Chapter 1

A Model of Dynamic Optimization

Economics is about intelligence──Prof. Wang

1.1 The Initial Model

1.1.1 The basic introduction

Once there was an agent, who was endowed with a certain amount of food M, and his life was divided into two periods: $t = 0$ and $t = 1$.

1. **Assumptions of the utility function:**

    (a) He has a utility function $u(c)$, which is same in both periods $t = 0$ and $t = 1$.

    (b) The utility function is strictly increasing: $u'(c) > 0$ for all c.

    (c) The marginal utility is decreasing: $u''(c) < 0$ for all c.

2. **Budget constraints:**

    (a) The food is **not** perishable and can be stored for the next period.

    (b) The agent cannot borrow food, so total consumption cannot exceed M:

$$c_0 + c_1 \leq M$$

3. **Optimization problem:**

$$\max_{c_0, c_1} u(c_0) + u(c_1) \tag{*}$$

subject to:

$$c_0 + c_1 \leq M \tag{1-1-1}$$

1.1.2 Observations

Observation 1. *Given the budget constraints, at optimality, (1-1-1) must hold with equality.*

证明. By contradiction: Suppose (c_0^*, c_1^*) is optimal but $c_0^* + c_1^* < M$. Let $\hat{c}_1 = c_1^* + \epsilon$ for some small $\epsilon > 0$. Then:

$$u(c_0^*) + u(\hat{c}_1) > u(c_0^*) + u(c_1^*)$$

which contradicts optimality. □

Observation 2. *The optimal solution is $c_0^* = c_1^* = \frac{M}{2}$.*

证明. By contradiction: Suppose $c_0^* > c_1^*$. Let:

$$\hat{c}_0 = c_0^* - \epsilon, \quad \hat{c}_1 = c_1^* + \epsilon \tag{1-2-1}$$

for small $\epsilon > 0$. Using Taylor expansion:

$$\lim_{\epsilon \to 0} [u(\hat{c}_0) + u(\hat{c}_1)] = u(c_0^*) + u(c_1^*) + \epsilon(u'(c_1^*) - u'(c_0^*)) \tag{1-2-2}$$

Since $u'(c_0^*) < u'(c_1^*)$ by assumption 1.3, (1-2-2) > (1-2-1), contradicting optimality. □

➢ **Remark 1.** *Let's consider the economic **story** behind the mathematical proof:*
Because the marginal utility is the utility of each additional unit of consumption. Whether the agent consumes more at $t = 0$ or $t = 1$, the agent could always sell or buy the good to make himself better off.

1.2 An advanced model

1.2.1 The new assumptions

• **Points 1.** *Introduce β as a discount factor, which means the utility of consumption at $t = 1$ is not as high as the utility of consumption at $t = 0$ to the agent's **lifetime utility**.*
The agent's lifetime utility is:

$$u(c_0) + \beta u(c_1) \qquad (2\text{-}1\text{-}1)$$

• **Points 2.** *How to understand the discount factor?*

1. *You should clearly know that the utility function has never changed.*

2. *The discount factor is a number between 0 and 1. Which means the agent is always **preferring** current consumption to future consumption, the future utility couldn't satisfy the agent's **lifetime utility**.*

3. *The discount factor isn't a constant, it depends on the agent's **patience**. The more patient the agent is, the larger the discount factor is.*

1.2.2 The observations

Observation 3. *Given the budget constraints, at optimality, (1-1-1) must hold with equality.*

Observation 4. *In optimal solution, $c_0^* > c_1^*$.*

证明. By way of contradiction: Suppose $c_0^* \leq c_1^*$. Let:

$$\hat{c}_0 = c_0^* + \epsilon, \quad \hat{c}_1 = c_1^* - \epsilon$$

for sufficiently small but positive $\epsilon > 0$.
We have to prove that:

$$u(\hat{c}_0) + \beta u(\hat{c}_1) > u(c_0^*) + \beta u(c_1^*) \qquad (2\text{-}2\text{-}1)$$

Using Taylor expansion:

$$\lim_{\epsilon \to 0} [u(\hat{c}_0) + \beta u(\hat{c}_1)] = u(c_0^*) + \beta u(c_1^*) + \epsilon(u'(c_0^*) - \beta u'(c_1^*)) \qquad \text{(2-2-2)}$$

Since $u'(c_0^*) \geq u'(c_1^*)$ by assumption 1.3, which means $u'(c_0^*) - \beta u'(c_1^*) > 0$. Obviously, (2-2-2) is strictly better than (2-2-1).

A contradiction. $\qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \square$

➤ **Remark 2.** *What's $u'(c)$ means?*

Mathematical explanation:

$$u'(c) = \lim_{\Delta c \to 0} \frac{u(c + \Delta c) - u(c)}{\Delta c}.$$

Economic explanation: $u'(c)$ *is the marginal utility of consumption at c, which means the rate of change of the utility function with respect to the consumption.*

Proposition 1. *Let (c_0^*, c_1^*) be the optimal solution of the optimization problem, then:*

$$\frac{u'(c_0^*)}{\beta u'(c_1^*)} = 1 \qquad \qquad \text{(2-2-3)}$$

证明. By the way of contradiction: Suppose (2-2-3) does not hold, then suppose:

$$\frac{u'(c_0^*)}{\beta u'(c_1^*)} < 1$$

In **Economic** sense, it means the marginal utility of **Lifetime utility** at $t = 0$ is strictly less than the marginal utility of **Lifetime utility** at $t = 1$.

Let:

$$\hat{c}_0 = c_0^* - \epsilon, \quad \hat{c}_1 = c_1^* + \epsilon$$

for sufficiently small but positive $\epsilon > 0$.

We have to prove that:

$$u(\hat{c}_0) + \beta u(\hat{c}_1) > u(c_0^*) + \beta u(c_1^*)$$

Using Taylor expansion:

$$\lim_{\epsilon \to 0} [u(\hat{c}_0) + \beta u(\hat{c}_1)] = u(c_0^*) + \beta u(c_1^*) - \epsilon(u'(c_0^*) - \beta u'(c_1^*))$$

Since $u'(c_0^*) < \beta u'(c_1^*)$, which means $u'(c_0^*) - \beta u'(c_1^*) < 0$.

A contradiction. Then a similar contradiction can be derived if $\frac{u'(c_0^*)}{\beta u'(c_1^*)} > 1$, by moving an ϵ amount of consumption from $t = 1$ to $t = 0$. □

证明. Another proof

Given that c_0^*, c_1^* is optimal solution, any plan $\hat{c}_0, \hat{c}_1$ with $\hat{c}_0 = c_0^* + \epsilon$ and $\hat{c}_1 = c_1^* - \epsilon$ for every ϵ is worse or equal than c_0^*, c_1^*.

That is (in **Mathematical** sense):

$$u(c_0^* + \epsilon) + \beta u(c_1^* - \epsilon) \leq u(c_0^*) + \beta u(c_1^*)$$

Using Taylor expansion: It's easy to get the result that (if ϵ is sufficiently small and positive):

$$u'(c_0^*) - \beta u'(c_1^*) \leq 0$$

In a similar way, we can get the result that (if ϵ is sufficiently small and negative):

$$u'(c_0^*) - \beta u'(c_1^*) \geq 0$$

Overall, it must hold that:

$$u'(c_0^*) - \beta u'(c_1^*) = 0$$

□

➤ **Remark 3.** *We have to understand the optimal solution in an **Economic** sense:*

In the proof of Contradiction, it's about moving an ϵ amount of consumption from $t = 1$ to $t = 0$.

And the contradiction means that you could always do better by moving, since the lifetime marginal utility is either smaller or larger than the other. In the another proof, it's also about moving an ϵ amount of consumption from $t = 1$ to $t = 0$.

But the optimal situation means that you shouldn't do better by moving, since the lifetime marginal utility is equal.

So you could reach the equation.

Two types of deviation wouldn't be better

1.3   The introduction of the financial market

1.3.1   The new assumptions

• **Points 3.** *Introduce a financial market, which means the agent can save the food for the next period to use at a certain interest rate r.*
Which means if the agent saves 1 unit of food at $t = 0$, he will have $1 + r$ units of food at $t = 1$.
So the agent's budget constraint is:

$$c_0 + \frac{c_1}{1 + r} = M \tag{3-1-1}$$

And the agent's utility function is still:

$$u(c_0) + \beta u(c_1) \tag{2-1-1}$$

• **Points 4.** *How to understand the interest rate?*

1. *The interest rate is a number $>\mathbf{0}$, which means the agent has "grasped" somehow a great **technology** to save the food for the next period.*

2. *In the next subsection, we will tell you the details about the interest rate. Please take it as an **important fundament**.*

1.3.2   The details about the interest rate

Let's come back to the agent's situation:
The agent has grasped a great **technology** to save the food for the next period.So there comes such a simple quation:

$$1 \text{ unit of good at } t = 0 \Leftrightarrow (1 + r) \text{ units of good at } t = 1 \tag{3-2-1}$$

This equivalence can be rewritten as:

$$\frac{1}{1 + r} \text{ units of good at } t = 0 \Leftrightarrow 1 \text{ unit of good at } t = 1 \tag{3-2-2}$$

➤ **Remark 4.** *These equations should be interpreted economically:*

- ***Saving Return**: Investing 1 unit today yields $(1 + r)$ units tomorrow*

- **_Borrowing Cost_**: *Receiving 1 unit today requires repaying* $\frac{1}{1+r}$ *units immediately*

To understand the value transformation, we must consider **Scarcity**:

- Scarcity implies limited availability

- Limited goods possess inherent value

Thus, we can express the value relationships:

$$V_0(1) = (1+r)V_1(1) \tag{3-2-3}$$

$$V_0\left(\frac{1}{1+r}\right) = V_1(1) \tag{3-2-4}$$

where $V_t(x)$ represents the value of x units at time t.

➤ **Remark 5.** *So let's think the transaction as a **contract**.*
*It's a trade between the **present** and the **future**.*
*In your opinion, you have a technology to save the food for the future, which means you will get more food at future. This will make the future suffer a **devaluation**.*
*Devaluation means that the future good is **less valuable** than the present good, and in that case, the amount of food you will get at future is **more** than the amount of food you will give at present.*

1.3.3 Findings and Proofs

Proposition 2. *Let's rewrite the question as:*

$$\max_{c_0, c_1} u(c_0) + \beta u(c_1)$$

subject to:

$$c_0 + \frac{c_1}{1+r} = M \tag{*}$$

Then the optimal solution is:

$$\frac{u'(c_0^*)}{\beta u'(c_1^*)} = 1 + r \tag{3-3-1}$$

➤ **Remark 6.** *let's concentrate on the left side of the equation:*
In (1.3.2), we have talked about the economic thoughts of 1+r, which could be explained in the value of goods.
*In **remark 2**, we have illustrate the meaning of u'(c), which is the marginal utility with changes in consumption. Let's think the utility as a good can be traded in the market, **a market in your mind**, we could get such two equations from the the contract with the "utility evil":*

$u'(c_0)$ *units of additional utility* $= 1$ *unit of additional good in* $t = 0$
$$(3\text{-}3\text{-}2)$$
$\beta u'(c_1)$ *units of additional utility* $= 1$ *unit of additional good in* $t = 1$
$$(3\text{-}3\text{-}3)$$

How to connect these two periods of life? Take 1 unit of additional utility as the right side of each equation:

$\dfrac{1}{u'(c_0)}$ *units of additional good in t0* $= 1$ *unit of additional utility* $(3\text{-}3\text{-}4)$

$\dfrac{1}{(\beta)u'(c_1)}$ *units of additional good in t1* $= 1$ *unit of additional utility*
$$(3\text{-}3\text{-}5)$$

证明. Economic proof So with the analysis above we could get the contract in your mind market, that is:

$$1 \text{ unit of good at } t = 0 \Leftrightarrow \frac{u'(c_0)}{\beta u'(c_1)} \text{ units of good at } t = 1 \qquad (3\text{-}3\text{-}6)$$

You should find it's similar to 3-2-1.
With your intuition, since the left side of each equation is the same, then the right side of each equation should also be the same!
Then we can reach the (*);
In economic proof:
Suppose not, and you should practice this by yourself! A key step: If the financial market comes to the situation that:
The good today is somehow more expensive, than your mind market. That means, you think the food shouldn't value that much in the financial market.
Then you will probably sold it out!

Make it more rare until the value in your mind market equals the value in financial market. □

证明. Mathematical proof Now that you have understood the economic proof, let's see the mathematical proof.

Suppose not, suppose the optimal solution (c_0^*, c_1^*) has:

$$\frac{u'(c_0^*)}{\beta u'(c_1^*)} > 1 + r$$

Then let $\hat{c}_0 = c_0^* + \Delta$, $\hat{c}_1 = c_1^* - \delta$, where δ and Δ is a small number. Then we have:

$$\Delta(1 + r) = \delta$$

Then using Taylor expansion:

$$u(\hat{c}_0) + \beta u(\hat{c}_1) = u(c_0^*) + \beta u(c_1^*) + \Delta(u'(c_0^*) - \beta(1 + r)u'(c_0^*))$$

Since $\frac{u'(c_0^*)}{\beta u'(c_1^*)} > 1 + r$, we have:

$$u(\hat{c}_0) + \beta u(\hat{c}_1) > u(c_0^*) + \beta u(c_1^*)$$

A contradiction. A similar contradiction can be derived if $\frac{u'(c_0^*)}{\beta u'(c_1^*)} < 1 + r$. □

证明. Another mathematical proof Solving the problem by Lagrange method:

$$\max_{c_0, c_1} u(c_0) + \beta u(c_1)$$

subject to:

$$c_0 + \frac{c_1}{1 + r} = M$$

The Lagrangian is:

$$\mathcal{L} = u(c_0) + \beta u(c_1) + \lambda(M - c_0 - \frac{c_1}{1 + r})$$

The first order conditions are:

$$\frac{\partial \mathcal{L}}{\partial c_0} = u'(c_0) - \lambda = 0 \quad (1)$$

$$\frac{\partial \mathcal{L}}{\partial c_1} = \beta u'(c_1) - \lambda \frac{1}{1+r} = 0 \quad (2)$$

From equation (1) and (2), we can derive:

$$\frac{u'(c_0)}{\beta u'(c_1)} = 1 + r$$

□

1.3.4   Substitution effect and wealth effect

Let's back to the financial market, and think the interest rate as a variable.

What will happen if the interest rate r rises?

Back to the optimal solution(3-3-1), we have:

$$\frac{u'(c_0^*)}{\beta u'(c_1^*)} = 1 + r$$

If r rises, the left side of the equation will rise.If the agent choose to save more food at present, then the consumption at present will defintely go down. And the consumption at future will rise.In mathemetical sense, it means that the marginal utility of consumption at present will rise, and the marginal utility of consumption at future will fall. So the left side of the equation will definitely rise.

That's what we call the **Substitution effect**.

Substitution effect:

A larger r will lead to a **lower** c_0 and a **higher** c_1.

A larger r will induce the agent to save more and consume less at period 0, and consume more and save at period 1.

However, it's not a must to save more at present, back to the constraint(*).

Since the interest rate is higher, $c_1 + c_2$ will be larger.

So if the agent choose to consume more at present, he still may consume more at future.

And if the agent just consume a little at present, the left side of the equation will still may rise, since the $c_1 + c_2$ will be larger.

This is what we call the **Wealth effect**.

Wealth effect:

A larger r will lead to a **higher** c_0 and a **higher** c_1. A larger r means that the agent can consume the same amount by saving less at period 0, and consume more in both periods.

1.3.5   Take different examples

• **Points 5.**

$u(c) = \sqrt{c}$. *rewrite (3-3-1) as:*

$$\frac{\frac{1}{2\sqrt{c_0^*}}}{\beta \frac{1}{2\sqrt{c_1^*}}} = 1 + r$$

which is equivalent to:

$$\frac{c_1^*}{c_0^*} = \beta(1+r)^2$$

Let's write c_0^* on the right side:

$$c_1^* = \beta(1+r)^2 c_0^*$$

Substitute it into the constraint(*):

$$c_0^* + \beta(1+r)c_0^* = M$$

$$c_0^*(1 + \beta(1+r)) = M$$

*With a larger r, c_0^* will fall, and c_1^* will rise. So in this case, the substitution effect is stronger than the wealth effect. Which **not** means that there is no wealth effect, but it's just that the substitution effect is stronger than the wealth effect.*

• **Points 6.**

$u(c) = \ln c$. *rewrite (3-3-1) as:*

$$\frac{\frac{1}{c_0^*}}{\beta \frac{1}{c_1^*}} = 1 + r$$

which is equivalent to:

$$\frac{c_1^*}{c_0^*} = \beta(1+r)$$

Let's write c_0^* on the right side:

$$c_1^* = \beta(1+r)c_0^*$$

Substitute it into the constraint(*):

$$c_0^* + \beta c_0^* = M$$

$$c_0^*(1+\beta) = M$$

With a larger r, c_0^* will be the same, but c_1^* will rise. So in this case, the substitution effect is equal to the wealth effect.

- **Points 7.**

$u(c) = -e^{-c}$. *rewrite (3-3-1) as:*

$$\frac{e^{-c_0^*}}{\beta e^{-c_1^*}} = 1+r$$

which is equivalent to:

$$c_1^* = c_0^* + \ln(\beta(1+r))$$

Substitute it into the constraint(*):

$$c_0^* + \frac{c_0^* + \ln(\beta(1+r))}{1+r} = M$$

$$c_0^*\left(1 + \frac{1 + \ln(\beta(1+r))}{1+r}\right) = M$$

*With a larger r, c_0^* will rise, and c_1^* will rise. So in this case, the substitution effect is **not** stronger than the wealth effect.*

1.4   Dividing the lifetime into infinite periods

However in real life, you can't divide the lifetime into just two periods. So we need to extend the model to the infinite periods. The agent again, has access to the financial market, which allows him to save/convert any X units of consumtion at period t into $(1+r)^t$ units of consumtion at period $t+1$, for all $t = 0, 1, 2, \cdots$.

1.4.1 The new assumptions

• **Points 8.** *We then divide the lifetime into infinite periods,that is* $t = 0, 1, 2, \cdots$. *And the* r *is then the interest rate, which is a constant in each periods.In the financial market, you just save the food for the next period to use at a certain interest rate.*

And the M *is the initial wealth, which is the amount of food you have at* $t = 0$, *it's a constant.*

And the c_t *is the amount of food you will consume at* t, *it's a variable. And the* β *is the discount factor, which is a constant. The budget constraint is:*

$$c_0 + \frac{c_1}{1+r} + \frac{c_2}{(1+r)^2} + \cdots = M$$

The utility function is:

$$u(c_0) + \beta u(c_1) + \beta^2 u(c_2) + \cdots$$

let's rewrite the question as:

$$\max_{c_0, c_1, c_2, \cdots} u(c_0) + \beta u(c_1) + \beta^2 u(c_2) + \cdots$$

subject to:

$$c_0 + \frac{c_1}{1+r} + \frac{c_2}{(1+r)^2} + \cdots = M$$

1.4.2 The optimal solution

The optimal solution is:

$$\frac{u'(c_t^*)}{\beta u'(c_{t+1}^*)} = 1 + r \tag{4-2-1}$$

It's similar to the previous question, but the difference is that the c_t is a variable.

Since t could be any period, so we still solve the problem by the same way.

1.4.3 A simple example to illusrate the two effects

• **Points 9.**

$$u(c) = \sqrt{c}$$

.

rewrite (4-2-1) as:

$$\frac{\frac{1}{2\sqrt{c_t^*}}}{\beta \frac{1}{2\sqrt{c_{t+1}^*}}} = 1 + r$$

which is equivalent to:

$$\frac{c_{t+1}^*}{c_t^*} = \beta^2 (1+r)^2$$

which is equivalent to:

$$c_{t+1}^* = \beta^2 (1+r)^2 c_t^* \tag{4-2-2}$$

for all $t = 0, 1, 2, \cdots$.

The simplification of problem

Let's rewrite the optimal question using (4-2-2): $c_1 = \beta^2(1+r)^2 c_0$ $c_2 = \beta^2(1+r)^2 c_1$ $c_3 = \beta^2(1+r)^2 c_2$ ... $c_{t+1} = \beta^2(1+r)^2 c_t$ ... Let $\Delta = \beta^2 1 + r^2$, then we have:

$$c_0 + \frac{\Delta c_0}{1+r} + \frac{\Delta^2 c_0}{(1+r)^2} + \cdots = M$$

which is equivalent to:

$$c_0 \left(1 + \frac{\Delta}{1+r} + \frac{\Delta}{(1+r)^2} + \cdots \right) = M$$

You know, if r is too large, you will definitely get an unbelievable return, that means the left side of the equation will be almost infinite, which is impossible.

So let's asuume $\frac{\Delta}{1+r} < 1$, then we have:

$$c_0 \left(\frac{1}{1 - \frac{\Delta}{1+r}}\right) = M$$

that is:

$$c_0 = M(1 - \beta^2(1+r))$$

So you should discover that the c_0 is related to the r and β.

In this example, you should find that with a larger r, c_0 will fall. So, the **substitution effect** is stronger than the **wealth effect**.

☞ **Tips 1.** *About mathemetical mechanism:*
Let $x < 1$, then we have:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

Let $Z = 1 + x + x^2 + x^3 + \cdots$, then we have:

$$Z = 1 + x(1 + x + x^2 + x^3 + \cdots) = 1 + xZ$$

$$Z(1 - x) = 1$$

$$Z = \frac{1}{1-x}$$

➤ **Remark 7.** *A more economic review of the model:*
Remember:

$$c_{t+1} = \beta^2 (1+r)^2 c_t$$

Which is equivalent to:

$$c_t = [\beta^2 (1+r)^2]^t (1 - \beta^2 (1+r))M$$

Suppose $r = 0$, then we have:

$$c_t = \beta^{2t}(1 - \beta^2)M$$

Let $\delta = \beta^2$, then we have:

$$c_t = \delta^t (1 - \delta)M$$

Taking the partial derivative of c_t with respect to t:

$$\frac{\partial c_t}{\partial t} = [t\delta^{t-1} + (t+1)\delta^t]M$$

Which is equivalent to:

$$\frac{\partial c_t}{\partial t} = \delta^t(\frac{t}{\delta} - 1)M$$

*So if δ is sufficiently small, then the consumption will rise over time. This is what we call the **time preference**. It means that if the agent is more patient, then he will consume more at present, and consume less at future.*

1.5   From time to the allocation of goods

1.5.1   The new assumptions

Let's come to the most classical model in economics, the **model of allocation of goods**.

• **Points 10.**

A consumer consumes 2 goods, whose consumption is x_1 and x_2.

You know, the utility function isn't concerned on the sort of goods, but on the number of consumption.
But the lifetime utility function is concerned on the different sort of goods. Take an example, you may prefer eating an apple than a banana, but the preference is showcased in the lifetime utility function, not in their individual utility function.

The consumer's lifetime utility function is:

$$u(x_1, x_2) = u(x_1) + \beta u(x_2)$$

In which β is a constant, and function $u(x)$ satisfies $u'(x) > 0$, $u''(x) < 0$, for all x.

The consumer's budget constraint is:

$$p_1 x_1 + p_2 x_2 = M$$

In which p_1 and p_2 are the prices of the two goods, and M is the initial wealth. That is:
The agent has access to a market in which both good1 and good2 are bought and sold, at prices p_1 and p_2, respectively, in units of cash.

And let's rewrite the question as:

$$\max_{x_1, x_2} u(x_1) + \beta u(x_2)$$

subject to:

$$p_1 x_1 + p_2 x_2 = M$$

1.5.2 The optimal solution

The optimal solution is:

$$\frac{u'(x_1^*)}{\beta u'(x_2^*)} = \frac{p_1}{p_2}$$

➤ **Remark 8.** *What's the economic meaning of p_1 and p_2?*
p_1 is the price of good1, p_2 is the price of good2.
The ratio of p_1 and p_2 is the relative price of good1 to good2. That means the agent could consume 1 unit of good1 by p_1 units of cash, consume 1 unit of good2 by p_2 units of cash.
That means you could get 1 unit of good1 in unit of $\frac{p_1}{p_2}$ units of good2.

1.5.3 Expand the model to the infinite goods

That means the agent could divide his target into infinite goods, and the utility function is:

$$\max_{c_1,c_2,\cdots} \pi_1 u(c_1) + \pi_2 u(c_2) + \cdots$$

And the budget constraint is:

$$p_1 c_1 + p_2 c_2 + \cdots = M$$

In which $\pi_1, \pi_2, \cdots$ is the **weight** of the utility function. A larger π_i means the agent prefers the good i.
And c_i is just the amount of the good i that the agent will consume.

- In this case, you should understand that you are just the taker of the given price in the competitive market.
  You **can't** change the price of the good and the utility function, but you could change the amount of the good you will consume.

1.5.4 The optimal solution

The optimal solution is:

$$\frac{\pi_1 u'(c_1^*)}{\pi_2 u'(c_2^*)} = \frac{p_1}{p_2}$$

And more generally, we have:

$$\frac{\pi_i u'(c_i^*)}{\pi_j u'(c_j^*)} = \frac{p_i}{p_j} \tag{*}$$

1.5.5   The substitution effect and wealth effect

Back to the story about the changes of price:

What if the p_1 rises?

You will probably consume less of good1 and more of good2.

Since you find that the relative price of good1 to good2 rises, you think that the good1 is relatively more expensive than good2.

The financial market deviates from your utility market, so why not buy less of good1 and more of good2, to get relatively more utility?

In simple words, you will **substitute** good1 with good2.

This is what we call the **substitution effect**.

However, the price rise will also make you feel that your wealth is reduced, so you will probably consume less of both goods.

As you are a more poor person, you tend to consume less of both goods.

This is what we call the **wealth effect**.

It's a similar story if you find that the price of good1 falls.

You will probably consume more of good1 and less of good2.

This is also a substitution effect.

However, the wealth effect will make you consume more of both goods, since you are a richer person.

1.5.6   From the Dynamic allocation model to the Dynamic optimization model

In the last section, we try to allocate different goods in different amount of resources, each goods have their own price.

So, we could say that in Dynamic Optimization model, we try to allocate different periods in different amount of resources, each goods have their own price.

Reformulate the question:

$$\max_{c_i} \sum_{i=0}^{\infty} \beta^i u(c_i)$$

subject to:

$$\sum_{i=0}^{\infty} p_i c_i = M$$

Since M is the initial wealth, so it's a constant.

And the p_i is the price of the good in period i **relative to the price of the good in period 0**.

So, it's easy to find that $p_0 = 1$.

And denoted by r, $p_i = \left(\frac{1}{1+r}\right)^i$.

Proposition 3. *If p_1 rises, then c_1 will **definitely** fall.*

证明. Let's consider different cases under p_1 rises:

- If c_1 holds constant, and c_2 also holds constant,then the (*) equation is violated.

- If c_1 rises, and c_2 rises, then the budget constraint is violated.

- If c_1 falls, and c_2 falls, then the (*) equation is violated.

That is:

No matter c_2 is increasing or decreasing, c_1 must be decreasing. In both cases, c_1 must be decreasing. □

You know, that offers a **theoretical** explanation for the **downward slopping** demand curve.