MATH 3D Notes

Collection of type setted notes from fall quarter of 2022
Math

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Week 1

1.1 Existence

Theorem 1.1 ▶ Picard's Theorem

If $y' = f(x_0, y_0)$ is continuous in x and y, and $\partial_y f$ exists and is continuous around (x_0, y_0) then:

$$\begin{cases} y' &= f(x,y) \\ y(x_0) &= y_0 \end{cases}$$

Picard's Theorem doesn't necessarily guarantee a global solution; only a local solution for some ϵ distance away from the input x_0.

1.2 Separability

$$\text{When } f(x,y) = \underbrace{h(x)g(y)}_{\text{Can be easily integrated}}$$

For the general case:

$$\begin{aligned} y' = f(x_0, y_0) &= h(x)g(y) \\ y' &= h(x)g(y) \\ \frac{y'}{g(y)} &= h(x) \\ \int \frac{y'}{g(y)}\mathrm{d}x &= \int h(x)\mathrm{d}x \\ \int \frac{1}{g(y)}dy &= \int h(x)\mathrm{d}x. \end{aligned}$$

Ex. Find the general solution of $y' = xy$

$$\begin{aligned} y' = xy \implies \frac{y'}{t} &= x \\ \int \frac{1}{y}dy &= \int x\mathrm{d}x \\ \ln|y| &= \frac{1}{2}x^2 + c \\ |y| &= e^{\frac{1}{2}x^2+c} \\ y &= Ae^{\frac{x^2}{2}}; A \in \mathbb{R}. \end{aligned}$$

Ex. Find the general solution of $y' = 1 - x^2 + y^2 - y^2x^2$; $y(1) = 0$

$$
\begin{aligned}
y' = (1+y^2)(1-x^2) \implies \int \frac{1}{1+y}\mathrm{d}t &= \int 1 - x^2 \mathrm{d}x \\
\arctan(y) &= x - \frac{x^3}{3} + c \\
y &= \tan\left(x - \frac{x^3}{3} + c\right)
\end{aligned}
$$

Solve with initial condition:

$$
\begin{aligned}
y(1) = 0 &= \tan\left(\frac{2}{3}\right) \\
\arctan(0) &= \frac{2}{3} + c \\
\{n\pi \,:\, n \in \mathbb{Z}\} &= \frac{2}{3} + c
\end{aligned}
$$

Therefore:

$$
\begin{aligned}
y &= \tan\left(x - \frac{x^3}{3} + c\right) \\
c &= \left\{n\pi - \frac{2}{3} \,:\, n \in \mathbb{Z}\right\}.
\end{aligned}
$$

1.3 Linear and Non-Linear ODE

Note: Linear vs Non-Linear ODE

$$
\begin{aligned}
\text{Linear} &\implies \text{Dependent variables and their derivatives appear linearly} \\
\text{Non-Linear} &\implies \text{Dependent variables and their derivatives have a power} \geq 2.
\end{aligned}
$$

1.3.1 Solving 1^{st} Order Linear ODE

Linear first order ODEs follow the form:

$$y' + p(x) \cdot y = f(x).$$

To solve such equations, utilize the **Integration Factor**:

$$r(x) = e^{\int p(x)\mathrm{d}x}.$$

Here is how the integration factor is utilized.

$$
\begin{aligned}
y' + p(x) \cdot y &= f(x) \\
y' r(x) + \underbrace{r(x) p(x)}_{\text{Equivalent to } r'(x)} \cdot y &= r(x) f(x) \\
\underbrace{y' r(x) + r'(x) \cdot y}_{\text{Use inverse product rule}} &= r(x) f(x) \\
\frac{\mathrm{d}}{\mathrm{d}x} \left[y \cdot r(x) \right] &= r(x) f(x) \\
\int \frac{\mathrm{d}}{\mathrm{d}x} \left[y \cdot r(x) \right] \mathrm{d}x &= \int r(x) f(x) \mathrm{d}x \\
y \cdot r(x) &= \int r(x) f(x) \mathrm{d}x \\
y &= \frac{1}{r(x)} \int r(x) f(x) \mathrm{d}x.
\end{aligned}
$$

Week 2

2.1 Finishing Last Lecture

From the previous lecture:

$$\frac{\mathrm{d}}{\mathrm{d}t}x + \frac{3x}{60+2t} = \frac{1}{2}.$$

How much salt is in the tank when it is full? First, find out how full the tank is (given that it holds 100 Litres):

$$60 + 2t = 100 \implies t = 20 \text{ minutes}$$

Now use the integration factor $r(t) = e^{\int p(x)\mathrm{d}x}$.

$$\begin{aligned} r(t) &= e^{\int \frac{3}{60+2t}\mathrm{d}t} \\ (u = 60 + 2t &\implies \mathrm{d}u = 2\mathrm{d}t) \\ r(t) &= e^{\frac{3}{2}\int \frac{1}{u}\mathrm{d}u} = e^{\frac{3}{2}\ln 60+2t} \\ r(t) &= (60+2t)^{\frac{3}{2}} \end{aligned}$$

Now utilize inverse product rule:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(60+2t)^{\frac{2}{3}} + \frac{3x}{60+2t}(60+2t)^{\frac{2}{3}} &= \frac{1}{2}(60+2t)^{\frac{2}{3}} \\ \int \frac{\mathrm{d}}{\mathrm{d}t}\left[x(t)(60+2t)\right]^{\frac{2}{3}} &= \int \frac{1}{2}(60+2t)^{\frac{2}{3}} \\ (60+2t)^{\frac{2}{3}} = \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{2}{5}(60+2t)^{\frac{5}{2}} + c &= \frac{1}{10}(60+2t)^{\frac{5}{2}} + c \\ x(t) &= \frac{1}{10}(60+2t) + C(60+2t)^{\frac{-3}{2}}. \end{aligned}$$

Then applying the initial condition:

$$x(0) = \frac{1}{10}(60) + C(60)^{-\frac{3}{2}} \implies \boxed{C \approx 1860.}$$

Now plug in $t = 20$

$$x(20) = \frac{1}{10}(60+40) + 1860(60+40)^{-\frac{3}{2}} \approx \boxed{11.86\text{kg.}}$$

2.1.1 Solving Tank Problems

Consider a tank of brine water or some dissolved substance in a solvent. Commonly it is brine water. The tank has both an input and output. The input liquid comes in at a rate r_1 with a concentration of c_1. The output is almost always considered to be homogeneous since it is at the

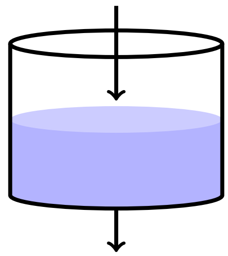

Figure 1: Singular Tank System

bottom of the tank. The output rate is r_2 and output concentration is c_2. The overall quantity of solute is $s(t)$ and tank volume is $v(t)$. The setup is represented by 1.
From the setup, one can determine that:

$$\Delta s = r_1 c_1 \Delta t - r_2 c_2 \Delta t.$$

Dividing by Δt and taking the limit arrives at the differential equation

$$\boxed{\frac{\mathrm{d}s}{\mathrm{d}t} = r_1 c_1 - r_2 c_2.}$$

2.2 Substitution

Nice types of ODES:

$$y' = f(x) \qquad y' = f(x,y) = h(x) \cdot g(y)$$

$$y' = f(y) \qquad \frac{\mathrm{d}y}{\mathrm{d}t} + p(t)x = f(t).$$

The last case represents a linear, first order differential equation. Via the integration factor, they are easy to solve. In certain cases, an equation may not look linear or separable; however, **change of variables** can resolve certain cases like this.

Note: Substitutions

General substitutions that work:

$y' = F(ax+by)$	$v = ax + by$
$y' = G(\frac{y}{x})$	$v = \frac{y}{x}$
$y' + p(x)y = q(x)y^n$	$v = y^{1-n}$

Ex. Find the general solution of $y' = (4x - y + 1)^2$

Let $v = 4x - y$. Rewrite in terms of v:

$$\begin{gathered} v' = 4 - y' \\ \downarrow \\ y' = 4 - v' \end{gathered}$$

Now:

$$4 - v' = (v+1)^2 \implies v' = 4 - (v+1)^2.$$

Note that it is now a separable equation.

$$\begin{aligned} \frac{v'}{4-(v+1)^2} = 1 \implies -\int \frac{v'}{4-(v+1)^2} dv &= \int 1 \mathrm{d}x \\ -\frac{1}{4}[\ln|v-1| - \ln|v+3|] &= x + c \\ \ln\left|\frac{v-1}{v+3}\right| &= -4x + c \\ \frac{v-1}{v+3} &= Ae^{-4x} \\ v &= \frac{1+3Ae^{-4x}}{1-Ae^{-4x}}. \end{aligned}$$

Now that the solution is found, rewrite v back in terms of y.

$$\begin{aligned} v = \frac{1+3Ae^{-4x}}{1-Ae^{-4x}} \implies 4x - y &= \frac{1+3Ae^{-4x}}{1-Ae^{-4x}} \\ y &= 4x - \frac{1+3Ae^{-4x}}{1-Ae^{-4x}}. \end{aligned}$$

Example where $v = \frac{y}{x}$:

Ex. Solve $x^2 y' = y^2 + xy$

First try dividing by the highest power of the independent variable (x^2)

$$y' = \frac{y^2}{x^2} + \frac{y}{x}$$

Now use $v = \frac{y}{x}$

$$y' = v^2 + v$$

Find y' in terms of v

$$
\begin{aligned}
v &= \underbrace{\frac{y}{x}}_{\text{Quotient rule}} \\
v' &= \frac{xy' - y}{x^2} \\
v' &= \frac{y'}{x} - \frac{y}{x^2} \\
v'x &= y' - \frac{y}{x} \\
v'x &= y' - v \\
y' &= v'x + v
\end{aligned}
$$

Substitute back into ODE:

$$
\begin{aligned}
xv' &= v^2 + v \\
v' &= \frac{v^2}{x} \\
\frac{v'}{v^2} &= \frac{1}{x} \\
\int \frac{1}{v^2} \mathrm{d}v &= \int \frac{1}{x} \mathrm{d}x \\
-\frac{1}{v} &= \ln(x) + c \\
v &= -\frac{1}{\ln(x) + c}
\end{aligned}
$$

Now plug back in y for v:

$$
\begin{aligned}
\frac{y}{x} &= -\frac{1}{\ln(x) + c} \\
y &= -\frac{x}{\ln(x) + c}.
\end{aligned}
$$

2.2.1 Solving a Bernoulli Equation

Given an equation in the form of a Bernoulli Equation, such as:

$$y' + \frac{4}{x}y = x^3y^2, x > 0$$

Since $n = 2$, let $v(x) = y^{-1}$. Find $v'(x)$

$$v'(x) = \frac{\mathrm{d}}{\mathrm{d}x}v = -y^{-2}y'.$$

Divide the ODE by y^n:

$$y^{-2}y' + \frac{4}{x}y^{-1} = x^3.$$

$$-v' + \frac{4}{x}v = x^3.$$

Rearrange into standard linear ODE form:

$$v' - \frac{4}{x}v = -x^3.$$

Utilize the integration factor $r(x) = e^{-\int \frac{4}{x}\mathrm{d}x} = x^{-4}$

$$\begin{aligned}
v' - \frac{4}{x}v &= -x^3 \\
v'r(x) - \frac{4}{x}vr(x) &= -x^3 r(x) \\
\underbrace{x^{-4}v' - x^{-4}\frac{4}{x}v}_{\text{Inverse Product Rule}} &= -\frac{1}{x} \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{-4}v\right) &= -\frac{1}{x} \\
x^{-4}v &= -\ln(x) + c \\
v &= -x^4\ln(x) + cx^4.
\end{aligned}$$

Plug y back in for v:

$$\boxed{y(x) = \frac{1}{-x^4\ln(x) + cx^4}.}$$

2.3 2nd Order Linear Equations

Note: General form of 2nd Order Linear ODE

$$A(x)y'' + B(x)y' + C(x)y = F(x).$$

Theorem 2.2 ▶ 2nd Order Linear ODE Solution Existence and Uniqueness

For an ODE of form $y'' + B(x)y' + C(x)y = D(x)$ with $B(x), C(x)$ and $D(x)$ as continuous functions on an interval I, for some $a \in I$ and some $b_0, b_1 \in \mathbb{R}$, a unique solution must exist and satisfy:

$$\begin{cases} y'' + B(x)y' + C(x)y = D(x) \\ y(a) = b_0 \\ y'(a) = b_1 \end{cases}$$

Note: Homogeneous Equation

$$ay'' + by' + cy = 0 \leftarrow \text{Since this is zero, it is homogeneous.}$$

2.3.1 Superposition Principle

Consider $y'' - k^2 y = 0$. A possible solution is $y_1 = e^{kx}$. Therefore $y_1' = ke^{kx}$ and $y_1'' = k^2 e^{kx}$. Plugging into the original ODE:

$$\boxed{y_1'' - k^2 y_1 = \left(k^2 e^{kx}\right) - k^2 \left(e^{kx}\right) = 0 \checkmark}$$

Another solution is $y_2 = e^{-kx}$. By the Superposition Principle, their linear combination is also a solution. In this instance, for $c_1, c_2 \in \mathbb{R}$:

Theorem 2.3 ▶ Superposition Principle

For any linearly homogeneous differential equation, if it has two solutions $y_1(t)$ and $y_2(t)$, then the function:

$$y(t) = c_1 y_1(t) + c_2 y_2(t).$$

Is also a solution.

2.3.2 The Wronskian

In the case of an 2^{nd} Order Linear Homogeneous equation, utilizing the Superposition Principle offers new solutions to the ODE. However, the linear combination may not always result in a general solution. Inspect the general form the ODE:

$$p(t)y'' + q(t)y' + r(t)y = 0 \implies \begin{cases} y(t_0) & = y_0 \\ y'(t_0) & = y_1 \end{cases}.$$

By the Superposition Principle

$$y(t) = c_1 y_1(t) + c_2 y_2(t).$$

Both y_1 and y_2 must satisfy the initial conditions in order to provide a general solution. Find the value of the constants c_1 and c_2.

$$\begin{aligned} y_0 &= y(t_0) = c_1 y_1(t_0) + c_2 y_2(t_0) \\ y_1 &= y'(t_0) = c_1 y_1'(t_0) + c_2 y_2'(t_0). \end{aligned}$$

Rewrite in terms of a system of matrices

$$\underbrace{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}}_{\text{A}} \underbrace{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}}_{\text{x}} = \underbrace{\begin{bmatrix} y_0 \\ y_1 \end{bmatrix}}_{\text{b}}$$

Note that c_1 and c_2 can be solved using Cramer's Rule

$$c_1 = \frac{\begin{vmatrix} y_0 & y_2(t_0) \\ y_1 & y_2'(t_0) \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}}. \qquad c_2 = \frac{\begin{vmatrix} y_1(t_0) & y_0 \\ y_1'(t_0) & y_1 \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}}.$$

This restatement of the imposed initial value conditions reveals a new, succinct condition to check for generality. If the denominator of either c_1 or c_2 is 0, the linear combination of y_1 and y_2 will not be the general solution of the ODE. This denominator is called TH The Wronskian.

Theorem 2.4 ▶ The Wronskian

$$W(f,g)(t) = \begin{vmatrix} f(t) & g(t) \\ f'(t) & g'(t) \end{vmatrix}.$$

For a 2$^{\text{nd}}$ Order Linear Homogeneous equation, if $W(y_1, y_2)(t) \neq 0$, then $y(x) = c_1 y_1 + c_2 y_2$ is a general solution to the ODE.

Note: Extension of the Wronskian

For any n^{th} order ODE, the Wronskian can be generalized. Generalizing by example, consider a 3$^{\text{rd}}$ order linearly homogeneous ODE.

$$a(t)y''' + b(t)y'' + c(t)y' + d(t)y = 0.$$

By the TH Superposition Principle, the solution is

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + c_3 y_3(t).$$

Like in TH 2.4

$$\begin{aligned} y_0 &= c_1 y_1(t_0) + c_2 y_2(t_0) + c_3 y_3(t_0) \\ y_1 &= c_1 y_1'(t_0) + c_2 y_2'(t_0) + c_3 y_3'(t_0) \\ y_2 &= c_1 y_1''(t_0) + c_2 y_2''(t_0) + c_3 y_3''(t_0) \end{aligned}$$

Which as prior can be written as a matrix multiplication. This holds for any order, therefore we can right the Wronskian generally as:

$$W(y_1, y_2, \dots, y_n)(t) = \begin{vmatrix} y_1(t) & y_2(t) & \dots & y_n(t) \\ y_1'(t) & y_2'(t) & \dots & y_n'(t) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^n(t) & y_2^n(t) & \dots & y_n^n(t) \end{vmatrix}$$

Week 3

3.1 Constant Coefficient 2nd Order ODEs

Consider the equation $y'' - 6y' + 8y = 0$.

The solution is going to be in the form of a function whose derivatives only effect its coefficients and not the function itself. Inspect the exponential function: $y = e^{rx} \implies y' = re^{rx} \implies y'' = r^2 e^{rx} \ldots$

$$y'' - 6y' + 8y = 0 \implies r^2 e^{rx} - 6re^{rx} + 8e^{rx} = 0.$$

Which turns into:

$$e^{rx}\left(r^2 - 6r + 8\right) = 0.$$

Now solve the internal quadratic for r:

$$(r-2)(r-4) = 0 \implies r = \{2, 4\}.$$

Therefore the solutions are:

$$y_1 = e^{2x} \qquad\qquad y_2 = e^{4x}.$$

Since both are linearly independent, all solutions are represented by:

$$y(x) = c_1 e^{2x} + c_2 e^{4x}\ ;\ \{c_1, c_2\} \in \mathbb{R}.$$

For any 2nd Order Linear Homogeneous ODE with constant coefficients, the solution can be determined by the roots of the characteristic equation:

$$ar^2 + br + c = 0.$$

Stated in a theorem:

Theorem 3.5 ▶ Constant Coefficient 2nd Order ODEs Solution

Let r_1 and r_2 be the roots of the characteristic polynomial. If both roots are distinct, the general solution is:

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}\ ;\ \{c_1, c_2\} \in \mathbb{R}.$$

If both roots are the same, the general solution is:

$$y(x) = e^{r_1 x}\left(c_1 + c_2 x\right).$$

If the roots are expressed as $r = \alpha \pm i\beta$:

$$\begin{aligned} y(x) &= Ae^{x(\alpha+i\beta)} + Be^{x(\alpha-i\beta)} \\ &= Ae^{\alpha x}e^{i\beta x} + Be^{\alpha x}e^{-i\beta x} \\ &= Ae^{\alpha x}(\cos\beta x + i\sin\beta x) + Be^{\alpha x}(\cos\beta x - i\sin\beta x) \\ y(x) &= c_1 e^{\alpha x}\cos(\beta x) + c_2 e^{\alpha x}\sin(\beta x)\ ;\ \{c_1, c_2\} \in \mathbb{R}. \end{aligned}$$

Note: Complex Root Selection

Note that in TH 3.5, one can just take one the complex values of r and take its real and imaginary components as linearly independent. Given $r = a + bi$,

$$\begin{aligned} y = e^{rt} &= e^{(a+bi)t} \\ &= e^{at}e^{bi\cdot t} \\ &= e^{at}\left(\cos(bt) + i\sin(bt)\right) \end{aligned}$$

$$\begin{aligned} y_1 &= \mathrm{Re}(y) \\ y_1 &= e^{at}\cos(bt) \end{aligned} \qquad \begin{aligned} y_2 &= \mathrm{Im}(y) \\ y_2 &= e^{at}\sin(bt). \end{aligned}$$

$$\boldsymbol{y(t) = c_1 e^{at}\cos(bt) + c_2 e^{at}\sin(bt)}.$$

Ex. Find solution to $y'' - 8y' + 16y = 0$, $y(0) = 2$, $y'(0) = 6$.

$$\begin{aligned} \text{Characteristic equation: } r^2 - 8r + 16 &= 0 \\ (r-4)^2 &= 0 \\ \text{General solution: } y(x) &= e^{4x}(c_1 + c_2 x). \end{aligned}$$

Using the initial condition:

For c_1:

$$\begin{aligned} y(0) = e^0(c_1 + c_2 \cdot 0) &= 2 \\ c_1 &= 2 \end{aligned}$$

For c_2:

$$\begin{aligned} y'(x) &= 4e^{4x}(c_1 + c_2 x) + c_2 e^{4x} \\ y'(0) &= 4e^{4\cdot 0}(c_1 + c_2 \cdot 0) + c_2 e^{4\cdot 0} \\ 4c_1 + c_2 &= 6 \\ 4c_1 &= 4 \\ c_1 &= 1. \end{aligned}$$

Note that TH 3.5 can be generalized to any nth order ODE as long as its linear and homogeneous:

$$\begin{bmatrix} y \\ y' \\ y'' \\ \vdots \\ y^n \end{bmatrix} \cdot \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = 0$$

Note that the parametrized solution $y(x) = e^{rx}$ works

$$e^{rx}\begin{bmatrix}1\\r\\r^2\\\vdots\\r^n\end{bmatrix}\cdot\begin{bmatrix}a_0\\a_1\\a_2\\\vdots\\a_n\end{bmatrix} = 0$$

Divide out by e^{rx} since it is always greater than 0

$$\begin{bmatrix}1\\r\\r^2\\\vdots\\r^n\end{bmatrix}\cdot\begin{bmatrix}a_0\\a_1\\a_2\\\vdots\\a_n\end{bmatrix} = 0$$

Expanding out the dot product

$$a_0 + a_1 r + a_2 r^2 + \ldots + a_{n-1} r^{n-1} + a_n r^n = 0$$

The resulting parametrized polynomial encodes the values of parameter r that define the solution. The final analytic solution will therefore be a superposition/linear combination of all the parametrized functions:

Note: Repeated Roots of r

If r is repeated k times, then the linearly independent solutions of k are:

$$e^{rx}, xe^{rx}, x^2e^{rx}, \ldots, x^k e^{rx}.$$

Ex. Find the general solution for $y^{(4)} - 3y''' - 3y'' - y' = 0$.

Utilize the parametrized solution $y = e^{rx}$

$$\begin{aligned} r^4 - 3r^3 - 3r^2 - r &= 0 \\ r(r^3 - 3r^2 - 3r - 1) &= 0 \\ r(r-1)^3 &= 0 \implies r = \{0, 1, 1, 1\} \end{aligned}$$

r is repeated three times, therefore:

$$y(x) = c_1 + c_2 e^x + c_3 x e^x + c_4 x^2 e^x$$

3.2 Non-Homogeneous Equation

If an ODE has the form $L(y) = f(x)$, then to find the solution you find:

Complementary Solution ($\implies y_c$) solves the associated linear homogeneous equation
Particular Solution ($\implies y_p$) solves the original non-homogeneous equation
Using these solutions, the general solution for the original ODE is

$$y(x) = y_c + y_p.$$

3.2.1 Method of Undetermined Coefficients

Theorem 3.6 ▶ Undetermined Coefficients

The general solution of a linear non-homogeneous ODE can be written as

$$y(t) = y_c(t) + y_p(t).$$

Where $y_c(t)$ is the general solution to the homogeneous form of the ODE and $y_p(t)$ is a solution with a form that is guessed by the form of the non-homogeneous term.

Ex. $y'' + 5y' + 6y = 2x + 1$, $y(0) = 0$ **and** $y'(0) = \frac{1}{3}$

Consider the associated homogeneous equation $y'' + 5y' + 6y = 0$

$$r^2 + 5r + 6 = 0 \implies r = \{-2, -3\}$$

Therefore the complementary solution is:

$$y_c = c_1 e^{-2x} + c_2 e^{-3x}.$$

To find the **particular solution**, take a guess about the form of y_p. Since the linear combination of 2nd derivative, 1st derivative, and itself is a linear polynomial, its possible that y_p is also a polynomial and linear. Therefore:

$$\text{Guess } y_p = Ax + B.$$

Substitute y_p into original ODE

$$\begin{aligned} y_p'' + 5y_p' + 6y_p &= 2x + 1 \\ 0 + 5A + 6(Ax + B) &= 2x + 1 \\ 6Ax + 5A + 6B &= 2x + 1. \end{aligned}$$

Now match coefficients

$$\begin{aligned} 6A &= 2 \\ 5A + 6B &= 1 \end{aligned}$$

Therefore $A = \frac{1}{3}$ and $B = -\frac{1}{9}$

$$y_p = \frac{1}{3}x - \frac{1}{9}.$$

The general solution is therefore

$$y(x) = c_1 e^{-2x} + c_2 e^{-3x} + \frac{1}{3}x - \frac{1}{9}.$$

Do not use initial condition in just y_c. To solve for c_1 and c_2, use the general solution

Note: Selecting a y_p

When guessing a form for y_p, take the most general form of the function and its derivatives. Some examples for $L(y) = f(x)$:

Given:	Ansatz:
$f(x) = x$	$y_p = Ax + B$
$f(x) = 3x^2 + 1$	$y_p = Ax^2 + Bx + C$
$f(x) = \cos(x)$	$y_p = A\cos(x) + B\sin(x)$
$f(x) = e^{kx}$	$y_p = Ae^{kx}$

What happens when y_p is a solution of the homogeneous equation (similar to that of a repeated root)?

Ex. Find the general solution of $y'' - 9y = e^{3x}$.

For the homogeneous system:

$$\begin{aligned} y'' - 9y &= 0 \\ r^2 - 9 &= 0 \implies r = \pm 3. \end{aligned}$$

$$\boxed{y_c = c_1e^{3x} + c_2e^{-3x}.}$$

For the particular system, guess that $y_p = Ae^{3x}$. Plug into the ODE:

$$\begin{aligned} 9Ae^{3x} - 9Ae^{3x} &= e^{3x} \\ e^{3x} &= 0. \end{aligned}$$

Note that the prediction leads to nonsense. Therefore, treat it like a repeated root of a characteristic equation and add a multiple of x. Now $y_p = Axe^{3x}$. Plugging into the ODE:

$$\begin{aligned} A(9xe^{3x} + 6e^{3x}) - 9Axe^{3x} &= e^{3x} \\ 6Ae^{3x} &= e^{3x} \implies A = \frac{1}{6}. \end{aligned}$$

While the Method of Undetermined Coefficients is really powerful, it fails to work in situations where the function of the independent variable has infinite linearly independent derivatives.

Week 4

4.1 Variation of Parameters

Theorem 4.7 ▶ Variation of Parameters

Given a 2nd Order ODE $y'' + p(t)y' + q(t)y = g(t)$, the particular solution of the ODE can be written in the form of

$$y_p = u_1 y_1 + u_2 y_2.$$

where y_1 and y_2 are the complementary function solutions to the homogeneous version of the ODE and u_1 and u_2 are functions of t that follow the following conditions:

$$\begin{aligned} u_1' y_1 + u_2' y_2 &= 0 \\ u_1' y_1' + u_2' y_2' &= g(t). \end{aligned}$$

The benefit to Variation of Parameters is that derivatives of $g(t)$ aren't the limiting factor, but rather the ability to find the functions u_1 and u_2 is the limiting factor. This means 4.7 is able to find particular solutions in cases where the method of undetermined coefficients is unable to do so.

Midterm

Week 5

5.1 System's of ODEs

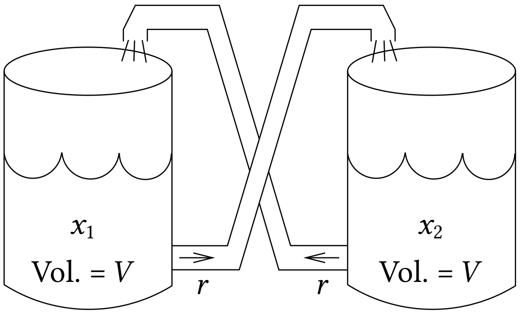

Figure 2: Connected Tank System

Ex. An interconnected system of two tanks

In this example, x_1 and x_2 are the quantities of salt in each tank, V is the volume in either tank, and r is the rate in and out of each tank. Examine each tank separately like in Solving Tank Problems

$$x_1' = \frac{x_2}{V}r - \frac{x_1}{V}r = \frac{r}{V}x_2 - \frac{r}{V}x_1 = \frac{r}{V}\left(x_2 - x_1\right).$$

Similarly for the rate x_2', the roles of x_1 and x_2 are reversed. All in all, the system of ODEs for this problem is

$$\begin{aligned} x_1' &= \frac{r}{V}\left(x_2 - x_1\right), \\ x_2' &= \frac{r}{V}\left(x_1 - x_2\right). \end{aligned}$$

To solve the system of equations, utilize a matrix/vector representation.

$$\vec{\mathbf{x}}(t) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \implies \vec{\mathbf{x}}'(t) = \begin{bmatrix} x_1' \\ x_2' \end{bmatrix}.$$

Note that we can define a matrix $\mathbf{A}$ such that

$$\vec{\mathbf{x}}'(t) = \mathbf{A}\vec{\mathbf{x}}(t).$$

For this system

$$\mathbf{A} = \frac{r}{V}\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}.$$

In order to solve this new system, eigenvalues can be used to describe the solution set.

5.1.1 Eigenvalues and Eigenvectors

Theorem 5.8 ▶ Eigenvalue Decomposition

A linearly homogeneous system can be represented by a constant coefficient matrix **A** such that

$$\vec{\mathbf{x}}'(t) = \mathbf{A}\vec{\mathbf{x}}(t).$$

Set $\vec{\mathbf{x}} = \vec{\mathbf{v}}e^{\lambda t}$ where $\vec{\mathbf{v}}$ can be any constant vector and λ is an eigenvalue of **A**

$$\lambda\vec{\mathbf{v}}e^{\lambda t} = \mathbf{A}\vec{\mathbf{v}}e^{\lambda t} \implies \mathbf{A}\vec{\mathbf{v}} = \lambda\vec{\mathbf{v}}.$$

By definition, $\vec{\mathbf{v}}$ must be a eigenvector of **A**. The final solution $\vec{\mathbf{x}}(t)$ can be written as a linear combination of the eigenvectors and exponential (by the Superposition Principle)

$$\boxed{\vec{\mathbf{x}}(t) = c_1\vec{\mathbf{v}}_1e^{\lambda_1 t} + c_2\vec{\mathbf{v}}_2e^{\lambda_2 t} \ldots + c_n\vec{\mathbf{v}}_ne^{\lambda_n t}.}$$

Continuing the previous example, Finding the eigenvectors is the same as finding

$$\frac{r}{V} \cdot \det(\mathbf{A} - \mathbf{I}\lambda) = 0.$$

Using the given matrix

$$\begin{vmatrix} -1-\lambda & 1 \\ 1 & -1-\lambda \end{vmatrix} = (\lambda+1)^2 - 1 = 0 \implies \lambda = \frac{r}{V} \cdot \{-2, 0\}.$$

Find the associated eigenvectors

$$\operatorname{Ker} \mathbf{A} + 2\mathbf{I} = \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right] \implies \vec{\mathbf{v}}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$
$$\operatorname{Ker} \mathbf{A} - 0\mathbf{I} = \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right] \implies \vec{\mathbf{v}}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Therefore the general solution of the system is

$$\vec{\mathbf{x}}(t) = c_1\begin{bmatrix} -1 \\ 1 \end{bmatrix}e^{-2t} + c_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Expanding out into solutions for x_1 and x_2

$$x_1(t) = c_2 - c_1e^{-\frac{2r}{V}t}$$
$$x_2(t) = c_2 + c_1e^{-\frac{2r}{V}t}.$$

Assume that $x_1(0) = s_0$ and $x_2 = s_1$.

$$x_1(t) = c_2 - c_1e^{-\frac{2r}{V}t} \implies s_0 = c_2 - c_1$$
$$x_2(t) = c_2 + c_1e^{-\frac{2r}{V}t} \implies s_1 = c_2 + c_1.$$

Therefore $c_1 = \frac{s_1 - s_0}{2}$ and $c_2 = \frac{s_0 + s_1}{2}$. Finally

$$\boxed{\begin{aligned} x_1(t) &= \frac{s_1+s_0}{2} - \frac{s_1-s_0}{2}e^{-\frac{2r}{V}t} \\ x_2(t) &= \frac{s_1+s_0}{2} + \frac{s_1-s_0}{2}e^{-\frac{2r}{V}t} \end{aligned}}$$

The solutions can be checked logically as it can be assumed that as time goes on, the mixture of salt in the solvent should even out since the volumes are identical and the rates are also identical. Taking the proper limits for both functions

$$\begin{aligned} \lim_{t\to\infty} x_1(t) &= \lim_{t\to\infty} \frac{s_1+s_0}{2} - \frac{s_1-s_0}{2}e^{-\frac{2r}{V}t} = \frac{s_1+s_0}{2} \\ \lim_{t\to\infty} x_2(t) &= \lim_{t\to\infty} \frac{s_1+s_0}{2} + \frac{s_1-s_0}{2}e^{-\frac{2r}{V}t} = \frac{s_1+s_0}{2}. \end{aligned}$$

As assumed by just physically guessing the behaviour of the system, the functions result in the same answer. As time approaches infinity, the amount of salt in each tank will even out, therefore becoming the average of the quantities of salt.

5.1.2 Complex Eigenvalues

Consider the system

$$\vec{\mathbf{x}}' = \begin{bmatrix} 0 & -2 \\ 1 & 2 \end{bmatrix}\vec{\mathbf{x}}.$$

Finding the eigenvalues

$$\det(\mathbf{A} - \mathbf{I}\lambda) = 0$$

$$\begin{vmatrix} -\lambda & -2 \\ 1 & 2-\lambda \end{vmatrix} = \lambda^2 - 2\lambda + 2 = 0 \implies \lambda = \{1+i, 1-i\}$$

Both of the eigenvalues for the matrix are complex. This is not a problem however. Continue using the same approach as before. Find the eigenvectors

$$\begin{aligned} \ker \mathbf{A} - (1+i)\mathbf{I} &= \left[\begin{array}{cc|c} -1-i & -2 & 0 \\ 1 & 1-i & 0 \end{array}\right] \implies \vec{\mathbf{v}}_1 = \begin{bmatrix} -1+i \\ 1 \end{bmatrix} \\ \ker \mathbf{A} - (1-i)\mathbf{I} &= \left[\begin{array}{cc|c} -1+i & -2 & 0 \\ 1 & 1+i & 0 \end{array}\right] \implies \vec{\mathbf{v}}_2 = \begin{bmatrix} -1-i \\ 1 \end{bmatrix}. \end{aligned}$$

Therefore the general solution of the system is

$$\vec{\mathbf{x}}(t) = c_1\begin{bmatrix} -1+i \\ 1 \end{bmatrix}e^{(1+i)t} + c_2\begin{bmatrix} -1-i \\ 1 \end{bmatrix}e^{(1-i)t}.$$

Combine into a singular vector

$$\begin{aligned}
\vec{\mathbf{x}}(t) &= \begin{bmatrix} c_1(-1+i)e^{(1+i)t} + c_2(-1-i)e^{(1-i)t} \\ c_1 e^{(1+i)t} + c_2 e^{(1-i)t} \end{bmatrix} \\
&= \begin{bmatrix} -c_1 e^{(1+i)t} + ic_1 e^{(1+i)t} + -c_2 e^{(1-i)t} - ic_2 e^{(1-i)t} \\ c_1 e^{(1+i)t} + c_2 e^{(1-i)t} \end{bmatrix} \\
&= \begin{bmatrix} -c_1 e^t(\cos(t) + i\sin(t)) + ic_1 e^t(\cos(t) + i\sin(t)) + -c_2 e^t(\cos(t) - i\sin(t)) - ic_2 e^t(\cos(t) - i\sin(t)) \\ e^t(\cos(t) + i\sin(t)) + c_2 e^t(\cos(t) - i\sin(t)) \end{bmatrix}
\end{aligned}$$

This becomes very unwieldy very quickly. Instead, remember that the real and imaginary components of a complex object are linearly independent. Therefore for each complex eigenvalue, only a single eigenvector needs to be considered as its conjugate pair will result in identical solutions. Take the eigenvector

$$\begin{bmatrix} -1+i \\ 1 \end{bmatrix}.$$

$$\begin{aligned}
\vec{\mathbf{x}}_1 &= \vec{\mathbf{v}}_1 e^{(1+i)t} \\
&= e^t(\cos(t) + i\sin(t)) \begin{bmatrix} -1+i \\ 1 \end{bmatrix} \\
&\Downarrow \\
\vec{\mathbf{x}}(t) &= c_1 \operatorname{Re}\left\{ e^t(\cos(t) + i\sin(t)) \begin{bmatrix} -1+i \\ 1 \end{bmatrix} \right\} + c_2 \operatorname{Im}\left\{ e^t(\cos(t) + i\sin(t)) \begin{bmatrix} -1+i \\ 1 \end{bmatrix} \right\} \\
\vec{\mathbf{x}}(t) &= c_1 \begin{bmatrix} -e^t\sin(t) - e^t\cos(t) \\ e^t\cos(t) \end{bmatrix} + c_2 \begin{bmatrix} e^t\cos(t) - e^t\sin(t) \\ e^t\sin(t) \end{bmatrix}.
\end{aligned}$$

$$\boxed{\vec{\mathbf{x}}(t) = c_1 e^t \begin{bmatrix} \cos(t) - \sin(t) \\ \sin(t) \end{bmatrix} + c_2 e^t \begin{bmatrix} 2\sin(t) \\ \cos(t) + \sin(t) \end{bmatrix}}$$

5.1.3 Algebraic and Geometric Multiplicity

Again by example, take the linear system

$$\vec{\mathbf{x}}' = \underbrace{\begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}}_{\mathbf{A}} \vec{\mathbf{x}}$$

$$\det\{\mathbf{A} - \mathbf{I}\lambda\} = 0 \implies \lambda = \{2, 2\}.$$

In this instance, the eigenvalue 2 appears twice. It is therefore said to have **Algebraic Multiplicity** of 2. Finding the associated eigenvalues results in

$$\text{Ker}\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix} = \left[\begin{array}{cc|c} 0 & 3 & 0\\ 0 & 0 & 0\end{array}\right] \implies \begin{cases} v_1 = v_1 \\ v_2 = 0\end{cases} \implies \vec{\lambda} = \begin{bmatrix}1\\0\end{bmatrix} \tag{1, 2}$$

Note that while the eigenvalue was present twice, both only correspond to a singular eigenvector. The count of the linearly independent eigenvectors associated with an eigenvalue is called the **Geometric Multiplicity** of the eigenvalue. In this instance it is 1.

Note: Algebraic and Geometric Multiplicity

For any eigenvalue, the following relationship holds

$$GM \leq AM.$$

If $GM < AM$, then the method of solving a system with Eigenvalue Decomposition will not work as normal, requiring a method to handle the 'repeated' root / eigenvalue

Note: Eigenvector Decomposition (Repeated Values)

In the case where an eigenvalue has an algebraic multiplicity larger than its geometric multiplicity, the solution set of a linear system cannot be accurately described. Therefore, extra dimensions must be constructed to allow the solution set to be complete and general. Take the system

$$\vec{\mathbf{x}}'(t) = \mathbf{A}\vec{\mathbf{x}}(t).$$

Assume an eigenvalue λ has an algebraic multiplicity of 2 and geometric multiplicity of 1. To resolve the lower dimensionality of the solution, assume the associated eigenvector solution is

$$\vec{\mathbf{x}}_\lambda(t) = \left(\vec{\mathbf{v}}_1 + t\vec{\mathbf{v}}_2\right)e^{\lambda t}.$$

Find the derivative and substitute into the original system equation

$$\begin{aligned}
\vec{\mathbf{x}}_\lambda'(t) &= e^{\lambda t}\vec{\mathbf{v}}_2 + \left(\vec{\mathbf{v}}_1 + t\vec{\mathbf{v}}_2\right)\lambda e^{\lambda t}\\
\mathbf{A}\vec{\mathbf{x}}(t) &= e^{\lambda t}\vec{\mathbf{v}}_2 + \left(\vec{\mathbf{v}}_1 + t\vec{\mathbf{v}}_2\right)\lambda e^{\lambda t}\\
\mathbf{A}(\vec{\mathbf{v}}_1 + t\vec{\mathbf{v}}_2 t)e^{\lambda t} &= e^{\lambda t}\vec{\mathbf{v}}_2 + \left(\vec{\mathbf{v}}_1 + t\vec{\mathbf{v}}_2\right)\lambda e^{\lambda t}\\
e^{\lambda t}\mathbf{A}\vec{\mathbf{v}}_1 + te^{\lambda t}\mathbf{A}\vec{\mathbf{v}}_2 &= e^{\lambda t}\vec{\mathbf{v}}_2 + \lambda e^{\lambda t}\vec{\mathbf{v}}_1 + t\lambda e^{\lambda t}\vec{\mathbf{v}}_2.
\end{aligned}$$

$$\begin{aligned}
e^{\lambda t}\vec{\mathbf{v}}_2 + \lambda e^{\lambda t}\vec{\mathbf{v}}_1 = e^{\lambda t}\mathbf{A}\vec{\mathbf{v}}_1 &\implies \vec{\mathbf{v}}_2 + \lambda\vec{\mathbf{v}}_1 = \mathbf{A}\vec{\mathbf{v}}_1 && \text{(Matching } e^{\lambda t}\text{)}\\
&\implies (\mathbf{A} - \mathbf{I}\lambda)\vec{\mathbf{v}}_1 = \vec{\mathbf{v}}_2.\\
\lambda te^{\lambda t}\vec{\mathbf{v}}_2 = te^{\lambda t}\mathbf{A}\vec{\mathbf{v}}_2 &\implies \lambda\vec{\mathbf{v}}_2 = \mathbf{A}\vec{\mathbf{v}}_2 && \text{(Matching } te^{\lambda t}\text{)}\\
&\implies (\mathbf{A} - \mathbf{I}\lambda)\vec{\mathbf{v}}_2 = 0.
\end{aligned}$$

Therefore, the solution works if the following conditions are met

$$\boxed{(\mathbf{A} - \mathbf{I}\lambda)\vec{\mathbf{v}}_2 = 0 \quad (\mathbf{A} - \mathbf{I}\lambda)\vec{\mathbf{v}}_1 = \vec{\mathbf{v}}_2.}$$

Theorem 5.9 ► Generalized Eigenvectors

For a system in the form of

$$\vec{\mathbf{x}}'(t) = \mathbf{A}\vec{\mathbf{x}}(t).$$

with an associated eigenvalue λ with $AM = N$

$$\vec{\mathbf{x}}_\lambda(t) = \left(\vec{\mathbf{x}}_1 + t\vec{\mathbf{x}}_2 + t^2\vec{\mathbf{x}}_3 + \ldots + t^{N-1}\vec{\mathbf{x}}_N\right)e^{\lambda t}.$$

Where

$$\boxed{\begin{aligned}(\mathbf{A} - \mathbf{I}\lambda)\vec{\mathbf{x}}_k &= k\vec{\mathbf{x}}_{k+1}.\\ (\mathbf{A} - \mathbf{I}\lambda)\vec{\mathbf{x}}_N &= 0.\end{aligned}}$$

The vector chain from $\vec{\mathbf{x}}_1$ to $\vec{\mathbf{x}}_{N-1}$ are the **Generalized Eigenvectors** associated with λ.

Week 6

6.1 Non-Homogeneous Linear System

Consider the system

$$x_1' = x_1 - 2x_2 + t$$
$$x_2' = x_1 - x_2$$

In matrix form, the system can be written as

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} x_1 - 2x_2 \\ x_1 - x_2 \end{bmatrix} + \begin{bmatrix} t \\ 0 \end{bmatrix} \implies \vec{\mathbf{x}}'(t) = \mathbf{A}\vec{\mathbf{x}}(t) + \vec{\mathbf{f}}(t).$$

Where

$$\mathbf{A} = \begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix}. \qquad \vec{\mathbf{f}}(t) = \begin{bmatrix} t \\ 0 \end{bmatrix}.$$

While potentially scarier than previous ODEs, previous methods such as Undetermined Coefficients and Variation of Parameters can be used, just utilizing vectors instead of scalars.

6.1.1 Method of Undetermined Coefficients

Using the prior example, apply the methods for Undetermined Coefficients using vectors instead of scalars. First, find a solution to the homogeneous system $\vec{\mathbf{x}}' = \mathbf{A}\vec{\mathbf{x}}$. Solve this system using Eigenvalue Decomposition

$$\det\left(\begin{bmatrix} 1-\lambda & -2 \\ 1 & -1-\lambda \end{bmatrix}\right) = (1-\lambda)(-1-\lambda) + 2$$
$$0 = \lambda^2 + 1 \implies \lambda = \pm i.$$

Use only a single eigenvalue to construct a eigenvector (due to conjugate pairing). Use $\lambda = i$

$$\ker(A - iI) = \left[\begin{array}{cc|c} 1-i & -2 & 0 \\ 1 & -1-i & 0 \end{array}\right] \implies \lambda_i = \begin{bmatrix} 1+i \\ 1 \end{bmatrix}.$$

Now, write the homogeneous solution as

$$\vec{\mathbf{x}}_c = c_1 \operatorname{Re}\left\{\lambda_i e^{it}\right\} + c_2 \operatorname{Im}\left\{\lambda_i e^{it}\right\}.$$

Find the real and imaginary components

$$\begin{aligned} e^{it}\vec{\lambda}_i &= (\cos(t) + i\sin(t))\vec{\lambda}_i \\ &= \begin{bmatrix} \cos(t) + i\sin(t) + i\cos(t) - \sin(t) \\ \cos(t) + i\sin(t) \end{bmatrix} \\ &= \underbrace{\begin{bmatrix} \cos(t) - \sin(t) \\ \cos(t) \end{bmatrix}}_{\text{Re}} + i \underbrace{\begin{bmatrix} \cos(t) + \sin(t) \\ \sin(t) \end{bmatrix}}_{\text{Im}}. \end{aligned}$$

Therefore the complementary solution can be written as

$$\vec{\mathbf{x}}_c(t) = c_1 \begin{bmatrix} \cos(t) - \sin(t) \\ \cos(t) \end{bmatrix} + c_2 \begin{bmatrix} \cos(t) + \sin(t) \\ \sin(t) \end{bmatrix}.$$

Just like before, along with a complementary solutions there is a particular solution with some the form of some Ansatz. Since in this case $\vec{\mathbf{f}}(t) = \begin{bmatrix} t \\ 0 \end{bmatrix}$, assume that

$$\vec{\mathbf{x}}_p = \vec{\mathbf{a}}t + \vec{\mathbf{b}} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} t + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

Taking the derivative is easy in this instance since the vectors are constants. Therefore

$$\vec{\mathbf{x}}_p' = \vec{\mathbf{a}}.$$

Now substitute into the original system

$$\begin{aligned}
\vec{\mathbf{x}}_p' &= A\vec{\mathbf{x}}_p + \vec{\mathbf{f}}(t) \\
\vec{\mathbf{a}} &= \mathrm{A}\left(\vec{\mathbf{a}}t + \vec{\mathbf{b}}\right) + \vec{\mathbf{f}}(t) \\
\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} &= \mathrm{A}\vec{\mathbf{a}}t + \mathrm{A}\vec{\mathbf{b}} + \vec{\mathbf{f}}(t) \\
\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} &= \mathrm{A}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} t + \mathrm{A}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} + \begin{bmatrix} t \\ 0 \end{bmatrix} \\
\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} &= \begin{bmatrix} a_1 - 2a_2 \\ a_1 - a_2 \end{bmatrix} t + \begin{bmatrix} b_1 - 2b_2 \\ b_1 - b_2 \end{bmatrix} + \begin{bmatrix} t \\ 0 \end{bmatrix} \\
\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} &= \begin{bmatrix} a_1 t - 2a_2 t + t + b_1 - 2b_2 \\ a_1 - a_2 + b_1 - b_2 \end{bmatrix}.
\end{aligned}$$

Matching coefficients can be used to determine $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$

Matching t

$$\begin{aligned}
0 &= a_1 - a_2 \implies a_1 = a_2 \\
0 &= a_1 - 2a_2 + 1 \\
&= a_1 - 2a_1 + 1 \implies a_1 = a_2 = 1.
\end{aligned}$$

Matching constants

$$\begin{aligned}
a_1 &= b_1 - 2b_2 \implies 1 = b_1 - 2b_2 \\
a_2 &= b_1 - b_2 \implies 1 = b_1 - b_2 \implies b_1 = 1 + b_2.
\end{aligned}$$

$$1 = (1 + b_2) - 2b_2 = 1 - b_2 \implies b_2 = 0 \implies b_1 = 1.$$

Therefore

$$\vec{\mathbf{x}}_p = \begin{bmatrix} 1 \\ 1 \end{bmatrix} t + \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

And the general solution is then

$$\vec{\mathbf{x}}(t) = c_1 \begin{bmatrix} \cos(t) - \sin(t) \\ \cos(t) \end{bmatrix} + c_2 \begin{bmatrix} \cos(t) + \sin(t) \\ \sin(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} t + \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

What if part of the particular solution appears in the complementary solution? Take for example

$$\vec{\mathbf{x}}' = \begin{bmatrix} 1 & 0 \\ 1 & -2 \end{bmatrix} \vec{\mathbf{x}} + \begin{bmatrix} 3t \\ e^t \end{bmatrix}.$$

Has a complementary solution

$$\vec{\mathbf{x}}_c = c_1 e^t \begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

If guessing naively, the particular solution would have the form

$$\vec{\mathbf{x}}_p = \vec{\mathbf{a}}t + \vec{\mathbf{b}} + \vec{\mathbf{c}}e^t.$$

The issue here is that $\vec{\mathbf{c}}e^t$ is not linearly independent to the complementary solution and therefore the Superposition Principle doesn't apply. Therefore, there must a new linearly independent term added to the particular solution. In this case, add the term $\vec{\mathbf{d}}te^t$ such that

$$\vec{\mathbf{x}}_p = \vec{\mathbf{a}}t + \vec{\mathbf{b}} + \vec{\mathbf{c}}e^t + \vec{\mathbf{d}}te^t.$$

This solution is now linearly independent from the complementary solution and can therefore be used to describe the general solution.

6.2 Laplace Transform

The Laplace transform is a transformation that can take a differential equation and convert it into an algebraic object. From there it can be solved with algebraic methods. The solution to that algebraic equation can then have the inverse Laplace transform applied to revert it back into a solution to the differential equation. The Laplace transformation is notated by

$$\mathcal{L}\{f(t)\} = F(s).$$

The Laplace transform is defined as

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt = F(s).$$

Note: Laplace Transform of Simple Functions

$$\begin{aligned}
\mathcal{L}\{1\} &= \int_0^\infty e^{-st} dt \\
&= \lim_{b\to\infty} -\frac{1}{s}e^{st}\Big|_0^b \\
&= \lim_{b\to\infty} -\frac{1}{s}e^{-bs} - \left(-\frac{1}{s}\right) \\
&= \frac{1}{s} \ (s > 0).
\end{aligned}$$

Theorem 6.10 ▶ Linearity of the Laplace Transform

Let A and B be constants. Let $f(t)$ and $g(t)$ be functions. Then the Laplace of their linear combination is

$$\mathcal{L}\{Af(t) + Bg(t)\} = AF(s) + BG(s).$$

Week 7

7.1 Inverse Laplace Transform

The inverse Laplace Transform is defined as

$$\mathcal{L}^{-1}\{F(s)\} \stackrel{\text{def}}{=} f(t).$$

Since it is the inverse of a linear transformation, the inverse is also linear in nature.

Ex. Find $\mathcal{L}^{-1}\left\{\frac{s^2+s+1}{s^3+s}\right\}$

Note that this doesn't fit any of the common forms for know Laplace Transforms. However, the polynomial division hints towards partial fraction decomposition.

$$\begin{gathered}
\frac{s^2+s+1}{s^3+s} = \frac{s^2+s+1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1} \\
\Downarrow \\
s^2+s+1 = A(s^2+1) + s(Bs+C) \\
s^2+s+1 = As^2 + A + Bs^2 + Cs \\
\Downarrow \\
(A,B,C) = (1,0,1) \\
\Downarrow \\
\frac{s^2+s+1}{s^3+s} = \frac{1}{s} + \frac{1}{s^2+1}.
\end{gathered}$$

Taking the Laplace transform of this is much easier, resulting in

$$\mathcal{L}^{-1}\left\{\frac{1}{s} + \frac{1}{s^2+1}\right\} = 1 + \sin(t).$$

7.1.1 Shifting Principle

In some cases, it is easier to write $F(s)$ as s shifted by some number a, such that the function in frequency space is $F(s+a)$. Note that

$$\begin{aligned}
\mathcal{L}\left\{e^{-at}f(t)\right\} &= \int_0^\infty e^{-st}e^{-at}f(t)dt \\
&= \int_0^\infty e^{-(s+a)t}f(t) \\
&= F(s+a).
\end{aligned}$$

Or equivalently,

$$\mathcal{L}^{-1}\{F(s+a)\} = e^{-at}f(t).$$

This is helpful in cases where a polynomial in a denominator can only be partially factored via complete the square

Ex. Find the inverse Laplace of $F(s) = \frac{1}{s^2+4s+13}$

In this case, $F(s)$ can not be factored in the denominator. However, it can be partially factored by completing the square.

$$\begin{aligned} F(s) &= \frac{1}{s^2+4s+13} \\ &= \frac{1}{(s^2+4s+4)+9} \\ &= \frac{1}{(s+2)^2+9}. \end{aligned}$$

Note that this result evokes the Laplace transformation of $\sin(\omega t)$

$$\mathcal{L}\{\sin(\omega t)\} = \frac{\omega}{s^2+\omega^2}.$$

The numerator doesn't fit the structure, but a simple manipulation resolves that issue

$$\frac{3}{3}\cdot\frac{1}{(s+2)^2+9} = \frac{1}{3}\cdot\frac{3}{(s+2)^2+9}.$$

Therefore the shifting principle can be utilized

$$\mathcal{L}^{-1}\left\{\frac{1}{3}\cdot\frac{3}{(s+2)^2+9}\right\} = \frac{1}{3}e^{-2t}\sin(3t).$$

Note that the shifting property appears when the Heaviside Function is involved. Consider

$$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}\mathcal{L}\{f(t)\} = e^{-as}F(s).$$

Another useful form is

$$\mathcal{L}\{f(t)u(t-a)\} = e^{-as}\mathcal{L}\{f(t+a)\}.$$

7.2 Laplace Transform of Derivatives

Since the Laplace transform is an integral transform, it would be advantageous to plug in time derivatives of functions to see their resultant function in frequency space. Assume a function $g(t)$ such that $\mathcal{L}\{g(t)\} = G(t)$.

$$\mathcal{L}\left\{g'(t)\right\} = \int_0^\infty e^{-st}g'(t)dt$$

Utilize integration by parts with $u = e^{-st}$ and $dv = g'(t)dt$

$$\begin{aligned} &= g(t)e^{-st}\Big|_0^\infty + s\int_0^\infty e^{st}g(t)dt \\ &= \lim_{t\to\infty} g(t)e^{-st} - g(0) + sG(s) \end{aligned}$$

In order for this to exist, $g(t)$ must grow slower than the exponential. Stated formally, $|g(t)| < Me^{ct}$ for appropriate positive constants M and c. Assuming this holds,

$$\mathcal{L}\{g'(t)\} = sG(s) - g(0).$$

7.2.1 Laplace Transform of Integrals

Laplace transforms act quite nicely when the input function is an integral. Consider an integral of the form

$$\int_0^t f(\tau)\mathrm{d}\tau.$$

Then the Laplace transform of $f(t)$ is

$$\mathcal{L}\left\{\int_0^t f(\tau)\mathrm{d}\tau\right\} = \frac{1}{s}\mathcal{L}\{f(t)\} = \frac{F(s)}{s}.$$

This can help solve harder Laplace transforms that would involve processes like partial fraction decomposition.

Ex. Find inverse Laplace of $\frac{1}{s(s^2+1)}$

Notice that

$$\frac{1}{s(s^2+1)} = \frac{1}{s}\mathcal{L}\{\sin(t)\}.$$

In this case, $f(t) = \sin(t)$, therefore,

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} &= \int_0^t \sin(\tau)\mathrm{d}\tau \\ &= -\cos(t)\Big|_0^t \\ &= 1-\cos(t).\end{aligned}$$

Appendix

Consider an ODE system of the form

$$\vec{\mathbf{x}}'(t) = \mathbf{A}\vec{\mathbf{x}}(t).$$

Where $\mathbf{A}$ is a finite constant square matrix. Assume $\mathbf{A}$ has an eigenvalue λ with an algebraic multiplicity N that is greater than its geometric multiplicity. Assume the associated vector solution has the form

$$\vec{\mathbf{x}}_\lambda(t) = \left(\vec{\mathbf{x}}_1 + \vec{\mathbf{x}}_2 t + \vec{\mathbf{x}}_3 t^2 + \ldots + \vec{\mathbf{x}}_N t^{N-1}\right)e^{\lambda t}.$$

where the number of vectors $\vec{\mathbf{x}}$ is finite. Define the sequence a_n

$$\begin{aligned}
a_n &= e^{\lambda t}\vec{\mathbf{x}}_n t^{n-1} \\
\frac{\mathrm{d}}{\mathrm{d}t}(a_n) &= \lambda e^{\lambda t}\vec{\mathbf{x}}_n t^{n-1} + (n-1)e^{\lambda t}\vec{\mathbf{x}}_n t^{n-2}
\end{aligned}$$

Define two new sequences b_n and c_n

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(a_n) &= \underbrace{\lambda e^{\lambda t}\vec{\mathbf{x}}_n t^{n-1}}_{b_n} + \underbrace{(n-1)e^{\lambda t}\vec{\mathbf{x}}_n t^{n-2}}_{c_n} \\
\frac{\mathrm{d}}{\mathrm{d}t}(a_n) &= b_n + c_n
\end{aligned}$$

Use the ODE system constraint

$$\begin{aligned}
\vec{\mathbf{x}}'(t) &= \mathbf{A}\vec{\mathbf{x}}(t) \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\vec{\mathbf{x}}(t)\right) &= \mathbf{A}\vec{\mathbf{x}}(t) \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{n=1}^{N} a_n\right) &= A\sum_{n=1}^{N} a_n \\
\sum_{n=1}^{N}\frac{\mathrm{d}}{\mathrm{d}t}(a_n) &= A\sum_{n=1}^{N} a_n \qquad (1) \\
\sum_{n=1}^{N} b_n + c_n &= A\sum_{n=1}^{N} a_n \\
\sum_{n=1}^{N} b_n + c_n &= \sum_{n=1}^{N} A\cdot a_n
\end{aligned}$$

Define the sequence $d_n = A\cdot a_n$

$$\sum_{n=1}^{N} b_n + c_n = \sum_{n=1}^{N} d_n$$

.

Match the coefficients of t between the sequences

$$\begin{aligned} b_n + c_{n+1} = d_n \implies \lambda e^{\lambda t}\vec{\mathbf{x}}_n t^{n-1} + n e^{\lambda t}\vec{\mathbf{x}}_{n+1} t^{n-1} &= A e^{\lambda t}\vec{\mathbf{x}}_n t^{n-1} \\ \lambda e^{\lambda t}\vec{\mathbf{x}}_n + n e^{\lambda t}\vec{\mathbf{x}}_{n+1} &= A e^{\lambda t}\vec{\mathbf{x}}_n \\ \lambda \vec{\mathbf{x}}_n + n\vec{\mathbf{x}}_{n+1} &= A\vec{\mathbf{x}}_n \\ A\vec{\mathbf{x}}_n - \lambda\vec{\mathbf{x}}_n &= n\vec{\mathbf{x}}_{n+1} \end{aligned}$$

Arriving at the condition

$$\boxed{(A - \mathbf{I}\lambda)\vec{\mathbf{x}}_n = n\vec{\mathbf{x}}_{n+1}}$$

In the boundary case where $n = N$

$$\begin{aligned} b_N &= \lambda e^{\lambda t}\vec{\mathbf{x}}_N t^{N-1} \\ c_N &= (N-1)e^{\lambda t}\vec{\mathbf{x}}_N t^{N-2} \\ d_N &= A e^{\lambda t}\vec{\mathbf{x}}_N t^{N-1}. \end{aligned}$$

Matching coefficients for t^{N-1}

$$\begin{aligned} \lambda e^{\lambda t}\vec{\mathbf{x}}_N t^{N-1} &= A e^{\lambda t}\vec{\mathbf{x}}_N t^{N-1} \\ \lambda\vec{\mathbf{x}}_N &= A\vec{\mathbf{x}}_N \\ A\vec{\mathbf{x}}_N - \lambda\vec{\mathbf{x}}_N &= 0. \end{aligned}$$

Arriving at the condition

$$\boxed{(A - \mathbf{I}\lambda)\vec{\mathbf{x}}_N = 0}$$

[1]In (1), the derivative can be brought into the summation since it is a linear operator and the sum is convergent due to the initial condition where $\vec{\mathrm{x}}$ and A are finite

Definitions

List of Theorems