

1 Formalization

1.1 Syntax

$f \in \text{TRANSFORMER NAMES}$	$t \in \text{TYPE NAMES}$
$a, x, y, z \in \text{IDENTIFIERS}$	$\alpha, \beta \in \text{TYPE VARIABLES}$
$\mathcal{Q}, \mathcal{R}, \mathcal{S} ::= ! \mid \text{any} \mid \text{nonempty} \mid \text{empty} \mid \text{every}$	(type quantities)
$M ::= \text{fungible} \mid \text{unique} \mid \text{immutable} \mid \text{consumable} \mid \text{asset}$	(type declaration modifiers)
$T ::= \text{bool} \mid \text{nat} \mid \alpha \mid t[\overline{T}] \mid \text{map } \tau \Rightarrow \tau \mid \{\overline{x} : \overline{\tau}\}$	(base types)
$\tau, \sigma, \pi ::= \mathcal{Q} T$	(types)
$T_V ::= \alpha \text{ is } \overline{M}$	(type variable declaration)
$\mathcal{L} ::= \text{true} \mid \text{false} \mid n$	
$\quad \mid x \mid \mathcal{L}.x \mid \text{var } x : T \mid [\overline{\mathcal{L}}] \mid \{\overline{x} : \tau \mapsto \overline{\mathcal{L}}\}$	
$\quad \mid \text{demote}(\mathcal{L}) \mid \text{copy}(\mathcal{L})$	
$\quad \mid \mathcal{L}[\mathcal{L}] \mid \mathcal{L}[\mathcal{Q} \text{ s.t. } f[\overline{T}](\overline{\mathcal{L}})] \mid \text{consume}$	
$\text{Decl} ::= \text{transformer } f[\overline{T}_V](\overline{x} : \tau) \rightarrow x : \tau \{ \text{Stmt} \}$	(transformers)
$\quad \mid \text{type } t[\overline{T}_V] \text{ is } \overline{M} T$	(type decl.)
$\text{Trfm} ::= \text{new } t[\overline{T}](\overline{\mathcal{L}}) \mid f[\overline{T}](\overline{\mathcal{L}})$	(transformer calls)
$\text{Stmt} ::= \mathcal{L} \rightarrow \mathcal{L} \mid \mathcal{L} \rightarrow \text{Trfm} \rightarrow \mathcal{L}$	(flows)
$\quad \mid \text{try } \{ \text{Stmt} \} \text{ catch } \{ \text{Stmt} \}$	(try-catch)
$\text{Prog} ::= \overline{\text{Decl}}; \overline{\text{Stmt}}$	(programs)

1.2 Statics

Define $\# : \mathbb{N} \cup \{\infty\} \rightarrow \mathcal{Q}$ so that $\#(n)$ is the best approximation by type quantity of n , i.e.,

$$\#(n) = \begin{cases} \text{empty} & \text{if } n = 0 \\ ! & \text{if } n = 1 \\ \text{nonempty} & \text{if } n > 1 \\ \text{every} & \text{if } n = \infty \end{cases}$$

$\boxed{\Gamma \vdash \mathcal{L} : \tau \dashv \Delta}$ **Locator Typing**

$$\frac{b \in \{\text{true}, \text{false}\}}{\Gamma \vdash b : ! \text{bool} \dashv \Gamma} \text{Bool}$$

$$\frac{}{\Gamma \vdash n : \#(n) \text{nat} \dashv \Gamma} \text{Nat}$$

[The idea is that both $\text{demote}(\mathcal{L})$ and $\text{copy}(\mathcal{L})$ give a demoted value, but $\text{demote}(\mathcal{L})$ gives a read-only value (so no copy needs to happen), whereas copy will actually copy all the data. The results below intentionally throw out the environment Δ , because we don't want to actually consume whatever references we used to get $\mathcal{L} : \tau$.]

$$\frac{\Gamma \vdash \mathcal{L} : \tau \dashv \Delta}{\Gamma \vdash \text{demote}(\mathcal{L}) : \text{demote}(\tau) \dashv \Gamma} \text{Demote}$$

$$\frac{\Gamma \vdash \mathcal{L} : \tau \dashv \Delta}{\Gamma \vdash \text{copy}(\mathcal{L}) : \text{demote}(\tau) \dashv \Gamma} \text{Copy}$$

[TODO: update the **FIELD** rule to modify the type of \mathcal{L} appropriately.]

$$\begin{array}{c}
\frac{\text{immutable} \notin \text{modifiers}(T)}{\Gamma, x : \mathcal{Q} \ T \vdash x : \mathcal{Q} \ T \vdash \Gamma, x : \text{empty} \ T} \text{VAR} \\
\\
\frac{\Gamma \vdash \mathcal{L} : ! \ T \vdash \Delta \quad \text{immutable} \notin \text{modifiers}(T) \quad \text{fields}(T) = \overline{z} : \overline{\sigma} \quad y : \tau \in \overline{z} : \overline{\sigma}}{\Gamma \vdash \mathcal{L}.y : \tau \vdash \Gamma} \text{FIELD} \\
\\
\frac{\Gamma = \Delta_0 \quad \forall 1 \leq i \leq n. \Delta_{i-1} \vdash \mathcal{L}_i : \tau \vdash \Delta_i}{\Gamma \vdash [\mathcal{L}_1, \dots, \mathcal{L}_n] : \#(n) \ \text{list} \ \tau \vdash \Delta_n} \text{LIST} \qquad \frac{\Gamma \vdash \overline{\mathcal{L}} : \tau \vdash \Delta}{\Gamma \vdash \{x : \tau \mapsto \overline{\mathcal{L}}\} : ! \ \{\overline{x} : \overline{\tau}\} \vdash \Delta} \text{RECORD} \\
\\
\frac{}{\Gamma \vdash (\text{var } x : T) : \text{empty} \ T \vdash \Gamma, x : \text{empty} \ T} \text{VARDEF} \qquad \frac{\tau \text{ consumable}}{\Gamma \vdash \text{consume} : \tau \vdash \Gamma} \text{CONSUME} \\
\\
\boxed{\Gamma \vdash S \ \text{ok} \vdash \Delta} \text{ Statement Well-formedness} \\
\\
\frac{\Gamma \vdash S : \mathcal{Q} \ T \vdash \Delta \quad \text{update}(\Delta, S, \Delta(S) \ominus \mathcal{Q}) \vdash \mathcal{D} : \mathcal{R} \ T \vdash \Xi}{\Gamma \vdash (S \rightarrow \mathcal{D}) \ \text{ok} \vdash \text{update}(\Xi, \mathcal{D}, \Xi(\mathcal{D}) \oplus \mathcal{Q})} \text{OK-FLOW-EVERY} \\
\\
\frac{\Gamma \vdash S : \mathcal{Q} \ T \vdash \Delta \quad \Delta \vdash x : \text{demote}(\mathcal{R} \ T) \vdash \Delta \quad \text{update}(\Delta, S, \Delta(S) \ominus \mathcal{Q}) \vdash \mathcal{D} : S \ T \vdash \Xi}{\Gamma \vdash (S \xrightarrow{x} \mathcal{D}) \ \text{ok} \vdash \text{update}(\Xi, \mathcal{D}, \Xi(\mathcal{D}) \oplus \mathcal{R})} \text{OK-FLOW-VAR} \\
\\
\frac{\Gamma \vdash S : \mathcal{Q} \ T \vdash \Delta \quad \text{typeof}(f, \overline{T}) = (\overline{x} : \overline{\sigma}, y : \text{demote}(\text{elemtype}(T))) \rightarrow z : ! \ \text{bool} \quad \forall i. \text{demote}(\Gamma(a_i)) = \sigma_i \quad \text{update}(\Delta, S, \Delta(S) \ominus \mathcal{Q}) \vdash \mathcal{D} : S \ T \vdash \Xi}{\Gamma \vdash (S \xrightarrow{\mathcal{R} \ \text{s.t.} \ f[\overline{T}](\overline{a})} \mathcal{D}) \ \text{ok} \vdash \text{update}(\Xi, \mathcal{D}, \Xi(\mathcal{D}) \oplus \min(\mathcal{Q}, \mathcal{R}))} \text{OK-FLOW-FILTER} \\
\\
\frac{\Gamma \vdash S : \mathcal{Q} \ T_1 \vdash \Delta \quad \text{typeof}(f, \overline{T}) = (\overline{x} : \overline{\sigma}, y : \text{elemtype}(T_1)) \rightarrow z : \mathcal{R} \ T_2 \ \{\overline{\text{Stmt}}\} \quad \forall i. \text{demote}(\Gamma(x_i)) = \sigma_i \quad \text{update}(\Delta, S, \Delta(S) \ominus \mathcal{Q}) \vdash \mathcal{D} : S \ T_2 \vdash \Xi}{\Gamma \vdash (S \rightarrow f[\overline{T}](\overline{x}) \rightarrow \mathcal{D}) \ \text{ok} \vdash \text{update}(\Xi, \mathcal{D}, \Xi(\mathcal{D}) \oplus \mathcal{Q})} \text{OK-FLOW-TRANSFORMER} \\
\\
\frac{\Gamma \vdash \overline{S}_1 \ \text{ok} \vdash \Delta \quad \Gamma \vdash \overline{S}_2 \ \text{ok} \vdash \Xi}{\Gamma \vdash (\text{try} \ \{\overline{S}_1\} \ \text{catch} \ \{\overline{S}_2\}) \ \text{ok} \vdash \Delta \sqcup \Xi} \text{OK-TRY}
\end{array}$$

$\vdash \text{Decl} \ \text{ok}$ Declaration Well-formedness

$$\frac{\overline{T}_V, \overline{x} : \overline{\tau} \vdash \overline{\text{Stmt}} \ \text{ok} \vdash \Gamma, y : \sigma \quad \forall \pi \in \text{img}(\Gamma). \neg \text{isAsset}(\overline{T}_V, \pi)}{\vdash (\text{transformer} \ f[\overline{T}_V](\overline{x} : \overline{\tau}) \rightarrow y : \sigma \ \{\overline{\text{Stmt}}\}) \ \text{ok}} \text{OK-TRANSFORMER}$$

$\text{Prog} \ \text{ok}$ Program Well-formedness

$$\frac{\vdash \overline{\text{Decl}} \ \text{ok} \quad \emptyset \vdash \overline{\text{Stmt}} \ \text{ok} \vdash \Gamma \quad \forall \tau \in \text{img}(\Gamma). \neg \text{isAsset}(\emptyset, \tau)}{(\overline{\text{Decl}}; \overline{\text{Stmt}}) \ \text{ok}} \text{OK-PROG}$$

1.3 Dynamics

$$\begin{aligned} V &::= \text{true} \mid \text{false} \mid n \mid \overline{\{x : \tau \mapsto \mathbb{N}\}} \\ \mathcal{V} &::= \overline{V} \\ \text{Stmt} &::= \dots \mid \text{revert} \mid \text{try}(\Sigma, \overline{\text{Stmt}}, \overline{\text{Stmt}}) \end{aligned}$$

Definition 1. An environment Σ is a tuple (μ, ρ) where $\mu : \text{IDENTIFIERNAMES} \rightarrow \mathbb{N}$ is the variable lookup environment, and $\rho : \mathbb{N} \rightarrow \mathcal{V}$ is the storage environment.

$$\boxed{\langle \Sigma, \overline{\text{Stmt}} \rangle \rightarrow \langle \Sigma, \overline{\text{Stmt}} \rangle}$$

Note that we abbreviate $\langle \Sigma, \cdot \rangle$ as Σ , which signals the end of evaluation.

$$\frac{\langle \Sigma, S_1 \rangle \rightarrow \langle \Sigma', \overline{S_3} \rangle}{\langle \Sigma, S_1 \overline{S_2} \rangle \rightarrow \langle \Sigma', \overline{S_3} \overline{S_2} \rangle} \text{SEQ} \qquad \frac{}{\langle \Sigma, (\text{revert}) \overline{S} \rangle \rightarrow \langle \Sigma, \text{revert} \rangle} \text{REVERT}$$

Locators.

$$\begin{aligned} &\frac{}{\langle \Sigma, x \rangle \rightarrow \langle \Sigma, \text{storage}(\mu(x), \rho(\mu(x))) \rangle} \text{LOC-ID} \\ &\frac{\ell \notin \text{dom}(\rho)}{\langle \Sigma, \text{var } x : T \rangle \rightarrow \langle \Sigma[\mu \mapsto \mu[x \mapsto \ell], \rho \mapsto \rho[\ell \mapsto []]], \ell \rangle} \text{LOC-VARDEF} \\ &\frac{\langle \Sigma, \mathcal{L} \rangle \rightarrow \langle \Sigma', \mathcal{L}' \rangle}{\langle \Sigma, \mathcal{L}.x \rangle \rightarrow \langle \Sigma', \mathcal{L}'.x \rangle} \text{LOC-FIELD-CONGR} \qquad \frac{\rho(\ell) = \overline{k} \quad \overline{\rho(k).x = j}}{\langle \Sigma, \ell.x \rangle \rightarrow \langle \Sigma', \overline{j} \rangle} \text{LOC-FIELD} \\ &\frac{\langle \Sigma, \mathcal{L} \rangle \rightarrow \langle \Sigma', \mathcal{L}' \rangle}{\langle \Sigma, [\overline{\ell}, \mathcal{L}, \overline{\mathcal{L}'}] \rangle \rightarrow \langle \Sigma', [\overline{\ell}, \mathcal{L}', \overline{\mathcal{L}'}] \rangle} \text{LOC-LIST-CONGR} \qquad \frac{k \notin \text{dom}(\rho)}{\langle \Sigma, [\overline{\ell}] \rangle \rightarrow \langle \Sigma[\rho \mapsto \rho[k \mapsto \overline{\ell}]], k \rangle} \text{LOC-LIST} \\ &\frac{\langle \Sigma, \mathcal{L} \rangle \rightarrow \langle \Sigma'', \mathcal{L}'' \rangle}{\langle \Sigma, \mathcal{L}[\mathcal{L}'] \rangle \rightarrow \langle \Sigma'', \mathcal{L}''[\mathcal{L}'] \rangle} \text{LOC-VAL-SRC-CONGR} \qquad \frac{\langle \Sigma, \mathcal{L}' \rangle \rightarrow \langle \Sigma'', \mathcal{L}'' \rangle}{\langle \Sigma, \ell[\mathcal{L}'] \rangle \rightarrow \langle \Sigma'', \ell[\mathcal{L}'] \rangle} \text{LOC-VAL-SEL-CONGR} \\ &\frac{}{\langle \Sigma, \overline{\ell}[\overline{k}] \rangle \rightarrow \langle \Sigma, \text{select}(\rho, \overline{\ell}, \overline{k}) \rangle} \text{LOC-VAL} \\ &\text{select}(\rho, \overline{\ell}, \overline{k}) = \begin{cases} [] & \text{if } \overline{k} = [] \\ \text{revert} & \text{if } \overline{\ell} = [] \text{ and } \overline{k} \neq [] \\ j, \text{select}(\rho, \overline{\ell} \setminus j, \overline{k}') & \text{if } \overline{k} = (i, \overline{k}') \text{ and } j \in \overline{\ell} \text{ and } \rho(i) = \rho(j) \\ \text{select}(\rho, \overline{\ell}, \overline{k}') & \text{if } \overline{k} = (i, \overline{k}') \text{ and } \forall j \in \overline{\ell}. \rho(i) \neq \rho(j) \end{cases} \end{aligned}$$

[TODO Finish this rule/figuring out exactly how all this reference stuff works...]

$$\frac{}{\langle \Sigma, \ell \rightarrow k \rangle \rightarrow \langle \Sigma[\rho \mapsto \rho[\ell \mapsto \rho(\ell)], k \mapsto \rho(k) + \rho(\ell)] \rangle} \text{FLOW}$$

$$\begin{array}{c}
\frac{\text{resolve}(\Sigma, \mathcal{S}) = (\Sigma', \ell)}{\langle \Sigma, \mathcal{S} \rightarrow \mathcal{D} \rangle \rightarrow \langle \Sigma'[\rho \mapsto \rho'[\ell \mapsto []], \text{put}(\rho'(\ell), \mathcal{D})] \rangle} \text{FLOW-EVERY} \\
\\
\frac{\text{resolve}(\Sigma, \mathcal{S}) = (\Sigma', \ell) \quad \rho'(\ell) = \mathcal{V} \quad \rho'(\mu'(x)) = \mathcal{W} \quad \mathcal{W} \leq \mathcal{V}}{\langle \Sigma, \mathcal{S} \xrightarrow{x} \mathcal{D} \rangle \rightarrow \langle \Sigma'[\rho \mapsto \rho'[\ell \mapsto \mathcal{V} - \mathcal{W}], \text{put}(\mathcal{W}, \mathcal{D})] \rangle} \text{FLOW-VAR} \\
\\
\frac{\text{resolve}(\Sigma, \mathcal{S}) = (\Sigma', \ell) \quad \rho'(\ell) = \mathcal{V} \quad \rho'(\mu'(x)) = \mathcal{W} \quad \mathcal{W} \not\leq \mathcal{V}}{\langle \Sigma, \mathcal{S} \xrightarrow{x} \mathcal{D} \rangle \rightarrow \langle \Sigma', \text{revert} \rangle} \text{FLOW-VAR-FAIL} \\
\\
\frac{\text{resolve}(\Sigma, \mathcal{S}) = (\Sigma', \ell) \quad \rho'(\ell) = \mathcal{V} \quad \mathcal{U} = [v \in \mathcal{V} \mid \langle \Sigma', f(\bar{x}, v) \rangle \rightarrow^* \langle \Sigma'', k \rangle \text{ and } \rho''(k) = \text{true}] \quad \text{compat}(|\mathcal{U}|, |\mathcal{V}|, \mathcal{Q})}{\langle \Sigma, \mathcal{S} \xrightarrow{\mathcal{Q} \text{ s.t. } f(\bar{x})} \mathcal{D} \rangle \rightarrow \langle \Sigma'[\rho \mapsto \rho'[\ell \mapsto \rho'(\ell) - \mathcal{U}], \text{put}(\mathcal{U}, \mathcal{D})] \rangle} \text{FLOW-FILTER} \\
\\
\frac{\text{resolve}(\Sigma, \mathcal{S}) = (\Sigma', \ell) \quad \rho'(\ell) = \mathcal{V} \quad \mathcal{U} = [v \in \mathcal{V} \mid \langle \Sigma', f(\bar{x}, v) \rangle \rightarrow^* \langle \Sigma'', k \rangle \text{ and } \rho''(k) = \text{true}] \quad \neg\text{compat}(|\mathcal{U}|, |\mathcal{V}|, \mathcal{Q})}{\langle \Sigma, \mathcal{S} \xrightarrow{\mathcal{Q} \text{ s.t. } f(\bar{x})} \mathcal{D} \rangle \rightarrow \langle \Sigma', \text{revert} \rangle} \text{FLOW-FILTER-FAIL} \\
\\
\frac{\text{resolve}(\Sigma, \mathcal{S}) = (\Sigma', \ell) \quad \rho'(\ell) = v, \mathcal{V} \quad \langle \Sigma'[\rho \mapsto \rho'[\ell \mapsto \mathcal{V}], f(\bar{x}, v) \rangle \rightarrow^* \langle \Sigma'', k \rangle}{\langle \Sigma, \mathcal{S} \rightarrow f(\bar{x}) \rightarrow \mathcal{D} \rangle \rightarrow \langle (\mu', \rho''), \text{put}([\rho''(k)], \mathcal{D}) (\mathcal{S} \rightarrow f(\bar{x}) \rightarrow \mathcal{D}) \rangle} \text{FLOW-TRANSFORMER} \\
\\
\frac{\text{resolve}(\Sigma, \mathcal{S}) = (\Sigma', \ell) \quad \rho'(\ell) = []}{\langle \Sigma, \mathcal{S} \rightarrow f(\bar{x}) \rightarrow \mathcal{D} \rangle \rightarrow \langle \Sigma, \text{put}([], \mathcal{D}) \rangle} \text{FLOW-TRANSFORMER-DONE}
\end{array}$$

[NOTE: It is important that we flow an empty list in the FLOW-TRANSFORMER-DONE rule, otherwise we may fail to allocate a variable as expected.]

$$\frac{\ell \notin \text{dom}(\rho) \quad \text{transformer } f(\overline{y} : \overline{\tau}) \rightarrow z : \sigma \{ \overline{S} \} \quad \mu' = \overline{y} \mapsto \overline{\mu(x)}, z \mapsto \ell}{\langle \Sigma, f(\bar{x}) \rangle \rightarrow \langle (\mu', \rho[\ell \mapsto []]), \overline{S} \ell \rangle} \text{CALL}$$

We introduce a new statement, $\text{try}(\Sigma, \overline{S}_1, \overline{S}_2)$, to implement the try-catch statement, which keeps track of the environment that we begin execution in so that we can revert to the original environment in the case of a `revert`.

$$\begin{array}{c}
\overline{\langle \Sigma, \text{try } \{\overline{S}_1\} \text{ catch } \{\overline{S}_2\} \rangle} \rightarrow \overline{\langle \Sigma, \text{try}(\Sigma, \overline{S}_1, \overline{S}_2) \rangle} \text{TRY-START} \\
\\
\frac{\langle \Sigma, \overline{S}_1 \rangle \rightarrow \langle \Sigma'', \overline{S}_1' \rangle}{\langle \Sigma, \text{try}(\Sigma', \overline{S}_1, \overline{S}_2) \rangle \rightarrow \langle \Sigma'', \text{try}(\Sigma', \overline{S}_1', \overline{S}_2) \rangle} \text{TRY-STEP} \\
\\
\overline{\langle \Sigma, \text{try}(\Sigma', \text{revert}, \overline{S}_2) \rangle} \rightarrow \overline{\langle \Sigma', \overline{S}_2 \rangle} \text{TRY-REVERT} \quad \overline{\langle \Sigma, \text{try}(\Sigma', \cdot, \overline{S}_2) \rangle} \rightarrow \Sigma \text{TRY-DONE}
\end{array}$$

[TODO: Need to be sure to handle uniqueness correctly]

1.4 Auxiliaries

Definition 2. Define $\mathbf{Quant} = \{\text{empty}, \text{any}, !, \text{nonempty}, \text{every}\}$, and call any $Q \in \mathbf{Quant}$ a type quantity. Define $\text{empty} < \text{any} < ! < \text{nonempty} < \text{every}$.

$\text{isAsset}(\overline{T_V}, \tau)$ **Asset Types**

$$\begin{aligned} \text{isAsset}(\overline{T_V}, Q\ T) \Leftrightarrow & Q \neq \text{empty} \text{ and } (\text{asset} \in \text{modifiers}(\overline{T_V}, T) \text{ or} \\ & (T = C\ \tau \text{ and } \text{isAsset}(\overline{T_V}, \tau)) \text{ or} \\ & (T = \{\overline{y} : \overline{\sigma}\} \text{ and } \exists x : \tau \in \overline{y} : \overline{\sigma}. \text{isAsset}(\overline{T_V}, \tau)) \text{ or} \end{aligned}$$

τ consumable **Consumable Types**

$$\begin{aligned} (Q\ T) \text{ consumable} \Leftrightarrow & \text{consumable} \in \text{modifiers}(T) \text{ or} \\ & \neg((Q\ T) \text{ asset}) \text{ or} \\ & (T = C\ \tau \text{ and } \tau \text{ consumable}) \text{ or} \\ & (T = \{\overline{y} : \overline{\sigma}\} \text{ and } \forall x : \tau \in \overline{y} : \overline{\sigma}. (\sigma \text{ consumable})) \end{aligned}$$

$Q \oplus \mathcal{R}$ represents the quantity present when flowing \mathcal{R} of something to a storage already containing Q . $Q \ominus \mathcal{R}$ represents the quantity left over after flowing \mathcal{R} from a storage containing Q .

Definition 3. Let $Q, \mathcal{R} \in \mathbf{Quant}$. Define the commutative operator \oplus , called combine, as the unique function $\mathbf{Quant}^2 \rightarrow \mathbf{Quant}$ such that

$$\begin{aligned} Q \oplus \text{empty} &= Q \\ Q \oplus \text{every} &= \text{every} \\ \text{nonempty} \oplus \mathcal{R} &= \text{nonempty} \quad \text{if } \text{empty} < \mathcal{R} < \text{every} \\ ! \oplus \mathcal{R} &= \text{nonempty} \quad \text{if } \text{empty} < \mathcal{R} < \text{every} \\ \text{any} \oplus \text{any} &= \text{any} \end{aligned}$$

Define the operator \ominus , called split, as the unique function $\mathbf{Quant}^2 \rightarrow \mathbf{Quant}$ such that

$$\begin{aligned} Q \ominus \text{empty} &= Q \\ \text{empty} \ominus \mathcal{R} &= \text{empty} \\ Q \ominus \text{every} &= \text{empty} \\ \text{every} \ominus \mathcal{R} &= \text{every} \quad \text{if } \mathcal{R} < \text{every} \\ \text{nonempty} \ominus \mathcal{R} &= \text{any} \quad \text{if } \text{empty} < \mathcal{R} < \text{every} \\ ! \ominus \mathcal{R} &= \text{empty} \quad \text{if } ! \leq \mathcal{R} \\ ! \ominus \text{any} &= \text{any} \\ \text{any} \ominus \mathcal{R} &= \text{any} \quad \text{if } \text{empty} < \mathcal{R} < \text{every} \end{aligned}$$

Note that we write $(Q\ T) \oplus \mathcal{R}$ to mean $(Q \oplus \mathcal{R})\ T$ and similarly $(Q\ T) \ominus \mathcal{R}$ to mean $(Q \ominus \mathcal{R})\ T$.

Definition 4. We can consider a type environment Γ as a function $\text{IDENTIFIERS} \rightarrow \text{TYPES} \cup \{\perp\}$ as follows:

$$\Gamma(x) = \begin{cases} \tau & \text{if } x : \tau \in \Gamma \\ \perp & \text{otherwise} \end{cases}$$

We write $\text{dom}(\Gamma)$ to mean $\{x \in \text{IDENTIFIERS} \mid \Gamma(x) \neq \perp\}$, and $\Gamma|_X$ to mean the environment $\{x : \tau \in \Gamma \mid x \in X\}$ (restricting the domain of Γ).

Definition 5. Let \mathcal{Q} and \mathcal{R} be type quantities, $T_{\mathcal{Q}}$ and $T_{\mathcal{R}}$ base types, and Γ and Δ type environments. Define the following orderings, which make types and type environments into join-semilattices. For type quantities, define the partial order \sqsubseteq as the reflexive closure of the strict partial order \sqsubset given by

$$\mathcal{Q} \sqsubset \mathcal{R} \Leftrightarrow (\mathcal{Q} \neq \text{any and } \mathcal{R} = \text{any}) \text{ or } (\mathcal{Q} \in \{!, \text{every}\} \text{ and } \mathcal{R} = \text{nonempty})$$

For types, define the partial order \leq by

$$\mathcal{Q} T_{\mathcal{Q}} \leq \mathcal{R} T_{\mathcal{R}} \Leftrightarrow T_{\mathcal{Q}} = T_{\mathcal{R}} \text{ and } \mathcal{Q} \sqsubseteq \mathcal{R}$$

For type environments, define the partial order \leq by

$$\Gamma \leq \Delta \Leftrightarrow \forall x. \Gamma(x) \leq \Delta(x)$$

Denote the join of Γ and Δ by $\Gamma \sqcup \Delta$.

$$\boxed{\text{elemtype}(T) = \tau}$$

$$\text{elemtype}(T) = \begin{cases} \text{elemtype}(T') & \text{if } T = \text{type } t \text{ is } \overline{M} T' \\ \tau & \text{if } T = \mathcal{C} \tau \\ ! T & \text{otherwise} \end{cases}$$

$$\boxed{\text{modifiers}(\overline{T_V}, T) = \overline{M}} \quad \textbf{Type Modifiers}$$

$$\text{modifiers}(\overline{T_V}, T) = \begin{cases} \overline{M} & \text{if } T = \text{type } t \text{ is } \overline{M} T' \\ \overline{M} & \text{if } (T \text{ is } \overline{M}) \in \overline{T_V} \\ \emptyset & \text{otherwise} \end{cases}$$

$$\boxed{\text{demote}(\tau) = \sigma} \quad \boxed{\text{demote}_*(T_1) = T_2}$$

Type Demotion demote and demote_* take a type and “strip” all the asset modifiers from it, as well as unfolding named type definitions. This process is useful, because it allows selecting asset types without actually having a value of the desired asset type. Note that demoting a transformer type changes nothing. This is because a transformer is **never** an asset, regardless of the types that it operators on, because it has no storage.

$$\text{demote}(\mathcal{Q} T) = \mathcal{Q} \text{demote}_*(T)$$

$$\text{demote}_*(\text{bool}) = \text{bool}$$

$$\text{demote}_*(\text{nat}) = \text{nat}$$

$$\text{demote}_*(\{\overline{x} : \overline{\tau}\}) = \{\overline{x} : \text{demote}(\overline{\tau})\}$$

$$\text{demote}_*(\text{type } t \text{ is } \overline{M} T) = \text{demote}_*(T)$$

$$\boxed{\text{fields}(T) = \overline{x} : \overline{\tau}} \quad \textbf{Fields}$$

$$\text{fields}(T) = \begin{cases} \overline{x} : \overline{\tau} & \text{if } T = \{\overline{x} : \overline{\tau}\} \\ \text{fields}(T) & \text{if } T = \text{type } t \text{ is } \overline{M} T \\ \emptyset & \text{otherwise} \end{cases}$$

$$\boxed{\text{update}(\Gamma, x, \tau)} \quad \textbf{Type environment modification}$$

$$\text{update}(\Gamma, x, \tau) = \begin{cases} \Delta, x : \tau & \text{if } \Gamma = \Delta, x : \sigma \\ \Gamma & \text{otherwise} \end{cases}$$

$\text{compat}(n, m, Q)$ The relation $\text{compat}(n, m, Q)$ holds when the number of values sent, n , is compatible with the original number of values m , and the type quantity used, Q .

$$\begin{aligned} \text{compat}(n, m, Q) \Leftrightarrow & (Q = \text{nonempty} \text{ and } n \geq 1) \text{ or} \\ & (Q = ! \text{ and } n = 1) \text{ or} \\ & (Q = \text{empty} \text{ and } n = 0) \text{ or} \\ & (Q = \text{every} \text{ and } n = m) \text{ or} \\ & Q = \text{any} \end{aligned}$$

$\text{values}(T) = \mathcal{V}$ The function values gives a list of all of the values of a given base type.

$$\begin{aligned} \text{values}(\text{bool}) &= [\text{true}, \text{false}] \\ \text{values}(\text{nat}) &= [0, 1, 2, \dots] \\ \text{values}(\text{list } T) &= [L \mid L \subseteq \text{values}(T), |L| < \infty] \\ \text{values}(\text{type } t \text{ is } \overline{M} T) &= \text{values}(T) \\ \text{values}(\{\overline{x : Q} T\}) &= [\{\overline{x : \tau \mapsto v} \mid v \in \text{values}(T)\}] \end{aligned}$$