Continuity

1.1 Continuous Functions

Definition 1.1 (Real Valued Function). Let $E \subset \mathbb{R}$. Then a mapping $f : E \to \mathbb{R}$ is a real valued function. If a domain E isn't specified, the largest possible subset of $\mathbb{R}$ is taken where $f(x)$ makes sense.

Definition 1.2 (Continuity). Let $f : E \to \mathbb{R}$ be a real valued function and $S \subset E$. Then

1. f is continuous at x_0 if $x_0 \in E$ iff
$$\lim f(x_n) = f(x_0)$$
for any sequence (x_n) in E that converges to x_0.
2. f is continuous on S iff f is continuous at x_0 for all $x_0 \in S$
3. f is continuous iff it is continuous on all of E

Theorem 1.1 (Epsilon-Delta Continuity). A real valued function f is continuous at some point $x_0 \in \mathrm{dom}(f)$ iff

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } |x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon.$$

Definition 1.3 (Operations on Real Valued Functions). Let $f : \mathrm{dom}(f) \subset \mathbb{R} \to \mathbb{R}$ and $g : \mathrm{dom}(g) \subset \mathbb{R} \to \mathbb{R}$. Then define

$$f \pm g : \mathrm{dom}(f) \cap \mathrm{dom}(g) \to \mathbb{R} : x \mapsto f(x) \pm g(x)$$
$$f \cdot g : \mathrm{dom}(f) \cap \mathrm{dom}(g) \to \mathbb{R} : x \mapsto f(x) \cdot g(x)$$

For division,

$$\frac{f}{g} : \mathrm{dom}(f) \cap \{x \in \mathrm{dom}(g) : g(x) \neq 0\} \to \mathbb{R} : x \mapsto \frac{f(x)}{g(x)}.$$

For maxima and minima,

$$\max(f, g) : \mathrm{dom}(f) \cap \mathrm{dom}(g) \to \mathbb{R} : x \mapsto \max\{f(x), g(x)\}$$
$$\min(f, g) : \mathrm{dom}(f) \cap \mathrm{dom}(g) \to \mathbb{R} : x \mapsto \min\{f(x), g(x)\}$$

Finally for composition,

$$g \circ f : \{x \in \mathrm{dom}(f) : f(x) \in \mathrm{dom}(g)\} \to \mathbb{R} : x \mapsto g(f(x)).$$

Theorem 1.2 (Basic Operations Preserve Continuity). Let f, g be real valued functions.

1. If f, g are continuous at x_0, then $f \pm g$ and $f \cdot g$ are continuous at x_0.
2. If f, g are continuous at x_0 and $g(x_0) \neq 0$, then $\frac{f}{g}$ is continuous at x_0.

Proof. Let f, g be real valued functions.

1. Assume that f, g are continuous at x_0. Let (x_n) be a sequence in $\text{dom}(f) \cap \text{dom}(g)$ that converges to x_0. Since f, g are continuous, then $f(x_n) \to x_0$ and $g(x_n) \to g(x_0)$ which by the limit theorems gives $f(x_n)+g(x_n) \to f(x)+g(x)$ meaning $f+g$ is continuous at x_0. The argument holds for $f \cdot g$.
2. Assume that f, g are continuous at x_0 and $g(x) \neq 0$ for all $x \in \text{dom}(f) \cap \text{dom}(f)$. Let (x_n) be a sequence in $\text{dom}(f) \cap \text{dom}(g)$ that converges to x_0. Since f, g are continuous, then $f(x_n) \to x_0$ and $g(x_n) \to g(x_0)$. Note that $g(x_n) \neq 0$ for all n by the assumption. Therefore by limit theorems it follows that $\frac{f(x_n)}{g(x_n)} \to \frac{f(x_0)}{g(x_0)}$, hence $\frac{f}{g}$ is continuous at x_0.

■

Theorem 1.3 (Composition Preserves Continuity). Let f, g be real valued functions. If f is continuous at x_0 and g is continuous at $f(x_0)$, then $g \circ f$ is continuous at x_0.

Proof. Let f, g be real valued functions and assume that f is continuous at x_0 and g is continuous at $f(x_0)$. Let (x_n) be a sequence in $\{x \in \text{dom}(f) : f(x) \in \text{dom}(g)\}$ such that $x_n \to x_0$. Since f is continuous at x_0, $f(x_n) \to x_0$. Let (y_n) be the sequence defined by $y_n = f(x_n)$. Then $y_0 = f(x_0)$. Therefore since g is continuous at $f(x_0)$, $g(y_n) \to g(y_0) = g(f(x_0))$. Therefore $g \circ f$ is continuous at x_0. ■

Theorem 1.4 (Maximum Preserves Continuity). Let f, g be real valued functions. If f is continuous at x_0 and g is continuous at $f(x_0)$, then $\max(f, g)$ is continuous at x_0.

Proof. First note that

$$\max(r, s) = \frac{1}{2}(r+s) + \frac{1}{2}|r-s|, \forall r, s \in \mathbb{R}.$$

Consider the case $r \geq s$. Then

$$\frac{1}{2}(r+s) + \frac{1}{2}|r-s| = \frac{1}{2}(r+s) + \frac{1}{2}(r-s) = r = \max(r, s).$$

If $r < s$, then

$$\frac{1}{2}(r+s) + \frac{1}{2}|r-s| = \frac{1}{2}(r+s) - \frac{1}{2}(r-s) = s = \max(r, s).$$

Therefore the original equation holds. Note then that

$$\max(f(x), g(x)) = \frac{1}{2}(f(x) + g(x)) - \frac{1}{2}|f(x) + g(x)|.$$

Since the absolute value function is continuous on all of $\mathbb{R}$, by 1.2 and 1.4 it follows that the maximum of two functions in also continuous. ■

1.2 Properties of Continuous Functions

Definition 1.4 (Function Boundedness). Let $f : \mathrm{dom}(f) \subset \mathbb{R} \to \mathbb{R}$ be a real valued function. f is bounded iff there is some $M \in \mathbb{R}$ such that

$$|f(x)| \leq M, \forall x \in \mathrm{dom}(f).$$

Example 1.1. Consider the function $\sqrt{x-1}$. Assume towards contradiction that it is bounded. That is, $\exists M \in \mathbb{R}$ such that

Theorem 1.5. Let $f : [a, b] \to \mathbb{R}$ be continuous. Then

1. f is bounded
2. f assumes its max and its min. That is $\exists x_m, x_M \in [a, b]$ such that

$$f(x_m) \leq f(x) \leq f(x_M), \forall x \in [a, b].$$

Proof. Let $f : [a, b] \to \mathbb{R}$ be continuous.

1. Assume towards contradiction that f is not bounded. Then $\forall n \in \mathbb{N}$, there is some $x_n \in [a, b]$ such that $|f(x_n)| \geq n$. Therefore $(x_n)_{n\in\mathbb{N}}$ is a sequence in $[a, b]$. Since (x_n) is bounded, there is some subsequence (n_j) such that (x_{n_j}) converges to $x_\infty \in [a, b]$. Since f is continuous, then $|f(x_{n_j})| \xrightarrow{j\to\infty} |f(x_\infty)|$. However, $n_j \leq |f(x_{n_j})|$ meaning the limit as $j \to \infty$ would be infinite. Hence a contradiction.

2. By the first claim, f is bounded. Therefore $m = \inf_{x\in[a,b]} f(x) > -\infty$. Then $\forall n \in \mathbb{N}, \exists x_n \in [a, b]$ such that $m \leq f(x_n) \leq m + \frac{1}{n}$. This gives a sequence (x_n) that is bounded (because it is in $[a, b]$). Therefore by Bolzano Weistrass, $\exists(n_j)$ such that $x_{n_j} \to x_{\min}$. Since f is continuous,

$$\lim_{j\to\infty} m \leq \lim_{j\to\infty} f(x_{n_j}) \leq \lim_{j\to\infty} m + \frac{1}{n_j} \implies f(x_{\min}) = m.$$

   Therefore the infimum m is achieved by f in its domain and therefore m is the minimum value and $x_{\min}$ is the minimum argument. The argument for the maximum follows the same by replacing inf with sup and flipping the inequality to squeeze towards the supremum.

■

Remark. If the interval is not closed, then the theorem is not true in general. Consider

$$f : (0, 1] \to \mathbb{R} : x \mapsto \frac{1}{x}.$$

Note that f is continuous but is unbounded and has no max. Furthermore

$$f : (-1, 1) \to \mathbb{R} : x \mapsto x^2.$$

f in this case is continuous and bounded, but it doesn't have a maximum.

Theorem 1.6 (Intermediate Value Theorem). Let $f : I \to \mathbb{R}$ be a continuous function where I is an interval in $\mathbb{R}$. If $y_0 \in \big(\min(f(a), f(b)), \max(f(a), f(b))\big)$ with $a < b$ and $a, b \in I$, then there exists $x_0 \in (a, b)$ such that $f(x_0) = y_0$.

Proof. WLOG, take $f(a) > y_0 > f(b)$. Let $S = \{x \in [a, b] : f(x) > y_0\}$. Note S is non empty since $a \in S$. Since S is bounded, let $x_0 = \sup S$. Therefore for all $n \in \mathbb{N}$, there is some $s_n \in S$ such that $x_0 \geq s_n \geq x_0 - \frac{1}{n}$ since $x_0 - \frac{1}{n}$ is not an upper bound. Therefore

$$\lim s_n = x_0, f(s_n) > y_0, \forall n \implies f(x_0) = \lim f(s_n) \geq y_0.$$

Next, take $x_0 \leq \xi_n = \min\left\{x_0 + \frac{1}{n}, b\right\}$. Then

$$f(x_0) = \lim f(\xi_n) \leq y_0.$$

Therefore $y_0 \leq f(x_0) \leq y_0 \implies f(x_0) = y_0$. ∎

Corollary 1.1. If $f : I \to \mathbb{R}$ where I is an interval in $\mathbb{R}$ is continuous, then

$$f(I) = \{f(x) : x \in I\}$$

is an interval or a singleton.

Proof. Let $J = f(I)$. Take $y_0, y_1 \in J$ with $y_0 < y_1$. Note that if $y_0 < y < y_1$, then by 1.6, $y \in J$. If $\inf J < \sup J$, then J is an interval and if they are the same then J is a singleton. ∎

Example 1.2. Let $f : [0, 1] \to [0, 1]$ be continuous. Then $\exists x_0 \in [0, 1]$ such that $f(x_0) = x_0$. That is, f has a fixed point.

Proof. Let $g : [0, 1] \to [0, 1] : x \mapsto f(x) - x$. Note then that $g(0) = f(0) - 0 \geq 0$ and $g(1) = f(1) - 1 \leq 0$. Therefore by 1.6, $\exists x_0 \in [0, 1]$ such that $g(x_0) = 0$ meaning $f(x_0) - x_0 = 0 \implies f(x_0) = x_0$. ∎

Example 1.3. If $y > 0$, then it has a positive m root.

Proof. Let $f(x) = x^m, x \geq 0$. Note that f is continuous and $\exists b > 0$ such that $y < b^m$. Then

$$f(0) < y \leq f(b) \implies \exists x \in (0, b) \text{ s.t. } f(x) = x^m = y.$$

■

Theorem 1.7. Let $g : J \to \mathbb{R}$ be a strictly increasing function over the interval J. Then if $g(J)$ is also an interval, g is continuous.

Proof. Take $x_0 \in J$ such that x_0 is not an endpoint. Then $g(x_0)$ is not an end point of $g(J) = I$ by monotonicity. Therefore it is possible to find a neighborhood $(g(x_0) - \epsilon_0, g(x_0) + \epsilon_0) \subset I$. Take ϵ such that $0 < \epsilon < \epsilon_0$ and for some x_1 and x_2 in J,

$$g(x_1) = g(x_0) - \epsilon, g(x_2) = g(x_0) + \epsilon.$$

By monotonicity,

$$x_1 < x_0 < x_2 \text{ and } g(x_0) - \epsilon \leq g(x_1) < g(x) < g(x_2) \leq g(x_0) + \epsilon, \forall x \in (x_1, x_2)$$

which implies $|g(x) - g(x_0)| < \epsilon$. Take $\delta = \min\{x_2 - x_0, x_1 - x_0\}$. Then

$$|x - x_0| < \delta \implies x_1 < x_0 < x_2 \implies |g(x) - g(x_0)| < \epsilon.$$

Therefore g is continuous. ■

Theorem 1.8. Let $f : I \to \mathbb{R}$ be continuous and strictly increasing where I is an interval. Then

1. $f(I) = J$ is an interval
2. $f^{-1} : J \to I$ exists and is strictly increasing and continuous.

Proof. ■

Theorem 1.9. Let $f : I \to \mathbb{R}$ be one to one and continuous where I is an interval. Then f is strictly increasing or strictly decreasing.

Proof. Let $f : I \to \mathbb{R}$ be one to one and continuous where I is an interval.

1. If $a < b < c$ in I, then $f(a) < f(b) < f(c)$. Assume towards contradiction that this is not the case. Then $f(b) > \max\{f(a), f(c)\}$ or $f(b) < \min\{f(a), f(c)\}$. Consider the second case. Take $f(b) < y < \min\{f(a), f(b)\}$ and use 1.6 on $[a, b]$ and $[b, c]$ to find $x_1 \in (a, b)$ and $x_2 \in (b, c)$ such that $f(x_1) = f(x_2) = y$. This contradicts the assumption that f is one to one since $x_1 \neq x_2$. The other case follows similarly.
2. Take $a_0 < b_0$ with $a_0, b_0 \in I$. WLOG, let $f(a_0) < f(b_0)$. Note that $f(x) < f(a_0)$

for $x < a_0$ since $x < a_0 < b_0$ and therefore follows from (1). Additionally, $f(a_0) < f(x) < f(b_0)$ for $a_0 < x < b_0$ and $f(x) > f(b_0)$ for $x > b_0$. It then follows that $f(x) < f(a_0)$ for all $x < a_0$ and $f(x) > f(a_0)$ for all $x > a_0$.

3. Take $x_1, x_2 \in I$ such that $x_1 < x_2$. If $x_1 \leq a_0 \leq x_2$, then by (2), $f(x_1) < f(x_2)$. If $x_1 < x_2 \leq a_0$, then $f(x_1) < f(a_0)$ and $f(x_1) < f(x_2)$. Lastly, if $a_0 \leq x_1 < x_2$, then $f(a_0) < f(x_2)$ and $f(x_1) < f(x_2)$. Therefore f is strictly increasing.

■

1.3 Uniform Continuity

Remark. Consider $f : \mathrm{dom}(f) \subset \mathbb{R} \to \mathbb{R}$ and assume that f is continuous on some $S \subset \mathrm{dom}(f)$ iff $\forall x_0 \in S, \forall \epsilon > 0, \exists \delta > 0$ such that $|x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon$ whenever $x \in \mathrm{dom}(f)$. Note that in general, δ is dependent on the value x_0 and ϵ.

Example 1.4. Consider $f : (0, \infty) \to \mathbb{R} : x \mapsto \frac{1}{x^2}$. Take $x_0 > 0$ and $\epsilon > 0$. Then

$$|f(x) - f(x_0)| = \left|\frac{1}{x^2} - \frac{1}{x_0^2}\right| = \frac{1}{x^2 x_0^2}(x - x_0)(x + x_0) = \frac{(x + x_0)}{x^2 x_0^2}(x - x_0).$$

If $|x - x_0| < \frac{x_0}{2}$, then $|x| > \frac{|x_0|}{2}$ and $|x| < \frac{3|x_0|}{2}$. Then, $|x + x_0| < \frac{5|x_0|}{2}$. Therefore

$$\frac{(x + x_0)}{x^2 x_0^2}(x - x_0) \leq \frac{\frac{5|x_0|}{2}}{\left(\frac{x_0}{2}\right)^2 x_0^2} \cdot |x - x_0| = \frac{10}{x_0^3}|x - x_0|.$$

By taking $\delta = \min\left\{\frac{x_0}{2}, \frac{x_0^2 \epsilon}{10}\right\}$, $|x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon$. In this case, we see that δ is reliant on both x_0 and ϵ.

Definition 1.5 (Uniform Continuity). A function $f : S \subset \mathbb{R} \to \mathbb{R}$ is uniformly continuous iff

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } |x - \tilde{x}| < \delta \implies |f(x) - f(\tilde{x})| < \epsilon, x, \tilde{x} \in S.$$

If f is said to be uniformly continuous, it is assumed to be uniformly continuous on its domain of definition unless specified.

Remark. Note that uniform continuity is a "stronger" notion of continuity. Note that

$$|x - \tilde{x}| < \delta$$

does not rely on some fixed argument $\tilde{x}$ unlike normal continuity. Fixing $\tilde{x}$ would produce an identical definition of continuity, therefore a function that is uniformly continuous is also continuous. Additionally, continuity is a property at a point while uniform continuity is property on a set. A function that is uniformly continuous at a point is meaningless.

Example 1.5. The function $f(x) = \frac{1}{x^2}$ is uniformly continuous on $[a, \infty)$ for any $a > 0$.

Proof. Note that

$$|f(x) - f(\tilde{x})| = \left|\frac{1}{x^2} - \frac{1}{\tilde{x}^2}\right| \leq \frac{x + \tilde{x}}{x^2\tilde{x}^2}|x - \tilde{x}| = \left(\frac{1}{x\tilde{x}^2} + \frac{1}{x^2\tilde{x}}\right)|x - \tilde{x}| \leq \frac{2}{a^3}|x - \tilde{x}|.$$

Take then $\epsilon > 0$ and let $\delta = \frac{a^3\epsilon}{2}$. Then

$$|x - \tilde{x}| < \delta \implies \frac{2}{a^3}|x - \tilde{x}| < \epsilon \implies |f(x) - f(\tilde{x})| < \epsilon.$$

Therefore f is uniformly continuous on $[a, \infty)$. ■

Example 1.6. The function $f(x) = \frac{1}{x^2}$ is not uniformly continuous on $(0, \infty)$.

Proof. Take $\epsilon = 1$ and show that $\forall \delta > 0$, there is $x, \tilde{x} \in (0, 1)$ such that $|x - \tilde{x}| < \delta$ but $|f(x) - f(\tilde{x})| > 1$. Take $\tilde{x} = x + \frac{\delta}{2}$. Note

$$\frac{1}{x^2} - \frac{1}{\tilde{x}^2} = \frac{1}{x^2} - \frac{1}{\left(x + \frac{\delta}{2}\right)^2} = \frac{\delta x + \frac{\delta^2}{4}}{x^2\left(x + \frac{\delta}{2}\right)^2} = \frac{\delta^2\frac{5}{4}}{\frac{9}{4}\delta^4} = \frac{5}{9}\frac{1}{\delta^2} > \frac{20}{9} > 1$$

for $\delta < \frac{1}{2}$. ■

Example 1.7. The function $f(x) = x^2$ is uniformly continuous on $[-7, 7]$.

Proof. Note that

$$|f(x) - f(\tilde{x})| = |x^2 - \tilde{x}| = |x + \tilde{x}||x - \tilde{x}| \leq 14|x - \tilde{x}|.$$

Therefore, take $\epsilon > 0$ and choose $\delta = \frac{\epsilon}{14}$. Then

$$|x - \tilde{x}| < \delta \implies 14|x - \tilde{x}| < \epsilon \implies |f(x) - f(\tilde{x})| < \epsilon$$

Hence f is uniformly continuous on $[-7, 7]$. ■

Theorem 1.10 (Closed Interval Implies Uniform Continuity). If f is continuous on $[a, b]$, then f is uniformly continuous on $[a, b]$.

Proof. Let $f : \text{dom}(f) \to \mathbb{R}$ be a real valued function and assume that it is continuous on the interval $[a, b]$. Assume towards contradiction that f is not uniformly continuous. Then $\exists \epsilon > 0$ such that $\forall \delta > 0$ there is $x, \tilde{x} \in [a, b]$ where $|x - \tilde{x}| < \delta$ and $|f(x) - f(\tilde{x})| \geq \epsilon$. Take $\delta_n = \frac{1}{n}$ to find a sequence of arguments (x_n) and $(\tilde{x}_n)$ in $[a, b]$ such that $|x_n - \tilde{x}_n| <$

δ_n and $|f(x_n) - f(\tilde{x_n})| \geq \epsilon$. By **??**, there exists a subsequence $(n_k)_{k\in\mathbb{N}} \subset \mathbb{N}$ such that

$$x_{n_k} \xrightarrow{n\to\infty} x_0$$
$$\tilde{x}_{n_k} \xrightarrow{n\to\infty} x_0 \text{ (since } |x_n - \tilde{x}_n| < \delta_n)$$

Since $[a, b]$ is closed, then the limit point $x_0 \in [a, b]$. Therefore since f is continuous on $[a, b]$,

$$f(x_{n_k}) \xrightarrow{n\to\infty} f(x_0)$$
$$f(\tilde{x}_{n_k}) \xrightarrow{n\to\infty} f(x_0)$$

which means that $|f(x_{n_k}) - f(\tilde{x}_{n_k})| \to 0$. However this contradicts the assumption that $|f(x_{n_k}) - f(\tilde{x}_{n_k})| \geq \epsilon > 0$ for all k. ■

Example 1.8. The following functions are uniformly continuous

$$[x \mapsto x^7 3], x \in [-15, 31]$$
$$[x \mapsto \sqrt{x}], x \in [0, 413]$$
$$[x \mapsto e^x], x \in [-1000, 1000]$$

Theorem 1.11 (Uniform Continuity Preserves Cauchy Sequences). If f is uniformly continuous on S, then a Cauchy sequence (s_n) in S is mapped to a Cauchy sequence $(f(s_n))$ in $\mathbb{R}$.

Proof. Take $\epsilon > 0$. Then find $\delta > 0$ such that $|x - \tilde{x}| \implies |f(x) - f(\tilde{x})| < \epsilon$ for $x, \tilde{x} \in S$. Since (s_n) is Cauchy, then $\exists N \in \mathbb{N}$ such that $|s_n - s_m| < \delta$ for all $n > N$. Then $|f(s_n) - f(s_m)| < \epsilon$ for all $n > M$ and hence $(f(s_n))$ is also Cauchy. ■

Example 1.9. Consider $f(x) = \frac{1}{x^2}$ on the interval $(0, 1]$. f is not uniformly continuous.

Proof. Consider the sequence $s_n = \frac{1}{n}$. (s_n) is convergent and in the domain of f, but $f(s_n) = n^2$ which is not Cauchy. Hence f is not uniformly continuous. ■

Definition 1.6 (Function Extension). $\tilde{f} : \text{dom}(\tilde{f}) \subset \mathbb{R} \to \mathbb{R}$ is an extension of f : $\text{dom}(f) \subset \mathbb{R} \to \mathbb{R}$ iff

1. $\text{dom}(f) \subset \text{dom}(\tilde{f})$
2. $\tilde{f}(x) = f(x)$ for $x \in \text{dom}(f)$

Example 1.10. Consider the function $f(x) = x\sin\frac{1}{x}$ on the interval $\left(0, \frac{1}{\pi}\right]$. Let

$$\tilde{f} = \begin{cases} f(x) & x \in \left(0, \frac{1}{\pi}\right] \\ r & x = 0 \end{cases}.$$

If $r = 0$, then $\tilde{f}$ is continuous on the closed interval $\left[0, \frac{1}{\pi}\right]$ and hence is uniformly continuous.

Example 1.11. Consider $f(x) = \sin\frac{1}{x}$ with $x \in \left(0, \frac{1}{\pi}\right]$. f can be extended to the closed interval by setting $f(0) = r \in \mathbb{R}$. However, no choice for r makes the extension continuous.

Theorem 1.12 (Uniform Continuity Extension Equivalency). $f : (a,b) \to \mathbb{R}$ is uniformly continuous on (a,b) iff f has a uniformly continuous extension $\tilde{f}$ on $[a,b]$.

Proof. Consider both implications

$\Leftarrow$) Assume that $\tilde{f}$ is uniformly continuous on $[a,b]$. Since $f(x) = \tilde{f}(x)$ for $x \in (a,b)$, f must be uniformly continuous.

$\Rightarrow$) Assume that f is uniformly continuous on (a,b). If f has a continuous extension $\tilde{f}$ on $[a,b]$, then it is uniformly continuous. Therefore it is sufficient to define $\tilde{f}$ at a and b. Consider b. It is possible to take $x_n \in (a,b)$ such that $\lim x_n = b$. Since (x_n) is convergent, it is also Cauchy. Since f is uniformly continuous, $(f(x_n))_{n\in\mathbb{N}}$ is also a Cauchy sequence and therefore is convergent. Therefore there is some $y \in \mathbb{R}$ such that $\lim f(x_n) = y$. Define then $\tilde{f}(b) = y$. It still needs to be verified that for any other sequence that converges to b that the functional sequence converges to y. Let $(\tilde{x}_n)$ be a sequence different that before that converges to b. Consider a new sequence $(s_n) = (x_1, \tilde{x_1}, x_2, \tilde{x}, \ldots)$. Note that (s_n) is Cauchy since $\lim s_n = b$. Therefore $(f(s_n))_{n\in\mathbb{N}}$ is also Cauchy, meaning $(f(s_n))_{n\in\mathbb{N}}$ has a limit. Therefore all its subsequential limits are the same, hence

$$\lim s_{2k} = \lim \tilde{x}_n = \lim s_{2k-1} = \lim x_n = y.$$

Therefore all convergent sequences to b will converge to y under f. The same construction follows for a.

Both implications therefore establish the equivalency. ■

Example 1.12. Consider $f(x) = \frac{\sin x}{x}$ with $x \neq 0$. Let

$$\tilde{f}(x) = \begin{cases} f(x) & x \neq 0 \\ 1 & x = 0 \end{cases}.$$

It turns out that $\tilde{f}$ is continuous on $\mathbb{R}$ and therefore is uniformly continuous on any closed interval.

Theorem 1.13. Let f be continuous on an interval I. If f restricted to $\mathring{I}$ is differentiable and the derivative is bounded, then f is uniformly continuous.

Proof. Apply MVT with $a < b$ and $a, b \in I$. Then

$$f(b) - f(a) = f'(x) \cdot (b - a), x \in (a, b).$$

Therefore $|f(b) - f(a)| \leq |f'(x)(b - a)| = |f'(x)|(b - a)$. Since $f'(x)$ is bounded, there is some $M \in \mathbb{R}$ such that $|f'(x)| \leq M$ for all x. Take $\epsilon > 0$ and let $\delta = \frac{\epsilon}{M}$. Then

$$|b - a| < \delta \implies |f(b) - f(a)| < \epsilon.$$

Hence f is uniformly continuous. ■