Sequences and Series of Functions

1.1 Power Series

Definition 1.1 (Power Series). A power series is a real valued function $f(x) = \sum a_n x^n$ for some sequence (a_n).

Theorem 1.1. For a power series $\sum a_n x^n$, let $\beta = \limsup |a_n|^{\frac{1}{n}}$ and $R = \frac{1}{\beta}$. The power series converges for $|x| < R$ and diverges for $|x| > R$

Proof. Apply the root test. Then

$$\limsup |c_n|^{\frac{1}{n}} = \limsup |a_n|^{\frac{1}{n}}|x| = \limsup |a_n|^{\frac{1}{n}}|x| = |x|\beta.$$

Note then that $|x| < R = \frac{1}{\beta}$ means that $\limsup |c_n|^{\frac{1}{n}} < 1$ and therefore the series converges. The opposite is true for $|x| > R$. ■

Example 1.1. Consider $\sum x^n$. Note that $a_n = 1$ for all $n \in \mathbb{N}$. Therefore $\limsup |a_n|^{\frac{1}{n}} = \limsup 1^{\frac{1}{n}} = 1$. Therefore the power series converges for all $|x| < 1$. Note that $x = 1$ gives a divergent series and $x = -1$ gives an alternating series whose non alternative part does not go to zero and hence also diverges.

Example 1.2. Consider $\sum \frac{x^n}{n!}$. In this instance $a_n = \frac{1}{n!}$. Then

$$\beta = \limsup |a_n|^{\frac{1}{n}} = \limsup \left|\frac{1}{n!}\right|^{\frac{1}{n}}.$$

This would be hard to compute. However, if this limit exists, then it matches the value of the ratio test and therefore

$$\beta = \limsup \left|\frac{a_{n+1}}{a_n}\right| = \limsup \frac{1}{(n+1)!} \cdot \frac{n!}{1} = \limsup \frac{1}{n} = 0.$$

Therefore $R = +\infty$ meaning the interval of convergence is all of $\mathbb{R}$.

Remark. Alternatively, one can use the Sterling approximation of the factorial to do the root test. The Sterling approximation is

$$n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}.$$

Hence

$$\limsup \left|\frac{1}{n!}\right|^{\frac{1}{n}} = \limsup \frac{1}{\left(\left(\frac{n}{e}\right)^n \sqrt{2\pi n}\right)^{\frac{1}{n}}} = \limsup \frac{1}{\frac{n}{e} \cdot \left(\sqrt{2\pi n}\right)^{\frac{1}{n}}} = \limsup \frac{1}{n} = 0.$$

Example 1.3. Consider $\sum \frac{x^n}{n^2}$. Then

$$\beta = \limsup \left(\frac{1}{n^2}\right)^{\frac{1}{n}} = \limsup \frac{1}{\sqrt[n]{n^2}} = 1.$$

Therefore the power series converges for $|x| < 1$. Importantly, for $x = 1$ and $x = -1$, you get convergent series and therefore the interval of convergence is $[-1, 1]$.

Example 1.4. Consider $\sum \frac{(-1)^{n+1}x^n}{n}$. Then $a_n = \frac{(-1)^{n+1}}{n}$ and

$$\beta = \limsup \left|\frac{(-1)^{n+1}}{n}\right|^{\frac{1}{n}} = \limsup \frac{1}{\sqrt[n]{n}} = \frac{1}{1} = 1.$$

Therefore the power series converges for $|x| < 1$. Checking $x = 1$,

$$\sum \frac{(-1)^{n+1}}{n} \text{ converges by alternating series test.}$$

And checking for $x = -1$,

$$\sum \frac{(-1)^{2n+1}}{n} = \sum \frac{-1}{n} = -\sum \frac{1}{n} \text{ which diverges.}$$

Therefore the interval of convergence is $(-1, 1]$.

Example 1.5. Consider $\sum \frac{(2n)!x^n}{(n!)^2}$. Then $a_n = \frac{(2n)!}{(n!)^2}$. Apply the ratio test to get β.

$$\beta = \limsup \left|\frac{a_{n+1}}{a_n}\right| = \limsup \frac{(2n+2)!}{((n+1)!)^2} \cdot \frac{(2n)!}{(n!)^2} = \limsup \frac{(2n+2)(2n+1)}{(n+1)(n+1)} = 4.$$

Therefore it converges on $|x| < \frac{1}{4}$. Checking the endpoints suck but $x = \frac{1}{4}$ diverges by using Sterlings approximation and $x = -\frac{1}{4}$ converges by the alternating series test by the previous method. Therefore the interval of convergence is $\left[-\frac{1}{4}, \frac{1}{4}\right)$.

1.2 Uniform Convergence

An initial, but weak, formulation of functional sequence convergence is by applying the a basic limit of a sequence.

Definition 1.2 (Pointwise Convergence). A sequence of real value functions $f_n : S \subset \mathbb{R} \to \mathbb{R}$ converges point wise to a function f on S if $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in S$

Definition 1.3 (Uniform Convergence). A sequence of real value functions $f_n : S \subset \mathbb{R} \to \mathbb{R}$ uniformly converges to a function f on S if $\forall \epsilon > 0$, there is some $N \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \epsilon, n > N, \forall x \in S.$$

Example 1.6. Consider the sequence of functions $f_n(x) = x^n$ on $[0,1]$. Note that for all n, $f_n(0) = 0$ and $f_n(1) = 1$. Furthermore, for $0 < x < 1$, $\lim x^n = 0$. Therefore

$$\lim f_n(x) = f(x) = \begin{cases} 0 & 0 \le x < 1 \\ 1 & x = 1 \end{cases}.$$

is the pointwise limit of the sequence. For uniform convergence, we want

$$|f_n(x) - f(x)| < \epsilon \Leftrightarrow \left| x^n - \begin{cases} 0 & 0 \le x < 1 \\ 1 & x = 1 \end{cases} \right| < \epsilon.$$

For $x = 1$, the absolute value goes to 0 and therefore only $0 \le x < 1$ matters. The question becomes when

$$x^n < \epsilon \implies n > \frac{\ln(\epsilon)}{\ln|x|}.$$

However, it is not possible to bound this quantity since $x \to 1$ leads to $\frac{1}{\ln|x|} \to -\infty$. Therefore the sequence does not uniformly converge to f.

Example 1.7. Let $g_n(x) = (1-|x|)^n$ on $(-1,1)$. Note that $\lim g_n(0) = 1$ since $g_n(0) = 1$ for all n. For any other x, $|x| < 1$ and therefore $1-|x| < 1$. Hence $\lim g_n(x) = 0$ for $x \neq 0$. Hence

$$\lim g_n(x) = g(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}.$$

Checking for uniform convergence,

$$|g_n(x) - g(x)| < \epsilon \Leftrightarrow |(1-|x|)^n \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}|.$$

We only have to care about $x \neq 0$, therefore

$$|(1-|x|)^n| < \epsilon \implies n > \frac{\ln(\epsilon)}{\ln(1-|x|)}.$$

However, $\sup_{x\in(-1,1)} \frac{\ln(\epsilon)}{\ln(1-|x|)} = +\infty$, therefore the sequence does not uniformly converge to $g(x)$.

Example 1.8. Let $h_n(x) = \frac{1}{n}\sin(nx)$. Since $\left|\frac{1}{n}\sin(nx)\right| \le \left|\frac{1}{n}\right| = \frac{1}{n}$, it follows that

$$0 \le \lim_{n\to\infty} \left|\frac{1}{n}\sin(nx)\right| \le \lim_{n\to\infty} \frac{1}{n} = 0.$$

Therefore $\lim h_n(x) = 0$. Checking for uniform convergence, we want

$$|h_n(x) - h(x)| < \epsilon \Leftrightarrow \left|\frac{1}{n}\sin(nx) - 0\right| < \epsilon.$$

Since $\left|\frac{1}{n}\sin(nx)\right| \le \frac{1}{n}$, choosing $n > \frac{1}{\epsilon}$ gives the desired inequality. Since the bound for n doesnt depend on x, the sequence uniformly converges to $h(x) = 0$.

Example 1.9. Let $j_n(x) = \frac{nx}{2n+1}$ on $S = [-2, 2]$. It's pointwise limit is

$$\lim j_n(x) = \lim \frac{nx}{2n+1} = x \lim \frac{n}{2n+1} = \frac{x}{2} = j(x).$$

Checking for uniform convergence, we want

$$\begin{aligned}\left|\frac{nx}{2n+1} - \frac{x}{2}\right| < \epsilon &\implies \left|\frac{2nx - (2n+1)x}{2(2n+1)}\right| < \epsilon \\ &\implies \frac{|x|}{2(2n+1)} < \epsilon \\ &\implies \frac{|x|}{2\epsilon} < 2n+1 \\ &\implies n > \frac{|x|}{4\epsilon} - \frac{1}{2}\end{aligned}$$

Since $|x| < 2$, $n > \frac{1}{2\epsilon} - \frac{1}{2} > \frac{|x|}{4\epsilon} - \frac{1}{2}$ gives the original inequality. Therefore the sequence uniformly converges to $j(x)$.

Example 1.10. Let

$$k_n(x) = \begin{cases} 1 & x > \frac{1}{n} \\ 0 & x \le \frac{1}{n} \end{cases}$$

on $S = [0, 1]$. Note that $0 \le \frac{1}{n}$ for all n, meaning $\lim k_n(0) = 0$. For similar reasoning $1 \ge \frac{1}{n}$ for all $n > 1$ and therefore $\lim k_n(1) = 1$. For any $0 < x < 1$, there will be some $N \in \mathbb{N}$ such that $n > N \implies \frac{1}{n} < x$. Hence $\lim k_n(x) = 1$ for all $0 < x < 1$. In total then, the pointwise convergence is

$$k(x) = \begin{cases} 0 & x = 0 \\ 1 & x \ne 0 \end{cases}.$$

Checking for uniform convergence, we want

$$|k_n(x) - k(x)| < \epsilon \implies \left|\begin{cases} 1 & x > \frac{1}{n} \\ 0 & x \le \frac{1}{n} \end{cases} - \begin{cases} 0 & x = 0 \\ 1 & x \ne 0 \end{cases}\right| = \left|\begin{cases} 0 - 0 & x = 0 \\ 0 - 1 & 0 < x \le \frac{1}{n} \\ 1 - 1 & \frac{1}{n} < x \le 1 \end{cases}\right| < \epsilon.$$

Note then that

$$\left|\begin{cases} 0-0 & x=0 \\ 0-1 & 0<x\le \frac{1}{n} \\ 1-1 & \frac{1}{n}<x\le 1 \end{cases}\right| = \begin{cases} 0 & x=0, \frac{1}{n}<x\le 1 \\ 1 & 0<x\le \frac{1}{n} \end{cases}.$$

Since $0 < x \le \frac{1}{n}$ the value is 1, it is not possible to get arbitrarily close to the pointwise convergence across all x.

Theorem 1.2. A sequence of functions f_n uniformly converges to f on $S \subset \mathbb{R}$ iff

$$\lim_{n\to\infty} \sup_{x\in S} \{f_n(x) - f(x)\}.$$

Theorem 1.3. If $f_n \to f$ uniformly on $[a, b]$ and f_n is continuous on $[a, b]$ for all n, then

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx.$$

Proof. We want to show that

$$\forall \epsilon > 0, \exists N, \text{ s.t. } \left|\int_a^b f_n(x)\mathrm{d}x - \int_a^b f(x)dx\right| < \epsilon.$$

Fix $\epsilon > 0$. Then

$$\begin{aligned} \left|\int_a^b f_n(x)\mathrm{d}x - \int_a^b f(x)dx\right| &= \left|\int_a^b f_n(x) - f(x)\mathrm{d}x\right| \\ &\le \int_a^b |f_n(x) - f(x)|\mathrm{d}x \end{aligned}$$

Since $f_n \to f$ uniformly on $[a, b]$, there is a N such that $|f_n(x) - f(x)| < \frac{\epsilon}{b-a}$ for $n > N$ and $x \in [a, b]$. Note then that

$$\int_a^b |f_n(x) - f(x)|\mathrm{d}x < \int_a^b \frac{\epsilon}{b-a}\mathrm{d}x = \epsilon.$$

Therefore for $n > N$,

$$\left|\int_a^b f_n(x)\mathrm{d}x - \int_a^b f(x)dx\right| \le \int_a^b |f_n(x) - f(x)|\mathrm{d}x < \epsilon.$$

■

1.3 Cauchy Function Sequences

Definition 1.4 (Uniformly Cauchy). A sequence of real valued functions f_n is called unfiormly Cauchy if

$$\forall \epsilon > 0, \exists N \text{ s.t. } |f_n(x) - f_m(x)| < \epsilon, \forall x \in S, n > m > N.$$

Theorem 1.4. If a sequence of real valued functions f_n is uniformly Cauchy on $S \subset \mathbb{R}$, then there exists some function $f(x)$ on S such that $f_n \to f$ uniformly on S.

Proof. Fix $x \in S$ and let $y_n = f_n(x)$. Note that this gives a Cauchy sequence since f_n is uniformly Cauchy. Therefore y_n converges to some $y \in \mathbb{R}$. Define $F(x) = y$. By construction, $f_n \to F$ pointwise. Fix $\epsilon > 0$. Since f_n is uniformly Cauchy

$$|f_n(x) - f_m(x)| < \frac{\epsilon}{2} \implies -\frac{\epsilon}{2} + f_m(x) < f_n(x) < f_m(x) + \frac{\epsilon}{2}.$$

Since $n > m$, n can be sent to infinity while fixing m, giving

$$-\frac{\epsilon}{2} + f_m(x) < F(x) < f_m(x) + \frac{\epsilon}{2} \implies -\frac{\epsilon}{2} < F(x) - f_m(x) < \frac{\epsilon}{2}.$$

Therefore

$$|f_m(x) - F(x)| < \frac{\epsilon}{2} < \epsilon.$$

Therefore f_m converges uniformly to F on S. ∎

Example 1.11. Consider the series $f_n(x) = \displaystyle\sum_{k=0}^{n} \frac{1}{1+x^k}$ on $[2, \infty)$. Trying to determine if this uniformy converges with a direct approach will not work as it requires knowledge about the function the infinite series represents. However, notice that for $n > m$

$$
\begin{aligned}
|f_n(x) - f_m(x)| &= \left|\sum_{k=0}^{n} \frac{1}{1+x^k} - \sum_{j=0}^{m} \frac{1}{1+x^j}\right| \\
&= \left|\sum_{k=m+1}^{n} \frac{1}{1+x^k}\right| \\
&\leq \left|\sum_{k=m+1}^{n} \frac{1}{1+2^k}\right| \\
&\leq \left|\sum_{k=m+1}^{n} \frac{1}{2^k}\right| \\
&= \frac{\frac{1}{2^{m+1}} - \frac{1}{2^n+1}}{1-\frac{1}{2}} \\
&= \frac{1}{2^m} - \frac{1}{2^n} < \frac{1}{2^m}
\end{aligned}
$$

Take $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that $\frac{1}{2^N} < \epsilon$. Then

$$|f_n(x) - f_m(x)| < \frac{1}{2^m} < \frac{1}{2^N} < \epsilon, n > m > N.$$

Therefore f_n is uniformly Cauchy on $[2, \infty)$. This means that $f_n \to f$ uniformly on $[2, \infty)$.

Example 1.12. Consider a power series $f(x) = \sum_{k=0}^{\infty} a_k x^k$ with the sequence of polynomials $f_n(x) = \sum_{k=0}^{n} a_k x^k$. Let $\beta = \limsup |a_n|^{\frac{1}{n}}$ and $R = \frac{1}{\beta}$. Consider some $0 < \tilde{R} < R$ and

$S = (-\tilde{R}, \tilde{R})$. Then

$$\begin{aligned} |f_n(x) - f_m(x)| &= \left| \sum_{k=m+1}^{n} a_k x^k \right| \\ &\leq \sum_{k=m+1}^{n} \left| a_k x^k \right| \\ &= \sum_{k=m+1}^{n} \left(|a_k|^{\frac{1}{k}} |x| \right)^k \\ &\leq \sum_{k=m+1}^{n} \left(|a_k|^{\frac{1}{k}} \tilde{R} \right)^k \end{aligned}$$

Since $\limsup |a_n|^{\frac{1}{n}} = \beta = \frac{1}{R}$, it is possible to find some $K \in \mathbb{N}$ such that

$$||a_k|^{\frac{1}{k}} - \beta| < \epsilon_1 \text{ with } (\beta + \epsilon_1)\tilde{R} < 1, k > K.$$

Let $\alpha = (\beta + \epsilon_1)\tilde{R} < 1$. Then

$$\sum_{k=m+1}^{n} \left(|a_k|^{\frac{1}{k}} \tilde{R} \right)^k < \sum_{k=m+1}^{n} \alpha^k \leq \frac{\alpha^{m+1}}{1 - \alpha} < \epsilon.$$

with $n > m > K$. This means that f_n is uniformly Cauchy and hence uniformly converges to f in the interval $(-\tilde{R}, \tilde{R})$. This result means that many useful properties about unfirom convergence apply to the interior of the interval of convergence.

1.4 Differentiation and Integration of Power Series

Theorem 1.5 (Continuity of Power Series). If $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is a power series with radius of convergence $R > 0$, then for any $0 < R' < R$ $f_n = \sum_{k=0}^{n} a_k x^k$ converges uniformly and f is continuous on $S = [-R', R']$.

Proof. Let $f_n(x) = \sum_{k=0}^{n} g_k(x)$ with $g_k(x) = a_k x^k$. Applying the M test gives

$$M_k = \sup_{x \in S} |g_k(x)| = |a_k|(R')^k.$$

Appying the root test to the series $\sum |a_k|(R')^k$,

$$\lim_{k\to\infty} |M_k|^{\frac{1}{k}} = \lim_{k\to\infty} |a_k|^{\frac{1}{k}} R' \leq \beta R' = \frac{1}{R} R' < 1.$$

Therefore this series converges and hence f_n converges uniformly on S by the M-test. This means that f itself is also continuous on S. ■

Corollary 1.1. If $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has a radius of convergence $R > 0$, then f is continuous on $(-R, R)$.

Example 1.13. Consider $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2 2^n}$. Applying the root test gives

$$\beta = \limsup |a_n|^{\frac{1}{n}} = \limsup \left|\frac{1}{n^2 2^n}\right|^{\frac{1}{n}} = \limsup \frac{1}{2\sqrt[n]{n^2}} = \frac{1}{2}.$$

Therefore $R = \frac{1}{\beta} = 2$ meaning f is continuous on $(-2, 2)$. Note that

$$f(2) = \sum_{n=1}^{\infty} \frac{2^n}{n^2 2^n} = \sum_{n=1}^{\infty} \frac{1}{n^2} \qquad \Longrightarrow \text{Converges}$$
$$f(-2) = \sum_{n=1}^{\infty} \frac{(-2)^n}{n^2 2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \quad \Longrightarrow \text{Converges}$$

Therefore the interval of convergence is $[-2, 2]$. Important to note that there is no guarantee about continuity at the endpoints even though convergence is established.

Example 1.14. Consider $g(x) = \sum_{n=0}^{\infty} 3^{-n} x^n$. Then

$$\beta = \limsup |a_n|^{\frac{1}{n}} = \limsup |3^{-n}|^{\frac{1}{n}} = \frac{1}{3}.$$

Therefore $R = \frac{1}{\beta} = 3$ and hence g is continuous on $(-3, 3)$. Note that

$$g(3) = \sum_{n=0}^{\infty} 3^{-n} 3^n = \sum_{n=0}^{\infty} 1 \qquad \Longrightarrow \text{Diverges}$$
$$g(-3) = \sum_{n=0}^{\infty} 3^{-n} (-3)^n = \sum_{n=0}^{\infty} (-1)^n \quad \Longrightarrow \text{Diverges}$$

Hence the interval of convergence is $(-3, 3)$.

Lemma 1.1. If $\sum a_n x^n$ has a radius of convergence R, then $\sum \frac{a_n}{n+1} x^{n+1}$ has a radius of convergence R.

Proof. Let $\beta = \frac{1}{R}$. The second series can be rewritten as

$$\sum \frac{a_n}{n+1} x^{n+1} = x \sum \frac{a_n}{n+1} x^n = x \sum b_n x^n.$$

Applying the root test to this new series gives

$$\tilde{\beta} = \limsup |b_n|^{\frac{1}{n}} = \limsup \left|\frac{a_n}{n+1}\right|^{\frac{1}{n}} = \frac{\limsup |a_n|^{\frac{1}{n}}}{\limsup |n+1|^{\frac{1}{n}}} = \frac{\beta}{1} = \beta.$$

Therefore $\tilde{R} = \frac{1}{\tilde{\beta}} = \frac{1}{\beta} = R$. ■

Theorem 1.6. If $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has radius of convergence R, then

$$\int_0^x f(t) dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$$

for all $|x| < R$.

Proof. First note that

$$\int_0^x f(t) \mathrm{d}t = \int_0^x \sum_{k=0}^{\infty} a_k t^k \mathrm{d}t = \int_0^x \lim_{n\to\infty} \sum_{k=0}^{n} a_k t^k \mathrm{d}t.$$

For $|x| < R$, the power series will uniformly converge so the limit can be swapped giving

$$\begin{aligned}\int_0^x \lim_{n\to\infty} \sum_{k=0}^{n} a_k t^k \mathrm{d}t = \lim_{n\to\infty} \int_0^x \sum_{k=0}^{n} a_k t^k \mathrm{d}t = \sum_{k=0}^{n} \int_0^x a_k t^k \mathrm{d}t \\ = \lim_{n\to\infty} \sum_{k=0}^{\infty} a_k \cdot \frac{x^{k+1}}{k+1} \\ = \sum_{k=0}^{\infty} \frac{a_k}{k+1} x^{k+1}\end{aligned}$$

■

Lemma 1.2. If $\sum a_n x^n$ has a radius of converge R, then $\sum n a_n x^{n-1}$ has a radius of convergence R.

Proof. Let $b_n = (n+1)a_{n+1}$ and $\beta = \frac{1}{R}$. Then

$$\begin{aligned}\tilde{\beta} = \limsup |b_n|^{\frac{1}{n}} &= \limsup |(n+1)a_n + 1|^{\frac{1}{n}} \\ &= \lim(n+1)^{\frac{1}{n}} \limsup |a_n + 1|^{\frac{1}{n}} \\ &= 1 \cdot \limsup \left(|a_{n+1}|^{\frac{1}{n}}\right)^{\frac{n+1}{n}} \\ &= \limsup |a_{n+1}|^{\frac{1}{n+1}} \cdot \limsup \left(|a_{n+1}|^{\frac{1}{n+1}}\right)^{\frac{1}{n}} = \beta \cdot 1 = \beta\end{aligned}$$

Therefore their radius of convergence are the same. ■

Theorem 1.7. If $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has radius of convergence R, the f is differentiable on $(-R, R)$ and $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$.

Proof. Let $g(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$. Then

$$G(x) = \int_0^x g(t)\mathrm{d}t = \int_0^x \sum_{n=1}^{\infty} n a_n t^{n-1}\mathrm{d}t = \sum_{n=1}^{\infty} n a_n \cdot \frac{x^n}{n} = \sum_{n=1}^{\infty} a_n x^n = f(x) - f(0).$$

Therefore $G'(x) = g(x) \implies f'(x) = g(x)$ for all $|x| < R$. ■