

# Penalized Regression

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Math 243: Stat Learning

October 11th, 2021

# Outline

In today's class, we will. . .

- Investigate the relationship between coefficient size and variance in linear models
- Discuss penalized regression models as means of improving MSE of linear models

## Section 1

# Penalized Regression

## Motivation

- Recall, for SLR,  $\hat{\beta}_0, \hat{\beta}_1$  are given by

$$\hat{\beta}_1 = \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^n (x_i - \bar{x})^2} \quad \hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$$

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- Under the standard assumptions, the coefficients produced by least squares regression are unbiased.
- That is, if the true relationship between  $Y$  and  $X$  is linear  $Y = \beta_0 + \beta_1 X + \epsilon$ , then

$$E[\hat{\beta}_0] = \beta_0 \quad E[\hat{\beta}_1] = \beta_1$$

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- Moreover, among all **unbiased** linear models, the least squares model has the lowest variance.
- Does this mean that the least squares model has the lowest MSE among all linear models?
  - No! MSE is a combination of bias and variance.
  - It is possible that small *increase* in bias, can correspond to large *decrease* in variance.

## Shrinking Coefficients

- Suppose the true relationship between  $Y$  and  $X_1, X_2$  is given by

$$Y = 1 + X_1 + 5X_2 + \epsilon \quad \epsilon \sim N(0, 1).$$

- Let  $\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2$  be the model coefficient estimates given by least squares regression. Which of the following models has higher variance in predictor estimates? Higher bias?

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Model 1:  $\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x_1 + \hat{\beta}_2 x_2$

Model 2:  $\hat{y} = \hat{\beta}_0 + 0.97 \cdot \hat{\beta}_1 x_1 + 0.98 \cdot \hat{\beta}_2 x_2$

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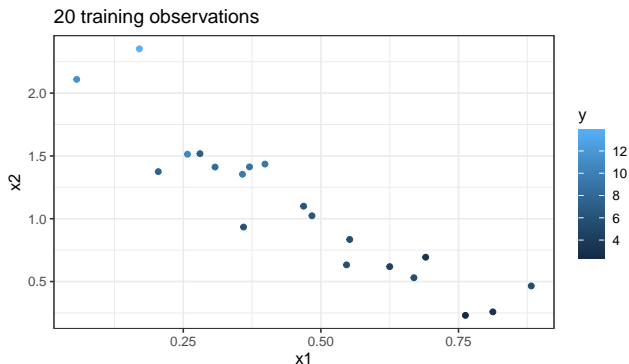
$$\text{Model 2: } \hat{y} = \hat{\beta}_0 + 0.97 \cdot \hat{\beta}_1 x_1 + 0.98 \cdot \hat{\beta}_2 x_2$$

- Model 2 has higher bias, but lower variance.

## A Linear Model

- Consider the following training data for the model:

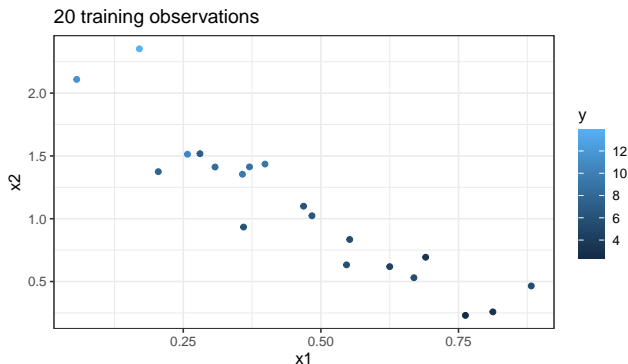
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# A Linear Model

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- What are some likely problems with the MLR model?

## Bias-Variance in Least Squares

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$$Y = 1 + X_1 + 5 \cdot X_2 = 1 + 0.25 + 5 \cdot 0.5 = 3.75$$

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- Using the least squares model from training data, the predicted value of  $Y$  is

$$\hat{Y} = -0.5 + 2.8X_1 + 5.8X_2 = -0.5 + 2.8 \cdot 0.25 + 5.8 \cdot 0.5 = 3.1$$

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- But how will the predicted value change if we repeat across 5000 simulations from the model?

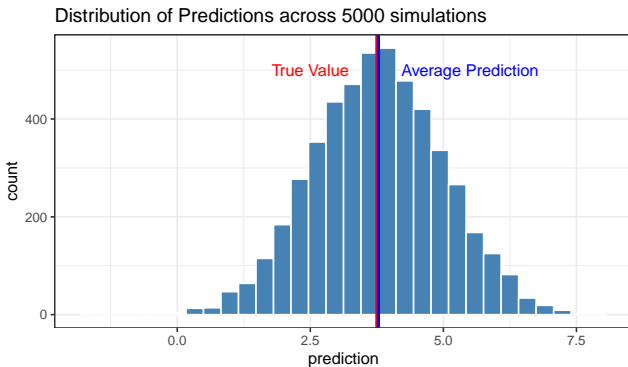
## Simulation

```
set.seed(1011)
test_point <- data.frame(x1 = 0.25, x2 = .5)

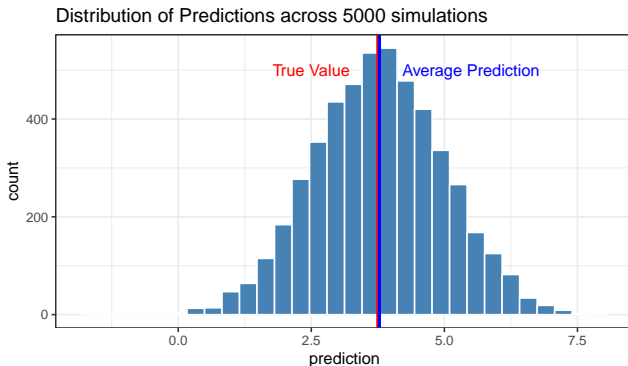
trials<-5000
prediction <- rep(NA, trials)
for (i in 1:trials){
  e<- rnorm(20,0,1)
  y<- 1 + x1 + 5*x2 + e
  sim_data <- data.frame(x1,x2,y)
  mod <- lm(y ~ x1 + x2, data = sim_data)
  prediction[i] <- predict(mod, test_point)
}

simulation <- data.frame(trial_num = 1:trials, prediction)
```

## Prediction Distribution



## Prediction Distribution



```
simulation %>% summarize(  
  mean = mean(prediction), variance = var(prediction))
```

```
##           mean variance  
## 1 3.772056 1.480935
```

## A Shrunk Model

- Now suppose we use the model algorithm

$$\hat{y} = \hat{\beta}_0 + 0.97 \cdot \hat{\beta}_1 x_1 + 0.98 \cdot \hat{\beta}_2 x_2$$

- Since  $\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2$  are unbiased, then the expected prediction for  $Y$  when  $X_1 = 0.25$  and  $X_2 = 0.5$  is

$$E[\hat{y}] = \beta_0 + 0.97 \cdot \beta_1 x_1 + 0.98 \cdot \beta_2 x_2 = 1 + 0.97 \cdot 0.25 + 0.98 \cdot 5 \cdot 0.5 = 3.69$$

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- Based on the first simulation, the model estimate is

$$\hat{Y} = -0.5 + 0.97 \cdot 2.8X_1 + 0.98 \cdot 5.8X_2 = -0.5 + 2.71X_1 + 5.68X_2$$

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- Based on the first simulation, the model estimate is

$$\hat{Y} = -0.5 + 0.97 \cdot 2.8X_1 + 0.98 \cdot 5.8X_2 = -0.5 + 2.71X_1 + 5.68X_2$$

- And the prediction when  $X_1 = 0.25$  and  $X_2 = 0.5$  is

$$\hat{y} = -0.5 + 2.71X_1 + 5.68X_2 = -0.5 + 2.71 \cdot 0.25 + 5.68 \cdot 0.5 = 3.525$$

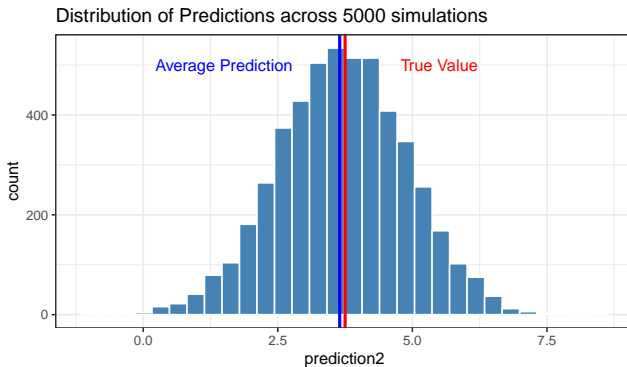
## Simulation II

```
set.seed(1001)

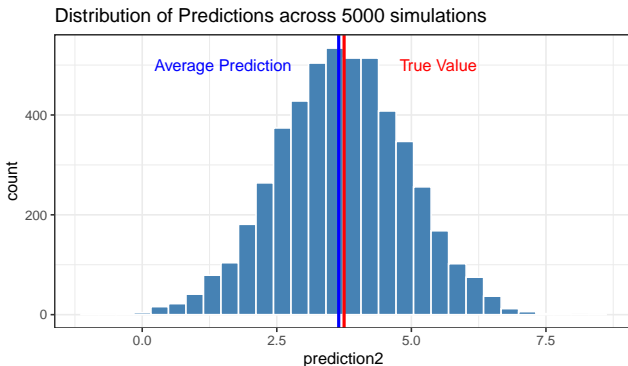
trials<-5000
prediction2 <- rep(NA, trials)
for (i in 1:trials){
  e<- rnorm(20,0,1)
  y<- 1 + x1 + 5*x2 + e
  sim_data <- data.frame(x1,x2,y)
  mod <- lm(y ~ x1 + x2, data = sim_data)
  b0 <- 1*coef(mod)[1]
  b1 <- .97*coef(mod)[2]
  b2 <- .98*coef(mod)[3]
  prediction2[i] <- b0 + b1*0.25 + b2*0.5
}

simulation2 <- data.frame(trial_num = 1:trials, prediction2)
```

## Prediction Distribution



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```
simulation2 %>% summarize(  
  mean = mean(prediction2), variance = var(prediction2))
```

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##      mean variance  
## 1 3.70387 1.434099
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## Model Comparison

- True relationship:  $Y = 1 + X_1 + 5X_2 + \epsilon$

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##          mean variance avg_error
## 1  3.772056  1.480935   1.481125
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- Model 2:  $\hat{y} = \hat{\beta}_0 + 0.97 \cdot \hat{\beta}_1 x_1 + 0.98 \cdot \hat{\beta}_2 x_2$

```
##          mean variance avg_error
## 1  3.70387  1.434099   1.435941
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##          mean variance avg_error
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```

- It looks like the model with smaller coefficients actually performed better!