CS1113
Quantifiers in Predicate Logic

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Quantifiers

$\forall$ -- the universal quantifier

$\exists$ -- the existential quantifier

truth values of quantified statements

translating specifications into quantified logic with predicates

$\neg\forall x\, P(x) \equiv \exists x\, \neg P(x)$

Previous Lecture

We introduced sets and relations, and allowed our propositions to talk about elements that satisfy a relation (or *predicate*).

is_connected_to(comp23,cs_domain)

(beats(Brazil,Mexico) ∧ beats(Brazil,Croatia) ∧ beats(Brazil,Cameroon)) →

winsGroup(Brazil)

We introduced *variables*, which could stand for any element of a set, but that means the statements are not propositions (because they are neither definitely true nor definitely false)

has_firewall(x)

less_than(x, y)

sameGroup(x,Australia) →beats(x,Australia)

The aim

We now want to be able to talk about these statements with variables, and have a procedure to determine whether what we say is true or false

- Is there a value for the variable that makes the statement true?
- Which values make the statement true?
- How many values make the statement true?
- Is the statement true for all possible values we could give to the variable?

"all computers on the network have an active copy of *FireGuard™* "

Is the statement hasActiveFireGuard(x) true no matter which computer on the network we replace x with?

From now on, we will assume all predicates are defined over some universal set U (or over cross products of the universal set $U \times U \times \ldots \times U$).

The universal quantifier

To state that a predicate is true for all possible assignments of values to its variables, we use a special symbol ∀, called the universal quantifier.

If we write firewall(x) to represent the statement that x has an active firewall, then to say all possible objects have an active firewall, we write:

$$\forall x \text{ firewall}(x)$$

Note: think of ∀ as an upside-down A, standing for "for All"

Exercise:
if Q(x) is the statement $x < x+1$, where $U = \mathbb{Z}$, what does

$$\forall x\, Q(x)$$

say? Is it true or false?

Truth value of the universal quantifier

$\forall x\, P(x)$ is true if in all possible assignments of a value to x , P(x) is true

$\forall x\, P(x)$ is false if there is one or more possible assignments of a value to x that makes P(x) false

<u>Examples</u>

R(x) is the statement $x^2 \geq 0$ where U = Z

$\forall x\, R(x)$ is a true statement

R(5) is true, since $5^2=25\geq 0$

R(-3) is true, since $(-3)^2=9\geq 0$

P(x) is the statement x>7 where U = Z

$\forall x\, P(x)$ is a false statement

P(3) is false, since $3 \not> 7$

The existential quantifier

To state that a predicate is true for at least one assignment of values to its variables, we use the special symbol ∃, called the existential quantifier.

If firewall(x) represents the statement that x has an active firewall, then to say at least one object has an active firewall, we write:

$$\exists x \text{ firewall}(x)$$

Note: think of ∃ as backwards E, standing for "there Exists"

Exercise: if Q(x) is the statement $x = x^2$, and U=Z, what does

$$\exists\, x\, Q(x)$$

say? Is it true or false?

Truth value of the existential quantifier

$\exists x\ P(x)$ is true if there is one or more assignments of a value to x such that P(x) is true

$\exists x\ P(x)$ is false if there no possible assignment of a value to x that makes P(x) true

Examples

R(x) is the statement $x \geq 0$ where U = Z

$\exists x\ R(x)$ is a true statement

R(5) is true, since $5 \geq 0$

P(x) is the statement $x^2 < 0$ where U = Z

$\exists x\ P(x)$ is a false statement

There is no integer whose square is less than 0

Examples

Consider students currently registered on the UCC database.
Represent the following using quantifiers and predicates:

1. All students have a student ID number.
2. Some students are American

Let U be the set of UCC students
Let *hasID(x)* state that x has a student ID number
Let *american(x)* state that x is american

1. $\forall x$ *hasID(x)*
2. $\exists x$ *american(x)*

Examples

Consider students currently registered on the UCC database, and degree programmes in the UCC Calendar
Represent the following using quantifiers and predicates:

1. There exists at least one student who studies both Computer Science and Chinese.

Let U be the set of UCC students and degree programmes

Let *student(x)* state that *x* is a student
Let *studies(x,y)* state that student x studies degree program y
Let CS and CH be elements of U, representing Computer Science and Chinese.

1. $\exists x\ (student(x) \land studies(x,CS) \land studies(x,CH))$

Example

Let P(x) be "x > 0" and let Q(x) be "2*x > 0", where U=Z

Then the wff

$\forall x\ (P(x) \rightarrow Q(x))$

says for all integers, if x > 0, then 2*x > 0.

This is a true statement. For any integer x, P(x) → Q(x) is true.

(using the truth table for →, we cannot find an integer which makes P(x) true and Q(x) false. i.e. we cannot find an integer x for which x > 0 but 2*x ≤ 0)

Example

For all students in UCC, if the student is qualified to enter 2nd year, then the student must have earned 50 credits.

Express this in logic.

Let U = all students in the UCC database

Let *credits(x)* mean *x* has achieved 50 credits, for $x \in U$
Let *qualified(x)* mean *x* is qualified to enter 2nd year, for $x \in U$

$$\forall x \ (qualified(x) \rightarrow credits(x))$$

Example

For all students in UCC CS 1^{st} year, either the student is registered for CS1105, or has a pass for CS1105, or has a pass for MA1015

Express this in logic.

Class Exercise

If U the set of all international soccer teams, and *european* is a unary predicate which is true if the team is european, what do the following say?

1) $\forall x\ (\textit{inWC2014}(x) \rightarrow \textit{european}(x))$

2) $\exists x\ (\textit{inWC2014}(x) \wedge \textit{sameGroup}(x, \text{Spain}))$

Are these statements true or false?

Two Equivalences

Let *P* be a predicate that acts on a single object from U

$$\neg\exists x\, P(x) \equiv \forall x\, \neg P(x)$$

"it is not true that there is an object x that has property P" is logically equivalent to "for every possible object x, the property P is not held by x

$$\neg\forall x\, P(x) \equiv \exists x\, \neg P(x)$$

"it is not true that all objects x have property P" is logically equivalent to "there is at least one object x which does not have property P

$$\neg\forall x\, \mathit{inWC2014}(x) \equiv \exists x\, \neg\mathit{inWC2014}(x)$$

Next lecture

the language of quantified logic (version 1)

logical equivalences

examples