Time Series Modeling
MA and ARMA Models (TS3)

1 Introduction

We have begun our study of Gaussian-based time series models with autoregressive (AR) models. We saw the data must be Gaussian and stationary before we can undertake an examination of the time series autocorrelation structure. These same conditions hold for moving average models constructed from white noise, and for autoregressive moving average (ARMA) models. ARMA models are the linear combination of both AR and MA models as is described below.

2 Stationarity and invertibility conditions for a linear process

When we construct a linear regression model, we test for conformance to a set of assumptions, viz., the Gauss-Markov assumptions. Time series models also require a set of assumptions to achieve an adequate, reliable forecast model. Just as with linear regression models which require conditions of constant variance for a stable inverted design matrix, time series models require a stationary process and the ability to invert a linearly-related set of polynomials.

3 Autoregressive Moving Average Models (ARMA)

We consider a class of time series models that require the time dependent process to be stationary. This class of models may be defined in terms of linear difference equations (not necessarily the first difference from above) with constant coefficients. This linear difference of constant coefficients defines a parametric family of stationary processes known as AutoRegressive Moving Average (ARMA) models. The linear structure of ARMA models allows for a simple theory of linear forecasting.

To understand ARMA models, we first recall the particulars of as white noise.

3.1 White Noise (WN)

A time series with mean zero and finite, constant variance with each realization through time independent of any other realization is called white noise. White noise is denoted $Z_t \sim \mathcal{WN}(0, \sigma^2)$, where Z_t is a time-ordered sequence of identically distributed random variables with zero mean and finite variance ($\sigma^2 < \infty$). If the white noise realizations are each independent in time, then we say the white noise is identically independently distributed ($Z_t \sim \text{iid } \mathcal{WN}(0, \sigma^2)$).

White noise processes are important to time series model construction as we can generate a large class of time series processes using white noise as a basis.

3.2 ARMA models

Suppose we have a mean zero time series $X_t,\ t = 0, \pm 1, \pm 2, \ldots$ This process is said to be $ARMA(p, q)$ if X_t is stationary and if, for all t,

$$X_t - \phi_1 X_{t-1} - \cdots - \phi_p X_{t-p} = Z_t + \theta_1 Z_{t-1} + \cdots + \theta_q Z_{t-q}, \tag{1}$$

where $Z_t \sim \mathcal{WN}(0, \sigma^2)$. It is clear that the process X_t is generated by a linear combination of constant coefficients of Z_t, a white noise series. A compact way to write Equation 1 is

$$\phi(B)X_t = \theta(B)Z_t, \quad t = 0, \pm 1, \pm 2, \ldots, \tag{2}$$

where ϕ and θ are the respective pth and qth polynomials

$$\phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p \tag{3}$$

and

$$\theta(z) = 1 + \theta_1 z + \cdots + \theta_p z^q. \tag{4}$$

B is the back shift operator

$$B^j X_t = X_{t-j},\ j = 0, \pm 1, \pm 2, \ldots \tag{5}$$

The coefficients $\phi_k,\ k = 1, 2, \ldots, p$ are the autoregressive (AR) polynomial coefficients and the coefficients $\theta_l,\ l = 1, 2, \ldots, q$ are the moving average (MA) polynomial coefficients.

3.3 Moving Average (MA) models

If $\phi(z) \equiv 1$, then

$$X_t = \theta(B)Z_t, \tag{6}$$

which is a moving average process of order q, and is denoted $MA(q)$. MA models are stationary models which are functions of white noise when the θ are correctly chosen.

3.4 Autoregressive (AR) models

If $\theta(z) \equiv 1$, then

$$\phi(B)X_t = Z_t, \tag{7}$$

which is an autoregressive (AR) process of order p and is denoted as $AR(p)$. Notice that if the coefficients ϕ are correctly chosen, the AR model is a white noise generator.

3.5 Duality between autoregressive and moving average models

If the process X_t is linearly dependent on Z_t through a set of coefficients whose sum is finite, and if X_t is an ARMA process for which none of the polynomials ϕ and θ have common zeros, then the

ARMA process X_t is considered to be causal. In addition, when we can write an ARMA process $\phi(B)X_t = \theta(B)Z_t$ such that

$$Z_t = \sum_{j=0}^{\infty} \pi_j X_{t-j}, \; t = 0, \pm 1, \pm 2, \ldots, \tag{8}$$

and $\sum_{j=0}^{\infty} \mid \pi_j \mid < \infty$, then the $ARMA(p, q)$ process is said to be invertible. A time series that is invertible is analogous to a linear regression model with a non-singular design matrix which enables inversion.

To obtain adequate, useful forecasting ARMA models, the ARMA process must be causal and invertible.

3.6 AutoCorrelation Function (ACF) and Partial AutoCorrelation Function (PACF)

The sample ACF can be determined for any time series and is not restricted to stationary processes. If a series contains a trend, the ACF will show a slow decay. If a series contains a periodic pattern, the ACF will exhibit the same cyclicality. We see, then, that the ACF may be used as an indicator of nonstationarity.

The ACF provides information on the order of an ARMA model's moving average order; i.e., the $MA(q)$.

The Partial AutoCorrelation Function (PACF), like the ACF, gives information on a time series' interdependencies. Unlike the ACF, the PACF at, say lag k, is the correlation between X_t and $X_{t-k}, \; k = 0, 1, 2, \ldots$, adjusted for the intervening realizations $X_{t-(k-1)}, X_{t-(k-2)}, \ldots, X_{t-1}$. The PACF is the correlation of X_t and X_{t-k} after regressing on the intervening realizations.

The PACF provides information on the order of an ARMA model's autoregressive order; i.e., the $AR(p)$.

Another way to denote an $ARMA(p, q)$ model is as $(AR(p), MA(q))$.

4 General properties of ARIMA models

If a time series has no apparent deviation from stationarity and has a rapidly decreasing ACF, we desire an ARMA process that represents the mean-corrected series. If these conditions do not exist, we often try a transformation to achieve stationarity and thereby a rapidly decreasing ACF. A commonly used transformation is a difference transformation, which we used in Equation **??** above.

If a differencing transformation results in a stationary process, we have a class of models known as AutoRegressive-Integrated Moving Average (ARIMA) processes. ARIMA models are denoted as $ARIMA(p, d, q)$, where p is the order of the autoregressive component, d is the order of the differencing (integrated component) required to achieve stationarity, and q is the order of the moving average component. The values of p, d, and q are positive integers. Note that once a suitable order of d is obtained, we may fit an $ARMA(p, q)$ model.

We shall investigate ARIMA models in more detail in upcoming modules.