3.5

Part A

Proof. Let $a, b \in \mathbb{R}$. Consider both directions.

$\Rightarrow$) Assume that $|b| \leq a$. If $b \geq 0$, then $|b| = b \leq a$. If $b < 0$, then $|b| = -b \leq a$ or equivalently $b \geq -a$. Therefore for any b, $-a \leq b \leq a$.

$\Leftarrow$) Assume that $-a \leq b \leq a$. Note this implies that $b \leq a$ and that $-b \leq a$ by using both sides of the inequality. Since both $-b$ and b are less than a, $|b| \leq a$.

■

Part B

Proof. Let $a, b \in \mathbb{R}$. Note that $|a| = |(a-b)+b| \leq |a-b|+|b|$ by the triangle inequality, giving $|a| - |b| \leq |a - b|$. Equivalently, $|b| - |a| \leq |b - a|$, meaning $|a| - |b| \geq -|a - b|$. Since $-|a - b| \leq |a| - |b| \leq |a - b|$, by 3.5,

$$||a| - |b|| \leq |a - b|$$

■

4.1 / 4.2

	A	B	E	F	H	I	K	N	T	U
LB	-1	-1	-1	-1	-1	-2	0	-2	DNE	-3
UB	5	4	2	1	DNE	3	DNE	2	10	DNE
inf	0	0	0	0	2	0	0	$-\sqrt{2}$	DNE	0
sup	1	1	1	0	DNE	1	DNE	$\sqrt{2}$	2	DNE

4.5

Proof. Let $S \subset \mathbb{R}$ be non-empty that is bounded above. Assume that $\sup S = M \in S$. Since M is the supremum, $s \leq M$ for all $s \in S$. Since $s \leq M$ for all $s \in S$ and $M \in S$, M is by definition the max of S, hence $\sup S = \max S$. ■

4.7

Part A

Proof. Let $S, T \subset \mathbb{R}$ be non empty and bounded. Assume that $S \subset T$. Let $x \in S$. Note that $\inf T \leq s$ for all $s \in S$ since every member of S is a member of T. This means that $\inf T$ is a lower bound of S implying $\inf T \leq \inf S$. Equivalently, $\sup T \geq s$ for all $s \in S$, meaning $\sup T$ is an upper bound of S and therefore $\sup T \geq \sup S$. Since $\inf S \leq \sup S$, these inequalities combine to

$$\inf T \leq \inf S \leq \sup S \leq \sup T$$

■

Part B

Proof. Let $S, T \subset \mathbb{R}$ be non empty and bounded. Note that $S, T \subset S \cup T$. Therefore from (A) it follows that $\sup S, \sup T \leq \sup(S \cup T)$ meaning $\max\{\sup S, \sup T\} \leq \sup(S \cup T)$. Let $s \in S$. Then $s \leq \sup S \leq \max\{\sup S, \sup T\}$. Let $t \in T$. Then $t \leq \sup T \leq \max\{\sup S, \sup T\}$. Therefore for an element $x \in S \cup T$, $x \leq \max\{\sup S, \sup T\}$. Therefore $\max\{\sup S, \sup T\}$ admits an upper bound on $S \cup T$ meaning $\sup(S \cup T) \leq \max\{\sup S, \sup T\}$. Since $\sup(S \cup T) \leq \max\{\sup S, \sup T\}$ and $\max\{\sup S, \sup T\} \leq \sup(S \cup T)$, $\sup(S \cup T) = \max\{\sup S, \sup T\}$. ■

4.10

Proof. Let $a > 0$. By applying the archimedean property twice, $\exists N_1, N_2$ such that $a > \frac{1}{N_1}$ and $a < N_2$. Let $N = \max\{N_1, N_2\}$. Note that $\frac{1}{N_1} \leq \frac{1}{N}$ and $N_2 \leq N$, therefore $\frac{1}{N} < a < N$. ■

4.11

Let $a, b \in \mathbb{R}$ with $a < b$. By the denseness of $\mathbb{Q}$, there is a rational r_1 such that $a < r_1 < b$. Since $r_1 \in \mathbb{R}$, the denseness of $\mathbb{Q}$ can be applied to the range $a < r_1$ to give an r_2 such that $a < r_2 < r_1$. This leads to an infinite sequence of r_n's that are between a and b. Hence there are infinitely many rationals between a and b.

4.12

Proof. Let $a, b \in \mathbb{R}$ such that $a < b$. Consider $r + \sqrt{2}$ where $r = \frac{p}{q} \in \mathbb{Q}$. Assume towards contradiction that $r + \sqrt{2}$ is rational. That is there is an $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ such that

$$\frac{p}{q} + \sqrt{2} = \frac{m}{n}$$

However, this implies that $\sqrt{2} = \frac{mq - np}{nq}$ which is rational. Therefore $r + \sqrt{2}$ is irrational for all $r \in \mathbb{Q}$. Since $a < b$, $a - \sqrt{2} < b - \sqrt{2}$. By the denseness of $\mathbb{Q}$, there exists $h \in \mathbb{Q}$ such that $a - \sqrt{2} < h < b - \sqrt{2}$, or equivalently $a < h + \sqrt{2} < b$. Note that $h + \sqrt{2}$ is irrational and hence there is an irrational between a and b. ■

5.1

$$\{x \in \mathbb{R} : x < 0\} = (-\infty, 0)$$
$$\left\{x \in \mathbb{R} : x^3 \leq 8\right\} = (-\infty, 2]$$
$$\left\{x^2 : x \in \mathbb{R}\right\} = [0, \infty)$$
$$\left\{x \in \mathbb{R} : x^2 < 8\right\} = (-\sqrt{8}, \sqrt{8})$$

5.2

$$\inf \{x \in \mathbb{R} : x < 0\} = -\infty \qquad \sup \{x \in \mathbb{R} : x < 0\} = 0$$
$$\inf \left\{x \in \mathbb{R} : x^3 \leq 8\right\} = -\infty \qquad \sup \left\{x \in \mathbb{R} : x^3 \leq 8\right\} = 2$$
$$\inf \left\{x^2 : x \in \mathbb{R}\right\} = 0 \qquad \sup \left\{x^2 : x \in \mathbb{R}\right\} = \infty$$
$$\inf \left\{x \in \mathbb{R} : x^2 < 8\right\} = -\sqrt{8} \qquad \sup \left\{x \in \mathbb{R} : x^2 < 8\right\} = \sqrt{8}$$

5.5

Proof. Let $S \subset \mathbb{R}$ be non-empty. If S is unbounded, $\inf S = -\infty$ and $\sup S = \infty$ meaning that $\inf S \leq \sup S$. If S is bounded below but unbounded above, $\inf S = m \leq \infty = \sup S$ meaning $\inf S \leq \sup S$. Alternatively, if S is bounded above but unbouneded below, $\sup S = M \geq -\infty = \inf S$ meaning $\inf S \leq \sup S$. If S is bounded, $\inf S = m$ and $\sup S = M$. For all $s \in S$, $m \leq s \leq M$, meaning $m \leq M$ and therefore $\inf S \leq \inf S$. ■