Time Series Modeling
ARMA Models with Seasonality (TS4)

1 Introduction

Model construction, including time series models, depends on EDA to uncover the properties of the data that allow the selection and construction of reliable effects models and prediction (forecast) models. For time series modeling, we have examined the foundations of parametric models such as ARMA models. Parametric time series models require the selection of orders of autoregression and moving averages as well as the order of differencing for obtaining a stationary series. Seasonal time series also require obtaining a stationary series to apply the appropriate seasonal parameterization. The efficacy of the model or models constructed then are assessed for fitness of use. This model construction process, when applied to time series data particularly, is known as the Box-Jenkins Methodology.

2 Box-Jenkins Methodology

Autoregressive Moving Average (ARMA) models enjoy extensive treatment in the statistics literature as well as other disciplines. Box and Jenkins (1970) introduced a comprehensive approach to time series modeling of univariate responses forming the basis of ARMA modeling. The theoretical underpinnings described by Box and Jenkins (1970) and Box, Jenkins, and Reinsel (1994) are mathematically sophisticated, but in the sections that follow, an introduction to the procedures needed for the data analyst to use ARMA models is provided. The steps described are model identification, estimation, diagnostic checking, and lack-of-fit tests. These steps are part of the generic model-building process including EDA, model selection, model construction, model diagnostics, forecast viability, and model description.

3 Model Order Identification

EDA provides descriptive information on the probability distribution of a time series, e.g., if the distribution is skewed or has tails different from what is expected for a normal distribution, and whether the series variance is constant. EDA informs on the correlation within the series, and among possible covariates. EDA allows us to detect trends and cycles in the series. The order of differencing to de-trend the series may be seen in EDA, and certainly, upon differencing the series, EDA must be performed in its entirety as we now have a new time series. However, the polynomial order of ARMA AR and MA processes must be identified using techniques other than traditional EDA techniques.

3.1 Transformations (log and differencing)

There are three basic conditions in time series data that may prevent a series from being stationary. The first condition is non-constant variance, the second is linear or nonlinear trends in the series, and the third is periodicity such as seasonality.

Non-constant variance in time series data is detected by comparing the variance of one sub-span of consecutive realizations with the variance of another, usually non-overlapping sub-span of consecutive realizations. If the time series is normally distributed, e.g., an F-test may be used to compare the span variances. If the variance is non-constant, then a commonly used transformation to stabilize the variance is the natural logarithm transformation. Other transformations such as a square root may be appropriate. You may find the Box-Cox transformation identification procedure useful. Regardless of the transformation used, the outcome of the transformation must be examined with EDA techniques to assure the variance is stabilized.

Once the variance is stabilized, we can address trends. A linear trend typically requires only a first difference, though fitting a regression model to the series as a response and time as a predictor can generate residuals that are stationary, and an ARMA model may then be fitted to these residuals. If the trend appears quadratic, a second order difference may be appropriate. Further, a cubic trend may require a third order difference, and so on. Once the difference order is chosen, the analyst must verify the resulting series is stationary.

3.2 ACF and PACF

Figure 1 has an ACF plot in the left panel and a PACF plot in the right panel. Although it is ancillary information, the process these plots represent was first de-trended with a first difference transformation. The ACF plot is used to ascertain the order of the MA process of the ARMA model, and the PACF plot is used to determine the order of the AR process of the ARMA model. The horizontal dashed blue line in each panel's plot represents a 95% confidence band which may be interpreted thusly: if a vertical line at a particular lag extends outside the band, then that lag number is used to determine the order of the MA process (ACF plot) or the AR process (PACF plot).

The left panel of Figure 1 suggests an MA order of 1. Note that the lag at zero is not used to determine MA order. Hence, it is reasonable to try a $MA(1)$ process for the ARMA model. The PACF plot is a bit more involved. The direct reading is order 4, suggesting an $AR(4)$ process. Thus, we have a $ARMA(4, 1)$.

The time series consists of 200 sample realizations. Considering that the PACF orders of 3 and 4 have less extension outside the confidence band, we may ask what is the probability these orders occur randomly? A 95% confidence band indicates that approximately 5% of the correlations may occur randomly. Hence, we have approximately $200 \times 0.05 = 10$ lags may extend beyond the confidence on a random basis.There are 6 lags that extend beyond the band, and therefore a possible autoregressive process may be $AR(2)$ if we assume the remaining lags are random. It is suggested both ARMA models be constructed and analyzed to see if there is significant improvement of one model over the other.

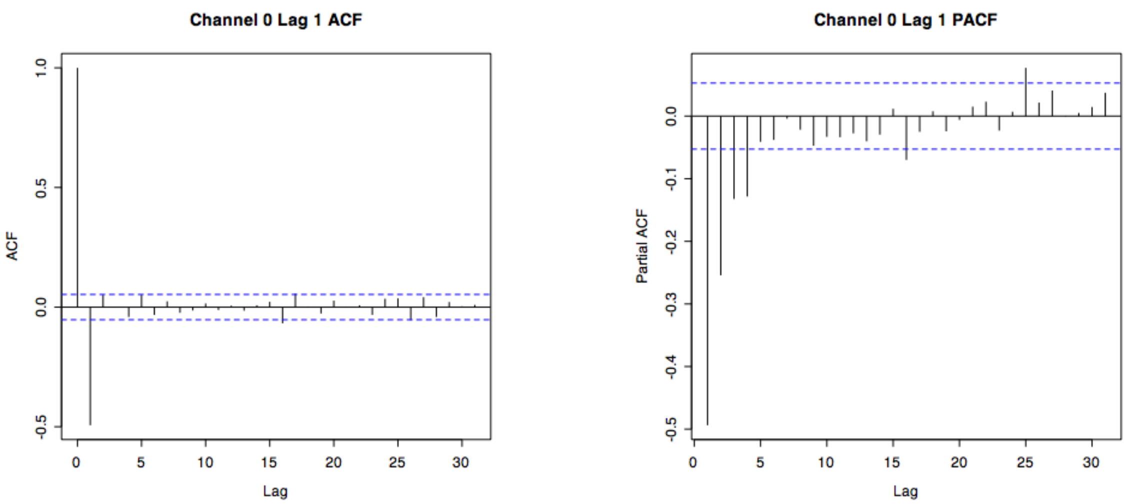

Figure 1: ACF and PACF plots after taking a first difference for trend removal.

3.3 ESCAF

The Extended Sample AutoCorrelation Function (ESACF) method can tentatively identify the orders of a stationary or non-stationary ARMA process based on iterated least squares estimates of the autoregressive parameters. For a stationary or non-stationary time series that is mean-corrected with a true autoregressive order of $p + d$ and with a true moving-average order of q. The ESACF method may be used to estimate the unknown orders $p + d$ and q in addition to analyzing the autocorrelation functions associated with the series. See Tsay and Tiao (1984), Box and Jenkins (1970), and Choi (1992).

3.4 SCAN

The Smallest CANonical (SCAN) correlation method can tentatively identify the orders of a stationary or non-stationary ARMA process. Given a stationary or non-stationary time series that is mean-corrected and has a true autoregressive order of $p + d$ and with a true moving-average order of q, the SCAN method utilizes eigenvalues of the correlation matrix of the ARMA process to find reasonable values of these orders. See Tsay and Tiao (1984), Box and Jenkins (1970), and Choi (1992).

3.5 MINIC

The MINimum Information Criterion (MINIC) method can tentatively identify the order of a stationary and invertible ARMA process. Given a stationary and invertible time series that is mean-corrected with a true autoregressive order of p and with a true moving-average order of q, the MINIC method computes the Akaike Information Criterion (AIC) for various autoregressive and moving average orders, and the orders producing the minimum AIC are presented. See Box et al. (1994) and Choi (1992).

4 Estimation

The ARMA procedure primarily uses the computational methods outlined by Box and Jenkins. MarquardtÕs method is used for the nonlinear least squares iterations. Numerical approximations of the derivatives of the sum-of-squares function are taken by using a fixed delta between derivative values. The methods do not always converge successfully for a given set of data, particularly if the starting values for the parameters are not close to the least squares estimates.

4.1 Initial estimates for the parameters of an ARMA model

If an autoregressive or moving average operator is specified with no missing lags, preliminary estimates of the parameters are computed by using the autocorrelation orders derived from the similarly to the ACF and PACF plots. Otherwise, the preliminary estimates may be arbitrarily set to values that produce stable polynomials. The model fit then must be tested.

Often the initial values of the coefficients for any given autoregressive or moving average coefficients are often set to 0.1 if the degree of the polynomial associated with the factor is 9 or less. Otherwise, the coefficients are determined by expanding the polynomial $(1 - 0.1B)$ to an appropriate power by using a recursive algorithm, where B is the backshift operator. These preliminary estimates are then the starting values in an iterative algorithm to compute estimates of the parameters.

4.2 Different approaches to the estimation of the model parameters

With Maximum Likelihood Estimation (MLE), the likelihood function is maximized via nonlinear least squares often using Marquardt's method. Maximum likelihood estimates are more expensive to compute than the conditional least squares estimates; however, they may be preferable as they are robust against estimations affected by scale. Also, the underlying assumptions are fewer than the least squares estimates.

Unconditional Least Squares (ULS) methods produce least squares estimates that are not dependent on any but the minimization of time series realizations to a set of polynomial coefficients. The ULS method is also referred to as the exact least squares (ELS) method. The unconditional least squares estimates are obtained by minimizing the sum of squared residuals rather than using the log-likelihood of MLEs as the criterion function.

Conditional Least Squares (CLS) produces least squares estimates conditional on the assumption that the past unobserved errors are equal to 0. The series realizations are represented in terms of the previous observations adjusted by a weight parameter, and the minimization is with respect to modifications of the parameters.

Each time series analysis package specifies which estimation methods are available as well as assigning a default method. Check the function or procedure documentation for details.

4.3 Model Order Adjustments

Note: t-ratio is arma (coefficient - 1) / se(coefficient).

For complex roots, see p 45, Tsay (2010), Analysis of Financial Time Series, R demo: Script beginning with

$$p1 < -c(1, -m2\$coer[1:3])$$
$$roots < -polyroot(p1)$$

etc. Mod is the modulus of a complex number and is defined as:

$$Mod(1.59 + 1.06) = sqrt(1.59^2 + 1.06^2) \approx 1.91$$

This then is used to calculate the frequency of the so-called business cycle which, in the Tsay example, is approximately 10.7 quarters for a full cycle. The inverse gives the period.

5 Model Diagnostics

The sufficiency and conformance of a model to the model specification orders, as well as model fit to the data, are assessed with model diagnostics. These diagnostics take the form of graphical representations and measures of fit. The graphical representations can identify non-homogeneity of variance, appropriate residuals distributions, and the like. The fit measures not only show the efficacy of the model, but also allow for model comparison to determine the "best" model among various versions. Diagnostics are mandatory on all models incarnations.

5.1 Residuals analysis

Residuals analysis include examinations of the residuals as subjected to distribution tests, outlier analysis, and checks for constant variance. The residuals often are standardized to more easily identify, particularly, outliers; i.e., those residual values that fall outside a region bounded by ± 3 standard deviations. Standardization also gives visual inspection of constant variance which permits formal testing of various spans as suggested by the plot. Also, the plot gives a visual check on how symmetric the residuals are around zero. A formal test of skewness and kurtosis may be made if asymmetry is suspected or observed.

The question as to how many residuals are likely to lie outside the ± 3 standard deviation region is determined by using a pre-established confidence level, say, 0.05, and multiplying this level by the number of realizations of the residuals. For an ARMA model from first-differenced data, the calculation is $0.05(n-1)$, truncated to the nearest integer. If more than this number are outliers, the model should be re-assessed for its viabiity as a forecast model.

5.2 Autocorrelation check

The residuals may be autocorrelated, which is a violation of the requirement for independence. Plots of the residual's ACF and PACF will reveal significant realization dependencies. Lag lines exceeding the confidence region boundaries indicate dependence. Should dependence be revealed, the model should be re-examined as a candidate for forecasting.

5.3 Use of residual analysis to modify models

Residuals analysis, as described above, show whether the residuals are independently identically distributed white noise. If not, the following remedial measures must be considered.

1. **Excessive outliers**: Consider using a transformation to the time series or find a more appropriate transformation than that already in use. If a transformation is inappropriate or a suitable one not available, the time series model type must be reconsidered.

2. **Significant skew or kurtosis**: Generally, skewness and kurtosis may be ameliorated with an appropriate series transformation. However, if this is not the case, then a different time series model type may be necessary.

3. **Non-constant variance**: As with the two situations above, variance stabilization usually is achieved via an appropriate transformation. If one is not available, a different model type should be considered.

4. **Non-stationariy**: Non-stationarity in the residuals may occur from residual realization dependence or insufficient trend adjustment. If the residuals exhibit dependencies, then the ACF and PACF lags can be used to correct the AR and MA orders of the ARMA model, and this model reconstructed accordingly. If a trend is apparent in the residuals plot, a different differencing order may correct the problem. If different differencing orders do not correct the issue, a regression fit to the time series may be needed.

When remedial measures are utilized, the model-building process may be a series of iterations to arrive at an adequate forecasting model. The construction and re-construction may occur several times, so the AIC may be introduced to assist in comparing the viability of the iterations. In any case, once the model is instituted for forecasting, monitoring of the forecasts for usefulness is necessary.

6 Portmanteau Lack-of-Fit Test

The test of residuals using the ACF essentially tests each lag individually for goodness of fit of the model. An alternative is to use a test of a set of lags simultaneously against a set of zeros. This type of test is known as portmanteau lack-of-fit test. Portmanteau tests are not restricted to lack of fit testing, and are often used to assess the significance of autocorrelation coefficients generally.

Box and Pierce (1970) developed a portmanteau test defined as

$$Q = n \sum_{k=1}^{h} \hat{\rho}_k^2 \tag{1}$$

where k is the maximum lag being considered and n is the number of observations in the series. If the residuals are white noise, Q follows a $\chi^2_{h-(p+q}$ distribution with degrees of freedom (df) equal to the number of lags h less the number of parameters p and q in the ARMA model. The value of Q then is compared to a χ^2 statistic with the appropriate df to obtain the significance of deviation from white noise.

More commonly used is the portmanteau test of Ljung and Box (1978) such that

$$Q^* = n(n+2)\sum_{n-k}^{h}(n-k)^{-1}\hat{\rho}_k^2 \tag{2}$$

which has a distribution closer to the χ^2 distribution than does the Box-Pierce Q statistic. If the residuals are white noise, the Ljung-Box Q^* statistic has exactly the same distribution as the Box-Pierce Q statistic, viz., a χ^2 distribution with $h-(p+q)$ df. Reject the hypothesis of white noise if either the Q statistic or the Q^* statistic lies in the 5% of the right-hand tail of the appropriate χ^2 distribution.

Be aware that these two portmanteau tests often fail to reject poorly fitting models. The portmanteau tests must be used in conjunction with the residuals analyses described above. However, if each of the residuals analyses suggest an adequate model, there usually is no need for further model refinement.

7 Conclusions

Once an adequate time series model is obtained, it then becomes necessary to assess the viability of the forecasts. Forecast viability may be examined by using a training data set and a test data set. If more than one model is under consideration, in addition to the AIC statistics, the forecast confidence band sizes should be compared. Once the forecasts are considered practicable, the model may be placed into service. However, any model must be monitored regularly to assure its viability remains without compromise.

References

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