Binary Structures

1.1 Isomorphic Binary Structures

Definition 1 (Binary Algebraic Structure)**.** $\langle S, * \rangle$ is a binary algebraic structure if S is a set and $*$ is a binary operation defined over S.

Definition 2 (Homomorphism Property)**.** Two binary structures $\langle S, * \rangle$ and $\langle S', *' \rangle$ are homomorphic if there exists a mapping $\phi : S \to S'$ such that

$$\phi(x * y) = \phi(x) *' \phi(y).$$

Remark. *If there exists a homorphic mapping between two algebraic structures that is additionally one-to-one, then the two algebraic structures are isomorphic.*

The following example structures are isomorphic

$$\begin{aligned}\langle [0,1], +_1 \rangle &\simeq \langle [0,c], +_c \rangle \\ \langle U_n, \cdot \rangle &\simeq \langle \mathbb{Z}_n, +_n \rangle \\ \langle \mathbb{Z}, + \rangle &\simeq \langle 2\mathbb{Z}, + \rangle.\end{aligned}$$

Consider the structures $\langle \mathbb{Q}, + \rangle$ and $\langle \mathbb{R}, + \rangle$. Are these two structures isomorphic? Indeed they cannot be because $|\mathbb{Q}| = \aleph_0$ and $|\mathbb{R}| \neq \aleph_0$. Therefore there cannot exist a onto-to-one and onto map between the structures, hence they cannot be isomorphic.

In general one can follow a process to show that two binary structures $\langle S, * \rangle$ and $\langle S', *' \rangle$ are isomorphic.

1. Define some function ϕ that will be shown to be an isomorphism from S to S'.
2. Show that ϕ is one-to-one.
3. Show that ϕ is onto.
4. Show that the homomorphic property holds under ϕ. That is that $\phi(x * y) = \phi(x) *' \phi(y)$ for all $x, y \in S$