

2.2.1

- a) $A \cap B$ is the set of students within a mile of the school that walk to class
- b) $A \cup B$ is the set of students who are within a mile of the school or walk to class
- c) $A - B$ is the set of students who are within a mile of the school but do not walk to class
- d) $B - A$ is the set of students who walk to class that are not within a mile of the school

2.2.3

- a) $\{0, 1, 2, 3, 4, 5, 6\}$
- b) $\{3\}$
- c) $\{1, 2, 4, 5\}$
- d) $\{0, 6\}$

2.2.9

- a) We will show that both sets are subsets of each other.

Proof. Let $x \in U$. If $x \in A$, then $x \in A \cup \bar{A}$. If $x \notin A$, then $x \in \bar{A}$ and hence $x \in A \cup \bar{A}$. Therefore $U \subseteq A \cup \bar{A}$. Since A and \bar{A} are subsets of U by definition, then $A \cup \bar{A} \subset U$. Therefore $A \cup \bar{A} = U$. ■

- b) We will show that $A \cap \bar{A}$ contains no elements.

Proof. Let $x \in A$. Then by the definition of the complement, $x \notin \bar{A}$. Therefore none of the elements in A are in \bar{A} and hence $A \cap \bar{A} = \emptyset$. ■

2.2.13

Proof. Let $x \in A$. Then $x \in A \cup B$ and hence $x \in A \cap (A \cup B)$. Therefore $A \subseteq A \cap (A \cup B)$. Let $x \in A \cap (A \cup B)$. Then $x \in A$ and hence $A \cap (A \cup B) \subseteq A$. Therefore $A \cap (A \cup B) = A$. ■

2.2.21**Part A**

Proof. Let $x \in A - B$. Then $x \in A$ and $x \notin B$. Therefore $x \in \bar{B}$ meaning $x \in A \cap \bar{B}$. Hence $A - B \subseteq A \cap \bar{B}$. Let $x \in A \cap \bar{B}$. Then $x \in A$ and $x \in \bar{B}$. Therefore $x \notin B$ meaning $x \in A - B$. Hence $A \cap \bar{B} \subseteq A - B$ giving $A - B = A \cap \bar{B}$. ■

Part B**Proof.** ■**2.2.27**

- a) $\{4, 6\}$
- b) $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
- c) $\{4, 5, 6, 8, 10\}$

2.2.51

Proof. Assume towards contradiction that A is an infinite set and $A \cup B$ is a finite set. Since $A \cup B$ is finite, it has n elements where n is a natural number. However, A has more than n elements since it is infinite. Since all the elements of A are in $A \cup B$, then $A \cup B$ has more than n elements, a contradiction. ■

2.3.7

- a) $\text{dom}(f) = \mathbb{Z}^+ \times \mathbb{Z}^+$ and $\text{range}(f) = \mathbb{Z}^+$
- b) $\text{dom}(f) = \mathbb{Z}^+$ and $\text{range}(f) = \{n \in \mathbb{Z} : 0 \leq n \leq 9\}$
- c) $\text{dom}(f) = \text{set of all bit strings}$ and $\text{range}(f) = \mathbb{N}_0$
- d) $\text{dom}(f) = \text{set of all bit strings}$ and $\text{range}(f) = \mathbb{N}_0$

2.3.11

Only (a) is onto since every element of $\{a, b, c, d\}$ is in the range of f .

2.3.13

Only (a) and (d) are onto. (b) is not onto since it maps integers to strictly positive integers. (c) is not onto since there is no integer whose cube is 2.

2.3.15

- a) The function is onto since for any $n \in \mathbb{Z}$ we have $f(n, 0) = n$.
- b) The function is not onto since there is no $m, n \in \mathbb{Z}$ such that $m^2 + n^2 = 10$.
- c) The function is onto for any $n \in \mathbb{Z}$ we have $f(n, 0) = n$.
- d) The function is not onto since $f[\mathbb{Z} \times \mathbb{Z}] = \mathbb{Z}^+ \neq \mathbb{Z}$

2.3.23

Only (a) and (c) are bijections since the image of the \mathbb{R} under both is \mathbb{R} itself. (b) and (d) are not bijections because they are not onto.

2.3.31

a) $f(S) = \{1, 0, 0, 0, 1, 3\}$

b) $f(S) = \{0, 0, 1, 3, 5, 8\}$

c) $f(S) = \{0, 8, 16, 40\}$

d) $f(S) = \{1, 12, 33, 65\}$

2.3.37

Proof. It is not true in general. Consider $f : 2\mathbb{Z} \rightarrow \mathbb{Z} : n \mapsto \frac{n}{2}$ and $g : \mathbb{Z} \rightarrow \mathbb{Z} : n \mapsto 2n$. Note that f is onto and $(f \circ g)(n) = n$ and is also onto. However g is not onto. ■

2.3.47

Proof. Note that

$$f^{-1}(\overline{S}) = \{a \in A : f(a) \notin S\} = \overline{\{a \in A : f(a) \in S\}} = \overline{f^{-1}(S)}.$$

Hence both are equal. ■

2.3.73

Proof. Let $x \in U$ and f_S be defined as given.

a) If $x \in A \cap B$, then $x \in A$ and $x \in B$. Therefore $f_{A \cap B}(x) = 1$ and $f_A(x) \cdot f_B(x) = 1 \cdot 1 = 1$. If $x \notin A \cap B$, then $x \notin A$ or $x \notin B$. Therefore $f_{A \cap B}(x) = 0$ and since one of $f_A(x)$ or $f_B(x)$ are zero meaning $f_A(x) \cdot f_B(x) = 0$. Therefore in either case $f_{A \cap B}(x) = f_A(x) \cdot f_B(x)$.

b) There are three cases to consider.

- Assume that $x \in A \cup B$ and $x \in A \cap B$. Then $f_{A \cup B}(x) = 1$ and

$$f_A(x) + f_B(x) - f_A(x) \cdot f_B(x) = 1 + 1 - f_{A \cap B}(x) = 1 + 1 - 1 = 1.$$

- Assume that $x \in A \cup B$ and $x \notin A \cap B$. Then $f_{A \cup B}(x) = 1$. Since $x \notin A \cap B$, then either $x \notin A$ or $x \notin B$. Therefore

$$f_A(x) + f_B(x) - f_A(x) \cdot f_B(x) = 0 + 1 + f_{A \cap B}(x) = 0 + 1 + 0 = 1.$$

- Assume that $x \notin A \cup B$. Then $f_{A \cup B}(x) = 0$. Since $x \notin A \cup B$, $x \notin A$ and $x \notin B$ meaning

$$f_A(x) + f_B(x) - f_A(x) \cdot f_B(x) = 0 + 0 - 0 \cdot 0 = 0.$$

In all possible cases, the two sides are equal.

c) Let $x \in \overline{A}$. Then $f_{\overline{A}}(x) = 1$. Since $x \in \overline{A}$, $x \notin A$ meaning $1 - f_A(x) = 1 - 0 = 1$. Same holds for $x \notin \overline{A}$.

d) Let $x \in A \oplus B$. Then $x \in A, x \notin B$ or $x \notin A, x \in B$. Therefore

$$f_A(x) + f_B(x) - 2f_A(x) \cdot f_B(x) = 1.$$

Same holds for $x \notin A \oplus B$.

■

2.4.3

a) $a_0 = 2, a_1 = 3, a_2 = 5, a_3 = 9$

b) $a_0 = 1, a_1 = 4, a_2 = 27, a_3 = 256$

c) $a_0 = 0, a_1 = 0, a_2 = 1, a_3 = 1$

d) $a_0 = 0, a_1 = 1, a_2 = 2, a_3 = 3$

2.4.7

The sequences $a_n = 2^n$, $a_n = 1 + \left\lfloor \frac{(n+1)^2}{3} \right\rfloor$, and $a_{n+1} = a_n + (n - 1)$ all satisfy this condition.

2.4.11

a) $a_0 = 6, a_1 = 17, a_2 = 49, a_3 = 143, a_4 = 421$

b) Note that

- $a_2 = 49 = 85 - 36 = 5(17) - 6(6) = 5a_1 - 6a_0$
- $a_3 = 143 = 245 - 102 = 5(49) - 6(17) = 5a_2 - 6a_1$
- $a_4 = 421 = 715 - 294 = 5(143) - 6(49) = 5a_3 - 6a_2$

c) For $n \geq 2$,

$$\begin{aligned} 5a_{n-1} - 6a_{n-2} &= 5(2^{n-1} + 5 \cdot 3^{n-1}) - 6(2^{n-2} + 5 \cdot 3^{n-2}) \\ &= 5 \cdot 2^{n-1} + 25 \cdot 3^{n-1} - 6 \cdot 2^{n-2} - 30 \cdot 3^{n-2} \\ &= 5 \cdot 2^{n-1} + 25 \cdot 3^{n-1} - 3 \cdot 2^{n-1} - 10 \cdot 3^{n-1} \\ &= 2 \cdot 2^{n-1} + 15 \cdot 3^{n-1} \\ &= 2^n + 5 \cdot 3^n \\ &= a_n \end{aligned}$$

2.4.13

a) Yes

b) No

c) No

d) Yes

e) Yes

f) Yes

g) No

h) No

2.4.15**2.4.25**

- a) 1 one followed by 1 zero, 2 ones followed by 2 zeroes, etc. Next three terms would be 1, 1, 1
- b) Each integer greater than 1 with each even integer repeated twice. Next three terms would be 9, 10, 10
- c) $a_n = 2^{\frac{n}{2}}$ for even n and 0 otherwise. Next three terms would be 32, 0, 64
- d) $a_n = 3 \cdot 2^n$. Next three terms would be 384, 768, 1536
- e) $a_n = 15 - 7n$. Next three terms would be -34, -41, -48
- f) $a_n = \frac{(n+1)^2 + n + 5}{2}$. Next three terms would be 57, 68, 80
- g) $a_n = 2n^3$. Next three terms would be 1024, 1458, 2000
- h) $a_n = 1 + n!$. Next three terms would be 362881, 3628801, 39916801

2.4.29

$$\text{a) } \sum_{k=1}^5 k + 1 = 5 + \sum_{k=1}^5 k = 5 + \frac{5 \cdot 6}{2} = 20$$

$$\text{b) } \sum_{j=0}^4 (-2)^j = 1 - 2 + 4 - 8 + 16 = 11$$

$$\text{c) } \sum_{i=1}^{10} 3 = 30$$

$$\text{d) } \sum_{j=0}^8 2^{j+1} - 2^j = \sum_{j=0}^8 2^j = 2^9 - 1$$

2.4.31

$$\text{a) } S_8 = 1533$$

$$\text{b) } S_8 - 1 = 510$$

$$\text{c) } S_8 - (1 - 3) = 4923$$

$$\text{d) } S_8 = 9842$$

2.4.33

$$\text{a) } 21$$

$$\text{b) } 78$$

$$\text{c) } 18$$

$$\text{d) } 18$$

2.4.35

Since for every term a_i with $0 < i < n$ appears both as a positive and negative term in the sum, they all cancel out leaving behind just $a_n - a_0$.

2.4.39

$$\sum_{k=100}^{200} k = \sum_{k=0}^{100} (k + 100) = 100 \cdot 101 + \frac{100(101)}{2} = 15150.$$

2.5.1

- | | |
|--|---|
| a) Countably infinite; $1, 2, 3, 4, \dots$ | b) Countably infinite; $0, 2, -2, 4, -4, \dots$ |
| c) Countably infinite; $99, 98, 97, \dots$ | d) Uncountable |
| e) Finite | f) Countably infinite; $0, 7, 7, 14, 14, \dots$ |

2.5.3

- | | |
|--|--|
| a) Countable; map n to the bit string contain n ones | b) Countable; follow the diagonalization argument excluding top three rows |
| c) Uncountable | d) Uncountable |

2.5.7

For each n , take the guest in room $2n$ and put them in room n of the original building and the guest in room $2n + 1$ into room n of the new building.

2.5.11

- a) $[-1, 0] \cap [0, 1] = \{0\}$
 b) $([-2, -1] \cup \mathbb{Z}) \cap ([1, 2] \cup \mathbb{Z}) = \mathbb{Z}$
 c) $[-2, 2] \cap [-1, 1] = [-1, 1]$

2.5.17

No. Notice that $[-4, 4] - [-1, 1] = [-4, -1) \cup (1, 4]$ which is still uncountable.

2.5.27

Proof. Let A_1, A_2, A_3, \dots be a list of countably countable sets. For a given set A_i , its elements can be listed out since it is countable in the manner

$$a_{i1}, a_{i2}, a_{i3}, \dots$$

Listing out the elements of $\bigcup A_i$ is simply listing out all elements a_{ij} where $i + j = 2$, then $i + j = 3$, and so forth. ■

2.5.31**2.6.1**

- | | | |
|---------------------------------|--|--|
| a) A is a 3×4 matrix | b) $\begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$ | c) $\begin{bmatrix} 2 & 0 & 4 & 6 \end{bmatrix}$ |
|---------------------------------|--|--|

d) 1

$$\text{e) } A^t = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 1 & 4 & 3 \\ 3 & 6 & 7 \end{bmatrix}$$

2.6.5

$$A = \begin{bmatrix} \frac{9}{5} & -\frac{6}{5} \\ -\frac{1}{5} & \frac{4}{5} \end{bmatrix}.$$

2.6.7

Note that for any entry in A_{ij} that

$$A_{ij} + \mathbf{0}_{ij} = \mathbf{0}_{ij} + A_{ij} = A_{ij} + 0 = A_{ij}.$$

Therefore the resultant matrix remains unchanged.

2.6.11

If AB has size $m \times n$, then BA has size $n \times m$.

2.6.15

$$A^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}.$$

This can be shown via induction using the base case

$$A^2 = AA = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

2.6.25

The system can be expressed as a matrix linear equation

$$\underbrace{\begin{bmatrix} 7 & -8 & 5 \\ -4 & 5 & -3 \\ 1 & -1 & 1 \end{bmatrix}}_A \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_x = \underbrace{\begin{bmatrix} 5 \\ -3 \\ 0 \end{bmatrix}}_b.$$

The solution is then $A^{-1}b$ where A^{-1} is given in exercise 18. Hence

$$x_1 = 1, x_2 = -1, x_3 = -2.$$

2.6.27

a)

$$A \vee B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

b)

$$A \wedge B = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

c)

$$A \odot B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

2.6.29

a)

$$A^{[2]} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

b)

$$A^{[3]} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

c)

$$A \vee A^{[2]} \vee A^{[3]} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

3.1.1

max := 1 → i := 2 → max := 8 → i := 3 → max := 12 → i := 4 → i := 5 → i := 6 → i := 7 →
 max := 14 → i := 8 → i := 9 → i := 10 → i := 11

3.1.3

Algorithm 1 Sum of sequece of integers a_1, \dots, a_n

```

1: function SUM_SEQUENCE( $a_1, \dots, a_n$ )
2:   sum :=  $a_1$ 
3:   for  $i = 2$  to  $n$  do
4:     sum = sum +  $a_i$ 
5:   end for
6:   return sum
7: end function

```

3.1.9

Algorithm 2 Determine if a given string is a palindrome

```

1: function IS_PALINDROME( $s_0s_1 \dots s_{n-1}$  : string)
2:   for  $i = 0$  to  $\lfloor \frac{n}{2} \rfloor - 1$  do
3:     if  $s_i \neq s_{n-i-1}$  then
4:       return False
5:     end if
6:   end for
7:   return True
8: end function

```

3.1.13

Linear Search) $i := 1 \rightarrow i := 2 \rightarrow i := 3 \rightarrow i := 4 \rightarrow i := 5 \rightarrow i := 6 \rightarrow i := 7 \rightarrow \text{location} := 7$

Binary Search) $i := 1 \rightarrow j := 8 \rightarrow m := 4 \rightarrow i := 5 \rightarrow m := 6 \rightarrow i := 7 \rightarrow m := 7 \rightarrow j := 7 \rightarrow \text{location} := 7$

3.2.1a) $C = 1, k = 10$ b) $C = 4, k = 7$ c) Is not $O(x)$ d) $C = 5, k = 1$ e) $C = 1, k = 0$ f) $C = 1, k = 2$ **3.2.3**

Since $x^4 + 9x^3 + 4x + 7 \leq 10000x^3$ for $x > 10$, we have witness $C = 10000, k = 10$.

3.2.7a) $n = 3$ b) $n = 3$ c) $n = 1$ d) $n = 0$ **3.2.11**

Since we are dealing with polynomials,

$$O(3x^4 + 1) = O(x^4) = O\left(\frac{x^4}{2}\right).$$

3.3.3

It is $O(n^2)$ since

$$\sum_{i=0}^n \sum_{k=i}^n 1 \sim n^2.$$

3.3.11**Part A**

Algorithm 3 Determine if a disjoint subset pair exists

```
1: function EXISTS_DISJOINT( $S_1, \dots, S_n$ : subsets of  $\{1, \dots, n\}$ )
2:   disjoint := False
3:   for  $i = 1$  to  $n$  do
4:     for  $j = i + 1$  to  $n$  do
5:       disjoint = True
6:       for  $k = 1$  to  $n$  do
7:         if  $k \in S_i \wedge k \in S_j$  then
8:           disjoint = False
9:         end if
10:      end for
11:    end for
12:  end for
13:  return disjoint
14: end function
```

Part B

Since there are $O(n^2)$ sets considered and $O(n)$ integers tested for each, then in total the answer is on the order of $O(n^3)$.