Set Theory II

1.1 Cartesian Product

Definition 1 (Cartesian Product)**.** Let A and B be sets. Their Cartesian product is defined as

$$A \times B = \{(a, b) : a \in A, b \in B\}.$$

For example, consider the 2-dimensional plane. Each point can be defined in Cartesian coordinates, and therefore as an ordered pair (x, y) where $x, y \in \mathbb{R}$. This means that all elements of the Cartesian plane can be expressed as elements of the set

$$\mathbb{R} \times \mathbb{R} = \mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}.$$

Theorem 1. If A and B are finite sets, then $|A \times B| = |A| \cdot |B|$

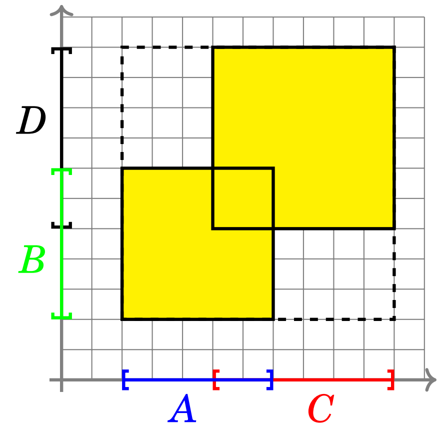
Proof. Let $|A| = m$ and $|B| = n$. Listing out $A \times B$ in a grid pattern results in

$$A \times B = \begin{array}{cccccccc} \{ & (a_1, b_1) & (a_1, b_2) & (a_1, b_3) & \dots & (a_1, b_n) & \\ & (a_2, b_1) & (a_2, b_2) & (a_2, b_3) & \dots & (a_2, b_n) & \\ & \vdots & \vdots & \vdots & \ddots & \vdots & \\ & (a_m, b_1) & (a_m, b_2) & (a_m, b_3) & \dots & (a_m, b_n) & \}. \end{array}$$

Every ordered pair is written only once. Since there are m rows and n columns, the number of elements is mn, therefore $|A \times B| = mn$. ■

Here is a basic set relationship involving the Cartesian product.

Theorem 2. Let A, B, C, D be any sets. Then $(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)$.



Visually, this can be seen as two regions in the Cartesian plane being contained within a larger region where the boundaries are equivalent to the union of the individual region's sides.

Proof. Let $(x, y) \in (A \times B) \cup (C \times D)$. If (x, y) is in $(A \times B)$, then $x \in A$ and $y \in B$. Therefore $x \in A \cup C$ and $y \in B \cup D$, meaning $(x, y) \in (A \cup C) \times (B \cup D)$. If (x, y) is in $(C \times D)$, then $x \in C$ and $y \in D$. Therefore $x \in A \cup C$ and $y \in B \cup D$, meaning $(x, y) \in (A \cup C) \times (B \cup D)$, as required. ■

Here is a proof of a more generalized version of Theorem 1 using induction.

Theorem 3. For all $n \in \mathbb{N}$, if $A_1, \ldots, A_n$ are finite sets, then

$$|A_1 \times \ldots \times A_n| = |A_1| \ldots |A_n|.$$

Proof. Proceed with induction to show that for all $n \in \mathbb{N}$, if $A_1, \ldots, A_n$ are finite sets, then $|A_1 \times \ldots \times A_n| = |A_1| \ldots |A_n|$. Consider the base case when $n = 1$. Then $|A_1| = |A_1|$, hence the base case is true. Assume for a fixed $n \in \mathbb{N}$ that $|A_1 \times \ldots \times A_n| = |A_1| \ldots |A_n|$. Consider then the Cartesian product $A_1 \times \ldots \times A_{n+1}$. This will result in every ordered pair in $A_1 \times \ldots \times A_n$ being repeated with a new element from A_{n+1} added in each time. Hence the number of ordered pairs in the set $A_1 \times \ldots \times A_{n+1}$ will be the same as the number of elements of $A_1 \times \ldots A_n$ multiplied by the number of elements in A_{n+1}. By the induction hypothesis, the number of elements in $A_1 \times \ldots \times A_n = |A_1| \ldots |A_n|$ and the number of elements in A_{n+1}

is $|A_n + 1|$. Hence

$$|A_1 \times \ldots \times A_{n+1}| = |A_1| \ldots |A_{n+1}|.$$

Therefore for all $n \in \mathbb{N}$, if $A_1, \ldots, A_n$ are finite sets, then

$$|A_1 \times \ldots \times A_n| = |A_1| \ldots |A_n|.$$

■

2 Power Sets

Definition 2 (Power Set)**.** The *power set* of a set A is the set $\mathcal{P}(A)$ of all subsets of A. That is

$$\mathcal{P}(A) = \{B : B \subseteq A\}.$$

Equivalently $B \in \mathcal{P}(A) \iff B \subseteq A$.

3 Indexed Collections

Definition 3 (Indexed Collection)**.** Given a family of indexed sets $\{A_n : n \in I\}$, the union or intersection of the family can be formed as

$$\bigcup_{n \in I} A_n = \{x : x \in A_n \text{ for some } n \in I\}$$
$$\bigcap_{n \in I} A_n = \{x : x \in A_n \text{ for all } n \in I\}.$$

Equivalently,

$$x \in \bigcup_{n \in I} A_n \iff \exists n \in I, x \in A_n$$

$$x \in \bigcap_{n \in I} A_n \iff \forall n \in I, x \in A_n$$