Consider an ODE system of the form

$$\vec{\mathbf{x}}'(t) = \mathbf{A}\vec{\mathbf{x}}(t).$$

Where $\mathbf{A}$ is a finite constant square matrix. Assume $\mathbf{A}$ has an eigenvalue λ with an algebraic multiplicity N that is greater than its geometric multiplicity. Assume the associated vector solution has the form

$$\vec{\mathbf{x}}_\lambda(t) = \left(\vec{\mathbf{x}}_1 + \vec{\mathbf{x}}_2 t + \vec{\mathbf{x}}_3 t^2 + \ldots + \vec{\mathbf{x}}_N t^{N-1}\right)e^{\lambda t}.$$

where the number of vectors $\vec{\mathbf{x}}$ is finite. Define the sequence a_n

$$\begin{aligned} a_n &= e^{\lambda t}\vec{\mathbf{x}}_n t^{n-1} \\ \frac{\mathrm{d}}{\mathrm{d}t}(a_n) &= \lambda e^{\lambda t}\vec{\mathbf{x}}_n t^{n-1} + (n-1)e^{\lambda t}\vec{\mathbf{x}}_n t^{n-2} \end{aligned}$$

Define two new sequences b_n and c_n

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(a_n) &= \underbrace{\lambda e^{\lambda t}\vec{\mathbf{x}}_n t^{n-1}}_{b_n} + \underbrace{(n-1)e^{\lambda t}\vec{\mathbf{x}}_n t^{n-2}}_{c_n} \\ \frac{\mathrm{d}}{\mathrm{d}t}(a_n) &= b_n + c_n \end{aligned}$$

Use the ODE system constraint

$$\begin{aligned} \vec{\mathbf{x}}'(t) &= \mathbf{A}\vec{\mathbf{x}}(t) \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\vec{\mathbf{x}}(t)\right) &= \mathbf{A}\vec{\mathbf{x}}(t) \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{n=1}^{N} a_n\right) &= A\sum_{n=1}^{N} a_n \\ \sum_{n=1}^{N}\frac{\mathrm{d}}{\mathrm{d}t}(a_n) &= A\sum_{n=1}^{N} a_n \\ \sum_{n=1}^{N} b_n + c_n &= A\sum_{n=1}^{N} a_n \\ \sum_{n=1}^{N} b_n + c_n &= \sum_{n=1}^{N} A\cdot a_n \end{aligned} \tag{1}$$

Define the sequence $d_n = A \cdot a_n$

$$\sum_{n=1}^{N} b_n + c_n = \sum_{n=1}^{N} d_n$$

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Match the coefficients of t between the sequences

$$\begin{aligned} b_n + c_{n+1} = d_n \implies \lambda e^{\lambda t}\vec{\mathbf{x}}_n t^{n-1} + n e^{\lambda t}\vec{\mathbf{x}}_{n+1} t^{n-1} &= A e^{\lambda t}\vec{\mathbf{x}}_n t^{n-1} \\ \lambda e^{\lambda t}\vec{\mathbf{x}}_n + n e^{\lambda t}\vec{\mathbf{x}}_{n+1} &= A e^{\lambda t}\vec{\mathbf{x}}_n \\ \lambda\vec{\mathbf{x}}_n + n\vec{\mathbf{x}}_{n+1} &= A\vec{\mathbf{x}}_n \\ A\vec{\mathbf{x}}_n - \lambda\vec{\mathbf{x}}_n &= n\vec{\mathbf{x}}_{n+1} \end{aligned}$$

Arriving at the condition

$$\boxed{(A - \mathbf{I}\lambda)\vec{\mathbf{x}}_n = n\vec{\mathbf{x}}_{n+1}}$$

In the boundary case where $n = N$

$$\begin{aligned} b_N &= \lambda e^{\lambda t}\vec{\mathbf{x}}_N t^{N-1} \\ c_N &= (N-1) e^{\lambda t}\vec{\mathbf{x}}_N t^{N-2} \\ d_N &= A e^{\lambda t}\vec{\mathbf{x}}_N t^{N-1}. \end{aligned}$$

Matching coefficients for t^{N-1}

$$\begin{aligned} \lambda e^{\lambda t}\vec{\mathbf{x}}_N t^{N-1} &= A e^{\lambda t}\vec{\mathbf{x}}_N t^{N-1} \\ \lambda\vec{\mathbf{x}}_N &= A\vec{\mathbf{x}}_N \\ A\vec{\mathbf{x}}_N - \lambda\vec{\mathbf{x}}_N &= 0. \end{aligned}$$

Arriving at the condition

$$\boxed{(A - \mathbf{I}\lambda)\vec{\mathbf{x}}_N = 0}$$

[1]In (1), the derivative can be brought into the summation since it is a linear operator and the sum is convergent due to the initial condition where $\vec{\mathbf{x}}$ and $\mathbf{A}$ are finite