department of Computer Science
University College Cork

CS1112
Relations on Single Sets

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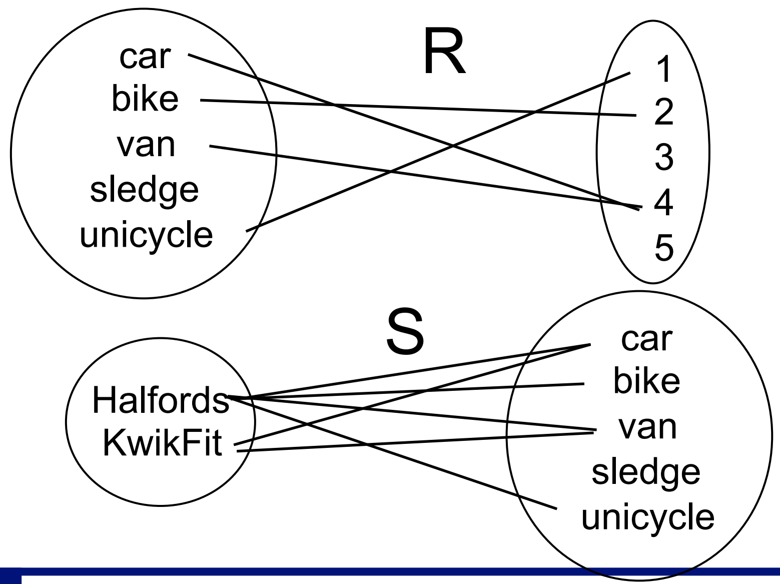
Previously …

- We have seen:
  - how to represent relations between two sets
  - various properties of relations
  - how to manipulate relations to get new ones
    - union, intersection, composition, inverse
  - examples using relations



R = {(car,4),(bike,2),(van,4),(unicycle,1)}
R^{-1} = {(4,car),(2,bike),(4,van),(1,unicycle)}
R∘S = {(halfords,4), (halfords,2), (halfords,1)
(kwikfit,4) }

Relations over a single set

Representing relations on a single set
Equivalence Relations
Closures

(Defn 3.9 – 3.21)

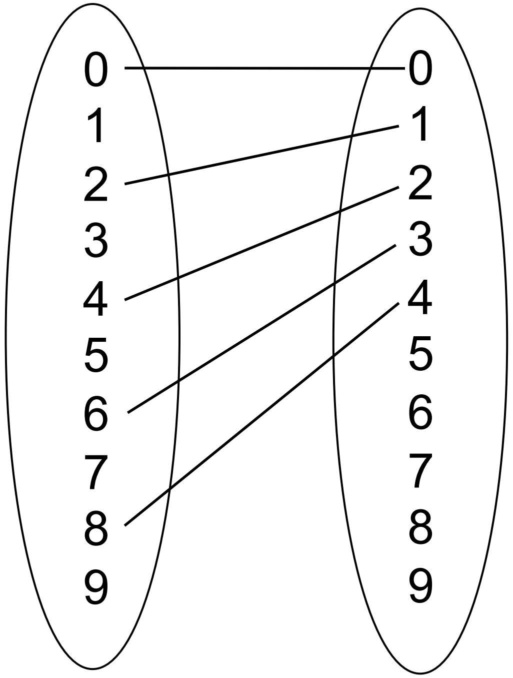
The two sets can be the same, or can be different:

	source		target
• Brian **is a friend of** Gordon on facebook	users	★	users
• Prof Kenny **is the mentor of** Mr Coveney in the DB	mentors		students
• Alice **is registered on** CS101 in the UCS database	students		modules
• Bob Geldof **is the father of** Peaches Geldof	people	★	people
• Susan **is registered for** an Economics degree	students		degrees
• CS1105 **is linked from** the UCC CS 1st portal	webpages	★	webpages
• the hall **is connected to** the corridor in a game	locations	★	locations
• Plastering **must be done before** painting	tasks	★	tasks
• CS1105 **is a pre-requisite for** CS2201	modules	★	modules
• Rod Flanders **wants to be a friend of** Bart Simpson	people	★	people
• Mr Coveney **is mentored by** Prof Kenny	students		mentors
• Painting **must be done after** plastering	tasks	★	tasks

Now consider relations with the two sets the same

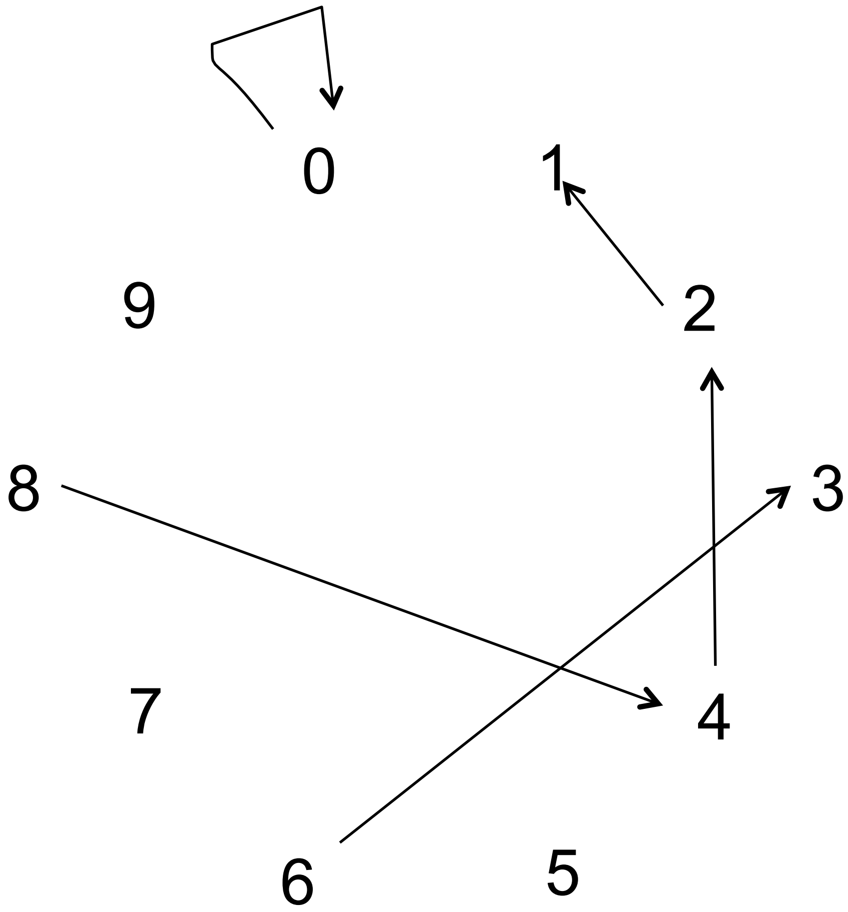
i.e. $R \subseteq A \times A$

Relations of this type are called homogeneous



$R = \{(0,0),(2,1),(4,2),(6,3),(8,4)\}$
$source(R) = \{0,1,2,3,4,5,6,7,8,9\}$
$target(R) = \{0,1,2,3,4,5,6,7,8,9\}$

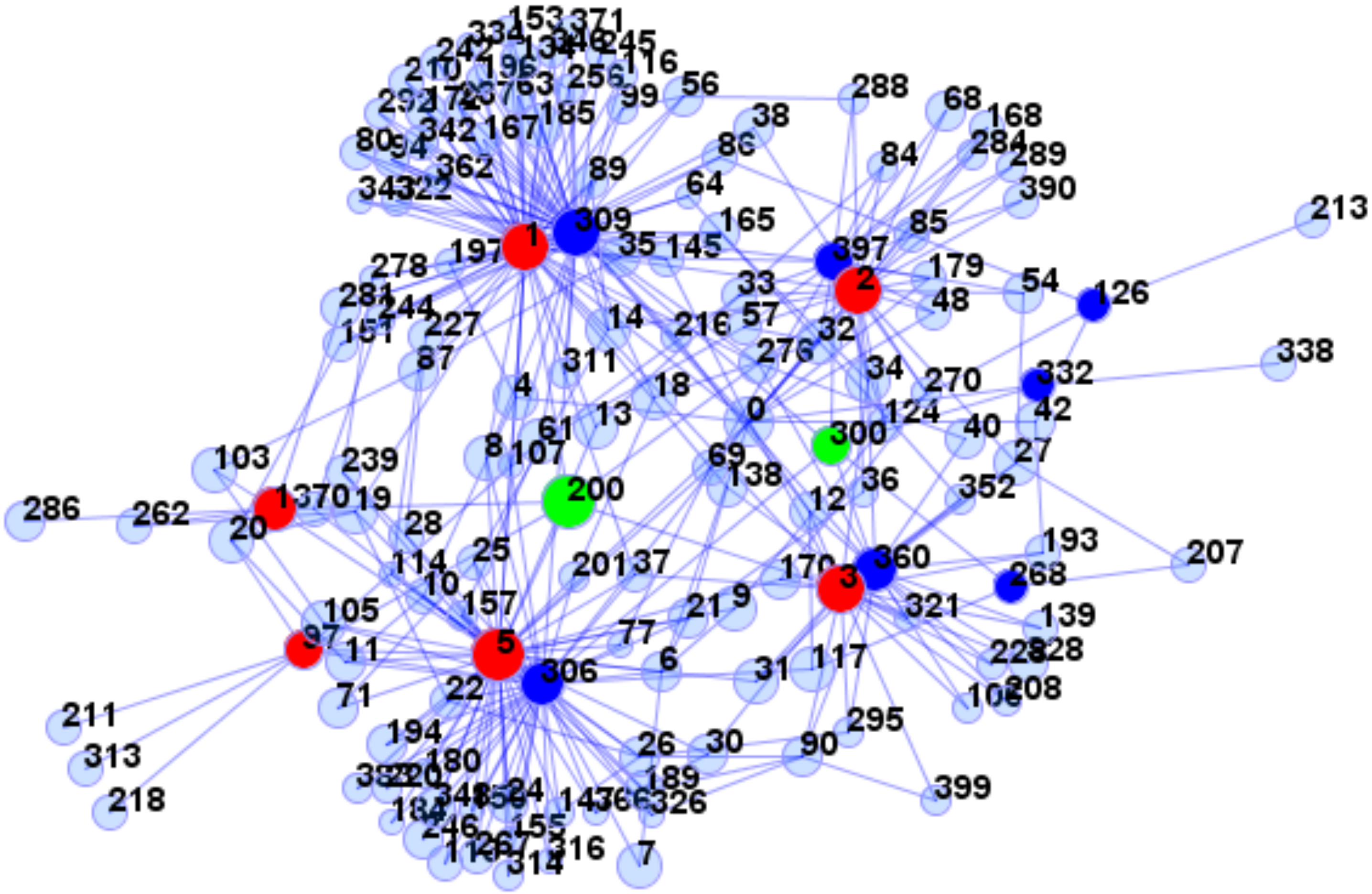
Normally we sketch homogeneous relations in a new style:



An arrow directed from a to b means aRb

(i.e. $(a,b) \in R$, or $R(a,b)$ is true)

(This is a directed graph – we will look at graphs after Christmas in CS1113)



153
371
245
116
56
99
288
68
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390
38
84
89
64
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309
1
197
35
145
397
2
179
54
126
278
33
48
57
32
14
227
87
311
276
34
270
332
338
4
18
124
42
40
0
13
61
300
27
103
8
107
239
69
138
352
36
12
286
262
19
200
28
193
207
360
20
25
114
201
37
170
3
268
10
157
321
139
21
9
105
77
117
11
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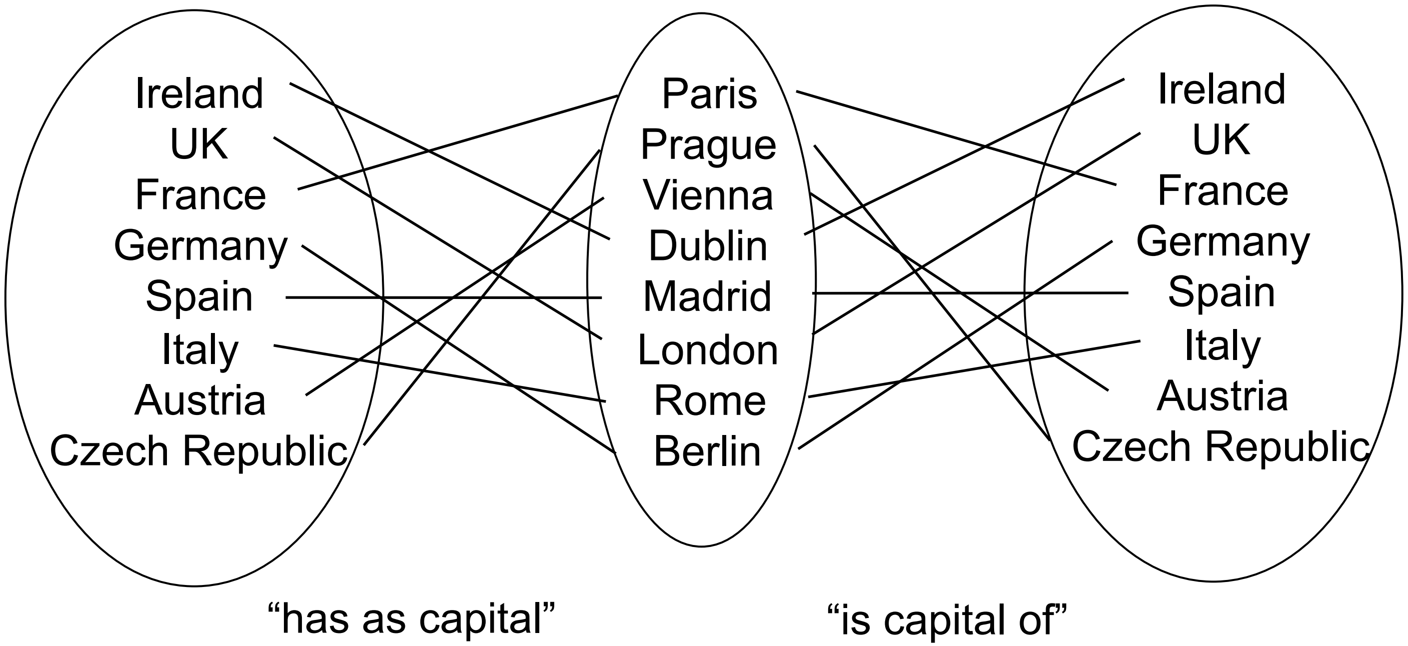
31
208
22
71
306
295
211
194
26
30
90
313
180
189
399
326
218
24
155
316
314
7

The identity relation, I, on any set A is the relation where every element is related to itself, and only to itself.

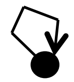
$I \subseteq A \times A$ where $I = \{(a,a) \mid a \in A\}$

Note: for any bijection, $R{:}A \rightarrow B$,

$R^{-1} \circ R = I$



A homogeneous relation $R \subseteq AxA$ is reflexive if and only if every element of A is related to itself

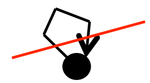


must have a self-arrow for every object

For all $a \in A$, $(a,a) \in R$

Example: "≤" is reflexive, since $x \leq x$, for any number x

A homogeneous relation $S \subseteq AxA$ is anti-reflexive if and only if no element of A is related to itself



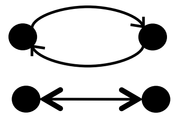
no self-arrow for any object

For all $a \in A$, $(a,a) \notin S$

Example: "<" is anti-reflexive, since $x \nless x$, for any number x

A homogeneous relation $R \subseteq AxA$ is symmetric if and only if whenever a is related to b in R, then b is also related to a

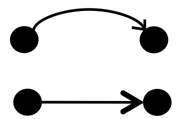
For all $a,b \in A$, $(a,b) \in R$ if and only if $(b,a) \in R$



all arrows must be 2-way

Example: "is married to" is a symmetric relation

A homogeneous relation $S \subseteq AxA$ is anti-symmetric if and only if for any *<u>two different elements</u>* $a \in A$ and $b \in A$ s.t. $(a,b) \in S$ then $(b,a) \notin S$.

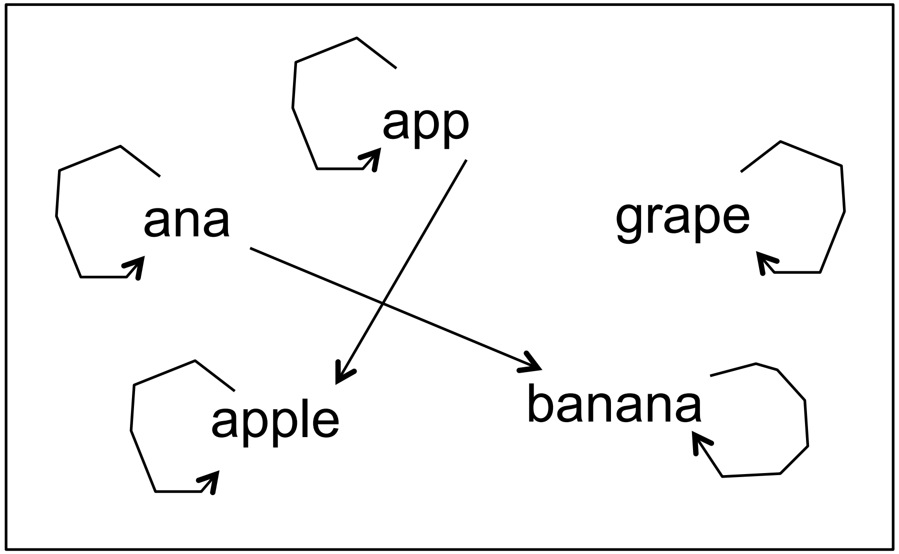


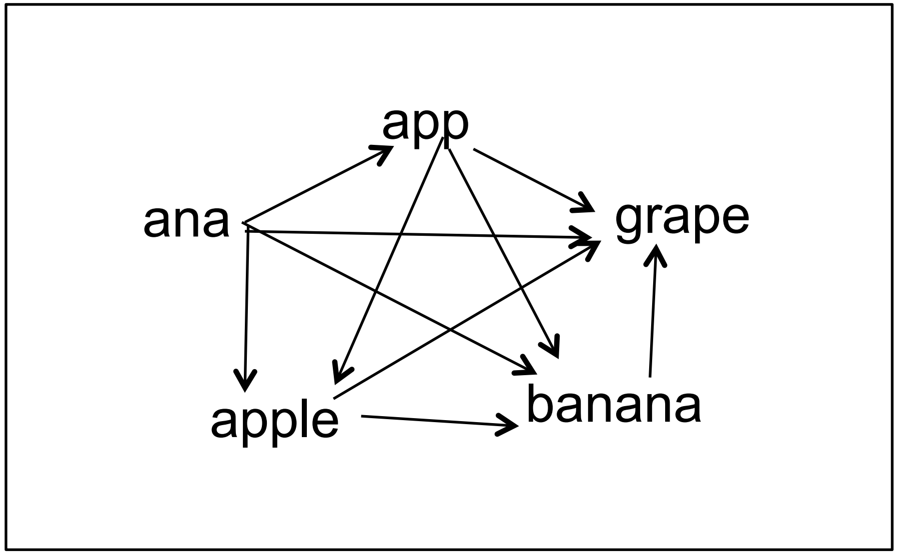
no arrows are 2-way

Note: if S is anti-symmetric, then for any $a \in A$, if $(a,b) \in S$ and $(b,a) \in S$, then $b=a$.

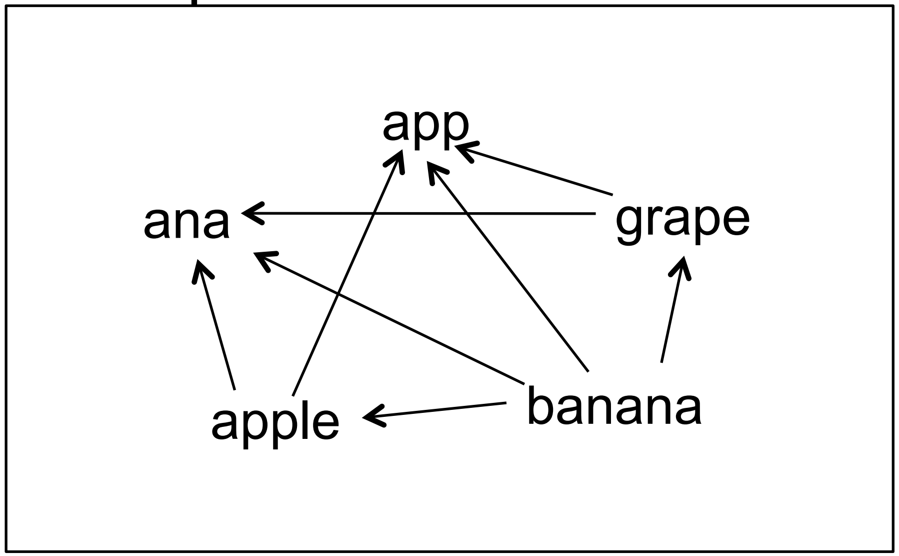
Example: "is a parent of" is an anti-symmetric relation

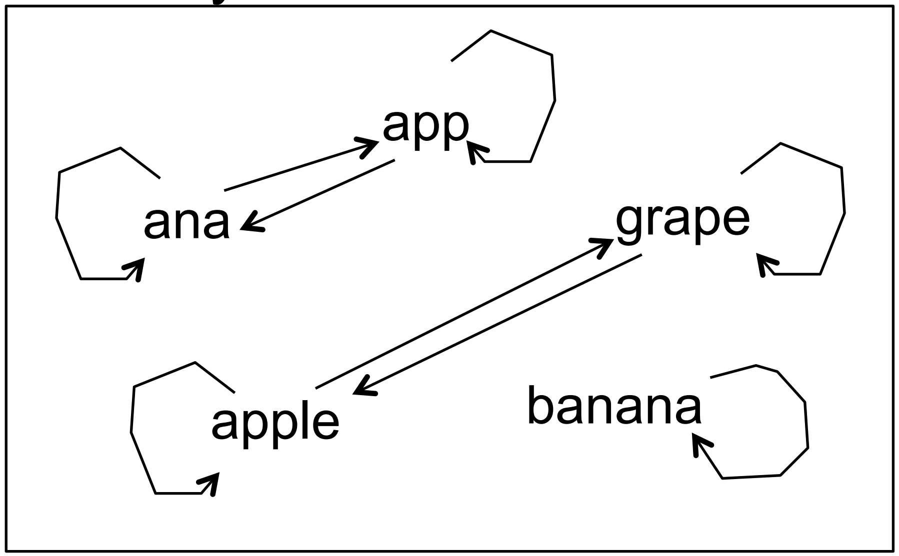
Example: which of these relations are reflexive?





Example: which of these relations are symmetric?

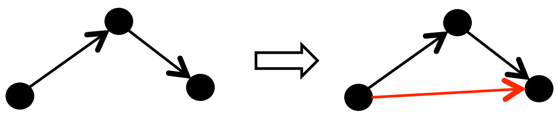




A homogeneous relation $R \subseteq AxA$ is transitive if and only if whenever a is related to b, and b is related to c in R, then a is also related to c

For all $a,b,c \in A$,

$$(a,b) \in R \text{ and } (b,c) \in R \text{ implies } (a,c) \in R$$



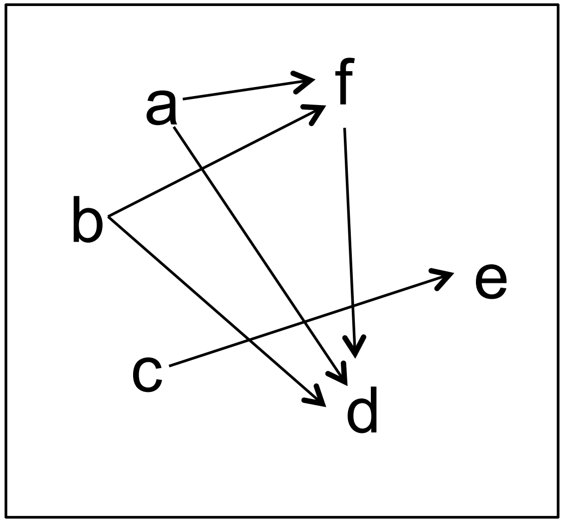
any chain of arrows must also have a single arrow connecting start to end

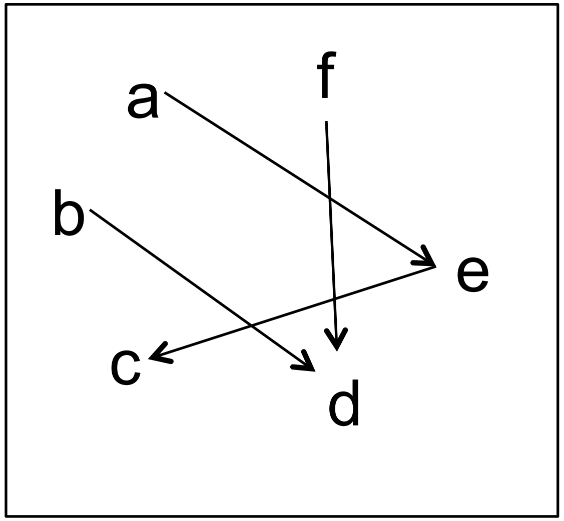
Note: if there is no chain, then the relation is transitive by default

Example: "is registered for the same degree as" is a transitive relation:

If Bob is registered for the same degree as Carol,
and Carol is registered for the same degree as Ted,
then Bob must be registered for the same degree as Ted.

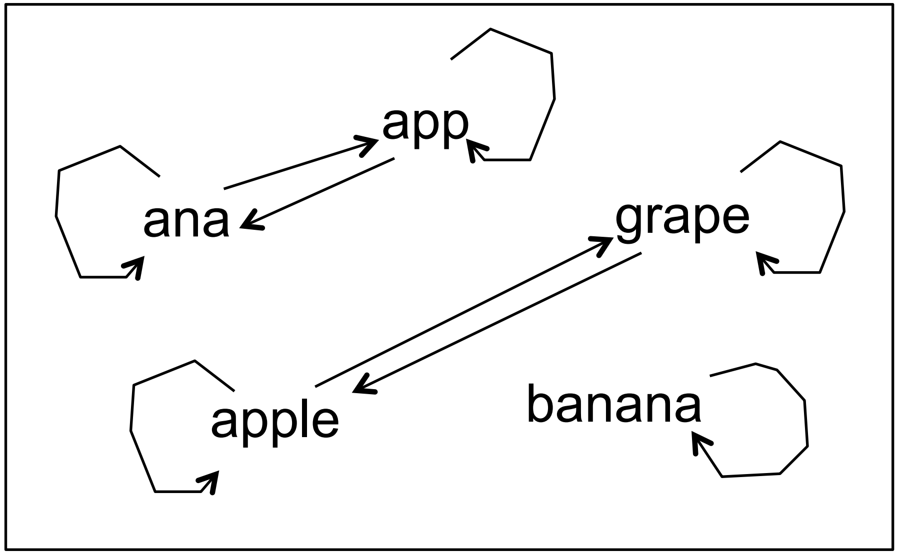
Example: are either of these relations transitive?





A homogeneous relation $R \subseteq AxA$ is an equivalence relation if and only if

(i) R is reflexive
(ii) R is symmetric
(iii) R is transitive



The concept of an equivalence relation is based on the relation "=" on numbers. If you want to define what it means for two things to be “equal”, you must satisfy all the conditions of an equivalence relation.

E.g. when we introduced sets, we defined what it meant for two sets to be "equal" – they have exactly the same members.

reflexive: for any set S, $S = S$

symmetric: for any sets S and T, if $S = T$, then S has exactly the same elements as T, so T has exactly the same elements as S, and so $T = S$.

transitive: for any sets S, T and V, if $S = T$ and $T = V$, then S has exactly the same elements as T, and T has exactly the same elements as V, so S must have exactly the same elements as V, and so $S = V$

Example – suppose we define nationality as meaning the country in which you were born.

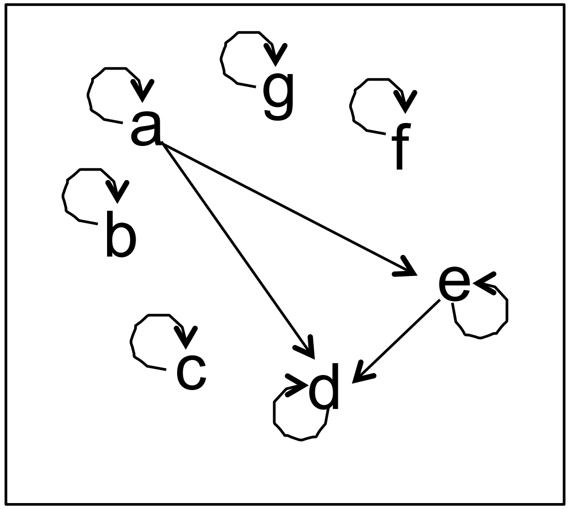
Then "*is the same nationality as*" is an equivalence relation:

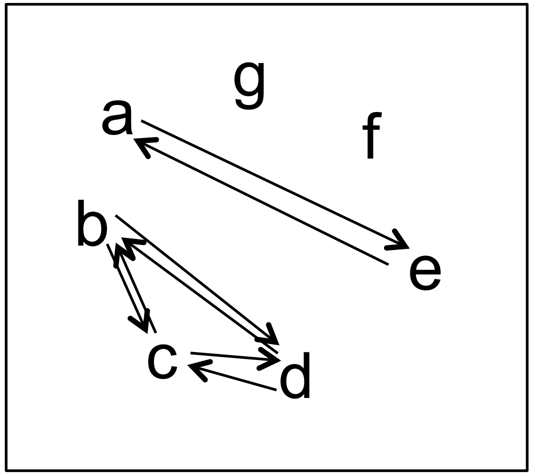
Reflexive: Any person is the same nationality as themselves

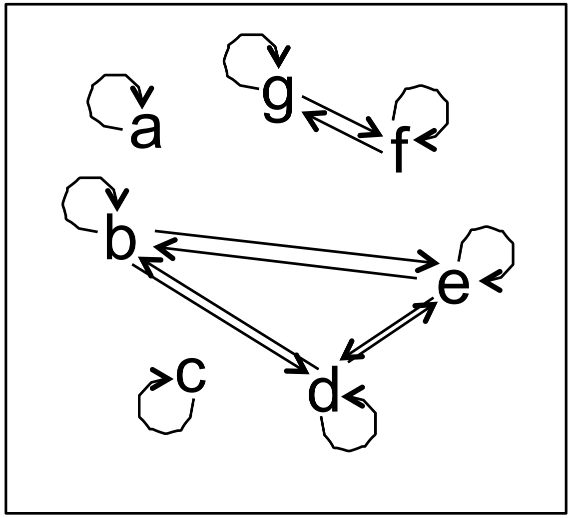
Symmetric: If Bob is the same nationality as Carol, then
Carol must be the same nationality as Bob

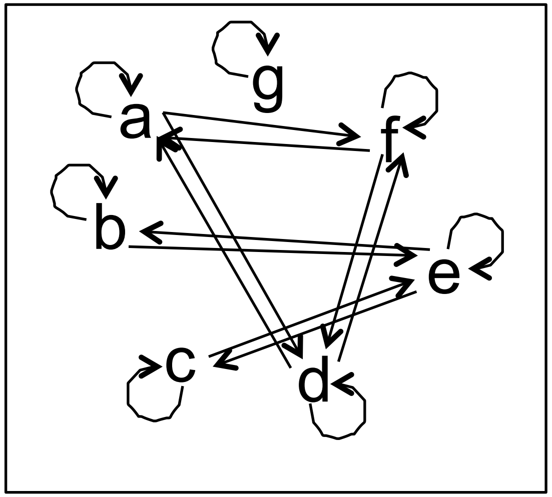
Transitive: If Bob is the same nationality as Carol, and
Carol is the same nationality as Ted, then
Bob must be the same nationality as Ted

Example: which of the following are equivalence relations?









Given an equivalence relation $R \subseteq AxA$, we can use R to create a partition of A.

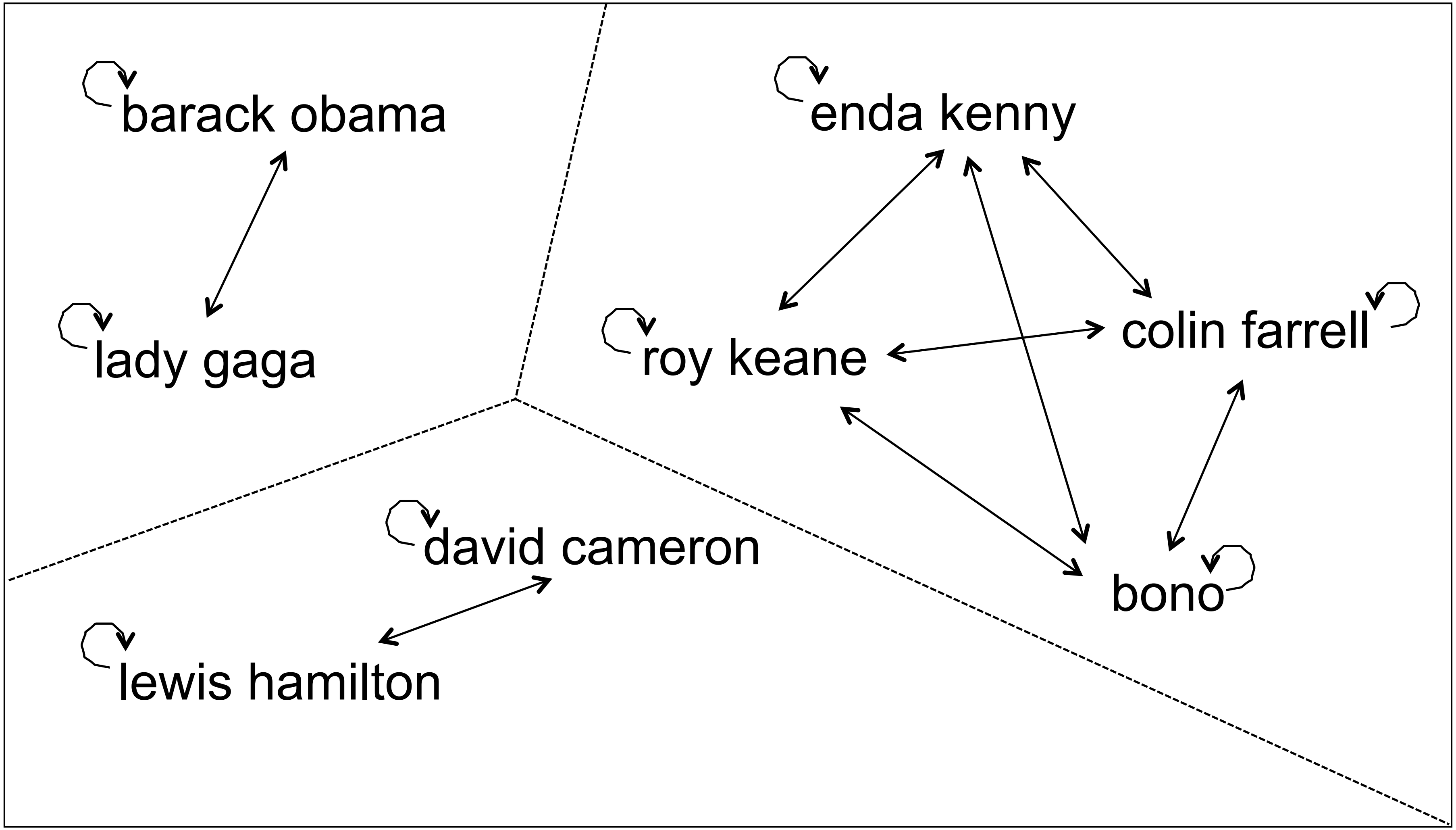
If $(a,b) \in R$, then a and b are both in the same partition subset.
If $(a,b) \notin R$, then a and b are in different partition subsets

When we are talking about equivalence relations, then a partition subset is called an equivalence class.

We define the R-equivalence class for an element a of A as being the set of all elements that a is related to under R, and we denote it as $E(R,a)$.

$$E(R,a) = \{ b \mid b \in A \text{ and } (a,b) \in R\}$$

"Is the same nationality as"



Sometimes, we will want to start with a simple relation, to save space, and then describe how to make it larger

The reflexive closure of a relation $R \subseteq AxA$ is $R \cup I$

i.e. add into R all pairs (x,x)

Example: $A = \{1,2,3,4\}$
$R = \{(1,2), (2,4)\}$
reflexive closure of $R = \{(1,1),(2,2),(3,3),(4,4),(1,2),(2,4)\}$

The symmetric closure of a relation $R \subseteq AxA$ is $R \cup R^{-1}$

i.e. if (x,y) is in R, add in (y,x)

Example: $A = \{1,2,3,4\}$
$R = \{(1,2), (2,4)\}$
symmetric closure of $R = \{ (1,2),(2,4), (2,1),(4,2)\}$

Notation: if R is a relation $R \subseteq AxA$, we will say $R^{(2)} = R^o R$, $R^{(3)} = R^o R^{(2)}$, etc and so $R^{(n)} = R^o R^{(n-1)}$.

Let A be a set s.t. $|A| = n$, and let R be a relation $R \subseteq AxA$
The transitive closure of R is $R \cup R^{(2)} \cup \ldots \cup R^{(n-1)}$

Example: $A = \{1,2,3,4\}$
$R = \{(1,2), (2,4)\}$
transitive closure of $R = \{ (1,2), (2,4), (1,4)\}$

$R = \{(1,2),(2,4)\}$
$R^{(2)} = R^o R = \{(1,4)\}$
$R^{(3)} = R^o R^{(2)} = \{ \}$

i.e. add in to R every pair (x,y) needed to make R transitive

transitive closure of $R = R \cup R^{(2)} \cup R^{(3)} = \{(1,2),(2,4),(1,4)\}$

The **transitive closure** is important, and the idea is useful throughout computer science and computer applications.

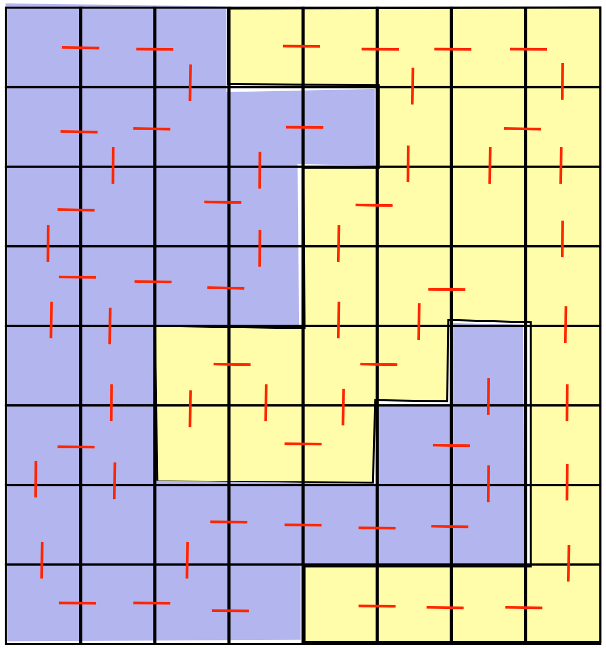
Example: let person *a* be connected to person *b* if *b* appears in the email address book of *a*.

Suppose we now have a virus, which emails itself to everybody in your address book.

The transitive closure of "connected to" is a relation that specifies who any individual may ultimately infect –
i.e. person *a* will be related to everyone in *a*'s address book, and everyone in their address books, and so on.

Example: consider the problem of finding a path for a robot from one area of a factory to another.

Or analysing secure zones in an airport – which areas are reachable from which other areas, without going through security control?

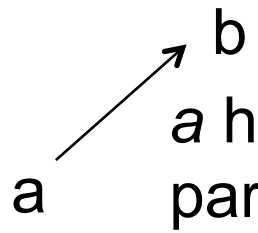


The red lines indicate doors that open between zones.

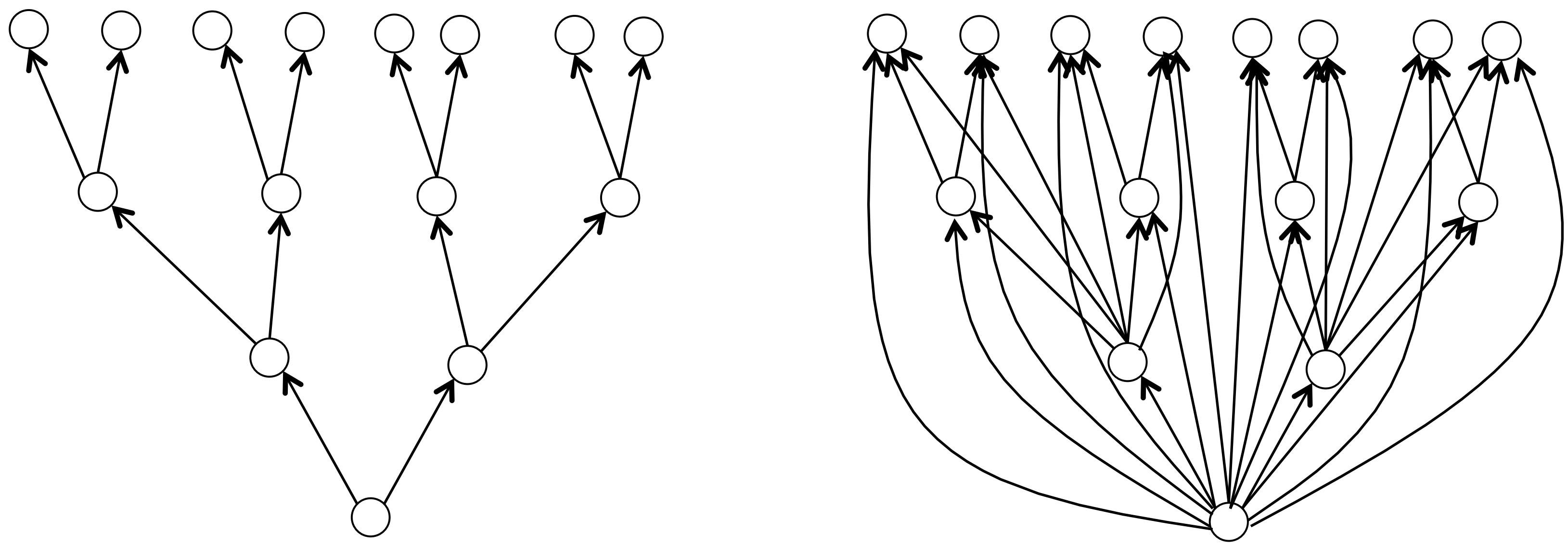
The transitive closure is the set of rooms that can be reached from each starting point.

In fact, the "link" relation and its transitive closure define a partition and equivalence relation

Example: family trees

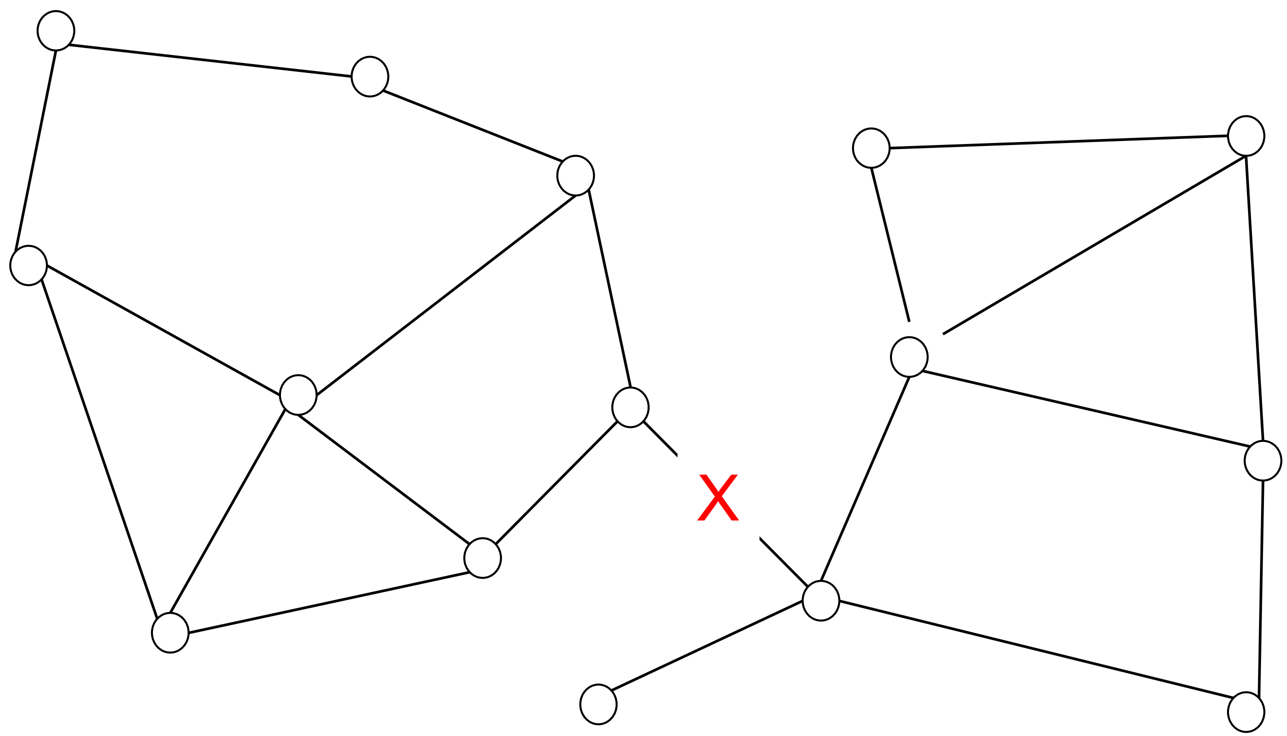


a has a parent *b*



"has an ancestor" is the transitive closure of "has a parent"

Example: network survivability analysis



"is directly linked to" (a symmetric relation)

transitive closure: "has a path to"

How has the transitive closure changed? What does it mean?

Next lecture ...

Order relations

(Defn 3.22 – 3.24)