2.1.1

a) True

b) False

c) False

d) True

e) False

f) False

g) True

h) False

2.1.5

Proof. Let $T : P_2(\mathbb{R}) \to P_3(\mathbb{R})$ defined by $T(f(x)) = x \cdot f(x) + f'(x)$. Let $f, g \in P_2(\mathbb{R})$ and $c \in \mathbb{R}$. Then

$$T(f+g) = x(f(x)+g(x)) + f'(x) + g'(x) = xf(x) + f'(x) + xg(x) + g'(x) = T(f) + T(g)$$

and

$$T(cf) = x(c \cdot f(x)) + cf'(x) = c(xf(x) + f'(x)) = cT(f).$$

Therefore T is a linear transformation. ■

$$\beta_{N(T)} = \{0\} \implies \dim(N(T)) = 0$$
$$\beta_{R(T)} = \left\{x, x^2+1, x^3\right\} \implies \dim(R(T)) = 3$$

Since $N(T) = \{0\}$, T is one-to-one but not onto since $\text{rank}(T) \neq \dim(P_3(\mathbb{R}))$.

2.1.9

1. $T(0,0) = (1,0) \neq (0,0)$
2. $cT(a_1, a_2) = (ca_1 ca_1^2) \neq (ca_1, c^2 a_1^2) = T(ca_1, ca_2)$
3. $T(2 \cdot \frac{\pi}{2}, 0) = (0,0) \neq (2,0) = 2 \cdot T(\frac{\pi}{2}, 0)$
4. $T((1,0) + (-1,0)) = (0,0) \neq (2,0) = T(1,0) + T(-1,0)$
5. $T(0,0) = (1,0) \neq (0,0)$

2.1.15

Since the only function when integrated equals zero is the zero function itself. Therefore $N(T) = \{0\}$, therefore T is one-to-one. Note as well that

$$T(a_nx^n + a_{n-1}x^{n-1} + \ldots + a_2x + a_1) = \frac{a_n}{n+1}x^{n+1} + \frac{a_{n-1}}{n-1}x^n + \ldots + \frac{a_2}{2}x^2 + a_1x$$

Since there is no constant term in the output, all constant polynomials dont have a corresponding polynomial that under T would equal it. Therefore T cannot be onto.

2.1.17

Part A

Since $\operatorname{rank} T \leq \dim V < \dim W$, $\operatorname{rank} T < \dim W$ and therefore T is not onto.

Part B

Since $\operatorname{nullity} T = \dim V - \operatorname{rank} T \geq \dim V - \dim W > 0$, $N(T) \neq \{0\}$ and therefore T cannot be one-to-one.

2.1.22

For $T : \mathbb{R}^3 \to \mathbb{R}$, let $a = T(1,0,0), b = T(0,1,0), c = T(0,0,1)$. Note then that

$$T(x,y,z) = xT(1,0,0) + yT(0,1,0) + zT(0,0,1) = ax + by + cz$$

Now generally:

Theorem 0.1. Let $T : \mathbb{F}^n \to \mathbb{F}$ be linear. Then there exists scalars $a_i \in \mathbb{F}$ such that $T(x_1, x_2, \ldots, x_n) = a_1x_1 + a_2x_2 + \ldots a_nx_n$.

Proof. Let $T : \mathbb{F}^n \to \mathbb{F}$ be linear. Let e_i denote the vector where the ith position is one and all other's are zero. Let $a_i = T(e_i)$ where $1 \leq i \leq n$. Note then that

$$T(x_1, x_2, x_3, \ldots, x_n) = \sum_{i=1}^{n} x_iT(e_i) = \sum_{i=1}^{n} a_ix_i.$$

■

2.2.1

a) True

b) True

c) False

d) True

e) True

f) False

2.2.4

$$[T]_\beta^\gamma = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

2.2.10

$$[T]_\beta = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & & 0 \\ 0 & 1 & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & 1 \end{pmatrix}$$

2.2.14

Proof. Let $V = P(\mathbb{R})$ and $T_j(f) = f^{(j)}(x)$. Let $n \in \mathbb{N}$ and assume that $\sum_{j=0}^{n} a_i T_i = 0$. Note that $T_j(x^n) = \frac{n!}{(n-j)!}x^{n-j}$. It is clear that for different j, the results are linearly independent since the degrees are different. Therefore $\sum_{j=0}^{n} a_i T_i(x^n) = 0$ implies that $a_i = 0$ for all i. Hence $\{T_1, T_2, \dots, T_n\}$ is linearly independent. ■

2.2.16

Proof. Let V and W be vector spaces such that $\dim V = \dim W = n$ and let $T : V \to W$ be linear. Let $\{u_1, u_2, u_3, \dots u_m\}$ be a basis for $N(T)$ and extend it to be a basis $\beta = \{u_1, \dots, u_m, v_{m+1}, v_{m+2}, \dots v_n\}$ of V. Examine the linear indepence of the set $\{T(v_{m+1}), T(v_{m+2}), \dots, T(v_n)\}$. Assume that

$$a_{m+1}T(v_{m+1}) + a_{m+2}T(v_{m+2}) + \dots + a_n T(v_n) = 0.$$

Then

$$T(a_{m+2}v_{m+1} + a_{m+2}v_{m+2} + \dots + a_n v_n) = 0.$$

Therefore $a_{m+2}v_{m+1} + a_{m+2}v_{m+2} + \dots + a_n v_n \in N(T)$ and therefore can be expressed as linear combination of basis vectors of $N(T)$. That is

$$\sum_{i=m+1}^{n} a_i v_i = \sum_{i=1}^{m} b_i u_i \implies \sum_{i=1}^{m} b_i u_i - \sum_{i=m+1}^{n} a_i v_i = w = 0.$$

However, note that w is a linear combination of the basis β. Therefore the summation is linearly independent meaning all coefficients, and therefore all a_i, must equal 0. Therefore $\{T(v_{m+1}), T(v_{m+2}), \dots, T(v_n)\}$ is linearly independent and hence can be extened to be a basis γ of W with $\gamma = \{w_1, w_2, \ldots, w_m, T(v_{m+1}), T(v_{m+2}), \ldots, T(v_n)\}$. Therefore

$$[T]_\beta^\gamma = \begin{pmatrix} 0 & & & & & \\ & \ddots & & & & \\ & & 0 & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}$$

which is a diagonal matrix. ■

2.3.3

Part A

$$[U]_\beta^\gamma = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

$$[T]_\beta = \begin{pmatrix} 2 & 3 & 0 \\ 0 & 3 & 8 \\ 0 & 0 & 2 \end{pmatrix}$$

$$[UT]_\beta^\gamma = \begin{pmatrix} 2 & 6 & 8 \\ 0 & 0 & 2 \\ 2 & 0 & -8 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \times \begin{pmatrix} 2 & 3 & 0 \\ 0 & 3 & 8 \\ 0 & 0 & 2 \end{pmatrix} \checkmark$$

2.3.9

Take $T(a, b) = (0, a)$ and $U(a, b) = (a, 0)$. Note then that

$$UT(a, b) = U(T(a, b)) = U(0, a) = (0, 0)$$

but that

$$TU(a, b) = T(U(a, b)) = T(a, 0) = (0, a) \neq (0, 0)$$

Therefore by using the standard basis for $\mathbb{F}^2$,

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

2.3.11

Proof. Let V be a vector space and $T : V \to V$ be linear.

$\Rightarrow$) Assume that $T^2 = T_0$. Let $w \in R(T)$. Then $\exists v \in V$ such that $w = T(v)$.Note then that $T(w) = T(T(v)) = 0$. Therefore $w \in N(T)$, meaning $R(T) \subseteq N(T)$.

$\Leftarrow$) Assume that $R(T) \subseteq N(T)$. That is, an element $w \in R(T)$ is in $N(T)$ meaning $T(w) = 0$. Since $w \in R(T)$, $\exists v \in V$ such that $T(v) = w$. Hence $T(T(v)) = 0$ meaning $T^2 = T_0$.

∎

2.3.16

Let V be a finite dimensional vector space and $T : V \to V$ be linear.

Part A

Proof. Assume that $\operatorname{rank} T = \operatorname{rank} T^2$. First, note that $N(T) \subseteq N(T^2)$ since for $x \in T(V)$, $T^2(x) = T(0) = 0$ hence $x \in N(T^2)$. Furthermore, $N(T) = N(T^2)$ since $\operatorname{nullity} T^2 = \dim V - \operatorname{rank} T^2 = \dim V - \operatorname{rank} T = \operatorname{nullity} T$. Let $v \in N(T) \cap R(T)$. Then $v \in R(T)$ meaning there is $u \in V$ such that $T(u) = v$. Since $v \in N(T)$ as well, $T(T(u)) = T(v) = 0$, meaning $u \in N(T^2)$ and therefore $u \in N(T)$. This means that $T(u) = 0 = v$ and therefore $v = 0$, hence $R(T) \cap N(T) = \{0\}$. ∎

Note that since $R(T) \cap N(T) = \{0\}$, it follows that $R(T) \oplus N(T)$ is well defined and that $R(T) \oplus N(T) = R(T) + N(T)$. Since $R(T) \subseteq V$ and $N(T) \subseteq V$, $R(T) + N(T) \subseteq V$. Note that $\dim(R(T)+N(T)) = \operatorname{rank} T + \operatorname{nullity} T - \dim(R(T) \cap N(t)) = \operatorname{rank} T + \operatorname{nullity} T + 0 = \dim V$. Therefore it follows that $V = R(T) \oplus N(T)$.

Part B

Proof. Note that $R(T^{k+1}) \subseteq R(T^k)$ for any k since $v \in R(T^{k+1})$ means there is some $a \in V$ such that $T^{k+1}(a) = T^k(T(a)) = v$ meaning $v \in R(T^k)$. Assume towards contradiction that there is no $k \in \mathbb{N}_0$ such that $R(T^k) \subseteq R(T^{k+1})$. Then $R(T^{k+1}) \subset R(T^k)$ meaning $\operatorname{rank} T^{k+1} < \operatorname{rank} T^k$ for all k. This gives rise to an infinite chain

$$0 \leq \ldots < \operatorname{rank} T^3 < \operatorname{rank} T^2 < \operatorname{rank} T$$

However, since $\operatorname{rank} T \leq \dim V$, it follows that there is an infinite chain of distinct powers of T between 0 and $\dim V$ which is only finite. Therefore there must be some $k \in \mathbb{N}_0$ such that $R(T^k) \subset R(T^{k+1})$ and therefore $R(T^k) = R(T^{k+1})$. Hence $\operatorname{rank} T^k = \operatorname{rank} T^{k+1}$. By using the arguement from part A and using the fact that $\operatorname{rank} T^k = \operatorname{rank} T^{k+1}$ instead of $\operatorname{rank} T = \operatorname{rank} T^2$, it will follow that $V = R(T^k) \oplus N(T^k)$. ∎