Problem 1

Let X denote the number of 6's that appear in 3 rolls. Then $X \sim \text{Binom}(3, \frac{1}{6})$. Therefore

$$\begin{aligned}\mathbb{P}(X \leq 1) &= \mathbb{P}(X = 0) + \mathbb{P}(X = 1) \\ &= \binom{3}{0}\left(\frac{5}{6}\right)^3 + \binom{3}{1}\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^2 \\ &= \frac{25}{27}.\end{aligned}$$

Problem 2

Since there is replacement, each drawing is independent. Let X denote the number of white balls drawn after 4 drawings. Then $X \sim \text{Binom}(4, \frac{1}{2})$. Therefore

$$\begin{aligned}\mathbb{P}(X = 2) &= \binom{4}{2} \cdot \frac{1}{2}^4 \\ &= 6 \cdot \frac{1}{16} = \frac{3}{8}.\end{aligned}$$

Problem 3

$$\begin{aligned}\mathbb{E}[X] &= 25\left(\frac{25}{148}\right) + 33\left(\frac{33}{148}\right) + 40\left(\frac{40}{148}\right) + 50\left(\frac{50}{148}\right) \\ &= \frac{2907}{74} \approx 39.28\end{aligned}$$

$$\begin{aligned}\mathbb{E}[Y] &= \frac{1}{4} \cdot (25 + 33 + 40 + 50) \\ &= \frac{148}{4} = 37\end{aligned}$$

Problem 4

Let I denote the revenue the company makes. Let X denote the profit the company makes. Note that $X = \{I, I - A\}$ with $\mathbb{P}(X = I - A) = p$ and $\mathbb{P}(X = I) = p$. Therefore the expected profit is

$$\mathbb{E}[X] = I(1 - p) + (I - A)p = I - pA.$$

Therefore since the company wants their expected profit to be 10% of A,

$$\begin{aligned}
\mathbb{E}[X] &= \frac{A}{10} \\
I - pA &= \frac{A}{10} \\
I &= \frac{A}{10} + pA \implies \boxed{I = A\left(p + \frac{1}{10}\right)}.
\end{aligned}$$

Problem 5

Let $X_1 \sim \text{Bern}(0.6)$ represent the first flip and $X_2 \sim \text{Bern}(0.7)$ represent the second flip. Then $X = X_1 + X_2$. Therefore

$$\begin{aligned}
\mathbb{P}(X = 1) &= \mathbb{P}(X_1 = 1, X_2 = 0) + \mathbb{P}(X_1 = 0, X_2 = 1) \\
&= (0.6)(1 - 0.7) + (0.7)(1 - 0.6) \\
&= 0.18 + 0.28 = 0.46.
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}[X] &= \mathbb{E}[X_1 + X_2] \\
&= \mathbb{E}[X_1] + \mathbb{E}[X_2] \\
&= 0.6 + 0.7 = 1.3.
\end{aligned}$$

Problem 6

The probability a tails appears on the n^{th} flip is $\left(\frac{1}{2}\right)^n$, therefore the expected value is

$$\begin{aligned}
\mathbb{E}[X] &= \sum_{n=1}^{\infty} 2^n \cdot \left(\frac{1}{2}\right)^n \\
&= \sum_{n=1}^{\infty} 2^n \cdot \frac{1}{2^n} \\
&= \sum_{n=1}^{\infty} 1 \to +\infty.
\end{aligned}$$

For (1), no because to get a net positive amount back one would have to flip 19 heads in a row which has a probability of 0.0000019073, meaning it's very unlikely one would recover their million dollars. However, for (2), it would advanteagous to pay a million because the expectation is infinite, meaning eventually after enough tries one could easily make more than a million dollars by playing the game contiuonsly.

Problem 7

Let X denote the winnings. Then

$$\mathbb{E}[X] = -1 \cdot \left(2 \cdot \frac{\binom{5}{2}}{\binom{10}{2}}\right) + 1.1 \cdot \left(2 \cdot \frac{\binom{5}{1}}{\binom{10}{2}}\right) = -0.2.$$

and

$$\text{Var}(X) = (-1)^2 \cdot \left(2 \cdot \frac{\binom{5}{2}}{\binom{10}{2}}\right) + (1.1)^2 \cdot \left(2 \cdot \frac{\binom{5}{1}}{\binom{10}{2}}\right) + 0.2 = 0.913.$$

Problem 8

Let $D \sim \text{Binom}\left(10, \frac{1}{3}\right)$ denote the daily demand. Let n denote the number of papers he buys and X_n denote the associated profit. If $D > n$, then $X = 0.05n$. If $D \leq n$, then $X = 0.04D - 0.10(n - X)$. Therefore

$$\mathbb{E}[X_n] = \sum_{i=0}^{n} (0.05i - 0.10(n-i))\mathbb{P}(D = i) + \sum_{k=n+1}^{10} (0.05n)\mathbb{P}(D = k).$$

Calculating $\mathbb{E}[X_n]$ for $0 \leq n \leq 10$ is shown in the table. Therefore the profit maximizing

0	1	2	3	4	5	6	7	8	9	10
0.000	0.047	0.082	0.087	0.053	-0.015	-0.100	-0.200	-0.300	-0.400	-0.500

amount of papers the boy should buy is 3.

Problem 9

Note that $\text{Var}(X) = \mathbb{E}\left[X^2\right] - \mathbb{E}[X]^2 \implies \mathbb{E}\left[X^2\right] = \text{Var}(X) + \mathbb{E}[X]^2$.

(1)

$$\begin{aligned}
\mathbb{E}\left[(2 + X^2)\right] &= \mathbb{E}\left[X^2 + 4X + 4\right] \\
&= \mathbb{E}\left[X^2\right] + 4 \cdot \mathbb{E}[X] + 4 \\
&= \text{Var}(X) + \mathbb{E}[X]^2 + 4 \cdot \mathbb{E}[X] + 4 \\
&= 5 + 1^2 + 4 \cdot 1 + 4 \\
&= 14.
\end{aligned}$$

(2)

$$\begin{aligned}
\text{Var}(4 + 3X) &= \text{Var}(3X) \\
&= 3^2 \cdot \text{Var}(X) \\
&= 9 \cdot 5 \\
&= 45.
\end{aligned}$$

Problem 10

Consider the case where $i = 2$. Let X_2 denote the number of games played before a team wins 2 times. Note that $\mathcal{R}_{X_2} = \{2, 3\}$ with the following possible game scenarios

$$\begin{array}{ll} AA & ABA \\ BB & BAA \\ & BAB \\ & ABB \end{array}$$

Therefore

$$\begin{aligned} \mathbb{E}[X_2] &= 2 \cdot \left[p^2 + (1-p)^2\right] + 3 \cdot \left[2p^2(1-p) + 2(1-p)^2 p\right] \\ &= 2 + 2p - 2p^2. \end{aligned}$$

Consider the case where $i = 3$. Let X_3 denote the number of games played before a teame wins 3 times. Note that $\mathcal{R}_{X_3} = \{3, 4, 5\}$ with the following possible game scenarios

$$\begin{array}{lll} AAA & AABA & BBAAA \\ BBB & ABAA & BABAA \\ & BAAA & BAABA \\ & BBAB & ABABA \\ & BABB & AABBA \\ & ABBB & AABBB \\ & & ABABB \\ & & ABBAB \\ & & BABAB \\ & & BBAAB \end{array}$$

Therefore

$$\begin{aligned} \mathbb{E}[E_3] &= 3 \cdot \left[p^3 + (1-p)^3\right] + 4 \cdot \left[3p^3(1-p) + 3p(1-p)^3\right] + 5 \cdot \left[4p^3(1-p)^2 + 4p^2(1-p)^3\right] \\ &= 3 + 3p - 7p^2 + 8p^3 - 4p^4. \end{aligned}$$

Consider the derivatives of each expectation.

$$\begin{aligned} \frac{\mathrm{d}\mathbb{E}[X_2]}{\mathrm{d}p} &= 2 - 4p = 0 \implies p = \frac{1}{2} \\ \frac{\mathrm{d}\mathbb{E}[X_3]}{\mathrm{d}p} &= 3 - 14p + 24p^2 - 15p^3 = 0 \implies p = \frac{1}{2}. \end{aligned}$$

Since both are concave down and $p = \frac{1}{2}$ is the only critical point for both, the expectations are maximized by $p = \frac{1}{2}$.