Remark 1. The intersection of an infinite collection of open sets is not neccesarily open. Consider the family of open intervals in $\mathbb{R}$ of the form

$$J_n = \left(-\frac{1}{n}, \frac{1}{n}\right).$$

Note that $\bigcap J_n = \{0\}$ which is not open.

Def. Neighborhood

Let $a \in \mathbb{R}^n$. A **neighborhood** of a is an open set $G \subseteq \mathbb{R}^n$ such that $a \in G$. Often the term *nbhd* is used as a shorthand.

Remark 2. If G is a nbhd of a, then $\exists r > 0$ such that $B_r(a) \subseteq G$.

Def. Interior

The **interior** of a set $A \subseteq \mathbb{R}^n$ is defined as

$$\text{int}(A) := \{x \in \mathbb{R}^n : x \text{ has a nbhd } G \subseteq A\}.$$

Example 1.

i) $\text{int}([a,b)) = (a,b)$ since any nbhd of a will contain points outside of the interval.

ii) Let $A = \left\{(x,y) \in \mathbb{R}^2 : x, y \geq 0\right\}$. Then $\text{int}(A) = \left\{(x,y) \in \mathbb{R}^2 : x, y > 0\right\}$ as any point along the axes fail by the same reasoning as above.

iii) $\text{int}(\mathbb{Q}) = \varnothing$ because there will always be an irrational x in any ball based around a rational number.

Theorem 1.

For any $A \subseteq \mathbb{R}^n$

i) $\text{int}(A)$ is open

ii) $\text{int}(A)$ is the largest open set contained in A

Proof. Let $x \in \operatorname{int}(A)$. Then there is some nbhd G such that $G \subseteq A$. Let $y \in G$. Since G is open, G is a nbhd of y as well hence $y \in \operatorname{int}(A)$. Therefore $G \subseteq \operatorname{int}(A)$ meaning $\operatorname{int}(A)$ is open. $\diamond$

Def. Closed set

A set $F \subseteq \mathbb{R}^n$ is **closed** if its complement F^c is open.

Example 2.

i) Both $\varnothing$ and $\mathbb{R}^n$ are closed

ii) $[a, b]$ is closed for all $a \neq b$

iii) $[a, \infty)$ is closed since $[a, \infty)^c = (-\infty, a)$ which is open

Theorem 2.

For every $a \in \mathbb{R}^n$ and $r > 0$, the closed ball $B_r[a] = \{x \in \mathbb{R}^n : |x - a| \leq r\}$ is closed in $\mathbb{R}^n$.

Proof. If $B_r[a]^c = \{x \in \mathbb{R}^n : |x - a| > r\}$ is open, then the desired result is achieved. Let $x \in B_r[a]^c$. Since $|x - a| > r$, then $\exists \rho > 0$ such that $|x - a| = r + \rho$. Take $y \in B_\rho(x)$. Then

$$\begin{aligned} |x - a| \leq |x - y| + |y - a| &\implies |y - a| \geq |x - a| - |x - y| \\ &\implies |y - a| > |x - a| - \rho = r \end{aligned}$$

Therefore $y \in Br[a]^c$, meaning $B_r[a]$ is open. $\diamond$