13.1

Part A

d_1 is a metric. Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^k$. Examine the requirements for d_1 to be a metric

1. Note that $d_1(x, x) = \max\{|x_i - x_i| : i = 1, \ldots, k\} = 0$. Additionally, $|x_i - y_i| > 0$ for all i and therefore the maximum is larger than 0, hence $d(x, y) > 0$.
2. Since $|x_i - y_i| = |y_i - x_i|$, it follows that $d_1(x, y) = d_2(y, x)$.
3. Note that one can rewrite $|x_i - z_i| = |x_i - y_i + y_i - z_i| \leq |x_i - y_i| + |y_i - z_i|$. This means that $\max\{|x_i - z_i| : i = 1, \ldots, k\} \leq \max\{|x_i - y_i| : i = 1, \ldots, k\} + \max\{|y_i - z_i| : i = 1, \ldots, k\}$ and therefore $d_1(x, z) \leq d_1(x, y) + d_1(y, z)$.

Therefore d_1 is a metric over $\mathbb{R}^k$. ■

d_2 is a metric. Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^k$. Examine the requirements for d_1 to be a metric.

1. Note that $d_2(\mathbf{x}, \mathbf{x}) = \sum_{i=0}^{k} |x_i - x_i| = 0$. Further more, $|x_i - y_i| > 0$ for all $i = 1, \ldots, k$ meaning their sum is positive, hence $d_2(\mathbf{x}, \mathbf{y}) > 0$.
2. Since $|x_i - y_i| = |y_i - x_i|$, it follows that $d_2(\mathbf{x}, \mathbf{y}) = d_2(\mathbf{y}, \mathbf{x})$.
3. Note that
$$\sum_{i=1}^{k} |x_i - z_i| = \sum_{i=1}^{k} |x_i - y_i + y_i - z_i| \leq \sum_{i=1}^{k} (|x_i - y_i| + |y_i - z_i|) = \sum_{i=1}^{k} |x_i - y_i| + \sum_{i=1}^{k} |y_i - z_i|$$
hence $d_2(\mathbf{x}, \mathbf{z}) \leq d_2(\mathbf{x}, \mathbf{y}) + d_2(\mathbf{y}, \mathbf{z})$.

Therefore d_2 is a metric over $\mathbb{R}^k$. ■

Part B

d_1 is a complete metric. Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^k$. We will first prove the following inequality
$$d_1(\mathbf{x}, \mathbf{y}) \leq d(\mathbf{x}, \mathbf{y}) \leq \sqrt{k} \cdot d_1(\mathbf{x}, \mathbf{y})$$
For the first inequality, note that $|x_j - y_j| \leq \sqrt{(x_j - y_j)^2} \leq \sqrt{\sum_{i=1}^{k} (x_i - y_i)^2}$ for all $i = j, \ldots, k$. Therefore $\max\{|x_i - y_i| : i = 1, \ldots, k\} \leq d(\mathbf{x}, \mathbf{y})$ hence $d_1(\mathbf{x}, \mathbf{y}) \leq d(\mathbf{x}, \mathbf{y})$. Now for the second inequality, note that
$$\begin{aligned} d(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_{i=1}^{k} (x_i - y_i)^2} &\leq \sqrt{\sum_{i=1}^{k} \left(\max\left\{|x_j - y_j| : j = 1, \ldots, k\right\}\right)^2} \\ &= \sqrt{k \cdot d_1^2(\mathbf{x}, \mathbf{y})} \\ &= \sqrt{k} \cdot d_1(\mathbf{x}, \mathbf{y}) \end{aligned}$$

Therefore $d(\mathbf{x}, \mathbf{y}) \leq \sqrt{k} \cdot d_1(\mathbf{x}, \mathbf{y})$ proving the original inequality. Let $(\mathbf{x}_n)$ be a sequence in $\mathbb{R}^k$ and assume it is Cauchy under d_1. Take $\epsilon > 0$. Then $\exists N \in \mathbb{N}$ such that

$$d_1(\mathbf{x}_n, \mathbf{x}_m) < \frac{\epsilon}{\sqrt{k}}, \forall m, n > N$$

and therefore

$$d(\mathbf{x}_n, \mathbf{x}_m) \leq \sqrt{k} \cdot d_1(\mathbf{x}_n, \mathbf{x}_m) < \epsilon, \forall m, n > N$$

meaning that $(\mathbf{x}_n)$ is Cauchy under d. Since $\mathbb{R}^k$ is complete under d there exists $\mathbf{s} \in \mathbb{R}^k$ such that $\lim_{n\to\infty} d(\mathbf{x}_n, \mathbf{s}) = 0$. Note then by the first inequality and squeeze lemma that

$$0 \leq \lim d_1(\mathbf{x_n}, \mathbf{s}) \leq \lim d(\mathbf{x_n}, \mathbf{s}) = 0 \implies \lim d_1(\mathbf{x_n}, \mathbf{s}) = 0$$

Therefore the sequence $(\mathbf{x}_n)$ converges under d_1 meaning $\mathbb{R}^k$ is complete under d_1. ■

d_2 is a complete metric. Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^k$. We will first prove the following inequality

$$d_1(\mathbf{x}, \mathbf{y}) \leq d_2(\mathbf{x}, \mathbf{y}) \leq k \cdot d_1(\mathbf{x}, \mathbf{y})$$

Consider $d_1(\mathbf{x}, \mathbf{y})$. Let q denote the index between $1, \ldots, k$ such that $d_1(\mathbf{x}, \mathbf{y}) = \max\{|x_i - y_i| : i = 1, \ldots, k\} = |x_q - y_q|$. Note then that

$$d_2(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{k} |x_i - y_i| = |x_q - y_q| + \sum_{i=1}^{q-1} |x_i - y_i| + \sum_{i=q+1}^{k} |x_i - y_i| \geq |x_q - y_q| = d_1(\mathbf{x}, \mathbf{y}).$$

Therefore the first inequality holds. Note that $|x_i - y_i| \leq d_1(\mathbf{x}, \mathbf{y})$ for all $i = 1, \ldots, k$. Therefore the sum of all i's is less than or equal to the sum of $d_1(\mathbf{x}, \mathbf{y})$ k times, meaning

$$d_2(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{k} |x_i - y_i| \leq k d_2(\mathbf{x}, \mathbf{y})$$

and hence the second inequality holds, proving the original inequality. Let $(\mathbf{x}_n)$ be a sequence in $\mathbb{R}^k$ and assume that is is Cauchy under d_2. Take $\epsilon > 0$. Then $\exists N \in \mathbb{N}$ such that

$$d_2(\mathbf{x}_n, \mathbf{x}_m) < \frac{\epsilon}{k}, \forall m, n > N$$

and therefore

$$d_1(\mathbf{x}_n, \mathbf{x}_m) < k \cdot d_2(\mathbf{x}_n, \mathbf{x}_m) < \epsilon, \forall m, n > N$$

meaning that $(\mathbf{x}_n)$ is Cauchy under d_1. Since d_1 is a complete metric by the previous proof, it follows that $\exists \mathbf{s} \in \mathbb{R}^k$ such that $\lim d_1(\mathbf{x}_n, \mathbf{s}) = 0$. Then by the first inequality and squeeze lemma

$$0 \leq \lim d_2(\mathbf{x}_n, \mathbf{s}) \leq k \cdot \lim d_1(\mathbf{x}_n, \mathbf{s}) = 0 \implies \lim d_2(\mathbf{x}_n, \mathbf{s}) = 0$$

Therefore the sequence converges under d_2 meaning $\mathbb{R}^k$ is complete under d_2. ■

13.3

Part A

Proof. Let $(x_n), (y_n), (z_n) \in B$. Examine the requirements for d to be a metric.

1. Note that $d(x_n, x_n) = \sup\{|x_i - x_i| : i = 1, 2, \ldots\} = 0$. Since d is the supremum of absolute values which are all positive, the metric will always be positive or 0.

2. Since $|x_i - y_i| = |y_i - x_i|$, it follows that $d(x_n, y_n) = d(y_n, x_n)$.

3. Note that

$$\begin{aligned} d(x_n, z_n) &= \sup\{|x_i - z_i| : i = 1, 2, \ldots\} \\ &= \sup\{|x_i - y_i + y_i - z_i| : i = 1, 2, \ldots\} \\ &\leq \sup\{|x_i - y_i| + |y_i - z_i| : i = 1, 2, \ldots\} \\ &\leq \sup\{|x_i - y_i| : i = 1, 2, \ldots\} + \sup\{|y_i - z_i| : i = 1, 2, \ldots\} \\ &= d(x_n, y_n) + d(y_n, z_n) \end{aligned}$$

   Therefore $d(x_n, z_n) \leq d(x_n, y_n) + d(y_n, z_n)$.

Therefore d is a metric over B. ■

Part B

d^* does not define a metric of B because it is not always finite. If $(x_n) = 1$ and $(y_n) = 1$, then $d^*(x_n, y_n) = \sum_{j=1}^{\infty} 1$ which is infinite.

13.4

iii

Proof. Let $V = \bigcup_n E_n$ be union of a collection of open sets. Let $x \in V$. Then $\exists m \in \mathbb{N}$ such that $x \in E_m$. Since E_m is open, $x \in \mathring{E}$ and therefore $\exists r > 0$ such that $\mathbb{B}(x, r) \subset E_m \subset V$. Therefore $x \in \mathring{V}$. Therefore $V \subset \mathring{V}$. Since by definition $\mathring{V} \subset V$, it follows that $V = \mathring{V}$ and therefore V is closed. ■

iv

Proof. Let $V = \bigcap_{i=1}^{n} E_i$ be a finite intersection of open sets. If $V = \varnothing$, then it is open since the empty set is open. Let $x \in V$. Then $x \in E_i, i = 1, \ldots, n$. Since each E_i is open, there is an associated r_i such that $\mathbb{B}(x, r_i) \subset E_i$. Let $r = \min\{r_i : i = 1, \ldots, n\}$. Note then that $\mathbb{B}(x, r) \subset \mathbb{B}(x, r_i) \subset E_i$ for all $i = 1, \ldots, n$. Therefore $\mathbb{B}(x, r) \subset V$ meaning $x \in \mathring{V}$. Therefore $V \subset \mathring{V}$ which by the same logic as the previous proof gives $V = \mathring{V}$. ■

13.5

Part A

Proof. Let S be a set and $\mathcal{U}$ be a collection of sets. Let $x \in \bigcap S \setminus U$. Then $x \in S$ and $x \notin U$ for all $U \in \mathcal{U}$. Therefore $x \notin \bigcup U$ meaning $x \in S \setminus \bigcup U$, hence $\bigcap S \setminus U \subset S \setminus \bigcup U$. Let $x \in S \setminus \bigcup U$. Then $x \in S$ and $x \notin \bigcup U$. Therefore $x \notin U$ meaning $s \in S \setminus U$ for all $U \in \mathcal{U}$. Therefore $x \in \bigcap S \setminus U$, hence $\bigcap S \setminus U \subset S \setminus \bigcup U$. Since both are subsets of each other,

$$\bigcap_{U \in \mathcal{U}} S \setminus U = S \setminus \bigcup_{U \in \mathcal{U}} U$$

■

Part B

Proof. First note that the previous result holds when the intersection and union are swapped. That is,

$$S \setminus \bigcap_{U \in \mathcal{U}} U = \bigcup_{U \in \mathcal{U}} S \setminus U$$

Let $\mathcal{U}$ be a collection of closed sets. Then

$$S \setminus \bigcap_{U \in \mathcal{U}} = \bigcup_{U \in \mathcal{U}} S \setminus U$$

Since each $U \in \mathcal{U}$ is closed, it follows that $S \setminus U$ is open. Therefore the right hand side is a union of open sets, which itself is open. Therefore the left hand side is open meaning the intersection of the closed sets must be closed. ■

13.6

Proof. Let E be a subset of a metric space (S, d).

1. $\Rightarrow$) Assume that E is closed. Then $\overline{E} = \subset \bigcap_{\substack{E \subset F \subset S \\ F \text{ closed}}} F \subset E$ since E is a subset of itself and is the smallest closed subset that contains itself. Since $E \subset \overline{E}$, it follows that $E = \overline{E}$.

   $\Leftarrow$) Assume that $E = \overline{E}$. Since $\overline{E}$ is the intersection of closed sets, it itself is closed. Therefore E must also be closed.

2. $\Rightarrow$) Assume that E is closed. Let (x_n) be a sequence in E that converges to some $x \in S$. Assume towards contradiction that $x \notin E$. Then $x \in S \setminus E$, meaning $\exists r > 0$ such that $\mathbb{B}(x, r) \subset S \setminus E$. However, this means that choosing an $\epsilon < r$ means $\exists N \in \mathbb{N}$ such that $d(x, x_n) < \epsilon < r$ for all $n > N$. Therefore $x_n \in \mathbb{B}(x, r)$ for $n > N$. But that means there are infinitely many terms of the sequence outside of E, a contradiction.

   $\Leftarrow$) Assume that E contains the limits of every convergent sequence in E. Let

$x \in S \setminus E$. Suppose that for any $r > 0$ that $\mathbb{B}(x,r) \cap E \neq \emptyset$. Then it is possible to construct a sequence (x_n) where $x_n \in \mathbb{B}(x, \frac{1}{n}) \cap E$. Note that $(x_n) \to x$ since $d(x_n, x) < \frac{1}{n}$ for each n. However, this sequence is in E but the limit point x is not in E, hence a contradiction. Therefore there must be some $r > 0$ such that $\mathbb{B}(x,r) \cap E = \emptyset$ which is the same as saying $\mathbb{B}(x,r) \subset S \setminus E$. This means that $S \setminus E$ is equal to its interior and therefore $S \setminus E$ is open. Therefore E is closed.

3. $\Rightarrow$) Assume that $x \in \overline{E}$. Note that it is sufficient to show that for any $r > 0$ that $\mathbb{B}(x,r) \cap E \neq \emptyset$. If this is true, then by the same logic in (b) it is possible to construct a sequence in E that will approach x. Take $r > 0$ and assume towards contradiction that $\mathbb{B}(x,r) \cap E = \emptyset$. Then $E \subset S \setminus \mathbb{B}(x,r)$. Since open balls are open, then $S \setminus \mathbb{B}(x,r)$ is a closed set containing E which means that $\overline{E} \subset S \setminus \mathbb{B}(x,r)$. But then by the assumption, $x \in S \setminus \mathbb{B}(x,r)$ and $x \in \mathbb{B}(x,r)$ which is a contradiction.

   $\Leftarrow$) Assume that x is the limit of a sequence (x_n) of points in E. By part (a), $\overline{E}$ is closed and by (b), $\overline{E}$ must contain the limit of any sequence of points in $\overline{E}$. Since $x_n \in E$ for all n, $x_n \in \overline{E}$ for all n. Therefore (x_n) is a sequence of points in $\overline{E}$ and hence its limit must also be in $\overline{E}$.

4. $\Rightarrow$) Assume that $x \in \partial E$. Therefore $x \in \overline{E}$ and $x \notin \mathring{E}$. Therefore it is sufficient to show that $x \in \overline{S \setminus E}$. Let $F \supset S \setminus E$ be a closed set. Note that then $S \setminus F$ is open and that $S \setminus F \subset E$. If $x \in S \setminus F$, then there is some $r > 0$ such that $\mathbb{B}(x,r) \subset S \setminus F$. Since $S \setminus F \subset E$, it follows that $\mathbb{B}(x,r) \subset E$. However, this implies that x is in the interior and is therefore a contradiction. Therefore $x \notin S \setminus F$ meaning $x \in F$. Since F was an arbitrary closed set containing $S \setminus E$, x is in every closed set containing $S \setminus E$ and therefore $x \in \overline{S \setminus E}$.

   $\Leftarrow$) Assume that $x \in \overline{E}$ and $x \in \overline{S \setminus E}$. It is sufficient to show that $x \notin \mathring{E}$ since x is assumed to be in the closure. Assume towards contradiction that $x \in \mathring{E}$. Then there exists some $r > 0$ such that $\mathbb{B}(x,r) \subset E$. This means that $S \setminus \mathbb{B}(x,r)$ is closed set with $S \setminus \mathbb{B}(x,r) \supset S \setminus E$ which requires that $x \in S \setminus \mathbb{B}(x,r)$. However this is not possible since x is contained in any ball centered around it. Therefore x cannot be interior to E.

■

13.8

Part A

(a,b) *is Open.* Note that to show a set is open is the same as showing the set is a subset of its interior. Let $x \in (a,b)$. Let $r_1 = b - x$ and $r_2 = x - a$ and take $r = \frac{1}{2}\min\{r_1, r_2\}$. Then $a < x - r < x + r < b$. Therefore $\mathbb{B}(x,r) \subset (a,b)$ meaning x is an interior point. Therefore (a,b) is a subset of its interior points and therefore is open. ■

Proof. Note that $[a,b]^{\complement} = (-\infty, a) \cup (b, \infty)$ which is the union of open sets. Therefore the complement of $[a,b]$ is open and therefore $[a,b]$ is closed. ■

Proof. Consider the interior of $[a,b]$. Note that a, b are not in the interior as for any $r > 0$, $a - r \notin [a,b]$ and $b + r \notin [a,b]$. Therefore the candidates for its interiorare (a,b). However, since (a,b) is open, it is equal to its interior and therefore the interior of $[a,b]$ is (a,b). ■

Proof. Note that the smallest closed superset of (a,b) is $[a,b]$. Therefore the set subtraction of $[a,b]$ and (a,b) gives $\{a,b\}$. Therefore the boundary of (a,b) is $\{a,b\}$. Since $[a,b]$ is closed, its closure is the same. Its interior is also (a,b) and so using the same logic as before its boundary is $\{a,b\}$. ■

Part B

Open Balls are Open. Let (S,d) be a metric space. Let $x \in S$ and $r > 0$. Consider the open ball $\{s \in S : d(s,x) < r\}$. Let $y \in \mathbb{B}(x,r)$. Choose $r' = r - d(x,y)$. Then if $z \in \mathbb{B}(y,r')$,

$$d(z,x) \leq d(z,y) + d(y,x) < r' + d(y,x) = r$$

Therefore $\mathbb{B}(y,r') \subset \mathbb{B}(x,r)$ hence the original ball is open. ■

Closed Balls are Closed. Let (S,d) be a metric space. Let $x \in S$ and $r > 0$. Consider the closed ball $D(x,r) = \{s \in S : d(s,x) \leq r\}$. Let $y \in S \setminus D(x,r)$. Then $d(y,s) > r$ and therefore $d(y,s) - r > 0$. Let $r' = d(y,s) - r$. Let $z \in \mathbb{B}(y,r')$. Then

$$d(s,y) \leq d(s,z) + d(z,y) \implies d(z,s) \geq d(s,y) - d(z,y) > d(s,y)$$

Therefore $z \in S \setminus D(s,r)$ which means $S \setminus D(s,r)$ is open and hence $D(s,r)$ is closed. ■

13.9

a) The closure is the set itself unioned with $\{0\}$

b) $\overline{\mathbb{Q}} = \mathbb{R}$

c) The closure is $[-\sqrt{2}, \sqrt{2}]$

13.11

Proof. Let $E \subset \mathbb{R}^k$.

$\Rightarrow$) Assume that E is compact. By the Heine-Borel theorem, E is bounded and closed. Therefore for any sequence in E, it is also bounded meaning the sequence has a convergent subsequence. Since E is also closed, it contains all its limit points and therefore the subsequence converges to a point in E.

$\Leftarrow$) Assume that every sequence in E has a subsequence that converges to a point in E. Note that E must be bounded otherwise there exists sequences that will have no subsequential limit in $\mathbb{R}^k$. Let (x_n) be a sequence in E that converges to some $x \in \mathbb{R}^k$. By the assumption, there exists a subsequence that converges to some $y \in E$. However since (x_n) converges, its subsequence also has the same limit and therefore $y = x \in E$. Therefore E contains its limit points and is closed. Since E is closed and bounded, by the Heine-Borel theorem it is compact.

■