MSDS 413 Time Series Analysis
Roots of Polynomials

Consider a circle in the x,y-plane centered at (0,0) with unit radius; i.e., $r = 1$. Then, by the Pythagorean Theorem,

$$\begin{aligned} r^2 &= x^2 + y^2 \\ &= 1 \quad \text{for a unit circle.} \end{aligned} \tag{1}$$

From trigonometry, for $x = \cos t$ and $y = \sin t$, we have

$$\begin{aligned} r^2 &= x^2 + y^2 \\ &= (\cos t)^2 + (\sin t)^2 \\ &= \cos^2 t + \sin^2 t \\ &= 1 \end{aligned} \tag{2}$$

Recall that a solution to a quadratic equation is through the use of the quadratic formula

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{3}$$

which yields expressions for the two solutions to $ax^2 + bx + c = 0$. However, this formula may involve square roots of negative numbers. We may treat these solutions as ordinary numbers if we utilize $\sqrt{-1} \cdot \sqrt{-1} = -1$ such that $i^2 = -1$, where i denotes accessing the imaginary numbers in the complex plane. Think of the complex plane in which the x-axis is the real number line and the y-axis is the imaginary number line.

Now let a cyclical function be such that $x = \cos t$ and $y = i \sin t$. Then, using de Moivre's Formula, we have that

$$z = r(\cos t + i \sin t) \tag{4}$$

where z is from the complex numbers. As we know $r = 1$ for the unit circle, then from Equation 4, we obtain by squaring z,

$$\begin{aligned} z^2 &= r^2(\cos(t+t) + i\sin(t+t)) \\ &= \cos 2t + i \sin 2t \\ &= \cos^2 t - \sin^2 t + i(2 \sin t \cos t) \\ &= \cos^2 t + \sin^2 t - i2 \sin t \cos t \\ &= 1 - iat \end{aligned} \tag{5}$$

$$\implies z = \sqrt{1 - iat} \tag{6}$$

Thus, we have at complex roots about the unit circle, and, for $a \neq 0$ we have evidence of cyclicity. This result applies to cycles in time series data which we often refer to as seasonality.

The null hypothesis of the Augmented Dickey-Fuller test is that at least one unit root is present which is interpreted as the process is nonstationary.

The business cycles are calculated for an AR(3) assuming $\phi_1 = 0.4386$, $\phi_2 = 0.2063$, $\phi_3 = -0.1559$, the roots (complex) are $1.6161 + 0.8642i$, $-1.0902 - 0i$, and $1.6161 - 0.88642i$. The moduli of the complex roots are, for Re(root) = real component and Im(root) = the imaginary part:

$$\begin{aligned}\text{mod }(1.6161 + 0.8642) &= |Re(1.6161 + 0.8642i) + Im(1.6161 + 0.8642i)| \\ &= \sqrt{1.6161^2 + 0.8642i^2} \\ &= \sqrt{2.611779 + 0.7468(1)} \text{ as } (\sqrt{-1})^2 = (-1)^2 = 1 \\ &= 1.8326\end{aligned} \tag{7}$$

Repeat for each complex root.

To calculate the business cycles, use Tsay [2010], p. 42, as:

$$\begin{aligned}k &= \frac{2\pi}{\cos^{-1}\left(\frac{\phi_1}{2\sqrt{-\phi_2}}\right)} \\ &= \frac{2\pi}{\cos^{-1}\left(\frac{1.6161}{2\sqrt{1.8326}}\right)} \\ &\approx 12.7952\end{aligned} \tag{8}$$

Similarly for the other complex roots.

References

Ruey S. Tsay. *Analysis of financial time series.* Wiley, Cambridge, MA, 2010.