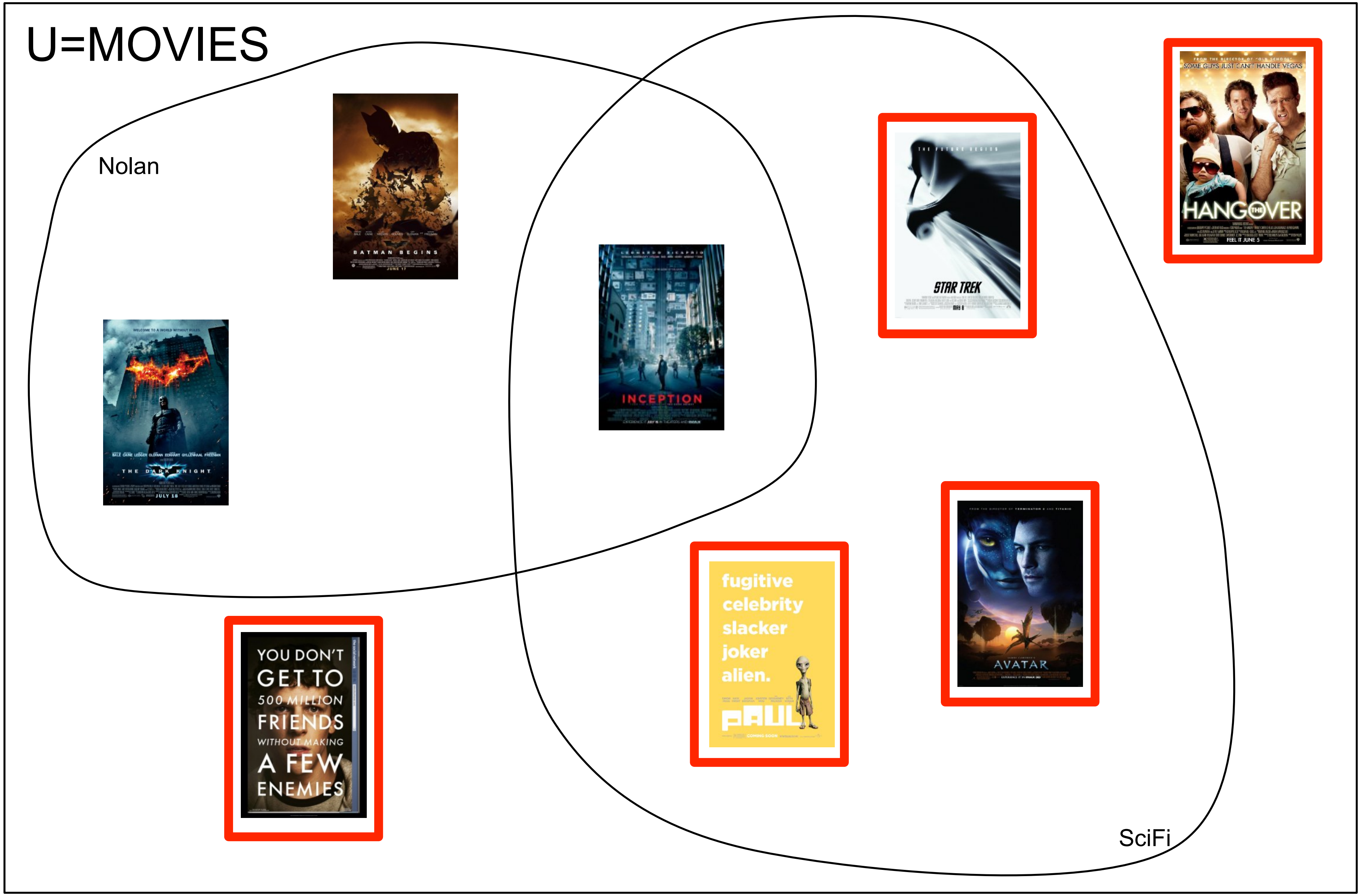
A’

For any element x of U,

$x \in A'$ if and only if $x \notin A$

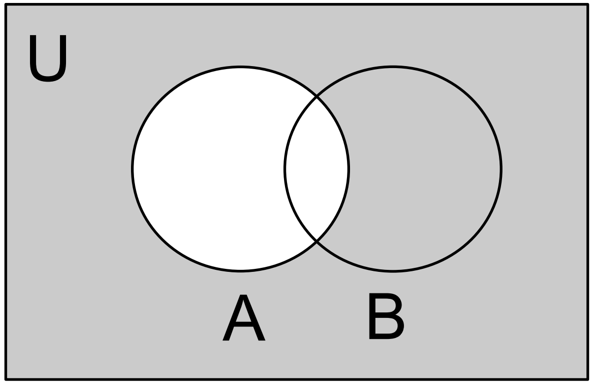
Note: A’ = U \ A

Nolan′


U=MOVIES
Nolan
SciFi
BATMAN BEGINS
THE DARK KNIGHT
INCEPTION
STAR TREK
SOME GUYS JUST CAN'T HANDLE VEGAS
HANGOVER
AVATAR
fugitive
celebrity
slacker
joker
alien.
PAUL
YOU DON'T
GET TO
500 MILLION
FRIENDS
WITHOUT MAKING
A FEW
ENEMIES

Note: complement, not compliment

In general:



A’

Set complement

Set compliment

The cardinality of a finite set is the number of elements in the set

We write cardinality using vertical bars on each side of the set,
so |{apple, banana, orange}| = 3

Examples:

|A|?

A={1,2,3,4}

A = {e,g,b,d,f}

A = {cork, kerry, cork, galway}

A= { }

Some simple facts about cardinality of sets:

If $A \subseteq B$, then $|A| \leq |B|$
If $A \subset B$, then $|A| < |B|$

E.g. The number of elements in the union of two sets can't be more than the number in the first plus the number in the second

$|(A \cup B)| \leq |A| + |B|$

$|(A \cap B)| \leq |A|$ and $|(A \cap B)| \leq |B|$

$|(A \cup B)| = |A| + |B| - |(A \cap B)|$

Add the number in the first set to the number in the second, and then take away the number that you have double counted.

$|(A \backslash B)| \leq |A|$

$|(A \backslash B)| = |A| - |(A \cap B)|$

These facts are not hard to understand – once you have got over the hurdle of reading and understanding the symbols, they follow immediately from the definitions we have already seen

Extended example: student record system

University College Skibbereen has 12 registered students (it is a small college): Alice, Bob, Carol, Darragh, Eileen, Frank, Oliver, Patsy, Ronan, Susan, Ted, and Una.

There are six modules:
CS101, CS102, CS103, MA101, EC101, EC102

Students are registered for different modules.

	Alice	Bob	Carol	Darragh	Eileen	Frank	Oliver	Patsy	Ronan	Susan	Ted	Una
CS101	X			X			X		X			
CS102		X	X		X							
CS103					X			X			X	
MA101				X			X					
EC101						X				X	X	
EC102	X	X						X		X		X

Extended example: extracting information from the database

Oliver is registered on CS101
Susan is not registered on MA101
The modules CS101 and EC101 do not have identical class lists
All students taking MA101 also take CS101
Not all students taking EC101 also take EC102
List all students taking an Economics module

Susan takes both Economics modules
How many students take both EC modules?

Some students take both CS102 and CS103
Bob and Carol are the CS102 students who do not also take CS103

List the students not taking EC102

From previous lecture:

- A set is a collection of things
- The "things" can be anything, physical or abstract
- What matters is that we can state clearly and precisely what "things" are in the collection

So that means we can have a set as a member of another set:

The example:

U={Alice, Bob, Carol, Darragh, Eileen, Frank, Oliver, Patsy, Ronan, Susan, Ted,Una}
CS101 = {Alice, Darragh, Oliver, Ronan}
CS102 = {Bob, Carol, Eileen}
CS103 = {Eileen, Patsy, Ted}
MA101 = {Darragh, Oliver}
EC101 = {Frank, Susan, Ted}
EC102 = {Alice, Bob, Una, Patsy, Susan}

Modules = {CS101, CS102, CS103, MA101, EC101, EC102}

We can mix and match the elements – they do not all have to be of the same type:

$S = \{ a, \{b\}, c, \{ \}, \{d,e\}\}$

In this example, {b} is both
- a thing (i.e. a member of S)
- a set

Note:

$a \in S$

$\{b\} \in S$

$\{ \} \in S$

$\{d,e\} \in S$

$b \notin S$

$d \notin S$

$\{a,c\} \subseteq S$

$\{b\} \not\subseteq S$

$\{\{b\}\} \subseteq S$

Exercises:

Is $c \in S$?

Is $\{c\} \in S$?

Is $\{\{b\},c\} \subseteq S$?

Is $\{ \} \subseteq S$?

What is $|S|$?

In normal set theory, for any x, $x \neq \{x\}$

The power set of a set A is the set containing all possible subsets of A.

We write this as P(A).

NOTE: the power set is a set.
So between "{" and "}", with commas.

Example:

$A = \{1,2,3\}$

$P(A) = \{ \{ \}, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\} \}$

$A = \{x\}$

$P(A) = \{ \{ \}, \{x\} \}$

$A = \{ \}$

$P(A) = \{ \{ \} \}$

For any A, $|P(A)| = 2^{|A|}$

For any thing x,

$x \in P(A)$ if and only if $x \subseteq A$

(and so if $x \in P(A)$, x is a set)

Extended example: allocating project groups

U={Alice, Bob, Carol, Darragh, Eileen, Frank, Oliver, Patsy, Ronan, Susan, Ted,Una}
CS101 = {Alice, Darragh, Oliver, Ronan}
CS102 = {Bob, Carol, Eileen}
CS103 = {Eileen, Patsy, Ted}
MA101 = {Darragh, Oliver}
EC101 = {Frank, Susan, Ted}
EC102 = {Alice, Bob, Una, Patsy, Susan}

Suppose UCS computer science department wants to introduce team projects into module CS102. What groups are possible?

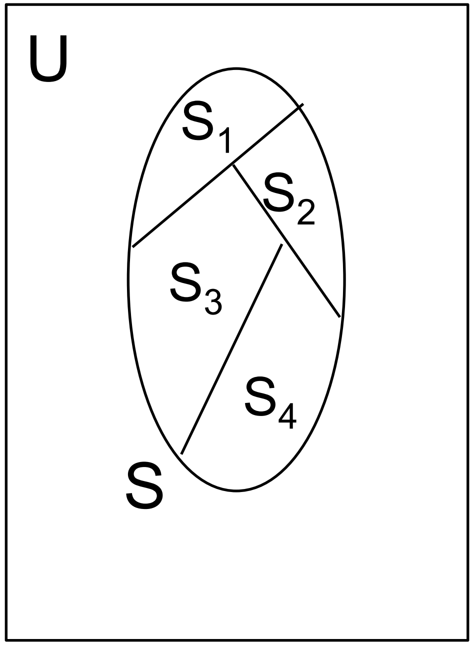
A partition of a set A is a division of all the members of A into non-empty non-intersecting subsets

Example: Suppose A = {1,2,3}.

One possible partition of A is { {1}, {2,3} }

The set {{1,2}, {2,3}} is not a partition :
the selected subsets must have no overlap

The set {{1}, { }, {2}, {3}} is not a partition:
the selected subsets must be non-empty



Let S be a set, and let P = $\{S_1, S_2, ..., S_n\}$.
P is a partition of S if and only if
(i) For each $S_i \in P$, $S_i \subseteq S$ and $S_i \neq \{ \}$
(ii) For any pair $S_i \in P$ and $S_j \in P$, $S_i \cap S_j = \{ \}$
(iii) For each $x \in S$, there is one $S_i \in P$ such that $x \in S_i$

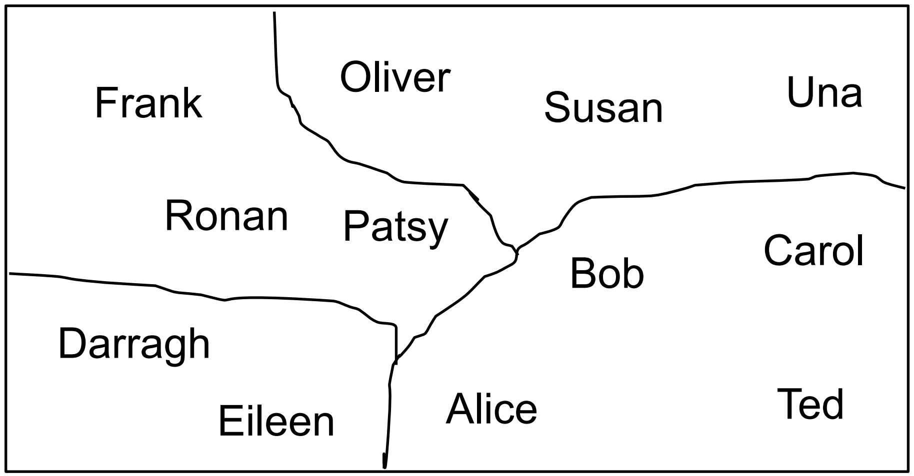
Extended example: allocating mentoring groups

U={Alice, Bob, Carol, Darragh, Eileen, Frank, Oliver, Patsy, Ronan, Susan, Ted,Una}
CS101 = {Alice, Darragh, Oliver, Ronan}
CS102 = {Bob, Carol, Eileen}
CS103 = {Eileen, Patsy, Ted}
MA101 = {Darragh, Oliver}
EC101 = {Frank, Susan, Ted}
EC102 = {Alice, Bob, Una, Patsy, Susan}

UCS now wants to introduce a mentoring scheme. Each student will be allocated to a group (which will also have a mentor), and no student will be in two groups.

MG_1 = {Alice, Bob, Carol, Ted}
MG_2 = {Darragh, Eileen}
MG_3 = {Frank, Patsy, Ronan}
MG_4 = {Oliver, Susan, Una}

UCS



The implications of these different set operations

- The set operations can be combined in different orders

- We need to understand what this means. For example, are the following two expressions describing the same set?

  - $A \cap B \cap C$
  - $C \cap A \cap B$

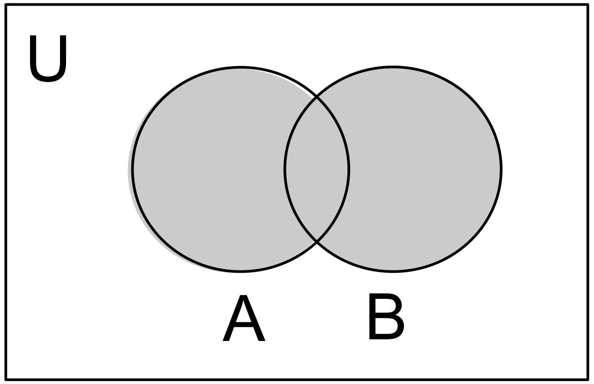
- And how about the following two?

  - $A \cap (B \cup C)$
  - $(A \cap B) \cup C$

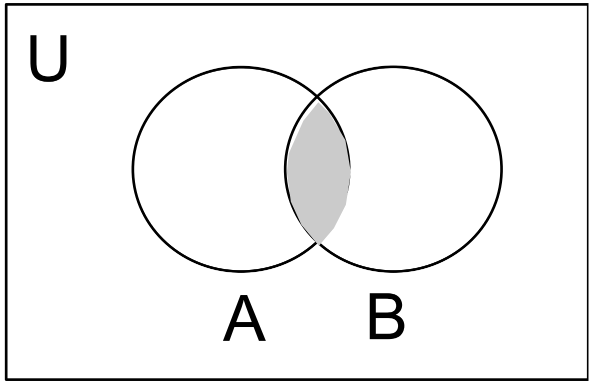
$A \cup B = B \cup A$

$A \cap B = B \cap A$

Union and intersection are commutative



$A \cup B$

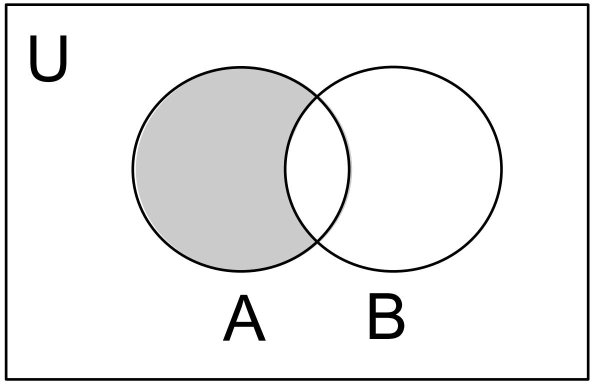


$A \cap B$

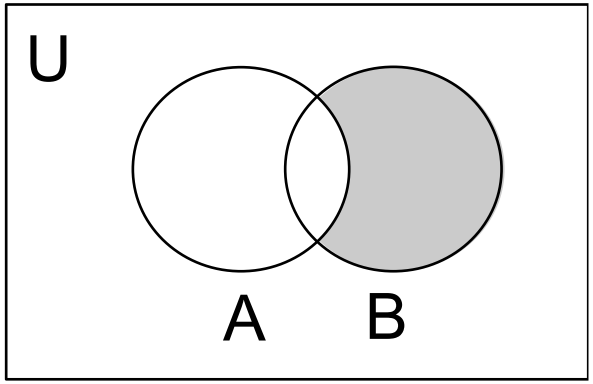
BUT, in general:

$A \setminus B \neq B \setminus A$

Set difference is <u>not</u> commutative



$A \setminus B$

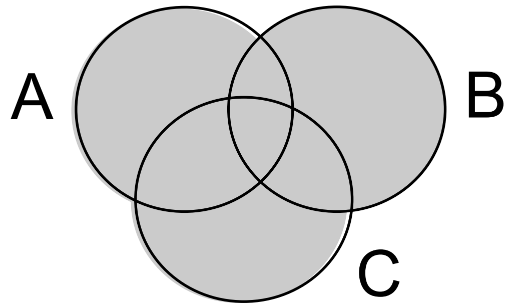


$B \setminus A$

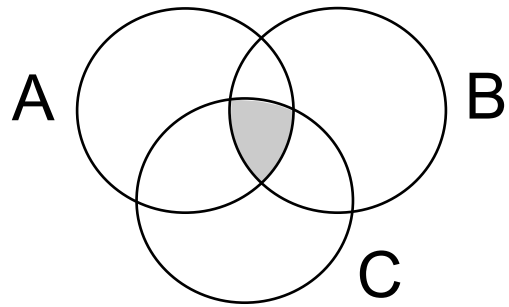
$(A \cup B) \cup C = A \cup (B \cup C)$

$(A \cap B) \cap C = A \cap (B \cap C)$

Union and intersection are associative



$A \cup B \cup C$

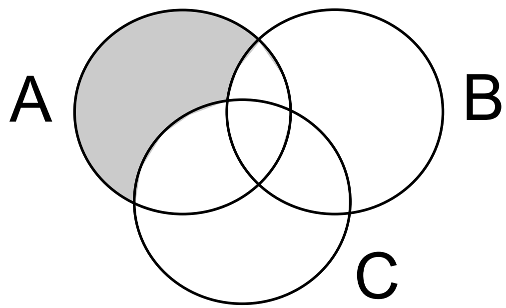


$A \cap B \cap C$

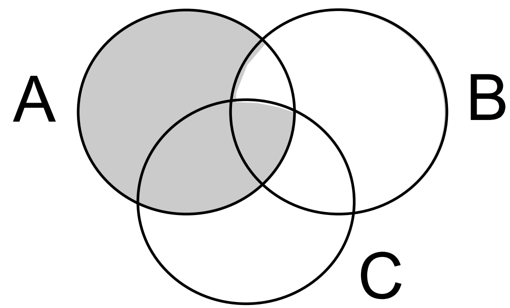
BUT, in general:

$(A \setminus B) \setminus C \neq A \setminus (B \setminus C)$

Set difference is <u>not</u> associative



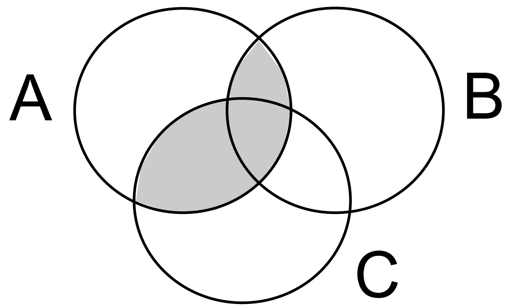
$(A \setminus B) \setminus C$



$A \setminus (B \setminus C)$

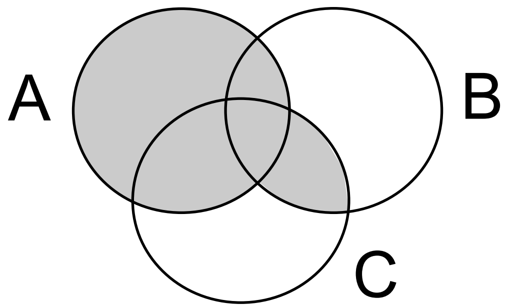
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

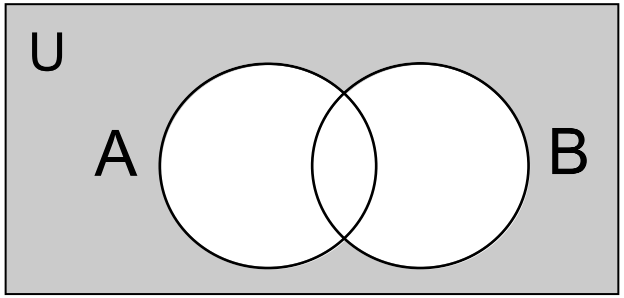
Intersection distributes over union



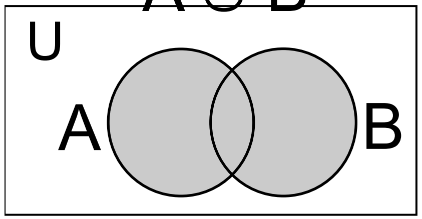
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Union distributes over intersection



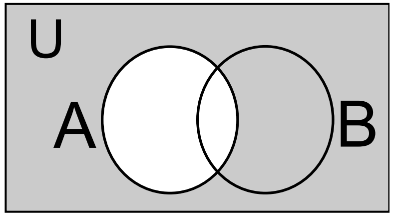


$A \cup B$



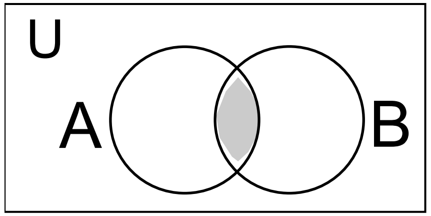
$$(A \cup B)' = (A' \cap B')$$

A'



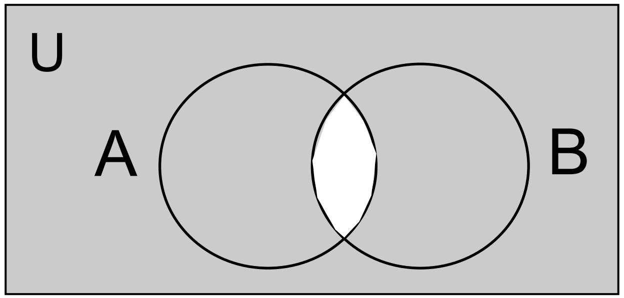
De Morgan's Laws

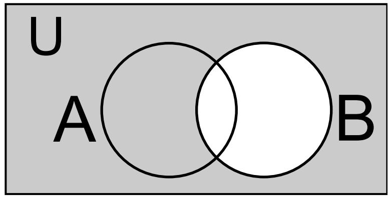
important



$A \cap B$

$$(A \cap B)' = (A' \cup B')$$





B'

Ordered Pairs

Sometimes, we will want to make connections between two or more elements or sets.

For example, in the student database, there are connections between the set of all students, and the set of all mentors.

To do this, we need the following definition:

An ordered pair is two objects listed inside round brackets, and separated by a comma.

Examples: (1,0), (dublin, ireland), (red, yellow)

- The order of the elements in an ordered pair is important
  - (1,0) and (0,1) are different ordered pairs
- We can repeat the same element on either side of an ordered pair
  - (red, red) is an ordered pair
- The elements can be from any set, and so can be sets themselves
  - ({1,2,3}, 3) is an ordered pair
- The definition extends to ordered n-tuples of any length
  - (3,5,2,1), (soup, fish, yoghurt) are orderd n-tuples

Film	Director	Star1	Star2	Year	Genre
Paul	Mottola	Pegg	Frost	2011	SciFi
Inception	Nolan	DiCaprio	Page	2010	Action
The Social Network	Fincher	Eisenberg	Garfield	2010	Drama
Avatar	Cameron	Worthington	Saldana	2009	SciFi
Star Trek	Abrams	Pine	Pegg	2009	SciFi
The Hangover	Phillips	Galifianakis	Cooper	2009	Comedy
The Dark Knight	Nolan	Bale	Ledger	2008	Action
Batman Begins	Nolan	Bale	Caine	2005	Action

The Cartesian Product of two sets A and B is the set of all possible ordered pairs, where the first element comes from A, and the second element comes from B.

We write this as $A \times B$

It is sometimes known as the cross product of A and B

Example: if A = {apple, banana, orange} and B = {0,1},

then $A \times B$ = {(apple, 0), (apple,1), (banana, 0), (banana,1), (orange,0), (orange,1) }

More formally, for any two objects x and y,

$(x,y) \in A \times B$ if and only if $x \in A$ and $y \in B$

The Cartesian Product of *n* sets $A_1, A_2, \ldots, A_n$ is the set of all possible ordered *n*-tuples, where the first element comes from A_1, the second element comes from A_2, ..., and the n^{th} element comes from A_n.

We write this as $A_1 \times A_2 \times \ldots \times A_n$

The size of the cartesian product is the product of the sizes of the original sets:

$$|(A \times B)| = |A| * |B|$$
$$|(A_1 \times A_2 \times \ldots \times A_n)| = |A_1| * |A_2| * \ldots * |A_n|$$

One important fact: If $A \neq B$, then $A \times B \neq B \times A$

Extended example: assigning mentors to groups

U={Alice, Bob, Carol, Darragh, Eileen, Frank, Oliver, Patsy, Ronan, Susan, Ted,Una}
CS101 = {Alice, Darragh, Oliver, Ronan}
CS102 = {Bob, Carol, Eileen}
CS103 = {Eileen, Patsy, Ted}
MA101 = {Darragh, Oliver}
EC101 = {Frank, Susan, Ted}
EC102 = {Alice, Bob, Una, Patsy, Susan}

MG_1 = {Alice, Bob, Carol, Ted}
MG_2 = {Darragh, Eileen}
MG_3 = {Frank, Patsy, Ronan}
MG_4 = {Oliver, Susan, Una}

Mentors = {brian, gordon, george}
Groups = {MG_1, MG_2, MG_3, MG_4}

What is the set of all possible assignments of a mentor to a group?

Let $B \subseteq A_1 \times A_2 \times \ldots \times A_n$
The projection of B onto A_i is the subset of elements of A_i that appear (in the right position) in one of the tuples of B

$x \in$ projection of B onto A_i
if and only if
there are elements $a_1 \in A_1$, $a_2 \in A_2$, …, $a_n \in A_n$ such that $(a_1, a_2, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n) \in B$

Sample Exam Question (CS1105), August 2010

1. (i) If $A = \{a,f,r,w,z\}$, $B = \{b,c,f,n,p,r,y\}$, and $C = \{a,d,n,q,r,w\}$, what is

(a) $A \setminus B$?

(b) P($A \cap C$) (i.e. the power set of $A \cap C$)?

(ii) If a set D has n elements, what is the size of P(D)?

(iii) Consider the expression $(X \cup Y)'$, for any pair of sets X and Y. Which of the following two expressions are equivalent to $(X \cup Y)'$? For the one that is not equivalent, find two sets that demonstrate it is not equivalent.

(a) $X' \cup Y'$

(b) $X' \cap Y'$

(5 marks)

Next lecture ...

Functions: mapping from one set to another