Math 140A: Elementary Analysis

Eli Griffiths

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Table of Contents

Introduction

1.1 The Natural Numbers

First examine the natural numbers. It is very common knowledge that 1 is a natural number and you obtain the rest by increasing the previous by 1. This is however not a rigorous construction of the natural numbers. An example of a rigorous construction is the **Peano axioms**

Definition 1.1 (Peano Axioms). The natural numbers are axiomatically defined by

1. $1 \in \mathbb{N}$
2. If $n \in \mathbb{N}$, then $n + 1 \in \mathbb{N}$
3. 1 is the first element, meaning it is not the sucessor of any element
4. If $S \subset \mathbb{N}$ such that $1 \in S$ and $n \in S$ implies $n + 1 \in S$, then $S = \mathbb{N}$

While the Peano Axioms are not strong enough for modern math, they are sufficient for lots of math and at least open up the world of rigorous axiomatic constructions. Consider axiom 4. Assume that it is not true. Then there is an $S \subset \mathbb{N}$ such that $1 \in S$ and $n \in S \implies n + 1 \in S$ but $S \neq \mathbb{N}$. Then let $n_0 = \min\{n \in \mathbb{N} : n \notin S\}$. Since $1 \in S$, $n_0 \neq 1$ and hence n_0 is the successor of $n_0 - 1$. However since $n \in S \implies n + 1 \in S$ and $n_0 - 1 \in S$, $n_0 \in S$ and therefore a contradiction.

While this is a persuasive and intuitive argument, it does not constitute a proof as the existence of n_0 is assumed because of the assumption of a minimum element in a non-empty subset of $\mathbb{N}$.

1.1.1 Mathematical Induction

Theorem 1.1 (Induction). If $S_1, S_2, S_3, \ldots$ are statements, all are true if

1. S_1 is true
2. $S_n \implies S_{n+1}$

For simplicity, the proof of induction shall be left more so as accepting the last Peano Axiom that declares its validity.

Example 1.1. Consider the statement $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$.

Proof. Consider the base case $n = 1$. Then $1 = \frac{1(2)}{2} = 1$, therefore the base case holds. Assume that for a fixed $n \in \mathbb{N}$ that $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$. Then it follows that

$$\begin{aligned} 1 + 2 + 3 + \ldots + n &= \frac{n(n+1)}{2} \\ 1 + 2 + 3 + \ldots + n + (n+1) &= \frac{n(n+1)}{2} + (n+1) \\ 1 + 2 + 3 + \ldots + (n+1) &= \frac{(n+1)(n+2)}{2} \end{aligned}$$

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Example 1.2. Consider the statement $|\sin(nx)| \leq |n \sin(x)|, \forall x \in \mathbb{R}$.

Proof. The base clearly holds. Assume that for a fixed $n \in \mathbb{N}$ that $|\sin(nx)| \leq |n \sin(x)|, \forall x \in \mathbb{R}$. Then

$$\begin{aligned} |\sin((n+1)x)| = |\sin(nx + x)| &= |\sin(nx)\cos(x) + \cos(nx)\sin(x)| \\ &\leq |\sin(nx)||\cos(x)| + |\cos(nx)||\sin(x)| \\ &\leq |\sin(nx)| + |\sin(x)| \\ &\leq n|\sin(x)| + |\sin(x)| \\ &\leq (n+1)|\sin(x)| \end{aligned}$$

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Extending the Naturals

2.1 Rational Numbers

Definition 2.2. The rational numbers is the set of numbers of the form $\frac{p}{q}$ where $p \in \mathbb{Z}$ and $q \in \mathbb{N}$.

Rational numbers are the first number system that provides a nice comprehensive structure. Multiplication, division, addition, and subtraction are all closed operations making it a strong number system.

Theorem 2.2 (Rational Root Theorem). Let $c_0, c_1, \ldots, c_n \in \mathbb{Z}$. If r solves $c_n x^n + c_{n-1}x^{n-1} + \ldots + c_1 x + c_0 = 0$, $c_n \neq 0 \neq c_1$ and $r = \frac{p}{q}$ where p and q are coprime

$$p|c_0, \quad q|c_n$$

Proof. Let r be a rational solution to the polynomial equation $c_n x^n + c_{n-1}x^{n-1} + \ldots + c_1 x + c_0 = 0$. Since $r \in \mathbb{Q}$, $r = \frac{p}{q}$ where $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. Then

$$\begin{aligned} c_n \left(\frac{p}{q}\right)^n + c_{n-1}\left(\frac{p}{q}\right)^{n-1} + \ldots + c_1\left(\frac{p}{q}\right) + c_0 &= 0 \\ c_n p^n + c_{n-1} q p^{n-1} + \ldots + c_1 q^{n-1} p + c_0 q^n &= 0 \\ -c_n p^n - c_{n-1} q p^{n-1} - \ldots - c_1 q^{n-1} p &= c_0 q^n \\ -p\left[c_n p^{n-1} - c_{n-1} q p^{n-2} - \ldots - c_1 q^{n-1}\right] &= c_0 q^n \end{aligned}$$

Therefore $p|c_0 q^n$. Since p and q are coprime, p must divide c_0. By solving for $c_n p^n$ instead, it follows that q divides c_n. ∎

While rationals are quite nice, there are many equations that have solutions that cannot be represented by a rational number.

Example 2.3 ($\sqrt{2}$). Consider the equation $x^2 - 2$. Its solutions by the Rational Root Theorem must be an integer. However no integer satisfies the equation and therefore there is no rational root for $x^2 - 2$.

2.2 Algebraic Numbers

Definition 2.3 (Algebraic Number). A number is called algebraic if it is the root of an integer coeffecient polynomial. That is, it is a solution to

$$c_n x^n + c_{n-1}x^{n-1} + \ldots + c_1 x + c_0 = 0$$

where $c_i \in \mathbb{Z}, c_i \neq 0$ and $n \geq 1$.

Many numbers that are used day to day are algebraic. It follows clearly that all integers are algebraic and all rationals are algebraic. Other numbers such as the $\sqrt{2}$ are algebraic. Even the number $\sqrt{2+\sqrt[3]{5}}$ is algebraic. However, there are infinitely many other numbers that are not algebraic such as π and e.

Real Numbers

As seen above, both the rationals and algebraic numbers can be very useful but fail to encapsulate important types of numbers. That is, both $\mathbb{Q}$ and the algebraic numbers have gaps in them, that is the irrationals for $\mathbb{Q}$ and transcendtals for algebraic numbers.

2.2.1 Ordering Structure

Definition 2.4 (Ordered Field). We say a field with a relation $(\mathbb{F}, +, \cdot, \leq)$ is an ordered field if it satisfies the following properties:

1. $p \leq q$ or $q \leq p$ for all $p, q \in \mathbb{F}$
2. $p \leq q$ and $q \leq p \implies p = q$
3. $p \leq q$ and $q \leq r \implies p \leq r$
4. $p \leq q \implies p + r \leq q + r$
5. $p \leq q \implies pr \leq qr$ for all $r \in \mathbb{F} \geq 0$

Certain properties are derivable from the properties and ordering of $\mathbb{R}$.

Theorem 2.3 (Properties of $\mathbb{R}$). For all $p, q, r \in \mathbb{R}$

1. $p + r = q + r \implies p = q$
2. $p \cdot 0 = 0 = 0 \cdot p$
3. $(-p)q = -(pq)$
4. $(-p)(-q) = pq$
5. $pr = qr \implies p = q$ if $r \neq 0$
6. $pq = 0 \implies p = 0$ or $q = 0$

Proof. Let $p, q, r \in \mathbb{R}$ for the following.

(1) Assume that $p+r = q+r$. Since additive inverses exist, $p+r+(-r) = q+r+(-r)$. By associativity, $p+(r+(-r)) = q+(r+(-r))$. By definition of inverses, $p+0 = q+0$. By the additive identity, $p = q$.

(2) Examine $p \cdot 0$. Note that $p \cdot 0 = p \cdot (0+0)$. By distribution, $p \cdot 0 + p \cdot 0 = p \cdot 0$. This means that $p \cdot 0$ does not change when added to itself, which is by definition

the additive identity. Therefore $p \cdot 0 = 0$.

(3) Consider the expression $pq + (-p)q$. By distributivity, $pq + (-p)q = (p + (-p))q$. By inverses, $pq + (-p)q = 0 \cdot q = 0$. Therefore $-pq = (-p)q$.

(4) To be completed

(5) To be completed

(6) Assume that $pq = 0$. WLOG, let $q \neq 0$. Since multiplicative inverses exist, $0 = q^{-1} \cdot 0 = 0 \cdot q^{-1} = pqq^{-1} = p(qq^{-1}) = p$. Therefore $p = 0$.

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When considering the ordered field of the reals, more properties are derivable.

Theorem 2.4 (Properties of Ordered Reals). Let $p, q, r \in \mathbb{R}$

1. $p \leq q \implies -q \leq -p$
2. $p \leq q, r \leq 0 \implies qr \leq pr$
3. $p \geq 0, q \geq 0 \implies pq \geq 0$
4. $p^2 \geq 0$
5. $0 < 1$
6. $p > 0 \implies p^{-1} > 0$
7. $0 < p < q \implies 0 < q^{-1} < p^{-1}$

Remark. $p < q$ is defined as $p \leq q$ and $p \neq q$.

Proof. Let $p, q, r \in \mathbb{R}$

(1) Assume that $p \leq q$. Let $r = (-p) + (-q)$. Since adding a number to both sides of a inequality preserves it, $p + r \leq q + r$. Then $p + (-p) + (-q) \leq q + (-p) + (-q)$. By commutativity and associativity, $(p + (-p)) + (-q) \leq (q + (-q)) + (-p)$. By inverses and additive identity, $-q \leq -p$.

(2) Assume that $p \leq q$ and that $r \leq 0$. By (1), $-r \leq 0$. Therefore, $p(-r) \leq q(-r)$ hence $-pr \leq -qr$. By (1), $qr \leq pr$.

(3) To complete

(4) By the properties of an ordered field, $p \leq 0$ or $p \geq 0$. If $p \geq 0$, then by (3), $p^2 = p \cdot p \geq 0$. If $p \leq 0$, then $-p \geq 0$. By 2.4.4, $p^2 = (-p)(-p) \geq 0$ by the first case.

(5) To complete

(6) Assume towards contradiction that $p > 0$ and $p^{-1} \leq 0$. By (1), $-p^{-1} \geq 0$. Since p and $-p^{-1}$ are non-negative, $p(-p^{-1}) \geq 0$. This means that $-1 \geq 0$ or equivalently $1 \leq 0$. By (5), this is a contradiction.

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2.2.2 Absolute Value

Definition 2.5 (Absolute Value). Let $p, q \in \mathbb{R}$.

$$|p| := \begin{cases} p & p \geq 0 \\ -p & p \leq 0 \end{cases}$$

Additionally, define the distance between two reals as

$$\text{dist}(p, q) = |p - q|$$

Theorem 2.5 (Properties of Absolute Value). Let $p, q \in \mathbb{R}$.

1. $|p| \geq 0$
2. $|pq| = |p||q|$
3. $|p + q| \leq |p| + |q|$

Proof. Let $p, q \in \mathbb{R}$.

(1) If $p \geq 0$, then $|p| \geq 0$. If $p \leq 0$, then $|p| \geq 0$. Therefore $|p| \geq 0$ for all p.

(2) If $p \geq 0, q \geq 0$. Then $|pq| = pq = |p||q|$. If $p \leq 0, q \leq 0$, then $-p \geq 0, -q \geq 0$ and $|p||q| = (-p)(-q) = pq = |pq|$.

(3) Note that $-|p| \leq p \leq |p|$. This is because p either is $|p|$ or $|p| = -p$ meaning $p = -|p|$. Same is true for q. Therefore

$$\begin{aligned} -|p| + (-|q|) \leq -|p| + q \leq p + q \leq |p| + q \leq |p| + |q| \\ -(|p| + |q|) \leq p + q \leq |p| + |q| \end{aligned}$$

The derived inequality shows that $p + q \leq |p| + |q|$ and $-(p + q) \leq |p| + |q|$. Since $|p + q|$ is either $p + q$ or $-(p + q)$, $|p + q| \leq |p| + |q|$.

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Corollary 2.1 (Distance Triangle Inequality).

$$\text{dist}(p, r) \leq \text{dist}(p, q) + \text{dist}(q, r)$$

2.5.3 is an important property of the absolute value, usually referred to as the *triangle* *inequality*.

Axiom of Completeness

3.1 Bounds

Definition 3.6 (Upper and Lower Bound). Let S be a non-empty subset of $\mathbb{R}$. An upper bound of S is a number M such that $s \leq M$ for all $s \in \mathbb{R}$. A lower bound of S is a number m such that $s \geq M$ for all $s \in \mathbb{R}$.

Note that any finite subset of $\mathbb{R}$ will admit an upper and lower bound as a larger/smaller number can always be chosen compared to any number in the set. There can potentially be infinitely many bounds on a set, but there is an important refinement that can be made.

Definition 3.7 (Supremum and Infimum). Let $S \subset \mathbb{R}$. If there exists an upper bound M for S such that for any other upper bound s $M \leq s$, M is called the *least upper bound* for S or equivalently the supremum (notated as $\sup S$). The same logic for lower bounds gives rise to the infimum or *greatest lower bound* (notated as $\inf S$).

Consider a finite subset of $\mathbb{R}$. Then it follows that the subset will have a minimum and maximum as each element can be checked against each other because it is finite. This does not generalize to an infinite subset of $\mathbb{R}$.

Example 3.4. Consider the set $S = \left\{r \in \mathbb{Q} : 0 \leq r, r^2 < 2\right\}$. In $\mathbb{R}$, 0 is a minimum and $\sqrt{2}$ is the supremum. Note that $\sqrt{2} \notin S$. Alternatively, if working over $\mathbb{Q}$, there is no supremum as $\sqrt{2} \notin \mathbb{Q}$.

Remark. The supremum or infimum of a set does not necessarily have to be an element of said set.

Theorem 3.6 (Uniqueness of Supremum and Infimum). If a set $S \subset \mathbb{R}$ has a supremum or infimum, then said supremum or infimum is unique.

Proof. Let $S \subset \mathbb{R}$. Assume that S has two supremum M and M'. By definition of a supremum, $M \leq M'$ and $M' \leq M$. Therefore $M = M'$. Same argument applies to the infimum. ■

Example 3.5. Consider the set

$$D = \left\{x \in \mathbb{R} : x^2 < 10\right\}.$$

Then $\sup D = \sqrt{10}$ and $\inf D = -\sqrt{10}$. Since $\pm\sqrt{10} \notin D$, there is no max or min.

3.2 The Completeness Axiom

The completeness axiom is a defining charactersitc of $\mathbb{R}$ that differentiates it from $\mathbb{Q}$. It can be interpreted as requiring there be no gaps between numbers.

Definition 3.8 (Axiom of Completeness). Let S be a non-empty subset of $\mathbb{R}$. If S is bounded above, then $\sup S$ exists.

Consider the set from example 3.4. When working over $\mathbb{Q}$, there exists upper bounds (such as 4), but it does not admit a least upper bound. In contrast, working over $\mathbb{R}$ admits a supremum. This distinction is what makes $\mathbb{R}$ useful for much of analysis and calculus. While the Axiom of Completeness only stipulates the existence of a supremum, it can be derived that the equivalent statement for lower bounds and infimum follows.

Corollary 3.2 (Axiom of Completeness Reversed). Let S be a non-empty subset of $\mathbb{R}$. If S is bounded below, then $\inf S$ exists.

Proof. Let S be a non-empty set that is bounded below. Therefore there exists an m such that $m \leq s$ for all $s \in S$. Equivalently, $-s \leq -m$ for all $s \in S$. Consider the set $-S = \{-s : s \in S\}$. $-s \leq -m$ for all $s \in S$ implies $-S$ is bounded above by $-m$ and therefore by the Axiom of Completeness $\sup(-S) = s_0$ exists. Therefore $r \leq s_0$ for all $r \in -S$ meaning $-s \leq s_0$ for all $s \in S$. Flipping the inequality gives $-s_0 \leq s$ for all $s \in S$, meaning $-s_0$ is a lower bound for S. ■

Theorem 3.7 (Archimedean Property). Let $a, b \in \mathbb{R} > 0$. Then $\exists n \in \mathbb{N}$ such that $an > b$.

Consider the special case when $b = 1$. Then $an > b \implies a > \frac{1}{n}$ for some $n \in \mathbb{N}$ meaning there is always a rational number smaller than any positive real number. In the case that $a = 1$, then $an > b \implies n > b$ for some $n \in \mathbb{N}$ meaning there is always a rational/integer larger than any positive real number.

Proof. Assume towards contradiction that $\exists a, b \in \mathbb{R} > 0$ such that $na \leq b$ for all $n \in \mathbb{N}$. Define the set $S = \{na : n \in \mathbb{N}\}$. Note that b is an upper bound of S. Therefore by the Axiom of Completeness, $\sup S = s_0$ exists. Since $a > 0$, then $a + s_0 > s_0$ or $s_0 - a < 0$. Note that $s_0 - a$ cannot be an upper bound as s_0 is the least upper bound of S. But note that then $s_0 - a < n_0 a$ for some $n_0 \in \mathbb{N}$ (because $s_0 - a$ is not an upper bound and therefore there is an element in the set S larger than it). However, this implies that $s_0 < (n_0 + 1)a$ and since $(n_0 + 1)a \in S$, s_0 is not a least upper bound. Therefore there cannot exist such a, b. ■

The Archimedean property shows that rational numbers are "everywhere", a concept further emboldended by the idea that the rationals are *dense* in $\mathbb{R}$.

Theorem 3.8 ($\mathbb{Q}$ is Dense in $\mathbb{R}$). Let $a, b \in \mathbb{R}$ with $a < b$. Then $\exists r \in \mathbb{Q}$ such that $a < r < b$.

Proof. Let $a, b \in \mathbb{R}$ such that $a < b$. This means that $b - a > 0$. By the Archimedean principle, there exists n such that $n(b - a) = nb - na > 1$. Therefore there exists $k \in \mathbb{N}$ such that $k > \max\{|an|, |bn|\}$, meaning $-k < an < bn < k$. Two things can be said about k

$$k \in K = \{j \in \mathbb{Z} : -k \leq j \leq k\}$$
$$k \in L = \{j \in K : an < j\}$$

Note that both sets are finite because the first has $2k + 1$ elements and the second is a subset of K. Define $m := \min L$ (which exists since L is non-empty and finite). Then $-k < an < m$. Therefore $m > -k$ meaning $m - 1 \in K$. Note that $an < m - 1$ is false since m is the minimum value where that inequality holds. Then $m - 1 \leq an$ meaning $m \leq an + 1 < bn$ (since $nb - na > 1$). Therefore since $an < m$, $an < m < bn$ or equivalently $a < \frac{m}{n} < b$. Since $\frac{m}{n} \in \mathbb{Q}$, there is a rational between a and b. ∎

Sequences

4.1 Limits of Sequences

Definition 4.9 (Sequence). A sequence is a mapping $s : \mathbb{N}_{\geq m} \to \mathbb{R}$ where m is typically 0 or 1. Alternatively, a sequence cant be thought of as an infinite tuple

$$s = (s_m, s_{m+1}, s_{m+2}, \ldots)$$

Define the image of a sequence as $S(\mathbb{N}_{\geq m}) := \{s_n : n \geq m\}$

Example 4.6. Consider $(s_n)_{n\in\mathbb{N}}$ given by $s_n = \frac{(-1)^n}{n^3}$. It is a sequence with $m = 1$ and looks like $(-1, \frac{1}{8}, -\frac{1}{27}, \ldots)$.

Example 4.7. Consider $(s_n)_{n\in\mathbb{N}_0} = (-1)^n$ which is the sequence $(1, -1, 1, -1, 1, \ldots)$. Note that the image $S(\mathbb{N}_0) = \{-1, 1\}$

Example 4.8. Consider $(s_n)_{n\in\mathbb{N}_0} = \left(1 + \frac{1}{n}\right)^n$ which gives a sequence of real numbers that 'appears' to get closer e as n grows large, as seen by the fact that $s_{1,000,000} = 2.718280469319377$.

4.1.1 Convergence of a Sequence

Definition 4.10 (Sequence Convergence). A sequence $(s_n)_{n\in\mathbb{N}_0}$ is said to converge to $s_0 \in \mathbb{R}$ if

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } |s_n - s| < \epsilon, \forall n > N$$

Pictorally, we are creating a neighborhood of 2ϵ around s_0. And if the sequence converges, there is an eventual s_N such that every subsequent number is within the neighbor hood around s_0.

Example 4.9. Consider $s_n = \frac{1}{n}$. Take $\epsilon > 0$. Note that $|s_n - 0| = \frac{1}{n} < \epsilon$ by the archimedean property. It is clearer if it is rewritten as $1 < n\epsilon$.

Example 4.10. Consider $s_n = (-1)^n, n \in \mathbb{N}$. Take $e > 0$.

Example 4.11. Consider $s_n = \frac{3n+1}{7n-4}, n \in \mathbb{N}$. A good guess for the limit is $\frac{3}{7}$ since the $3n$

and $7n$ terms 'dominate' as $n \to \infty$. Take $\epsilon > 0$. Then

$$\begin{aligned}
\left|\frac{3n+1}{7n-1} - \frac{3}{7}\right| &= \left|\frac{21n+7-21n+12}{7(7n-4)}\right| \\
&= \left|\frac{19}{7} \cdot \frac{1}{7n-4}\right| \\
&= \frac{19}{7} \cdot \frac{1}{7n-4} \\
&\leq \frac{19}{49} \cdot \frac{1}{n-1}
\end{aligned}$$

Since $\frac{1}{n-1} \to 0$ as $n \to \infty$.

Example 4.12. Consider $s_n = \sqrt[n]{n}, n \in \mathbb{N}$. Take $s_0 = 1$, and prove this much later.

Theorem 4.9 (Uniqueness of Limits). If a limit of a sequence exists, then it is unique.

Proof. Let s_n be a sequence that converges to s and s' as $n \to \infty$. Then

$$\begin{aligned}
\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } |s_n - s| < \frac{\epsilon}{2}, \forall n > N \\
\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } |s_n - s'| < \frac{\epsilon}{2}, \forall n > N
\end{aligned}$$

Then

$$\begin{aligned}
|s - s'| &= |s - s_n + s_n - s'| \\
&\leq |s_n - s| + |s_n - s'| < \epsilon
\end{aligned}$$

Therefore $0 \geq |s - s'| < \epsilon$ for all $e > 0$, meaning $s = s'$. ∎

Example 4.13. Consider $\lim_{n\to\infty} \frac{1}{n^2}$. Let $s_0 = 0$.

Example 4.14. Consider $\lim_{n\to\infty} \frac{4n^3+3n}{n^3-6} \stackrel{?}{=} 4$. Take $\epsilon > 0$. Then

$$\begin{aligned}
\left|\frac{4n^3+3n}{n^3-6} - 4\right| &= \left|\frac{4n^3+3n-4n^3+24}{n^3-6}\right| \\
&= \frac{3n+24}{|n^3-6|}
\end{aligned}$$

Note that $3n + 24 \leq 27n$ for all $n \in \mathbb{N}$ and $n^3 - 6 \geq \frac{n^3}{4}$ for $n \geq 2$.

$$\begin{aligned} &\leq 4 \cdot \frac{27n}{n^3} \\ &= \frac{108}{n^2} < \epsilon \end{aligned}$$

Take $N \in \mathbb{N} \geq \sqrt{\frac{108}{\epsilon}}$. Then

$$\left| \frac{4n^3 + 3n}{n^3 - 6} - 4 \right| \leq 108$$

Theorem 4.10. Let $(s_n)_{n \in \mathbb{N}}$ be a sequence $s_n \geq 0$ for all n and $s = \lim_{n \to \infty} s_n$. Then $\lim_{n \to \infty} \sqrt{s_n} = \sqrt{s}$

Proof. Consider $|\sqrt{s_n} - \sqrt{s}|$.

$$\begin{aligned} |\sqrt{s_n} - \sqrt{s}| &= \left| \frac{(\sqrt{s_n} - \sqrt{s})(\sqrt{s_n} + \sqrt{s})}{\sqrt{s_n} + \sqrt{s}} \right| \\ &= \frac{|s_n - s|}{\sqrt{s_n} - \sqrt{s}} \end{aligned}$$

If $s > 0$, then $\frac{1}{\sqrt{s_n} + \sqrt{s}} \leq \frac{1}{\sqrt{s}}$ meaning we would want $\frac{|s_n - s|}{\sqrt{s}} < \epsilon$ or equivalentely $|s_n - s| < \epsilon\sqrt{s}$. If $s = 0$, we want $\sqrt{s_n} < \epsilon$ or $s_n < \epsilon^2$. Now, formally:

∎

Theorem 4.11. Let $(s_n)_{n \in \mathbb{N}}$ be convergent to $s \neq 0$ with $\forall n \in \mathbb{N}, s_n \neq 0$. Then $\inf\{|s_n| : n \in \mathbb{N}\} > 0$.

Proof. The idea is that given a neighborhood around s, there is a finite amount of values of the sequence outside of it. By choosing a neighborhood size of $\frac{|s|}{2}$, 0 is avoided. Therefore proceed with the formal proof by letting $\epsilon = \frac{|s|}{2}$. Since s_n converges and $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that $|s_n - s| < \frac{|s|}{2}$ for all $n > N$. Note that

$$||s_n| - |s|| \leq |s_n - s| < \frac{|s|}{2}, \forall n > N$$

and that

$$|s_n| \in (s - \epsilon, s + \epsilon), \forall n > N$$

∎

Definition 4.11 (Bounded Series). A series $(s_n)_{n \in \mathbb{N}}$ is bounded if the image is bounded. Equivalently, it is bounded if $\exists M \in \mathbb{R}$ such that $s_n \leq M$ for all $n \in \mathbb{N}$.

Theorem 4.12 (Convergence Implies Boundedness). Let $(s_n)_{n\in\mathbb{N}}$ be a series that converges to s. Then the series is bounded.

Proof. Let $(s_n)_{n\in\mathbb{N}}$ be a series and assume it converges to s. Take $\epsilon = 1$ and find $N \in \mathbb{N}$ such that $|s_n - s| < 1$ for all $n > N$. Therefore s_n and s are at most 1 apart, therefore $|s_n| \leq |s| + 1$. This provides an upperbound on s_n for $n > N$. For $n \leq N$, construct the set $M = \{s_1, s_2, \ldots, s_N, |s| + 1\}$. Then note that

$$s_n \leq \max M < \infty, \forall n \in \mathbb{N}$$

Therefore the series is bounded. ∎

Theorem 4.13 (Properties of Limits). The following properties hold for all limits of sequences.

a) $\lim_{n\to\infty} s_n = s, c \in \mathbb{R} \implies \lim_{n\to\infty} c \cdot s_n = c \lim_{n\to\infty} s_n$

b) $\lim_{n\to\infty} s_n = s, \lim_{n\to\infty} t_n = t \implies \lim_{n\to\infty}(s_n + t_n) = s + t$

c) $\lim_{n\to\infty} s_n = s, \lim_{n\to\infty} t_n = t \implies \lim_{n\to\infty}(s_n \cdot t_n) = st$

d) $\lim_{n\to\infty} s_n = s, s_n \neq 0 \forall n \in \mathbb{N}, s \neq 0 \implies \lim_{n\to\infty} \frac{1}{s_n} = \frac{1}{s}$

Proof. Let s_n and t_n be sequences that converge to s and t respectively.

a) TODO

b) Since both sequences are converget, they both admit $N_1, N_2 \in \mathbb{N}$ such that for an $\epsilon > 0$

$$|s_n - s| < \frac{\epsilon}{2}, \forall n > N_1$$
$$|t_n - t| < \frac{\epsilon}{2}, \forall n > N_2$$

Note then that

$$|(s_n - s) + (t_n - t)| \leq |s_n - s| + |t_n - t| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \forall n > \max\{N_1, N_2\}$$

c)

d) Note that

$$|s_n t_n - st| = |s_n(t_n - t) + (s_n - s)t| \leq |s_n||t_n - t| + |t||s_n - s|$$

Since s_n and t_n converge, they are bounded. Therefore, take $\epsilon > 0$ and note

$$\begin{aligned}
&\exists M > 0, |s_n| \leq M, |t_n| \leq M, \forall n \in \mathbb{N} \\
&\exists N_1, |s_n - s| < \frac{\epsilon}{2M}, \forall n > N_1 \\
&\exists N_2, |t_n - t| < \frac{\epsilon}{2M}, \forall n > N_2
\end{aligned}$$

Therefore

$$|s_n t_n - st| \leq M \cdot \frac{\epsilon}{2M} + M \cdot \frac{\epsilon}{2M} = \epsilon, \forall n > \max\{N_1, N_2\}$$

e) Consider the target expression $\left|\frac{1}{S_n} - \frac{1}{s}\right|$. This can be reformed into $\frac{1}{s_n \cdot s}|s_n - s|$. Since $s_n \neq 0$ and $s \neq 0$, $|s_n|$ is bounded below by a positive number m for all n. This also means that $s \geq m$. Therefore

$$\left|\frac{1}{s_n} - \frac{1}{s}\right| \leq \frac{1}{m^2}|s_n - s|$$

Formally, take $\epsilon > 0$. Since s_n converges, $\exists N \in \mathbb{N}$ such that

$$|s_n - s| < \epsilon m^2, \forall n > N$$

By the previous derivation,

$$\left|\frac{1}{s_n} - \frac{1}{s}\right| \leq \frac{m^2 \epsilon}{m^2} = \epsilon$$

Hence $\frac{1}{s_n}$ converges to $\frac{1}{s}$.

■

Example 4.15 (Example Limits). The following are basic example limits.

1. $\lim_{n\to\infty} \frac{1}{n^p} = 0, \forall p > 0$
2. $\lim_{n\to\infty} r^n = 0, |r| < 1$
3. $\lim_{n\to\infty} \sqrt[n]{n} = 1$
4. $\lim_{n\to\infty} \sqrt[n]{r} = 1, r > 0$

Proof. 1. Take $\epsilon > 0$ and $N > \sqrt[p]{\frac{1}{\epsilon}}$. Then $\frac{1}{n^p} < \epsilon$ for all $n > N$.

2. If $r = 0$, then clearly r^n is 0 for all n. Consider then $r \neq 0$. If $|r| < 1$, then $\exists S$ such that $|r| = \frac{1}{1+S}$. Therefore $|r^n| = \frac{1}{(1+S)^n} \leq \frac{1}{1+Sn}$. Take $\epsilon > 0$ and $N > \frac{1}{S\epsilon}$. Then for all

$n > N$, $|r^n| < \epsilon$.

3. Let $s_n = \sqrt[n]{n} - 1 \geq 0$. Note that

$$n = (1+s_n)^n \geq \underbrace{1 + ns_n + \frac{1}{2}n(n-1)s_n^2}_{\text{truncated binomial theorem}} > \frac{1}{2}n(n-1)s_n^2$$

Therefore $0 \leq s_n < \sqrt{\frac{2}{n-1}}$. Since $\lim_{n\to\infty}\sqrt{\frac{2}{n-1}} = 0$, by the squeeze theorem $\lim_{n\to\infty} s_n = 0$, meaning $\lim_{n\to\infty}\sqrt[n]{n} = 1$.

4. Consider $r \geq 1$. Then there is always an $n \geq r$, meaning $1 \leq r \leq n$. Therefore $1 \leq r^{\frac{1}{n}} \leq n^{\frac{1}{n}} = 1$, hence $\lim_{n\to\infty}\sqrt[n]{r} = 1$. If $0 < r < 1$, then $\frac{1}{r} > 1$ meaning $\left(\frac{1}{r}\right)^n > 1$. ■

Example 4.16. Consider $\lim_{n\to\infty}\frac{n^3+6n^2+7}{4n^3+3n-4}$. This can be rewritten as $\frac{1+\frac{6}{n}+\frac{7}{n^3}}{4+\frac{3}{n^2}-\frac{4}{n^3}}$. Then

$$\lim_{n\to\infty}\frac{n^3+6n^2+7}{4n^3+3n-4} = \lim_{n\to\infty}\frac{1+\frac{6}{n}+\frac{7}{n^3}}{4+\frac{3}{n^2}-\frac{4}{n^3}} = \frac{1+0+0}{4+0+0} = \frac{1}{4}$$

Example 4.17. Consider $\lim_{n\to\infty}\frac{n-5}{n^2+7}$. This can be rewritten as $\frac{\frac{1}{n}-\frac{5}{n^2}}{1+\frac{7}{n^2}}$. Then

$$\lim_{n\to\infty}\frac{n-5}{n^2+7} = \lim_{n\to\infty}\frac{\frac{1}{n}-\frac{5}{n^2}}{1+\frac{7}{n^2}} = \frac{0}{1} = 0$$

4.1.2 Unbounded Limits

Definition 4.12 (Infinite Limit). $\lim_{n\to\infty} s_n = \infty$ if $\forall M > 0, \exists N$ s.t. $s_n > M, \forall n > N$. Likewise, $\lim_{n\to\infty} s_n = -\infty$ if $\forall M \leq 0, \exists N$ s.t. $s_n < M, \forall n > N$.

Theorem 4.14 (Implication of Infinite Limits). Let s_n and t_n be sequences.

1. If $\lim_{n\to\infty} s_n = \infty$ and $\lim_{n\to\infty} t_n > 0$, then $\lim_{n\to\infty} s_n t_n = \infty$.
2. $\lim_{n\to\infty} s_n = \infty \Leftrightarrow \lim_{n\to\infty}\frac{1}{s_n} = 0$

Proof. ■

4.1.3

Definition 4.13 (hi). Let $(s_n)_{n\in\mathbb{N}}$ be a real sequence. The statement

$$\sup_{n\geq N} s_n$$

Is the supremum of the tail of the sequence (since it only acts on terms greater than N). In the limiting case where $N \to \infty$, this can be written as

$$\lim_{N\to\infty} \sup_{n\geq N} s_n = \limsup_{n\to\infty} s_n$$

Remark. If (s_n) is not bounded above $\limsup_{n\to\infty} s_n = \infty$ and if it not bounded below then $\liminf_{n\to\infty} s_n = -\infty$

List of Theorems

List of Definitions