46.1

Part A

$$\begin{aligned}
\int_C f(z)\mathrm{d}z &= \int_0^{\pi} \left(1 + \frac{2}{2e^{i\theta}}\right)\frac{\mathrm{d}}{\mathrm{d}\theta}\left(2e^{i\theta}\right)\mathrm{d}\theta \\
&= \int_0^{\pi} \left(1 + e^{-i\theta}\right)\left(2ie^{i\theta}\right)\mathrm{d}\theta \\
&= \int_0^{\pi} \left(2ie^{i\theta} + 2i\right)\mathrm{d}\theta \\
&= \left[2e^{i\theta} + 2i\theta\right]_0^{\pi} \\
&= 2e^{i\pi} + 2\pi i - 2e^0 + 0 \\
&= -4 + 2\pi i
\end{aligned}$$

Part B

$$\begin{aligned}
\int_C f(z)\mathrm{d}z &= \int_{\pi}^{2\pi} \left(1 + \frac{2}{2e^{i\theta}}\right)\frac{\mathrm{d}}{\mathrm{d}\theta}\left(2e^{i\theta}\right)\mathrm{d}\theta \\
&= \int_{\pi}^{2\pi} \left(1 + e^{-i\theta}\right)\left(2ie^{i\theta}\right)\mathrm{d}\theta \\
&= \int_{\pi}^{2\pi} \left(2ie^{i\theta} + 2i\right)\mathrm{d}\theta \\
&= \left[2e^{i\theta} + 2i\theta\right]_{\pi}^{2\pi} \\
&= 2e^{2\pi i} + 4\pi i - 2e^{\pi i} - 2\pi i \\
&= 2 + 4\pi i + 2 - 2\pi i \\
&= 4 + 2\pi i
\end{aligned}$$

Part C

$$\begin{aligned}
\int_C f(z)\mathrm{d}z &= \int_0^{2\pi} \left(1 + \frac{2}{2e^{i\theta}}\right)\frac{\mathrm{d}}{\mathrm{d}\theta}\left(2e^{i\theta}\right)\mathrm{d}\theta \\
&= \int_0^{2\pi} \left(1 + e^{-i\theta}\right)\left(2ie^{i\theta}\right)\mathrm{d}\theta \\
&= \int_0^{2\pi} \left(2ie^{i\theta} + 2i\right)\mathrm{d}\theta \\
&= \left[2e^{i\theta} + 2i\theta\right]_0^{2\pi} \\
&= 2e^{2\pi i} + 4\pi i - 2e^0 - 0 \\
&= 2 + 4\pi i - 2 \\
&= 4\pi i
\end{aligned}$$

46.8

C can be parameterized by the function

$$z(\theta) = Re^{i\theta}, -\pi \le \theta \le \pi$$

meaning

$$f(z(\theta)) = \exp\left[(a-1)\,\mathrm{Log}(Re^{i\theta})\right] = \exp[(a-1)(\ln R + i\theta)].$$

Therefore

$$\begin{aligned}
\int_C f(z)\mathrm{d}z &= \int_{-\pi}^{\pi} \exp[(a-1)(\ln R + i\theta)]\left(iRe^{i\theta}\right)\mathrm{d}\theta \\
&= iRe^{(a-1)\ln R}\int_{-\pi}^{\pi} e^{ia\theta}\mathrm{d}\theta \\
&= iRR^{a-1}\cdot\frac{1}{ia}\left[e^{ia\theta}\right]_{-\pi}^{\pi} \\
&= \frac{R^a}{a}(e^{a\pi i} - e^{-a\pi i}) \\
&= i\frac{2R^a}{a}\sin(a\pi)
\end{aligned}$$

46.10

Parameterize the contour C with $z(\theta) = e^{i\theta}$ with $0 \le \theta \le 2\pi$. Then

$$\begin{aligned}
\int_C z^m\overline{z}^n\mathrm{d}z = \int_0^{2\pi}\left(e^{im\theta}e^{-in\theta}\right)\left(ie^{i\theta}\right)\mathrm{d}\theta &= i\int_0^{2\pi} e^{i(m+1)\theta}e^{-in\theta}\mathrm{d}\theta \\
&= i\begin{cases} 0 & m+1 \ne n \\ 2\pi & m+1 = n \end{cases} \\
&= \begin{cases} 0 & m+1 \ne n \\ 2\pi i & m+1 = n \end{cases}
\end{aligned}$$

46.13

Proof. Let $f(z) = (z-z_0)^{n-1}$ and $z(t) = z_0 + Re^{it}$. Note then that

$$f(z(t)) = \left(z_0 + Re^{it} - z_0\right)^{n-1} = R^{n-1}e^{it(n-1)}$$

and

$$z'(t) = Rie^{it}.$$

Therefore

$$\begin{aligned}
\int_{C_0}(z-z_0)^{n-1}\mathrm{d}z &= \int_{-\pi}^{\pi}\left(R^{n-1}e^{it(n-1)}\right)\left(Rie^{it}\right)\mathrm{d}t \\
&= iR^n\int_{-\pi}^{\pi} e^{itn}\mathrm{d}t
\end{aligned}$$

Consider two cases

$(n = 0)$ The integral equals

$$iR^0 \int_{-\pi}^{\pi} e^{it(0)} \mathrm{d}t = i \int_{\pi}^{\pi} \mathrm{d}t = 2\pi i.$$

$(n \neq 0)$ The integral equals

$$iR^n \int_{-\pi}^{\pi} e^{itn} = iR^n \left[\frac{e^{itn}}{in} \right]_{-\pi}^{\pi} = \frac{R^n}{n} \left[e^{in(\pi)} - e^{in(-\pi)} \right] = \frac{R^n}{n} [-1 - (-1)] = 0.$$

Therefore

$$\int_{C_0} (z - z_0)^{n-1} \mathrm{d}z = \begin{cases} 0 & n \in \mathbb{Z} \setminus \{0\} \\ 2\pi i & n = 0 \end{cases}$$ ■

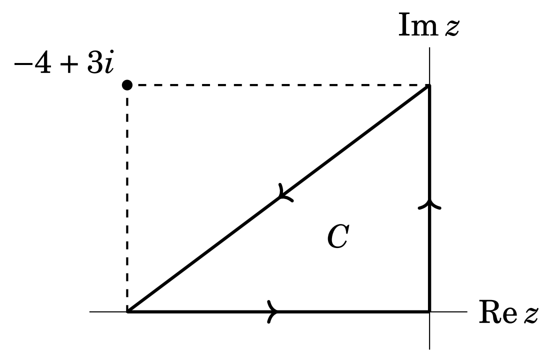
47·3

The entire contour can be bounded by the rectangle from 0 to $-4 + 3i$. Therefore for $z = x + iy$ it follows on C that $-4 \leq x \leq 0$ and $0 \leq y \leq 3$. Since $|e^z - \overline{z}| \leq e^x + \sqrt{x^2 + y^2}$

$$|e^z - \overline{z}| \leq e^x + \sqrt{x^2 + y^2} \leq e^0 + \sqrt{(-4)^2 + 3^2} = 6$$

when z is in the rectangle. Since the length of the path is $4 + 3 + \sqrt{3^2 + (-4)^2} = 3 + 4 + 5 = 12$, it follows that

$$\left| \int_C e^z - \overline{z} \right| \leq 5 \cdot 12 = 60.$$



47·6

Proof. When on C_ρ, $|z^{\frac{1}{2}}| = \sqrt{\rho}$ and therefore $\left| z^{-\frac{1}{2}} \right| = \frac{\sqrt{\rho}}{\rho}$. Since $f(z)$ is analytic on the disk $|z| \leq 1$, then there exists some $M \in \mathbb{R} > 0$ such that $|f(z)| \leq M$ for all z on the disk. Therefore

$$\left| z^{-\frac{1}{2}} f(z) \right| \leq \frac{M\sqrt{\rho}}{\rho}.$$

$z^{-\frac{1}{2}}$ is analytic on any branch taken and so $z^{-\frac{1}{2}} f(z)$ is analytic and hence piecewise continuous on C_ρ. Therefore

$$\int_{C_\rho} \left(z^{-\frac{1}{2}} f(z) \right) \leq 2\pi\rho \cdot \frac{M\sqrt{\rho}}{\rho} = 2\pi M \sqrt{\rho}.$$

Therefore since $\lim_{\rho\to 0} 2\pi M\sqrt{\rho} = 0$, then

$$\lim_{\rho\to 0}\left|\int_{C_\rho} z^{-\frac{1}{2}} f(z)\right| = 0 \implies \lim_{\rho\to 0}\int_{C_\rho} z^{-\frac{1}{2}} f(z) = 0.$$

■

49.2

Part A

Note that $F(z) = \frac{z^3}{3}$ is an antiderivative of z^2 since $\frac{\mathrm{d}}{\mathrm{d}z}\frac{z^3}{3} = 3\cdot\frac{x^2}{3} = x^2$. Therefore the integral is

$$\int_0^{1+i} z^2 \mathrm{d}z = \left[\frac{z^3}{3}\right]_0^{1+i} = \frac{(1+i)^3}{3} = \frac{-2-2i}{3} = \frac{2}{3}(-1-i).$$

Part B

Note that $F(z) = 2\sin\left(\frac{z}{2}\right)$ is an antiderivative of $\cos\left(\frac{z}{2}\right)$ since $\frac{\mathrm{d}}{\mathrm{d}z}2\sin\left(\frac{z}{2}\right) = 2\cdot\frac{1}{2}\cdot\cos\left(\frac{z}{2}\right) = \cos\left(\frac{z}{2}\right)$. Therefore the integral is

$$\begin{aligned}\int_0^{\pi+2i}\cos\left(\frac{z}{2}\right)\mathrm{d}z &= 2\left[\sin\left(\frac{z}{2}\right)\right]_0^{\pi+2i}\\ &= 2\sin\left(\frac{\pi}{2}+i\right)\\ &= 2\left(\frac{e^{i\left(\frac{\pi}{2}+i\right)} - e^{-i\left(\frac{\pi}{2}+i\right)}}{2i}\right)\\ &= \frac{e^{-1}e^{i\frac{\pi}{2}} - e^{1}e^{-i\frac{\pi}{2}}}{i}\\ &= \frac{e^{-1}i + e^{1}i}{i}\\ &= \frac{1}{e} + e\end{aligned}$$

Part C

Note that $F(z) = \frac{1}{4}(z-2)^4$ is an antiderivative of $(z-2)^3$ since $\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{1}{4}(z-2)^4\right) = 4\cdot\frac{1}{4}(z-2)^3 = (z-2)^3$. Therefore the integral is

$$\int_1^3 (z-2)^3\mathrm{d}z = \frac{1}{4}\left[(z-2)^4\right]_1^3 = \frac{1}{4}[1-1] = 0.$$

49.5

Let γ be a path from -1 to 1 above the real axis. Note then that for any $z\in\gamma$ that $0 \le \operatorname{Arg} z \le \pi$. Therefore for all $z\in\gamma$ except $z=-1$, the branches $(-\pi,\pi)$ and $(-\frac{\pi}{2},\frac{3\pi}{2})$ agree value wise. Using this branch of log gives an anti derivative valid on all of γ that agrees with the principal branch

$$F(z) = \frac{1}{i}\exp[i\log z] = \frac{1}{i+1}z^{i+1}.$$

Therefore

$$\begin{aligned}
\int_{-1}^{1} z^i \mathrm{d}z &= F(1) - F(-1) \\
&= \frac{1}{i+1}(\exp[(1+i)\log 1] - \exp[(1+i)\log(-1)]) \\
&= \frac{1}{i+1}(\exp[0] - \exp[(1+i)(\ln 1 + i\pi)]) \\
&= \frac{1}{i+1}\left(1 - e^{\ln 1}e^{i\pi}e^{-\pi}e^{i\ln 1}\right) \\
&= \frac{1}{i+1}(1 - 1(-1)e^{-\pi}(1)) \\
&= \frac{1}{i+1}(1+e^{-\pi}) \\
&= \frac{i-1}{(i+1)(i-1)}(1+e^{-\pi}) \\
&= \frac{i-1}{2}(1+e^{-\pi}) \\
&= \frac{1+e^{-\pi}}{2}(i-1)
\end{aligned}$$

53.2

Part A

Since $f(z)$ is a composition of an entire function and a $\frac{1}{z}$, $f(z)$ will be analytic everywhere except when $3z^2 + 1 = 0 \implies z^2 = -\frac{1}{3} \implies z = \pm\frac{i}{\sqrt{3}}$. Since these points are not in closed region between C_1 and C_2, then curve deformation can be applied to get $\int_{C_1} f(z)\mathrm{d}z = \int_{C_2} f(z)\mathrm{d}z$.

Part B

The only place where $f(z)$ is not analytic is when $\sin\left(\frac{z}{2}\right) = 0$ which is when $z = 2\pi k$ for $k \in \mathbb{Z}$. The closest zeroes are then $0, 2\pi$ and -2π which are all not inside the region between C_1 and C_2. Therefore f is analytic in the region between the contours and hence the integral paths can be deformed into each other.

Part C

The only place where $f(z)$ is not analytic is when $1 - e^z = 0$ which is at $z = 0$. Since $z = 0$ is not in the region between C_1 and C_2, f is analytic in the region and therefore the integral paths can be deformed into each other.

53·3

Consider a circular contour C_0 of radius $R = 10$ centered around $2 + i$. Note that the given rectangular contour is contained inside the circle. Since $(z - 2 - i)^{n-1}$ is entire, the region between these contours is entire and therefore the integral around the rectangular path equals the integral around the circular path. Therefore

$$\int_C (z - 2 - i)^{n-1} \mathrm{d}z = \int_{C_0} (z - z_0)^{n-1} \mathrm{d}z = \begin{cases} 0 & n \in \mathbb{Z} \\ 2\pi i & n = 0 \end{cases}.$$

53·7

Proof. Since $\overline{z} = x - iy$ for $z = x + iy$, then its partials are

$$\begin{aligned} u_x &= 1 & v_x &= 0 \\ u_y &= 0 & v_y &= -1 \end{aligned}$$

Let $\mathcal{R}$ denote the region enclosed by the contour C. Then

$$\begin{aligned} \frac{1}{2i} \int_C \overline{z} \mathrm{d}z &= \frac{1}{2i} \left[\iint_{\mathcal{R}} (-v_x - u_y) \mathrm{d}A + i \iint_{\mathcal{R}} (u_x - u_y) \mathrm{d}A \right] \\ &= \frac{1}{2i} \left[\iint_{\mathcal{R}} 0 \mathrm{d}A + i \iint_{\mathcal{R}} 2 \mathrm{d}A \right] \\ &= \frac{2}{2i} \cdot i \iint_{\mathcal{R}} \mathrm{d}A \\ &= \iint_{\mathcal{R}} \mathrm{d}A = \text{area of } \mathcal{R} \end{aligned}$$

■

57·1

Rewriting Cauchy Integral Formula and it's extension gives

$$\int_C \frac{f(z)}{z - z_0} \mathrm{d}z = 2\pi i f(z_0)$$

and

$$\int_C \frac{f(z)}{(z - z_0)^{n+1}} \mathrm{d}z = \frac{2\pi i}{n!} f^{(n)}(z_0).$$

Part A

Let $f(z) = e^{-z}$ and $z_0 = \frac{\pi i}{2}$. Note $f(z)$ is entire and therefore analytic on and in the contour C. Then

$$\int_C \frac{e^{-z}}{z - \frac{\pi i}{2}} \mathrm{d}z = \int_C \frac{f(z)}{z - z_0} \mathrm{d}z = 2\pi i f(z_0) = 2\pi i e^{-\frac{\pi i}{2}} = 2\pi i(-i) = 2\pi.$$

Part B

Let $f(z) = \frac{\cos z}{z^2+8}$ and $z_0 = 0$. Since $\cos z$ is entire, $f(z)$ is not analytic only when $z^2 + 8 = 0$ which occurs at $z = \pm\sqrt{8}$ which is outside of the contour. Therefore $f(z)$ is analytic on and in the contour. Then

$$\int_C \frac{\cos z}{z(z^2+8)} \mathrm{d}z = \int_C \frac{f(z)}{z - z_0} \mathrm{d}z = 2\pi i f(z_0) = 2\pi i \cdot \frac{\cos 0}{0^2+8} = \frac{2\pi i}{8} = \frac{\pi i}{4}.$$

Part C

Let $f(z) = \frac{z}{2}$ and $z_0 = -\frac{1}{2}$. Since $f(z)$ is entire, $f(z)$ is analytic on and in the contour. Then

$$\int_C \frac{z}{2z+1} \mathrm{d}z = \int_C \frac{\frac{z}{2}}{z+\frac{1}{2}} \mathrm{d}z = \int_C \frac{\frac{z}{2}}{z - \left(-\frac{1}{2}\right)} \mathrm{d}z = \int_C \frac{f(z)}{z - z_0} \mathrm{d}z = 2\pi i f(z_0) = -2\pi i \cdot \frac{1}{4} = -\frac{\pi i}{2}.$$

Part D

Let $f(z) = \cosh(z)$ and $z_0 = 0$. Since $f(z)$ is entire, then it's derivatives are entire and hence analytic on and in the contour. Note that $f^{(3)}(z) = \frac{\mathrm{d}^3}{\mathrm{d}z^3} \cosh z = \sinh z$. Then

$$\int_C \frac{\cosh z}{z^4} \mathrm{d}z = \int_C \frac{\cosh z}{(z-0)^{3+1}} \mathrm{d}z = \int_C \frac{f(z)}{(z-z_0)^{3+1}} \mathrm{d}z = \frac{2\pi i}{3!} f^{(3)}(z_0) = \frac{\pi i}{3} \cdot \sinh 0 = 0.$$

Part E

Let $f(z) = \tan\left(\frac{z}{2}\right)$ and $z_0 = x_0$. Since $f(z)$ is analytic for $-\pi < \operatorname{Re} z < \pi$ and $-\pi < -2 \leq \operatorname{Re} z \leq 2 < \pi$ in and on C, it follows that $f(z)$ is analytic on and in C. Then

$$\int_C \frac{\tan\left(\frac{x}{2}\right)}{(z-x_0)^2} \mathrm{d}z = \int_C \frac{f(z)}{(z-z_0)^{1+1}} \mathrm{d}z = \frac{2\pi i}{1!} f'(z_0) = 2\pi i \cdot \frac{1}{2} \sec^2\left(\frac{z_0}{2}\right) = \pi i \sec^2\left(\frac{x_0}{2}\right).$$

57·3

$$g(2) = \int_C \frac{2s^2 - s - 2}{s - 2} \mathrm{d}s$$

Since $2s^2 - s - 2$ is entire, then it is analytic on and inside of C meaning

$$\begin{aligned} &= 2\pi i\left(2 \cdot 2^2 - 2 - 2\right) \\ &= 2\pi i \cdot 2^2 \\ &= 8\pi i \end{aligned}$$

If $|z| > 3$, then $\frac{2s^2-s-2}{s-z}$ will be analytic in an on the curve since the boundary and interior of the curve is when $|z| < 3$. Therefore since the integrand is analytic and the integral is over a closed contour, then the integral will be 0 meaning $g(z) = 0$ for $|z| > 3$.

57·6

Proof. Let z be a point on the interior of C and $d = \inf\{|z - s| : s \in C\}$. Choose Δz such that $0 < |\Delta z| < d$ since $d \neq 0$. It follows then that

$$\frac{g(z+\Delta z) - g(z)}{\Delta z} = \frac{1}{2\pi i}\int_C \left(\frac{1}{s-z-\Delta z} - \frac{1}{s-z}\right)\frac{f(s)}{\Delta z}\mathrm{d}s.$$

Since $\frac{1}{s-z-\Delta z} - \frac{1}{s-z} = \frac{\Delta z}{(s-z-\Delta z)(s-z)}$, it follows

$$\frac{g(z+\Delta z) - g(z)}{\Delta z} = \frac{1}{2\pi i}\int_C \frac{f(s)}{(s-z-\Delta z)(s-z)}\mathrm{d}s.$$

Rewriting $\frac{1}{(s-z-\Delta z)(s-z)} = \frac{1}{(s-z)^2} + \frac{\Delta z}{(s-z-\Delta z)(s-z)^2}$ gives

$$\frac{g(z+\Delta z) - g(z)}{\Delta z} - \frac{1}{2\pi i}\int_C \frac{f(s)}{(s-z)^2}\mathrm{d}s = \frac{1}{2\pi i}\int_C \frac{\Delta z f(s)}{(s-z-\Delta z)(s-z)^2}\mathrm{d}s. \qquad (\star)$$

Since $f(z)$ is continuous on the simple closed contour C and a simple closed contour is a compact set, it follows that $|f(z)|$ is bounded by some M. Since $|s - z| \geq d$ and $|\Delta z| < d$ for $s \in C$, the inequality

$$|s - z - \Delta z| = |(s-z) - \Delta z| \geq ||s-z| - |\Delta z|| \geq d - |\Delta z| > 0$$

holds. Therefore

$$\left|\frac{1}{2\pi i}\int_C \frac{\Delta z f(s)}{(s-z-\Delta z)(s-z)^2}\mathrm{d}s\right| \leq \frac{|\Delta z| ML}{(d - |\Delta z|)d^2}$$

where L denotes the length of C. Since

$$\lim_{\Delta z \to 0} \frac{|\Delta z| ML}{(d - |\Delta z|)d^2} = 0$$

it follows that the left hand side of $(\star)$ has the same limit meaning

$$g'(z) = \lim_{\Delta z \to 0} \frac{g(z+\Delta z) - g(z)}{\Delta z} = \frac{1}{2\pi i}\int_C \frac{f(s)}{(s-z)^2}\mathrm{d}s.$$

Therefore since z was an arbitrary interior point to C, $g'(z)$ exists on the interior of C. Furthermore, since every $z \in C$ has an open ball around it in C, then every z has a neighborhood that is differentiable meaning $g(z)$ is analytic on the interior of C. ■

57.10

Proof. Consider a positively oriented circular contour C_R around some $z_0 \in \mathbb{C}$ with radius R. Note that for all $z \in C_R$ that $|z| \leq |z_0| + R$. Therefore $A|z| \leq A(|z_0| + R)$ on the contour, giving an upper bound of f on C_R. Since f is entire, then it is analytic in and on C_R meaning

$$|f''(z_0)| \leq \frac{2! \cdot A(|z_0| + R)}{R^n} = 2\left(\frac{A|z_0|}{R^n} + \frac{1}{R^{n-1}}\right).$$

Since R was chosen arbitrarily and $\lim_{R\to\infty} 2\left(\frac{A|z_0|}{R^n} + \frac{1}{R^{n-1}}\right) = 2(0+0) = 0$, it follows that $|f''(z_0)| = 0 \implies f''(z_0) = 0$ for all $z_0 \in \mathbb{C}$. Therefore $f(z) = a_1x + b_1$ for some $a_1, b_1 \in \mathbb{C}$. Since $|f(0)| \leq A \cdot 0$, it follows that $b_1 = 0$. Therefore $f(z) = a_1x$ for all z. ■

59.1

Proof. Take $f(z)$ to be entire and assume that there is an upper bound u_0 for $u(x, y) = \text{Re} f$ on all of $\mathbb{C}$. Consider the function $g(z) = e^{f(z)}$. Since e^z and $f(z)$ are entire, then $g(z)$ is also entire. Note that

$$g(z) = \exp[f(z)] = \exp[u(x, y) + iv(x, y)] = \exp[u(x, y)] \exp[iv(x, y)] \leq \exp[u(x, y)]$$

Since $u(x, y) \leq u_0$, $e^{u(x,y)} \leq e^{u_0}$ meaning $g(z)$ is bounded. By Liouville's Theorem, $g(z)$ is constant. Therefore

$$g'(z) = f'(z)e^{f(z)} = 0 \implies f'(z) = 0$$

for all z. Therefore $f(z)$ is a constant meaning its real component is constant. ■