Simple Linear Regression

Nate Wells

Math 243: Stat Learning

September 13th, 2021

Outline

In today's class, we will...

- Discuss theoretical foundation for linear regression
- Perform inference for simple linear models
- Implement simple linear regression in R

Section 1

Foundations

Linear Regression

- Suppose we have one or more predictors $(X_1, X_2, \ldots, X_p)$ and a *quantitative* response variable Y, and that
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- We'll use **Simple Linear Regression** (SLR) to build intuition about all linear models

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- So we are **estimating** an **approximation** to a relationship between response and predictors.

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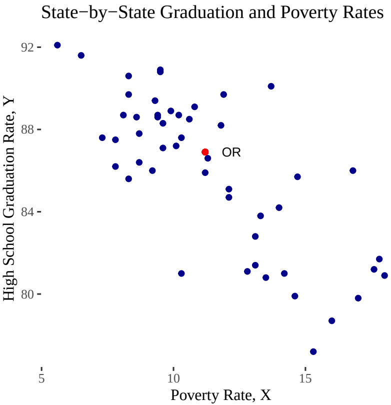

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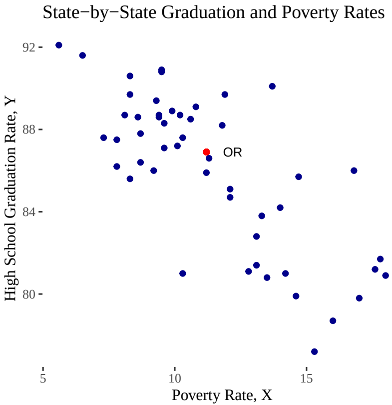

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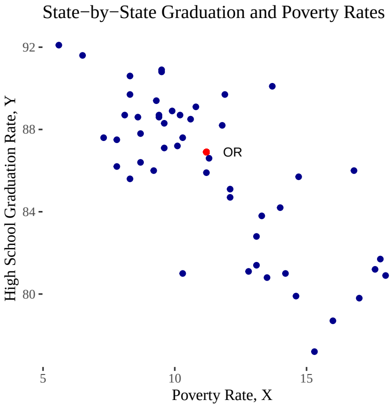

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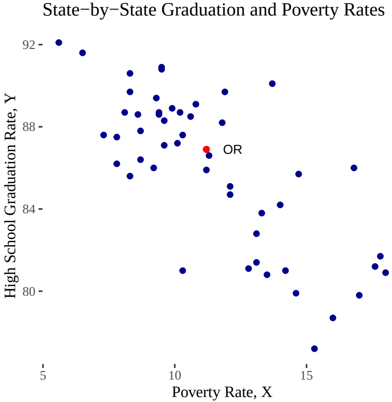

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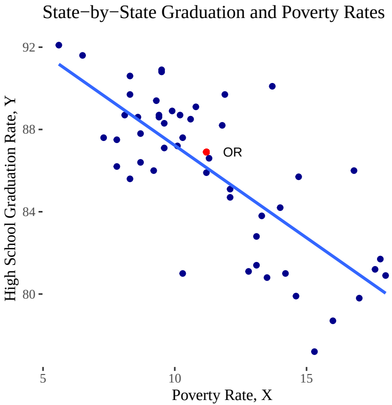

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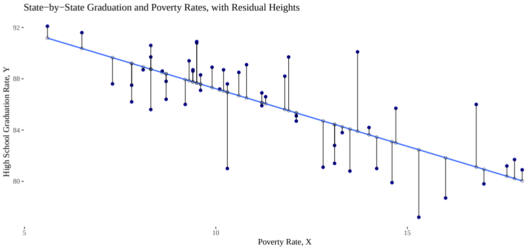

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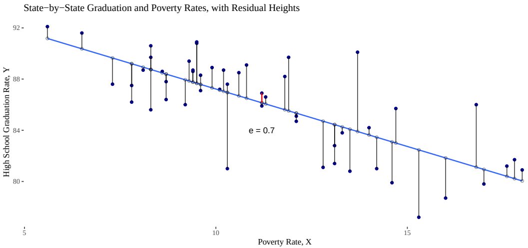

- Oregon's residual is

$$e = y - \hat{y} = 86.9 - 86.2 = 0.7$$

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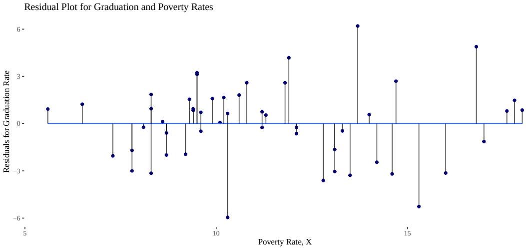

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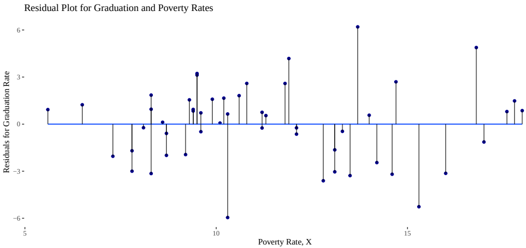

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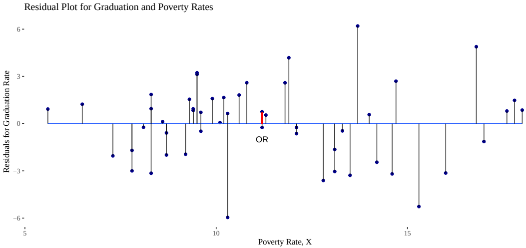

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- Using calculus or linear algebra, we can show that RSS is minimized when

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n}(x_i - \bar{x})^2} \qquad \hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$$

Section 2

Inference for Linear Models

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- **Tools**: confidence intervals, hypothesis tests
- **The Problems**: Our model will change if built using a different random sample. So in addition to estimates, we need to know about variability

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- The value $\mathrm{SE}(\hat{\theta})$ is the standard error of $\hat{\theta}$, or the standard deviation of the sampling distribution

Common Regression Assumptions

In order to safely use simple linear regression, we require these assumptions:

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If one or more of these conditions do not hold, our predictions may not be accurate and we should be skeptical of inferential claims.

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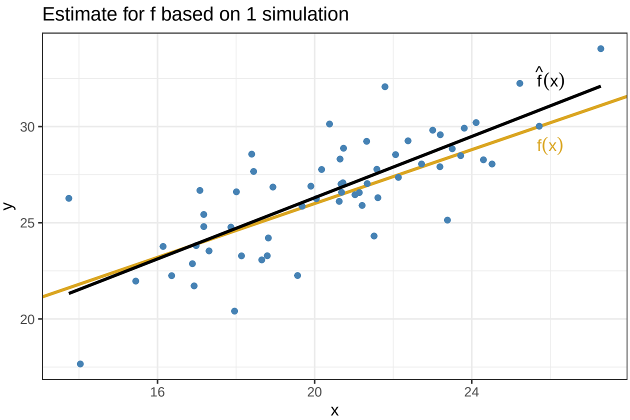

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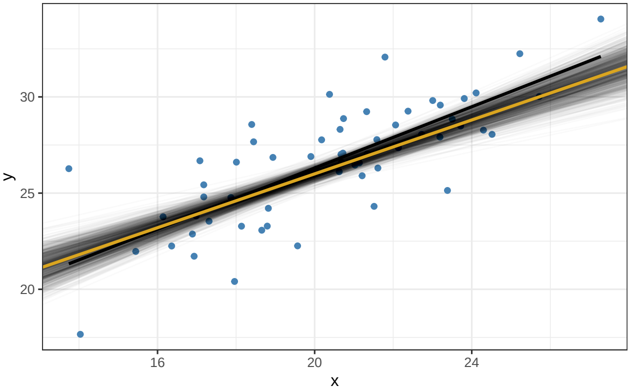
Estimates for f based on 1000 simulations



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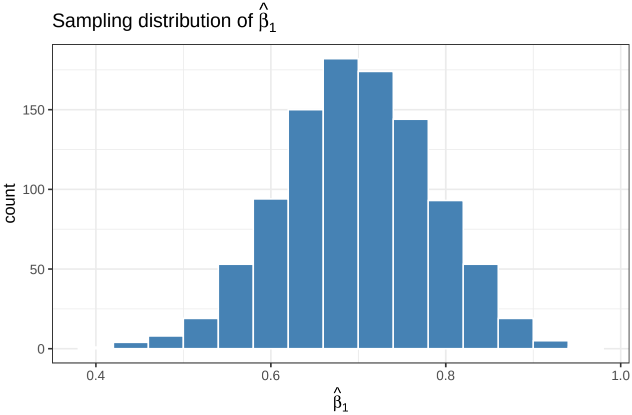

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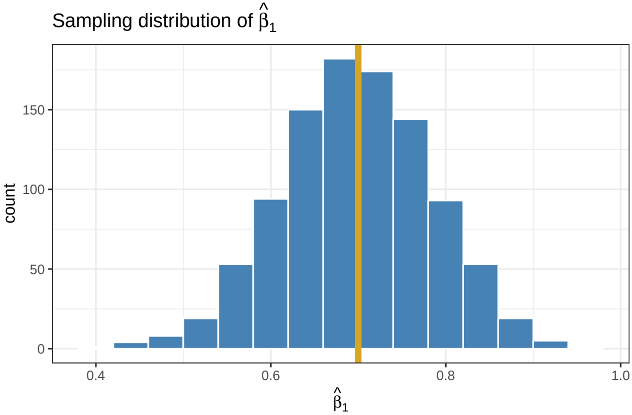

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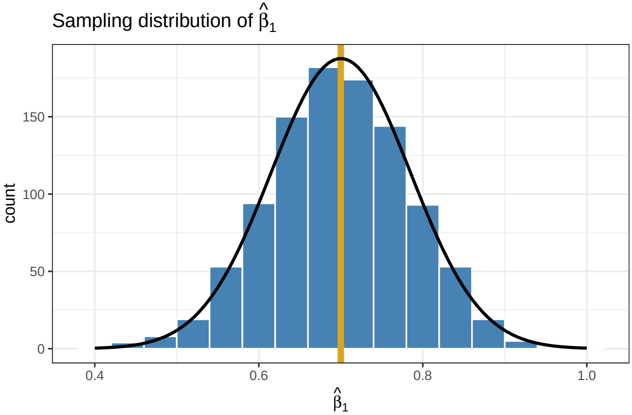

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Interpretation We are *95% confident* that the true slope relating x and y lies between lower and upper bound of this interval.

Hypothesis test for $\hat{\beta}_1$

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- An observed t with p-value less than a desired significance level (often $\alpha = 0.05$) gives good evidence against the null-hypothesis.

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    - The associated statistic is again t-distributed, although with more complicated SE.
    - For details, see DeGroot and Schervish "Probability and Statistics" (or take Math 392)