0.1 Finishing Last Lecture

From the previous lecture:

$$\frac{\mathrm{d}}{\mathrm{d}t}x + \frac{3x}{60+2t} = \frac{1}{2}.$$

How much salt is in the tank when it is full? First, find out how full the tank is (given that it holds 100 Litres):

$$60 + 2t = 100 \implies t = 20 \text{ minutes}$$

Now use the integration factor $r(t) = e^{\int p(x)\mathrm{d}x}$.

$$\begin{aligned}
r(t) &= e^{\int \frac{3}{60+2t}\mathrm{d}t} \\
(u = 60 + 2t &\implies \mathrm{d}u = 2\mathrm{d}t) \\
r(t) &= e^{\frac{3}{2}\int \frac{1}{u}\mathrm{d}u} = e^{\frac{3}{2}\ln 60+2t} \\
r(t) &= (60+2t)^{\frac{3}{2}}
\end{aligned}$$

Now utilize inverse product rule:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(60+2t)^{\frac{2}{3}} + \frac{3x}{60+2t}(60+2t)^{\frac{2}{3}} &= \frac{1}{2}(60+2t)^{\frac{2}{3}} \\
\int \frac{\mathrm{d}}{\mathrm{d}t}\left[x(t)(60+2t)\right]^{\frac{2}{3}} &= \int \frac{1}{2}(60+2t)^{\frac{2}{3}} \\
(60+2t)^{\frac{2}{3}} = \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{2}{5}(60+2t)^{\frac{5}{2}} + c &= \frac{1}{10}(60+2t)^{\frac{5}{2}} + c \\
x(t) &= \frac{1}{10}(60+2t) + C(60+2t)^{\frac{-3}{2}}.
\end{aligned}$$

Then applying the initial condition:

$$x(0) = \frac{1}{10}(60) + C(60)^{-\frac{3}{2}} \implies \boxed{C \approx 1860.}$$

Now plug in $t = 20$

$$x(20) = \frac{1}{10}(60+40) + 1860(60+40)^{-\frac{3}{2}} \approx \boxed{11.86\text{kg.}}$$

0.1.1 Solving Tank Problems

Consider a tank of brine water or some dissolved substance in a solvent. Commonly it is brine water. The tank has both an input and output. The input liquid comes in at a rate r_1 with a concentration of c_1. The output is almost always considered to be homogeneous since it is at the bottom of the tank. The output rate is r_2 and output concentration is c_2. The overall quantity of solute is $s(t)$ and tank volume is $v(t)$. The setup is represented by **??**.

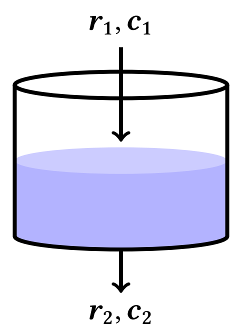

Figure 1: Singular Tank System

From the setup, one can determine that:

$$\Delta s = r_1 c_1 \Delta t - r_2 c_2 \Delta t.$$

Dividing by Δt and taking the limit arrives at the differential equation

$$\boxed{\frac{\mathrm{d}s}{\mathrm{d}t} = r_1 c_1 - r_2 c_2.}$$

0.2 Substitution

Nice types of ODES:

$$y' = f(x) \qquad y' = f(x, y) = h(x) \cdot g(y)$$

$$y' = f(y) \qquad \frac{\mathrm{d}x}{\mathrm{d}t} + p(t)x = f(t).$$

The last case represents a linear, first order differential equation. Via the integration factor, they are easy to solve. In certain cases, an equation may not look linear or separable; however, **change of variables** can resolve certain cases like this.

Note: Substitutions

General substitutions that work:

$y' = F(ax + by)$	$v = ax + by$
$y' = G(\frac{y}{x})$	$v = \frac{y}{x}$
$y' + p(x)y = q(x)y^n$	$v = y^{1-n}$

Ex. Find the general solution of $y' = (4x - y + 1)^2$

Let $v = 4x - y$. Rewrite in terms of v:

$$v' = 4 - y'$$
$$\downarrow$$
$$y' = 4 - v'$$

Now:

$$4 - v' = (v+1)^2 \implies v' = 4 - (v+1)^2.$$

Note that it is now a separable equation.

$$\begin{aligned}\frac{v'}{4-(v+1)^2} = 1 \implies -\int \frac{v'}{4-(v+1)^2}dv &= \int 1dx \\ -\frac{1}{4}[\ln|v-1| - \ln|v+3|] &= x + c \\ \ln\left|\frac{v-1}{v+3}\right| &= -4x + c \\ \frac{v-1}{v+3} &= Ae^{-4x} \\ v &= \frac{1+3Ae^{-4x}}{1-Ae^{-4x}}.\end{aligned}$$

Now that the solution is found, rewrite v back in terms of y.

$$\begin{aligned}v = \frac{1+3Ae^{-4x}}{1-Ae^{-4x}} \implies 4x - y &= \frac{1+3Ae^{-4x}}{1-Ae^{-4x}} \\ y &= 4x - \frac{1+3Ae^{-4x}}{1-Ae^{-4x}}.\end{aligned}$$

Example where $v = \frac{y}{x}$:

Ex. Solve $x^2y' = y^2 + xy$

First try dividing by the highest power of the independent variable (x^2)

$$y' = \frac{y^2}{x^2} + \frac{y}{x}$$

Now use $v = \frac{y}{x}$

$$y' = v^2 + v$$

Find y' in terms of v

$$v = \underbrace{\frac{y}{x}}_{\text{Quotient rule}}$$
$$v' = \frac{xy' - y}{x^2}$$
$$v' = \frac{y'}{x} - \frac{y}{x^2}$$
$$v'x = y' - \frac{y}{x}$$
$$v'x = y' - v$$
$$y' = v'x + v$$

Substitute back into ODE:

$$xv' = v^2 + v$$
$$v' = \frac{v^2}{x}$$
$$\frac{v'}{v^2} = \frac{1}{x}$$
$$\int \frac{1}{v^2}\mathrm{d}v = \int \frac{1}{x}\mathrm{d}x$$
$$-\frac{1}{v} = \ln(x) + c$$
$$v = -\frac{1}{\ln(x) + c}$$

Now plug back in y for v:

$$\frac{y}{x} = -\frac{1}{\ln(x) + c}$$
$$y = -\frac{x}{\ln(x) + c}.$$

0.2.1 Solving a Bernoulli Equation

Given an equation in the form of a Bernoulli Equation, such as:

$$y' + \frac{4}{x}y = x^3y^2, x > 0$$

Since $n = 2$, let $v(x) = y^{-1}$. Find $v'(x)$

$$v'(x) = \frac{\mathrm{d}}{\mathrm{d}x}v = -y^{-2}y'.$$

Divide the ODE by y^n:

$$y^{-2}y' + \frac{4}{x}y^{-1} = x^3.$$

$$-v' + \frac{4}{x}v = x^3.$$

Rearrange into standard linear ODE form:

$$v' - \frac{4}{x}v = -x^3.$$

Utilize the integration factor $r(x) = e^{-\int \frac{4}{x}\mathrm{d}x} = x^{-4}$

$$\begin{aligned}
v' - \frac{4}{x}v &= -x^3 \\
v'r(x) - \frac{4}{x}vr(x) &= -x^3 r(x) \\
\underbrace{x^{-4}v' - x^{-4}\frac{4}{x}v}_{\text{Inverse Product Rule}} &= -\frac{1}{x} \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{-4}v\right) &= -\frac{1}{x} \\
x^{-4}v &= -\ln(x) + c \\
v &= -x^4\ln(x) + cx^4.
\end{aligned}$$

Plug y back in for v:

$$\boxed{y(x) = \frac{1}{-x^4\ln(x) + cx^4}.}$$

0.3 2^{nd} Order Linear Equations

Note: General form of 2^{nd} Order Linear ODE

$$A(x)y'' + B(x)y' + C(x)y = F(x).$$

Theorem 0.1 ▶ 2^{nd} Order Linear ODE Solution Existence and Uniqueness

For an ODE of form $y'' + B(x)y' + C(x)y = D(x)$ with $B(x), C(x)$ and $D(x)$ as continuous functions on an interval I, for some $a \in I$ and some $b_0, b_1 \in \mathbb{R}$, a unique solution must exist and satisfy:

$$\begin{cases} y'' + B(x)y' + C(x)y = D(x) \\ y(a) = b_0 \\ y'(a) = b_1 \end{cases}$$

Note: Homogeneous Equation

$$ay'' + by' + cy = 0 \leftarrow \text{Since this is zero, it is homogeneous.}$$

0.3.1 Superposition Principle

Consider $y'' - k^2y = 0$. A possible solution is $y_1 = e^{kx}$. Therefore $y_1' = ke^{kx}$ and $y_1'' = k^2e^{kx}$. Plugging into the original ODE:

$$\boxed{y_1'' - k^2y_1 = \left(k^2e^{kx}\right) - k^2\left(e^{kx}\right) = 0 \checkmark}$$

Another solution is $y_2 = e^{-kx}$. By the ??, their linear combination is also a solution. In this instance, for $c_1, c_2 \in \mathbb{R}$:

Theorem 0.2 ▶ Superposition Principle

For any linearly homogeneous differential equation, if it has two solutions $y_1(t)$ and $y_2(t)$, then the function:

$$y(t) = c_1y_1(t) + c_2y_2(t).$$

Is also a solution.

0.3.2 The Wronskian

In the case of an 2[nd] Order Linear Homogeneous equation, utilizing the ?? offers new solutions to the ODE. However, the linear combination may not always result in a general solution. Inspect the general form the ODE:

$$p(t)y'' + q(t)y' + r(t)y = 0 \implies \begin{cases} y(t_0) & = y_0 \\ y'(t_0) & = y_1 \end{cases}.$$

By the **??**

$$y(t) = c_1y_1(t) + c_2y_2(t).$$

Both y_1 and y_2 must satisfy the initial conditions in order to provide a general solution. Find the value of the constants c_1 and c_2.

$$\begin{aligned} y_0 &= y(t_0) = c_1y_1(t_0) + c_2y_2(t_0) \\ y_1 &= y'(t_0) = c_1y_1'(t_0) + c_2y_2'(t_0). \end{aligned}$$

Rewrite in terms of a system of matrices

$$\underbrace{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} y_0 \\ y_1 \end{bmatrix}}_{\mathbf{b}}$$

Note that c_1 and c_2 can be solved using Cramer's Rule

$$c_1 = \frac{\begin{vmatrix} y_0 & y_2(t_0) \\ y_1 & y_2'(t_0) \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}}. \qquad c_2 = \frac{\begin{vmatrix} y_1(t_0) & y_0 \\ y_1'(t_0) & y_1 \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}}.$$

This restatement of the imposed initial value conditions reveals a new, succinct condition to check for generality. If the denominator of either c_1 or c_2 is 0, the linear combination of y_1 and y_2 will not be the general solution of the ODE. This denominator is called ??.

Theorem 0.3 ▶ The Wronskian

$$W(f,g)(t) = \begin{vmatrix} f(t) & g(t) \\ f'(t) & g'(t) \end{vmatrix}.$$

For a 2^{nd} Order Linear Homogeneous equation, if $W(y_1, y_2)(t) \neq 0$, then $y(x) = c_1 y_1 + c_2 y_2$ is a general solution to the ODE.

Note: Extension of the Wronskian

For any n^{th} order ODE, the Wronskian can be generalized. Generalizing by example, consider a 3^{rd} order linearly homogeneous ODE.

$$a(t)y''' + b(t)y'' + c(t)y' + d(t)y = 0.$$

By the ??, the solution is

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + c_3 y_3(t).$$

Like in ??

$$\begin{aligned} y_0 &= c_1 y_1(t_0) + c_2 y_2(t_0) + c_3 y_3(t_0) \\ y_1 &= c_1 y_1'(t_0) + c_2 y_2'(t_0) + c_3 y_3'(t_0) \\ y_2 &= c_1 y_1''(t_0) + c_2 y_2''(t_0) + c_3 y_3''(t_0) \end{aligned}$$

Which as prior can be written as a matrix multiplication. This holds for any order, therefore we can right the Wronskian generally as:

$$W(y_1, y_2, \ldots, y_n)(t) = \begin{vmatrix} y_1(t) & y_2(t) & \ldots & y_n(t) \\ y_1'(t) & y_2'(t) & \ldots & y_n'(t) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^n(t) & y_2^n(t) & \ldots & y_n^n(t) \end{vmatrix}$$