Problem 1

Proof. We can expand the right hand side to get

$$\begin{aligned}\text{RHS} &= \left(\frac{x+y}{2}\right)^* A\left(\frac{x+y}{2}\right) - \left(\frac{x-y}{2}\right)^* A\left(\frac{x-y}{2}\right) + i\left(\frac{x+iy}{2}\right)^* A\left(\frac{x+iy}{2}\right) - i\left(\frac{x-iy}{2}\right)^* A\left(\frac{x-iy}{2}\right) \\ &= \frac{1}{4}\Big[(x^*+y^*)(Ax+Ay) - (x^*-y^*)(Ax-Ay) + i(x^*-iy^*)(Ax+iAy) - i(x^*+iy^*)(Ax-iAy)\Big]\end{aligned}$$

We expand each term to get

$$(x^*+y^*)(Ax+Ay) = x^*Ax + x^*Ay + y^*Ax + y^*Ay$$

$$(x^*-y^*)(Ax-Ay) = x^*Ax - x^*Ay - y^*Ax + y^*Ay$$

$$(x^*-iy^*)(Ax+iAy) = x^*Ax + ix^*Ay - iy^*Ax + y^*Ay$$

$$(x^*+iy^*)(Ax-iAy) = x^*Ax - ix^*Ay + iy^*Ax + iy^*Ay$$

Substituing we have

$$\begin{aligned}\text{RHS} &= \frac{1}{4}\left(\Big[2x^*Ay + 2y^*Ax\Big] + i\Big[2ix^*Ay - 2iy^*Ax + y^*Ay - iy^*Ay\Big]\right) \\ &= \frac{1}{4}(2x^*Ay + 2y^*Ax - 2x^*Ay + 2y^*Ax) \\ &= y^*Ax\end{aligned}$$

Since $y^*Ax = \langle Ax, y\rangle$ we have the desired identity.

Suppose $W(A) = \{0\}$. Then for any $x \in \mathbb{C}^n \neq 0$ we have

$$\frac{Q_A(x)}{x^*x} = 0 \implies Q_A(x) = 0.$$

By definition we also have that $Q_A(0) = 0$. Now take $x, y \in \mathbb{C}^n$ with $x \neq 0$. Using the polarization identity we have

$$\langle Ax, y\rangle = Q_A(\ldots) - Q_A(\ldots) + iQ_A(\ldots) - iQ_A(\ldots) = 0 - 0 + 0 - 0 = 0.$$

Since x was taken to be non zero and y as any vector, we must have that $A = 0$.

■

Problem 2

Proof. Note that we can rewrite $Q_A(x) = \langle x, Ax\rangle$ and $x^*x = \langle x, x\rangle$ using the standard inner product on $\mathbb{C}^n$. The lower bound follows quickly from Cauchy Schwarz,

$$r(A) = \sup_{x\in\mathbb{C}^n\setminus\{0\}} \left|\frac{\langle x, Ax\rangle}{\langle x, x\rangle}\right| \leq \sup_{x\in\mathbb{C}^n\setminus\{0\}} \frac{\|Ax\|_2\|x\|_2}{\|x\|_2^2} = \sup_{x\in\mathbb{C}^n\setminus\{0\}} \frac{\|Ax\|_2}{\|x\|_2} = \|A\|.$$

Note first that $|Q_A(x)| \le r(A)\|x\|_2^2$. Therefore if $x \in \mathbb{C}^n$ from (1) and the triangle inequality we have

$$\begin{aligned}
|\langle Ax, y\rangle| &\le \left|Q_A\left(\frac{x+y}{2}\right)\right| + \left|Q_A\left(\frac{x-y}{2}\right)\right| + \left|Q_A\left(\frac{x+iy}{2}\right)\right| + \left|Q_A\left(\frac{x-iy}{2}\right)\right| \\
&\le r(A)\left(\left\|\frac{x+y}{2}\right\|_2^2 + \left\|\frac{x-y}{2}\right\|_2^2\right) + r(A)\left(\left\|\frac{x+iy}{2}\right\|_2^2 + \left\|\frac{x-iy}{2}\right\|_2^2\right) \\
&= r(A)\left(2\left\|\frac{x}{2}\right\|_2^2 + 2\left\|\frac{y}{2}\right\|_2^2\right) + r(A)\left(2\left\|\frac{x}{2}\right\|_2^2 + 2\left\|\frac{iy}{2}\right\|_2^2\right) \\
&= r(A)\left(2\left\|\frac{x}{2}\right\|_2^2 + 2\left\|\frac{y}{2}\right\|_2^2\right) + r(A)\left(2\left\|\frac{x}{2}\right\|_2^2 + 2\left\|\frac{y}{2}\right\|_2^2\right) \\
&= 2r(A)\left(2\left\|\frac{x}{2}\right\|_2^2 + 2\left\|\frac{y}{2}\right\|_2^2\right) \\
&= 2r(A)\left(\frac{1}{2}\|x\|_2^2 + \frac{1}{2}\|y\|_2^2\right) \\
&= r(A)\left(\|x\|_2^2 + \|y\|_2^2\right) \qquad (*)
\end{aligned}$$

Using Cauchy Schwarz we can reformulate the matrix norm as

$$\|A\| = \sup_{\|x\|_2 = \|y\|_2 = 1} |\langle Ax, y\rangle|.$$

By taking this supremum of the extreme sides in the previous derivation, we have

$$\|A\| = \sup_{\|x\|_2 = \|y\|_2 = 1} |\langle Ax, y\rangle| \le \sup_{\|x\|_2 = \|y\|_2 = 1} r(A)\left(\|x\|_2^2 + \|y\|_2^2\right) = r(A)(1^2 + 1^2) = 2r(A).$$

Hence $\|A\| \le 2r(A)$. ∎

Problem 3

Example 0.1 ($\|A\| = r(A)$). Note that for the identity matrix I that

$$\frac{Q_I(x)}{x^*x} = \frac{x^*Ix}{x^*x} = \frac{x^*x}{x^*x} = 1$$

when $x \neq 0$. Therefore $W(I) = \{1\}$ and hence $r(I) = 1$. We also have

$$\frac{\|Ix\|_2}{\|x\|_2} = \frac{\|x\|_2}{\|x\|_2} = 1$$

when $x \neq 0$. Therefore $\|I\| = 1$ as well meaning $\|I\| = r(I)$.

Example 0.2 ($\|A\| = 2r(A)$). Consider the matrix

$$A = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}.$$

Take $x \in \mathbb{C}^2 \neq 0$ with components x_1 and x_2. We have

$$\left|\frac{Q_A(x)}{x^*x}\right| = \left|\frac{x^*Ax}{x^*x}\right| = \left|\frac{\begin{pmatrix}\overline{x_1} & \overline{x_2}\end{pmatrix}\begin{pmatrix}2x_1\\0\end{pmatrix}}{|x_1|^2+|x_2|^2}\right| = \left|\frac{2x_1\overline{x_2}}{|x_1|^2+|x_2|^2}\right| = \frac{2|x_1||x_2|}{|x_1|^2+|x_2|^2}.$$

Note that

$$0 \leq (|x_1|-|x_2|)^2 = |x_1|^2 - 2|x_1||x_2| + |x_2|^2 \implies 0 \leq \frac{2|x_1||x_2|}{|x_1|^2+|x_2|^2} \leq 1.$$

The upper bound is achieved with $x = 1$, hence $r(A) = 1$. Consider $\|A\|$. Note that

$$\|A\| = \left(\sup_{x\in\mathbb{C}^2\setminus\{0\}} \frac{\|Ax\|_2^2}{\|x\|_2^2}\right)^{\frac{1}{2}}$$

by the monotonicity of the square root. We then have

$$\frac{\|Ax\|_2^2}{\|x\|_2^2} = \frac{\left\|\begin{pmatrix}2x_2\\0\end{pmatrix}\right\|_2}{|x_1|^2+|x_2|^2} = \frac{4|x_2|^2}{|x_1|^2+|x_2|^2}.$$

Note that $|x_2|^2 \leq |x_2|^2 + |x_1|^2$ meaning

$$\frac{|x_2|^2}{|x_1|^2+|x_2|^2} \leq 1 \implies \frac{4|x_2|^2}{|x_1|^2+|x_2|^2} \leq 4.$$

This upper bound is achieved with $x_1 = 0$ and $x_2 = 1$ hence $\|A\| = \sqrt{4} = 2$. Therefore $\|A\| = 2r(A)$.

Problem 4

Proof. Consider the characteristic polynomial $p_N(t)$. Since N is purely upper triangular, we have $p_N(t) = \det(N - tI) = (-1)^n t^n$. By the Cayley-Hamilton theorem, we have that $\mathbf{0} = p_N(N) = (-1)^n N^n$ which means $N^n = \mathbf{0}$. Therefore N is nilpotent. ■