9.1

$$\begin{aligned} O_\sigma(1) = O_\sigma(2) = O_\sigma(5) &= \{1, 5, 2\} \\ O_\sigma(4) = O_\sigma(6) &= \{4, 6\} \\ O_\sigma(3) &= \{3\}. \end{aligned}$$

9.6

$$\begin{aligned} O_\sigma(3n) &= 3\mathbb{Z} \\ O_\sigma(3n+1) &= 3\mathbb{Z}+1 \\ O_\sigma(3n+2) &= 3\mathbb{Z}+2. \end{aligned}$$

9.9

$$(1,2)(4,7,8)(2,1)(7,2,8,1,5) = (1,5,8)(2,4,7).$$

9.13(a)

Let $\sigma = (1,4,5,7)$.

$$\begin{aligned} \sigma^2 &= (1,5)(4,7) \\ \sigma^3 &= (1,7,5,4) \\ \sigma^4 &= e. \end{aligned}$$

Therefore $|\sigma| = 4$.

9.16

The maximum order of an element is going to be the maximum value that can be obtained from $\operatorname{lcm}(a,b)$ where $a+b=7$ and $a,b \geq 0$. This is because the order of an element is the least common multiple of the order's of it's disjoint cycles, which is maximized with 2 cycles. Consider all the ways to add 2 numbers to get seven and their lcm:

$$\begin{aligned} 1+6 &\implies 6 \\ 2+5 &\implies 10 \\ 3+4 &\implies 12. \end{aligned}$$

Therefore the maximum order of an element in S_7 is 12.

9.29

Proof. Let $H \leq S_n$ with $n \geq 2$. If H does not contain any odd permutations, then all the permutations of H are even. Examine the case if there exists an odd permutation in H, that is some $\sigma \in H$ that is odd. Define the mapping $\phi : H \to H : \mu \mapsto \sigma\mu$. Let $\mu_1, \mu_2 \in H$ and assume that $\phi(\mu_1) = \phi(\mu_2)$. Then $\sigma\mu_1 = \sigma\mu_2$ which by cancellation implies $\mu_1 = \mu_2$, hence ϕ is one-to-one. Let $\mu \in H$. Note that $\phi(\sigma^{-1}\mu) = \mu$, hence ϕ is onto. Therefore ϕ is a bijection on H. Note that ϕ takes an odd permutation to an even permutation and takes an even permutation to an odd permutation. Therefore since ϕ is bijective, there must be an equal amount of even elements as odd elements otherwise the swapping would not be one-to-one and onto. Therefore if H has an odd permutation, exactly half of its permutations are even. ■

9.34

Proof. It can be assumed without loss of generality that an odd cycle can be represented as $\sigma = (1, 2, 3, \ldots, m)$ where m is an odd number. Computing its square results in

$$\sigma^2 = (1, 3, 5, \ldots, m, 2, 4, 6, \ldots, m-1)$$

which is a cycle. ■