Problem 5.2.2

Prove by induction that for each natural number n, we have $\sum_{j=0}^{n} 2^j = 2^{n+1} - 1$.

Solution

Proof. Proceed with induction. Let $P(n) : \sum_{j=0}^{n} 2^j = 2^{n+1} - 1$. Consider the base case when $n = 1$. Then

$$\begin{aligned}\sum_{j=0}^{1} 2^j &= 2^2 - 1 \\ 2^0 + 2^1 &= 2^2 - 1 \\ 3 &= 3.\end{aligned}$$

$P(1)$ is true. Assume for some fixed $n \in \mathbb{N}$ that $P(n)$ is true. Then,

$$\begin{aligned}\sum_{j=0}^{n+1} 2^j &= 2^{n+1} + \sum_{j=0}^{n} 2^j \\ &= 2^{n+1} + 2^{n+1} - 1 \\ &= 2^{(n+1)+1} - 1.\end{aligned}$$

Therefore $P(n+1)$ is true, meaning that for each natural number n, we have

$$\sum_{j=0}^{n} 2^j = 2^{n+1} - 1.$$

■

Problem 5.2.5

Show by induction that for every $n \in \mathbb{N}$ we have: $n \equiv 5 \pmod 3$ or $n \equiv 6 \pmod 3$ or $n \equiv 7 \pmod 3$.

Solution

Proof. Proceed with induction. Let $P(n) : (n \equiv 5 \mod 3) \vee (n \equiv 6 \mod 3) \vee (n \equiv$

$7 \mod 3)$. By the properties of modular arithmetic, $P(n)$ can be restated as

$$P(n) : (n \equiv 0 \bmod 3) \vee (n \equiv 1 \bmod 3) \vee (n \equiv 2 \bmod 3).$$

Consider the base case when $n = 1$. Then $n \equiv 1 \pmod 3$, therefore $P(1)$ is true. Assume for a fixed $n \in \mathbb{N}$ that $P(n)$ is true. Consider then three cases.

1. If $n \equiv 0 \pmod 3$, then $n + 1 \equiv 1 \pmod 3$, meaning that $P(n + 1)$ is true.
2. If $n \equiv 1 \pmod 3$, then $n + 1 \equiv 2 \pmod 3$, meaning that $P(n + 1)$ is true.
3. If $n \equiv 2 \pmod 3$, then $n + 1 \equiv 0 \pmod 3$, meaning that $P(n + 1)$ is true.

Therefore $P(n)$ implies $P(n + 1)$, meaning for every $n \in \mathbb{N}$ we have: $n \equiv 5 \pmod 3$ or $n \equiv 6 \pmod 3$ or $n \equiv 7 \pmod 3$. ■

Problem 5.2.6

Prove by induction that, for all $n \in \mathbb{N}$, $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + n(n+1) = \frac{1}{3}n(n+1)(n+2)$.

Solution

Proof. Proceed with induction. Let $P(n) : \sum_{j=0}^{n} j(j+1) = \dfrac{1}{3}n(n+1)(n+2)$. Consider the base case $n = 1$. Then

$$\begin{aligned}\sum_{j=0}^{1} j(j+1) &= \frac{1}{3}(1)(1+1)(1+2) \\ 2 &= \frac{1}{3}(6) \\ 2 &= 2.\end{aligned}$$

$P(1)$ is true. Assume for some fixed $n \in \mathbb{N}$ that $P(n)$ is true. Then it follows that

$$\begin{aligned}\sum_{j=0}^{n+1} j(j+1) &= (n+1)(n+2) + \sum_{j=0}^{n} j(j+1) \\ &= (n+1)(n+2) + \frac{1}{3}(n)(n+1)(n+2) \\ &= (\frac{1}{3}n + 1)(n+1)(n+2) \\ &= \frac{1}{3}(n+1)(n+2)(n+3).\end{aligned}$$

Therefore $P(n+1)$ is true, meaning for all $n \in \mathbb{N}$, $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + n(n+1) = \frac{1}{3}n(n+1)(n+2)$ ■

Problem 5.3.2

Suppose that $n \geq 3$. Prove that $\left(\frac{n+1}{n}\right)^2 < 2$.

Solution

Proof. Proceed with induction. Let $P(n) : \left(\frac{n+1}{n}\right)^2 < 2$. Consider the base case when $n = 3$. Then $\left(\frac{3+1}{3}\right)^2 = \left(\frac{4}{3}\right)^2 = \frac{16}{9} < 2$. Therefore $P(3)$ is true. Assume for a fixed $n \in \mathbb{N} \geq 3$ that $P(n)$ is true. Then

$$\begin{aligned}
\left(\frac{n+2}{n+1}\right)^2 &= \left(\frac{n+2}{n+1}\right)^2 \left(\frac{n+1}{n}\right)^2 \left(\frac{n}{n+1}\right)^2 \\
&< 2 \cdot \left(\frac{n+2}{n+1}\right)^2 \left(\frac{n}{n+1}\right)^2 \\
&= 2 \cdot \left(\frac{n^2(n+2)^2}{(n+1)^4}\right) \\
&= 2 \cdot \left(\frac{n^4 + 4n^3 + 4n^2}{n^4 + 4n^3 + 6n^2 + 4n + 1}\right) \qquad (*)
\end{aligned}$$

Note that $n^4 + 4n^3 + 4n^2 \leq n^4 + 4n^3 + 4n^2 + a$ when $a \geq 0$. Let $a = 2n^2 + 4n + 1$. Since n is positive, $2n^2 + 4n + 1$ will always be greater than or equal to zero. Therefore $a \geq 0$. This means that

$$\begin{aligned}
n^4 + 4n^3 + 4n^2 &\leq n^4 + 4n^3 + 4n^2 + a \\
n^4 + 4n^3 + 4n^2 &\leq n^4 + 4n^3 + 4n^2 + 2n^2 + 4n + 1 \\
\frac{n^4 + 4n^3 + 4n^2}{n^4 + 4n^3 + 4n^2 + 2n^2 + 4n + 1} &\leq 1.
\end{aligned}$$

Therefore returning back to $(*)$,

$$\begin{aligned}
\left(\frac{n+2}{n+1}\right)^2 &< 2 \cdot \left(\frac{n^4 + 4n^3 + 4n^2}{n^4 + 4n^3 + 6n^2 + 4n + 1}\right) \\
&< 2 \cdot 1 \\
&< 2.
\end{aligned}$$

Therefore $P(n+1)$ is true, meaning that for all $n \geq 3$, $\left(\frac{n+1}{n}\right)^2 < 2$. ■

Problem 5.3.3

Consider the following result. For every natural number $n \geq 2$,

$$\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\left(1-\frac{1}{16}\right)\cdots\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n}.$$

(a) If the statement is written in the form $\forall n \in \mathbb{N} \geq 2, P(n)$, what is the proposition $P(n)$?

(b) Rewrite the statement using Π-notation.

(c) Prove the result by induction.

Solution

Part A

$$P(n):\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)\left(1-\frac{1}{16}\right)\cdots\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n}.$$

Part B

$$P(n):\prod_{i=2}^{n}\left(1-\frac{1}{i^2}\right)=\frac{n+1}{2n}.$$

Part C

Proof. Proceed with induction. Let $P(n):\prod_{i=2}^{n}\left(1-\frac{1}{i^2}\right)=\frac{n+1}{2n}$. Consider the base case when $n=2$. Then

$$\begin{aligned}\prod_{i=2}^{2}\left(1-\frac{1}{i^2}\right)&=\frac{2+1}{2(2)}\\ \left(1-\frac{1}{4}\right)&=\frac{3}{4}\\ \frac{3}{4}&=\frac{3}{4}\end{aligned}$$

$P(2)$ is true. Assume for some fixed $n \in \mathbb{N} \geq 2$ that $P(n)$ is true. Then

$$\begin{aligned}
\prod_{i=2}^{n+1}\left(1-\frac{1}{i^2}\right) &= \left(1-\frac{1}{(n+1)^2}\right)\cdot\prod_{i=2}^{n}\left(1-\frac{1}{i^2}\right) \\
&= \frac{(n+1)^2-1}{(n+1)^2}\cdot\frac{n+1}{2n} \\
&= \frac{n^2+2n+1-1}{2n(n+1)} \\
&= \frac{n(n+2)}{2n(n+1)} \\
&= \frac{n+2}{2(n+1)}.
\end{aligned}$$

Therefore $P(n+1)$ is true, meaning $\forall n \in \mathbb{N} \geq 2, \prod_{i=2}^{n}\left(1-\frac{1}{i^2}\right) = \frac{n+1}{2n}$. ■

Problem 5.3.4

Recall the geometric series formula from calculus: if $r \neq 1$ is constant, and $n \in \mathbb{N}_0$, then

$$\sum_{k=0}^{k} r^n = \frac{1-r^{n-1}}{1-r} \tag{$*$}$$

(a) Explain why the given proof by induction is incorrect.

(b) Provide a correct proof of $(*)$.

Part A

The given proof is incorrect as it starts with $P(n+1)$. $P(n+1)$ is the goal of the proof, therefore attempting to prove $P(n) \implies P(n+1)$ by starting with $P(n+1)$ is incorrect.

Part B

Proof. Proceed with induction. Let $P(n) : \sum_{k=0}^{n} r^k = \frac{1-r^{n-1}}{1-r}$. Consider the base case when $n = 0$. Then $\sum_{k=0}^{0} r^k = r^0 = 1 = \frac{1-r^{0+1}}{1-r}$, meaning $P(0)$ is true. Assume for some fixed $n \in \mathbb{N}_0$ that $P(n)$ is true. Then

$$\begin{aligned}\sum_{k=0}^{n+1} r^k &= r^{n+1} + \sum_{k=0}^{n} r^k \\ &= r^{n+1} + \frac{1-r^{n-1}}{1-r} \\ &= \frac{r^{n+1}-r^{n-2}}{1-r} + \frac{1-r^{n-1}}{1-r} \\ &= \frac{1-r^{n-2}}{1-r}.\end{aligned}$$

Therefore $P(n+1)$ is true, meaning if $r \neq 1$ is constant, and $n \in \mathbb{N}_0$, then $\sum_{k=0}^{k} r^n = \frac{1-r^{n-1}}{1-r}$ is true. ■

Problem 5.3.8

Prove that if $A \subseteq \mathbb{R}$ is a *finite* set, then A is well-ordered.

Solution

Proof that any finite subset of the real numbers contains a minimum element, hence any finite subset of A will contain a minimum element, meaning A is well-ordered.

Proof. Proof via induction that any finite subset of the real numbers has a minimum element. Let $X_n \subseteq \mathbb{R}$ such that it is finite and contains $n \in \mathbb{N}$ elements. Consider the base case of X_1. Then $\exists a \in \mathbb{R}$ such that $X_1 = \{a\}$. It is obvious then that X_1 contains a minimum element since $a \leq a$. Assume for a fixed $n \in \mathbb{N}$ that X_n has a miminum element p. Consider the set X_{n+1}. There exists $q \in \mathbb{R} \neq p$ such that $X_{n+1} = \{q\} \cup X_n$. There are now two cases.

$(q < p)$: If q is smaller than p, then the minimum element of X_{n+1} will be q since it is smaller than the minimum element of X_n.

$(q > p)$: If q is greater than or equal to p, then the minimum element of X_{n+1} will be p since p is smaller than q.

In both cases, X_{n+1} will have a minimum element. Therefore all finite subsets of the real numbers contain a minimum element. ■

Problem 5.4.1

Define a sequence $(b_n)_{n=1}^{\infty}$ as follows:

$$\begin{cases} b_n = b_{n-1} + b_{n-2} \\ b_1 = 3, b_2 = 6 \end{cases}.$$

Prove: $\forall n \in \mathbb{N}, b_n$ is divisible by 3.

Solution

Proof. Proceed with strong induction. Consider the base cases where $n = 1$ and $n = 2$. Then $b_1 = 3 = 3(1)$ which is divisible by 3 and $b_2 = 6 = 3(2)$ which is divisible by 3. Fix $n \in \mathbb{N}_{\geq 2}$ and assume that b_k is divisible by 3 for all $k \in \mathbb{N}, 1 \leq k \leq n$. Then

$$b_{n+1} = b_n + b_{n-1}.$$

By the induction hypothesis, b_n and b_{n-1} are both divisible by 3. Therefore there exists integers a, b such that $b_n = 3a$ and $b_{n-1} = 3b$. Therefore

$$\begin{aligned} b_{n+1} &= b_n + b_{n-1} \\ &= 3a + 3b \\ &= 3(a + b). \end{aligned}$$

Since $a + b \in \mathbb{Z}$, then b_{n+1} is divisible by 3. By strong induction we see that b_n is divisible by 3 for all $n \in \mathbb{N}$. ∎