

Math 300 - Final Exam Study Guide

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Cramer's Rule

$$\mathbf{A}_i(b) = [a_1, \dots, b, \dots, a_n]$$

Let \mathbf{A} be an invertible $n \times n$ matrix. For any $b \in \mathbb{R}$, the unique solution \vec{x} of $\mathbf{A}\vec{x} = b$ has entries given by

$$\vec{x}_i = \frac{\det \mathbf{A}_i(b)}{\det \mathbf{A}}$$

An inverse formula extending from Cramer's Rule is

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \cdot \text{adj}(\mathbf{A})$$

Determinants as Area

If \mathbf{A} is a 2×2 matrix, the area of the parallelogram determined by the columns of \mathbf{A} is $|\det \mathbf{A}|$. If \mathbf{A} is a 3×3 matrix, the volume of the parallelepiped determined by the columns of \mathbf{A} is $|\det \mathbf{A}|$.

Eigenvalues & Eigenvectors

An **eigenvector** of \mathbf{A} (corresponding to λ) is a nonzero vector x such that

$$\mathbf{A}x = \lambda x$$

A scalar λ is an **eigenvalue** of \mathbf{A} if there exists a nonzero vector x such that

$$\mathbf{A}x = \lambda x$$

More formally, let \mathbf{V} be a vector space. An **eigenvector** of a linear transformation $\mathbf{T} : \mathbf{V} \rightarrow \mathbf{V}$ is a nonzero vector $x \in \mathbf{V}$ such that $\mathbf{T}(x) = \lambda x$ for some scalar λ . This scalar λ is called an **eigenvalue** of \mathbf{T} if there is a nontrivial solution x of $\mathbf{T}(x) = \lambda x$; such an x is called an **eigenvector** corresponding to λ .

The Diagonalization Theorem

An $n \times n$ matrix \mathbf{A} is diagonalizable if and only if \mathbf{A} has n linearly independent eigenvectors.

In fact, $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$, with diagonal matrix \mathbf{D} , if and only if the columns of \mathbf{P} are n linearly independent eigenvectors of \mathbf{A} . In this case, the diagonal entries of \mathbf{D} are eigenvalues of \mathbf{A} that correspond respectively to the eigenvectors in \mathbf{P} .

Theorem: An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

Invertible Matrix Theorem

1. \mathbf{A} is row-equivalent to the $n \times n$ identity matrix \mathbf{I}_n .
2. \mathbf{A} has n pivot positions.
3. The equation $\mathbf{A}x = 0$ has only the trivial solution $x = 0$.
4. The columns of \mathbf{A} form a linearly independent set.
5. The linear transformation $x \mapsto \mathbf{A}x$ is one-to-one.
6. For each column vector $b \in \mathbb{R}^n$, the equation $\mathbf{A}x = b$ has a unique solution.
7. The columns of \mathbf{A} span \mathbb{R}^n .
8. The linear transformation $x \mapsto \mathbf{A}x$ is a surjection (onto).
9. There is an $n \times n$ matrix \mathbf{C} such that $\mathbf{CA} = \mathbf{I}_n$.
10. There is an $n \times n$ matrix \mathbf{D} such that $\mathbf{AD} = \mathbf{I}_n$.
11. The transpose matrix \mathbf{A}^T is invertible.
12. The columns of \mathbf{A} form a basis for \mathbb{R}^n .
13. The column space of \mathbf{A} is equal to \mathbb{R}^n .
14. The dimension of the column space of \mathbf{A} is n .
15. The rank of \mathbf{A} is n .
16. The null space of \mathbf{A} is $\{0\}$.
17. The dimension of the null space of \mathbf{A} is 0 .
18. 0 fails to be an eigenvalue of \mathbf{A} .
19. The determinant of \mathbf{A} is not 0 .
20. The orthogonal complement of the column space of $\mathbf{A} = \mathbf{A}^\perp$ is $\{0\}$.
21. The orthogonal complement of the null space of \mathbf{A} is \mathbb{R}^n .
22. The row space of \mathbf{A} is \mathbb{R}^n .
23. The matrix \mathbf{A} has n non-zero singular values (*not studied in MATH-300*).

Complex Eigenvalue Decomposition

Let \mathbf{A} be a real 2×2 matrix with a complex eigenvalue $\lambda = a - bi$ ($b \neq 0$) and an associated eigenvector $v \in \mathbb{C}^2$. Then

$$\mathbf{A} = \mathbf{P}\mathbf{C}\mathbf{P}^{-1}, \text{ where } \mathbf{P} = [\operatorname{Re} v \quad \operatorname{Im} v] \text{ and } \mathbf{C} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

Inner Product

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{v}$$

Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be vectors in \mathbb{R}^n and let c be a scalar. Then

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2. $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$
3. $(c\mathbf{u}) \cdot \mathbf{v} = (c\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (c\mathbf{v})$
4. $\mathbf{u} \cdot \mathbf{u} \geq 0$, and $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

Distance

For \mathbf{u} and $\mathbf{v} \in \mathbb{R}^n$, the **distance between \mathbf{u} and \mathbf{v}** , written as $\operatorname{dist}(\mathbf{u}, \mathbf{v})$, is the length of the vector $\mathbf{u} - \mathbf{v}$. That is,

$$\operatorname{dist}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$$

Orthogonality

Two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ are **orthogonal** to each other if $\mathbf{u} \cdot \mathbf{v} = 0$.

Two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ are **orthogonal** to each other if $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$.

Orthogonal Complements

1. A vector x is in \mathbf{W}^\perp if and only if x is orthogonal to every vector in a set that spans \mathbf{W} .
2. \mathbf{W} is a subspace of \mathbb{R}^n .

Theorem: Let \mathbf{A} be an $m \times n$ matrix. The orthogonal complement of the row space of \mathbf{A} is the null space of \mathbf{A} , and the orthogonal complement of the column space of \mathbf{A} is the null space of \mathbf{A}^T :

$$(\operatorname{Row} \mathbf{A})^\perp = \operatorname{Null} \mathbf{A} \quad \text{and} \quad (\operatorname{Col} \mathbf{A})^\perp = \operatorname{Null} \mathbf{A}^T$$

To find the angle between two nonzero vectors in either \mathbb{R}^2 or \mathbb{R}^3 , the inner product can be used:

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos \theta$$