17.9

Part A

Proof. Let $f : \mathbb{R} \to \mathbb{R} : x \mapsto x^2$ and $x_0 = 2$. We want to show that for all $\epsilon > 0$ that $\exists \delta > 0$ such that $|x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon$. Note that

$$|f(x) - f(x_0)| = |x^2 - 4| = |x + 2||x - 2|.$$

If we let $\delta < 1$, then $|x - x_0| < \delta < 1$ meaning $x_0 - 1 < x < x_0 + 1 \implies |x + 2| < x_0 + 3$. Therefore

$$|x + 2||x - 2| < |x - 2| \cdot (x_0 + 3) < \epsilon \implies |x - 2| < \frac{\epsilon}{x_0 + 3}.$$

So then formally, take $\epsilon > 0$ and let $\delta = \min\left\{1, \frac{\epsilon}{x_0+3}\right\}$. Then

$$|x - x_0| = |x - 2| < \delta \implies |f(x) - f(x_0)| = |x^2 - 4| < \epsilon.$$

Hence f is continuous at $x_0 = 2$. ■

Part B

Proof. Note that

$$|f(x) - f(0)| = |\sqrt{x}| = \sqrt{x} < \epsilon \implies x < \epsilon^2 \implies |x| < \epsilon^2.$$

Therefore take $\epsilon > 0$ and let $\delta = \epsilon^2$. Then

$$|x - 0| = |x| < \delta \implies |x| < \epsilon^2 \implies x < \epsilon^2 \implies |\sqrt{x}| < \epsilon.$$

Hence f is continuous at $x_0 = 0$. ■

Part C

Proof. Take $\epsilon > 0$ and let $\delta = \epsilon$. Note then that

$$|x| < \delta \implies |x| < \epsilon.$$

Since $|\sin x| \le 1$ for all $x \in \mathbb{R}$, $|\sin \frac{1}{x}| \le 1$ for $x \neq 0$ and therefore

$$|x||\sin \frac{1}{x}| \le |x| < \epsilon.$$

Therefore

$$|x \sin \frac{1}{x}| < \epsilon.$$

Hence f is continuous at $x = 0$. ■

Part D

Proof. Take $\epsilon > 0$. Let $\delta = \min\left\{1, \frac{\varepsilon}{3|x_0|^2+3|x_0|+1}\right\}$. Assume that $|x - x_0| < \delta$. Then

$$|x - x_0| < \delta \implies |x - x_0| < \frac{\varepsilon}{3|x_0|^2 + 3|x_0| + 1}$$
$$|x - x_0|(3|x_0|^2 + 3|x_0| + 1) < \varepsilon$$
$$|x - x_0|((1 + |x_0|)^2 + |x_0|(1 + |x_0|) + |x_0|^2) < \epsilon$$

Since $|x - x_0| < 1$, $|x| = |x - x_0 + x_0| \le 1 + |x_0|$. Therefore

$$|x - x_0|(|x|^2 + |x_0 x| + |x_0^2|) < \varepsilon$$
$$|x^3 - x_0^3| < \epsilon$$

Therefore $|x - x_0| < \delta \implies |x^3 - x_0^3| < \epsilon$, meaning $f(x) = x^3$ is continuous over all of $\mathbb{R}$. ■

17.14

Proof. First show that f is discontinuous at any point in $\mathbb{Q}$. Let $x \in \mathbb{Q}$ with $x = \frac{p}{q}$. Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Construct the sequence

$$(x_n)_{n\in\mathbb{N}}, x_n = x + \frac{\alpha}{n}, \forall n.$$

Note that $\lim x_n = x$ and $x_n \notin \mathbb{Q}$ for all n. Assume towards contradiction that f is continuous at any point in $\mathbb{Q}$. Then since $x \in \mathbb{Q}$ and (x_n) is a sequence that converges to x, $\lim f(x_n) = f(x) = \frac{1}{q} > 0$. However, since $x_n \notin \mathbb{Q}$ for all n, $f(x_n) = 0$ for all n. However, this means that $\lim f(x_n) = 0$ which contradicts the assumption. Therefore f is not continuous at points in $\mathbb{Q}$. Let $x_0 \in \mathbb{R} \setminus \mathbb{Q}$. Then for every $q \in \mathbb{N}$, there exists some δ_q such that $|x_0 - y| \ge \delta_q$ for all $y \in \left\{\frac{p}{q} : p \in \mathbb{Z}\right\}$. Take $\epsilon > 0$ and choose $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$. Let $\delta = \min\{\delta_1, \ldots, \delta_N\}$. Then

$$|x - x_0| < \delta \implies x = \frac{p}{q'}, q' > N \implies |f(x) - f(x_0)| = |f(x)| < \frac{1}{N} < \epsilon.$$

Therefore f is continuous on all points in $\mathbb{R} \setminus \mathbb{Q}$. ■

18.1

Proof. Let f be the function outlined. Assume that $-f$ assumes its maximum at $x_0 \in [a,b]$. That is,

$$-f(x) \leq -f(x_0), \forall x \in [a,b] \implies f(x) \geq f(x_0), \forall x \in [a,b].$$

Therefore x_0 is the minimum of f. ■

18.2

The proof breaks down when the interval is open because the sequence (x_n) that is constructed can converge to some x_0 that is the boundary point of the interval, which if open are not a part of the functions domain of continuity.

18.4

Proof. Let $S \subset \mathbb{R}$ and assume that there is a sequence (x_n) in S that converges to a number x_0 not in S. Let $f : S \to \mathbb{R} : x \mapsto \frac{1}{x-x_0}$. Note that f is continuous on S since $x_0 \notin S$ and is unbounded. ■

18.6

Proof. Let $g : [0, \frac{\pi}{2}] \to \mathbb{R} : x \mapsto x - \cos(x)$. Note that

$$g(0) = -1 < 0, g(\frac{\pi}{2}) = \frac{\pi}{2} > 0.$$

Therefore by the IVT, there exists some $x_0 \in (0, \frac{\pi}{2})$ such that $g(x_0) = 0$ and therefore $x_0 - \cos x_0 = 0 \implies x_0 = \cos x_0$. ■

18.9

Proof. Let f be a polynomial of odd degree n. WLOG let the sign of the highest order odd term be positive. Note then that $\lim_{x\to\infty} f(x) = +\infty$ and $\lim x \to -\infty f(x) = -\infty$ since the highest order term is odd. This means that there is some $x_1, x_2 \in \mathbb{R}$ such that $f(x_1) < 0$ and $f(x_2) > 0$. By the intermediate value theorem, there must be some $x \in (x_1, x_2)$ such that $f(x) = 0$. Therefore f has at least a single real root. ■