27.4

The prime and maximal ideals of $\mathbb{Z}_2 \times \mathbb{Z}_4$ will be the same. Furthermore the possible ideals for $\mathbb{Z}_2 \times \mathbb{Z}_4$ will be of the form $I \times J$ where I is an ideal of $\mathbb{Z}_2$ and J an ideal of $\mathbb{Z}_4$. The possible prime ideal choices for I are $\langle 1 \rangle$ and $\langle 0 \rangle$. For J the possible choices are $\langle 1 \rangle$ and $\langle 2 \rangle$. By eliminating non-prime candidates it leaves

$$\langle (0,1) \rangle, \langle (1,2) \rangle.$$

27.8

Note that $\mathbb{Z}_5[x]/\langle x^2 + x + c \rangle$ will be a field when $\langle x^2 + x + c \rangle$ is maximal since $\mathbb{Z}_5[x]$ is a commutative ring with unity. Furthermore $\langle x^2+x+c \rangle$ will be maximal when it is irreducible over $\mathbb{Z}_5$. Therefore the values of c are those that make $x^2 + x + c$ irreducible in $\mathbb{Z}_5$. Let $f(x) = x^2 + x$. Note that

$$\begin{aligned} f(0) &= 0 \\ f(1) &= 2 \\ f(2) &= 1 \\ f(-2) &= 2 \\ f(-1) &= 0 \end{aligned}$$

Therefore c has to be chosen such that $f(x) + c \neq 0$ for all $x \in \mathbb{Z}_5$. This is only possible for $c = 1$ and $c = 2$. Therefore $c = 1$ or 2 makes $\mathbb{Z}_5[x]/\langle x^2 + x + c \rangle$ a field.

27.15

Since $\mathbb{Z} \times \mathbb{Z}$ is a commutative ring with unity, any maximal ideal lends itself to a factor ring that is a field and vice versa. Therefore $\mathbb{Z} \times 3\mathbb{Z}$ is a maximal ideal of $\mathbb{Z} \times \mathbb{Z}$ since $\mathbb{Z} \times \mathbb{Z}/\mathbb{Z} \times 3\mathbb{Z} \simeq \mathbb{Z}_3$ which is a field.

27.16

Note that $\mathbb{Z} \times \{0\}$ is a prime ideal. However $\mathbb{Z} \times \mathbb{Z}/\mathbb{Z} \times \{0\} \simeq \mathbb{Z}$ which is not a field and therefore $\mathbb{Z} \times \{0\}$ cannot be maximal.

27.17

Note that $\mathbb{Z} \times 4\mathbb{Z}$ is a nontrivial proper ideal of $\mathbb{Z} \times \mathbb{Z}$. However, since $\mathbb{Z} \times \mathbb{Z}/\mathbb{Z} \times 4\mathbb{Z} \simeq \mathbb{Z}_4$ which is not an integral domain, $\mathbb{Z} \times 4\mathbb{Z}$ is not a prime ideal.

27.18

$\mathbb{Q}[x]/\langle x^2 - 5x + 6 \rangle$ will be a field if $x^2 - 5x + 6$ is irreducible in $\mathbb{Q}$. The roots of the polynomial are

$$\frac{5 \pm \sqrt{25 - 4(1)(6)}}{2} = \frac{5 \pm 1}{2}.$$

Since both roots are rational numbers, $x^2 - 5x + 6$ is not irreducible in $\mathbb{Q}$ and therefore the original factor ring is not a field.

27.30

Proof. Note that every ideal of $F[x]$ must be principal since F is a field. Every proper non-trivial ideal prime ideal will be in the form $\langle f(x) \rangle \neq 0$. Assume towards contradiction that $\langle f(x) \rangle$ is not maximal. Then that means that $f(x)$ is reducible over F meaning there exists some $g(x)$ and $h(x)$ with degrees less than $f(x)$ such that $f(x) = g(x)h(x)$. Since every polynomial in $\langle f(x) \rangle$ has degree greater than or equal to $f(x)$, then $g(x)$ and $h(x)$ cannot be in $\langle f(x) \rangle$. However, since $\langle f(x) \rangle$ is a prime ideal, then since $f(x) = h(x)g(x)$ either $h(x)$ or $g(x)$ must be in $\langle f(x) \rangle$. This is a contradiction and therefore $\langle f(x) \rangle$ must be maximal. ■

27.34

Part A

Proof. Let $x \in A + B$. Then $x = a + b$ for some $a \in A$ and $b \in B$. Since A and B are ideals, then for any $r \in R$ it follows $ra, ar \in A$ and $rb, br \in B$. Therefore both $ra + rb$ and $ar + br$ are in $A + B$. Since

$$rx = r(a + b) = ra + rb$$

and

$$xr = (a + b)r = ar + br$$

it follows that $rx, xr \in A + B$. Note for $(a_1 + b_1), (a_2 + b_2) \in A + B$ that

$$(a_1 + b_1) + (a_2 + b_2) = (a_1 + a_2) + (b_1 + b_2) \in A + B$$

and $0 + 0 \in A + B$. Furthermore $(-a_1 - b_1) \in A + B$ and $(a_1 + b_1) + (-a_1 - b_1) = 0$. Therefore $A + B$ is an ideal of R. ■

Part B

Proof. Since $0 \in R$ and for any ideal N of R it follows $rx \in N$ for any $x \in R$, $0 \in N$ by choosing $r = 0$. Therefore $0 \in A$ and $0 \in B$ meaning

$$A = \{a + 0 : a \in A\} \subseteq \{a + b : a \in A, b \in B\} = A + B$$

and

$$B = \{0 + b : b \in B\} \subseteq \{a + b : a \in A, b \in B\} = A + B.$$ ■

27.35

Part A

Proof. AB is closed under addition since

$$\sum_{i=1}^{n} a_i b_i + \sum_{j=1}^{m} a_j b_j = \sum_{k=1}^{m+n} a_k b_k \in AB.$$

Since A and B are both ideals, then for any $r \in R$

$$r \sum_{i=1}^{n} a_i b_i = \sum_{i=1}^{n} (r a_i) b_i \in AB$$

since $r a_i \in A$ and

$$\left(\sum_{i=1}^{n} a_i b_i \right) r = \sum_{i=1}^{n} a_i (b_i r) \in AB$$

since $b_i r \in B$. Furthermore since $0 \in A, B$ then $(0)(0) = 0 \in AB$ and $-(a_i b_i) = (-a_i) b_i$ then AB has both an additive identity and inverses. Therefore AB is an ideal of R. ∎

Part B

Proof. Since $a_i b_i$ must be in A since A is an ideal and must also simultaneously be in B since it is an ideal, then $a_i b_i$ is in $A \cap B$. Since A and B are closed under addition, then any sum of $a_i b_i$ must therefore be in A and B simultaneously meaning any element of AB must be in $A \cap B$. Hence $AB \subseteq A \cap B$. ∎