Math 130A: Intro to Probability Theory

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Basic Probability

Definition 1.1 (Sample Space). The collection of all possible outcomes in an experiment.

Probability can be understood from the perspective of gambling. The simplest way to gamble to flip a coin. There are only two possibilities, **H** or **T**. Therefore

$$S_{\text{coin}} = \{\mathbf{H}, \mathbf{T}\}.$$

Considering a singular die, there are only six outcomes corresponding to the 6 sides of the die. Therefore

$$S_{\text{die}} = \{1, 2, 3, 4, 5, 6\}.$$

Definition 1.2. An event is a subset of a sample space. Equivalently a set E is an event of a sample space if $E \subseteq S$.

An event in the first example would be tossing a coin and getting tails ($E = \{\mathbf{T}\}$). Given two events E and F, their intersection is denoted as

$$EF \Leftrightarrow E \cap F.$$

Definition 1.3 (Probability). Given some event $E \subset S$ where S is a sample space, $\mathbb{P}(E)$ is a number assigned to the event such that

1. $0 \leq \mathbb{P}(E) \leq 1$
2. $\mathbb{P}(S) = 1$
3. For mutually exclusive events $E_1, E_2, E_3, \ldots, E_n, \ldots$ then

$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mathbb{P}(E_n).$$

Combinatorics

Combinatorics concerns itself with how to count things. Consider a problem such as the following: What is the chance that you draw a heart from deck after you have already drawn 4 hearts? Since 4 hearts have been drawn already, that means that there are 9 remaining in the deck. There is also now only 48 cards in the deck since 4 have been removed. Therefore the probability of drawing a heart is $\frac{9}{48} = 18.75\%$.

Basic Probability

3.1 Axioms of Probability

There are 3 fundamental axioms of probability

Conditional Probability and Independence

4.1 Independent Events

Definition 4.4 (Pairwise Independence). Two events E and F are said to be Independent if

$$P(EF) = P(E) \cdot P(F).$$

or equivalently

$$P(E|F) = P(E).$$

Pairwise independence can be understood as the two events having no impact on the other. This intuition follows from the definition of independence using conditional probability since an event E's probability of occurring given F occurred is just the probability of E, meaning F occurring had no impact on E.

Example 4.1. Consider tossing a coin 2 times. Let $E_1 = \{HH, TT\}, E_2 = \{TH, HH\}$. Then

$$E_1 E_2 = \{HH\}.$$

The probability $P(E_1 E_2) = \frac{1}{4}$. Calculating the probability of each individual event results in $P(E_1) = \frac{1}{2}$ and $E_2 = \frac{1}{2}$, hence

$$P(E_1) \cdot P(E_2) = \frac{1}{2} \cdot \frac{1}{2} = P(E_1 E_2).$$

Therefore the two events are Independent of each other.

4.2 Bayes Theorem

Theorem 4.1. Given two events E and F, the following holds

1. Law of Total Probability

$$P(E|F)P(F) + (1 - P(F)) \cdot P(E|F^{\mathsf{C}}).$$

2. Bayes Rule

$$P(E|F) = \frac{P(F|E) \cdot P(E)}{P(F)}.$$

Proof. Let E and F be events. Note that EF and EF^{C} are disjoint events. Therefore

$$\begin{aligned} P(E) &= P(EF) + P(EF^{\mathsf{C}}) \\ &= P(E|F)P(F) + P(E|F^{\mathsf{C}})P(F^{\mathsf{C}}) \\ &= P(E|F)P(F) + (1 - P(F)) \cdot P(E|F^{\mathsf{C}}) \end{aligned}$$

hence the Law of Total Probability. Consider now the following

$$\begin{aligned} P(E|F) &= \frac{P(FE)}{P(E)} \\ &= \frac{P(E|F)P(E)}{P(F)} \end{aligned}$$

hence Baye's Rules. ■

Random Variables

5.1 Discrete Random Variables

List of Theorems

List of Definitions