Problem 1

We have the following probability mass functions for X and Y

$$\mathbb{P}[X = i] = \binom{n}{i} p^i (1-p)^{n-i} \qquad \mathbb{P}[Y = j] = \binom{m}{j} p^j (1-p)^{m-j}.$$

Therefore considering the probability mass function of $X + Y$ we have

$$\begin{aligned}
\mathbb{P}[X + Y = k] &= \sum_{i=0}^{k} \mathbb{P}[X = k - i, Y = i] \\
&= \sum_{i=0}^{k} \mathbb{P}[X = k - i]\mathbb{P}[Y = i] \\
&= \sum_{i=0}^{k} \left(\binom{n}{k-i} p^{k-i} (1-p)^{n-k+i} \right) \left(\binom{m}{i} p^i (1-p)^{m-i} \right) \\
&= p^k (1-p)^{n+m-k} \sum_{i=0}^{k} \binom{n}{k-i} \binom{m}{i} \\
&= \binom{n+m}{k} p^k (1-p)^{n+m-k}
\end{aligned}$$

which is the probability mass function of $\text{Binomial}(n + m, p)$.

Problem 2

Part A

By the definition of a conditional distribution,

$$p_{X|Y}(x|3) = \frac{p(x,3)}{p_Y(3)} = \frac{p(x,3)}{0.05 + 0.1 + 0.35} = 2p(x,3).$$

Therefore

$$p_{X|Y}(1 \mid 3) = \frac{1}{10}$$
$$p_{X|Y}(2 \mid 3) = \frac{2}{10}$$
$$p_{X|Y}(3 \mid 3) = \frac{7}{10}$$

Part B

Again by the definition of a conditional distribution,

$$p_{Y|X}(y|2) = \frac{p(2,y)}{p_X(2)} = \frac{p(2,y)}{0.2 + 0.1 + 0.05} = \frac{20p(2,y)}{7}.$$

Therefore

$$p_{Y|X}(1 \mid 2) = \frac{2}{10} \cdot \frac{20}{7} = \frac{4}{7}$$
$$p_{Y|X}(3 \mid 2) = \frac{1}{10} \cdot \frac{20}{7} = \frac{2}{7}$$
$$p_{Y|X}(5 \mid 2) = \frac{1}{20} \cdot \frac{20}{7} = \frac{1}{7}$$

Part C

No they are not the same. We have

$$p_{Y|X}(3 \mid 2) = \frac{1}{7} \neq \frac{2}{10} = p_{X|Y}(2 \mid 3).$$

Problem 3

Part A

We can obtain the marginal density by integrating the joint density over the possible values of y. Thus

$$\begin{aligned} f_X(x) &= \int_{0<x<y} f(x,y)\mathrm{d}y \\ &= \int_x^\infty e^{-y}\mathrm{d}y \\ &= -e^{-y}\Big|_x^\infty \\ &= e^x, \quad x > 0 \end{aligned}$$

Part B

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = e^{-(x+y)}, \quad y > x.$$

Problem 4

Part A

Since U is uniform on $(0,1)$, its probability density is simply $f_U(u) = 1$ for $0 < u < 1$. Note that

$$\mathbb{P}[U > a] = \int_a^1 1\mathrm{d}u = 1 - a.$$

Therefore for $a < u < 1$

$$f_{U|U>a}(u) = \frac{f_U(u)}{\mathbb{P}[U > a]} = \frac{1}{1-a}.$$

Part B

In a similar manner to (A), we have

$$\mathbb{P}[U < a] = \int_0^a 1 \mathrm{d}u = a.$$

Therefore for $0 < u < a$

$$f_{U|U<a}(u) = \frac{f_U(u)}{\mathbb{P}[U < a]} = \frac{1}{a}.$$