Chapter 2

A model of insurance

> One's destiny, of course, depends on one's self-struggle, but one must also take into account the course of history.
> —President Jiang

2.1 An interesting economic story

You know, the world isn't always so perfect that your situation remains constant. You always encounter some risks and uncertainty. Let's make the story more theoretical.

2.1.1 Assumptions

The agent lives in a period divided into two parts: the **ex post** state and the **ex ante** state.

The ex post period is after event X occurs, while the ex ante period is before event X occurs.

How does the event affect you? After the event happens, you will receive different incomes: M_1 and M_2.

These different incomes occur with different probabilities: $\pi_i = \text{Prob}\{\theta = \theta_i\}, i = 1, 2$, where $\pi_i > 0$ and $\pi_1 + \pi_2 = 1$.

Assume that in each of your ex post income states, the utility function the agent uses to measure utility is $u(c)$, which is monotonically increasing and weakly concave, meaning: $u'(c) > 0$, $u''(c) \leq 0$.

What does the agent know in the current (ex ante) state?

- The probability of each ex post income state
- The possible incomes (c_1, c_2)
- The utility function

What does the agent not know? Which state he will find himself in - that's the risk!

Therefore, his utility in ex post life is:

$$u(c_1, c_2) = \pi_1 u(c_1) + \pi_2 u(c_2)$$

We call this function "expected utility", representing the expected value of his utility across the two different ex post states of the world.

Note that c represents consumption, not wage, in each state.

2.1.2 Expansion to multiple states

Clearly, if $\theta = \{\theta_1, \theta_2, \ldots, \theta_N\}$, $\pi_i = \text{Prob}\{\theta = \theta_i\}, i = 1, 2, \ldots, N$.

Then:

$$u(c_1, c_2, \ldots, c_N) = \sum_{i=1}^{N} \pi_i u(c_i)$$

2.1.3 Insurance

You know, we fear risks. From the previous model, we should conclude that a relatively controllable and even allocation is always better than a random allocation.

However, unlike the previous model - where we were superstars mastering a technology to reallocate food between our young and old selves - now we must face risks.

But you know, risks might make you rich or might make you poor. It doesn't depend on you, but you have a desire to transfer some goods from your rich self (one day you might find money) to your poor self (one day you might be robbed).

The desire exists, but you can't reallocate food this time.

How can you implement a plan to transfer consumption from your rich self to your poor self?

That's what insurance is!

• **Points 1.** *My view Insurance means receiving consumption when I'm poor and paying when I'm rich.*
Through a contract, I achieve balance with myself.

• **Points 2.** *Market view Insurance is traded now. The underlying logic is that the market has a somewhat different utility function (linear).*

2.1.4 Underlying logic of insurance

From the assumptions, you should derive two straightforward conclusions: the concave nature of the utility function, and the expected ex ante utility function.

From these assumptions we can conclude:

$$u(\pi_1 c_1 + \pi_2 c_2) > \pi_1 u(c_1) + \pi_2 u(c_2)$$

This is the underlying logic, and we will now provide a proof.

2.2 Proof of the underlying logic

2.2.1 Mathematical proof

The inequality we want to prove is:

$$u(\pi_1 c_1 + \pi_2 c_2) \geq \pi_1 u(c_1) + \pi_2 u(c_2)$$

This directly applies **Jensen's Inequality**, which states that for a concave function u:

$$u\left(\sum_{i=1}^{n} \lambda_i x_i\right) \geq \sum_{i=1}^{n} \lambda_i u(x_i)$$

where $\lambda_i \geq 0$ and $\sum_{i=1}^{n} \lambda_i = 1$.

证明. For the two-state case:

1. Since u is concave, by definition the secant line lies below the function:

$$u(\lambda c_1 + (1-\lambda)c_2) \geq \lambda u(c_1) + (1-\lambda)u(c_2)$$

for any $\lambda \in [0,1]$.

2. Let $\lambda = \pi_1$ and $1-\lambda = \pi_2$, then:

$$u(\pi_1 c_1 + \pi_2 c_2) \geq \pi_1 u(c_1) + \pi_2 u(c_2)$$

3. Equality holds only when u is linear or when $c_1 = c_2$.

□

This shows that risk-averse individuals (with concave utility) prefer certain consumption $\bar{c} = \pi_1 c_1 + \pi_2 c_2$ over uncertain consumption (c_1, c_2) with the same expected value.

Consider $u(c) = \sqrt{c}$. For all $c_1, c_2 > 0$, $c_1 < c_2$, can you show:

$$\sqrt{\pi_1 c_1 + \pi_2 c_2} > \pi_1 \sqrt{c_1} + \pi_2 \sqrt{c_2}$$

证明. This is equivalent to proving:

$$\pi_1 c_1 + \pi_2 c_2 > (\pi_1 \sqrt{c_1} + \pi_2 \sqrt{c_2})^2$$

Expanding the right side:

$$\pi_1 c_1 + \pi_2 c_2 > \pi_1^2 c_1 + 2\pi_1 \pi_2 \sqrt{c_1 c_2} + \pi_2^2 c_2$$

Rearranging terms:

$$0 > \pi_1(\pi_1 - 1)c_1 + \pi_2(\pi_2 - 1)c_2 + 2\pi_1 \pi_2 \sqrt{c_1 c_2}$$

Since $\pi_1 + \pi_2 = 1$, we have $\pi_1 - 1 = -\pi_2$ and $\pi_2 - 1 = -\pi_1$:

$$0 > -\pi_1 \pi_2 c_1 - \pi_1 \pi_2 c_2 + 2\pi_1 \pi_2 \sqrt{c_1 c_2}$$

Dividing both sides by $\pi_1 \pi_2$:

$$0 > -c_1 - c_2 + 2\sqrt{c_1 c_2}$$

Which can be rewritten as:

$$c_1 + c_2 - 2\sqrt{c_1 c_2} > 0$$

This is a perfect square:

$$(\sqrt{c_1} - \sqrt{c_2})^2 > 0$$

Since $c_1 \neq c_2$, the inequality always holds. □

2.2.2 Economic proof

➢ **Remark 1.** *An economic explanation of the above inequality: The **LHS** represents the ex ante utility of average consumption, where the function's variable is the expected amount of consumption in your ex post life. Since all probabilities and incomes are given, it's a constant. The **RHS** represents the average of utilities across states. Therefore, as a risk-averse agent, one will always prefer the certain plan over the lottery.*

Now I will provide an economic proof: Take $\pi_1 = \pi_2$. Suppose someone offers a trade: Each time you transfer a little from your rich self to your poor self, utility improves, until you can no longer do so. This involves transferring 1 unit of consumption from state 2 to state 1. A large gap between c_1 and c_2 means facing greater risk.

证明. Suppose we start with $c_1 < c_2$. Assume $\pi_1 = \pi_2$. Suppose someone offers to give you a small amount of consumption δ, $\delta > 0$, in state 1, but asks you to pay back the same δ units of consumption in state 2. This trade will make you better off. By the property of diminishing marginal utility:

$$u'(c_1) > u'(c_2)$$

Therefore, transferring 1 unit of consumption from state 2 to state 1 increases total utility:

$$\pi_1 u(c_1 + \delta) + \pi_2 u(c_2 - \delta) = \pi_1 u(c_1) + \pi_2 u(c_2) + \pi_1 (u'(c_1) - u'(c_2))$$

Since $c_1 < c_2$, $u'(c_1) > u'(c_2)$. Thus:

$$\pi_1 u(c_1 + \delta) + \pi_2 u(c_2 - \delta) > \pi_1 u(c_1) + \pi_2 u(c_2)$$

This process continues until $c_1 = c_2$, when optimality is achieved. □

Hence, we get a conclusion: A smoother consumption means a better utility.

2.3 Introduction to a Insurance corporation

As a contract, we should get a fair deal with the corporation who provides good or take goods.
As a fair deal, on average, how much you get from agent must equal to how much you dive the agent. It's on the **average**.

2.3.1 Risk neutral

Like the definition of "Risk averse", "Risk neutral" means the agent don't care about risk. In Mathematical definition, if the agent has $u'(c) > 0$, $u''(c) = 0$, for all c.This suppose the agent's utility is linear, with $u(c) = \alpha + \beta c$, for some α and β, $\beta > 0$.
So we rewrite the optimaility:

$$u(\pi_1 c_1 + \pi_2 c_2) = \pi_1 u(c_1) + \pi_2 u(c_2)$$

This equation now holds for all c_1 and c_2.

➢ **Remark 2.** *What does this equation mean?*
It means a risk-neutral agent is indifferent between the utility of average ex post consumption and the average of expected utilities across ex post states. Therefore, the agent neither prefers nor dislikes risk - they are simply neutral to it.

So we can introduce the insurance company into the question:
A risk neutral insurance company serves "insurance contract" in a market populated by a large amount number of risk averse consumers. Who each has a random income of θ, which offers agent "consumption" θ_1 at prob π_1, θ_2 at prob π_2, where $c_1 < c_2$, $\pi_1 + \pi_2 = 1$, $\pi_1 > 0$, $\pi_2 > 0$. The value of θ_1 θ_2 π_1 π_2 are all known and given to the people in the market.

The consumer are all expected utility maximizers, and their utility function is u(c), and $u'(c) > 0$, $u''(c) < 0$, for all c.

2.3.2 The new situation

There is an "insurance contract", which is sold by a monopolist seller.For any i unit of consumption in state θ_1, the price is p units of consumption in state θ_2
That is:
The contract says if you promise to pay me p units of good in state θ_2, I'll pay you 1 unit of good in state θ_1.
So in order to optimize the insurance contract:

$$\max_p X(p) = u(\pi_1(-1) + \pi_2(p))$$

- **Points 3.**
  - *$X(p)$ represents the demand function from the consumer's perspective. When the insurance company sets the price at p, $X(p)$ determines how many insurance contracts the consumer will purchase. This is entirely the consumer's choice.*
  - *The insurance company's utility comes from paying out in state θ_1 and receiving payments in state θ_2. The net utility can be expressed as $u(c) = u(\pi_1(-1) + \pi_2(p))$, where -1 represents the payout and p represents the premium received.*

However we also need to consider ibn the consumer's shoes, since X(p) is decided by consumers. As a consumer, I also want to maximize my utility, you know my utility optimal question is

$$\max_X \pi_1 u(\theta_1 + X(p)) + \pi_2 u(\theta_2 - pX(p))$$

So here X(p) is the optimal amout of insurance(i.e. consumption at θ_1), to buy from the insurance company, which could be solved from the above inequality.