Problem 5.4.2

Define a sequence $\{c_n\}_{n=0}^{\infty}$ as follows:

$$\begin{cases} c_{n+1} = \frac{49}{8}c_n - \frac{225}{8}c_{n-2}, \ n \geq 2 \\ c_0 = 0, c_1 = 2, c_2 = 16 \end{cases}$$

Prove that $c_n = 5^n - 3^n$ for all $n \in \mathbb{N}_0$.

Solution

Proof. Proceed with strong induction. Let $P(n) : c_n = 5^n - 3^n$. Consider three base cases: $n = 0, n = 1, n = 2$.

1. $c_0 = 0 = 5^0 - 3^0 = 1 - 1$, therefore $P(0)$ is true.
2. $c_1 = 2 = 5^1 - 3^1 = 5 - 2$, therefore $P(1)$ is true.
3. $c_2 = 16 = 5^2 - 3^2 = 25 - 9$, therefore $P(2)$ is true.

Fix $n \geq 2$ and suppose that $c_k = 5^k - 3^k$ for all $2 \leq k \leq n$. Then

$$\begin{aligned} a_{n+1} &= \frac{49}{8}c_n - \frac{225}{8}c_{n-2} \\ &= \frac{49}{8}(5^n - 3^n) - \frac{225}{8}(5^{n-2} - 3^{n-2}) \\ &= \frac{49}{8}(5^n - 3^n) - \frac{5^2 \cdot 3^2}{8}(5^{n-2} - 3^{n-2}) \\ &= \frac{49}{8}(5^n - 3^n) - \frac{1}{8}(9 \cdot 5^n - 25 \cdot 3^n) \\ &= \frac{49}{8}5^n - \frac{49}{8}3^n - \frac{9}{8}5^n + \frac{25}{8}3^n \\ &= \frac{40}{8}5^n - \frac{24}{8}3^n \\ &= 5 \cdot 5^n - 3 \cdot 3^n \\ &= 5^{n+1} - 3^{n+1}. \end{aligned}$$

By strong induction, $c_n = 5^n - 3^n$ for all $n \in \mathbb{N}_0$. ■

Problem 5.4.6

Show that for every positive integer n, $(3 + \sqrt{5})^n + (3 - \sqrt{5})^n$ is an even integer.

Solution

Proof. Let $u_n = (3+\sqrt{5})^n + (3-\sqrt{5})^n$. Proceed with strong induction to prove that for all positive integers n that u_n is an even integer. Consider the two base cases where $n = 1$ and $n = 2$. It follows then that

$$u_1 = 3 + \sqrt{5} + 3 - \sqrt{5} = 6 = 2(3).$$

and

$$u_2 = 3 + 6\sqrt{5} + 9 + 3 - 6\sqrt{5} + 9 = 28 = 2(14).$$

Both are even integers hence the base cases are true. Fix $n \geq 2$ and suppose that u_k is even for all $2 \leq k \leq n$. Then

$$\begin{aligned}(3+\sqrt{5})^{n+1} + (3-\sqrt{5})^{n+1} &= (3+\sqrt{5})(3+\sqrt{5})^n + (3-\sqrt{5})(3-\sqrt{5})^n \\ &= (3+\sqrt{5}+3-\sqrt{5})((3+\sqrt{5})^n + (3-\sqrt{5})^n) - a\end{aligned}$$

Where $a = (3+\sqrt{5})(3-\sqrt{5})\left((3+\sqrt{5})^{n-1} + (3-\sqrt{5})^{n-1}\right)$. Note that $(3+\sqrt{5})(3-\sqrt{5}) = 4$. Therfore

$$\begin{aligned}&= 6((3+\sqrt{5})^n + (3-\sqrt{5})^n) - 4((3+\sqrt{5})^{n-1} + (3-\sqrt{5})^{n-1}) \\ &= 6u_n - 4u_{n-1}\end{aligned}$$

By the induction hypothesis, there are integers m and n such that $u_n = 2m$ and $u_{n-1} = 2n$. Therefore

$$\begin{aligned}&= 12m - 8n \\ &= 2(6m - 4n).\end{aligned}$$

Therefore $u_n + 1$ is an even integer. Hence for every positive integer n, $(3+\sqrt{5})^n + (3-\sqrt{5})^n$ is an even integer. ■

Problem 6.1.1

(a) Suppose that $A = \{1,2\}$ and $B = \{3,4,5\}$. State the set $A \times B$ in roster notation.

(b) Sketch both $A \times B$ and $B \times A$ using dots on the plane. What do you observe about your pictures?

(c) If A, B, C are any sets, we may define the triple Cartesian product as

$$A \times B \times C = \{(a,b,c) : a \in A, b \in B, c \in C\}.$$

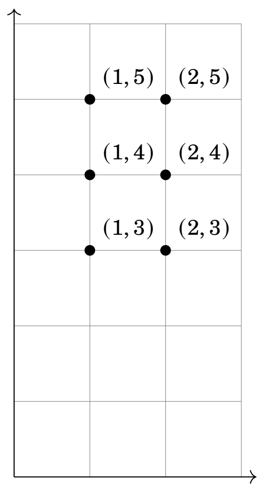
If $C = \{6,7\}$ and A, B are as above, state the set $A \times B \times C$ in roster notation.

Solution

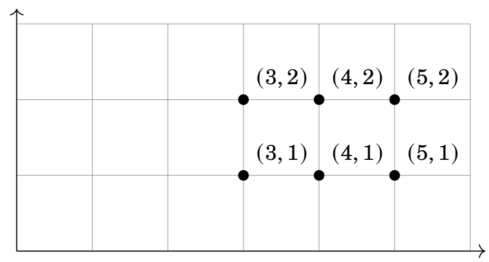
Part A

$$A \times B = \{(1,3),(1,4),(1,5),(2,3),(2,4),(2,5)\}.$$

Part B



$A \times B$



$B \times A$

The pictures look like a rotation and reflection of each other. It's similar to a reflection across the line $y = x$ like in the case for functions and their inverses.

Part C

$$A \times B \times C = \left\{ \begin{array}{cccc} (1,3,6), & (1,3,7), & (1,4,6), & (1,4,7), \\ (1,5,6), & (1,5,7), & (2,3,6), & (2,3,7), \\ (2,4,6), & (2,4,7), & (2,5,6), & (2,5,7) \end{array} \right\}.$$

Problem 6.1.7

Prove that $A \cap B = \varnothing \iff (A \times B) \cap (B \times A) = \varnothing$.

Solution

Proof. Let A and B be sets.

($\Rightarrow$) Proceed with proof by contrapositive. Assume that $(A \times B) \cap (B \times A) \neq \varnothing$. Let $(x,y) \in (A \times B) \cap (B \times A)$. It follows then that by the intersection that $(x,y) \in A \times B$ and $(x,y) \in B \times A$. Then by the definition of the Cartesian product, $(x,y) \in A \times B \implies x \in A, y \in B$ and $(x,y) \in B \times A \implies x \in B, y \in A$. Looking at the element x, it is both in A and B. Therefore it is in $A \cap B$.

Therefore since there is an element in $A \cap B$, it follows that $A \cap B$ cannot be empty, or equivalently $A \cap B \neq \varnothing$.

($\Leftarrow$) Assume towards contradiction that $(A \times B) \cap (B \times A) = \varnothing$ and $A \cap B \neq \varnothing$. Since $A \cap B \neq \varnothing$, there is an element $x \in A \cap B$. It follows by the intersection that $x \in A, x \in B$. Now consider the ordered pair (x, x). Since x is in A and B, $(x, x) \in A \times B$ and $(x, x) \in B \times A$. Since (x, x) is in both $A \times B$ and $B \times A$, their intersection is non-empty, or equivalently $(A \times B) \cap (B \times A) \neq \varnothing$. However it was assumed that $(A \times B) \cap (B \times A) = \varnothing$, hence a contradiction.

Therefore $A \cap B = \varnothing$ if and only if $(A \times B) \cap (B \times A) = \varnothing$. ■

Problem 6.1.9

Prove the following by induction. For all $n \in \mathbb{N}$, if $A_1, \ldots, A_n$ are finite sets, then $|A_1 \times \ldots \times A_n| = |A_1| \ldots |A_n|$.

Solution

Proof. Proceed with induction to show that for all $n \in \mathbb{N}$, if $A_1, \ldots, A_n$ are finite sets, then $|A_1 \times \ldots \times A_n| = |A_1| \ldots |A_n|$. Consider the base case when $n = 1$. Then $|A_1| = |A_1|$, hence the base case is true. Assume for a fixed $n \in \mathbb{N}$ that $|A_1 \times \ldots \times A_n| = |A_1| \ldots |A_n|$. Consider then the Cartesian product $A_1 \times \ldots \times A_{n+1}$. This will result in every ordered pair in $A_1 \times \ldots \times A_n$ being repeated with a new element from A_{n+1} added in each time. Hence the number of ordered pairs in the set $A_1 \times \ldots \times A_{n+1}$ will be the same as the number of elements of $A_1 \times \ldots A_n$ multiplied by the number of elements in A_{n+1}. By the induction hypothesis, the number of elements in $A_1 \times \ldots \times A_n = |A_1| \ldots |A_n|$ and the number of elements in A_{n+1} is $|A_n + 1|$. Hence

$$|A_1 \times \ldots \times A_{n+1}| = |A_1| \ldots |A_{n+1}|.$$

Therefore for all $n \in \mathbb{N}$, if $A_1, \ldots, A_n$ are finite sets, then

$$|A_1 \times \ldots \times A_n| = |A_1| \ldots |A_n|.$$

■

Problem 6.2.2

Let $A = \{1, 3\}$ and $B = \{2, 4\}$

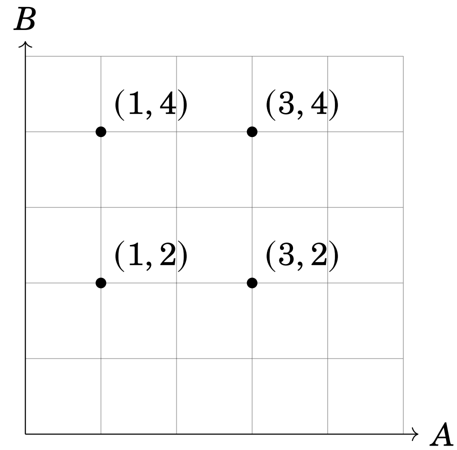
(a) Draw a picture of the set $A \times B$

(b) Compute $\mathcal{P}(A \times B)$

(c) What is the cardinality of the set $\mathcal{P}(A) \times \mathcal{P}(B)$

Solution

Part A



Part B

$$\mathcal{P}(A \times B) = \left\{ \begin{array}{l} \varnothing, \\ \{(1,2)\}, \\ \{(1,4)\}, \\ \{(3,2)\}, \\ \{(3,4)\}, \\ \{(1,2),(1,4)\}, \\ \{(1,2),(3,2)\}, \\ \{(1,2),(3,4)\}, \\ \{(1,4),(3,2)\}, \\ \{(1,4),(3,4)\}, \\ \{(3,2),(3,4)\}, \\ \{(1,2),(1,4),(3,2)\}, \\ \{(1,2),(1,4),(3,4)\}, \\ \{(1,2),(3,2),(3,4)\}, \\ \{(1,4),(3,2),(3,4)\}, \\ \{(1,2),(1,4),(3,2),(3,4)\} \end{array} \right\}.$$

Part C

The cardinality of $|\mathcal{P}(A)| = 2^{|A|}$ and similarly $|\mathcal{P}(B)| = 2^{|B|}$. The cardinality of the Cartesian product of two sets is their cardinalities multiplied. Therefore $\mathcal{P}(A) \times \mathcal{P}(B) = 2^{|A|} \cdot 2^{|B|} = 2^4 = 16$.

Problem 6.2.6

(a) Prove that $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$. Provide a counter-example to show that we do not expect equality.

(b) Does anything change if you replace $\cup$ with $\cap$ in part (a)? Justify your answer.

Part A

Proof. Let A and B be sets. It is true that $A \subseteq A \cup B$ and $B \subseteq A \cup B$. Hence for both $\mathcal{P}(A) \subseteq \mathcal{P}(A \cup B)$ and $\mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$. Therefore since both $\mathcal{P}(A)$ and $\mathcal{P}(B)$ are subsets of $\mathcal{P}(A \cup B)$, then $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$. ■

Part B

The proposition still holds if all instances of $\cup$ are replaced with $\cap$.

Proof. Let A and B be sets. Consider the set element S in $\mathcal{P}(A) \cap \mathcal{P}(B)$. By the definition of the intersection, S is in both $\mathcal{P}(A)$ and $\mathcal{P}(B)$, or equivalently S is a subset of both A and B. This means that every element within S is contained in both A and B, hence $S \subset A \cap B$ meaning $S \in \mathcal{P}(A \cap B)$. Therefore since S was an arbitrary element of $\mathcal{P}(A) \cap \mathcal{P}(B)$ and is in $\mathcal{P}(A \cap B)$, it follows that $\mathcal{P}(A) \cap \mathcal{P}(B) \subseteq \mathcal{P}(A \cap B)$. ∎

Problem 6.2.8

We use the following notation for the binomial coefficient: $\binom{n}{r} = \frac{n!}{r!(n-r)!}$. This symbol denotes the number of distinct ways one can choose r objects from a set of n objects.

(a) Use the definition of the binomial coefficient to prove the following:

$$\text{If } 1 \leq r \leq n, \text{ then } \binom{n+1}{r} = \binom{n}{r} + \binom{n}{r-1}.$$

(b) Prove by induction that $\forall n \in \mathbb{N}_0, \displaystyle\sum_{r=0}^{n} \binom{n}{r} = 2^n$.

(c) Explain why part (b) provides an alternative proof of Theorem 6.6.

Solution

Part A

Proof. Let $n, r \in \mathbb{N}$ with $1 \leq r \leq n$. Then

$$\begin{aligned}
\binom{n}{r} + \binom{n}{r-1} &= \frac{n!}{r!(n-r)!} + \frac{n!}{(r-1)!(n-r+1)!} \\
&= \frac{n!(n-r+1)}{r!(n-r)!(n-r+1)} + \frac{n!r}{r(r-1)!(n-r+1)!} \\
&= \frac{n!(n-r+1)}{r!(n-r+1)!} + \frac{n!r}{r!(n-r+1)!} \\
&= \frac{n!(n-r+1) + n!r}{r!(n-r+1)!} \\
&= \frac{n!(n+1)}{r!(n-r+1)!} \\
&= \frac{(n+1)!}{r!(n-r+1)!} \\
&= \binom{n+1}{r}.
\end{aligned}$$

■

Part B

Proof. Proceed with induction to prove that for all $n \in \mathbb{N}_0$ that $\sum_{r=0}^{n} \binom{n}{r} = 2^n$. In order to use the previous result, $n \geq 1$, so consider the case when $n = 0$. Then $\sum_{r=0}^{0} \binom{n}{r} = \binom{0}{0} = 1 = 2^0$ hence the proposition holds for $n = 0$. Consider the base case when $n = 1$. Then $\sum_{r=0}^{1} \binom{1}{r} = \binom{1}{0} + \binom{1}{1} = 1 + 1 = 2^1$

Hence the base case is true. Assume for a fixed $n \in \mathbb{N}$ that $\sum_{r=0}^{n} \binom{n}{r} = 2^n$. Then

$$\begin{aligned}
\sum_{r=0}^{n+1} \binom{n+1}{r} &= \binom{n+1}{0} + \binom{n+1}{1} + \ldots + \binom{n+1}{n} + \binom{n+1}{n+1} \\
&= \binom{n+1}{0} + \binom{n+1}{n+1} + \sum_{r=1}^{n} \binom{n+1}{r} \\
&= 2 + \sum_{r=1}^{n} \binom{n+1}{r} \\
&= 2 + \sum_{r=1}^{n} \binom{n}{r} + \sum_{j=1}^{n} \binom{n}{j-1} \\
&= 2 + \sum_{r=1}^{n} \binom{n}{r} + \sum_{j=0}^{n-1} \binom{n}{j} \\
&= 2 - \binom{n}{n} - \binom{0}{0} + \sum_{r=0}^{n} \binom{n}{r} + \sum_{j=0}^{n} \binom{n}{j} \\
&= 2 - 2 + 2\sum_{r=0}^{n} \binom{n}{r} \\
&= 2\sum_{r=0}^{n} \binom{n}{r} \\
&= 2(2^n) \\
&= 2^{n+1}.
\end{aligned}$$

Hence including the case when $n = 0$ and the induction across all $n \in \mathbb{N}$, for all

$n \in \mathbb{N}_0$ it is true that $\sum_{r=0}^{n} \binom{n}{r} = 2^n$. ■

Part C

Theorem 6.6 states that for a set A, $|\mathcal{P}(A)| = 2^{|A|}$. Counting up the number of elements in $\mathcal{P}(A)$ is equivalent to adding up an entire row of Pascal's triangle where the row is the number of elements in A. The binomial coefficient can be inrepreted as grabbing a value from pascals triangle where in $\binom{n}{r}$, n represents the row of Pascal's triangle and r represents the rth element of that row. Therefore the equivalent question of the cardinality of the powerset of a set is the same as the sum of the $|A|$th row of Pascal's triangle. The result just proved gives the sum of a row of Pascal's triangle, hence being an alternative proof to Theorem 6.6.