1.1

Proof. Proceed with induction. Consider the base when $n = 1$. Then $1^2 = 1 = \frac{1}{6} \cdot 6 = \frac{1}{6}1(1+1)(2+1)$. Therefore the base case holds. Assume for a fixed $n \in \mathbb{N}$ that $1^2 + 2^2 + \ldots + n^2 = \frac{1}{6}n(n+1)(2n+1)$. Note that

$$\frac{1}{6}n(n+1)(2n+1) = \frac{2n^3 + 3n^2 + n}{6}$$

Consider the equation

$$\begin{aligned}
\frac{1}{6}(n+1)(n+2)(2n+3) &= \frac{1}{6}\left[(n^3 + 3n + 2)(2n+3)\right] \\
&= \frac{1}{6}\left[2n^3 + 6n^2 + 4n + 3n^2 + 9n + 6\right] \\
&= \frac{1}{6}\left[2n^3 + 9n^2 + 13n + 6\right] \\
&= \frac{2n^3 + 3n^2 + n}{6} + n^2 + 2n + 1 \\
&= \frac{2n^3 + 3n^2 + n}{6} + (n+1)^2
\end{aligned}$$

Applying the induction hypothesis,

$$= 1^2 + 2^2 + \ldots + n^2 + (n+1)^2$$

therefore the $n+1$ case holds. Therefore for all $n \in \mathbb{N}$, $1^2 + 2^2 + \ldots + n^2 = \frac{1}{6}n(n+1)(2n+1)$. ■

1.4

Part A

$$1 + 3 + \ldots + (2n - 1) = n^2$$

Part B

Proof. Proceed with induction. Consider the base case where $n = 1$. Then $2n - 1 = 2 - 1 = 1 = 1^2$, therefore the base case holds. Assume for a fixed $n \in \mathbb{N}$ that $1 + 3 + \ldots + (2n - 1) = n^2$. Then

$$(n + 1)^2 = n^2 + 2n + 1$$

Applying the induction hypothesis to n^2,

$$(n + 1)^2 = 1 + 3 + \ldots + (2n - 1) + (2n + 1).$$

Since $2n + 1 = (2(n + 1) - 1)$, the $n + 1$ case holds. Therefore for all $n \in \mathbb{N}$, $1 + 3 + \ldots + (2n - 1) = n^2$. ∎

1.9

Part A

$2^n > n^2$ for all $n \geq 5$.

Part B

Proof. Proceed with induction. Consider the base case where $n = 5$. Then $2^5 = 32 > 25 = 5^2$, therefore the base case holds. Assume for a fixed $n \in \mathbb{N} \geq 5$ that $2^n > n^2$. Then

$$2^{n+1} = 2(2^n) > 2n^2 > n^2 + 2n + 1 = (n + 1)^2$$

$2n^2 > n^2 + 2n + 1$ is true because in 1.8 it is established that $n^2 > n + 1$, which leads to $2n^2 > 2n + 2 > 2n + 1$. $2n^2 > n^2 + 2n + 1$ can be rewritten as $n^2 > 2n + 1$ and hence is true by the previous derivation. Transversing the inequalities gives $2^{n+1} > (n + 1)^2$, meaning the $n + 1$ case holds. Therefore for all $n \in \mathbb{N} \geq 5$, $2^{n+1} > (n + 1)^2$. ∎

2.1

1. Let $x = \sqrt{3}$. Then $x^2 - 3 = 0$. By Corollary 2.3, the only rational solutions are integers that divide -3, meaning $\pm 1, \pm 3$. $(\pm 1)^2 - 3 = -2$ and $(\pm 3)^2 - 3 = 6$. None of the possible rational solutions work, so $\sqrt{3}$ is not rational.

2. Let $x = \sqrt{5}$. Then $x^2 - 5 = 0$. By Corollary 2.3, the only rational solutions are integers that divide -5, meaning $\pm 1, \pm 5$. $(\pm 1)^2 - 5 = -4$ and $(\pm 5)^2 - 5 = 20$. None of the possible rational solutions work, so $\sqrt{5}$ is not rational.

3. Let $x = \sqrt{7}$. Then $x^2 - 7 = 0$. By Corollary 2.3, the only rational solutions are integers that divide -7, meaning $\pm 1, \pm 7$. $(\pm 1)^2 - 7 = -6$ and $(\pm 7)^2 - 7 = 42$. None of the possible rational solutions work, so $\sqrt{7}$ is not rational.

4. Let $x = \sqrt{24}$. Then $x^2 - 24 = 0$. By Corollary 2.3, the only rational solutions are

integers that divide -24, meaning $\pm 1, \pm 2, \pm 3, \pm 6, \pm 8, \pm 12, \pm 24$. Checking each:

$$\begin{aligned}
(\pm 1)^2 - 24 &= 1 - 24 \neq 0 \\
(\pm 2)^2 - 24 &= 4 - 24 \neq 0 \\
(\pm 3)^2 - 24 &= 9 - 24 \neq 0 \\
(\pm 4)^2 - 24 &= 16 - 24 \neq 0 \\
(\pm 6)^2 - 24 &= 36 - 24 \neq 0 \\
(\pm 8)^2 - 24 &= 64 - 24 \neq 0 \\
(\pm 12)^2 - 24 &= 144 - 24 \neq 0
\end{aligned}$$

None of the possible rational solutions work, so $\sqrt{24}$ is not rational.

5. Let $x = \sqrt{31}$. Then $x^2 - 31 = 0$. By Corollary 2.3, the only rational solutions are integers that divide -31, meaning $\pm 1, \pm 31$. $(\pm 1)^2 - 31 = -30$ and $(\pm 31)^2 - 31 = 930$. None of the possible rational solutions work, so $\sqrt{31}$ is not rational.

2.5

Let $x = \left(3 + \sqrt{2}\right)^{\frac{2}{3}}$. Then $x^6 - 22x^3 + 49 = 0$. By Corollary 2.3, the only rational solutions are integers that divide 49, meaning $\pm 1, \pm 7$. Checking each:

$$\begin{aligned}
1^6 - 22 \cdot 1^3 + 49 &= 28 \neq 0 \\
(-1)^6 - 22 \cdot (-1)^3 + 49 &= 72 \neq 0 \\
7^6 - 22 \cdot 7^3 + 49 &>> 0 \\
(-7)^6 - 22 \cdot (-7)^3 + 49 &>> 0
\end{aligned}$$

Since none of the possible rational solutions work, $\left(3 + \sqrt{2}\right)^{\frac{2}{3}}$ is not rational.

2.7

Part A

Let $x = \sqrt{4 + 2\sqrt{3}} - \sqrt{3}$. Then $x^4 - 14x^2 + 24x - 11 = 0$. By Corollary 2.3, the rational solutions are integers that divide -11 meaning $\pm 1, \pm 11$. Consider 1.

$$1^4 - 14 \cdot 1^2 + 24 - 11 = 1 - 14 + 24 - 11 = 0$$

Therefore 1 is a rational solution. Note then that

$$\begin{aligned} 1 &= \sqrt{4+2\sqrt{3}} - \sqrt{3} \\ 1+\sqrt{3} &= \sqrt{4+2\sqrt{3}} \\ 1+2\sqrt{3}+3 &= 4+2\sqrt{3} \\ 1+3 &= 4 \\ 4 &= 4 \end{aligned}$$

Therefore $x = 1$ and is hence rational.

Part B

Let $x = \sqrt{6+4\sqrt{2}} - \sqrt{2}$. Then $x^4 - 16x^2 + 32x - 16 = 0$. By Corollary 2.3, the rational solutions are integers that divide -16 meaning $\pm 1, \pm 2, \pm 4, \pm 8$. Consider 2.

$$2^4 - 16 \cdot 2^2 + 32 \cdot 2 - 16 = 16 - 64 + 64 - 16 = 0$$

Therefore 2 is a rational solution. Note then that

$$\begin{aligned} 2 &= \sqrt{6+4\sqrt{2}} - \sqrt{2} \\ 2+\sqrt{2} &= \sqrt{6+4\sqrt{2}} \\ 6+4\sqrt{2} &= 6+4\sqrt{2} \\ 6 &= 6 \end{aligned}$$

Therefore $x = 2$ and is hence rational.

2.8

By Corollary 2.3, the rational solutions of $x^8 - 4x^5 + 13x^3 - 7x + 1 = 0$ are the integers that divide 1, meaning ± 1. Checking 1 and -1:

$$1^8 - 4 \cdot 1^5 + 13 \cdot 1^3 - 7 \cdot 1 + 1 = 1 - 4 + 13 - 7 + 1 = 4 \neq 0$$
$$(-1)^8 - 4 \cdot (-1)^5 + 13 \cdot (-1)^3 - 7 \cdot (-1) + 1 = 1 + 4 - 13 + 7 + 1 = 0$$

Therefore the only rational solution is $x = -1$.

3.1

For $\mathbb{N}$ A4 and M4 fails and for $\mathbb{Z}$ only M4 fails.

3.3

iv.)

Proof. Let $a, b \in \mathbb{F}$. Consider the equation

$$\begin{aligned} -ab + (-a)(-b) &= -(a)b + (-a)(-b) && \text{(By iii)} \\ &= -(a)(b + (-b)) && \text{(By DL)} \\ &= -(a) \cdot 0 && \text{(By A4)} \\ &= 0 && \text{(By ii)} \end{aligned}$$

Therefore $-ab + (-a)(-b) = 0$, meaning $(-a)(-b) = ab$. ∎

v.)

Proof. Let $a, b, c \in \mathbb{F}$ with $c \neq 0$. Consider the equation $ac = bc$. Since c is non-zero it has an inverse c^{-1}. Then

$$\begin{aligned} ac &= bc \\ (ac)c^{-1} &= (bc)c^{-1} \\ a(cc^{-1}) &= b(cc^{-1}) && \text{(By M1)} \\ a \cdot 1 &= b \cdot 1 && \text{(By M4)} \\ a &= b && \text{(By M3)} \end{aligned}$$

Therefore $a = b$. ∎

3.4

v.)

Proof. By (iv), for all $a \in \mathbb{F}$, $0 <= a^2$. Therefore $0 <= 1^2$. Since $1^2 = 1 \cdot 1 = 1$ by M3, $0 \leq 1$. Since $0 \neq 1$, $0 < 1$. ∎

vii.)

Proof. Let $a, b \in \mathbb{F}$ and assume that $0 < a < b$. Since $a > 0$ and $b > 0$, they have inverses and, with $c = a^{-1}b^{-1}$, $c > 0$. Since $a < b$ and $c > 0$, $ac < bc$ meaning

$$aa^{-1}b^{-1} < ba^{-1}b^{-1}$$

which by commutivity, associativity, and inverses results in

$$b^{-1} < a^{-1}.$$

Since $a > 0$ and $b > 0$, their inverses are also greater than 0 meaning overall that $0 < b^{-1} < a^{-1}$. ∎