8

It does not form a ring because $\langle \mathbb{Z}_+, + \rangle$ doesn't have an identity and therefore cannot be a group.

10

It does form a ring since $n\mathbb{Z}$ is a ring for $n \geq 1$ and the direct product of rings is also a ring. $n\mathbb{Z}$ is commutative for $n \geq 1$, meaning $2\mathbb{Z} \times \mathbb{Z}$ is also commutative. It does not have unity since $2\mathbb{Z}$ doesn't have unity. Since $2\mathbb{Z} \times \mathbb{Z}$ doesn't have unity, it cannot be a field.

12

Let $\mathcal{R} = \left\{a + b\sqrt{2} : a, b \in \mathbb{Q}\right\}$. First, check that the binary operations are closed. Let $a, b, c, d \in \mathbb{Q}$. Then

$$\left(a + b\sqrt{2}\right) + \left(c + d\sqrt{2}\right) = (a + c) + (b + d)\sqrt{2}$$

and

$$\left(a + b\sqrt{2}\right)\left(c + d\sqrt{2}\right) = ac + ad\sqrt{2} + bc\sqrt{2} + 2bd = (ac + 2bd) + (ad + bc)\sqrt{2}.$$

Since the resulting components are also rational numbers, both operations are closed. Next, check if $\langle \mathcal{R}, + \rangle$ is an abelian group.

$\mathcal{G}_1$) The given addition operation is associative and hence is also associative on $\mathcal{R}$

$\mathcal{G}_2$) The additive identity $0 = 0 + 0\sqrt{2}$ is in $\mathcal{R}$

$\mathcal{G}_3$) Since for any $a + b\sqrt{2}$ $-a, -b \in \mathbb{Q}$, every element has an inverse

Abelian) The given addition is commutative and therefore the group is abelian

Since multiplication of real numbers is associative and $\mathcal{R} \subset \mathbb{R}$, multiplication is also associative. The given multiplication is also commutative and satisfies the distributive laws. $\mathcal{R}$ has unity since $1 + 0\sqrt{2} \in \mathcal{R}$. Let $a, b \in \mathbb{Q}$ such that $a + b\sqrt{2} \neq 0$. Note that

$$\frac{1}{a + b\sqrt{2}} = \frac{1}{a + b\sqrt{2}} \cdot \frac{a - b\sqrt{2}}{a - b\sqrt{2}} = \frac{a - b\sqrt{2}}{a^2 - 2b^2} = \frac{a}{a - 2b^2} - \frac{b}{a^2 - 2b^2}\sqrt{2}.$$

Therefore every element is a unit. Therefore, in total $\mathcal{R}$ is a commutative division ring with unity and also a field.

18

1 and -1 are the only units of $\mathbb{Z}$, and every non zero element in $\mathbb{Q}$ is a unit meaning the units of $\mathbb{Z} \times \mathbb{Q} \times \mathbb{Z}$ are of the form $(\pm 1, q, \pm 1)$ with $q \in \mathbb{Q}^*$.

19

The units are 1 and 3 since $1 \cdot 1 = 1$ and $3 \cdot 3 = 1$

24

If $\phi : \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ is a ring homomorphism, then $\phi(1)^2 = \phi(1)\phi(1) = \phi(1^2) = \phi(1)$. Therefore $\phi(1)$ must be an element in $\mathbb{Z} \times \mathbb{Z}$ where its square is itself. Since $(0,0)$, $(1,0)$, $(0,1)$ and $(1,1)$ are the only elements with this property in $\mathbb{Z} \times \mathbb{Z}$, the possible ring homomorphisms are

$$\begin{aligned} \phi_{(0,0)}(n) &= (0,0) \\ \phi_{(1,0)}(n) &= (n,0) \\ \phi_{(0,1)}(n) &= (0,n) \\ \phi_{(1,1)}(n) &= (n,n) \end{aligned}$$

$\phi_{(0,0)}$ is trivially a ring homorphism. For $\phi_{(1,0)}$,

$$\begin{aligned} \phi_{(1,0)}(a+b) &= (a+b,0) = \phi_{(1,0)}(a) + \phi_{(1,0)}(b)) \\ \phi_{(1,0)}(ab) &= (ab \quad, 0) = \phi_{(1,0)}(a)\phi_{(1,0)}(b) \end{aligned}$$

Therefore $\phi_{(1,0)}$ is a homorphism and by a similar argument so is $\phi_{(0,1)}$. In the case of $\phi_{(1,1)}$

$$\begin{aligned} \phi_{(1,1)}(a+b) &= (a+b, a+b) = \phi_{(1,1)}(a) + \phi_{(1,1)}(b) \\ \phi_{(1,1)}(ab) &= (ab \quad, ab) \quad\ = \phi_{(1,1)}(a)\phi_{(1,1)}(b) \end{aligned}$$

25

If $\phi : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ is a ring homomorphism, then the elements $(1,0)$ and $(0,1)$ (which are squares of themselves) must map to an element in $\mathbb{Z}$ whose square is itself. Since 0 and 1 satisfy this condition, there are 4 candidate homomorphisms.

$$\begin{aligned} \phi_1((1,0)) = 1, \phi_1((0,1)) = 1 &\implies \phi_1((a,b)) = a+b \\ \phi_2((1,0)) = 1, \phi_2((0,1)) = 0 &\implies \phi_2((a,b)) = a \\ \phi_3((1,0)) = 0, \phi_3((0,1)) = 1 &\implies \phi_3((a,b)) = b \\ \phi_4((1,0)) = 0, \phi_4((0,1)) = 0 &\implies \phi_4((a,b)) = 0 \end{aligned}$$

ϕ_4 is trivially a ring homomorphism. For ϕ_2,

$$\phi_2((a,b)\cdot(c,d)) = \phi_2((ac,bd)) = ac = \phi((a,b))\phi((c,d))$$

therefore ϕ_2 is also a ring homomorphism. By the same argument, ϕ_3 is as well. However, note that

$$\phi_1((1,2)\cdot(2,3)) = \phi_1((2,6)) = 8 \neq 10 = \phi((1,2))\phi((2,3))$$

therefore ϕ_1 is not a ring homomorphism. Hence the only ring homomorphisms from $\mathbb{Z} \times \mathbb{Z}$ to $\mathbb{Z}$ are ϕ_2, ϕ_3 and ϕ_4.

37

Proof. Let $x, y \in U$. Assume towards contradiction that $xy \notin U$. Then xy is not a unit and therefore has no multiplicative inverse. However, since x and y are units, they have multiplicative inverses. Therefore $(xy)(y^{-1}x^{-1}) = 1$. However this is a contradiction meaning $xy \in U$, hence U is closed under $\cdot$. Examining the group axioms

$\mathcal{G}_1$.) Since R is a ring, the operator $\cdot$ is associative and therefore is associative over U.

$\mathcal{G}_2$.) Since R is a ring with unity, it has a multiplicative identity 1 meaning that U will have an identity (specifically 1).

$\mathcal{G}_3$.) Let $x \in U$. Since it is a unit, it has a multiplicative inverse x^{-1}. But since x is the inverse of x^{-1}, x^{-1} is also a unit and so $x^{-1} \in U$. Therefore every element in U has an inverse.

Since U is closed under $\cdot$ and satisfies the group axioms, $\langle G, \cdot \rangle$ is a group. ■

40

Take $\phi : 2\mathbb{Z} \to 3\mathbb{Z}$ to be a ring isomorphism. Since ϕ must be a group homorphism over addition, $\phi(2)$ must be equal to either 3 or -3. Therefore $\phi(2n) = \phi(3n)$ or $\phi(2n) = \phi(-3n)$. Consider

$$\phi(2 \cdot 2) = \phi(4) = \pm 6 \neq 9 = \phi(2) \cdot \phi(2).$$

Therefore ϕ cannot be a ring homomorphism and hence cannot be a ring isomorphism. $\mathbb{R}$ and $\mathbb{C}$ are not isomorphic since every element in $\mathbb{C}$ can be written as the square of another element in $\mathbb{C}$, but the same does not hold in $\mathbb{R}$.

48

Proof.

$\Rightarrow$) Assume S is a subring of R. Then the S is an abelian group under $+$ meaning it must have the additive identity 0. Furthermore, every element has an additive inverse and S is closed under $+$ meaning $(a - b) \in S$ for all $a, b \in S$. S must be closed under $\cdot$ by the assumption and therefore it follows $ab \in S$ for all $a, b \in S$.

$\Leftarrow$) Examine the condtions for S to be a subring of R

Closure) Multiplication is closed since $ab \in S$ for all $a, b \in S$. Since $0 \in S$ and $a - b \in S$, $-b \in S$ for all $b \in S$ meaning S contains its additive inverses. Therefore $a - (-b) \in S$ and hence $a + b \in S$ for all $a, b \in S$.

$\mathcal{R}_1$) Since addition from R is associative and commutative and S contains an additive identity and inverses, $\langle S, + \rangle$ is an abelian group.

$\mathcal{R}_2$) Multiplication from R is associative meaning it is associative on S

$\mathcal{R}_3$) The left and right distributive laws hold for the multiplication operator and hence they hold on S.

■

50

Proof. Since $a \cdot 0 = 0$ for any a, $0 \in I_a$. Let $x, y \in I_a$. Then $ax = 0$ and $ay = 0$. Therefore $ax - ay = 0 \implies a(x-y) = 0$ meaning $x - y \in I_a$. Furthermore, $a(xy) = (ax)y = 0y = 0$, meaning $xy \in I_a$. Therefore by Exercise 48, I_a is a subring of R. ■