CS1112
More about Functions

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More Functions

properties of functions

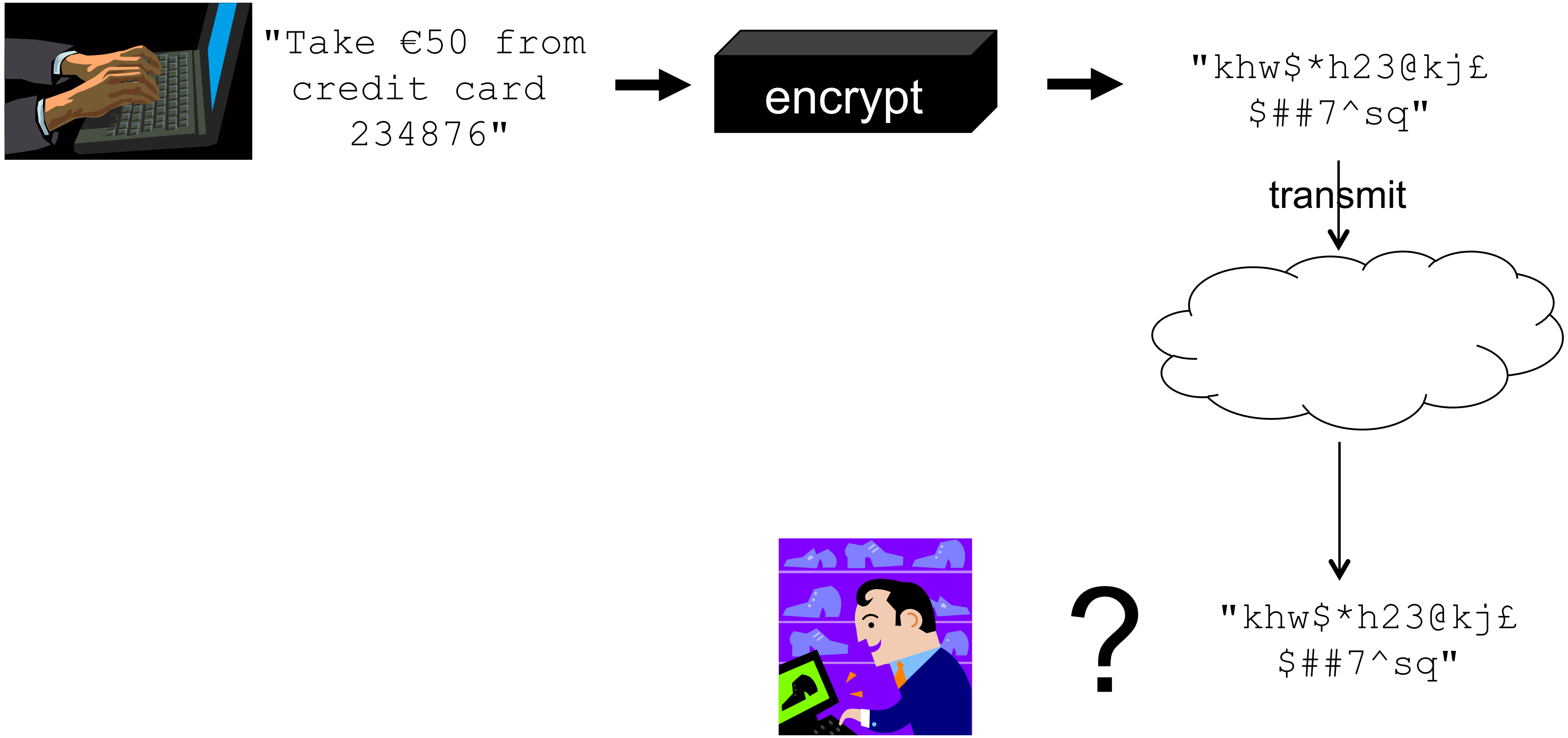
combining two functions

a new definition of set cardinality

(Defn 2.6 $\rightarrow$ Note 2.16)

from last lecture:

- given the text of an email, a function "encrypt" returns an encrypted version of the text

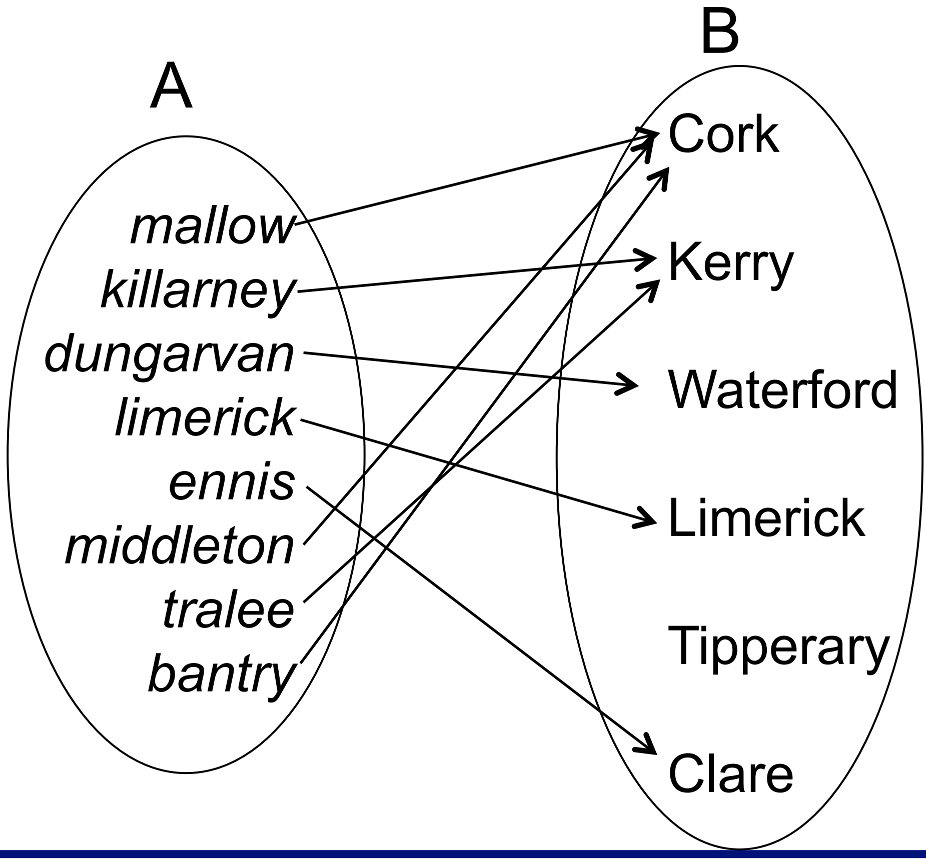


Reminder

Let *A* and *B* be two non-empty sets. A function *f* from *A* to *B* specifies for each element of *A* exactly one element of *B*.
We write $f : A \rightarrow B$, and if *f* specifies *b* for *a*, we write $f(a)=b$, or $f : A \rightarrow B : a \mapsto b.$

NOTE: one input from A must give one output from B

If $f : A \rightarrow B$,
A is the domain,
B is the codomain.



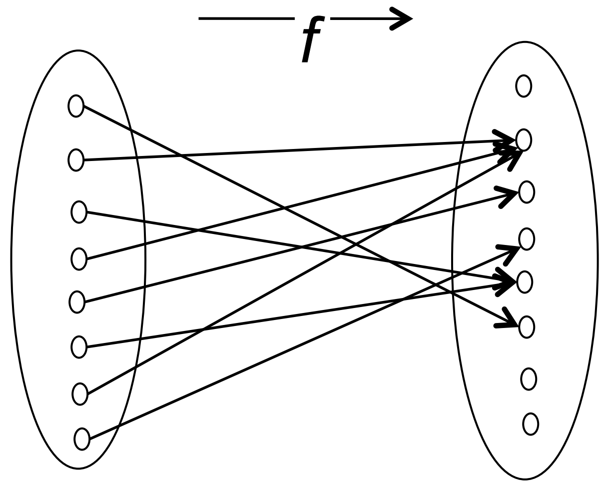
Properties of functions

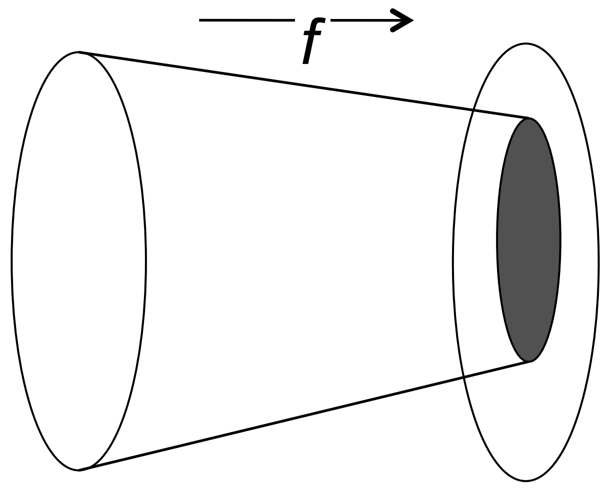
Consider a function f:$A \rightarrow B$

If $f(a) = b$, then b is the image of a under f.

The range of f is the subset of B to which elements of A are mapped.

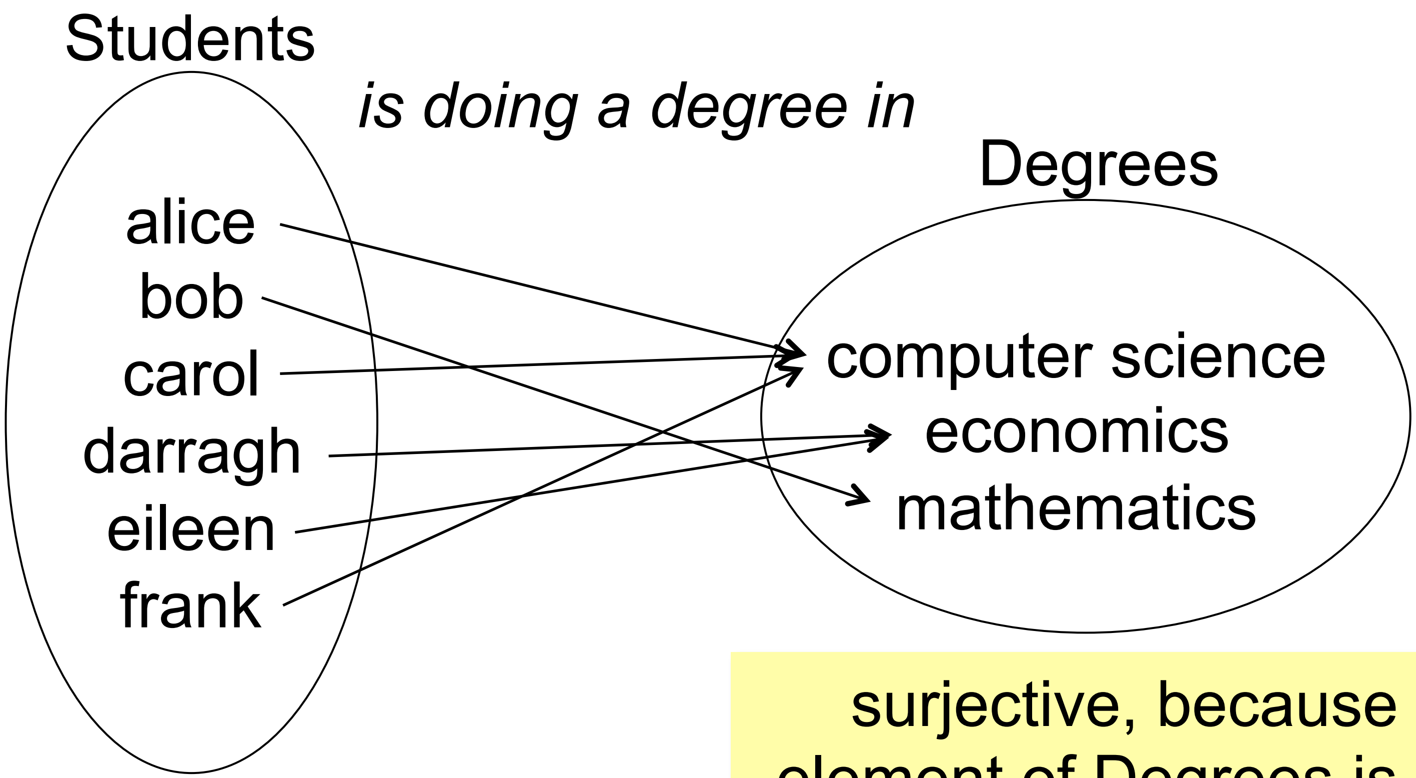
range(f) = {$b \in$ B | there is an $a \in$ A s.t. $f(a) = b$}





A function f:A->B is surjective, or onto, if range(f) = B

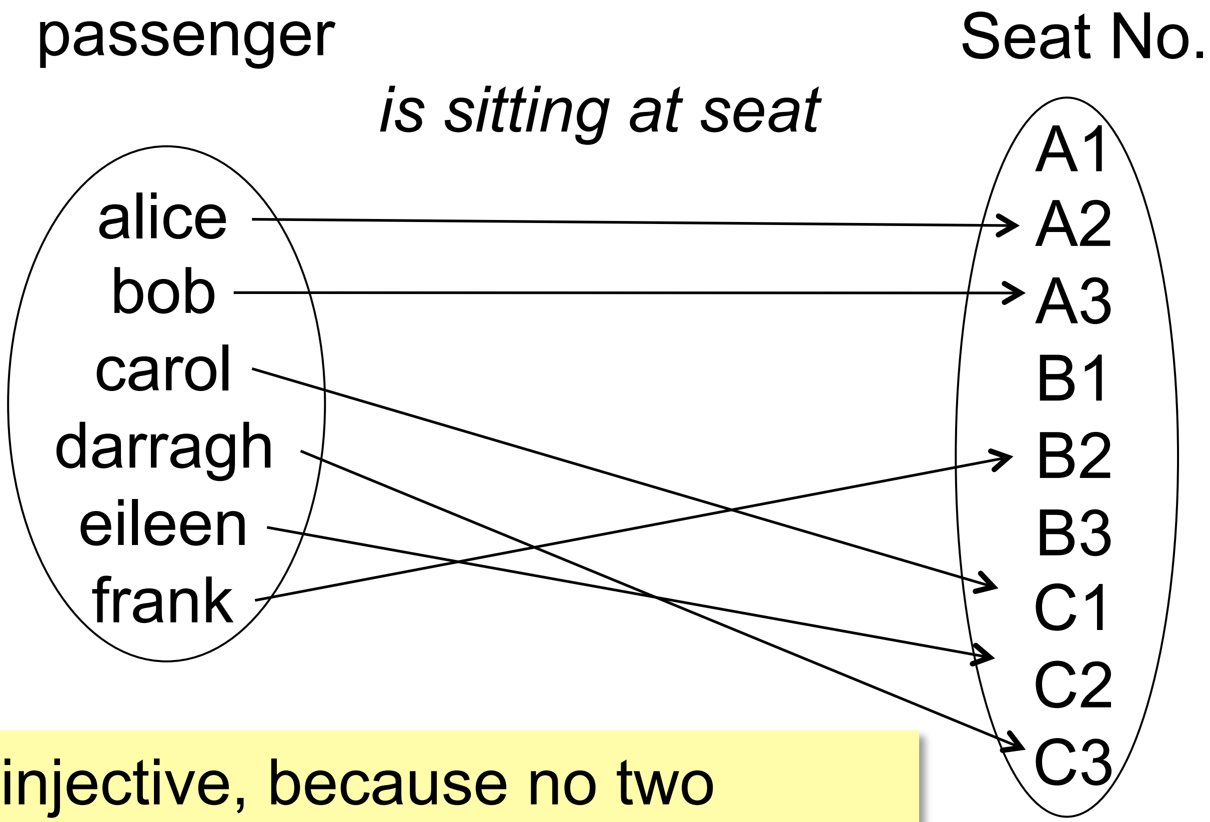
that is, for every element $b \in$ B, there is an element $a \in$ A such that $f(a) = b$



surjective, because every element of Degrees is mapped onto by something in Students

A function *f*:A->B is injective if and only if no two elements of A map onto the same element of B.

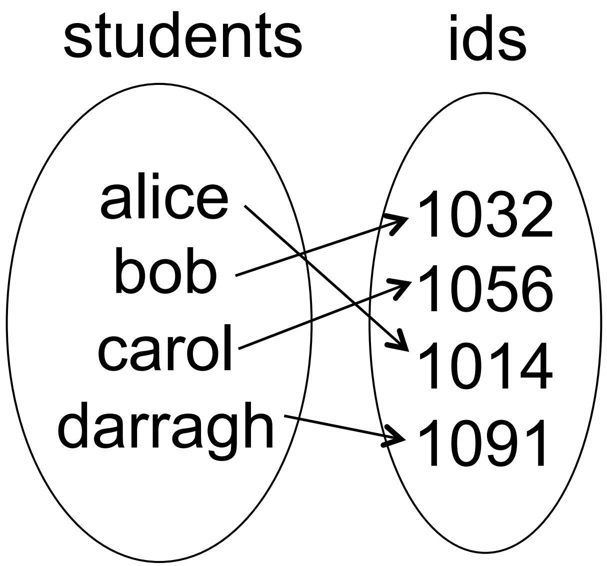
That is, for all elements a_1 and $a_2 \in A$ s.t. $a_1 \neq a_2$, $f(a_1) \neq f(a_2)$



injective, because no two passengers have the same seat no.

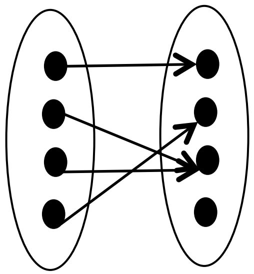
Injective: different input must give different output

A function *f*:A->B is bijective, if and only if it is both surjective and injective



It is surjective, because each ID is owned by a student
It is injective, because no students have the same ID

Exercise: classify these as injective function, surjective function, bijective function, plain function or not a function:



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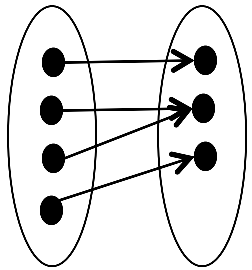

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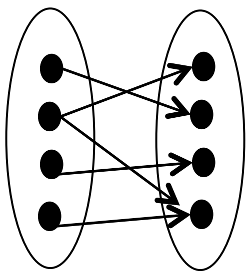

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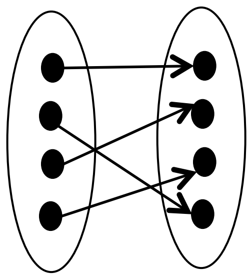

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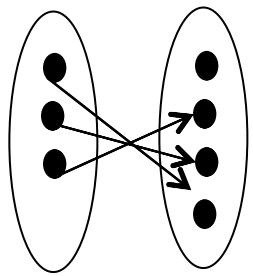

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Functions have a direction

Let $f : A \rightarrow B$ be a function, $a \in A$, $b \in B$ and $f(a)=b$. What is $f(b)$?

$f(b)$ **may not be defined**. f is a function that takes elements of A as input, and produces elements of B as output.

We are only allowed to apply f to elements of A. If we write $f(\ldots)$, then the thing we put in the brackets must be a member of A. We have not been told $b \in A$.

Even if $b \in A$, we **cannot** say that $f(b) = a$, unless we are given more information. The function has a direction.

Example: $f : EvenIntegers \rightarrow \mathrm{Z} : x \mapsto x/2$

$f(12) = ?$ $f(6) = ?$ $f(3) = ?$

If f is a function that maps some object a onto $f(a)$, then an inverse function for f transforms $f(a)$ back to a. We write the inverse function for f as f^{-1}.

<u>Example</u>

$f : EvenIntegers \rightarrow Z : x \mapsto x/2$

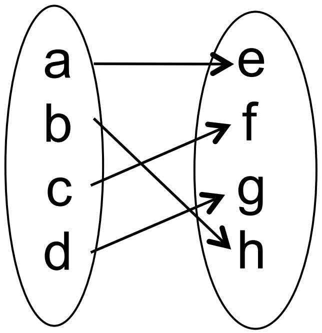
The inverse function for f is $f^{-1} : Z \rightarrow EvenIntegers : y \mapsto 2y$

if $f : A \rightarrow B : a \mapsto f(a)$, and f^{-1} is the inverse of f, then $f^{-1} : B \rightarrow A : f(a) \mapsto a$

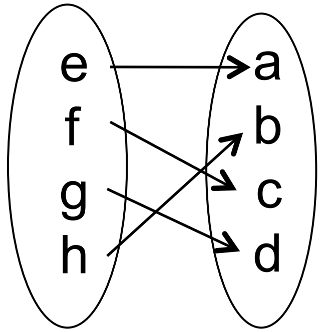
Note: if f is the inverse of g, then g must be the inverse of f.

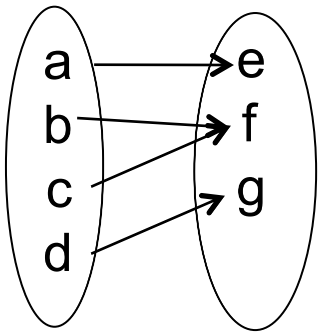
A function f has an inverse function f^{-1} if and only if f is a bijection

If $f{:}A \rightarrow B$ is a bijection, then $f^{-1}{:}B \rightarrow A{:}\ f(x) \mapsto x$

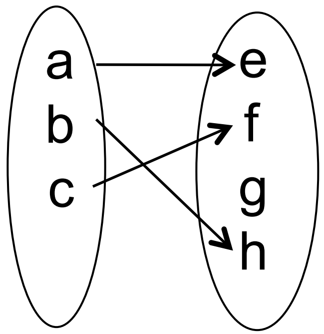


has the inverse function





has no inverse



has no inverse

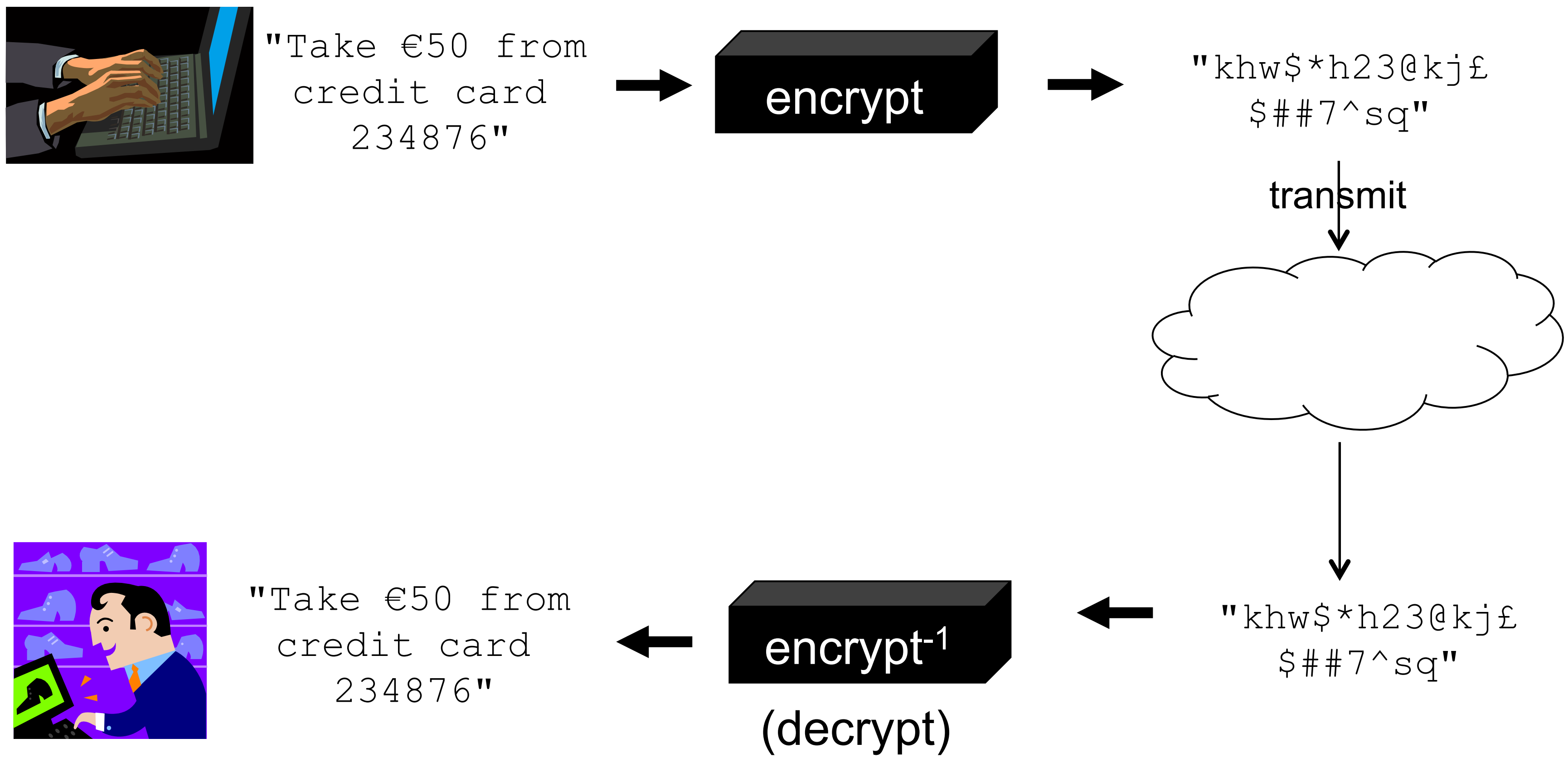
It is important that we know whether or not a function has an inverse.

- If a function has an inverse, we can start with some data, transform it using the function, and then later transform it back again to get the original data.

- If the function does not have an inverse, we cannot transform it back again, and we have lost the original data.

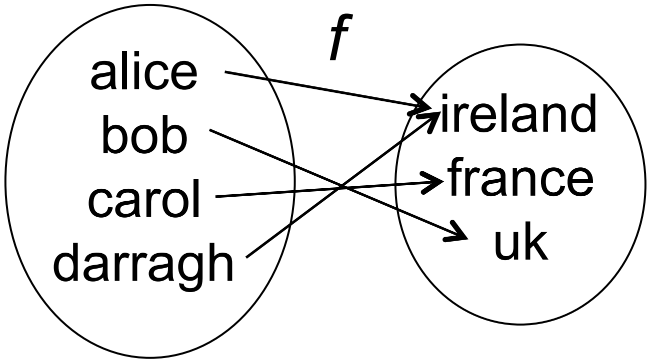
from last lecture:

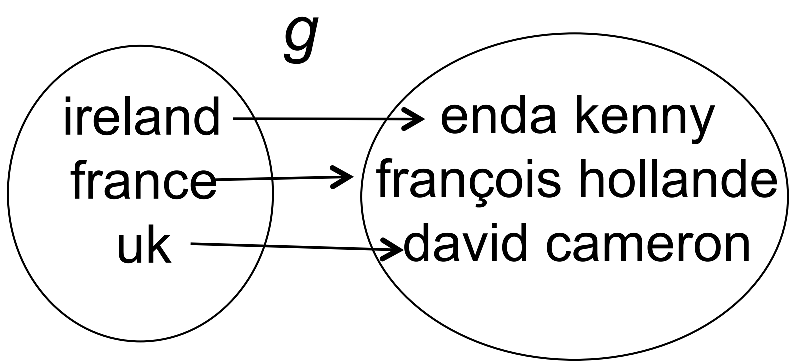
- given the text of an email, a function "encrypt" returns an encrypted version of the text

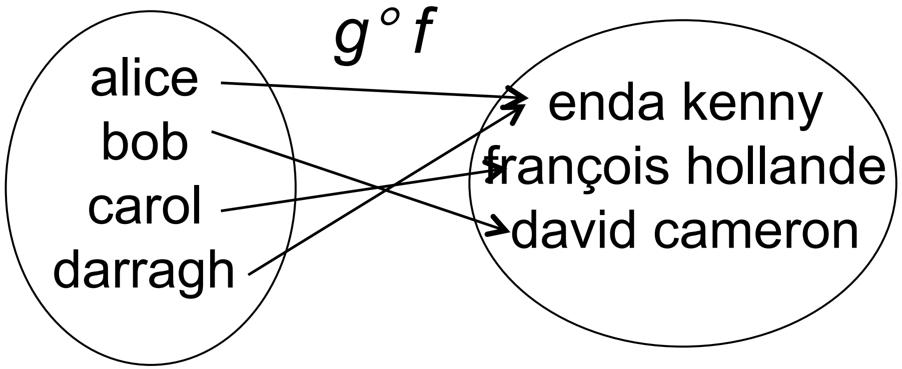


Composing functions

If *f* is a function $f{:}A \rightarrow B$, and *g* is a function $g{:}B \rightarrow C$
then we can compose the two functions to get a new function from *A* to *C*. We write this new function as $g^\circ f$ [g *after* f]
$g^\circ f$: $A \rightarrow C$: $g^\circ f(a) = g(f(a))$







Can we be sure that $g \circ f$ is actually a function from A to C?

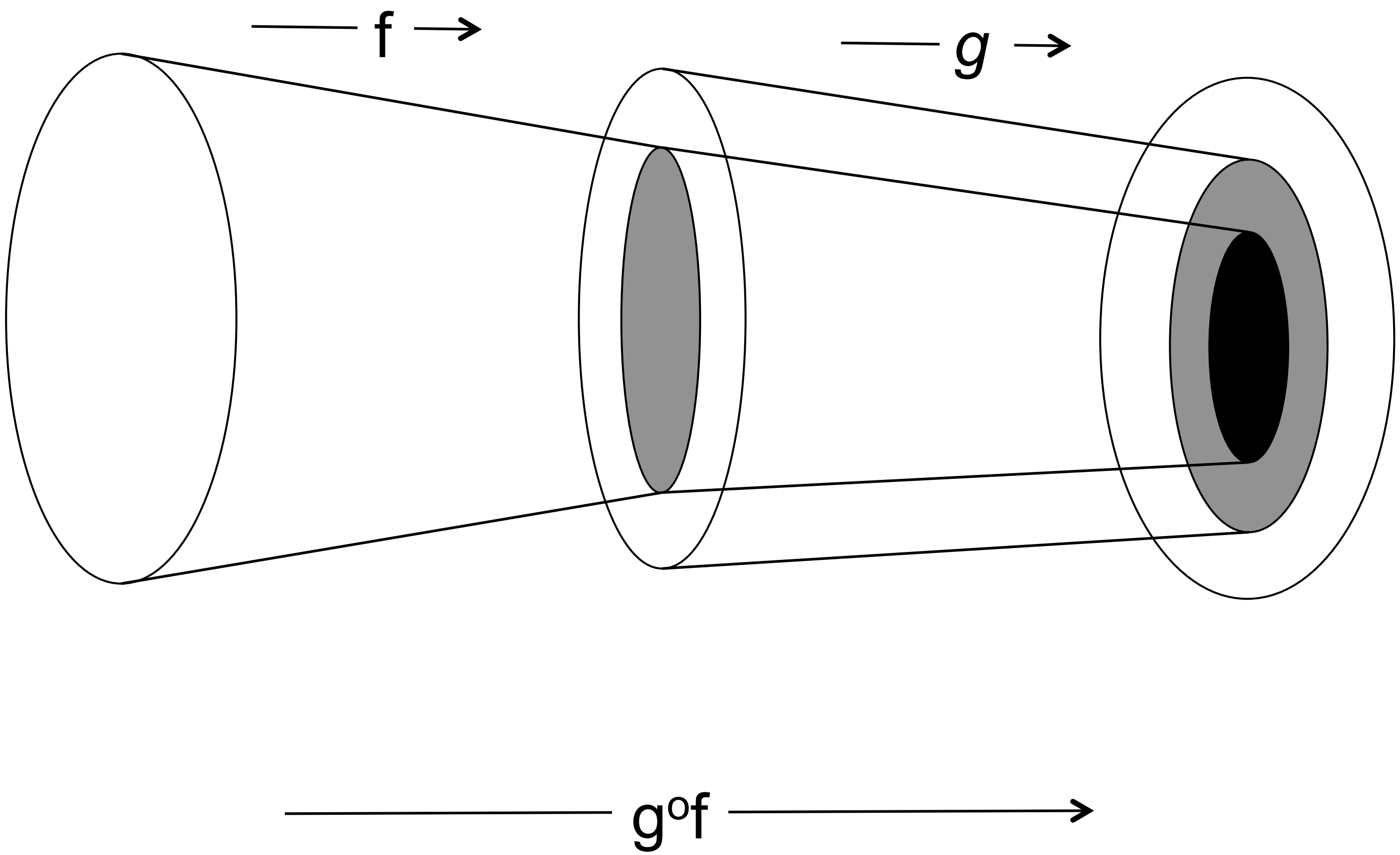
It is a function if it specifies, for each (input) element of A, a single (image or output) element in C.

Proof that it is a function: Let a be an arbitrary element of A.

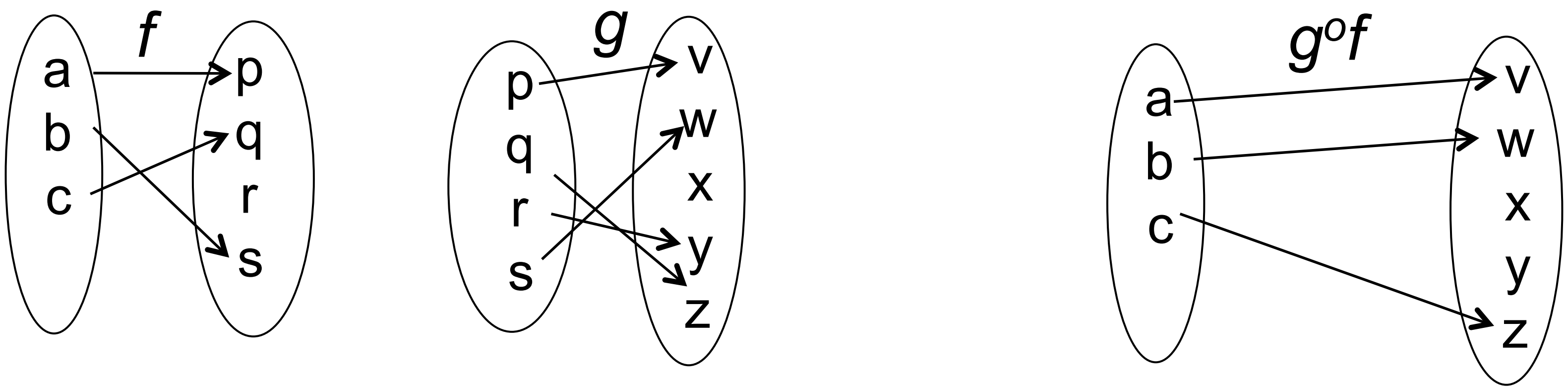
(i) f is a function from A to B, so a has at least one image $f(a)$ in B. g is a function from B to C so $f(a)$ has at least one image $g(f(a))$ in C. So a has at least one image $g(f(a))$ in C.

(ii) Since f is a function, there can be only one element of B that a is mapped to, which is $f(a)$. Since g is a function, there can be only one element of C that $f(a)$ is mapped to, which is $g(f(a))$. So a has only one element that $g \circ f$ maps it to, which is $g(f(a))$.

$$\mathrm{range}(g^o f) \subseteq \mathrm{range}(g)$$

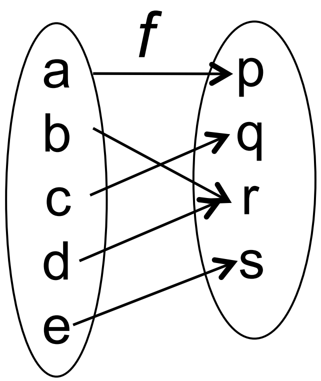


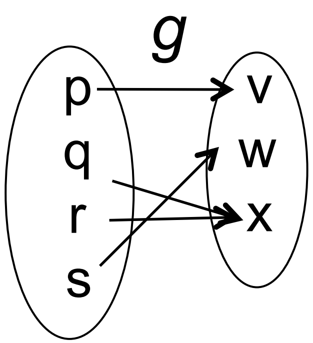
if f and g are injective, so is $g^o f$

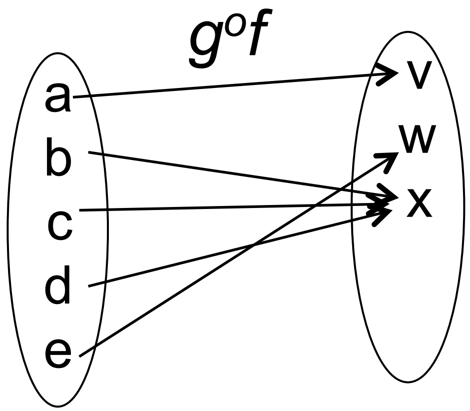


To show this in general, let $f{:}A \rightarrow B$ and $g{:}B \rightarrow C$
Consider any two elements of A, say a_1 and a_2, s.t. $a_1 \neq a_2$.
f is injective, so $f(a_1) \neq f(a_2)$. But g is also injective, so $g(f(a_1)) \neq g(f(a_2))$. Therefore $g^o f(a_1) \neq g^o f(a_2)$. But a_1 and a_2 were arbitrary elements of A. So $g^o f$ must be injective.

if f and g are surjective, so is $g^o f$

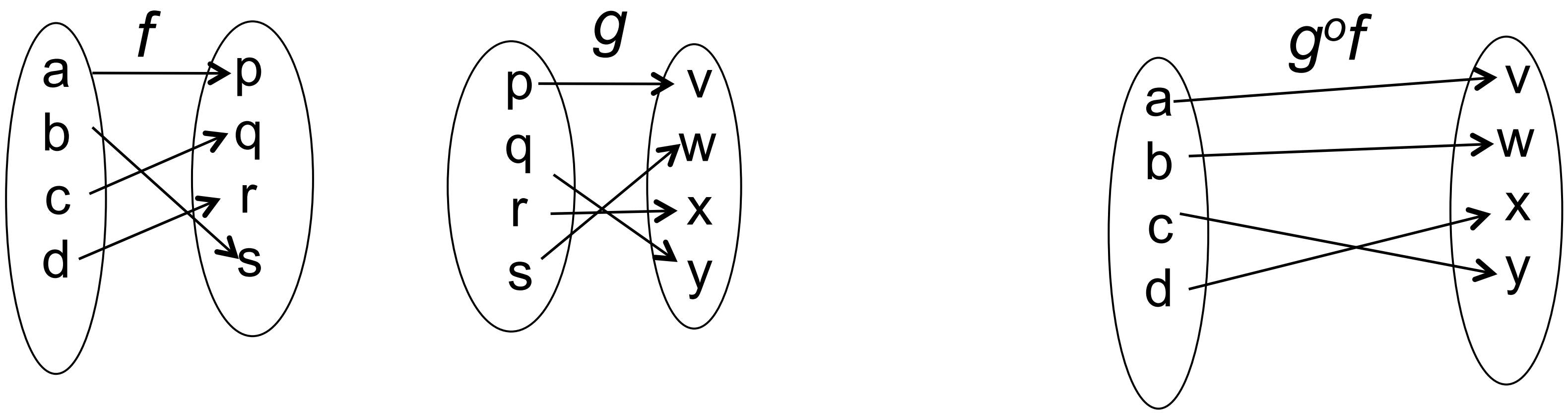






To show this in general, let f:$A \rightarrow B$ and g:B→C
Consider any element of C, say c.
g is surjective, so there is some b in B s.t. $g(b) = c$. But f is also surjective, so there is some a in A s.t. $f(a) = b$.
Therefore, $g(f(a)) = c$.
But c was an arbitrary element of C. So $g^o f$ must be surjective.

if f and g are bijective, so is $g^o f$



Exercise: demonstrate this is always true

If f and g are bijective, then $g^o f$ has an inverse. $(g \circ f)^{-1} = f^{-1} {}^o g^{-1}$

If $f{:}A \rightarrow B$ and $g{:}B \rightarrow C$

$(g^o f)^{-1}{:}C \rightarrow A{:}\ c \mapsto f^{-1}(g^{-1}(c))$

Why do we care about properties of composed functions?

If we have a number of bijective functions, then we know we can apply them one after another to our data to get some transformed output, but we can then apply the inverse of each function in reverse order to get back the original data.

In file compression, we are often interested in "lossless" algorithms – this means they do not lose any detail of the original file, and when we uncompress, we get the file back.

In multimedia modules, you will see the difference between "lossy" compression and "lossless" compression

Functions between more than 2 sets

The Cartesian Product of *n* sets is still just a set.

So we can define a function

$$f : A_1 \times A_2 \times \ldots \times A_n \rightarrow B_1 \times B_2 \times \ldots \times B_m$$

Each element of the domain will be an ordered *n*-tuple from $A_1 \times A_2 \times \ldots \times A_n$ and each element of the image will be an ordered *m*-tuple from $B_1 \times B_2 \times \ldots \times B_m$

So we might have $f(a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_m)$

Compare this to functional dependency in relational databases.

Set cardinality

- Back when we discussed sets, we said the cardinality of a set is the number of elements it has. But that only applied to finite sets.

- We now extend the definition.

Two sets A and B have the same cardinality if and only if it is possible to create a bijection from A to B.

We can take an obvious bijection for finite sets – simply list the elements in order, and then map the first element of A to the first element of B, the second element of A to the second element of B, and so on.