4.2.1

a) False

b) True

c) True

d) True

e) False

f) False

g) False

h) True

4.2.4

$$k = \underbrace{(-1)}_{-R_1} \times \underbrace{(2)}_{R_1 \to R_1 + R_2 + R_2} \times \underbrace{(1)}_{R_2 -> R_2 - R_1} \times \underbrace{(1)}_{R_3 -> R_3 - R_1} \times \underbrace{(-1)}_{R_3 \leftrightarrow R_2} = 2.$$

4.2.7

$$\det A = 1 \cdot \begin{vmatrix} 1 & 2 \\ 3 & 0 \end{vmatrix} + 0 + 3 \cdot \begin{vmatrix} 0 & 1 \\ 2 & 3 \end{vmatrix} = -6 + 3(-2) = -12.$$

4.2.25

Proof. Let $A \in M_{n\times n}(\mathbb{F})$. Note that kA is the same as multiplying every row of A by k. Therefore since there are n rows in A,

$$\det(kA) = k^n \det A.$$

■

4.2.30

By swapping the ith row with the $n + 1 - i$ row for $i = 1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor$, it follows that

$$\det B = (-1)^{\left\lfloor \frac{n}{2} \right\rfloor} \det A.$$

4.3.1

a) False

b) True

c) False

d) True

e) False

f) True

g) True

h) False

i) False

4.3.4

$$A = \begin{pmatrix} 2 & 1 & -3 \\ 1 & -2 & 1 \\ 3 & 4 & -2 \end{pmatrix}, \det A = \begin{vmatrix} 2 & 1 & -3 \\ 1 & -2 & 1 \\ 0 & 5 & 0 \end{vmatrix} = -5(2+3) = -25.$$

$$x_1 = -\frac{1}{25}\begin{vmatrix} 1 & 1 & -3 \\ 0 & -2 & 1 \\ -5 & 4 & -2 \end{vmatrix} = -\frac{1}{25}\left(\begin{vmatrix} -2 & 1 \\ 4 & -2 \end{vmatrix} - 5\begin{vmatrix} 1 & -3 \\ -2 & 1 \end{vmatrix}\right) = -1$$

$$x_2 = -\frac{1}{25}\begin{vmatrix} 2 & 1 & -3 \\ 1 & 0 & 1 \\ 3 & -5 & -2 \end{vmatrix} = -\frac{1}{25}\begin{vmatrix} 2 & 1 & -3 \\ 1 & 0 & 1 \\ 0 & -6 & 0 \end{vmatrix} = -\frac{1}{25}\left(6 \cdot \begin{vmatrix} 2 & -3 \\ 1 & 1 \end{vmatrix}\right) = -\frac{6}{5}$$

$$x_3 = -\frac{1}{25}\begin{vmatrix} 2 & 1 & 1 \\ 1 & -2 & 0 \\ 3 & 4 & -5 \end{vmatrix} = -\frac{1}{25}\begin{vmatrix} 2 & 1 & 1 \\ 1 & -2 & 0 \\ 0 & 35 & 0 \end{vmatrix} = -\frac{1}{25}\begin{vmatrix} 1 & -2 \\ 0 & 35 \end{vmatrix} = -\frac{7}{5}$$

4.3.10

Proof. Let $A \in M_{n\times n}(\mathbb{F})$ and assume that A is nilpotent. That is $\exists k \in \mathbb{Z}$ such that $A^k = O$. Note then that

$$(\det A)^k = \det\left(A^k\right) = 0 \implies \det A = 0.$$

Therefore any nilpotent matrix has a zero determinant. ■

4.3.17

Proof. Let $A, B \in M_{n\times n}(\mathbb{F})$ where $AB = -AB$. Assume that n is odd and that $\mathbb{F}$ has characteristic not equal to 2. Then

$$\begin{aligned}
\det(AB) &= \det(-BA) \\
\det(A)\det(B) &= (-1)^n \det(B)\det(A) \\
(1-(-1)^n)\det(A)\det(B) &= 0 \\
(1+1)\det(A)\det(B) &= 0
\end{aligned}$$

Since $\mathbb{F}$ doesn't have characteristic 2, $1+1 \neq 0$ so either A or B must a zero determinant and hence A or B are not invertible. ■