2.4.1

a) False

b) True

c) False

d) False

e) True

f) False

g) True

h) True

i) True

2.4.3

a) Not isomoprhic since their dimensions are not equal ($3 \neq 4$)

b) Yes since they are the same dimension and any vector space is isomorphic to $\mathbb{F}^n$ where n is the dimension of the vector space

c) Yes since theyre finitely dimensional with the same dimension

d) Not isomorphic since their dimensions are not equal ($2 \neq 4$)

2.4.16

Let $\Phi^{-1}(A) = BAB^{-1}$. Note then that

$$\Phi(\Phi^{-1}(A)) = B^{-1}BAB^{-1}B = A$$
$$\Phi^{-1}(\Phi(A)) = BB^{-1}AB^{-1}B = A$$

Therefore $\Phi\Phi^{-1} = \Phi^{-1}\Phi = I$. Therefore Φ is invertible and hence an isomorphism between $M_{n\times n}(\mathbb{F})$ and itself.

2.4.17

Part A

Proof. Since T is an isomorphism, it is linear. Let $y_1, y_2 \in T(V_0)$ where $y_1 = T(x_1)$ and $y_2 = T(x_2)$. Then $y_1 + y_2 = T(x_1) + T(x_2) = T(x_1 + x_2) \in T(V_0)$ since $x_1, x_2 \in V_0$. Additionally, $cy_1 = cT(x_1) = T(cx_1) \in T(V_0)$ by linearity of T. $0_W \in T(V_0)$ since V_0 is a subspace and hence $0_V \in V_0$ and $T(0_V) = 0_W$. Therefore $T(V_0)$ is a subspace of W. ■

Part B

Proof. Let $T' : V_0 \to W$ with $T'(x) = T(x)$. Since T is invertible, it is one-to-one and onto and consequently so is T'. Therefore nullity $T' = 0$ and

$$\operatorname{rank} T' = \dim(V_0) \implies \dim(T(V_0)) = \dim(V_0)$$

■

2.4.22

Proof. Note that

$$\begin{aligned} T(f+cg) &= ((f+cg)(c_0), \ldots (f+cg)(c_n)) \\ &= (f(c_0) + c \cdot g(c_0), \ldots, f(c_n) + c \cdot g(c_n)) \\ &= (f(c_0), \ldots, f(c_n)) + c(g(c_0), \ldots, g(c_n)) \\ &= T(f) + cT(g) \end{aligned}$$

The only functions that will map to 0 are functions that have $n+1$ zeroes, which must be the zero function. Therefore $N(T) = \{0\}$. Since $\dim P_n(\mathbb{F}) = \dim F^{n+1}$ and T is injective since $N(T) = \{0\}$, T is also onto and therefore a bijection. Hence T is invertible and therefore an isomorphism. ■

2.5.1

a) False

b) True

c) True

d) False

e) True

2.5.4

$$
\begin{aligned}
[T]_{\beta'} &= [I]_{\beta}^{\beta'}[T]_{\beta}[I]_{\beta'}^{\beta} \\
&= \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \\
&= \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \\
&= \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ -2 & -5 \end{pmatrix} \\
&= \begin{pmatrix} 8 & 13 \\ -5 & -9 \end{pmatrix}
\end{aligned}
$$

2.5.5

$$
\begin{aligned}
[T]_{\beta'} &= [I]_{\beta}^{\beta'}[T]_{\beta}[I]_{\beta'}^{\beta} \\
&= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}
\end{aligned}
$$

2.5.6

1. $[L_A]_\beta = \begin{pmatrix} 6 & 11 \\ -2 & -4 \end{pmatrix}$ and $Q = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$
2. $[L_A]_\beta = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$ and $Q = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$
3. $[L_A]_\beta = \begin{pmatrix} 2 & 2 & 2 \\ -2 & -3 & -4 \\ 1 & 1 & 2 \end{pmatrix}$ and $Q = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix}$
4. $[L_A]_\beta = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 18 \end{pmatrix}$ and $Q = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ -2 & 0 & 1 \end{pmatrix}$