Math 140A: Elementary Analysis

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Introduction

1.1 The Natural Numbers

First examine the natural numbers. It is very common knowledge that 1 is a natural number and you obtain the rest by increasing the previous by 1. This is however not a rigorous construction of the natural numbers. An example of a rigorous construction is the **Peano axioms**

Definition 1.1 (Peano Axioms). The natural numbers are axiomatically defined by

1. $1 \in \mathbb{N}$
2. If $n \in \mathbb{N}$, then $n + 1 \in \mathbb{N}$
3. 1 is the first element, meaning it is not the sucessor of any element
4. If $S \subset \mathbb{N}$ such that $1 \in S$ and $n \in S$ implies $n + 1 \in S$, then $S = \mathbb{N}$

While the Peano Axioms are not strong enough for modern math, they are sufficient for lots of math and at least open up the world of rigorous axiomatic constructions. Consider axiom 4. Assume that it is not true. Then there is an $S \subset \mathbb{N}$ such that $1 \in S$ and $n \in S \implies n+1 \in S$ but $S \neq \mathbb{N}$. Then let $n_0 = \min\{n \in \mathbb{N} : n \notin S\}$. Since $1 \in S$, $n_0 \neq 1$ and hence n_0 is the successor of $n_0 - 1$. However since $n \in S \implies n+1 \in S$ and $n_0 - 1 \in S$, $n_0 \in S$ and therefore a contradiction.
While this is a persuasive and intuitive argument, it does not constitute a proof as the existence of n_0 is assumed because of the assumption of a minimum element in a non-empty subset of $\mathbb{N}$.

1.1.1 Mathematical Induction

Theorem 1.1 (Induction). If $S_1, S_2, S_3, \ldots$ are statements, all are true if

1. S_1 is true
2. $S_n \implies S_{n+1}$

For simplicity, the proof of induction shall be left more so as accepting the last Peano Axiom that declares its validity.

Example 1.1. Consider the statement $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$.

Proof. Consider the base case $n = 1$. Then $1 = \frac{1(2)}{2} = 1$, therefore the base case holds. Assume that for a fixed $n \in \mathbb{N}$ that $1+2+3+\ldots+n = \frac{n(n+1)}{2}$. Then it follows that

$$\begin{aligned} 1+2+3+\ldots+n &= \frac{n(n+1)}{2} \\ 1+2+3+\ldots+n+(n+1) &= \frac{n(n+1)}{2}+(n+1) \\ 1+2+3+\ldots+(n+1) &= \frac{(n+1)(n+2)}{2} \end{aligned}$$

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Example 1.2. Consider the statement $|\sin(nx)| \leq |n\sin(x)|, \forall x \in \mathbb{R}$.

Proof. The base clearly holds. Assume that for a fixed $n \in \mathbb{N}$ that $|\sin(nx)| \leq |n\sin(x)|, \forall x \in \mathbb{R}$. Then

$$\begin{aligned} |\sin((n+1)x)| = |\sin(nx+x)| &= |\sin(nx)\cos(x)+\cos(nx)\sin(x)| \\ &\leq |\sin(nx)||\cos(x)|+|\cos(nx)||\sin(x)| \\ &\leq |\sin(nx)|+|\sin(x)| \\ &\leq n|\sin(x)|+|\sin(x)| \\ &\leq (n+1)|\sin(x)| \end{aligned}$$

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Extending the Naturals

2.1 Rational Numbers

Definition 2.2. The rational numbers is the set of numbers of the form $\frac{p}{q}$ where $p \in \mathbb{Z}$ and $q \in \mathbb{N}$.

Rational numbers are the first number system that provides a nice comprehensive structure. Multiplication, division, addition, and subtraction are all closed operations making it a strong number system.

Theorem 2.2 (Rational Root Theorem). Let $c_0, c_1, \ldots, c_n \in \mathbb{Z}$. If r solves $c_n x^n + c_{n-1}x^{n-1} + \ldots + c_1 x + c_0 = 0$, $c_n \neq 0 \neq c_1$ and $r = \frac{p}{q}$ where p and q are coprime

$$p | c_0, \quad q | c_n$$

Proof. Let r be a rational solution to the polynomial equation $c_n x^n + c_{n-1}x^{n-1} + \ldots + c_1 x + c_0 = 0$. Since $r \in \mathbb{Q}$, $r = \frac{p}{q}$ where $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. Then

$$\begin{aligned}
c_n \left(\frac{p}{q}\right)^n + c_{n-1}\left(\frac{p}{q}\right)^{n-1} + \ldots + c_1\left(\frac{p}{q}\right) + c_0 &= 0 \\
c_n p^n + c_{n-1} q p^{n-1} + \ldots + c_1 q^{n-1} p + c_0 q^n &= 0 \\
-c_n p^n - c_{n-1} q p^{n-1} - \ldots - c_1 q^{n-1} p &= c_0 q^n \\
-p\left[c_n p^{n-1} - c_{n-1} q p^{n-2} - \ldots - c_1 q^{n-1}\right] &= c_0 q^n
\end{aligned}$$

Therefore $p | c_0 q^n$. Since p and q are coprime, p must divide c_0. By solving for $c_n p^n$ instead, it follows that q divides c_n. ■

While rationals are quite nice, there are many equations that have solutions that cannot be represented by a rational number.

Example 2.3 ($\sqrt{2}$). Consider the equation $x^2 - 2$. Its solutions by the Rational Root Theorem must be an integer. However no integer satisfies the equation and therefore there is no rational root for $x^2 - 2$.

2.2 Algebraic Numbers

Definition 2.3 (Algebraic Number). A number is called algebraic if it is the root of an integer coeffecient polynomial. That is, it is a solution to

$$c_n x^n + c_{n-1}x^{n-1} + \ldots + c_1 x + c_0 = 0$$

where $c_i \in \mathbb{Z}$, $c_i \neq 0$ and $n \geq 1$.

Many numbers that are used day to day are algebraic. It follows clearly that all integers are algebraic and all rationals are algebraic. Other numbers such as the $\sqrt{2}$ are algebraic. Even the number $\sqrt{2+\sqrt[3]{5}}$ is algebraic. However, there are infinitely many other numbers that are not algebraic such as π and e.

Real Numbers

As seen above, both the rationals and algebraic numbers can be very useful but fail to encapsulate important types of numbers. That is, both $\mathbb{Q}$ and the algebraic numbers have gaps in them, that is the irrationals for $\mathbb{Q}$ and transcendtals for algebraic numbers.

2.2.1 Ordering Structure

Definition 2.4 (Ordered Field). We say a field with a relation $(\mathbb{F}, +, \cdot, \leq)$ is an ordered field if it satisfies the following properties:

1. $p \leq q$ or $q \leq p$ for all $p, q \in \mathbb{F}$
2. $p \leq q$ and $q \leq p \implies p = q$
3. $p \leq q$ and $q \leq r \implies p \leq r$
4. $p \leq q \implies p + r \leq q + r$
5. $p \leq q \implies pr \leq qr$ for all $r \in \mathbb{F} \geq 0$

Certain properties are derivable from the properties and ordering of $\mathbb{R}$.

Theorem 2.3 (Properties of $\mathbb{R}$). For all $p, q, r \in \mathbb{R}$

1. $p + r = q + r \implies p = q$
2. $p \cdot 0 = 0 = 0 \cdot p$
3. $(-p)q = -(pq)$
4. $(-p)(-q) = pq$
5. $pr = qr \implies p = q$ if $r \neq 0$
6. $pq = 0 \implies p = 0$ or $q = 0$

Proof. Let $p, q, r \in \mathbb{R}$ for the following.

(1) Assume that $p+r = q+r$. Since additive inverses exist, $p+r+(-r) = q+r+(-r)$. By associativity, $p+(r+(-r)) = q+(r+(-r))$. By definition of inverses, $p+0 = q+0$. By the additive identity, $p = q$.

(2) Examine $p \cdot 0$. Note that $p \cdot 0 = p \cdot (0+0)$. By distribution, $p \cdot 0 + p \cdot 0 = p \cdot 0$. This means that $p \cdot 0$ does not change when added to itself, which is by definition

the additive identity. Therefore $p \cdot 0 = 0$.

(3) Consider the expression $pq + (-p)q$. By distributivity, $pq + (-p)q = (p + (-p))q$. By inverses, $pq + (-p)q = 0 \cdot q = 0$. Therefore $-pq = (-p)q$.

(4) To be completed

(5) To be completed

(6) Assume that $pq = 0$. WLOG, let $q \neq 0$. Since multiplicative inverses exist, $0 = q^{-1} \cdot 0 = 0 \cdot q^{-1} = pqq^{-1} = p(qq^{-1}) = p$. Therefore $p = 0$.

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When considering the ordered field of the reals, more properties are derivable.

Theorem 2.4 (Properties of Ordered Reals). Let $p, q, r \in \mathbb{R}$

1. $p \leq q \implies -q \leq -p$
2. $p \leq q, r \leq 0 \implies qr \leq pr$
3. $p \geq 0, q \geq 0 \implies pq \geq 0$
4. $p^2 \geq 0$
5. $0 < 1$
6. $p > 0 \implies p^{-1} > 0$
7. $0 < p < q \implies 0 < q^{-1} < p^{-1}$

Remark. $p < q$ is defined as $p \leq q$ and $p \neq q$.

Proof. Let $p, q, r \in \mathbb{R}$

(1) Assume that $p \leq q$. Let $r = (-p) + (-q)$. Since adding a number to both sides of a inequality preserves it, $p + r \leq q + r$. Then $p + (-p) + (-q) \leq q + (-p) + (-q)$. By commutativity and associativity, $(p + (-p)) + (-q) \leq (q + (-q)) + (-p)$. By inverses and additive identity, $-q \leq -p$.

(2) Assume that $p \leq q$ and that $r \leq 0$. By (1), $-r \leq 0$. Therefore, $p(-r) \leq q(-r)$ hence $-pr \leq -qr$. By (1), $qr \leq pr$.

(3) To complete

(4) By the properties of an ordered field, $p \leq 0$ or $p \geq 0$. If $p \geq 0$, then by (3), $p^2 = p \cdot p \geq 0$. If $p \leq 0$, then $-p \geq 0$. By 2.4.4, $p^2 = (-p)(-p) \geq 0$ by the first case.

(5) To complete

(6) Assume towards contradiction that $p > 0$ and $p^{-1} \leq 0$. By (1), $-p^{-1} \geq 0$. Since p and $-p^{-1}$ are non-negative, $p(-p^{-1}) \geq 0$. This means that $-1 \geq 0$ or equivalently $1 \leq 0$. By (5), this is a contradiction.

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2.2.2 Absolute Value

Definition 2.5 (Absolute Value). Let $p, q \in \mathbb{R}$.

$$|p| := \begin{cases} p & p \geq 0 \\ -p & p \leq 0 \end{cases}$$

Additionally, define the distance between two reals as

$$\operatorname{dist}(p, q) = |p - q|$$

Theorem 2.5 (Properties of Absolute Value). Let $p, q \in \mathbb{R}$.

1. $|p| \geq 0$
2. $|pq| = |p||q|$
3. $|p + q| \leq |p| + |q|$

Proof. Let $p, q \in \mathbb{R}$.

(1) If $p \geq 0$, then $|p| \geq 0$. If $p \leq 0$, then $|p| \geq 0$. Therefore $|p| \geq 0$ for all p.

(2) If $p \geq 0, q \geq 0$. Then $|pq| = pq = |p||q|$. If $p \leq 0, q \leq 0$, then $-p \geq 0, -q \geq 0$ and $|p||q| = (-p)(-q) = pq = |pq|$.

(3) Note that $-|p| \leq p \leq |p|$. This is because p either is $|p|$ or $|p| = -p$ meaning $p = -|p|$. Same is true for q. Therefore

$$-|p| + (-|q|) \leq -|p| + q \leq p + q \leq |p| + q \leq |p| + |q|$$
$$-(|p| + |q|) \leq p + q \leq |p| + |q|$$

The derived inequality shows that $p + q \leq |p| + |q|$ and $-(p + q) \leq |p| + |q|$. Since $|p + q|$ is either $p + q$ or $-(p + q)$, $|p + q| \leq |p| + |q|$.

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Corollary 2.1 (Distance Triangle Inequality).

$$\operatorname{dist}(p, r) \leq \operatorname{dist}(p, q) + \operatorname{dist}(q, r)$$

2.5.3 is an important property of the absolute value, usually referred to as the *triangle* *inequality*.

Axiom of Completeness

3.1 Bounds

Definition 3.6 (Upper and Lower Bound). Let S be a non-empty subset of $\mathbb{R}$. An upper bound of S is a number M such that $s \leq M$ for all $s \in \mathbb{R}$. A lower bound of S is a number m such that $s \geq M$ for all $s \in \mathbb{R}$.

Note that any finite subset of $\mathbb{R}$ will admit an upper and lower bound as a larger/smaller number can always be chosen compared to any number in the set. There can potentially be infinitely many bounds on a set, but there is an important refinement that can be made.

Definition 3.7 (Supremum and Infimum). Let $S \subset \mathbb{R}$. If there exists an upper bound M for S such that for any other upper bound s $M \leq s$, M is called the *least upper bound* for S or equivalently the supremum (notated as $\sup S$). The same logic for lower bounds gives rise to the infimum or *greatest lower bound* (notated as $\inf S$).

Consider a finite subset of $\mathbb{R}$. Then it follows that the subset will have a minimum and maximum as each element can be checked against each other because it is finite. This does not generalize to an infinite subset of $\mathbb{R}$.

Example 3.4. Consider the set $S = \left\{r \in \mathbb{Q} : 0 \leq r, r^2 < 2\right\}$. In $\mathbb{R}$, 0 is a minimum and $\sqrt{2}$ is the supremum. Note that $\sqrt{2} \notin S$. Alternatively, if working over $\mathbb{Q}$, there is no supremum as $\sqrt{2} \notin \mathbb{Q}$.

Remark. The supremum or infimum of a set does not necessarily have to be an element of said set.

Theorem 3.6 (Uniqueness of Supremum and Infimum). If a set $S \subset \mathbb{R}$ has a supremum or infimum, then said supremum or infimum is unique.

Proof. Let $S \subset \mathbb{R}$. Assume that S has two supremum M and M'. By definition of a supremum, $M \leq M'$ and $M' \leq M$. Therefore $M = M'$. Same argument applies to the infimum. ■

Example 3.5. Consider the set

$$D = \left\{x \in \mathbb{R} : x^2 < 10\right\}.$$

Then $\sup D = \sqrt{10}$ and $\inf D = -\sqrt{10}$. Since $\pm\sqrt{10} \notin D$, there is no max or min.

3.2 The Completeness Axiom

The completeness axiom is a defining charactersitc of $\mathbb{R}$ that differentiates it from $\mathbb{Q}$. It can be interpreted as requiring there be no gaps between numbers.

Definition 3.8 (Axiom of Completeness). Let S be a non-empty subset of $\mathbb{R}$. If S is bounded above, then $\sup S$ exists.

Consider the set from example 3.4. When working over $\mathbb{Q}$, there exists upper bounds (such as 4), but it does not admit a least upper bound. In contrast, working over $\mathbb{R}$ admits a supremum. This distinction is what makes $\mathbb{R}$ useful for much of analysis and calculus. While the Axiom of Completeness only stipulates the existence of a supremum, it can be derived that the equivalent statement for lower bounds and infimum follows.

Corollary 3.2 (Axiom of Completeness Reversed). Let S be a non-empty subset of $\mathbb{R}$. If S is bounded below, then $\inf S$ exists.

Proof. Let S be a non-empty set that is bounded below. Therefore there exists an m such that $m \leq s$ for all $s \in S$. Equivalently, $-s \leq -m$ for all $s \in S$. Consider the set $-S = \{-s : s \in S\}$. $-s \leq -m$ for all $s \in S$ implies $-S$ is bounded above by $-m$ and therefore by the Axiom of Completeness $\sup(-S) = s_0$ exists. Therefore $r \leq s_0$ for all $r \in -S$ meaning $-s \leq s_0$ for all $s \in S$. Flipping the inequality gives $-s_0 \leq s$ for all $s \in S$, meaning $-s_0$ is a lower bound for S. ■

Theorem 3.7 (Archimedean Property). Let $a, b \in \mathbb{R} > 0$. Then $\exists n \in \mathbb{N}$ such that $an > b$.

Consider the special case when $b = 1$. Then $an > b \implies a > \frac{1}{n}$ for some $n \in \mathbb{N}$ meaning there is always a rational number smaller than any positive real number. In the case that $a = 1$, then $an > b \implies n > b$ for some $n \in \mathbb{N}$ meaning there is always a rational/integer larger than any positive real number.

Proof. Assume towards contradiction that $\exists a, b \in \mathbb{R} > 0$ such that $na \leq b$ for all $n \in \mathbb{N}$. Define the set $S = \{na : n \in \mathbb{N}\}$. Note that b is an upper bound of S. Therefore by the Axiom of Completeness, $\sup S = s_0$ exists. Since $a > 0$, then $a + s_0 > s_0$ or $s_0 - a < 0$. Note that $s_0 - a$ cannot be an upper bound as s_0 is the least upper bound of S. But note that then $s_0 - a < n_0 a$ for some $n_0 \in \mathbb{N}$ (because $s_0 - a$ is not an upper bound and therefore there is an element in the set S larger than it). However, this implies that $s_0 < (n_0 + 1)a$ and since $(n_0 + 1)a \in S$, s_0 is not a least upper bound. Therefore there cannot exist such a, b. ■

The Archimedean property shows that rational numbers are "everywhere", a concept further emboldended by the idea that the rationals are *dense* in $\mathbb{R}$.

Theorem 3.8 ($\mathbb{Q}$ is Dense in $\mathbb{R}$). Let $a, b \in \mathbb{R}$ with $a < b$. Then $\exists r \in \mathbb{Q}$ such that $a < r < b$.

Proof. Let $a, b \in \mathbb{R}$ such that $a < b$. This means that $b - a > 0$. By the Archimedean principle, there exists n such that $n(b - a) = nb - na > 1$. Therefore there exists $k \in \mathbb{N}$ such that $k > \max\{|an|, |bn|\}$, meaning $-k < an < bn < k$. Two things can be said about k

$$k \in K = \{j \in \mathbb{Z} : -k \leq j \leq k\}$$
$$k \in L = \{j \in K : an < j\}$$

Note that both sets are finite because the first has $2k + 1$ elements and the second is a subset of K. Define $m := \min L$ (which exists since L is non-empty and finite). Then $-k < an < m$. Therefore $m > -k$ meaning $m - 1 \in K$. Note that $an < m - 1$ is false since m is the minimum value where that inequality holds. Then $m - 1 \leq an$ meaning $m \leq an + 1 < bn$ (since $nb - na > 1$). Therefore since $an < m$, $an < m < bn$ or equivalently $a < \frac{m}{n} < b$. Since $\frac{m}{n} \in \mathbb{Q}$, there is a rational between a and b. ■

4.1 Limits of Sequences

Definition 4.9 (Sequence). A sequence is a mapping $s : \mathbb{N}_{\geq m} \to \mathbb{R}$ where m is typically 0 or 1. Alternatively, a sequence cant be thought of as an infinite tuple

$$s = (s_m, s_{m+1}, s_{m+2}, \ldots)$$

Define the image of a sequence as $S(\mathbb{N}_{\geq m}) := \{s_n : n \geq m\}$

Example 4.6. Consider $(s_n)_{n\in\mathbb{N}}$ given by $s_n = \frac{(-1)^n}{n^3}$. It is a sequence with $m = 1$ and looks like $(-1, \frac{1}{8}, -\frac{1}{27}, \ldots)$.

Example 4.7. Consider $(s_n)_{n\in\mathbb{N}_0} = (-1)^n$ which is the sequence $(1, -1, 1, -1, 1, \ldots)$. Note that the image $S(\mathbb{N}_0) = \{-1, 1\}$

Example 4.8. Consider $(s_n)_{n\in\mathbb{N}_0} = \left(1 + \frac{1}{n}\right)^n$ which gives a sequence of real numbers that 'appears' to get closer e as n grows large, as seen by the fact that $s_{1,000,000} = 2.718280469319377$.

4.1.1 Convergence of a Sequence

Definition 4.10 (Sequence Convergence). A sequence $(s_n)_{n\in\mathbb{N}_0}$ is said to converge to $s_0 \in \mathbb{R}$ if

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } |s_n - s| < \epsilon, \forall n > N$$

Pictorially, we are creating a neighborhood of 2ϵ around s_0. And if the sequence converges, there is an eventual s_N such that every subsequent number is within the neighbor hood around s_0.

Example 4.9. Consider $s_n = \frac{1}{n}$. Take $\epsilon > 0$. Note that $|s_n - 0| = \frac{1}{n} < \epsilon$ by the archimedean property. It is clearer if it is rewritten as $1 < n\epsilon$.

Example 4.10. Consider $s_n = (-1)^n, n \in \mathbb{N}$. Take $e > 0$.

Example 4.11. Consider $s_n = \frac{3n+1}{7n-4}, n \in \mathbb{N}$. A good guess for the limit is $\frac{3}{7}$ since the $3n$

and $7n$ terms 'dominate' as $n \to \infty$. Take $\epsilon > 0$. Then

$$\begin{aligned}\left|\frac{3n+1}{7n-1} - \frac{3}{7}\right| &= \left|\frac{21n+7-21n+12}{7(7n-4)}\right| \\ &= \left|\frac{19}{7} \cdot \frac{1}{7n-4}\right| \\ &= \frac{19}{7} \cdot \frac{1}{7n-4} \\ &\leq \frac{19}{49} \cdot \frac{1}{n-1}\end{aligned}$$

Since $\frac{1}{n-1} \to 0$ as $n \to \infty$.

Example 4.12. Consider $s_n = \sqrt[n]{n}, n \in \mathbb{N}$. Take $s_0 = 1$, and prove this much later.

Theorem 4.9 (Uniqueness of Limits). If a limit of a sequence exists, then it is unique.

Proof. Let s_n be a sequence that converges to s and s' as $n \to \infty$. Then

$$\begin{aligned}\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } |s_n - s| < \frac{\epsilon}{2}, \forall n > N \\ \forall \epsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } |s_n - s'| < \frac{\epsilon}{2}, \forall n > N\end{aligned}$$

Then

$$\begin{aligned}|s - s'| &= |s - s_n + s_n - s'| \\ &\leq |s_n - s| + |s_n - s'| < \epsilon\end{aligned}$$

Therefore $0 \leq |s - s'| < \epsilon$ for all $e > 0$, meaning $s = s'$. ■

Example 4.13. Consider $\lim_{n\to\infty} \frac{1}{n^2}$. Let $s_0 = 0$.

Example 4.14. Consider $\lim_{n\to\infty} \frac{4n^3+3n}{n^3-6} \stackrel{?}{=} 4$. Take $\epsilon > 0$. Then

$$\begin{aligned}\left|\frac{4n^3+3n}{n^3-6} - 4\right| &= \left|\frac{4n^3+3n-4n^3+24}{n^3-6}\right| \\ &= \frac{3n+24}{|n^3-6|}\end{aligned}$$

Note that $3n + 24 \leq 27n$ for all $n \in \mathbb{N}$ and $n^3 - 6 \geq \frac{n^3}{4}$ for $n \geq 2$.

$$\begin{aligned} &\leq 4 \cdot \frac{27n}{n^3} \\ &= \frac{108}{n^2} < \epsilon \end{aligned}$$

Take $N \in \mathbb{N} \geq \sqrt{\frac{108}{\epsilon}}$. Then

$$\left|\frac{4n^3 + 3n}{n^3 - 6} - 4\right| \leq 108$$

Theorem 4.10. Let $(s_n)_{n\in\mathbb{N}}$ be a sequence $s_n \geq 0$ for all n and $s = \lim_{n\to\infty} s_n$. Then $\lim_{n\to\infty} \sqrt{s_n} = \sqrt{s}$

Proof. Consider $|\sqrt{s_n} - \sqrt{s}|$.

$$\begin{aligned} |\sqrt{s_n} - \sqrt{s}| &= \left|\frac{(\sqrt{s_n} - \sqrt{s})(\sqrt{s_n} + \sqrt{s})}{\sqrt{s_n} + \sqrt{s}}\right| \\ &= \frac{|s_n - s|}{\sqrt{s_n} - \sqrt{s}} \end{aligned}$$

If $s > 0$, then $\frac{1}{\sqrt{s_n}+\sqrt{s}} \leq \frac{1}{\sqrt{s}}$ meaning we would want $\frac{|s_n - s|}{\sqrt{s}} < \epsilon$ or equivalently $|s_n - s| < \epsilon\sqrt{s}$. If $s = 0$, we want $\sqrt{s_n} < \epsilon$ or $s_n < \epsilon^2$. Now, formally:

■

Theorem 4.11. Let $(s_n)_{n\in\mathbb{N}}$ be convergent to $s \neq 0$ with $\forall n \in \mathbb{N}, s_n \neq 0$. Then $\inf\{|s_n| : n \in \mathbb{N}\} > 0$.

Proof. The idea is that given a neighborhood around s, there is a finite amount of values of the sequence outside of it. By choosing a neighborhood size of $\frac{|s|}{2}$, 0 is avoided. Therefore proceed with the formal proof by letting $\epsilon = \frac{|s|}{2}$. Since s_n converges and $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that $|s_n - s| < \frac{|s|}{2}$ for all $n > N$. Note that

$$||s_n| - |s|| \leq |s_n - s| < \frac{|s|}{2}, \forall n > N$$

and that

$$|s_n| \in (s - \epsilon, s + \epsilon), \forall n > N$$

■

Definition 4.11 (Bounded Series). A series $(s_n)_{n\in\mathbb{N}}$ is bounded if the image is bounded. Equivalently, it is bounded if $\exists M \in \mathbb{R}$ such that $s_n \leq M$ for all $n \in \mathbb{N}$.

Theorem 4.12 (Convergence Implies Boundedness). Let $(s_n)_{n\in\mathbb{N}}$ be a series that converges to s. Then the series is bounded.

Proof. Let $(s_n)_{n\in\mathbb{N}}$ be a series and assume it converges to s. Take $\epsilon = 1$ and find $N \in \mathbb{N}$ such that $|s_n - s| < 1$ for all $n > N$. Therefore s_n and s are at most 1 apart, therefore $|s_n| \leq |s| + 1$. This provides an upper bound on s_n for $n > N$. For $n \leq N$, construct the set $M = \{s_1, s_2, \ldots, s_N, |s| + 1\}$. Then note that

$$s_n \leq \max M < \infty, \forall n \in \mathbb{N}$$

Therefore the series is bounded. ■

Theorem 4.13 (Properties of Limits). The following properties hold for all limits of sequences.

a) $\lim_{n\to\infty} s_n = s, c \in \mathbb{R} \implies \lim_{n\to\infty} c \cdot s_n = c \lim_{n\to\infty} s_n$

b) $\lim_{n\to\infty} s_n = s, \lim_{n\to\infty} t_n = t \implies \lim_{n\to\infty}(s_n + t_n) = s + t$

c) $\lim_{n\to\infty} s_n = s, \lim_{n\to\infty} t_n = t \implies \lim_{n\to\infty}(s_n \cdot t_n) = st$

d) $\lim_{n\to\infty} s_n = s, s_n \neq 0 \forall n \in \mathbb{N}, s \neq 0 \implies \lim_{n\to\infty} \frac{1}{s_n} = \frac{1}{s}$

Proof. Let s_n and t_n be sequences that converge to s and t respectively.

a) TODO

b) Since both sequences are converges, they both admit $N_1, N_2 \in \mathbb{N}$ such that for an $\epsilon > 0$

$$|s_n - s| < \frac{\epsilon}{2}, \forall n > N_1$$
$$|t_n - t| < \frac{\epsilon}{2}, \forall n > N_2$$

Note then that

$$|(s_n - s) + (t_n - t)| \leq |s_n - s| + |t_n - t| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \forall n > \max\{N_1, N_2\}$$

c)

d) Note that

$$|s_n t_n - st| = |s_n(t_n - t) + (s_n - s)t| \leq |s_n||t_n - t| + |t||s_n - s|$$

Since s_n and t_n converge, they are bounded. Therefore, take $\epsilon > 0$ and note

$$\begin{aligned}
&\exists M > 0, |s_n| \leq M, |t_n| \leq M, \forall n \in \mathbb{N} \\
&\exists N_1, |s_n - s| < \frac{\epsilon}{2M}, \forall n > N_1 \\
&\exists N_2, |t_n - t| < \frac{\epsilon}{2M}, \forall n > N_2
\end{aligned}$$

Therefore

$$|s_n t_n - st| \leq M \cdot \frac{\epsilon}{2M} + M \cdot \frac{\epsilon}{2M} = \epsilon, \forall n > \max\{N_1, N_2\}$$

e) Consider the target expression $\left|\frac{1}{S_n} - \frac{1}{s}\right|$. This can be reformed into $\frac{1}{s_n \cdot s}|s_n - s|$. Since $s_n \neq 0$ and $s \neq 0$, $|s_n|$ is bounded below by a positive number m for all n. This also means that $s \geq m$. Therefore

$$\left|\frac{1}{s_n} - \frac{1}{s}\right| \leq \frac{1}{m^2}|s_n - s|$$

Formally, take $\epsilon > 0$. Since s_n converges, $\exists N \in \mathbb{N}$ such that

$$|s_n - s| < \epsilon m^2, \forall n > N$$

By the previous derivation,

$$\left|\frac{1}{s_n} - \frac{1}{s}\right| \leq \frac{m^2 \epsilon}{m^2} = \epsilon$$

Hence $\frac{1}{s_n}$ converges to $\frac{1}{s}$.

■

Example 4.15 (Example Limits). The following are basic example limits.

1. $\lim_{n\to\infty} \frac{1}{n^p} = 0, \forall p > 0$
2. $\lim_{n\to\infty} r^n = 0, |r| < 1$
3. $\lim_{n\to\infty} \sqrt[n]{n} = 1$
4. $\lim_{n\to\infty} \sqrt[n]{r} = 1, r > 0$

Proof. 1. Take $\epsilon > 0$ and $N > \sqrt[p]{\frac{1}{\epsilon}}$. Then $\frac{1}{n^p} < \epsilon$ for all $n > N$.

2. If $r = 0$, then clearly r^n is 0 for all n. Consider then $r \neq 0$. If $|r| < 1$, then $\exists S$ such that $|r| = \frac{1}{1+S}$. Therefore $|r^n| = \frac{1}{(1+S)^n} \leq \frac{1}{1+Sn}$. Take $\epsilon > 0$ and $N > \frac{1}{S\epsilon}$. Then for all

$n > N$, $|r^n| < \epsilon$.

3. Let $s_n = \sqrt[n]{n} - 1 \geq 0$. Note that

$$n = (1+s_n)^n \geq \underbrace{1 + ns_n + \frac{1}{2}n(n-1)s_n^2}_{\text{truncated binomial theorem}} > \frac{1}{2}n(n-1)s_n^2$$

Therefore $0 \leq s_n < \sqrt{\frac{2}{n-1}}$. Since $\lim_{n\to\infty} \sqrt{\frac{2}{n-1}} = 0$, by the squeeze theorem $\lim_{n\to\infty} s_n = 0$, meaning $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

4. Consider $r \geq 1$. Then there is always an $n \geq r$, meaning $1 \leq r \leq n$. Therefore $1 \leq r^{\frac{1}{n}} \leq n^{\frac{1}{n}} = 1$, hence $\lim_{n\to\infty} \sqrt[n]{r} = 1$. If $0 < r < 1$, then $\frac{1}{r} > 1$ meaning $\left(\frac{1}{r}\right)^n > 1$. ■

Example 4.16. Consider $\lim_{n\to\infty} \frac{n^3+6n^2+7}{4n^3+3n-4}$. This can be rewritten as $\frac{1+\frac{6}{n}+\frac{7}{n^3}}{4+\frac{3}{n^2}-\frac{4}{n^3}}$. Then

$$\lim_{n\to\infty} \frac{n^3+6n^2+7}{4n^3+3n-4} = \lim_{n\to\infty} \frac{1+\frac{6}{n}+\frac{7}{n^3}}{4+\frac{3}{n^2}-\frac{4}{n^3}} = \frac{1+0+0}{4+0+0} = \frac{1}{4}$$

Example 4.17. Consider $\lim_{n\to\infty} \frac{n-5}{n^2+7}$. This can be rewritten as $\frac{\frac{1}{n}-\frac{5}{n^2}}{1+\frac{7}{n^2}}$. Then

$$\lim_{n\to\infty} \frac{n-5}{n^2+7} = \lim_{n\to\infty} \frac{\frac{1}{n}-\frac{5}{n^2}}{1+\frac{7}{n^2}} = \frac{0}{1} = 0$$

4.1.2 Unbounded Limits

Definition 4.12 (Infinite Limit). $\lim_{n\to\infty} s_n = \infty$ if $\forall M > 0, \exists N$ s.t. $s_n > M, \forall n > N$. Likewise, $\lim_{n\to\infty} s_n = -\infty$ if $\forall M \leq 0, \exists N$ s.t. $s_n < M, \forall n > N$.

Theorem 4.14 (Implication of Infinite Limits). Let s_n and t_n be sequences.

1. If $\lim_{n\to\infty} s_n = \infty$ and $\lim_{n\to\infty} t_n > 0$, then $\lim_{n\to\infty} s_n t_n = \infty$.
2. $\lim_{n\to\infty} s_n = \infty \Leftrightarrow \lim_{n\to\infty} \frac{1}{s_n} = 0$

Proof. ■

Theorem 4.15. If (s_n) is unbounded and increasing, then $\lim_{n\to\infty} s_n = +\infty$.

Proof. Fix $M > 0$. Since s_n is increasing, it must have a lower bound s_1 and therefore is unbounded above. Since it is unbounded, it is possible to find some N such that $S_N > M$. Since s is increasing, $s_n \geq s_N > M$ for all $n > N$. Therefore the limit is $+\infty$. ■

4.1.3 Limits of Supremum and Infimum

Definition 4.13 (Limsup and Liminf). Let $(s_n)_{n\in\mathbb{N}}$ be a real sequence. The statement

$$\sup_{n\geq N} s_n$$

Is the supremum of the tail of the sequence (since it only acts on terms greater than N). In the limiting case where $N \to \infty$, this can be written as

$$\lim_{N\to\infty} \sup_{n\geq N} s_n = \limsup_{n\to\infty} s_n$$

The same applies for the infimum

$$\lim_{N\to\infty} \inf_{n\geq N} s_n = \liminf_{n\to\infty} s_n$$

Remark. If (s_n) is not bounded above $\limsup_{n\to\infty} s_n = \infty$ and if it not bounded below then $\liminf_{n\to\infty} s_n = -\infty$

Theorem 4.16. $\limsup_{n\to\infty} s_n \leq \sup\{s_n : n \in \mathbb{N}\}$

Theorem 4.17. If $\liminf_{n\to\infty} s_n = +\infty$, then $\lim_{n\to\infty} s_n = +\infty$.

Theorem 4.18. Let $(s_n)_{n\in\mathbb{N}}$ be a real sequence. Then $\lim_{n\to\infty} s_n$ exists or equals $\pm\infty$ if and only if

$$\limsup_{n\to\infty} s_n = \liminf_{n\to\infty} s_n = \lim_{n\to\infty} s_n$$

Proof. Let $l_N = \inf_{n\geq N} s_n$ and $u_N = \sup_{n\geq N} s_n$. Let $l = \lim_{N\to\infty} l_N$ and $u = \lim_{N\to\infty} u_N$.

$\Rightarrow$) Assume that s_n converges. That is $s = \lim_{n\to\infty} s_n$. Consider the case where $s = \infty$. Then

$$\forall M > 0, \exists N \in \mathbb{N} \text{ s.t. } s_n > M, \forall n > N$$

This means that $l_m \geq l_N \geq M$ for all $m > N$. Therefore $l = \infty$. Since $u \geq l$, $u = \infty$. Consider the case where $s \in \mathbb{R}$. Then by definition of convergence

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } |s_n - s| < \epsilon, \forall n > N.$$

Therefore $s_n < s + \epsilon$ for all $n > N$. Then $u_m \leq u_N \leq s + \epsilon$ for all $m > N$, meaning $u \leq s + \epsilon$. Since ϵ is arbitrary, it can be taken as 0 and hence $u \leq s$. Returning back to the definition of convergence, it is true that $s_n > s - \epsilon$ for all $n > N$. By the same logic as above, $l \geq s - \epsilon$ and by choosing $\epsilon = 0$ it follows that $l \geq s$. Then overall

$$s \leq l \leq u \leq s$$

meaning $l = u$.

$\Longleftarrow$) Consider the case where $\limsup s_n = \limsup s_n = s \in \mathbb{R}$. Then

$$\forall \epsilon > 0, \exists M_1 \in \mathbb{N} \text{ s.t. } |s - u_N| < \epsilon, \forall N > M_1.$$

This means that $\sup \{s_n : n > M_1\} < s + \epsilon$. Therefore $s_n < s + \epsilon$ for all $n > M_1$. Considering the infimum,

$$\forall \epsilon > 0, \exists M_2 \in \mathbb{N} \text{ s.t. } |s - l_N| < \epsilon, \forall N > M_2.$$

This means that $\inf \{s_n : n > M_2\} > s - \epsilon$ and therefore $s_n > s - \epsilon$ for all $n > M_2$. Therefore

$$s - \epsilon < s_n < s + \epsilon, \forall n > \max \{M_1, M_2\}$$

meaning s_n is convergent.

■

Definition 4.14 (Cauchy Sequence). A sequence $(s_n)_{n \in \mathbb{N}}$ in $\mathbb{R}$ is a Cauchy Sequence if $\forall \epsilon > 0, \exists N \in \mathbb{N}$ such that $|s_n - s_m| < \epsilon$ for all $m, n > N$.

Theorem 4.19 (Properties of Cauchy Sequences). Let $(s_n)_{n \in \mathbb{N}}$ be a Cauchy sequence.

1. Convergent sequences are Cauchy sequences
2. s_n is bounded

Proof. Let $(s_n)_{n \in \mathbb{N}}$ be a sequence.

1. Assume that s_n is convergent. That is

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } |s_n - s| < \frac{\epsilon}{2}, \forall n > N$$

   Note that $|s_n - s_m| \leq |s_n - s| + |s - s_m| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ for all $n, m > N$.

2. Assume that s_n is a Cauchy sequence. Choose $\epsilon = 1$. Then $\exists N \in \mathbb{N}$ such that $|s_n - s_m| < 1$ for all $m, n > N$. Choosing $m = N$, it follows that $|s_n| < |s_N| + 1$. This means that $|s_n| < \max \{|s_1|, |s_2|, \ldots, |s_N|, |s_N| + 1\}$. Therefore s_n is bounded.

■

Theorem 4.20. A sequence of real numbers converges if and only if it is Cauchy.

Proof. Let s_n be a sequence of reals.

$\Rightarrow$) Assume that s_n converges. By Theorem 4.19, s_n is Cauchy.

$\Leftarrow$) Assume that s_n is Cauchy. By Theorem 4.19, s_n is bounded. Since s_n is Cauchy, $\forall \epsilon > 0, \exists N \in \mathbb{N}$ such that $|s_n - s_m| < \epsilon$ for all $m, n > N$. This means that $s_n < s_m + \epsilon$ for all $m, n > N$.

■

4.2 Subsequences

Definition 4.15 (Subsequence). Let $(s_n)_{n\in\mathbb{N}}$ be a real sequence. If $(n_k)_{k\in\mathbb{N}}$ is a strictly increasing sequence of natural numbers, then $(s_{n_k})_{k\in\mathbb{N}}$ is a sub sequence.

Example 4.18. Consider the sequence $s_n = n^2(-1)^n$ with $n_k = 2k, k \in \mathbb{N}$. Substituting into s_n results in the subsequence $s_{2k} = 4k^2$.

Theorem 4.21 (Properties of Subsequences). Let $(s_n)_{n\in\mathbb{N}}$ be a real sequence.

1. There exists a subsequence that converges to $t \in \mathbb{R}$ if and only if the set $\{n \in \mathbb{N} : |s_n - t| < \epsilon\}$ is infinite for all $\epsilon > 0$
2. If s_n is unbounded above, it has a subsequence with limit $+\infty$
3. If s_n is unbounded below, it has a subsequence with limit $-\infty$

Proof. Let $(s_n)_{n\in\mathbb{N}}$ be a real sequence.

1. $\Rightarrow$)

   $\Leftarrow$) Assume that $\{n \in \mathbb{N} : |s_n - t| < \epsilon\}$ for all ϵ. Consider the set $N = \{n \in \mathbb{N} : s_n = t\}$. If it is infinite, it gives rise to a subsequence of the indices of N such that
   Consider when N is then finite. Let

   $$S_\epsilon = \underbrace{\{n \in \mathbb{N} : t - \epsilon < s_n < t\}}_{S_\epsilon^-} \cup \underbrace{\{n \in \mathbb{N} : t < s_n < t + \epsilon\}}_{S_\epsilon^+}$$

   Note then that $S_{\epsilon_1}^{\pm} \subset S_{\epsilon_2}^{\pm}$ for all $0 < \epsilon_1 < \epsilon_2$. Therefore $S_{\epsilon_1}^{\pm}$ is finite if $S_{\epsilon_2}^{\pm}$ is finite. Since S_ϵ is infinite, either S_ϵ^+ or S_ϵ^- is infinite. WLOG, assume that S_ϵ^+

is infinite. Choose

$$\begin{aligned}
n_1 &\in S^+_1 \text{ such that } t < s_n < t+1 \\
n_2 &\in S^+_{\frac{1}{2}} \text{ such that } t < s_n < t+\frac{1}{2} \\
n_3 &\in S^+_{\frac{1}{3}} \text{ such that } t < s_n < t+\frac{1}{3} \\
&\vdots \\
n_k &\in S^+_{\frac{1}{k}} \text{ such that } t < s_n < t+\frac{1}{k}
\end{aligned}$$

Note that then $t < s_{n_k} < t + \frac{1}{k}$ when $n_k > n_{k-1}$. Therefore $t < s_{n_k} < t + \frac{1}{k}$ for all $k \in \mathbb{N}$ and therefore the subsequence s_{n_k} converges to t.

2. Assume that s_n is unbounded above. Let $n_1 = 1$.

3.

■

Example 4.19. For any real number $r \in \mathbb{R}$, it is possible to find a subsequence of rationals $q_{n_k^r} \to r$ as $k \to \infty$.

Example 4.20. Let $(s_n)_{n\in\mathbb{N}}$ with $\inf\{s_n : n \in \mathbb{N}\} = 0$ and $s_n > 0$ for all n. Note that s_n is not necessairly bounded or convergent as seen by the following cases

$$s_n = \left(1, \frac{1}{2}, 1, \frac{1}{3}, 1, \frac{1}{4}, \ldots\right)$$
$$s_n = \left(1, \frac{1}{2}, 3, \frac{1}{3}, 4, \frac{1}{5}, \ldots\right)$$

However, note that both have a subsequence that converges to 0.

Theorem 4.22 (Convergence Implies Subsequence Convergence). If $(s_n)_{n\in\mathbb{N}}$ converges, then every subsequence of s_n converges to the same limit

Proof. Let $(s_n)_{n\in\mathbb{N}}$ be a convergent sequence with $s = \lim_{n\to\infty} s_n$. Let $(s_{n_k})_{k\in\mathbb{N}}$ be a subsquence of s_n. Note that $n_k \geq k$ for all k. By the convergence of s_n, take $\epsilon > 0$ and

$$\exists N \in \mathbb{N}, \text{ s.t. } |s_n - s| < \epsilon, \forall n > N$$

This implies that

$$\exists N \in \mathbb{N}, \text{ s.t. } |s_{n_k} - s| < \epsilon, \forall k > N$$

and since $n_k \geq k > N$, $\lim_{k\to\infty} s_{n_k} = s$. ■

Theorem 4.23 (Sequence's Have Monotonic Subsequences). Every sequence has a monotonic subsequence.

Proof. The n-th term will be called dominant if $x_m < x_n$ for all $m > n$. Assume that s_n has infinitely many dominant terms. Then let S_{n_k} be the subsequence of all dominant terms. Note that $s_{n_{k+1}} < s_{n_k}$ by the definition of dominant. Hence s_{n_k} is a monotonic decreasing sequence. Consider the case where there are finitely many dominant terms. ■

Theorem 4.24 (Bolzano-Weistrass Theorem). Every bounded sequence of real numbers has a convergent subsequence.

Proof. Let (s_n) be a real sequence and assume it is bounded. By 4.23, s_n has a monotonic subsequence. By 4.19 ■

Definition 4.16 (Subsequential Limit). A subsequential limit is any real number or $\pm\infty$ that is the limit of a subsequence.

Example 4.21. Let (q_n) be the sequence of all rational numbers (which is possible since $\mathbb{Q}$ is countable). Then r is a subsequential limit of (q_n) if $r \in \mathbb{R}$.

Theorem 4.25 (Limsup and Liminf Monotone Subsequences). Let (s_n) be a sequence of reals. Then there are monotone subsequences that converge to $\limsup s_n$ and $\liminf s_n$.

Proof. If s_n is not bounded (either above or below), then by 4.21 there are monotone subsequences that will converge to either $+\infty$ or $-\infty$. Assume that s_n is then bounded below. Then $\liminf s_n = l > -\infty$. Therefore $\exists N_0$ such that

$$\inf \{n \in \mathbb{N} : n > N\} > l - \epsilon, \forall \epsilon > 0, \forall n \geq N_0$$

Note that $\{n \in \mathbb{N} : l - \epsilon < s_n < l + \epsilon\}$ is infinite (otherwise you can derive a contradiction). Therefore using 4.21 again it is possible to create a monotone subsequence. The same argument applies for bounded above. ■

4.2.1 Subsequential Limits

Theorem 4.26 (Set of Subsequential Limits). Let (s_n) be a real sequence and $S = \{r \in \mathbb{R} : r \text{ is a subsequential limit for } s_n\}$. Then

1. $S \neq \emptyset$
2. $\inf S = \liminf s_n$ and $\sup S = \limsup s_n$
3. (s_n) converges if and only if S is a singleton

Proof. Let (s_n) be a real sequence and $S = \{r \in \mathbb{R} : r \text{ is a subsequential limit for } s_n\}$.

1. Follows from 4.25
2. Let $s = \liminf s_{k_n} = \limsup s_{k_n}$. Note that $n_k \geq k$ for all k as well as $\{s_{n_k} : k > N\} \subseteq \{s_n : n > N\}$ for all $N \in \mathbb{N}$. Therefore

$$\liminf s_n \leq \liminf s_{n_k} = s = \limsup s_{n_k} \leq \limsup s_n$$

meaning

$$\liminf s_n \leq \inf S \leq \sup S \leq \limsup s_n.$$

And $\limsup s_n, \liminf s_n \in S$.

3. Trivial?

■

Theorem 4.27. Let S be the set of subsequential limits of a sequence (s_n). Suppose that (t_n) is a sequence in $\mathbb{R} \cap S$. Then $\lim t_n \in S$.

Proof. Suppose that $t = \lim t_n$ is finite. Consider an interval $(t - \epsilon, t + \epsilon)$. Then there is a t_n in the interval. Let $\delta = \min t + \epsilon - t_n, t_n - t + \epsilon$. Then $(t_n - \delta, t_n + \delta) \subseteq (t - \epsilon, t + \epsilon)$. Since t_n is in S, it is a subsequential limit, $\{n \in \mathbb{N} : s_n \in (t - \epsilon, t + \epsilon)\}$ is infinite. Therefore by 4.21.1, t is a limit of a subsequence and hence a subsequential limit. ■

Example 4.22. If it is the case that for a sequence s that $\liminf s < \limsup s$, then it could be the case that $\{n \in \mathbb{N} : \liminf s \leq s_n \leq \limsup s_n\}$ is empty. This is true for the sequence $s_n = (-1)^n(1 + \frac{1}{n})$. Note that $S = \{-1, 1\}$. But

$$s_{2k} = 1 + \frac{1}{2k} > 1, \forall k$$
$$s_{2k+1} = -\left(1 + \frac{1}{2k}\right) < -1, \forall k$$

Metric Spaces and Topological Concepts

5.1 Expanding $\mathbb{R}$

Most of the focus so far has been on $\mathbb{R}$. Importantly, on $\mathbb{R}$ it was possible to define an ordering relation from which the absolute value and distance functions could arise. A natural question to ask is if this conceptual construction of distance can be constructed over different sets.

Definition 5.17 (Metric Space). Let S be a set. If there exists some mapping $d : S \times S \to \mathbb{R}$ (called a metric or distance) such that it satisfies

1. $d(x, x) = 0, \forall x \in S$ and $d(x, y) > 0, \forall x, y \in S, x \neq y$
2. $d(x, y) = d(y, x), \forall x, y \in S$
3. $d(x, z) \leq d(x, y) + d(y, z), \forall x, y, z \in S$

then (S, d) is a metric space.

Clearly $(\mathbb{R}, \text{dist})$ is a metric space. However, there are alternative metrics that still admit a metric space over $\mathbb{R}$.

Example 5.23. The following are some examples of metric spaces

a) $S = \mathbb{R}, d(x, y) = |x - y|$

b) $S = \mathbb{R}^k = \{(x_1, \ldots, x_k) : x_i \in \mathbb{R}, \forall i = 1, \ldots, k\}, d(x, y) = \sqrt{\sum_{i=1}^k (y_i - x_i)^2}$

Consider specifically the case of $\mathbb{R}^k$.

Proof. Consider the metric $d(x, y) = \sqrt{\sum_{i=1}^k (y_i - x_i)^2}$ over $\mathbb{R}^k$. Check that it satisfies the properties of being a metric.

1. The metric is zero when $y_i = x_i$ and therefore $x = y$, hence $d(x, x) = 0$ for all $x \in \mathbb{R}^k$
2. Since the summation terms are squared, the order of x_i and y_i does not matter, hence $d(x, y) = d(y, x)$ for all $x, y \in \mathbb{R}^k$
3. Firstly, an equivalence is

$$d(x, z) \leq d(x, y) + d(y, z) \Leftrightarrow d(x, z)^2 \leq d(x, y)^2 + 2d(x, y)d(y, z) + d(y, z)^2$$

By using the scalar product and its properties from vector spaces,

$$\begin{aligned} d(x, z)^2 = (x - z) \cdot (x - z) &= (x - y + y - z) \cdot (x - y + y - z) \\ &= (x - y) \cdot (x - y) + 2(x - y) \cdot (y - z) + (y - z) \cdot (y - z) \\ &= d(x, y)^2 + 2(x - y) \cdot (y - z) + d(y, z)^2 \qquad (*) \end{aligned}$$

Note that $\forall t > 0$

$$\begin{aligned} 0 &\leq ((x-y) - t(y-z)) \cdot ((x-y) - t(y-z)) \\ &= d(x,y)^2 + d(y,z)^2 t^2 - 2t(x-y)(y-z) \end{aligned}$$

Therefore by rearranging

$$(x-y) \cdot (y-z) \leq \frac{1}{2t} d(x,y)^2 + \frac{t}{2} d(y,z)^2$$

Since t was arbitrary, choosing $t = \frac{d(x,y)}{d(y,z)}$ gives

$$(x-y) \cdot (y-z) \leq d(x,y) d(y,z) \qquad \text{(Cauchy Schwarz Inequality)}$$

Going back to $(*)$,

$$\begin{aligned} d(x,z)^2 &= d(x,y)^2 + 2(x-y) \cdot (y-z) + d(y,z)^2 \\ &\leq d(x,y)^2 + 2d(x,y)d(y,z) + d(y,z)^2 \\ &= (d(x,y) + d(y,z))^2 \end{aligned}$$

and therefore by taking the root of each side,

$$d(x,z) \leq d(x,y) + d(y,z), \forall x,y,z \in \mathbb{R}^k$$

Since d satisfies all the properties of a metric, $(\mathbb{R}^k, d)$ is a metric space ■

Having a metric space provides enough machinery to define concepts like convergence.

Definition 5.18 (Metric Space Equivalents). Let $(s_n)_{n\in\mathbb{N}}$ be a sequence in (S,d) and $s \in S$. Then

1. Convergence is defined as

$$\lim_{n\to\infty} s_n = s \overset{\text{def}}{\iff} \lim_{n\to\infty} d(s_n, s) = 0$$

2. Cauchy is defined as

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } d(s_n, s_m) < \epsilon, \forall m,n > N$$

3. (S,d) is **complete** iff all Cauchy sequences converge.

The last idea of completeness is different in form than the Axiom of Completeness, however $(\mathbb{R}, \text{dist})$ satisfies this alternative definition of completeness (and is in fact equivalent to the Axiom of Completeness).

Theorem 5.28 ($\mathbb{R}^k$ is a Metric Space). $(\mathbb{R}^k, d)$ is a complete metric space.

It will be useful to show that convergence of a sequence in $\mathbb{R}^k$ can be determined by element wise sequences converging (and equivalently for determining if a sequence is Cauchy). For notation sake, the superscript refers to the index into a sequence and the subscript is the position index of the original sequence. That is a sequence $(x_n)_{n\in\mathbb{N}} \in \mathbb{R}^k$ is

$$(x_n)_{n\in\mathbb{N}} = \begin{pmatrix} x_1^n \\ \vdots \\ x_k^n \end{pmatrix}$$

Lemma 5.1 (Element Wise Implies Sequence Wise). A sequence $(x^n)_{n\in\mathbb{N}}$ in $\mathbb{R}^k$ converges iff (x_j^n) converges in $\mathbb{R}$ for $1 \le j \le k$. Additionally, $(x^n)_{n\in\mathbb{N}}$ is Cauchy iff (x_j^n) is Cauchy in $\mathbb{R}$ for $1 \le j \le k$.

Proof. ■

Proof of 5.28. Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathbb{R}^k$. Then by 5.1, $\left(x_n^j\right)$ is a Cauchy sequence in $\mathbb{R}$. Since $\mathbb{R}$ is complete, $\left(x_n^j\right)$ converges. Therefore all component sequences of (x_n) converge which by 5.1 implies the convergence of (x_n). ■

An interesting fact is that the Bolzano-Weistrass Theorem generalizes to $\mathbb{R}^k$ as long as boundedness is properly defined.

Definition 5.19 (Boundedness in $\mathbb{R}^k$). Let $S \subset \mathbb{R}^k$. S is bounded iff there exists $M \in \mathbb{R}$ such that $d(0, s) \le M$ for all $s \in S$.

Theorem 5.29. Each bounded sequence in $\mathbb{R}^k$ has a convergent subsequence.

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence in $\mathbb{R}^k$. Note that $\left|\left(x_j^n\right)\right| \le d(0, x_n)$ for all $j = 1, \ldots, k$ and $n \in \mathbb{N}$. Therefore each element wise sequence is bounded. Then

$$\begin{aligned}
(x_1^n) \text{ is bounded } &\implies \exists \left(n_l^1\right)_{l\in\mathbb{N}} \text{ s.t. } x_1^{n_l^1} \to x_1^\infty \\
\left(x_2^{n_l^1}\right) \text{ is bounded } &\implies \exists \left(n_l^2\right)_{l\in\mathbb{N}} \subset \left(n_l^1\right)_{l\in\mathbb{N}} \text{ s.t. } x_2^{n_l^2} \to x_2^\infty \\
&\vdots \\
\left(x_k^{n_l^{k-1}}\right) \text{ is bounded } &\implies \exists \left(n_l^k\right)_{l\in\mathbb{N}} \subset \left(n_l^{k-1}\right)_{l\in\mathbb{N}} \text{ s.t. } x_k^{n_l^k} \to x_k^\infty
\end{aligned}$$

Therefore a convergent subsequence of (x_n) can be constructed. ■

Definition 5.20 (Openness). Let (S, d) be a metric space and $E \subset S$. Then

1. $x \in E$ is an interior point of E iff $\{s \in S : d(x, s) < r\} \subset E$ for some $r > 0$.
2. $\mathring{E} = \{s \in E : s \text{ is an interior point }\}$
3. E is open iff $E = \mathring{E}$

Theorem 5.30 (Properties of Openness). Let $E \subset S$ where (S, d) is a metric space.

1. S is open in S
2. $\varnothing$ is open in S
3. E_α is open $\forall \alpha \in A$, then $\bigcup_{\alpha \in A} E_\alpha$ is open
4. E_j is open $\forall j = 1, \ldots, n$ then $\bigcap_{j=1}^n E_j$ is open

Definition 5.21. Let (S, d) be a metric space. Then

1. $E \subset S$ is closed iff $E^c = S \setminus E$ is open
2. The closure of E is $\overline{E} = \bigcap\limits_{\substack{E \subset F \subset S \\ F \text{ closed}}} F$
3. The boundary of E is $\partial E = \overline{E} \setminus \mathring{E}$

Remark. $\overline{E}$ is a closed set and is the smallest closed set that contains E.

Example 5.24. The following are examples of openness and boundaries

1. (a, b) is open and $[a, b]$ is closed in $\mathbb{R}$
2. $(a, b]$ and $[a, b)$ are neither open nor closed
3. With $I = \{(a, b), [a, b], [a, b), (a, b]\}$
   (a) $\overline{I} = [a, b]$
   (b) $\mathring{I} = (a, b)$
   (c) $\partial I = \{a, b\}$
4. Let $x \in \mathbb{R}^k$ and $r > 0$. Let $\mathbb{B}(x, r) = \left\{y \in \mathbb{R}^k : d(x, y) < r\right\}$
   (a) $\mathbb{B}(x, r)$ is open
   (b) $\overline{\mathbb{B}}(x, r)$ is closed
   (c) $\partial \mathbb{B}(x, r) = \left\{y \in \mathbb{R}^k : d(x, y) = r\right\}$

Theorem 5.31. Let (S, d) be a metric space and $E \subset S$. Then

1. E is closed iff $E = \overline{E}$
2. E is closed iff E contains the limit of every convergent sequence in E
3. $x \in \overline{E}$ iff there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in E that converges to x
4. $x \in \partial E$ iff $x \in \overline{E} \cap \overline{S \setminus E}$

Proof. Let E be a subset of a metric space (S, d).

1. $\Rightarrow$) Assume that E is closed. Then $\overline{E} = \subset \bigcap_{\substack{E\subset F\subset S \\ F \text{ closed}}} F \subset E$ since E is a subset of itself and is the smallest closed subset that contains itself. Since $E \subset \overline{E}$, it follows that $E = \overline{E}$.

   $\Leftarrow$) Assume that $E = \overline{E}$. Since $\overline{E}$ is the intersection of closed sets, it itself is closed. Therefore E must also be closed.

2. $\Rightarrow$) Assume that E is closed. Let (x_n) be a sequence in E that converges to some $x \in S$. Assume towards contradiction that $x \notin E$. Then $x \in S \setminus E$, meaning $\exists r > 0$ such that $\mathbb{B}(x, r) \subset S \setminus E$. However, this means that choosing an $\epsilon < r$ means $\exists N \in \mathbb{N}$ such that $d(x, x_n) < \epsilon < r$ for all $n > N$. Therefore $x_n \in \mathbb{B}(x, r)$ for $n > N$. But that means there are infinitely many terms of the sequence outside of E, a contradiction.

   $\Leftarrow$) Assume that E contains the limits of every convergent sequence in E. Let $x \in S \setminus E$. Suppose that for any $r > 0$ that $\mathbb{B}(x, r) \cap E \neq \varnothing$. Then it is possible to construct a sequence (x_n) where $x_n \in \mathbb{B}(x, \frac{1}{n}) \cap E$. Note that $(x_n) \to x$ since $d(x_n, x) < \frac{1}{n}$ for each n. However, this sequence is in E but the limit point x is not in E, hence a contradiction. Therefore there must be some $r > 0$ such that $\mathbb{B}(x, r) \cap E = \varnothing$ which is the same as saying $\mathbb{B}(x, r) \subset S \setminus E$. This means that $S \setminus E$ is equal to its interior and therefore $S \setminus E$ is open. Therefore E is closed.

3. $\Rightarrow$) Assume that $x \in \overline{E}$. Note that it is sufficient to show that for any $r > 0$ that $\mathbb{B}(x, r) \cap E \neq \varnothing$. If this is true, then by the same logic in (b) it is possible to construct a sequence in E that will approach x. Take $r > 0$ and assume towards contradiction that $\mathbb{B}(x, r) \cap E = \varnothing$. Then $E \subset S \setminus \mathbb{B}(x, r)$. Since open balls are open, then $S \setminus \mathbb{B}(x, r)$ is a closed set containing E which means that $\overline{E} \subset S \setminus \mathbb{B}(x, r)$. But then by the assumption, $x \in S \setminus \mathbb{B}(x, r)$ and $x \in \mathbb{B}(x, r)$ which is a contradiction.

   $\Leftarrow$) Assume that x is the limit of a sequence (x_n) of points in E. By part (a), $\overline{E}$ is closed and by (b), $\overline{E}$ must contain the limit of any sequence of points in $\overline{E}$. Since $x_n \in E$ for all n, $x_n \in \overline{E}$ for all n. Therefore (x_n) is a sequence of points in $\overline{E}$ and hence its limit must also be in $\overline{E}$.

4. $\Rightarrow$) Assume that $x \in \partial E$. Therefore $x \in \overline{E}$ and $x \notin \mathring{E}$. Therefore it is sufficient to show that $x \in \overline{S \setminus E}$. Let $F \supset S \setminus E$ be a closed set. Note that then $S \setminus F$ is open and that $S \setminus F \subset E$. If $x \in S \setminus F$, then there is some $r > 0$ such that $\mathbb{B}(x,r) \subset S \setminus F$. Since $S \setminus F \subset E$, it follows that $\mathbb{B}(x,r) \subset E$. However, this implies that x is in the interior and is therefore a contradiction. Therefore $x \notin S \setminus F$ meaning $x \in F$. Since F was an arbitrary closed set containing $S \setminus E$, x is in every closed set containing $S \setminus E$ and therefore $x \in \overline{S \setminus E}$.

   $\Leftarrow$) Assume that $x \in \overline{E}$ and $x \in \overline{S \setminus E}$. It is sufficient to show that $x \notin \mathring{E}$ since x is assumed to be in the closure. Assume towards contradiction that $x \in \mathring{E}$. Then there exists some $r > 0$ such that $\mathbb{B}(x,r) \subset E$. This means that $S \setminus \mathbb{B}(x,r)$ is closed set with $S \setminus \mathbb{B}(x,r) \supset S \setminus E$ which requires that $x \in S \setminus \mathbb{B}(x,r)$. However this is not possible since x is contained in any ball centered around it. Therefore x cannot be interior to E.

■

Theorem 5.32. Let F_n denote a sequence of sets such that $\varnothing \neq F_n = \overline{F_n} \subset \mathbb{R}^k$, $F_{n+1} \subset F_n$, and F_n is bounded for all n. Then

$$\bigcap_{n \in \mathbb{N}} F_n \neq \varnothing \text{ and is closed and bounded}$$

Proof. First, prove closure of the intersection. Let (x_k) be a sequence in $\bigcap_{n\in\mathbb{N}} F_n$ such that $x_k \to x$. Therefore (x_k) is a sequence in F_n for all $n \in \mathbb{N}$. Since every F_n is closed, the limit $x \in F_n, \forall n$. Therefore $x \in \bigcap_{n\in\mathbb{N}} F_n$, hence closure is proven. Note that $\bigcap_{n\in\mathbb{N}} F_n \subset F_1$. Since F_1 is bounded, the intersection is also bounded. For all $n \in \mathbb{N}$, $\exists x_n \in F_n$ since $F_n \neq \varnothing$. Therefore a sequence (x_n) can be made of these terms. The sequence is bounded since F_n is bounded for all n. Then by 4.24, there exists some subsequence $\left(x_{n_l}\right)$ that converges to some $x \in \mathbb{R}^k$. Then for all $N \in \mathbb{N}$, $x_{n_l} \in F_N$ for all $n_l \geq l > N$ meaning F_N is closed and $x \in \cap_{n\in\mathbb{N}} F_n$ (since N is arbitrary). ■

Example 5.25 (Cantor Set). Start with the closed interval $F_1 = [0,1]$. Then split into thirds and discard the middle, giving $F_2 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. Keep going for all $n \in \mathbb{N}$. Since $F_{n+1} \subset F_n$, each set is closed, bounded, and non-empty meaning $\bigcap_{n\in\mathbb{N}} F_n = C \neq \varnothing$. Note that the length of each interval is $l(F_n) = \left(\frac{2}{3}\right)^{n-1}$ which converges to 0 as $n \to \infty$. However, C cannot be put into a sequence.

Definition 5.22 (Covering and Compactness). Let (S,d) be a metric space.

1. A collection of open sets $\mathcal{U}$ in S is called an **open cover** of $E \subset S$ iff $E \subset \bigcup_{U\in\mathcal{U}} U$
2. A subcover of $\mathcal{U}$ is any subcollection that covers E
3. A cover (or subcover) is finite iff it consists of finitely many sets
4. A set is called **compact** iff every open cover poesses a finite subcover

Theorem 5.33 (Heine-Borel Theorem). A subset E of $\mathbb{R}^k$ is compact iff E is closed and bounded.

Proof. Let $E \subset \mathbb{R}^k$.

$\Rightarrow$) Assume that E is compact. Let $\mathcal{U}_m = \mathbb{B}(0, m)$. Note that $E \subset \bigcup_{m \in \mathbb{N}} \mathcal{U}_m = \mathbb{R}^k$. Therefore there exists and finite subcover $\mathcal{U}_{m_1}, \dots, \mathcal{U}_{m_n}$ with $m_1 < \dots < m_n$. Therefore $E \subset \mathcal{U}_{m_n}$ since every ball m_k is a subset of $\mathcal{U}_{m_n}$, which means that E is bounded. Next take $x \in S \setminus E$ and let $V_m = \overline{\mathbb{B}}(x, \frac{1}{m})^{\complement}$. Let $\mathcal{V} = \bigcap_{m \in \mathbb{N}} V_m$. Note that $\mathcal{V}$ is an open cover of E since $V = \mathbb{R}^k \setminus \{x\}$. Therefore there exists a finite open subcover such that $E \subset \bigcup_{l=1}^{n} V_{m_l}$ for some $m_1 < \dots < m_n$. Therefore $\mathbb{B}\left(()x, \frac{1}{m_n}\right) \subset E^{\complement}$. Since x was arbitrary, it follows that $S \setminus E$ is open and therefore E is closed.

In order to prove the reverse implication, some mathematical machinery will be needed.

Definition 5.23 (K-Cell). A k-cell (parallelalpiped) F is a set of the form $[a_1, b_1] \times \dots \times [a_k, b_k] \subset \mathbb{R}^k$. The diameter of F is defined as $\delta F = \sup\{d(\mathbf{x}, \mathbf{y}) : \mathbf{x}, \mathbf{y} \in F\}$ which can be calculated via the normal geometric methods.

Lemma 5.2. k-cells are compact.

Proof. Assume towards contradiction that a k-cell is not compact. Let F be a k-cell and $\mathcal{U}$ be an open cover of F. Then there is an open cover that does admit a finite subcover over F. Approach F as the union of the original k-cell halved. That is

$$F = \bigcup_{j=1}^{2^k} F_j^1, \delta F_j^1 = \frac{1}{2}\delta F.$$

Since there is no finite cover over F, at least one of sub cells cannot be finitely covered. Say that $F_{j_1}^1$ is the subcell that does not have a finite cover. Then do the same halving process to $F_{j_1}^1$, giving

$$F_{j_1}^1 = \bigcup_{j=1}^{2^k} F_j^2, \delta F_j^2 = \frac{1}{2}F_{j_1}^1.$$

Therefore by the same logic, there is at least one F^2 subcell that cannot be finitely covered. By continuing this process, it follows that there is a sequence of subsets such that

$$F_{j_1}^1 \supset F_{j_2}^2 \ldots \supset F_{j_n}^n, \delta F_{j_n}^n = \frac{\delta F}{2^n}$$

Therefore by 5.32,

$$\bigcap F_{j_n}^n \neq \emptyset \text{ and is closed and bounded}$$

■

$\Leftarrow$) Assume that E is bounded and closed. Since E is bounded, there must be a "square" Q such that

$$\underbrace{[-m, m] \times \ldots [-m, m]}_{k-\text{cell}} = Q_m \supset E$$

If $\mathcal{U}$ is an open cover of E, then $\mathcal{U} \cup E^\complement$ is also an open cover since $E^\complement$ is open. Furthermore, this is an open cover for Q_m. Since Q_m is compact, then $\mathcal{U} \cup E^\complement$ admits a finite subcover for Q_m and therefore a finite subcover for E.

■

Example 5.26 (Distance to set). Let (S, d) be a metric space and $\emptyset \neq E \subset S$. Then define

$$d(x, E) := \inf\{d(x, e) : e \in E\}, x \in S$$

Note that $|d(x_1, E) - d(x_2, E)| \leq d(x_1, x_2)$ for all $x_1, x_2 \in S$. Consider the following claim

Theorem 5.34. If $E \subset S$ is compact and $E \subset U$ where U is open, then $\exists \delta > 0$ such that $\{x \in S : d(x,E) < \delta\} \subset U$.

Proof. $\forall x \in E \subset U$, there is some $r_x > 0$ such that $\mathbb{B}(x, r_x) \subset U$ since U is open. Note that then

$$\left\{\mathbb{B}\left(x, \frac{r_x}{2}\right) : x \in E\right\}$$

is an open cover of E. Since E is compact, there is a finite subcover

$$\mathbb{B}\left(x_1, \frac{r_1}{2}\right), \ldots, \mathbb{B}\left(x_n, \frac{r_n}{2}\right) \implies E \subset \bigcup_{i=1}^{n} \mathbb{B}\left(x_i, \frac{r_i}{2}\right).$$

Choose $\delta \coloneqq \min\left\{\frac{r_1}{2}, \ldots, \frac{r_n}{2}\right\}$. Take $x \in S$ such that $d(x,E) < \delta$. Then $d(x,y) < \delta$ for some $y \in E$, meaning $d(x_j, y) < \delta$ for some $j = 1, \ldots, n$. Note that

$$\begin{aligned} d(x, x_j) &\leq d(x,y) + d(y, x_j) \\ &\leq \delta + \frac{r_{x_j}}{2} \\ &\leq \frac{r_{x_j}}{2} \end{aligned}$$

■

Definition 6.24 (Summation). Given a sequence (a_n) starting at m, then

$$S_n := \sum_{k=m}^{n} a_k, n \geq m$$

and $(S_n)_{n\geq m}$ is the sequence of partial sums. Then

$$\sum_{k=m}^{\infty} a_k \text{ converges } \Leftrightarrow (S_n)_{n\geq m} \text{ converges.}$$

Furthermore, if $\lim S_n = s$, then

$$\lim_{n\to\infty} \sum_{k=m}^{n} a_k = s.$$

Remark. Note the following properties for the sequence of partial sums

a) $a_k \geq 0$ for all $k \geq m$, then $(S_n)_{n\geq m}$ is an increasing sequence and either converges or diverges to ∞.

b) As a consequence, $\sum_{k=m}^{n} a_k$ is always meaningful.

The last property motivates defining another form of convergence.

Definition 6.25 (Absolute Convergence). $\sum_{k=m}^{\infty} a_k$ converges absolutely if $\sum_{k=m}^{\infty} |a_k|$ converges.

Example 6.27. Note that $(1-r)(1+r+r^2+...+r^n) = 1+r^{n+1}$. Therefore

$$(1+r+r^2+...+r^n) = \sum_{k=0}^{n} r^k = \frac{1+r^{n+1}}{1-r}, \forall n \geq 0.$$

Example 6.28. Consider the sum

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges if and only if $p > 1$.

Definition 6.26 (Cauchy Series). A series $\sum a_k$ satisfies the Cauchy criterion if its sequence of partial sums (S_n) is Cauchy.

Theorem 6.35 (Cauchy Criterion). Let $\sum a_k$ be a series.

1. $\sum a_k$ converges iff it satisfies the Cauchy Criterion
2. If $\sum a_k$ converges, then $\lim a_n = 0$.

Proof. Let $\sum a_k$ be a series.

1. $\Rightarrow$) Assume that $\sum a_k$ converges. Then the sequence of partial sums is convergent and therefore Cauchy. Hence by the definition of the Cauchy criterion $\sum a_k$ is Cauchy.

   $\Leftarrow$) Assume that $\sum a_k$ is Cauchy. Therefore the sequence of partial sums are Cauchy and hence converge. Therefore $\sum a_k$ converges.

■

Remark. Note that in the prior theorem that the second statement is not an if or only if. Consider the Harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

Since

$$\sum_{k=n}^{2n} \frac{1}{k} \geq n \cdot \frac{1}{2n} = \frac{1}{2} \text{ for all } n$$

(S_n) cannot be Cauchy and therefore the partial sums do not converge, meaning the series doesn't converge. But importantly note that $a_n \to 0$, hence why the theorem is not and iff.

Theorem 6.36. Let $\sum a_k$ and $\sum b_k$ be series with $a_n \geq 0$ for all n. Then

1. If $\sum a_k$ converges and $|b_k| \leq a_k$ for all k, then $\sum b_k$ also converges
2. If $\sum a_k = \infty$ and $b_k \geq a_k$ for all k, then $\sum b_k = \infty$
3. Absolute convergence implies convergence

Proof. Let $\sum a_k$ and $\sum b_k$ be series with $a_n \geq 0$ for all n. Then

1. Assume that $\sum a_k$ converges and $|b_k| \leq a_k$ for all k. Note that

   $$0 \leq \left|\sum_{k=m}^{n} b_k\right| \leq \sum_{k=m}^{n} |b_k| \leq \sum_{k=m}^{n} a_k.$$

   Since $\sum a_k$ converges, for all $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that $\sum_{k=m}^{n} a_k < \epsilon$ which by the previous inequality implies that $\sum b_k$ satisfies the Cauchy criterion. Hence $\sum b_k$ converges.

2.

3. If $\sum b_k$ absolutely converges, then

$$0 \leq \left| \sum_{k=m}^{n} b_k \right| \leq \sum_{k=m}^{n} |b_k|.$$

Therefore $\sum b_k$ satisfies the Cauchy criterion and hence converges.

■

Theorem 6.37 (Ratio Test). Let $\sum a_k$ be a series where $a_n \neq 0$ for all n.

1. $\sum a_k$ absolutely converges if $\limsup \left| \frac{a_{n+1}}{a_n} \right| < 1$
2. $\sum a_k$ diverges if $\liminf \left| \frac{a_{n+1}}{a_n} \right| > 1$
3. If $\liminf \left| \frac{a_{n+1}}{a_n} \right| \leq 1 \leq \limsup \left| \frac{a_{n+1}}{a_n} \right|$, then nothing can be concluded about $\sum a_k$

Theorem 6.38. For some series $\sum a_k$, define $\alpha := \limsup |a_n|^{\frac{1}{n}}$. Then

1. If $\alpha < 1$, $\sum a_k$ absolutely converges
2. If $\alpha > 1$, $\sum a_k$ diverges
3. If $\alpha = 1$, then nothing can be concluded about $\sum a_k$

Proof. Let $\sum a_k$ be a series and define $\alpha := \limsup |a_n|^{\frac{1}{n}}$.

1. Assume that $\alpha < 1$. Then choose $0 < \epsilon < 1 - \alpha$. Then $\exists N \in \mathbb{N}$ such that

$$\alpha - \epsilon < \sup_{n>N} |a_n|^{\frac{1}{n}} < \alpha + \epsilon$$

meaning that for all $n > N$ that

$$|a_n| < (\alpha + \epsilon)^n \tag{$*$}$$

By the choice of ϵ, $\alpha + \epsilon < 1$ which means that

$$\sum_n (\alpha + \epsilon)^n$$

is convergent. Therefore by $(*)$, $\sum |a_k|$ converges.

2. Assume that $\alpha > 1$. Then by the definition of the limit superior, $\exists$ a subsequence (n_k) such that $|a_{n_k}|^{\frac{1}{n_k}} > 1$ for all k. Then $|a_{n_k}| > 1$ for all k. Therefore $a_{n_k} \not\to 0$ which means $\sum a_k$ diverges by the converse of statement 2 in 6.35.

3. Take $a_n = \frac{1}{n}$. Then $\alpha = \limsup \frac{1}{\sqrt[n]{n}} = 1$. Take $a_n = \frac{1}{n^2}$. Then $\alpha = \limsup \frac{1}{n} = 1$. Therefore a series with $\alpha = 1$ may or may not diverge, meaning nothing can be concluded about $\sum a_k$ in general.

■

Remark. Let $\sum a_k$ be a series with $a_n \neq 0$ for all n. If $\lim \left|\frac{a_n+1}{a_n}\right| = 1$ implies that $\limsup |a_n|^{\frac{1}{n}} = 1$.

Example 6.29. Consider the series $\sum \frac{n}{n^2+3}$. Since

$$\frac{n}{n^2+3} \geq \frac{n}{n^2+3n^2} \geq \frac{1}{4} \cdot \frac{1}{n}$$

the series diverges. Note that

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n+1}{n} \cdot \frac{n^2+3}{n^2+2n+4} \xrightarrow{n\to\infty} 0.$$

Example 6.30. Consider the series $\sum \frac{n}{3^n}$. Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n+1}{n} \cdot \frac{1}{3} \xrightarrow{n\to\infty} \frac{1}{3} < 1.$$

Therefore the series converges by the ratio test.

Example 6.31. Consider the series

$$\sum_{n\geq 0} 2^{(-1)^n - n} = 2 + \frac{1}{4} + \frac{1}{2} + \frac{1}{16} + \ldots$$

Since $a_n < \frac{1}{2^{n-1}}$ for all n, the series converges by comparison. Additionally,

$$\frac{2^{(-1)^{n+1}-n-1}}{2^{(-1)^n-n}} = \begin{cases} \frac{1}{8} & n \text{ even} \\ 2 & n \text{ odd} \end{cases}$$

which means that $\liminf a_n = \frac{1}{8} < 1 < 2 = \limsup a_n$, therefore nothing can be concluded by the ratio test. Attempt to use the root test. Then

$$|a_n|^{\frac{1}{n}} = \begin{cases} 2^{\frac{1}{n}-1} & n \text{ even} \\ 2^{-\frac{1}{n}-1} & n \text{ odd} \end{cases}$$

Therefore $\liminf |a_n|^{\frac{1}{n}} = \limsup |a_n|^{\frac{1}{n}} = \lim |a_n|^{\frac{1}{n}} = \frac{1}{2}$ since $2^{\frac{1}{n}} \to 1$ and $2^{-\frac{1}{n}} \to 1$, meaning the series converges.

Example 6.32. Consider the sereies

$$\sum_{n\geq 1}\frac{(-1)^n}{\sqrt{n}}.$$

For this series, the root test, ratio test, and comparison are useless because of the alternating sign of each term.

Continuity

7.1 Continuous Functions

Definition 7.27 (Real Valued Function). Let $E \subset \mathbb{R}$. Then a mapping $f : E \to \mathbb{R}$ is a real valued function. If a domain E isn't specified, the largest possible subset of $\mathbb{R}$ is taken where $f(x)$ makes sense.

Definition 7.28 (Continuity). Let $f : E \to \mathbb{R}$ be a real valued function and $S \subset E$. Then

1. f is continuous at x_0 if $x_0 \in E$ iff
$$\lim f(x_n) = f(x_0)$$
for any sequence (x_n) in E that converges to x_0.

2. f is continuous on S iff f is continuous at x_0 for all $x_0 \in S$

3. f is continuous iff it is continuous on all of E

Theorem 7.39 (Epsilon-Delta Continuity). A real valued function f is continuous at some point $x_0 \in \mathrm{dom}(f)$ iff
$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } |x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon.$$

Definition 7.29 (Operations on Real Valued Functions). Let $f : \mathrm{dom}(f) \subset \mathbb{R} \to \mathbb{R}$ and $g : \mathrm{dom}(g) \subset \mathbb{R} \to \mathbb{R}$. Then define
$$f \pm g : \mathrm{dom}(f) \cap \mathrm{dom}(g) \to \mathbb{R} : x \mapsto f(x) \pm g(x)$$
$$f \cdot g : \mathrm{dom}(f) \cap \mathrm{dom}(g) \to \mathbb{R} : x \mapsto f(x) \cdot g(x)$$

For division,
$$\frac{f}{g} : \mathrm{dom}(f) \cap \{x \in \mathrm{dom}(g) : g(x) \neq 0\} \to \mathbb{R} : x \mapsto \frac{f(x)}{g(x)}.$$

For maxima and minima,
$$\max(f, g) : \mathrm{dom}(f) \cap \mathrm{dom}(g) \to \mathbb{R} : x \mapsto \max\{f(x), g(x)\}$$
$$\min(f, g) : \mathrm{dom}(f) \cap \mathrm{dom}(g) \to \mathbb{R} : x \mapsto \min\{f(x), g(x)\}$$

Finally for composition,
$$g \circ f : \{x \in \mathrm{dom}(f) : f(x) \in \mathrm{dom}(g)\} \to \mathbb{R} : x \mapsto g(f(x)).$$

Theorem 7.40 (Basic Operations Preserve Continuity). Let f, g be real valued functions.

1. If f, g are continuous at x_0, then $f \pm g$ and $f \cdot g$ are continuous at x_0.
2. If f, g are continuous at x_0 and $g(x_0) \neq 0$, then $\frac{f}{g}$ is continuous at x_0.

Proof. Let f, g be real valued functions.

1. Assume that f, g are continuous at x_0. Let (x_n) be a sequence in $\text{dom}(f) \cap \text{dom}(g)$ that converges to x_0. Since f, g are continuous, then $f(x_n) \to x_0$ and $g(x_n) \to g(x_0)$ which by the limit theorems gives $f(x_n) + g(x_n) \to f(x) + g(x)$ meaning $f + g$ is continuous at x_0. The argument holds for $f \cdot g$.
2. Assume that f, g are continuous at x_0 and $g(x) \neq 0$ for all $x \in \text{dom}(f) \cap \text{dom}(f)$. Let (x_n) be a sequence in $\text{dom}(f) \cap \text{dom}(g)$ that converges to x_0. Since f, g are continuous, then $f(x_n) \to x_0$ and $g(x_n) \to g(x_0)$. Note that $g(x_n) \neq 0$ for all n by the assumption. Therefore by limit theorems it follows that $\frac{f(x_n)}{g(x_n)} \to \frac{f(x_0)}{g(x_0)}$, hence $\frac{f}{g}$ is continuous at x_0.

■

Theorem 7.41 (Composition Preserves Continuity). Let f, g be real valued functions. If f is continuous at x_0 and g is continuous at $f(x_0)$, then $g \circ f$ is continuous at x_0.

Proof. Let f, g be real valued functions and assume that f is continuous at x_0 and g is continuous at $f(x_0)$. Let (x_n) be a sequence in $\{x \in \text{dom}(f) : f(x) \in \text{dom}(g)\}$ such that $x_n \to x_0$. Since f is continuous at x_0, $f(x_n) \to x_0$. Let (y_n) be the sequence defined by $y_n = f(x_n)$. Then $y_0 = f(x_0)$. Therefore since g is continuous at $f(x_0)$, $g(y_n) \to g(y_0) = g(f(x_0))$. Therefore $g \circ f$ is continuous at x_0. ■

Theorem 7.42 (Maximum Preserves Continuity). Let f, g be real valued functions. If f is continuous at x_0 and g is continuous at $f(x_0)$, then $\max(f, g)$ is continuous at x_0.

Proof. First note that

$$\max(r, s) = \frac{1}{2}(r + s) + \frac{1}{2}|r - s|, \forall r, s \in \mathbb{R}.$$

Consider the case $r \geq s$. Then

$$\frac{1}{2}(r + s) + \frac{1}{2}|r - s| = \frac{1}{2}(r + s) + \frac{1}{2}(r - s) = r = \max(r, s).$$

If $r < s$, then

$$\frac{1}{2}(r + s) + \frac{1}{2}|r - s| = \frac{1}{2}(r + s) - \frac{1}{2}(r - s) = s = \max(r, s).$$

Therefore the original equation holds. Note then that

$$\max(f(x), g(x)) = \frac{1}{2}(f(x) + g(x)) - \frac{1}{2}|f(x) + g(x)|.$$

Since the absolute value function is continuous on all of $\mathbb{R}$, by 7.40 and 7.42 it follows that the maximum of two functions in also continuous. ∎

7.2 Properties of Continuous Functions

Definition 7.30 (Function Boundedness). Let $f : \mathrm{dom}(f) \subset \mathbb{R} \to \mathbb{R}$ be a real valued function. f is bounded iff there is some $M \in \mathbb{R}$ such that

$$|f(x)| \leq M, \forall x \in \mathrm{dom}(f).$$

Example 7.33. Consider the function $\sqrt{x-1}$. Assume towards contradiction that it is bounded. That is, $\exists M \in \mathbb{R}$ such that

Theorem 7.43. Let $f : [a, b] \to \mathbb{R}$ be continuous. Then

1. f is bounded
2. f assumes its max and its min. That is $\exists x_m, x_M \in [a, b]$ such that

$$f(x_m) \leq f(x) \leq f(x_M), \forall x \in [a, b].$$

Proof. Let $f : [a, b] \to \mathbb{R}$ be continuous.

1. Assume towards contradiction that f is not bounded. Then $\forall n \in \mathbb{N}$, there is some $x_n \in [a, b]$ such that $|f(x_n)| \geq n$. Therefore $(x_n)_{n\in\mathbb{N}}$ is a sequence in $[a, b]$. Since (x_n) is bounded, there is some subsequence (n_j) such that (x_{n_j}) converges to $x_\infty \in [a, b]$. Since f is continuous, then $|f(x_{n_j})| \xrightarrow{j\to\infty} |f(x_\infty)|$. However, $n_j \leq |f(x_{n_j})|$ meaning the limit as $j \to \infty$ would be infinite. Hence a contradiction.

2. By the first claim, f is bounded. Therefore $m = \inf_{x\in[a,b]} f(x) > -\infty$. Then $\forall n \in \mathbb{N}, \exists x_n \in [a, b]$ such that $m \leq f(x_n) \leq m + \frac{1}{n}$. This gives a sequence (x_n) that is bounded (because it is in $[a, b]$). Therefore by Bolzano Weistrass, $\exists(n_j)$ such that $x_{n_j} \to x_{\min}$. Since f is continuous,

$$\lim_{j\to\infty} m \leq \lim_{j\to\infty} f(x_{n_j}) \leq \lim_{j\to\infty} m + \frac{1}{n_j} \implies f(x_{\min}) = m.$$

Therefore the infimum m is achieved by f in its domain and therefore m is the minimum value and $x_{\min}$ is the minimum argument. The argument for the maximum follows the same by replacing inf with sup and flipping the inequality to squeeze towards the supremum.

∎

Remark. If the interval is not closed, then the theorem is not true in general. Consider

$$f : (0, 1] \to \mathbb{R} : x \mapsto \frac{1}{x}.$$

Note that f is continuous but is unbounded and has no max. Furthermore

$$f : (-1, 1) \to \mathbb{R} : x \mapsto x^2.$$

f in this case is continuous and bounded, but it doesn't have a maximum.

Theorem 7.44 (Intermediate Value Theorem). Let $f : I \to \mathbb{R}$ be a continuous function where I is an interval in $\mathbb{R}$. If $y_0 \in \big(\min(f(a), f(b)), \max(f(a), f(b))\big)$ with $a < b$ and $a, b \in I$, then there exists $x_0 \in (a, b)$ such that $f(x_0) = y_0$.

Proof. WLOG, take $f(a) > y_0 > f(b)$. Let $S = \{x \in [a, b] : f(x) > y_0\}$. Note S is non empty since $a \in S$. Since S is bounded, let $x_0 = \sup S$. Therefore for all $n \in \mathbb{N}$, there is some $s_n \in S$ such that $x_0 \geq s_n \geq x_0 - \frac{1}{n}$ since $x_0 - \frac{1}{n}$ is not an upper bound. Therefore

$$\lim s_n = x_0, f(s_n) > y_0, \forall n \implies f(x_0) = \lim f(s_n) \geq y_0.$$

Next, take $x_0 \leq \xi_n = \min\left\{x_0 + \frac{1}{n}, b\right\}$. Then

$$f(x_0) = \lim f(\xi_n) \leq y_0.$$

Therefore $y_0 \leq f(x_0) \leq y_0 \implies f(x_0) = y_0$. ∎

Corollary 7.3. If $f : I \to \mathbb{R}$ where I is an interval in $\mathbb{R}$ is continuous, then

$$f(I) = \{f(x) : x \in I\}$$

is an interval or a singleton.

Proof. Let $J = f(I)$. Take $y_0, y_1 \in J$ with $y_0 < y_1$. Note that if $y_0 < y < y_1$, then by 7.44, $y \in J$. If $\inf J < \sup J$, then J is an interval and if they are the same then J is a singleton. ∎

Example 7.34. Let $f : [0, 1] \to [0, 1]$ be continuous. Then $\exists x_0 \in [0, 1]$ such that $f(x_0) = x_0$. That is, f has a fixed point.

Proof. Let $g : [0, 1] \to [0, 1] : x \mapsto f(x) - x$. Note then that $g(0) = f(0) - 0 \geq 0$ and $g(1) = f(1) - 1 \leq 0$. Therefore by 7.44, $\exists x_0 \in [0, 1]$ such that $g(x_0) = 0$ meaning $f(x_0) - x_0 = 0 \implies f(x_0) = x_0$. ∎

Example 7.35. If $y > 0$, then it has a positive m root.

Proof. Let $f(x) = x^m, x \geq 0$. Note that f is continuous and $\exists b > 0$ such that $y < b^m$. Then

$$f(0) < y \leq f(b) \implies \exists x \in (0,b) \text{ s.t. } f(x) = x^m = y.$$

■

Theorem 7.45. Let $g : J \to \mathbb{R}$ be a strictly increasing function over the interval J. Then if $g(J)$ is also an interval, g is continuous.

Proof. Take $x_0 \in J$ such that x_0 is not an endpoint. Then $g(x_0)$ is not an end point of $g(J) = I$ by monotonicity. Therefore it is possible to find a neighborhood $(g(x_0) - \epsilon_0, g(x_0) + \epsilon_0) \subset I$. Take ϵ such that $0 < \epsilon < \epsilon_0$ and for some x_1 and x_2 in J,

$$g(x_1) = g(x_0) - \epsilon, g(x_2) = g(x_0) + \epsilon.$$

By monotonicity,

$$x_1 < x_0 < x_2 \text{ and } g(x_0) - \epsilon \leq g(x_1) < g(x) < g(x_2) \leq g(x_0) + \epsilon, \forall x \in (x_1, x_2)$$

which implies $|g(x) - g(x_0)| < \epsilon$. Take $\delta = \min\{x_2 - x_0, x_1 - x_0\}$. Then

$$|x - x_0| < \delta \implies x_1 < x_0 < x_2 \implies |g(x) - g(x_0)| < \epsilon.$$

Therefore g is continuous. ■

Theorem 7.46. Let $f : I \to \mathbb{R}$ be continuous and strictly increasing where I is an interval. Then

1. $f(I) = J$ is an interval
2. $f^{-1} : J \to I$ exists and is strictly increasing and continuous.

Proof. ■

Theorem 7.47. Let $f : I \to \mathbb{R}$ be one to one and continuous where I is an interval. Then f is strictly increasing or strictly decreasing.

Proof. Let $f : I \to \mathbb{R}$ be one to one and continuous where I is an interval.

1. If $a < b < c$ in I, then $f(a) < f(b) < f(c)$. Assume towards contradiction that this is not the case. Then $f(b) > \max\{f(a), f(c)\}$ or $f(b) < \min\{f(a), f(c)\}$. Consider the second case. Take $f(b) < y < \min\{f(a), f(b)\}$ and use 7.44 on $[a,b]$ and $[b,c]$ to find $x_1 \in (a,b)$ and $x_2 \in (b,c)$ such that $f(x_1) = f(x_2) = y$. This contradicts the assumption that f is one to one since $x_1 \neq x_2$. The other case follows similarly.

2. Take $a_0 < b_0$ with $a_0, b_0 \in I$. WLOG, let $f(a_0) < f(b_0)$. Note that $f(x) < f(a_0)$ for $x < a_0$ since $x < a_0 < b_0$ and therefore follows from (1). Additionally, $f(a_0) < f(x) < f(b_0)$ for $a_0 < x < b_0$ and $f(x) > f(b_0)$ for $x > b_0$. It then follows that $f(x) < f(a_0)$ for all $x < a_0$ and $f(x) > f(a_0)$ for all $x > a_0$.

3. Take $x_1, x_2 \in I$ such that $x_1 < x_2$. If $x_1 \leq a_0 \leq x_2$, then by (2), $f(x_1) < f(x_2)$. If $x_1 < x_2 \leq a_0$, then $f(x_1) < f(a_0)$ and $f(x_1) < f(x_2)$. Lastly, if $a_0 \leq x_1 < x_2$, then $f(a_0) < f(x_2)$ and $f(x_1) < f(x_2)$. Therefore f is strictly increasing.

■

7.3 Uniform Continuity

Remark. Consider $f : \text{dom}(f) \subset \mathbb{R} \to \mathbb{R}$ and assume that f is continuous on some $S \subset \text{dom}(f)$ iff $\forall x_0 \in S, \forall \epsilon > 0, \exists \delta > 0$ such that $|x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon$ whenever $x \in \text{dom}(f)$. Note that in general, δ is dependent on the value x_0 and ϵ.

Example 7.36. Consider $f : (0, \infty) \to \mathbb{R} : x \mapsto \frac{1}{x^2}$. Take $x_0 > 0$ and $\epsilon > 0$. Then

$$|f(x) - f(x_0)| = \left|\frac{1}{x^2} - \frac{1}{x_0^2}\right| = \frac{1}{x^2 x_0^2}(x - x_0)(x + x_0) = \frac{(x + x_0)}{x^2 x_0^2}(x - x_0).$$

If $|x - x_0| < \frac{x_0}{2}$, then $|x| > \frac{|x_0|}{2}$ and $|x| < \frac{3|x_0|}{2}$. Then, $|x + x_0| < \frac{5|x_0|}{2}$. Therefore

$$\frac{(x + x_0)}{x^2 x_0^2}(x - x_0) \leq \frac{\frac{5|x_0|}{2}}{\left(\frac{x_0}{2}\right)^2 x_0^2} \cdot |x - x_0| = \frac{10}{x_0^3}|x - x_0|.$$

By taking $\delta = \min\left\{\frac{x_0}{2}, \frac{x_0^2 \epsilon}{10}\right\}$, $|x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon$. In this case, we see that δ is reliant on both x_0 and ϵ.

Definition 7.31 (Uniform Continuity). A function $f : S \subset \mathbb{R} \to \mathbb{R}$ is uniformly continuous iff

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } |x - \tilde{x}| < \delta \implies |f(x) - f(\tilde{x})| < \epsilon, x, \tilde{x} \in S.$$

If f is said to be uniformly continuous, it is assumed to be uniformly continuous on its domain of definition unless specified.

Remark. Note that uniform continuity is a "stronger" notion of continuity. Note that

$$|x - \tilde{x}| < \delta$$

does not rely on some fixed argument $\tilde{x}$ unlike normal continuity. Fixing $\tilde{x}$ would produce an identical definition of continuity, therefore a function that is uniformly continuous is also continuous. Additionally, continuity is a property at a point while uniform continuity is property on a set. A function that is uniformly continuous at a point is meaningless.

Example 7.37. The function $f(x) = \frac{1}{x^2}$ is uniformly continuous on $[a, \infty)$ for any $a > 0$.

Proof. Note that

$$|f(x) - f(\tilde{x})| = \left|\frac{1}{x^2} - \frac{1}{\tilde{x}^2}\right| \leq \frac{x + \tilde{x}}{x^2\tilde{x}^2}|x - \tilde{x}| = \left(\frac{1}{x\tilde{x}^2} + \frac{1}{x^2\tilde{x}}\right)|x - \tilde{x}| \leq \frac{2}{a^3}|x - \tilde{x}|.$$

Take then $\epsilon > 0$ and let $\delta = \frac{a^3\epsilon}{2}$. Then

$$|x - \tilde{x}| < \delta \implies \frac{2}{a^3}|x - \tilde{x}| < \epsilon \implies |f(x) - f(\tilde{x})| < \epsilon.$$

Therefore f is uniformly continuous on $[a, \infty)$. ■

Example 7.38. The function $f(x) = \frac{1}{x^2}$ is not uniformly continuous on $(0, \infty)$.

Proof. Take $\epsilon = 1$ and show that $\forall\delta > 0$, there is $x, \tilde{x} \in (0, 1)$ such that $|x - \tilde{x}| < \delta$ but $|f(x) - f(\tilde{x})| > 1$. Take $\tilde{x} = x + \frac{\delta}{2}$. Note

$$\frac{1}{x^2} - \frac{1}{\tilde{x}^2} = \frac{1}{x^2} - \frac{1}{\left(x + \frac{\delta}{2}\right)^2} = \frac{\delta x + \frac{\delta^2}{4}}{x^2\left(x + \frac{\delta}{2}\right)^2} = \frac{\delta^2\frac{5}{4}}{\frac{9}{4}\delta^4} = \frac{5}{9}\frac{1}{\delta^2} > \frac{20}{9} > 1$$

for $\delta < \frac{1}{2}$. ■

Example 7.39. The function $f(x) = x^2$ is uniformly continuous on $[-7, 7]$.

Proof. Note that

$$|f(x) - f(\tilde{x})| = |x^2 - \tilde{x}| = |x + \tilde{x}||x - \tilde{x}| \leq 14|x - \tilde{x}|.$$

Therefore, take $\epsilon > 0$ and choose $\delta = \frac{\epsilon}{14}$. Then

$$|x - \tilde{x}| < \delta \implies 14|x - \tilde{x}| < \epsilon \implies |f(x) - f(\tilde{x})| < \epsilon$$

Hence f is uniformly continuous on $[-7, 7]$. ■

Theorem 7.48 (Closed Interval Implies Uniform Continuity). If f is continuous on $[a, b]$, then f is uniformly continuous on $[a, b]$.

Proof. Let $f : \mathrm{dom}(f) \to \mathbb{R}$ be a real valued function and assume that it is continuous on the interval $[a, b]$. Assume towards contradiction that f is not uniformly continuous. Then $\exists\epsilon > 0$ such that $\forall\delta > 0$ there is $x, \tilde{x} \in [a, b]$ where $|x - \tilde{x}| < \delta$ and $|f(x) - f(\tilde{x})| \geq \epsilon$. Take $\delta_n = \frac{1}{n}$ to find a sequence of arguments (x_n) and $(\tilde{x}_n)$ in $[a, b]$ such that $|x_n - \tilde{x}_n| <$

δ_n and $|f(x_n) - f(\tilde{x_n})| \geq \epsilon$. By 4.24, there exists a subsequence $(n_k)_{k\in\mathbb{N}} \subset \mathbb{N}$ such that

$$x_{n_k} \xrightarrow{n\to\infty} x_0$$
$$\tilde{x}_{n_k} \xrightarrow{n\to\infty} x_0 \text{ (since } |x_n - \tilde{x}_n| < \delta_n)$$

Since $[a, b]$ is closed, then the limit point $x_0 \in [a, b]$. Therefore since f is continuous on $[a, b]$,

$$f(x_{n_k}) \xrightarrow{n\to\infty} f(x_0)$$
$$f(\tilde{x}_{n_k}) \xrightarrow{n\to\infty} f(x_0)$$

which means that $|f(x_{n_k}) - f(\tilde{x}_{n_k})| \to 0$. However this contradicts the assumption that $|f(x_{n_k}) - f(\tilde{x}_{n_k})| \geq \epsilon > 0$ for all k. ■

Example 7.40. The following functions are uniformly continuous

$$[x \mapsto x^7 3], x \in [-15, 31]$$
$$[x \mapsto \sqrt{x}], x \in [0, 413]$$
$$[x \mapsto e^x], x \in [-1000, 1000]$$

Theorem 7.49 (Uniform Continuity Preserves Cauchy Sequences). If f is uniformly continuous on S, then a Cauchy sequence (s_n) in S is mapped to a Cauchy sequence $(f(s_n))$ in $\mathbb{R}$.

Proof. Take $\epsilon > 0$. Then find $\delta > 0$ such that $|x - \tilde{x}| \implies |f(x) - f(\tilde{x})| < \epsilon$ for $x, \tilde{x} \in S$. Since (s_n) is Cauchy, then $\exists N \in \mathbb{N}$ such that $|s_n - s_m| < \delta$ for all $n > N$. Then $|f(s_n) - f(s_m)| < \epsilon$ for all $n > M$ and hence $(f(s_n))$ is also Cauchy. ■

Example 7.41. Consider $f(x) = \frac{1}{x^2}$ on the interval $(0, 1]$. f is not uniformly continuous.

Proof. Consider the sequence $s_n = \frac{1}{n}$. (s_n) is convergent and in the domain of f, but $f(s_n) = n^2$ which is not Cauchy. Hence f is not uniformly continuous. ■

Definition 7.32 (Function Extension). $\tilde{f} : \text{dom}(\tilde{f}) \subset \mathbb{R} \to \mathbb{R}$ is an extension of $f : \text{dom}(f) \subset \mathbb{R} \to \mathbb{R}$ iff

1. $\text{dom}(f) \subset \text{dom}(\tilde{f})$
2. $\tilde{f}(x) = f(x)$ for $x \in \text{dom}(f)$

Example 7.42. Consider the function $f(x) = x \sin \frac{1}{x}$ on the interval $\left(0, \frac{1}{\pi}\right]$. Let

$$\tilde{f} = \begin{cases} f(x) & x \in \left(0, \frac{1}{\pi}\right] \\ r & x = 0 \end{cases}.$$

If $r = 0$, then $\tilde{f}$ is continuous on the closed interval $\left[0, \frac{1}{\pi}\right]$ and hence is uniformly continuous.

Example 7.43. Consider $f(x) = \sin \frac{1}{x}$ with $x \in \left(0, \frac{1}{\pi}\right]$. f can be extended to the closed interval by setting $f(0) = r \in \mathbb{R}$. However, no choice for r makes the extension continuous.

Theorem 7.50 (Uniform Continuity Extension Equivalency). $f : (a, b) \to \mathbb{R}$ is uniformly continuous on (a, b) iff f has a uniformly continuous extension $\tilde{f}$ on $[a, b]$.

Proof. Consider both implications

$\Leftarrow$) Assume that $\tilde{f}$ is uniformly continuous on $[a, b]$. Since $f(x) = \tilde{f}(x)$ for $x \in (a, b)$, f must be uniformly continuous.

$\Rightarrow$) Assume that f is uniformly continuous on (a, b). If f has a continuous extension $\tilde{f}$ on $[a, b]$, then it is uniformly continuous. Therefore it is sufficient to define $\tilde{f}$ at a and b. Consider b. It is possible to take $x_n \in (a, b)$ such that $\lim x_n = b$. Since (x_n) is convergent, it is also Cauchy. Since f is uniformly continuous, $(f(x_n))_{n \in \mathbb{N}}$ is also a Cauchy sequence and therefore is convergent. Therefore there is some $y \in \mathbb{R}$ such that $\lim f(x_n) = y$. Define then $\tilde{f}(b) = y$. It still needs to be verified that for any other sequence that converges to b that the functional sequence converges to y. Let $(\tilde{x}_n)$ be a sequence different that before that converges to b. Consider a new sequence $(s_n) = (x_1, \tilde{x_1}, x_2, \tilde{x}, \ldots)$. Note that (s_n) is Cauchy since $\lim s_n = b$. Therefore $(f(s_n))_{n \in \mathbb{N}}$ is also Cauchy, meaning $(f(s_n))_{n \in \mathbb{N}}$ has a limit. Therefore all its subsequential limits are the same, hence

$$\lim s_{2k} = \lim \tilde{x}_n = \lim s_{2k-1} = \lim x_n = y.$$

Therefore all convergent sequences to b will converge to y under f. The same construction follows for a.

Both implications therefore establish the equivalency. ■

Example 7.44. Consider $f(x) = \frac{\sin x}{x}$ with $x \neq 0$. Let

$$\tilde{f}(x) = \begin{cases} f(x) & x \neq 0 \\ 1 & x = 0 \end{cases}.$$

It turns out that $\tilde{f}$ is continuous on $\mathbb{R}$ and therefore is uniformly continuous on any closed interval.

Theorem 7.51. Let f be continuous on an interval I. If f restricted to $\mathring{I}$ is differentiable and the derivative is bounded, then f is uniformly continuous.

Proof. Apply MVT with $a < b$ and $a, b \in I$. Then

$$f(b) - f(a) = f'(x) \cdot (b - a), x \in (a, b).$$

Therefore $|f(b) - f(a)| \leq |f'(x)(b - a)| = |f'(x)|(b - a)$. Since $f'(x)$ is bounded, there is some $M \in \mathbb{R}$ such that $|f'(x)| \leq M$ for all x. Take $\epsilon > 0$ and let $\delta = \frac{\epsilon}{M}$. Then

$$|b - a| < \delta \implies |f(b) - f(a)| < \epsilon.$$

Hence f is uniformly continuous. ■

1. ***Proof.*** Fuck? ■

List of Theorems

List of Definitions