5.2.1

a) False

b) False

c) False

d) True

e) True

f) False

g) True

h) True

i) False

5.2.2

Part D

$$\det(A-\lambda I)=\det\begin{pmatrix}7-\lambda & -4 & 0\\ 8 & -5-\lambda & 0\\ 6 & -6 & 3-\lambda\end{pmatrix}=-(\lambda-3)^2(1+\lambda)\implies \lambda=\{-1,3\}.$$

For $\lambda=-1$,

$$E_{-1}=N(A+I)=\operatorname{span}\left\{\begin{pmatrix}2\\4\\3\end{pmatrix}\right\}.$$

For $\lambda=3$,

$$E_3=N(A-3I)=\operatorname{span}\left\{\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}.$$

Therefore A is diagonalizable with

$$D=\begin{pmatrix}-1&0&0\\0&3&0\\0&0&3\end{pmatrix},Q=\begin{pmatrix}2&1&0\\4&1&0\\3&0&1\end{pmatrix}.$$

Part E

Since

$$\det(A-\lambda I)=(\lambda^2+1)(1-\lambda).$$

The characteristic polynomial does not split and therefore A is not diagonalizable.

Part F

Since A is upper triangular, its eigenvalues are $\lambda = \{1, 3\}$. Note that

$$N(A - I) = E_1 = \operatorname{span}\left\{\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}.$$

Since $\dim E_1 < 2$, A is not diagonalizable.

5.2.3

Part A

$$[T]_e = \begin{pmatrix}0 & 1 & 2 & 0\\0 & 0 & 2 & 6\\0 & 0 & 0 & 3\\0 & 0 & 0 & 0\end{pmatrix}.$$

Since $[T]_e$ is upper triangular, its singular eigenvalue is 0. Then

$$E_0 = N(A) = \operatorname{span}\left\{\begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right\}.$$

Therefore since $\dim E_0 < 4$, T is not diagonalizable.

Part B

$$[T]_e = \begin{pmatrix}0 & 0 & 1\\0 & 1 & 0\\1 & 0 & 0\end{pmatrix}.$$

$$\det([T]_e - \lambda I) = \begin{pmatrix}-\lambda & 0 & 1\\0 & 1-\lambda & 0\\1 & 0 & -\lambda\end{pmatrix} = -(\lambda - 1)^2(\lambda + 1) \implies \lambda = \{-1, 1\}.$$

For $\lambda = -1$,

$$E_{-1} = N(A + I) = \operatorname{span}\left\{\begin{pmatrix}1\\0\\-1\end{pmatrix}\right\}.$$

For $\lambda = 1$,

$$E_1 = N(A - I) = \operatorname{span}\left\{\begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}\right\}.$$

Therefore A is diagonalizable with

$$D = \begin{pmatrix}-1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}, Q = \begin{pmatrix}1 & 0 & 1\\0 & 1 & 0\\-1 & 0 & 1\end{pmatrix}.$$

5.2.8

Proof. Since λ_1 and λ_2 are distinct, there must be an eigenvector $\vec{\lambda}$ associated with λ_2 that is independent of the vectors in E_{λ_1}. Therefore the set of all eigenvectors between them will have dimension $n-1+1=n$ and therefore A will be diagonalizable. ■

5.2.18

Part A

Proof. Let β be the ordered basis such that $[T]_\beta$ and $[U]_\beta$ are diagonal matrices. Since diagonal matrices commute,

$$[T]_\beta[U]_\beta = [U]_\beta[T]_\beta.$$

Therefore since the matrix representations commute, $TU = UT$. ■

Part B

Proof. Let Q^{-1} be the matrix that makes A and B simultaneously diagonalizable. Then

$$(Q^{-1}AQ)(Q^{-1}BQ) = (Q^{-1}BQ)(Q^{-1}AQ)$$

since Q is invertible, hence A and B commute. ■

5.2.19

Proof. Since T and T^m have the same eigenvectors, if T is diagonalizable then T^m is diagonalizable under the same basis. ■

5.4.1

a) False

b) True

c) False

d) False

e) True

f) True

g) True

5.4.2

Part A

Yes. For any polynomial in $P_2(\mathbb{F})$, it follows that

$$T(ax^2 + bx + c) = 2ax + b \in P_2(\mathbb{F}).$$

Part B

No. For any polynomial in $W = P_2(\mathbb{F})$, it follows that

$$T(ax^2 + bx + c) = ax^3 + bx^2 + cx \notin W.$$

Part C

Yes. With $(t, t, t) \in W$,

$$T((t,t,t)) = (t + t + t, t + t + t, t + t + t) = 3 \cdot (t, t, t) \in W.$$

Part D

Yes. With $at + b \in W$,

$$T(at + b) = t \int_0^1 ax + b\mathrm{d}x = \left(\frac{a}{2} + b\right)t \in W.$$

Part E

No. Note that $A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \in W$, but

$$T(A) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}.$$

5.4.4

Proof. Let T be a linear operator on V and W be a T-invariant subspace. Let $g(t) = a_n t^n + \ldots + a_1 t + a_0$. Note that W is a T-invariant subspace of any scalar multiple and positive integer power of T. For some $w \in W$ and $a \in \mathbb{F}$ and $k \in \mathbb{Z}_+$,

$$T(w) \in W \implies T(T(w)) \in W \implies T^k(w) \in W.$$

and

$$aw \in W \implies T(aw) = aT(w) \in W.$$

Furthermore,

$$T(w_1 + w_2) \in W \implies T(w_1) + T(w_2) \in W.$$

Therefore linear combinations of vectors in W under T, whether T is scaled or raised to a power, are still in W. Hence the polynomial of T preserves T-invariance. ■

5.4.17

Proof. Let $f(t) = (-1)^n t^n + \ldots + a_0$ be the characteristic polynomial of A. Then

$$f(A) = (-1)^n A^n + \ldots + Aa_0 = O.$$

Therefore A^n is a linear combination of $\left\{I, A, \ldots, A^{n-1}\right\}$. Note then that

$$Af(A) = (-1)^n A^{n+1} + \ldots A^2 A_0 = 0.$$

Therefore A^{n+1} is a linear combination of $\{I, A, \ldots, A^n\}$ which reduces to $\left\{I, A, \ldots, A^{n-1}\right\}$. This is true for any succesive power, therefore

$$\dim\left\{I, A, A^2, \ldots\right\} = \dim\left\{I, A, \ldots, A^{n-1}\right\} \leq n.$$

■

5.4.18

Part A

Since $f(t) = \det(A - tI)$, if $f(0) = 0$, then $\det(A) = 0$ which would mean A is not invertible. Therefore $f(0) = a_0 \neq 0$.

Part B

Note that

$$\begin{aligned}
AA^{-1} &= -A \cdot \frac{1}{a_0}\Big((-1)^n A^{n-1} + a_{n-1}A^{n-2} + \ldots + a_1 I_n\Big) \\
&= -\frac{1}{a_0}\Big((-1)^n A^n + a_{n-1}A^{n-1} + \ldots a_1 A\Big) \\
&= -\frac{1}{a_0}(-a_0 I) = I
\end{aligned}$$

Therefore the formula for A^{-1} is valid.