Problem 1

Part 1

For $f(x)$ to be a valid probability density function, it must be true that $\int_{\mathbb{S}_X} f(x)\mathrm{d}x = 1$. Therefore

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)\mathrm{d}x &= 1 \\
\int_{-1}^{1} c(1-x^2)\mathrm{d}x &= 1 \\
c\int_{-1}^{1} 1-x^2\mathrm{d}x &= 1 \\
c\left[x - \frac{x^3}{3}\right]_{-1}^{1} &= 1 \\
c\left[\frac{2}{3} - \left(-\frac{2}{3}\right)\right] &= 1 \\
c \cdot \frac{4}{3} &= 1 \implies \boxed{c = \frac{3}{4}}.
\end{aligned}$$

Part 2

$$\begin{aligned}
F(a) = \mathbb{P}(X \le a) = \frac{3}{4}\int_{-1}^{a} 1-x^2\mathrm{d}x = \frac{3}{4}\left[x - \frac{x^3}{3}\right]_{-1}^{a} &= \frac{3}{4}\left(a - \frac{a^3}{3} + \frac{2}{3}\right) \\
&= \boxed{\frac{1}{2} + \frac{3a}{4} - \frac{a^3}{4}}.
\end{aligned}$$

Problem 2

$$\begin{aligned}
\mathbb{E}[X] &= \frac{3}{5} \\
\int_0^1 x\left(ax + bx^2\right)\mathrm{d}x &= \frac{3}{5} \\
\int_0^1 ax^2 + bx^3\mathrm{d}x &= \frac{3}{5} \\
\frac{ax^3}{3} + \frac{bx^4}{4}\Big|_0^1 &= \frac{3}{5} \\
\frac{a}{3} + \frac{b}{4} &= \frac{3}{5} \\
4a + 3b &= \frac{36}{5}.
\end{aligned}$$

Additionally,

$$\begin{aligned}\int_0^1 ax + bx^2 \mathrm{d}x &= 1\\ \frac{ax^2}{2} + \frac{bx^3}{3}\bigg|_0^1 &= 1\\ \frac{a}{2} + \frac{b}{3} &= 1\\ 3a + 2b &= 6.\end{aligned}$$

Therefore we have a system of equations

$$\begin{aligned}4a + 3b &= \frac{36}{5}\\ 3a + 2b &= 6.\end{aligned}$$

Solving this system gives $a = \frac{18}{5}, b = -\frac{12}{5}$.

Problem 3

By the definition of expectation of a continuous random variable,

$$\mathbb{E}[X] = \int_0^\infty x^2 e^{-x} \mathrm{d}x.$$

Using the tabular form of integration by parts results in

$$\begin{array}{c|c} dv & u \\ \hline x^2 & e^{-x} \\ 2x & -e^{-x} \\ 2 & e^{-x} \\ 0 & -e^{-x} \end{array}.$$

meaning that

$$\mathbb{E}[X] = -x^2 e^{-x} - 2xe^{-x} - 2e^{-x}\bigg|_0^\infty = 2.$$

Problem 4

Part 1

$$\begin{aligned}\int_0^2 c\left(4x-2x^2\right)\mathrm{d}x &= 1\\ c\int_0^2 4x-2x^2\mathrm{d}x &= 1\\ c\left[2x^2-\frac{2}{3}x^3\right]_0^2 &= 1\\ c\cdot\frac{8}{3} &= 1 \implies \boxed{c=\frac{3}{8}}.\end{aligned}$$

Part 2

$$\begin{aligned}\mathbb{P}\left(\frac{1}{2}<X<\frac{3}{2}\right) &= \frac{3}{8}\int_{\frac{1}{2}}^{\frac{3}{2}} 4x-2x^2\mathrm{d}x\\ &= \frac{3}{8}\left[2x^2-\frac{2}{3}x^3\right]_{\frac{1}{2}}^{\frac{3}{2}}\\ &= \frac{3}{8}\left[2\cdot\frac{9}{4}-\frac{2}{3}\cdot\frac{27}{8}-\left(2\cdot\frac{1}{4}-\frac{2}{3}\cdot\frac{1}{8}\right)\right]\\ &= \frac{3}{8}\left[\frac{9}{2}-\frac{9}{4}-\left(\frac{1}{2}-\frac{1}{12}\right)\right]\\ &= \frac{3}{8}\left[\frac{9}{2}-\frac{9}{4}-\frac{5}{12}\right]\\ &= \frac{3}{8}\cdot\frac{11}{6}=\boxed{\frac{11}{16}}.\end{aligned}$$

Problem 5

$$\begin{aligned}&F_Y(x)=\mathbb{P}(Y\le x)=\mathbb{P}\left(e^X\le x\right)=\mathbb{P}(X\le\ln(x))\\ &\Downarrow\frac{\mathrm{d}}{\mathrm{d}x}\\ &f_Y(x)=F_X'(\ln(x))\cdot\frac{1}{x}=f_X(\ln(x))\cdot\frac{1}{x}=\frac{1}{x}.\end{aligned}$$

Therefore

$$f_Y(x)=\begin{cases}\frac{1}{x} & 1<x<e\\ 0 & \text{otherwise}\end{cases}.$$

Problem 6

Part 1

$$\begin{aligned}\mathbb{P}\left(|X| > \frac{1}{2}\right) &= \mathbb{P}\left(\left(X > \frac{1}{2}\right) \cup \left(X < -\frac{1}{2}\right)\right) \\ &= \mathbb{P}\left(X > \frac{1}{2}\right) + \mathbb{P}\left(X < -\frac{1}{2}\right) \\ &= \frac{1}{2} \cdot \left(\frac{1}{2} + \frac{1}{2}\right) = \frac{1}{2}.\end{aligned}$$

Part 2

Let $Y = |X|$.

$$\begin{aligned}&F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}(|X| \leq y) = \mathbb{P}(-y \geq X \leq y) = \int_{-y}^{0} \frac{1}{2}\mathrm{d}y + \int_{0}^{y} \frac{1}{2}\mathrm{d}y \\ &\Downarrow \frac{\mathrm{d}}{\mathrm{d}y} \\ &f_Y(y) = \frac{\mathrm{d}}{\mathrm{d}y}\left[\int_{-y}^{0} \frac{1}{2}\mathrm{d}y + \int_{0}^{y} \frac{1}{2}\mathrm{d}y\right] = \frac{\mathrm{d}}{\mathrm{d}y}(y) = 1\end{aligned}$$

Therefore

$$f_Y(x) = \begin{cases} 1 & x \in [0,1) \\ 0 & x \notin [0,1) \end{cases}.$$

Problem 7

Part 1

$$\mathbb{P}(X > 10) = \int_{10}^{30} \frac{1}{30}\mathrm{d}x = \frac{2}{3}.$$

Part 2

$$\mathbb{P}(X > 25 | X > 15) = \frac{\mathbb{P}(X > 25)}{\mathbb{P}(X > 15)} = \frac{\int_{25}^{30} \mathrm{d}x}{\int_{15}^{30} \mathrm{d}x} = \frac{30 - 25}{30 - 15} = \frac{1}{3}.$$

Problem 8

$$
\begin{aligned}
\mathbb{E}[X^n] &= \int_0^1 x^n \mathrm{d}x \\
&= \int_0^1 x^n \mathrm{d}x \\
&= \frac{x^{n+1}}{n+1}\bigg|_0^1 = \frac{1}{n+1}.
\end{aligned}
$$

Problem 9

Let c be the capacity of the tank.

$$
\begin{aligned}
\mathbb{P}(X \geq c) &= 0.01 \\
\int_c^1 5(1-x)^4 \mathrm{d}x &= 0.01 \\
5\int_c^1 (1-x)^4 \mathrm{d}x &= 0.01 \\
-5\int_{1-c}^0 u^4 \mathrm{d}u &= 0.01 \\
5\int_0^{1-c} u^4 \mathrm{d}u &= 0.01 \\
5 \cdot \frac{(1-c)^5}{5} &= 0.01 \\
(1-c)^5 &= 0.01 \\
c &= 1 - \sqrt[5]{0.01} \implies \boxed{c \approx 0.6019}
\end{aligned}
$$

Problem 10

The position of the point that is randomnly chosen can be interpreted as a uniform random variable X from 0 to L. Consider the position of a point denoted by a. There are two seperate ratios to consider. If 0 to a is the short segment, then the ratio is

$$
\frac{a}{L-a}.
$$

If the a to $L-a$ segment is the short segment, then the ratio is

$$
\frac{L-a}{a}.
$$

These events are mutually exclusive. Therefore

$$\begin{aligned}
\mathbb{P}\left(\frac{\text{short}}{\text{long}} < \frac{1}{4}\right) &= \mathbb{P}\left(\left(\frac{X}{L-X} < \frac{1}{4}\right) \cup \left(\frac{L-X}{X} < \frac{1}{4}\right)\right) \\
&= \mathbb{P}\left(X < \frac{L}{5}\right) + \mathbb{P}\left(X > \frac{4L}{5}\right) \\
&= \frac{1}{5} + \frac{1}{5} = \boxed{\frac{2}{5}}
\end{aligned}$$