Relations

Relations will serve useful in concretizing the idea of functions and in general how elements of sets are related to each other.

Definition 1 (Relation)**.** A relation $\mathcal{R}$ on a set A is defined as $\mathcal{R} \subseteq A \times A$ with 3 possible properties

Reflexive	$\forall a \in A, (a, a) \in \mathcal{R}$
Symmetric	$\forall a, b \in A, (a, b) \in \mathcal{R} \implies (b, a) \in \mathcal{R}$
Transitive	$\forall a, b, c \in A, (a, b), (b, c) \in \mathcal{R} \implies (a, c) \in \mathcal{R}.$

Consider the relation $\mathcal{R}$ defined as $(\leq, \mathbb{R})$. Which properties of a relation does it satisfy?

Proof. Let ■

If a relation $\mathcal{R}$ obeys all 3 possible properties of a relation, it is called an **Equivalence Relation** often denoted by a $\sim$. Let $\mathcal{R}$ be the relation $\sim$ on $\mathbb{Z}$ such that

$$x \sim y \iff x - y \text{ is even.}$$

Is $\mathcal{R}$ an equivalence relation?

Proof. Proceed to show that $\mathcal{R}$ is an equivalence relation.

(Reflexivity) Let $a \in \mathbb{Z}$. It follows that

$$\begin{aligned} a \sim a &\implies 2|a - a \\ &\implies 2|0 \end{aligned}$$

which is true. Therefore $\mathcal{R}$ is reflexive.

(Symmetry) Let $a, b \in \mathbb{Z}$. It follows that

$$\begin{aligned} a \sim b &\implies a - b = 2k \\ &\implies b - a = 2(-k) \\ &\implies b \sim a \end{aligned}$$ ■

hence $\mathcal{R}$ is symmetric.