Problem 1

Part A

Proof. Let v be an eigenvector of M with associated eigenvalue $\lambda \in \mathbb{R}$ (since M is self adjoint). Since M is positive definite, we have

$$0 < v^* M v = v^*(Mv) = v^*(\lambda v) = \lambda(v^* v)$$

Note that $v^* v$ is the standard inner product on $\mathbb{R}^n$ or $\mathbb{C}^n$ of v with itself. Since v is non-zero we must have $v^* v > 0$. But this means that $\lambda > 0$. Therefore all the eigenvalues of M must be strictly positive. ■

Part B

Proof. Suppose M is self adjoint and has n positive eigenvalues. Since M is self adjoint, there exists an orthonormal basis $\beta = \{v_1, \ldots, v_n\}$ of eigenvectors of M. Let $x \in V$. We can write x as

$$x = \sum_{i=1}^{n} a_i v_i.$$

Note that we can rewrite $x^* M x = \langle x, Mx \rangle = \langle Mx, x \rangle$ since M is self adjoint. Furthermore

$$\langle Mx, x \rangle = \left\langle M \sum_{i=1}^{n} a_i v_i, \sum_{i=1}^{n} a_i v_i \right\rangle = \left\langle \sum_{i=1}^{n} a_i \lambda_i v_i, \sum_{i=1}^{n} a_i v_i \right\rangle = \sum_{i=1}^{n} |a_i|^2 \lambda_i.$$

Since each $|a_i| \geq 0$ and $\lambda_i > 0$, this sum must be non-negative. Furthermore, since x is non zero, there is at least one a_i such that $|a_i| > 0$, hence the sum must itself be strictly positive. Therefore $x^* M x = \langle Mx, x \rangle > 0$, hence M is positive definite. ■

Part C

Proof.

a) The eigenvalues of a positive semidefinite matrix M are non-negative.

Let v be an eigenvector of M with associated eigenvalue $\lambda \in \mathbb{R}$ (since M is self adjoint). Since M is positive semidefinite, we have

$$0 \leq v^* M v = v^*(Mv) = v^*(\lambda v) = \lambda(v^* v)$$

Note that $v^* v$ is the standard inner product on $\mathbb{R}^n$ or $\mathbb{C}^n$ of v with itself. Since v is non-zero we must have $v^* v > 0$. But this means that $\lambda \geq 0$. Therefore all the eigenvalues of M must be non-negative.

b) If a matrix M is self adjoint and has n non-negative eigenvalues, it is positive semidefinite.

Suppose M is self adjoint and has n non-negative eigenvalues. Since M is self adjoint, there exists an orthonormal basis $\beta = \{v_1, \ldots, v_n\}$ of eigenvectors of M. Let $x \in V$. We can write x as

$$x = \sum_{i=1}^{n} a_i v_i.$$

Note that we can rewrite $x^* M x = \langle x, Mx \rangle = \langle Mx, x \rangle$ since M is self adjoint. Furthermore

$$\langle Mx, x \rangle = \left\langle M \sum_{i=1}^{n} a_i v_i, \sum_{i=1}^{n} a_i v_i \right\rangle = \left\langle \sum_{i=1}^{n} a_i \lambda_i v_i, \sum_{i=1}^{n} a_i v_i \right\rangle = \sum_{i=1}^{n} |a_i|^2 \lambda_i.$$

Since each $|a_i| \geq 0$ and $\lambda_i \geq 0$, this sum must be non-negative. Therefore $x^*Mx = \langle Mx, x\rangle \geq 0$, hence M is positive semidefinite.

Problem 2

Proof. Suppose M is positive semidefinite. Then M is self adjoint and from $(1c)$ all of its eigenvalues are non-negative. Because it is self adjoint, M can be decomposed as $M = UDU^*$ where U is unitary and D is the diagonal matrix of its eigenvalues. Since all of the eigenvalues are real and non-negative, we can define the matrix $\widetilde{D}$ as the diagonal matrix with the square root of the entries of D on the diagonal. Note then that $\widetilde{D}^2 = D$. Choosing $Q = \widetilde{D}U^*$ we have $Q^*Q = (\widetilde{D}U^*)^*(\widetilde{D}U^*) = (U\widetilde{D})(\widetilde{D}U^*) = UDU^* = M$.

Suppose that we can write $M = Q^*Q$ for some matrix Q. Let x be a non zero vector in the appropriate space $\mathbb{R}^n$ or $\mathbb{C}^n$. Then we have

$$\langle Mx, x\rangle = \langle Q^*Qx, x\rangle = \langle Qx, Qx\rangle = \|Qx\| \geq 0.$$

Therefore M is positive semidefinite. ■

Problem 3

Proof. By SVD, there exists a decomposition of $A = U\Sigma V^*$ where U and V are unitary and Σ is a diagonal matrix of the singular values $\sigma_1, \sigma_2, \ldots, \sigma_n$ of A. Since V is unitary, we can rewrite

$$A = U\Sigma V^* = UV^*V\Sigma V^* = MP$$

where $M = UV^*$ and $P = V\Sigma V^*$. Since $\sigma_i \geq 0$ and is real, we can construct the matrix $\widetilde{\Sigma}$ as a diagonal matrix with entries $\sqrt{\sigma_i}$. Note that $\widetilde{\Sigma}^* = \widetilde{\Sigma}$ and $\widetilde{\Sigma}^2 = \Sigma$. Therefore

$$P = V\Sigma V^* = V\widetilde{\Sigma}^*\widetilde{\Sigma}V^* = V\widetilde{\Sigma}^*V^*V\widetilde{\Sigma}V^* = (V\widetilde{\Sigma}V^*)^*(V\widetilde{\Sigma}V^*).$$

By (2), P must then be positive semidefinite. Since M is a product of unitary matrices, it must also be unitary. Therefore $A = MP$ where M is unitary and P is positive semidefinite. ■

Problem 4

Proof. Suppose M is an $n \times n$ positive semidefinite matrix with rank r. Since M is self adjoint, we can rewrite $M = UDU^*$ where U is unitary and D is the diagonal matrix of the eigenvalues λ_i of M. Note then that $M^*M = MM = M^2$ and $M^*M = UD^2U^*$. Therefore we know that $M^2 = UD^2U^*$ and hence the eigenvalues of M^2 are λ_i^2. Consider the positive singular values $\sigma_1, \ldots, \sigma_r$. These are equal to the square root of the positive eigenvalues of $M^*M = M^2$. But this means that if $\lambda_j^2 > 0$, then $\sigma_j = \sqrt{\lambda_j^2} = |\lambda_j|$. Since M is positive semidefinite, we know that each $\lambda_i \geq 0$ therefore we can conclude that $\sigma_j = |\lambda_j| = \lambda_j$. Consider the singular values that are zero $\sigma_r + 1, \ldots, \sigma_n$. In this instance there are $n - r$ zero singular values. Since M has rank r and is positive semidefinite, it must have 0 as an eigenvalue repeated $n - r$ times. Therefore we have that the positive singular values of M coincide with its positive eigenvalues, and the singular values that are zero is the same count as the times 0 is an eigenvalue for M, and this in total covers all the eigenvalue of M. Hence the eigenvalues of M are its singular values. ■