Problem 6.3.1

For each integer n, consider the set $B_n = \{n\} \times \mathbb{R}$

(a) Draw a picture of $\bigcup_{n=2}^{4} B_n$ (in the Cartesian plane).

(b) Draw a picture of the set $C = [1, 5] \times \{-2, 2\}$

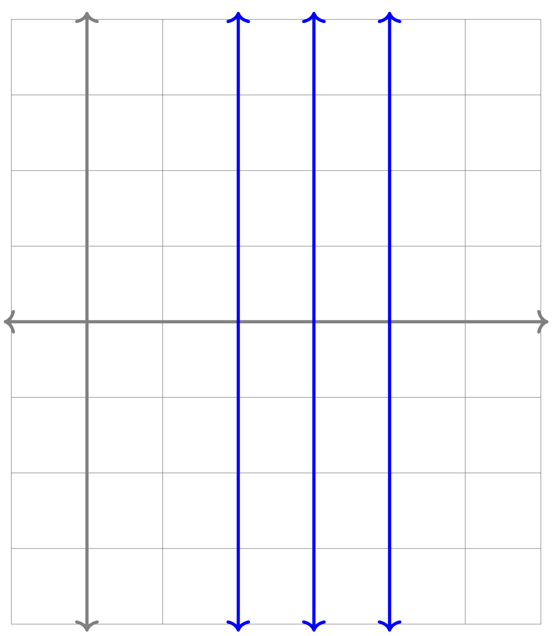
(c) Compute $\left(\bigcup_{n=2}^{4} B_n \right) \cap C$

(d) Compute $\bigcup_{n=2}^{4} (B_n \cap C)$

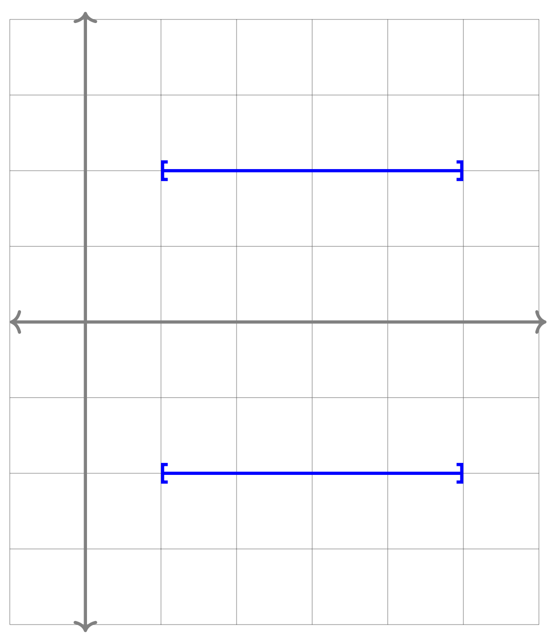
(e) Compare $\left(\bigcup_{n=2}^{4} B_n \right) \cap C$ and $\bigcup_{n=2}^{4} (B_n \cap C)$

Solution

Part A



Part B



Part C

$$\left(\bigcup_{n=2}^{4} B_n \right) \cap C = \{(2, 2), (3, 2), (4, 2), (2, -2), (3, -2), (4, -2)\}.$$

Part D

$$\bigcup_{n=2}^{4}(B_n \cap C) = \{(2,2),(3,2),(4,2),(2,-2),(3,-2),(4,-2)\}.$$

Part E

Both sets are the same regardless whether the intersection with C happens within the indexed collection or outside the indexed collection.

Problem 6.3.2

For each real number r, define the interval $S_r = [r-1, r+3]$. Let $I = \{1,3,4\}$. Determine $\bigcup_{r\in I} S_r$ and $\bigcap_{r\in I} S_r$.

Solution

$$\bigcup_{r\in I} S_r = [0,4] \cup [2,6] \cup [3,7] = [0,7]$$
$$\bigcap_{r\in I} S_r = [0,4] \cap [2,6] \cap [3,7] = [3,4].$$

Problem 6.3.3

Give an example of four different subsets A, B, C and D of $\{1,2,3,4\}$ such that all intersections of two subsets are different.

Solution

$$A = \{1,2,3,4\}$$
$$B = \{2,3\}$$
$$C = \{3,4\}$$
$$D = \{4,1\}.$$

$A \cap B = \{2,3\}$	$A \cap C = \{3,4\}$	$A \cap D = \{4,1\}$
$B \cap C = \{3\}$	$B \cap D = \varnothing$	$C \cap D = \{4\}$

Problem 6.3.4

For each of the following collections of intervals, define an interval An for each $n \in \mathbb{N}$ such that indexed collection $\{A_n\}_{n\in\mathbb{N}}$ is the given collection of sets. Then find both the union and intersection of the indexed collections of sets.

(a) $\{[1, 2+1), \left[1, 2+\frac{1}{2}\right), \left[1, 2+\frac{1}{3}\right), \ldots\}$

(b) $\{(-1, 2), (-\frac{3}{2}, 4), (-\frac{5}{3}, 6), (-\frac{7}{4}, 8), \ldots\}$

(c) $\{(\frac{1}{4}, 1), (\frac{1}{8}, \frac{1}{2}), (\frac{1}{16}, \frac{1}{4}), (\frac{1}{32}, \frac{1}{8}), (\frac{1}{64}, \frac{1}{16}), \ldots\}$

Solution

Part A

$$A_n = \left[1, 2 + \frac{1}{n}\right) \text{ with } \bigcup_{n\in\mathbb{N}} A_n = [1, 3)$$
$$\bigcap_{n\in\mathbb{N}} A_n = [1, 2].$$

Proof. Firstly consider $\bigcup_{n\in\mathbb{N}} A_n$. Let $x \in \bigcup_{n\in\mathbb{N}} A_n$. Then $x \in \left[1, 2+\frac{1}{n}\right)$ for some $n \geq 1$. Therefore $1 \leq x < 2 + \frac{1}{n}$. If $n = 1$, then $1 \leq x < 3$, therefore $x \in [1, 3)$, meaning $\bigcup_{n\in\mathbb{N}} A_n \subseteq [1, 3)$. Let $x \in [1, 3)$. It follows that $x \in A_1$ and therefore $x \in \bigcup_{n\in\mathbb{N}} A_n$, meaning $[1, 3) \subseteq \bigcup_{n\in\mathbb{N}}$. Therefore $\bigcup_{n\in\mathbb{N}} A_n = [1, 3)$.

Consider now $\bigcap_{n\in\mathbb{N}} A_n$. Let $x \in \bigcap_{n\in\mathbb{N}} A_n$. Then $\forall n \in \mathbb{N}, x \in [1, 2+\frac{1}{n})$. If $x < 1$, then $x \notin A_1$. For all $n \in \mathbb{N}$, if $x = 2$ then $x \in A_n$ since $2 < 2+\frac{1}{n}$. If $x > 2$, then $\frac{1}{N} \leq x$ with $N = \lceil \frac{1}{x} \rceil$, hence $x \notin A_N$. Therefore $\bigcap_{n\in\mathbb{N}} A_n \subseteq [1, 2]$. Let $x \in [1, 2]$. For all $n \in \mathbb{N}$, $x \in [1, 2+\frac{1}{n})$, hence $x \in \bigcap_{n\in\mathbb{N}} A_n$. Therefore $\bigcap_{n\in\mathbb{N}} A_n = [1, 2]$. ■

Part B

$$A_n = \left(\frac{1-2n}{n}, 2n\right) \text{ with } \bigcup_{n\in\mathbb{N}} A_n = (-2, \infty)$$
$$\bigcap_{n\in\mathbb{N}} A_n = (-1, 2).$$

Proof. First consider $\bigcup_{n\in\mathbb{N}} A_n$. Let $x \in \bigcup_{n\in\mathbb{N}} A_n$. Assume that $x \leq -2$. Note that $-2 < -2 + \frac{1}{n} = \frac{1-2n}{n}$ for all $n \in \mathbb{N}$. Therefore x cannot be in any A_n, meaning $x \notin \bigcup_{n\in\mathbb{N}} A_n$, meaning $x \nleq -2$. Therefore $x \in (-2, \infty)$, hence $\bigcup_{n\in\mathbb{N}} A_n \subseteq (-2, \infty)$. Let $x \in (-2, \infty)$. If $x = 0$, then $x \in A_n$ for all $n \in \mathbb{N}$ since the lower bound is always negative and the upper bound is always positive. If $x > 0$, then choose $N = \lceil x \rceil$. It follows that $x \in A_N$, meaning $x \in \bigcup_{n\in\mathbb{N}} A_n$. If $-2 < x < 0$, then since $\lim_{n\to\infty} \frac{1-2n}{n} = -2$, then a $N \in \mathbb{N}$ can be chosen such that $\frac{1-2N}{N} < x$. Therefore $x \in A_N$.

Now consider $\bigcap_{n\in\mathbb{N}} A_n$. Let $x \in \bigcap_{n\in\mathbb{N}} A_n$. Let $x \in (-1, 2)$. Therefore $-1 < x < 2$. Note that for all $n \in \mathbb{N}$ that

$$\begin{aligned} -2 + \frac{1}{n} < -1 < x < 2 \leq 2n \\ \frac{1-2n}{n} < x < 2n. \end{aligned}$$

Therefore $x \in A_n$ for all $n \in \mathbb{N}$, meaning $x \in \bigcap_{n\in\mathbb{N}} A_n$. Let $x \in \bigcap_{n\in\mathbb{N}} A_n$. Note that for all $n \in \mathbb{N}$ that $A_1 \subseteq A_n$. Since $x \in \bigcap_{n\in\mathbb{N}} A_n$, then $x \in A_n$ for all $n \in \mathbb{N}$. Therefore since $A_1 \subseteq A_n$ and x is in all sets A_n, $x \in A_1$. Hence $x \in (-1, 2)$. Therefore $\bigcap\limits_{n\in\mathbb{N}} A_n = (-1, 2)$. ■

Part C

$$A_n = \left(\frac{1}{2^{n+1}}, \frac{1}{2^{n-1}}\right) \text{ with } \bigcup_{n\in\mathbb{N}} A_n = (0,1)$$
$$\bigcap_{n\in\mathbb{N}} A_n = \varnothing.$$

Solution

Proof. First consider $\bigcup_{n\in\mathbb{N}} A_n$. Let $x \in \bigcup_{n\in\mathbb{N}} A_n$. Assume that $x \leq 0$. Note that for all $n \in \mathbb{N}$ that the lower bound $\frac{1}{2^{n+1}}$ is strictly positive or otherwords strictly greater than zero. Therefore x cannot be in A_n for any n and hence $x \nleq 0$. Assume that $x = 1$. Note that the upper bound $\frac{1}{2^{n-1}}$ only equals 1 when $n = 1$. However the range is non-inclusive and hence $1 \notin \bigcup_{n\in\mathbb{N}} A_n$, meaning $x \neq 1$. Assume $x > 1$. Note that for all $n \in \mathbb{N}$ that $1 \geq \frac{1}{2^{n-1}}$. Therefore $x > \frac{1}{2^{n-11}}$, meaning $x \notin \bigcup_{n\in\mathbb{N}} A_n$ and therefore $x \ngtr 1$. Therefore $x \in (0, 1)$. Let $x \in (0, 1)$.

Now consider $\bigcap_{n\in\mathbb{N}} A_n$. Assume towards contradiction that there is an element

$x \in \bigcap_{n\in\mathbb{N}} A_n$. This means that for all $n \in \mathbb{N}$ that $x \in A_n$. Let $N \in \mathbb{N}$ be the index of the set A_N that x is in. Consider now the set A_L where $L = N+3$. Then $A_N = \left(\frac{1}{2^{N+1}}, \frac{1}{2^{N-1}}\right)$ and $A_L = \left(\frac{1}{2^{N+4}}, \frac{1}{2^{N+2}}\right)$. It follows that $A_N \cap A_L = \varnothing$. Therefore x is not in every set A_n, hence a contradiction. Therefore there are no elements in the intersection meaning $\bigcap_{n\in\mathbb{N}} A_n = \varnothing$. ■

Problem 6.4.7

Use Definition 6.7 to prove the following results about nested sets.

(a) $A_1 \supseteq A_2 \supseteq A_3 \supseteq \dots \implies \bigcup_{n\in\mathbb{N}} A_n = A_1$

(b) $A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots \implies \bigcap_{n\in\mathbb{N}} A_n = A_1$

Solution

Part A

Proof. Let $A_1, A_2, A_3, \dots$ be sets and assume that $A_1 \supseteq A_2 \supseteq A_3 \supseteq \dots$. Let $x \in \bigcup_{n\in\mathbb{N}} A_n$. Then $\exists l \in \mathbb{N}$ such that $x \in A_l$. By the transitivity of subsets, $\forall n \in \mathbb{N}, A_n \subseteq A_1$. Therefore since x is in A_l, x is also in A_1. Therefore $\bigcup_{x\in\mathbb{N}} A_n \subseteq A_1$. Let $x \in A_1$. By the definition of the indexed union, since 1 is in the index set $\mathbb{N}$, $x \in \bigcup_{n\in\mathbb{N}} A_n$. Therefore $A_1 \subseteq \bigcup_{n\in\mathbb{N}} A_n$, meaning $\bigcup_{n\in\mathbb{N}} A_n = A_1$. ■

Part B

Proof. Let $A_1, A_2, A_3, \dots$ be sets and assume that $A_1 \supseteq A_2 \supseteq A_3 \supseteq \dots$. Let $x \in \bigcap_{n\in\mathbb{N}} A_n$. Then $\forall n \in \mathbb{N}$, x is in A_n meaning $x \in A_1$. Therefore $\bigcap_{n\in\mathbb{N}} A_n \subseteq A_1$. Let $x \in A_1$. By the transitivity of subsets, $A_1 \subseteq A_n$ for all $n \in \mathbb{N}$. Therefore since $x \in A_1$, it follows that $x \in A_n$ for all $n \in \mathbb{N}$, which by definition of the indexed intersection means that $x \in \bigcap_{n\in\mathbb{N}} A_n$. Therefore $A_1 \subseteq \bigcap_{n\in\mathbb{N}} A_n$. Hence $\bigcap_{n\in\mathbb{N}} A_n = A_1$. ■

Additional Problem #1

Let A and B be disjoint sets and define a function

$$f : \mathcal{P}(A) \times \mathcal{P}(B) \to \mathcal{P}(A \cup B) : (X, Y) \mapsto X \cup Y.$$

Prove that f is bijective.

Solution

Proof. Let A and B be disjoint sets associated with the function f as defined.

(Injectivity) Suppose that $(X_1, Y_1), (X_2, Y_2) \in \mathcal{P}(A) \times \mathcal{P}(B)$. Therefore $X_1, X_2 \subseteq A$ and $Y_1, Y_2 \subseteq B$. Since A and B are disjoint, both X_1 and X_2 are disjoint to both Y_1 and Y_2 and vice versa. Assume that $f(X_1, Y_1) = f(X_2, Y_2)$.

$$f(X_1, Y_1) = f(X_2, Y_2)$$
$$X_1 \cup Y_1 = X_2 \cup Y_2.$$

Since X_1 and X_2 are disjoint to Y_1 and Y_2, this implies that X_1 must equal X_2. The same argument implies that $Y_1 = Y_2$. Hence f is injective

(Surjectivity) Let $S \in \mathcal{P}(A \cup B)$. Therefore $S \subseteq A \cup B$. Let $K_1 = S \cap A$ and $K_2 = S \cap B$. Note that $K_1 \subseteq A$ and $K_2 \subseteq B$, therefore $K_1 \in \mathcal{P}(A)$ and $K_2 \in \mathcal{P}(B)$. It also follows that

$$\begin{aligned} f(K_1, K_2) &= K_1 \cup K_2 \\ &= (S \cap A) \cup (S \cap B) \\ &= S \cap (A \cup B) \\ &= S. \end{aligned}$$

Therefore f is surjective.

Since f is both injective and surjective, it follows that f is bijective. ■

Additional Problem #2

Let $A_n = \left\{x \in \mathbb{R} : \left|x^2\right| < \frac{1}{n}\right\}$. Determine $\bigcup_{n\in\mathbb{N}} A_n$ and $\bigcap_{n\in\mathbb{N}} A_n$ and prove your claims.

Solution

$$\bigcup_{n\in\mathbb{N}} A_n = (-1, 1) \qquad \bigcap_{n\in\mathbb{N}} A_n = \{0\}.$$

Proof. First note that A_n can be rewritten as $A_n = \left(-\sqrt{\frac{1}{n}}, \sqrt{\frac{1}{n}}\right)$. Let $x \in (-1, 1)$. It follows that $x \in A_1$ since $A_1 = (-1, 1)$. Therefore $x \in \bigcup_{n\in\mathbb{N}} A_n$, meaning $(-1, 1) \subseteq \bigcup_{n\in\mathbb{N}} A_n$. Now let $x \in \bigcup_{n\in\mathbb{N}} A_n$. Note that for all $n \in \mathbb{N}$, $A_n \subseteq A_1$ since $\frac{1}{n} \leq 1$. Therefore $x \in (-1, 1)$, meaning $\bigcup_{n\in\mathbb{N}} A_n \subseteq (-1, 1)$, therefore $\bigcup_{n\in\mathbb{N}} A_n = (-1, 1)$.

Now consider $\bigcap_{n\in\mathbb{N}} A_n$. Let $x \in \bigcap_{n\in\mathbb{N}} A_n$. Therefore $x \in A_n$ for all $n \in \mathbb{N}$. Note that $x = 0$ works because $0 \in A_n$ for all n. Assume towards contradiction that $x > 0$ or $x < 0$. If $x > 0$, since $\lim_{n\to\infty} \sqrt{\frac{1}{n}} = 0$, there exists an $N \in \mathbb{N}$ such that $\frac{1}{N} < x$, meaning $x \notin A_N$. The same argument applies if $x < 0$. Therefore x can only be zero, meaning $\bigcap_{n\in\mathbb{N}} A_n = \{0\}$. ■

Additional Problem #3

Suppose you are given access to an infinite number of 3-pound weights and 10-pound weights. Prove that you can stack these weights to get a total of N-pounds of weight for any N greater than or equal to 18. (e.g., you can form a 19-pound weight by combining 10+3+3+3 but you cannot form a 7-pound weight).

Solution

Proof. We proceed with strong induction. Consider the following base cases.

$$\begin{aligned} n = 18 &\implies n = 6(3) + 0(10), \text{ hence 18 pounds is possible} \\ n = 19 &\implies n = 3(3) + 1(10), \text{ hence 19 pounds is possible} \\ n = 20 &\implies n = 0(6) + 2(10), \text{ hence 20 pounds is possible} \end{aligned}$$

Now fix an $n \in \mathbb{N}$ where $n \geq 21$ and assume that for all $k \in \mathbb{N}$ with $18 \leq k \leq n$ that a k-pound weight can be made of 3 and 10 pound weights. Otherwisely stated, $\exists a, b \in \mathbb{N}_0$ such that $k = 3a + 10b$. Consider the $n + 1$ case. Then

$$n + 1 - 3 = n - 2$$

By the induction hypothesis, $\exists a, b \in \mathbb{N}_0$ such that $n - 2 = 3a + 10b$. Therefore

$$\begin{aligned} n + 1 - 3 &= 3a + 10b \\ n + 1 &= 3(a + 1) + 10b. \end{aligned}$$

Therefore for all $n \in \mathbb{N}$ greater than or equal to 18, an n-pound can be made from 3 and 10 pound weights. ■