Problem 4.1.2

Describe the following sets in set-builder notation (look for a pattern).

(a) $\{\dots, -3, 0, 3, 6, 9, \dots\}$

(b) $\{-3, 1, 5, 9, 13, \dots\}$

(c) $\{1, \frac{1}{3}, \frac{1}{7}, \frac{1}{15}, \frac{1}{31}, \dots\}$

Solution

(a) $\{3n : n \in \mathbb{Z}\}$

(b) $\{4n + 1 : n \in \mathbb{Z}\}$

(c) $\left\{\dfrac{1}{4n-1} : n \in \mathbb{Z}\right\}$

Problem 4.1.5

Compare the sets $A = \{3x \in \mathbb{Z} : x \in 2\mathbb{Z}\}$ and $B = \{x \in \mathbb{Z} : x \equiv 12 \pmod 6\}$. Are they equal?

Solution

Set B can be rewritten as $\{x \in \mathbb{Z} : x \equiv 0 \pmod 6\}$. Additionally, set A can be rewritten as $\{6x : x \in \mathbb{Z}\}$. Set A is all the integer multiples of 6. This is equivalent to saying the set of all integers that reduce to $0 \pmod 6$, meaning $A = \{x \in \mathbb{Z} : x \equiv 0 \pmod 6\} = B$. Therefore both set A and B have the same elements and are therefore equal.

Problem 4.1.7

Let $A = \{1, 2, 3, 4\}$, and let B be the set $B = \big\{\{x, y\} : x, y \in A\big\}$.

(a) Describe B in roster notation.

(b) Now compute the cardinality of the sets

$$C = \Big\{\big\{x, \{y\}\big\} : x, y \in A\Big\}.$$

and

$$D = \left\{\Big\{\big\{x, \{y\}\big\} : x, y \in A\Big\}\right\}.$$

Compare them to B.

Solution

Part A

$$B = \big\{\{1,1\},\ \{2,2\},\ \{3,3\},\ \{4,4\},\ \{1,2\},\ \{1,3\},\ \{1,4\},\ \{2,3\},\ \{2,4\},\ \{3,4\}\big\}.$$

Part B

Set D will have a cardinality of 1 since it contains a single element within it, the set C. Set C will have the same cardinality as B since the number of unique elements is still the same, even though in C, one of elements of the inner sets is itself a set.

Problem 4.2.2

Let $A = \big\{x \in \mathbb{R} : x^3 + x^2 - x - 1 = 0\big\}$ and $B = \big\{x \in \mathbb{R} : x^4 - 5x^2 + 4 = 0\big\}$. Are either of the relations $A \subseteq B$ or $B \subseteq A$ true? Explain.

Solution

The roots for the polynomial $x^4 - 5x^2 + 4$ are the square root of the roots of the polynomial $y^2 - 5y + 4 \implies y = \{1,4\} \implies x = \{-4,-1,1,4\}$. Similarly, the roots for the polynomial $x^3 + x^2 - x - 1$ are $x = -1, 1$. Therefore $A = \{-1,1\}$ and $B = \{-4,-1,1,4\}$. Therefore the only relation that is true is $A \subseteq B$ since all the elements of A are within B. $B \subseteq A$ is not true since the element -4 is not within A.

Problem 4.2.4

Given $A \subseteq Z$ and $x \in \mathbb{Z}$, we say that x is A-mirrored if and only if $-x \in A$. We also define:

$$M_A := \{x \in \mathbb{Z} : x \text{ is A-mirrored}\}.$$

(a) What is the negation of x is A-mirrored.

(b) Find M_B for $B = \{0, 1, -6, -7, 7, 100\}$.

(c) Assume that $A \subseteq \mathbb{Z}$ is closed under addition (i.e., for all $x, y \in A$, we have $x + y \in A$). Show that M_A is closed under addition.

(d) In your own words, under which conditions is $A = M_A$?

Solution

Part A

The negation of x is A-mirrored is x is not A-mirrored, which in symbols means that x is not A-mirrored $\iff -x \notin A$.

Part B

$M_B = \{0, -7, 7\}$

Part C

> ***Proof.*** Let $A \subseteq \mathbb{Z}$ and assume that A is closed under addition. Consider the set M_A. Let $x, y \in M_A$. The construction of M_A then implies that $x, y \in A$ and $-x, -y \in A$. Since A is closed under addition, then $(-x) + (-y) = -x - y \in A$. Therefore it follows that $x + y \in M_A$. ■

Part D

In order for A and M_A to be equal, then every element in A must have its negative present.

Problem 4.2.5

Define the set $[1]$ by: $[1] = \{x \in \mathbb{Z} : x \equiv 1 \pmod 5\}$.

(a) Describe the set $[1]$ in roster notation.

(b) Compute the set $M_{[1]}$, as defined in Exercise 4.2.4

(c) Are the sets $[1]$ and $M_{[1]}$ equal? Prove/Disprove.

(d) Now consider the set $[10] = \{x \in \mathbb{Z} : x \equiv 10 \pmod 5\}$. Are the sets $[10]$ and $M_{[10]}$ equal? Prove/Disprove.

Solution

Part A

$$[1] = \{\ldots, -14, -9, -4, 1, 6, 11, 16, \ldots\}.$$

Part B

$$M_{[1]} = \emptyset.$$

Part C

Proof by contradiction that the sets $[1]$ and $M_{[1]}$ are not equal.

> ***Proof.*** Let $[1] = \{x \in \mathbb{Z} : x \equiv 1 \pmod 5\}$. Assume that $[1] = M_{[1]}$. Let $x = 1$. It follows that $x \in [1]$ since $1 \equiv 1 \pmod 5$. However, $-1 \notin [1]$ since $-1 \equiv 4 \pmod 5$, $1 \notin M_{[1]}$. Therefore, $[1] \neq M_{[1]}$; a contradiction. ■

Part D

Problem 4.3.2

Let $A = \{1, 3, 5, 7, 9, 11\}$ and $B = \{1, 4, 7, 10, 13\}$. What are the following sets?

(a) $A \cap B$

(b) $A \cup B$

(c) $A \setminus B$

(d) $(A \cup B) \setminus (A \cap B)$

Solution

(a) $A \cap B = \{1, 7\}$

(b) $A \cup B = \{1, 3, 4, 5, 7, 9, 10, 11, 13\}$

(c) $A \setminus B = \{3, 5, 9, 11\}$

(d) $(A \cup B) \setminus (A \cap B) = \{3, 4, 5, 9, 10, 11, 13\}$

Problem 4.3.5

Prove that $B \setminus A = B \iff A \cap B = \emptyset$.

Solution

> ***Proof.*** Let A and B be sets.
>
> ($\Rightarrow$) Assume that $B \setminus A = B$. Let $x \in B$. It follows that $x \in B \setminus A$. $B \setminus A$ is equivalent to $B \cap A^{\complement}$. Therefore $x \in B \cap A^{\complement}$. By definition of the intersection, $x \in A^{\complement}$ and therefore $x \notin A$. Therefore since x was an arbitrary element in B and it is not in A, $A \cap B$ will have no elements and therefore be equal to $\emptyset$. ■
>
> ($\Leftarrow$) Proof by contrapositive. Assume that $B \setminus A \neq B$. This can be rewritten as $B \cap A^{\complement} \neq B$. Let $x \in B$. It follows that $x \notin A^{\complement} \implies x \in A$. Since x is both in A and B, their interesection is non-empty. Alternatively, $A \cap B \neq \emptyset$. ■

Problem 4.3.7

Write out a formal proof of the set identity

$$A = (A \setminus B) \cup (A \cap B).$$

by showing that each side is a subset of the other. Now repeat your argument using only results from set algebra (Theorems 4.9 and 4.10).

Solution

Proof by showing that each side is a subset of the other.

> ***Proof.*** Let A and B be sets. Let $x \in A$. Consider the case where $x \notin B$. It follows that $x \in B^{\complement}$. It follows that for the set $A \setminus B$, or equivalently $A \cap B^{\complement}$ that $x \in A \cap B^{\complement}$ since x is in both A and $B^{\complement}$. Therefore the set $(A \setminus B) \cup (A \cap B)$ will contain the element x. Consider the case where $x \in B$. It follows that $x \in A \cap B$ since x is in both A and B. Since $x \in A \cap B$, it follows that $x \in (A \setminus B) \cup (A \cap B)$. Since x is an arbitrary element of A and is always an element of $(A \setminus B) \cup (A \cap B)$, it follows that $A \subseteq (A \setminus B) \cup (A \cap B)$. ■

Problem 4.4.1

For each of the following functions $f : A \to B$ determine whether f is injective, surjective or bijective. Prove your assertions.

(a) $f : [0, 3] \to \mathbb{R}$ where $f(x) = 2x$.

(b) $f : [3, 12) \to [0, 3)$ where $f(x) = \sqrt{x - 3}$.

(c) $f : (-4, 1] \to (-5, -3]$ where $f(x) = -\sqrt{x^2 + 9}$.

Solution

Part A

f is injective but not surjective.

> ***Proof.*** Let $f : [0, 3] \to \mathbb{R}$ where $f(x) = 2x$. Let $a, b \in [0, 3]$. Then
>
> $$\begin{aligned} f(a) &= f(b) \\ 2a &= 2b \\ a &= b. \end{aligned}$$
>
> It follows that f is injective. Now assume that f is surjective. Therefore for all $c \in \mathbb{R}$, there is $d \in [0, 3]$ such that $f(d) = c$. Consider the case where $c = 1000$. It

follows

$$\begin{aligned} f(d) &= c \\ f(d) &= 1000 \\ 2d &= 1000 \\ d &= 500. \end{aligned}$$

However, it was assumed that $d \in [0, 3]$, hence a contradiction. ■

Part B

f is bijective.

Proof. Let $f : [3, 13) \to [0, 3)$ where $f(x) = \sqrt{x-3}$. Let $a, b \in [3, 13)$. Then

$$\begin{aligned} f(a) &= f(b) \\ \sqrt{a-3} &= \sqrt{b-3} \\ a-3 &= b-3 \\ a &= b. \end{aligned}$$

Therefore f is injective. Consider now an aribitary element $y \in [0, 3)$. Let $x = y^2+3$, then $x \in [3, 12)$

$$\begin{aligned} 0 &\leq y < 3 \\ 0 &\leq y^2 < 9 \\ 3 &\leq y^2 + 3 < 12. \end{aligned}$$

It also follows that

$$\begin{aligned} f(x) &= \sqrt{x-3} \\ &= \sqrt{(y^2+3)-3} \\ &= \sqrt{y^2} \\ &= \pm y. \end{aligned}$$

Since $f(x) > 0$ for all input, then $f(x) = y$. Therefore f is surjective, meaning f is bijective. ■

Part C

f is surjective but not injective.

Problem 4.4.5

You may assume that $g : [2, \infty) \to \mathbb{R} : x \to \sqrt{x^3 - 8}$ is an injective function. Find a function $f : \mathbb{R} \to \mathbb{R}$ which is not injective, but for which the composition $f \circ g : [2, \infty) \to \mathbb{R}$ is injective. Justify your answer.

Solution

Let $f(x) = x^2$. $f(x)$ is not injective from $\mathbb{R} \to \mathbb{R}$. Let $a, b \in \mathbb{R}$ such that $b = -a$. Then $f(a) = a^2 = b^2 = (-b)^2 = f(b)$. However, $f \circ g : [2, \infty) \to \mathbb{R}$ is injective. Consider $x_1, x_2 \in [2, \infty)$. Let $h(x) = g(f(x)) = x^3 - 8$. Assume $h(x_1) = h(x_2)$. Then

$$\begin{aligned} h(x_1) &= h(x_2) \\ x_1^3 - 8 &= x_2^3 - 8 \\ x_1^3 &= x_2^3 \\ x_1 &= x_2. \end{aligned}$$

Therefore establishing an injection.

Problem 4.4.10

Suppose that $g \circ f$ is injective. Prove that f is injective.

Solution

> ***Proof.*** Let X, Y and Z be sets. Let $f : X \to Y$ and $g : Y \to Z$. Assume towards contradiction that $g \circ f$ is injective and f is not injective. Since f is not injective, then there exists $a, b \in X$ such that $f(a) = f(b)$. However, this implies that $g(f(a)) = g(f(b))$, meaning $g \circ f$ is not injective. Hence a contradiction. ■