Problem 1

Part A

$f(x)$ will be a valid density if

$$\int_0^2 f(x)\mathrm{d}x = 1.$$

Therefore

$$\begin{aligned}
\int_0^2 f(x)dx &= \int_0^2 cx^3\mathrm{d}x \\
1 &= \int_0^2 cx^3\mathrm{d}x \\
1 &= \int_0^2 cx^3\mathrm{d}x \\
1 &= \frac{c}{4}\cdot x^4\Big|_0^2 \\
1 &= \frac{c}{4}\cdot 2^4 \\
1 &= 4c \implies \boxed{c = \frac{1}{4}}.
\end{aligned}$$

Part B

$$\mathbb{P}(X = 1) = \int_1^1 f(x)\mathrm{d}x = 0.$$

Part C

$$\mathbb{P}(X = 1 \text{ or } X = 2) = \mathbb{P}(X = 1) + \mathbb{P}(X = 2) = \int_1^1 f(x)\mathrm{d}x + \int_2^2 f(x)\mathrm{d}x = 0.$$

Part D

$$\mathbb{E}[X] = \int_0^2 xf(x)\mathrm{d}x = \int_0^2 \frac{x^4}{4}\mathrm{d}x = \frac{8}{5}.$$

Part E

$$\mathbb{P}(0.5 < X < 1.5) = \int_{0.5}^{1.5} \frac{x^3}{4}\mathrm{d}x = \frac{x^4}{16}\Big|_{0.5}^{1.5} = \frac{5}{16}.$$

Part F

$$\mathbb{P}(0.5 < X < 2.5) = \mathbb{P}(0.5 < X < 2) = \int_{0.5}^{2} \frac{x^3}{4}\mathrm{d}x = \frac{x^4}{16}\Big|_{0.5}^{2} = \frac{255}{256}.$$

Problem 2

Part A

$$\mathbb{P}(X > 3) = 1 - \mathbb{P}(X < 3) = 1 - \left(1 - e^{-10(3)}\right) = e^{-30} \approx 9.35762 \times 10^{-14}.$$

Part B

$$\begin{aligned}
\mathbb{E}[X] &= \int_0^\infty xe^{-10x}\mathrm{d}x \\
&= -xe^{-10x}\Big|_0^\infty + \int_0^\infty e^{-10x}\mathrm{d}x \\
&= 0 + \frac{e^{-10x}}{-10}\Big|_0^\infty \\
&= \frac{1}{10} \text{ minutes.}
\end{aligned}$$

Part C

$$\mathbb{P}(X > 5) = 1 - \mathbb{P}(X < 5) = 1 - \left(1 - e^{-10(5)}\right) = e^{-50} \approx 1.92875 \times 10^{-22}.$$

Part D

Expected time to wait is

$$\mathbb{E}[10X] = 10 \cdot \mathbb{E}[X] = 10 \cdot \frac{1}{10} = 1 \text{ minute.}$$

The variance of the true wait time is

$$\mathrm{Var}(10X) = 100 \cdot \mathrm{Var}(X) = 100 \cdot \frac{1}{100} = 1 \text{ minute}$$

Problem 3

Part A

$f(x)$ will be a valid probability mass function if

$$\sum_{x=-\infty}^{\infty} f(x) = 1.$$

Therefore

$$\begin{aligned}
\sum_{x=-\infty}^{\infty} f(x) &= 1 \\
\sum_{x=-\infty}^{0} f(x) + \sum_{x=0}^{\infty} f(x) &= 1 \\
0 + \sum_{x=1}^{6} f(x) + \sum_{x=7}^{\infty} &= 1 \\
\sum_{x=0}^{6} \frac{c}{x} &= 1 \\
c \cdot \sum_{x=0}^{6} \frac{1}{x} &= 1 \\
c \cdot \frac{49}{20} &= 1 \implies \boxed{c = \frac{20}{49}}
\end{aligned}$$

Part B

$$\begin{aligned}
\mathbb{P}(1 < X < 6) &= \sum_{x=1}^{5} f(x) \\
&= \sum_{x=1}^{5} \frac{20}{49x} \\
&= \frac{20}{49} \cdot \sum_{x=2}^{5} \frac{1}{x} \\
&= \frac{20}{49} \cdot \frac{77}{60} = \boxed{\frac{11}{21}}
\end{aligned}$$

Problem 4

Part A

$$\mathbb{P}(X < 12) = 1 - e^{-0.03(12)^{1.2}} \approx 0.4466.$$

Part B

$$\mathbb{P}(X > 12) = 1 - \mathbb{P}(X < 12) \approx 1 - 0.4466 = 0.5534.$$

Part C

$$\mathbb{P}(X > 12 | X > 6) = \frac{\mathbb{P}(X > 12)}{\mathbb{P}(X > 6)} = \frac{\mathbb{P}(X > 12)}{e^{-0.03\cdot(6)^{1.2}}} \approx \frac{0.5534}{0.7729} = 0.716.$$

Part D

$$\begin{aligned}
\mathbb{P}(X \leq a) &= 0.5 \\
1 - e^{-0.03\cdot a^{1.2}} &= 0.5 \\
e^{-0.03\cdot a^{1.2}} &= 0.5 \\
-0.03 \cdot a^{1.2} &= \ln(0.5) \\
a^{1.2} &= \frac{\ln(0.5)}{-0.03} \\
a &= \left(\frac{\ln(0.5)}{-0.03}\right)^{\frac{1}{1.2}} \implies \boxed{a \approx 13.6905}.
\end{aligned}$$

Part E

$$\begin{aligned}
\mathbb{P}(\text{Only one last more than 12 months}) &= 2 \cdot \mathbb{P}(X > 12) \cdot \mathbb{P}(X < 12) \\
&\approx 2 \cdot (0.4466)(0.5534) = 0.4943.
\end{aligned}$$

Part F

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x} F(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(1 - e^{-0.03\cdot x^{1.2}}\right) \\
&= 0.03(1.2)(x^{0.2})e^{-0.03\cdot x^{1.2}} \\
f(x) &= 0.036 \cdot x^{0.2} e^{-0.03\cdot x^{1.2}}.
\end{aligned}$$