

Math 147A: Complex Analysis

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Complex Numbers

1.1 What are the Complex Numbers?

Definition 1.1 (Complex Number). Formally, a complex number $z \in \mathbb{C}$ is a pair of reals (x, y) that are written in the form $z = x + iy$ where "informally" $i = \sqrt{-1}$.

The complex numbers are fairly analogous to the \mathbb{R}^2 plane. \mathbb{C} makes up a field where addition and multiplication are defined as follows

$$z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$$

$$\begin{aligned} z_1 z_2 &= (x_1 + iy_1)(x_2 + iy_2) \\ &= x_1 x_2 + ix_1 y_2 + ix_2 y_1 - y_1 y_2 \\ &= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1) \end{aligned}$$

Theorem 1.1 (Properties of Complex Numbers). Let $z_1, z_2, z_3 \in \mathbb{C}$. Then

1. $z_1 + z_2 = z_2 + z_1$
2. $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$
3. $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$
4. $z_1 + 0 = z_1$ and $1 \cdot z_1 = z_1$
5. $\forall z \in \mathbb{C}, \exists w \in \mathbb{C}$ such that $z + w = 0$
- (★) 6. $\forall z \in \mathbb{C} \neq 0, \exists w \in \mathbb{C}$ such that $zw = 1$.

It does not follow directly that (★) is true. Through some brute force computation though, it is equivalent to finding some u, v for all $x, y \in \mathbb{R}$ such that

$$\begin{aligned} xu - yv &= 1 \\ xv + yu &= 0 \end{aligned}$$

The corresponding solution to this for some $z = x + iy$ is then

$$z^{-1} = \frac{1}{z} = \frac{x - iy}{x^2 + y^2}.$$

While not that elegant, it can be rewritten in terms of other important properties of complex numbers.

1.2 Conjugate and Modulus

Definition 1.2 (Conjugate). The conjugate of some $z \in \mathbb{C}$ is denoted as \bar{z} and is the mirror image of z across the real axis. That is, if $z = x + iy$, then $\bar{z} = x - iy$

Theorem 1.2 (Properties of Conjugate). Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then

1. $\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$
2. $\overline{z_1 z_2} = \bar{z}_1 \cdot \bar{z}_2$
3. $\frac{\bar{z}_1}{z_2} = \frac{\bar{z}_1}{\bar{z}_2}$ when $z_2 \neq 0$
4. $z_1 + \bar{z}_1 = 2 \operatorname{Re} z_1$ or equivalently $\operatorname{Re} z_1 = \frac{z_1 + \bar{z}_1}{2}$
5. $z_1 - \bar{z}_1 = 2i \operatorname{Im} z_1$ or equivalently $\operatorname{Im} z_1 = \frac{z_1 - \bar{z}_1}{2i}$

Note that for any $z \in \mathbb{C}$ that $z\bar{z} = x^2 + y^2$. Geometrically, this quantity represents the squared "length" of z , notated as $|z|^2$. This quantity is also referred to as the squared *modulus of z* . Since $z \neq 0 \implies |z|^2 \neq 0$, then

$$z\bar{z} = |z|^2 \implies z^{-1} = \frac{1}{z} = \frac{\bar{z}}{|z|^2}.$$

Definition 1.3 (Modulus). Let $z = x + iy$. The modulus of z is defined as

$$|z| := \sqrt{x^2 + y^2} = \sqrt{(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2}.$$

Remark. The modulus squared $|z|^2$ is often worked with to avoid square roots.

The modulus captures some important objects, mainly complex disks.

Example 1.1. Consider the set of complex numbers z that satisfy $|z - z_0| = R$ where $z, z_0 \in \mathbb{C}$ and $R \in \mathbb{R}$. This is the set of all points z a distance R away from z_0 , hence the boundary of a disk centered at z_0 with radius R .

The modulus also has some important properties.

Theorem 1.3 (Properties of Modulus). Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then

1. $|\bar{z}_1| = |z_1|$
 2. $|z_1 z_2| = |z_1| |z_2|$
 3. $\left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|}$
 4. $|z^n| = |z|^n$
- (★) $|z_1 + z_2| \leq |z_1| + |z_2|$ and generally $|z_1 + z_2 + \dots + z_n| \leq |z_1| + |z_2| + \dots + |z_n|$

Proof.

1. Let $z = x + iy$. Then $|z| = \sqrt{x^2 + y^2} = \sqrt{x^2 + (-y)^2} = |\bar{z}|$
2. First note that since $|z| \geq 0$ for all $z \in \mathbb{C}$, the statement is equivalent to showing $|z_1 z_2|^2 = |z_1|^2 |z_2|^2$. Then

$$\begin{aligned} |z_1 z_2|^2 &= (z_1 z_2)(\overline{z_1 z_2}) \\ &= (z_1 z_2)(\overline{z_1} \cdot \overline{z_2}) \\ &= z_1 \cdot z_2 \cdot \overline{z_1} \cdot \overline{z_2} \\ &= z_1 \overline{z_1} z_2 \overline{z_2} \\ &= |z_1|^2 |z_2|^2 \end{aligned}$$

Hence the original proposition holds.

- (★) Since the moduli are all positive, it is possible to square both sides and maintain the inequality. Therefore

$$\begin{aligned} |z_1 + z_2|^2 &= (z_1 + z_2) \cdot \overline{(z_1 + z_2)} \\ &= z_1 \overline{z_1} + z_1 \overline{z_2} + \overline{z_1} z_2 + z_2 \overline{z_2} \\ &= |z_1|^2 + z_1 \overline{z_2} + \overline{z_1} z_2 + |z_2|^2 \\ &= |z_1|^2 + 2 \cdot \operatorname{Re}(z_1 \overline{z_2}) + |z_2|^2 \end{aligned}$$

Since $|\operatorname{Re} z| \leq |z|$, the middle is bounded and hence

$$\begin{aligned} &\leq |z_1|^2 + 2|z_1 \overline{z_2}| + |z_2|^2 \\ &= |z_1|^2 + 2|z_1 z_2| + |z_2|^2 \\ &= |z_1|^2 + 2|z_1||z_2| + |z_2|^2 \\ &= (|z_1| + |z_2|)^2 \end{aligned}$$

Therefore $|z_1 + z_2|^2 \leq (|z_1| + |z_2|)^2$ meaning $|z_1 + z_2| \leq |z_1| + |z_2|$. The general case follows by a simple inductive argument. ■

Theorem 1.4 (Further Properties of \mathbb{C}). Let $z_1, z_2 \in \mathbb{C}$. Then

1. If $z_1, z_2 \neq 0$, then $z_1 z_2 \neq 0$
2. $z_1 - z_2 := z_1 + (-z_2) = (x_1 - x_2) + i(y_1 - y_2)$
3. $\frac{z_1}{z_2} := z_1 z_2^{-1} = \frac{z_1 \overline{z_2}}{|z_2|^2}$

1.3 Polar/Exponential Form

Since there is a natural connection between complex numbers and vectors in \mathbb{R}^2 , it is natural to ask what representations of \mathbb{R}^2 would work as representations for \mathbb{C} . In the case of a vector in \mathbb{R}^2 , it can be described as a Cartesian coordinate, or in polar form. For a vector $(x, y) \in \mathbb{R}^2$, its Cartesian coordinates can be encapsulated by a polar pair (r, θ) such that

$$\begin{aligned}x &= r \cos \theta \\y &= r \sin \theta\end{aligned}$$

Therefore if $z = x + iy$, it can be rewritten as

$$z = r \cos \theta + ir \sin \theta = r(\cos \theta + i \sin \theta) = r \operatorname{cis} \theta.$$

Remark. If $z = r \operatorname{cis} \theta$, then $\bar{z} = r \operatorname{cis}(-\theta)$.

Note however, that θ is not a unique value since adding $2\pi k$ for $k \in \mathbb{Z}$ results in the same complex number.

Definition 1.4 (Argument). The argument of $z \in \mathbb{C}$ is the set of all θ such that $z = r \operatorname{cis} \theta$. That is,

$$\arg z := \{\theta \in \mathbb{R} : z = r \operatorname{cis} \theta\}.$$

This set is guaranteed to be infinite, hence there is motivation to pick out a "preferred" value of θ as a representation of z .

Definition 1.5 (Principal Argument). The principal argument of some $z \in \mathbb{C}$ is defined as the unique θ in $\arg z$ between $(-\pi, \pi]$. That is

$$\operatorname{Arg} z := \text{Unique element in } \{\theta \in (-\pi, \pi] : z = r \operatorname{cis} \theta\}.$$

Note that $\arg z = \{\operatorname{Arg} z + 2\pi k : k \in \mathbb{Z}\}$.

This polar form of complex numbers is also termed the exponential form because of the following theorem and corresponding representation.

Theorem 1.5 (Euler's Formula). Given some $\theta \in \mathbb{R}$, $e^{i\theta} = \operatorname{cis} \theta = \cos \theta + i \sin \theta$.

Definition 1.6 (Exponential Form). A complex number $z \in \mathbb{C}$ can be represented as $z = re^{i\theta}$ where $r = |z|$ and $\theta \in \arg z$. The angle θ is generally taken to be $\operatorname{Arg} z$.

Example 1.2. $e^{i\pi}$ corresponds to the complex number with polar representation $(1, \pi)$. Hence $e^{i\pi} = -1$.

Example 1.3. A circle of radius R around some $z_0 \in \mathbb{C}$ can be represented as all points z such that

$$z = z_0 + Re^{i\theta}.$$

for $\theta \in (-\pi, \pi]$.

1.4 Products and Powers

A benefit to the polar/exponential form of a complex number is its simplicity as an algebraic object. Therefore it is often easier to do manipulations on a complex numbers polar representation compared to its Cartesian alternative.

Example 1.4. Consider the product $z_1 z_2$. Let $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$. Then

$$\begin{aligned} z_1 z_2 &= r_1 e^{i\theta_1} r_2 e^{i\theta_2} \\ &= r_1 r_2 [(\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 + i \sin \theta_2)] \\ &= r_1 r_2 [(\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) + i(\cos \theta_1 \sin \theta_2 + \sin \theta_1 \cos \theta_2)] \\ &= r_1 r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)] \\ &= r_1 r_2 e^{i(\theta_1 + \theta_2)} \end{aligned}$$

Note that multiplication is therefore multiplying the lengths and adding the angles which is comparatively easier than Cartesian multiplication.

Remark. For $z_1, z_2 \in \mathbb{C}$ and $z_2 \neq 0$, $\frac{z_1}{z_2} = \frac{|z_1|}{|z_2|} e^{i(\text{Arg } z_1 - \text{Arg } z_2)} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$

Exponentiation of complex numbers is also easier in polar form as

$$z^n = |z|^n e^{in\theta}, n \geq 0.$$

This can be extended to all integer powers by defining $z^{-n} := (z^{-1})^n$. Therefore $z^{-n} = (z^{-1})^n = (z^n)^{-1} = r^{-n} e^{-in\theta}$

Theorem 1.6 (De Moivre's Formula).

$$(r \cos \theta + ir \sin \theta)^n = r^n \cos(n\theta) + r^n \sin(n\theta).$$

Theorem 1.7 (Properties of Products and Powers). Let $z_1, z_2 \in \mathbb{C}$.

1. $z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$
2. $z_1^k = r_1^k e^{ik\theta_1}$ for all $k \in \mathbb{Z}$
3. $\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$
4. $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$
5. $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

1.5 Roots of Complex Numbers

Given $z_0 \in \mathbb{C}$ with $z_0 \neq 0$, for $n = 0, 1, 2, \dots$ which $z \in \mathbb{C}$ satisfy $z^n = z_0$. That is, what are the n th roots of z_0 ?

Theorem 1.8. For some $z_0 \in \mathbb{C}$, there are $n \in \mathbb{N}$ complex solutions to the equation $z^n = z_0$.

Proof. Let $z_0 = r_0 e^{i\theta_0}$ and $z = r e^{i\theta}$. Then

$$z^n = z_0 \Leftrightarrow r^n e^{in\theta} = r_0^n e^{i\theta_0} \Leftrightarrow r^n = r_0, n\theta = \theta_0 + 2\pi k, k \in \mathbb{Z}.$$

Therefore

$$r = \sqrt[n]{r_0}, \theta = \frac{\theta_0}{n} + \frac{2k\pi}{n}, k \in \mathbb{N}.$$

Hence the n th roots of a complex number z_0 are of the form

$$\sqrt[n]{r_0} \exp\left(i\left(\frac{\theta_0}{n} + \frac{2k\pi}{n}\right)\right).$$

Note that when $k = n$, the solution wrap's back around and therefore there are no unique roots from n onward. Furthermore, $\frac{\theta_0}{n} + \frac{2k\pi}{n} = \frac{\theta_0}{n} + \frac{2\pi(1-k)}{n}$ meaning the unique solutions are captured by $k = 0, \dots, n-1$. Hence there are n unique roots.

Remark. This multivalued root motivates defining $z_0^{\frac{1}{n}}$ as the set of all z_0 's n th roots. That is

$$z_0^{\frac{1}{n}} := \{c_0, \dots, c_{n-1}\}.$$

where c_i is the i th solution to $z^n = z_0$. ■

Definition 1.7 (Principal Root). The principal n th root of $z_0 \in \mathbb{C}$ is defined as

$$c_0 = \sqrt[n]{r_0} \exp\left(i\frac{\text{Arg } z_0}{n}\right).$$

From the principal root, all other roots can be recovered using

$$c_k = c_0 \exp\left(i\frac{2k\pi}{n}\right), k = 1, \dots, n-1.$$

The previous definition offers the object $\exp\left(i\frac{2k\pi}{n}\right)$, which is independent of the complex number z_0 . Furthermore, they can be interpreted as the n th roots of 1. These objects are useful enough to be defined

Definition 1.8 (Primitive Roots). The primitive n th roots are the n th roots of 1. That is

$$\omega_k = \exp\left(i\frac{2k\pi}{n}\right).$$

1.6 To Be Filed

Theorem 1.9. Let $p(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$ with $a_i \in \mathbb{C}$ and $a_n \neq 0$. There is a $R > 0$ such that

$$\left| \frac{1}{p(z)} \right| \leq \frac{2}{|a_n| R^n}$$

for $|z| > R$.

Proof. Let $w(z) = \frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \dots + \frac{a_{n-1}}{z} = \frac{p(z)}{z^n} - a_n$. Therefore $p(z) = (a_n + w(z))z^n$ for $z \neq 0$. Then

$$\begin{aligned} w(z)z^n &= a_0 + a_1 z + \dots + a_{n-1} z^{n-1} \\ |w(z)z^n| &= |a_0 + a_1 z + \dots + a_{n-1} z^{n-1}| \\ |w(z)||z|^n &\leq |a_0| + |a_1||z| + \dots + |a_{n-1}||z|^{n-1} \\ |w(z)| &\leq \frac{|a_0|}{|z|^n} + \frac{|a_1|}{|z|^{n-1}} + \dots + \frac{|a_{n-1}|}{|z|} \end{aligned}$$

Since the quantities $\frac{1}{|z|^k}$ get arbitrarily small for large $|z|$ and any positive integer k , take R to be large enough such that for $|z| > R$

$$\frac{|a_0|}{|z|^n}, \frac{|a_1|}{|z|^{n-1}}, \dots, \frac{|a_{n-1}|}{|z|} < \frac{|a_n|}{2n}. \quad (\text{Not a sum})$$

Therefore

$$|w(z)| < n \cdot \frac{|a_n|}{2n} = \frac{|a_n|}{2}.$$

Since $|p(z)| = |a_n + w(z)||z|^n$, for $|z| > R$

$$\begin{aligned} |p(z)| &= |a_n + w(z)||z|^n \\ &\geq ||a_n| - |w(z)||z|^n \\ &> \frac{|a_n|}{2}|z|^n \\ &> \frac{|a_n|}{2}R^n \end{aligned} \quad (\star)$$

The reason (\star) is true is that the distance between $|a_n|$ and $|w(z)|$ is at least $\frac{|a_n|}{2}$ because $|w(z)|$ is less than $\frac{|a_n|}{2}$. Therefore

$$\left| \frac{1}{p(z)} \right| \leq \frac{2}{|a_n| R^n}.$$

Hence the original proposition holds. ■

Complex Regions

Definition 2.9 (ϵ -Neighborhood). An ϵ -neighborhood of a point $z_0 \in \mathbb{C}$ is the set of points given by

$$|z - z_0| < \epsilon.$$

This is often denoted by $B_\epsilon(z_0)$ or $B(z_0, \epsilon)$.

Definition 2.10 (Interior, Exterior, and Boundary Points). Given a set $S \subset \mathbb{C}$ and a point $z_0 \in \mathbb{C}$, there are 3 possibilities in how it sits in relation to S .

1. There is an ϵ -neighborhood of z_0 that is contained entirely in S . In this case, z_0 is an **interior point**
2. There is an ϵ -neighborhood of z_0 that is disjoint from S . In this case, z_0 is an **exterior point**
3. For all ϵ -neighborhood's of z_0 , there are points that are in S and not in S . In this case, z_0 is a **boundary point**

Definition 2.11 (Open and Closed Sets). Let $S \subset \mathbb{C}$. S is **open** if all its points are interior points. That is

$$\forall z \in S, \exists \epsilon > 0 \text{ s.t. } B_\epsilon(z) \subset S.$$

S is **closed** if it contains its boundary points.

Theorem 2.10 (Closure and Complement). A set $S \subset \mathbb{C}$ is open iff $\mathbb{C} \setminus S$ is closed.

Proof.

- \Rightarrow) Suppose S is open. Let z_0 be a boundary point of $\mathbb{C} \setminus S$. This means that for every ϵ -neighborhood of z_0 , there is a point in $\mathbb{C} \setminus S$ and a point outside of $\mathbb{C} \setminus S$. This means that there is a point always in S and a point outside of S , hence z_0 is also a boundary point of S . Since S is open, z_0 is not in S and therefore it is in $\mathbb{C} \setminus S$ and therefore $\mathbb{C} \setminus S$ contains its boundary. Hence it is closed.
- \Leftarrow) Suppose that $\mathbb{C} \setminus S$ is closed. Let $z_0 \in S$. Since z_0 is always in any ϵ -neighborhood around itself, it can't be an exterior point. Assume towards contradiction that z_0 is a boundary point of S . Then by the previous direction, it is also a boundary point of $\mathbb{C} \setminus S$. Since $\mathbb{C} \setminus S$ is closed, it contains z_0 and hence a contradiction. Therefore z_0 is neither an exterior or boundary point and must be an interior point of S .

■

Something important to note is that sets are not in a binary of open or closed. Sets can fall into 4 different categories

	Closed	Not Closed
Open	\emptyset, \mathbb{C}	$B_\epsilon(z_0)$
Not Open	$\overline{B_\epsilon(z_0)}$	$\{z \in \mathbb{C} : r < z \leq R\}$

Definition 2.12 (Closure). Let $S \subset \mathbb{C}$. Then the closure of S is $\overline{S} = S \cup \partial S$

Definition 2.13 (Connectedness). An open set $S \subset \mathbb{C}$ is connected if given $u, v \in S$ there exists a finite set of points $u = w_1, w_2, \dots, w_n = v$ such that $\overline{w_i w_{i+1}} \subset S$ for $i = 1, 2, \dots, n-1$. That is there exists a path of finite line segments between the two points contained in S .

Definition 2.14 (Domain). A set $S \subset \mathbb{C}$ is a domain if it is a connected open set.

Definition 2.15 (Region). $S \subset \mathbb{C}$ is a region if it is a domain unioned with a subset of its boundary.

Definition 2.16 (Boundedness). A set $S \subset \mathbb{C}$ is bounded if there is an $R \in \mathbb{R}$ such that $S \subset B_R(0)$.

Example 2.5. Consider the set $S = \{z \in \mathbb{C} : \frac{\pi}{4} < \arg z < \frac{\pi}{2}\}$

Definition 2.17 (Accumulation Point). Let $S \subset \mathbb{C}$. z_0 is an accumulation point of S if

$$(B_\epsilon(z_0) \setminus \{z_0\}) \cap S \neq \emptyset, \forall \epsilon > 0.$$

That is, z_0 is an accumulation point if every neighborhood contains a point in S that isn't z_0 .

An accumulation point can be thought of as a point that can be continually well approximated by points inside some set S . This idea also applies to things such as the supremum on \mathbb{R} or the limit of a sequence over a topology.

Analytic Functions

3.1 Complex Functions

Definition 3.18 (Complex Function). A complex function on $S \subset \mathbb{C}$ is a rule that assigns to each $z \in S$ a value $f(z) = w \in \mathbb{C}$, denoted by $f : S \rightarrow \mathbb{C}$.

Example 3.6. There are (surprise!) many complex functions.

1. The function $f(z) = \frac{1}{z}$ is well defined everywhere except $z = 0$, therefore it's domain of definition is $\mathbb{C} \setminus \{0\}$.
2. Any complex polynomial $f(z) = c_n z^n + \dots + c_1 z + c_0$ with $c_i \in \mathbb{C}$ is a complex function over all of \mathbb{C} .
3. Any rational function $\frac{f(x)}{g(x)}$ where the domain is $\mathbb{C} \setminus \{z \in \mathbb{C} : g(z) = 0\}$

A complex function can also often be represented in the form

$$f(x + iy) = u(x, y) + iv(x, y).$$

Consider the case of $\frac{1}{z}$. Then

$$\frac{1}{x + iy} = \frac{x - iy}{x^2 + y^2} = \frac{x}{x^2 + y^2} - i \cdot \frac{y}{x^2 + y^2}.$$

Therefore in this case $u(x, y) = \frac{x}{x^2 + y^2}$ and $v(x, y) = \frac{y}{x^2 + y^2}$.

Definition 3.19 (Limits in \mathbb{C}). The limit of a function $f : \text{dom } f \rightarrow \mathbb{C}$

$$\lim_{z \rightarrow z_0} f(z) = w_0$$

means that for any $\epsilon > 0$, there is some $\delta > 0$ such that

$$|z - z_0| < \delta \implies |f(z) - w_0| < \epsilon.$$

That is, for any ϵ neighborhood of w_0 , there is some deleted δ neighborhood around z_0 such that every z in the δ neighborhood maps into the ϵ neighborhood.

Example 3.7. Consider the function $f(z) = \frac{i}{2}\bar{z}$. One can guess that

$$\lim_{z \rightarrow 1} f(z) = \frac{i}{2}1 = \frac{i}{2}.$$

For this to happen,

$$\begin{aligned} \left| \frac{i}{2}\bar{z} - \frac{i}{2} \right| < \epsilon &\implies \left| \frac{i}{2} \right| |\bar{z} - 1| < \epsilon \\ \frac{1}{2} |\bar{z} - 1| &< \epsilon \\ \frac{1}{2} |z - 1| &< \epsilon \\ |z - 1| &< 2\epsilon \end{aligned}$$

Therefore choosing $\delta = 2\epsilon$ gives the desired result.

Example 3.8. Consider $f(z) = \frac{\bar{z}}{z}$. Does $f(z)$ have a limit at $z_0 = 0$? Note that along the real axis, $z = x$ and $\bar{z} = x$, hence the limit is $\lim_{x \rightarrow 0} \frac{x}{x} = 1$. Along the imaginary axis, $z = y$ and $\bar{z} = -y$, meaning the limit is $\lim_{y \rightarrow 0} \frac{-y}{y} = -1$. Therefore there is no limit.

Theorem 3.11 (Limit Equivalence). If $f(z) = u(z) + iv(z)$ where u and v are real valued functions, then

$$\lim_{z \rightarrow z_0} f(z) = u_0 + iv_0 \iff \begin{aligned} \lim_{z \rightarrow z_0} u(z) &= u_0 \\ \lim_{z \rightarrow z_0} v(z) &= v_0 \end{aligned}.$$

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