Extending the Naturals

1.1 Rational Numbers

Definition 1.1. The rational numbers is the set of numbers of the form $\frac{p}{q}$ where $p \in \mathbb{Z}$ and $q \in \mathbb{N}$.

Rational numbers are the first number system that provides a nice comprehensive structure. Multiplication, division, addition, and subtraction are all closed operations making it a strong number system.

Theorem 1.1 (Rational Root Theorem). Let $c_0, c_1, \ldots, c_n \in \mathbb{Z}$. If r solves $c_n x^n + c_{n-1}x^{n-1} + \ldots + c_1 x + c_0 = 0$ and $r = \frac{p}{q}$ where p and q are coprime

$$p | c_0, \quad q | c_n$$

Proof. Let r be a rational solution to the polynomial equation $c_n x^n + c_{n-1}x^{n-1} + \ldots + c_1 x + c_0 = 0$. Since $r \in \mathbb{Q}$, $r = \frac{p}{q}$ where $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. Then

$$\begin{aligned} c_n\left(\frac{p}{q}\right)^n + c_{n-1}\left(\frac{p}{q}\right)^{n-1} + \ldots + c_1\left(\frac{p}{q}\right) + c_0 &= 0 \\ c_n p^n + c_{n-1}qp^{n-1} + \ldots + c_1 q^{n-1}p + c_0 q^n &= 0 \\ -c_n p^n - c_{n-1}qp^{n-1} - \ldots - c_1 q^{n-1}p &= c_0 q^n \\ -p\left[c_n p^{n-1} - c_{n-1}qp^{n-2} - \ldots - c_1 q^{n-1}\right] &= c_0 q^n \end{aligned}$$

Therefore $p | c_0 q^n$. Since p and q are coprime, p must divide c_0. By solving for $c_n p^n$ instead, it follows that q divides c_n. ■

While rationals are quite nice, there are many equations that have solutions that cannot be represented by a rational number.

Example 1.1 ($\sqrt{2}$). Consider the equation $x^2 - 2$. Its solutions by the Rational Root Theorem must be an integer. However no integer satisfies the equation and therefore there is no rational root for $x^2 - 2$.

1.2 Algebraic Numbers

Definition 1.2 (Algebraic Number). A number is called algebraic if it is the root of an integer coeffecient polynomial. That is, it is a solution to

$$c_n x^n + c_{n-1}x^{n-1} + \ldots + c_1 x + c_0 = 0$$

where $c_i \in \mathbb{Z}, c_i \neq 0$ and $n \geq 1$.

Many numbers that are used day to day are algebraic. It follows clearly that all integers are algebraic and all rationals are algebraic. Other numbers such as the $\sqrt{2}$ are algebraic. Even the number $\sqrt{2+\sqrt[3]{5}}$ is algebraic. However, there are infinitely many other numbers that are not algebraic such as π and e.

Real Numbers

As seen above, both the rationals and algebraic numbers can be very useful but fail to encapsulate important types of numbers. That is, both $\mathbb{Q}$ and the algebraic numbers have gaps in them, that is the irrationals for $\mathbb{Q}$ and transcendtals for algebraic numbers.

1.2.1 Ordering Structure

Definition 1.3 (Ordered Field). We say a field with a relation $(\mathbb{F}, +, \cdot, \leq)$ is an ordered field if it satisfies the following properties:

1. $p \leq q$ or $q \leq p$ for all $p, q \in \mathbb{F}$
2. $p \leq q$ and $q \leq p \implies p = q$
3. $p \leq q$ and $q \leq r \implies p \leq r$
4. $p \leq q \implies p + r \leq q + r$
5. $p \leq q \implies pr \leq qr$ for all $r \in \mathbb{F} \geq 0$

Certain properties are derivable from the properties and ordering of $\mathbb{R}$.

Theorem 1.2 (Properties of $\mathbb{R}$). For all $p, q, r \in \mathbb{R}$

1. $p + r = q + r \implies p = q$
2. $p \cdot 0 = 0 = 0 \cdot p$
3. $(-p)q = -(pq)$
4. $(-p)(-q) = pq$
5. $pr = qr \implies p = q$ if $r \neq 0$
6. $pq = 0 \implies p = 0$ or $q = 0$