Problem 1

Proof. Let n denote the size of C. We proceed with induction on n. Consider the base case $n = 1$. Then

$$\det(C - tI) = \left|-a_0 - t\right| = (-1)^n(t + a_0)$$

hence the base case holds. If we therefore we consider the case of $n+1$ expanding along the first row gives

$$\det(C - tI) = -t\begin{vmatrix} -t & 0 & \cdots & -a_1 \\ 1 & -t & \cdots & -a_2 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 1 & -a_{n-1} - t \end{vmatrix} - a_0 I.$$

∎

Problem 2

It is true for all matrices A.

Proof. Assume that $a_0 \neq 0$. Note that

$$p_A(0) = (-1)^n(0^n + a_{n-1} \cdot 0^{n-1} + \ldots + a_0) = (-1)^n a_0 \neq 0.$$

By the definition of the characteristic polynomial, we know $p_A(t) = \det(A - tI)$. Therefore

$$p_A(0) = \det(A - 0(I)) = \det A = (-1)^n a_0 \neq 0.$$

Since the determinant of A is non-zero, it must be invertible. ∎

Problem 3

Part A

Proof. We will show $\langle A, B\rangle_F$ satisfies the 4 requirements of being an inner product. Let $A, B \in M_{n\times n}(\mathbb{R})$ and $s \in \mathbb{R}$.

1. We want to show linearity of the inner product. Take $C \in M_{n\times n}(\mathbb{R})$. Note that both the transpose and trace are linear maps, therefore

$$\begin{aligned} \langle A + C, B\rangle_F &= \mathrm{tr}\Big((A + C)^T B\Big) \\ &= \mathrm{tr}\Big((A^T + C^T)B\Big) \\ &= \mathrm{tr}\Big(A^T B + C^T B\Big) \\ &= \mathrm{tr}\Big(A^T B\Big) + \mathrm{tr}\Big(C^T B\Big) = \langle A, B\rangle_F + \langle C, B\rangle_F \end{aligned}$$

   Therefore $\langle\cdot,\cdot\rangle_F$ satisfies linearity.

2. We want to show $\langle sA, B\rangle = s\,\langle A, B\rangle$.

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Part B

Proof. Let $A \in M_{n\times n}(\mathbb{R})$ and assume that A is diagonalizable. Then $A = PDP^{-1}$ where P is unitary and D is a diagonal matrix with its entries being the eigenvalues of A. Note that

$$\begin{aligned} A^TA &= \left(PDP^{-1}\right)^T \left(PDP^{-1}\right) \\ &= \left(P^{-1}\right)^T D^T P^T PDP^{-1} \end{aligned}$$

Since P is unitary, its transpose is its inverse giving

$$\begin{aligned} &= PD^TIDP^{-1} \\ &= PD^2P^{-1}. \end{aligned}$$

Therefore A^TA is also diagonalizable. Note that D^2 will be a diagonal matrix as well with entries λ_i^2 where λ_i are the original entries from D. Since D^2 is the diagonal matrix in the decomposition of A^TA, its entries λ_i^2 are the eigenvalues of A^TA. Since the trace of a matrix is equal to the sum of its eigenvalues, it follows

$$\operatorname{tr}\left(A^TA\right) = \sum_{i=1}^n \lambda_i^2 \implies \|A\|_F = \sqrt{\sum_{i=1}^n \lambda_i^2}$$

which was to be shown. ■