INTEGRATION REVIEW SHEET
BC Calculus
03/04/12

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1. Formulas for basic antiderivatives

$$\int x^n dx = \frac{x^{n+1}}{n+1},\ (n \neq -1)$$
$$\int 0\ dx = c$$
$$\int k\ dx = kx + c$$
$$\int kf(x)dx = k\int f(x)dx$$
$$\int [f(x) \pm g(x)]\, dx = \int f(x)dx \pm \int g(x)dx$$

2. Chain rule of integration

$$\int f'(g(x)) \cdot g'(x) = f(g(x)) + c$$

3. Trig integration

   (a) Big 6

$$\int \cos(x)\ dx = \sin(x) + c$$
$$\int \sin(x)\ dx = -\cos(x) + c$$
$$\int \csc^2 x\ dx = -\cot(x) + c$$
$$\int \sec^2 x\ dx = \tan(x) + c$$
$$\int \sec(x)\tan(x)\ dx = \sec(x) + c$$
$$\int \csc(x)\cot(x)\ dx = -\csc(x) + c$$

   (b) Big 4

$$\int \tan^2 x\ dx = \int (\sec^2 x - 1)\ dx = \tan(x) - x + c$$
$$\int \cot^2 x\ dx = \int (\csc^2 x - 1)\ dx = -\cot(x) - x + c$$
$$\int \sin^2 x\ dx = \int \frac{1-\cos(2x)}{2}\ dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + c$$
$$\int \cos^2 x\ dx = \int \frac{1+\cos(2x)}{2}\ dx = \frac{1}{2}x + \frac{1}{4}\sin 2x + c$$

(c) Example of chain rule

$$\int \sec^2(5x)\ dx = \frac{\tan(5x)}{5} + c$$

4. Example of an initial condition problem (find c)

$$\begin{aligned}
F'(x) &= \frac{1}{x^2}, \quad x > 0 \\
F(x) &= \int \frac{1}{x^2}\ dx \\
&= \frac{x^{-1}}{-1} + c \\
&= -\frac{1}{x} + c, \quad x > 0 \\
F(1) &= -1 + c = 0 \Rightarrow c = 1 \\
F(x) &= \frac{-1}{x} + 1, \quad x > 0
\end{aligned}$$

5. Solve a differential equation (separate and integrate)
Find the general solution of the differential equation $\frac{dy}{dx} = x^2 + 1$.

$$\begin{aligned}
dy &= (x^2+1)dx \\
\int dy &= \int x^2 + 1\ dx \\
y &= \frac{x^3}{3} + x + c
\end{aligned}$$

6. Integrate from acceleration to position finding both constants from initial points

$$\begin{aligned}
s(0) &= 80 \quad s'(0) = 64 \quad s''(t) = -32 \\
s'(t) &= \int s''(t)dt = \int -32\ dt = -32t + c_1 \\
s'(0) &= 64 = -32(0) + c_1 \Rightarrow c_1 = 64 \\
s(t) &= \int s'(t)dt = \int -32t + 64\ dt = -16t^2 + 64t + c_2 \\
80 &= -16(0)^2 + 64(0) + c_2 \Rightarrow c_2 = 80 \\
s(t) &= -16t^2 + 64t + 80
\end{aligned}$$

7. Distance travelled
A particle moves along the x-axis at a velocity of $v(t) = \frac{1}{\sqrt{t}}$. Find the distance traveled from $t = 0$ to $t = 1$.

$$\begin{aligned}
x(t) &= \int v(t)dt \\
&= \int \frac{1}{\sqrt{t}}dt \\
&= 2\sqrt{t} \\
&= 2 - 0 = 2
\end{aligned}$$

8. Formula for definite integral. Meaning of definite integral.

   The definite integral of $f(x)$ from a to b represents the area under the curve $f(x)$ from a to b. The definition is as follows.

$$\lim_{||\Delta|| \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i = \int_a^b f(x)dx$$

9. LRAM, RRAM, MRAM – what affects order?

$$LRAM = \sum_{i=1}^{n} f(x_{i-1})\Delta x$$
$$RRAM = \sum_{i=1}^{n} f(x_i)\Delta x$$
$$MRAM = \sum_{i=1}^{n} f(\frac{x_i + x_{i-1}}{2})\Delta x$$

   For an increasing function $LRAM \leq MRAM \leq RRAM$ and for decreasing functions $RRAM \leq MRAM \leq LRAM$. Otherwise the order cannot be determined.

10. Trapeziod rule – what affects over or under approximation?

$$\int_a^b f(x)dx \approx \sum_{i=1}^{n} \left(\frac{f(x_i) + f(x_{i-1})}{2} \right) \Delta x$$

   Whether or not the curve is concave up or concave down affects affects the approximation of this Riemann Sum. It under approximates functions which are concave down and over approximates functions which are concave up.

11. Summation as integral – example with constants and n rectangles

$$n = 4$$
$$\int_1^3 x^2 + 2\ dx \approx \frac{1}{2} \sum_{i=1}^{4} \left(\frac{i}{2} \right)^2$$

12. Properties of integrals (all 10)

   (a) $\int_a^a f(x)dx = 0$

   (b) $\int_a^b f(x)dx = -\int_b^a f(x)dx$

   (c) if $f(x)$ is odd then $\int_{-a}^a f(x)dx = 0$

   (d) if $f(x)$ is even then $\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx$

   (e) $\int_a^b f(x)dx = \int_{a-c}^{b-c} f(x+c)dx$

   (f) $\int_a^b kf(x)dx = k\int_a^b f(x)dx$

   (g) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

   (h) $\int_a^b [f(x) \pm g(x)]\, dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

(i) $\int_a^b [f(x) \pm k]\,dx = \int_a^b f(x)dx \pm \int_a^b k dx$

(j) if $f(x) \geq 0$ then $\int_a^b f(x)dx \geq 0$
if $f(x) \leq 0$ then $\int_a^b f(x)dx \leq 0$

13. First Fundamental Theorem

If a function f is continuous on the closed interval $[a, b]$ and F is the antiderivative of f on the interval $[a, b]$ then

$$\int_a^b f(x)\ dx = F(b) - F(a)$$

14. Second Fundamental Theorem

If f is continuous on an open interval i containing a, then for every x in the interval,

$$\frac{d}{dx}\left[\int_a^x f(t)\ dt\right] = f(x)$$

(a)

$$\frac{d}{dx}\left[\int_0^x \sqrt{t^2+1}\ dt\right] = \sqrt{x^2+1}$$

(b)

$$\frac{d}{dx}\left[\int_{\pi/2}^{x^3} \cos t\ dt\right] = 3x^2 \cos x^3$$

15. Accumulation Theorem

$$F(a) + \int_a^b f(x)\ dx = F(b)$$

(a) Graphical interpretation

(b) Verbal interpretation

16. Area between 2 curves

$$A = \int_a^b [f(x) - g(x)]\ dx$$

(a) Vertical Rectangles

Find the area of the region bounded by the graphs of $f(x) = 2 - x^2$ and $g(x) = x$.

$$2 - x^2 = x \Rightarrow x = -2, 1$$
$$a = -2, \quad b = 1$$
$$A = \int_{-2}^1 (2 - x^2) - x\ dx$$
$$= \frac{9}{2}$$

(b) Horizontal Rectangles

Find the area of the region bounded by the graphs of $x = 3 - y^2$ and $x = y + 1$.

$$3 - y^2 = y + 1 \Rightarrow y = -2, 1$$
$$A = \int_{-2}^1 (3 - y^2) - (y + 1)\ dy$$
$$= \frac{9}{2}$$

17. Volume – Disks

To find the volume of a solid of revolution with the disk method:

$$\begin{aligned} V &= \pi \int_a^b [R(x)]^2 \; dx \quad \textit{horizontal axis of revolution} \\ &= \pi \int_a^b [R(y)]^2 \; dy \quad \textit{vertical axis of revolution} \end{aligned}$$

(a) About x-axis

Find the volume of the solid formed by revolving the region bounded by the graph of $f(x) = \sqrt{\sin x}$ and the x-axis $(0 \leq x \leq \pi)$ about the x-axis.

$$\begin{aligned} V &= \pi \int_0^\pi (\sqrt{\sin x})^2 \; dx \\ &= \pi \int_0^\pi \sin x \; dx \\ &= [-\cos x]_0^\pi \; dx \\ &= 2\pi \end{aligned}$$

(b) About y-axis

Find the volume of the solid formed by revolving the region bound by the graph of $y = x$ and $y = 2$ around the y-axis.

$$\begin{aligned} V &= \pi \int_0^2 y^2 \; dy \\ &= \frac{8\pi}{3} \end{aligned}$$

18. Volume washers

Volume of washer $= \pi(R^2 - r^2)w$.

$$V = \pi \int_a^b \left([R(x)]^2 - [r(x)]^2 \right) \; dx$$

(a) About x axis

Find the volume of the solid formed by revolving the region bounded by the graphs of $y = \sqrt{x}$ and $y = x^2$ about the x-axis.

$$\begin{aligned} V &= \pi \int_0^1 \left[\left(\sqrt{x}\right)^2 - (x^2)^2 \right] dx \\ &= \pi \int_0^1 (x - x^4) dx \\ &= \frac{3\pi}{10} \end{aligned}$$

(b) About y axis

Find the volume of the solid formed by revolving the region bounded by the graphs of $y = x^2 + 1$,

$y = 0$, $x = 0$, and $x = 1$ about the y-axis.

$$r(y) = \begin{cases} 0 & 0 \le y \le 1 \\ \sqrt{y-1} & 1 \le y \le 2 \end{cases}$$

$$\begin{aligned} V &= \pi \int_0^1 (1^2 - 0^2)dy + \pi \int_1^2 \left[1^2 - \left(\sqrt{y-1}\right)^2\right] dy \\ &= \pi \int_0^1 (1)dy + \pi \int_1^2 (2-y)dy \\ &= \frac{3\pi}{2} \end{aligned}$$

19. Volume known cross sections

$$\begin{aligned} V &= \int_a^b A(x)\ dx \quad \textit{perpendicular to } x\textit{-axis} \\ &= \int_a^b A(y)\ dy \quad \textit{perpendicular to } y\textit{-axis} \end{aligned}$$

(a) Example with a square

$$\begin{aligned} y &= \sqrt{16 - x^2} \\ V &= 2\int_0^4 \left(\sqrt{16-x^2}\right)^2 \end{aligned}$$

(b) Example with a right triangle

$$\begin{aligned} y &= \sqrt{16 - x^2} \\ V &= \int_0^4 \left(\sqrt{16-x^2}\right)^2 \end{aligned}$$

(c) Example with a semicircle

$$\begin{aligned} y &= \sqrt{16 - x^2} \\ V &= 2\pi \int_0^4 \left(\frac{\sqrt{16-x^2}}{2}\right)^2 \end{aligned}$$