Power Series

Definition 1.1 (Power Series). A power series is a real valued function $f(x) = \sum a_n x^n$ for some sequence (a_n).

Theorem 1.1. For a power series $\sum a_n x^n$, let $\beta = \limsup |a_n|^{\frac{1}{n}}$ and $R = \frac{1}{\beta}$. The power series converges for $|x| < R$ and diverges for $|x| > R$

Proof. Apply the root test. Then

$$\limsup |c_n|^{\frac{1}{n}} = \limsup |a_n|^{\frac{1}{n}} |x| = \limsup |a_n|^{\frac{1}{n}} |x| = |x|\beta.$$

Note then that $|x| < R = \frac{1}{\beta}$ means that $\limsup |c_n|^{\frac{1}{n}} < 1$ and therefore the series converges. The opposite is true for $|x| > R$. ■

Example 1.1. Consider $\sum x^n$. Note that $a_n = 1$ for all $n \in \mathbb{N}$. Therefore $\limsup |a_n|^{\frac{1}{n}} = \limsup 1^{\frac{1}{n}} = 1$. Therefore the power series converges for all $|x| < 1$. Note that $x = 1$ gives a divergent series and $x = -1$ gives an alternating series whose non alternative part does not go to zero and hence also diverges.

Example 1.2. Consider $\sum \frac{x^n}{n!}$. In this instance $a_n = \frac{1}{n!}$. Then

$$\beta = \limsup |a_n|^{\frac{1}{n}} = \limsup \left|\frac{1}{n!}\right|^{\frac{1}{n}}.$$

This would be hard to compute. However, if this limit exists, then it matches the value of the ratio test and therefore

$$\beta = \limsup \left|\frac{a_{n+1}}{a_n}\right| = \limsup \frac{1}{(n+1)!} \cdot \frac{n!}{1} = \limsup \frac{1}{n} = 0.$$

Therefore $R = +\infty$ meaning the interval of convergence is all of $\mathbb{R}$.
Remark. Alternatively, one can use the Sterling approximation of the factorial to do the root test. The Sterling approximation is

$$n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}.$$

Hence

$$\limsup \left|\frac{1}{n!}\right|^{\frac{1}{n}} = \limsup \frac{1}{\left(\left(\frac{n}{e}\right)^n \sqrt{2\pi n}\right)^{\frac{1}{n}}} = \limsup \frac{1}{\frac{n}{e} \cdot \left(\sqrt{2\pi n}\right)^{\frac{1}{n}}} = \limsup \frac{1}{n} = 0.$$

Example 1.3. Consider $\sum \frac{x^n}{n^2}$. Then

$$\beta = \limsup \left(\frac{1}{n^2}\right)^{\frac{1}{n}} = \limsup \frac{1}{\sqrt[n]{n^2}} = 1.$$

Therefore the power series converges for $|x| < 1$. Importantly, for $x = 1$ and $x = -1$, you get convergent series and therefore the interval of convergence is $[-1, 1]$.

Example 1.4. Consider $\sum \frac{(-1)^{n+1}x^n}{n}$. Then $a_n = \frac{(-1)^{n+1}}{n}$ and

$$\beta = \limsup \left|\frac{(-1)^{n+1}}{n}\right|^{\frac{1}{n}} = \limsup \frac{1}{\sqrt[n]{n}} = \frac{1}{1} = 1.$$

Therefore the power series converges for $|x| < 1$. Checking $x = 1$,

$$\sum \frac{(-1)^{n+1}}{n} \text{ converges by alternating series test.}$$

And checking for $x = -1$,

$$\sum \frac{(-1)^{2n+1}}{n} = \sum \frac{-1}{n} = -\sum \frac{1}{n} \text{ which diverges.}$$

Therefore the interval of convergence is $(-1, 1]$.

Example 1.5. Consider $\sum \frac{(2n)!x^n}{(n!)^2}$. Then $a_n = \frac{(2n)!}{(n!)^2}$. Apply the ratio test to get β.

$$\beta = \limsup |\frac{a_{n+1i}}{a_n}| = \limsup \frac{(2n+2)!}{((n+1)!)^2} \cdot \frac{(2n)!}{(n!)^2} = \limsup \frac{(2n+2)(2n+1)}{(n+1)(n+1)} = 4.$$

Therefore it converges on $|x| < \frac{1}{4}$. Checking the endpoints suck but $x = \frac{1}{4}$ diverges by using Sterlings approximation and $x = -\frac{1}{4}$ converges by the alternating series test by the previous method. Therefore the interval of convergence is $\left[-\frac{1}{4}, \frac{1}{4}\right)$.