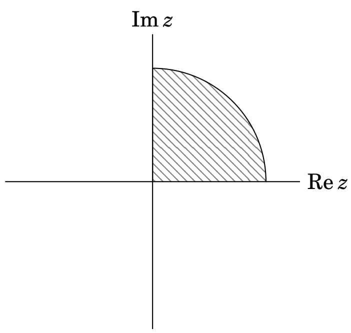
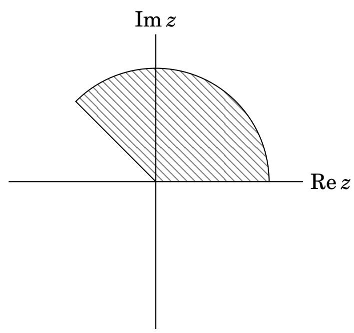
14.8

For $z = re^{i\theta}$, it follows that $z^n = r^n e^{ni\theta}$. Hence for each mapping,

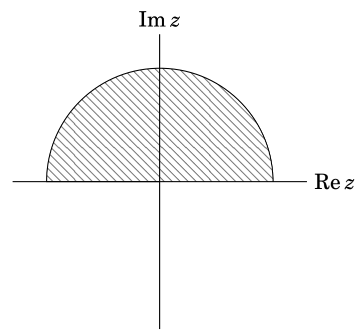
$$\begin{aligned} z^2 &\implies 0 \le \theta \le \frac{\pi}{2}, \quad 0 \le r \le 1 \\ z^3 &\implies 0 \le \theta \le \frac{3\pi}{4}, \; 0 \le r \le 1 \\ z^4 &\implies 0 \le \theta \le \pi, \quad 0 \le r \le 1 \end{aligned}$$



(a) $w = z^2$



(b) $w = z^3$



(c) $w = z^4$

18.1

Part C

Proof. Take $\epsilon > 0$ and let $\delta = \epsilon$. Note then that for $z \in C$ in the δ deleted neighborhood of 0 (that is $z \neq 0$)

$$\begin{aligned} |z - 0| < \delta &\implies |z| < \delta \\ &\implies \frac{|z|^2}{|z|} < \delta && \text{(Since } |z| \neq 0) \\ &\implies \frac{|\overline{z}|^2}{|z|} < \delta \\ &\implies \left|\frac{\overline{z}^2}{z}\right| < \delta \implies \left|\frac{\overline{z}^2}{z} - 0\right| < \delta = \epsilon \end{aligned}$$

Therefore by definition $\lim\limits_{z\to 0} \dfrac{\overline{z}^2}{z} = 0$. ■

18.7

Proof. Assume that $\lim_{z\to z_0} f(z) = w_0$. Take $\epsilon > 0$. Then there is some δ such that $|z - z_0| < \delta \implies |f(z) - w_0| < \epsilon$. Since $||f(z)| - |w_0|| \le |f(z) - w_0| < \epsilon$. Therefore

$$|z - z_0| < \delta \implies ||f(z)| - |w_0|| < \epsilon.$$

Therefore by definition $\lim_{z \to z_0} |f(z)| = |w_0|$. ∎

20.8

Part A

Proof. Let $z_0 = x_0 + iy_0$. Let $\Delta z = z - z_0$ and $\Delta f = f(z + \Delta z) - f(z)$. Then

$$\frac{\Delta f}{\Delta z} = \frac{\mathrm{Re}\{z + \Delta z\} - \mathrm{Re}\{z\}}{\Delta z} = \frac{\mathrm{Re}\{z\} + \mathrm{Re}\{\Delta z\} - \mathrm{Re}\{z\}}{\Delta z} = \frac{\mathrm{Re}\{\Delta z\}}{\Delta z}.$$

For $f'(z_0)$ to exist, this quantity must be the same no matter how $\Delta z \to 0$. If Δz approaches 0 along the real axis, then $\Delta z = \Delta x + i0 = \Delta x$ and $\mathrm{Re}\{\Delta z\} = \Delta x$. Therefore

$$\frac{\Delta f}{\Delta z} = \frac{\Delta x}{\Delta x} = 1.$$

If Δz approaches 0 along the imaginary axis, then $\Delta z = 0 + i\Delta y = i\Delta y$ and $\mathrm{Re}\{\Delta z\} = 0$. Therefore

$$\frac{\Delta f}{\Delta z} = \frac{0}{i\Delta y} = 0.$$

Therefore since the value is not the same on every path for $\Delta z \to 0$, the limit cannot exist at z_0 which was an arbitaray point in $\mathbb{C}$. Therefore f is not differentiable anywhere on $\mathbb{C}$. ∎

Part B

Proof. Let $z_0 = x_0 + iy_0$. Let $\Delta z = z - z_0$ and $\Delta f = f(z + \Delta z) - f(z)$. Then

$$\frac{\Delta f}{\Delta z} = \frac{\mathrm{Im}\{z + \Delta z\} - \mathrm{Im}\{z\}}{\Delta z} = \frac{\mathrm{Im}\{z\} + \mathrm{Im}\{\Delta z\} - \mathrm{Im}\{z\}}{\Delta z} = \frac{\mathrm{Im}\{\Delta z\}}{\Delta z}.$$

For $f'(z_0)$ to exist, this quantity must be the same no matter how $\Delta z \to 0$. If Δz approaches 0 along the real axis, then $\Delta z = \Delta x + i0 = \Delta x$ and $\mathrm{Im}\{\Delta z\} = 0$. Therefore

$$\frac{\Delta f}{\Delta z} = \frac{0}{\Delta x} = 0.$$

If Δz approaches 0 along the imaginary axis, then $\Delta z = 0 + i\Delta y = i\Delta y$ and $\mathrm{Im}\{\Delta z\} = \Delta y$. Therefore

$$\frac{\Delta f}{\Delta z} = \frac{\Delta y}{i\Delta y} = -i.$$

Therefore since the value is not the same on every path for $\Delta z \to 0$, the limit cannot exist at z_0 which was an arbitaray point in $\mathbb{C}$. Therefore f is not differentiable anywhere on $\mathbb{C}$. ∎

24.1

Part A

Proof. $f(z) = \overline{z}$ will be non-differentiable at all points where f does not satisfy the Cauchy-Riemann equations. For $z = x + iy$, $f(z) = x - iy$. Therefore the partials for f are

$$\begin{aligned} u_x &= 1 & u_y &= 0 \\ v_x &= 0 & v_y &= -1 \end{aligned}$$

Applying the Cauchy Riemann equations, $1 = -1$ and $0 = 0$. Since $1 = -1$ is never true, it follows that the Cauchy Riemann equations dont hold for any $z \in \mathbb{C}$ and hence f is not differentiable anywhere on $\mathbb{C}$. ■

Part B

Proof. $f(z) = z - \overline{z}$ will be non-differentiable at all points where f does not satisfy the Cauchy-Riemann equations. For $z = x + iy$, $f(z) = 2iy$. The partials for f therefore are

$$\begin{aligned} u_x &= 0 & u_y &= 0 \\ v_x &= 0 & v_y &= 2i \end{aligned}$$

Applying the Cauchy Riemann equations, $0 = 0$ and $0 = 2i$. Since $0 = 2i$ is never true, it follows that the Cauchy Riemann equations dont hold for any $z \in \mathbb{C}$ and hence f is not differentiable anywhere on $\mathbb{C}$. ■

Part C

Proof. $f(x + iy) = 2x + ixy^2$ will be non-differentiable at all points where f does not satisfy the Cauchy-Riemann equations. The partials for f are

$$\begin{aligned} u_x &= 2 & u_y &= 0 \\ v_x &= iy^2 & v_y &= 2ixy \end{aligned}$$

Applying the Cauchy Riemann equations, $2 = 2ixy$ and $0 = -iy^2$. Therefore

$$\begin{aligned} 0 = -iy^2 &\implies y = 0 \\ 2 = 2ixy &\implies x = 0 \text{ or } y = 0 \end{aligned}$$

Therefore the only candidate point for differentiability is $x = y = 0$. ■

Part D

Proof. f will be potentially differentiable only at points where the Cauchy Riemann equations hold. Rewriting f in terms of its components gives

$$f(z) = e^x e^{-iy} = e^x[\cos(-y) + i\sin(-y)] = e^x \cos y - ie^x \sin y.$$

Therefore the partials are

$$\begin{aligned} u_x &= e^x \cos y & u_y &= -e^x \sin y \\ v_x &= -e^x \sin y & v_y &= -e^x \cos y \end{aligned}$$

Applying the Cauchy Riemann equations gives

$$u_x = v_y \implies e^x \cos y = -e^x \cos y \implies \cos y = -\cos y$$

which is true when $y = \frac{\pi}{2} + \pi k$ for $k \in \mathbb{Z}$ and that

$$u_y = -v_x \implies -e^x \sin y = e^x \sin y \implies \sin y = -\sin y$$

which is true when $y = \pi m$ for $m \in \mathbb{Z}$. However, both equations cannot be satisifed simultaneously since there is no y such that both $\sin y = \cos y = 0$. This means that f is not differentiable at any $x + iy$ and therefore cannot be differentiable anywhere. ■

24.3

Part A

Proof. The candidate points where f is differentiable are those that satisfy the Cauchy Riemann equations. Since $f(z) = \frac{1}{z} = \frac{\bar{z}}{|z|^2} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$ for $z \neq 0$, the partials are

$$\begin{aligned} u_x &= \frac{y^2 - x^2}{(x^2+y^2)^2} & u_y &= -\frac{2xy}{(x^2+y^2)^2} \\ v_x &= \frac{2xy}{(x^2+y^2)^2} & v_y &= \frac{y^2 - x^2}{(x^2+y^2)^2} \end{aligned}$$

Applying the Cauchy Riemann equations gives

$$u_x = v_y \implies \frac{y^2 - x^2}{(x^2+y^2)^2} = \frac{y^2 - x^2}{(x^2+y^2)^2} \implies 0 = 0$$

and

$$u_y = -v_x \implies -\frac{2xy}{(x^2+y^2)^2} = -\frac{2xy}{(x^2+y^2)^2} \implies 0 = 0.$$

Therefore the Cauchy Riemann equations hold for every $z \neq 0$. Since the partials are also conintuous for $x+iy \neq 0$, then f' exists at all $z \neq 0$. Therefore

$$
\begin{aligned}
f'(x+iy) = u_x + iv_x &= \frac{y^2-x^2}{(x^2+y^2)^2} + i\frac{2xy}{(x^2+y^2)^2} \\
&= \frac{y^2-x^2+i2xy}{|z|^4} \\
&= \frac{(x-iy)^2}{|z|^4} \\
&= \frac{\overline{z}^2}{|z|^4} \\
&= \left(\frac{\overline{z}}{|z|^2}\right)^2 \\
&= \left(z^{-1}\right)^2 = \frac{1}{z^2}
\end{aligned}
$$

■

Part B

Proof. The candidate points where f is differentiable are those that satisfy the Cauchy Riemann equations. The partials are

$$
\begin{aligned}
u_x &= 2x \quad & u_y &= 0 \\
v_x &= 0 \quad & v_y &= 2y
\end{aligned}
$$

Therefore applying the Cauchy Riemann equations gives $2x = 2y$ and $0 = 0$. Therefore f is not differentiable anywhere off the line $y = x$. Note that for any ϵ-neighborhood of some z_0 on this line, the partials exist and are also continuous at z_0 since they exist and are continuous everywhere. Therefore f is differentiable on this line and

$$f'(z) = u_x + iv_x = 2x + i0 = 2x.$$

■

Part C

Proof. The candidate points where f is differentiable are those that satisfy the Cauchy Riemann equations. Note that $f(x+iy) = (x+iy)y = xy + iy^2$. Therefore the partials are

$$
\begin{aligned}
u_x &= y \quad & u_y &= x \\
v_x &= 0 \quad & v_y &= 2y
\end{aligned}
$$

Therefore applying the Cauchy Riemann equations gives $y = 2y$ and $x = 0$. Therefore $x = y = 0$ is the only possible point that f is differentiable. Since the partials exist and are continuous everywhere, then for any ϵ-neighborhood around $z = 0$ the partials are continuous. Therefore $f'(0) = u_x(0,0) + iv_x(0,0) = 0$. ■

24.8

Part A

Proof. Since $z = x + iy$ and $\overline{z} = x - iy$, then $x = \frac{z+\overline{z}}{2}$ and $y = \frac{z-\overline{z}}{2i}$. $F(x,y)$ can be changed to a function of a single imaginary input $\overline{z}$. Therefore using the multivariable chain rule gives

$$\frac{\partial F}{\partial \overline{z}} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial \overline{z}} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial \overline{z}} = \frac{\partial F}{\partial x}\left(\frac{1}{2}\right) + \frac{\partial F}{\partial y}\left(-\frac{1}{2i}\right) = \frac{1}{2}\left(\frac{\partial F}{\partial x} + i\frac{\partial F}{\partial y}\right).$$

■

Part B

Proof. Let $f(z) = u(x,y) + iv(x,y)$ satisfy the Cauchy Riemann equations. That is $u_x = v_y$ and $u_y = -v_x$. Applying the operator $\frac{\partial}{\partial \overline{z}}$ gives

$$\begin{aligned}\frac{\partial f}{\partial \overline{z}} = \frac{\partial}{\partial \overline{z}}f(z) &= \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(u(x,y) + iv(x,y))\\ &= \frac{1}{2}\left[u_x - v_y + iu_y + iv_x\right]\\ &= \frac{1}{2}\left[u_x - v_y + i(u_y + v_x)\right]\end{aligned}$$

By substituting u_x for v_y and u_y for $-v_x$,

$$\begin{aligned}&= \frac{1}{2}\left[v_y - v_y + i(-v_x + v_x)\right]\\ &= 0\end{aligned}$$

Therefore when f satisfies the Cauchy Riemann equations, $\frac{\partial f}{\partial \overline{z}} = 0$. ■

26.5

Proof. Note that when $|z| > 0$ and $|\operatorname{Arg} z| < \frac{\pi}{2}$, $\operatorname{Re}\{z\} > 0$. Since $g(z)$ is analytic when $|z| > 0$ and $|\operatorname{Arg} z| < \frac{\pi}{2}$, $g(z)$ is analytic when $\operatorname{Re}\{z\} > 1$. Since $f(z) = 2z - 2 + i$ is analytic everywhere in $\mathbb{C}$, $g(f(z))$ will be analytic when $\operatorname{Re}\{f(z)\} > 0$. That is,

$$\operatorname{Re} f(z) = \operatorname{Re}\{2z - 2 + i\} = 2x - 2 > 0 \implies x > 1.$$

Therefore $g(f(z))$ will be analytic in the half plane $x > 1$. It follows by the chain rule that

$$G'(x) = \frac{\mathrm{d}}{\mathrm{d}z}g(f(z)) = g'(f(z)) \cdot f'(z) = 2 \cdot \frac{1}{2 \cdot g(2z - 2 + i)} = \frac{1}{g(2z - 2 + i)}.$$

■

26.7

Proof. Since $f(z)$ is real valued, then if $f(z) = u + iv$, it follows $v(x, y) = 0$. Since f is also analytic everywhere in $\mathcal{D}$, it satisfies the Cauchy Riemann equations in its entire domain. Therefore

$$u_x = v_y \implies u_x = 0.$$

Therefore $f'(z) = u_x + iv_x = 0 + i0 = 0$ on all of $\mathcal{D}$. Therefore $f(z)$ is constant throughout $\mathcal{D}$. ■

27.2

Proof. Assume that $z_0 = (x_0, y_0) \in \mathcal{D}$ where $u(x_0, y_0) = v(x_0, y_0)$, that $f'(z_0) \neq 0$, and that f is analytic in $\mathcal{D}$. Since $u(x, y)$ and $v(x, y)$ can be considered as real multivariate functions, the techniques of multivariable calculus can be applied to their level curves $u(x, y) = c_1$ and $v(x, y) = c_2$. The tangent lines on both level curves are perpendicular if their normal vectors are also perpendicular. From multivariable calculus, the gradient of a function is always perpendicular to it's level curves and therefore the tangents are perpendicular when the gradients

$$\nabla u = \langle u_x, u_y \rangle$$
$$\nabla v = \langle v_x, v_y \rangle$$

are perpendicular. Since $f'(z_0) \neq 0$, then

$$f'(z_0) = u_x + iv_x = u_x - iu_y \neq 0 \implies \nabla u \neq \vec{0}$$
$$f'(z_0) = u_x + iv_x = v_y + iv_x \neq 0 \implies \nabla v \neq \vec{0}$$

at z_0. The gradients are normal when their dot product is zero. Since f is analytic, the Cauchy Riemann equations hold at z_0 and therefore

$$\nabla u \cdot \nabla v = u_x v_x + u_y v_y = v_x v_y - v_x v_y = 0.$$

Therefore the normal vectors of each curve at z_0 are perpendicular and hence their tangents are perpendicular at z_0. ■

30.1

Part A

By splitting the exponential into a real power and complex power

$$e^{2\pm 3\pi i} = e^2 e^{\pm 3\pi i}.$$

Note that

$$\begin{aligned} e^{3\pi i} &= \cos(3\pi) \quad + i\sin(3\pi) \quad = -1 + i(0) = -1 \\ e^{-3\pi i} &= \cos(-3\pi) + i\sin(-3\pi) = -1 + i(0) = -1 \end{aligned}$$

Therefore $e^{\pm 3\pi i} = -1$ meaning $e^{2\pm 3\pi i} = -e^2$.

Part B

By splitting the exponential into a real power and complex power

$$e^{\frac{2+\pi i}{4}} = e^{\frac{2}{4}} \cdot e^{\frac{\pi i}{4}} = \sqrt{e}e^{\frac{\pi i}{4}}.$$

Since $e^{\frac{\pi i}{4}} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$, it follows

$$e^{\frac{2+\pi i}{4}} = \sqrt{e}\left[\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right] = \sqrt{\frac{e}{2}}(1+i).$$

Part C

Let $z = x + iy$. Then

$$e^{z+\pi i} = e^{x+iy+\pi i} = e^x e^{i(y+\pi)}.$$

Note that $e^{i(y+\pi)} = \cos(y+\pi) + i\sin(y+\pi) = -\cos y - i\sin y = -e^{iy}$. Therefore

$$e^{z+\pi i} = -e^x e^{iy} = -e^{x+iy} = -e^z.$$

30.3

Proof. f will be potentially analytic only at points where the Cauchy Riemann equations hold. Rewriting f in terms of its components gives

$$f(z) = e^{\overline{z}} = e^{x-iy} = e^x e^{-iy} = e^x[\cos(-y) + i\sin(-y)] = e^x\cos y - ie^x\sin y.$$

Therefore the partials are

$$\begin{aligned} u_x &= e^x\cos y \quad & u_y &= -e^x\sin y \\ v_x &= -e^x\sin y \quad & v_y &= -e^x\cos y \end{aligned}$$

Applying the Cauchy Riemann equations gives

$$u_x = v_y \implies e^x\cos y = -e^x\cos y \implies \cos y = -\cos y$$

which is true when $y = \frac{\pi}{2} + \pi k$ for $k \in \mathbb{Z}$ and that

$$u_y = -v_x \implies -e^x\sin y = e^x\sin y \implies \sin y = -\sin y$$

which is true when $y = \pi m$ for $m \in \mathbb{Z}$. However, both equations cannot be satisifed simultaneously since there is no y such that both $\sin y = \cos y = 0$. This means that f is not analytic at any $x + iy$ and therefore cannot be analytic anywhere. ■

30.10

Part A

Proof. Let $z = x + iy$ and assume e^z is purely real. Splitting the exponential gives

$$e^z = e^{x+iy} = e^x e^{iy} = e^x[\cos y + i \sin y].$$

Since e^z is real, then $\operatorname{Im} e^z = 0$ meaning $e^x \sin y = 0$. Since e^x is never 0, this is true when $\sin y = 0$. Therefore $y = n\pi$ for $n \in \mathbb{Z}$. This means that

$$\operatorname{Im} z = \operatorname{Im}\{x + iy\} = \operatorname{Im}\{x + i(n\pi)\} = n\pi, n \in \mathbb{Z}.$$

■

Part B

The restriction on z is that $\operatorname{Im} z = \frac{\pi}{2} + n\pi$ for some $n \in \mathbb{Z}$

Proof. Let $z = x + iy$ and assume e^z is purely imaginary. By part A, $\operatorname{Re}\{e^z\} = e^x \cos y$. Since e^z is purely imaginary, $\operatorname{Re}\{e^z\} = 0$. Therefore $e^x \cos y = 0$. Since e^x is never 0, this is true when $\cos y = 0$. Therefore $y = \frac{\pi}{2} + n\pi$ for $n \in \mathbb{Z}$. This means that

$$\operatorname{Im} z = \operatorname{Im}\{x + iy\} = \operatorname{Im}\left\{x + i\left(\frac{\pi}{2} + n\pi\right)\right\} = \frac{\pi}{2} + n\pi, n \in \mathbb{Z}.$$

■