0.1 Inverse Laplace Transform

The inverse Laplace Transform is defined as

$$\mathcal{L}^{-1}\{F(s)\} \stackrel{\text{def}}{=} f(t).$$

Since it is the inverse of a linear transformation, the inverse is also linear in nature.

Ex. Find $\mathcal{L}^{-1}\left\{\frac{s^2+s+1}{s^3+s}\right\}$

Note that this doesn't fit any of the common forms for know Laplace Transforms. However, the polynomial division hints towards partial fraction decomposition.

$$\frac{s^2+s+1}{s^3+s} = \frac{s^2+s+1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1}$$
$$\Downarrow$$
$$s^2+s+1 = A(s^2+1) + s(Bs+C)$$
$$s^2+s+1 = As^2 + A + Bs^2 + Cs$$
$$\Downarrow$$
$$(A, B, C) = (1, 0, 1)$$
$$\Downarrow$$
$$\frac{s^2+s+1}{s^3+s} = \frac{1}{s} + \frac{1}{s^2+1}.$$

Taking the Laplace transform of this is much easier, resulting in

$$\mathcal{L}^{-1}\left\{\frac{1}{s} + \frac{1}{s^2+1}\right\} = 1 + \sin(t).$$

0.1.1 Shifting Principle

In some cases, it is easier to write $F(s)$ as s shifted by some number a, such that the function in frequency space is $F(s+a)$. Note that

$$\begin{aligned}\mathcal{L}\left\{e^{-at}f(t)\right\} &= \int_0^\infty e^{-st}e^{-at}f(t)dt \\ &= \int_0^\infty e^{-(s+a)t}f(t) \\ &= F(s+a).\end{aligned}$$

Or equivalently,

$$\mathcal{L}^{-1}\{F(s+a)\} = e^{-at}f(t).$$

This is helpful in cases where a polynomial in a denominator can only be partially factored via complete the square

Ex. Find the inverse Laplace of $F(s) = \frac{1}{s^2+4s+13}$

In this case, $F(s)$ can not be factored in the denominator. However, it can be partially factored by completing the square.

$$\begin{aligned} F(s) &= \frac{1}{s^2+4s+13} \\ &= \frac{1}{(s^2+4s+4)+9} \\ &= \frac{1}{(s+2)^2+9}. \end{aligned}$$

Note that this result evokes the Laplace transformation of $\sin(\omega t)$

$$\mathcal{L}\{\sin(\omega t)\} = \frac{\omega}{s^2+\omega^2}.$$

The numerator doesn't fit the structure, but a simple manipulation resolves that issue

$$\frac{3}{3} \cdot \frac{1}{(s+2)^2+9} = \frac{1}{3} \cdot \frac{3}{(s+2)^2+9}.$$

Therefore the shifting principle can be utilized

$$\mathcal{L}^{-1}\left\{\frac{1}{3} \cdot \frac{3}{(s+2)^2+9}\right\} = \frac{1}{3} e^{-2t} \sin(3t).$$

Note that the shifting property appears when the Heaviside Function is involved. Consider

$$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}\mathcal{L}\{f(t)\} = e^{-as}F(s).$$

Another useful form is

$$\mathcal{L}\{f(t)u(t-a)\} = e^{-as}\mathcal{L}\{f(t+a)\}.$$

0.2 Laplace Transform of Derivatives

Since the Laplace transform is an integral transform, it would be advantageous to plug in time derivatives of functions to see their resultant function in frequency space. Assume a function $g(t)$ such that $\mathcal{L}\{g(t)\} = G(t)$.

$$\mathcal{L}\left\{g'(t)\right\} = \int_0^\infty e^{-st} g'(t) dt$$

Utilize integration by parts with $u = e^{-st}$ and $dv = g'(t)dt$

$$\begin{aligned} &= g(t)e^{-st}\Big|_0^\infty + s\int_0^\infty e^{st} g(t) dt \\ &= \lim_{t\to\infty} g(t)e^{-st} - g(0) + sG(s) \end{aligned}$$

In order for this to exist, $g(t)$ must grow slower than the exponential. Stated formally, $|g(t)| < Me^{ct}$ for appropriate positive constants M and c. Assuming this holds,

$$\mathcal{L}\left\{g'(t)\right\} = sG(s) - g(0).$$

0.2.1 Laplace Transform of Integrals

Laplace transforms act quite nicely when the input function is an integral. Consider an integral of the form

$$\int_0^t f(\tau)\mathrm{d}\tau.$$

Then the Laplace transform of $f(t)$ is

$$\mathcal{L}\left\{\int_0^t f(\tau)\mathrm{d}\tau\right\} = \frac{1}{s}\mathcal{L}\{f(t)\} = \frac{F(s)}{s}.$$

This can help solve harder Laplace transforms that would involve processes like partial fraction decomposition.

Ex. Find inverse Laplace of $\frac{1}{s(s^2+1)}$

Notice that

$$\frac{1}{s(s^2+1)} = \frac{1}{s}\mathcal{L}\{\sin(t)\}.$$

In this case, $f(t) = \sin(t)$, therefore,

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} &= \int_0^t \sin(\tau)\mathrm{d}\tau \\ &= -\cos(t)\Big|_0^t \\ &= 1 - \cos(t).\end{aligned}$$