0.1 Existence

Theorem 0.1 ▶ Picard's Theorem

If $y' = f(x_0, y_0)$ is continuous in x and y, and $\partial_y f$ exists and is continuous around (x_0, y_0) then:

$$\begin{cases} y' &= f(x, y) \\ y(x_0) &= y_0 \end{cases}$$

Picard's Theorem doesn't necessarily guarantee a global solution; only a local solution for some ϵ distance away from the input x_0.

0.2 Separability

$$\text{When } f(x, y) = \underbrace{h(x)g(y)}_{\text{Can be easily integrated}}$$

For the general case:

$$\begin{aligned} y' = f(x_0, y_0) &= h(x)g(y) \\ y' &= h(x)g(y) \\ \frac{y'}{g(y)} &= h(x) \\ \int \frac{y'}{g(y)} \mathrm{d}x &= \int h(x)\mathrm{d}x \\ \int \frac{1}{g(y)} dy &= \int h(x)\mathrm{d}x. \end{aligned}$$

Ex. Find the general solution of $y' = xy$

$$\begin{aligned} y' = xy \implies \frac{y'}{t} &= x \\ \int \frac{1}{y} dy &= \int x\mathrm{d}x \\ \ln|y| &= \frac{1}{2}x^2 + c \\ |y| &= e^{\frac{1}{2}x^2 + c} \\ y &= Ae^{\frac{x^2}{2}}; A \in \mathbb{R}. \end{aligned}$$

Ex. Find the general solution of $y' = 1 - x^2 + y^2 - y^2x^2$; $y(1) = 0$

$$
\begin{aligned}
y' = (1+y^2)(1-x^2) \implies \int \frac{1}{1+y}\mathrm{d}t &= \int 1 - x^2 \mathrm{d}x \\
\arctan(y) &= x - \frac{x^3}{3} + c \\
y &= \tan\left(x - \frac{x^3}{3} + c\right)
\end{aligned}
$$

Solve with initial condition:

$$
\begin{aligned}
y(1) = 0 &= \tan\left(\frac{2}{3}\right) \\
\arctan(0) &= \frac{2}{3} + c \\
\{n\pi \, : \, n \in \mathbb{Z}\} &= \frac{2}{3} + c
\end{aligned}
$$

Therefore:

$$
\begin{aligned}
y &= \tan\left(x - \frac{x^3}{3} + c\right) \\
c &= \left\{n\pi - \frac{2}{3} \, : \, n \in \mathbb{Z}\right\}.
\end{aligned}
$$

0.3 Linear and Non-Linear ODE

Note: Linear vs Non-Linear ODE

$$
\begin{aligned}
\text{Linear} &\implies \text{Dependent variables and their derivatives appear linearly} \\
\text{Non-Linear} &\implies \text{Dependent variables and their derivatives have a power} \geq 2.
\end{aligned}
$$

0.3.1 Solving 1^{st} Order Linear ODE

Linear first order ODEs follow the form:

$$y' + p(x) \cdot y = f(x).$$

To solve such equations, utilize the **Integration Factor**:

$$r(x) = e^{\int p(x)\mathrm{d}x}.$$

Here is how the integration factor is utilized.

$$
\begin{aligned}
y' + p(x) \cdot y &= f(x) \\
y'r(x) + \underbrace{r(x)p(x)}_{\text{Equivalent to } r'(x)} \cdot y &= r(x)f(x) \\
\underbrace{y'r(x) + r'(x) \cdot y}_{\text{Use inverse product rule}} &= r(x)f(x) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left[y \cdot r(x)\right] &= r(x)f(x) \\
\int \frac{\mathrm{d}}{\mathrm{d}x}\left[y \cdot r(x)\right] \mathrm{d}x &= \int r(x)f(x)\mathrm{d}x \\
y \cdot r(x) &= \int r(x)f(x)\mathrm{d}x \\
y &= \frac{1}{r(x)} \int r(x)f(x)\mathrm{d}x.
\end{aligned}
$$