2.2

Let $z = x + iy$ such that $\mathrm{Re}(z) = x$ and $\mathrm{Im}(z) = y$.

Part A

$$\mathrm{Re}(iz) = \mathrm{Re}(i(x+iy)) = \mathrm{Re}(ix - y) = \mathrm{Re}(-y + ix) = -y = -\mathrm{Im}(z).$$

Part B

$$\mathrm{Im}(iz) = \mathrm{Im}(i(x+iy)) = \mathrm{Im}(-y + ix) = x = \mathrm{Re}(z).$$

3.1

Part A

$$\begin{aligned}
\frac{1+2i}{3-4i} + \frac{2-i}{5i} &= \frac{(1+2i)(\overline{3-4i})}{3^2+(-4)^2} + \frac{(2-i)(\overline{5i})}{0^2+5^2} \\
&= \frac{(1+2i)(3+4i)}{25} + \frac{(2-i)(-5i)}{25} \\
&= \frac{3+4i+6i+8i^2}{25} + \frac{-10i+5i^2}{25} \\
&= \frac{-5+10i}{25} + \frac{-5-10i}{25} \\
&= -\frac{10}{25} = -\frac{2}{5}
\end{aligned}$$

Part B

$$\begin{aligned}
\frac{5i}{(1-i)(2-i)(3-i)} &= \frac{5i}{(2-i-2i+i^2)(3-i)} \\
&= \frac{5i}{(1-3i)(3-i)} \\
&= \frac{5i}{3-i-9i+3i^2} \\
&= \frac{5i}{-10i} \\
&= -\frac{5}{10} = -\frac{1}{2}
\end{aligned}$$

Part C

$$(1-i)^2 = 1 - 2i + i^2 = -2i \implies (1-i)^4 = (-2i)^2 = 4i^2 = -4.$$

6.7

$$\begin{aligned}
|\operatorname{Re}(2+\overline{z}+z^3)| &= \left|\frac{2+\overline{z}+z^3+(\overline{2+\overline{z}+z^3})}{2}\right| \\
&= \left|\frac{2+\overline{z}+z^3+2+z+\overline{z}^3}{2}\right| \\
&= \left|\frac{4+z+\overline{z}+z^3+2}{2}\right| \\
&\le \frac{2+|z|+|\overline{z}|+|z|^3+|\overline{z}|^3}{2} \\
&= \frac{2+2|z|+2|z|^3}{2} \\
&\le \frac{2+2+2}{2} = 3 \le 4
\end{aligned}$$

6.10

Proof. Let $z = x + iy$.

$\Rightarrow$) Assume that z is real. That is, $y = 0$. Then $z = x + 0y = x = x - 0y = \overline{z}$. Therefore $z = \overline{z}$

$\Leftarrow$) Assume that $z = \overline{z}$. Then

$$x + iy = x - iy.$$

Equating the imaginary components gives $iy = -iy$ or equivalently $y = -y$. This is only true if $y = 0$. Therefore $z = x + 0y = x$ and hence z is real.

Both directions hence prove the if and only if. ■

6.13

$$\begin{aligned}
|z-z_0| = R \implies |z-z_0|^2 &= R^2 \\
(z-z_0)\overline{(z-z_0)} &= R^2 \\
(z-z_0)(\overline{z}-\overline{z_0}) &= R^2 \\
z\overline{z} - z\overline{z_0} - \overline{z}z_0 + z_0\overline{z_0} &= R^2 \\
|z|^2 - z\overline{z_0} - \overline{z\overline{z_0}} + |z_0|^2 &= R^2 \\
|z|^2 - (z\overline{z_0} + \overline{z\overline{z_0}}) + |z_0|^2 &= R^2 \\
|z|^2 - 2\operatorname{Re}(z\overline{z_0}) + |z_0|^2 &= R^2
\end{aligned}$$

9.5

Part A

Since

$$
\begin{aligned}
i &\Leftrightarrow e^{i\frac{\pi}{2}} \\
1 - i\sqrt{3} &\Leftrightarrow 2e^{-i\frac{\pi}{3}} \\
\sqrt{3} + i &\Leftrightarrow 2e^{i\frac{\pi}{6}}
\end{aligned}
$$

it follows that

$$
\begin{aligned}
i(1 - i\sqrt{3})(\sqrt{3} + i) &= e^{i\frac{\pi}{2}} \cdot 2e^{-i\frac{\pi}{3}} \cdot 2e^{i\frac{\pi}{6}} \\
&= 4e^{i\left(\frac{\pi}{2} - \frac{\pi}{3} + \frac{\pi}{6}\right)} \\
&= 4e^{i\frac{\pi}{3}} \\
&= 4 \cdot (\frac{1}{2} + i\frac{\sqrt{3}}{2}) = 2 \cdot (1 + i\sqrt{3})
\end{aligned}
$$

Part B

Since

$$
\begin{aligned}
5i &\Leftrightarrow 5e^{i\frac{\pi}{2}} \\
2 + i &\Leftrightarrow \sqrt{5}e^{i\,\mathrm{arctan}\left(\frac{1}{2}\right)}
\end{aligned}
$$

Let $\theta = \mathrm{arctan}\left(\frac{1}{2}\right)$. It follows

$$
\begin{aligned}
\frac{5i}{2 + i} &= 5e^{\pi\frac{\pi}{2}} \cdot \frac{1}{\sqrt{5}}e^{-i\theta} \\
&= \frac{5}{\sqrt{5}}e^{i\left(\frac{\pi}{2} - \theta\right)} \\
&= \frac{5}{\sqrt{5}}(\cos\left(\frac{\pi}{2} - \theta\right) + i\sin\left(\frac{\pi}{2} - \theta\right)) \\
&= \frac{5}{\sqrt{5}}(\sin\theta - i\cos\theta) \\
&= \frac{5}{\sqrt{5}}(\sin\theta - i\cos\theta)
\end{aligned}
$$

Part C

Part D

9.6

9.8

11.3

11.5

12.1

12.4