26.4

$+$	$8\mathbb{Z}$	$2+8\mathbb{Z}$	$4+8\mathbb{Z}$	$6+8\mathbb{Z}$
$8\mathbb{Z}$	$8\mathbb{Z}$	$2+8\mathbb{Z}$	$4+8\mathbb{Z}$	$6+8\mathbb{Z}$
$2+8\mathbb{Z}$	$2+8\mathbb{Z}$	$4+8\mathbb{Z}$	$6+8\mathbb{Z}$	$8\mathbb{Z}$
$4+8\mathbb{Z}$	$4+8\mathbb{Z}$	$6+8\mathbb{Z}$	$8\mathbb{Z}$	$2+8\mathbb{Z}$
$6+8\mathbb{Z}$	$6+8\mathbb{Z}$	$8\mathbb{Z}$	$2+8\mathbb{Z}$	$4+8\mathbb{Z}$

$\times$	$8\mathbb{Z}$	$2+8\mathbb{Z}$	$4+8\mathbb{Z}$	$6+8\mathbb{Z}$
$8\mathbb{Z}$	$8\mathbb{Z}$	$8\mathbb{Z}$	$8\mathbb{Z}$	$8\mathbb{Z}$
$2+8\mathbb{Z}$	$8\mathbb{Z}$	$4+8\mathbb{Z}$	$8\mathbb{Z}$	$4+8\mathbb{Z}$
$4+8\mathbb{Z}$	$8\mathbb{Z}$	$8\mathbb{Z}$	$8\mathbb{Z}$	$8\mathbb{Z}$
$6+8\mathbb{Z}$	$8\mathbb{Z}$	$4+8\mathbb{Z}$	$8\mathbb{Z}$	$4+8\mathbb{Z}$

The rings cannot be isomorphic since $2\mathbb{Z}/8\mathbb{Z}$ doesn't have unity but $\mathbb{Z}_4$ does.

26.9

Let $\phi : \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z} : n \mapsto (n, 0)$. ϕ is a homormorphism, however the unity of $\mathbb{Z}$ is 1 and $\phi(1) = (1, 0)$ which is not the unity of $\mathbb{Z} \times \mathbb{Z}$.

26.12

The factor ring $\mathbb{Z}/2\mathbb{Z}$ is a field since $\mathbb{Z}/2\mathbb{Z}$ is isomorphic to $\mathbb{Z}_2$ which is a field.

26.13

The factor ring $\mathbb{Z}/6\mathbb{Z}$ is isomorphic to $\mathbb{Z}_6$ which has the zero divisor 3.

26.14

In $\mathbb{Z} \times \mathbb{Z}$, $(1, 0)(0, 1) = 0$ meaning it has zero divisors, however the factor ring $(\mathbb{Z} \times \mathbb{Z})/(\mathbb{Z} \times \{0\})$ is isomorphic to $\mathbb{Z}$ which has no zero divisors.

26.15

The subring $\{(n, n) : n \in \mathbb{Z}\}$ is a subring, however $(a, b)(n, n) = (an, bn)$ will not be an element in the subring if $a \neq b$.

26.18

Proof. Let $\phi : F \to R$ be a ring homorphism and $N = \ker \phi$. Assume that $N \neq \{0\}$. Then there is some element $a \in F$ and $a \in N$. Since F is a field, a is a unit, meaning $a^{-1} \in F$. Since N is also an ideal, then $a^{-1}a \in N$ meaning N contains unity. Therefore $x1 = x \in N$ for all $x \in F$, hence $N = F$. Therefore every element of F is mapped to 0. If $N = \{0\}$, then ϕ must be one-to-one. ■

26.20

Proof. By the binomial theorem,

$$(a+b)^p = \sum_{n=0}^{p} \binom{p}{n} a^n b^{p-n}.$$

Since p is prime, the coefficients $\binom{p}{n}$ for $1 \leq n \leq p-1$ will all be multiples of p. Since R is of characteristic p, these terms all will go to zero. Therefore for any $a, b \in R$, it follows $(a+b)^p = a^p + b^p$. Therefore

$$\phi_p(a+b) = (a+b)^p = a^p + b^p = \phi_p(a) + \phi_p(b).$$

Trivially $\phi_p(ab) = (ab)^p = a^p b^p = \phi_p(a)\phi_p(b)$. Therefore ϕ_p is a ring homomorphism. ■

26.22

Part A

Proof. Since an ideal is also a subring, then $\phi[N]$ is a subring of R'. Let $r \in R$ and $n \in N$. Since N is an ideal, then $rn \in N$ and $nr \in N$. Since ϕ is a ring homomorphism

$$\phi(r)\phi(n) = \phi(rn) \in \phi[N]$$
$$\phi(n)\phi(r) = \phi(nr) \in \phi[N]$$

Therefore $\phi[N]$ is an ideal of $\phi[R]$. ■

Part B

Take $\phi : \mathbb{Z} \to \mathbb{Q} : x \mapsto x$ and the ideal $3\mathbb{Z}$ of $\mathbb{Z}$. Since

$$\frac{1}{3} \cdot 3 = 1$$

then if $3\mathbb{Z}$ was an ideal of $\mathbb{Q}$, it would have to contain unity but it does not. So $3\mathbb{Z}$ is an ideal of $\mathbb{Z}$ but $\phi[3\mathbb{Z}]$ is not an ideal of $\mathbb{Q}$.

Part C

Proof. Note that N' is a subring of R. Let $r \in R$ and $n \in \phi^{-1}[N']$. That is $\phi(n) \in N'$. Then $\phi(rn) = \phi(r)\phi(n)$. Since N' is an ideal of $\phi[R]$ or R', then $\phi(r)\phi(n) \in N'$ and the same argument shows $\phi(nr) \in N'$. Therefore $\phi^{-1}[N']$ is an ideal of R. ■

26.26

Proof. Let $x, y \in I_a$. Then $ax = 0$ and $ay = 0$. Note that then

$$ax + ay = 0 \implies a(x + y) = 0.$$

Therefore $x + y \in I_a$. Since $a0 = 0$, I_a contains the additive identity. Furthermore, $a(-x) = -(ax) = 0$, therefore I_a contains additive inverses. Let $r \in R$. Note that $rax = r0 = 0$ and $axr = 0r = 0$ meaning $rI_a \subseteq I_a$ and $I_a r \subseteq I_a$. Therefore I_a is an ideal of R. ■