Problem 1

Proof. Let $\left(a^{(k)}\right)$ be a sequence in A that converges to some $a \in \mathbb{R}^n$. Then for each $a^{(k)}$ there exists some $x^{(k)} \in F$ such that $\left\|x^{(k)} - a^{(k)}\right\| = \delta$. Since $\left(a^{(k)}\right)$ converges, it is bounded by some $M > 0$. Therefore

$$\left\|x^{(k)}\right\| = \left\|x^{(k)} - a^{(k)} + a^{(k)}\right\| \leq \left\|x^{(k)} - a^{(k)}\right\| + \left\|a^{(k)}\right\| \leq \delta + M$$

meaning $\left(x^{(k)}\right)$ is also bounded. Thus by Bolzano-Weierstrass, there exists a convergent subsequence $\left(x^{(k_j)}\right)$ with limit x as $j \to \infty$. Since F is closed, it follows that $x \in F$. Furthermore $|\cdot|$ is continuous meaning limits pass through it. Since $\left|x^{(k_j)} - a^{(k)}\right| = \delta$ for all k,

$$\lim_{k\to\infty} \left|x^{(k_j)} - a^{(k)}\right| = |x - a| = \delta.$$

Therefore $a \in A$ since $x \in F$, hence A is closed. ◇

Problem 2

Proof. Suppose that A is closed. Let $(x^{(k)})$ be a Cauchy sequence in A. Since $(x^{(k)})$ is also a Cauchy sequence in $\mathbb{R}^n$, it must converge to some point x. If $x \in A$, we are done. Suppose towards contradiction then that $x \notin A$. Since A is closed, there exists some radius $r > 0$ such that $B_r(x) \cap A = \varnothing$. But since $(x^{(k)})$ is convergent, it follows that there exists $K \in \mathbb{N}$ such that for $k \geq K$,

$$\left\|x^{(k)} - x\right\| < r.$$

meaning $x^{(k)} \notin A$ for $k \geq K$, a contradiction. Therefore every Cauchy sequence in A converges to a point in A.

Suppose that A is complete. Let $x \in \overline{A}$. Then there exists a sequence $(x^{(k)})$ in A that converges to x. Since convergent sequences are Cauchy, it follows by completeness of A that $x \in A$. Thus $\overline{A} \subseteq A$. Every set is a subset of its closure, meaning $A \subseteq \overline{A}$. Therefore $A = \overline{A}$ and A is closed. ◇

Problem 3

a) The limit does not exist. Consider the sequences $(\frac{1}{k}, 0)$ and $(-\frac{1}{k}, 0)$. Both converge to $(0,0)$ as $k \to \infty$, but

$$f\left(\frac{1}{k}, 0\right) = k \to \infty \quad (k \to \infty)$$

and

$$f\left(-\frac{1}{k}, 0\right) = -k \to -\infty \quad (k \to \infty).$$

Thus the limit cannot exist.

b) The limit does exist and is 0. Note that

$$\left|\frac{x+y}{x^2+y^2}\right| \leq \frac{|x|+|y|}{x^2+y^2}$$

and both $|x| \leq \sqrt{x^2+y^2}$ and $|y| \leq \sqrt{x^2+y^2}$. Therefore

$$\left|\frac{x+y}{x^2+y^2}\right| \leq \frac{|x|+|y|}{x^2+y^2} \leq \frac{2\sqrt{x^2+y^2}}{x^2+y^2} = \frac{2}{\sqrt{x^2+y^2}}.$$

As $|p| \to \infty$, $\sqrt{x^2+y^2} \to \infty$ meaning

$$\left|\frac{x+y}{x^2+y^2}\right| \to 0.$$

c) The limit does not exist. Consider the sequence $(0, 0, \frac{1}{k})$ and $(\frac{1}{k}, \frac{1}{k}, 0)$. Both converge to $(0,0,0)$ as $k \to \infty$ but

$$f\left(0, 0, \frac{1}{k}\right) = \frac{-\frac{1}{k^2}}{\frac{1}{k^2}} - 1$$

and

$$f\left(\frac{1}{k}, \frac{1}{k}, 0\right) = \frac{\frac{1}{k^2}}{2\frac{1}{k^2}} = \frac{1}{2}.$$

Thus the limit cannot exist.

d) The limit does not exist. Consider the sequences $(0, 0, k)$ and $(k, k, 0)$. Both diverge in magnitude to ∞ as $k \to \infty$ but

$$f(0,0,k) = \frac{-k^2}{k^2} = -1$$

and

$$f(k,k,0) = \frac{k^2}{2k^2} = \frac{1}{2}.$$

Thus the limit cannot exist.

Problem 4

a) Note that $|x^2| \leq |2x^2 + y^2|$, therefore

$$\left|\frac{x^2}{2x^2+y^2}\right| \leq 1 \implies 0 \leq \underbrace{\left|\frac{x^2 y}{2x^2+y^2}\right|}_{|F(x,y)|} \leq |y|.$$

Thus if $(x, y) \to (0, 0)$, then $y \to 0$ meaning $F(x, y) \to 0$.

b) The limit does not exist. Consider the sequences $\left(\frac{1}{k}, 0\right)$ and $\left(\frac{1}{k}, \frac{1}{k^2}\right)$. Both converge to 0 as $k \to \infty$ but

$$f\left(\frac{1}{k}, 0\right) = \frac{\frac{1}{k^2} \cdot y}{\frac{3}{k^4} + 2 \cdot 0^2} = 0$$

and

$$f\left(\frac{1}{k}, \frac{1}{k^2}\right) = \frac{\frac{1}{k^2} \cdot \frac{1}{k^2}}{\frac{3}{k^4} + \frac{2}{k^4}} = \frac{\frac{1}{k^4}}{\frac{5}{k^4}} = \frac{1}{5}.$$

Thus the limit cannot exist.

Problem 5

Consider the following paths:

- $(x_1, 0)$ where $x_1 \to 0$ gives $f(x_1, 0) = \frac{x_1^2(0)^2}{x_1^2+0^2} = 0$
- $(0, x_2)$ where $x_2 \to 0$ gives $f(0, x_2) = \frac{(0)^2(x_2)^2}{0^2+x_2^2} = 0$
- For any $a, b \in \mathbb{R}$, (ax_1, bx_1) where $x_1 \to 0$ gives $f(ax_1, bx_1) = \frac{a^2b^2x_1^4}{2a^2b^2x_1^2} = \frac{x_1^2}{2} \to 0$

All of these paths converge to 0 and under f converge to $f(0) = 0$, but this doesn't prove continuity. That is because continuity requires that every possible path converging to 0 under f also converges to 0, something that cannot be hand checked in a case by case manner.

It is indeed the case that f is continuous there. Note for $(x, y) \neq (0, 0)$ that

$$\frac{x^2y^2}{x^2+y^2} \leq \frac{x^2y^2}{x^2} = y^2. \tag{$\star$}$$

Take $\varepsilon > 0$ and $\delta = \sqrt{\varepsilon}$. Note then if $\|(x, y) - (0, 0)\| \leq \delta$ that

$$\|(x, y)\| = \sqrt{x^2 + y^2} \leq \delta \implies x^2 + y^2 \leq \delta^2 \implies y^2 \leq \delta^2 = \varepsilon.$$

Thus by ($\star$)

$$\left|\frac{x^2y^2}{x^2+y^2} - 0\right| = \frac{x^2y^2}{x^2+y^2} \leq y^2 \leq \varepsilon.$$

Hence f is continuous at $(0, 0)$.

Problem 6

Proof. Suppose towards contradiction that f is continuous. Consider the sequence $x^{(k)} = \left(\frac{1}{k^3}, \frac{1}{k}\right)$. Since f is continuous at $(0, 0)$ it must be the case that $\lim f(x^{(k)}) = f(\lim x^{(k)}) = f(0, 0) = 0$. Note that

$$f(x^{(k)}) = \frac{\frac{1}{k^5}}{2 \cdot \frac{1}{k^6}} = \frac{k}{2}.$$

Therefore $f(x^{(k)}) \to \infty$ as $k \to \infty$ and not 0, hence f is not continuous at $(0, 0)$. $\diamond$

Problem 7

Proof.

i) Let $A \subseteq \mathbb{R}^n$ and $y \in f(\overline{A})$. Then $\exists x \in \overline{A}$ such that $f(x) = y$. Since $x \in \overline{A}$, there exists a sequence $(x^{(k)})$ in A that converges to x. By continuity of f, $\lim_{k\to\infty} f(x^{(k)}) = f(x) = y$. Note that the sequence $(f(x^{(k)}))$ is in $f(A)$, thus since it converges to y, $y \in \overline{f(A)}$. Therefore $f(\overline{A}) \subseteq \overline{f(A)}$. $\diamond$

ii) Consider $f : \mathbb{R} \to \mathbb{R}$ where

$$f(x) = \begin{cases} 0 & x \leq 0 \\ x & x > 0 \end{cases}$$

f in this case is continuous. Note that if $A = (-1, 1)$ that $f(\text{int}(A)) = f((-1, 1)) = [0, 1)$ but $\text{int}(f(A)) = \text{int}([0, 1)) = (0, 1)$. Thus it cannot be said generally that $f(\text{int}(A)) \subseteq \text{int}(f(A))$.

Now consider $f : \mathbb{R}^2 \to \mathbb{R}$ where $f(x, y) = x$ and

$$A = \left\{(x, y) \in \mathbb{R}^2 : xy = 1\right\}.$$

Then $\text{int}(A) = \varnothing$, hence $f(\text{int}(A)) = \varnothing$. But $f(A) = \mathbb{R} \setminus \{0\}$. The interior of this is clearly non empty, thus it is not true in general that $\text{int}(f(A)) \subseteq f(\text{int}(A))$. $\diamond$

Problem 8

Proof. Let $a \in \mathbb{R}^n \setminus A$. Since $\overline{A} = \mathbb{R}^n$, there exists a sequence $(x^{(k)})$ in A such that $x^{(k)} \to a$ when $k \to \infty$. By continuity of f and g, it follows that $f(x^{(k)}) \to f(a)$ and $g(x^{(k)}) \to g(a)$ as $k \to \infty$. Take $\varepsilon > 0$. Then $\exists K_1, K_2 \in \mathbb{N}$ such that $\left\|f(x^{(k)}) - f(a)\right\| < \frac{\varepsilon}{2}$ for $k \geq K_1$ and $\left\|g(x^{(k)}) - g(a)\right\| < \frac{\varepsilon}{2}$ for $k \geq K_2$. Note that $f(x^{(k)}) = g(x^{(k)})$ since $x^{(k)}$ is a sequence in A, and thus for $k \geq \max\{K_1, K_2\}$

$$\begin{aligned}
\|f(a) - g(a)\| &= \left\| f(a) - f(x^{(k)}) + g(x^{(k)}) - g(a) + f(x^{(k)} - g(x^{(k)}) \right\| \\
&\leq \left\| f(x^{(k)}) - f(a) \right\| + \left\| g(x^{(k)}) - g(a) \right\| + \left\| f(x^{(k)}) - g(x^{(k)}) \right\| \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} + 0 \\
&= \varepsilon
\end{aligned}$$

Therefore $\|f(a) - g(a)\|$ can be made arbitrarily small, meaning $f(a) = g(a)$. $\diamond$