2.2.1

a) $A \cap B$ is the set of students within a mile of the school that walk to class

b) $A \cup B$ is the set of students who are within a mile of the school or walk to class

c) $A - B$ is the set of students who are within a mile of the school but do not walk to class

d) $B - A$ is the set of students who walk to class that are not within a mile of the school

2.2.3

a) $\{0, 1, 2, 3, 4, 5, 6\}$

b) $\{3\}$

c) $\{1, 2, 4, 5\}$

d) $\{0, 6\}$

2.2.9

a) We will show that both sets are subsets of each other.

> ***Proof.*** Let $x \in U$. If $x \in A$, then $x \in A \cup \overline{A}$. If $x \notin A$, then $x \in \overline{A}$ and hence $x \in A \cup \overline{A}$. Therefore $U \subseteq A \cup \overline{A}$. Since A and $\overline{A}$ are subsets of U by definition, then $A \cup \overline{A} \subset U$. Therefore $A \cup \overline{A} = U$. ■

b) We will show that $A \cap \overline{A}$ contains no elements.

> ***Proof.*** Let $x \in A$. Then by the definition of the complement, $x \notin \overline{A}$. Therefore none of the elements in A are in $\overline{A}$ and hence $A \cap \overline{A} = \varnothing$. ■

2.2.13

> ***Proof.*** Let $x \in A$. Then $x \in A \cup B$ and hence $x \in A \cap (A \cup B)$. Therefore $A \subseteq A \cap (A \cup B)$. Let $x \in A \cap (A \cup B)$. Then $x \in A$ and hence $A \cap (A \cup B) \subseteq A$. Therefore $A \cap (A \cup B) = A$. ■

2.2.21

Part A

> ***Proof.*** Let $x \in A - B$. Then $x \in A$ and $x \notin B$. Therefore $x \in \overline{B}$ meaning $x \in A \cap \overline{B}$. Hence $A - B \subseteq A \cap \overline{B}$. Let $x \in A \cap \overline{B}$. Then $x \in A$ and $x \in \overline{B}$. Therefore $x \notin B$ meaning $x \in A - B$. Hence $A \cap \overline{B} \subseteq A - B$ giving $A - B = A \cap \overline{B}$. ■

Part B

Proof. Let $x \in A$. Then either $x \in B$ or $x \notin B$ meaning $x \in A \cap B$ or $x \in A \cap \overline{B}$. Therefore $x \in (A \cap B) \cup (A \cap \overline{B})$ hence $A \subseteq (A \cap B) \cup (A \cap \overline{B})$. Let $x \in (A \cap B) \cup (A \cap \overline{B})$. Then $x \in A \cap B$ or $A \cap \overline{B}$. In either case $x \in A$ meaning $A \subseteq (A \cap B) \cup (A \cap \overline{B})$. Since both are subsets of each other, $(A \cap B) \cup (A \cap \overline{B}) = A$. ■

2.2.27

a) $\{4, 6\}$

b) $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

c) $\{4, 5, 6, 8, 10\}$

2.2.51

Proof. Assume towards contradiction that A is an infinite set and $A \cup B$ is a finite set. Since $A \cup B$ is finite, it has n elements where n is a natural number. However, A has more than n elements since it is infinite. Since all the elements of A are in $A \cup B$, then $A \cup B$ has more than n elements, a contradiction. ■

2.3.7

a) $\text{dom}(f) = \mathbb{Z}^+ \times \mathbb{Z}^+$ and $\text{range}(f) = \mathbb{Z}^+$

b) $\text{dom}(f) = \mathbb{Z}^+$ and $\text{range}(f) = \{n \in \mathbb{Z} : 0 \leq n \leq 9\}$

c) $\text{dom}(f) =$ set of all bit strings and $\text{range}(f) = \mathbb{N}_0$

d) $\text{dom}(f) =$ set of all bit strings and $\text{range}(f) = \mathbb{N}_0$

2.3.11

Only (a) is onto since every element of $\{a, b, c, d\}$ is in the range of f.

2.3.13

Only (a) and (d) are onto. (b) is not onto since it maps integers to strictly positive integers. (c) is not onto since there is no integer whose cube is 2.

2.3.15

a) The function is onto since for any $n \in \mathbb{Z}$ we have $f(n, 0) = n$.

b) The function is not onto since there is no $m, n \in \mathbb{Z}$ such that $m^2 + n^2 = 10$.

c) The function is onto for any $n \in \mathbb{Z}$ we have $f(n, 0) = n$.

d) The function is not onto since $f[\mathbb{Z} \times \mathbb{Z}] = \mathbb{Z}^+ \neq \mathbb{Z}$

2.3.23

Only (a) and (c) are bijections since the image of the $\mathbb{R}$ under both is $\mathbb{R}$ itself. (b) and (d) are not bijections because they are not onto.

2.3.31

a) $f(S) = \{1,0,0,0,1,3\}$

b) $f(S) = \{0,0,1,3,5,8\}$

c) $f(S) = \{0,8,16,40\}$

d) $f(S) = \{1,12,33,65\}$

2.3.37

Proof. It is not true in general. Consider $f : 2\mathbb{Z} \to \mathbb{Z} : n \mapsto \frac{n}{2}$ and $g : \mathbb{Z} \to \mathbb{Z} : n \mapsto 2n$. Note that f is onto and $(f \circ g)(n) = n$ and is also onto. However g is not onto. ■

2.3.47

Proof. Note that

$$f^{-1}(\overline{S}) = \{a \in A : f(a) \notin S\} = \overline{\{a \in A : f(a) \in S\}} = \overline{f^{-1}(S)}.$$

Hence both are equal. ■

2.3.73

Proof. Let $x \in U$ and f_S be defined as given.

a) If $x \in A \cap B$, then $x \in A$ and $x \in B$. Therefore $f_{A\cap B}(x) = 1$ and $f_A(x) \cdot f_B(x) = 1 \cdot 1 = 1$. If $x \notin A \cap B$, then $x \notin A$ or $x \notin B$. Therefore $f_{A\cap B}(x) = 0$ and since one of $f_A(x)$ or $f_B(x)$ are zero meaning $f_A(x) \cdot f_B(x) = 0$. Therefore in either case $f_{A\cap B}(x) = f_A(x) \cdot f_B(x)$.

b) There are three cases to consider.

- Assume that $x \in A \cup B$ and $x \in A \cap B$. Then $f_{A\cup B}(x) = 1$ and

$$f_A(x) + f_B(x) - f_A(x) \cdot f_B(x) = 1 + 1 - f_{A\cap B}(x) = 1 + 1 - 1 = 1.$$

- Assume that $x \in A \cup B$ and $x \notin A \cap B$. Then $f_{A\cup B}(x) = 1$. Since $x \notin A \cap B$, then either $x \notin A$ or $x \notin B$. Therefore

$$f_A(x) + f_B(x) - f_A(x) \cdot f_B(x) = 0 + 1 + f_{A\cap B}(x) = 0 + 1 + 0 = 1.$$

- Assume that $x \notin A \cup B$. Then $f_{A\cup B}(x) = 0$. Since $x \notin A \cup B$, $x \notin A$ and $x \notin B$ meaning

$$f_A(x) + f_B(x) - f_A(x) \cdot f_B(x) = 0 + 0 - 0 \cdot 0 = 0.$$

In all possible cases, the two sides are equal.

c) Let $x \in \overline{A}$. Then $f_{\overline{A}}(x) = 1$. Since $x \in \overline{A}$, $x \notin A$ meaning $1 - f_A(x) = 1 - 0 = 1$. Same holds for $x \notin \overline{A}$.

d) Let $x \in A \bigoplus B$. Then $x \in A, x \notin B$ or $x \notin A, x \in B$. Therefore

$$f_A(x) + f_B(x) - 2f_A(x) \cdot f_B(x) = 1.$$

Same holds for $x \notin A \bigoplus B$.

■

2.4.3

a) $a_0 = 2, a_1 = 3, a_2 = 5, a_3 = 9$

b) $a_0 = 1, a_1 = 4, a_2 = 27, a_3 = 256$

c) $a_0 = 0, a_1 = 0, a_2 = 1, a_3 = 1$

d) $a_0 = 0, a_1 = 1, a_2 = 2, a_3 = 3$

2.4.7

The sequences $a_n = 2^n$, $a_n = 1 + \left\lfloor \frac{(n+1)^2}{3} \right\rfloor$, and $a_{n+1} = a_n + (n-1)$ all satisfy this condition.

2.4.11

a) $a_0 = 6, a_1 = 17, a_2 = 49, a_3 = 143, a_4 = 421$

b) Note that

- $a_2 = 49 = 85 - 36 = 5(17) - 6(6) = 5a_1 - 5a_0$
- $a_3 = 143 = 245 - 102 = 5(49) - 6(17) = 5a_2 - 6a_1$
- $a_4 = 421 = 715 - 294 = 5(143) - 6(49) = 5a_3 - 6a_2$

c) For $n \geq 2$,

$$\begin{aligned} 5a_{n-1} - 6a_{n-2} &= 5(2^{n-1} + 5 \cdot 3^{n-1}) - 6(2^{n-2} + 5 \cdot 3^{n-2}) \\ &= 5 \cdot 2^{n-1} + 25 \cdot 3^{n-1} - 6 \cdot 2^{n-2} - 30 \cdot 3^{n-2} \\ &= 5 \cdot 2^{n-1} + 25 \cdot 3^{n-1} - 3 \cdot 2^{n-1} - 10 \cdot 3^{n-1} \\ &= 2 \cdot 2^{n-1} + 15 \cdot 3^{n-1} \\ &= 2^n + 5 \cdot 3^n \\ &= a_n \end{aligned}$$

2.4.13

a) Yes

b) No

c) No

d) Yes

e) Yes

f) Yes

g) No

h) No

2.4.15

a) $a_{n-1} + 2a_{n-2} + 2n - 9 = -n + 1 + 2 + 2(-n+4) + 2n - 9 = -n + 2 = a_n$

b)

$$\begin{aligned} a_{n-1} + 2a_{n-2} + 2n - 9 &= 5(-1)^{n-1} - n + 3 + 2\left(5(-1)^{n-2} - n + 4\right) + 2n - 9 \\ &= 5(-1)^{n-1} + 10(-1)^{n-2} - n + 2 \\ &= (-5)(-1)^n + 10(-1)^n - n + 2 \\ &= 5(-1)^n - n + 2 \\ &= a_n \end{aligned}$$

c)

$$\begin{aligned} a_{n-1} + 2a_{n-2} + 2n - 9 &= 3(-1)^{n-1} + 2^{n-1} - n + 3 + 2\left(3(-1)^{n-2} + 2^{n-2} - n + 4\right) + 2n - 9 \\ &= 3(-1)^{n-1} + 2^{n-1} + 6(-1)^{n-2} + 2^{n-1} - n + 2 \\ &= -3(-1)^n + 6(-1)^n + 2^n - n + 2 \\ &= 3(-1)^n + 2^n - n + 2 \\ &= a_n \end{aligned}$$

d)

$$\begin{aligned} a_{n-1} + 2a_{n-2} + 2n - 9 &= 7 \cdot 2^{n-1} - n + 3 + 2\left(7 \cdot 2^{n-2} - n + 4\right) + 2n - 9 \\ &= 7 \cdot 2^{n-1} + 7 \cdot 2^{n-1} - n + 2 \\ &= 7 \cdot 2 \cdot 2^{n-1} - n + 2 \\ &= 7 \cdot 2^n - n + 2 \\ &= a_n \end{aligned}$$

2.4.25

a) 1 one followed by 1 zero, 2 ones followed by 2 zeroes, etc. Next three terms would be $1, 1, 1$

b) Each integer greater than 1 with each even integer repeated twice. Next three terms would be $9, 10, 10$

c) $a_n = 2^{\frac{n}{2}}$ for even n and 0 otherwise. Next three terms would be $32, 0, 64$

d) $a_n = 3 \cdot 2^n$. Next three terms would be $384, 768, 1536$

e) $a_n = 15 - 7n$. Next three terms would be $-34, -41, -48$

f) $a_n = \frac{(n+1)^2+n+5}{2}$. Next three terms would be $57, 68, 80$

g) $a_n = 2n^3$. Next three terms would be $1024, 1458, 2000$

h) $a_n = 1 + n!$. Next three terms would be $362881, 3628801, 39916801$

2.4.29

a) $\displaystyle\sum_{k=1}^{5} k + 1 = 5 + \sum_{k=1}^{5} k = 5 + \frac{5 \cdot 6}{2} = 20$

b) $\displaystyle\sum_{j=0}^{4} (-2)^j = 1 - 2 + 4 - 8 + 16 = 11$

c) $\displaystyle\sum_{i=1}^{10} 3 = 30$

d) $\displaystyle\sum_{j=0}^{8} 2^{j+1} - 2^j = \sum_{j=0}^{8} 2^j = 2^9 - 1$

2.4.31

a) $S_8 = 1533$

b) $S_8 - 1 = 510$

c) $S_8 - (1 - 3) = 4923$

d) $S_8 = 9842$

2.4.33

a) 21

b) 78

c) 18

d) 18

2.4.35

Since for every term a_i with $0 < i < n$ appears both as a positive and negative term in the sum, they all cancel out leaving behind just $a_n - a_0$.

2.4.39

$$\sum_{k=100}^{200} k = \sum_{k=0}^{100} (k + 100) = 100 \cdot 101 + \frac{100(101)}{2} = 15150.$$

2.5.1

a) Countably infinite; $1, 2, 3, 4, \ldots$

b) Countably infinite; $0, 2, -2, 4, -4, \ldots$

c) Countably infinite; $99, 98, 97, \ldots$

d) Uncountable

e) Finite

f) Countably infinite; $0, 7, 7, 14, 14, \ldots$

2.5.3

a) Countable; map n to the bit string contain n ones

b) Countable; follow the diagonalization arguement excluding top three rows

c) Uncountable

d) Uncountable

2.5.7

For each n, take the guest in room $2n$ and put them in room n of the original building and the guest in room $2n+1$ into room n of the new building.

2.5.11

a) $[-1,0] \cap [0,1] = \{0\}$
b) $([-2,-1] \cup \mathbb{Z}) \cap ([1,2] \cup \mathbb{Z}) = \mathbb{Z}$
c) $[-2,2] \cap [-1,1] = [-1,1]$

2.5.17

No. Notice that $[-4,4] - [-1,1] = [-4,-1) \cup (1,4]$ which is still uncountable.

2.5.27

Proof. Let $A_1, A_2, A_3, \ldots$ be a list of countably countable sets. For a given set A_i, its elements can be listed out since it is countable in the manner

$$a_{i1}, a_{i2}, a_{i3}, \ldots.$$

Listing out the elements of $\bigcup A_i$ is simply listing out all elements a_{ij} where $i+j=2$, then $i+j=3$, and so forth. ∎

2.5.31

Proof. Let $m+n$ be fixed. That is $m+n=z$ for some $z \in \mathbb{Z}^+$. Note then that

$$\frac{(z-2)(z-1)}{2} + 1 \leq f(m,n) \leq \frac{(z-2)(z-1)}{2} + (z-1).$$

Showing f is bijective means showing that $x+1$ leads to a range of values that picks up where x left off from. Equivalently, this amounts to showing $f(x-1,1)+1 = f(1,x)$. Note that

$$f(x-1,1)+1 = \frac{(x-2)(x-1)}{2} + (x-1) + 1 = \frac{x^2-x+2}{2} = \frac{x(x-1)}{2} = f(1,x).$$

Therefore f is bijective. ∎

2.6.1

a) A is a 3×4 matrix

b) $\begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$

c) $\begin{bmatrix} 2 & 0 & 4 & 6 \end{bmatrix}$

d) 1 e) $A^t = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 1 & 4 & 3 \\ 3 & 6 & 7 \end{bmatrix}$

2.6.5

$$A = \begin{bmatrix} \frac{9}{5} & -\frac{6}{5} \\ -\frac{1}{5} & \frac{4}{5} \end{bmatrix}.$$

2.6.7

Note that for any entry in A_{ij} that

$$A_{ij} + \mathbf{0}_{ij} = \mathbf{0}_{ij} + A_{ij} = A_{ij} + 0 = A_{ij}.$$

Therefore the resultant matrix remains unchanged.

2.6.11

If AB has size $m \times n$, then BA has size $n \times m$.

2.6.15

$$A^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}.$$

This can be shown via induction using the base case

$$A^2 = AA = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

2.6.25

The system can be expressed as a matrix linear equation

$$\underbrace{\begin{bmatrix} 7 & -8 & 5 \\ -4 & 5 & -3 \\ 1 & -1 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 5 \\ -3 \\ 0 \end{bmatrix}}_{b}.$$

The solution is then $A^{-1}b$ where A^{-1} is given in exercise 18. Hence

$$x_1 = 1, x_2 = -1, x_3 = -2.$$

2.6.27

a)

$$A \vee B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

b)

$$A \wedge B = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

c)

$$A \odot B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

2.6.29

a)

$$A^{[2]} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

b)

$$A^{[3]} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

c)

$$A \vee A^{[2]} \vee A^{[3]} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

3.1.1

max := 1 → i := 2 → max := 8 → i := 3 → max := 12 → i := 4 → i := 5 → i := 6 → i := 7 → max := 14 → i := 8 → i := 9 → i := 10 → i := 11

3.1.3

Algorithm 1 Sum of sequece of integers $a_1, \ldots, a_n$

function SUM_SEQUENCE($a_1, \ldots, a_n$)
    sum := a_1
    for $i = 2$ to n **do**
        sum = sum + a_i
    end for
    return sum
end function

3.1.9

Algorithm 2 Determine if a given string is a palindrome

```
function IS_PALINDROME(s_0 s_1 ... s_{n-1} : string)
    for i = 0 to ⌊n/2⌋ − 1 do
        if s_i ≠ s_{n−i−1} then
            return False
        end if
    end for
    return True
end function
```

3.1.13

Linear Search) i := 1 → i := 2 → i := 3 → i := 4 → i := 5 → i := 6 → i := 7 → location := 7

Binary Search) i := 1 → j := 8 → m := 4 → i := 5 → m := 6 → i := 7 → m := 7 → j := 7 → location := 7

3.2.1

a) $C = 1, k = 10$
b) $C = 4, k = 7$
c) Is not $O(x)$
d) $C = 5, k = 1$
e) $C = 1, k = 0$
f) $C = 1, k = 2$

3.2.3

Since $x^4 + 9x^3 + 4x + 7 \leq 10000x^3$ for $x > 10$, we have witness $C = 10000, k = 10$.

3.2.7

a) $n = 3$
b) $n = 3$
c) $n = 1$
d) $n = 0$

3.2.11

Since we are dealing with polynomials,

$$O\left(3x^4 + 1\right) = O\left(x^4\right) = O\left(\frac{x^4}{2}\right).$$

3.3.3

It is $O(n^2)$ since

$$\sum_{i=0}^{n} \sum_{k=i}^{n} 1 \sim n^2.$$

3.3.11

Part A

Algorithm 3 Determine if a disjoint subset pair exists

```
function EXISTS_DISJOINT(S_1, ..., S_n: subsets of {1, ..., n})
    disjoint := False
    for i = 1 to n do
        for j = i + 1 to n do
            disjoint = True
            for k = 1 to n do
                if k ∈ S_i ∧ k ∈ S_j then
                    disjoint = False
                end if
            end for
        end for
    end for
    return disjoint
end function
```

Part B

Since there are $O(n^2)$ sets considered and $O(n)$ integers tested for each, then in total the answer is on the order of $O(n^3)$.

4.1.3

Proof. Let $a, b, c \in \mathbb{Z}$ such that $a|b$ and $a \neq 0$. Then $\exists m \in \mathbb{Z}$ such that $b = am$. Note then that $bc = amc = a(mc)$. Therefore bc is a multiple of a meaning $a|bc$ ■

4.1.7

Proof. Let $a, b, c \in \mathbb{Z}$ with $a, c \neq 0$ and $ac|bc$. Then $\exists m \in \mathbb{Z}$ such that $bc = m(ac)$. Therefore $b = ma$ since $c \neq 0$ meaning b is a multiple of a, hence $a|b$. ■

4.1.11

Proof. Let $a \in \mathbb{Z}$ such that $3 \nmid a$. Then $a = 3k + 1$ or $a = 3k + 2$ for some $k \in \mathbb{Z}$. Therefore

$$(a + 1)(a + 2) = 3(3k + 2)(k + 1).$$

or

$$(a + 1)(a + 2) = 3(k + 1)(3k + 4).$$

In either case $(a + 1)(a + 2)$ is divisible by 3. ■

4.1.13

a) 2, 5
b) 11, 10
c) 34, 7
d) 77, 0
e) 0, 0
f) 0, 3
g) 1, 2
h) 4, 0

4.1.15

a) $11 + 80 \equiv 91 \equiv 7 \pmod{12} \implies 7:00$

b) $12 - 40 \equiv -4 \equiv 8 \pmod{12} \implies 8:00$

c) $100 + 6 \equiv -2 \equiv 10 \pmod{12} \implies 10:00$

4.1.17

a) 10
b) 8
c) 0
d) 9
e) 6
f) 11

4.1.27

a) $13 \equiv 1 \pmod{3}$

b) $-97 \equiv 2 \pmod{11}$

c) $155 \equiv 190 - 35 \equiv 0 + 3 \equiv 3 \pmod{19}$

d) $-221 \equiv -230 + 9 \equiv 9 \pmod{23}$

4.1.31

a) $a = -15$

b) $a = 24 - 31 = -7$

c) $a = 99 + 41 = 140$

4.1.33

$$-76, -51, -26, -1, 24, 4974, 99.$$

4.1.43

a) $0 \cdot 2 \equiv 1 \cdot 2 \equiv 0 \pmod{2}$ but $0 \not\equiv 1 \pmod{2}$

b) Choosing $m = 5, a = b = 3, c = 1$ and $d = 6$ gives $3 \equiv 3 \pmod{5}$ and $1 \equiv 6 \pmod{5}$, but $31 \equiv 3 \not\equiv 4 \equiv 729 \equiv 36 \pmod{5}$.