Lecture Problem

With the given definition

Definition 1. We say a mapping $f : A \to B$ is well-defined if

(a) $\forall a \in A, f(a) \in B$

(b) $\forall a_1, a_2 \in A, a_1 = a_2 \implies f(a_1) = f(a_2)$

Part A

1. Give and example of f that doesn't satisfy (a)
2. Give and example of f that doesn't satisfy (b)
3. Give and example of f that satisfies (a) and (b) and prove it

Part 1

Let f be defined as

$$f : [0, 2] \to [0, 1] : x \mapsto x^2.$$

In this case f doesn't satisfy (a) since $2 \in [0, 2]$, but $f(2) = 4 \notin [0, 1]$.

Part 2

Let f

$$f : [0, 1] \to [-1, 1] : x \mapsto \pm\sqrt{x}.$$

In this case f doesn't satisfy (b) since $f(1) = 1$ and $f(1) = -1$.

Part 3

Let f be defined as

$$f : A \to B.$$

Where $A = \{0, 1\}$ and $B = \{2, 3\}$ where

$$\begin{aligned} 0 &\mapsto 2 \\ 1 &\mapsto 3. \end{aligned}$$

Proof. Proof that f satisfies both conditions (a) and (b). Note that $\forall a \in A, f(a) \in B$ since $f(0) = 2 \in B$ and $f(1) = 3 \in B$. Let $a_1, a_2 \in A$ and assume $a_1 = a_2$. If $a_1 = a_2 = 0$, then $f(a_1) = f(a_2) = 2$ and if $a_1 = a_2 = 1$, then $f(a_1) = f(a_2) = 3$. Therefore both (a) and (b) are true for f. ∎

Part B

Let

$$f : \mathbb{Q} \to \mathbb{Q} : \frac{a}{b} \mapsto a.$$

Show that f is not well-defined

Solution

Proof. Let $q_1 = \frac{1}{2}$ and $q_2 = \frac{2}{4}$. Then $q_1 = q_2$ but $f(q_1) = 1 \neq 2 = f(q_2)$, hence f is not well-defined. ∎

Part C

Let

$$f : \mathbb{Q} \to \mathbb{Q} : \frac{a}{b} \mapsto \left(\frac{a}{b}\right)^2$$

Show that f is well-defined.

Proof. Let $q = \frac{a}{b} \in \mathbb{Q}$. Then $f(q) = f(\frac{a}{b}) = \frac{a^2}{b^2} \in \mathbb{Q}$. Let $q_1 = \frac{a_1}{b_1} \in \mathbb{Q}$ and $q_2 = \frac{a_2}{b_2} \in \mathbb{Q}$. Assume $q_1 = q_2$. Then

$$\begin{gathered}
q_1 = q_2 \\
\frac{a_1}{b_1} = \frac{a_2}{b_2} \\
\left(\frac{a_1}{b_1}\right)^2 = \left(\frac{a_2}{b_2}\right)^2 \\
f(q_1) = f(q_2).
\end{gathered}$$

∎