Sequences and Series of Functions

1.1 Power Series

Definition 1.1 (Power Series). A power series is a real valued function $f(x) = \sum a_n x^n$ for some sequence (a_n).

Theorem 1.1. For a power series $\sum a_n x^n$, let $\beta = \limsup |a_n|^{\frac{1}{n}}$ and $R = \frac{1}{\beta}$. The power series converges for $|x| < R$ and diverges for $|x| > R$

Proof. Apply the root test. Then

$$\limsup |c_n|^{\frac{1}{n}} = \limsup |a_n|^{\frac{1}{n}} |x| = \limsup |a_n|^{\frac{1}{n}} |x| = |x|\beta.$$

Note then that $|x| < R = \frac{1}{\beta}$ means that $\limsup |c_n|^{\frac{1}{n}} < 1$ and therefore the series converges. The opposite is true for $|x| > R$. ■

Example 1.1. Consider $\sum x^n$. Note that $a_n = 1$ for all $n \in \mathbb{N}$. Therefore $\limsup |a_n|^{\frac{1}{n}} = \limsup 1^{\frac{1}{n}} = 1$. Therefore the power series converges for all $|x| < 1$. Note that $x = 1$ gives a divergent series and $x = -1$ gives an alternating series whose non alternative part does not go to zero and hence also diverges.

Example 1.2. Consider $\sum \frac{x^n}{n!}$. In this instance $a_n = \frac{1}{n!}$. Then

$$\beta = \limsup |a_n|^{\frac{1}{n}} = \limsup \left|\frac{1}{n!}\right|^{\frac{1}{n}}.$$

This would be hard to compute. However, if this limit exists, then it matches the value of the ratio test and therefore

$$\beta = \limsup \left|\frac{a_{n+1}}{a_n}\right| = \limsup \frac{1}{(n+1)!} \cdot \frac{n!}{1} = \limsup \frac{1}{n} = 0.$$

Therefore $R = +\infty$ meaning the interval of convergence is all of $\mathbb{R}$.
Remark. Alternatively, one can use the Sterling approximation of the factorial to do the root test. The Sterling approximation is

$$n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}.$$

Hence

$$\limsup \left|\frac{1}{n!}\right|^{\frac{1}{n}} = \limsup \frac{1}{\left(\left(\frac{n}{e}\right)^n \sqrt{2\pi n}\right)^{\frac{1}{n}}} = \limsup \frac{1}{\frac{n}{e} \cdot \left(\sqrt{2\pi n}\right)^{\frac{1}{n}}} = \limsup \frac{1}{n} = 0.$$

Example 1.3. Consider $\sum \frac{x^n}{n^2}$. Then

$$\beta = \limsup \left(\frac{1}{n^2}\right)^{\frac{1}{n}} = \limsup \frac{1}{\sqrt[n]{n^2}} = 1.$$

Therefore the power series converges for $|x| < 1$. Importantly, for $x = 1$ and $x = -1$, you get convergent series and therefore the interval of convergence is $[-1, 1]$.

Example 1.4. Consider $\sum \frac{(-1)^{n+1}x^n}{n}$. Then $a_n = \frac{(-1)^{n+1}}{n}$ and

$$\beta = \limsup \left|\frac{(-1)^{n+1}}{n}\right|^{\frac{1}{n}} = \limsup \frac{1}{\sqrt[n]{n}} = \frac{1}{1} = 1.$$

Therefore the power series converges for $|x| < 1$. Checking $x = 1$,

$$\sum \frac{(-1)^{n+1}}{n} \text{ converges by alternating series test.}$$

And checking for $x = -1$,

$$\sum \frac{(-1)^{2n+1}}{n} = \sum \frac{-1}{n} = -\sum \frac{1}{n} \text{ which diverges.}$$

Therefore the interval of convergence is $(-1, 1]$.

Example 1.5. Consider $\sum \frac{(2n)!x^n}{(n!)^2}$. Then $a_n = \frac{(2n)!}{(n!)^2}$. Apply the ratio test to get β.

$$\beta = \limsup \left|\frac{a_{n+1}}{a_n}\right| = \limsup \frac{(2n+2)!}{((n+1)!)^2} \cdot \frac{(2n)!}{(n!)^2} = \limsup \frac{(2n+2)(2n+1)}{(n+1)(n+1)} = 4.$$

Therefore it converges on $|x| < \frac{1}{4}$. Checking the endpoints suck but $x = \frac{1}{4}$ diverges by using Sterlings approximation and $x = -\frac{1}{4}$ converges by the alternating series test by the previous method. Therefore the interval of convergence is $\left[-\frac{1}{4}, \frac{1}{4}\right)$.

1.2 Uniform Convergence

An initial, but weak, formulation of functional sequence convergence is by applying the a basic limit of a sequence.

Definition 1.2 (Pointwise Convergence). A sequence of real value functions $f_n : S \subset \mathbb{R} \to \mathbb{R}$ converges point wise to a function f on S if $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in S$

Definition 1.3 (Uniform Convergence). A sequence of real value functions $f_n : S \subset \mathbb{R} \to \mathbb{R}$ uniformly converges to a function f on S if $\forall \epsilon > 0$, there is some $N \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \epsilon, n > N, \forall x \in S.$$

Example 1.6. Consider the sequence of functions $f_n(x) = x^n$ on $[0,1]$. Note that for all n, $f_n(0) = 0$ and $f_n(1) = 1$. Furthermore, for $0 < x < 1$, $\lim x^n = 0$. Therefore

$$\lim f_n(x) = f(x) = \begin{cases} 0 & 0 \le x < 1 \\ 1 & x = 1 \end{cases}.$$

is the pointwise limit of the sequence. For uniform convergence, we want

$$|f_n(x) - f(x)| < \epsilon \Leftrightarrow \left| x^n - \begin{cases} 0 & 0 \le x < 1 \\ 1 & x = 1 \end{cases} \right| < \epsilon.$$

For $x = 1$, the absolute value goes to 0 and therefore only $0 \le x < 1$ matters. The question becomes when

$$x^n < \epsilon \implies n > \frac{\ln(\epsilon)}{\ln|x|}.$$

However, it is not possible to bound this quantity since $x \to 1$ leads to $\frac{1}{\ln|x|} \to -\infty$. Therefore the sequence does not uniformly converge to f.

Example 1.7. Let $g_n(x) = (1-|x|)^n$ on $(-1,1)$. Note that $\lim g_n(0) = 1$ since $g_n(0) = 1$ for all n. For any other x, $|x| < 1$ and therefore $1-|x| < 1$. Hence $\lim g_n(x) = 0$ for $x \neq 0$. Hence

$$\lim g_n(x) = g(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}.$$

Checking for uniform convergence,

$$|g_n(x) - g(x)| < \epsilon \Leftrightarrow |(1-|x|)^n \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}|.$$

We only have to care about $x \neq 0$, therefore

$$|(1-|x|)^n| < \epsilon \implies n > \frac{\ln(\epsilon)}{\ln(1-|x|)}.$$

However, $\sup_{x \in (-1,1)} \frac{\ln(\epsilon)}{\ln(1-|x|)} = +\infty$, therefore the sequence does not uniformly converge to $g(x)$.

Example 1.8. Let $h_n(x) = \frac{1}{n}\sin(nx)$. Since $\left|\frac{1}{n}\sin(nx)\right| \le \left|\frac{1}{n}\right| = \frac{1}{n}$, it follows that

$$0 \le \lim_{n\to\infty} \left|\frac{1}{n}\sin(nx)\right| \le \lim_{n\to\infty} \frac{1}{n} = 0.$$

Therefore $\lim h_n(x) = 0$. Checking for uniform convergence, we want

$$|h_n(x) - h(x)| < \epsilon \Leftrightarrow \left|\frac{1}{n}\sin(nx) - 0\right| < \epsilon.$$

Since $\left|\frac{1}{n}\sin(nx)\right| \leq \frac{1}{n}$, choosing $n > \frac{1}{\epsilon}$ gives the desired inequality. Since the bound for n doesnt depend on x, the sequence uniformly converges to $h(x) = 0$.

Example 1.9. Let $j_n(x) = \frac{nx}{2n+1}$ on $S = [-2, 2]$. It's pointwise limit is

$$\lim j_n(x) = \lim \frac{nx}{2n+1} = x \lim \frac{n}{2n+1} = \frac{x}{2} = j(x).$$

Checking for uniform convergence, we want

$$\begin{aligned}
\left|\frac{nx}{2n+1} - \frac{x}{2}\right| < \epsilon \implies & \left|\frac{2nx - (2n+1)x}{2(2n+1)}\right| < \epsilon \\
\implies & \frac{|x|}{2(2n+1)} < \epsilon \\
\implies & \frac{|x|}{2\epsilon} < 2n + 1 \\
\implies & n > \frac{|x|}{4\epsilon} - \frac{1}{2}
\end{aligned}$$

Since $|x| < 2$, $n > \frac{1}{2\epsilon} - \frac{1}{2} > \frac{|x|}{4\epsilon} - \frac{1}{2}$ gives the original inequality. Therefore the sequence uniformly converges to $j(x)$.

Example 1.10. Let

$$k_n(x) = \begin{cases} 1 & x > \frac{1}{n} \\ 0 & x \leq \frac{1}{n} \end{cases}$$

on $S = [0, 1]$. Note that $0 \leq \frac{1}{n}$ for all n, meaning $\lim k_n(0) = 0$. For similar reasoning $1 \geq \frac{1}{n}$ for all $n > 1$ and therefore $\lim k_n(1) = 1$. For any $0 < x < 1$, there will be some $N \in \mathbb{N}$ such that $n > N \implies \frac{1}{n} < x$. Hence $\lim k_n(x) = 1$ for all $0 < x < 1$. In total then, the pointwise convergence is

$$k(x) = \begin{cases} 0 & x = 0 \\ 1 & x \neq 0 \end{cases}.$$

Checking for uniform convergence, we want

$$|k_n(x) - k(x)| < \epsilon \implies \left|\begin{cases} 1 & x > \frac{1}{n} \\ 0 & x \leq \frac{1}{n} \end{cases} - \begin{cases} 0 & x = 0 \\ 1 & x \neq 0 \end{cases}\right| = \left|\begin{cases} 0 - 0 & x = 0 \\ 0 - 1 & 0 < x \leq \frac{1}{n} \\ 1 - 1 & \frac{1}{n} < x \leq 1 \end{cases}\right| < \epsilon.$$

Note then that

$$\left|\begin{cases} 0-0 & x=0 \\ 0-1 & 0<x\leq \frac{1}{n} \\ 1-1 & \frac{1}{n}<x\leq 1 \end{cases}\right| = \begin{cases} 0 & x=0, \frac{1}{n}<x\leq 1 \\ 1 & 0<x\leq \frac{1}{n} \end{cases}.$$

Since $0 < x \leq \frac{1}{n}$ the value is 1, it is not possible to get arbitrarily close to the pointwise convergence across all x.