Math 300 - Final Exam Study Guide

Reece McMillin

May 12, 2021

Contents

Cramer's Rule

$$\mathbf{A}_i(b) = \left[a_1, \ldots, b, \ldots, a_2\right]$$

Let $\mathbf{A}$ be an invertible $n \times n$ matrix. For any $b \in \mathbb{R}^n$, the unique solution $\vec{x}$ of $\mathbf{A}\vec{x} = b$ has entries given by

$$\vec{x}_i = \frac{\det \mathbf{A}_i(b)}{\det \mathbf{A}}$$

An inverse formula extending from Cramer's Rule is

$$\mathbf{A}^{-1} = \frac{1}{\det A} \cdot \operatorname{adj}(\mathbf{A})$$

Determinants as Area

If $\mathbf{A}$ is a 2×2 matrix, the area of the parallelogram determined by the columns of $\mathbf{A}$ is $|\det A|$. If $\mathbf{A}$ is a 3×3 matrix, the volume of the parallelopiped determined by the columns of $\mathbf{A}$ is $|\det A|$.

Eigenvalues & Eigenvectors

An **eigenvector** of $\mathbf{A}$ (corresponding to λ) is a nonzero vector x such that

$$\mathbf{A}x = \lambda x$$

A scalar λ is an **eigenvalue** of $\mathbf{A}$ if there exists a nonzero vector x such that

$$\mathbf{A}x = \lambda x$$

More formally, let $\mathbf{V}$ be a vector space. An **eigenvector** of a linear transformation $\mathbf{T} : \mathbf{V} \to \mathbf{V}$ is a nonzero vector $x \in \mathbf{V}$ such that $\mathbf{T}(x) = \lambda x$ for some scalar λ. This scalar λ is called an **eigenvalue** of $\mathbf{T}$ if there is a nontrivial solution x of $\mathbf{T}(x) = \lambda x$; ssuch an x is called an **eigenvector** corresponding to λ.

The Diagonalization Theorem

An $n \times n$ matrix $\mathbf{A}$ is diagonalizable if and only if $\mathbf{A}$ has n linearly independent eigenvectors.

In fact, $\mathbf{A} = \mathbf{PDP}^{-1}$, with diagonal matrix $\mathbf{D}$, if and only if the columns of $\mathbf{P}$ are n linearly independent eigenvectors of A. In this case, the diagonal entries of $\mathbf{D}$ are eigenvalues of $\mathbf{A}$ that correspond respectively to the eigenvectors in $\mathbf{P}$.

Theorem: An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

Invertible Matrix Theorem

1. $\mathbf{A}$ is row-equivalent to the $n \times n$ identity matrix $\mathbf{I}_n$.
2. $\mathbf{A}$ has n pivot positions.
3. The equation $\mathbf{A}x = 0$ has only the trivial solution $x = 0$.
4. The columns of $\mathbf{A}$ form a linearly independent set.
5. The linear transformation $x \mapsto \mathbf{A}x$ is one-to-one.
6. For each column vector $b \in \mathbb{R}^n$, the equation $\mathbf{A}x = b$ has a unique solution.
7. The columns of $\mathbf{A}$ span $\mathbb{R}^n$.
8. The linear transformation $x \mapsto \mathbf{A}x$ is a surjection (onto).
9. There is an $n \times n$ matrix $\mathbf{C}$ such that $\mathbf{CA} = \mathbf{I}_n$.
10. There is an $n \times n$ matrix $\mathbf{D}$ such that $\mathbf{AD} = \mathbf{I}_n$.
11. The transpose matrix $\mathbf{A}^{\mathrm{T}}$ is invertible.
12. The columns of $\mathbf{A}$ form a basis for $\mathbb{R}^n$.
13. The column space of $\mathbf{A}$ is equal to $\mathbb{R}^n$.
14. The dimension of the column space of $\mathbf{A}$ is n.
15. The rank of $\mathbf{A}$ is n.
16. The null space of $\mathbf{A}$ is 0.
17. The dimension of the null space of $\mathbf{A}$ is 0.
18. 0 fails to be an eigenvalue of $\mathbf{A}$.
19. The determinant of $\mathbf{A}$ is not 0.
20. The orthogonal complement of the column space of $\mathbf{A} = \mathbf{A}^{\perp}$ is 0.
21. The orthogonal complement of the null space of $\mathbf{A}$ is $\mathbb{R}^n$.
22. The row space of $\mathbf{A}$ is $\mathbb{R}^n$.
23. The matrix $\mathbf{A}$ has n non-zero singular values (*not studied in MATH-300*).