9.1

Part B

$$
\begin{aligned}
\lim \frac{3n+7}{6n-5} &= \lim \frac{3+\frac{7}{n}}{6-\frac{5}{n}} \\
&= \frac{\lim\left(3+\frac{7}{n}\right)}{\lim\left(6-\frac{5}{n}\right)} \\
&= \frac{3+7\lim\frac{1}{n}}{6-5\lim\frac{1}{n}} \\
&= \frac{3+7(0)}{6-5(0)} \\
&= \frac{3}{6} = \frac{1}{2}.
\end{aligned}
$$

Part C

$$
\begin{aligned}
\lim \frac{17n^5+73n^4-18n^2+3}{23n^5+13n^3} &= \lim \frac{17+\frac{73}{n}-\frac{18}{n^3}+\frac{3}{n^5}}{23+\frac{13}{n^2}} \\
&= \frac{17+73\cdot\lim\frac{1}{n}-18\cdot\lim\frac{1}{n^3}+3\cdot\frac{1}{n^5}}{23+13\cdot\lim\frac{1}{n^2}} \\
&= \frac{17+73(0)-18(0)+3(0)}{23+13(0)} \\
&= \frac{17}{23}
\end{aligned}
$$

9.3

Since $b_n^2+1>0$ for all $n\in\mathbb{N}$,

$$
\begin{aligned}
\lim s_n = \frac{\lim a_n^3+4a_n}{\lim b_n^2+1} &= \frac{\lim a_n^3+4\lim a_n}{b^2+1} \\
&= \frac{(\lim a_n)^3+4a}{b^2+1} = \frac{a^3+4a}{b^2+1}
\end{aligned}
$$

9.4

Part A

$$
\begin{aligned}
s_1 &= 1 \\
s_2 &= \sqrt{2} \\
s_3 &= \sqrt{\sqrt{2}+1} \\
s_4 &= \sqrt{\sqrt{\sqrt{2}+1}+1}
\end{aligned}
$$

Part B

Since s_n converges, let $\lim_{n\to\infty} s_n = s$. Then

$$\lim_{n\to\infty} s_{n+1} = \lim_{n\to\infty} \sqrt{s_n+1}$$

meaning

$$
\begin{aligned}
s = \sqrt{s+1} &\implies s^2 - s - 1 = 0 \\
&\implies s = \frac{1\pm\sqrt{5}}{2}
\end{aligned}
$$

Since $s_n > 0$ for all $n \in \mathbb{N}$, it follows that $s = \frac{1+\sqrt{5}}{2}$.

9.9

Suppose that $\exists N_0 \in \mathbb{N}$ such that $s_n \leq t_n$ for all $n > N_0$.

Part A

Proof. Assume that $\lim s_n = +\infty$. That is

$$\forall M > 0, \exists N_1 \in \mathbb{N} \text{ such that } s_n > M, \forall n > N_1$$

Take $N = \max\{N_0, N_1\}$. If $n > N$, then $t_n \geq s_n > M$. Therefore $\lim t_n = +\infty$. ■

Part B

Proof. Assume that $\lim t_n = -\infty$. That is

$$\forall M < 0, \exists N_1 \in \mathbb{N} \text{ such that } t_n < M, \forall n > N_1$$

Take $N = \min\{N_0, N_1\}$. If $n > N$, then $s_n \leq t_n < M$. Therefore $\lim s_n = -\infty$. ■

Part C

Proof. Assume that $\lim s_n = s$ and $\lim t_n = t$ exist. Consider the case that s and t are finite. Let $\epsilon > 0$. Then there exists $N_1, N_2 \in \mathbb{N}$ such that

$$|s_n - s| < \epsilon, \forall n > N_1$$
$$|t_n - t| < \epsilon, \forall n > N_2$$

Therefore considering $N = \max\{N_0, N_1, N_2\}$,

$$s - \epsilon < s_n \leq t_n < t + \epsilon \implies s < t + 2\epsilon$$

Since ϵ is arbitrary, it follows then that $s \leq t$. If $s = -\infty$, then $t \geq s$ no matter what t is. If $s = \infty$, then by part A it follows that $t = \infty \geq \infty$. ■

9.12

Part A

Proof. Assume that $L < 1$. Let $a \in (L, 1) > 0$ such that $L < a < 1$. Take $\epsilon = a - L > 0$. Since $\left|\frac{s_{n+1}}{s_n}\right|$ converges to L, there exists $N \in \mathbb{N}$ such that

$$L - \epsilon < \left|\frac{s_{n+1}}{s_n}\right| < L + \epsilon, \forall n > N$$

meaning

$$-a + 2L < \left|\frac{s_{n+1}}{s_n}\right| < a \implies \left|\frac{s_{n+1}}{s_n}\right| < a, \forall n > N.$$

Therefore $|s_{n+1}| < a|s_n|$ for all $n > N$. Proceed with induction to show that $|s_n| < a^{n-N}|s_N|$ for $n > N$. Consider the base case $n = N + 1$. By the previous result, $|s_{N+1}| < a|s_N| = a^{N+1-N}|s_N|$, hence the base case holds. Assume for some fixed $n > N$ that $|s_n| < a^{n-N}|s_N|$. Since $a > 0$,

$$a|s_n| < a^{(n+1)-N}|s_N|$$

And since $|s_{n+1}| < a|s_n|$ for all $n > N$,

$$|s_{n+1}| < a|s_n| < a^{(n+1)-N}|s_N| \implies |s_{n+1}| < a^{(n+1)-N}|s_N|$$

Therefore the statement holds for all $n > N$. Note then that

$$0 \leq |s_n| \leq a^{n-N}|s_N|, \forall n > N$$

Since $0 < a < 1$, $a^{n-N}|s_N|$ converges to 0. By the squeeze theorem, $\lim |s_n| = 0$ hence $\lim s_n = 0$. ■

Part B

Proof. Assume that $L > 1$ and let $t_n = \frac{1}{|s_n|}$. Note that then $\lim \left|\frac{t_{n+1}}{t_n}\right| = \frac{1}{L}$ when $L < \infty$ and 0 when $L = +\infty$. Therefore $\lim \left|\frac{t_{n+1}}{t_n}\right| < 1$, which by part A means $\lim t_n = 0$. By Theorem 9.10, $\lim s_n = +\infty$. ∎

10.1

Empty means false.

	A	B	C	D	E	F
Increasing			✓			
Decreasing	✓					✓
Bounded	✓	✓		✓		✓

10.3

Proof. Let $K.d_1d_2d_3\ldots$ be a decimal expansion of a real number. Note that for all $n \in \mathbb{N}$,

$$\frac{d_1}{10} + \frac{d_2}{10^2} + \ldots + \frac{d_n}{10^n} \le \frac{9}{10} + \frac{9}{10^2} + \ldots + \frac{9}{10^n} = 1 - \frac{1}{10^n} < 1$$

hence

$$\frac{d_1}{10} + \frac{d_2}{10^2} + \ldots + \frac{d_n}{10^n} < 1 \implies K + \frac{d_1}{10} + \frac{d_2}{10^2} + \ldots + \frac{d_n}{10^n} < K + 1$$

for all n. By the definition (s_n), it follows that $s_n < K + 1, \forall n \in \mathbb{N}$ ∎

10.4

Both theorems rely on the completeness axiom to ensure the existence of a supremum which does not hold for $\mathbb{Q}$.

10.6

Part A

Proof. Let (s_n) be a sequence and assume that $|s_{n+1} - s_n| < 2^{-n}$ for all $n \in \mathbb{N}$. Let $m, w \in \mathbb{N}$ and WLOG assume $m \ge w$. Note that

$$|s_m - s_w| = |s_m - s_{m-1} + s_{m-1} - s_{m-2} + \ldots + s_{w+1} - s_w|$$

which by triangle inequality

$$
\begin{aligned}
|s_m - s_w| \leq |s_{m+1} - s_m| + |s_m - s_{m-1}| + \ldots + |s_{w+1} - s_w| &< 2^{-m} + 2^{1-m} + \ldots + 2^{-w} \\
&= \frac{1}{2^m} + \ldots + \frac{1}{2^w} \\
&= \frac{1}{2^m}\left(1 + \frac{1}{2} + \ldots \frac{1}{2^{w-m}}\right)
\end{aligned}
$$

Since $1 + \frac{1}{2} + \ldots + \frac{1}{2^{w-m}} < 2$,

$$|s_m - s_w| < \frac{1}{2^m}\left(1 + \frac{1}{2} + \ldots \frac{1}{2^{w-m}}\right) < \frac{1}{2^{w-1}}$$

Take $\epsilon > 0$. Note that for any $n \in \mathbb{N}$, $n < 2^n$ or equivalently $2^{-n} < \frac{1}{n}$ for all n. By the archimedean property, $\exists N_0$ such that $\frac{1}{N} < \epsilon$. Then $2^{-N} < \epsilon$. If $m, w > N$, then $\frac{1}{2^{m-1}} \leq \frac{1}{2^{-N}}$. Therefore

$$|s_m - s_w| < \frac{1}{2^{m-1}} \leq \frac{1}{2^{-N}} < \epsilon$$

Therefore s_n is Cauchy and converges since all Cauchy sequences converge. ■

Part B

The result would not be true if we assume it is less than $\frac{1}{n}$. A key part of the proof is that the series $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots$ has an upper bound. In the case of $\frac{1}{n}$, the series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}$ is not bounded and therefore it is not possible to obtain an N large enough that for any distance between m and n the difference $|s_m - s_n|$ stays below a fixed upper bound ϵ.

10.8

Proof. Note that

$$\begin{aligned}
\sigma_{n+1} - \sigma_n &= \frac{s_1 + \ldots + s_{n+1}}{n+1} - \frac{s_1 + \ldots + s_n}{n} \\
&= \frac{(n+1)(s_1 + \ldots + s_{n+1})}{n(n+1)} - \frac{n(s_1 + \ldots + s_n)}{n(n+1)} \\
&= \frac{(s_{n+1} - s_n) + (s_{n+1} - s_{n-1}) + \ldots + (s_{n+1} - s_1)}{n(n+1)}
\end{aligned}$$

Since s_n is increasing, $s_n \geq s_m \implies s_n - s_m \geq 0$ for all $n \geq m$, meaning

$$\geq \frac{0 + \ldots + 0}{n(n+1)} = 0$$

Therefore $\sigma_{n+1} - \sigma_n \geq 0$ meaning $\sigma_{n+1} \geq \sigma_n$, hence σ_n is an increasing sequence. ■

10.11

Part A

Proof. First note that t_n is a decreasing sequence since t_{n+1} is t_n multiplied by a number between 0 and 1. Additionally, it is bounded above by 1 since it is decreasing and below by 0 since each successive term is positive and $t_1 > 0$. Therefore since t_n is a bounded monotonic sequence, it converges. ■

Part B

Intuitively, $t_n > 0.5$. By creating a desmos simulation, $t_{574} \approx 0.636$, and using Mathematica to solve the recurrence relation gives $\lim t_n = \frac{2}{\pi}$.

$$t_n = \frac{\left(\frac{1}{2}\right)_{n-1} \cdot \left(\frac{3}{2}\right)_{n-1}}{(1)^2_{n-1}}$$