Chapter 1

A Model of Dynamic Optimization

1.1 The Initial Model

1.1.1 The basic introduction

Once there was an agent, who was endowed with a certain amount of food M, and his life was divided into two periods: $t = 0$ and $t = 1$.

1. **Assumptions of the utility function:**
   (a) He has a utility function $u(c)$, which is same in both periods $t = 0$ and $t = 1$.
   (b) The utility function is strictly increasing: $u'(c) > 0$ for all c.
   (c) The marginal utility is decreasing: $u''(c) < 0$ for all c.

2. **Budget constraints:**
   (a) The food is **not** perishable and can be stored for the next period.
   (b) The agent cannot borrow food, so total consumption cannot exceed M:

$$c_0 + c_1 \leq M$$

3. **Optimization problem:**

$$\max_{c_0, c_1} u(c_0) + u(c_1) \tag{*}$$

subject to:

$$c_0 + c_1 \leq M \tag{1-1-1}$$

1.1.2 Observations

Observation 1. *Given the budget constraints, at optimality, (1-1-1) must hold with equality.*

Proof. By contradiction: Suppose (c_0^*, c_1^*) is optimal but $c_0^* + c_1^* < M$. Let $\hat{c}_1 = c_1^* + \epsilon$ for some small $\epsilon > 0$. Then:

$$u(c_0^*) + u(\hat{c}_1) > u(c_0^*) + u(c_1^*)$$

which contradicts optimality. □

Observation 2. *The optimal solution is $c_0^* = c_1^* = \frac{M}{2}$.*

Proof. By contradiction: Suppose $c_0^* > c_1^*$. Let:

$$\hat{c}_0 = c_0^* - \epsilon, \quad \hat{c}_1 = c_1^* + \epsilon \tag{1-2-1}$$

for small $\epsilon > 0$. Using Taylor expansion:

$$\lim_{\epsilon \to 0} [u(\hat{c}_0) + u(\hat{c}_1)] = u(c_0^*) + u(c_1^*) + \epsilon(u'(c_1^*) - u'(c_0^*)) \tag{1-2-2}$$

Since $u'(c_0^*) < u'(c_1^*)$ by assumption 1.3, (1-2-2) > (1-2-1), contradicting optimality. □

➢ **Remark 1.** *Let's consider the economic **story** behind the mathematical proof:*
Because the marginal utility is the utility of each additional unit of consumption. Whether the agent consumes more at $t = 0$ or $t = 1$, the agent could always sell or buy the good to make himself better off.

1.2 An advanced model

1.2.1 The new assumptions

• **Points 1.** *Introduce β as a discount factor, which means the utility of consumption at $t = 1$ is not as high as the utility of consumption at $t = 0$ to the agent's **lifetime utility**.*
The agent's lifetime utility is:

$$u(c_0) + \beta u(c_1) \tag{2-1-1}$$

• **Points 2.** *How to understand the discount factor?*

1. *You should clearly know that the utility function has never changed.*
2. *The discount factor is a number between 0 and 1. Which means the agent is always **preferring** current consumption to future consumption, the future utility couldn't satisfy the agent's **lifetime utility**.*
3. *The discount factor isn't a constant, it depends on the agent's **patience**. The more patient the agent is, the larger the discount factor is.*

1.2.2 The observations

Observation 3. *Given the budget constraints, at optimality, (1-1-1) must hold with equality.*

Observation 4. *In optimal solution, $c_0^* > c_1^*$.*

Proof. By way of contradiction: Suppose $c_0^* \leq c_1^*$. Let:

$$\hat{c}_0 = c_0^* + \epsilon, \quad \hat{c}_1 = c_1^* - \epsilon$$

for sufficiently small but positive $\epsilon > 0$.
We have to prove that:

$$u(\hat{c}_0) + \beta u(\hat{c}_1) > u(c_0^*) + \beta u(c_1^*) \tag{2-2-1}$$

Using Taylor expansion:

$$\lim_{\epsilon \to 0} [u(\hat{c}_0) + \beta u(\hat{c}_1)] = u(c_0^*) + \beta u(c_1^*) + \epsilon(u'(c_0^*) - \beta u'(c_1^*)) \tag{2-2-2}$$

Since $u'(c_0^*) \geq u'(c_1^*)$ by assumption 1.3, which means $u'(c_0^*) - \beta u'(c_1^*) > 0$.
Obviously, (2-2-2) is strictly better than (2-2-1).
A contradiction. □

➢ **Remark 2.** *What's $u'(c)$ means?*
Mathematical explanation:

$$u'(c) = \lim_{\Delta c \to 0} \frac{u(c + \Delta c) - u(c)}{\Delta c}.$$

Economic explanation: *$u'(c)$ is the marginal utility of consumption at c, which means the rate of change of the utility function with respect to the consumption.*

Proposition 1. *Let (c_0^*, c_1^*) be the optimal solution of the optimization problem, then:*

$$\frac{u'(c_0^*)}{\beta u'(c_1^*)} = 1 \tag{2-2-3}$$

Proof. By the way of contradiction: Suppose (2-2-3) does not hold, then suppose:

$$\frac{u'(c_0^*)}{\beta u'(c_1^*)} < 1$$

In **Economic** sense, it means the marginal utility of **Lifetime utility** at $t = 0$ is strictly less than the marginal utility of **Lifetime utility** at $t = 1$.
Let:

$$\hat{c}_0 = c_0^* - \epsilon, \quad \hat{c}_1 = c_1^* + \epsilon$$

for sufficiently small but positive $\epsilon > 0$.
We have to prove that:

$$u(\hat{c}_0) + \beta u(\hat{c}_1) > u(c_0^*) + \beta u(c_1^*)$$

Using Taylor expansion:

$$\lim_{\epsilon \to 0} [u(\hat{c}_0) + \beta u(\hat{c}_1)] = u(c_0^*) + \beta u(c_1^*) - \epsilon(u'(c_0^*) - \beta u'(c_1^*))$$

Since $u'(c_0^*) < \beta u'(c_1^*)$, which means $u'(c_0^*) - \beta u'(c_1^*) < 0$.
A contradiction. Then a similar contradiction can be derived if $\frac{u'(c_0^*)}{\beta u'(c_1^*)} > 1$, by moving an ϵ amount of consumption from $t = 1$ to $t = 0$. □

Proof. Another proof
Given that c_0^*, c_1^* is optimal solution, any plan $\hat{c}_0, \hat{c}_1$ with $\hat{c}_0 = c_0^* + \epsilon$ and $\hat{c}_1 = c_1^* - \epsilon$ for every ϵ is worse or equal than c_0^*, c_1^*.
That is (in **Mathematical** sense):

$$u(c_0^* + \epsilon) + \beta u(c_1^* - \epsilon) \leq u(c_0^*) + \beta u(c_1^*)$$

Using Taylor expansion: It's easy to get the result that (if ϵ is sufficiently small and positive):

$$u'(c_0^*) - \beta u'(c_1^*) \leq 0$$

In a similar way, we can get the result that (if ϵ is sufficiently small and negative):

$$u'(c_0^*) - \beta u'(c_1^*) \geq 0$$

Overall, it must hold that:

$$u'(c_0^*) - \beta u'(c_1^*) = 0$$

□

➢ **Remark 3.** *We have to understand the optimal solution in an* ***Economic*** *sense:*
In the proof of Contradiction, it's about moving an ϵ amount of consumption from $t = 1$ to $t = 0$.
And the contradiction means that you could always do better by moving, since the lifetime marginal utility is either smaller or larger than the other.
In the another proof, it's also about moving an ϵ amount of consumption from $t = 1$ to $t = 0$.
But the optimal situation means that you shouldn't do better by moving, since the lifetime marginal utility is equal.
So you could reach the equation.
Two types of deviation wouldn't be better

1.3 The introduction of the financial market

1.3.1 The new assumptions

• **Points 3.** *Introduce a financial market, which means the agent can save the food for the next period to use at a certain interest rate r.*

Which means if the agent saves 1 *unit of food at* $t = 0$*, he will have* $1 + r$ *units of food at* $t = 1$.
So the agent's budget constraint is:

$$c_0 + \frac{c_1}{1+r} = M \tag{3-1-1}$$

And the agent's utility function is still:

$$u(c_0) + \beta u(c_1) \tag{2-1-1}$$

- **Points 4.** *How to understand the interest rate?*

1. *The interest rate is a number* ***>0****, which means the agent has "grasped" somehow a great* ***technology*** *to save the food for the next period.*
2. *In the next subsection, we will tell you the details about the interest rate. Please take it as an* ***important fundament****.*

1.3.2 The details about the interest rate

Let's come back to the agent's situation:
The agent has grasped a great **technology** to save the food for the next period.So there comes such a simple quation:

$$1 \text{ unit of good at } t = 0 \Leftrightarrow (1 + r) \text{ units of good at } t = 1 \tag{3-2-1}$$

This equivalence can be rewritten as:

$$\frac{1}{1+r} \text{ units of good at } t = 0 \Leftrightarrow 1 \text{ unit of good at } t = 1 \tag{3-2-2}$$

➢ **Remark 4.** *These equations should be interpreted economically:*

- ***Saving Return****: Investing 1 unit today yields* $(1 + r)$ *units tomorrow*
- ***Borrowing Cost****: Receiving 1 unit today requires repaying* $\frac{1}{1+r}$ *units immediately*

To understand the value transformation, we must consider **Scarcity**:

- Scarcity implies limited availability
- Limited goods possess inherent value

Thus, we can express the value relationships:

$$V_0(1) = (1 + r)V_1(1) \tag{3-2-3}$$

$$V_0\left(\frac{1}{1+r}\right) = V_1(1) \tag{3-2-4}$$

where $V_t(x)$ represents the value of x units at time t.

➢ **Remark 5.** *So let's think the transaction as a* ***contract****.*
It's a trade between the ***present*** *and the* ***future****.*
In your opinion, you have a technology to save the food for the future, which means you will get more food at future.This will make the future suffer a ***devaluation****.*
Devaluation means that the future good is ***less valuable*** *than the present good,and in that case, the amount of food you will get at future is* ***more*** *than the amount of food you will give at present.*

1.3.3 Findings and Proofs

Proposition 2. *Let's rewrite the question as:*

$$\max_{c_0,c_1} u(c_0) + \beta u(c_1)$$

subject to:

$$c_0 + \frac{c_1}{1+r} = M \tag{*}$$

Then the optimal solution is:

$$\frac{u'(c_0^*)}{\beta u'(c_1^*)} = 1 + r \tag{3-3-1}$$

➢ **Remark 6.** *let's concentrate on the left side of the equation:*
In (1.3.2), we have talked about the economic thoughts of 1+r, which could be explained in the value of goods.
In ***remark 2****, we have illustrate the meaning of u'(c), which is the marginal utility with changes in consumption. Let's think the utility as a good can be traded in the market,* ***a market in your mind****, we could get such two equations from the the contract with the "utility evil":*

$$u'(c_0) \textit{ units of additional utility} = 1 \textit{ unit of additional good in } t = 0 \tag{3-3-2}$$

$$\beta u'(c_1) \textit{ units of additional utility} = 1 \textit{ unit of additional good in } t = 1 \tag{3-3-3}$$

How to connect these two periods of life? Take 1 unit of additional utility as the right side of each equation:

$$\frac{1}{u'(c_0)} \textit{ units of additional good in t0} = 1 \textit{ unit of additional utility} \tag{3-3-4}$$

$$\frac{1}{(\beta)u'(c_1)} \textit{ units of additional good in t1} = 1 \textit{ unit of additional utility} \tag{3-3-5}$$

Proof. Economic proof So with the analysis above we could get the contract in your mind market, that is:

$$1 \text{ unit of good at } t = 0 \Leftrightarrow \frac{u'(c_0)}{\beta u'(c_1)} \text{ units of good at } t = 1 \tag{3-3-6}$$

You should find it's similar to 3-2-1.
With your intuition, since the left side of each equation is the same, then the right side of each equation should also be the same!
Then we can reach the (*);
In economic proof:
Suppose not, and you should practice this by yourself! A key step: If the financial market comes to the situation that:
The good today is somehow more expensive, than your mind market. That means, you think the food shouldn't value that much in the financial market. Then you will probably sold it out!
Make it more rare until the value in your mind market equals the value in financial market. □

Proof. Mathematical proof Now that you have understood the economic proof, let's see the mathematical proof.
Suppose not, suppose the optimal solution (c_0^*, c_1^*) has:

$$\frac{u'(c_0^*)}{\beta u'(c_1^*)} > 1 + r$$

Then let $\hat{c_0} = c_0^* + \Delta$, $\hat{c_1} = c_1^* - \delta$, where δ and Δ is a small number. Then we have:

$$\Delta(1 + r) = \delta$$

Then using Taylor expansion:

$$u(\hat{c_0}) + \beta u(\hat{c_1}) = u(c_0^*) + \beta u(c_1^*) + \Delta(u'(c_0^*) - \beta(1 + r)u'(c_0^*))$$

Since $\frac{u'(c_0^*)}{\beta u'(c_1^*)} > 1 + r$, we have:

$$u(\hat{c_0}) + \beta u(\hat{c_1}) > u(c_0^*) + \beta u(c_1^*)$$

A contradiction. A similar contradiction can be derived if $\frac{u'(c_0^*)}{\beta u'(c_1^*)} < 1 + r$. □

Proof. Another mathematical proof Solving the problem by Lagrange method:

$$\max_{c_0, c_1} u(c_0) + \beta u(c_1)$$

subject to:

$$c_0 + \frac{c_1}{1 + r} = M$$

The Lagrangian is:

$$\mathcal{L} = u(c_0) + \beta u(c_1) + \lambda(M - c_0 - \frac{c_1}{1 + r})$$

The first order conditions are:

$$\frac{\partial \mathcal{L}}{\partial c_0} = u'(c_0) - \lambda = 0 \quad (1)$$

$$\frac{\partial \mathcal{L}}{\partial c_1} = \beta u'(c_1) - \lambda \frac{1}{1+r} = 0 \quad (2)$$

From equation (1) and (2), we can derive:

$$\frac{u'(c_0)}{\beta u'(c_1)} = 1 + r$$

□

1.3.4 Substitution effect and wealth effect

Let's back to the financial market, and think the interest rate as a variable.
What will happen if the interest rate r rises?
Back to the optimal solution(3-3-1), we have:

$$\frac{u'(c_0^*)}{\beta u'(c_1^*)} = 1 + r$$

If r rises, the left side of the equation will rise.If the agent choose to save more food at present, then the consumption at present will defintely go down. And the consumption at future will rise.In mathematical sense, it means that the marginal utility of consumption at present will rise, and the marginal utility of consumption at future will fall. So the left side of the equation will definitely rise.
That's what we call the **Substitution effect**.
Substitution effect:
A larger r will lead to a **lower** c_0 and a **higher** c_1.
A larger r will induce the agent to save more and consume less at period 0, and consume more and save at period 1.
However, it's not a must to save more at present, back to the constraint(*).
Since the interest rate is higher, $c_1 + c_2$ will be larger.
So if the agent choose to consume more at present, he still may consume more at future.
And if the agent just consume a little at present, the left side of the equation will still may rise, since the $c_1 + c_2$ will be larger.
This is what we call the **Wealth effect**.
Wealth effect:
A larger r will lead to a **higher** c_0 and a **higher** c_1. A larger r means that the agent can consume the same amount by saving less at period 0, and consume more in both periods.

1.3.5 Take different examples

- **Points 5.**

$$u(c) = \sqrt{c}$$

.

rewrite (3-3-1) as:

$$\frac{\frac{1}{2\sqrt{c_0^*}}}{\beta\frac{1}{2\sqrt{c_1^*}}} = 1 + r$$

which is equivalent to:

$$\frac{c_1^*}{c_0^*} = \beta(1+r)^2$$

Let's write $c_0^* ontherightside$:

$c_1^* = \beta(1+r)^2 c_0^*$ Substitute it into the constraint(*):

$$c_0^* + \beta(1+r)c_0^* = M$$

$$c_0^*(1 + \beta(1+r)) = M$$

With a larger r, c_0^* will fall, and c_1^* will rise.
So in this case, the substitution effect is stronger than the wealth effect. Which **not** means that there is no wealth effect, but it's just that the substitution effect is stronger than the wealth effect.

- **Points 6.**

$$u(c) = \ln c$$

.

rewrite (3-3-1) as:

$$\frac{\frac{1}{c_0^*}}{\beta\frac{1}{c_1^*}} = 1 + r$$

which is equivalent to:

$$\frac{c_1^*}{c_0^*} = \beta(1+r)$$

Let's write $c_0^* ontherightside$:

$c_1^* = \beta(1+r)c_0^*$ Substitute it into the constraint(*):

$$c_0^* + \beta c_0^* = M$$

$$c_0^*(1+\beta) = M$$

With a larger r, c_0^* will be the same, but c_1^* will rise.
So in this case, the substitution effect is equal to the wealth effect.

• **Points 7.**

$$u(c) = -e^{-c}$$

.
rewrite (3-3-1) as:

$$\frac{e^{-c_0^*}}{\beta e^{-c_1^*}} = 1 + r$$

which is equivalent to:

$$c_1^* = c_0^* + \ln(\beta(1+r))$$

Substitute it into the constraint(*):

$$c_0^* + \frac{c_0^* + \ln(\beta(1+r))}{1+r} = M$$

$$c_0^*(1 + \frac{1 + \ln(\beta(1+r))}{1+r}) = M$$

With a larger r, c_0^* will rise, and c_1^* will rise.
So in this case, the substitution effect is ***not*** *stronger than the wealth effect.*

1.4 Dividing the lifetime into infinite periods

However in real life, you can't divide the lifetime into just two periods. So we need to extend the model to the infinite periods. The agent again, has access to the financial market, which allows him to save/convert any X units of consumtion at period t into $(1+r)^t$ units of consumtion at period $t+1$, for all $t = 0, 1, 2, \cdots$.

1.4.1 The new assumptions

• **Points 8.** *We then divide the lifetime into infinite periods,that is $t = 0, 1, 2, \cdots$. And the r is then the interest rate, which is a constant in each periods.In the financial market, you just save the food for the next period to use at a certain interest rate.*
And the M is the initial wealth, which is the amount of food you have at $t = 0$, it's a constant.
And the c_t is the amount of food you will consume at t, it's a variable. And the β is the discount factor, which is a constant. The budget constraint is:

$$c_0 + \frac{c_1}{1+r} + \frac{c_2}{(1+r)^2} + \cdots = M$$

The utility function is:

$$u(c_0) + \beta u(c_1) + \beta^2 u(c_2) + \cdots$$

let's rewrite the question as:

$$\max_{c_0,c_1,c_2,\cdots} u(c_0)+\beta u(c_1)+\beta^2 u(c_2)+\cdots$$

subject to:

$$c_0+\frac{c_1}{1+r}+\frac{c_2}{(1+r)^2}+\cdots=M$$

1.4.2 The optimal solution

The optimal solution is:

$$\frac{u'(c_t^*)}{\beta u'(c_{t+1}^*)}=1+r \tag{4-2-1}$$

It's similar to the previous question, but the difference is that the c_t is a variable. Since t could be any period, so we still solve the problem by the same way.

1.4.3 A simple example to illusrate the two effects

- **Points 9.**

$$u(c)=\sqrt{c}$$

.

rewrite (4-2-1) as:

$$\frac{\frac{1}{2\sqrt{c_t^*}}}{\beta\frac{1}{2\sqrt{c_{t+1}^*}}}=1+r$$

which is equivalent to:

$$\frac{c_{t+1}^*}{c_t^*}=\beta^2(1+r)^2$$

which is equivalent to:

$$c_{t+1}^*=\beta^2(1+r)^2c_t^* \tag{4-2-2}$$

for all $t=0,1,2,\cdots$.

The simplification of problem

Let's rewrite the optimal question using (4-2-2): $c_1=\beta^2(1+r)^2c_0$ $c_2=\beta^2(1+r)^2c_1$ $c_3=\beta^2(1+r)^2c_2$ ... $c_{t+1}=\beta^2(1+r)^2c_t$ ... Let $\Delta=\beta^2 1+r^2$, then we have:

$$c_0+\frac{\Delta c_0}{1+r}+\frac{\Delta^2 c_0}{(1+r)^2}+\cdots=M$$

which is equivalent to:

$$c_0(1+\frac{\Delta}{1+r}+\frac{\Delta}{(1+r)^2}+\cdots)=M$$

You know, if r is too large, you will definitely get an unbelievable return, that means the left side of the equation will be almost infinite, which is impossible. So let's asuume $\frac{\Delta}{1+r} < 1$, then we have:

$$c_0(\frac{1}{1-\frac{\Delta}{1+r}}) = M$$

that is:

$$c_0 = M(1-\beta^2(1+r))$$

So you should discover that the c_0 is related to the r and β.
In this example, you should find that with a larger r, c_0 will fall. So, the **substitution effect** is stronger than the **wealth effect**.

☞ **Tips 1.** *About mathematical mechanism:*
Let $x < 1$, then we have:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

Let $Z = 1 + x + x^2 + x^3 + \cdots$, then we have:

$$Z = 1 + x(1 + x + x^2 + x^3 + \cdots) = 1 + xZ$$
$$Z(1-x) = 1$$
$$Z = \frac{1}{1-x}$$

➢ **Remark 7.** *A more economic review of the model:*
Remember:

$$c_{t+1} = \beta^2(1+r)^2 c_t$$

Which is equivalent to:

$$c_t = [\beta^2(1+r)^2]^t(1-\beta^2(1+r))M$$

Suppose $r = 0$, then we have:

$$c_t = \beta^{2t}(1-\beta^2)M$$

Let $\delta = \beta^2$, then we have:

$$c_t = \delta^t(1-\delta)M$$

Taking the partial derivative of c_t with respect to t:

$$\frac{\partial c_t}{\partial t} = [t\delta^{t-1} + (t+1)\delta^t]M$$

Which is equivalent to:

$$\frac{\partial c_t}{\partial t} = \delta^t(\frac{t}{\delta} - 1)M$$

*So if δ is sufficiently small, then the consumption will rise over time. This is what we call the **time preference**. It means that if the agent is more patient, then he will consume more at present, and consume less at future.*