Problem 1

Show that for any given integers a, b, c, if a is even and b is odd, then $7a-ab+12c+b^2+4$ is odd.

Solution

A direct proof that for any given integers a, b, c, if a is even and b is odd, then $7a-ab+12c+b^2+4$ is odd.

> ***Proof.*** Let $a, b, c \in \mathbb{Z}$. Suppose a is an even integer and b is an odd integer. There exists $m, k \in \mathbb{Z}$ such that $a = 2m$ and $b = 2k+1$. Then
>
> $$\begin{aligned} 7a - ab + 12c + b^2 + 4 &= 7(2m) - (2m)(2k+1) + 12c + (2k+1)^2 + 4 \\ &= 14m - 4mk - 2m + 12c + 4k^2 + 4k + 1 + 4 \\ &= 2(6m - 2mk + 6c + 2k^2 + 2k + 2) + 1. \end{aligned}$$
>
> is by definition an odd integer because $6m - 2mk + 6c + 2k^2 + 2k + 2 \in \mathbb{Z}$. ∎

Problem 3

Prove or disprove the following conjectures:

(a) The sum of any 3 consecutive integers is divisible by 3.

(b) The sum of any 4 consecutive integers is divisible by 4.

Solution

Part A

A direct proof that the sum of any 3 consecutive integers is divisible by 3.

> ***Proof.*** Let $a, b, c \in \mathbb{Z}$. Suppose that a, b and c are consecutive. Without loss of generality they can be expressed as $a = a$, $b = a+1$, and $c = a+2$ by the definition consecutive integers. It then follows
>
> $$\begin{aligned} a + b + c &= (a) + (a+1) + (a+2) \\ &= 3a + 3 \\ &= 3(a+1). \end{aligned}$$
>
> is a multiple of 3. ∎

Part B

A direct proof that the sum of any 4 consecutive integers is not divisible by 4, disproving the second conjecture.

> ***Proof.*** Let $a, b, c, d \in \mathbb{Z}$. Suppose that a, b, c and d are consecutive. Without loss of generality they can be written as $a = a$, $b = a + 1$, $c = a + 2$, and $c = a + 3$. Then
>
> $$\begin{aligned} a + b + c + d &= (a) + (a + 1) + (a + 2) + (a + 3) \\ &= 4a + 6. \end{aligned}$$
>
> is not a multiple of 4. ∎

Problem 5

Prove that if n is a natural number greater than 1, then $n! + 2$ is even.

Solution

A direct proof that if n is a natural number greater than 1, then $n! + 2$ is even.

> ***Proof.*** Let $n \in \mathbb{N}$ such that $n > 1$. By definition of the factorial, $n! = n \cdot (n - 1) \cdot (n - 2) \ldots 3 \cdot 2 \cdot 1$. Then
>
> $$\begin{aligned} n! + 2 &= n \cdot (n - 1) \cdot (n - 2) \ldots 3 \cdot 2 \cdot 1 + 2 \\ &= 2(n \cdot (n - 1) \cdot (n - 2) \ldots 3 \cdot 1 + 1) \end{aligned}$$
>
> Since $n \cdot (n - 1) \cdot (n - 2) \ldots 3 \cdot 1 + 1$ is an integer, $n! + 2$ is an even number. ∎

Problem 7

(a) Let $x \in \mathbb{Z}$. Prove that $5x + 3$ is even if and only if $7x - 2$ is odd.

(b) Can you conclude anything about $7x - 2$ if $5x + 3$ is odd?

Solution

Part A

A direct proof of both directions.

> ***Proof.*** Let $x \in \mathbb{Z}$. Suppose that $5x + 3$ is even. By definition there exists $k \in \mathbb{Z}$

> such that $5x + 3 = 2k$. It follows that
>
> $$\begin{aligned} 5x + 3 &= 2k \\ &\Updownarrow \\ 5x - 2 &= 2k - 5 \\ &\Updownarrow \\ 7x - 2 &= 2k + 2x - 5 \\ &= 2k + 2x - 6 + 1 \\ &= 2(k + x - 3) + 1 \end{aligned}$$
>
> Since $k + x - 3 \in \mathbb{Z}$, $7x - 2$ is by definition an odd integer. ■

Part B

Yes. One can conclude that if $5x+3$ is odd then $7x-2$ is even. Consider the backwards direction of Part A. That is: "If $7x - 2$ is odd, then $5x + 3$ is even". This is a true statement as established in Part A. Therefore its contrapositive is also true. Therefore the statement: "If $5x + 3$ is odd, then $7x - 2$ is even" is true.

Problem 10

Definition 1. A real number x is rational if it may be written in the form $x = \frac{p}{q}$ where p is an integer and q is a positive integer. x is irrational if it is not rational.

Prove or disprove the following conjecture.

> **Conjecture 1.** If x and y are real numbers such that $3x + 5y$ is irrational, then at least one of x and y is irrational.

Solution

A proof by contrapositive that if x and y are real numbers such that $3x+5y$ is irrational, then at least one of x and y is irrational.

> ***Proof.*** Let $x, y \in \mathbb{R}$. Suppose both are rational. Therefore both can be written in the form $x = \frac{p}{q}$ and $y = \frac{m}{n}$ where p, q, m, and n are integers with q and n being

positive. Then

$$\begin{aligned} 3x + 5y &= 3\left(\frac{p}{q}\right) + 5\left(\frac{m}{n}\right) \\ &= \frac{3p}{q} + \frac{5m}{n} \\ &= \frac{3pn + 5mq}{qn} \end{aligned}$$

is by definition a rational number since the top is an integer and the bottom is a product of positive integers and therefore also a positive integer. This proves the contrapositive and therefore the original proposition. ∎

Problem 11

Let x and y be integers. Prove: For $x^2 + y^2$ to be even, it is necessary that x and y have the same parity (i.e. both even or both odd).

Solution

Proof by contrapositive that if $x^2 + y^2$ is even then x and y have the same parity.

Proof. Suppose there are two integers x and y with different parity. That is, one of x or y is even with the other being odd. Without loss of generality, assume that x is an even integer and y is an odd integer. Therefore there exists integers m and n such that $x = 2m$ and $y = 2n + 1$. Then

$$\begin{aligned} x^2 + y^2 &= (2m)^2 + (2n+1)^2 \\ &= 4m^2 + 4n^2 + 4n + 1 \\ &= 2(2m^2 + 2n^2 + 2n) + 1. \end{aligned}$$

is an odd integer. This proves the contrapositive and therefore the original proposition. ∎

Problem 12

Prove that if x and y are positive real numbers, then $\sqrt{x+y} \neq \sqrt{x} + \sqrt{y}$. *Argue by contradiction.*

Solution

Proof by contradiction that if x and y are positive real numbers, then $\sqrt{x+y} \neq \sqrt{x} + \sqrt{y}$.

Proof. Let x and y be real numbers. Assume towards contradiction that x and y are positive and that $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$. It follows that

$$\begin{aligned}
\sqrt{x+y} &= \sqrt{x} + \sqrt{y} \\
\left(\sqrt{x+y}\right)^2 &= \left(\sqrt{x} + \sqrt{y}\right)^2 \\
x + y &= x + \sqrt{xy} + y \\
\sqrt{xy} &= 0 \\
xy &= 0.
\end{aligned}$$

which implies that either x or y are 0. However since it was assumed both x and y are positive, they are both stricly greater than 0 and hence a contradiction. ■