Problem 1

Proof. If $a, b \in \mathbb{R}$ with $a \neq b$, then

$$\left|\frac{f(a) - f(b)}{a - b}\right| \leq M|a - b|.$$

Therefore taking $x \in \mathbb{R}$ and $h \in \mathbb{R} \setminus \{0\}$ gives

$$0 \leq \left|\frac{f(x+h) - f(x)}{x + h - x}\right| = \left|\frac{f(x+h) - f(x)}{h}\right| \leq M|h|.$$

Hence by the squeeze lemma

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = 0.$$

But this means that $f'(x) = 0$ everywhere meaning f must be a constant function. ◇

Problem 2

Part A

The partial derivatives of f when $(x, y) \neq 0$ (which can be found by normal differentiation treating x or y as a constant where needed) are

$$\frac{\partial f}{\partial x}(x, y) = \frac{y}{x^2 + y^2} - \frac{2x^2 y}{(x^2 + y^2)^2}$$
$$\frac{\partial f}{\partial y}(x, y) = \frac{x}{x^2 + y^2} - \frac{2xy^2}{(x^2 + y^2)^2}$$

and when $(x, y) = 0$,

$$\frac{\partial f}{\partial x}(0, 0) = \lim_{t \to 0} \frac{f(t, 0) - f(0, 0)}{t} = \lim_{t \to 0} \frac{0}{t} = 0$$
$$\frac{\partial f}{\partial x}(0, 0) = \lim_{t \to 0} \frac{f(0, t) - f(0, 0)}{t} = \lim_{t \to 0} \frac{0}{t} = 0$$

Thus the partial derivatives exist everywhere. However, they are not both continuous at $(0,0)$. Consider the sequences $(0,\frac{1}{k})$ and $(0,-\frac{1}{k})$ for $k \in \mathbb{N}$. Both converge to $(0,0)$ but as $k \to \infty$,

$$\frac{\partial f}{\partial x}\left(0,\frac{1}{k}\right) = k \to \infty$$
$$\frac{\partial f}{\partial x}\left(0,-\frac{1}{k}\right) = -k \to -\infty$$

Thus $\frac{\partial f}{\partial x}$ is not continuous at 0 and therefore not continuous on all of $\mathbb{R}^2$, meaning f cannot be C^1.

Part B

Since f has partial derivatives at the origin, then every directional derivative at the origin also exists. That is because if $v \in \mathbb{R}^2 \setminus \{0\}$, by linearity of the differential

$$\begin{aligned} f'(0,0)v &= f'(0,0)(v_1e_1) + f'(0,0)(v_2e_2) \\ &= v_1 f'(0,0)e_1 + v_2 f'(0,0)e_2 \\ &= v_1\frac{\partial f}{\partial x}(0,0) + v_2\frac{\partial f}{\partial y}(0,0) = 0 \end{aligned}$$

Part C

No, f is not continuous at the origin. Consider the paths (x_1,x_1) and $(-x_1,x_1)$ where $x_1 \to 0$. Note that

$$f(x_1,x_1) = \frac{x_1^2}{2x_1^2} = \frac{1}{2}$$

but

$$f(-x_1,x_1) = \frac{-x_1^2}{2x_1^2} = -\frac{1}{2}.$$

Since these paths give different limits, f cannot be continuous at the origin.

Problem 3

Proof. Suppose D is bounded and f is uniformly continuous. Take $\varepsilon = 1$. Since f is uniformly continuous, there exists some $\delta > 0$ such that $\|x - y\| \le \delta \implies \|f(x) - f(y)\| \le \varepsilon$ for all $x, y \in D$. Since D is bounded, it is possible to cover D with a finite number of open balls with radius δ centered at some set of points $x_1, \dots, x_n \in D$. Note then that for any $x \in D$ that $x \in B_1(x_k)$ for some x_k, thus $\|x - x_k\| \le \delta$. Therefore

$$\|f(x)\| \le \|f(x) - f(x_k)\| + \|f(x_k)\| \le \|f(x_k)\| + 1.$$

Take then $M = \max\{\|f(x_1)\|, \dots, \|f(x_n)\|\} + 1$. Since for any $x \in D$, $\|f(x)\| \le M$, f is bounded on D. $\diamond$

Problem 4

Proof. Let $S = f(\mathbb{R}^n)$ and $\left(y^{(k)}\right)$ be a sequence in S that converges to some $y \in \mathbb{R}^n$. Note then there exists a sequence $\left(x^{(k)}\right)$ such that $f(x^{(k)}) = y^{(k)}$ for all $k \in \mathbb{N}$. Therefore since f is continuous

$$\lim_{k\to\infty} f(y^{(k)}) = f(y).$$

But note that $f(y^{(k)}) = f(f(x^{(k)})) = f(x^{(k)}) = y_k$. Therefore

$$\lim_{k\to\infty} f(y^{(k)}) = y.$$

Thus $y = f(y)$ meaning $y \in S$ and S is closed. $\diamond$

Problem 5

Proof. Let $x \in A + B$. Then there exists $a \in A$ and $b \in B$ such that $x = a + b$. Since A is open, there exists some $r > 0$ such that $B_r(a) \subseteq A$. Take a point $y \in B_r(x)$. Then there exists some h such that $y = x + h$ and $\|h\| < r$. Note then that $a + h \in B_r(a)$, hence $a + h \in A$. Thus $y = x + h = (a + h) + b \in A + B$. Therefore $A + B$ is open. $\diamond$

Problem 6

Proof. Let $(a, b) \in \mathbb{R}^2 \setminus \mathbb{Q}^2$ not equal to $(0, 0)$. Since $\mathbb{Q}^2$ is dense in $\mathbb{R}^2$, there exists a sequence $\left(x^{(k)}, y^{(k)}\right)$ in $\mathbb{Q}^2 \setminus \{0\}$ that converges to (a, b). Note then that $f(x^{(k)}, y^{(k)}) = (x^{(k)})^2 + (y^{(k)})^2 \neq 0$, but $f(a, b) = 0$. Therefore

$$\lim_{k\to\infty} f(x^{(k)}, y^{(k)}) = a^2 + b^2 \neq 0 = f(a, b)$$

hence f cannot be continuous at irrational pairs. Since $\mathbb{R} \setminus \mathbb{Q}^2$ is dense in $\mathbb{Q}^2$, the same argument as above applies for $(a, b) \in \mathbb{Q}^2$ and $\left(x^{(k)}, y^{(k)}\right)$ in $\mathbb{R}^2 \setminus \mathbb{Q}^2$ but instead $f(x^{(k)}, y^{(k)}) = 0$ and $f(a, b) = a^2 + b^2 \neq 0$. Therefore f cannot be continuous away from the origin, and thus also can only be differentiable at the origin.

Consider the proposed differential $\mathcal{D}(x, y) = 0$. Note then that

$$\lim_{h\to 0} \frac{\|f(0+h) - f(0) - \mathcal{D}h\|}{\|h\|} = \lim_{h\to 0} \frac{\|f(h)\|}{\|h\|} \leq \lim_{h\to 0} \frac{\|h\|^2}{\|h\|} = \lim_{h\to 0} \|h\| = 0.$$

Since this limit is zero, the proposed differential is the actual differential of f, and thus f is differentiable at 0, implying also continuity. Therefore f is continuous and differentiable exclusively at 0. $\diamond$