Problem 0.11

$$\{a,b,c\}\times\{1,2,c\}=\{(a,1),(a,2),(a,c),(b,1),(b,2),(b,c),(c,1),(c,2),(c,c)\}.$$

Problem 0.12

Part A

The relation is a function mapping A into B. The function is not one-to-one since $(1,4)$ and $(2,4)$ are in the relation. The function is not onto B either since there is no $a \in A$ that maps to a $b = 2$.

Part B

The relation is a function mapping A into B. The function is not one-to-one since $(1,4)$ and $(3,4)$ are in the relation. The function is not onto B either since there is no $a \in A$ that maps to a $b = 2$.

Problem 0.18

Proof. Let A be any set and let B^A be the set of all functions mapping A into the set $B = \{0,1\}$. Define a map $\phi : B^A \to \mathcal{P}(A)$ where given a function $f \in B^A$, it is defined that $\phi(f) = \{a \in A : f(a) = 1\}$. Proceed to show that ϕ is bijective.

(One-to-One) Let $f, g \in B^A$ and assume that $\phi(f) = \phi(g)$. Therefore for some element $x \in A$, $f(x) = 1$ if and only if $g(x) = 0$. Since f and g only take on two possible values, it also follows that $f(x) = 0$ if and only if $g(x) = 0$. Therefore $f = g$, meaning ϕ is one-to-one.

(Onto) Let $S \in \mathcal{P}(A)$. Therefore $S \subseteq A$. Define $\theta : A \to \{0,1\}$ by

$$\theta(x) = \begin{cases} 1, & x \in S \\ 0, & x \notin S \end{cases}.$$

Note that $\phi(\theta) = S$, hence ϕ is onto.

Since ϕ is both one-to-one and onto, it is a bijection between B^A and $\mathcal{P}(A)$ meaning they have the same cardinality. ■

Problem 0.19

Proof. Assume towards contradiction there exists a one-to-one map $\phi : A \to \mathcal{P}(A)$. Define $S = \{x \in A : x \notin \phi(x)\}$. Let $a \in A$ and consider two cases. If $a \in \phi(a)$, then $a \notin S$.

If $a \notin \phi(a)$, then $a \in S$. Therefore S and $\phi(a)$ are different subsets of A since a is only ever in one but not the either. Hence ϕ cannot be one-to-one, which is a contradiction. Therefore ϕ cannot exist. ■

The set of everything is not a logically acceptable. If T denotes this supposed set of everything, then $\mathcal{P}(T)$ would be larger than T, contradicting the fact that it already the set of everything.

Problem 0.32

Proof. Proof that $\mathcal{R}$ is not an equivalence relation. Examine transitivity. Let $a, b, c \in \mathbb{R}$ with $a = 1, b = 3$ and $c = 6$. Note that $a\mathcal{R}b$ since $|a - b| = |1 - 3| = 2 \leq 3$ and $b\mathcal{R}c$ since $|b - c| = |3 - 6| = 3 \leq 3$. However a is not related to c since $|a - c| = |1 - 6| = 5 \nleq 3$. Therefore $\mathcal{R}$ is not transitive, meaning it is not an equivalence relation. ■

Problem 1.4

$$(-i)^{35} = (-1)^{35} \cdot i^{35} = -(i^3) = i.$$

Problem 1.20

$$\begin{aligned} z^6 = 1 \implies & |z^6|(\cos(6\theta) + i\sin(6\theta)) = 1 \\ & |z^6|(\cos(6\theta) + i\sin(6\theta)) = 1(1 + 0 \cdot i). \end{aligned}$$

Therefore $|z| = 1, \cos(6\theta) = 1$ and $\sin(6\theta) = 0$. This implies that $\theta = \frac{\pi n}{3}$ for $n \in \mathbb{Z}$. Thus the solutions when $\theta \in [0, 2\pi)$ are

$$z = \left\{e^{\frac{\pi n}{3}} : n \in \{0, 1, 2, 3, 4, 5\}\right\}.$$

Problem 1.22

$$10 +_{17} 16 = 9.$$

Problem 1.25

$$\frac{1}{2} +_1 \frac{7}{8} = \frac{3}{8}.$$

Problem 1.26

$$\frac{3\pi}{4} +_{2\pi} \frac{3\pi}{2} = \frac{\pi}{4}.$$

Problem 1.29

Solutions will be when $x + 7 = 15 + 3$, meaning $x = 11$. All other possible values of $x \in \mathbb{Z}_15$ do not work.

Problem 1.36

$$
\begin{aligned}
\zeta &\leftrightarrow 4 \\
\zeta^2 = \zeta \cdot \zeta &\leftrightarrow 4 +_7 4 = 1 \\
\zeta^3 = \zeta^2 \cdot \zeta &\leftrightarrow 4 +_7 1 = 5 \\
\zeta^4 = \zeta^2 \cdot \zeta^2 &\leftrightarrow 1 +_7 1 = 2 \\
\zeta^5 = \zeta^4 \cdot \zeta &\leftrightarrow 5 +_7 1 = 6 \\
\zeta^6 = \zeta^5 \cdot \zeta &\leftrightarrow 6 +_7 4 = 3 \\
\zeta^0 = \zeta^5 \cdot \zeta^2 &\leftrightarrow 6 +_7 1 = 0
\end{aligned}
$$

Problem 1.37

If there was an isomorphism between U_6 and $\mathbb{Z}_6$, then $\zeta^2 = \zeta \cdot \zeta \leftrightarrow 4 +_6 4 = 2$ and $\zeta^4 = \zeta^2 \cdot \zeta^2 \leftrightarrow 2 +_6 2 = 4$. However that means ζ and ζ^4 correspond to the same value which contradicts the requirement that an ismorphism is one-to-one.

Problem 2.1

$$b * d = e$$

$$c * c = b$$

$$
\begin{aligned}
[(a * c) * e] * a &= [c * e] * a \\
&= a * a \\
&= a.
\end{aligned}
$$

Problem 2.3

$$
\begin{aligned}
(b * d) * c &= e * c \\
&= a.
\end{aligned}
$$

$$\begin{aligned} b * (d * c) &= b * b \\ &= c. \end{aligned}$$

This computation does imply that $*$ in this instance is not associative since the results of each computation are not equal to each other.

Problem 2.5

$*$	a	b	c	d
a	a	b	c	d
b	b	d	a	c
c	c	a	d	b
d	d	c	b	a

Problem 2.8

$*$ is commutatitive but not associative.

Proof. First examine commutativity. Let $a, b \in \mathbb{Q}$. Note then that

$$a * b = ab + 1 = ba + 1 = b * a.$$

Therefore $*$ is commutative. Now examine associativity. Let $a = 1, b = 2, c = 3$. Then note that

$$\begin{aligned} (a * b) * c &= 3 * c \\ &= 3 * 3 \\ &= 10. \end{aligned}$$

and that

$$\begin{aligned} a * (b * c) &= a * 7 \\ &= 1 * 7 \\ &= 8. \end{aligned}$$

In this case $(a * b) * c \neq a * (b * c)$, meaning $*$ is not associative. ■

Problem 2.10

$*$ is commutative but not associative.

Proof. First examine commutativity. Let $a, b \in \mathbb{Z}^+$. Note then that

$$a * b = 2^{ab} = 2^{ba} = b * a.$$

Therefore $*$ is commutative. Now examine associativity. Let $a = 1, b = 2, c = 3$. Then note that

$$\begin{aligned}(a * b) * c &= 2^2 * c \\ &= 4 * 3 \\ &= 2^{12}.\end{aligned}$$

and that

$$\begin{aligned}a * (b * c) &= a * 2^6 \\ &= 1 * 64 \\ &= 2^{64}.\end{aligned}$$

In this case $(a * b) * c \neq a * (b * c)$, meaning $*$ is not associative. ■

Problem 2.12

Consider the tabular representation of $*$. Given the set S that $*$ is over, define $n = |S|$. The table will therefore have n^2 entries in it. Each entry has n choices as it can be any element of S. Therefore since you have n choices n^2 times

$$\textit{Number of possible binary operations} = n^{(n^2)}.$$

Problem 2.17

$*$ does not give a binary operation since it breaks Condition 2. Consider $a = 1, b = 2$. Both 1 and 2 are in $\mathbb{Z}^+$, however $a * b = 1 - 2 = -1 \notin \mathbb{Z}^+$.

Problem 2.20

$*$ is a binary operation since it obeys both conditions.

Problem 2.23

H is closed under under both matrix addition and matrix multiplication.

Proof. Consider first matrix addition on H. Let $a, b, c, d \in \mathbb{R}$

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} + \begin{bmatrix} c & -d \\ d & c \end{bmatrix} = \begin{bmatrix} (a+c) & -(b+d) \\ (b+d) & (a+c) \end{bmatrix}.$$

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} c & -d \\ d & c \end{bmatrix} = \begin{bmatrix} ac-bd & -(ad+bc) \\ ad+bc & ac-bd \end{bmatrix}.$$

∎

Problem 2.24

Problem 2.36