15.1

Part A

The series converges by the alternating series test since $\frac{1}{n} > \frac{1}{n+1}$ and $\lim \frac{1}{n} = 0$.

Part B

The series diverges by the ratio test since $\left|\frac{a_{n+1}}{a_n}\right| = \frac{n+1}{2} \longrightarrow \infty$.

15.2

Part B

Since $-1 < \sin\left(\frac{\pi n}{7}\right) < 1$ for all n, $|\sin\left(\frac{\pi n}{7}\right)| < 1$. Since it is also periodic, it is possible to choose $1 > r > \max\left\{\sin\left(\frac{\pi n}{7}\right) : n = 1, \ldots, 14\right\}$. Note then that $|\sin\left(\frac{\pi n}{7}\right)|^n < r^n$. Since $0 \leq r < 1$, by the comparison test it follows that the original series converges absolutely and hence converges.

15.3

Proof. Suppose that $p > 1$. Then

$$\begin{aligned}
\int_2^n \frac{1}{x(\log x)^p} \mathrm{d}x &= \int_{\log 2}^{\log n} \frac{1}{u^p} \mathrm{d}u \\
&= -\frac{1}{p-1}\left(\frac{1}{(\log p)^{p-1}} - \frac{1}{(\log 2)^{p-1}}\right) \\
&= \frac{1}{p-1}\left(\frac{1}{(\log 2)^{p-1}} - \frac{1}{(\log p)^{p-1}}\right) \xrightarrow{n\to\infty} \frac{1}{p-1} \cdot \frac{1}{(\log 2)^{p-1}}
\end{aligned}$$

Therefore the interval converges and therefore by the integral test the series converges. If $p = 1$, then

$$\int_2^n \frac{1}{x \log x} = \int_{\log 2}^{\log n} \frac{1}{u} \mathrm{d}u = \log(\log n) - \log(\log 2) \xrightarrow{n\to\infty} \infty$$

Hence the series diverges by the integral test. Assume that $0 < p < 1$. Then by the first case of $p > 1$,

$$\begin{aligned}
\int_2^n \frac{1}{x(\log x)^p} \mathrm{d}x &= \frac{1}{p-1}\left(\frac{1}{(\log 2)^{p-1}} - \frac{1}{(\log p)^{p-1}}\right) \\
&= \frac{1}{1-p}\left((\log n)^{1-p} - (\log 2)^{1-p}\right) \xrightarrow{n\to\infty} \infty
\end{aligned}$$

Therefore the series diverges by the integral test. If $p \leq 0$, then the series terms do not converge to 0 and therefore the series does not converge. ■

15.5

It wouldn't be useful to use the comparison test as it would require using an exponent larger than p to compare with, which is a part of the result that is trying to be proven.

15.7

Part A

Proof. Let (a_n) be a sequence and assume that it is decreasing and that $\sum a_n$ converges. Note that this means $a_n > 0$ for all n and that $a_n \to 0$. Let $\epsilon > 0$. Since $\sum a_n$ converges, by the Cauchy criterion there exists $M \in \mathbb{N}$ such that for $n > M$

$$a_{M+1} + \dots a_n < \frac{\epsilon}{2}.$$

Since (a_n) converges, there is some $P \in \mathbb{N}$ such that $n \geq P$ implies $a_n < \frac{\epsilon}{2M}$. Let $N = \max\{M, P\}$. Then for $n > N$

$$\begin{aligned} n \cdot a_n &= \underbrace{a_n + \dots + a_n}_{n \text{ times}} \\ &\leq \underbrace{a_P + \dots + a_P}_{m \text{ times}} + \underbrace{a_{M+1} + \dots + a_n}_{n-m \text{ times}} \\ &< M a_P + \frac{\epsilon}{2} \\ &< M \cdot \frac{\epsilon}{2M} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

Therefore $\lim n a_n = 0$. ■

17.1

Part A

$$\begin{aligned} \mathrm{dom}(f+g) &\implies (-\infty, 4] \\ \mathrm{dom}(fg) &\implies (-\infty, 4] \\ \mathrm{dom}(f \circ g) &\implies [-2, 2] \\ \mathrm{dom}(g \circ f) &= (-\infty, 4] \end{aligned}$$

Part B

$$(f \circ g)(0) = 2$$
$$(g \circ f)(0) = 4$$
$$(f \circ g)(1) = \sqrt{3}$$
$$(g \circ f)(1) = 3$$
$$(f \circ g)(2) = 0$$
$$(g \circ f)(2) = 2$$

Part C

The functions are not equal.

Part D

Only $(g \circ f)(3)$ is meaningful as the x value is in its domain.

17.2

Part A

$$(f+g)(x) = \begin{cases} x^2 & x < 0 \\ 4 + x^2 & x \geq 0 \end{cases}$$
$$(fg)(x) = \begin{cases} 0 & x < 0 \\ 4x^2 & x \geq 0 \end{cases}$$
$$(f \circ g)(x) = 4, \forall x \in \mathbb{R}$$
$$(g \circ f)(x) = \begin{cases} 0 & x < 0 \\ 16 & x \geq 0 \end{cases}$$

Part B

Only g, fg and $f \circ g$ are continuous.

17.5

Proof. Consider the real valued function $f(x) = x^m$ on all of $\mathbb{R}$ where $m \in \mathbb{N}$. Let (x_n) be a sequence in $\mathbb{R}$ that converges to $x_0 \in \mathbb{R}$. Note then that $\lim f(x_n) = \lim(x_n)^m = (\lim x_n)^m = x_0^m = f(x_0)$. Therefore $f(x)$ is continuous at x_0 and hence on all of $\mathbb{R}$. ■

17.6

Proof. Let $p(x)$ and $q(x)$ be real polynomials and consider the domain $D = \{x \in \mathbb{R} : q(x) \neq 0\}$. Let (x_n) be a sequence in D that converges to $x_0 \in D$. Note then that

$$\lim \frac{p(x_n)}{q(x_n)} \xRightarrow{q(x)\neq 0} \frac{\lim p(x_n)}{\lim q(x_n)} = \frac{p(x_0)}{q(x_0)}.$$

Therefore the rational function $\frac{p}{q}$ on the domain D is continuous. ■