CS1113
Counting Combinations

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Counting Combinations

counting the number of choices we have in different situations

(and what you should know when you play the lottery)

Example: picking project teams

Suppose we have a group of five software engineers, {alice, bob, carol, darragh, eileen}, and we want to pick a team of three of them to work on a project.

How many different possible teams are there?

By the permutation rule, there are ${}^{5}P_{3} = 5!/(5-3)! = 5!/2! = (5*4*3*2*1)/(2*1) = 5*4*3 = 60$ different *permutations* of 3 people from a group of 5.

But many of these teams will be the same: e.g. <alice,bob,carol> is the same team as <carol,alice,bob>

So how many teams have we double counted?

Example (continued)

By the permutation rule, each team of three people can be represented as 3! = 6 different permutations.

So out of our total of 60, for every permutation we keep, we need to throw away 5 duplicates.

So there are 60/6 = 10 unique teams.

{alice,bob,carol}	{alice,darragh,eileen}
{alice,bob,darragh}	{bob,carol,darragh}
{alice,bob,eileen}	{bob,carol,eileen}
{alice,carol,darragh}	{bob,darragh,eileen}
{alice,carol,eileen}	{carol,darragh,eileen}

Unordered Combinations: "n choose r"

Often, when choosing some elements from a set, the order in which we list them does not matter.

To choose *r* elements from a set of size *n*, there are ${}^{n}P_{r} = n!/(n-r)!$ permutations.

But if we ignore the order in which we list the elements, then each permutation is one of a set of *r*! duplicates. We only need to keep 1 out of every set of *r*!, and so we need to divide our total by *r*! to get the number of unordered sets.

There are $n!/r!(n-r)!$ ways of choosing r elements from n.

We write $n!/r!(n-r)!$ as either ${}^{n}C_{r}$ or $\binom{n}{r}$ and say "*n* choose *r*".

Example: Poker hands

How many different
poker hands are possible?

There are 52 cards in a pack. Each card is different. A standard poker hand has 5 cards. The order in which the cards are placed in the hand does not matter.

How many different ways can we choose 5 items from 52?

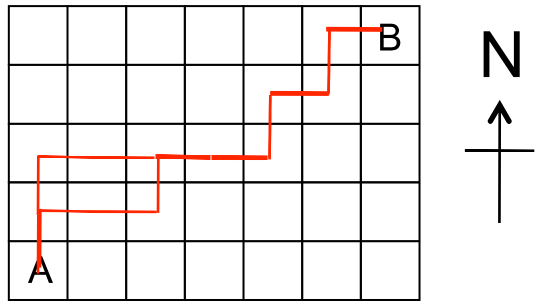
$$\binom{52}{5} = \frac{52!}{5!(52-5)!} = \frac{52!}{5!*47!} = \frac{52*51*50*49*48*47*46*...*1}{(5*4*3*2*1)*(47*46*...*1)} = \frac{52*51*50*49*48}{5*4*3*2*1} = 2598960$$

Example: Robot route planning

In the simple grid below, a robot can move from one square into an adjacent square (but not diagonally). How many possible paths are there from A to B, if the robot never moves to a square that is further away from B than its current square?

Each move is either N or E (and never S or W). There must be exactly 4 N steps and 6 E steps, and so 10 steps in total. So we can represent the route as an array of 10 directions. E.g.

route 1: NNEEEEENENE
1234567890
route 2: NEENEENENE
1234567890



robot route planning (continued)

route 1: NNEEEEENENE route 2: NEENEEENENE
1234567890 1234567890

We can represent the routes by the positions of the four Ns:
route 1: 1-2-7-9 route 2: 1-4-7-9

So two routes are different if the set of four positions for the Ns are different. Therefore, choosing a route is the essentially choosing four numbers from {0,1,2,...,9}, so the number of ways of doing this is:

$$\binom{10}{4} = \frac{10!}{4!(10-4)!} = \frac{10!}{4!6!} = \frac{10*9*8*7}{4*3*2*1} = 210$$

Computing choice combinations

Sometimes, you will need to compute combinations inside programs. Be careful! As we saw before, factorials can grow very quickly, producing numbers that are too large to be represented accurately.

For example, ${}^{20}C_3$ = 20!/3!*17!
If you try to compute 20! first, you will need to store a value larger than 10^{18}.

Instead, do the cancellation first:

$$\frac{20!}{3!*17!} = \frac{20*19*18*17*16*15*...*1}{(3*2*1)*(17*16*15*...*1)} = \frac{20*19*18}{3*2*1}$$

Combinations and products

The rules for combinations, products and sums can be used together (but you need to think carefully about what you are doing).

Suppose we want to enter a mixed soccer team into a 5-a-side tournament. The tournament rules say we must have 2 women and 3 men. There are 5 women to choose from, and 10 men. How many possible teams are there?

We choose 2 women from 5, and for each of these sets, we can then choose 3 men from 10. Therefore, there are:

$$\binom{5}{2} * \binom{10}{3} = \frac{5!}{2!3!} * \frac{10!}{3!7!} = \frac{5*4}{2*1} * \frac{10*9*8}{3*2} = \frac{20}{2} * \frac{720}{6} = 10*120 = 1200$$

Example: Poker hands (cont)

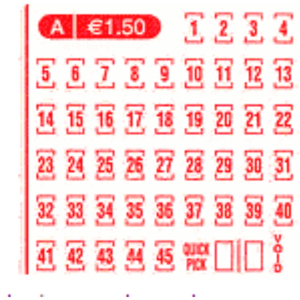
How many different
full house poker hands are possible?

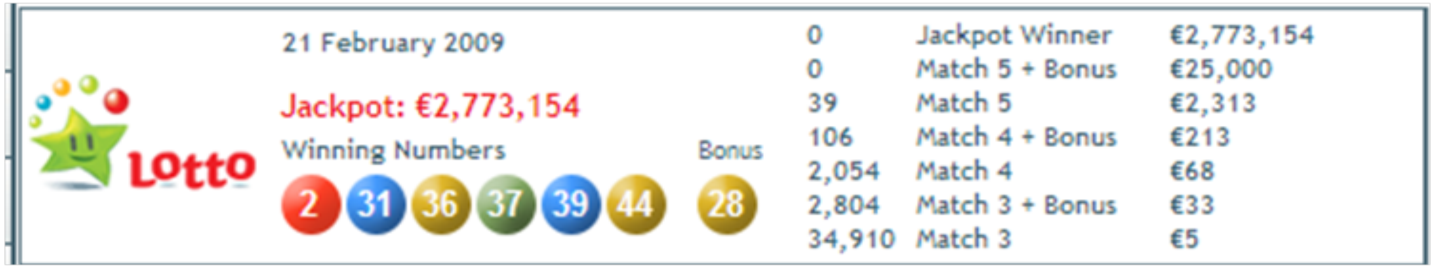
A full house consists of 3 cards of one rank, and 2 cards of another rank.

We must choose the rank of the 3-of-a-kind, then choose the 3 suits, then choose the rank of the pair, and choose the 2 suits. There are 13 possible ranks, and 4 possible suits.

$$\binom{13}{1} * \binom{4}{3} * \binom{12}{1} * \binom{4}{2} = \frac{13!}{1!(13-1)!} * \frac{4!}{3!(4-3)!} * \frac{12!}{1!(12-1)!} * \frac{4!}{2!(4-2)!} = 13 * 4 * 12 * \frac{4!}{2! * 2!} == 3744$$

Examples





In the national lottery (Lotto), each panel asks you to choose 6 numbers from 1 to 45. How many different 6-number entries can you make? What are your chances of winning a share in the jackpot?

Matching 3 numbers wins €5. How many different results (i.e. set of balls drawn in the lottery) would give you €5? What are your chances of winning €5?

Equivalence

$$\binom{n}{r} = \binom{n}{n-r}$$

Proof:

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n!}{(n-r)!r!} = \frac{n!}{(n-r)!(n-n+r)!} = \frac{n!}{(n-r)!(n-(n-r))!} = \binom{n}{n-r}$$

Examples

You must select 6 staff for a project team. Regulations state that there must be more experienced staff on the team than junior staff. You have 8 experienced staff members, and 12 junior staff. How many ways could you form the team?

Exercise: how many bit strings of length 8 have at least two 1s and at least two 0s?

more on assigning jobs to processors

Suppose now we have 3 processors, and 4 identical jobs. How many ways can we distribute the jobs?

Let the processors be called A, B and C.
If we put 1 job on A, 2 on B and 1 on C, we write ABBC, etc.
The full set of choices is:

AAAA	AABB	ABBB	ACCC	BBCC
AAAB	AABC	ABBC	BBBB	BCCC
AAAC	AACC	ABCC	BBBC	CCCC

which gives 15 choices in total.

re-expressing the choices

We can adapt the way we write the choices to give us a general rule.

We can rewrite ABBC as *|**|* where we interpret the number of stars before the 1st "|" as being the number of jobs on processor A, then the number of stars between the 1st and 2nd "|" as being the jobs on B, etc..

We know there are exactly 4 jobs, so we have 4 stars, and 2 "|".
Each different sequence gives a different distribution of jobs.
e.g. ***||* means 3 jobs on A, 0 jobs on B, and 1 job on C.

We can now rewrite this as the sequence of positions in which a star appears.
So 1236 gives ***||* which is AAAC, while 1346 gives *|**|*, giving ABBC

So the number of different sets of star positions is the number of different ways we can distribute the jobs over the processors.

number of ways to distribute 4 jobs over 3 processors

We have 4 jobs and 3 processors, and we represent one assignment by stating where the 4 stars appear in a sequence of 6 symbols.

So the number of ways of choosing 4 positions from 6 is the number of different assignments:

$$\binom{6}{4} = \frac{6!}{4!*2!} = \frac{6*5}{2*1} = 15$$ ways of distributing 4 jobs on 3 processors

How did we get the values '6' and '4'?
There were 4 jobs (which gives 4 stars), and 3 processors needed to be separated by 2 "|". This gives 4+2=6 positions.
So we chose 4 from 6.

Choosing *r* items with repeats from *n* types

If we must choose a set of *r* items where we have n types to choose from, and we can repeat some of the items,

we have $\binom{r+n-1}{r}$ different ways of doing it.

Note: if we must choose a *permutation* of *r* items where we have *n* types to choose from, and we can repeat items, then we have n^r ways of doing it.

Example

We all have to eat 5 portions of fruit or vegetables each day. Suppose we only eat 4 types: apples, oranges, bananas and carrots. How many different menus can we create?

Combinations and permutations: summary

	repetitions?	Formula	Notation
#r-permutations	no	$\dfrac{n!}{(n-r)!}$	${}^{n}P_{r}$
	yes	n^r	
#r-combinations	no	$\dfrac{n!}{(n-r)!r!}$	${}^{n}C_{r}$ or $\dbinom{n}{r}$
	yes	$\dfrac{(r+n-1)!}{r!(n-1)!} = \dbinom{r+n-1}{r}$	

Binomial Expansion

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \binom{n}{3}x^{n-3}y^3 + \cdots + \binom{n}{n-2}x^{n-(n-2)}y^{n-2} + \binom{n}{n-1}x^{n-(n-1)}y^{n-1} + \binom{n}{n}y^n$$

$$(x+y)^2 = \binom{2}{0}x^2 + \binom{2}{1}xy + \binom{2}{2}y^2 = \frac{2!}{0!2!}x^2 + \frac{2!}{1!1!}xy + \frac{2!}{2!0!}y^2 = x^2 + 2xy + y^2$$

$$(x+y)^3$$

$$= \binom{3}{0}x^3 + \binom{3}{1}x^2y + \binom{3}{2}xy^2 + \binom{3}{3}y^3$$

$$= \frac{3!}{0!3!}x^3 + \frac{3!}{1!2!}x^2y + \frac{3!}{2!1!}xy^2 + \frac{3!}{3!0!}xy^3$$

$$= x^3 + 3x^2y + 3xy^2 + y^3$$

Next lecture ...

classifying algorithm runtime