Assessing Model Accuracy

Nate Wells

Math 243: Stat Learning

September 8th, 2021

Outline

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- Analyze data from the 'guess my age' activity

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- Investigate the Bias-Variance trade-off

Section 1

How Old?

Reflection

The task: Consider photos for 8 math and stats faculty at Reed. Estimate the age of each faculty member (at the time photo was taken).








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- Were you interested primarily in prediction or inference?

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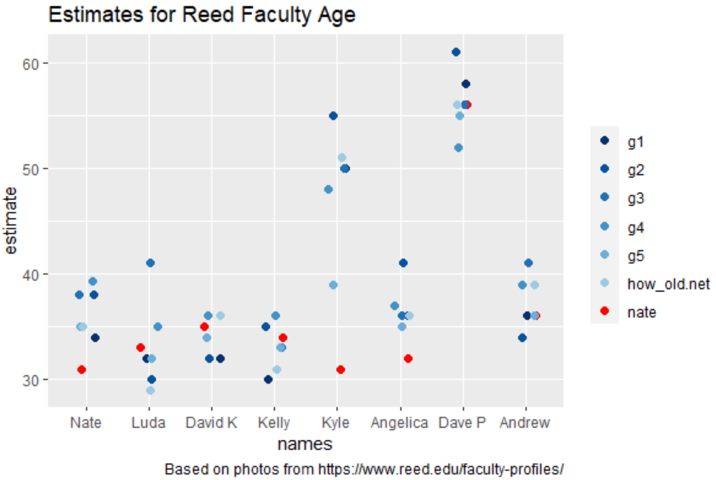
- Was the *How Old?* activity supervised or unsupervised?
- Did it represent a classification or regression problem?
- Were you interested primarily in prediction or inference?
- Did you use parametric or non-parametric methods?

The Results



Estimates for Reed Faculty Age
estimate
60
50
40
30
Nate
Luda
David K
Kelly
Kyle
Angelica
Dave P
Andrew
names
g1
g2
g3
g4
g5
how_old.net
nate
Based on photos from https://www.reed.edu/faculty-profiles/

Debrief

- How should we quantify error?
- What are some sources for error in our estimates?
- How should we assess the overall accuracy of a group's predictions?
- Did any groups seem to consistently over- or under-estimate ages? By how much?
- Do any faculty member ages seem to consistently be over- or under-estimated?
- Are there any faculty members where the guesses seem to be in a particularly large or small range?

Section 2

Mean Squared Error

How do we measure quality of a model?

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$$\mathrm{MSE}(\hat{f}) = \frac{1}{n}\sum_{i=1}^{n}\left(y_i - \hat{f}(x_i)\right)^2$$

  where $\hat{f}$ is the model, the x_i are the observed predictor values, and the y_i are the corresponding observed response values.

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- Under what circumstances is MSE small?
- What are the problems with trying to minimize MSE on the set of observed data $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$?

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- If we have training and test data, we can construct a number of models on the training data, and compare their performance on the test data in order to select the best model

An Example

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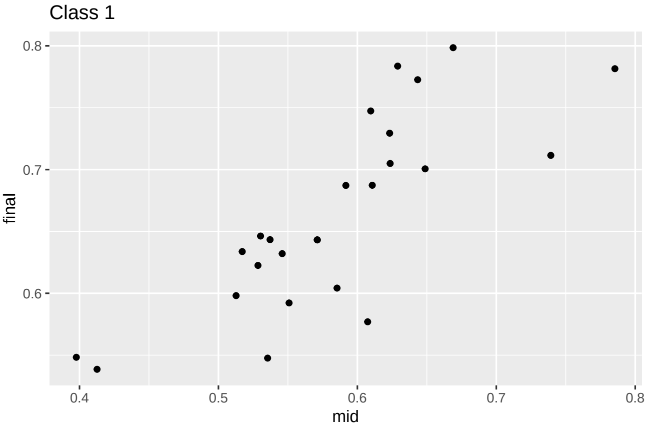
- Suppose e don't care about how well our model predicts exam scores for the previous classes. We want to know how well it predicts future scores.
  - Use the first class as training data
  - Use the second class as test data

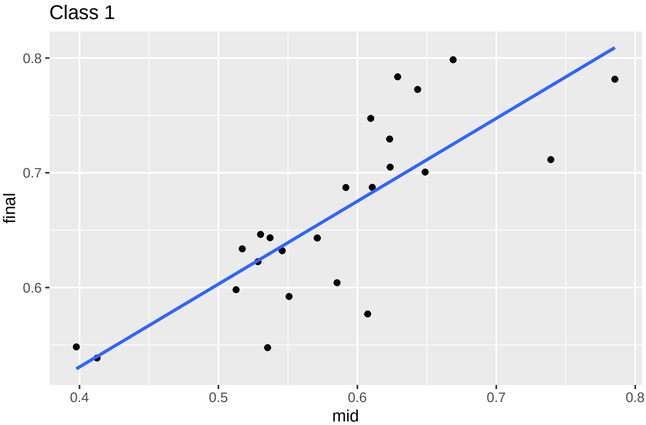
Training Set

```
##
##
scores %>% ggplot( aes(x = mid, y = final)) +
  geom_point()+labs(title = "Class 1")
```



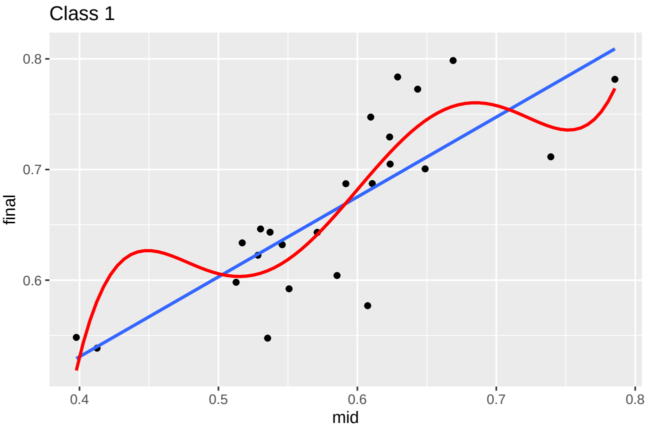
Model 1

```
##
scores %>% ggplot( aes(x = mid, y = final)) + geom_point()+
  labs(title = "Class 1") +
  geom_smooth( method = "lm" , se = FALSE)
```

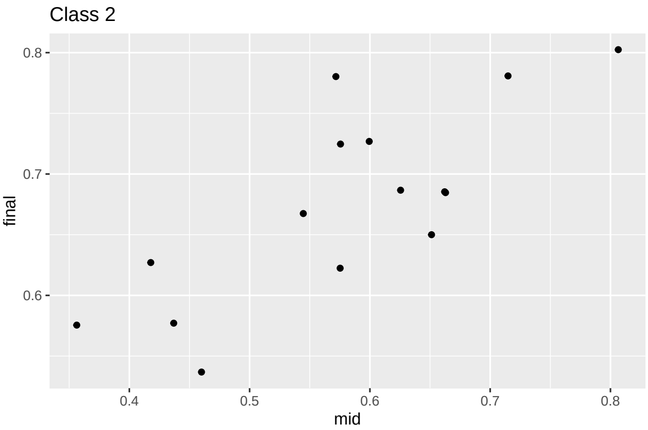


Model 1 and 2

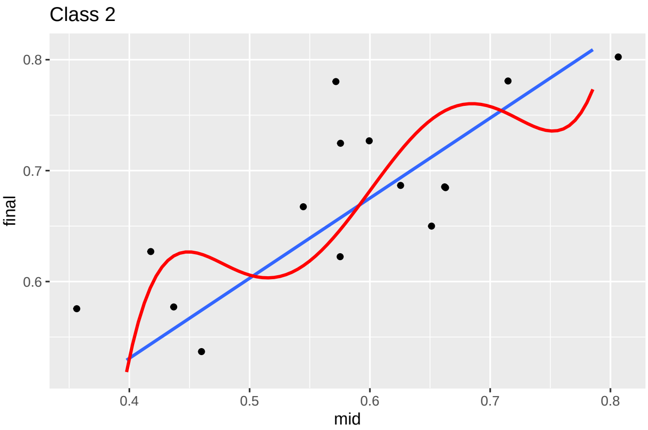
```
scores %>% ggplot( aes(x = mid, y = final)) + geom_point() +
  labs(title = "Class 1") +
  geom_smooth( method = "lm" , se = FALSE) +
  geom_smooth( method = "lm" ,formula = y ~ poly(x, 5), se = FALSE, color = "red")
```



Test Set



Test Set with models



MSE

Prediction accuracy

```
## # A tibble: 15 x 5
##    actual lin_pred poly_pred lin_sq_error poly_sq_error
##     <dbl>    <dbl>     <dbl>        <dbl>         <dbl>
##  1  0.537    0.574    0.625     0.00139       0.00771
##  2  0.687    0.694    0.718     0.0000487     0.000988
##  3  0.576    0.499    0.0801    0.00582       0.245
##  4  0.727    0.675    0.681     0.00271       0.00211
##  5  0.685    0.720    0.754     0.00121       0.00469
##  6  0.781    0.758    0.751     0.000515      0.000871
##  7  0.627    0.544    0.595     0.00695       0.00101
##  8  0.622    0.657    0.647     0.00122       0.000585
##  9  0.725    0.658    0.647     0.00450       0.00603
## 10  0.780    0.655    0.642     0.0157        0.0191
## 11  0.667    0.635    0.614     0.00104       0.00283
## 12  0.685    0.721    0.754     0.00129       0.00485
## 13  0.802    0.824    0.864     0.000478      0.00381
## 14  0.577    0.557    0.623     0.000387      0.00211
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Overall MSE

```
## # A tibble: 1 x 2
##   lin_mse poly_mse
##     <dbl>    <dbl>
## 1 0.00315   0.0208
```

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But no guarantee that model which minimizes MSE on training data will also do so on test data.

In fact, when selecting a complex model that minimizes MSE on the training data, the test MSE will often be very large!

Demo in RStudio

See .Rmd file (Wednesday 9-9 Demo) on the schedule page of the course website

Section 3

Bias-Variance Trade-off

Training vs Test MSE

Suppose we consider a variety of model shapes to predict Y, with each model of increasing complexity. What happens to the training MSE and the test MSE as model complexity increases?

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Expected test MSE can be decomposed as the sum of 3 quantities:

$$\mathrm{E}(y_0 - \hat{f}(x_0)) = \mathrm{Var}(\hat{f}(x_0)) + \left[\mathrm{Bias}(\hat{f}(x_0))\right]^2 + \mathrm{Var}(\epsilon)$$

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To minimize MSE, we need to *simultaneously* minimize both variance and bias.

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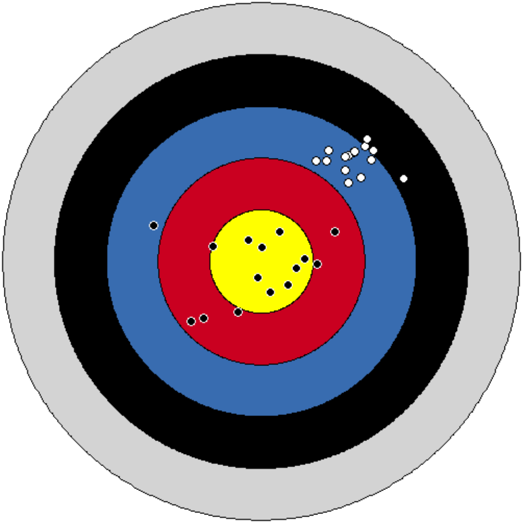
Target Practice



The Trade-off

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How do we solve it?