Problem 2.3.3

Suppose that $P(x), Q(y)$ and $R(x, y, z)$ are propositional functions. Compute the negation of the following quantified propositions:

(a) $\forall x, \exists y, P(x) \wedge Q(y)$

(b) $\forall x, \exists y, \forall z, R(x, y, z)$

Solution

(a) $\exists x, \forall y, \neg P(x) \vee \neg Q(y)$

(b) $\exists x, \forall y, \exists z, \neg R(x, y, z)$

Problem 2.3.10

Consider the propositional function $P(x, y, z) : (x-3)^2 + (y-2)^2 + (z-7)^2 > 0$ where the domain of each of the variables x, y and z is $\mathbb{R}$.

(a) Express the quantified statement $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, \forall z \in \mathbb{R}, P(x, y, z)$ in words.

(b) Is the quantified statement in (a) true or false? Explain.

(c) Express the negation of the quantified statement in (a) in symbols.

(d) Express the negation of the quantified statement in (a) in words.

(e) Is the negation of the quantified statement in (a) true or false? Explain.

Solution

(a) For all real numbers x, y and z, $(x-3)^2 + (y-2)^2 + (z-7)^2$ is strictly greater than 0.

(b) The quantified statement is false. Consider the case where $x = 3, y = 2, z = 7$. Therefore $(x-3)^2 + (y-2)^2 + (z-7)^2 \implies 0 > 0$ which is false.

(c) $\exists x \in \mathbb{R}, \exists y \in \mathbb{R}, \exists z \in \mathbb{R}, \neg P(x, y, z)$

(d) There exists 3 real numbers x, y, z such that $(x-3)^2 + (y-2)^2 + (z-7)^2$ is less than or equal to 0.

(e) The negation of the quantified statement in a is true. Consider the same case as in (b). That is, $x = 3, y = 2, z = 7$. Therefore $(x-3)^2 + (y-2)^2 + (z-7)^2 \implies 0 \leq 0$ which is true.

Problem 2.3.11

The following statements are about positive real numbers. Which one is true? Explain your answer.

(a) $\forall x, \exists y$ such that $xy < y^2$.

(b) $\exists x$ such that $\forall y, xy < y^2$.

Solution

A is true since it can be simplified to for any positive real number x there exists a positive real number y such that $x < y$. Since the positive real numbers are unbounded, for any real number there is another larger real number. Therefore for every positive real number x, there exists a larger real number y $\implies x < y$.

Problem 2.3.16

You are given the following definition (you do not have to know what is meant by a field).

Let x be an element of a field $\mathbb{F}$. An inverse of x is an element y in $\mathbb{F}$ such that $xy = 1$.

Consider the following proposition:

All non-zero elements in a field have an inverse.

(a) Restate the proposition using both of the quantifiers $\forall$ and $\exists$.

(b) Find the negation of the proposition, again using quantifiers.

Solution

(a) $\forall x \neq 0, \exists y$ such that $xy = 1$.

(b) $\exists x \neq 0, \forall y$ we have $xy \neq 1$

Problem 2.3.19

Recall from calculus the definitions of the limit of a sequence $(x_n) = (x_1, x_2, x_3, \ldots)$.

'x_n diverges to ∞' means: $\forall M > 0, \exists N \in \mathbb{N}$ such that $n > N \implies x_n > M$.
'x_n converges to L' means: $\forall \epsilon > 0, \exists N \in \mathbb{N}$ such that $n > N \implies | x_n - L | < \epsilon$.

Here we assume that all elements of (x_n) are real numbers.

- State what it means for a sequence x_n not to converge at all.

Solution

In symbols: $\forall L, \exists \epsilon > 0, \forall N \in \mathbb{N}, (n > N) \wedge (| x_n - L | \geq \epsilon)$. Or in words, there exists an $\epsilon > 0$ such that for all natural numbers N, there exists a natural number n larger than N with $| x_n - L | \geq \epsilon$.

Problem 3.1.13

If $a \mid b$ and $b \mid c$, prove that $a \mid c$.

Solution

Proof. Let $a, b, c \in \mathbb{Z}$. Assume that $a \mid b$ and $b \mid c$. By definition a and b are non zero and there exists $m, n \in \mathbb{Z}$ such that $b = ma$ and $c = nb$. It follows that $b = \frac{c}{n}$. Therefore

$$\begin{aligned} b &= ma \\ \frac{c}{n} &= ma \\ c &= nma \end{aligned}$$

Since $nm \in \mathbb{Z}$, by definition $a \mid c$. ■

Problem 3.1.15

Here are two conjectures. Decide whether each conjecture is true or false and prove/dis-

$$\begin{aligned} \textit{Conjecture 1}: \quad & a \mid b \text{ and } a \mid c \implies a \mid bc \\ \textit{Conjecture 2}: \quad & a \mid c \text{ and } b \mid c \implies ab \mid c \end{aligned}$$

prove your assertions.

Solution

Proof that conjecture 1 is true.

Proof. Let $a, b, c \in \mathbb{Z}$. Assume that $a \mid b$ and $a \mid c$. By definition there exists $m, n \in \mathbb{Z}$ such that $b = ma$ and $c = na$. It follows then

$$\begin{aligned} bc &= (ma)(na) \\ &= (mna)a \end{aligned}$$

> Since $mna \in \mathbb{Z}$, then by definition $a \mid bc$. ∎

Proof that conjecture 2 is false by counterexample.

> ***Proof.*** Let $a = 3, b = 6$ and $c = 12$. It is true that $a \mid c \iff 3 \mid 12$ and that $b \mid c \iff 6 \mid 12$. However, it is not true that $ab \mid c$ since $ab = 18 \implies 18 \nmid 12$. ∎