Problem 1

Part 1

$$\begin{aligned}\mathbb{P}(X > 5) &= \mathbb{P}\left(\frac{X-10}{6} > \frac{5-10}{6}\right)\\ &= \mathbb{P}\left(Z > -\frac{5}{6}\right)\\ &= 1 - \Phi\left(-\frac{5}{6}\right) \approx 0.79767.\end{aligned}$$

Part 2

$$\begin{aligned}\mathbb{P}(4 < X < 16) &= \mathbb{P}\left(\frac{4-10}{6} < \frac{X-10}{6} < \frac{16-10}{6}\right)\\ &= \mathbb{P}(-1 < Z < 1)\\ &= \Phi(1) - \Phi(-1) = 2 \cdot \Phi(1) - 1 \approx 0.68268.\end{aligned}$$

Part 3

$$\begin{aligned}\mathbb{P}(X < 8) &= \mathbb{P}\left(\frac{X-10}{6} < \frac{8-10}{6}\right)\\ &= \mathbb{P}\left(Z < -\frac{1}{3}\right)\\ &= \Phi\left(-\frac{1}{3}\right) \approx 0.36944.\end{aligned}$$

Part 4

$$\begin{aligned}\mathbb{P}(X < 20) &= \mathbb{P}\left(\frac{X-10}{6} < \frac{20-10}{6}\right)\\ &= \mathbb{P}\left(Z < \frac{5}{3}\right)\\ &= \Phi\left(\frac{5}{3}\right) \approx 0.95221.\end{aligned}$$

Part 5

$$\begin{aligned}\mathbb{P}(X > 16) &= \mathbb{P}\left(\frac{X-10}{6} > \frac{16-10}{6}\right)\\ &= \mathbb{P}(Z > 1)\\ &= 1 - \Phi(1) \approx 0.15866.\end{aligned}$$

Problem 2

Assuming that the annual rainfall does not change from year to year and that the rainfall from each year is independent, the probability is going to be

$$\mathbb{P}(X \le 50)^{10} = \mathbb{P}\left(\frac{X-40}{4} \le \frac{50-40}{4}\right)^{10} = \mathbb{P}(Z \le 2.5)^{10} = \Phi(2.5)^{10} \approx 0.93961.$$

Problem 3

Let X denote the salaries of the physicians in thousands of dollars. By the given information,

$$\mathbb{P}(X < 180) = \mathbb{P}(X > 320) = 0.25.$$

Therefore

$$\begin{aligned}
\mathbb{P}(X < 180) &= 0.25 \\
\mathbb{P}\left(\frac{X-\mu}{\sigma} < \frac{180-\mu}{\sigma}\right) &= 0.25 \\
\mathbb{P}\left(Z < \frac{180-\mu}{\sigma}\right) &= 0.25 \\
\Phi\left(\frac{180-\mu}{\sigma}\right) &= 0.25 \implies \frac{180-\mu}{\sigma} = -0.67449
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{P}(X > 320) &= 0.25 \\
\mathbb{P}\left(\frac{X-\mu}{\sigma} > \frac{320-\mu}{\sigma}\right) &= 0.25 \\
\mathbb{P}\left(Z > \frac{320-\mu}{\sigma}\right) &= 0.25 \\
1-\Phi\left(\frac{320-\mu}{\sigma}\right) &= 0.25 \implies \frac{320-\mu}{\sigma} = 0.67449
\end{aligned}$$

Therefore this gives a system of 2 equations

$$\begin{aligned}
180-\mu &= -0.67449 \cdot \sigma \\
320-\mu &= 0.67449 \cdot \sigma
\end{aligned}.$$

Solving gives $\mu = 250$ and $\sigma = 103.704$.

Part 1

$$\begin{aligned}
\mathbb{P}(X < 200) &= \mathbb{P}\left(\frac{X-250}{103.704} < \frac{200-250}{103.704}\right) \\
&= \mathbb{P}(Z < -0.4821414796) \\
&= \Phi{-0.4821414796} \approx 0.31485.
\end{aligned}$$

Part 2

$$\begin{aligned}\mathbb{P}(280 < X < 320) &= \mathbb{P}\left(\frac{280-250}{103.704} < \frac{X-250}{103.704} < \frac{320-250}{103.704}\right) \\ &= \mathbb{P}(0.28928 < Z < 0.67499) \\ &= \Phi(0.67499) - \Phi(0.28928) \approx 0.13634\end{aligned}$$

Problem 4

$$\begin{aligned}\mathbb{P}(X > c) &= 0.1 \\ \mathbb{P}\left(\frac{X-12}{2} > \frac{c-12}{2}\right) &= 0.1 \\ \mathbb{P}\left(Z > \frac{c-12}{2}\right) &= 0.1 \\ 1 - \Phi(Z > \frac{c-12}{2}) &= 0.1 \\ \Phi(Z > \frac{c-12}{2}) &= 0.9\end{aligned}$$

Using a lookup table for when Φ is 0.9 gives 1.2816, therefore

$$\begin{aligned}\frac{c-12}{2} &= 1.2816 \\ c &= 2(1.2816) + 12 \implies \boxed{c = 14.5632}\end{aligned}$$

Problem 5

$$\mathbb{P}(X > 2) = 1 - \mathbb{P}(X \leq 2) = 1 - \int_0^2 e^{-x}\mathrm{d}x = 1 - [-e^{-x}]_0^2 = 1 - 1 + e^{-2} = e^{-2}.$$

Problem 6

Since the exponential distribution is memoryless, the probability it will last an additional 8 years will be the same as the probability that a new radio would last 8 years. Therefore the probability is

$$\mathbb{P}(X > 8) = 1 - \mathbb{P}(X < 8) = 1 - \int_0^8 18e^{-18x}\mathrm{d}x = 1 - \left[-e^{-18x}\right]_0^8 = 1 - 1 + e^{-144} = e^{-144}.$$

Problem 7

A quadratic's roots are both real when its discriminant is positive. The discriminant in this case is $16Y^2 - 4(4)(Y+2)$. It follows that

$$\begin{aligned} 16Y^2 - 16Y - 32 &\geq 0 \\ Y^2 - Y - 2 &\geq 0 \\ (Y+1)(Y-2) &\geq 0. \end{aligned}$$

Since $0 < Y < 5$, this only holds when $Y - 2 \geq 0$. Therefore

$$\begin{aligned} \mathbb{P}(Y - 2 \geq 0) &= \mathbb{P}(Y \geq 2) \\ &= \int_2^5 \frac{1}{5} \mathrm{d}x = \frac{3}{5}. \end{aligned}$$

Problem 8

$$\begin{aligned} &F_Y(t) = \mathbb{P}(Y \leq t) = \mathbb{P}(\log(X) \leq t) = \mathbb{P}\big(X \leq e^t\big) = F_X(e^t) \\ &\Downarrow \frac{\mathrm{d}}{\mathrm{d}t} \\ &f_Y(t) = f_X(e^t) \cdot e^t \implies f_Y(t) = e^{t - e^t}, t \in \mathbb{R}. \end{aligned}$$

Problem 9

Part 1

$$\begin{aligned} \mathbb{E}[|X - a|] &= \int_0^A \frac{1}{A} \cdot |x - a| \mathrm{d}x \\ &= \frac{1}{A} \cdot \left[\int_0^a (a - x)\mathrm{d}x + \int_a^A (x - a)\mathrm{d}x \right] \\ &= \frac{1}{A} \cdot \left[ax - \frac{x^2}{2} \bigg|_0^a + \frac{x^2}{2} - ax \bigg|_a^A \right] \\ &= \frac{1}{A} \cdot \left[\frac{a^2}{2} + \frac{A^2}{2} - Aa - \frac{a^2}{2} + a^2 \right] \\ &= \frac{1}{A} \cdot \left[\frac{A^2}{2} - Aa + a^2 \right] \\ &= \frac{a^2}{A} - a + \frac{A}{2}. \end{aligned}$$

Therefore by minimizing with respect to a,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}a}\mathbb{E}[|X-a|] &= 0\\ \frac{\mathrm{d}}{\mathrm{d}a}\left(\frac{a^2}{A}-a+\frac{A}{2}\right) &= 0\\ \frac{2a}{A}-1=0 &\implies a=\frac{A}{2}.\end{aligned}$$

The concavity at $\frac{a}{2}$ is positive, therefore $\frac{a}{2}$ minimizes the expected distance from the fire.

Part 2

$$\begin{aligned}
\mathbb{E}[|X-a|] &= \lambda\left[\int_0^\infty |x-a|\cdot e^{-\lambda x}\mathrm{d}x\right]\\
&= \lambda\left[\int_0^a (a-x)e^{-\lambda x}\mathrm{d}x + \int_a^\infty (x-a)e^{-\lambda x}\mathrm{d}x\right]\\
&= \lambda\left[\int_0^a ae^{-\lambda x} - xe^{-\lambda x}\mathrm{d}x + \int_a^\infty xe^{-\lambda x} - ae^{-\lambda x}\right]\\
&= \lambda\left[a\int_0^a e^{-\lambda x}\mathrm{d}x - \int_0^a xe^{-\lambda x}\mathrm{d}x + \int_0^\infty xe^{-\lambda x}\mathrm{d}x - a\int_a^\infty e^{-\lambda x}\mathrm{d}x\right]\\
&= \lambda\left[a\left[-\frac{e^{-\lambda x}}{\lambda}\right]_0^a - a\left[-\frac{e^{-\lambda x}}{\lambda}\right]_a^\infty - \int_0^a xe^{-\lambda x}\mathrm{d}x + \int_0^\infty xe^{-\lambda x}\mathrm{d}x\right]\\
&= \lambda\left[\frac{a}{\lambda}\left[-e^{-\lambda a}+1+0-e^{-\lambda a}\right] - \int_0^a xe^{-\lambda x}\mathrm{d}x + \int_a^\infty xe^{-\lambda x}\mathrm{d}x\right]\\
&= \lambda\left[\frac{a}{\lambda}\left[1-2e^{-\lambda a}\right] - \left(-\frac{xe^{-\lambda x}}{\lambda}\bigg|_0^a + \int_0^a \frac{e^{-\lambda x}}{\lambda}\mathrm{d}x\right) + \left(-\frac{xe^{-\lambda x}}{\lambda}\bigg|_a^\infty + \int_a^\infty \frac{e^{-\lambda x}}{\lambda}\mathrm{d}x\right)\right]\\
&= \lambda\left[\frac{a}{\lambda}\left[1-2e^{-\lambda a}\right] - \left(-\frac{xe^{-\lambda x}}{\lambda}\bigg|_0^a + \left(-\frac{e^{-\lambda x}}{\lambda^2}\bigg|_0^a\right)\right) + \left(-\frac{xe^{-\lambda x}}{\lambda}\bigg|_a^\infty + \left(-\frac{e^{-\lambda x}}{\lambda^2}\bigg|_a^\infty\right)\right)\right]\\
&= \lambda\left[\frac{a}{\lambda}\left[1-2e^{-\lambda a}\right] - \left(-\frac{ae^{-\lambda a}}{\lambda} - \frac{e^{-\lambda a}}{\lambda^2} + \frac{1}{\lambda^2}\right) + \left(0 + \frac{ae^{-\lambda a}}{\lambda} + \frac{e^{-\lambda a}}{\lambda^2}\right)\right]\\
&= \lambda\left[\frac{a}{\lambda}\left[1-2e^{-\lambda a}\right] + \frac{ae^{-\lambda a}}{\lambda} + \frac{e^{-\lambda a}}{\lambda^2} - \frac{1}{\lambda^2} + \frac{ae^{-\lambda a}}{\lambda} + \frac{e^{-\lambda a}}{\lambda^2}\right]\\
&= \lambda\left[\frac{a}{\lambda}\left[1-2e^{-\lambda a}\right] + \frac{2ae^{-\lambda a}}{\lambda} + \frac{2e^{-\lambda a}}{\lambda^2} - \frac{1}{\lambda^2}\right]\\
&= a\left(1-2e^{-\lambda a}\right) + 2ae^{-\lambda a} + \frac{2e^{-\lambda a}}{\lambda} - \frac{1}{\lambda}\\
&= a - 2ae^{-\lambda a} + 2ae^{-\lambda a} + \frac{2e^{-\lambda a}}{\lambda} - \frac{1}{\lambda}\\
&= a + \frac{2e^{-\lambda a}}{\lambda} - \frac{1}{\lambda}.
\end{aligned}$$

Therefore by minimizing with respect to a,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}a}\left(a + \frac{2e^{-\lambda a}}{\lambda} - \frac{1}{\lambda}\right) &= 0 \\
1 - 2e^{-\lambda a} &= 0 \\
e^{-\lambda a} &= \frac{1}{2} \\
\lambda a = \log(2) &\implies \boxed{a = \frac{\log(2)}{\lambda}}.
\end{aligned}$$

The concavity is always positive so $\frac{\log(2)}{\lambda}$ minimizes the expected distance from the fire.

Problem 10

Let $Y \sim \mathrm{Binom}\left(1000, \frac{1}{6}\right)$ be the number of 6's that appear in 1000 rolls. Since np and nq are large, Y can be approximated by a normal random variable $X \sim N(np, \sqrt{npq}) = N\left(\frac{500}{3}, \frac{25\sqrt{2}}{3}\right)$. Let $\mu = \frac{500}{3}$ and $\sigma = \frac{25\sqrt{2}}{3}$. Then

$$\begin{aligned}
\mathbb{P}(150 \le Y \le 200) &\approx \mathbb{P}(150 \le X \le 200) \\
&= \mathbb{P}\left(\frac{150-\mu}{\sigma} \le \frac{X-\mu}{\sigma} \le \frac{200-\mu}{\sigma}\right) \\
&= \mathbb{P}\left(-\sqrt{2} \le Z \le 2\sqrt{2}\right) \\
&= \Phi\left(-\sqrt{2}\right) - \Phi\left(2\sqrt{2}\right) \approx 0.919.
\end{aligned}$$

Consider now if 6 has appeared 200 times. Then the number of times 5 appears in the remaining 800 rolls can be represented by the random variable $Y \sim \mathrm{Binom}\left(800, \frac{1}{5}\right)$ since there are 800 trials, and since 6 will not appear again, 5 has a $\frac{1}{5}$ chance of appearing. Since np and nq are large, Y can be approximated by $X \sim N(np, \sqrt{npq}) = N\left(160, 8\sqrt{2}\right)$. Therefore

$$\begin{aligned}
\mathbb{P}(Y < 150) &\approx \mathbb{P}(X < 150) \\
&= \mathbb{P}\left(\frac{X-160}{8\sqrt{2}} < \frac{150-160}{8\sqrt{2}}\right) \\
&= \mathbb{P}\left(Z < -\frac{5}{4\sqrt{2}}\right) \\
&= \Phi\left(-\frac{5}{4\sqrt{2}}\right) \approx 0.18838.
\end{aligned}$$