Problem 2

$$
\begin{aligned}
4\mathbb{Z} + 0 &= \{\ldots, -8, -4, 0, 4, 8, \ldots\} \\
4\mathbb{Z} + 2 &= \{\ldots, -6, -2, 2, 4, 6, \ldots\}.
\end{aligned}
$$

Problem 4

$$
\begin{aligned}
\langle 2 \rangle + 0 &= \{0, 2, 4, 6, 8, 10\} \\
\langle 2 \rangle + 1 &= \{1, 3, 5, 7, 9, 11\}.
\end{aligned}
$$

Problem 12

The cosets of $\langle 3 \rangle$ in $\mathbb{Z}_{24}$ are

$$
\begin{aligned}
\langle 3 \rangle + 0 &= \{0, 3, 6, 9, 12, 15, 18, 21\} \\
\langle 3 \rangle + 1 &= \{1, 4, 7, 10, 13, 16, 19, 22\} \\
\langle 3 \rangle + 2 &= \{2, 5, 8, 11, 14, 17, 20, 23\}.
\end{aligned}
$$

Therefore

$$[\mathbb{Z}_{24} : \langle 3 \rangle] = 3.$$

Problem 15

Rewriting σ as disjoint cycles gives

$$\sigma = (1, 2, 3, 5, 4).$$

meaning that $|\sigma| = 5$, hence

$$[S_5 : \langle \sigma \rangle] = \frac{|S_5|}{|\sigma|} = \frac{5!}{5} = 24.$$

Problem 27

> ***Proof.*** Let G be a group, H be a subgroup of G, and $g \in G$. Define the mapping $\phi : H \to Hg$ where $\phi(h) = hg$. Let $a, b \in H$ and assume $\phi(a) = \phi(b)$. Then $ag = bg$ which be the cancellation law implies $a = b$, hence ϕ is injective. Let $y \in Hg$. Then there exists some $x \in H$ such that $y = xg$. $\phi(x) = g$, meaning ϕ is onto. Therefore ϕ is onto and one-to-one. ∎

Problem 28

Proof. Let H be a subgroup of a group G such that $g^{-1}hg \in H$ for all $g \in G$ and $h \in H$. Let $g \in G$ and consider its cosets. Let $x \in Hg$. Then $\exists h_1 \in H$ such that $x = hg = gg^{-1}hg$. Let $r = g^{-1}$. Then $x = gr^{-1}h_1r$, meaning $x \in H$ and therefore $x \in gH$. Therefore $Hg \subseteq gH$. Let $x \in gH$. Then $\exists h_2 \in H$ such that $x = gh_2 = gh_2g^{-1}g = r^{-1}h_2rg$, meaning $x \in Hg$. Therefore $gH \subseteq Hg$. Hence $gH = Hg$ for any g, meaning all left cosets and right cosets of H are the same. ■

Problem 30

Consider the subgroup $\{\rho_0, \mu_1\}$ of S_3. It follows that

$$\rho_1\{\rho_0, \mu_1\} = \mu_3\{\rho_0, \mu_1\} = \{\rho_1, \mu_2\}.$$

However,

$$\{\rho_0, \mu_1\}\rho_1 = \{\rho_1, \mu_2\} \neq \{\rho_2, \mu_3\} = \{\rho_0, \mu_1\}\mu_3.$$

Problem 32

It is true. Since $H \leq G$, $\left\{h^{-1} : h \in H\right\} = H$. Therefore

$$Ha^{-1} = \left\{ha^{-1} : h \in H\right\} = \left\{h^{-1}a^{-1} : h \in H\right\} = \left\{(ah)^{-1} : h \in H\right\}.$$

That is Ha^{-1} contains all the inverses of aH. If $aH = bH$, then the inverse of all the elements in both cosets are the same, meaning $Ha^{-1} = Hb^{-1}$.

Problem 35

Proof. Let G be a group and $H \leq G$. Define the mapping ϕ where $\phi(aH) = Ha^{-1}$ for all $a \in G$. First check if ϕ is well defined. Assume that $aH = bH$. Then $a^{-1}bH = H$ meaning $a^{-1}b \in H$. Note that $a^{-1}b = a^{-1}\left(b^{-1}\right)^{-1} \in H$. Since $a^{-1}\left(b^{-1}\right)^{-1} \in H$, $Ha^{-1}\left\{b^{-1}\right\}^{-1} = H$ implying $Ha^{-1} = Hb^{-1}$. Therefore ϕ is well defined. Let $x, y \in G$ and assume $\phi(xH) = \phi(yH)$. Then $Hx^{-1} = Hy^{-1}$, meaning there is some $h \in H$ such that $x^{-1} = hy^{-1}$. Therefore $h = x^{-1}y$ and $h^{-1} = y^{-1}x$. Since $H \leq G$, $h^{-1} \in H$ and therefore $y^{-1}x \in H$. Therefore $y^{-1}xH = H$ meaning $xH = yH$, hence ϕ is injective. Let Ha be a right coset of G. Note that $\phi(a^{-1}H) = Ha$, hence ϕ is surjective. Therefore ϕ is a bijection between the left and right cosets of H, meaning the left and right cosets of H are equinumerous. ■

Problem 36

Proof. Let G be an abelian group of order $2n$ where n is an odd number. Suppose that there are two elements a and b of G both with order 2. That is $a^2 = e$ and $b^2 = e$. Note that $(ab)^2 = a^2b^2 = e$. $ab \neq e$ since a is its own inverse and $b \neq a$. Therefore ab also has an order of 2. It also follows that $\{e, a, b, ab\}$ is a subgroup with order 4. Since n is odd, $\exists k \in \mathbb{Z}$ such that $n = 2k + 1$, therefore $|G| = 4k + 2$. Therefore $4 \nmid |G|$, meaning the subgroup $\{e, a, b, ab\}$ can not exist, cotnradicting the assumption that there are 2 elements of order 2. Since the group has an even order, there must exist some element that is it's own inverse, meaning there does exist a unique element of order 2. ■