4.17

The set of $n \times n$ upper-triangular matrices with determinant 1 under matrix multiplication is a group.

Proof. Let M_n denote the set of $n \times n$ upper triangular matrices with determinant 1. First note that the multiplication of two upper triangular matrices also results in an upper triangular matrix. Let $A, B \in M_n$ and $C = AB$. An entry C_{ij} from C with $i > j$ is given by

$$C_{ij} = \sum_{k=1}^{n} a_{ik} b_{jk}.$$

The sum can be split into two parts, resulting in

$$\begin{aligned} C_{ij} &= \sum_{k=1}^{n} a_{ik} b_{jk} \\ &= \sum_{k=1}^{i-1} a_{ik} b_{jk} + \sum_{k=i}^{n} a_{ik} b_{jk} \\ &= 0 + 0 = 0. \end{aligned}$$

Therefore the entries below the diagonal of C are 0, meaning C is also upper-triangular. Additionally, $\det(C) = \det(AB) = \det(A)\det(B) = 1$. Therefore M_n is closed under matrix multiplication. Consider now the three group axioms.

$\mathcal{G}_1$.) Associativity is satisfied since matrix multiplication is associative.

$\mathcal{G}_2$.) The identity matrix I_n is an upper-triangular matrix with $\det(I_n) = 1$, therefore M_n has an identity element.

$\mathcal{G}_3$.) Let $A \in M_n$. Note that the inverse of A can be found by row reducing the augmented matrix $[A|I]$ to $\left[I|A^{-1}\right]$. This will look like

$$\left(\begin{array}{cccc|cccc} a_{11} & a_{12} & \cdots & a_{1n} & 1 & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & a_{2n} & 0 & 1 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & a_{nn} & 0 & 0 & 0 & 1 \end{array}\right).$$

Since A is in upper triangular form, its augmented form can be row-reduced using back substitution which will maintain the upper triangular form on the right side. Therefore once the matrix is in the form $\left[I|A^{-1}\right]$, the inverse matrix will be upper-triangular as well. Additionally, $\det\left(A^{-1}\right) = \frac{1}{\det(A)} = 1$. Therefore $A^{-1} \in M_n$, meaning every element in M_n has an inverse.

Since M_n under matrix multiplication satisfies the group axioms, it is a group. ■

4.18

All $n \times n$ matrices with determinant either 1 or -1 under matrix multiplication forms a group

Proof. Let M_n denote all $n \times n$ matrices with determinant 1 or -1. Let $A, B \in M_n$. Their product is an $n \times n$ matrix since both are $n \times n$. Additionally $\det(AB) = \det(A)\det(B)$. Therefore the determinant of their product is also ± 1, hence M_n is closed under matrix multiplication. Consider now the three group axioms.

$\mathcal{G}_1$.) Associativity is satisfied since matrix multiplication is associative.

$\mathcal{G}_2$.) The identity matrix I_n is an $n \times n$ matrix and has $\det(I_n) = 1$ meaning $I_n \in M_n$, hence M_n has an identity element.

$\mathcal{G}_3$.) Let $A \in M_n$. Since $\det(A) \neq 0$ and $\det\left(A^{-1}\right) = \frac{1}{\det(A)}$ which is either 1 or -1, A has an inverse A^{-1} such that $AA^{-1} = A^{-1}A = I_n$ with $A^{-1} \in M_n$. Therefore M_n has an inverse for each element.

Since M_n under matrix multiplication satisfies the group axioms, it is a group. ■

4.19

0.1 Part A

$*$ is a binary operation on S.

Proof. Let S be the set $\mathbb{R} \setminus \{-1\}$ and define the mapping $* : S \times S \to S$ where $a * b = a + b + ab$. Examine if $*$ is a well defined map. Since the addition and multiplication of real numbers is well defined, $*$ can only ever be not well-defined if there exists $a, b \in S$ such that $a * b = -1$. Assume towards contradiction that these a and b exist. Then

$$\begin{aligned} a + b + ab &= -1 \\ a + ab + b + 1 &= 0 \\ (a+1)(b+1) &= 0. \end{aligned}$$

However, this implies that one of a or b is -1, contradicting the assumption that $a, b \in S$ since elements in S cannot be equal to -1. Note also that $a + b + ab$ results in a singular value. Therefore since $*$ maps into S exclusively and has only one associated value for every input, it is a well-defined map and hence a binary operation. ■

0.2 Part B

$\langle S, * \rangle$ is a group.

Proof. Define the binary algebraic structure $\langle S, * \rangle$ with the prior S and $*$. Examine the axioms for S to be a group under $*$.

$\mathcal{G}_1$.) Let $a, b, c \in S$. It follows that

$$\begin{aligned} a * (b * c) &= a * (b + c + bc) \\ &= a + b + c + bc + ab + ac + abc. \end{aligned}$$

Additionally,

$$\begin{aligned} (a * b) * c &= (a + b + ab) * c \\ &= a + b + ab + c + ca + cb + cab \\ &= a + b + c + bc + ab + ac + abc. \end{aligned}$$

Since $a * (b * c) = (a * b) * c$, associativity is satisfied.

$\mathcal{G}_2$.) Consider the element $0 \in S$. Let $a \in S$. Then

$$a * 0 = 0 * a = a + 0 + a(0) = a.$$

Therefore 0 is the identity element of S.

$\mathcal{G}_3$.) Let $a \in S$. Choose $a' = -\frac{a}{1+a}$. Note then that

$$\begin{aligned} a * a' = a' * a &= a - \frac{a}{1+a} - a \cdot \frac{a}{1+a} \\ &= \frac{a(1+a)}{1+a} - \frac{a}{1+a} - \frac{a^2}{1+a} \\ &= \frac{a + a^2 - a - a^2}{1+a} \\ &= \frac{0}{1+a} \\ &= 0. \end{aligned}$$

Since $a \neq -1$, the inverse is well defined and therefore there is an inverse for every element in S.

Since S under $*$ satisfies the group axioms, it is a group. $\blacksquare$

Part C

Note the operation is commutative (because $a + b + ab = b + a + ba$).

$$\begin{aligned}
2 * x * 3 &= 7 \\
x * 3 * 2 &= 7 \\
x * (3 + 2 + 3 \cdot 2) &= 7 \\
x * 11 &= 7 \\
x * 11 * 11' &= 7 * 11' \\
x * 0 &= 7 * 11' \\
x &= 7 * 11' \\
x &= 7 * \left(-\frac{11}{12}\right) \\
x &= 7 - \frac{11}{12} - \frac{77}{12} \\
x &= \frac{84}{12} - \frac{11}{12} - \frac{77}{12} \\
x &= \frac{84 - 11 - 77}{12} \\
x &= -\frac{4}{12} \\
x &= -\frac{1}{3}.
\end{aligned}$$

4.20

Displayed are all of the 4 element groups.

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	a	e
c	c	b	e	a

The second table can be made into the third table by swapping all instances of a with b, resulting in

	e	b	a	c
e	e	b	a	c
b	a	a	c	e
a	b	c	e	b
c	c	e	b	a

and then rearranging the order back to e, a, b, c provides

	e	a	b	c
e	e	a	b	c
a	b	e	c	b
b	a	c	a	e
c	c	b	e	a

Part A

Every table is symmetric across its diagonal, hence every group of 4 elements is abelian.

Part B

The second table is isomorphic to U_4 with the mapping

$$\begin{aligned} e &\to 1 \\ a &\to i \\ b &\to -1 \\ c &\to -i. \end{aligned}$$

This is true since the table

	1	i	-1	$-i$
1	1	i	-1	$-i$
i	i	-1	$-i$	1
-1	-1	$-i$	1	i
$-i$	$-i$	1	i	-1

is equivalent under the mapping outlined above.

Part C

Consider the first table. Choose $n = 2$ and define the following matrices

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},\ B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},\ C = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

These are all in the group outlined in Example 14 since all their determinants are 1 or -1. If the following mapping is used

$$\begin{aligned} e &\to E \\ a &\to A \\ b &\to B \\ c &\to C \end{aligned}$$

then the same structure is achieved between the two groups. This can be checked by the fact that the table is the Klein-4 group, therefore if $A^2 = B^2 = C^2 = E$, the isomorphism is correct.

$$
\begin{aligned}
A^2 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = E \\
B^2 &= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = E \\
C^2 &= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = E.
\end{aligned}
$$

4.21

Let S be a set of 3 elements. That is $S = \{x_1, x_2, x_3\}$. For a group structure to emerge from a binary operation on S, one of the elements must be chosen as an identity element. Therefore there are 3 possible choices for an identity element. There is only one group structure for a given identity element as seen in the following table:

	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

Therefore since there is only one associated group structure for every choice of an identity element and there are 3 choices for an identity element, there are 3 binary operations that give a group structure over a set of 3 elements.