Math 300 - Final Exam Study Guide

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Contents

Cramer's Rule

$$\mathbf{A}_i(b) = \left[a_1, \ldots, b, \ldots, a_2\right]$$

Let $\mathbf{A}$ be an invertible $n \times n$ matrix. For any $b \in \mathbb{R}^n$, the unique solution $\vec{x}$ of $\mathbf{A}\vec{x} = b$ has entries given by

$$\vec{x}_i = \frac{\det \mathbf{A}_i(b)}{\det \mathbf{A}}$$

An inverse formula extending from Cramer's Rule is

$$\mathbf{A}^{-1} = \frac{1}{\det A} \cdot \mathrm{adj}(\mathbf{A})$$

Determinants as Area

If $\mathbf{A}$ is a 2×2 matrix, the area of the parallelogram determined by the columns of $\mathbf{A}$ is $|\det A|$. If $\mathbf{A}$ is a 3×3 matrix, the volume of the parallelopiped determined by the columns of $\mathbf{A}$ is $|\det A|$.

Eigenvalues & Eigenvectors

An **eigenvector** of $\mathbf{A}$ (corresponding to λ) is a nonzero vector x such that

$$\mathbf{A}x = \lambda x$$

A scalar λ is an **eigenvalue** of $\mathbf{A}$ if there exists a nonzero vector x such that

$$\mathbf{A}x = \lambda x$$

More formally, let $\mathbf{V}$ be a vector space. An **eigenvector** of a linear transformation $\mathbf{T} : \mathbf{V} \to \mathbf{V}$ is a nonzero vector $x \in \mathbf{V}$ such that $\mathbf{T}(x) = \lambda x$ for some scalar λ. This scalar λ is called an **eigenvalue** of $\mathbf{T}$ if there is a nontrivial solution x of $\mathbf{T}(x) = \lambda x$; ssuch an x is called an **eigenvector** corresponding to λ.

The Diagonalization Theorem

An $n \times n$ matrix $\mathbf{A}$ is diagonalizable if and only if $\mathbf{A}$ has n linearly independent eigenvectors.

In fact, $\mathbf{A} = \mathbf{PDP}^{-1}$, with diagonal matrix $\mathbf{D}$, if and only if the columns of $\mathbf{P}$ are n linearly independent eigenvectors of A. In this case, the diagonal entries of $\mathbf{D}$ are eigenvalues of $\mathbf{A}$ that correspond respectively to the eigenvectors in $\mathbf{P}$.

Theorem: An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

Invertible Matrix Theorem

1. $\mathbf{A}$ is row-equivalent to the $n \times n$ identity matrix $\mathbf{I}_n$.
2. $\mathbf{A}$ has n pivot positions.
3. The equation $\mathbf{A}x = 0$ has only the trivial solution $x = 0$.
4. The columns of $\mathbf{A}$ form a linearly independent set.
5. The linear transformation $x \mapsto \mathbf{A}x$ is one-to-one.
6. For each column vector $b \in \mathbb{R}^n$, the equation $\mathbf{A}x = b$ has a unique solution.
7. The columns of $\mathbf{A}$ span $\mathbb{R}^n$.
8. The linear transformation $x \mapsto \mathbf{A}x$ is a surjection (onto).
9. There is an $n \times n$ matrix $\mathbf{C}$ such that $\mathbf{CA} = \mathbf{I}_n$.
10. There is an $n \times n$ matrix $\mathbf{D}$ such that $\mathbf{AD} = \mathbf{I}_n$.
11. The transpose matrix $\mathbf{A}^{\mathrm{T}}$ is invertible.
12. The columns of $\mathbf{A}$ form a basis for $\mathbb{R}^n$.
13. The column space of $\mathbf{A}$ is equal to $\mathbb{R}^n$.
14. The dimension of the column space of $\mathbf{A}$ is n.
15. The rank of $\mathbf{A}$ is n.
16. The null space of $\mathbf{A}$ is 0.
17. The dimension of the null space of $\mathbf{A}$ is 0.
18. 0 fails to be an eigenvalue of $\mathbf{A}$.
19. The determinant of $\mathbf{A}$ is not 0.
20. The orthogonal complement of the column space of $\mathbf{A} = \mathbf{A}^{\perp}$ is 0.
21. The orthogonal complement of the null space of $\mathbf{A}$ is $\mathbb{R}^n$.
22. The row space of $\mathbf{A}$ is $\mathbb{R}^n$.
23. The matrix $\mathbf{A}$ has n non-zero singular values (*not studied in MATH-300*).

Complex Eigenvalue Decomposition

Let $\mathbf{A}$ be a real 2×2 matrix with a complex eigenvalue $\lambda = a - bi$ ($b \neq 0$) and an associated eigenvector $v \in \mathbb{C}^2$. Then

$$\mathbf{A} = \mathbf{PCP}^{-1}, \text{ where } \mathbf{P} = \begin{bmatrix} \text{Re } v & \text{Im } v \end{bmatrix} \text{ and } \mathbf{C} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

Inner Product

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^{\mathbf{T}}\mathbf{v}$$

Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be vectors in $\mathbb{R}^n$ and let c be a scalar. Then

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2. $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$
3. $(c\mathbf{u}) \cdot \mathbf{v} = (c\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (c\mathbf{v})$
4. $\mathbf{u} \cdot \mathbf{u} \geq 0$, and $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = 0$.

Distance

For $\mathbf{u}$ and $\mathbf{u} \in \mathbb{R}^n$, the **distance between u and v**, written as $\text{dist}(\mathbf{u}, \mathbf{v})$, is the length of the vector $\mathbf{u} - \mathbf{v}$. That is,

$$\text{dist}(\mathbf{u}, \mathbf{v}) = ||\mathbf{u} - \mathbf{v}||$$

Orthogonality

Two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ are **orthogonal** to each other if $\mathbf{u} \cdot \mathbf{v} = 0$.

Two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ are **orthogonal** to each other if $||\mathbf{u}+\mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2$.

Orthogonal Complements

1. A vector x is in $\mathbf{W}^{\perp}$ if and only if x is orthogonal to every vector in a set that spans $\mathbf{W}$.
2. $\mathbf{W}$ is a subspace of $\mathbb{R}^n$.

Theorem: Let $\mathbf{A}$ be an $m \times n$ matrix. The orthogonal complement of the row space of $\mathbf{A}$ is the null space of $\mathbf{A}$, and the orthogonal complement of the column space of $\mathbf{A}$ is the null space of $\mathbf{A}^T$:

$$(\text{Row } \mathbf{A})^{\perp} = \text{Null } \mathbf{A} \quad \text{and} \quad (\text{Col } \mathbf{A})^{\perp} = \text{Null } \mathbf{A}^{\perp}$$

To find the angle between two nonzero vectors in either $\mathbb{R}^2$ or $\mathbb{R}^3$, the inner product can be used:

$$\mathbf{u} \cdot \mathbf{v} = ||\mathbf{u}||||\mathbf{v}|| \cos \theta$$

Orthogonal Sets

Theorem: If $S = u_1, \dots, u_p$ is an orthogonal set of nonzero vectors in $\mathbb{R}^n$, then S is linearly independent and hence is a basis for the subspace spanned by S.

An **orthogonal basis** for a subspace $\mathbf{W}$ of $\mathbb{R}^n$ is a basis for $\mathbf{W}$ that is also an orthogonal set.

Theorem: Let $S = u_1, \dots, u_p$ be an orthogonal basis for a subspace $\mathbf{W}$ of $\mathbb{R}^n$. For each $y \in \mathbf{W}$, the weights in the linear combination

$$y = c_1 u_1 + \cdots + c_p u_p$$

are given by

$$c_j = \frac{y \cdot u_j}{u_j \cdot u_j} \quad (j = 1, \dots, p)$$

Orthogonal Projection

Given some nonzero vector u in $\mathbb{R}^n$, consider the problem of decomposing a vector $y \in \mathbb{R}^n$ into the sum of two vectors, one a multiple of u and the other orthogonal to u. We want to write

$$y = \hat{y} + z, \quad z \perp u$$

We can consider $\hat{y}$ to be the shadow of y onto W ($\hat{y} = \text{proj}_W y$) and z to be the remaining vertical component of y. This projection is determined by the *subspace* L spanned by u (the line through u and 0). $\hat{y}$ is denoted by $\text{proj}_L y$ and is called the **orthogonal projection of y onto L**. That is,

$$\hat{y} = \text{proj}_L y = \frac{y \cdot u}{u \cdot u} u$$

The Best Approximation Theorem

Let $\mathbf{W}$ be a subspace of $\mathbb{R}^n$, let y be any vector in $\mathbb{R}^n$, and let $\hat{y}$ be the orthogonal projection of y onto $\mathbf{W}$. Then $\hat{y}$ is the closest point in $\mathbf{W}$ to y, in the sense that

$$||y - \hat{y}|| < ||y - v||$$

for all $v \in \mathbf{W}$ distinct from $\hat{y}$.

The Gram-Schmidt Process

Given a basis $x_1, \dots, x_p$ for a nonzero subspace $\mathbf{W}$ of $\mathbb{R}^n$, define

$$\begin{aligned}
v_1 &= x_1 \\
v_2 &= x_2 - \frac{x_2 \cdot v_1}{v_1 \cdot v_1} v_1 \\
v_3 &= x_3 - \frac{x_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{x_3 \cdot v_2}{v_2 \cdot v_2} v_2 \\
&\vdots \\
v_p &= x_p - \frac{x_p \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{x_p \cdot v_2}{v_2 \cdot v_2} v_2 - \cdots - \frac{x_p \cdot v_{p-1}}{v_{p-1} \cdot v_{p-1}} v_{p-1}
\end{aligned}$$

Then $v_1, \dots, v_p$ is an orthogonal basis for $\mathbf{W}$. In addition,

$$\text{Span}\{v_1, \dots, v_k\} = \text{Span}\{x_1, \dots, x_k\} \qquad \text{for } 1 \leq k \leq p$$