Metric Spaces and Topological Concepts

1.1 Expanding $\mathbb{R}$

Most of the focus so far has been on $\mathbb{R}$. Importantly, on $\mathbb{R}$ it was possible to define an ordering relation from which the absolute value and distance functions could arise. A natural question to ask is if this conceptual construction of distance can be constructed over different sets.

Definition 1.1 (Metric Space). Let S be a set. If there exists some mapping $d : S \times S \to \mathbb{R}$ (called a metric or distance) such that it satisfies

1. $d(x, x) = 0, \forall x \in S$ and $d(x, y) > 0, \forall x, y \in S, x \neq y$
2. $d(x, y) = d(y, x), \forall x, y \in S$
3. $d(x, z) \leq d(x, y) + d(y, z), \forall x, y, z \in S$

then (S, d) is a metric space.

Clearly $(\mathbb{R}, \text{dist})$ is a metric space. However, there are alternative metrics that still admit a metric space over $\mathbb{R}$.

Example 1.1. The following are some examples of metric spaces

a) $S = \mathbb{R}, d(x, y) = |x - y|$

b) $S = \mathbb{R}^k = \{(x_1, \ldots, x_k) : x_i \in \mathbb{R}, \forall i = 1, \ldots, k\}, d(x, y) = \sqrt{\sum_{i=1}^k (y_i - x_i)^2}$

Consider specifically the case of $\mathbb{R}^k$.

Proof. Consider the metric $d(x, y) = \sqrt{\sum_{i=1}^k (y_i - x_i)^2}$ over $\mathbb{R}^k$. Check that it satisfies the properties of being a metric.

1. The metric is zero when $y_i = x_i$ and therefore $x = y$, hence $d(x, x) = 0$ for all $x \in \mathbb{R}^k$
2. Since the summation terms are squared, the order of x_i and y_i does not matter, hence $d(x, y) = d(y, x)$ for all $x, y \in \mathbb{R}^k$
3. Firstly, an equivalence is

$$d(x, z) \leq d(x, y) + d(y, z) \Leftrightarrow d(x, z)^2 \leq d(x, y)^2 + 2d(x, y)d(y, z) + d(y, z)^2$$

By using the scalar product and its properties from vector spaces,

$$\begin{aligned} d(x, z)^2 = (x - z) \cdot (x - z) &= (x - y + y - z) \cdot (x - y + y - z) \\ &= (x - y) \cdot (x - y) + 2(x - y) \cdot (y - z) + (y - z) \cdot (y - z) \\ &= d(x, y)^2 + 2(x - y) \cdot (y - z) + d(y, z)^2 \qquad (*) \end{aligned}$$

Note that $\forall t > 0$

$$\begin{aligned} 0 &\leq ((x-y) - t(y-z)) \cdot ((x-y) - t(y-z)) \\ &= d(x,y)^2 + d(y,z)^2 t^2 - 2t(x-y)(y-z) \end{aligned}$$

Therefore by rearranging

$$(x-y) \cdot (y-z) \leq \frac{1}{2t} d(x,y)^2 + \frac{t}{2} d(y,z)^2$$

Since t was arbitrary, choosing $t = \frac{d(x,y)}{d(y,z)}$ gives

$$(x-y) \cdot (y-z) \leq d(x,y)d(y,z) \qquad \text{(Cauchy Schwarz Inequality)}$$

Going back to $(*)$,

$$\begin{aligned} d(x,z)^2 &= d(x,y)^2 + 2(x-y) \cdot (y-z) + d(y,z)^2 \\ &\leq d(x,y)^2 + 2d(x,y)d(y,z) + d(y,z)^2 \\ &= (d(x,y) + d(y,z))^2 \end{aligned}$$

and therefore by taking the root of each side,

$$d(x,z) \leq d(x,y) + d(y,z), \forall x, y, z \in \mathbb{R}^k$$

Since d satisfies all the properties of a metric, $(\mathbb{R}^k, d)$ is a metric space ■

Having a metric space provides enough machinery to define concepts like convergence.

Definition 1.2 (Metric Space Equivalents). Let $(s_n)_{n\in\mathbb{N}}$ be a sequence in (S, d) and $s \in S$. Then

1. Convergence is defined as

$$\lim_{n\to\infty} s_n = s \overset{\text{def}}{\iff} \lim_{n\to\infty} d(s_n, s) = 0$$

2. Cauchy is defined as

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } d(s_n, s_m) < \epsilon, \forall m, n > N$$

3. (S, d) is **complete** iff all Cauchy sequences converge.

The last idea of completeness is different in form than the **??**, however $(\mathbb{R}, \text{dist})$ satisfies this alternative definition of completeness (and is in fact equivalent to the **??**).

Theorem 1.1 ($\mathbb{R}^k$ is a Metric Space). $(\mathbb{R}^k, d)$ is a complete metric space.

It will be useful to show that convergence of a sequence in $\mathbb{R}^k$ can be determined by element wise sequences converging (and equivalently for determining if a sequence is Cauchy). For notation sake, the superscript refers to the index into a sequence and the subscript is the position index of the original sequence. That is a sequence $(x_n)_{n\in\mathbb{N}} \in \mathbb{R}^k$ is

$$(x_n)_{n\in\mathbb{N}} = \begin{pmatrix} x_1^n \\ \vdots \\ x_k^n \end{pmatrix}$$

Lemma 1.1 (Element Wise Implies Sequence Wise). A sequence $(x^n)_{n\in\mathbb{N}}$ in $\mathbb{R}^k$ converges iff (x_j^n) converges in $\mathbb{R}$ for $1 \le j \le k$. Additionally, $(x^n)_{n\in\mathbb{N}}$ is Cauchy iff (x_j^n) is Cauchy in $\mathbb{R}$ for $1 \le j \le k$.

Proof. ■

Proof of 1.1. Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathbb{R}^k$. Then by 1.1, $\left(x_n^j\right)$ is a Cauchy sequence in $\mathbb{R}$. Since $\mathbb{R}$ is complete, $\left(x_n^j\right)$ converges. Therefore all component sequences of (x_n) converge which by 1.1 implies the convergence of (x_n). ■

An interesting fact is that the Bolzano-Weistrass Theorem generalizes to $\mathbb{R}^k$ as long as boundedness is properly defined.

Definition 1.3 (Boundedness in $\mathbb{R}^k$). Let $S \subset \mathbb{R}^k$. S is bounded if there exists $M \in \mathbb{R}$ such that $d(0, s) \le M$ for all $s \in S$.

Theorem 1.2. Each bounded sequence in $\mathbb{R}^k$ has a convergent subsequence.

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence in $\mathbb{R}^k$. Note that $\left|\left(x_j^n\right)\right| \le d(0, x_n)$ for all $j = 1, \dots, k$ and $n \in \mathbb{N}$. Therefore each element wise sequence is bounded. Then

$$\begin{aligned}
(x_1^n) \text{ is bounded} &\implies \exists \left(n_l^1\right)_{l\in\mathbb{N}} \text{ s.t. } x_1^{n_l^1} \to x_1^\infty \\
\left(x_2^{n_l^1}\right) \text{ is bounded} &\implies \exists \left(n_l^2\right)_{l\in\mathbb{N}} \subset \left(n_l^1\right)_{l\in\mathbb{N}} \text{ s.t. } x_2^{n_l^2} \to x_2^\infty \\
&\vdots \\
\left(x_k^{n_l^{k-1}}\right) \text{ is bounded} &\implies \exists \left(n_l^k\right)_{l\in\mathbb{N}} \subset \left(n_l^{k-1}\right)_{l\in\mathbb{N}} \text{ s.t. } x_k^{n_l^k} \to x_k^\infty
\end{aligned}$$

Therefore a convergent subsequence of (x_n) can be constructed. ■

Definition 1.4 (Openness). Let (S, d) be a metric space and $E \subset S$. Then

1. $s_0 \in E$ is an interior of E iff $\{s \in S : d(s, s_0) < r\} \subset E$ for some $r > 0$.
2. $\mathring{E} = \{s \in E : s \text{ is an interior point }\}$
3. If $E = \mathring{E}$, then E is open

Remark. The following are important properties of openness

1. S is open in S
2. $\varnothing$ is open in S
3. E_α is open $\forall \alpha \in A$, then $\bigcup_{\alpha \in A} E_\alpha$ is open
4. E_j is open $\forall j = 1, \ldots, n$ then $\bigcap_{j=1}^{n} E_j$ is open

Definition 1.5. Let (S, d) be a metric space. Then

1. $E \subset S$ is closed if $E^c = S \setminus E$ is open
2. $\bar{E} = \displaystyle\bigcap_{\substack{E \subset F \subset S \\ F \text{ closed}}} F$ is the closure of E
3. $\partial E = E \setminus \mathring{E}$ is the boundary of E

Remark. $\bar{E}$ is a closed set and is the smallest closed set that contains E.

Example 1.2. The following are examples of openness and boundaries

1. (a, b) is open and $[a, b]$ is closed in $\mathbb{R}$
2. $(a, b]$ and $[a, b)$ are neither open nor closed
3. With $I = \{(a, b), [a, b], [a, b), (a, b]\}$
   (a) $\bar{I} = [a, b]$
   (b) $\mathring{I} = (a, b)$
   (c) $\partial I = \{a, b\}$
4. Let $x \in \mathbb{R}^k$ and $r > 0$. Let $\mathbb{B}(x, r) = \left\{y \in \mathbb{R}^k : d(x, y) < r\right\}$
   (a) $\mathbb{B}(x, r)$ is open
   (b) $\bar{\mathbb{B}}(x, r)$ is closed
   (c) $\partial \mathbb{B}(x, r) = \left\{y \in \mathbb{R}^k : d(x, y) = r\right\}$

Theorem 1.3. Let (S, d) be a metric space and $E \subset S$. Then

1. If E is closed, $E = \bar{E}$
2. E is closed iff E contains the limit of every convergent sequence in E
3. $x \in \bar{E}$ iff there exists a sequence $(x_n)_{n \in \mathbb{N}}$ in E that converges to x
4. $x \in \partial E$ iff $x \in \bar{E} \cap \overline{S \setminus E}$