33.2

Part A

Since $e = e \cdot e^{2\pi i n}$ for $n \in \mathbb{Z}$, it follows that

$$\log e = \ln e + 2\pi i n = 1 + 2\pi i n, n \in \mathbb{Z}.$$

Part B

Since $i = e^{i\frac{\pi}{2}+2\pi n}$ for $n \in \mathbb{Z}$, it follows that

$$\log i = \ln 1 + i\left(\frac{\pi}{2} + 2\pi n\right) = \left(2n + \frac{1}{2}\right)\pi i, n \in \mathbb{Z}.$$

Part C

Since $-1 + i\sqrt{3} = 2\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = 2e^{i\left(\frac{2\pi}{3}+2\pi n\right)}$ for $n \in \mathbb{Z}$, it follows that

$$\log\left(-1 + i\sqrt{3}\right) = \ln 2 + i\left(\frac{2\pi}{3} + 2\pi n\right) = \ln 2 + 2\pi i\left(n + \frac{1}{3}\right), n \in \mathbb{Z}.$$

33.4

Proof. Since $i^2 = e^{i\pi}$ and π is its argument in the branch, $\log\left(i^2\right) = \ln 1 + i\pi = i\pi$. Furthermore, $i = e^{i\frac{\pi}{2}}$ which has argument $\frac{5\pi}{2}$ in the branch meaning $2\log i = 2(\ln 1 + i\frac{5\pi}{2}) = 5\pi i \neq i\pi$ ∎

33.11

Proof. The second partials of $F(x, y) = \ln\left(x^2 + y^2\right)$ are

$$\frac{\partial^2}{\partial x^2}\ln\left(x^2 + y^2\right) = \frac{\partial}{\partial x}\frac{2x}{x^2 + y^2} = \frac{(x^2 + y^2) \cdot 2 - 2x(2x)}{(x^2 + y^2)^2} = 2 \cdot \frac{y^2 - x^2}{(x^2 + y^2)^2}$$
$$\frac{\partial^2}{\partial y^2}\ln\left(x^2 + y^2\right) = \frac{\partial}{\partial y}\frac{2y}{x^2 + y^2} = \frac{(x^2 + y^2) \cdot 2 - 2y(2y)}{(x^2 + y^2)^2} = 2 \cdot \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

Therefore

$$F_{xx} + F_{yy} = 2\left(\frac{y^2 - x^2 + x^2 - y^2}{(x^2 + y^2)^2}\right) = 0.$$

Hence $\ln\left(x^2 + y^2\right)$ is harmonic everywhere excluding the origin. ∎

Proof. Note that $\ln\left(x^2 + y^2\right)$ is the real component of $2 \cdot \log z$ for $z = x + iy$ on any branch and is therefore harmonic on the branches domain. Since the branches

$$\alpha = \left\{-\pi, \frac{\pi}{2}\right\}$$

together cover all $z \neq 0$ and $\ln(x^2+y^2)$ is then harmonic on both, it follows that $\ln(x^2+y^2)$ is harmonic everywhere except the origin. ■

34.1

Proof. Let $z_1, z_2 \in \mathbb{C}$ with $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ and $-\pi < \theta_1, \theta_2 < \pi$. Note that $-2\pi < \theta_1 + \theta_2 < 2\pi$. Let

$$N = \begin{cases} -1 & \theta_1 + \theta_2 > \pi \\ 1 & \theta_1 + \theta_2 \leq -\pi \\ 0 & \text{otherwise} \end{cases}.$$

and therefore $\operatorname{Arg} z_1 z_2 = \theta_1 + \theta_2 + 2\pi N$. It follows that

$$\begin{aligned} \operatorname{Log}(z_1 z_2) = \ln(r_1 r_2) + i \operatorname{Arg} z_1 z_2 &= \ln r_1 + \ln r_2 + i(\theta_1 + \theta_2 + 2\pi N) \\ &= \ln r_1 + i\theta_1 + \ln r_2 + i\theta_2 + 2\pi N i \\ &= \operatorname{Log} z_1 + \operatorname{Log} z_2 + 2\pi N i \end{aligned}$$ ■

34.2

Proof. Let $z_1, z_2 \in \mathbb{C}$ with $z_2 \neq 0$. First note that for all $z \neq 0$

$$\begin{aligned} \log \frac{1}{z} &= \log\left(\frac{1}{|z|} e^{-i \arg z}\right) \\ &= \ln \frac{1}{|z|} - i \arg z \\ &= -\ln|z| - i \arg z \\ &= -(\ln|z| + i \arg z) = -\log z \end{aligned}$$

Therefore

$$\begin{aligned} \log \frac{z_1}{z_2} &= \log\left(z_1 \cdot \frac{1}{z_2}\right) \\ &= \log(z_1) + \log\left(\frac{1}{z_2}\right) \\ &= \log(z_1) - \log(z_2) \end{aligned}$$ ■

34.5

Proof. Let $z = re^{i\theta}$ where $r > 0$ and $\theta \in (-\pi, \pi]$. Then

$$z^{\frac{1}{n}} = \left\{ r^{\frac{1}{n}} e^{i\frac{\theta + 2k\pi}{n}} : 0 \leq k \leq n-1 \right\}.$$

Therefore

$$\begin{aligned}
\log z^{\frac{1}{n}} &= \left\{\ln\left(r^{\frac{1}{n}}\right) + i\left(\frac{\theta + 2\pi k}{n} + 2\pi p\right) : 0 \leq k \leq n-1, p \in \mathbb{Z}\right\} \\
&= \left\{\frac{1}{n}\ln r + i\left(\frac{\theta + 2\pi(np+k)}{n}\right) : 0 \leq k \leq n-1, p \in \mathbb{Z}\right\} \\
&= \frac{1}{n}\{\ln r + i(\theta + 2\pi(np+k)) : 0 \leq k \leq n-1, p \in \mathbb{Z}\}
\end{aligned}$$

Note that any integer $q \in \mathbb{Z}$ can be written as $p \equiv k \pmod{n}$ and therefore $q = np + k$ meaning

$$\begin{aligned}
&= \frac{1}{n}\{\ln r + i(\theta + 2\pi q) : q \in \mathbb{Z}\} \\
&= \frac{1}{n}\log z
\end{aligned}$$

■

36.2

Part A

$$(-i)^i = \exp(i\,\mathrm{Log}(-i)) = \exp\left(i\,\mathrm{Log}(e^{-i\frac{\pi}{2}})\right) = \exp\left(i\left(\ln 1 - i\frac{\pi}{2}\right)\right) = e^{\frac{\pi}{2}}.$$

Part B

$$\begin{aligned}
\left(\frac{e}{2}(-1-\sqrt{3}i)\right)^{3\pi i} &= \exp\left[3\pi i \cdot \mathrm{Log}\left(\frac{e}{2}(-1 - i\sqrt{3})\right)\right] \\
&= \exp\left[3\pi i(\mathrm{Log}\left(\frac{e}{2}\right) + \mathrm{Log}(-1 - i\sqrt{3}))\right] \\
&= \exp\left[3\pi i\left(1 - \ln 2 + \mathrm{Log}\left(2\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)\right)\right)\right] \\
&= \exp\left[3\pi i\left(1 - \ln 2 + \ln 2 + i\left(-\frac{2\pi}{3}\right)\right)\right] \\
&= \exp\left[3\pi i\left(1 - i\frac{2\pi}{3}\right)\right] \\
&= \exp\left[2\pi^2 + 3\pi i\right] \\
&= \exp\left[2\pi^2\right] \cdot \exp[3\pi i] = -\exp\left(2\pi^2\right)
\end{aligned}$$

Part C

$$\begin{aligned}
(1-i)^{4i} &= \exp[4i\,\mathrm{Log}(1-i)] \\
&= \exp\left[4i\,\mathrm{Log}\left(\sqrt{2}\left(\frac{1}{\sqrt{2}}\right)-\frac{1}{\sqrt{2}}i\right)\right] \\
&= \exp\left[4i\left(\frac{1}{2}\ln 2 - i\left(\frac{\pi}{4}\right)\right)\right] \\
&= \exp[\pi + 2i\ln 2] \\
&= \exp[\pi]\exp[2i\ln 2] \\
&= \exp[\pi](\cos(2\ln 2) + i\sin(2\ln 2))
\end{aligned}$$

36.5

Proof. Let $z_0 = re^{i\theta} \in \mathbb{C}$ with $\theta \in (-\pi, \pi]$. Then the principal root of $z_0^{\frac{1}{n}}$ from Section 10 is

$$c_0 = r^{\frac{1}{n}}e^{i\frac{\theta}{n}}.$$

Using the complex power function,

$$\begin{aligned}
z_0^{\frac{1}{n}} = \exp\left[\frac{1}{n}\,\mathrm{Log}\,z_0\right] &= \exp\left[\frac{1}{n}(\ln r + i\theta)\right] \\
&= \exp\left[\ln r^{\frac{1}{n}} + i\frac{\theta}{n}\right] \\
&= \exp\left[\ln r^{\frac{1}{n}}\right]\cdot\exp\left[i\frac{\theta}{n}\right] = r^{\frac{1}{n}}e^{i\frac{\theta}{n}} = c_0
\end{aligned}$$ ■

36.9

$$\frac{\mathrm{d}}{\mathrm{d}z}c^{f(z)} = \frac{\mathrm{d}}{\mathrm{d}z}\exp[f(z)\log c] = \exp[f(z)\log c]f'(z)\log c = \boxed{c^{f(z)}\cdot f'(z)\log c}.$$

38.11

Proof. Let $z = x + iy$. Since

$$\begin{aligned}
\sin \overline{z} &= \frac{e^{i\overline{z}} - e^{-i\overline{z}}}{2i} \\
&= \frac{e^{i(x-iy)} - e^{-i(x-iy)}}{2i} \\
&= \frac{e^{y+ix} - e^{-y-ix}}{2i} \\
&= \frac{e^{y}e^{ix} - e^{-y}e^{-ix}}{2i} \\
&= \frac{e^{y}(\cos x + i\sin x) - e^{-y}(\cos(-x) + i\sin(-x))}{2i} \\
&= \frac{e^{y}(\cos x + i\sin x) - e^{-y}(\cos x - i\sin x)}{2i} \\
&= \frac{(e^{y} - e^{-y})\cos x + i(e^{y} + e^{-y})\sin x}{2i} \\
&= \frac{1}{2}(e^{y} + e^{-y})\sin x - \frac{i}{2}(e^{y} - e^{-y})\cos x
\end{aligned}$$

Therefore the partials are

$$\begin{aligned}
u_x &= \frac{1}{2}(e^{y} + e^{-y})\cos x & u_y &= \frac{1}{2}(e^{y} - e^{-y})\sin x \\
v_x &= \frac{1}{2}(e^{y} - e^{-y})\sin x & v_y &= -\frac{1}{2}(e^{y} + e^{-y})\cos x
\end{aligned}$$

Applying the C.R. equations gives

$$u_x = v_y \implies (e^{y} + e^{-y})\cos x = 0 \implies \cos x = 0$$

and

$$u_y = -v_x \implies (e^{y} - e^{-y})\sin x = 0 \implies y = 0 \text{ or } \sin x = 0$$

Since there is no x such that $\cos x = \sin x = 0$, the only places where C.R. holds is when $y = 0$ and $\cos x = 0$. However, there are only countably distinct points that satsify this and therefore no neighborhood around them can be differentiable. Hence the function is nowhere analytic. ■

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&= \frac{e^{i(x-iy)} + e^{-i(x-iy)}}{2} \\
&= \frac{e^{y+ix} + e^{-y-ix}}{2} \\
&= \frac{e^{y}e^{ix} + e^{-y}e^{-ix}}{2} \\
&= \frac{e^{y}(\cos x + i\sin x) + e^{-y}(\cos(-x) + i\sin(-x))}{2} \\
&= \frac{e^{y}(\cos x + i\sin x) + e^{-y}(\cos x - i\sin x)}{2} \\
&= \frac{1}{2}(e^{y} + e^{-y})\cos x + \frac{i}{2}(e^{y} - e^{-y})\sin x
\end{aligned}$$

Therefore the partials are

$$\begin{aligned}
u_x &= -\frac{1}{2}(e^{y} + e^{-y})\sin x & \quad u_y &= \frac{1}{2}(e^{y} - e^{-y})\cos x \\
v_x &= \frac{1}{2}(e^{y} + e^{-y})\cos x & \quad v_y &= \frac{1}{2}(e^{y} + e^{-y})\sin x
\end{aligned}$$

Applying the C.R. equations gives

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Since there is no x such that $\cos x = \sin x = 0$, the only places where C.R. holds is when $y = 0$ and $\cos x = 0$. However, there are only countably distinct points that satsify this and therefore no neighborhood around them can be differentiable. Hence the function is nowhere analytic. ■

38.14

Part A

Proof. Note that

$$\begin{aligned}
\overline{\cos(iz)} &= \overline{\frac{e^{i(iz)} + e^{-i(iz)}}{2}} \\
&= \overline{\frac{e^{-z} + e^{z}}{2}} \\
&= \frac{e^{\overline{-z}} + e^{\overline{z}}}{2} \\
&= \frac{e^{-\overline{z}} + e^{\overline{z}}}{2}
\end{aligned}$$

and

$$\begin{aligned}
\cos(i\overline{z}) &= \frac{e^{i(i\overline{z})} + e^{-i(i\overline{z})}}{2} \\
&= \frac{e^{-\overline{z}} + e^{\overline{z}}}{2}
\end{aligned}$$

Therefore $\overline{\cos(iz)} = \cos(i\overline{z})$. ■

Part B

Proof. Note that

$$\begin{aligned}
\overline{\sin(iz)} &= \overline{\left(\frac{e^{i(iz)} - e^{-i(iz)}}{2i}\right)} \\
&= \overline{\left(\frac{e^{-z} - e^{z}}{2i}\right)} \\
&= -\frac{e^{\overline{-z}} - e^{\overline{z}}}{2i} \\
&= \frac{e^{\overline{z}} - e^{-\overline{z}}}{2i}
\end{aligned}$$

and

$$\begin{aligned}
\sin(i\overline{z}) &= \frac{e^{i(i\overline{z})} - e^{-i(i\overline{z})}}{2i} \\
&= \frac{e^{-\overline{z}} - e^{\overline{z}}}{2i}
\end{aligned}$$

Therefore $\overline{\sin(iz)} = \sin(i\overline{z})$ when

$$\begin{aligned}
\frac{e^{\overline{z}} - e^{-\overline{z}}}{2i} &= \frac{e^{-\overline{z}} - e^{\overline{z}}}{2i} \\
2e^{\overline{z}} &= 2e^{-\overline{z}} \\
2e^{\overline{z}} &= 2e^{-\overline{z}} \\
e^{\overline{z}} &= e^{-\overline{z}} \\
e^{z} &= e^{-z}
\end{aligned}$$

Let $z = x + iy$. Then

$$\begin{aligned}
e^{z} &= e^{-z} \\
e^{x}(\cos y + i \sin y) &= e^{-x}(\cos y - i \sin y)
\end{aligned}$$

which is true when $e^x \cos y = e^{-x} \cos y$ and $e^x \sin y = -e^{-x} \sin y$ meaning $x = 0$ and

$$\sin y = -\sin y \implies \sin y = 0 \implies y = n\pi i, n \in \mathbb{Z}$$

Hence z must be $n\pi i$ for $n \in \mathbb{Z}$. ■

42.2

Part A

$$\begin{aligned}
\int_0^1 (1+it)^2 \mathrm{d}t &= \int_0^1 (1 - t^2 + 2it)\mathrm{d}t \\
&= \int_0^1 (1-t^2)\mathrm{d}t + i\int_0^1 2t\mathrm{d}t \\
&= \left[t - \frac{t^3}{3}\right]_0^1 + i\left[t^2\right]_0^1 \\
&= \frac{2}{3} + i
\end{aligned}$$

Part B

$$\begin{aligned}
\int_1^2 \left(\frac{1}{t} - i\right)^2 \mathrm{d}t &= \int_1^2 \left(\frac{1}{t^2} - \frac{2i}{t} - 1\right)\mathrm{d}t \\
&= \int_1^2 \left(\frac{1}{t^2} - 1\right)\mathrm{d}t - 2i\int_1^2 \frac{1}{t}\mathrm{d}t \\
&= \left[-\frac{1}{t} - t\right]_1^2 - 2i[\ln t]_1^2 \\
&= -\frac{1}{2} - 2i\ln 2 = -\frac{1}{2} - i\ln 4
\end{aligned}$$

Part C

$$\int_0^{\frac{\pi}{6}} e^{i2t}\mathrm{d}t = \frac{1}{2i}e^{i2t}\Big|_0^{\frac{\pi}{6}} = \frac{1}{2i}e^{i\frac{\pi}{3}} - \frac{1}{2i}e^0 = \frac{1}{2i}\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}-1\right) = -\frac{i}{2}\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{4}+\frac{i}{4}.$$

Part D

Since $\operatorname{Re} z > 0$, $z \neq 0$. Then

$$\int_0^{\infty} e^{-zt} = -\frac{1}{z}e^{-zt}\Big|_0^{\infty} = \frac{1}{z}\left(1-\lim_{t\to\infty} e^{-zt}\right) = \frac{1}{z}(1-0) = \frac{1}{z}.$$

42.3

Proof. Let $m, n \in \mathbb{Z}$. Note that $e^{im\theta}e^{-in\theta} = e^{i\theta(m-n)}$ Consider two cases

$(m = n)$ Since $m = n$, $e^{i\theta(m-n)} = e^0 = 1$. Therefore

$$\int_0^{2\pi} e^{im\theta}e^{-in\theta}\mathrm{d}\theta = \int_0^{2\pi} 1\mathrm{d}\theta = 2\pi.$$

$(m \neq n)$ Since $m \neq n$, $e^{i\theta(m-n)}$ is non constant and $\frac{1}{m-n}$ is defined. Therefore

$$\begin{aligned}\int_0^{2\pi} e^{im\theta}e^{-in\theta}\mathrm{d}\theta &= \int_0^{2\pi} e^{i\theta(m-n)}\mathrm{d}\theta \\ &= -\frac{i}{\theta(m-n)}e^{i\theta(m-n)}\Big|_0^{2\pi} \\ &= -\frac{i}{\theta(m-n)}\left(e^{2\pi i(m-n)} - e^0\right)\end{aligned}$$

Since $m-n \in \mathbb{Z}\setminus\{0\}$, $e^{2\pi i(m-n)} = e^{2\pi i}$ which is 1 it follows that

$$= -\frac{i}{\theta(m-n)}(1-1) = 0$$

Hence

$$\int_0^{2\pi} e^{im\theta}e^{-in\theta}\mathrm{d}\theta = \begin{cases} 0 & m \neq n \\ 2\pi & m = n \end{cases}$$ ■

43.5

Proof. Let $f(z) = u(x,y) + iv(x,y)$ and $z(t) = x(t) + iy(t)$ for $a \le t \le b$. Then

$$w(t) = f(z(t)) = u(x(t), y(t)) + iv(x(t), y(t)).$$

Therefore by the multivariable chain rule,

$$w'(t) = (u_x x' + u_y y') + i(v_x x' + v_y y').$$

Since $w(t_0) = f(z(t_0))$ is analytic, then the C.R. equations hold at t_0 meaning at t_0

$$\begin{aligned} w'(t) &= u_x x' + u_y y' + i(v_x x' + v_y y') \\ &= u_x x' - v_x y' + i(v_x x' + u_x y') \\ &= (u_x + iv_x)(x'(t) + iy'(t)) \\ &= f'(z(t))z'(t) \end{aligned}$$

■