0.1 More on Power Series

Consider the geometric power series

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}.$$

The equivalent power series then for its square would be

$$\left(\frac{1}{1-x}\right)^2 = \sum_{k=0}^{\infty} x^k \cdot \sum_{k=0}^{\infty} x^k.$$

Given two power series multiplied, they can be expressed as a new power series of the form

$$f(x) \cdot g(x) = \sum_{n=1}^{\infty} c_n x^k$$
$$c_n = \sum_{j=1}^{n} a_j b_{n-j}.$$

Therefore for this case, $c_n = k + 1$, meaning

$$\left(\frac{1}{1-x}\right)^2 = \sum_{k=0}^{\infty} (k+1) x^k.$$

Consider a more difficult example

Ex. Find the power series to $\frac{x}{1-4x+4x^2}$

Notice first that the x can be ignored in then numerator as it will just be a multiplier in the final answer. Note that

$$\frac{1}{1-4x+4x^2} = \frac{1}{(1-2x)^2}.$$

While the power series for $\frac{1}{1-2x}$ is unknown, it is simply a scaled input to the power series previously mentioned. Therefore

$$\frac{1}{1-2x} = \sum_{k=0}^{\infty} 2^k x^k.$$

Squaring the result necessitates finding the c_n terms.

$$a_n = 2^n$$
$$b_n = 2^n$$
$$a_j b_{n-j} = 2^n$$
$$c_n = \sum_{k=1}^{n} 2^k$$
$$c_n = (n+1)2^n.$$

Therefore the final power series representation is

$$\frac{x}{1 - 4x + 4x^2} = \sum_{n=1}^{\infty} (n+1) 2^n x^{n+1}.$$