Math 147A: Complex Analysis

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Table of Contents

Complex Numbers

1.1 What are the Complex Numbers?

Definition 1.1 (Complex Number). Formally, a complex number $z \in \mathbb{C}$ is a pair of reals (x, y) that are written in the form $z = x + iy$ where "informally" $i = \sqrt{-1}$.

The complex numbers are fairly analogous to the $\mathbb{R}^2$ plane. $\mathbb{C}$ makes up a field where addition and multiplication are defined as follows

$$z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$$

$$\begin{aligned} z_1 z_2 &= (x_1 + iy_1)(x_2 + iy_2) \\ &= x_1x_2 + ix_1y_2 + ix_2y_1 - y_1y_2 \\ &= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1) \end{aligned}$$

Theorem 1.1 (Properties of Complex Numbers). Let $z_1, z_2, z_3 \in \mathbb{C}$. Then

1. $z_1 + z_2 = z_2 + z_1$
2. $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$
3. $z_1(z_2 + z_3) = z_1z_2 + z_1z_3$
4. $z_1 + 0 = z_1$ and $1 \cdot z_1 = z_1$
5. $\forall z \in \mathbb{C}, \exists w \in \mathbb{C}$ such that $z + w = 0$

($\star$) 6. $\forall z \in \mathbb{C} \neq 0, \exists w \in \mathbb{C}$ such that $zw = 1$.

It does not follow directly that ($\star$) is true. Through some brute force computation though, it is equivalent to finding some u, v for all $x, y \in \mathbb{R}$ such that

$$\begin{aligned} xu - yv &= 1 \\ xv + yu &= 0 \end{aligned}$$

The corresponding solution to this for some $z = x + iy$ is then

$$z^{-1} = \frac{1}{z} = \frac{x - iy}{x^2 + y^2}.$$

While not that elegant, it can be rewritten in terms of other important properties of complex numbers.

1.2 Conjugate and Modulus

Definition 1.2 (Conjugate). The conjugate of some $z \in \mathbb{C}$ is denoted as $\overline{z}$ and is the mirror image of z across the real axis. That is, if $z = x + iy$, then $\overline{z} = x - iy$

Theorem 1.2 (Properties of Conjugate). Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then

1. $\overline{z_1 \pm z_2} = \overline{z_1} \pm \overline{z_2}$
2. $\overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2}$
3. $\overline{\dfrac{z_1}{z_2}} = \dfrac{\overline{z_1}}{\overline{z_2}}$ when $z_2 \neq 0$
4. $z_1 + \overline{z_1} = 2 \operatorname{Re} z_1$ or equivalently $\operatorname{Re} z_1 = \frac{z_1 + \overline{z_1}}{2}$
5. $z_1 - \overline{z_1} = 2i \operatorname{Im} z_1$ or equivalently $\operatorname{Im} z_1 = \frac{z_1 - \overline{z_1}}{2i}$

Note that for any $z \in \mathbb{C}$ that $z\overline{z} = x^2 + y^2$. Geometrically, this quantity represents the squared "length" of z, notated as $|z|^2$. This quantity is also referred to as the squared *modulus of z*. Since $z \neq 0 \implies |z|^2 \neq 0$, then

$$z\overline{z} = |z|^2 \implies z^{-1} = \frac{1}{z} = \frac{\overline{z}}{|z|^2}.$$

Definition 1.3 (Modulus). Let $z = x + iy$. The modulus of z is defined as

$$|z| := \sqrt{x^2 + y^2} = \sqrt{(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2}.$$

***Remark*.** The modulus squared $|z|^2$ is often worked with to avoid square roots.

The modulus captures some important objects, mainly complex disks.

Example 1.1. Consider the set of complex numbers z that satisfy $|z - z_0| = R$ where $z, z_0 \in \mathbb{C}$ and $R \in \mathbb{R}$. This is the set of all points z a distance R away from z_0, hence the boundary of a disk centered at z_0 with radius R.

The modulus also has some important properties.

Theorem 1.3 (Properties of Modulus). Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then

1. $|\overline{z_1}| = |z_1|$
2. $|z_1 z_2| = |z_1||z_2|$
3. $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$
4. $|z^n| = |z|^n$

($\star$) $|z_1 + z_2| \leq |z_1| + |z_2|$ and generally $|z_1 + z_2 + \ldots z_n| \leq |z_1| + |z_2| + \ldots + |z_n|$

Proof.

1. Let $z = x + iy$. Then $|z| = \sqrt{x^2 + y^2} = \sqrt{x^2 + (-y)^2} = |\overline{z}|$

2. First note that since $|z| \geq 0$ for all $z \in \mathbb{C}$, the statement is equivalent to showing $|z_1z_2|^2 = |z_1|^2|z_2|^2$. Then

$$\begin{aligned}
|z_1z_2|^2 &= (z_1z_2)(\overline{z_1z_2}) \\
&= (z_1z_2)(\overline{z_1} \cdot \overline{z_2}) \\
&= z_1 \cdot z_2 \cdot \overline{z_1} \cdot \overline{z_2} \\
&= z_1\overline{z_1}z_2\overline{z_2} \\
&= |z_1|^2|z_2|^2
\end{aligned}$$

Hence the original proposition holds.

(⋆) Since the moduli are all positive, it is possible to square both sides and maintain the inequality. Therefore

$$\begin{aligned}
|z_1 + z_2|^2 &= (z_1 + z_2) \cdot \overline{(z_1 + z_2)} \\
&= z_1\overline{z_1} + z_1\overline{z_2} + \overline{z_1}z_2 + z_2\overline{z_2} \\
&= |z_1|^2 + z_1\overline{z_2} + \overline{\overline{z_1}z_2} + |z_2|^2 \\
&= |z_1|^2 + 2 \cdot \operatorname{Re}(z_1\overline{z_2}) + |z_2|^2
\end{aligned}$$

Since $|\operatorname{Re} z| \leq |z|$, the middle is bounded and hence

$$\begin{aligned}
&\leq |z_1|^2 + 2|z_1\overline{z_2}| + |z_2|^2 \\
&= |z_1|^2 + 2|z_1z_2| + |z_2|^2 \\
&= |z_1|^2 + 2|z_1||z_2| + |z_2|^2 \\
&= (|z_1| + |z_2|)^2
\end{aligned}$$

Therefore $|z_1 + z_2|^2 \leq (|z_1| + |z_2|)^2$ meaning $|z_1 + z_2| \leq |z_1| + |z_2|$. The general case follows by a simple inductive argument.

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Theorem 1.4 (Further Properties of $\mathbb{C}$). Let $z_1, z_2 \in \mathbb{C}$. Then

1. If $z_1, z_2 \neq 0$, then $z_1z_2 \neq 0$

2. $z_1 - z_2 := z_1 + (-z_2) = (x_1 - x_2) + i(y_1 - y_2)$

3. $\dfrac{z_1}{z_2} := z_1z_2^{-1} = \dfrac{z_1\overline{z}_2}{|z_2|^2}$

1.3 Polar/Exponential Form

Since there is a natural connection between complex numbers and vectors in $\mathbb{R}^2$, it is natural to ask what representations of $\mathbb{R}^2$ would work as representations for $\mathbb{C}$. In the case of a vector in $\mathbb{R}^2$, it can be described as a Cartesian coordinate, or in polar form. For a vector $(x, y) \in \mathbb{R}^2$, its Cartesian coordinates can be encapsulated by a polar pair (r, θ) such that

$$x = r\cos\theta$$
$$y = r\sin\theta$$

Therefore if $z = x + iy$, it can be rewritten as

$$z = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta) = r\operatorname{cis}\theta.$$

Remark. If $z = r\operatorname{cis}\theta$, then $\overline{z} = r\operatorname{cis}(-\theta)$.

Note however, that theta is not a unique value since adding $2\pi k$ for $k \in \mathbb{Z}$ results in the same complex number.

Definition 1.4 (Argument). The argument of $z \in \mathbb{C}$ is the set of all θ theta such that $z = r\operatorname{cis}\theta$. That is,

$$\arg z := \{\theta \in \mathbb{R} : z = r\operatorname{cis}\theta\}.$$

This set is guaranteed to be infinite, hence there is motivation to pick out a "preferred" value of θ as a representation of z.

Definition 1.5 (Principal Argument). The principal argument of some $z \in \mathbb{C}$ is defined as the unique θ in $\arg z$ between $(-\pi, \pi]$. That is

$$\operatorname{Arg} z := \text{Unique element in } \{\theta \in (-\pi, \pi] : z = r\operatorname{cis}\theta\}.$$

Note that $\arg z = \{\operatorname{Arg} z + 2\pi k : k \in \mathbb{Z}\}$.

This polar form of complex numbers is also termed the exponential form because of the following theorem and corresponding representation.

Theorem 1.5 (Euler's Formula). Given some $\theta \in \mathbb{R}$, $e^{i\theta} = \operatorname{cis}\theta = \cos\theta + i\sin\theta$.

Definition 1.6 (Exponential Form). A complex number $z \in \mathbb{C}$ can be represented as $z = re^{i\theta}$ where $r = |z|$ and $\theta \in \arg z$. The angle θ is generally taken to be $\operatorname{Arg} z$.

Example 1.2. $e^{i\pi}$ corresponds to the complex number with polar representation $(1, \pi)$. Hence $e^{i\pi} = -1$.

Example 1.3. A circle of radius R around some $z_0 \in \mathbb{C}$ can be represented as all points z such that

$$z = z_0 + Re^{i\theta}.$$

for $\theta \in (-\pi, \pi]$.

1.4 Products and Powers

A benefit to the polar/exponential form of a complex number is its simplicity as an algebraic object. Therefore it is often easier to do manipulations on a complex numbers polar representation compared to its Cartesian alternative.

Example 1.4. Consider the product $z_1 z_2$. Let $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$. Then

$$\begin{aligned} z_1 z_2 &= r_1 e^{i\theta_1} r_2 e^{i\theta_2} \\ &= r_1 r_2 [(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)] \\ &= r_1 r_2 [(\cos\theta\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)] \\ &= r_1 r_2 [\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)] \\ &= r_1 r_2 e^{i(\theta_1+\theta_2)} \end{aligned}$$

Note that multiplication is therefore multiplying the lengths and adding the angles which is comparatively easier than Cartesian multiplication.

Remark. For $z_1, z_2 \in \mathbb{C}$ and $z_2 \neq 0$, $\dfrac{z_1}{z_2} = \dfrac{|z_1|}{|z_2|} e^{i(\operatorname{Arg} z_1 - \operatorname{Arg} z_2)} = \dfrac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$

Exponentiation of complex numbers is also easier in polar form as

$$z^n = |z|^n e^{in\theta}, n \geq 0.$$

This can be extended to all integer powers by defining $z^{-n} := (z^{-1})^n$. Therefore $z^{-n} = (z^{-1})^n = (z^n)^{-1} = r^{-n} e^{-in\theta}$

Theorem 1.6 (De Moivre's Formula).

$$(r\cos\theta + ir\sin\theta)^n = r^n\cos(n\theta) + r^n\sin(n\theta).$$

Theorem 1.7 (Properties of Products and Powers). Let $z_1, z_2 \in \mathbb{C}$.

1. $z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$
2. $z_1^k = r_1^k e^{ik\theta_1}$ for all $k \in \mathbb{Z}$
3. $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$
4. $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$
5. $\arg\left(\dfrac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

1.5 Roots of Complex Numbers

Given $z_0 \in \mathbb{C}$ with $z_0 \neq 0$, for $n = 0, 1, 2, \ldots$ which $z \in \mathbb{C}$ satisfy $z^n = z_0$. That is, what are the nth roots of z_0?

Theorem 1.8. For some $z_0 \in \mathbb{C}$, there are $n \in \mathbb{N}$ complex solutions to the equation $z^n = z_0$.

Proof. Let $z_0 = r_0 e^{i\theta_0}$ and $z = re^{i\theta}$. Then

$$z^n = z_0 \Leftrightarrow r^n e^{in\theta} = r_0^n e^{i\theta_0} \Leftrightarrow r^n = r_0, n\theta = \theta_0 + 2\pi k, k \in \mathbb{Z}.$$

Therefore

$$r = \sqrt[n]{r_0}, \theta = \frac{\theta_0}{n} + \frac{2k\pi}{n}, k \in \mathbb{N}.$$

Hence the nth roots of a complex number z_0 are of the form

$$\sqrt[n]{r_0} \exp\left(i\left(\frac{\theta_0}{n} + \frac{2k\pi}{n}\right)\right).$$

Note that when $k = n$, the solution wrap's back around and therefore there are no unique roots from n onward. Furthermore, $\frac{\theta_0}{n} + \frac{2k\pi}{n} = \frac{\theta_0}{n} + \frac{2\pi(1-k)}{n}$ meaning the unique solutions are captured by $k = 0, \ldots, n-1$. Hence there are n unique roots.

Remark. This multivalued root motivates defining $z_0^{\frac{1}{n}}$ as the set of all z_0's nth roots. That is

$$z_0^{\frac{1}{n}} := \{c_0, \ldots, c_{n-1}\}.$$

where c_i is the ith solution to $z^n = z_0$.

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Definition 1.7 (Principal Root). The principal nth root of $z_0 \in \mathbb{C}$ is defined as

$$c_0 = \sqrt[n]{r_0} \exp\left(i\frac{\operatorname{Arg} z_0}{n}\right).$$

From the principal root, all other roots can be recovered using

$$c_k = c_0 \exp\left(i\frac{2k\pi}{n}\right), k = 1, \ldots, n-1.$$

The previous definition offers the object $\exp\left(i\frac{2k\pi}{n}\right)$, which is independent of the complex number z_0. Furthermore, they can be interpreted as the nth roots of 1. These objects are useful enough to be defined

Definition 1.8 (Primitive Roots). The primitive nth roots are the nth roots of 1. That is

$$\omega_k = \exp\left(i\frac{2k\pi}{n}\right).$$

1.6 To Be Filed

Theorem 1.9. Let $p(z) = a_n z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0$ with $a_i \in C$ and $a_n \neq 0$. There is a $R > 0$ such that
$$\left|\frac{1}{p(z)}\right| \leq \frac{2}{|a_n|R^n}$$
for $|z| > R$.

Proof. Let $w(z) = \frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \ldots + \frac{a_{n-1}}{z} = \frac{p(z)}{z^n} - a_n$. Therefore $p(z) = (a_n + w(z))z^n$ for $z \neq 0$. Then
$$\begin{aligned} w(z)z^n &= a_0 + a_1 z + \ldots + a_{n-1}z^{n-1} \\ |w(z)z^n| &= |a_0 + a_1 z + \ldots + a_{n-1}z^{n-1}| \\ |w(z)||z|^n &\leq |a_0| + |a_1||z| + \ldots + |a_{n-1}||z^{n-1}| \\ |w(z)| &\leq \frac{|a_0|}{|z|^n} + \frac{|a_1|}{|z|^{n-1}} + \ldots + \frac{|a_{n-1}|}{|z|} \end{aligned}$$
Since the quantities $\frac{1}{|z|^k}$ get arbitrarily small for large $|z|$ and any positive integer k, take R to be large enough such that for $|z| > R$
$$\frac{|a_0|}{|z|^n}, \frac{|a_1|}{|z|^{n-1}}, \ldots, \frac{|a_{n-1}|}{|z|} < \frac{|a_n|}{2n}. \qquad \text{(Not a sum)}$$
Therefore
$$|w(z)| < n \cdot \frac{|a_n|}{2n} = \frac{|a_n|}{2}.$$
Since $|p(z)| = |a_n + w(z)||z|^n$, for $|z| > R$
$$\begin{aligned} |p(z)| &= |a_n + w(z)||z|^n \\ &\geq ||a_n| - |w(z)|||z|^n \\ &> \frac{|a_n|}{2}|z|^n \qquad (\star) \\ &> \frac{|a_n|}{2}R^n \end{aligned}$$
The reason $(\star)$ is true is that the distance between $|a_n|$ and $|w(z)|$ is at least $\frac{|a_n|}{2}$ because $|w(z)|$ is less than $\frac{|a_n|}{2}$. Therefore
$$\left|\frac{1}{p(z)}\right| \leq \frac{2}{|a_n|R^n}.$$
Hence the original proposition holds. ∎

List of Theorems

List of Definitions