2.1.1

1. True
2. False
3. False
4. True
5. False
6. False
7. True
8. False

2.1.5

Proof. Let $T : P_2(\mathbb{R}) \to P_3(\mathbb{R})$ defined by $T(f(x)) = x \cdot f(x) + f'(x)$. Let $f, g \in P_2(\mathbb{R})$ and $c \in \mathbb{R}$. Then

$$T(f+g) = x(f(x)+g(x)) + f'(x) + g'(x) = xf(x) + f'(x) + xg(x) + g'(x) = T(f) + T(g)$$

and

$$T(cf) = x(c \cdot f(x)) + cf'(x) = c(xf(x) + f'(x)) = cT(f).$$

Therefore T is a linear transformation. ■

$$\beta_{N(T)} = \{0\} \qquad \Longrightarrow \dim(N(T)) = 0$$
$$\beta_{R(T)} = \left\{x, x^2+1, x^3\right\} \qquad \Longrightarrow \dim(R(T)) = 3$$

Since $N(T) = \{0\}$, T is one-to-one but not onto since $\operatorname{rank}(T) < \dim(P_4(\mathbb{R}))$.

2.1.9

1. $T(0,0) = (1,0) \neq (0,0)$
2. $cT(a_1, a_2) = (ca_1ca_1^2) \neq (ca_1, c^2a_1^2) = T(ca_1, ca_2)$
3. $T(2 \cdot \frac{\pi}{2}, 0) = (0,0) \neq (2,0) = 2 \cdot T(\frac{\pi}{2}, 0)$
4. $T((1,0) + (-1,0)) = (0,0) \neq (2,0) = T(1,0) + T(-1,0)$
5. $T(0,0) = (1,0) \neq (0,0)$

2.1.15

Since the only function when integrated equals zero is the zero function itself. Therefore $N(T) = \{0\}$, therefore T is one-to-one. Note as well that

$$T(a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x + a_1) = \frac{a_n}{n+1}x^{n+1} + \frac{a_{n-1}}{n-1}x^n + \ldots + \frac{a_2}{2}x^2 + a_1 x$$

Since there is no constant term in the output, all constant polynomials dont have a corresponding polynomial that under T would equal it. Therefore T cannot be onto.

2.1.17

Part A

Since $\operatorname{rank} T \leq \dim V < \dim W$, $\operatorname{rank} T < \dim W$ and therefore T is not onto.

Part A

Since $\operatorname{nullity} T = \dim V - \operatorname{rank} T \geq \dim V - \dim W > 0$, $N(T) \neq \{0\}$ and therefore T cannot be one-to-one.

2.1.22

For $T : \mathbb{R}^3 \to \mathbb{R}$, let $a = T(1,0,0), b = T(0,1,0), c = T(0,0,1)$. Note then that

$$T(x,y,z) = xT(1,0,0) + yT(0,1,0) + zT(0,0,1) = ax + by + cz$$

Now generally:

Theorem 0.1. Let $T : \mathbb{F}^n \to \mathbb{F}$ be linear. Then there exists scalars $a_i \in \mathbb{F}$ such that $T(x_1, x_2, \ldots, x_n) = a_1 x_1 + a_2 x_2 + \ldots a_n x_n$.

Proof. Let $T : \mathbb{F}^n \to \mathbb{F}$ be linear. Let e_i denote the vector where the ith position is one and all other's are zero. Let $a_i = T(e_i)$ where $1 \leq i \leq n$. Note then that

$$T(x_1, x_2, x_3, \ldots, x_n) = \sum_{i=1}^{n} x_i T(e_i) = \sum_{i=1}^{n} a_i x_i.$$

■

2.2.1

1. True
2. True
3. False
4. True

5. True
6. False

2.2.4

$$[T]_\beta^\gamma = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

2.2.10

$$[T]_\beta = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & & 0 \\ 0 & 1 & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & 1 \end{pmatrix}$$

2.2.14

Proof. Let $V = P(\mathbb{R})$ and $T_j(f) = f^{(j)}(x)$. Let $n \in \mathbb{N}$ and assume that $\sum_{j=0}^{n} a_i T_i = 0$. Note that $T_j(x^n) = \frac{n!}{(n-j)!}x^{n-j}$. It is clear that for different j, the results are linearly independent since the degrees are different. Therefore $\sum_{j=0}^{n} a_i T_i(x^n) = 0$ implies that $a_i = 0$ for all i. Hence $\{T_1, T_2, \ldots, T_n\}$ is linearly independent. ■

2.2.16

2.3.3

Part A

$$[U]_\beta^\gamma = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

$$[T]_\beta = \begin{pmatrix} 2 & 3 & 0 \\ 0 & 3 & 8 \\ 0 & 0 & 2 \end{pmatrix}$$

$$[UT]_\beta^\gamma = \begin{pmatrix} 2 & 6 & 8 \\ 0 & 0 & 2 \\ 2 & 0 & -8 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \times \begin{pmatrix} 2 & 3 & 0 \\ 0 & 3 & 8 \\ 0 & 0 & 2 \end{pmatrix} \checkmark$$

2.3.9

Take $T(a,b) = (0,a)$ and $U(a,b) = (a,0)$. Note then that

$$UT(a,b) = U(T(a,b)) = U(0,a) = (0,0)$$

but that

$$TU(a,b) = T(U(a,b)) = T(a,0) = (0,a) \neq (0,0)$$

Therefore by using the standard basis for $\mathbb{F}^2$,

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$
$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

2.3.11

2.3.16