Problem 1

a) True. Since all entries are real, $A^T = A^*$ and hence $AA^* = AA^T = A^TA = A^*A$. Therefore real symmetric matrices are normal.

b) False. Consider

$$A = \begin{pmatrix} 0 & i \\ i & 1 \end{pmatrix}.$$

A is symmetric but

$$AA^* = \begin{pmatrix} 1 & i \\ -i & 2 \end{pmatrix} \neq \begin{pmatrix} 1 & -i \\ i & 2 \end{pmatrix} = A^*A.$$

c) True. All entries being real means $A^T = A^*$ and hence $A = A^T = A^*$ meaning A is self adjoint.

d) False. Take the counterexample A from (b). A^* will have $-i$ instead of i meaning $A \neq A^*$.

Problem 2

Proof. We prove each direction individually

$\Rightarrow$) Suppose T_1T_2 is self adjoint. Then clearly $T_1T_2 = (T_1T_2)^* = T_2^*T_1^* = T_2T_1$. Hence they commute

$\Leftarrow$) Suppose T_1 and T_2 commute. Then $(T_1T_2)^* = T_2^*T_1^*$. Since T_1 and T_2 are self adjoint, then $(T_1T_2)^* = T_2T_1$. But they also commute giving $(T_1T_2)^* = T_1T_2$.

Hence T_1T_2 is self adjoint if and only if T_1 and T_2 commute. ■

Problem 3

The statement is true.

Proof. Suppose $\langle Tx, x\rangle = 0$ for all $x \in V$. Let $u, v \in V$ and $a \in \mathbb{C}$. Note then

$$\begin{aligned} 0 = \langle T(u+av), u+av\rangle &= \langle Tu + aTv, u+av\rangle \\ &= \langle Tu, u+av\rangle + \langle aTv, u+av\rangle \\ &= \langle Tu, u\rangle + \langle Tu, av\rangle + \langle aTv, u\rangle + \langle aTv, av\rangle \\ &= \overline{a}\,\langle Tu, v\rangle + a\,\langle Tv, u\rangle \end{aligned}$$

Therefore we have $\overline{a}\,\langle Tu, v\rangle + a\,\langle Tv, u\rangle = 0$ for any $a \in C$. By setting $a = 1$ and $a = i$ we then get that

$$\begin{aligned} \langle Tu, v\rangle + \langle Tv, u\rangle &= 0 \\ \langle Tu, v\rangle - \langle Tv, u\rangle &= 0 \end{aligned}$$

But these can be added to get $\langle Tu, v\rangle = 0$ for any $u, v \in V$. Therefore

$$\langle Tu, Tu\rangle = 0 \implies \|Tu\| = 0$$

for any $u \in V$, hence T must be the zero operator. ■

Problem 4

It is not the case for either (a) or (b). Consider

$$T : \mathbb{R}^2 \to \mathbb{R}^2 : (a, b) \mapsto \left(\frac{b}{\sqrt{2}}, a + \frac{b}{\sqrt{2}}\right).$$

Note that

$$\begin{aligned} \|T(1,0)\| &= \|(0,1)\| &= 1 = \|(1,0)\| \\ \|T(0,1)\| &= \left\|\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right\| &= 1 = \|(0,1)\| \end{aligned}$$

Therefore T satisfies the condition on the standard basis of $\mathbb{R}^2$ which is orthonormal. However, T is not orthogonal because

$$\langle T(1,0), T(0,1)\rangle = \frac{1}{\sqrt{2}} \neq 0 = \langle (1,0), (0,1)\rangle \,.$$

Problem 5

Proof. Let T be a normal operator on a finite complex vector space V. Then it has a matrix repsentation A that is diagonalizable. That is $A = X\Lambda X^T$ with X unitary and Λ diagonal. Note

$$p_T(t) = \det(A - tI) = \det\left(X\Lambda X^T - tXX^T\right) = \det\left(X(\Lambda - tI)X^T\right) = \det(\Lambda - tI).$$

Since Λ is just a diagonal matrix with the eigenvalues of T, if $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of T we have

$$p_T(t) = \prod_{i=1}^{n}(\lambda_i - t).$$

Therefore $p_T(\Lambda) = \mathbf{0}$ since for each $\lambda_i I - \Lambda$ term there will be a zero on the ith diagonal position and hence the product across all i terms will give the zero matrix. Note that

$$A^k = (X\Lambda X^T)^k = (X\Lambda X^T)(X\Lambda X^T)\ldots(X\Lambda X^T) = X\Lambda^k X^T$$

and hence for any polynomial $f(t) = a_0 + a_1 t + \ldots a_{n-1}t^{n-1}$

$$f(A) = X(a_0 + a_1\Lambda + \ldots + a_{n-1}\Lambda^{n-1})X^T = Xf(\Lambda)X^T.$$

But then we have

$$p_T(A) = Xp_T(\Lambda)X^T = X\mathbf{0}X^T = \mathbf{0}.$$

Therefore since T has a zero matrix representation, it must be the zero transformation. ■