Theorem 0.1. Connectedness is a topological property.

Proof. Assume $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ are topological spaces and are homeomorphic, and $(Y, \mathcal{T}_Y)$ is not connected. Then by the negation of Definition 10.1, there exist two nonempty open sets U, V such that $U \cap V = \emptyset$ and $U \cup V = Y$. Then by Definition 9.7, there exists a homeomorphism $f : X \to Y$ which is bijective, and both f and f^{-1} are continuous. We want to show that $(X, \mathcal{T}_X)$ is not connected, hence examine the four conditions that $(X, \mathcal{T}_X)$ is not connected.

1. Note f is continuous and U and V are open sets. Then by Definition 9.1, their preimages, $f^{-1}(U)$ and $f^{-1}(U)$, are also open.

2. Since $U, V \neq \emptyset$, $f^{-1}(U) \neq \emptyset$ and $f^{-1}(V) \neq \emptyset$

3. Assume for the sake of contradiction that $f^{-1}(U) \cap f^{-1}(V)$ is nonempty. Then there exists an $x \in X$ such that $x \in f^{-1}(U) \cap f^{-1}(V)$. Then by Definition 3.11, $x \in f^{-1}(U)$ and $x \in f^{-1}(V)$. Because f is invertible, it is the case that $f(x) \in U$ and $f(x) \in V$. Then by Definition 3.11, $f(x) \in U \cap V$. This contradicts our previous assumption that $U \cap V = \emptyset$, so $f^{-1}(U) \cap f^{-1}(V)$ must be empty. Equivalently, $f^{-1}(U) \cap f^{-1}(V) = \emptyset$.

4. Now we will show that $f^{-1}(U) \cup f^{-1}(V)$ and X are subsets of each and therefore are equal.

   a) First we will show that $f^{-1}(U) \cup f^{-1}(V) \subseteq X$. Let $a \in f^{-1}(U) \cup f^{-1}(V)$. Then by definition 3.11, $a \in f^{-1}(U)$ or $a \in f^{-1}(V)$. Note that $f^{-1}(U)$ and $f^{-1}(V)$ are both subsets of X. Therefore in both cases, $a \in X$. Since $a \in X$ for all $a \in a \in f^{-1}(U) \cup f^{-1}(V)$, by Definition 3.4, $f^{-1}(U) \cup f^{-1}(V) \subseteq X$

   b) We will now show that $X \subseteq f^{-1}(U) \cup f^{-1}(V)$. Let $b \in X$. Then $f(b) \in Y$. Note that $Y = U \cup V$. It follows that $f(b) \in U \cup V$. By Definition 3.11, $f(b) \in U$ or $f(b) \in V$. Since f is invertible, either $b \in f^{-1}(U)$ or $b \in f^{-1}(V)$. Then by Definition 3.11, $b \in f^{-1}(U) \cup f^{-1}(V)$. Since $b \in f^{-1}(U) \cup f^{-1}(V)$ for all $b \in X$, by Definition 3.4, $X \subseteq f^{-1}(U) \cup f^{-1}(V)$.

   Note we have shown that $f^{-1}(U) \cup f^{-1}(V) \subseteq X$ and $X \subseteq f^{-1}(U) \cup f^{-1}(V)$. By the theorem of equality of sets (RQ 3), $f^{-1}(U) \cup f^{-1}(V) = X$

Note $f^{-1}(U)$ and $f^{-1}(V)$ exist, are elements of X, and $f^{-1}(U) \cap f^{-1}(V) = \emptyset$ and $f^{-1}(U) \cup f^{-1}(V) = X$. Therefore $(X, \mathcal{T}_X)$ is not connected by Definition 10.1.
So far we have shown that if $(Y, \mathcal{T}_Y)$ is not connected, any space $(X, \mathcal{T}_X)$ homeomorphic to $(Y, \mathcal{T}_Y)$ cannot be connected. Recall Definition 9.10, which states P is a topological property if whenever $(X, \mathcal{T}_X)$ does not have property P than neither do any spaces homeomorphic to $(X, \mathcal{T}_X)$. Thus, Connectedness is a topological property. ■