Sequences and Series of Functions

1.1 Power Series

Definition 1.1 (Power Series). A power series is a real valued function $f(x) = \sum a_n x^n$ for some sequence (a_n).

Theorem 1.1. For a power series $\sum a_n x^n$, let $\beta = \limsup |a_n|^{\frac{1}{n}}$ and $R = \frac{1}{\beta}$. The power series converges for $|x| < R$ and diverges for $|x| > R$

Proof. Apply the root test. Then

$$\limsup |c_n|^{\frac{1}{n}} = \limsup |a_n|^{\frac{1}{n}}|x| = \limsup |a_n|^{\frac{1}{n}}|x| = |x|\beta.$$

Note then that $|x| < R = \frac{1}{\beta}$ means that $\limsup |c_n|^{\frac{1}{n}} < 1$ and therefore the series converges. The opposite is true for $|x| > R$. ■

Example 1.1. Consider $\sum x^n$. Note that $a_n = 1$ for all $n \in \mathbb{N}$. Therefore $\limsup |a_n|^{\frac{1}{n}} = \limsup 1^{\frac{1}{n}} = 1$. Therefore the power series converges for all $|x| < 1$. Note that $x = 1$ gives a divergent series and $x = -1$ gives an alternating series whose non alternative part does not go to zero and hence also diverges.

Example 1.2. Consider $\sum \frac{x^n}{n!}$. In this instance $a_n = \frac{1}{n!}$. Then

$$\beta = \limsup |a_n|^{\frac{1}{n}} = \limsup \left|\frac{1}{n!}\right|^{\frac{1}{n}}.$$

This would be hard to compute. However, if this limit exists, then it matches the value of the ratio test and therefore

$$\beta = \limsup \left|\frac{a_{n+1}}{a_n}\right| = \limsup \frac{1}{(n+1)!} \cdot \frac{n!}{1} = \limsup \frac{1}{n} = 0.$$

Therefore $R = +\infty$ meaning the interval of convergence is all of $\mathbb{R}$.
Remark. Alternatively, one can use the Sterling approximation of the factorial to do the root test. The Sterling approximation is

$$n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}.$$

Hence

$$\limsup \left|\frac{1}{n!}\right|^{\frac{1}{n}} = \limsup \frac{1}{\left(\left(\frac{n}{e}\right)^n \sqrt{2\pi n}\right)^{\frac{1}{n}}} = \limsup \frac{1}{\frac{n}{e} \cdot \left(\sqrt{2\pi n}\right)^{\frac{1}{n}}} = \limsup \frac{1}{n} = 0.$$

Example 1.3. Consider $\sum \frac{x^n}{n^2}$. Then

$$\beta = \limsup \left(\frac{1}{n^2}\right)^{\frac{1}{n}} = \limsup \frac{1}{\sqrt[n]{n^2}} = 1.$$

Therefore the power series converges for $|x| < 1$. Importantly, for $x = 1$ and $x = -1$, you get convergent series and therefore the interval of convergence is $[-1, 1]$.

Example 1.4. Consider $\sum \frac{(-1)^{n+1}x^n}{n}$. Then $a_n = \frac{(-1)^{n+1}}{n}$ and

$$\beta = \limsup \left|\frac{(-1)^{n+1}}{n}\right|^{\frac{1}{n}} = \limsup \frac{1}{\sqrt[n]{n}} = \frac{1}{1} = 1.$$

Therefore the power series converges for $|x| < 1$. Checking $x = 1$,

$$\sum \frac{(-1)^{n+1}}{n} \text{ converges by alternating series test.}$$

And checking for $x = -1$,

$$\sum \frac{(-1)^{2n+1}}{n} = \sum \frac{-1}{n} = -\sum \frac{1}{n} \text{ which diverges.}$$

Therefore the interval of convergence is $(-1, 1]$.

Example 1.5. Consider $\sum \frac{(2n)!x^n}{(n!)^2}$. Then $a_n = \frac{(2n)!}{(n!)^2}$. Apply the ratio test to get β.

$$\beta = \limsup \left|\frac{a_{n+1}}{a_n}\right| = \limsup \frac{(2n+2)!}{((n+1)!)^2} \cdot \frac{(2n)!}{(n!)^2} = \limsup \frac{(2n+2)(2n+1)}{(n+1)(n+1)} = 4.$$

Therefore it converges on $|x| < \frac{1}{4}$. Checking the endpoints suck but $x = \frac{1}{4}$ diverges by using Sterlings approximation and $x = -\frac{1}{4}$ converges by the alternating series test by the previous method. Therefore the interval of convergence is $\left[-\frac{1}{4}, \frac{1}{4}\right)$.

1.2 Uniform Convergence

An initial, but weak, formulation of functional sequence convergence is by applying the a basic limit of a sequence.

Definition 1.2 (Pointwise Convergence). A sequence of real value functions $f_n : S \subset \mathbb{R} \to \mathbb{R}$ converges point wise to a function f on S if $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in S$

Definition 1.3 (Uniform Convergence). A sequence of real value functions $f_n : S \subset \mathbb{R} \to \mathbb{R}$ uniformly converges to a function f on S if $\forall \epsilon > 0$, there is some $N \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \epsilon, n > N, \forall x \in S.$$

Example 1.6. Consider the sequence of functions $f_n(x) = x^n$ on $[0,1]$. Note that for all n, $f_n(0) = 0$ and $f_n(1) = 1$. Furthermore, for $0 < x < 1$, $\lim x^n = 0$. Therefore

$$\lim f_n(x) = f(x) = \begin{cases} 0 & 0 \le x < 1 \\ 1 & x = 1 \end{cases}.$$

is the pointwise limit of the sequence. For uniform convergence, we want

$$|f_n(x) - f(x)| < \epsilon \Leftrightarrow \left| x^n - \begin{cases} 0 & 0 \le x < 1 \\ 1 & x = 1 \end{cases} \right| < \epsilon.$$

For $x = 1$, the absolute value goes to 0 and therefore only $0 \le x < 1$ matters. The question becomes when

$$x^n < \epsilon \implies n > \frac{\ln(\epsilon)}{\ln|x|}.$$

However, it is not possible to bound this quantity since $x \to 1$ leads to $\frac{1}{\ln|x|} \to -\infty$. Therefore the sequence does not uniformly converge to f.

Example 1.7. Let $g_n(x) = (1-|x|)^n$ on $(-1,1)$. Note that $\lim g_n(0) = 1$ since $g_n(0) = 1$ for all n. For any other x, $|x| < 1$ and therefore $1-|x| < 1$. Hence $\lim g_n(x) = 0$ for $x \ne 0$. Hence

$$\lim g_n(x) = g(x) = \begin{cases} 1 & x = 0 \\ 0 & x \ne 0 \end{cases}.$$

Checking for uniform convergence,

$$|g_n(x) - g(x)| < \epsilon \Leftrightarrow |(1-|x|)^n \begin{cases} 1 & x = 0 \\ 0 & x \ne 0 \end{cases}|.$$

We only have to care about $x \ne 0$, therefore

$$|(1-|x|)^n| < \epsilon \implies n > \frac{\ln(\epsilon)}{\ln(1-|x|)}.$$

However, $\sup_{x\in(-1,1)} \frac{\ln(\epsilon)}{\ln(1-|x|)} = +\infty$, therefore the sequence does not uniformly converge to $g(x)$.

Example 1.8. Let $h_n(x) = \frac{1}{n}\sin(nx)$. Since $\left|\frac{1}{n}\sin(nx)\right| \le \left|\frac{1}{n}\right| = \frac{1}{n}$, it follows that

$$0 \le \lim_{n\to\infty} \left|\frac{1}{n}\sin(nx)\right| \le \lim_{n\to\infty} \frac{1}{n} = 0.$$

Therefore $\lim h_n(x) = 0$. Checking for uniform convergence, we want

$$|h_n(x) - h(x)| < \epsilon \Leftrightarrow \left|\frac{1}{n}\sin(nx) - 0\right| < \epsilon.$$

Since $\left|\frac{1}{n}\sin(nx)\right| \leq \frac{1}{n}$, choosing $n > \frac{1}{\epsilon}$ gives the desired inequality. Since the bound for n doesnt depend on x, the sequence uniformly converges to $h(x) = 0$.

Example 1.9. Let $j_n(x) = \frac{nx}{2n+1}$ on $S = [-2, 2]$. It's pointwise limit is

$$\lim j_n(x) = \lim \frac{nx}{2n+1} = x \lim \frac{n}{2n+1} = \frac{x}{2} = j(x).$$

Checking for uniform convergence, we want

$$\begin{aligned}\left|\frac{nx}{2n+1} - \frac{x}{2}\right| < \epsilon &\implies \left|\frac{2nx - (2n+1)x}{2(2n+1)}\right| < \epsilon \\ &\implies \frac{|x|}{2(2n+1)} < \epsilon \\ &\implies \frac{|x|}{2\epsilon} < 2n+1 \\ &\implies n > \frac{|x|}{4\epsilon} - \frac{1}{2}\end{aligned}$$

Since $|x| < 2$, $n > \frac{1}{2\epsilon} - \frac{1}{2} > \frac{|x|}{4\epsilon} - \frac{1}{2}$ gives the original inequality. Therefore the sequence uniformly converges to $j(x)$.

Example 1.10. Let

$$k_n(x) = \begin{cases} 1 & x > \frac{1}{n} \\ 0 & x \leq \frac{1}{n} \end{cases}$$

on $S = [0, 1]$. Note that $0 \leq \frac{1}{n}$ for all n, meaning $\lim k_n(0) = 0$. For similar reasoning $1 \geq \frac{1}{n}$ for all $n > 1$ and therefore $\lim k_n(1) = 1$. For any $0 < x < 1$, there will be some $N \in \mathbb{N}$ such that $n > N \implies \frac{1}{n} < x$. Hence $\lim k_n(x) = 1$ for all $0 < x < 1$. In total then, the pointwise convergence is

$$k(x) = \begin{cases} 0 & x = 0 \\ 1 & x \neq 0 \end{cases}.$$

Checking for uniform convergence, we want

$$|k_n(x) - k(x)| < \epsilon \implies \left|\begin{cases} 1 & x > \frac{1}{n} \\ 0 & x \leq \frac{1}{n} \end{cases} - \begin{cases} 0 & x = 0 \\ 1 & x \neq 0 \end{cases}\right| = \left|\begin{cases} 0-0 & x = 0 \\ 0-1 & 0 < x \leq \frac{1}{n} \\ 1-1 & \frac{1}{n} < x \leq 1 \end{cases}\right| < \epsilon.$$

Note then that

$$\left|\begin{cases} 0-0 & x=0 \\ 0-1 & 0<x\leq \frac{1}{n} \\ 1-1 & \frac{1}{n}<x\leq 1 \end{cases}\right| = \begin{cases} 0 & x=0, \frac{1}{n}<x\leq 1 \\ 1 & 0<x\leq \frac{1}{n} \end{cases}.$$

Since $0 < x \leq \frac{1}{n}$ the value is 1, it is not possible to get arbitrarily close to the pointwise convergence across all x.

Theorem 1.2. A sequence of functions f_n uniformly converges to f on $S \subset \mathbb{R}$ iff

$$\lim_{n\to\infty} \sup_{x\in S} \{f_n(x) - f(x)\}.$$

Theorem 1.3. If $f_n \to f$ uniformly on $[a, b]$ and f_n is continuous on $[a, b]$ for all n, then

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx.$$

Proof. We want to show that

$$\forall \epsilon > 0, \exists N, \text{ s.t. } \left|\int_a^b f_n(x)\mathrm{d}x - \int_a^b f(x)dx\right| < \epsilon.$$

Fix $\epsilon > 0$. Then

$$\begin{aligned} \left|\int_a^b f_n(x)\mathrm{d}x - \int_a^b f(x)dx\right| &= \left|\int_a^b f_n(x) - f(x)\mathrm{d}x\right| \\ &\leq \int_a^b |f_n(x) - f(x)|\mathrm{d}x \end{aligned}$$

Since $f_n \to f$ uniformly on $[a, b]$, there is a N such that $|f_n(x) - f(x)| < \frac{\epsilon}{b-a}$ for $n > N$ and $x \in [a, b]$. Note then that

$$\int_a^b |f_n(x) - f(x)|\mathrm{d}x < \int_a^b \frac{\epsilon}{b-a}\mathrm{d}x = \epsilon.$$

Therefore for $n > N$,

$$\left|\int_a^b f_n(x)\mathrm{d}x - \int_a^b f(x)dx\right| \leq \int_a^b |f_n(x) - f(x)|\mathrm{d}x < \epsilon.$$

■

1.3 Cauchy Function Sequences

Definition 1.4 (Uniformly Cauchy). A sequence of real valued functions f_n is called unfiormly Cauchy if

$$\forall \epsilon > 0, \exists N \text{ s.t. } |f_n(x) - f_m(x)| < \epsilon, \forall x \in S, n > m > N.$$

Theorem 1.4. If a sequence of real valued functions f_n is uniformly Cauchy on $S \subset \mathbb{R}$, then there exists some function $f(x)$ on S such that $f_n \to f$ uniformly on S.

Proof. Fix $x \in S$ and let $y_n = f_n(x)$. Note that this gives a Cauchy sequence since f_n is uniformly Cauchy. Therefore y_n converges to some $y \in \mathbb{R}$. Define $F(x) = y$. By construction, $f_n \to F$ pointwise. Fix $\epsilon > 0$. Since f_n is uniformly Cauchy

$$|f_n(x) - f_m(x)| < \frac{\epsilon}{2} \implies -\frac{\epsilon}{2} + f_m(x) < f_n(x) < f_m(x) + \frac{\epsilon}{2}.$$

Since $n > m$, n can be sent to infinity while fixing m, giving

$$-\frac{\epsilon}{2} + f_m(x) < F(x) < f_m(x) + \frac{\epsilon}{2} \implies -\frac{\epsilon}{2} < F(x) - f_m(x) < \frac{\epsilon}{2}.$$

Therefore

$$|f_m(x) - F(x)| < \frac{\epsilon}{2} < \epsilon.$$

Therefore f_m converges uniformly to F on S. ■

Example 1.11. Consider the series $f_n(x) = \sum_{k=0}^{n} \frac{1}{1+x^k}$ on $[2, \infty)$. Trying to determine if this uniformy converges with a direct approach will not work as it requires knowledge about the function the infinite series represents. However, notice that for $n > m$

$$\begin{aligned}
|f_n(x) - f_m(x)| &= \left|\sum_{k=0}^{n} \frac{1}{1+x^k} - \sum_{j=0}^{m} \frac{1}{1+x^j}\right| \\
&= \left|\sum_{k=m+1}^{n} \frac{1}{1+x^k}\right| \\
&\leq \left|\sum_{k=m+1}^{n} \frac{1}{1+2^k}\right| \\
&\leq \left|\sum_{k=m+1}^{n} \frac{1}{2^k}\right| \\
&= \frac{\frac{1}{2^{m+1}} - \frac{1}{2^n+1}}{1-\frac{1}{2}} \\
&= \frac{1}{2^m} - \frac{1}{2^n} < \frac{1}{2^m}
\end{aligned}$$

Take $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that $\frac{1}{2^N} < \epsilon$. Then

$$|f_n(x) - f_m(x)| < \frac{1}{2^m} < \frac{1}{2^N} < \epsilon, n > m > N.$$

Therefore f_n is uniformly Cauchy on $[2, \infty)$. This means that $f_n \to f$ uniformly on $[2, \infty)$.

Example 1.12. Consider a power series $f(x) = \sum_{k=0}^{\infty} a_k x^k$ with the sequence of polynomials $f_n(x) = \sum_{k=0}^{n} a_k x^k$. Let $\beta = \limsup |a_n|^{\frac{1}{n}}$ and $R = \frac{1}{\beta}$. Consider some $0 < \tilde{R} < R$ and

$S = (-\tilde{R}, \tilde{R})$. Then

$$
\begin{aligned}
|f_n(x) - f_m(x)| &= \left| \sum_{k=m+1}^{n} a_k x^k \right| \\
&\leq \sum_{k=m+1}^{n} \left| a_k x^k \right| \\
&= \sum_{k=m+1}^{n} \left(|a_k|^{\frac{1}{k}} |x| \right)^k \\
&\leq \sum_{k=m+1}^{n} \left(|a_k|^{\frac{1}{k}} \tilde{R} \right)^k
\end{aligned}
$$

Since $\limsup |a_n|^{\frac{1}{n}} = \beta = \frac{1}{R}$, it is possible to find some $K \in \mathbb{N}$ such that

$$||a_k|^{\frac{1}{k}} - \beta| < \epsilon_1 \text{ with } (\beta + \epsilon_1)\tilde{R} < 1, k > K.$$

Let $\alpha = (\beta + \epsilon_1)\tilde{R} < 1$. Then

$$\sum_{k=m+1}^{n} \left(|a_k|^{\frac{1}{k}} \tilde{R} \right)^k < \sum_{k=m+1}^{n} \alpha^k \leq \frac{\alpha^{m+1}}{1 - \alpha} < \epsilon.$$

with $n > m > K$. This means that f_n is uniformly Cauchy and hence uniformly converges to f in the interval $(-\tilde{R}, \tilde{R})$. This result means that many useful properties about unfirom convergence apply to the interior of the interval of convergence.