

Math 300 - Final Exam Study Guide

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Cramer's Rule

$$\mathbf{A}_i(b) = [a_1, \dots, b, \dots, a_n]$$

Let \mathbf{A} be an invertible $n \times n$ matrix. For any $b \in \mathbb{R}$, the unique solution \vec{x} of $\mathbf{A}\vec{x} = b$ has entries given by

$$\vec{x}_i = \frac{\det \mathbf{A}_i(b)}{\det \mathbf{A}}$$

An inverse formula extending from Cramer's Rule is

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \cdot \text{adj}(\mathbf{A})$$

Determinants as Area

If \mathbf{A} is a 2×2 matrix, the area of the parallelogram determined by the columns of \mathbf{A} is $|\det \mathbf{A}|$. If \mathbf{A} is a 3×3 matrix, the volume of the parallelepiped determined by the columns of \mathbf{A} is $|\det \mathbf{A}|$.

Eigenvalues & Eigenvectors

An **eigenvector** of \mathbf{A} (corresponding to λ) is a nonzero vector x such that

$$\mathbf{A}x = \lambda x$$

A scalar λ is an **eigenvalue** of \mathbf{A} if there exists a nonzero vector x such that

$$\mathbf{A}x = \lambda x$$

More formally, let \mathbf{V} be a vector space. An **eigenvector** of a linear transformation $\mathbf{T} : \mathbf{V} \rightarrow \mathbf{V}$ is a nonzero vector $x \in \mathbf{V}$ such that $\mathbf{T}(x) = \lambda x$ for some scalar λ . This scalar λ is called an **eigenvalue** of \mathbf{T} if there is a nontrivial solution x of $\mathbf{T}(x) = \lambda x$; such an x is called an **eigenvector** corresponding to λ .

The Diagonalization Theorem

An $n \times n$ matrix \mathbf{A} is diagonalizable if and only if \mathbf{A} has n linearly independent eigenvectors.

In fact, $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$, with diagonal matrix \mathbf{D} , if and only if the columns of \mathbf{P} are n linearly independent eigenvectors of \mathbf{A} . In this case, the diagonal entries of \mathbf{D} are eigenvalues of \mathbf{A} that correspond respectively to the eigenvectors in \mathbf{P} .

Theorem: An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

Invertible Matrix Theorem

1. \mathbf{A} is row-equivalent to the $n \times n$ identity matrix \mathbf{I}_n .
2. \mathbf{A} has n pivot positions.
3. The equation $\mathbf{A}x = 0$ has only the trivial solution $x = 0$.
4. The columns of \mathbf{A} form a linearly independent set.
5. The linear transformation $x \mapsto \mathbf{A}x$ is one-to-one.
6. For each column vector $b \in \mathbb{R}^n$, the equation $\mathbf{A}x = b$ has a unique solution.
7. The columns of \mathbf{A} span \mathbb{R}^n .
8. The linear transformation $x \mapsto \mathbf{A}x$ is a surjection (onto).
9. There is an $n \times n$ matrix \mathbf{C} such that $\mathbf{CA} = \mathbf{I}_n$.
10. There is an $n \times n$ matrix \mathbf{D} such that $\mathbf{AD} = \mathbf{I}_n$.
11. The transpose matrix \mathbf{A}^T is invertible.
12. The columns of \mathbf{A} form a basis for \mathbb{R}^n .
13. The column space of \mathbf{A} is equal to \mathbb{R}^n .
14. The dimension of the column space of \mathbf{A} is n .
15. The rank of \mathbf{A} is n .
16. The null space of \mathbf{A} is $\{0\}$.
17. The dimension of the null space of \mathbf{A} is 0 .
18. 0 fails to be an eigenvalue of \mathbf{A} .
19. The determinant of \mathbf{A} is not 0 .
20. The orthogonal complement of the column space of $\mathbf{A} = \mathbf{A}^\perp$ is $\{0\}$.
21. The orthogonal complement of the null space of \mathbf{A} is \mathbb{R}^n .
22. The row space of \mathbf{A} is \mathbb{R}^n .
23. The matrix \mathbf{A} has n non-zero singular values (*not studied in MATH-300*).

Complex Eigenvalue Decomposition

Let \mathbf{A} be a real 2×2 matrix with a complex eigenvalue $\lambda = a - bi$ ($b \neq 0$) and an associated eigenvector $v \in \mathbb{C}^2$. Then

$$\mathbf{A} = \mathbf{P}\mathbf{C}\mathbf{P}^{-1}, \text{ where } \mathbf{P} = [\operatorname{Re} v \quad \operatorname{Im} v] \text{ and } \mathbf{C} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

Inner Product

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{v}$$

Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be vectors in \mathbb{R}^n and let c be a scalar. Then

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2. $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$
3. $(c\mathbf{u}) \cdot \mathbf{v} = (c\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (c\mathbf{v})$
4. $\mathbf{u} \cdot \mathbf{u} \geq 0$, and $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

Distance

For \mathbf{u} and $\mathbf{v} \in \mathbb{R}^n$, the **distance between \mathbf{u} and \mathbf{v}** , written as $\operatorname{dist}(\mathbf{u}, \mathbf{v})$, is the length of the vector $\mathbf{u} - \mathbf{v}$. That is,

$$\operatorname{dist}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$$

Orthogonality

Two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ are **orthogonal** to each other if $\mathbf{u} \cdot \mathbf{v} = 0$.

Two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ are **orthogonal** to each other if $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$.

Orthogonal Complements

1. A vector x is in \mathbf{W}^\perp if and only if x is orthogonal to every vector in a set that spans \mathbf{W} .
2. \mathbf{W} is a subspace of \mathbb{R}^n .

Theorem: Let \mathbf{A} be an $m \times n$ matrix. The orthogonal complement of the row space of \mathbf{A} is the null space of \mathbf{A} , and the orthogonal complement of the column space of \mathbf{A} is the null space of \mathbf{A}^T :

$$(\operatorname{Row} \mathbf{A})^\perp = \operatorname{Null} \mathbf{A} \quad \text{and} \quad (\operatorname{Col} \mathbf{A})^\perp = \operatorname{Null} \mathbf{A}^T$$

To find the angle between two nonzero vectors in either \mathbb{R}^2 or \mathbb{R}^3 , the inner product can be used:

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos \theta$$

Orthogonal Sets

Theorem: If $S = u_1, \dots, u_p$ is an orthogonal set of nonzero vectors in \mathbb{R}^n , then S is linearly independent and hence is a basis for the subspace spanned by S .

An **orthogonal basis** for a subspace \mathbf{W} of \mathbb{R}^n is a basis for \mathbf{W} that is also an orthogonal set.

Theorem: Let $S = u_1, \dots, u_p$ be an orthogonal basis for a subspace \mathbf{W} of \mathbb{R}^n . For each $y \in \mathbf{W}$, the weights in the linear combination

$$y = c_1 u_1 + \dots + c_p u_p$$

are given by

$$c_j = \frac{y \cdot u_j}{u_j \cdot u_j} \quad (j = 1, \dots, p)$$

Orthogonal Projection

Given some nonzero vector u in \mathbb{R}^n , consider the problem of decomposing a vector $y \in \mathbb{R}^n$ into the sum of two vectors, one a multiple of u and the other orthogonal to u . We want to write

$$y = \hat{y} + z, \quad z \perp u$$

We can consider \hat{y} to be the shadow of y onto W ($\hat{y} = \text{proj}_W y$) and z to be the remaining vertical component of y . This projection is determined by the subspace L spanned by u (the line through u and 0). \hat{y} is denoted by $\text{proj}_L y$ and is called the **orthogonal projection of y onto L** . That is,

$$\hat{y} = \text{proj}_L y = \frac{y \cdot u}{u \cdot u} u$$

The Best Approximation Theorem

Let \mathbf{W} be a subspace of \mathbb{R}^n , let y be any vector in \mathbb{R}^n , and let \hat{y} be the orthogonal projection of y onto \mathbf{W} . Then \hat{y} is the closest point in \mathbf{W} to y , in the sense that

$$\|y - \hat{y}\| < \|y - v\|$$

for all $v \in \mathbf{W}$ distinct from \hat{y} .

The Gram-Schmidt Process

Given a basis x_1, \dots, x_p for a nonzero subspace \mathbf{W} of \mathbb{R}^n , define

$$\begin{aligned} v_1 &= x_1 \\ v_2 &= x_2 - \frac{x_2 \cdot v_1}{v_1 \cdot v_1} v_1 \\ v_3 &= x_3 - \frac{x_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{x_3 \cdot v_2}{v_2 \cdot v_2} v_2 \\ &\vdots \\ v_p &= x_p - \frac{x_p \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{x_p \cdot v_2}{v_2 \cdot v_2} v_2 - \dots - \frac{x_p \cdot v_{p-1}}{v_{p-1} \cdot v_{p-1}} v_{p-1} \end{aligned}$$

Then v_1, \dots, v_p is an orthogonal basis for \mathbf{W} . In addition,

$$\text{Span}\{v_1, \dots, v_k\} = \text{Span}\{x_1, \dots, x_k\} \quad \text{for } 1 \leq k \leq p$$

Least-Squares Problems

If \mathbf{A} is $m \times n$ and $b \in \mathbb{R}^n$, a **least-squares solution** of $\mathbf{A}x = b$ is some $\hat{X} \in \mathbb{R}^n$ such that

$$\|b - \mathbf{A}\hat{x}\| \leq \|b - \mathbf{A}x\|$$

for all $x \in \mathbb{R}^n$.

Theorem: The set of least-squares solutions of $\mathbf{A}x = b$ coincides with the nonempty set of solutions of the normal equations $\mathbf{A}^T \mathbf{A}x = \mathbf{A}^T b$.

Definitions

- **Adjugate:** The matrix $\text{adj } \mathbf{A}$ formed from a square matrix \mathbf{A} by replacing the (i, j) -entry by the (j, i) -cofactor, for all i and j , and then transposing the resulting matrix.
- **Basis:** An indexed set $\beta = \{v_1, \dots, v_p\}$ in V such that β is a linearly independent set and the subspace spanned by β coincides with $H = \text{Span}\{v_1, \dots, v_p\}$.
- **Best Approximation:** The closest point in a given subspace to a given vector.
- **Characteristic Equation:** $\det(\mathbf{A} - \lambda \mathbf{I}) = 0$
- **Column Space:** The set $\text{Col } \mathbf{A}$ of all linear combinations of the columns of \mathbf{A} . If $\mathbf{A} = [a_1, \dots, a_n]$, then $\text{Col } \mathbf{A} = \text{Span}(a_1, \dots, a_n)$.
- **Diagonalizable:** A matrix that can be written in factored form as PDP^{-1} , where D is a diagonal matrix and P is an invertible matrix.
- **Eigenspace:** The set of *all* solutions of $Ax = \lambda x$, where λ is an eigenvalue of A .
- **Elementary Matrix:** An invertible matrix that results by performing exactly one elementary row operation on an identity matrix.
- **Linear Combination:** A sum of scalar multiples of vectors. The scalars are called the *weights*.
- **Linear Dependence Relation:** A homogeneous vector equation where the weights are all specified and at least one weight is nonzero.
- **Linearly Dependent:** An indexed set $\{v_1, \dots, v_p\}$ with the property that there exist weights c_1, \dots, c_p , not all zero, such that $c_1 v_1 + \dots + c_p v_p = 0$. That is, the vector equation $c_1 v_1 + \dots + c_p v_p = 0$ has a *nontrivial* solution.
- **Linearly Independent:** An indexed set $\{v_1, \dots, v_p\}$ with the property that the vector equation $c_1 v_1 + \dots + c_p v_p = 0$ has *only the trivial solution*.
- **LU Factorization:** The representation of a matrix \mathbf{A} in the form $\mathbf{A} = \mathbf{L}\mathbf{U}$ where \mathbf{L} is a square lower triangular matrix with ones on the diagonal (a unit lower triangular matrix) and \mathbf{U} is an echelon form of \mathbf{A} .
- **Matrix Transformation:** A mapping $x \mapsto \mathbf{A}x$ where \mathbf{A} is an $m \times n$ matrix and x represents any vector in \mathbb{R}^n .
- **Norm:** The length of vector v , or $\|v\| = \sqrt{v \cdot v}$.
- **Normal Vector:** A vector $n \in \mathbb{R}^n$ such that $n \cdot x = 0$ for all $x \in V$.
- **Null Space:** The set $\text{Nul } \mathbf{A}$ of all solutions to the homogeneous equation $\mathbf{A}x = 0$.
- **One-to-One:** A mapping $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ such that each $b \in \mathbb{R}^m$ is the image of *at most* one $x \in \mathbb{R}^n$.
- **Onto:** A mapping $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ such that each $b \in \mathbb{R}^m$ is the image of *at least* one $x \in \mathbb{R}^n$.

- **Orthogonal Basis:** A basis that is also an orthogonal set.
- **Orthogonal Complement:** The set W^\perp of all vectors orthogonal to W .
- **Orthogonal Decomposition:** The representation of a vector y as the sum of two vectors, one in a specified subspace W and the other in W^\perp .
- **Orthonormal Basis:** A basis that is an orthogonal set of unit vectors.
- **Plane through \mathbf{u} , \mathbf{v} , and the origin:** A set whose parametric equation is $x = su + tv$ ($s, t \in \mathbb{R}$), with u and v linearly independent.
- **Proper Subspace:** Any subspace of a vector space V other than V itself.
- **Range:** The set of all vectors of the form $T(x)$ for some x in the domain of T .
- **Reduced Echelon Matrix:** A rectangular matrix in echelon form that has these additional properties: The leading entry in each nonzero row is 1, and each leading 1 is the only nonzero entry in its column.
- **Row Equivalent:** Two matrices for which there exists a finite sequence of row operations that transforms one matrix into the other.
- **Row Space:** The set $\text{Row } \mathbf{A}$ of all linear combinations of the vectors formed from the rows of \mathbf{A} ; also denoted $\text{Col } \mathbf{A}^T$.
- **Span:** The set of all linear combinations of v_1, \dots, v_p . Also, the subspace generated by v_1, \dots, v_p .
- **Subspace:** A subset H of some vector space V such that H has these properties:
 1. The zero vector of V is in H .
 2. H is closed under vector addition.
 3. H is closed under multiplication by scalars.
- **Linear System:** A collection of one or more linear equations involving the same set of variables, say x_1, \dots, x_n .
- **Vector Space:** A set of objects, called vectors, on which two operations are defined: addition and multiplication by scalars.