11.1

Part A

$$1, 5, 1, 5, 1, 5, 1, 5$$

Part B

$$\sigma(k) = 2k, (s_{n_k}) = 5, \forall k \in \mathbb{N}$$

11.2

	a_n	b_n	c_n	d_n
Monotone	$\sigma(k) = 2k$	$\sigma(k) = 2k$	$\sigma(k) = 2k - 1$	$\sigma(k) = 3k$
Sub. Limits	$\{1, -1\}$	$\{0\}$	$\{+\infty\}$	$\left\{\frac{6}{7}\right\}$
Liminf	-1	0	$+\infty$	$\frac{6}{7}$
Limsup	1	0	$+\infty$	$\frac{6}{7}$
Bounded	✓	✓		✓
Limit	DNE	0	$+\infty$	$\frac{6}{7}$

11.5

The set of subsequential limits is $[0, 1] \subset \mathbb{R}$.

$$\limsup_{n\to\infty} q_n = 1$$
$$\liminf_{n\to\infty} q_n = 0$$

11.8

$$\begin{aligned}
\liminf s_n &= \lim_{N\to\infty} \inf\{s_n : n > N\} \\
&= -\lim_{N\to\infty} \sup\{-s_n : n > N\} \\
&= -\limsup(-s_n)
\end{aligned}$$

11.9

Part A

Proof. Let (s_n) be a sequence of reals in $[a, b]$ with $\lim s_n = s$. Since $a \leq s_n \leq b$ for all n, it follows that $a \leq s \leq b$, hence $[a, b]$ is closed. ■

Part B

No since $(0, 1)$ is not closed.

12.1

Proof. Let $a_N = \inf \{s_n : n > N\}$ and $b_N = \inf \{t_n : n > N\}$. For $n > N > N_0$, $a_N \leq s_n \leq t_n$ hence $a_N \leq b_N$ for all $N > N_0$. Therefore by excercise 9.9, $\lim a_N \leq \lim b_N$ or equivalently $\liminf s_n \leq \liminf t_n$. The same argument works for sup. ■

12.3

a) 0

b) 1

c) 2

d) 3

e) 4

f) 0

g) 2

12.4

Proof. Since s_n and t_n are bounded, their sups exist and note that $s_n + t_n \leq \sup \{s_n : n > N\} + \sup \{t_n : n > N\}$ for all $n > N \in \mathbb{N}$. Therefore $\sup \{s_n : n > N\} + \sup \{t_n : n > N\}$ is an upper bound for $s_n + t_n$ meaning

$$\sup \{s_n + t_n : n > N\} \leq \sup \{s_n : n > N\} + \sup \{t_n : n > N\}, \forall n > N$$

Since N is arbitrary, it holds for all $N \in \mathbb{N}$ meaning along with the results from 9.9, $\limsup(s_n + t_n) \leq \limsup(s_n) + \limsup(t_n)$ ■

12.10

Proof. Let (s_n) be a sequence.

$\Rightarrow$) Assume that s_n is bounded. That is $\exists M \in \mathbb{R}$ such that $|s_n| \leq M, \forall n \in \mathbb{N}$. Then $\sup\{|s_n| : n > N\} \leq M$ for all $N \in \mathbb{N}$, hence $\limsup |s_n| \leq M < +\infty$.

$\Leftarrow$) Proof by contrapositive. Assume that s_n is not bounded. That is, for all $M \in \mathbb{R}$, $\exists N \in \mathbb{N}$ such that $|s_n| > M$ for all $n > N$. Therefore $\sup |s_n| : n > N > M$. That means that the supremum is larger than any real number, hence $\limsup s_n = +\infty$.

■