3.1.1

a) True

b) False

c) True

d) False

e) True

f) False

g) True

h) False

i) True

3.1.12

Proof. Let $A \in M_{m\times n}(\mathbb{F})$. Follow the following process with i representing current row. Start with row 1 and go till out of rows.

1. If row i is all zero's, skip to the next row
2. If the row is non-empty, swap rows i and j where j is the first column that is non-empty
3. Add $-\frac{A_{ij}}{A_{ii}}$ of row i to row j
4. Go to the next row

The operations involved are of type 1 and 3. ■

3.2.1

a) False

b) False

c) True

d) True

e) False

f) True

g) True

h) True

i) True

3.2.5

Part A

$$\operatorname{rank} A = 2$$

$$\left(\begin{array}{cccc} 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right) \xrightarrow{R_2-R_1} \left(\begin{array}{cccc} 1 & 2 & 1 & 0 \\ 0 & -1 & -1 & 1 \end{array}\right) \xrightarrow{R_1+2R_2} \left(\begin{array}{cccc} 1 & 0 & -1 & 0 \\ 0 & -1 & -1 & 1 \end{array}\right) \xrightarrow{-R_2} \left(\begin{array}{cccc} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & -1 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$$

Part B

$$\operatorname{rank} A = 1 \implies \text{Not invertible}$$

Part C

$$\operatorname{rank} A = 2 \implies \text{Not invertible}$$

Part D

$$\operatorname{rank} A = 3$$

$$\left(\begin{array}{cccccc} 0 & -2 & 4 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 1 & 0 \\ 2 & 4 & -5 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2}$$

$$\left(\begin{array}{cccccc} 1 & 1 & -1 & 0 & 1 & 0 \\ 0 & -2 & 4 & 1 & 0 & 0 \\ 2 & 4 & -5 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_3-2R_2}$$

$$\left(\begin{array}{cccccc} 1 & 1 & -1 & 0 & 1 & 0 \\ 0 & -2 & 4 & 1 & 0 & 0 \\ 0 & 2 & -3 & 0 & -2 & 1 \end{array}\right) \xrightarrow{\frac{-R_2}{2}}$$

$$\left(\begin{array}{cccccc} 1 & 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & -2 & -\frac{1}{2} & 0 & 0 \\ 0 & 2 & -3 & 0 & -2 & 1 \end{array}\right) \xrightarrow{R_3-2R_2}$$

$$\left(\begin{array}{cccccc} 1 & 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & -2 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right) \xrightarrow{R_2+2R_3}$$

$$\left(\begin{array}{cccccc} 1 & 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & \frac{3}{2} & -4 & 2 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right) \xrightarrow{R_1-R_2+R_3}$$

$$\left(\begin{array}{cccccc} 1 & 0 & 0 & -\frac{1}{2} & 3 & -1 \\ 0 & 1 & 0 & \frac{3}{2} & -4 & 2 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} -\frac{1}{2} & 3 & -1 \\ \frac{3}{2} & -4 & 2 \\ 1 & -2 & 1 \end{pmatrix}$$

Part E

$$\operatorname{rank} A = 3$$

$$\left(\begin{array}{cccccc} 1 & 2 & 1 & 1 & 0 & 0 \\ -1 & 1 & 2 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_2+R_1}$$
$$\left(\begin{array}{cccccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 3 & 3 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_3-R_1}$$
$$\left(\begin{array}{cccccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 3 & 3 & 1 & 1 & 0 \\ 0 & -2 & 0 & -1 & 0 & 1 \end{array}\right) \xrightarrow{\frac{R_2}{3}}$$
$$\left(\begin{array}{cccccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & -2 & 0 & -1 & 0 & 1 \end{array}\right) \xrightarrow{R_2 \leftrightarrow R_3}$$
$$\left(\begin{array}{cccccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -2 & 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & \frac{1}{3} & \frac{1}{3} & 0 \end{array}\right) \xrightarrow{-\frac{R_2}{2}}$$
$$\left(\begin{array}{cccccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 1 & \frac{1}{3} & \frac{1}{3} & 0 \end{array}\right) \xrightarrow{R_1-2R_3}$$
$$\left(\begin{array}{cccccc} 1 & 0 & -1 & \frac{1}{3} & -\frac{2}{3} & 0 \\ 0 & 1 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 1 & \frac{1}{3} & \frac{1}{3} & 0 \end{array}\right) \xrightarrow{R_3-R_2}$$
$$\left(\begin{array}{cccccc} 1 & 0 & -1 & \frac{1}{3} & -\frac{2}{3} & 0 \\ 0 & 1 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{array}\right) \xrightarrow{R_1+R_3}$$
$$\left(\begin{array}{cccccc} 1 & 0 & 0 & \frac{1}{6} & -\frac{1}{3} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} \frac{1}{6} & -\frac{1}{3} & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{pmatrix}$$

Part F

$$\operatorname{rank} A = 2 \implies \text{Not invertible}$$

Part G

$$\operatorname{rank} A = 3$$

$$\left(\begin{array}{cccccccc} 1 & 2 & 1 & 0 & 1 & 0 & 0 & 0 \\ 2 & 5 & 5 & 1 & 0 & 1 & 0 & 0 \\ -2 & -3 & 0 & 3 & 0 & 0 & 1 & 0 \\ 3 & 4 & -2 & -3 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_2-2R_1}$$

$$\left(\begin{array}{cccccccc} 1 & 2 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 1 & -2 & 1 & 0 & 0 \\ -2 & -3 & 0 & 3 & 0 & 0 & 1 & 0 \\ 3 & 4 & -2 & -3 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_3+2R_1}$$

$$\left(\begin{array}{cccccccc} 1 & 2 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 & 2 & 0 & 1 & 0 \\ 3 & 4 & -2 & -3 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_4-3R_1}$$

$$\left(\begin{array}{cccccccc} 1 & 2 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 & 2 & 0 & 1 & 0 \\ 0 & -2 & -5 & -3 & -3 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_3-R_2}$$

$$\left(\begin{array}{cccccccc} 1 & 2 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & -1 & 2 & 4 & -1 & 1 & 0 \\ 0 & -2 & -5 & -3 & -3 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_4+2R_2}$$

$$\left(\begin{array}{cccccccc} 1 & 2 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & -1 & 2 & 4 & -1 & 1 & 0 \\ 0 & -2 & -5 & -3 & -3 & 0 & 0 & 1 \end{array}\right) \xrightarrow{-R_3}$$

$$\left(\begin{array}{cccccccc} 1 & 2 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & -4 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & -7 & 2 & 0 & 1 \end{array}\right) \xrightarrow{R_4-R_3}$$

$$\left(\begin{array}{cccccccc} 1 & 2 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & -4 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -3 & 1 & 1 & 1 \end{array}\right) \xrightarrow{R_3+2R_4}$$

$$\left(\begin{array}{cccccccc} 1 & 2 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -10 & 3 & 1 & 2 \\ 0 & 0 & 0 & 1 & -3 & 1 & 1 & 1 \end{array}\right) \xrightarrow{R_2-3R_3-R_4}$$

$$\left(\begin{array}{cccccccc} 1 & 2 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 31 & -9 & -4 & -7 \\ 0 & 0 & 1 & 0 & -10 & 3 & 1 & 2 \\ 0 & 0 & 0 & 1 & -3 & 1 & 1 & 1 \end{array}\right) \xrightarrow{R_1-2R_2-R_3}$$

$$\left(\begin{array}{cccccccc} 1 & 0 & 0 & 0 & -51 & 15 & 7 & 12 \\ 0 & 1 & 0 & 0 & 31 & -9 & -4 & -7 \\ 0 & 0 & 1 & 0 & -10 & 3 & 1 & 2 \\ 0 & 0 & 0 & 1 & -3 & 1 & 1 & 1 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} -51 & 15 & 7 & 12 \\ 31 & -9 & -4 & -7 \\ -10 & 3 & 1 & 2 \\ -3 & 1 & 1 & 1 \end{pmatrix}$$

Part H

$$\operatorname{rank} A = 3 \implies \text{Not invertible}$$

3.2.6

Let β and γ indicate the standard basis for their appropriate vector space.

Part A

$$[T]_\beta = \begin{pmatrix} -1 & 2 & 2 \\ 0 & -1 & 4 \\ 0 & 0 & -1 \end{pmatrix}$$

Since $\operatorname{rank} T = 3$ and the vector spaces are the same, T is invertible and therefore

$$[T]_\beta^{-1} = \begin{pmatrix} -1 & -2 & -10 \\ 0 & -1 & -4 \\ 0 & 0 & -1 \end{pmatrix}$$

Part B

$$[T]_\beta = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix}$$

T is not invertible since 0 is a column vector.

Part C

$$[T]_\beta = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

T is invertible, meaning

$$[T]_\beta^{-1} = \begin{pmatrix} \frac{1}{2} & -1 & -\frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 1 & \frac{3}{2} \end{pmatrix}$$

Part D

$$[T]_\beta^\gamma = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

T is invertible, meaning

$$\left([T]_\beta^\gamma\right)^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & -1 \end{pmatrix}$$

Part E

$$[T]_\beta^\gamma = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

T is invertible, meaning

$$\left([T]_\beta^\gamma\right)^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{pmatrix}$$

Part F

$$[T]_\beta^\gamma = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

T is not invertible.

3.2.14

Part A

Proof. Let $r \in R(T+U)$. Then $\exists v \in V$ such that $(T+U)(v) = r$, meaning $T(x)+U(x) = r$. Since $T(x) \in R(T)$ and $U(x) \in R(U)$, $r = T(x)+U(x) \in R(T)+R(U)$ hence $R(T+U) \subseteq R(T)+R(U)$. ■

Part B

Proof. First observe from the previous result that $R(T+U) \subseteq R(T)+R(U)$. Since both are subspaces, it follows that

$$\begin{aligned} \operatorname{rank}(T+U) &\leq \dim(R(T)+R(U)) \\ &= \operatorname{rank}(T)+\operatorname{rank}(U)-\operatorname{rank}(T\cap U) \\ &\leq \operatorname{rank}(T)+\operatorname{rank}(U) \end{aligned}$$

Therefore $\operatorname{rank}(T+U) \leq \operatorname{rank}(T)+\operatorname{rank}(U)$ ■

Part C

Proof. Let L_A and L_B be the left multiplication transforms of A and B. By the previous result,

$$\operatorname{rank}(L_A + L_B) \leq \operatorname{rank}(L_A) + \operatorname{rank}(L_B)$$

Since $\operatorname{rank} A = \operatorname{rank}(L_A)$ (and same for B), then $\operatorname{rank}(A + B) \leq \operatorname{rank} A + \operatorname{rank} B$. ■

3.2.21

Proof. Since $\operatorname{rank} A = m$, A represents a surjective linear transformation. That means that for each standard basis vector in $\mathbb{F}^m$, there is a corresponding vector $v_i \in \mathbb{F}^n$ that is mapped to it. Therefore by making a matrix B of these v_i as the columns, $AB = I_m$. ■