Analytic Functions

1.1 Complex Functions

Definition 1.1 (Complex Function). A complex function on $S \subset \mathbb{C}$ is a rule that assigns to each $z \in S$ a value $f(z) = w \in \mathbb{C}$, denoted by $f : S \to \mathbb{C}$.

Example 1.1. There are (surprise!) many complex functions.

1. The function $f(z) = \frac{1}{z}$ is well defined everywhere except $z = 0$, therefore it's domain of definition is $\mathbb{C} \setminus \{0\}$.
2. Any complex polynomial $f(z) = c_n z^n + \ldots + c_1 z + c_0$ with $c_i \in \mathbb{C}$ is a complex function over all of $\mathbb{C}$.
3. Any rational function $\frac{f(x)}{g(x)}$ where the domain is $\mathbb{C} \setminus \{z \in \mathbb{C} : g(z) = 0\}$

A complex function can also often be represented in the form

$$f(x + iy) = u(x, y) + iv(x, y).$$

Consider the case of $\frac{1}{z}$. Then

$$\frac{1}{x + iy} = \frac{x - iy}{x^2 + y^2} = \frac{x}{x^2 + y^2} - i \cdot \frac{y}{x^2 + y^2}.$$

Therefore in this case $u(x, y) = \frac{x}{x^2+y^2}$ and $v(x, y) = \frac{y}{x^2+y^2}$.

Definition 1.2 (Limits in $\mathbb{C}$). The limit of a function $f : \operatorname{dom} f \to \mathbb{C}$

$$\lim_{z \to z_0} f(z) = w_0$$

means that

$$\forall \epsilon > 0, \exists \delta > 0, \text{ s.t. } |z - z_0| < \delta \implies |f(z) - w_0| < \epsilon, \forall z.$$

That is, for any ϵ neighborhood of w_0, there is some deleted δ neighborhood around z_0 such that every z in the δ neighborhood maps into the ϵ neighborhood.

Example 1.2. Consider the function $f(z) = \frac{i}{2}\bar{z}$. One can guess that

$$\lim_{z \to 1} f(z) = \frac{i}{2} 1 = \frac{i}{2}.$$

For this to happen,

$$\left|\frac{i}{2}\overline{z} - \frac{i}{2}\right| < \epsilon \implies \left|\frac{i}{2}\right||\overline{z} - 1| < \epsilon$$
$$\frac{1}{2}|\overline{z} - 1| < \epsilon$$
$$\frac{1}{2}|z - 1| < \epsilon$$
$$|z - 1| < 2\epsilon$$

Therefore choosing $\delta = 2\epsilon$ gives the desired result.

Example 1.3. Consider $f(z) = \frac{\overline{z}}{z}$. Does $f(z)$ have a limit at $z_0 = 0$? Note that along the real axis, $z = x$ and $\overline{z} = x$, hence the limit is $\lim_{x\to 0} \frac{x}{x} = 1$. Along the imaginary axis, $z = y$ and $\overline{z} = -y$, meaning the limit is $\lim_{y\to 0} \frac{-y}{y} = -1$. Therefore there is no limit.

Theorem 1.1 (Limit Equivalence). If $f(z) = u(z) + iv(z)$ where u and v are real valued functions, then

$$\lim_{z\to z_0} f(z) = u_0 + iv_0 \iff \begin{aligned} \lim_{z\to z_0} u(z) &= u_0 \\ \lim_{z\to z_0} v(z) &= v_0 \end{aligned}.$$

1.2 Continuity

Definition 1.3 (Continuity). A function $f : \operatorname{dom} f \to \mathbb{C}$ is continuous at $z_0 \in \mathbb{C}$ if

$$\lim_{z\to z_0} f(z) = f(z_0).$$

That is, the limit exists, $f(z_0)$ exists, and that they are equal. The epsilon-delta form is

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } |z - z_0| < \delta \implies |f(z) - f(z_0)| < \epsilon.$$

Example 1.4. Is $f(z) = \overline{z}$ continuous? That is does $\lim_{z\to z_0} f(z) = \overline{z_0}$? Fix $\epsilon > 0$ and take $\delta = \epsilon$. Note then that

$$|z - z_0| < \delta \implies |\overline{z - z_0}| < \epsilon \implies |\overline{z} - \overline{z_0}| < \epsilon.$$

Therefore $f(z)$ is continuous for all $z \in \mathbb{C}$.

Example 1.5. Consider $f(z) = \operatorname{Arg} z$. Intuitively, it is not continuous since it is always possible to find two points on opposites side the real axis that get arbitrarily close but will have a difference of 2π.

Theorem 1.2 (Continuity Results). Let f, g be continuous functions at z_0. Then

1. $f + g$ is continuous at z_0
2. $f \cdot g$ is continuous at z_0
3. $\frac{f}{g}$ is continuous at z_0 if $g(z_0) \neq 0$
4. If g is continuous at $f(z_0)$, then $g \circ f$ is continuous at z_0

Theorem 1.3. If $f(z)$ is continuous at z_0 and $f(z_0) \neq 0$, then there is some neighborhood of z_0 where $f(z) \neq 0$.

Proof. Let $\epsilon = \frac{|f(z_0)|}{2}$. Since f is continuous at z_0, there is some $\delta > 0$ such that $|z - z_0| < \delta \implies |f(z) - f(z_0)| < \epsilon$. Assume towards contradiction that $f(z) = 0$ for some z where $|z - z_0| < \delta$. Then

$$|f(z) - f(z_0)| = |f(z_0)| < \epsilon = \frac{|f(z_0)|}{2} \implies 1 < \frac{1}{2}.$$

This is a contradiction. Therefore $f(z) \neq 0$ when $|z - z_0| < \delta$. ∎

Theorem 1.4. If $f(z) = u(z) + iv(z)$ and $z_0 = x_0 + iy_0$, then f is continuous at $f(z_0)$ iff $u(z)$ and $v(z)$ are continuous at z_0.

Theorem 1.5. Suppose f is continuous on a closed and bounded region $\mathcal{D}$. Then there is some $M \geq 0$ such that

$$|f(z)| \leq M, \forall z \in \mathcal{D}$$

and there is some $z \in \mathcal{D}$ such that $|f(z)| = M$.

Proof. Let $f(z) = u(x, y) + iv(x, y)$ be continuous on a closed and bounded region $\mathcal{D}$. Therefore

$$(x, y) \mapsto \sqrt{u(x, y)^2 + v(x, y)^2}$$

is also continuous from $\mathcal{D} \to \mathbb{R}$. Since this is a real function on a closed and bounded region, then there is some maximum value $M \geq 0$ that it obtains. Since the function is the modulus, then

$$|f(z)| \leq M, \forall z \in \mathcal{D}$$

and there is a $z \in \mathcal{D}$ where $|f(z)| = M$. ∎

1.3 Differentiability

Theorem 1.6 (Cauchy Riemann Equations). Let $f(z) = u + iv$. If f is differentiable at z_0, then

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}\end{aligned}$$

at z_0.

Example 1.6. Consider $f(x+iy) = 2x + ixy^2$. Then

$$\begin{aligned}u(x,y) &= 2x \\ v(x,y) &= xy^2\end{aligned}$$

Therefore

$$\begin{aligned}\frac{\partial u}{\partial x} &= 2, \ \frac{\partial u}{\partial y} = 0 \\ \frac{\partial v}{\partial x} &= y^2, \frac{\partial v}{\partial y} = 2xy\end{aligned}$$

From the first Cauchy Riemann equation, $2 = 2xy \implies xy = 1$. From the second, $0 = -y^2 \implies y = 0$. Notice then that $xy = 0$ for all x. Hence the equations are never satisfied and f is differentiable nowhere.

Example 1.7. Consider $f(z) = e^{\bar{z}}$. Let $z = x + iy$. Then

$$e^{\bar{z}} = e^{x-iy} = e^x e^{-iy} = e^x(\cos(-y) + i\sin(-y)) = e^x(\cos y - i\sin y)$$

Therefore

$$\begin{aligned}u(x,y) &= e^x \cos y \\ v(x,y) &= -e^x \sin y\end{aligned}$$

The partials are

$$\begin{aligned}\frac{\partial u}{\partial x} &= e^x\cos y, \ \frac{\partial u}{\partial y} = -e^x \sin y \\ \frac{\partial v}{\partial x} &= -e^x \sin y, \frac{\partial v}{\partial y} = -e^x\cos y\end{aligned}$$

Checking the first Cauchy Riemann equation gives

$$e^x\cos y = -e^x\cos y \implies 2e^x \cos y = 0 \implies \cos y = 0.$$

Therefore $y = \frac{\pi}{2} + k\pi$ for $k \in \mathbb{Z}$. Checking the second equation gives

$$-e^x \sin y = e^x \sin y \implies 2e^x \sin y = 0 \implies \sin y = 0.$$

This is only true when $y = k\pi$ for $k \in \mathbb{Z}$. However there is no y that satisifies both conditions so f is differentiable nowhere.

1.3.1 Polar Cauchy Riemann Equations

Proof. Let $f(x+iy) = u(x,y) + iv(x,y)$ and $z_0 \in \mathbb{C} \neq 0$. Subsitute $x = r\cos\theta$ and $y = r\sin\theta$. Thus u and v can be considered functions of r and θ. Using the multivariable chain rule gives

$$\begin{aligned}\frac{\partial u}{\partial r} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} \\ &= \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta\end{aligned}$$

$$\begin{aligned}\frac{\partial u}{\partial \theta} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} \\ &= \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\end{aligned}$$

Suppose that the Cauchy Riemann equations are satisfied for f. Then

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}\end{aligned}$$

Therefore

$$\begin{aligned}\frac{\partial v}{\partial r} &= -\frac{\partial u}{\partial y}\cos\theta + \frac{\partial u}{\partial x}\sin\theta &= r\frac{\partial u}{\partial r} \\ \frac{\partial v}{\partial \theta} &= \frac{\partial u}{\partial y} r\sin\theta + \frac{\partial u}{\partial x} r\cos\theta &= -\frac{1}{r}\frac{\partial u}{\partial \theta}\end{aligned}$$

Therefore the following are equivalent to the Cauchy Riemann equations

$$\begin{aligned}\frac{\partial v}{\partial r} &= r\frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial \theta} &= -\frac{1}{r}\frac{\partial u}{\partial \theta}\end{aligned}$$

■

1.3.2 Converse of Cauchy Riemann

Theorem 1.7 (Converse of C.R.). If $f = u + iv$ is defined in an ϵ-neighborhood of some $z_0 = x_0 + iy_0$ and

1. The Cauchy Riemann equations hold at z_0
2. u_x, u_y, v_x, v_y exist in the ϵ-neighborhood and are continuous at z_0

then f is differentiable at z_0 and $f'(z_0) = u_x(z_0) + iv_x(z_0)$.

1.3.3

1.4 Uniqueness Theorem

Theorem 1.8 (Unqiueness Theorem). Suppose f is defined in a domain $\mathcal{D}$ and

1. f is analytic in $\mathcal{D}$
2. $f(z) = 0$ for all z in some $\mathbb{B}(z_0, \delta) \subset \mathcal{D}$ or a line segment $L \subset \mathcal{D}$

Then $f(z) = 0$ for all $z \in \mathcal{D}$.

Open Neighborhood. Let $p \in \mathcal{D}$. Since $\mathcal{D}$ is connected, there is a piecewise linear curve γ connecting z_0 and p. Let $d = \min\{\delta, \text{distance from } \gamma \text{ to } \partial\mathcal{D}\}$. Construct a finite sequence of points $\{z_n\} \subset \gamma$ that starts at z_0 and ends at p such that

$$|z_k - z_{k-1}| < d, k > 1.$$

For each point z_i, let $N_i = \mathbb{B}(z_i, d)$. Since $d \leq \delta$, $N_0 \subset \mathbb{B}(z_0, \delta)$ and therefore f is zero on N_0. Since $|z_1 - z_0| < \delta$, $z_1 \in \mathbb{B}(z_0, \delta)$ and therefore $f(z_1) = 0$. There is a later result that will finish this proof.

Theorem 1.9. If f is analytic in a neighborhood N_0 of some z_0 and $f \equiv 0$ on a domain $\mathcal{D}$ or line segment L in N_0, then $f \equiv 0$ on N_0.

Therefore $f(z)$ is zero on N_1. This same process can be applied iteratevly, and since p is in the last constructed neighborhood, $f(p) = 0$. ■

Corollary 1.1. Suppose f, g are analytic functions on some domain $\mathcal{D}$ and $f \equiv g$ in some domain $\mathcal{D}' \subset \mathcal{D}$ or line segment $L \subset \mathcal{D}$. Then $f \equiv g$ on $\mathcal{D}$.