Math 147A: Complex Analysis

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Complex Numbers

1.1 What are the Complex Numbers?

Definition 1.1 (Complex Number). Formally, a complex number $z \in \mathbb{C}$ is a pair of reals (x, y) that are written in the form $z = x + iy$ where "informally" $i = \sqrt{-1}$.

The complex numbers are fairly analogous to the $\mathbb{R}^2$ plane. $\mathbb{C}$ makes up a field where addition and multiplication are defined as follows

$$z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$$

$$\begin{aligned} z_1 z_2 &= (x_1 + iy_1)(x_2 + iy_2) \\ &= x_1x_2 + ix_1y_2 + ix_2y_1 - y_1y_2 \\ &= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1) \end{aligned}$$

Theorem 1.1 (Properties of Complex Numbers). Let $z_1, z_2, z_3 \in \mathbb{C}$. Then

1. $z_1 + z_2 = z_2 + z_1$
2. $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$
3. $z_1(z_2 + z_3) = z_1z_2 + z_1z_3$
4. $z_1 + 0 = z_1$ and $1 \cdot z_1 = z_1$
5. $\forall z \in \mathbb{C}, \exists w \in \mathbb{C}$ such that $z + w = 0$

($\star$) 6. $\forall z \in \mathbb{C} \neq 0, \exists w \in \mathbb{C}$ such that $zw = 1$.

It does not follow directly that ($\star$) is true. Through some brute force computation though, it is equivalent to finding some u, v for all $x, y \in \mathbb{R}$ such that

$$\begin{aligned} xu - yv &= 1 \\ xv + yu &= 0 \end{aligned}$$

The corresponding solution to this for some $z = x + iy$ is then

$$z^{-1} = \frac{1}{z} = \frac{x - iy}{x^2 + y^2}.$$

While not that elegant, it can be rewritten in terms of other important properties of complex numbers.

1.2 Conjugate and Modulus

Definition 1.2 (Conjugate). The conjugate of some $z \in \mathbb{C}$ is denoted as $\overline{z}$ and is the mirror image of z across the real axis. That is, if $z = x + iy$, then $\overline{z} = x - iy$

Theorem 1.2 (Properties of Conjugate). Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then

1. $\overline{z_1 \pm z_2} = \overline{z_1} \pm \overline{z_2}$
2. $\overline{z_1 z_2} = \overline{z_1} \cdot \overline{z_2}$
3. $\overline{\dfrac{z_1}{z_2}} = \dfrac{\overline{z_1}}{\overline{z_2}}$ when $z_2 \neq 0$
4. $z_1 + \overline{z_1} = 2\operatorname{Re} z_1$ or equivalently $\operatorname{Re} z_1 = \frac{z_1 + \overline{z_1}}{2}$
5. $z_1 - \overline{z_1} = 2i \operatorname{Im} z_1$ or equivalently $\operatorname{Im} z_1 = \frac{z_1 - \overline{z_1}}{2i}$

Note that for any $z \in \mathbb{C}$ that $z\overline{z} = x^2 + y^2$. Geometrically, this quantity represents the squared "length" of z, notated as $|z|^2$. This quantity is also referred to as the squared *modulus of z*. Since $z \neq 0 \implies |z|^2 \neq 0$, then

$$z\overline{z} = |z|^2 \implies z^{-1} = \frac{1}{z} = \frac{\overline{z}}{|z|^2}.$$

Definition 1.3 (Modulus). Let $z = x + iy$. The modulus of z is defined as

$$|z| := \sqrt{x^2 + y^2} = \sqrt{(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2}.$$

***Remark*.** The modulus squared $|z|^2$ is often worked with to avoid square roots.

The modulus captures some important objects, mainly complex disks.

Example 1.1. Consider the set of complex numbers z that satisfy $|z - z_0| = R$ where $z, z_0 \in \mathbb{C}$ and $R \in \mathbb{R}$. This is the set of all points z a distance R away from z_0, hence the boundary of a disk centered at z_0 with radius R.

The modulus also has some important properties.

Theorem 1.3 (Properties of Modulus). Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then

1. $|\overline{z_1}| = |z_1|$
2. $|z_1 z_2| = |z_1||z_2|$
3. $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$
4. $|z^n| = |z|^n$

($\star$) $|z_1 + z_2| \leq |z_1| + |z_2|$ and generally $|z_1 + z_2 + \ldots z_n| \leq |z_1| + |z_2| + \ldots + |z_n|$

Proof.

1. Let $z = x + iy$. Then $|z| = \sqrt{x^2 + y^2} = \sqrt{x^2 + (-y)^2} = |\overline{z}|$

2. First note that since $|z| \geq 0$ for all $z \in \mathbb{C}$, the statement is equivalent to showing $|z_1 z_2|^2 = |z_1|^2 |z_2|^2$. Then

$$\begin{aligned}
|z_1 z_2|^2 &= (z_1 z_2)(\overline{z_1 z_2}) \\
&= (z_1 z_2)(\overline{z_1} \cdot \overline{z_2}) \\
&= z_1 \cdot z_2 \cdot \overline{z_1} \cdot \overline{z_2} \\
&= z_1 \overline{z_1} z_2 \overline{z_2} \\
&= |z_1|^2 |z_2|^2
\end{aligned}$$

Hence the original proposition holds.

(⋆) Since the moduli are all positive, it is possible to square both sides and maintain the inequality. Therefore

$$\begin{aligned}
|z_1 + z_2|^2 &= (z_1 + z_2) \cdot \overline{(z_1 + z_2)} \\
&= z_1 \overline{z_1} + z_1 \overline{z_2} + \overline{z_1} z_2 + z_2 \overline{z_2} \\
&= |z_1|^2 + z_1 \overline{z_2} + \overline{\overline{z_1} z_2} + |z_2|^2 \\
&= |z_1|^2 + 2 \cdot \mathrm{Re}(z_1 \overline{z_2}) + |z_2|^2
\end{aligned}$$

Since $|\operatorname{Re} z| \leq |z|$, the middle is bounded and hence

$$\begin{aligned}
&\leq |z_1|^2 + 2|z_1 \overline{z_2}| + |z_2|^2 \\
&= |z_1|^2 + 2|z_1 z_2| + |z_2|^2 \\
&= |z_1|^2 + 2|z_1||z_2| + |z_2|^2 \\
&= (|z_1| + |z_2|)^2
\end{aligned}$$

Therefore $|z_1 + z_2|^2 \leq (|z_1| + |z_2|)^2$ meaning $|z_1 + z_2| \leq |z_1| + |z_2|$. The general case follows by a simple inductive argument.

■

Theorem 1.4 (Further Properties of $\mathbb{C}$). Let $z_1, z_2 \in \mathbb{C}$. Then

1. If $z_1, z_2 \neq 0$, then $z_1 z_2 \neq 0$

2. $z_1 - z_2 := z_1 + (-z_2) = (x_1 - x_2) + i(y_1 - y_2)$

3. $\dfrac{z_1}{z_2} := z_1 z_2^{-1} = \dfrac{z_1 \overline{z}_2}{|z_2|^2}$

1.3 Polar/Exponential Form

Since there is a natural connection between complex numbers and vectors in $\mathbb{R}^2$, it is natural to ask what representations of $\mathbb{R}^2$ would work as representations for $\mathbb{C}$. In the case of a vector in $\mathbb{R}^2$, it can be described as a Cartesian coordinate, or in polar form. For a vector $(x, y) \in \mathbb{R}^2$, its Cartesian coordinates can be encapsulated by a polar pair (r, θ) such that

$$x = r\cos\theta$$
$$y = r\sin\theta$$

Therefore if $z = x + iy$, it can be rewritten as

$$z = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta) = r\operatorname{cis}\theta.$$

Remark. If $z = r\operatorname{cis}\theta$, then $\overline{z} = r\operatorname{cis}(-\theta)$.

Note however, that theta is not a unique value since adding $2\pi k$ for $k \in \mathbb{Z}$ results in the same complex number.

Definition 1.4 (Argument). The argument of $z \in \mathbb{C}$ is the set of all θ theta such that $z = r\operatorname{cis}\theta$. That is,

$$\arg z := \{\theta \in \mathbb{R} : z = r\operatorname{cis}\theta\}.$$

This set is guaranteed to be infinite, hence there is motivation to pick out a "preferred" value of θ as a representation of z.

Definition 1.5 (Principal Argument). The principal argument of some $z \in \mathbb{C}$ is defined as the unique θ in $\arg z$ between $(-\pi, \pi]$. That is

$$\operatorname{Arg} z := \text{Unique element in } \{\theta \in (-\pi, \pi] : z = r\operatorname{cis}\theta\}.$$

Note that $\arg z = \{\operatorname{Arg} z + 2\pi k : k \in \mathbb{Z}\}$.

This polar form of complex numbers is also termed the exponential form because of the following theorem and corresponding representation.

Theorem 1.5 (Euler's Formula). Given some $\theta \in \mathbb{R}$, $e^{i\theta} = \operatorname{cis}\theta = \cos\theta + i\sin\theta$.

Definition 1.6 (Exponential Form). A complex number $z \in \mathbb{C}$ can be represented as $z = re^{i\theta}$ where $r = |z|$ and $\theta \in \arg z$. The angle θ is generally taken to be $\operatorname{Arg} z$.

Example 1.2. $e^{i\pi}$ corresponds to the complex number with polar representation $(1, \pi)$. Hence $e^{i\pi} = -1$.

Example 1.3. A circle of radius R around some $z_0 \in \mathbb{C}$ can be represented as all points z such that

$$z = z_0 + Re^{i\theta}.$$

for $\theta \in (-\pi, \pi]$.

1.4 Products and Powers

A benefit to the polar/exponential form of a complex number is its simplicity as an algebraic object. Therefore it is often easier to do manipulations on a complex numbers polar representation compared to its Cartesian alternative.

Example 1.4. Consider the product $z_1 z_2$. Let $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$. Then

$$\begin{aligned}
z_1 z_2 &= r_1 e^{i\theta_1} r_2 e^{i\theta_2} \\
&= r_1 r_2 [(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)] \\
&= r_1 r_2 [(\cos\theta\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)] \\
&= r_1 r_2 [\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)] \\
&= r_1 r_2 e^{i(\theta_1 + \theta_2)}
\end{aligned}$$

Note that multiplication is therefore multiplying the lengths and adding the angles which is comparatively easier than Cartesian multiplication.

Remark. For $z_1, z_2 \in \mathbb{C}$ and $z_2 \neq 0$, $\dfrac{z_1}{z_2} = \dfrac{|z_1|}{|z_2|} e^{i(\operatorname{Arg} z_1 - \operatorname{Arg} z_2)} = \dfrac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$

Exponentiation of complex numbers is also easier in polar form as

$$z^n = |z|^n e^{in\theta}, n \geq 0.$$

This can be extended to all integer powers by defining $z^{-n} := (z^{-1})^n$. Therefore $z^{-n} = (z^{-1})^n = (z^n)^{-1} = r^{-n} e^{-in\theta}$

Theorem 1.6 (De Moivre's Formula).

$$(r\cos\theta + ir\sin\theta)^n = r^n\cos(n\theta) + r^n\sin(n\theta).$$

Theorem 1.7 (Properties of Products and Powers). Let $z_1, z_2 \in \mathbb{C}$.

1. $z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$
2. $z_1^k = r_1^k e^{ik\theta_1}$ for all $k \in \mathbb{Z}$
3. $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$
4. $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$
5. $\arg\left(\dfrac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

1.5 Roots of Complex Numbers

Given $z_0 \in \mathbb{C}$ with $z_0 \neq 0$, for $n = 0, 1, 2, \ldots$ which $z \in \mathbb{C}$ satisfy $z^n = z_0$. That is, what are the nth roots of z_0?

Theorem 1.8. For some $z_0 \in \mathbb{C}$, there are $n \in \mathbb{N}$ complex solutions to the equation $z^n = z_0$.

Proof. Let $z_0 = r_0 e^{i\theta_0}$ and $z = re^{i\theta}$. Then

$$z^n = z_0 \Leftrightarrow r^n e^{in\theta} = r_0^n e^{i\theta_0} \Leftrightarrow r^n = r_0, n\theta = \theta_0 + 2\pi k, k \in \mathbb{Z}.$$

Therefore

$$r = \sqrt[n]{r_0}, \theta = \frac{\theta_0}{n} + \frac{2k\pi}{n}, k \in \mathbb{N}.$$

Hence the nth roots of a complex number z_0 are of the form

$$\sqrt[n]{r_0} \exp\left(i\left(\frac{\theta_0}{n} + \frac{2k\pi}{n}\right)\right).$$

Note that when $k = n$, the solution wrap's back around and therefore there are no unique roots from n onward. Furthermore, $\frac{\theta_0}{n} + \frac{2k\pi}{n} = \frac{\theta_0}{n} + \frac{2\pi(1-k)}{n}$ meaning the unique solutions are captured by $k = 0, \ldots, n-1$. Hence there are n unique roots.

Remark. This multivalued root motivates defining $z_0^{\frac{1}{n}}$ as the set of all z_0's nth roots. That is

$$z_0^{\frac{1}{n}} \coloneqq \{c_0, \ldots, c_{n-1}\}.$$

where c_i is the ith solution to $z^n = z_0$.

∎

Definition 1.7 (Principal Root). The principal nth root of $z_0 \in \mathbb{C}$ is defined as

$$c_0 = \sqrt[n]{r_0} \exp\left(i\frac{\operatorname{Arg} z_0}{n}\right).$$

From the principal root, all other roots can be recovered using

$$c_k = c_0 \exp\left(i\frac{2k\pi}{n}\right), k = 1, \ldots, n-1.$$

The previous definition offers the object $\exp\left(i\frac{2k\pi}{n}\right)$, which is independent of the complex number z_0. Furthermore, they can be interpreted as the nth roots of 1. These objects are useful enough to be defined

Definition 1.8 (Primitive Roots). The primitive nth roots are the nth roots of 1. That is

$$\omega_k = \exp\left(i\frac{2k\pi}{n}\right).$$

1.6 To Be Filed

Theorem 1.9. Let $p(z) = a_n z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0$ with $a_i \in C$ and $a_n \neq 0$. There is a $R > 0$ such that

$$\left|\frac{1}{p(z)}\right| \leq \frac{2}{|a_n|R^n}$$

for $|z| > R$.

Proof. Let $w(z) = \frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \ldots + \frac{a_{n-1}}{z} = \frac{p(z)}{z^n} - a_n$. Therefore $p(z) = (a_n + w(z))z^n$ for $z \neq 0$. Then

$$\begin{aligned} w(z)z^n &= a_0 + a_1 z + \ldots + a_{n-1}z^{n-1} \\ |w(z)z^n| &= |a_0 + a_1 z + \ldots + a_{n-1}z^{n-1}| \\ |w(z)||z|^n &\leq |a_0| + |a_1||z| + \ldots + |a_{n-1}||z^{n-1}| \\ |w(z)| &\leq \frac{|a_0|}{|z|^n} + \frac{|a_1|}{|z|^{n-1}} + \ldots + \frac{|a_{n-1}|}{|z|} \end{aligned}$$

Since the quantities $\frac{1}{|z|^k}$ get arbitrarily small for large $|z|$ and any positive integer k, take R to be large enough such that for $|z| > R$

$$\frac{|a_0|}{|z|^n}, \frac{|a_1|}{|z|^{n-1}}, \ldots, \frac{|a_{n-1}|}{|z|} < \frac{|a_n|}{2n}. \qquad \text{(Not a sum)}$$

Therefore

$$|w(z)| < n \cdot \frac{|a_n|}{2n} = \frac{|a_n|}{2}.$$

Since $|p(z)| = |a_n + w(z)||z|^n$, for $|z| > R$

$$\begin{aligned} |p(z)| &= |a_n + w(z)||z|^n \\ &\geq ||a_n| - |w(z)|||z|^n \\ &> \frac{|a_n|}{2}|z|^n \qquad (\star) \\ &> \frac{|a_n|}{2}R^n \end{aligned}$$

The reason $(\star)$ is true is that the distance between $|a_n|$ and $|w(z)|$ is at least $\frac{|a_n|}{2}$ because $|w(z)|$ is less than $\frac{|a_n|}{2}$. Therefore

$$\left|\frac{1}{p(z)}\right| \leq \frac{2}{|a_n|R^n}.$$

Hence the original proposition holds. ■

Complex Regions

Definition 2.9 (ϵ-Neighborhood). An ϵ-neighborhood of a point $z_0 \in \mathbb{C}$ is the set of points given by

$$|z - z_0| < \epsilon.$$

This is often denoted by $B_\epsilon(z_0)$ or $B(z_0, \epsilon)$.

Definition 2.10 (Interior, Exterior, and Boundary Points). Given a set $S \subset \mathbb{C}$ and a point $z_0 \in \mathbb{C}$, there are 3 possibilities in how it sits in relation to S.

1. There is an ϵ-neighborhood of z_0 that is contained entirely in S. In this case, z_0 is an **interior point**
2. There is an ϵ-neighborhood of z_0 that is disjoint from S. In this case, z_0 is an **exterior point**
3. For all ϵ-neighborhood's of z_0, there are points that are in S and not in S. In this case, z_0 is a **boundary point**

Definition 2.11 (Open and Closed Sets). Let $S \subset \mathbb{C}$. S is **open** if all its points are interior points. That is

$$\forall z \in S, \exists \epsilon > 0 \text{ s.t. } B_\epsilon(z) \subset S.$$

S is **closed** if it contains its boundary points.

Theorem 2.10 (Closure and Complement). A set $S \subset \mathbb{C}$ is open iff $\mathbb{C} \setminus S$ is closed.

Proof.

$\Rightarrow$) Suppose S is open. Let z_0 be a boundary point of $\mathbb{C} \setminus S$. This means that for every ϵ-neighborhood of z_0, there is a point in $\mathbb{C} \setminus S$ and a point outside of $\mathbb{C} \setminus S$. This means that there is a point always in S and a point outside of S, hence z_0 is also a boundary point of S. Since S is open, z_0 is not in S and therefore it is in $\mathbb{C} \setminus S$ and therefore $\mathbb{C} \setminus S$ contains it's boundary. Hence it is closed.

$\Leftarrow$) Suppose that $\mathbb{C} \setminus S$ is closed. Let $z_0 \in S$. Since z_0 is always in any ϵ-neighborhood around itself, it cant be an exterior point. Assume towards contradiction that z_0 is a boundary point of S. Then by the previous direction, it is also a boundary point of $\mathbb{C} \setminus S$. Since $\mathbb{C} \setminus S$ is closed, it contains z_0 and hence a contradiction. Therefore z_0 is neither an exterior or boundary point and must be an interior point of S.

■

Something important to note is that sets are not in a binary of open or closed. Sets can fall into 4 different categories

	Closed	Not Closed
Open	$\varnothing, \mathbb{C}$	$B_\epsilon(z_0)$
Not Open	$\overline{B}_\epsilon(z_0)$	$\{z \in \mathbb{C} : r < \|z\| \leq R\}$

.

Definition 2.12 (Closure). Let $S \subset \mathbb{C}$. Then the closure of S is $\overline{S} = S \cup \partial S$

Definition 2.13 (Connectedness). An open set $S \subset \mathbb{C}$ is connected if given $u, v \in S$ there exists a finite set of points $u = w_1, w_2, \ldots, w_n = v$ such that $\overline{w_i w_{i+1}} \subset S$ for $i = 1, 2, \ldots, n-1$. That is there exists a path of finite line segments between the two points contained in S.

Definition 2.14 (Domain). A set $S \subset \mathbb{C}$ is a domain if it is a connected open set.

Definition 2.15 (Region). $S \subset \mathbb{C}$ is a region if it is a domain unioned with a subset of its boundary.

Definition 2.16 (Boundedness). A set $S \subset \mathbb{C}$ is bounded if there is an $R \in \mathbb{R}$ such that $S \subset B_R(0)$.

Example 2.5. Consider the set $S = \left\{z \in \mathbb{C} : \frac{\pi}{4} < \arg z < \frac{\pi}{2}\right\}$

Definition 2.17 (Accumulation Point). Let $S \subset \mathbb{C}$. z_0 is an accumulation point of S if

$$(B_\epsilon(z_0) \setminus z_0) \cap S \neq \varnothing, \forall \epsilon > 0.$$

That is, z_0 is an accumulation point if every neighborhood contains a point in S that isnt z_0.

An accumulation point can be thought of as a point that can be continually well approximated by points inside some set S. This idea also applies to things such as the suprememum on $\mathbb{R}$ or the limit of a sequence over a toplogy.

Analytic Functions

3.1 Complex Functions

Definition 3.18 (Complex Function). A complex function on $S \subset \mathbb{C}$ is a rule that assigns to each $z \in S$ a value $f(z) = w \in \mathbb{C}$, denoted by $f : S \to \mathbb{C}$.

Example 3.6. There are (surprise!) many complex functions.

1. The function $f(z) = \frac{1}{z}$ is well defined everywhere except $z = 0$, therefore it's domain of definition is $\mathbb{C} \setminus \{0\}$.
2. Any complex polynomial $f(z) = c_n z^n + \ldots + c_1 z + c_0$ with $c_i \in \mathbb{C}$ is a complex function over all of $\mathbb{C}$.
3. Any rational function $\frac{f(x)}{g(x)}$ where the domain is $\mathbb{C} \setminus \{z \in \mathbb{C} : g(z) = 0\}$

A complex function can also often be represented in the form

$$f(x + iy) = u(x, y) + iv(x, y).$$

Consider the case of $\frac{1}{z}$. Then

$$\frac{1}{x + iy} = \frac{x - iy}{x^2 + y^2} = \frac{x}{x^2 + y^2} - i \cdot \frac{y}{x^2 + y^2}.$$

Therefore in this case $u(x, y) = \frac{x}{x^2+y^2}$ and $v(x, y) = \frac{y}{x^2+y^2}$.

Definition 3.19 (Limits in $\mathbb{C}$). The limit of a function $f : \operatorname{dom} f \to \mathbb{C}$

$$\lim_{z \to z_0} f(z) = w_0$$

means that

$$\forall \epsilon > 0, \exists \delta > 0, \text{ s.t. } |z - z_0| < \delta \implies |f(z) - w_0| < \epsilon, \forall z.$$

That is, for any ϵ neighborhood of w_0, there is some deleted δ neighborhood around z_0 such that every z in the δ neighborhood maps into the ϵ neighborhood.

Example 3.7. Consider the function $f(z) = \frac{i}{2}\overline{z}$. One can guess that

$$\lim_{z \to 1} f(z) = \frac{i}{2} 1 = \frac{i}{2}.$$

For this to happen,

$$\begin{aligned}\left|\frac{i}{2}\overline{z}-\frac{i}{2}\right|<\epsilon \implies \left|\frac{i}{2}\right||\overline{z}-1|&<\epsilon\\ \frac{1}{2}|\overline{z}-1|&<\epsilon\\ \frac{1}{2}|z-1|&<\epsilon\\ |z-1|&<2\epsilon\end{aligned}$$

Therefore choosing $\delta = 2\epsilon$ gives the desired result.

Example 3.8. Consider $f(z) = \frac{\overline{z}}{z}$. Does $f(z)$ have a limit at $z_0 = 0$? Note that along the real axis, $z = x$ and $\overline{z} = x$, hence the limit is $\lim_{x\to 0}\frac{x}{x} = 1$. Along the imaginary axis, $z = y$ and $\overline{z} = -y$, meaning the limit is $\lim_{y\to 0}\frac{-y}{y} = -1$. Therefore there is no limit.

Theorem 3.11 (Limit Equivalence). If $f(z) = u(z)+iv(z)$ where u and v are real valued functions, then

$$\lim_{z\to z_0} f(z) = u_0 + iv_0 \iff \begin{aligned}\lim_{z\to z_0} u(z) &= u_0\\ \lim_{z\to z_0} v(z) &= v_0\end{aligned}.$$

3.2 Continuity

Definition 3.20 (Continuity). A function $f : \operatorname{dom} f \to \mathbb{C}$ is continuous at $z_0 \in \mathbb{C}$ if

$$\lim_{z\to z_0} f(z) = f(z_0).$$

That is, the limit exists, $f(z_0)$ exists, and that they are equal. The epsilon-delta form is

$$\forall\epsilon > 0, \exists\delta > 0 \text{ s.t. } |z - z_0| < \delta \implies |f(z) - f(z_0)| < \epsilon.$$

Example 3.9. Is $f(z) = \overline{z}$ continuous? That is does $\lim_{z\to z_0} f(z) = \overline{z_0}$? Fix $\epsilon > 0$ and take $\delta = \epsilon$. Note then that

$$|z - z_0| < \delta \implies |\overline{z - z_0}| < \epsilon \implies |\overline{z} - \overline{z_0}| < \epsilon.$$

Therefore $f(z)$ is continuous for all $z \in \mathbb{C}$.

Example 3.10. Consider $f(z) = \operatorname{Arg} z$. Intuitively, it is not continuous since it is always possible to find two points on opposites side the real axis that get arbitrarily close but will have a difference of 2π.

Theorem 3.12 (Continuity Results). Let f, g be continuous functions at z_0. Then

1. $f + g$ is continuous at z_0
2. $f \cdot g$ is continuous at z_0
3. $\frac{f}{g}$ is continuous at z_0 if $g(z_0) \neq 0$
4. If g is continuous at $f(z_0)$, then $g \circ f$ is continuous at z_0

Theorem 3.13. If $f(z)$ is continuous at z_0 and $f(z_0) \neq 0$, then there is some neighborhood of z_0 where $f(z) \neq 0$.

Proof. Let $\epsilon = \frac{|f(z_0)|}{2}$. Since f is continuous at z_0, there is some $\delta > 0$ such that $|z - z_0| < \delta \implies |f(z) - f(z_0)| < \epsilon$. Assume towards contradiction that $f(z) = 0$ for some z where $|z - z_0| < \delta$. Then

$$|f(z) - f(z_0)| = |f(z_0)| < \epsilon = \frac{|f(z_0)|}{2} \implies 1 < \frac{1}{2}.$$

This is a contradiction. Therefore $f(z) \neq 0$ when $|z - z_0| < \delta$. ■

Theorem 3.14. If $f(z) = u(z) + iv(z)$ and $z_0 = x_0 + iy_0$, then f is continuous at $f(z_0)$ iff $u(z)$ and $v(z)$ are continuous at z_0.

Theorem 3.15. Suppose f is continuous on a closed and bounded region $\mathcal{D}$. Then there is some $M \geq 0$ such that

$$|f(z)| \leq M, \forall z \in \mathcal{D}$$

and there is some $z \in \mathcal{D}$ such that $|f(z)| = M$.

Proof. Let $f(z) = u(x, y) + iv(x, y)$ be continuous on a closed and bounded region $\mathcal{D}$. Therefore

$$(x, y) \mapsto \sqrt{u(x, y)^2 + v(x, y)^2}$$

is also continuous from $\mathcal{D} \to \mathbb{R}$. Since this is a real function on a closed and bounded region, then there is some maximum value $M \geq 0$ that it obtains. Since the function is the modulus, then

$$|f(z)| \leq M, \forall z \in \mathcal{D}$$

and there is a $z \in \mathcal{D}$ where $|f(z)| = M$. ■

3.3 Differentiability

Theorem 3.16 (Cauchy Riemann Equations). Let $f(z) = u + iv$. If f is differentiable at z_0, then

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \end{aligned}$$

at z_0.

Example 3.11. Consider $f(x+iy) = 2x + ixy^2$. Then

$$\begin{aligned} u(x,y) &= 2x \\ v(x,y) &= xy^2 \end{aligned}$$

Therefore

$$\begin{aligned} \frac{\partial u}{\partial x} &= 2, \ \frac{\partial u}{\partial y} = 0 \\ \frac{\partial v}{\partial x} &= y^2, \frac{\partial v}{\partial y} = 2xy \end{aligned}$$

From the first Cauchy Riemann equation, $2 = 2xy \implies xy = 1$. From the second, $0 = -y^2 \implies y = 0$. Notice then that $xy = 0$ for all x. Hence the equations are never satisfied and f is differentiable nowhere.

Example 3.12. Consider $f(z) = e^{\bar{z}}$. Let $z = x + iy$. Then

$$e^{\bar{z}} = e^{x-iy} = e^x e^{-iy} = e^x(\cos(-y) + i\sin(-y)) = e^x(\cos y - i \sin y)$$

Therefore

$$\begin{aligned} u(x,y) &= e^x \cos y \\ v(x,y) &= -e^x \sin y \end{aligned}$$

The partials are

$$\begin{aligned} \frac{\partial u}{\partial x} &= e^x \cos y, \ \frac{\partial u}{\partial y} = -e^x \sin y \\ \frac{\partial v}{\partial x} &= -e^x \sin y, \frac{\partial v}{\partial y} = -e^x \cos y \end{aligned}$$

Checking the first Cauchy Riemann equation gives

$$e^x \cos y = -e^x \cos y \implies 2e^x \cos y = 0 \implies \cos y = 0.$$

Therefore $y = \frac{\pi}{2} + k\pi$ for $k \in \mathbb{Z}$. Checking the second equation gives

$$-e^x \sin y = e^x \sin y \implies 2e^x \sin y = 0 \implies \sin y = 0.$$

This is only true when $y = k\pi$ for $k \in \mathbb{Z}$. However there is no y that satisifies both conditions so f is differentiable nowhere.

3.3.1 Polar Cauchy Riemann Equations

Proof. Let $f(x+iy) = u(x,y) + iv(x,y)$ and $z_0 \in \mathbb{C} \neq 0$. Subsitute $x = r\cos\theta$ and $y = r\sin\theta$. Thus u and v can be considered functions of r and θ. Using the multivariable chain rule gives

$$\begin{aligned}\frac{\partial u}{\partial r} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} \\ &= \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta\end{aligned}$$

$$\begin{aligned}\frac{\partial u}{\partial \theta} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} \\ &= \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\end{aligned}$$

Suppose that the Cauchy Riemann equations are satisfied for f. Then

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}\end{aligned}$$

Therefore

$$\begin{aligned}\frac{\partial v}{\partial r} &= -\frac{\partial u}{\partial y}\cos\theta + \frac{\partial u}{\partial x}\sin\theta &= r\frac{\partial u}{\partial r} \\ \frac{\partial v}{\partial \theta} &= \frac{\partial u}{\partial y} r\sin\theta + \frac{\partial u}{\partial x} r\cos\theta = -\frac{1}{r}\frac{\partial u}{\partial \theta}\end{aligned}$$

Therefore the following are equivalent to the Cauchy Riemann equations

$$\begin{aligned}\frac{\partial v}{\partial r} &= r\frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial \theta} &= -\frac{1}{r}\frac{\partial u}{\partial \theta}\end{aligned}$$

■

3.3.2 Converse of Cauchy Riemann

Theorem 3.17 (Converse of C.R.). If $f = u + iv$ is defined in an ϵ-neighborhood of some $z_0 = x_0 + iy_0$ and

1. The Cauchy Riemann equations hold at z_0
2. u_x, u_y, v_x, v_y exist in the ϵ-neighborhood and are continuous at z_0

then f is differentiable at z_0 and $f'(z_0) = u_x(z_0) + iv_x(z_0)$.

3.3.3

3.4 Uniqueness Theorem

Theorem 3.18 (Unqiueness Theorem). Suppose f is defined in a domain $\mathcal{D}$ and

1. f is analytic in $\mathcal{D}$
2. $f(z) = 0$ for all z in some $\mathbb{B}(z_0, \delta) \subset \mathcal{D}$ or a line segment $L \subset \mathcal{D}$

Then $f(z) = 0$ for all $z \in \mathcal{D}$.

Open Neighborhood. Let $p \in \mathcal{D}$. Since $\mathcal{D}$ is connected, there is a piecewise linear curve γ connecting z_0 and p. Let $d = \min\{\delta, \text{distance from } \gamma \text{ to } \partial\mathcal{D}\}$. Construct a finite sequence of points $\{z_n\} \subset \gamma$ that starts at z_0 and ends at p such that

$$|z_k - z_{k-1}| < d, k > 1.$$

For each point z_i, let $N_i = \mathbb{B}(z_i, d)$. Since $d \leq \delta$, $N_0 \subset \mathbb{B}(z_0, \delta)$ and therefore f is zero on N_0. Since $|z_1 - z_0| < \delta$, $z_1 \in \mathbb{B}(z_0, \delta)$ and therefore $f(z_1) = 0$. There is a later result that will finish this proof.

Theorem 3.19. If f is analytic in a neighborhood N_0 of some z_0 and $f \equiv 0$ on a domain $\mathcal{D}$ or line segment L in N_0, then $f \equiv 0$ on N_0.

Therefore $f(z)$ is zero on N_1. This same process can be applied iteratevly, and since p is in the last constructed neighborhood, $f(p) = 0$. ∎

Corollary 3.1. Suppose f, g are analytic functions on some domain $\mathcal{D}$ and $f \equiv g$ in some domain $\mathcal{D}' \subset \mathcal{D}$ or line segment $L \subset \mathcal{D}$. Then $f \equiv g$ on $\mathcal{D}$.

Elementary Functions

4.1 Logarithm

Consider an angle subset of the logarithm. That is taking a specific "principal value" to base it around. Then for some $z = re^{i\theta}$ with $r > 0$ and $\alpha \in \mathbb{R}$,

$$\log z = \ln r + i\theta. \qquad (\theta \in (\alpha, \alpha + 2\pi))$$

The problem with this formulation of log is that the line $\theta = \alpha$ represents a discontinuous section. This discontinuity is specifically a "branch" of $\log z$ and must be excluded for $\log z$ to be analytical on some domain. Applying the Cauchy Riemann equations to log on this branch cut, then

$$\begin{aligned} u_r &= \frac{1}{r}, v_r = 0 \\ u_\theta &= 0, v_\theta = 1 \end{aligned}$$

which when applied gives statements that hold everywhere with continuous partials. Therefore $\log z$ is analytic on this domain or "branch". Therefore

$$\frac{\mathrm{d}}{\mathrm{d}x} \log z = e^{-i\theta}\left(\frac{1}{r} + i\theta\right) = \frac{1}{re^{i\theta}} = \frac{1}{z}.$$

Remark. When $\alpha = \pi$, the values of theta are $(-\pi, \pi)$ which is called the principal branch of $\log z$ or the principal logarithm $\operatorname{Log} z$

4.1.1 Identities with Logs

Theorem 4.20 (Properties of Logs). Let $z_1, z_2 \in \mathbb{C}$. Then

1. $\log z_1 z_2 = \log z_1 + \log z_2$ $(\star)$
2. $\log \frac{z_1}{z_2} = \log z_1 - \log z_2$

Proof.

1. Note that

$$\begin{aligned} \log z_1 z_2 &= \ln |z_1 z_2| + i \arg z_1 z_2 \\ &= (\ln |z_1| + \ln |z_2|) + i(\arg z_1 + \arg z_2) \\ &= \log z_1 + \log z_2 \end{aligned}$$

■

Remark. It is important that for $(\star)$ that the principal logarithm is not used (same as with arg vs Arg). Consider $z_1 = z_2 = -1$. Then

$$\operatorname{Log} z_1 z_2 = \operatorname{Log} 1 = 0$$

but

$$\operatorname{Log} z_1 + \operatorname{Log} z_2 = i\pi + i\pi = i2\pi.$$

4.2 Power's

At this point, z^n, z^{-n} and $z^{\frac{1}{n}}$ is well defined only when $n \in \mathbb{N}$. Therefore it is natural to ask what z^c looks like when $c \in \mathbb{C}$. The trick to finding the answer is to employ the logarithm.

Theorem 4.21. For $n \in \mathbb{Z}$ and $z \in \mathbb{C}$, the following equalities hold

$$\begin{aligned} z^n &= e^{n \log z} \\ z^{\frac{1}{n}} &= e^{\frac{1}{n} \log z} \end{aligned}$$

Proof. Pick $z \neq 0$ and $n \in \mathbb{Z}$. Consider $e^{n \log z}$. Let $z = re^{i\theta}$ for some $\theta \in \arg z$. Then

$$z^n = r^n e^{in\theta}.$$

From the previous formulation of log,

$$\begin{aligned} \log z = \ln r + i\theta \implies & n \log z = \ln r^n + in\theta \\ \implies & e^{n \log z} = r^n \cdot e^{in\theta} = z^n \end{aligned}$$

Consider now $e^{\frac{1}{n} \log z}$. Then

$$\begin{aligned} \exp\left(\frac{1}{n} \log z\right) &= \exp\left(\frac{1}{n}(\ln r + i(\theta + 2k\pi))\right) \\ &= \exp\left(\ln r^{\frac{1}{n}} + i\left(\frac{\theta + 2k\pi}{n}\right)\right) \\ &= z^{\frac{1}{n}} \end{aligned}$$

■

This reformulation of the previous idea of powers motivates the following definition to fill in the "gaps" for powers.

Definition 4.21 (Complex Power). Let $c \in \mathbb{C}$ and $z \in \mathbb{C} \neq 0$. Then

$$z^c := e^{c \log z}.$$

Remark. This is a multivalued definition since $\log z$ is used.

This definition behaves in ways that are expected. For example

$$\frac{1}{z} = \frac{1}{\exp(c \log z)} = \exp(-c \log z) = z^{-c}.$$

Just like $\log z$ having a branch based on some α, z^c can be taken to be on a branch based on some α, and on such a branch it will be analytic due to the chain rule.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}z} z^c &= \frac{\mathrm{d}}{\mathrm{d}z} \exp(c \log z) \\
&= \exp(c \log z) \cdot c \cdot \frac{1}{z} \\
&= \exp(c \log z) \cdot c \cdot \exp(-\log z) \\
&= \exp((c-1) \log z) \cdot c \\
&= c e^{(c-1)\log z} \\
&= c z^{c-1}
\end{aligned}$$

If working with $\operatorname{Log} z$, then this is called the principal value of z^c.

Definition 4.22 (Exponential with Base). Let $c \in \mathbb{C} \neq 0$ and $z \in \mathbb{C}$. Then

$$c^z := e^{z \log c}$$

Remark. Note for $c = e$, this definition would imply e^z is multivalued. By choosing the principal branch of log, this fixes the problem.

If one fixes $\log c$ in some manner, then the derivative of the exponential is single valued and entire. That is due to

$$\frac{\mathrm{d}}{\mathrm{d}z} c^z = \frac{\mathrm{d}}{\mathrm{d}z} e^{z \log c} = e^{z \log c} \log c = c^z \log c.$$

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