Where Time Meets Choice

Economic intuition with Mathematical Proofs

Reed He

School Of Economics/Fudan University

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Preface

"Economics is about intelligence"
—Prof. Wang

"Destiny depends on both self-struggle and historical course"
—President Jiang

This compilation represents my earnest attempt to synthesize Professor Wang's brilliant lectures with my own humble understanding, structuring key economic insights into an interconnected framework for review and discussion.

The journey begins with dynamic optimization models not merely by convention, but because they reveal the fundamental grammar of economic reality. These models crystallize the essence of scarcity—the inescapable tension between present and future that governs all rational choice. Through their mathematical elegance, we trace the DNA of economic thinking: opportunity costs quantified by Lagrange multipliers, marginal analysis made operational through derivatives, and equilibrium emerging as the calculus of competing incentives.

Yet equations alone remain sterile without economic intuition. Every mathematical proposition in these notes is paired with its economic story—for when Jacobians fade from memory, it is the narrative of substitution effects and intertemporal tradeoffs that endure. The formalisms matter precisely because they give precise voice to Adam Smith's "invisible hand" and Keynes' "animal spirits."

As a learner navigating these profound ideas, I've undoubtedly committed both syntactic errors and substantive misunderstandings. Readers are warmly invited to submit corrections via GitHub Issues—this text belongs equally to Professor Wang's pedagogical legacy and our collective pursuit of economic wisdom.

April 17, 2025 *Reed He*
Shanghai, China

Chapter 1

A Model of Dynamic Optimization

Economics is about intelligence——Prof. Wang

1.1 The Initial Model

1.1.1 The basic introduction

Once there was an agent, who was endowed with a certain amount of food M, and his life was divided into two periods: $t = 0$ and $t = 1$.

1. **Assumptions of the utility function:**

   (a) He has a utility function $u(c)$, which is same in both periods $t = 0$ and $t = 1$.

   (b) The utility function is strictly increasing: $u'(c) > 0$ for all c.

   (c) The marginal utility is decreasing: $u''(c) < 0$ for all c.

2. **Budget constraints:**

   (a) The food is **not** perishable and can be stored for the next period.

   (b) The agent cannot borrow food, so total consumption cannot exceed M:

$$c_0 + c_1 \leq M$$

3. **Optimization problem:**

$$\max_{c_0, c_1} u(c_0) + u(c_1) \tag{*}$$

subject to:

$$c_0 + c_1 \leq M \tag{1-1-1}$$

1.1.2 Observations

Observation 1. *Given the budget constraints, at optimality, (1-1-1) must hold with equality.*

证明. By contradiction: Suppose (c_0^*, c_1^*) is optimal but $c_0^* + c_1^* < M$. Let $\hat{c}_1 = c_1^* + \epsilon$ for some small $\epsilon > 0$. Then:

$$u(c_0^*) + u(\hat{c}_1) > u(c_0^*) + u(c_1^*)$$

which contradicts optimality. □

Observation 2. *The optimal solution is* $c_0^* = c_1^* = \frac{M}{2}$.

证明. By contradiction: Suppose $c_0^* > c_1^*$. Let:

$$\hat{c}_0 = c_0^* - \epsilon, \quad \hat{c}_1 = c_1^* + \epsilon \tag{1-2-1}$$

for small $\epsilon > 0$. Using Taylor expansion:

$$\lim_{\epsilon \to 0} [u(\hat{c}_0) + u(\hat{c}_1)] = u(c_0^*) + u(c_1^*) + \epsilon(u'(c_1^*) - u'(c_0^*)) \tag{1-2-2}$$

Since $u'(c_0^*) < u'(c_1^*)$ by assumption 1.3, (1-2-2) > (1-2-1), contradicting optimality. □

➢ **Remark 1.** *Let's consider the economic* ***story*** *behind the mathematical proof:*
Because the marginal utility is the utility of each additional unit of consumption. Whether the agent consumes more at $t = 0$ *or* $t = 1$*, the agent could always sell or buy the good to make himself better off.*

1.2 An advanced model

1.2.1 The new assumptions

• **Points 1.** *Introduce β as a discount factor, which means the utility of consumption at $t = 1$ is not as high as the utility of consumption at $t = 0$ to the agent's* ***lifetime utility****.*
The agent's lifetime utility is:

$$u(c_0) + \beta u(c_1) \tag{2-1-1}$$

• **Points 2.** *How to understand the discount factor?*

1. *You should clearly know that the utility function has never changed.*
2. *The discount factor is a number between 0 and 1. Which means the agent is always* ***preferring*** *current consumption to future consumption, the future utility couldn't satisfy the agent's* ***lifetime utility****.*
3. *The discount factor isn't a constant, it depends on the agent's* ***patience****. The more patient the agent is, the larger the discount factor is.*

1.2.2 The observations

Observation 3. *Given the budget constraints, at optimality, (1-1-1) must hold with equality.*

Observation 4. *In optimal solution, $c_0^* > c_1^*$.*

证明. By way of contradiction: Suppose $c_0^* \leq c_1^*$. Let:

$$\hat{c}_0 = c_0^* + \epsilon, \quad \hat{c}_1 = c_1^* - \epsilon$$

for sufficiently small but positive $\epsilon > 0$.
We have to prove that:

$$u(\hat{c}_0) + \beta u(\hat{c}_1) > u(c_0^*) + \beta u(c_1^*) \tag{2-2-1}$$

Using Taylor expansion:

$$\lim_{\epsilon \to 0} [u(\hat{c}_0) + \beta u(\hat{c}_1)] = u(c_0^*) + \beta u(c_1^*) + \epsilon(u'(c_0^*) - \beta u'(c_1^*)) \tag{2-2-2}$$

Since $u'(c_0^*) \geq u'(c_1^*)$ by assumption 1.3, which means $u'(c_0^*) - \beta u'(c_1^*) > 0$. Obviously, (2-2-2) is strictly better than (2-2-1).
A contradiction. □

➢ **Remark 2.** *What's $u'(c)$ means?*
Mathematical explanation:

$$u'(c) = \lim_{\Delta c \to 0} \frac{u(c + \Delta c) - u(c)}{\Delta c}.$$

Economic explanation: *$u'(c)$ is the marginal utility of consumption at c, which means the rate of change of the utility function with respect to the consumption.*

Proposition 1. *Let (c_0^*, c_1^*) be the optimal solution of the optimization problem, then:*

$$\frac{u'(c_0^*)}{\beta u'(c_1^*)} = 1 \tag{2-2-3}$$

证明. By the way of contradiction: Suppose (2-2-3) does not hold, then suppose:

$$\frac{u'(c_0^*)}{\beta u'(c_1^*)} < 1$$

In **Economic** sense, it means the marginal utility of **Lifetime utility** at $t = 0$ is strictly less than the marginal utility of **Lifetime utility** at $t = 1$.
Let:

$$\hat{c}_0 = c_0^* - \epsilon, \quad \hat{c}_1 = c_1^* + \epsilon$$

for sufficiently small but positive $\epsilon > 0$.
We have to prove that:

$$u(\hat{c}_0) + \beta u(\hat{c}_1) > u(c_0^*) + \beta u(c_1^*)$$

Using Taylor expansion:

$$\lim_{\epsilon \to 0} [u(\hat{c}_0) + \beta u(\hat{c}_1)] = u(c_0^*) + \beta u(c_1^*) - \epsilon(u'(c_0^*) - \beta u'(c_1^*))$$

Since $u'(c_0^*) < \beta u'(c_1^*)$, which means $u'(c_0^*) - \beta u'(c_1^*) < 0$.
A contradiction. Then a similar contradiction can be derived if $\frac{u'(c_0^*)}{\beta u'(c_1^*)} > 1$, by moving an ϵ amount of consumption from $t = 1$ to $t = 0$. □

证明. Another proof
Given that c_0^*, c_1^* is optimal solution, any plan $\hat{c}_0, \hat{c}_1$ with $\hat{c}_0 = c_0^* + \epsilon$ and $\hat{c}_1 = c_1^* - \epsilon$ for every ϵ is worse or equal than c_0^*, c_1^*.
That is (in **Mathematical** sense):

$$u(c_0^* + \epsilon) + \beta u(c_1^* - \epsilon) \leq u(c_0^*) + \beta u(c_1^*)$$

Using Taylor expansion: It's easy to get the result that (if ϵ is sufficiently small and positive):

$$u'(c_0^*) - \beta u'(c_1^*) \leq 0$$

In a similar way, we can get the result that (if ϵ is sufficiently small and negative):

$$u'(c_0^*) - \beta u'(c_1^*) \geq 0$$

Overall, it must hold that:

$$u'(c_0^*) - \beta u'(c_1^*) = 0$$

□

➢ **Remark 3.** *We have to understand the optimal solution in an* ***Economic*** *sense:*
In the proof of Contradiction, it's about moving an ϵ amount of consumption from $t = 1$ to $t = 0$.
And the contradiction means that you could always do better by moving, since the lifetime marginal utility is either smaller or larger than the other.
In the another proof, it's also about moving an ϵ amount of consumption from $t = 1$ to $t = 0$.
But the optimal situation means that you shouldn't do better by moving, since the lifetime marginal utility is equal.
So you could reach the equation.
Two types of deviation wouldn't be better

1.3 The introduction of the financial market

1.3.1 The new assumptions

• **Points 3.** *Introduce a financial market, which means the agent can save the food for the next period to use at a certain interest rate r.*
Which means if the agent saves 1 unit of food at $t = 0$, he will have $1 + r$ units of food at $t = 1$.
So the agent's budget constraint is:

$$c_0 + \frac{c_1}{1+r} = M \tag{3-1-1}$$

And the agent's utility function is still:

$$u(c_0) + \beta u(c_1) \tag{2-1-1}$$

• **Points 4.** *How to understand the interest rate?*

1. *The interest rate is a number* ***>0****, which means the agent has "grasped" somehow a great* ***technology*** *to save the food for the next period.*
2. *In the next subsection, we will tell you the details about the interest rate. Please take it as an* ***important fundament****.*

1.3.2 The details about the interest rate

Let's come back to the agent's situation:
The agent has grasped a great **technology** to save the food for the next period.So there comes such a simple quation:

$$1 \text{ unit of good at } t = 0 \Leftrightarrow (1 + r) \text{ units of good at } t = 1 \tag{3-2-1}$$

This equivalence can be rewritten as:

$$\frac{1}{1+r} \text{ units of good at } t = 0 \Leftrightarrow 1 \text{ unit of good at } t = 1 \tag{3-2-2}$$

➢ **Remark 4.** *These equations should be interpreted economically:*

- ***Saving Return****: Investing 1 unit today yields $(1 + r)$ units tomorrow*

- ***Borrowing Cost****: Receiving 1 unit today requires repaying* $\frac{1}{1+r}$ *units immediately*

To understand the value transformation, we must consider **Scarcity**:

- Scarcity implies limited availability
- Limited goods possess inherent value

Thus, we can express the value relationships:

$$V_0(1) = (1+r)V_1(1) \tag{3-2-3}$$

$$V_0\left(\frac{1}{1+r}\right) = V_1(1) \tag{3-2-4}$$

where $V_t(x)$ represents the value of x units at time t.

➢ **Remark 5.** *So let's think the transaction as a* ***contract****.*
It's a trade between the ***present*** *and the* ***future****.*
In your opinion, you have a technology to save the food for the future, which means you will get more food at future.This will make the future suffer a ***devaluation****.*
Devaluation means that the future good is ***less valuable*** *than the present good,and in that case, the amount of food you will get at future is* ***more*** *than the amount of food you will give at present.*

1.3.3 Findings and Proofs

Proposition 2. *Let's rewrite the question as:*

$$\max_{c_0,c_1} u(c_0) + \beta u(c_1)$$

subject to:

$$c_0 + \frac{c_1}{1+r} = M \tag{*}$$

Then the optimal solution is:

$$\frac{u'(c_0^*)}{\beta u'(c_1^*)} = 1+r \tag{3-3-1}$$

➢ **Remark 6.** *let's concentrate on the left side of the equation:*
In (1.3.2), we have talked about the economic thoughts of 1+r, which could be explained in the value of goods.
In ***remark 2****, we have illustrate the meaning of u'(c), which is the marginal utility with changes in consumption. Let's think the utility as a good can be traded in the market,* ***a market in your mind****, we could get such two equations from the the contract with the "utility evil":*

$$u'(c_0) \textit{ units of additional utility} = 1 \textit{ unit of additional good in } t = 0 \tag{3-3-2}$$

$$\beta u'(c_1) \textit{ units of additional utility} = 1 \textit{ unit of additional good in } t = 1 \tag{3-3-3}$$

How to connect these two periods of life? Take 1 unit of additional utility as the right side of each equation:

$$\frac{1}{u'(c_0)} \textit{ units of additional good in t0} = 1 \textit{ unit of additional utility} \tag{3-3-4}$$

$$\frac{1}{(\beta)u'(c_1)} \textit{ units of additional good in t1} = 1 \textit{ unit of additional utility} \tag{3-3-5}$$

证明. Economic proof So with the analysis above we could get the contract in your mind market, that is:

$$1 \text{ unit of good at } t = 0 \Leftrightarrow \frac{u'(c_0)}{\beta u'(c_1)} \text{ units of good at } t = 1 \tag{3-3-6}$$

You should find it's similar to 3-2-1.
With your intuition, since the left side of each equation is the same, then the right side of each equation should also be the same!
Then we can reach the (*);
In economic proof:
Suppose not, and you should practice this by yourself! A key step: If the financial market comes to the situation that:
The good today is somehow more expensive, than your mind market. That means, you think the food shouldn't value that much in the financial market. Then you will probably sold it out!

Make it more rare until the value in your mind market equals the value in financial market. □

证明. Mathematical proof Now that you have understood the economic proof, let's see the mathematical proof.
Suppose not, suppose the optimal solution (c_0^*, c_1^*) has:

$$\frac{u'(c_0^*)}{\beta u'(c_1^*)} > 1 + r$$

Then let $\hat{c_0} = c_0^* + \Delta$, $\hat{c_1} = c_1^* - \delta$, where δ and Δ is a small number. Then we have:

$$\Delta(1 + r) = \delta$$

Then using Taylor expansion:

$$u(\hat{c_0}) + \beta u(\hat{c_1}) = u(c_0^*) + \beta u(c_1^*) + \Delta(u'(c_0^*) - \beta(1 + r)u'(c_0^*))$$

Since $\frac{u'(c_0^*)}{\beta u'(c_1^*)} > 1 + r$, we have:

$$u(\hat{c_0}) + \beta u(\hat{c_1}) > u(c_0^*) + \beta u(c_1^*)$$

A contradiction. A similar contradiction can be derived if $\frac{u'(c_0^*)}{\beta u'(c_1^*)} < 1+r$. □

证明. Another mathematical proof Solving the problem by Lagrange method:

$$\max_{c_0, c_1} u(c_0) + \beta u(c_1)$$

subject to:

$$c_0 + \frac{c_1}{1 + r} = M$$

The Lagrangian is:

$$\mathcal{L} = u(c_0) + \beta u(c_1) + \lambda(M - c_0 - \frac{c_1}{1 + r})$$

The first order conditions are:

$$\frac{\partial \mathcal{L}}{\partial c_0} = u'(c_0) - \lambda = 0 \quad (1)$$

$$\frac{\partial \mathcal{L}}{\partial c_1} = \beta u'(c_1) - \lambda \frac{1}{1+r} = 0 \quad (2)$$

From equation (1) and (2), we can derive:

$$\frac{u'(c_0)}{\beta u'(c_1)} = 1 + r$$

$\square$

1.3.4 Substitution effect and wealth effect

Let's back to the financial market, and think the interest rate as a variable.
What will happen if the interest rate r rises?
Back to the optimal solution(3-3-1), we have:

$$\frac{u'(c_0^*)}{\beta u'(c_1^*)} = 1 + r$$

If r rises, the left side of the equation will rise.If the agent choose to save more food at present, then the consumption at present will defintely go down. And the consumption at future will rise.In mathemetical sense, it means that the marginal utility of consumption at present will rise, and the marginal utility of consumption at future will fall. So the left side of the equation will definitely rise.
That's what we call the **Substitution effect**.
Substitution effect:
A larger r will lead to a **lower** c_0 and a **higher** c_1.
A larger r will induce the agent to save more and consume less at period 0, and consume more and save at period 1.
However, it's not a must to save more at present, back to the constraint(*).
Since the interest rate is higher, $c_1 + c_2$ will be larger.
So if the agent choose to consume more at present, he still may consume more at future.
And if the agent just consume a little at present, the left side of the equation will still may rise, since the $c_1 + c_2$ will be larger.
This is what we call the **Wealth effect**.

Wealth effect:

A larger r will lead to a **higher** c_0 and a **higher** c_1. A larger r means that the agent can consume the same amount by saving less at period 0, and consume more in both periods.

1.3.5 Take different examples

• **Points 5.**

$u(c) = \sqrt{c}$. *rewrite (3-3-1) as:*

$$\frac{\frac{1}{2\sqrt{c_0^*}}}{\beta\frac{1}{2\sqrt{c_1^*}}} = 1 + r$$

which is equivalent to:

$$\frac{c_1^*}{c_0^*} = \beta(1+r)^2$$

Let's write c_0^* on the right side:

$$c_1^* = \beta(1+r)^2 c_0^*$$

Substitute it into the constraint(*):

$$c_0^* + \beta(1+r)c_0^* = M$$

$$c_0^*(1 + \beta(1+r)) = M$$

*With a larger r, c_0^* will fall, and c_1^* will rise. So in this case, the substitution effect is stronger than the wealth effect. Which **not** means that there is no wealth effect, but it's just that the substitution effect is stronger than the wealth effect.*

• **Points 6.**

$u(c) = \ln c$. *rewrite (3-3-1) as:*

$$\frac{\frac{1}{c_0^*}}{\beta\frac{1}{c_1^*}} = 1 + r$$

which is equivalent to:

$$\frac{c_1^*}{c_0^*} = \beta(1+r)$$

Let's write c_0^* on the right side:

$$c_1^* = \beta(1+r)c_0^*$$

Substitute it into the constraint(*):

$$c_0^* + \beta c_0^* = M$$

$$c_0^*(1+\beta) = M$$

With a larger r, c_0^* will be the same, but c_1^* will rise. So in this case, the substitution effect is equal to the wealth effect.

- **Points 7.**

$u(c) = -e^{-c}$. *rewrite (3-3-1) as:*

$$\frac{e^{-c_0^*}}{\beta e^{-c_1^*}} = 1 + r$$

which is equivalent to:

$$c_1^* = c_0^* + \ln(\beta(1+r))$$

Substitute it into the constraint(*):

$$c_0^* + \frac{c_0^* + \ln(\beta(1+r))}{1+r} = M$$

$$c_0^*\left(1 + \frac{1 + \ln(\beta(1+r))}{1+r}\right) = M$$

With a larger r, c_0^* will rise, and c_1^* will rise. So in this case, the substitution effect is ***not*** *stronger than the wealth effect.*

1.4 Dividing the lifetime into infinite periods

However in real life, you can't divide the lifetime into just two periods. So we need to extend the model to the infinite periods. The agent again, has access to the financial market, which allows him to save/convert any X units of consumtion at period t into $(1+r)^t$ units of consumtion at period $t+1$, for all $t = 0, 1, 2, \cdots$.

1.4.1 The new assumptions

• **Points 8.** *We then divide the lifetime into infinite periods,that is* $t = 0, 1, 2, \cdots$*. And the* r *is then the interest rate, which is a constant in each periods.In the financial market, you just save the food for the next period to use at a certain interest rate.*
And the M *is the initial wealth, which is the amount of food you have at* $t = 0$*, it's a constant.*
And the c_t *is the amount of food you will consume at* t*, it's a variable. And the* β *is the discount factor, which is a constant. The budget constraint is:*

$$c_0 + \frac{c_1}{1+r} + \frac{c_2}{(1+r)^2} + \cdots = M$$

The utility function is:

$$u(c_0) + \beta u(c_1) + \beta^2 u(c_2) + \cdots$$

let's rewrite the question as:

$$\max_{c_0, c_1, c_2, \cdots} u(c_0) + \beta u(c_1) + \beta^2 u(c_2) + \cdots$$

subject to:

$$c_0 + \frac{c_1}{1+r} + \frac{c_2}{(1+r)^2} + \cdots = M$$

1.4.2 The optimal solution

The optimal solution is:

$$\frac{u'(c_t^*)}{\beta u'(c_{t+1}^*)} = 1 + r \tag{4-2-1}$$

It's similar to the previous question, but the difference is that the c_t is a variable.
Since t could be any period, so we still solve the problem by the same way.

1.4.3 A simple example to illusrate the two effects

• **Points 9.**

$$u(c) = \sqrt{c}$$

.

rewrite (4-2-1) as:

$$\frac{\frac{1}{2\sqrt{c_t^*}}}{\beta\frac{1}{2\sqrt{c_{t+1}^*}}} = 1 + r$$

which is equivalent to:

$$\frac{c_{t+1}^*}{c_t^*} = \beta^2(1+r)^2$$

which is equivalent to:

$$c_{t+1}^* = \beta^2(1+r)^2 c_t^* \tag{4-2-2}$$

for all $t = 0, 1, 2, \cdots$.

The simplification of problem

Let's rewrite the optimal question using (4-2-2): $c_1 = \beta^2(1+r)^2 c_0$ $c_2 = \beta^2(1+r)^2 c_1$ $c_3 = \beta^2(1+r)^2 c_2$ ... $c_{t+1} = \beta^2(1+r)^2 c_t$ ... Let $\Delta = \beta^2 1 + r^2$, then we have:

$$c_0 + \frac{\Delta c_0}{1+r} + \frac{\Delta^2 c_0}{(1+r)^2} + \cdots = M$$

which is equivalent to:

$$c_0(1 + \frac{\Delta}{1+r} + \frac{\Delta}{(1+r)^2} + \cdots) = M$$

You know, if r is too large, you will definitely get an unbelievable return, that means the left side of the equation will be almost infinite, which is impossible.

So let's asuume $\frac{\Delta}{1+r} < 1$, then we have:

$$c_0(\frac{1}{1-\frac{\Delta}{1+r}}) = M$$

that is:

$$c_0 = M(1 - \beta^2(1+r))$$

So you should discover that the c_0 is related to the r and β.

In this example, you should find that with a larger r, c_0 will fall. So, the **substitution effect** is stronger than the **wealth effect**.

☞ **Tips 1.** *About mathemetical mechanism:*
Let $x < 1$, then we have:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

Let $Z = 1 + x + x^2 + x^3 + \cdots$, then we have:

$$Z = 1 + x(1 + x + x^2 + x^3 + \cdots) = 1 + xZ$$

$$Z(1-x) = 1$$

$$Z = \frac{1}{1-x}$$

➢ **Remark 7.** *A more economic review of the model:*
Remember:

$$c_{t+1} = \beta^2(1+r)^2 c_t$$

Which is equivalent to:

$$c_t = [\beta^2(1+r)^2]^t(1-\beta^2(1+r))M$$

Suppose $r = 0$, then we have:

$$c_t = \beta^{2t}(1-\beta^2)M$$

Let $\delta = \beta^2$, then we have:

$$c_t = \delta^t(1-\delta)M$$

Taking the partial derivative of c_t with respect to t:

$$\frac{\partial c_t}{\partial t} = [t\delta^{t-1} + (t+1)\delta^t]M$$

Which is equivalent to:

$$\frac{\partial c_t}{\partial t} = \delta^t(\frac{t}{\delta} - 1)M$$

*So if δ is sufficiently small, then the consumption will rise over time. This is what we call the **time preference**. It means that if the agent is more patient, then he will consume more at present, and consume less at future.*

1.5 From time to the allocation of goods

1.5.1 The new assumptions

Let's come to the most classical model in economics, the **model of allocation of goods**.

- **Points 10.**

A consumer consumes 2 goods, whose consumption is x_1 and x_2.

You know, the utility function isn't concerned on the sort of goods, but on the number of consumption.
But the lifetime utility function is concerned on the different sort of goods. Take an example, you may prefer eating an apple than a banana, but the preference is showcased in the lifetime utility function, not in their individual utility function.

The consumer's lifetime utility function is:

$$u(x_1, x_2) = u(x_1) + \beta u(x_2)$$

In which β is a constant, and function $u(x)$ satisfies $u'(x) > 0$, $u''(x) < 0$, for all x.

The consumer's budget constraint is:

$$p_1 x_1 + p_2 x_2 = M$$

In which p_1 and p_2 are the prices of the two goods, and M is the initial wealth. That is:
The agent has access to a market in which both good1 and good2 are bought and sold, at prices p_1 and p_2, respectively, in units of cash.

And let's rewrite the question as:

$$\max_{x_1, x_2} u(x_1) + \beta u(x_2)$$

subject to:

$$p_1 x_1 + p_2 x_2 = M$$

1.5.2 The optimal solution

The optimal solution is:

$$\frac{u'(x_1^*)}{\beta u'(x_2^*)} = \frac{p_1}{p_2}$$

➢ **Remark 8.** *What's the economic meaning of p_1 and p_2?*
p_1 is the price of good1, p_2 is the price of good2.
The ratio of p_1 and p_2 is the relative price of good1 to good2. That means the agent could consume 1 unit of good1 by p_1 units of cash, consume 1 unit of good2 by p_2 units of cash.
That means you could get 1 unit of good1 in unit of $\frac{p_1}{p_2}$ units of good2.

1.5.3 Expand the model to the infinite goods

That means the agent could divide his target into infinite goods, and the utility function is:

$$\max_{c_1, c_2, \cdots} \pi_1 u(c_1) + \pi_2 u(c_2) + \cdots$$

And the budget constraint is:

$$p_1 c_1 + p_2 c_2 + \cdots = M$$

In which $\pi_1, \pi_2, \cdots$ is the **weight** of the utility function. A larger π_i means the agent prefers the good i.
And c_i is just the amount of the good i that the agent will consume.

- In this case, you should understand that you are just the taker of the given price in the competitive market.
  You **can't** change the price of the good and the utility function, but you could change the amount of the good you will consume.

1.5.4 The optimal solution

The optimal solution is:

$$\frac{\pi_1 u'(c_1^*)}{\pi_2 u'(c_2^*)} = \frac{p_1}{p_2}$$

And more generally, we have:

$$\frac{\pi_i u'(c_i^*)}{\pi_j u'(c_j^*)} = \frac{p_i}{p_j} \tag{*}$$

1.5.5 The substitution effect and wealth effect

Back to the story about the changes of price:
What if the p_1 rises?
You will probably consume less of good1 and more of good2.
Since you find that the relative price of good1 to good2 rises, you think that the good1 is relatively more expensive than good2.
The financial market deviates from your utility market, so why not buy less of good1 and more of good2, to get relatively more utility?
In simple words, you will **substitute** good1 with good2.
This is what we call the **substitution effect**.
However, the price rise will also make you feel that your wealth is reduced, so you will probably consume less of both goods.
As you are a more poor person, you tend to consume less of both goods.
This is what we call the **wealth effect**.
It's a similar story if you find that the price of good1 falls.
You will probably consume more of good1 and less of good2.
This is also a substitution effect.
However, the wealth effect will make you consume more of both goods, since you are a richer person.

1.5.6 From the Dynamic allocation model to the Dynamic optimization model

In the last section, we try to allocate different goods in different amount of resources, each goods have their own price.
So, we could say that in Dynamic Optimization model, we try to allocate different periods in different amount of resources, each goods have their own price.
Reformulate the question:

$$\max_{c_i} \sum_{i=0}^{\infty} \beta^i u(c_i)$$

subject to:

$$\sum_{i=0}^{\infty} p_i c_i = M$$

Since M is the initial wealth, so it's a constant.
And the p_i is the price of the good in period i **relative to the price of the good in period 0**.
So, it's easy to find that $p_0 = 1$.
And denoted by r, $p_i = \left(\frac{1}{1+r}\right)^i$.

Proposition 3. *If p_1 rises, then c_1 will **definitely** fall.*

证明. Let's consider different cases under p_1 rises:

- If c_1 holds constant, and c_2 also holds constant,then the (*) equation is violated.
- If c_1 rises, and c_2 rises, then the budget constraint is violated.
- If c_1 falls, and c_2 falls, then the (*) equation is violated.

That is:
No matter c_2 is increasing or decreasing, c_1 must be decreasing. In both cases, c_1 must be decreasing. □

You know, that offers a **theoretical** explanation for the **downward slopping** demand curve.

Chapter 2

A model of insurance

> One's destiny, of course, depends on one's self-struggle, but one must also take into account the course of history.
> —President Jiang

2.1 An interesting economic story

You know, the world isn't always so perfect that your situation remains constant. You always encounter some risks and uncertainty. Let's make the story more theoretical.

2.1.1 Assumptions

The agent lives in a period divided into two parts: the **ex post** state and the **ex ante** state.

The ex post period is after event X occurs, while the ex ante period is before event X occurs.

How does the event affect you? After the event happens, you will receive different incomes: M_1 and M_2.

These different incomes occur with different probabilities: $\pi_i = \text{Prob}\{\theta = \theta_i\}, i = 1, 2$, where $\pi_i > 0$ and $\pi_1 + \pi_2 = 1$.

Assume that in each of your ex post income states, the utility function the agent uses to measure utility is $u(c)$, which is monotonically increasing and weakly concave, meaning: $u'(c) > 0$, $u''(c) \leq 0$.

What does the agent know in the current (ex ante) state?

- The probability of each ex post income state
- The possible incomes (c_1, c_2)
- The utility function

What does the agent not know? Which state he will find himself in - that's the risk!

Therefore, his utility in ex post life is:

$$u(c_1, c_2) = \pi_1 u(c_1) + \pi_2 u(c_2)$$

We call this function "expected utility", representing the expected value of his utility across the two different ex post states of the world.

Note that c represents consumption, not wage, in each state.

2.1.2 Expansion to multiple states

Clearly, if $\theta = \{\theta_1, \theta_2, \dots, \theta_N\}$, $\pi_i = \text{Prob}\{\theta = \theta_i\}, i = 1, 2, \dots, N$.

Then:

$$u(c_1, c_2, \dots, c_N) = \sum_{i=1}^{N} \pi_i u(c_i)$$

2.1.3 Insurance

You know, we fear risks. From the previous model, we should conclude that a relatively controllable and even allocation is always better than a random allocation.

However, unlike the previous model - where we were superstars mastering a technology to reallocate food between our young and old selves - now we must face risks.

But you know, risks might make you rich or might make you poor. It doesn't depend on you, but you have a desire to transfer some goods from your rich self (one day you might find money) to your poor self (one day you might be robbed).

The desire exists, but you can't reallocate food this time.

How can you implement a plan to transfer consumption from your rich self to your poor self?

That's what insurance is!

• **Points 1.** *My view Insurance means receiving consumption when I'm poor and paying when I'm rich.*
Through a contract, I achieve balance with myself.

• **Points 2.** *Market view Insurance is traded now. The underlying logic is that the market has a somewhat different utility function (linear).*

2.1.4 Underlying logic of insurance

From the assumptions, you should derive two straightforward conclusions: the concave nature of the utility function, and the expected ex ante utility function.

From these assumptions we can conclude:

$$u(\pi_1 c_1 + \pi_2 c_2) > \pi_1 u(c_1) + \pi_2 u(c_2)$$

This is the underlying logic, and we will now provide a proof.

2.2 Proof of the underlying logic

2.2.1 Mathematical proof

The inequality we want to prove is:

$$u(\pi_1 c_1 + \pi_2 c_2) \geq \pi_1 u(c_1) + \pi_2 u(c_2)$$

This directly applies **Jensen's Inequality**, which states that for a concave function u:

$$u\left(\sum_{i=1}^{n} \lambda_i x_i\right) \geq \sum_{i=1}^{n} \lambda_i u(x_i)$$

where $\lambda_i \geq 0$ and $\sum_{i=1}^{n} \lambda_i = 1$.

证明. For the two-state case:

1. Since u is concave, by definition the secant line lies below the function:

$$u(\lambda c_1 + (1-\lambda)c_2) \geq \lambda u(c_1) + (1-\lambda)u(c_2)$$

   for any $\lambda \in [0,1]$.

2. Let $\lambda = \pi_1$ and $1-\lambda = \pi_2$, then:

$$u(\pi_1 c_1 + \pi_2 c_2) \geq \pi_1 u(c_1) + \pi_2 u(c_2)$$

3. Equality holds only when u is linear or when $c_1 = c_2$.

□

This shows that risk-averse individuals (with concave utility) prefer certain consumption $\bar{c} = \pi_1 c_1 + \pi_2 c_2$ over uncertain consumption (c_1, c_2) with the same expected value.

Consider $u(c) = \sqrt{c}$. For all $c_1, c_2 > 0$, $c_1 < c_2$, can you show:

$$\sqrt{\pi_1 c_1 + \pi_2 c_2} > \pi_1\sqrt{c_1} + \pi_2\sqrt{c_2}$$

证明. This is equivalent to proving:

$$\pi_1 c_1 + \pi_2 c_2 > (\pi_1\sqrt{c_1} + \pi_2\sqrt{c_2})^2$$

Expanding the right side:

$$\pi_1 c_1 + \pi_2 c_2 > \pi_1^2 c_1 + 2\pi_1\pi_2\sqrt{c_1 c_2} + \pi_2^2 c_2$$

Rearranging terms:

$$0 > \pi_1(\pi_1 - 1)c_1 + \pi_2(\pi_2 - 1)c_2 + 2\pi_1\pi_2\sqrt{c_1 c_2}$$

Since $\pi_1 + \pi_2 = 1$, we have $\pi_1 - 1 = -\pi_2$ and $\pi_2 - 1 = -\pi_1$:

$$0 > -\pi_1\pi_2 c_1 - \pi_1\pi_2 c_2 + 2\pi_1\pi_2\sqrt{c_1 c_2}$$

Dividing both sides by $\pi_1\pi_2$:

$$0 > -c_1 - c_2 + 2\sqrt{c_1 c_2}$$

Which can be rewritten as:

$$c_1 + c_2 - 2\sqrt{c_1 c_2} > 0$$

This is a perfect square:

$$(\sqrt{c_1} - \sqrt{c_2})^2 > 0$$

Since $c_1 \neq c_2$, the inequality always holds. □

2.2.2 Economic proof

➢ **Remark 1.** *An economic explanation of the above inequality: The* ***LHS*** *represents the ex ante utility of average consumption, where the function's variable is the expected amount of consumption in your ex post life. Since all probabilities and incomes are given, it's a constant. The* ***RHS*** *represents the average of utilities across states. Therefore, as a risk-averse agent, one will always prefer the certain plan over the lottery.*

Now I will provide an economic proof: Take $\pi_1 = \pi_2$. Suppose someone offers a trade: Each time you transfer a little from your rich self to your poor self, utility improves, until you can no longer do so. This involves transferring 1 unit of consumption from state 2 to state 1. A large gap between c_1 and c_2 means facing greater risk.

证明. Suppose we start with $c_1 < c_2$. Assume $\pi_1 = \pi_2$. Suppose someone offers to give you a small amount of consumption δ, $\delta > 0$, in state 1, but asks you to pay back the same δ units of consumption in state 2. This trade will make you better off. By the property of diminishing marginal utility:

$$u'(c_1) > u'(c_2)$$

Therefore, transferring 1 unit of consumption from state 2 to state 1 increases total utility:

$$\pi_1 u(c_1 + \delta) + \pi_2 u(c_2 - \delta) = \pi_1 u(c_1) + \pi_2 u(c_2) + \pi_1 (u'(c_1) - u'(c_2))$$

Since $c_1 < c_2$, $u'(c_1) > u'(c_2)$. Thus:

$$\pi_1 u(c_1 + \delta) + \pi_2 u(c_2 - \delta) > \pi_1 u(c_1) + \pi_2 u(c_2)$$

This process continues until $c_1 = c_2$, when optimality is achieved. □

Hence, we get a conclusion: A smoother consumption means a better utility.

2.3 Introduction to an Insurance corporation

As a contract, we should get a fair deal with the corporation who provides good or take goods.
As a fair deal, on average, how much you get from agent must equal to how much you dive the agent. It's on the **average**.

2.3.1 Risk neutral

Like the definition of "Risk averse", "Risk neutral" means the agent don't care about risk. In Mathematical definition, if the agent has $u'(c) > 0$, $u''(c) = 0$, for all c.This suppose the agent's utility is linear, with $u(c) = \alpha + \beta c$, for some α and β, $\beta > 0$.
So we rewrite the optimaility:

$$u(\pi_1 c_1 + \pi_2 c_2) = \pi_1 u(c_1) + \pi_2 u(c_2)$$

This equation now holds for all c_1 and c_2.

➢ **Remark 2.** *What does this equation mean?*
It means a risk-neutral agent is indifferent between the utility of average ex post consumption and the average of expected utilities across ex post states. Therefore, the agent neither prefers nor dislikes risk - they are simply neutral to it.

So we can introduce the insurance company into the question:
A risk neutral insurance company serves "insurance contract" in a market populated by a large amount number of risk averse consumers. Who each has a random income of θ, which offers agent "consumption" θ_1 at prob π_1, θ_2 at prob π_2, where $c_1 < c_2$, $\pi_1 + \pi_2 = 1$, $\pi_1 > 0$, $\pi_2 > 0$. The value of θ_1 θ_2 π_1 π_2 are all known and given to the people in the market.

The consumer are all expected utility maximizers, and their utility function is u(c), and $u'(c) > 0$, $u''(c) < 0$, for all c.

➢ **Remark 3.** *Let's review on what we have talked about:*

- *Firstly, we described the world as a lottery, or random variables.In mathematical language, you are in a space of ex post possibilities as* $\theta \in \Theta$。
- *Then we talked about risk-averse agent, which is when I give you θ, and you give me in return the mean of θ.*
- *A risk-averse agent will prefer $E\theta$ strictly over $\theta.u(E\theta_i) > Eu(\theta_i)$.*
- *A risk-neutral agent on the other hand, is in different between θ and $E\theta$.*
- *A corollary of this is that you are willing to pay a price, say ϵ, as the price of service,for selling θ for $E.\theta$, i.e. $u(E\theta - \epsilon) \geq E(u(\theta))$*
- *As long as ϵ is small enough, the inequality $u(E\theta - \epsilon) \geq E(u(\theta))$ holds.*

2.3.2 The new situation

There is an "insurance contract", which is sold by a monopolist seller.For any i unit of consumption in state θ_1, the price is p units of consumption in state θ_2
That is:
The contract says if you promise to pay me p units of good in state θ_2, I'll pay you 1 unit of good in state θ_1.
So in order to optimize the insurance contract:

$$\max_p -\pi_1 X^*(p) + \pi_2 X^*(p)p \tag{2.1}$$

- **Points 3.**
  - *$X(p)$ represents the demand function from the consumer's perspective. When the insurance company sets the price at p, $X(p)$ determines how many insurance contracts the consumer will purchase in units of good1 in state θ_1. This is entirely the consumer's choice.*

- *The insurance company's utility comes from paying out in state θ_1 and receiving payments in state θ_2. Since his utility function is linear, the profits totally depends on the what he paid out and he gained in, where he paid out $X^*(p)$ and gained $X^*(p)p$.*

However we also need to consider in the consumer's shoes, since X(p) is decided by consumers. As a consumer, I also want to maximize my utility, you know my utility optimal question is

$$\max_X \pi_1 u(\theta_1 + X(p)) + \pi_2 u(\theta_2 - pX(p)) \tag{2.2}$$

So here $X^*(p)$ is the optimal amount of insurance (i.e. consumption at θ_1) to buy from the insurance company, which could be solved from the above inequality.

2.3.3 Solve the problem

Let's solve for the problem:
Note you can rewrite the (2.2) as:

$$\max_x \pi_1 u(\theta_1 + x) + \pi_2 u(\theta_2 - px)$$

Thinking strategically—that is: before I make up my mind, I need to figure out what my opponents will do; pick up the 'p' to induce the opponents, serving for my purpose, then I should know what my opponents do.

➢ **Remark 4.** *You should find that the (2.2) is a dynamic optimazation model in essence–it's entirely about allocating good between a poor self and rich self.*
So just solve it with our conclusions in Chap1.

$$\frac{\pi_1 u'(\theta_1 + x)}{\pi_2 u'(\theta_2 - px)} = p \tag{*}$$

• **Points 4.** *We should get:*
The (*) equation is entirely the format we've reached at the optimization model. An ***economic proof****:The LHS is still the price of good 1 in expected*

utility market over the price of good 2 in expected utility.
Since 1 unit of good 1 values $\pi_1 u'(\theta_1 + x)$ units of utility, and 1 unit of utility values $\frac{1}{\pi_2 u'(\theta_2 - px)}$ units of good 2.
So the LHS means the price of good 1 in units of consumption in state θ_2, in expected utility market. Which should equal to the price of good 1 in units of consumption in state θ_2 in financial market.

- **Points 5.** *A* ***Mathematical proof***:

证明. The (*) equation could be derived from taking the first order of the agents' objective function w.r.t. x:

$$\pi_1 u'(\theta_1 + x) + \pi_2 u'(\theta_2 - px)(-p) = 0$$

□

Taking a look at the (*) equation, you should find that substitution effect once again works out–What if I raise p?
you know when the RHS becomes larger, the LHS should also becomes larger, then by the way of contradiction:
If x stays constant then the LHS should become smaller.If x becomes larger then the LHS should definitely become smaller even more.
Then x should becomes smaller, which caters to our experience in real life: if price becomes higher, then we'll tend to cut down our willing to buy something.
So the (*) equation in essence tells us what the x^* is, which is the optimaility information.
So what's more, a few more conclusion we could see when the format of utility function is $u(c) = \sqrt{c}$, which will be solved in next subsection.

Up to now, we've solved the relationship between x^* and p, which is derived from the consumer's optimal solution.So we should now reconsider the company's optimality.
From equation(2.1), it's now an entirely optimal problem only concerning variable p, using the equation(*).
So we take the first order derivative of the objective(2.1) with respect to p:

$$-\pi_1 x'(p) + \pi_2 p x'(p) + \pi_2 x(p) = 0$$

which is equivalent to:

$$\frac{x(p)}{x'(p)} = \frac{\pi_1}{\pi_2} - p \tag{2.3}$$

2.3.4 An example for analysis

Example1: $u(c) = \sqrt{c}$

Solve the (*) equation for $x^*(p)$

First compute the derivatives:

$$u'(c) = \frac{1}{2\sqrt{c}}$$

Substitute into (*):

$$\frac{\pi_1 \frac{1}{2\sqrt{\theta_1 + x}}}{\pi_2 \frac{1}{2\sqrt{\theta_2 - px}}} = p$$

Simplify:

$$\frac{\pi_1 \sqrt{\theta_2 - px}}{\pi_2 \sqrt{\theta_1 + x}} = p$$

Square both sides:

$$\left(\frac{\pi_1}{\pi_2}\right)^2 \frac{\theta_2 - px}{\theta_1 + x} = p^2$$

Cross multiply:

$$\pi_1^2(\theta_2 - px) = \pi_2^2 p^2 (\theta_1 + x)$$

Expand and collect terms:

$$\pi_1^2\theta_2 - \pi_1^2 px = \pi_2^2 p^2 \theta_1 + \pi_2^2 p^2 x$$

Gather x terms:

$$x(-\pi_1^2 p - \pi_2^2 p^2) = \pi_2^2 p^2 \theta_1 - \pi_1^2 \theta_2$$

Solve for x:

$$x^*(p) = \frac{\pi_1^2\theta_2 - \pi_2^2 p^2 \theta_1}{\pi_1^2 p + \pi_2^2 p^2}$$

This gives us the optimal transfer amount x^* as a function of price p for the square root utility case. And what about rewrite it as:

$$x^*(p) = \frac{\theta_2 - \left(\frac{\pi_2}{\pi_1}\right)^2 p^2\theta_1}{p + \left(\frac{\pi_2}{\pi_1}\right)^2 p^2} \tag{2.4}$$

Then we could get some other conclusions:

- If θ_2 goes up, then from (2.4), you should get the x^* going up.It's easily to find out, since when your rich self becomes more rich, you tend to take more care of your young self.
- If p is too high that x^* is negative, that means you even want to sell the insurance for the market.
- If $\frac{\pi_2}{\pi_1}$ goes down, that menas the poor self is less important, then you tend to buy less insurance.

2.4 A comparison of "fair contract" and "profitable contract"

2.4.1 A fair contract

Last time we have talked about a fair contract from the perspective of "consumer", this time we talked about it from the insurance company.
That is: the insurance company makes no profits from the contract.
In other words, for any amount of contract, say x, that the risk-averse agent purchases.

$$-\pi_1 x + \pi_2 p_f x = 0$$

At p_f then what agent pays the insurance company equal in expected value that the insurance company pays the agent.
You should find out that $p_f = \frac{\pi_1}{\pi_2}$.
What happens at p_f for the agent?His utility at p_f would be $\pi_1 u(\theta_1 + x) + \pi_2 u(\theta_2 - px)$ His optimal question is:

$$\max_x \pi_1 u(\theta_1 + x) + \pi_2 u(\theta_2 - px)$$

F.O.C:

$$\pi_1 u'(\theta_1 + x_f) - p\pi_2 u(|theta_2 - px_f) = 0$$

So we should get that $x_f = (\theta_2 - \theta_1)\pi_2$

The risk-averse agents' consumption is constant across the state, provided as $p = p_f$.Consumption is perfectly smooth.
When the above equation holds,we say the agent's consumption is perfectly smooth across the ex post states of income.
Equation1:$x_f = (\theta_2 - \theta_1)\pi_2$

- if π_2 rises, the x_f also rises.There is a wealth effect.
- if $\theta_2 - \theta_1$ rises, the rich self will take more care of his poor self.

2.4.2 A profitable contract

Question: Can the insurance company do make a positive profits from the agent? **Answer:**Yes, by way of charging a price $p > p_f$.
$\pi(p) = (-\pi_1 X^*(p) + \pi_2 p)X^*(p)$ is the insurance company's profit function. Then $\pi(p_f) = 0$, and as long as $X^*(p) > 0$, which is the agent's optimal demand for insurance at p, profits will be positive.
If p is so high that people would rather choose to don't buy anything, profits will be negative.
Example:$u(c) = ln(c)$ Rewrite the question1 from the consumer's question:

$$\max_x \pi_1 u(\theta_1 + x) + \pi_2 u(\theta_2 - px)$$

Optimality:

$$\frac{\pi_1 u'(\theta_1 + x)}{\pi_2 u'(\theta_2 - px)} = p$$

So the quetsion will be:

$$\frac{\frac{pi_1}{\theta_1+x}}{\frac{\pi_2}{\theta_2-px}} = p$$

And x^* could be derived from the equation as:

$$x^* = \frac{\theta_2 + p\theta_1}{p}\pi_1 - \theta_1$$

which is equivalent to:

$$x^* = \frac{\theta_2}{p}\pi_1 - \pi_2\theta_1$$

So you could check the answer by making the θ_2 higher and θ_1 smaller.As the risk is more inevitable. the agent tends to buy more insurance.And when p rises, a substitution effect happens, the agent tends to substitute good1 for good2.

Rewrite the company's optimal question.

$$\max_p -\pi_1 x^*(p) + \pi_2 p x^*(p)$$

Our optimality:

$$\frac{x'(p)}{x(p)} = \frac{\pi_2}{\pi_1 - \pi_2 p}$$

And in that case:
You should easily get the answer as:

$$p^* = \frac{pi_1}{pi_2}\sqrt{\frac{\theta_2}{\theta_1}}$$

Which makes sense as it's economic explanation is obvious.

Meanwhile, if you take the F.O.C of $\pi(p)$, you should find it as a concave function.Which could be derived by take the S.O.C.

2.5 An intro to social insurance

The idea of social insurance is almost the same as the insurance of individuals.
We still have $\theta_1, \theta_2, \pi_1, \pi_2$, and this time we need to maximize social welfare, which is described by:

$$\pi_1 u(\theta_1 + x) + \pi_2 u(\theta_2 - px)$$

State 1, in this model, means the people who are unemployed, and they are π_1 of the citizens. State 2 is opposite. A lot of people will simultaneously have the same life, we still need to maximize ex ante utility.

Say this time,

$$\max_b, t\pi_1 u(\theta_1 + b) + pi_2 u(\theta_2 - t)$$

b represents benefits received in state 1, t represents the tax paid in state 2.
And the optimality will be the same.