Time Series Modeling
Volatility Models (TS6)

1 Introduction

The models we constructed in sessions 1 through 5 were mean models; i.e., we modeled the behavior of a time series' mean behavior. Mean models include the variants of AR, MA, ARMA, and ARIMA models. Mean models also include the seasonal models SARMA and SARIMA, among others. Similarly, in previous courses, we have constructed the expected value of a response using generalized linear models, including the subset of normal linear regression models. We also used generalized linear modeling to build models of the response variance. Volatility models in time series analysis refer to models of the variance of a time series.

2 Volatility

Changes in time series variance are important for understanding the series behavior, and for providing accurate and useful forecasts. Changing variance affects the validity and efficiency of statistical inference about the model parameters that describe the series.

The mean models assume the residuals, a_t, are unconditional identically distributed and stationary. However, a series with changing variance such as portrayed in Figure 1 suggests the residuals have variance conditional (dependent) on time.

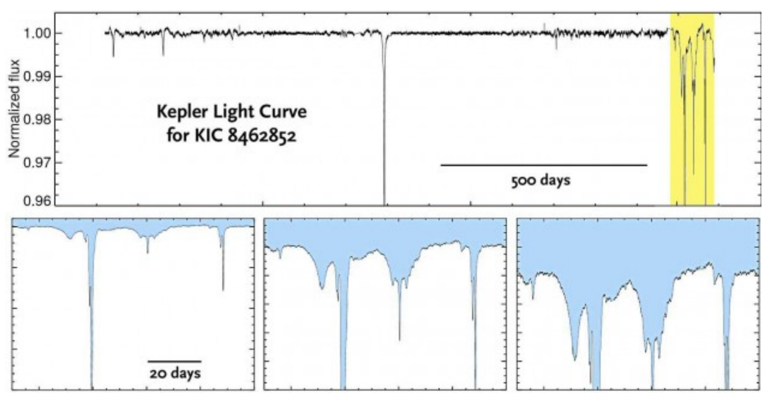

Figure 1: The top panel shows four years of Kepler observations of the 12^{th}-magnitude star KIC 8462852 in Cygnus. Several sporadic dips in its light output (normalized to 100%) suggest partial blocking of the star's light. The portion highlighted in yellow, recorded in February to April 2013, is shown at three different time scales along the bottom. The random, irregular shape of each dip could not be caused by a transiting exoplanet, which would show periodic dips. T. Boyajian et.al., MNRAS, 2015.

One approach for modeling volatility is to model the square of the residuals, a_t^2, as an AR(m) process such that

$$a_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2 + \cdots + \alpha_m a_{t-m}^2 + w_t, \tag{1}$$

where w_t is the associated white noise

$$\mathcal{E}(w_t) = 0$$
$$\mathcal{E}(w_t w_s) = \begin{cases} \xi & \text{for } t = s \\ 0 & else. \end{cases} \tag{2}$$

A white noise process a_t as described above is considered an AutoRgressive Conditional Heteroscedastic (ARCH) process of order m, and is denoted as $a_t \sim ARCH(m)$. As a_t is random, and a_t^2 is nonnegative, a_t^2 is weak stationary only if there are no unit roots.

3 Test for ARCH Effects

Testing residuals for time-varying heteroscedasticity is straightforward, and without need to estimate the ARCH parameters. Engle (1982) derived a test based on the Lagrange multiplier. Essentially, $\hat{a}_t^2$ is regressed on an intercept and m of its lagged values. The sample size, T, multiplies the centered regression R^2 which then converges to a χ_m^2 variable under $H_0 : a_t \sim \mathcal{N}(0, \sigma^2)$.

Another approach is to use the techniques we've studied based on the Box-Jenkins methods; viz., using residuals versus time plots, residuals ACF, and the Ljung-Box test (Box, Jenkins, and Reinsel, 1994; Box and Pierce, 1970).

4 ARCH and GARCH Models

ARCH models may be thought of as the equivalent AR model for a mean time series model. Note that it has but one parameter, m, which, in many cases, is insufficient to describe the complexity of the variance of a time series. This stands to reason as many probability distributions must be specified by no fewer than 2 parameters: the classic distribution of 2 parameters is, of course, the normal distribution.

If, then, the ARCH model is too simplistic, we may try a GARCH model. A GARCH model has two parameters. These two may be thought of as characterizing the autoregressive tendency of the variance of the time series, and a moving average tendency. A GARCH model is specified as $GARCH(r, m)$, where r specifies the autoregressive tendency and m specifies the moving average tendency. If the GARCH model is adequate, then the residuals are described by a white noise process. Hence, the ARIMA model methods used for diagnostics will apply to the GARCH model.

The GARCH model may be thought of a the time series variance version of an ARMA model. To construct a GARCH model, transform the time series to stabilize the variance if necessary. Then use the ACF and PACF as you would for an ARMA model. You will test for two parameters as mentioned above. Fit the suggested GARCH model, then use the diagnostic methods to test for WN.

An IGARCH model is the variance analog to the ARIMA mean model. Model construction is as in the previous paragraph with the addition of a differencing term. The idea is to produce a variance series that has mean zero and unit variance.

An EGARCH model, specified as $EGARCH(p, m)$, is used when the variance is not symmetrically distributed about zero (after variance stabilization and mean centering via differencing). Asymmetric variance is quite common, and the EGARCH model incorporates a parameter in the generalized error distribution that identifies the magnitude and direction (positive or negative) of the asymmetry.

Finance theory suggests that an asset with a higher perceived risk would pay a higher return than what is considered an average return. This characteristic implies that the mean return is tied to the level of the variance, i.e., the mean is a function of the variance. The model to manage this asymmetry is the GARCH-M model, which is specified as $GARCH - M(p, m)$.

A TGARCH model ($TGARCH(p, m)$ accounts for, simplistically, a stepwise shift in the variance through time. TGARCH estimation segregates the variance distribution into disjoint intervals. The disjoint intervals then allow for variance estimation by interval. Commonly, the standard deviation rather the variance is modeled.

Each of these models may be tested for persistence by summing the model coefficients.

5 Stochastic Volatility Model

The Stochastic Volatility (SV) model accounts for variance that fluctuates in time. The SV model accounts for fluctuations by introducing a Poisson shock term at the moment of a fluctuation onset. The term may be either a fixed period term, or a stochastic term. The period specification accounts for cyclical intervals s for the fluctuations whereas the stochastic term allows for random intervals for the shocks.

6 Summary

As we have seen, there are many useful and powerful models for describing the time dependence of the a series variance. We introduced only a few of the myriad of volatility models available. While there are many models for accounting for time series series volatility, the residuals of an adequate volatility model are expected to be close to a white noise process.

References

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