Problem 7.3.1

A relation $\mathcal{R}$ is antisymmetric if $((x,y) \in \mathcal{R}) \wedge ((y,x) \in \mathcal{R}) \Rightarrow x = y$. Give examples of relations $\mathcal{R}$ on $A = \{1,2,3\}$ having the stated property.

(a) $\mathcal{R}$ is both symmetric and antisymmetric.

(b) $\mathcal{R}$ is neither symmetric nor antisymmetric.

Solution

Part A

$$\mathcal{R} = \{(1,1)\}$$
$$\mathcal{R} = \{(3,3)\}$$
$$\mathcal{R} = \{(1,1),(2,2),(3,3)\}.$$

Part B

$$\mathcal{R} = \{(1,2),(2,3),(2,1)\}$$
$$\mathcal{R} = \{(1,3),(1,2),(3,1)\}$$
$$\mathcal{R} = \{(1,2),(3,1),(1,3)\}.$$

Problem 7.3.4

(a) Let $\sim$ be the relation defined on $\mathbb{Z}$ by $a \sim b \iff a + b$ is even. Show that $\sim$ is an equivalence relation and determine the distinct equivalence classes.

(b) Suppose that 'even' is replaced by 'odd' in part (a). Which of the properties reflexive, symmetric, transitive does $\sim$ posses?

Solution

Part A

Proof. Proceed to show that $\sim$ is an equivalence relation on $\mathbb{Z}$.

(Reflexivity) Let $a \in \mathbb{Z}$. Then $a + a = 2a$, which by definition is even. Therefore since $a + a$ is even it follows that $a \sim a$, hence $\sim$ is reflexive.

(Symmetry) Let $a, b \in \mathbb{Z}$ and assume $a \sim b$. Therefore $a + b$ is even. Since $a + b = b + a$, it follows that $b + a$ is also even and therefore $b \sim a$. Hence $\sim$ is symmetric.

(Transitivity) Let $a, b, c \in \mathbb{Z}$ and assume that $a \sim b$ and $b \sim c$. Therefore $a+b$ and $b+c$ are both even, meaning $\exists m, n \in \mathbb{Z}$ such that $a+b = 2m$ and $b + c = 2n$. Then

$$\begin{aligned}(a+b)+(b+c) &= 2m+2n \\ a+2b+c &= 2m+2n \\ a+c &= 2m+2n-2b \\ a+c &= 2(m+n-b).\end{aligned}$$

Since $m+n-b \in \mathbb{Z}, a+c$ is even and therefore $a \sim c$. Hence $\sim$ is transitive.

Since $\sim$ is reflexive, symmetric, and transitive it is an equivalence relation. ■

The two possible equivalence classes are $2\mathbb{Z}$ and $\mathbb{Z}\setminus 2\mathbb{Z}$. Note that if x is an odd integer, then

$$[a] = \{b : a+b \text{ is odd}\}.$$

Since a is odd, then b has to also be odd for $a+b$ to be even, meaning the equivalence class of an odd integer is the odd integers. If a is an even integer, then

$$[a] = \{b : a+b \text{ is even}\}.$$

Since a is now an even integer, b must also be an even integer for $a+b$ to be even, meaning the equivalence class of an even integer is the even integers.

Part B

If *even* is replaced with *odd*, then $\sim$ will be symmetric but not reflexive or transitive. Consider reflexivity. Let $a = 1$ and note that $a + a = 2$ which is not odd, hence $\sim$ would not be reflexive. Consider transitivity. Note $1 \sim 2$ and $2 \sim 3$ since $1+2 = 3$ and $2+3 = 5$, but $1 \nsim 3$ because $1+3 = 4$ which is even. Consider symmetry. Let $a, b \in \mathbb{Z}$ and assume $a \sim b$. Then $\exists n \in \mathbb{N}$ such that $a+b = 2n+1$. Equivalently $b+a = 2n+1$, therefore $b \sim a$. Hence $\sim$ would be symmetric.

Problem 7.3.6

For the purposes of this question, we call a real number x small if $|x| \leq 1$. Let $\mathcal{R}$ be the relation on the set of real numbers defined by

$$x\mathcal{R}y \iff x-y \text{ is small}.$$

Prove or disprove: $\mathcal{R}$ is an equivalence relation on $\mathbb{R}$.

Solution

Proof. Proof that $\mathcal{R}$ is not an equivalence relation on $\mathbb{R}$. Let $a = 1, b = 2, c = 3$. Note that $|a - b| = 1 \leq 1$ and $|b - c| = 1 \leq 1$, therefore $a\mathcal{R}b$ and $b\mathcal{R}c$. However $|a - c| = 2 \nleq 1$, meaning a does not relate c. Therefore $\mathcal{R}$ is not transitive and hence not an equivalence relation. ■

Problem 7.3.10

Let $A = \{2m : m \in \mathbb{Z}\}$. A relation $\sim$ is defined on the set $\mathbb{Q}^+$ of positive rational numbers by

$$a \sim b \iff \frac{a}{b} \in A.$$

(a) Show that $\sim$ is an equivalence relation.

(b) Describe the elements in the equivalence class $[3]$.

Solution

Part A

Proof. Proceed to show that $\sim$ is an equivalence relation on $\mathbb{Q}^+$.

(Reflexivity) Let $a \in \mathbb{Q}^+$. Note that $\frac{a}{a} = 1 = 2^0$. Since $2^0 \in A$, $\frac{a}{a} \in A$ meaning $a \sim a$. Therefore $\sim$ is reflexive.

(Symmetry) Let $a, b \in \mathbb{Q}^+$ and assume that $a \sim b$. Therefore $\frac{a}{b} \in A$, meaning $\exists m \in \mathbb{Z}$ such that $\frac{a}{b} = 2^m$. Note that $\frac{b}{a} = 2^{-m}$. Since $2^{-m} \in A$, $\frac{b}{a} \in A$ meaning $b \sim a$. Therefore $\sim$ is symmetric.

(Transitivity) Let $a, b, c \in \mathbb{Q}^+$ and assume that $a \sim b$ and $b \sim c$. Therefore $\frac{a}{b} \in A$ and $\frac{b}{c} \in A$, meaning $\exists m, n \in \mathbb{Z}$ such that $\frac{a}{b} = 2^m$ and $\frac{b}{c} = 2^n$. Then it follows that

$$\begin{aligned} \frac{a}{b} \cdot \frac{b}{c} &= 2^m \cdot 2^n \\ \frac{a}{c} &= 2^{m+n}. \end{aligned}$$

Since $m + n \in \mathbb{Z}$, then $\frac{a}{c} \in A$. Therefore $a \sim c$ meaning $\sim$ is transitive.

Since $\sim$ is reflexive, symmetric, and transitive it is an equivalence relation. ■

Part B

The equivalence class of 3 can be described as

$$\begin{aligned}[3] &= \{y : 3 \sim y\} \\ &= \{y : y \sim 3\} \\ &= \left\{y : \frac{y}{3} = 2^m, \forall m \in \mathbb{Z}\right\} \\ &= \{y : y = 3 \cdot 2^m, \forall m \in \mathbb{Z}\} \\ &= \left\{3, \frac{3}{2}, 6, \frac{3}{4}, 12, \ldots\right\}.\end{aligned}$$

Problem 7.4.3

Let $X = \{1, 2, 3\}$. Define a relation $\mathcal{R} = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (3, 1), (3, 3)\}$ on X.

(a) Which of the properties reflexive, symmetric, transitive are satisfied by $\mathcal{R}$?

(b) Compute A_1, A_2, A_3 where $A_n = \{x \in X : x\mathcal{R}n\}$. Show that $\{A_1, A_2, A_3\}$ do not form a partition of X.

(c) Repeat parts (a) and (b) for the relations $\mathcal{S}$ and $\mathcal{T}$ on X, where

$$\begin{aligned}\mathcal{S} &= \{(1, 1), (1, 3), (3, 1), (3, 3)\} \\ \mathcal{T} &= \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 3)\}.\end{aligned}$$

Solution

Part A

$\mathcal{R}$ is reflexive and symmetric but not transitive.

Part B

$$\begin{aligned}A_1 &= \{1, 2, 3\} \\ A_2 &= \{1, 2\} \\ A_3 &= \{1, 3\}.\end{aligned}$$

Since $A_1 \cap A_2 \neq \varnothing$, A_1, A_2, A_3 do not form a partition of X.

Part C

Relation $\mathcal{S}$

$\mathcal{S}$ is symmetric but not transitive or reflexive.

$$A_1 = \{1, 3\}$$
$$A_2 = \varnothing$$
$$A_3 = \{1, 3\}.$$

These sets do not form a partition of X since $A_1 \cup A_2 \cup A_3 = \{1, 3\} \neq X$.

Relation $\mathcal{T}$

$\mathcal{T}$ is reflexive but not transitive or symmetric

$$A_1 = \{1, 2\}$$
$$A_2 = \{1, 2\}$$
$$A_3 = \{1, 2, 3\}.$$

Since $A_1 \cap A_2 \neq \varnothing$, A_1, A_2, A_3 do not form a partition of X.