29.6

Note that

$$\alpha = \sqrt{3-\sqrt{6}} \implies \alpha^2 - 3 = -\sqrt{6} \implies \alpha^4 - 6\alpha^2 + 3 = 0$$

meaning α is a zero of $f(x) = x^4 - 6x^2 + 3$ in $\mathbb{Q}[x]$. Since the Eisenstein criterion holds for $p = 3$, $f(x)$ is irreducible. Therefore $\operatorname{irr}(\alpha, \mathbb{Q}) = f(x)$ and $\deg(\alpha, \mathbb{Q}) = 4$

29.8

Note that

$$\alpha = \sqrt{2} + i \implies \alpha^2 = 2 + 2\sqrt{2}i - 1 \implies \alpha^4 - 2\alpha^2 + 9 = 0$$

meaning α is a zero of $f(x) = x^4 - 2x^2 + 9$ in $\mathbb{Q}[x]$. If f was reducible over $\mathbb{Q}$, then it must have a zero in $\mathbb{Z}$ that divides 9. Checking $\pm 1, \pm 3$ gives no such zero, hence f is irreducible. Therefore $\operatorname{irr}(\alpha, \mathbb{Q}) = f(x)$ and $\deg(\alpha, \mathbb{Q}) = 4$.

29.12

Since $\pi \in \mathbb{R}$ then $\sqrt{\pi} \in \mathbb{R}$. Therefore it is algebraic in $\mathbb{R}$ with $\deg(\sqrt{\pi}, \mathbb{R}) = 1$ since it is a zero of the linear polynomial $f(x) = x - \sqrt{\pi}$.

29.16

Since $(\pi^2)^3 - (\pi^3)^2 = 0$, then π^2 is a zero of the polynomial $f(x) = x^3 - \pi^6 \in \mathbb{Q}(\pi^3)$. This polynomial is irreducible hence π^2 is algebraic in $\mathbb{Q}(\pi^3)$ with $\deg(\pi^2, \mathbb{Q}(\pi^3)) = 3$.

29.18

Part A

Proof. Note that

$$\begin{aligned} x = 0 &\implies 0^2 + 1 = 1 \neq 0 \\ x = 1 &\implies 1^2 + 1 = 2 \neq 0 \\ x = 2 &\implies 2^2 + 1 = 2 \neq 0 \end{aligned}$$

Therefore $f(x)$ has no zero in $\mathbb{Z}_3$ and hence is irreducible. ■

Part B

$+$	0	1	2	α	2α	$1+\alpha$	$1+2\alpha$	$2+\alpha$	$2+2\alpha$
0	0	1	2	α	2α	$1+\alpha$	$1+2\alpha$	$2+\alpha$	$2+2\alpha$
1	1	2	0	$1+\alpha$	$1+2\alpha$	$2+\alpha$	$2+2\alpha$	α	2α
2	2	0	1	$2+\alpha$	$2+2\alpha$	α	2α	$1+\alpha$	$1+2\alpha$
α	α	$1+\alpha$	$2+\alpha$	2α	0	$1+2\alpha$	1	$2+2\alpha$	2
2α	2α	$1+2\alpha$	$2+2\alpha$	0	α	1	$1+\alpha$	2	$2+\alpha$
$1+\alpha$	$1+\alpha$	$2+\alpha$	α	$1+2\alpha$	1	$2+2\alpha$	2	2α	0
$1+2\alpha$	$1+2\alpha$	$2+2\alpha$	2α	1	$1+\alpha$	2	$2+\alpha$	0	α
$2+\alpha$	$2+\alpha$	α	$1+\alpha$	$2+2\alpha$	2	2α	0	$1+2\alpha$	1
$2+2\alpha$	$2+2\alpha$	2α	$1+2\alpha$	2	$2+\alpha$	0	α	1	$1+\alpha$

.

$\cdot$	0	1	2	α	2α	$1+\alpha$	$1+2\alpha$	$2+\alpha$	$2+2\alpha$
0	0	0	0	0	0	0	0	0	0
1	0	1	2	α	2α	$1+\alpha$	$1+2\alpha$	$2+\alpha$	$2+2\alpha$
2	0	2	1	2α	α	$2+2\alpha$	$2+\alpha$	$1+2\alpha$	$1+\alpha$
α	0	α	2α	2	1	$2+\alpha$	$1+\alpha$	$2+2\alpha$	$1+2\alpha$
2α	0	2α	α	1	2	$1+2\alpha$	$2+2\alpha$	$1+\alpha$	$2+\alpha$
$1+\alpha$	0	$1+\alpha$	$2+2\alpha$	$2+\alpha$	$1+2\alpha$	2α	2	1	α
$1+2\alpha$	0	$1+2\alpha$	$2+\alpha$	$1+\alpha$	$2+2\alpha$	2	α	2α	1
$2+\alpha$	0	$2+\alpha$	$1+2\alpha$	$2+2\alpha$	$1+\alpha$	1	2α	α	2
$2+2\alpha$	0	$2+2\alpha$	$1+\alpha$	$1+2\alpha$	$2+\alpha$	α	1	2	2α

.

29.25

Part A

Proof. Let $f(x) = x^3 + x^2 + 1$. Since $f(0) = 1$ and $f(-1) = 1$, f has no zeroes in $\mathbb{Z}_2$ and is hence irreducible. ■

Part B

Part (1) shows that $x - \alpha$ is a linear factor of $f(x)$. Checking α^2 shows that is a zero of the remainder, hence doing another long division as demonstrated in (2) gives another factor of $x - \alpha^2$. Therefore

$$x^3+x^2+1 = (x-\alpha)(x-\alpha^2)(x+1+\alpha+\alpha^2).$$

in $\mathbb{Z}_2(\alpha)$.

①

$$\begin{array}{r l}
 & x^2 + (1+\alpha)x + (\alpha^2+\alpha) \\
x-\alpha \,\big) & x^3 + x^2 + 1 \\
 & -(x^3 - \alpha x^2) \\
 & (1+\alpha)x^2 \\
 & -((1+\alpha)x^2 - (\alpha^2+\alpha)x) \\
 & (\alpha^2+\alpha)x + 1 \\
 & -[(\alpha^2+\alpha)x - (\alpha^3+\alpha^2)] \\
 & 1 - (\alpha^3+\alpha^2) \\
 & 0
\end{array}$$

$$\alpha^3 = \alpha^2 + 1 \;\Downarrow\; \alpha^3 + \alpha^2 = 1$$

②

$$\begin{array}{r l}
 & x + (1+\alpha+\alpha^2) \\
x-\alpha^2 \,\big) & x^2 + (1+\alpha)x + (\alpha^2+\alpha) \\
 & x^2 - \alpha^2 x \\
 & (1+\alpha+\alpha^2)x \\
 & (1+\alpha+\alpha^2)x - \alpha^2(1+\alpha+\alpha^2) \\
 & \alpha^2+\alpha+\alpha^2+\alpha^3+\alpha^4 = 0
\end{array}$$

29.26

Since $\langle \mathbb{Z}_2(\alpha), + \rangle$ is abelian of order 8 and $a + a = 0$ for all a in it, it is isomorphic to just $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Furthermore, since $\langle \mathbb{Z}_2(\alpha), \cdot \rangle$ is abelian of order 7, and 7 is prime, then it must be isomorphic to just $\mathbb{Z}_7$.

29.29

Proof. Since α is algebraic in $F(\beta)$, there is a polynomial $f(x)$ with coefficients in $F(\beta)$ such that $f(\alpha) = 0$. The coefficients of f have the form of a ration of two polynomials in $F[x]$. By mutliplying all the denominators together, a polynomial is achieved such that when multiplied with f, f still remains 0 but with coefficients in β. Since indeterminates are order-free, it follows β is algebraic in $F(\beta)$ ■

29.30

Proof. Note that every element of $F(\alpha)$ can be expressed as

$$b_0 + b_1\alpha + \ldots + b_{n-1}\alpha^{n-1}.$$

for $b_i \in F$. Since F contains q elements, there are q choices for each coefficient that give each a unique element in $F(\alpha)$. Since there are n coefficients, there are then n choices meaning q^n elements in $F(\alpha)$. ■