0.1 Constant Coefficient 2nd Order ODEs

Consider the equation $y'' - 6y' + 8y = 0$.

The solution is going to be in the form of a function whose derivatives only effect its coefficients and not the function itself. Inspect the exponential function: $y = e^{rx} \implies y' = re^{rx} \implies y'' = r^2 e^{rx} \ldots$

$$y'' - 6y' + 8y = 0 \implies r^2 e^{rx} - 6re^{rx} + 8e^{rx} = 0.$$

Which turns into:

$$e^{rx}\left(r^2 - 6r + 8\right) = 0.$$

Now solve the internal quadratic for r:

$$(r-2)(r-4) = 0 \implies r = \{2, 4\}.$$

Therefore the solutions are:

$$y_1 = e^{2x} \qquad\qquad y_2 = e^{4x}.$$

Since both are linearly independent, all solutions are represented by:

$$y(x) = c_1 e^{2x} + c_2 e^{4x}\ ;\ \{c_1, c_2\} \in \mathbb{R}.$$

For any 2nd Order Linear Homogeneous ODE with constant coefficients, the solution can be determined by the roots of the characteristic equation:

$$ar^2 + br + c = 0.$$

Stated in a theorem:

Theorem 0.1 ▶ Constant Coefficient 2nd Order ODEs Solution

Let r_1 and r_2 be the roots of the characteristic polynomial. If both roots are distinct, the general solution is:

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}\ ;\ \{c_1, c_2\} \in \mathbb{R}.$$

If both roots are the same, the general solution is:

$$y(x) = e^{r_1 x}\left(c_1 + c_2 x\right).$$

If the roots are expressed as $r = \alpha \pm i\beta$:

$$\begin{aligned}
y(x) &= Ae^{x(\alpha+i\beta)} + Be^{x(\alpha-i\beta)} \\
&= Ae^{\alpha x}e^{i\beta x} + Be^{\alpha x}e^{-i\beta x} \\
&= Ae^{\alpha x}(\cos\beta x + i\sin\beta x) + Be^{\alpha x}(\cos\beta x - i\sin\beta x) \\
y(x) &= c_1 e^{\alpha x}\cos(\beta x) + c_2 e^{\alpha x}\sin(\beta x)\ ;\ \{c_1, c_2\} \in \mathbb{R}.
\end{aligned}$$

Note: Complex Root Selection

Note that in ??, one can just take one the complex values of r and take its real and imaginary components as linearly independent. Given $r = a + bi$,

$$\begin{aligned} y = e^{rt} &= e^{(a+bi)t} \\ &= e^{at} e^{bi \cdot t} \\ &= e^{at} (\cos(bt) + i \sin(bt)) \end{aligned}$$

$$\begin{aligned} y_1 &= \operatorname{Re}(y) & y_2 &= \operatorname{Im}(y) \\ y_1 &= e^{at} \cos(bt) & y_2 &= e^{at} \sin(bt). \end{aligned}$$

$$\boldsymbol{y(t) = c_1 e^{at} \cos(bt) + c_2 e^{at} \sin(bt)}.$$

Ex. Find solution to $y'' - 8y' + 16y = 0$, $y(0) = 2$, $y'(0) = 6$.

$$\begin{aligned} \text{Characteristic equation: } r^2 - 8r + 16 &= 0 \\ (r-4)^2 &= 0 \\ \text{General solution: } y(x) = e^{4x}(c_1 + c_2 x). \end{aligned}$$

Using the initial condition:

For c_1:

$$\begin{aligned} y(0) = e^0 (c_1 + c_2 \cdot 0) &= 2 \\ c_1 &= 2 \end{aligned}$$

For c_2:

$$\begin{aligned} y'(x) &= 4e^{4x}(c_1 + c_2 x) + c_2 e^{4x} \\ y'(0) &= 4e^{4 \cdot 0}(c_1 + c_2 \cdot 0) + c_2 e^{4 \cdot 0} \\ 4c_1 + c_2 &= 6 \\ 4c_1 &= 4 \\ c_1 &= 1. \end{aligned}$$

Note that ?? can be generalized to any nth order ODE as long as its linear and homogeneous:

$$\begin{bmatrix} y \\ y' \\ y'' \\ \vdots \\ y^n \end{bmatrix} \cdot \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = 0$$

Note that the parametrized solution $y(x) = e^{rx}$ works

$$e^{rx} \begin{bmatrix} 1 \\ r \\ r^2 \\ \vdots \\ r^n \end{bmatrix} \cdot \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = 0$$

Divide out by e^{rx} since it is always greater than 0

$$\begin{bmatrix} 1 \\ r \\ r^2 \\ \vdots \\ r^n \end{bmatrix} \cdot \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = 0$$

Expanding out the dot product

$$a_0 + a_1 r + a_2 r^2 + \ldots + a_{n-1} r^{n-1} + a_n r^n = 0$$

The resulting parametrized polynomial encodes the values of parameter r that define the solution. The final analytic solution will therefore be a superposition/linear combination of all the parametrized functions:

Note: Repeated Roots of r

If r is repeated k times, then the linearly independent solutions of k are:

$$e^{rx}, xe^{rx}, x^2 e^{rx}, \ldots, x^k e^{rx}.$$

Ex. Find the general solution for $y^{(4)} - 3y''' - 3y'' - y' = 0$.

Utilize the parametrized solution $y = e^{rx}$

$$r^4 - 3r^3 - 3r^2 - r = 0$$
$$r(r^3 - 3r^2 - 3r - 1) = 0$$
$$r(r-1)^3 = 0 \implies r = \{0, 1, 1, 1\}$$

r is repeated three times, therefore:

$$y(x) = c_1 + c_2 e^x + c_3 x e^x + c_4 x^2 e^x$$

0.2 Non-Homogeneous Equation

If an ODE has the form $L(y) = f(x)$, then to find the solution you find:
Complementary Solution ($\implies y_c$) solves the associated linear homogeneous equation
Particular Solution ($\implies y_p$) solves the original non-homogeneous equation
Using these solutions, the general solution for the original ODE is

$$y(x) = y_c + y_p.$$

0.2.1 Method of Undetermined Coefficients

Ex. $y'' + 5y' + 6y = 2x + 1$, $y(0) = 0$ and $y'(0) = \frac{1}{3}$

Consider the associated homogeneous equation $y'' + 5y' + 6y = 0$

$$r^2 + 5r + 6 = 0 \implies r = \{-2, -3\}$$

Therefore the complementary solution is:

$$y_c = c_1 e^{-2x} + c_2 e^{-3x}.$$

To find the **particular solution**, take a guess about the form of y_p. Since the linear combination of 2^{nd} derivative, 1^{st} derivative, and itself is a linear polynomial, its possible that y_p is also a polynomial and linear. Therefore:

$$\text{Guess } y_p = Ax + B.$$

Substitute y_p into original ODE

$$\begin{aligned} y_p'' + 5y_p' + 6y_p &= 2x + 1 \\ 0 + 5A + 6(Ax + B) &= 2x + 1 \\ 6Ax + 5A + 6B &= 2x + 1. \end{aligned}$$

Now match coefficients

$$\begin{aligned} 6A &= 2 \\ 5A + 6B &= 1 \end{aligned}$$

Therefore $A = \frac{1}{3}$ and $B = -\frac{1}{9}$

$$y_p = \frac{1}{3}x - \frac{1}{9}.$$

The general solution is therefore

$$y(x) = c_1 e^{-2x} + c_2 e^{-3x} + \frac{1}{3}x - \frac{1}{9}.$$

Do not use initial condition in just y_c. To solve for c_1 and c_2, use the general solution

Note: Selecting a y_p

When guessing a form for y_p, take the most general form of the function and its derivatives. Some examples for $L(y) = f(x)$:

Given:	Ansatz:
$f(x) = x$	$y_p = Ax + B$
$f(x) = 3x^2 + 1$	$y_p = Ax^2 + Bx + C$
$f(x) = \cos(x)$	$y_p = A\cos(x) + B\sin(x)$
$f(x) = e^{kx}$	$y_p = Ae^{kx}$

What happens when y_p is a solution of the homogeneous equation (similar to that of a repeated root)?

Ex. Find the general solution of $y'' - 9y = e^{3x}$.

For the homogeneous system:

$$\begin{aligned} y'' - 9y &= 0 \\ r^2 - 9 &= 0 \implies r = \pm 3. \end{aligned}$$

$$\boxed{y_c = c_1 e^{3x} + c_2 e^{-3x}.}$$

For the particular system, guess that $y_p = Ae^{3x}$. Plug into the ODE:

$$\begin{aligned} 9Ae^{3x} - 9Ae^{3x} &= e^{3x} \\ e^{3x} &= 0. \end{aligned}$$

Note that the prediction leads to nonsense. Therefore, treat it like a repeated root of a characteristic equation and add a multiple of x. Now $y_p = Axe^{3x}$. Plugging into the ODE:

$$\begin{aligned} A(9xe^{3x} + 6e^{3x}) - 9Axe^{3x} &= e^{3x} \\ 6Ae^{3x} &= e^{3x} \implies A = \frac{1}{6}. \end{aligned}$$

While the Method of Undetermined Coefficients is really powerful, it fails to work in situations where the function of the independent variable has infinite linearly independent derivatives.