Problem 1

Proof. We can first rewrite the norm in terms of the inner product,

$$\begin{aligned}\left\|\sum_i a_i v_i\right\|^2 &= \left(\sqrt{\left\langle \sum_i a_i v_i, \sum_i a_i v_i \right\rangle}\right)^2 \\ &= \left\langle \sum_i a_i v_i, \sum_i a_i v_i \right\rangle\end{aligned}$$

Since the inner product is linear, we can pass the sums through

$$\begin{aligned}&= \sum_i \left\langle a_i v_i, \sum_j a_j v_j \right\rangle \\ &= \sum_i a_i \sum_j \langle v_i, a_j v_j \rangle\end{aligned}$$

S is orthogonal as well meaning $\langle v_i, v_j \rangle = 0$ if $i = j$, hence

$$\begin{aligned}&= \sum_i a_i \langle v_i, a_i v_i \rangle \\ &= \sum_i a_i \overline{a_i} \langle v_i, v_i \rangle \\ &= \sum_i |a_i|^2 \|v_i\|^2\end{aligned}$$

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Problem 2

Proof. Since we are dealing with a norm induced by an inner product, we can rewrite the left hand side as

$$\|x + y\|^2 + \|x - y\|^2 = \langle x + y, x + y \rangle + \langle x - y, x - y \rangle$$

We can use the linearity of the inner product to split apart the terms,

$$= \langle x, x + y \rangle + \langle y, x + y \rangle + \langle x, x - y \rangle + \langle -y, x - y \rangle$$

and the -1 attached to $-y$ can be moved into the second position since $\overline{-1} = -1$,

$$\begin{aligned}&= \langle x, x + y \rangle + \langle y, x + y \rangle + \langle x, x - y \rangle + \langle y, y - x \rangle \\ &= \langle x, (x + y) + (x - y) \rangle + \langle y, (x + y) + (y - x) \rangle \\ &= \langle x, 2x \rangle + \langle y, 2y \rangle \\ &= 2 \langle x, x \rangle + 2 \langle y, y \rangle \\ &= 2\|x\|^2 + 2\|y\|^2.\end{aligned}$$

Hence the equality holds. ■

Problem 3

No, it does not hold in general. Consider $\mathbb{R}^2$ with the infinity norm $\|x\|_\infty = \max_i |x_i|$. Then if $x = (1,0)$ and $y = (0,1)$ we have

$$\|x+y\|_\infty^2 + \|x-y\|_\infty^2 = 1^2 + 1^2 = 2 \neq 4 = 2 + 2 = 2\|x\|_\infty^2 + 2\|y\|_\infty^2.$$

Problem 4

Proof. Let $y \in V$ and

$$y_\parallel := \sum_{i=1}^{k} \langle y, w_i \rangle w_i.$$

We will show that

$$\operatorname*{argmin}_{x \in W} \|y - x\| = y_\parallel$$

Let $z = y - y_\parallel$. Then for any $w_j \in S$

$$\begin{aligned}
\left\langle z, w_j \right\rangle &= \left\langle \left(y - \sum_{i=1}^{k} \langle y, w_i \rangle w_i \right), w_j \right\rangle \\
&= \left\langle y, w_j \right\rangle - \left\langle \sum_{i=1}^{k} \langle y, w_i \rangle w_i, w_j \right\rangle \\
&= \left\langle y, w_j \right\rangle - \sum_{i=1}^{k} \left\langle \langle y, w_i \rangle w_i, w_j \right\rangle \\
&= \left\langle y, w_j \right\rangle - \sum_{i=1}^{k} \langle y, w_i \rangle \left\langle w_i, w_j \right\rangle.
\end{aligned}$$

Since S is orthonormal, we have $\left\langle w_i, w_j \right\rangle = \delta_{ij}$ and

$$\left\langle z, w_j \right\rangle = \left\langle y, w_j \right\rangle - \sum_{i=1}^{k} \langle y, w_i \rangle \left\langle w_i, w_j \right\rangle = \left\langle y, w_j \right\rangle - \left\langle y, w_j \right\rangle = 0.$$

Therefore $z \in W^\perp$. Now consider some $x \in W$. Since $y_\parallel$ is defined as a linear combination of vectors from W, it must also be in W meaning $y_\parallel - x \in W$. Then $y_\parallel - x$ is orthogonal to z. With this orthogonality,

$$\begin{aligned}
\|y - x\|^2 &= \left\| y_\parallel + z - x \right\|^2 \\
&= \left\| (y_\parallel - x) + z \right\| \\
&= \left\| y_\parallel - x \right\|^2 + \|z\|^2 \geq \|z\|^2 = \left\| y - y_\parallel \right\|.
\end{aligned}$$

Reading the extreme ends of both sides and taking the square root gives $\|y - x\| \geq \left\| y - y_\parallel \right\|$. Thus any choice of x will give a value greater than or equal to the proposed $y_\parallel$. Therefore we know that at least $y_\parallel$ is in the argmin. If $\|y - x\| = \left\| y - y_\parallel \right\|$ then we have from the previous derivation equality and

$$\left\| y_\parallel - x \right\|^2 + \|x\|^2 = \|x\|^2 \implies \left\| y_\parallel - x \right\| = 0.$$

But this means $y_\parallel = x$. Therefore we have $y_\parallel$ as the unique vector in the argmin. ■

Problem 5

Proof. Let $x \in R(T^*)^\perp$. Then for any $y = T^*y' \in R(T^*)'$ with $y' \in V_1$ it follows

$$0 = \langle x, y \rangle = \langle x, T^*y' \rangle = \langle Tx, y' \rangle .$$

Therefore Tx must be zero since it holds for any y' meaning $x \in N(T)$. Now let $x \in N(T)$. Then

$$0 = \langle 0, y \rangle = \langle Tx, y \rangle = \langle x, T^*y \rangle$$

for any $y \in V_1$. Therefore $x \in R(T^*)^\perp$, hence $R(T^*)^\perp = N(T)$. Since V_1 and V_2 are finite dimensional, then we have $(W^\perp)^\perp = W$ for any subspace W in both spaces which gives $N(T)^\perp = (R(T^*)^\perp)^\perp = R(T^*)$. ■