Operations

1.1 Binary Operation

Definition 1 (Binary Operation)**.** $*$ is a binary operation if it denotes the mapping $* : S \times S \to S$ into some set S

For example, addition on the reals is a binary operation as it is a mapping defined by

$$+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R} : (a, b) \mapsto a + b.$$

Often in abstract algebra, imposing or analyzing structure provides the greatest insight. Therefore there are certain algebraic properties commonly used to identify binary operations. Consider for example the concept of *closure*.

Definition 2 (Closure)**.** Let $*$ be a binary operation on S. Let $H \subseteq S$. H is closed under $*$ if for all $(u, v) \in H \times H$ that $u * v \in H$.

As an example, consider the normal addition and multiplication on the integers

$$+, \cdot : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}.$$

Consider the subset $H = \{2n + 1 : n \in \mathbb{Z}\} \subseteq \mathbb{Z}$. Firstly one has to ask if either operations are indeed a binary operation on H. In the case of multiplication, one can consider two elemenets $a, b \in H$. Therefore $\exists m, n \in \mathbb{Z}$ such that $a = 2n + 1$ and $b = 2m + 1$. Multiplying them together results in $2(2m^2 + 2mn) + 1$ which is indeed in H. For addition, $5 \in H$ and $3 \in H$, however $3 + 5 = 8 \notin H$.

Remark. *Given an arbitrary binary operation $*$, it is not always the case that $a * b = b * a$.*

If a binary operation indeed does have $a * b = b * a$, it is *commutative*.

Definition 3 (Commutative Operation)**.** A binary operation $*$ on S is commutative if $\forall a, b \in S$ that $a * b = b * a$.

Pulling from other well known operations, we can generalize the notion of associativity from multiplication and addition to a general binary operation.

Definition 4 (Associativity)**.** A binary operation $*$ on S is associative if $\forall a, b, c \in S$ that $(a * b) * c = a * (b * c)$.

Consider a more complex example of a (potential) binary operation. Define the set $F = \{f \mid f : \mathbb{R} \to \mathbb{R}\}$. Define the operation $*$ by

$$f * g \mapsto f \circ g.$$

It is fairly obvious that $*$ is indeed a binary operation as the composition of two real valued functions should still remain real valued. One may want to say $*$ is commutative, however consider the following functions

$$f(x) = x + 1$$
$$g(x) = x^2.$$

It follows fairly quickly that $f \circ g \neq g \circ f$ in this instance, meaning $*$ can not be commutative. Now a harder question is if $*$ is associative. This would require that for all possible real valued functions f, g, h that $f \circ (g \circ h) = (f \circ g) \circ h$. Surprisingly this is true. Note that

$$\begin{aligned} f \circ (g \circ h) &= f \circ (g(h(x))) \\ &= f(g(h(x))). \end{aligned}$$

and that

$$\begin{aligned} (f \circ g) \circ h &= (f(g(x))) \circ h \\ &= f(g(h(x))). \end{aligned}$$

Hence both are equivalent meaning $*$ is indeed associative.

1.1.1 Tabular Representation

If given a finite set S, a binary operation $*$ on S can be defined by tabulating all possible combinations of elements $a, b \in S$. Consider for example $S = \{a, b\}$. The operation can then be defined as

$*$	a	b
a	b	b
b	a	a

Consider then what the outcome of $a * b$ would be. Using the table, the first element will index the row and the second element will index the column. Therefore $a * b = b$. Consider the following question:

How many possible binary operations can be defined on a finite set?

The tabular representation of a binary operation is useful in this instance. Given the set S that $*$ is over, define $n = |S|$. The table will therefore have n^2 entries in it. Each entry has n choices as it can be any element of S. Therefore since you have n choices n^2 times, therefore

$$\textit{Number of possible relations} = n^{(n^2)}.$$

Remark. *Not every binary operation is well defined*

Consider for example $* : \mathbb{R} \times \mathbb{R} \to \mathbb{R} : (a, b) \mapsto a^b$. Note that $-1 * \sqrt{2} = (-1)^{\sqrt{2}} \notin \mathbb{R}$, hence $*$ in this case is not well defined.

1.1.2 Isomorphic Binary Structures

Definition 5 (Binary Algebraic Structure)**.** $\langle S, * \rangle$ is a binary algebraic structure if S is a set and $*$ is a binary operation defined over S.

Definition 6. Two binary algebraic structures $\langle A, * \rangle$ and $\langle B, *' \rangle$ are isomorphic if there exists a one-to-one and onto map $f : A \to B$ such that $\forall u, v \in A$ that $f(u * v) = f(u) *' f(v)$. Two isomorphic structures are denoted by $A \simeq B$. In the case that there only exists an onto function but no one-to-one function, A and B are homomorphisms.