7.1/7.2

Part A

$$s_1 = \frac{1}{4}$$
$$s_2 = \frac{1}{7}$$
$$s_3 = \frac{1}{10}$$
$$s_4 = \frac{1}{13}$$
$$s_5 = \frac{1}{16}$$

The sequence converges to 0.

Part D

$$s_1 = \frac{\sqrt{2}}{2}$$
$$s_2 = 1$$
$$s_3 = \frac{\sqrt{2}}{2}$$
$$s_4 = 0$$
$$s_5 = -\frac{\sqrt{2}}{2}$$

The sequence does not converge as it cycles.

7.3

A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T
1	1	0	1	DNC	1	∞	DNC	0	$\frac{7}{2}$	∞	DNC	DNC	DNC	0	2	0	1	$\frac{4}{3}$	0

7.4

Part A

$$s_n = \frac{\sqrt{2}}{n} \in \mathbb{I}, \lim_{n\to\infty} s_n = 0 \in \mathbb{Q}$$

Part B

Let F_n denote the n'th fibonacci number with $F_1 = F_2 = 1$.

$$s_n = \frac{F_{n+1}}{F_n} \in \mathbb{Q}, \lim_{n\to\infty} s_n = \phi = \frac{1+\sqrt{5}}{2} \in \mathbb{I}$$

7.5

Part B

Note that $\sqrt{n^2+n} - n = \frac{n}{\sqrt{n^2+n}+n} \sim \frac{n}{2n}$. Therefore

$$\lim_{n\to\infty} = \sqrt{n^2+n} - n = \frac{1}{2}$$

8.1

Part C

Proof. Take $\epsilon > 0$. Let $N \in \mathbb{N} > \frac{3}{5\epsilon}$. Note that

$$\left|\frac{2n-1}{3n+2} - \frac{2}{3}\right| = \left|\frac{7}{3} \cdot \frac{1}{3n+2}\right| = \frac{7}{3} \cdot \frac{1}{3n+2} \leq \frac{3}{5n}.$$

For $n > N$, $\frac{3}{5n} \leq \frac{3}{5N} < \epsilon$. Therefore

$$\left|\frac{2n-1}{3n+2} - \frac{2}{3}\right| < \epsilon, \forall n > N$$

hence $\lim_{n\to\infty} \frac{2n-1}{3n+2} = \frac{2}{3}$. ■

Part D

Proof. Take $\epsilon > 0$. Let $N \in \mathbb{N} > \max\left\{6, 6 + \frac{1}{\epsilon}\right\}$. Note that $N > 6$, meaning for all $n > N$

$$\left|\frac{n+6}{n^2-6}\right| = \frac{n+6}{n^2-6}.$$

Since $0 < n^2 - 36 < n^2 - 6$ for all $n > N$,

$$\frac{n+6}{n^2-6} \leq \frac{n+6}{n^2-36} = \frac{1}{n-6} < \frac{1}{N-6} < \epsilon, \forall n > N.$$

Therefore, $\left|\frac{n+6}{n^2-6}\right| < \epsilon$ for all $n > N$, hence $\lim_{n\to\infty} \frac{n+6}{n^2-6} = 0$. ■

8.2

Part B

$$\lim_{n\to\infty} \frac{7n-19}{3n+7} = \frac{7}{3}$$

Proof. Take $\epsilon > 0$. Let $N \in \mathbb{N} > \frac{5}{9\epsilon}$. Note that

$$\left|\frac{7n-19}{3n+7} - \frac{7}{3}\right| = \left|\frac{-5}{9n+21}\right| = \frac{5}{9n+21}.$$

Since $9n \leq 9n + 21$,

$$\frac{5}{9n+21} \leq \frac{5}{9n} < \frac{5}{9N} < \epsilon, \forall n > N.$$

Therefore

$$\left|\frac{7n-19}{3n+7} - \frac{7}{3}\right| < \epsilon, \forall n > N$$

hence $\lim_{n\to\infty} \frac{7n-19}{3n+7} = \frac{7}{3}$. ■

Part E

$$\lim_{n\to\infty} \frac{1}{n}\sin(n) = 0$$

Proof. Take $\epsilon > 0$. Take $N \in \mathbb{N} > \epsilon$. Note that since $-1 \leq \sin(x) \leq 1$ for all $x \in \mathbb{R}$, $|\sin(n)| \leq 1$ for all $x \in \mathbb{R}$. Therefore for $n \in \mathbb{N}$,

$$\left|\frac{1}{n}\sin(n)\right| = \frac{1}{n}|\sin(n)| \leq \frac{1}{n}.$$

For $n > N$, $\frac{1}{n} < \frac{1}{N} < \epsilon$. Therefore

$$\left|\frac{1}{n}\sin(n)\right| < \epsilon, \forall n < N$$

hence $\lim_{n\to\infty} \frac{1}{n}\sin(n) = 0$. ■

8.4

Proof. Let t_n be a bounded sequence and s_n be a sequence that converges to 0. Since t_n is bounded, $|t_n| \leq M \in \mathbb{R}$ for all $n \in \mathbb{N}$. Since s_n converges to 0,

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } |s_n| < \frac{\epsilon}{M}, \forall n > N$$

Note that $|s_n \cdot t_n| = |s_n||t_n| \leq |s_n|M < \epsilon$ for all $n > N$. Therefore $\lim_{n\to\infty} t_n s_n = 0$. ■

8.5

Part A

Proof. Let a_n, b_n, s_n be sequences such that $a_n \leq s_n \leq b_n$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = s \in \mathbb{R}$. Since a_n and b_n converge, for any $\epsilon > 0$,

$$\begin{aligned} \exists N_1 \in \mathbb{N} \text{ s.t. } |a_n - s| < \epsilon, \forall n > N_1 &\implies a_n > s - \epsilon, \forall n > N_1 \\ \exists N_2 \in \mathbb{N} \text{ s.t. } |b_n - s| < \epsilon, \forall n > N_2 &\implies b_n < s + \epsilon, \forall n > N_2 \end{aligned}$$

By taking $N = \max\{N_1, N_2\}$, it follows that that $a_n > s - \epsilon$ and $b_n < s + \epsilon$ for all $n > N$. Note then that

$$s - \epsilon < a_n \leq s_n \leq b_n < s + \epsilon \implies |s_n - s| < \epsilon, \forall n > N.$$

Therefore $\lim_{n\to\infty} s_n = s$. ■

8.9

Part A

Proof. Let s_n be a convergent sequence such that $s_n \geq a \in \mathbb{R}$ for all but finitely many n. Let $s = \lim_{n\to\infty} s_n$ and let $S = \{n \in \mathbb{N} : s_n < a\}$. Since S is finite, choose $N = \max S$. Note that then for all $n > N$, $s_n \geq a$. Assume towards contradiction that $s < a$. Then $a - s > 0$. Choose $\epsilon > 0$ such that $0 < \epsilon < a - s$. Since s_n converges, $\exists N_0 \in \mathbb{N}$ such that $s_n < s + \epsilon < a$ for all $n > N_0$. This also holds for all $n > \max N, N_0$. However, this means that there is an $n > N$ such that $s_n < a$, which contradicts the fact that N is the maximal index such that $s_N < a$. Therefore $s \geq a$. ■