1.4.9

Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2\times 2}(\mathbb{F})$ with $a, b, c, d \in \mathbb{F}$. Note that it can be written as $a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Therefore $M_{2\times 2}(\mathbb{F})$ is generated by those matrices.

1.4.10

Consider a linear combination of the matrices with coefficients $a, b, c \in \mathbb{F}$

$$a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + c\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a & c \\ c & b \end{pmatrix}$$

Notice the linear combination is a symmetric matrix. Therefore span $\{M_1, M_2, M_3\}$ is the set of all symmetric 2×2 matrices.

1.4.12

Proof. Let V be a vector space and $W \subset V$.

$\Rightarrow$) Assume that W is a subspace of V. This means that W is a subset of V and is closed under linear combinations. This means that the set of all linear combinations of vectors in W stay within W. Equivalently, span $\{W\} = W$.

$\Leftarrow$) Assume that span $\{W\} = W$. Since $W \subset V$, by theorem 1.5 (the span of subset of a vector space is a subspace) the span of W is a subspace of V.

■

1.4.15

Proof. Let V be a vector space and $S_1, S_2 \subset V$. Let $v \in$ span $\{S_1 \cap S_2\}$ such that $v = c_1x_1 + c_2x_2 + \ldots + c_nx_n$ where $x_i \in S_1 \cap S_2$. Therefore $x_i \in S_1$ and $x_i \in S_2$. This means that the linear combination is in both span $\{S_1\}$ and span $\{S_2\}$. Therefore $v \in$ span $\{S_1\} \cap$ span $\{S_2\}$. ■

An example where both are equal is $S_1 = S_2 = \mathbb{R}$. An example where they are not equal is $S_1 = \{0, 1\}$ and $S_2 = \{0, 2\}$.

1.5.1

a) False

b) True

c) False

d) False

e) True

f) True

1.5.8

Part A

Proof. Let $S = \{(1,1,0),(1,0,1),(0,1,1)\} \subset \mathbb{R}^3$. Assume towards contradiction that S is linearly dependent. Then there exists $a,b,c \in \mathbb{R} \neq 0$ such that

$$a(1,1,0) + b(1,0,1) + c(0,1,1) = 0$$

This means that

$$\begin{aligned} a + b &= 0 \\ a + c &= 0 \\ b + c &= 0 \end{aligned}$$

However, $a + b = 0 \implies a = -b$ and therefore $a + c = -b + c \implies b = c$. Since $b + c = 0$, $b + b = 0 \implies b = 0$, contradicting the assumption that S is dependent. Therefore S is independent. ■

Part B

Proof. Let $S = \{(1,1,0),(1,0,1),(0,1,1)\} \subset \mathbb{F}^3$ where $\mathbb{F}$ has characteristic 2. Note then that

$$(1,1,0) + (0,1,1) = (1,0,1)$$

Therefore S is linearly dependent. ■

1.5.11

Since there are two scalar options for each vector (0 or 1), there are 2^n linear combination choices and therefore there are 2^n vectors in the span.

1.5.19

Proof. Let $\{A_1, A_2, \ldots, A_k\}$ be a subset of $M_{n\times n}(\mathbb{F})$ and assume that is linearly independent. Therefore $c_1A_1 + c_2A_2 + \ldots + c_kA_k = 0$ only when $c_i = 0$. Note that

$$\begin{aligned} c_1A_1 + c_2A_2 + \ldots c_kA_k &= 0 \\ (c_1A_1 + c_2A_2 + \ldots c_kA_k)^\intercal &= 0^\intercal \\ (c_1A_1)^\intercal + (c_2A_2)^\intercal + \ldots (c_kA_k)^\intercal &= 0 \\ c_1A_1^\intercal + c_2A_2^\intercal + \ldots c_kA_k^\intercal &= 0 \end{aligned}$$

Therefore the linear combination of the transposes is also equal to zero only when $c_i = 0$, meaning the set of transposes is a linearly independent subset. ■

1.5.20

Proof. Let $f, g \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ where $f(t) = e^{rt}$ and $g(t) = e^{st}$ with $r \neq s$. Assume towards contradiction that they are linearly dependent. That is $c_1 e^{rt} + c_2 e^{st} = 0$ with $c_1, c_2 \neq 0$. Then

$$\begin{aligned} c_1 e^{rt} + c_2 e^{st} &= 0 \\ c_1 + c_2 e^{(s-r)t} &= 0 \\ c_2 e^{(s-r)t} &= -c_1 \\ e^{(s-r)t} &= -\frac{c_1}{c_2} \end{aligned}$$

However, this implies that $e^{(s-r)t}$ is a constant function which it is not. Therefore $c_1, c_2 = 0$. ■

1.6.2

All are bases except for b and e.

1.6.4

The polynomials do not generate the space as there are only 3 independent vectors which cannot generate a 4 dimensional space.

1.6.9

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = a_1 u_1 + (a_2 - a_1) u_2 + (a_3 - a_2) u_3 + (a_4 - a_3) u_4$$

1.6.24

Proof. Let $f \in \mathcal{P}_n(\mathbb{R})$ such that f has degree n. The question of if any g in $\mathcal{P}_n(\mathbb{R})$ can be written as

$$g(x) = c_0 f(x) + c_1 f'(x) + \ldots + c_n f^{(n)}(x)$$

is the same as the question does span $\{f, f', \ldots, f^{(n)}\} = \mathcal{P}_n(\mathbb{R})$. Note that the set $\{f, f', \ldots, f^{(n)}\}$ is linearly independent since each entry is of a different order. Since the dimension of $\mathcal{P}_n(\mathbb{R})$ is $n+1$ and $\{f, f', \ldots, f^{(n)}\}$ is a set of $n+1$ linearly independent vectors in $\mathcal{P}_n(\mathbb{R})$, $\{f, f', \ldots, f^{(n)}\}$ forms a basis and therefore span $\{f, f', \ldots, f^{(n)}\} =$

$\mathcal{P}_n(\mathbb{R})$. This implies that any function $g \in \mathcal{P}_n(\mathbb{R})$ can be represented as

$$g(x) = c_0 f(x) + c_1 f'(x) + \ldots + c_n f^{(n)}(x).$$

■