2.2

Let $z = x + iy$ such that $\text{Re}(z) = x$ and $\text{Im}(z) = y$.

Part A

$$\text{Re}(iz) = \text{Re}(i(x+iy)) = \text{Re}(ix - y) = \text{Re}(-y + ix) = -y = -\text{Im}(z).$$

Part B

$$\text{Im}(iz) = \text{Im}(i(x+iy)) = \text{Im}(-y + ix) = x = \text{Re}(z).$$

3.1

Part A

$$\begin{aligned}
\frac{1+2i}{3-4i} + \frac{2-i}{5i} &= \frac{(1+2i)(\overline{3-4i})}{3^2+(-4)^2} + \frac{(2-i)(\overline{5i})}{0^2+5^2} \\
&= \frac{(1+2i)(3+4i)}{25} + \frac{(2-i)(-5i)}{25} \\
&= \frac{3+4i+6i+8i^2}{25} + \frac{-10i+5i^2}{25} \\
&= \frac{-5+10i}{25} + \frac{-5-10i}{25} \\
&= -\frac{10}{25} = -\frac{2}{5}
\end{aligned}$$

Part B

$$\begin{aligned}
\frac{5i}{(1-i)(2-i)(3-i)} &= \frac{5i}{(2-i-2i+i^2)(3-i)} \\
&= \frac{5i}{(1-3i)(3-i)} \\
&= \frac{5i}{3-i-9i+3i^2} \\
&= \frac{5i}{-10i} \\
&= -\frac{5}{10} = -\frac{1}{2}
\end{aligned}$$

Part C

$$(1-i)^2 = 1 - 2i + i^2 = -2i \implies (1-i)^4 = (-2i)^2 = 4i^2 = -4.$$

6.7

$$
\begin{aligned}
|\operatorname{Re}(2+\overline{z}+z^3)| &= \left|\frac{2+\overline{z}+z^3+(\overline{2+\overline{z}+z^3})}{2}\right| \\
&= \left|\frac{2+\overline{z}+z^3+2+z+\overline{z}^3}{2}\right| \\
&= \left|\frac{4+z+\overline{z}+z^3+2}{2}\right| \\
&\leq \frac{2+|z|+|\overline{z}|+|z|^3+|\overline{z}|^3}{2} \\
&= \frac{2+2|z|+2|z|^3}{2}
\end{aligned}
$$

Since $|z| \leq 1$, this quantity is bounded above and therefore

$$
\leq \frac{2+2+2}{2} = 3 \leq 4
$$

6.10

Part A

Proof. Let $z = x + iy$.

$\Rightarrow$) Assume that z is real. That is, $y = 0$. Then $z = x + 0y = x = x - 0y = \overline{z}$. Therefore $z = \overline{z}$

$\Leftarrow$) Assume that $z = \overline{z}$. Then

$$x + iy = x - iy.$$

Equating the imaginary components gives $iy = -iy$ or equivalently $y = -y$. This is only true if $y = 0$. Therefore $z = x + 0y = x$ and hence z is real.

Both directions hence prove the if and only if. ■

Part B

Proof. Let $z = x + iy$.

$\Rightarrow$) Assume that z is real or pure imaginary. Consider the case that z is real. That is $y = 0$ and $x \neq 0$. Then

$$\overline{z}^2 = (x - iy)^2 = x^2 = (x + iy)^2 = z^2.$$

In the case z is purely imaginary, then $x = 0$ and $y \neq 0$ meaning

$$\overline{z}^2 = (x - iy)^2 = -y^2 = (x + iy)^2 = z.$$

Hence $\overline{z} = z$ when z is purely imaginary or real.

$\Leftarrow$) Assume towards contradiction that z is not purely real or imaginary and that $\overline{z}^2 = z^2$. That is $x, y \neq 0$. Note then that

$$\begin{aligned} z^2 &= (x+iy)^2 = x^2 - y^2 + 2ixy \\ \overline{z}^2 &= (x-iy)^2 = x^2 + y^2 - 2ixy \end{aligned}$$

Since $\overline{z}^2 = z^2$,

$$\begin{aligned} x^2 - y^2 + 2ixy &= x^2 + y^2 - 2ixy \\ 4ixy &= 2y^2 \\ ix &= 2y \\ 2 \cdot \frac{y}{x} &= i \end{aligned}$$

However, this is a contradiction since $x, y \in \mathbb{R}$ and therefore $2 \cdot \frac{y}{x}$ cannot be imaginary.

■

6.13

$$\begin{aligned} |z - z_0| = R \implies |z - z_0|^2 &= R^2 \\ (z - z_0)\overline{(z - z_0)} &= R^2 \\ (z - z_0)(\overline{z} - \overline{z_0}) &= R^2 \\ z\overline{z} - z\overline{z_0} - \overline{z}z_0 + z_0\overline{z_0} &= R^2 \\ |z|^2 - z\overline{z_0} - \overline{z\overline{z_0}} + |z_0|^2 &= R^2 \\ |z|^2 - (z\overline{z_0} + \overline{z\overline{z_0}}) + |z_0|^2 &= R^2 \\ |z|^2 - 2\,\mathrm{Re}(z\overline{z_0}) + |z_0|^2 &= R^2 \end{aligned}$$

9.5

Part A

Since

$$\begin{aligned} i &\Leftrightarrow e^{i\frac{\pi}{2}} \\ 1 - i\sqrt{3} &\Leftrightarrow 2e^{-i\frac{\pi}{3}} \\ \sqrt{3} + i &\Leftrightarrow 2e^{i\frac{\pi}{6}} \end{aligned}$$

it follows that

$$\begin{aligned}
i(1-i\sqrt{3})(\sqrt{3}+i) &= e^{i\frac{\pi}{2}} \cdot 2e^{-i\frac{\pi}{3}} \cdot 2e^{i\frac{\pi}{6}} \\
&= 4e^{i\left(\frac{\pi}{2}-\frac{\pi}{3}+\frac{\pi}{6}\right)} \\
&= 4e^{i\frac{\pi}{3}} \\
&= 4 \cdot (\frac{1}{2} + i\frac{\sqrt{3}}{2}) = 2 \cdot (1 + i\sqrt{3})
\end{aligned}$$

Part B

Since

$$\begin{aligned}
5i &\Leftrightarrow 5e^{i\frac{\pi}{2}} \\
2+i &\Leftrightarrow \sqrt{5}e^{i\arctan\left(\frac{1}{2}\right)}
\end{aligned}$$

Let $\theta = \arctan\left(\frac{1}{2}\right)$. It follows

$$\begin{aligned}
\frac{5i}{2+i} &= 5e^{\pi\frac{\pi}{2}} \cdot \frac{1}{\sqrt{5}}e^{-i\theta} \\
&= \frac{5}{\sqrt{5}}e^{i\left(\frac{\pi}{2}-\theta\right)} \\
&= \frac{5}{\sqrt{5}}(\cos\left(\frac{\pi}{2}-\theta\right) + i\sin\left(\frac{\pi}{2}-\theta\right)) \\
&= \frac{5}{\sqrt{5}}(\sin\theta + i\cos\theta) \\
&= \frac{5}{\sqrt{5}}\left(\frac{1}{\sqrt{5}} + i \cdot \frac{2}{\sqrt{5}}\right) \\
&= 1 + 2i
\end{aligned}$$

Part C

Let $z = \sqrt{3} + i$ and $r = |z| = \sqrt{3 + 1^2} = 2$. Since

$$\frac{z}{|z|} = \frac{\sqrt{3}}{2} + \frac{i}{2} \implies \theta = \frac{\pi}{6}.$$

Therefore

$$z^6 = \left(2e^{i\frac{\pi}{6}}\right)^6 = 2^6e^{i\pi} = -64.$$

Part D

Let $z = 1 + i\sqrt{3}$ and $r = |z| = \sqrt{1 + 3^2} = 2$. Since

$$\frac{z}{|z|} = \frac{1}{2} + i\frac{\sqrt{3}}{2} \implies \theta = \frac{\pi}{3}.$$

Therefore

$$z^{-11} = (2e^{i\frac{\pi}{3}})^{-10} = 2^{-10}e^{-i\frac{10\pi}{3}} = 2^{-10}e^{-i\frac{\pi}{3}} = 2^{-10}\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = 2^{-11}(1 - i\sqrt{3}).$$

9.6

Since $\operatorname{Re} z_1$ and $\operatorname{Re} z_2$, both their principal arguments lie in the right half of the unit circle and therefore $\operatorname{Arg} z_1, \operatorname{Arg} z_2 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. This means that their sum is bounded by

$$-\frac{\pi}{2} - \frac{\pi}{2} = -\pi < \operatorname{Arg} z_1 + \operatorname{Arg} z_2 < \pi = \frac{\pi}{2} + \frac{\pi}{2}.$$

Therefore since their sum lies in the interval for the principal arguement of $z_1 z_2$, it follows that

$$\operatorname{Arg} z_1 z_2 = \operatorname{Arg} z_1 + \operatorname{Arg} z_2.$$

9.8

Proof. Let $z_1, z_2 \in \mathbb{C}$ with $r_1 = |z_1|, r_2 = |z_2|$ and $\theta_1 = \operatorname{Arg} z_1, \theta_2 = \operatorname{Arg} z_2$.

$\Rightarrow$) Assume that z_1 and z_2 have the same moduli. That is $r_1 = r_2$. Let

$$c_1 = r_1 \exp\left(i\frac{\theta_1 + \theta_2}{2}\right)$$
$$c_2 = \exp\left(i\frac{\theta_1 - \theta_2}{2}\right)$$

Note that $c_1 c_2 = r_1 e^{i\theta_1} = z_1$ and $c_1\overline{z_2} = r_1 e^{i\theta_2} = r_2 e^{i\theta_2} = z_2$. Therefore $z_1 = c_1 c_2$ and $z_2 = z_1\overline{z_2}$.

$\Leftarrow$) Assume that there are complex numbers c_1, c_2 such that $z_1 = c_1 c_2$ and $z_2 = c_1\overline{z_2}$. Then

$$|z_1| = |c_1 c_2| = |c_1||c_2| = |c_1||\overline{c_2}| = |c_1\overline{c_2}| = |z_2|.$$

Therefore $|z_1| = |z_2|$.

$\blacksquare$

11.3

First, convert $z = -8 - 8\sqrt{3}i$ to exponential form. Then

$$|z| = \sqrt{8^2 + 3 \cdot 8^2} = \sqrt{4 \cdot 8^2} = 2 \cdot 8 = 16$$

Note that $\frac{z}{|z|} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$ which corresponds to the angle $\theta = -\frac{2\pi}{3}$ on the unit circle. Since $2 = \sqrt[4]{16}$, the the roots of z are

$$\begin{aligned}
c_0 &= 2e^{-i\frac{\pi}{6}} = 2\left(\frac{\sqrt{3}}{2} - i\frac{1}{2}\right) &&= \sqrt{3} - i \\
c_1 &= c_0 e^{i\frac{\pi}{2}} = 2e^{i\frac{\pi}{3}} = 2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) &&= 1 + i\sqrt{3} \\
c_2 &= c_0 e^{i\pi} = -c_0 &&= -\sqrt{3} + i \\
c_3 &= c_0 e^{i\frac{3\pi}{2}} = 2e^{-i\frac{2\pi}{3}} = 2\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) &&= -1 - i\sqrt{3}
\end{aligned}$$

Therefore the roots are $\pm(\sqrt{3} - i)$ and $\pm(1 + i\sqrt{3})$.

11.5

Let $z_0 = -4\sqrt{2} + 4\sqrt{2}i$. Then $r = |z_0| = \sqrt{2(4\sqrt{2})^2} = \sqrt{64} = 8$. It follows that

$$\frac{z}{|z_0|} = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i \implies \theta = \frac{3\pi}{4}.$$

Therefore the principal cube root of z_0 is

$$c_0 = 2e^{i\frac{\pi}{4}} = 2\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = \sqrt{2}(1+i).$$

The other cube roots are $c_0\omega_3$ and $c_0\omega_3^2$. Hence

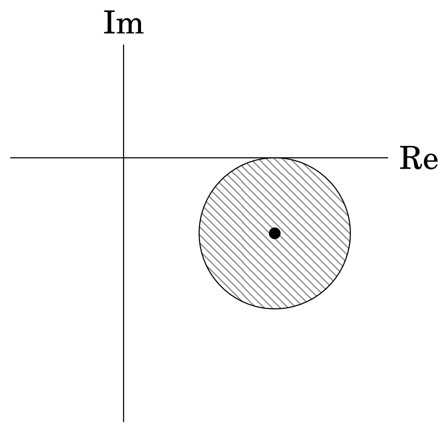
$$\begin{aligned}
c_0\omega_3 = 2e^{i\frac{\pi}{4}}e^{i\frac{2\pi}{3}} = 2\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right) &= \left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)\left(-1 + i\sqrt{3}\right) \\
&= \frac{-(\sqrt{3}+1) + (\sqrt{3}-1)i}{\sqrt{2}}
\end{aligned}$$

$$\begin{aligned}
c_0\omega_3^2 = 2e^{i\frac{\pi}{4}}e^{i\frac{4\pi}{3}} = 2\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)\left(-\frac{\sqrt{3}}{2} - i\frac{1}{2}\right) = \left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)(-\sqrt{3} - i) &= -c_0\omega_3 \\
&= \frac{(\sqrt{3}+1) - (\sqrt{3}-1)i}{\sqrt{2}}
\end{aligned}$$

12.1

Let $z = x + iy$.

Part A



This is not a domain since it is a closed set (closed disk).

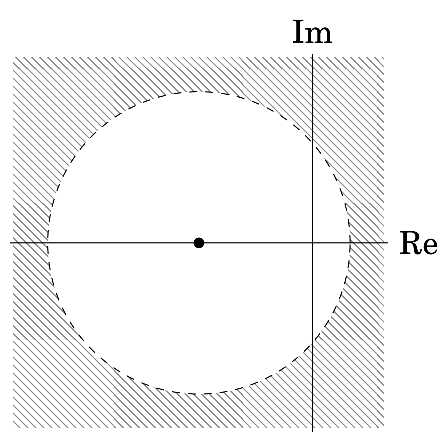
Part B

Since

$$\begin{aligned} |2z+3| &= |2x+2iy+3| \\ &= \sqrt{(2x+3)^2+(2y)^2} \end{aligned}$$

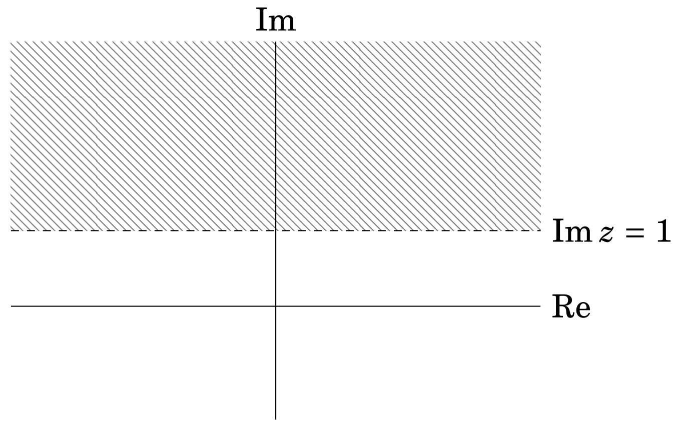
it follows that

$$\begin{aligned} |2z+3| > 4 \implies (2x+3)^2+(2y)^2 &> 16 \\ 4x^2+12x+9+4y^2 &> 16 \\ x^2+3x+\frac{9}{4}+y^2 &> 4 \\ \left(x+\frac{3}{2}\right)^2+y^2 &> 4 \end{aligned}$$



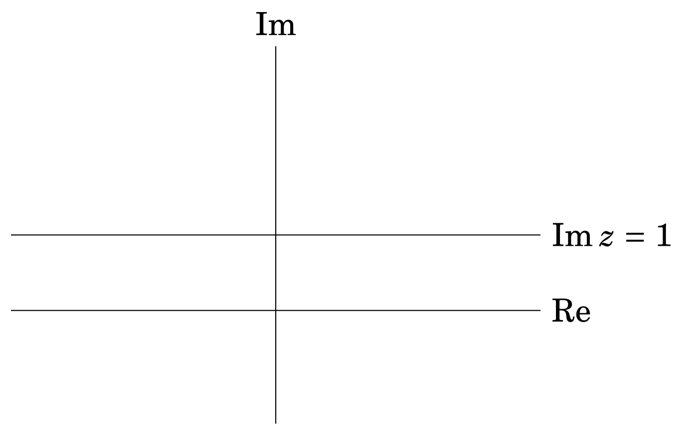
This is a domain since it is open and connected.

Part C



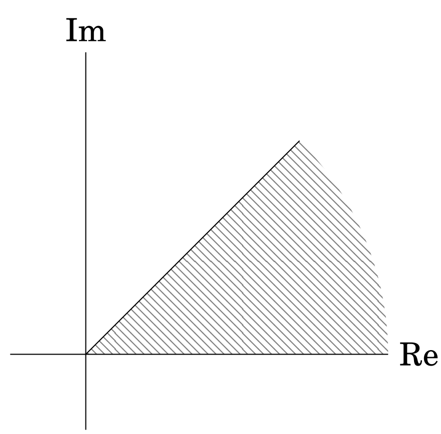
It is a domain since it is open and connected.

Part D



It is not a domain since it is not open.

Part E

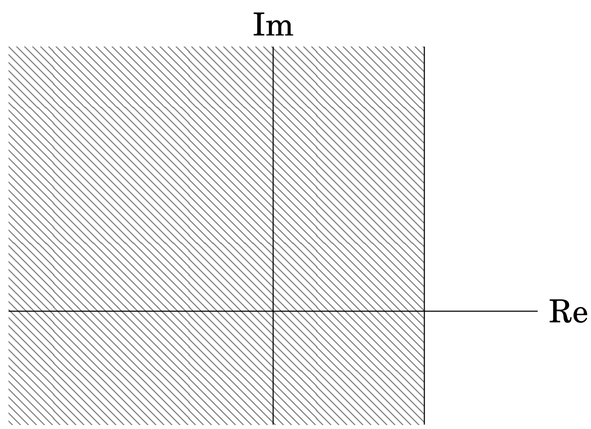


It is not a domain since it is closed.

Part F

$$
\begin{aligned}
|z-4| \geq |z| \implies |(x-4)+iy| &\geq |x+iy| \\
\sqrt{(x-4)^2+y^2} &\geq \sqrt{x^2+y^2} \\
(x-4)^2+y^2 &\geq x^2+y^2 \\
(x-4)^2 &\geq x^2 \\
x^2-8x+16 &\geq x^2 \\
8x &\leq 16 \\
x &\leq 2
\end{aligned}
$$

Therefore the set looks like

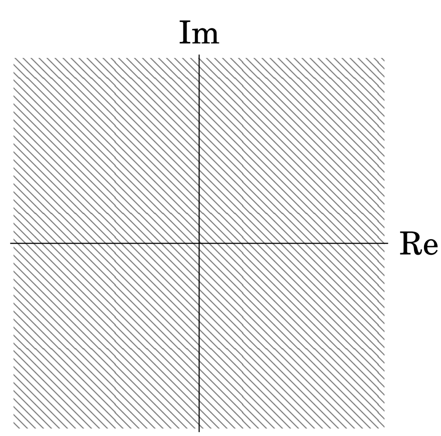


It is not a domain since it is closed

12.4

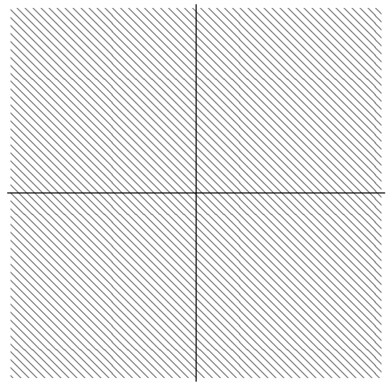
Part A

The original set is every complex number except those with a principal argument of π which means the set is the complex plane minus the negative real line. Therefore it's closure is the entire complex plane.



Part B

Since the inequality is true for all $z \in \mathbb{C}$, its closure is $\mathbb{C}$ since $\mathbb{C}$ is closed.



Part C

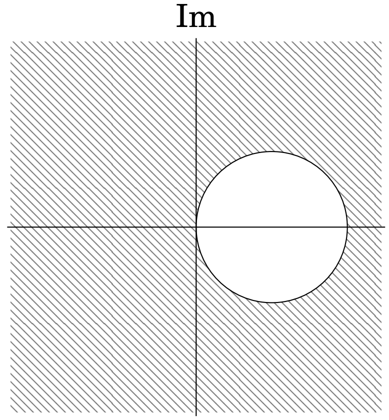
Since

$$\begin{aligned}\operatorname{Re}\frac{1}{z} &= \operatorname{Re}\frac{\overline{z}}{|z|^2} \\ &= \operatorname{Re}\left(\frac{x}{x^2+y^2} - i\cdot\frac{y}{x^2+y^2}\right) \\ &= \frac{x}{x^2+y^2}\end{aligned}$$

The inequality is the same as

$$\begin{aligned}\frac{x}{x^2+y^2} \leq \frac{1}{2} \implies x^2+y^2 &\geq 2x \\ x^2-2x+1+y^2 &\geq 1 \\ (x-1)^2+y^2 &\geq 1\end{aligned}$$

This is all the points outside an open unit disk centered at $z = 1$. Since it is the complement of an open disk, it is a closed set and hence the closure is itself



Part D

$$\operatorname{Re} z^2 = \operatorname{Re}\left(x^2 + 2ixy - y^2\right) = x^2 - y^2$$

Therefore

$$\operatorname{Re} z^2 > 0 \implies x^2 - y^2 > 0 \implies -x < y < x.$$

The boundary of this set are the points where $y = x$ and $y = -x$. Hence the closure of the set is $-x \leq y \leq x$.