11.1

$$\mathbb{Z}_2 \times \mathbb{Z}_4 = \left\{ \begin{array}{c} (0,0),(0,1),(0,2),(0,3) \\ (1,0),(1,1),(1,2),(1,3) \end{array} \right\}.$$

$$\begin{array}{ll} |(0,0)| = 1 & |(1,0)| = 2 \\ |(0,1)| = 4 & |(1,1)| = 4 \\ |(0,2)| = 2 & |(1,2)| = 2 \\ |(0,3)| = 4 & |(1,3)| = 4 \end{array}.$$

11.2

$$\mathbb{Z}_3 \times \mathbb{Z}_4 = \left\{ \begin{array}{c} (0,0),(0,1),(0,2),(0,3) \\ (1,0),(1,1),(1,2),(1,3) \\ (2,0),(2,1),(2,2),(2,3) \end{array} \right\}.$$

$$\begin{array}{lll} |(0,0)| = 1 & |(1,0)| = 3 & |(2,0)| = 3 \\ |(0,1)| = 4 & |(1,1)| = 12 & |(2,1)| = 12 \\ |(0,2)| = 2 & |(1,2)| = 6 & |(2,2)| = 6 \\ |(0,3)| = 4 & |(1,3)| = 12 & |(2,3)| = 12 \end{array}.$$

11.4

$$|(2,3)| = \operatorname{lcm}(3,5) = 15.$$

11.9

$$\mathbb{Z}_2 \times \mathbb{Z}_2 = \left\{ \begin{array}{c} (0,0),(0,1) \\ (1,0),(1,1) \end{array} \right\}.$$

The proper non-trivial subgroups will be those of order 2, hence

$$\begin{gathered} \{(0,0),(1,1)\} \\ \{(0,0),(1,0)\} \\ \{(0,0),(0,1)\}. \end{gathered}$$

11.16

Yes they are isomorphic since

$$\mathbb{Z}_2 \times \mathbb{Z}_{12} \simeq \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4 \simeq \mathbb{Z}_4 \times \mathbb{Z}_6.$$

11.20

Yes they are isomorphic since

$$\mathbb{Z}_4 \times \mathbb{Z}_{18} \times \mathbb{Z}_{15} \simeq \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4 \times \mathbb{Z}_5 \times \mathbb{Z}_9 \simeq \mathbb{Z}_3 \times \mathbb{Z}_{36} \times \mathbb{Z}_{10}.$$

11.24

Note that $720 = 2^4 \cdot 3^2 \cdot 5$. From the table in 11.29, there are 5 finite abelian groups of order 2^4, 2 finite abelian groups of order 3^2, and there is one finite abelian group of order 5^1. Therefore there are $1 \cdot 2 \cdot 5 = 10$ finite abelian groups of order 720.

11.29

Part A

n	# of Groups
2	2
3	3
4	5
5	7
6	11
7	15
8	22

Part B

$$\begin{aligned} p^3q^4r^7 &\implies 3 \cdot 5 \cdot 15 = 225 \\ q^7r^7 &\implies 15^2 = 225 \\ q^8r^4 &\implies 22 \cdot 5 = 110. \end{aligned}$$

11.46

> ***Proof.*** Let $G_1, G_2, \ldots G_n$ be a collection of abelian groups. Consider $G_1 \times G_2 \times \ldots \times G_n$. Let a, b be elements of the direct product and $*_n$ denote the binary operation of the nth group in the collection.
>
> $$\begin{aligned} ab &= (a_1, a_2, \ldots, a_n)(b_1, b_2, \ldots, b_n) \\ &= (a_1 *_1 b_1, a_2 *_2 b_2, \ldots, a_n *_n b_n) \\ &= (b_1 *_1 a_1, b_2 *_2 a_2, \ldots, b_n *_n a_n) \\ &= (b_1, b_2, \ldots, b_n)(a_1, a_2, \ldots, a_n) = ba \end{aligned}$$
>
> Therefore the direct product of abelian groups is abelian. ■

11.54

Proof. Let G, H, K be finitely generated abelian groups. By the Fundamental Theorem of Finitely Generated Abelian Groups, each group will have a unqiue decomposition. Since $G \times K \simeq H \times K$, the decomposition of $G \times K$ and $H \times K$ must be the same. The decomposition of both can be written by placing the decomposition of K at the end. Considering then the decomposition excluding K's decomposition leaves behind that the decomposition of G and H are isomorphic. Therefore G and H are isomorphic since their decompositions are isomorphic. ■