K-Nearest Neighbor

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Math 243: Stat Learning

September 10th, 2021

Outline

In today's class, we will...

- Discuss the Bayes Classifier
- Implement KNN as estimate for Bayes Classifier

Section 1

The Bayes Classifier

The Task

Suppose Y is categorical response variable with several levels $A_1, \ldots, A_k$.

Goal: Build a model f to classify an observation into levels A or B based on the values of several predictors $X_1, X_2, \ldots, X_p$ (quantitative or categorical)

$$Y = f(X_1, X_2, \ldots, X_p) + \epsilon \qquad \text{where } f, \epsilon \text{ take values in } \{A_1, \ldots, A_k\}$$

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- Training data: Compute error rate on observations in training data:

$$\text{Training Error} = \frac{1}{n} \sum_{i=1}^{n} I(y_i \neq \hat{y}_i)$$

where $I(y_i \neq \hat{y}_i)$ is the indicator variable that equals 1 if $y_i \neq \hat{y}_i$ and 0 otherwise.

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- Test data: Compute average proportion of errors on test data

$$\text{Test Error} = \text{Avg. } I(y_0 \neq \hat{y}_0)$$

  where $\hat{y}_0$ is the predicted class for a test observation with predictor x_0.

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- In practice, we cannot build this optimal model, since we don't know $P(Y = A_j \mid X = x_0)$

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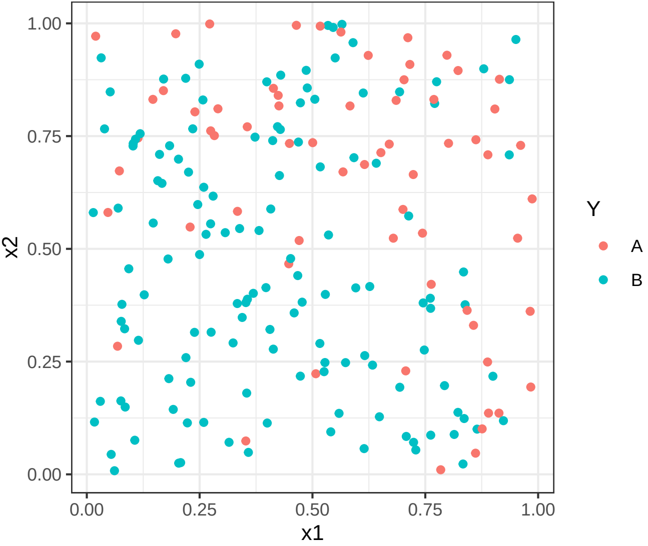
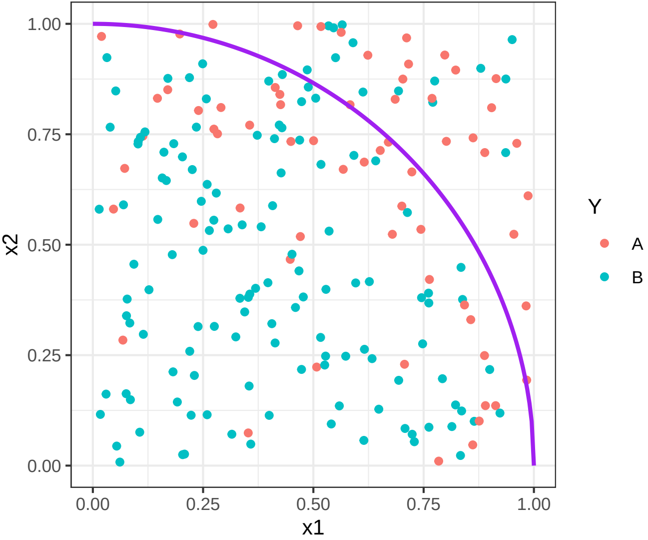
Then

$$f(x_0) = \operatorname{argmax}_j P(Y = A_j \mid X = x_0) = \begin{cases} A, & \text{if } x_1^2 + x_2^2 \geq 1 \\ B, & \text{if } x_1^2 + x_2^2 < 1 \end{cases}$$

Plot 1



Plot 2



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This is the theoretical lower bound on average error for a classification problem.

Section 2

K-Nearest Neighbors

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Given a positive integer K and a test observation x_0, let N_0 denote the K nearest training observations to x_0. Then

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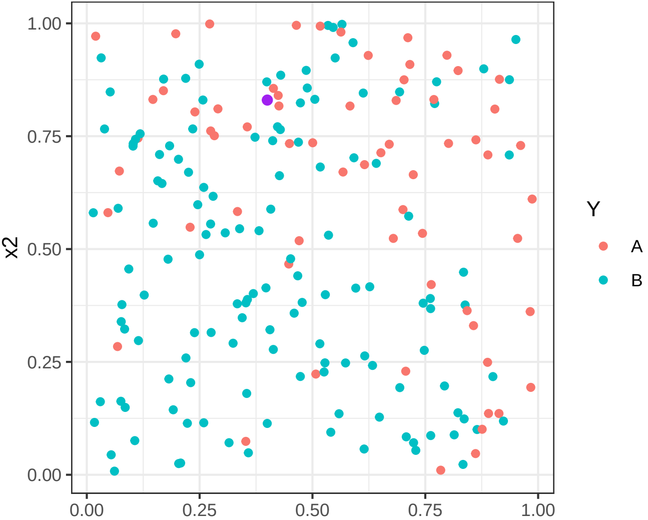
$$P(Y = A_j \mid X = x_0) \approx \frac{1}{K} \sum_{i \in N_0} I(y_i = A_j)$$

- Our model is therefore $f(x_0) = \frac{1}{K} \sum_{i \in N_0} I(y_i = A_j)$.

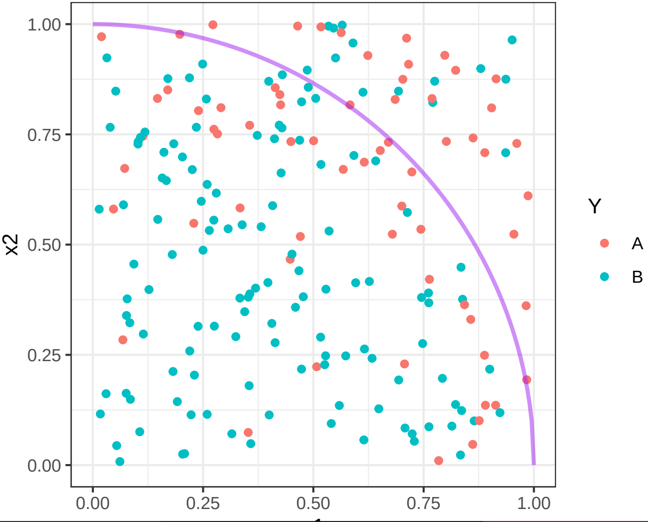
Simulation

Classify x_0 for $K = 1, 2, 3, 5, 10, 200$.



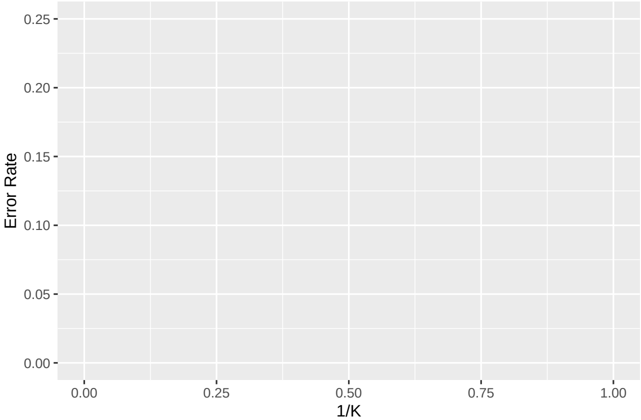
Simulation

Sketch the classification boundaries for a variety of values of K.



Error Rates

Sketch the graph of KNN error rates as function of K^{-1}



Extra Practice

Use the first part of the .Rmd file on the course website to generate 5 random points and form classification boundaries for $K = 1$ and $K = 2$ KNN.

Then use the second part of the .Rmd file to classify 5 randomly generated points.