"Discrete Mathematics"

Boolean Algebra, Logic Gates and Switching Circuits

Prepared By:
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Introduction

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- On studying Logic, we started with the Propositional Logic. We were interested in formalizing simple and clear statements expressing the surrounding universe, that includes mysterious, complicated, interrelated and interacted tremendous number of ideas and facts.

- In fact that study was mental with no applied verification. An important question will arise: How it will be, if we need such a verification?

- Surely, another platform -to achieve such a verification- is essentially needed. This platform will include:

1. Analysis, that will be achieved using: The *Boolean Algebra.*
2. Representation/Design, that will be achieved using: The *Logic Gates*.
3. Implementation, that will be achieved using: using The *Switching Circuits.*

The Basic Elements of Boolean Algebra

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- Like Algebra of Propositional Logic, but use some mathematical operators.
- <u>States/Numerical-Values</u>:

0 and 1 in place of: False and True, No and Yes, Off and On…

- <u>Operands/Variables</u>:

Upper case letters of which the can only be 1 or 0, instead of true or false, respectively. It is preferable to say the state of the variable rather than the variable value.

- <u>Operators</u>:

"."	≡ "Dot"	in place of: AND	≡ "$\wedge$".
"+"	≡ "Plus"	in place of: OR	≡ "$\vee$".
"$\overline{\ }$"	≡ "Bar"/Complement	in place of: NOT	≡ "$\neg$".

Basic Operators of Boolean Algebra

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- NOT — Simulated as a horizontal bar "¯" above the number
  - 0 = 1
  - 1 = 0

- OR — Simulated as a plus "+"
  - 0 + 0 = 0
  - 0 + 1 = 1
  - 1 + 0 = 1
  - 1 + 1 = 1

- AND — Simulated as a Dot "."
  - 0 . 0 = 0
  - 0 . 1 = 0
  - 1 . 0 = 0
  - 1 . 1 = 1

Boolean Algebra: Variables and Algebraic Expressions (Single-Variable)

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The variable x is called a **Boolean variable** if it assumes only the possible values: 0 and 1.

$B_n = \{(x_1, x_2, \ldots, x_n) \mid x_i \in B \text{ for } 1 \le i \le n\}$ is the set of all possible n-tuples of 0s and 1s. A function from B_n to B is called a **Boolean function of degree n.**

<u>DOT/AND (Intersection)</u>

- $x \,.\, 0 = 0$
- $x \,.\, 1 = x$
- $x \,.\, x = x$
- $x \,.\, x' = 0$

<u>PLUS/OR (Union)</u>

- $x + 0 = x$
- $x + 1 = 1$
- $x + x = x$
- $x + x' = 1$

Boolean Algebra: Variables and Algebraic Expressions (Multi-Variables)

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- $x + x\,y \quad = x$
- $x \,.\, y \quad = y \,.\, x$
- $x + y \quad = y + x$
- $x + x'\,y \quad = x + y$
- $x' + x\,y \quad = x' + y$
- $x \,.\, (y + z) \quad = x \,.\, y + x \,.\, z$
- $x \,.\, (y \,.\, z) = (x \,.\, y) \,.\, z \quad = x \,.\, y \,.\, z$
- $x + (y + z) = (x + y) + z \quad = x + y + z$
- $(w + x) \,.(y + z) = w \,.\, y + x \,.\, y + w \,.\, z + x \,.\, z$

Laws and Theorems of Boolean Algebra

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Annulment Law إلغاء

1a. $X \cdot 0 = 0$ 1b. $X + 1 = 1$

Identity Law

2a. $X \cdot 1 = X$ 2b. $X + 0 = X$

Idempotent Law

3a. $X \cdot X = X$ 3b. $X + X = X$

Complement Law

4a. $X \cdot \overline{X} = 0$ 4b. $X + \overline{X} = 1$

Double Negation Law إرتداد

5. $\overline{\overline{X}} = X$

Laws and Theorems of Boolean Algebra

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Associative Law

ترابطي

7a. $X\,(Y\,Z) = (X\,Y)\,Z = (X\,Z)\,Y = X\,Y\,Z$

7b. $X + (Y + Z) = (X + Y) + Z = (X + Z) + Y = X + Y + Z$

Distributive Law

توزيعي

8a. $X \cdot (Y + Z) = X\,Y + X\,Z$

8b. $X + Y\,Z = (X + Y) \cdot (X + Z)$

De Morgan's Theorem

9a. $\overline{X \cdot Y} = \overline{X} + \overline{Y}$

9b. $\overline{X + Y} = \overline{X} \cdot \overline{Y}$

Laws and Theorems of Boolean Algebra

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Absorption Law

10a. $X \cdot (X + Y) = X$

10b. $X + X\,Y = X$

Redundancy Law

زائد عن الحاجة

11a. $(X + Y) \cdot (X + \overline{Y}) = X$

11b. $X\,Y + X\,\overline{Y} = X$

Redundancy Law

12a. $(X + \overline{Y}) \cdot Y = XY$

12b. $X\,\overline{Y} + Y = X + Y$

Consensus Law

إجماع

13a. $X\,Y + \overline{X}\,Z + Y\,Z = X\,Y + \overline{X}\,Z$

13b. $(X + Y) \cdot (\overline{X} + Z) \cdot (Y + Z) = (X + Y) \cdot (\overline{X} + Z)$

Basic Logic Gates

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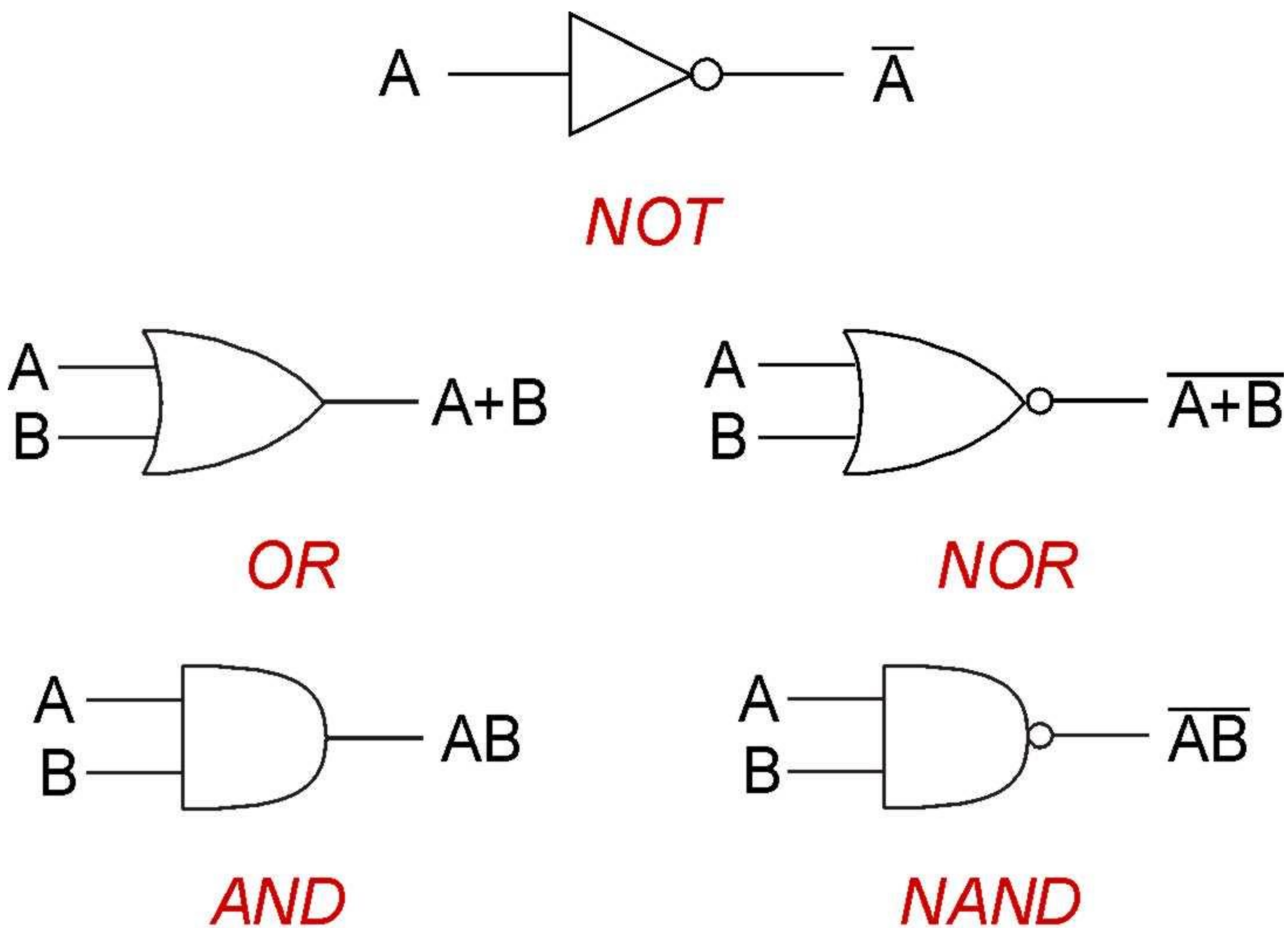
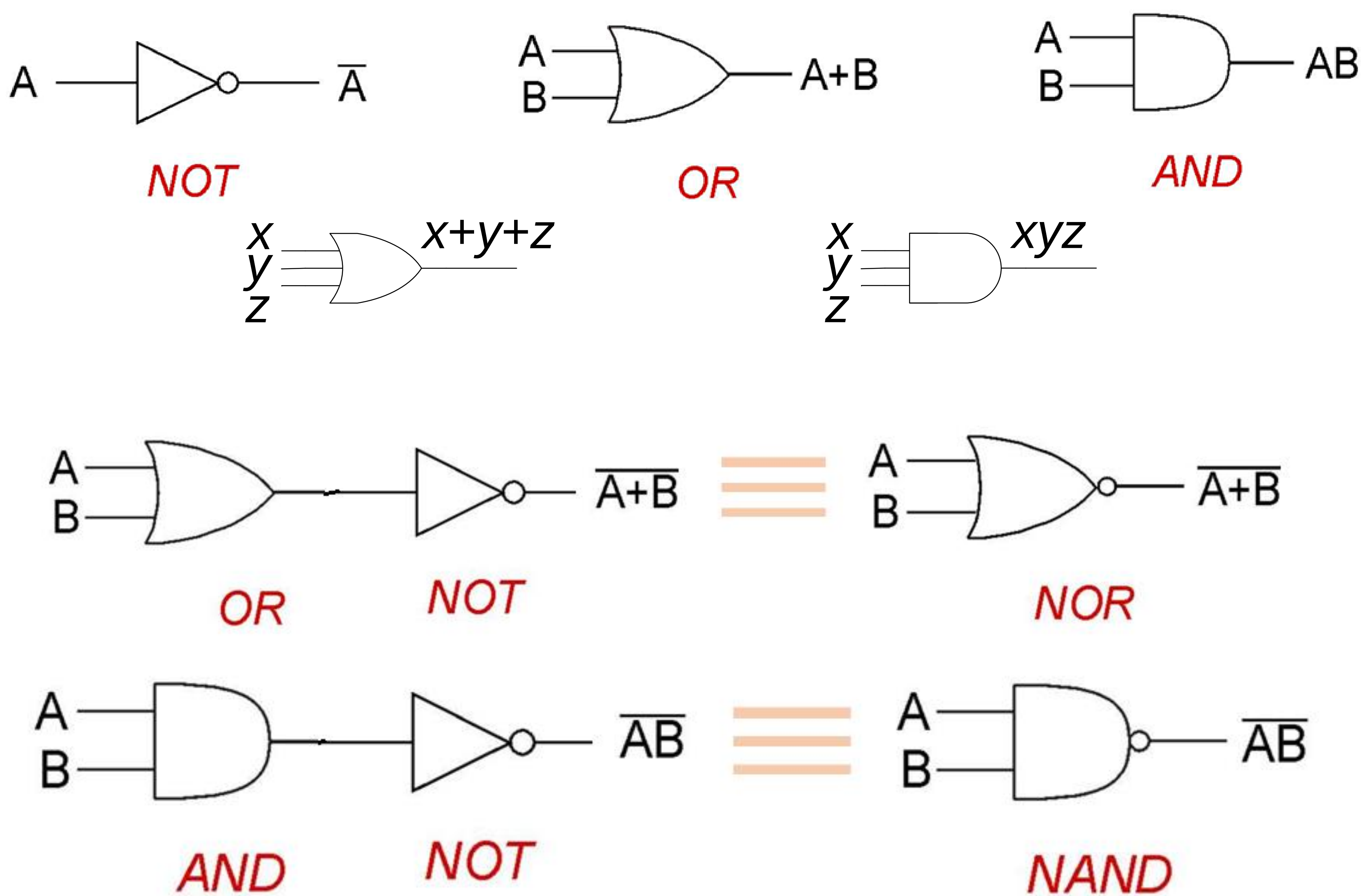


Basic Logic Gates



Gates

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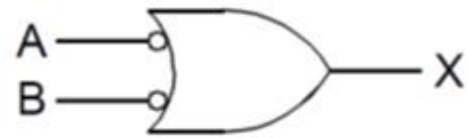
Standard | De Morgan's

NAND

$X = \overline{A \cdot B}$

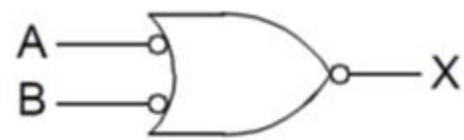
$X = \overline{A} + \overline{B}$



AND

$X = A \cdot B$

$X = \overline{\overline{A} + \overline{B}}$



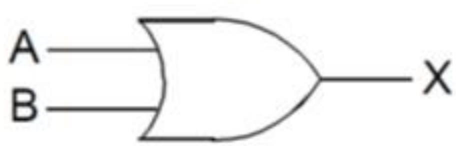
NOR

$X = \overline{A + B}$

$X = \overline{A} \cdot \overline{B}$

OR

$X = A + B$



$X = \overline{\overline{A} \cdot \overline{B}}$

Important Compound Gates

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XOR Gate

$$14a.\ X \oplus Y = (X + Y) \cdot (\overline{X} + \overline{Y})$$

$$14b.\ X \oplus Y = (X + Y) \cdot (\overline{X \cdot Y})$$

$$14c.\ X \oplus Y = \overline{X} \cdot Y + X \cdot \overline{Y}$$

XNOR Gate

$$15a.\ X \odot Y = \overline{X}\,\overline{Y} + X\,Y$$

$$15b.\ X \odot Y = (X + \overline{Y}) \cdot (\overline{X} + Y)$$

$$15c.\ X \odot Y = \overline{(X \cdot \overline{Y}) + (\overline{X} \cdot Y)}$$

$$X \odot Y = \overline{X \oplus Y} \qquad X \oplus Y = \overline{X \odot Y}$$

The Common Logic Functions and their Equivalent Boolean Notation

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Logic Function	Boolean Notation
AND	$A.B$
OR	$A+B$
NOT	$\overline{A}$
NAND	$\overline{A.B}$
NOR	$\overline{A+B}$
EX-OR	$(A.\overline{B}) + (\overline{A}.B)$ or $A \oplus B$
EX-NOR	$(A.B) + (\overline{A}.\overline{B})$ or $\overline{A \oplus B}$

Truth Table of the Logical Functions of the 2-input Logic Gates

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Inputs		Truth Table Outputs For Each Gate					
A	B	AND	NAND	OR	NOR	EX-OR	EX-NOR
0	0	0	1	0	1	0	1
0	1	0	1	1	0	1	0
1	0	0	1	1	0	1	0
1	1	1	0	1	0	0	1

De Morgan's Theorem #1

$$\overline{A \cdot B} = \overline{A} + \overline{B}$$

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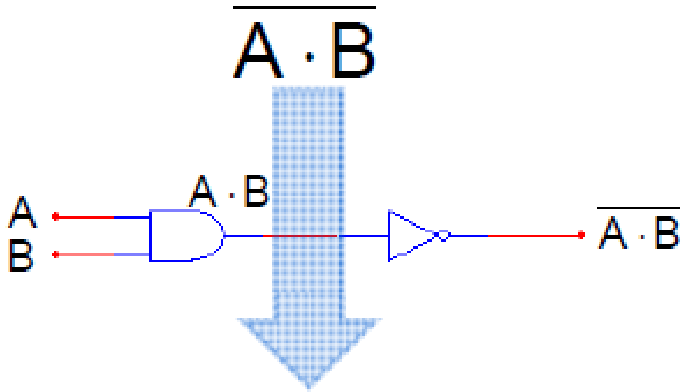

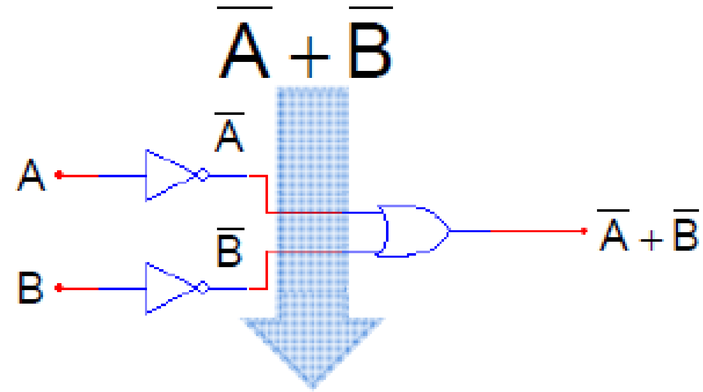

A	B	$A \cdot B$	$\overline{A \cdot B}$
0	0	0	1
0	1	0	1
1	0	0	1
1	1	1	0

A	B	$\overline{A}$	$\overline{B}$	$\overline{A} + \overline{B}$
0	0	1	1	1
0	1	1	0	1
1	0	0	1	1
1	1	0	0	0

The truth-tables are equal; therefore, the Boolean equations must be equal.

De Morgan's Theorem #2

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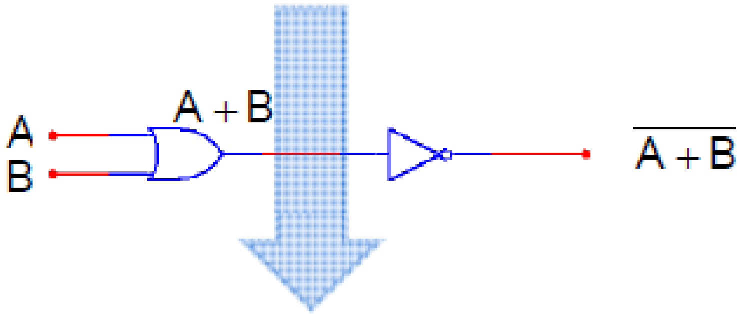
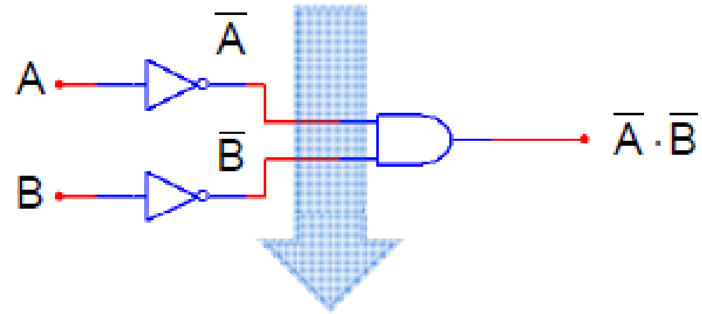
$$\overline{A+B} = \overline{A} \cdot \overline{B}$$

$\overline{A+B}$



$\overline{A} \cdot \overline{B}$



A	B	$A+B$	$\overline{A+B}$
0	0	0	1
0	1	1	0
1	0	1	0
1	1	1	0

A	B	$\overline{A}$	$\overline{B}$	$\overline{A} \cdot \overline{B}$
0	0	1	1	1
0	1	1	0	0
1	0	0	1	0
1	1	0	0	0

The truth-tables are equal; therefore, the Boolean equations must be equal.

De Morgan Shortcut

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Break The Line, Change The Sign

Break the *LINE* over the two variables,
and change the *SIGN* directly under the line.

$$\overline{A \cdot B} = \overline{A} + \overline{B}$$

For Theorem #14A, break the line, and change the AND function to an OR function. Be sure to keep the lines over the variables.

$$\overline{A + B} = \overline{A} \cdot \overline{B}$$

For Theorem #14B, break the line, and change the OR function to an AND function. Be sure to keep the lines over the variables.

Representations of Boolean Functions Using The Truth Table (Canonical Form)

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Inputs			Output	
A	B	C	f(A,B,C)	
0	0	0	1	A'B'C'
0	0	1	1	+ A'B'C
0	1	0	0	
0	1	1	1	+ A'BC
1	0	0	0	
1	0	1	0	
1	1	0	1	+ ABC'
1	1	1	1	+ ABC

Simplifications of Boolean Expressions

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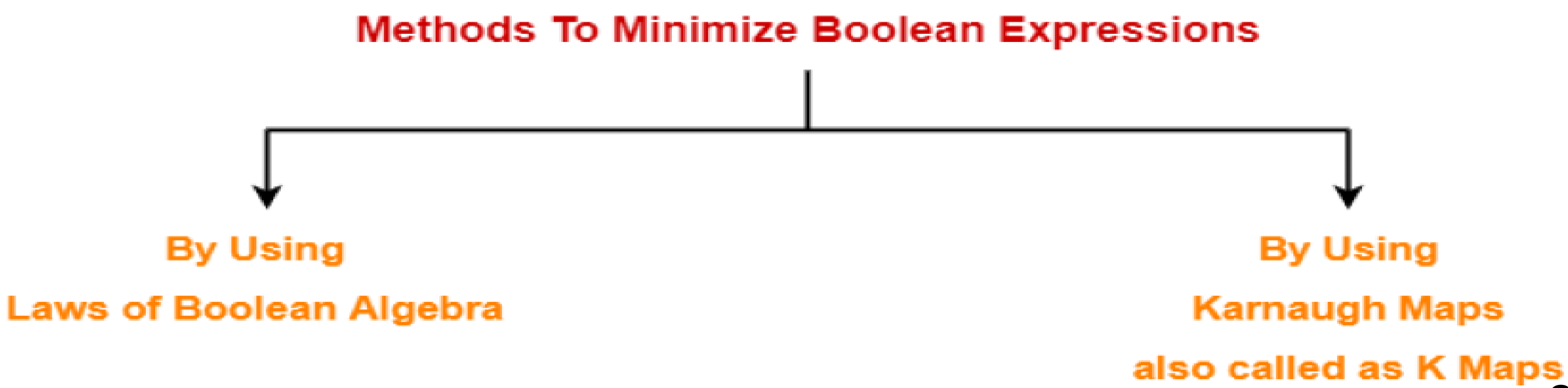
In many digital circuits and practical problems we need to find expression with minimum variables.

There are following two methods of minimizing or reducing the Boolean expressions, using:

1. Laws of Boolean Algebra, as shown above.
2. Karnaugh Maps, also called as K Maps.



Simplification Using Boolean Algebra Example 01

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A	B	C	F
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	0

$F=A'B'C+A'BC+AB'C'+AB'C+ABC'$

$= A'B'C+A'BC+AB'(C+C')+ABC'$

$= A'C(B+B') \quad +AB' \qquad +ABC'$

$= A'C+AB'+AC'(B+B')$;we can reuse $AB'C'$

$=A'C+AB'+AC'$

SOP

Simplification Using Boolean Algebra Example 02

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A	B	C	F
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

F=A'B'C+A'BC+AB'C+ABC'+ABC

= A'C(B+B')+AB'C+ABC'+ABC

= A'C + AC(B'+B) +ABC'

= A'C+AC+ AB(C+C')

=A'C+AC+AB

= C(A+A')+AB

= C+AB

Simplification Using Boolean Algebra Example 03

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$\overline{A}BC + A\overline{B}C + AB\overline{C} + ABC$

$\downarrow$ Factoring BC out of 1st and 4th terms

$BC(\overline{A} + A) + A\overline{B}C + AB\overline{C}$

$\downarrow$ Applying identity $A + \overline{A} = 1$

$BC(1) + A\overline{B}C + AB\overline{C}$

$\downarrow$ Applying identity $1A = A$

$BC + A\overline{B}C + AB\overline{C}$

$\downarrow$ Factoring B out of 1st and 3rd terms

$B(C + A\overline{C}) + A\overline{B}C$

$\downarrow$ Applying rule $A + \overline{A}B = A + B$ to the $C + A\overline{C}$ term

$B(C + A) + A\overline{B}C$

Example 03 "Cont."

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$$B(C + A) + A\overline{B}C$$

$\downarrow$ Distributing terms

$$BC + AB + A\overline{B}C$$

$\downarrow$ Factoring A out of 2nd and 3rd terms

$$BC + A(B + \overline{B}C)$$

$\downarrow$ Applying rule $A + \overline{A}B = A + B$ to the $B + \overline{B}C$ term

$$BC + A(B + C)$$

$\downarrow$ Distributing terms

$$BC + AB + AC$$

or — Simplified result

$$AB + BC + AC$$

Applied Electronics Outcome

Combinational التوافقية Logic Circuit Design

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When designing a system to suit a need you should proceed in the following order.

1) Describe the problem clearly in words.
2) Write out a Truth Table for the system.
3) Derive the Boolean expression from the Truth Table.
4) Simplify this expression if possible.
5) Draw a logic circuit diagram for the system using AND, OR and NOT gates.
6) Convert the circuit to NAND gates only.

It is entirely possible that not every problem will require all of these steps to be followed, however this will be a useful guide for most.

"Discrete Mathematics"

Illustrative Examples on:

Boolean Algebra, Logic Gates and Switching Circuits

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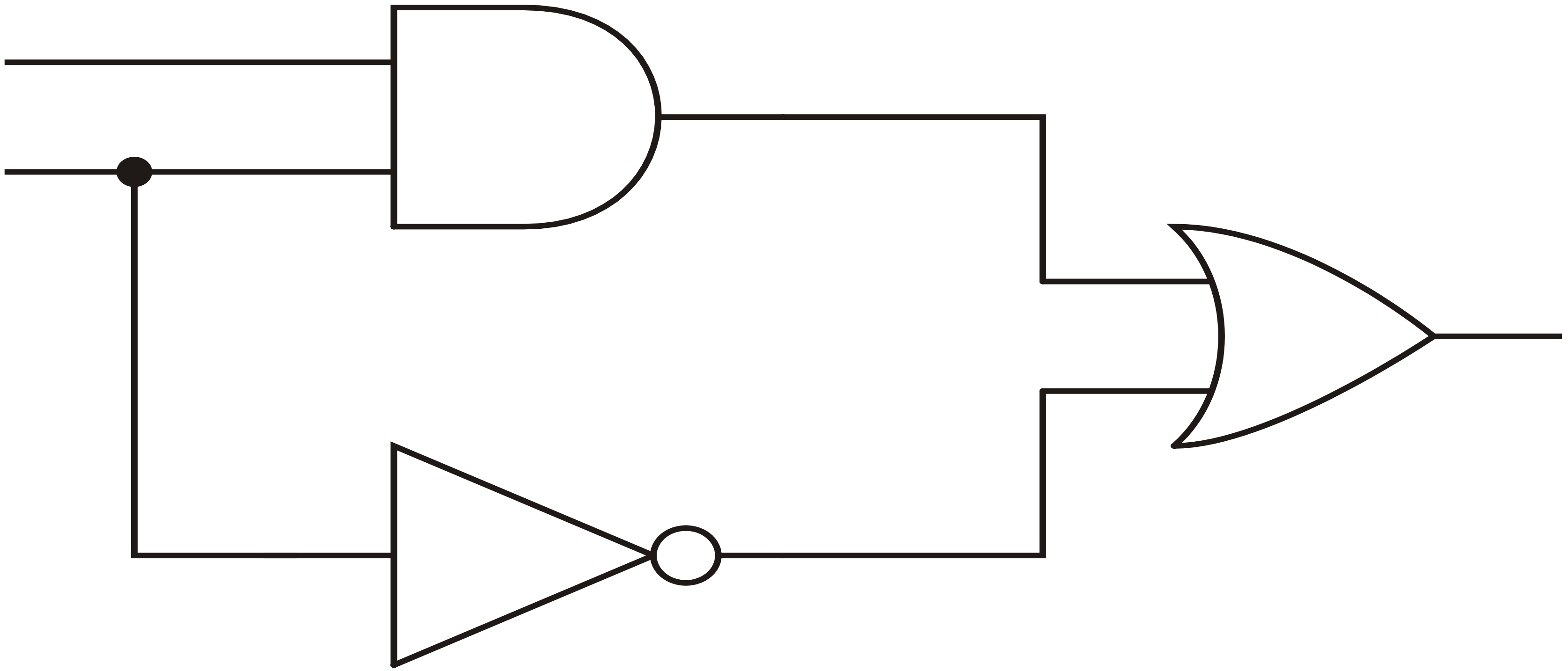
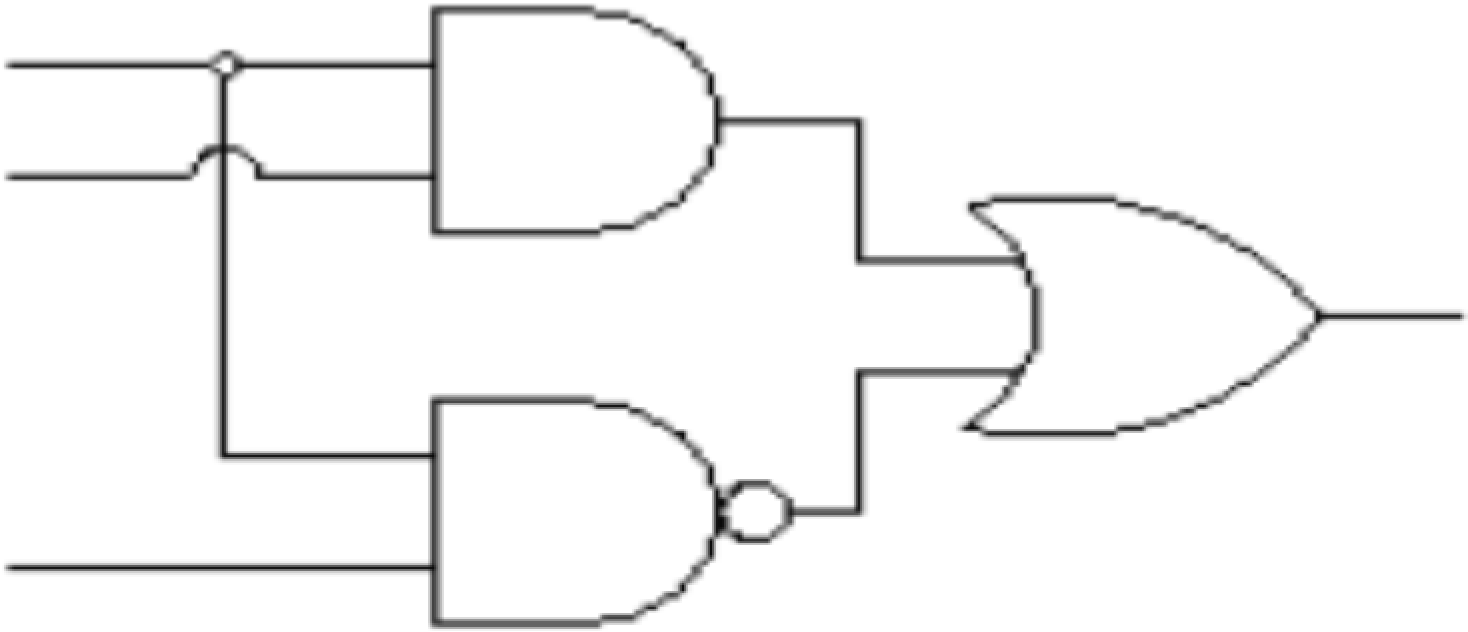
Problems Set 01

Find the output of the following logic circuits

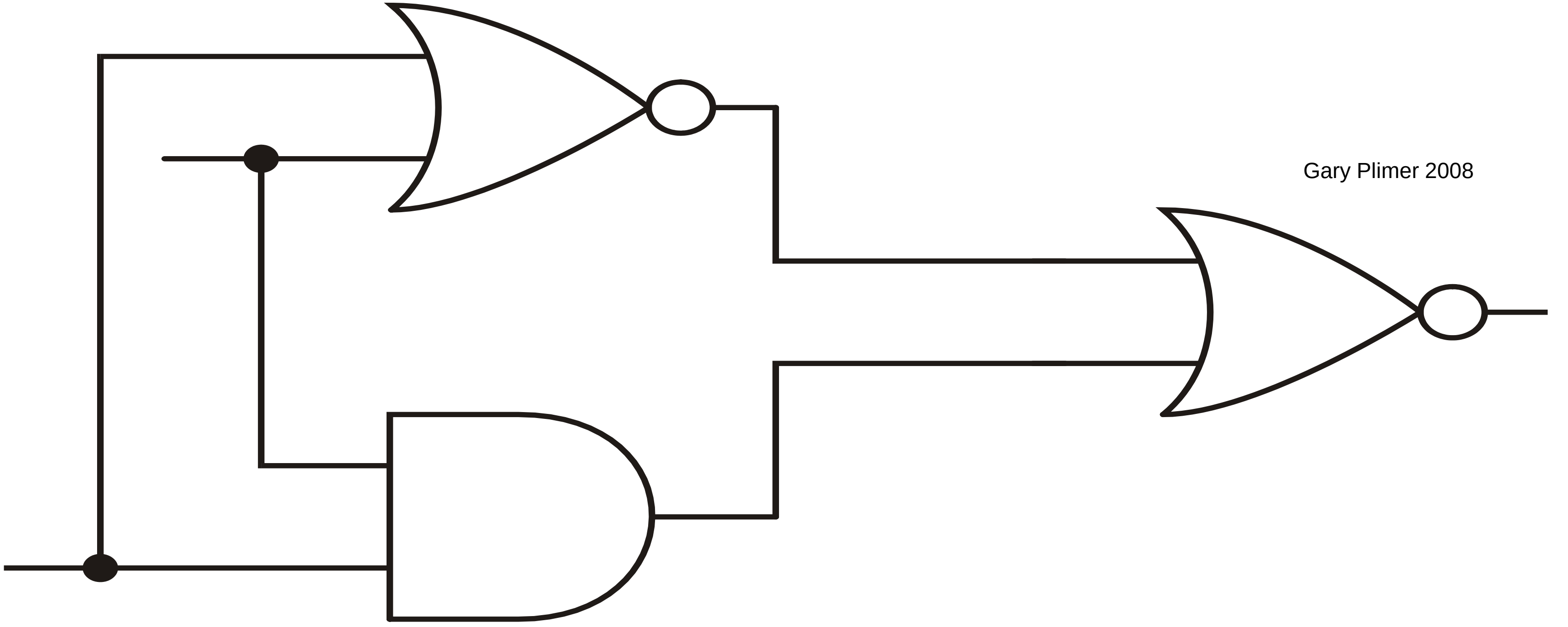
1.1



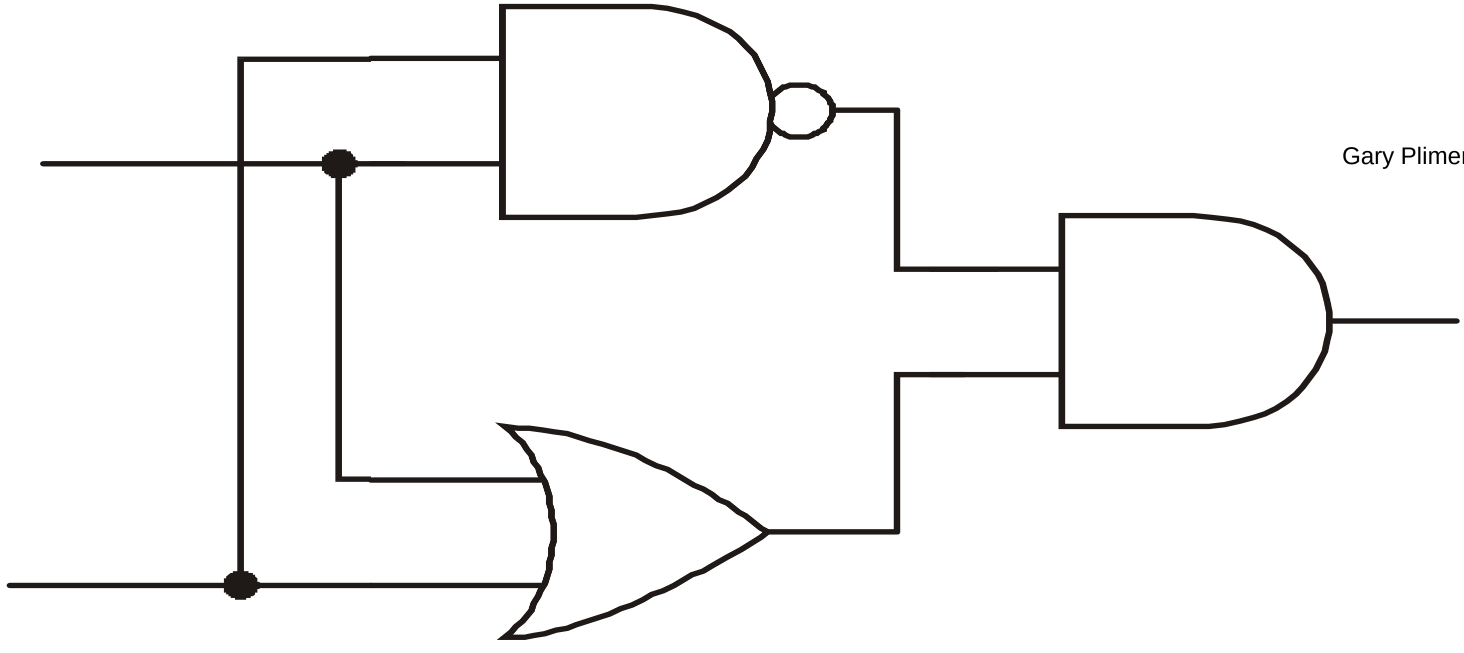
1.3



1.4



1.5



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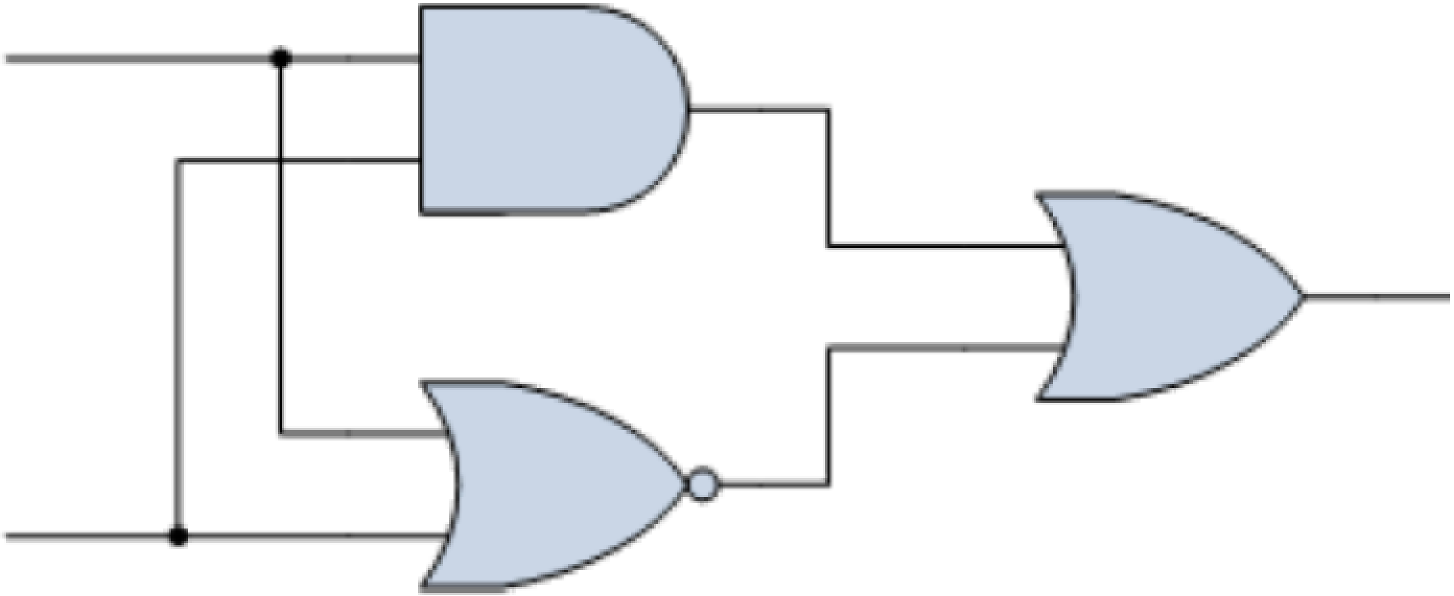
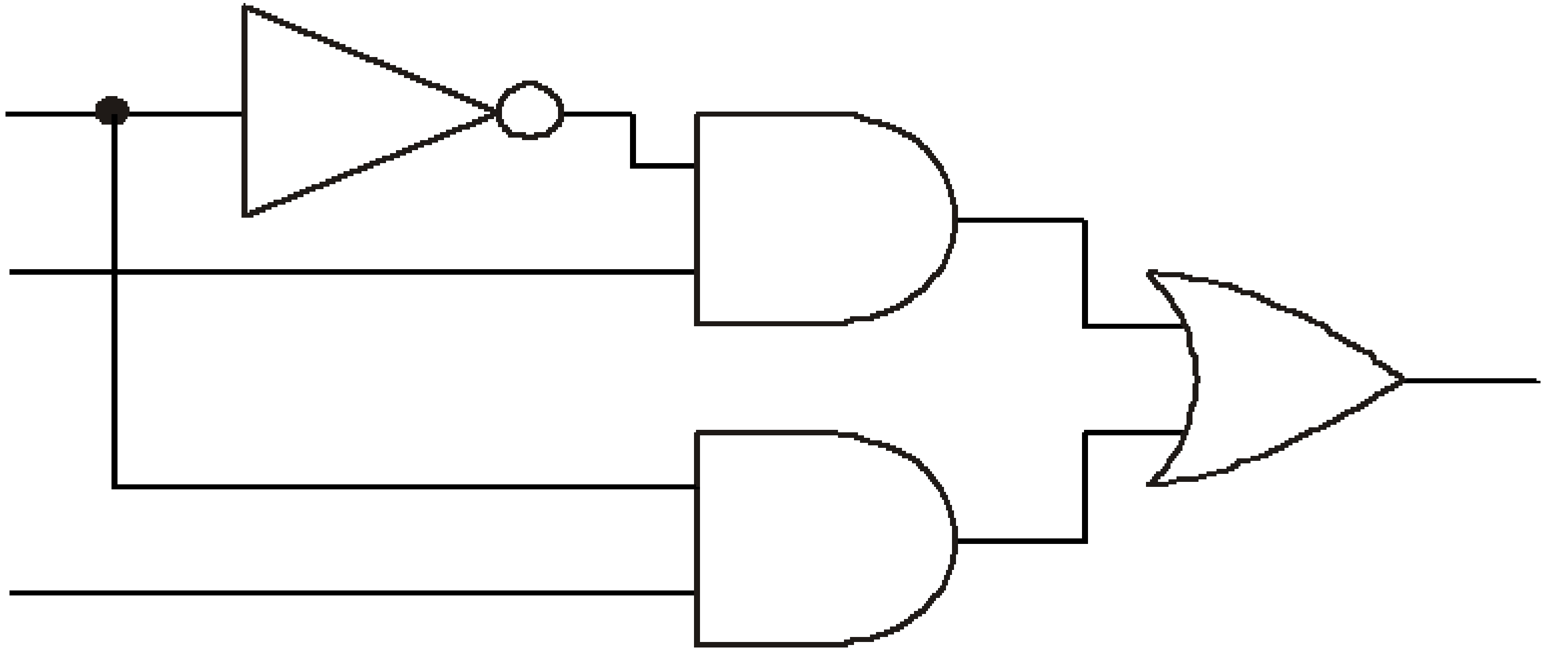
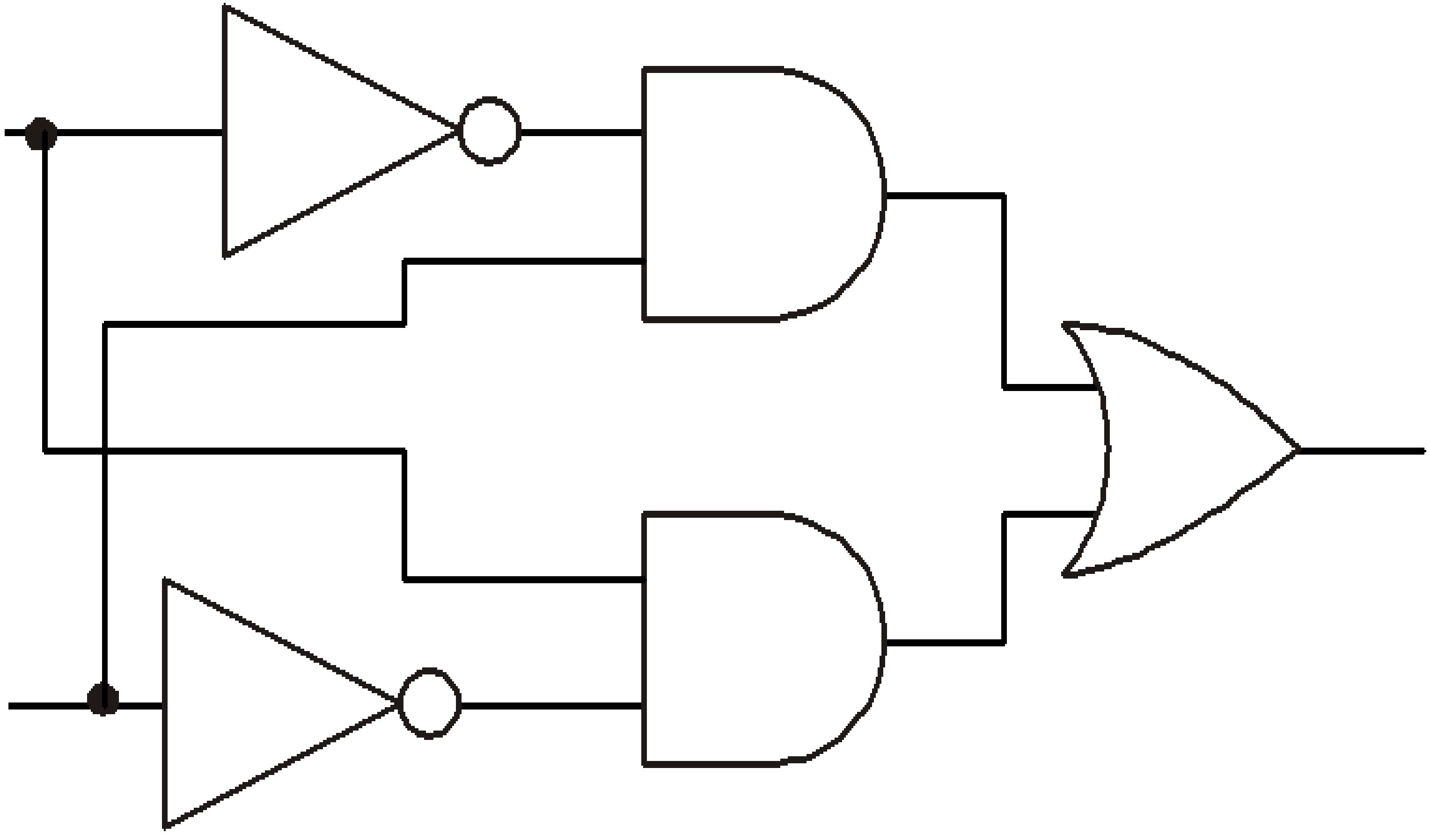
1.7



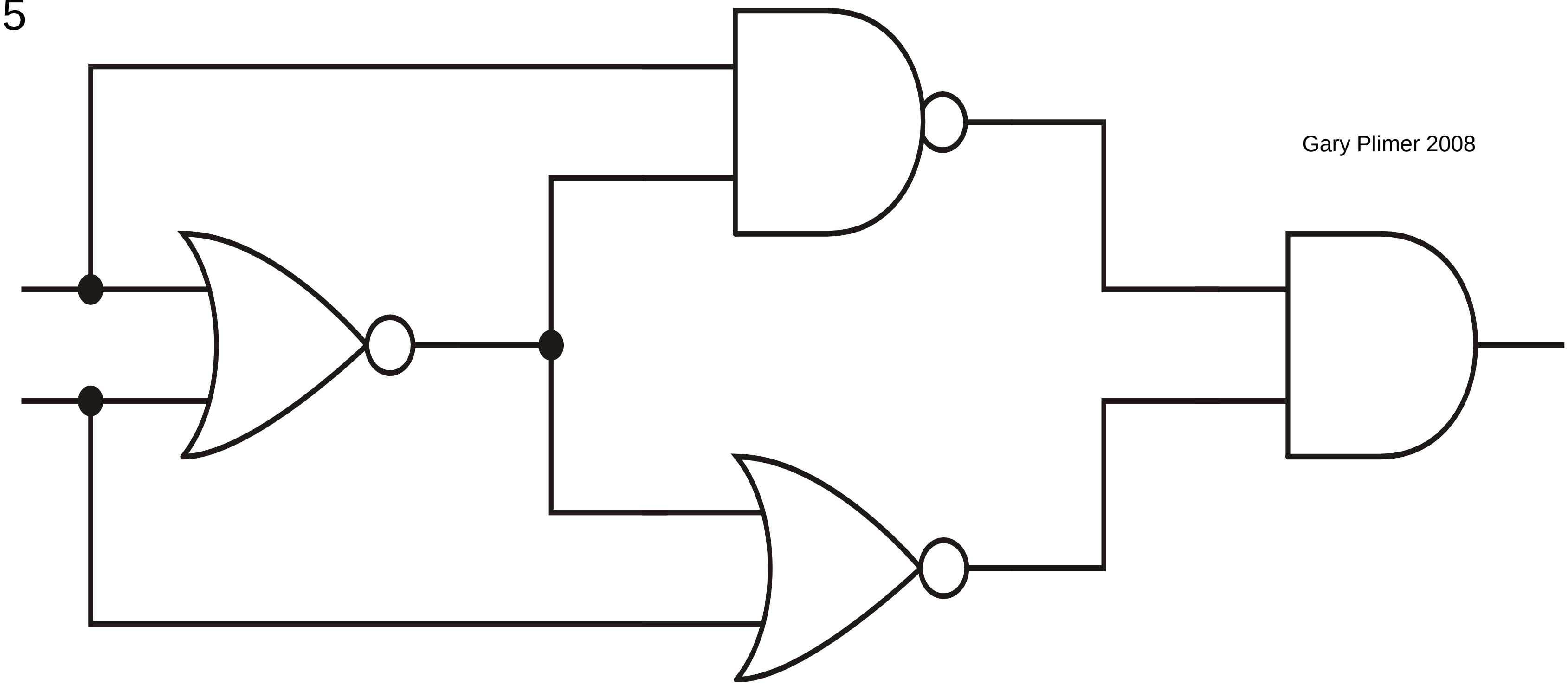
1.10



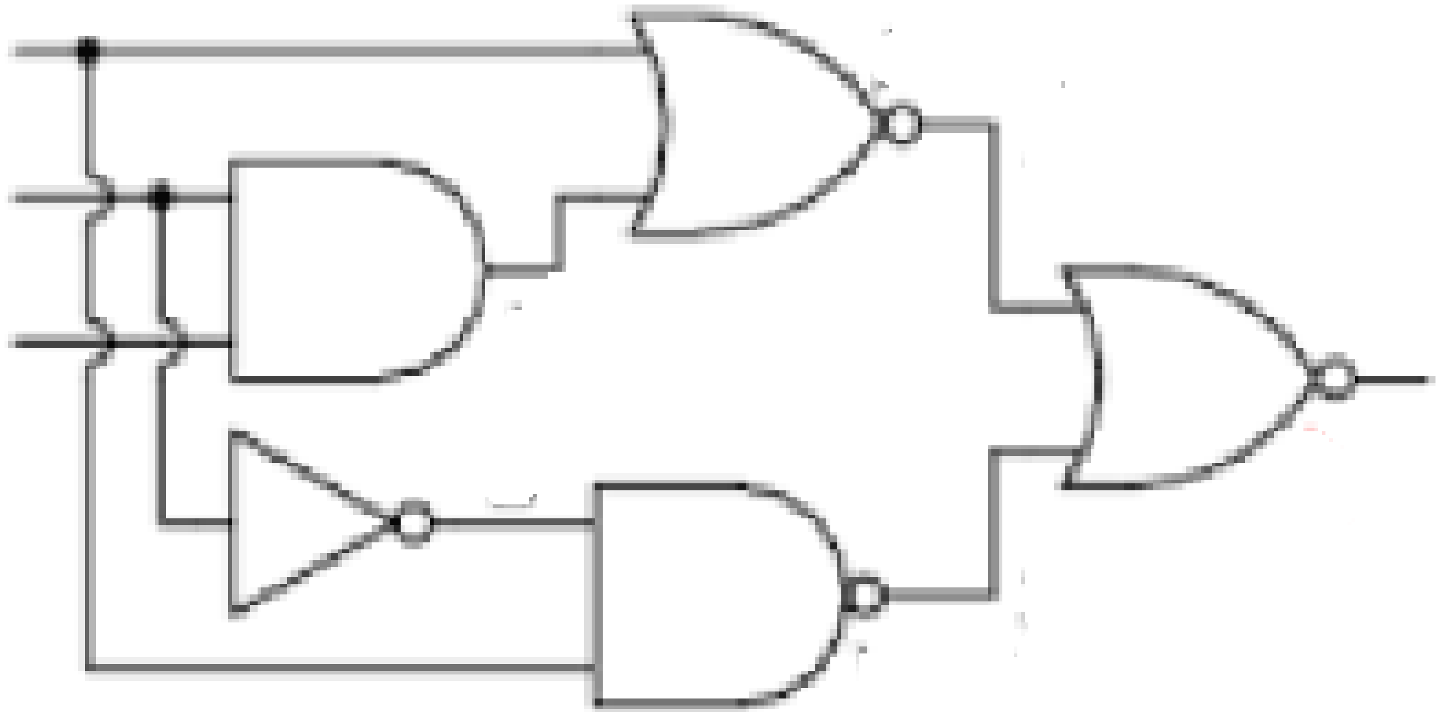
1.13



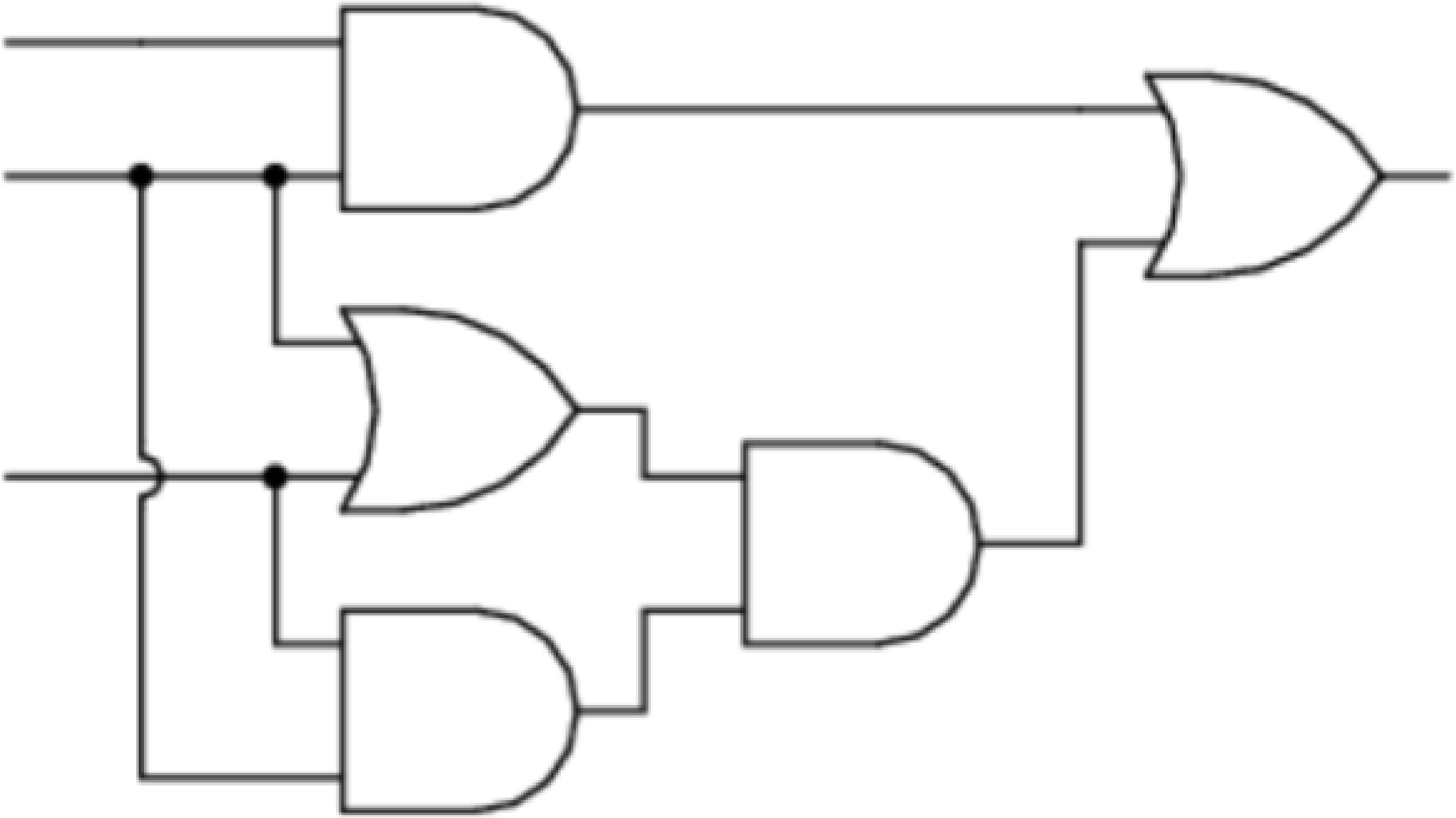
1.15



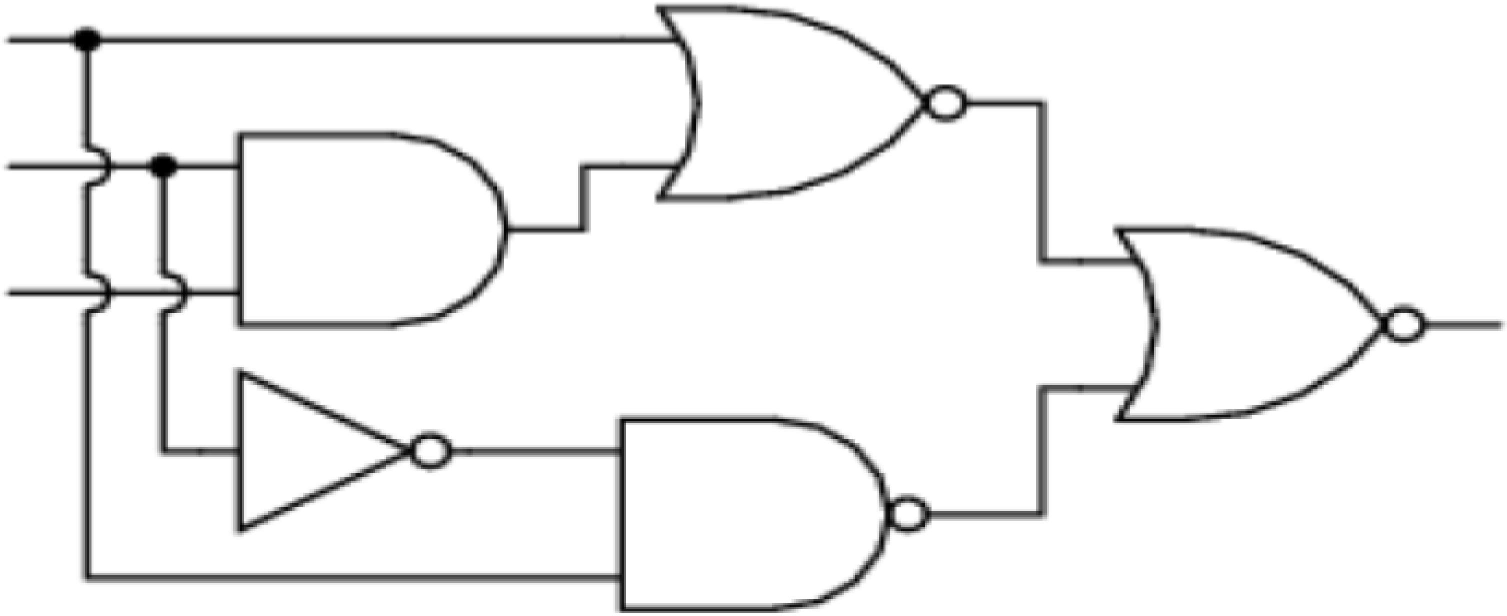
1.17



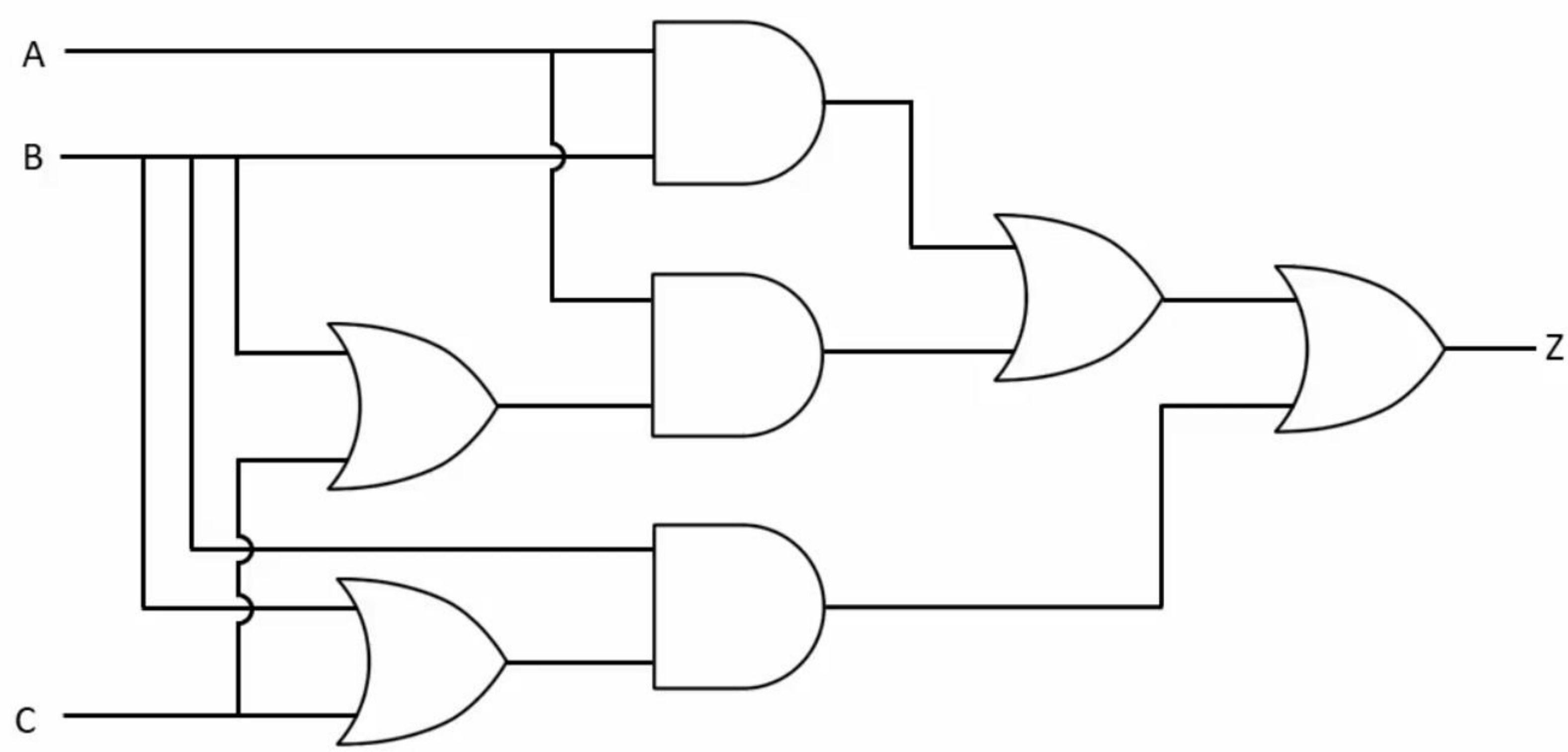
1.20



1.21



1.22



Problems Set 02

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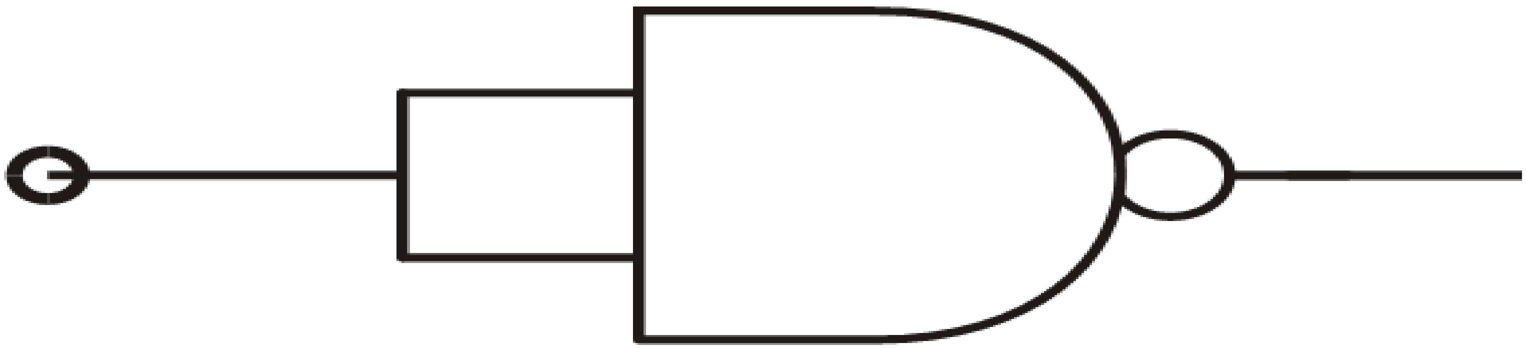
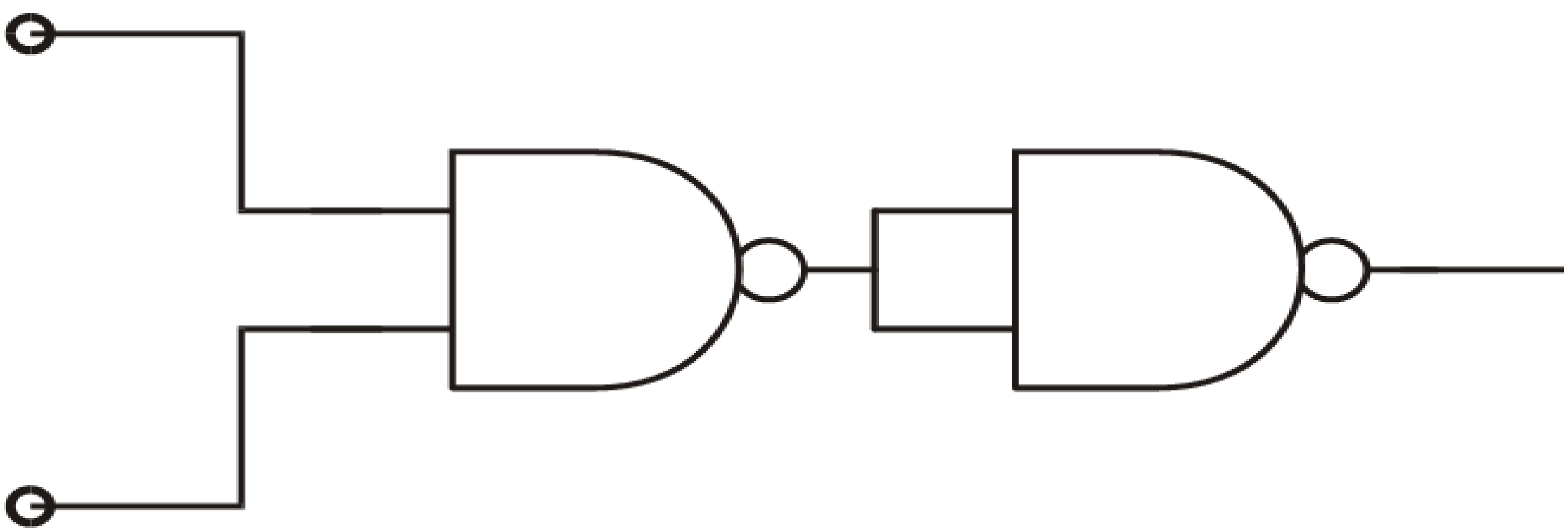
What are the following circuits equivalent of?

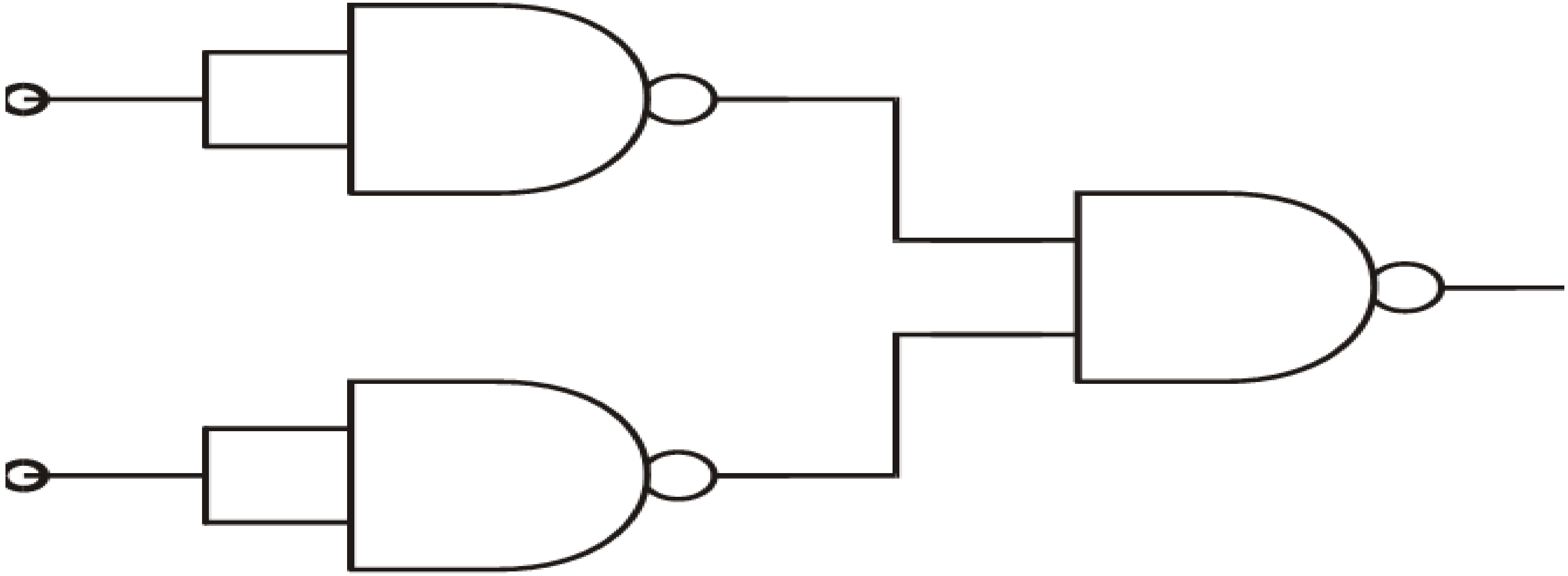
2.1



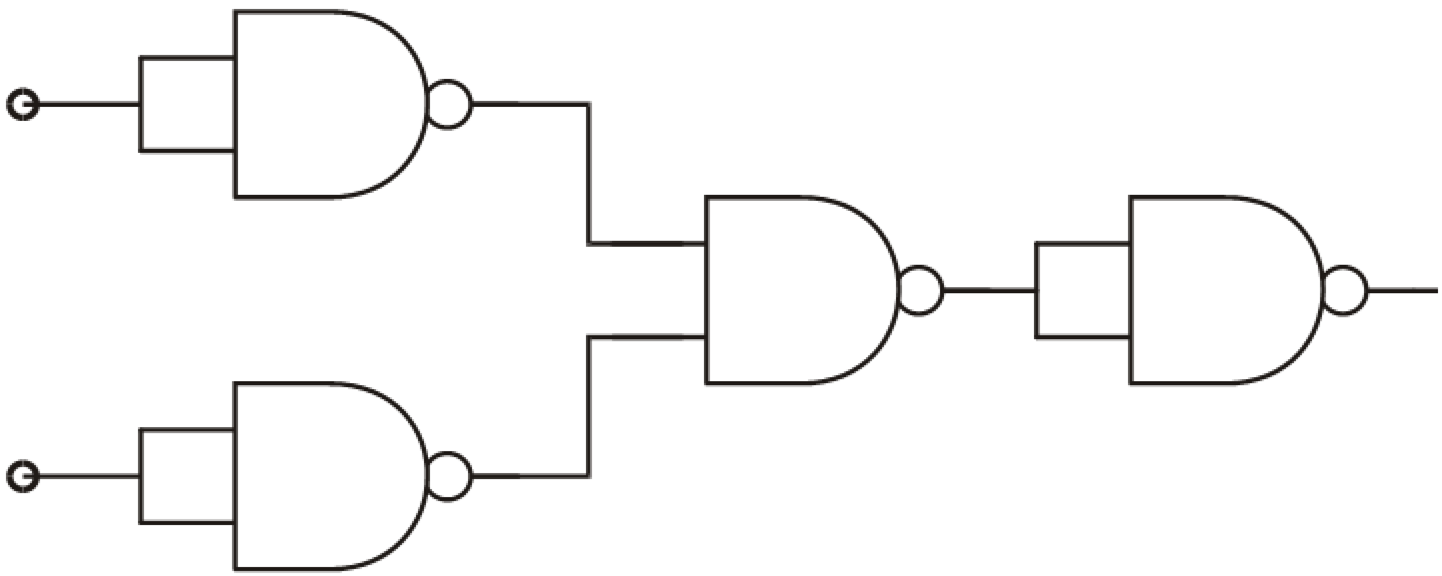
2.2



2.3

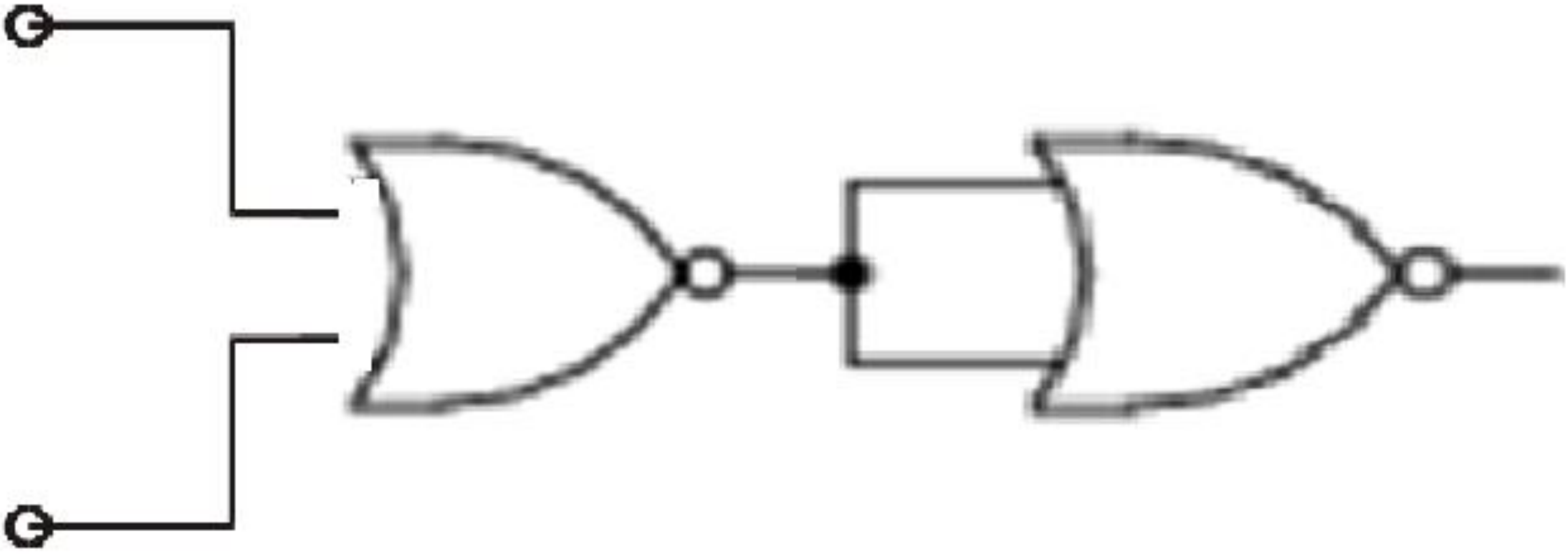


2.4

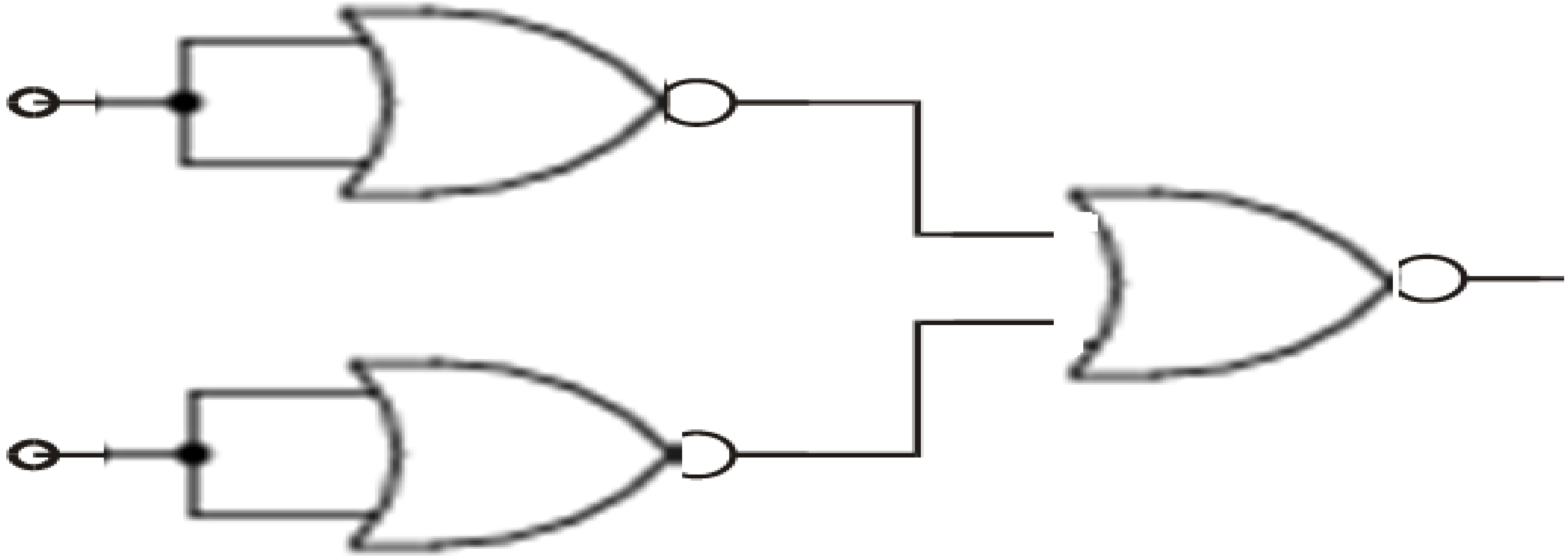


2.5

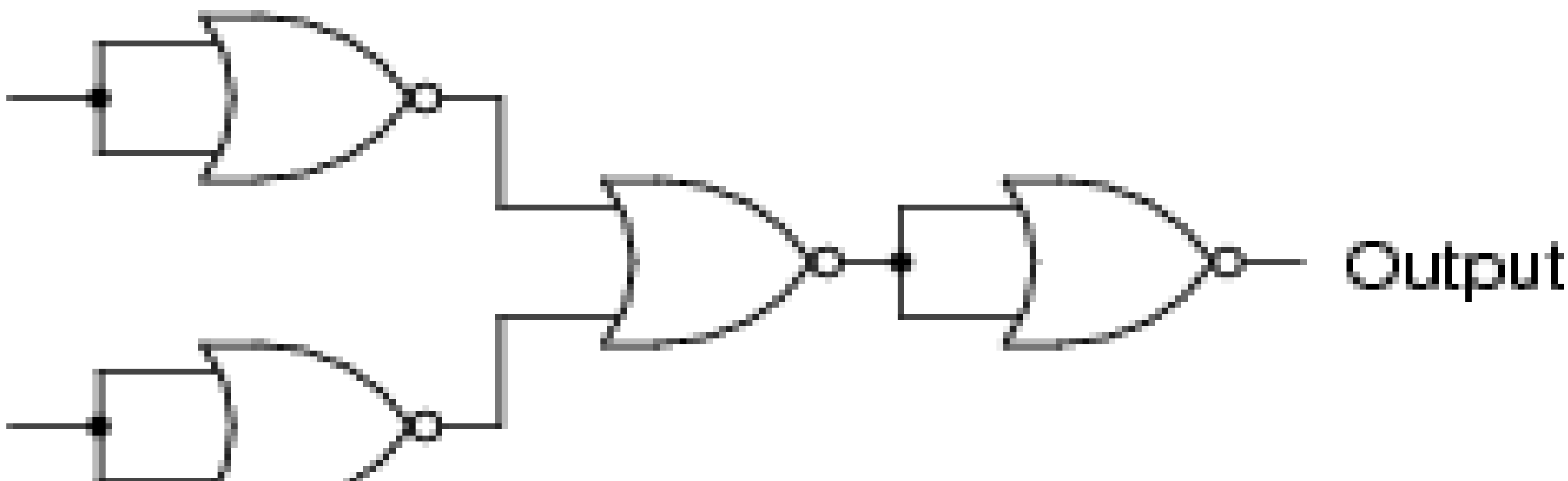
2.6



2.7



2.8



Converting to NAND's

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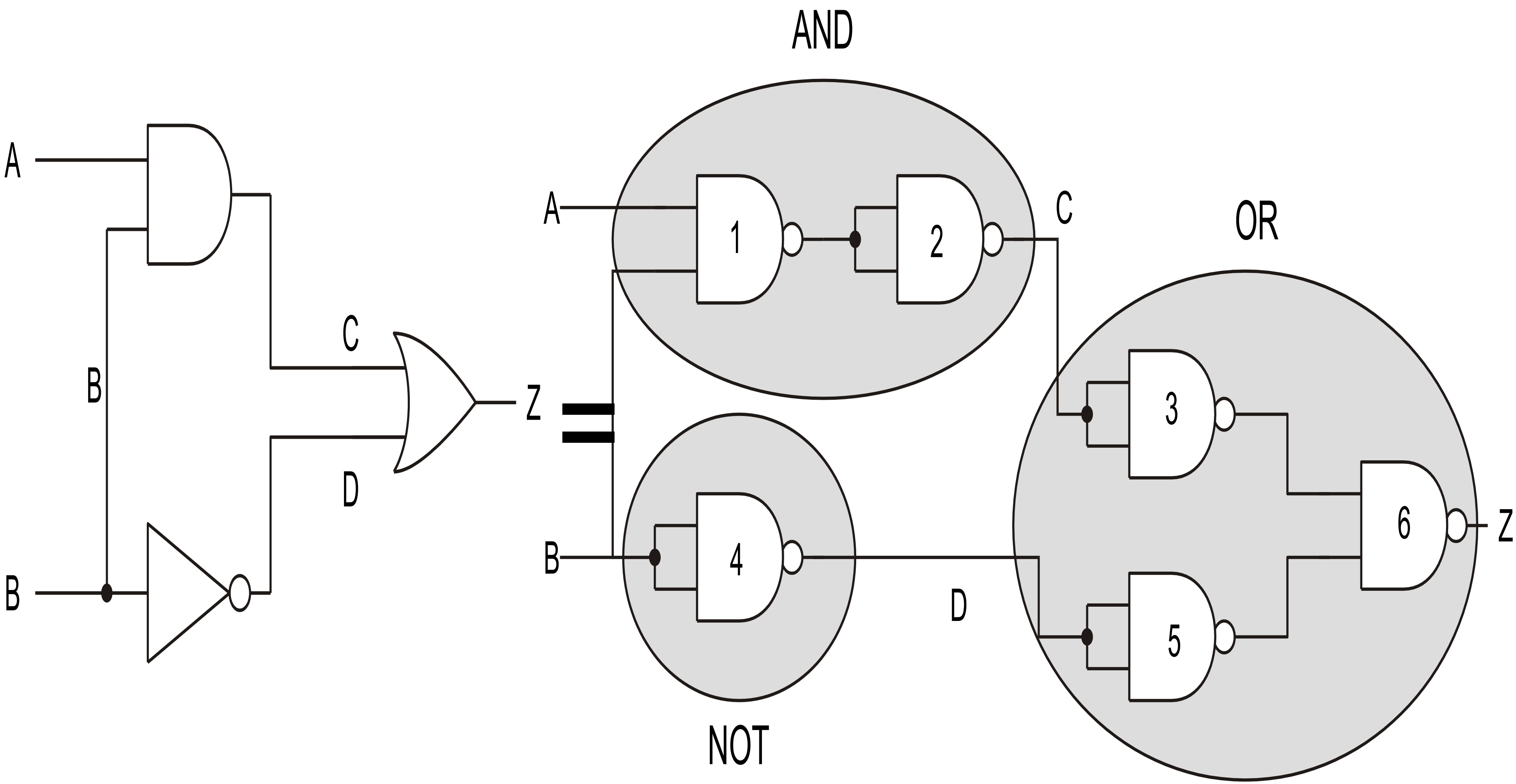

Converting to NOR's

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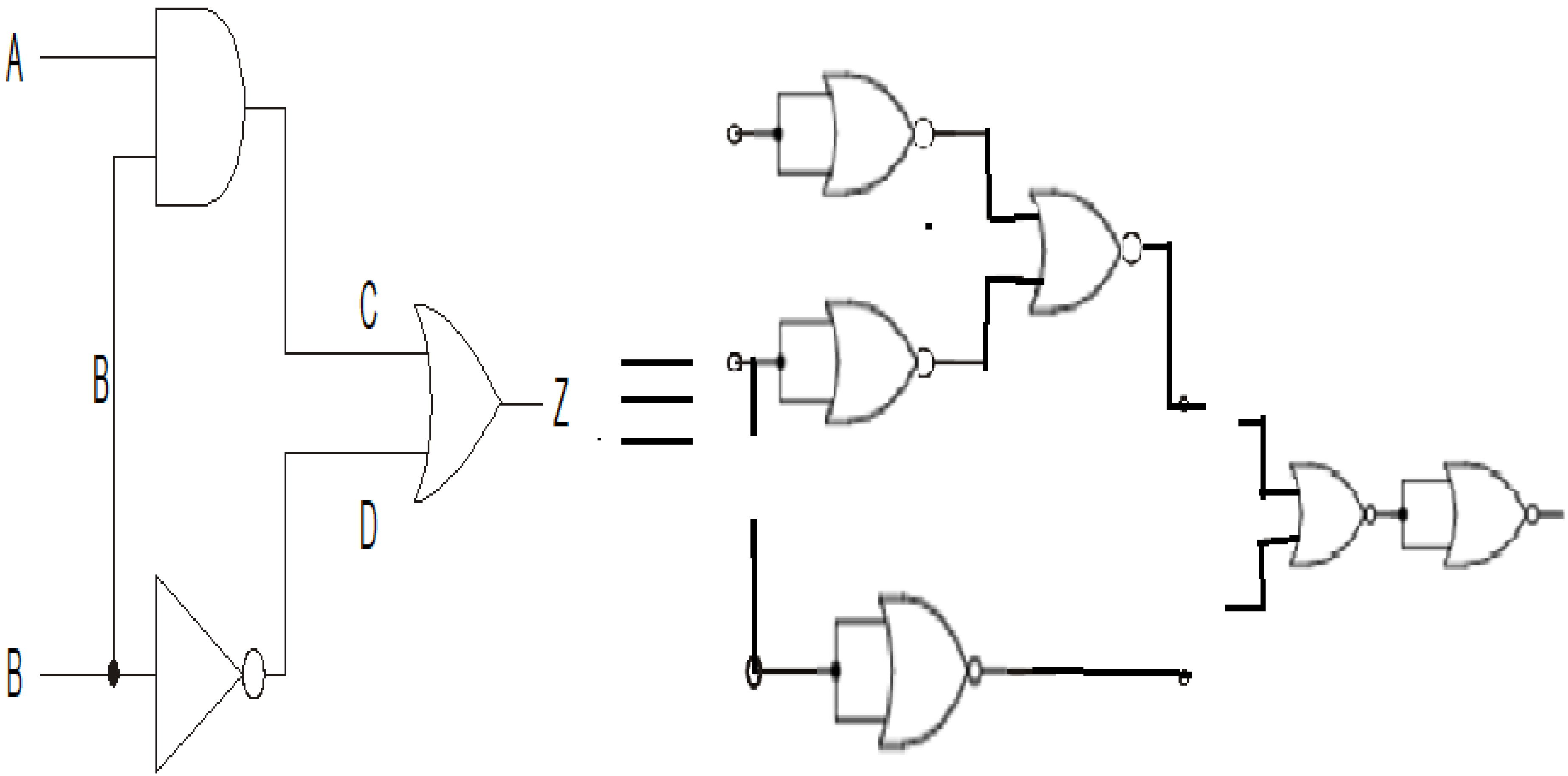
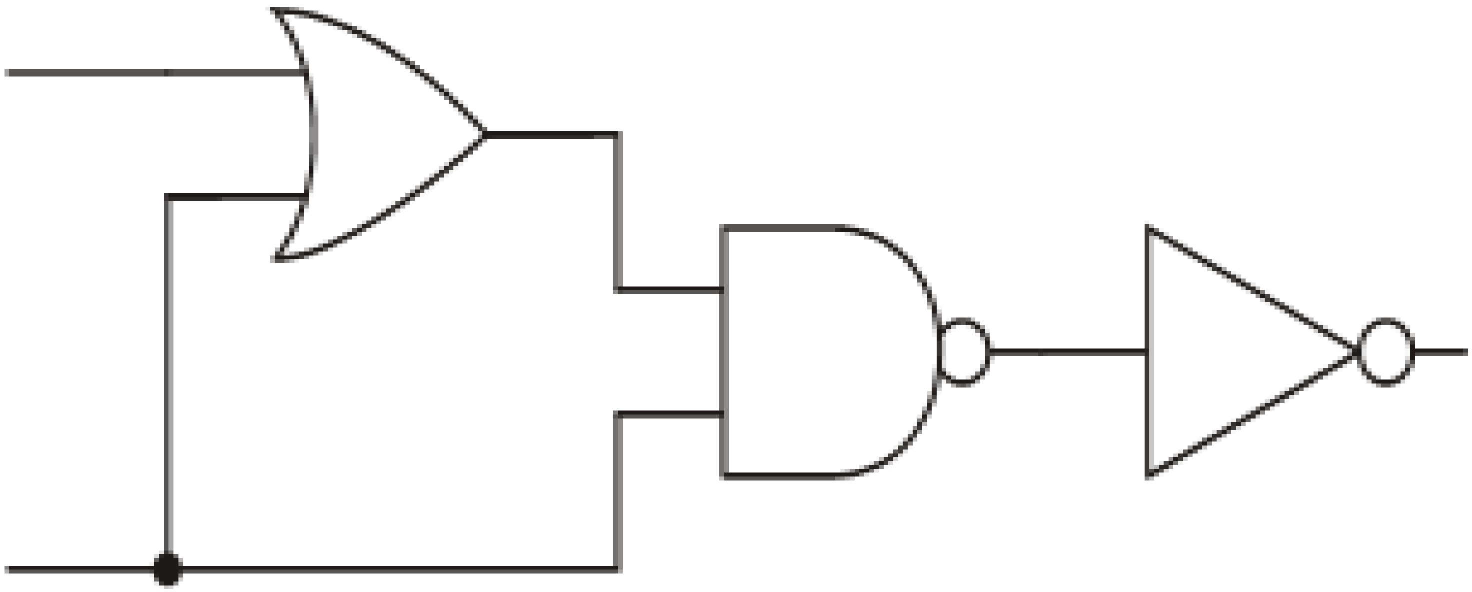


Problems Set 03

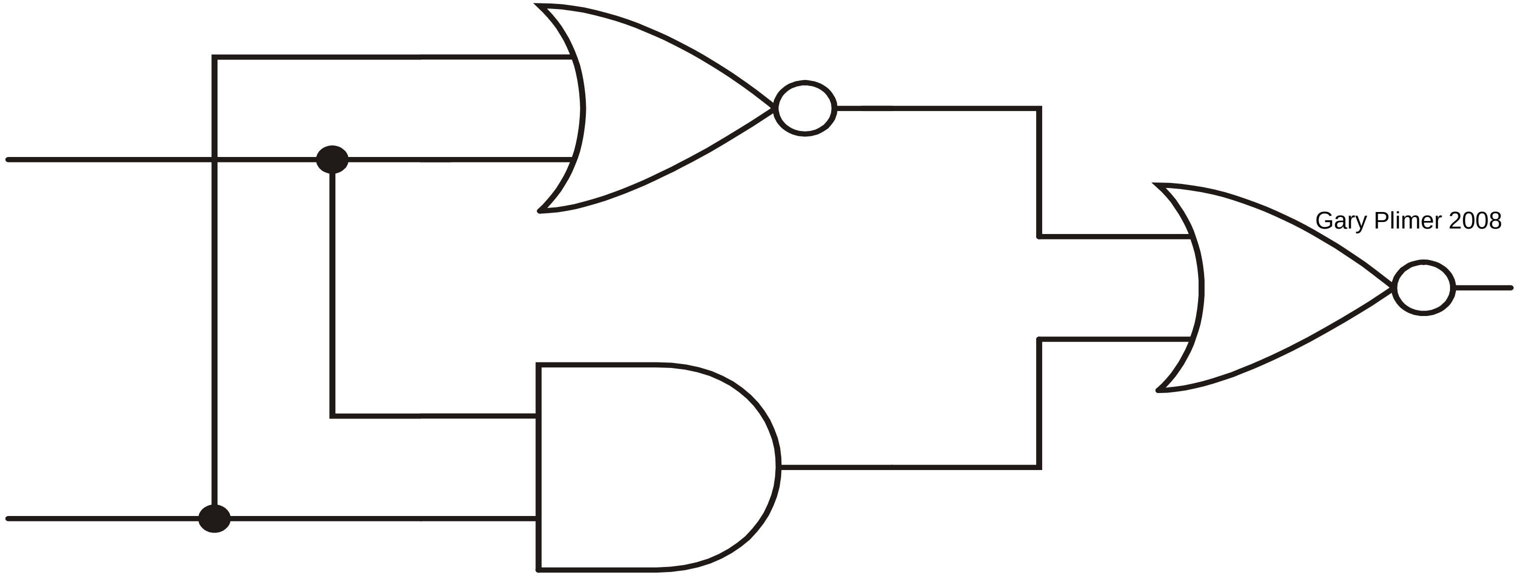
Simulate each of the following circuits. Gary Plimer 2008

To the NANDs (NORs) Ones?

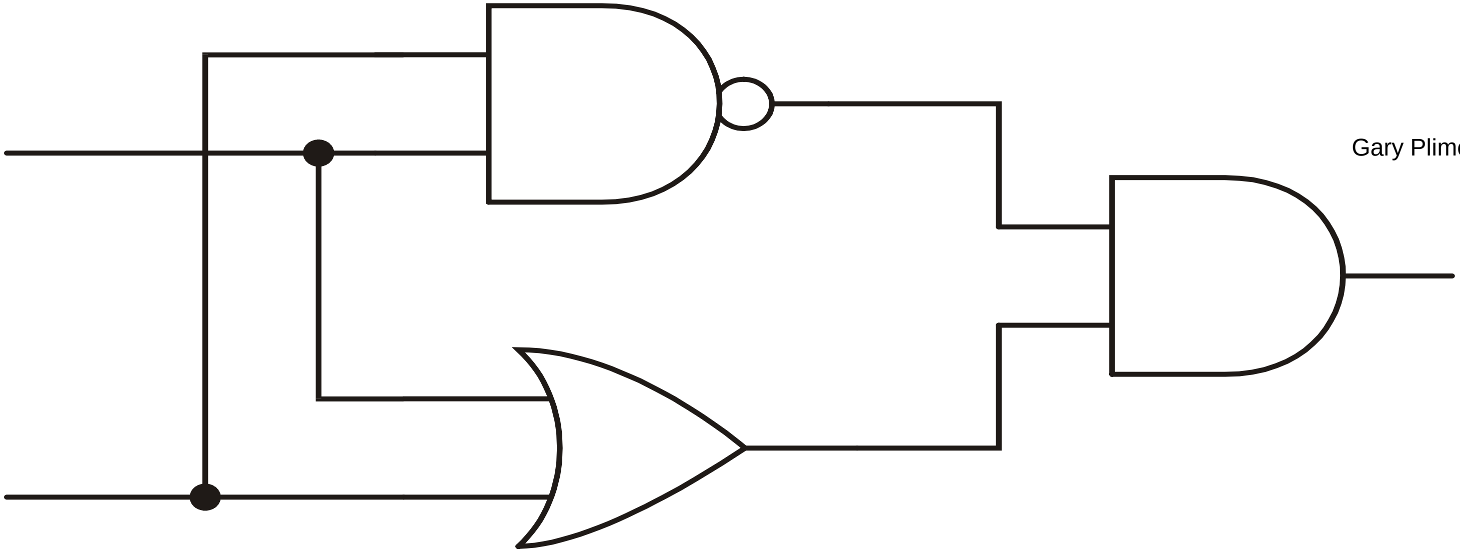
3.1



3.2



3.3



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Problems Set 04

For each pair of circuits shown below:

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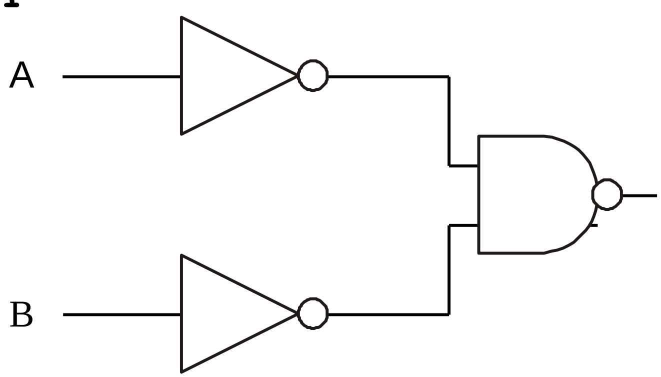
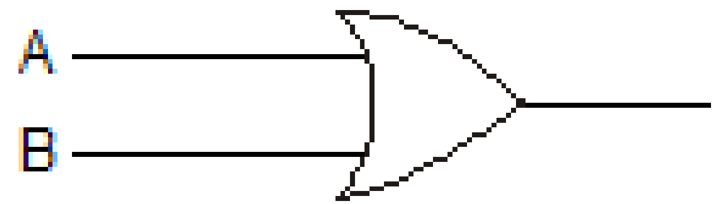
a) Write a Boolean expression for each of these circuits;
b) By constructing a truth table for each of them, show whether they are equivalent;
c) Draw the equivalent arrangements using only 2-input NAND gates.

4.1





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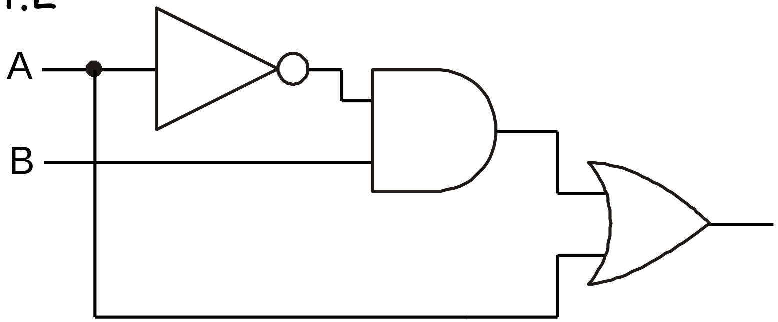

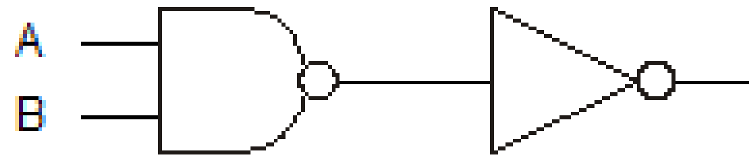

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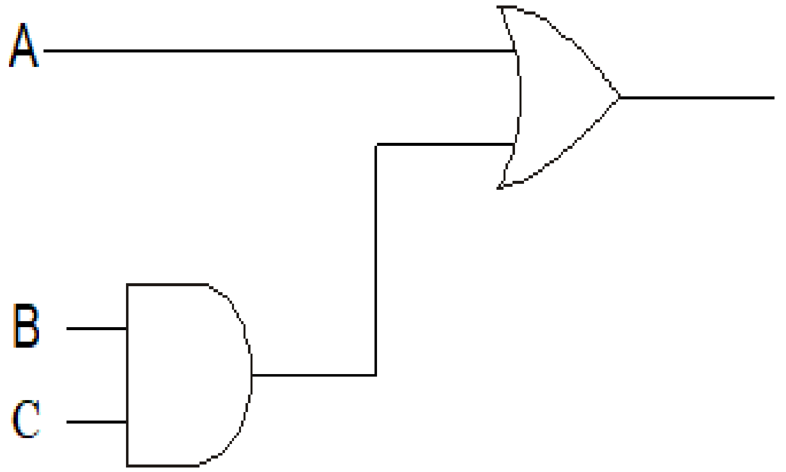

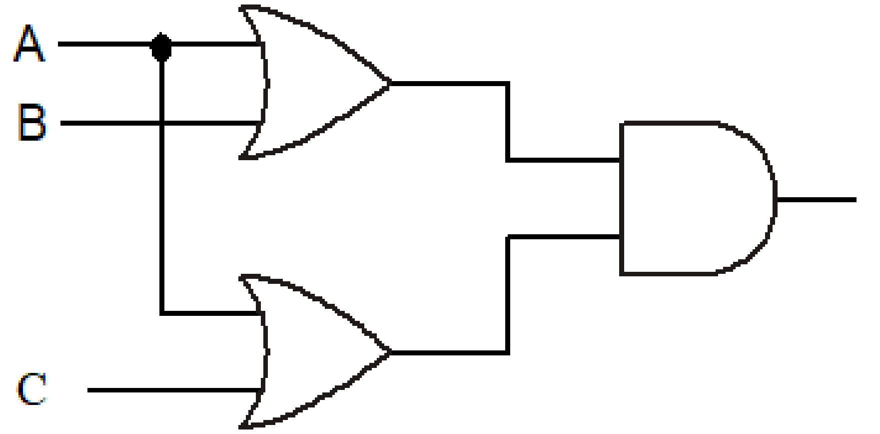

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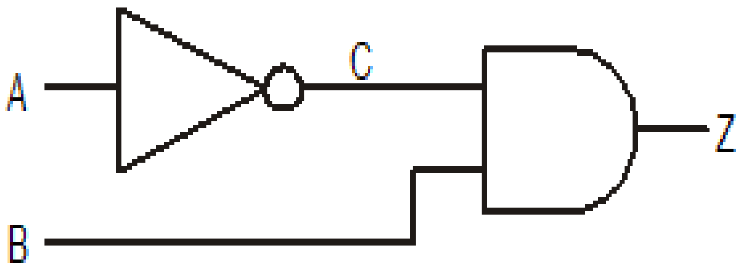

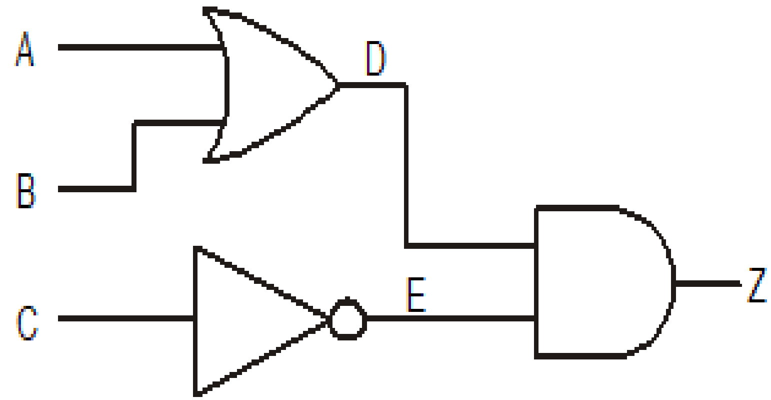

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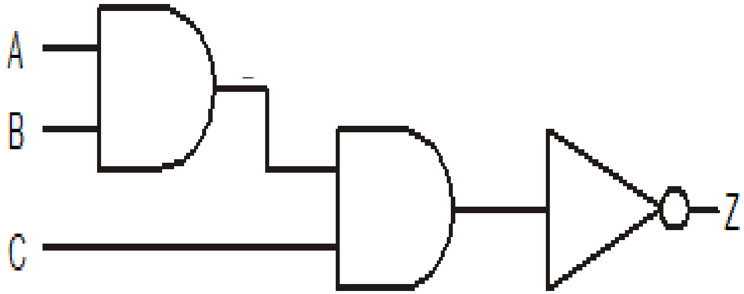

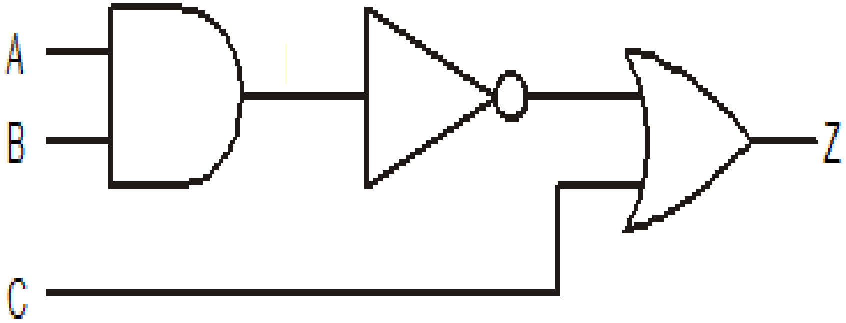

Problems Set 05

For each of the following Boolean expressions, simplify -if possible- then draw a logic diagram

5.1 $Z = A + A \bullet B$

5.2 $Z = A + \overline{B} + C$

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5.3 $Z = A \bullet B + A \bullet C$

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5.4 $Z = A \bullet B + \bar{A} \bullet C + B \bullet C$

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5.5 $Z = A \bullet B \bullet C + D \bullet E + F \bullet G$

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5.6 $Z = \overline{A} \bullet \overline{B} + A \bullet \overline{B} + A \bullet B$

5.7 $Z = \overline{(A + B + C) \cdot D}$

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5.8 $$F = \overline{(\overline{X \cdot \overline{Y}}) \cdot (\overline{Y} + Z)}$$

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5.9 $$F = \overline{(\overline{X} + Z)\cdot(\overline{X\cdot Y})}$$

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5.10 $Y = \overline{(A + \overline{B})(\overline{B + C})}$

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5.11 $Z = \overline{A \cdot B \cdot C + D \cdot E \cdot F}$

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5.12 $$Z = \overline{A \cdot \overline{B} + C \cdot \overline{D} + E \cdot F}$$

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5.13 $Z = [A \cdot \overline{B} \cdot (C + B \cdot D) + \overline{A} \cdot \overline{B}] \cdot C$

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5.14 $$Z = A\cdot\overline{B} + A\cdot\overline{(B + C)} + B\cdot\overline{(B + C)}.$$

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5.15

$$Z = \overline{\overline{A + B \cdot \overline{C}} + D \cdot (\overline{E + \overline{F}})}$$

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5.16 $Z = (A + C) . (A . D + A . \overline{D}) + A.C + C$

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5.17 $Z = \overline{A}\cdot B\cdot C + A\cdot \overline{B}\cdot \overline{C} + \overline{A}\cdot \overline{B}\cdot \overline{C} + A\cdot \overline{B}\cdot C + A\cdot B\cdot C$

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Problems Set 06

Derive the Boolean expression for the following truth tables

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Develop a Boolean equation and draw a logic circuit diagram containing AND, OR and NOT gates to yield the truth table shown.

6.1

A	B	Z
0	0	0
0	1	1
1	0	1
1	1	1

6.2

A	B	Z
0	0	1
0	1	0
1	0	1
1	1	1

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6.3

A	B	Z
0	0	0
0	1	1
1	0	0
1	1	0

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6.4

A	B	Z
0	0	1
0	1	0
1	0	0
1	1	1

6.5

A	B	Z
0	0	0
0	1	1
1	0	1
1	1	1

6.6

A B C	Z
0 0 0	0
0 0 1	0
0 1 0	1
0 1 1	1
1 0 0	0
1 0 1	1
1 1 0	0
1 1 1	1

6.7

A	B	C	Z
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	0

6.8

A	B	C	Z
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

6.9

A	B	C	Z
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

6.10

A	B	C	Z
0	0	0	1
0	0	1	0
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

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Problems Set 07

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(7.1) Paper can be fed through a computer printer either by pressing the button on the printer (line feed) or by sending a signal from the computer.

Which logic gate should be used for this operation ?

(7.2) The motor in a washing machine should not operate until a high signal is sent from the control program and the water level in the drum is high enough.

Which logic gate should be used for this operation ?

(7.3) To avoid accidents at times of poor visibility, a warning indicator in a car operates if the light level is too low (logic level 0) and the headlamps are switched off.

Which logic gate should be used for this operation ?

(7.4) In the maternity unit قسم الولادة of a hospital, the temperature and pulse rate of premature babies الطفل المولود قبل اوانه has to be continually monitored. *A* warning alarm should sound if either the temperature or the pulse rate of the baby falls too LOW.

Which logic gate should be used for this operation ?

(7.5)An electric guillotine must be adequately guarded. In order to safeguard the operator the machine has two switches, A and B, set about one meter apart, both of which need to be pressed before the machine will operate. Design a logic circuit that will give a green light if, both switches are not pressed and both switches are pressed. If only one switch is pressed a red light should come on.

(7.6) A garage door is operated by a motor which is controlled by three switches. The motor runs either: When a pressure pad switch A, in the drive is closed and a light dependent resistor switching circuit, B, is simultaneously activated by the car's headlights

Or: When the keyswitch, C, in the garage door is operated. Prepare a truth table for all possible combinations of switching conditions for switched A, B and C. Take switch open as logic 0. From the truth table, prepare a logic diagram using the least number of gates.

(7.7) A given logic circuit has two logic inputs A and B. It is required to produce two logic outputs X and Y according to the following rules:

1. X is to be at logic 1 if (A or B) but NOT (A and B) are at logic 1.
2. Y is to be at logic 1 if (A and B) but NOT (A or B) are at logic 1.

(7.8)A domestic burglar alarm system is designed such that a bell will operate when the power switch is closed and a pressure switch under a carpet is closed or a switch is opened as a window is lifted.

Assuming all switches to be logic state zero (0) when open Draw a logic diagram for the design, allocating capital letters to the inputs to each gate and to the output to the bell.

Prepare a truth table for the design. Your table must be headed by the appropriate letters Show by use of a logic diagrams how you would\modify or combine 3-input NAND gates to provide AND and OR gates.

(7.9) A house doorbell is to ring if a push button at the front door, a push button at the back door or both buttons are operated. Draw a logic diagram and write a Boolean equation.

(7.10) A lift motor is to start only when by closing the door, a switch is activated and a passenger has pressed a button. Prepare a truth table, a logic diagram and a Boolean equation for this system.

(7.11) The driver of a dustcart is to be able to operate the loading claw مخلب by pressing a button, but only when the senior loader at the rear of the cart has pressed a button to give the 'all clear'. Draw a logic diagram and write a Boolean equation for this system.

(7.12) An automatic central heating system is to heat the radiators (R) if the mains switch (M) is on, the timing control switch (T) is closed and the override تجاوز button (O) is not selected. Draw a logic diagram, truth table and Boolean statement for this system.

(7.13) A drill حفار is to operate if an isolator is closed, a guard is in place (closing a micro switch), either 'HI' or 'LOW' speed is selected and a foot pedal is operated. Draw a suitable logic diagram for this system. Write the Boolean Expression.

(7.14) A large hall has three temperature sensors. A logic system is to operate the radiator when any two of the temperature sensors fall below a preset level. Draw up a truth table for this system and draw a logic diagram.

(7.15) A burglar alarm will operate if the mains switch is on and either an electronic beam is broken, a pressure pad is stood on or a window is opened. Draw a logic diagram for this system.

Puzzle 01: Logicians and Hats

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Problem: Three logicians are each wearing a black hat or a white hat, but not all white. Nobody can see their own hat. However, A can see the hats of B and C, and B can see the hats of A and C. C is blind. You go and ask them one by one in the order A, B, C, whether they know the color of their own hat. A answers "No". B answers "No". Then C answers "Yes". Explain how this is possible.

Puzzle 02: Criminals and Hats

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Problem: A warden plans to line up 100 prisoners in order tomorrow. The warden will place a white or black hat on each prisoner's head so that no prisoner can see the hat on his or her own head, but they can see the hats of the prisoners **in front** of them. From the back of the line, the warden will ask each prisoner "Is your hat black?" If the prisoner answers correctly, he or she is set free. Incorrect answers lead to immediate and noisy beheading. If they answer anything other than "yes" or "no", **all** prisoners are beheaded. They get one hour as a group to plan their strategy.

How many prisoners can be saved?