Complex Regions

Definition 1.1 (ϵ-Neighborhood). An ϵ-neighborhood of a point $z_0 \in \mathbb{C}$ is the set of points given by

$$|z - z_0| < \epsilon.$$

This is often denoted by $B_\epsilon(z_0)$ or $B(z_0, \epsilon)$.

Definition 1.2 (Interior, Exterior, and Boundary Points). Given a set $S \subset \mathbb{C}$ and a point $z_0 \in \mathbb{C}$, there are 3 possibilities in how it sits in relation to S.

1. There is an ϵ-neighborhood of z_0 that is contained entirely in S. In this case, z_0 is an **interior point**

2. There is an ϵ-neighborhood of z_0 that is disjoint from S. In this case, z_0 is an **exterior point**

3. For all ϵ-neighborhood's of z_0, there are points that are in S and not in S. In this case, z_0 is a **boundary point**

Definition 1.3 (Open and Closed Sets). Let $S \subset \mathbb{C}$. S is **open** if all its points are interior points. That is

$$\forall z \in S, \exists \epsilon > 0 \text{ s.t. } B_\epsilon(z) \subset S.$$

S is **closed** if it contains its boundary points.

Theorem 1.1 (Closure and Complement). A set $S \subset \mathbb{C}$ is open iff $\mathbb{C} \setminus S$ is closed.

Proof.

$\Rightarrow$) Suppose S is open. Let z_0 be a boundary point of $\mathbb{C} \setminus S$. This means that for every ϵ-neighborhood of z_0, there is a point in $\mathbb{C} \setminus S$ and a point outside of $\mathbb{C} \setminus S$. This means that there is a point always in S and a point outside of S, hence z_0 is also a boundary point of S. Since S is open, z_0 is not in S and therefore it is in $\mathbb{C} \setminus S$ and therefore $\mathbb{C} \setminus S$ contains it's boundary. Hence it is closed.

$\Leftarrow$) Suppose that $\mathbb{C} \setminus S$ is closed. Let $z_0 \in S$. Since z_0 is always in any ϵ-neighborhood around itself, it cant be an exterior point. Assume towards contradiction that z_0 is a boundary point of S. Then by the previous direction, it is also a boundary point of $\mathbb{C} \setminus S$. Since $\mathbb{C} \setminus S$ is closed, it contains z_0 and hence a contradiction. Therefore z_0 is neither an exterior or boundary point and must be an interior point of S.

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Definition 1.4 (Connectedness). A set $S \subset \mathbb{C}$ is connected if for all $z_1, z_2 \in S$ there is a finite sequence of line segments in S that join z_1 and z_2.