Ex 1.2

Proof. Let $E = B_1 \cup \ldots \cup B_n$ and $F = V_1 \cup \ldots \cup V_m$ where B_i, V_j are boxes in $\mathbb{R}^d$.

(i) By definition, E and F are the finite union of boxes, and thus $E \cup F$ is the finite union of boxes meaning it is elementary.

(ii) Note that the intersection of two intervals is always an interval (assuming $\varnothing$ is also an interval). That is because if the intersection is non-empty and the endpoints of the intervals are a_1, b_1 and a_2, b_2, then the endpoints of their intersection are $\min(a_1, a_2)$ and $\max(b_1, b_2)$. Thus the intersection of two boxes is itself a box. Note

$$E \cap F = \bigcup_i \bigcup_j B_i \cap V_j$$

which is the union of the intersection of boxes, which themselves are boxes. Hence $E \cap F$ is elementary.

(iii) Note that

$$\begin{aligned} E \setminus F &= [\bigcup_i B_i] \setminus F \\ &= \bigcup_i B_i \setminus F \\ &= \bigcup_i B_i \setminus [\bigcup_j V_j] \\ &= \bigcup_i \bigcap_j B_i \setminus V_j \end{aligned}$$

Therefore if the subtraction of two boxes is an elementary set, then by (i) and (ii) $E \setminus F$ will also be elementary. The difference of two boxes will themselves be a union of boxes, as one can divide the resulting set $B_i \setminus V_j$ along the hyperplanes of V_i to obtain a disjoint decomposition. This means that $B_i \setminus V_j$ is an elementary set, which means that $E \setminus F$ is elementary.

(iv) Since $E \setminus F$ and $F \setminus E$ are elementary from (iii), then their union is also elementary from (i). Thus $E \Delta F$ is elementary.

(v) Note that $E+x = \bigcup_i (B_i+x)$ thus we only need to consider B_i+x. Suppose $B_i = [a_1,b_1]\times\ldots\times[a_d,b_d]$ (since the following argument works regardless of endpoint inclusion). Note then that

$$B_i + x = [a_1+x_1, b_1+x_1] \times \ldots \times [a_d+x_d, b_d+x_d]$$

which is still a box. Thus $E+x$ is the union of a finite number of boxes and therefore elementary.

◇

Ex 1.7

Proof. Let $c = \tilde{m}([0,1)^d)$. Note that finite additivity and non-negativity of $\tilde{m}$ give monotonicity because if $A \subset B$, then $\tilde{m}(B \setminus A) \geq 0$ and

$$\tilde{m}(B) = \tilde{m}(A \cup (B \setminus A)) \geq \tilde{m}(A) + \tilde{m}(B \setminus A) \geq \tilde{m}(A).$$

We can now extend the known value of $\tilde{m}$ to general boxes.

- Take $n \in \mathbb{N}$. Then $[0,1)^d$ can be subdivided into n^d cubes of side length $\frac{1}{n}$. By translation invariance, each of these cubes must have the same measure, thus $\tilde{m}\left([0,\frac{1}{n})^d\right) = \frac{c}{n^d}$.
- Let B be a box with rational endpoints. By translation invariance it can be assumed $B = \prod_{i=1}^d [0, r_i)$ where $r_i \in \mathbb{Q}$. For some $q \in \mathbb{N}$, we can then write $r_i = \frac{p_i}{q}$ for some $p_i \in \mathbb{Z}$. It is therefore possible to partition B into $p_1p_2\ldots p_d$ boxes of side length $\frac{1}{q}$. It follows from the previous part then that
$$\tilde{m}(B) = p_1 \ldots p_d \cdot \frac{c}{q^d} = c \cdot \prod_{i=1}^d r_i = cm(B).$$
- Now let B be a box with real endpoints. Again by translation invariance it can be assumed $B = \prod_{i=1}^d [0, t_i)$. Take $q \in \mathbb{N}$ and set $p_i = \lfloor qt_i \rfloor$. Define then
$$B_L = \prod_{i=1}^d \left[0, \frac{p_i}{q}\right) \qquad B_U = \prod_{i=1}^d \left[0, \frac{p_i+1}{q}\right).$$

Note that $B_L \subset B \subset B_U$ and

$$\frac{p_1 \dots p_d}{q} \leq t_1 \dots t_d \leq \frac{(p_1+1)\dots(p_d+1)}{q} \tag{$\star$}$$

which in the limit as $q \to \infty$ leads to both sides converging to $t_1 \dots t_d$. Monotonicity gives $\tilde{m}(B_L) \leq \tilde{m}(B) \leq \tilde{m}(B_U)$. From the previous part, we have

$$\tilde{m}(B_L) = c \cdot \frac{p_1 \dots p_d}{q} \qquad \tilde{m}(B_U) = c \cdot \frac{(p_1+1)\dots(p_d+1)}{q}.$$

Therefore

$$c \cdot \frac{p_1 \dots p_d}{q} \leq \tilde{m}(B) \leq c \cdot \frac{(p_1+1)\dots(p_d+1)}{q}.$$

In the limit as $q \to \infty$, it follows then in conjunction with ($\star$) that $\tilde{m}(B) = c \cdot t_1 \dots t_2 = cm(B)$.

Now consider an elementary set E. Then there is a disjoint decomposition of E into boxes $B_1 \cup \dots B_n$. By finite additivity of $\tilde{m}$, it follows

$$\tilde{m}(B) = \sum_i \tilde{B}_i = \sum_i c \cdot m(B_i) = c \sum_i m(B_i) = cm(E)$$

which was to be shown. ◇

Ex 1.11

Proof. We will prove $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 1$.

$1 \Rightarrow 2)$ Suppose E is Jordan measurable and take $\varepsilon > 0$. By the definition of the inner and outer Jordan measure, it follows that there exists elementary sets A and B such that $A \subset E$ and $E \subset B$ as well as

$$m(A) + \frac{\varepsilon}{2} \leq m_*^J(E) \qquad m_J^*(E) \leq m(B) - \frac{\varepsilon}{2}.$$

Subtracting the first inequality from the second gives

$$m(B) - \frac{\varepsilon}{2} - m(A) - \frac{\varepsilon}{2} \leq m_J^*(E) - m_*^J(E) = 0 \implies m(B) - m(A) \leq \varepsilon.$$

Since $B \setminus A$ and A are disjoint and $(B \setminus A) \cup A = A$,

$$\begin{aligned} m(A) &= m((B \setminus A) \cup A) \\ &= m(B \setminus A) + m(A) \implies m(B \setminus A) = m(B) - m(A) \end{aligned}$$

Therefore $m(B \setminus A) \leq \varepsilon$.

$2 \Rightarrow 3)$ Take $\varepsilon > 0$ and suppose there exists elementary sets $A \subset E \subset B$ such that $m(B \setminus A) \leq \varepsilon$. Note that $A \setminus E = \emptyset$, thus $A \Delta E = E \setminus A$. Since $E \setminus A \subset B$ and $(E \setminus A) \cap A = \emptyset$, $E \setminus A \subset B \setminus A$ and thus $A \Delta E \subset B \setminus A$. Therefore $m^*_J(A \Delta E) \leq m(B \setminus A) \leq \varepsilon$.

$3 \Rightarrow 1)$ Take $\varepsilon > 0$ and suppose there exists and elementary set A such that $m^*_J(A \Delta E) \leq \varepsilon$. Then for any $\delta > 0$ there exists some elementary set C such that $A \Delta E \subset C$ and $m(C) \leq m^*_J(A \Delta E) + \delta \leq \varepsilon + \delta$. Note that $A \cup C \supset E$ and $A \setminus C \subset E$ are both elementary sets. Therefore we have

$$m^*_J(E) \leq m^*_J(A \cup C) = m(A \cup C) \leq m(A) + m(C) \leq m(A) + (\varepsilon + \delta)$$

and

$$m^J_*(E) \geq m^J_*(A \setminus C) = m(A \setminus C) \geq m(A) - m(C) \geq m(A) - (\varepsilon + \delta).$$

Putting these together gives

$$m(A) - (\varepsilon + \delta) \leq m^J_*(E) \leq m^*_J(E) \leq m(A) + (\varepsilon + \delta).$$

Since δ was arbitrary, we can reduce the inequality bounds to $m(A) \pm \varepsilon$. Furthermore since ε was arbitrary, $m^J_*(E) = m^*_J(E) = m(A)$ and thus E is measurable.

◇

Ex. 1.13

Proof.

(i) Take $\varepsilon > 0$. Then there exists elementary sets A_1, A_2, B_1, B_2 such that $A_1 \subset E_1 \subset B_1$, $A_2 \subset E_2 \subset B_2$ and $m(B_1 \setminus A_1), m(B_2 \setminus A_2) \leq \varepsilon$.

- Note that $A_1 \cup A_2 \subset E_1 \cup E_2 \subset B_1 \cup B_2$ and

$$\begin{aligned}(B_1 \cup B_2) \setminus (A_1 \cup A_2) &= (B_1 \setminus (A_1 \cup A_2)) \cup (B_2 \setminus (A_1 \cup A_2))\\ &\subset (B_1 \setminus A_1) \cup (B_2 \setminus A_2)\end{aligned}$$

  Therefore $m((B_1 \cup B_2) \setminus (A_1 \cup A_2)) \leq m(B_1 \cup A_1) + m(B_2 \setminus A_2) \leq 2\varepsilon$. Since $A_1 \cup A_2$ and $B_1 \cup B_2$ are elementary, $E_1 \cup E_2$ is Jordan measurable.

- Note that $A_1 \cap A_2 \subset E_1 \cap E_2 \subset B_1 \cap B_2$ and

$$\begin{aligned}m((B_1 \cap B_2) \setminus (A_1 \cap A_2)) &= m(((B_1 \cap B_2) \setminus A_1) \cup ((B_1 \cap B_2) \setminus A_2))\\ &\leq m((B_1 \cap B_2) \setminus A_1) + m((B_1 \cap B_2) \setminus A_2)\\ &\leq m(B_1 \setminus A_1) + m(B_2 \setminus A_2)\\ &\leq 2\varepsilon\end{aligned}$$

  Since $A_1 \cap A_2$ and $B_1 \cap B_2$ are elementary, $E_1 \cap E_2$ is Jordan measurable.

- Note that $A_1 \setminus B_2 \subset E_1 \setminus E_2 \subset B_1 \setminus A_2$, as well as

$$\begin{aligned}m((B_1 \setminus A_2) \setminus (A_1 \setminus B_2)) &= m(B_1 \setminus A_2) - m(A_1 \setminus B_2)\\ &\leq (m(B_1) - m(A_2)) - (m(A_1) - m(B_2))\\ &= (m(B_1) - m(A_1)) + (m(B_2) - m(A_2))\\ &\leq 2\varepsilon\end{aligned}$$

  Since $A_1 \setminus B_2$ and $B_1 \setminus A_2$ are elementary, $E_1 \setminus E_2$ is Jordan measurable.

- Since $E_1 \Delta E_2 = (E_1 \setminus E_2) \cup (E_2 \setminus E_1)$, $E_1 \Delta E_2$ is Jordan measurable by (i) and (iii).

(ii) Since the elementary measure is non-negative and $m(E_1) = m_*^J(E_1) = \sup\left\{m(A) : A \subset E_1, A \in \mathcal{E}(\mathbb{R}^d)\right\} \geq 0$

(iv) If $A \subset E_1$ is elementary, then $A \subset E_2$, thus the supremum over all inner elementary sets for E_1 is a subset of the inner elementary sets for E_2. Thus $m_*^J(E_1) \leq m_*^J(E_2)$, which gives $m(E_1) \leq m(E_2)$ since E_1 and E_2 are Jordan measurable.

(v) Pick $\varepsilon > 0$ and elementary sets $B_1 \supset E_1, B_2 \supset E_2$ such that $m(B_1) - m(E_1) \leq \varepsilon$ and $m(B_2) - m(E_2) \leq \varepsilon$. Since $E_1 \cup E_2 \subset B_1 \cup B_2$ by (iv) it follows

$$m(E_1 \cup E_2) \leq m(B_1 \cup B_2) \leq m(B_1) + m(B_2) \leq m(E_1) + m(E_2) + 2\varepsilon.$$

Since ε was arbitrary, it follows $m(E_1 \cup E_2) \leq m(E_1) + m(E_2)$.

(iii) Take $\varepsilon > 0$. Then there exists elementary sets $A_1 \subset E_1, A_2 \subset E_2$ such that $m(E_1) - m(A_1) \leq \varepsilon$ and $m(E_2) - m(A_2) \leq \varepsilon$. Since $E_1 \cap E_2 = \emptyset$, $A_1 \cap A_2 = \emptyset$ which means $m(A_1 \cup A_2) = m(A_1) + m(A_2)$. Note that $A_1 \cup A_2 \subset E_1 \cup E_2$, which by (iv) gives

$$m(E_1 \cup E_2) \geq m(A_1 \cup A_2) = m(A_1) + m(A_2) \geq m(E_1) + m(E_2) - 2\varepsilon.$$

Since ε was arbitrary, it follows $m(E_1 \cup E_2) \geq m(E_1) + m(E_2)$ which in conjunction with (v) gives $m(E_1 \cup E_2) = m(E_1) + m(E_2)$.

(vi) Take $\varepsilon > 0$. Then there exists elementary sets $A \subset E_1 \subset B$ such that $m(B \setminus A) \leq \varepsilon$. Note that $A + x \subset E_1 + x \subset B + x$. Since $m(\cdot)$ is translation invariant for elementary sets, it follows $m(A + x) = m(A)$ and $m(B + x) = m(B)$. Note then by (iv) that

$$m(A) \leq m(E_1) \leq m(B)$$

and by the definition of the outer/inner Jordan measure that

$$m(A) = m(A + x) \leq m_*^J(E_1 + x) \leq m_J^*(E_1 + x) \leq m(B + x) = m(B).$$

Since $m(B \setminus A) = m(B) - m(A) \leq \varepsilon$ we have both $|m(E_1) - m_J^*(E_1 + x)| \leq \varepsilon$ and $|m(E_1) - m_*^J(E_1 + x)|$. The choice of ε was arbitrary, hence $m_J^*(E_1 + x) = m_*^J(E_1 + x) = m(E)$.

◇

Ex. 1.14

Proof.

i) Take $\varepsilon > 0$. Since B is closed and bounded, it is compact meaning f is uniformly continuous over B. Therefore $\exists \delta > 0$ such that $|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$. Partition B into a disjoint set of closed boxes Q_i whose diameters are smaller than δ. Associate then the set of points x_i with Q_i where $x_i \in Q_i$. Note then that $x \in Q_i$ gives $|x - x_i| < \delta \implies |f(x) - f(x_i)| < \varepsilon$. Thus we have

$$\{(x, f(x)) : x \in Q_i\} \subset Q_i \times [f(x_i) - \varepsilon, f(x_i) + \varepsilon].$$

Since all the Q_i cover B, it follows

$$G(f) \subset \bigcup_i Q_i \times [f(x_i) - \varepsilon, f(x_i) + \varepsilon].$$

The measure of the union can be bounded above by $2\varepsilon \cdot M$ where M is the total size of all the Q_i. The total size of all the Q_i is constant since it is simply the size of B. Therefore we have

$$0 \le m_*^J(G(f)) \le m_J^*(G(f)) \le 2\varepsilon \cdot M.$$

Since ε is arbitrary, it follows that $m(G(f)) = 0$.

ii) Take $\varepsilon > 0$. Partition B into the boxes Q_i as described above. Let $m_i = \inf_{x \in B_i} f(x)$ and $M_i = \sup_{x \in B_i} f(x)$, and define the elementary sets

$$L = \bigcup_i Q_i \times [0, m_i] \qquad\qquad U = \bigcup_i Q_i \times [0, M_i].$$

Note that $A \subset B(f) \subset B$ and that $0 \le M_i - m_i \le \varepsilon$. Therefore

$$m(U \setminus L) = m\left(\bigcup_i Q_i \times [m_i, M_i]\right) \le \sum_i |Q_i|(M_i - m_i) \le \varepsilon \sum_i |Q_i|.$$

Similar to above, the total size of all the Q_i is constant giving $m(U \setminus L) \le \varepsilon \cdot M$, hence $B(f)$ is Jordan measurable.

$\diamond$

Ex 1.18

Proof. Let $c := m(L([0,1]^d))$. Note that for sets $A, B \subset \mathbb{R}^d$ that $L(A \cup B) = L(A) \cup L(B)$.

- Take $n \in \mathbb{N}$. Then $[0,1)^d$ can be subdivided into n^d cubes of side length $\frac{1}{n}$. Note that each cube can be written as $x + [0, \frac{1}{n})^d$ for some $x \in \mathbb{R}^d$, and by linearity that $L(x + [0, \frac{1}{n}))^d) = L(x) + L([0, \frac{1}{n})^d)$. Therefore by translation invariance
$$m\left(L\left(\left[0, \frac{1}{n}\right)^d\right)\right) = \frac{c}{n^d}.$$
- Let B be a box with rational endpoints. By translation invariance it can be assumed $B = \prod_{i=1}^d [0, r_i)$ where $r_i \in \mathbb{Q}$. For some $q \in \mathbb{N}$ we can then write $r_i = \frac{p_i}{q}$ for some $p_i \in \mathbb{Z}$. It is therefore possible to partition B into $p_1 p_2 \ldots p_d$ boxes of side length $\frac{1}{q}$. From the previous part it then follows
$$m(L(B)) = p_1 \ldots p_d \cdot m\left(L\left(\left[0, \frac{1}{q}\right)^d\right)\right) = \frac{p_1 \ldots p_d}{q^d}.$$
- Now let B be a box with real endpoints, which again by translation invariance can be assumed to be $B = \prod_{i=1}^d [0, t_i)$ for $t_i \in \mathbb{R}$. Take $q \in \mathbb{N}$ and let $p_i = \lfloor q t_i \rfloor$. Define
$$B_L = \prod_{i=1}^d \left[0, \frac{p_i}{q}\right) \qquad B_U = \prod_{i=1}^d \left[0, \frac{p_i + 1}{q}\right).$$
Note that $B_L \subset B \subset B_U$, and
$$\lim_{q \to \infty} \frac{p_1 \ldots p_d}{q^d} = \lim_{q \to \infty} \frac{(p_1 + 1) \ldots (p_d + 1)}{q^d} = t_1 \ldots t_d = |B|.$$
Montonicity gives $m(L(B_L)) \le m(L(B)) \le m(L(B_U))$, and from the previous part
$$m(L(B_L)) = c \cdot \frac{p_1 \ldots p_d}{q^d} \qquad m(L(B_U)) = c \cdot \frac{(p_1 + 1) \ldots (p_d + 1)}{q^d}.$$
Therefore
$$c \cdot \frac{p_1 \ldots p_d}{q^d} \le m(L(B)) \le c \cdot \frac{(p_1 + 1) \ldots (p_d + 1)}{q^d}.$$
Thus in the limit as $q \to \infty$ it follows $m(L(B)) = c \cdot |B|$.

Let $E \subset \mathbb{R}^d$ be an elementary set. Then E has a finite disjoint decomposition into boxes $E = \coprod_i B_i$. Note then that

$$m(L(E)) = m\left(L\left(\coprod_i B_i\right)\right) = m\left(\coprod_i L(B_i)\right)$$
$$= \sum_i m(L(B_i)) = c \cdot \sum_i |B_i| = cm(E)$$

proving (i).

Now let E be Jordan measurable, and take $\varepsilon > 0$. Then there exists elementary sets A, B such that $A \subset E \subset B$ and $m(B \setminus A) \leq \varepsilon$. Therefore $m(B) - m(A) \leq \varepsilon$, or equivalently

$$\frac{m(L(B))}{c} - \frac{m(L(A))}{c} \leq \varepsilon \implies m(L(B)) - m(L(A)) \leq c\varepsilon.$$

Thus since $L(A) \subset L(E) \subset L(B)$, it follows

$$cm(A) = m(L(A)) \leq m_*^J(L(E)) \leq m_J^*(L(E)) \leq m(L(B)) = cm(B).$$

Letting ε go to 0 means $L(E)$ is measurable, and both $m(A)$ and $m(B)$ will converge to $m(E)$, giving $m(L(E)) = cm(E)$.

$\diamond$

Ex 1.23

Proof. Define A_n be the union of all dyadic cubes of side length 2^{-n} contained in E, B_n the ones that intersect E. Let $a_n := m(A_n) = 2^{-dn} e_*(E, n)$ and $b_n := m(B_n) = 2^{-dn} e^*(E, n)$. When the limits exist, denote $\alpha = \lim_{n\to\infty} a_n$ and $\beta = \lim_{n\to\infty}$.

Suppose E is measurable and take $\varepsilon > 0$. Note that

$$m(B_n \setminus A_n) = m(B_n) - m(A_n) = 2^{-dn}(e^*(E, n) - e_*(E, n)).$$

Since E is measurable, $m^*(\partial E) = 0$ and therefore there exists an elementary set $C \supset \partial E$ such that $m(C) < \varepsilon$. We can assume C to be open, and since ∂E is closed and bounded, it is also compact. Therefore we can create an open cover of cubes contained in C for every point in ∂E, of which gives a finite subcover

O. It is then possible to take n large enough such that $B_n \setminus A_n \subset O \subset C$ (taking δ to be smallest diameter of the non-empty pairwise intersections of the open cubes in the finite subcover and choosing $2^{-n} < \delta$ accomplishes this). Therefore $m(B_n \setminus A_n) \leq m(C) \leq \varepsilon$. As ε goes to 0, n goes to infinity giving

$$\lim_{n\to\infty} m(B_n \setminus A_n) = 0 \implies \lim_{n\to\infty} 2^{-dn}(e^*(E,n) - e_*(E,n)) = 0.$$

Suppose then that

$$\lim_{n\to\infty} 2^{-dn}(e^*(E,n) - e_*(E,n)) = \lim_{n\to\infty}(b_n - a_n) = \beta - \alpha = 0.$$

Then $\alpha = \beta$, and since $a_n \leq m_*^J(E) \leq m_J^*(E) \leq b_n$, it follows E is measurable and

$$m(E) = \alpha = \beta$$

which was to be shown. $\diamond$

Ex 1.25

Proof. Let $E \subset \mathbb{R}^{d_1}$ and $F \subset \mathbb{R}^{d_2}$ be elementary sets. Then there are finite disjoint decompositions $E = \bigcup_i B_i$ and $F = \bigcup_i V_i$ where B_i and V_i are boxes. Note then that

$$E \times F = \bigcup_i \bigcup_j B_i \times V_i$$

and $|B_i \times V_i| = |B_i||V_i|$. Therefore

$$\begin{aligned} m_{d_1+d_2}(E \times F) &= \sum_i \sum_j |B_i \times V_i| \\ &= \sum_i \sum_j |B_i||V_i| \\ &= \left(\sum_i |B_i|\right)\left(\sum_j |V_i|\right) \\ &= m_{d_1}(E) m_{d_2}(F) \end{aligned}$$

For any elementary sets U, V such that $U \supset E_1$ and $V \supset E_2$, it follows $U \times V \supset E_1 \times E_2$ and

$$m_J^*(E_1 \times E_2) \leq m(U \times V) = m(U)m(V).$$

Thus taking the infimum over all these possible elementary sets gives

$$m_J^*(E_1 \times E_2) \leq m_J^*(E_1) m_J^*(E_2).$$

A similar argument using subsets and the supremum gives

$$m_*^J(E_1 \times E_2) \geq m_*^J(E_1) m_*^J(E_2).$$

Therefore if E_1 and E_2 are both measurable,

$$m_{d_1}(E_1) m_{d_2}(E_2) \leq m_*^J(E_1 \times E_2) \leq m_J^*(E_1 \times E_2) \leq m_{d_1}(E_1) m_{d_2}(E_2).$$

Thus $E_1 \times E_2$ is measurable and $m_{d_1+d_2}(E_1 \times E_2) = m_{d_1}(E_1) m_{d_2}(E_2)$. $\diamond$

Ex 1.26

Proof.

i) Since $E \subset \overline{E}$ it follows $m_J^*(E) \leq m_J^*(\overline{E})$, thus it suffices to show the reverse inequality. Take $\varepsilon > 0$. Then there exists boxes $B_1, \ldots, B_n$ such that $E \subset \bigcup B_i$ and $\sum |B_i| \leq m_J^*(E) + \varepsilon$. Note that $|\overline{B_i}| = |B_i|$ and

$$\overline{E} \subset \overline{\bigcup_i B_i} = \bigcup_i \overline{B_i}.$$

Therefore $m_J^*(\overline{E}) \leq \sum |\overline{B_i}| = \sum |B_i| \leq m_J^*(E) + \varepsilon$, giving $m_J^*(\overline{E}) \leq m_J^*(E) + \varepsilon$. Since ε was arbitrary, we have equality.

ii) We use a similar argument as above. Clearly $m_*^J(\mathring{E}) \leq m_*^J(E)$. Take $\varepsilon > 0$. Then there exists boxes $B_1, \ldots, B_n$ such that $\bigcup B_i \subset E$ with $\sum |B_i| \geq m_*^J(E) - \varepsilon$. Note that $|\mathring{B}_i| = |B_i|$ and

$$\bigcup_i \mathring{B}_i \subset \operatorname{int}\left(\bigcup_i B_i\right) \subset \mathring{E}.$$

Therefore $m_*^J(E) \geq \sum |\mathring{B}_i| = \sum |B_i| \geq$. Since ε was arbitrary, we have equality.

iii) Suppose E is Jordan measurable and take $\varepsilon > 0$. Then there exists $A \subset E \subset B$ elementary such that $m(B \setminus A) = m(B) - m(A) \le \varepsilon$. Note then that $\partial E \subset \overline{B} \setminus \mathring{A}$. From the previous parts, it follows that $m(\mathring{A}) = m(A)$ and $m(\overline{B}) = m(B)$, thus

$$m_J^*(\partial E) \le m(\overline{B} \setminus \mathring{A}) = m(\overline{B}) - m(\mathring{A}) = m(B) - m(A) \le \varepsilon.$$

Since ε was arbitrary, it follows $m_J^*(\partial E) = 0$.

Suppose then $m_J^*(\partial E) = 0$ and take $\varepsilon > 0$. Then there exists an elementary set $C \supset \partial E$ with $m(C) \le \varepsilon$. We can assume C to be open, thus $\overline{E} \setminus C$ is a compact set. Since $\overline{E} \setminus C \subset \mathring{E}$, we can find an open box around every point $x \in \overline{E} \setminus C$ contained in $\mathring{E}$. This defines an open cover of $\overline{E} \setminus C$, thus there is a finite subcover A by compactness. Note that A is elementary since it is the union of finitely many open boxes. Furthermore $A \subset \mathring{E}$, meaning

$$m_J^*(\overline{E}) \le m(A \cup C) \le m(A) + m(C) \le m_*^J(\mathring{E}) + \varepsilon.$$

Letting ε go to 0 gives $m_J^*(\overline{E}) = m_*^J(\mathring{E})$. It follows then from the previous two parts that $m_*^J(E) = m_J^*(E)$, thus E is measurable.

iv) Since $\overline{[0,1]^2 \setminus \mathbb{Q}^2} = [0,1]^2 = \overline{[0,1]^2 \cap \mathbb{Q}^2}$, it follows $m_J^*([0,1]^2 \setminus \mathbb{Q}^2) = m_J^*([0,1]^2 \cap \mathbb{Q}^2) = m_J^*([0,1]^2) = 1$. But $\mathrm{int}([0,1]^2 \setminus) = \varnothing = \mathrm{int}([0,1]^2 \cap \mathbb{Q}^2)$, thus $m_*^J([0,1]^2 \setminus \mathbb{Q}) = m_*^J([0,1]^2 \cap \mathbb{Q}^2) = m_*^J(\varnothing) = 0$. Therefore neither $[0,1]^2 \setminus \mathbb{Q}^2$ or $[0,1]^2 \cap \mathbb{Q}^2$ are Jordan measurable.

◇

Ex. 1.28

Proof. Since $E = (E \cap F) \cup (E \setminus F)$, it follows that $m_J^*(E) \le m_J^*(E \cap F) + m_J^*(E \setminus F)$. For the other inequality, take $\varepsilon > 0$. There then exists some elementary set $A \supset E$ such that $m(A) \le m_J^*(E) + \varepsilon$. Since F is elementary, it follows $A \cap F$ and $A \setminus F$ are both elementary, and since both are disjoint $m(A) = m(A \cap F) + m(A \setminus F)$. Thus

$$m_J^*(E) + \varepsilon \ge m(A) = m(A \cap F) + m(A \setminus F) \ge m_J^*(E \cap F) + m_J^*(E \setminus F).$$

Since ε was arbitrary, we have $m_J^*(E) \ge m_J^*(E \cap F) + m_J^*(E \setminus F)$. Combining both inequalities gives the desired result. ◇

Ex 2.13

Proof.

i $\Rightarrow$ ii) Suppose E is Lebesgue measurable and take $\varepsilon > 0$. Then there exists $O \overset{o}{\supset} E$ such that $m^*(O \setminus E) \le \varepsilon$. Note that $O \Delta E = (O \setminus E) \cup (E \setminus O) = O \setminus E$ since $O \supset E$. Thus $m^*(O \Delta E) = m^*(O \setminus E) \le \varepsilon$.

ii $\Rightarrow$ i) Suppose there is an open set O such that $m^*(O \Delta E) \le \varepsilon$. This means that $m^*(O \setminus E) \le \varepsilon$ and thus there exists an open set C covering $O \setminus E$ and $m^*(C) \le m^*(O \setminus E) + \varepsilon = 2\varepsilon$ (C can come from taking a countable collection of open boxes from the definition of outer measure). Taking $O' = O \cup C$, we have O' is open and $O' \setminus E \subset (O \setminus E) \cup C$. Thus by subadditivity

$$m^*(O \setminus E) \le m^*(O \setminus E) + m^*(C) \le 3\varepsilon$$

which was to be shown.

i $\Leftrightarrow$ iii) Suppose E is measurable. Then E^c is measurable, hence there exists and open set $O \supset E^c$ such that $m^*(O \setminus E^c) \le \varepsilon$. Let $F = O^c$ and note that F is closed $E \setminus F = E \cap O = O \setminus E^c$. Therefore

$$m^*(E \setminus F) = m(O \setminus E^c) \le \varepsilon.$$

which was to be shown.

Suppose then there is a closed set $F \subset E$ such that $m^*(E \setminus F) \le \varepsilon$. Then $O = F^c$ is an open set and $O \setminus E = E^c \cap O = E \setminus F$. Therefore

$$m^*(O \setminus E) = m(E \setminus F) \le \varepsilon$$

which was to be shown.

iii $\Rightarrow$ iv) Take $\varepsilon > 0$ and suppose there exists a closed set $F \subset E$ such that $m^*(E \setminus F) \le \varepsilon$. Note that $F \Delta E = (F \setminus E) \cup (E \setminus F) = E \setminus F$ since $F \subset E$. Thus $m^*(F \Delta E) = m^*(E \setminus F) \le \varepsilon$.

iv $\Rightarrow$ i) Take $\varepsilon > 0$ and suppose there exists a closed set F such that $m^*(E \Delta F) \le \varepsilon$. Note that $O = F^c$ is an open set and $E \Delta F = E^c \Delta O$, thus

$$m^*(O \Delta E^c) = m^*(E \Delta F) \le \varepsilon.$$

From the established equivalency for (ii) $\Leftrightarrow$ (i), it follows E^c is measurable and thus so is E.

i $\Rightarrow$ v) Taking $E_\varepsilon = E$ gives the result directly since $m^*(E\Delta E_\varepsilon) = m^*(\emptyset) = 0 < \varepsilon$ for any $\varepsilon > 0$.

v $\Rightarrow$ ii) Suppose there exists a Lebesgue measurable set E_ε such that $m^*(E\Delta E_\varepsilon) \leq \varepsilon$. Since E_ε is measurable, by the equivalency of (ii) $\Leftrightarrow$ (i) there exists some open set O such that $m^*(O\Delta E) \leq \varepsilon$. Since $A\Delta C = (A\Delta B) \cup (B\Delta C)$ for any sets A, B, C, it follows

$$m^*(O\Delta E) \leq m^*(O\Delta E_\varepsilon) + m^*(E_\varepsilon \Delta E) \leq 2\varepsilon$$

which was to be shown.

$\diamond$

Ex 2.18

Proof.

1. Take $x \in E$. From pointwise convergence it follows $\exists N \in \mathbb{N}$ such that $|\mathbb{1}_{E_n}(x) - \mathbb{1}_E(x)| < \frac{1}{2}$ for every $n \geq N$. But this means that $\mathbb{1}_{E_n}(x) = \mathbb{1}_E(x)$ since any indicator function only takes on the values $\{0, 1\}$. Therefore $x \in E_n$ for every $n \geq N$. Thus

$$E \subset \bigcup_{N\in\mathbb{N}} \bigcap_{n\geq N} E_n =: L.$$

Now take $x \in L$. Then there is some $N \in \mathbb{N}$ such that $x \in E_n$ for all $n \geq N$. Thus $\mathbb{1}_{E_n}(x) = 1$ for every $n \geq N$. It follows then $\mathbb{1}_E = \lim_{n\to\infty} \mathbb{1}_{E_n}(x) = 1$, meaning $x \in E$. Therefore we have the other subset inclusion giving $E = L$.

Since every E_n is measurable, the countable intersection of them is also measurable. This collection of countable intersections are all measurable, thus the countable union of them is also measurable. Hence L is measurable, which means E is measurable.

2. Since from (a) we know E is measurable, we have $|m(E_n) - m(E)| \leq m(E_n \Delta E)$. Let $A_k = \bigcup_{n \geq k} E_n \Delta E$. From (a), we know that for sufficiently large N, it is either the case that $x \in E_n$ for all $n > N$, or $x \notin E_n$ for all $n > N$, of which precisely are the cases where $x \in E$ or $x \notin E$ respectively. Therefore
$$A := \bigcap_{k \in \mathbb{N}} A_k = \varnothing.$$
Since each A_k is the countable union of measurable sets, each A_k is measurable, and A is measurable since it is the countable intersection of measurable sets. Since each $E_n \subset F$, it follows $E \subset F$ and thus $A_k \subset F$. Therefore $m(A_k) < \infty$ meaning by continuity from above
$$\lim_{k \to \infty} m(A_k) = m(A) = 0.$$
Note that for $n \geq k$ that $E_n \Delta E \subset A_k$, so $m(E_n \Delta E) \leq m(A_k)$ which combined with previous result means $\lim_{n \to \infty} m(E_n \Delta E) = 0$. Therefore $m(E_n) = m(E)$.

3. Let $E_n = n + [0, 1]$. Note that $m(E_n) = 1$ for all n by translation invariance. However, for any $x \in \mathbb{R}$, there exists some $N \in \mathbb{N}$ such that $x < n$ for all $n > N$. Thus $x \notin E_n$ for $n > N$, and hence $\mathbb{1}_{E_n}(x) \neq 1$ for $n > N$. This means that $\lim_{n \to \infty} \mathbb{1}_{E_n} = \mathbb{1}_{\varnothing}$, and since $m(\varnothing) = 0 \neq 1$, (b) does not hold.

$\diamond$

Ex 2.19

Proof.

1. Note that for each $n \in \mathbb{N}$, there exists an open set $O_n \supset E$ such that $m^*(O_n) \leq m^*(E) + \frac{1}{n}$. Let $O = \bigcap_{i \in \mathbb{N}} O_n$. Since open sets are measurable, and O is the countable intersection of open sets, it follows O is measurable. Note that $O \supset E$ meaning $m^*(E) \leq m(O)$, and thus if $m^*(E) = \infty$ then $m(O) = \infty$ and we are done. Suppose then $m^*(E) < \infty$. Since $O \supset O_n$ for every n, then
$$m(O) = m^*(O) \leq m^*(O_n) \leq m^*(E) + \frac{1}{n}$$

which in the limit as $n \to \infty$ gives $m(O) \leq m^*(E)$. Since we know from the infinite case $m^*(E) \leq m(O)$, we have $m(O) = m^*(E)$, which was to be shown.

2. Suppose E is bounded and take $\varepsilon > 0$. Since E is measurable, there then exists a closed set $F_\varepsilon \subset E$ such that $m(E \setminus F_\varepsilon) \leq \varepsilon$. Note that $m(E \setminus F_\varepsilon) = m(E) - m(F_\varepsilon) \leq \varepsilon$, and thus $m(E) - \varepsilon \leq m(F_\varepsilon)$. Since E is bounded, every F_ε is bounded and thus compact. Thus since there exists a compact set $K \subset E$ for every $\varepsilon > 0$ where $m(E) - \varepsilon \leq m(K)$, it follows

$$m(E) = \sup_{K \subset E, K \text{ compact}} m(K).$$

Suppose then E is unbounded. Let $E_R := E \cap B_R(0)$. Each E_R is measurable since it is the intersection of two measurable sets. Since $E = \bigcup_{R \in \mathbb{N}_0} E_R$, it follows that

$$m(E) = \lim_{R \to \infty} m(E_R).$$

Any compact subset of E_R is also a compact subset of E, so from the previous result for bounded sets we know that

$$m(E_R) = \sup_{K \subset E_R, K \text{ compact}} m(K) \leq \sup_{K \subset E, K \text{ compact}} m(K).$$

Note then that if $m(E) = \infty$, then taking the limit as $R \to \infty$ of the above inequality gives the desired result. Assume then $m(E) < \infty$. By monotonicity, any compact subset K of E will have $m(K) \leq m(E)$, thus combining this result with the previous inequality gives

$$m(E_R) \leq \sup_{K \subset E, K \text{ compact}} m(K) \leq m(E).$$

In the limit as $R \to \infty$, the lower bound becomes $m(E)$ and thus the desired result is achieved.

◇

0.1 Exercise 2.20

Proof.

i $\Rightarrow$ ii) Suppose E is measurable and $m(E) < \infty$. Take $\varepsilon > 0$. Then by outer regularity, there exists an open set O such that $m^*(O) \le m^*(E) + \varepsilon$. Since $O \supset E$, it follows

$$m^*(O \setminus E) \le m^*(O) - m^*(E) \le \varepsilon.$$

Since E is measurable and O is open and hence also measurable, $m(O) \le m(E) + \varepsilon < \infty$.

ii $\Rightarrow$ iii) Suppose there exists an open set $O \supset E$ where $m(O) < \infty$ and $m^*(O \setminus E) \le \varepsilon$. Let $O_R = O \cap B_R(0)$. Since $O = \bigcup_{R \in \mathbb{N}} O_R$, by continuity it follows $m(O) = \lim_{R \to \infty} O_R$. Since $m(O)$ is finite, there then exists some $N > 0$ such that $m(O \setminus O_N) \le \varepsilon$. Note that $O_N \setminus E \subset O \setminus E$ and $E \setminus O_N \subset O \setminus O_N$, thus

$$\begin{aligned} m^*(E \Delta O_N) &\le m^*(E \setminus O_N) + m^*(O_N \setminus E) \\ &\le m^*(O \setminus O_N) + m^*(O \setminus E) \\ &\le \varepsilon + \varepsilon \\ &= 2\varepsilon \end{aligned}$$

Since O_N is the intersection of an open set with an open and bounded set, it itself is also open and bounded, giving the desired set for (iii).

iii $\Rightarrow$ i) Suppose there exists a bounded open set O such that $m^*(E \Delta O) \le \varepsilon$. Note that $m^*(E \setminus O) \le m^*(E \Delta O) \le \varepsilon$ (the same argument holds for $O \setminus E$), thus there exists an open set $C \supset E \setminus O$ such that $m^*(C) \le m^*(E \setminus O) + \varepsilon \le 2\varepsilon$. Let $O' = O \cup C$, and note that $O' \supset E$ and $O' \setminus E \subset (O \setminus E) \cup C$. Therefore

$$m^*(O' \setminus E) \le m^*(O \setminus E) + m^*(C) \le \varepsilon + 2\varepsilon = 3\varepsilon$$

which was to be shown.

i $\Rightarrow$ iv) Suppose E is measurable. From the previous homework problem, it follows that there exists some compact set $K \subset E$ such that $m(K) \ge m(E) - \varepsilon$. Thus

$$m^*(E \setminus K) \le m^*(E) - m^*(K) = m(E) - m(K) \le \varepsilon$$

which was to be shown

iv ⇒ v) If F is the compact set from (iv), then the set of difference between E and F, which is $E\Delta F$, has outer measure $m^*(E\Delta F) = m^*(E \setminus F) \leq \varepsilon$ (since $F \subset E$), which was to be shown.

v ⇒ vi) This is simply a restatement of (v) since any compact set is also bounded.

vi ⇒ vii) Suppose there exists a bounded measurable set F such that $m^*(E\Delta F) \leq \varepsilon$. Note since F is bounded, it is contained in some bounded box, which has finite measure. Thus by monotonicity, $m(F) < \infty$. (vii) thus follows from the supposition.

vii ⇒ viii) Suppose there exists a measurable set F with finite measure such that $m^*(E\Delta F) \leq \varepsilon$. By the continuity of the Lebesgue measure, there exists $R > 0$ such that $m(F \setminus B_R(0)) \leq \varepsilon$. Let $F_R := F \cap B_R(0)$. Note then F_R is both bounded and measurable, as well as

$$m^*(F\Delta F_R) = m^*(F \setminus F_R) = m(F \setminus B_R(0)) \leq \varepsilon.$$

Since $A\Delta C = (A\Delta B) \cup (B\Delta C)$ for any sets A, B, C, it follows

$$m^*(E\Delta F_R) \leq m^*(E\Delta F) + m^*(F\Delta F_R) \leq 3\varepsilon.$$

F_R is bounded and measurable, so we can find an open set $O \supset F_R$ such that $m(O \setminus F_R) \leq \varepsilon$, and this open set can be described as the countable union of boxes B_i. Let then $A_N = \bigcup_{i=1}^N B_i$. Since $\bigcup_{N\in\mathbb{N}} A_N = O$, by continuity $\lim_{N\to\infty} m(A_N) = m(O)$. Thus there is some N such that $m(O \setminus A_N) \leq \varepsilon$. Note that A_N is a finite union of boxes and is thus elementary, and since $A_N \subset O$

$$\begin{aligned}
m^*(E\Delta A_N) &\leq m^*(E\Delta O) + m^*(O\Delta A_N) \\
&\leq m^*(E\Delta F_R) + m^*(F_R\Delta O) + m^*(O\Delta A_N) \\
&\leq m^*(E\Delta F_R) + m(O \setminus F_R) + m(O \setminus A_N) \\
&\leq 3\varepsilon + \varepsilon + \varepsilon \\
&= 5\varepsilon
\end{aligned}$$

which was to be shown.

viii $\Rightarrow$ ix) Suppose there exists an elementary set A such that $m^*(E\Delta A) \leq \varepsilon$. Let D_n denote the set of all dyadic cubes of sidelength 2^{-n} contained in A. Since A is elementary, it is Jordan measurable and thus $\lim_{n\to\infty} m(D_n) = m(A)$ (this specifically follows from Exercise 1.23). Therefore we can take $N \in \mathbb{N}$ such that $m(A \setminus D_n) \leq \varepsilon$ since $D_n \subset A$. Thus we have

$$\begin{aligned} m^*(E\Delta D_n) &\leq m^*(E\Delta A) + m^*(A\Delta D_n) \\ &\leq m^*(E\Delta A) + m(A \setminus D_n) \\ &\leq \varepsilon + \varepsilon \\ &= 2\varepsilon \end{aligned}$$

which was to be shown.

ix $\Rightarrow$ i) Suppose there exists finite union F of closed dyadic cubes of sidelength 2^{-n} such that $m^*(E\Delta F) \leq \varepsilon$. Since F is a finite union of cubes, it is thus elementary and Jordan measurable. Therefore $m_J(\partial F) = m_J^*(\partial F) = 0$. Since the Jordan and Lebesgue measure agree on Jordan measurable sets, there then exists an open set $C \supset \partial F$ such that $m(C) \leq \varepsilon$. Define then $N = F \cup C$ and note that N is an open set, and $m^*(N \setminus F) = m(C) \leq \varepsilon$.

Since $m^*(E \setminus F) \leq m^*(E\Delta F) \leq \varepsilon$, there exists an open set $U \supset E \setminus F$ such that $m^*(U) \leq m^*(E \setminus F) + 2\varepsilon$ (U can be constructed from an countable open covering of $E \setminus F$ with boxes from the definition of outer measure).

Now let $O = N \cup U$. Note that O is open since N and U are open and contains E. Since $O \setminus E \subset (N \setminus F) \cup (F \setminus E) \cup U$ and $m^*(F \setminus E) \leq m^*(E\Delta F) \leq \varepsilon$, by subadditivity it follows

$$m^*(O \setminus E) \leq m^*(N \setminus F) + m^*(F \setminus E) + m^*(U) \leq \varepsilon + \varepsilon + 2\varepsilon = 4\varepsilon$$

which was to be shown.

$\diamond$

Exercise 2.21

Proof.

i $\Rightarrow$ ii) Suppose E is Lebesgue measurable. Then if A is elementary, it is also Lebesgue measurable. Since $A = (A \cap E) \cup (A \setminus E)$ and the Lebesgue measure has countable additivity, it follows

$$m(A) = m(A \cap E) + m(A \setminus E) = m^*(A \cap E) + m^*(A \setminus E).$$

ii $\Rightarrow$ iii) Since every box B is an elementary set and $m(B) = |B|$, the result follows directly from (ii).

iii $\Rightarrow$ i) Take $\varepsilon > 0$. Then there exists countable collection of open boxes $B = \cup_i B_i$ such that $E \subset B$ and $\sum_i |B_i| \leq m^*(E) + \varepsilon$. We can convert the collection B into a collection $U = \bigcup_i V_i$ of almost disjoint boxes. Since the boxes are almost disjoint, from the supposition we have

$$m^*(U) = \sum_i |V_i| = \sum_i \left[m^*(V_i \cap E) + m^*(V_i \setminus E)\right].$$

Since

$$U \cap E = \bigcup_i V_i \cap E \qquad U \setminus E = \bigcup_i V_i \setminus E$$

it follows from subadditivity that

$$m^*(U \cap E) + m^*(U \setminus E) \leq \sum_i \left[m^*(V_i \cap E) + m^*(V_i \setminus E)\right] = m^*(U). \quad (\star_1)$$

But $U = (U \cap E) \cup (U \setminus E)$, which means

$$m^*(U) \leq m^*(U \cap E) + m^*(U \setminus E). \quad (\star_2)$$

Therefore combining $(\star_1)$ and $(\star_2)$ gives $m^*(U) = m^*(U \cap E) + m^*(U \setminus E)$. Since $E \subset U$, it follows $m^*(U) = m^*(E) + m^*(U \setminus E)$, or equivalently $m^*(U \setminus E) = m^*(U) - m^*(E)$. From the construction of U we have $m^*(U) = \sum_i |B_i| \leq m^*(E) + \varepsilon$. Thus

$$m^*(U \setminus E) = m^*(U) - m^*(E) \leq m^*(E) - m^*(E) + \varepsilon = \varepsilon.$$

Since U is equivalent to B which is the union of open boxes, U itself is open. Therefore E is Lebesgue measurable.

$\diamond$

Exercise 2.25

Proof. Suppose E is measurable.

1. For each $n \in \mathbb{N}$, take $U_n \supset E$ open such that $m^*(U_n \setminus E) \leq \frac{1}{n}$. Let $G = \bigcap_{n\in\mathbb{N}} U_n$. Note that G is a G_δ set and $m^*(G \setminus E) \leq m^*(U_n \setminus E) \leq \frac{1}{n}$ for every $n \in \mathbb{N}$. Therefore $m^*(G \setminus E) = 0$. If $N = G \setminus E$, then $m(N) \leq m^*(N) = 0$ so N is a null set and $E = G \setminus N$.

2. The previous argument can be applied to E^c (which is measurable since E is measurable), giving $E^c = G' \setminus N'$ where G' is a G_δ set and N' is a null set. Then $E = (G')^c \cup N'$, and the complement of a G_δ set is a F_σ set.

Suppose then (i) or (ii) holds. Note that any G_δ or F_σ set is measurable since open sets and closed sets are measurable, and thus any countable union or intersection of them is also measurable. Therefore since E can be written as the union or subtraction of two measurable sets, and union and subtraction preserve measurability, E must be measurable. ◇

Exercise 2.27

Proof. If T is not invertible, then $T(E)$ for any $E \subset \mathbb{R}^d$ will lay in a subspace of $\mathbb{R}^d$ with dimension less than d. Therefore $m(T(E)) = 0$ which is the desired result. Assume then going forward T is invertible. Let $A \subset \mathbb{R}^d$. Note for any countable cover $\bigcup_i B_i$ of A with boxes that

$$T(A) \subset \bigcup_i T(B_i).$$

From previous HW we know that $m(T(B_i)) = |\det T| m(B_i)$, therefore

$$m^*(T(A)) \leq \sum_i m^*(T(B_i)) = |\det T| \sum_i m^*(B_i).$$

Taking the infimum over all possible coverings of A gives $m^*(T(A)) \leq |\det T| m^*(A)$. Since T is invertible,

$$m^*(A) = m^*(T^{-1}(T(A))) \leq |\det T^{-1}| m^*(T(A)) = \frac{1}{|\det T|} m^*(T(A))$$

thus $m^*(T(A)) = |\det T| m^*(A)$.

Suppose then E is Lebesgue measurable. Fix $A \subset \mathbb{R}^d$ elementary. Since E is measurable, by Carathéodory's criterion

$$m(A) = m^*(A \cap E) + m^*(A \setminus E).$$

Note then that

$$\begin{aligned}
m^*(T(A)) &= |\det T| m^*(A) \\
&= |\det T|(m^*(A \cap E) + m^*(A \setminus E)) \\
&= m^*(T(A \cap E)) + m^*(T(A \setminus E)) \\
&= m^*(T(A) \cap T(E)) + m^*(T(A) \setminus T(E))
\end{aligned}$$

Since T is invertible and thus bijective, $T(A)$ can represent any elementary subset of $\mathbb{R}^d$, and thus $T(E)$ is measurable by Carathéodory's criterion. Therefore

$$m(T(E)) = m^*(T(E)) = |\det T| m^*(E) = |\det T| m(E)$$

which was to be shown. ◇

Exercise 2.29

Proof.

1. Let $\varepsilon > 0$. Take $\{B_i\}$ and $\{V_i\}$ to be covers of E and F respectively of boxes such that

   $$\sum_{i \in \mathbb{N}} |B_i| \leq m^*_{d_1}(E) + \varepsilon \qquad \sum_{j \in \mathbb{N}} |V_j| \leq m^*_{d_2}(F) + \varepsilon.$$

   Note that

   $$E \times F \subset \bigcup_{(i,j) \in \mathbb{N}^2} B_i \times V_j$$

is a countable covering of boxes for $E \times F$, therefore

$$\begin{aligned}
m^*_{d_1+d_2}(E \times F) &\le \sum_{(i,j)\in\mathbb{N}^2} |B_i \times V_j| \\
&= \sum_{(i,j)\in\mathbb{N}^2} |B_i||V_j| \\
&= \left(\sum_i |B_i|\right)\left(\sum_j |V_j|\right) \\
&\le (m^*_{d_1}(E) + \varepsilon)(m^*_{d_2}(F) + \varepsilon) \\
&= m^*_{d_1}(E) \cdot m^*_{d_2}(F) + \varepsilon(\ldots)
\end{aligned}$$

Since ε was arbitrary, it follows that $m^*_{d_1+d_2}(E \times F) \le m^*_{d_1}(E) \cdot m^*_{d_2}(F)$.

2. By Problem (1), there exists F_σ sets $G \subset \mathbb{R}^{d_1}$ and $H \subset \mathbb{R}^{d_2}$ such that $m(E \setminus G) = 0$ and $m(F \setminus H) = 0$. Note then that $G \times H$ is also a G_δ set and that
$$(G \times H) \setminus (E \times F) = ((G \setminus E) \times H) \cup ((H \setminus F) \times G).$$
From (a), we know $m((G \setminus E) \times H) = 0$ and $m((H \setminus F) \times G) = 0$ since $G \setminus E$ and $H \setminus F$ are null sets. Therefore
$$m^*_{d_1+d_2}((G \times H) \setminus (E \times F)) = 0$$
thus $E \times F$ is measurable since it differs from a G_δ set by a null set.

A result we will need to prove is that it the desired equality holds for open sets. Let $A \subset \mathbb{R}^{d_1}$ and $B \subset \mathbb{R}^{d_2}$ be open. Then both can be written as the countable union of open boxes $A = \bigcup_i B_i$ and $B = \bigcup_j V_j$. Let $A_n = \bigcup_{i=1}^n B_i$ and $B_n = \bigcup_{j=1}^n V_j$. Since each A_n and B_n are Jordan measurable, then we have $m_{d_1+d_2}(A_n \times B_n) = m_{d_1}(A_n) m_{d_2}(B_n)$. Thus by continuity of the Lebesgue measure we have
$$\begin{aligned}
m_{d_1+d_2}(A \times B) &= \lim_{n\to\infty} m_{d_1+d_2}(A_n \times B_n) \\
&= \lim_{n\to\infty} m_{d_1}(A_n) m_{d_2}(B_n) \\
&= m_{d_1}(A) m_{d_2}(B)
\end{aligned}$$
Thus we have equality for open sets.

Let $G = \bigcap_n G_n$ and $H = \bigcap_n H_n$ where $G_{n+1} \subset G_n$ and $H_{n+1} \subset H_n$ (these come directly from taking the partial intersections of open sets in their definition). Thus by continuity from above we have

$$m_{d_1+d_2}(G\times H) = \lim_{n\to\infty} m_{d_1+d_2}(G_n\times H_n) = \lim_{n\to\infty} m_{d_1}(G_n)m_{d_2}(H_n) = m(G)m(H).$$

Since $m(G \times H) = m(E \times F)$, $m(G) = m(E)$ and $m(H) = m(F)$, we have the desired equality.

$\diamond$

Exercise 2.31

Proof. Since we are working with sets in 2^A and A is elementary and thus bounded, we can assume all measures and outer measures to be finite when they exist.

1. Since $E\Delta E = \varnothing$ which is a null set, $\sim$ is reflexive. Furthermore $E\Delta F = F\Delta E$, thus $\sim$ is also symmetric. Suppose then that $A \sim B$ and $B \sim C$. Note that $A\Delta C = (A\Delta B) \cup (B\Delta C)$. Since $A\Delta B$ and $B\Delta C$ are null sets, then $A\Delta C$ is also a null set and hence $A \sim C$. Therefore $\sim$ is an equivalence relation.

2. Let $E_1 \sim E_2$ and $F_1 \sim F_2$. Note that

   $$E_1\Delta F_1 = (E_1\Delta E_2) \cup (E_2\Delta F_2) \cup (F_2\Delta F_1)$$

   Therefore

   $$m^*(E_1\Delta F_1) \le m^*(E_1\Delta E_2) + m^*(E_2\Delta F_2) + m^*(F_2\Delta F_1) = m^*(E_2\Delta F_2).$$

   The same argument can be applied to obtain the reverse inequality, giving $m^*(E_1\Delta F_1) = m^*(E_2\Delta F_2)$, hence $d([E],[F])$ is well defined.

   Consider then the axioms for a metric

   - $d(\cdot,\cdot)$ is always non-negative since the outer measure of any set is non-negative
   - Symmetry follows since $E\Delta F = F\Delta E$.

- Take E, F, G in 2^A. Then

$$\begin{aligned} d([E],[G]) &= m^*(E\Delta G) \\ &= m^*((E\Delta F)\cup(F\Delta G)) \\ &\le m^*(E\Delta F)+m^*(F\Delta G) \\ &= d([E],[F])+d([F],[G]) \end{aligned}$$

  Therefore $d(\cdot,\cdot)$ satisfies the triangle inequality.

Thus $d(\cdot,\cdot)$ is a metric on $2^A/\sim$.

For completeness, let $[E_n]$ be a Cauchy sequence in $2^A/\sim$.

3. Let $E \in \mathcal{L}$. Then for $\varepsilon > 0$ there exists an open set $U \supset E$ such that $U \subset A$ and $m^*(U \setminus E) \le \varepsilon$. Since U is the countable union of boxes, by continuity from below there is a finite union of boxes $B \subset U$ such that $m^*(U\setminus B) \le \varepsilon$. Note then B is elementary and $E\Delta B \subset (U \setminus E)\cup(U\setminus B)$, meaning

$$\begin{aligned} d([E],[B]) &= m^*(E\Delta B) \\ &\le m^*(U\setminus E)+m^*(U\setminus B) \\ &\le 2\varepsilon \end{aligned}$$

   Thus it is possible for any ε to produce some $[B] \in \mathcal{E}/\sim$ such that $d([E],[B]) \le \varepsilon$, hence $\mathcal{L}/\sim$ is the closure of $\mathcal{E}/\sim$ with respect to d.

4. Let $E, F \in \mathcal{L}$. If $E \sim F$, then $m(E\setminus F) \le m^*(E\Delta F) = 0$ and $m(F\setminus E) \le m^*(E\Delta F) = 0$. Since $E = (E\setminus F)\cup(E\cap F)$ and $F = (F\setminus F)\cup(E\cap F)$ which are both disjoint decompositions,

$$\begin{aligned} m(E) = m(E\setminus F)+m(E\cap F) = m(E\cap F) \\ m(F) = m(F\setminus E)+m(F\cap E) = m(E\cap F) \end{aligned}$$

   Therefore $m(E) = m(F)$, so $m(E)$ is constant for any representative of $[E]$ and thus $m([E])$ is well defined. Note that $|m(E)-m(F)| \le m(E\Delta F)$, thus

$$|m([E]) - m([F])| \le d([E],[F]).$$

   This means $m : \mathcal{L}/\sim \longrightarrow \mathbb{R}_{\ge 0}$ is Lipschitz continuous and thus continuous. Note the above inequality holds as well for $m : \mathcal{E}/\sim \longrightarrow \mathbb{R}_{\ge 0}$ meaning it is also continuous. Since $\mathcal{E}/\sim$ is dense in $\mathcal{L}/\sim$ and both measures agree on $\mathcal{E}/\sim$, it follows $m : \mathcal{L}/\sim \longrightarrow \mathbb{R}_{\geq 0}$ is the unique continuous extension.

◇

Exercise 3.5

Proof. Let

$$f = \sum_{j=1}^{n} c_j \mathbb{1}_{E_j} = \sum_{k=1}^{m} d_k \mathbb{1}_{F_k}$$

be two separate representations of the same simple function f. Let $G_{j,k} = E_j \cap F_k$. Note that each $G_{j,k}$ are disjoint and measurable, and that $\bigcup G_{j,k} = \bigcup E_j = \bigcup F_k$. If $x \in G_{j,k}$, then $x \in E_j$ and $x \in F_k$. Thus $f(x) = c_j$ and $f(x) = d_k$ meaning $c_j = d_k$. Therefore

$$\sum_{j=1}^{n} c_j m(E_j) = \sum_{j=1}^{n} \sum_{k=1}^{m} c_j m(G_{j,k}) = \sum_{j=1}^{n} \sum_{k=1}^{m} d_k m(G_{j,k}) = \sum_{k=1}^{m} d_k m(F_k).$$

Therefore $\mathcal{I}_S(f)$ is well defined and invariant to the representation of f. ◇

Exercise 3.8

Proof. Let $f = \sum_{j=1}^{n} c_j \mathbb{1}_{E_j}$ and $g = \sum_{k}^{m} d_k \mathbb{1}_{F_k}$.

1. Let $G_{j,k} = E_j \cap F_k$. Note that

$$f = \sum_{j=1}^{n} \sum_{k=1}^{m} c_j \mathbb{1}_{G_{j,k}} \qquad g = \sum_{j=1}^{n} \sum_{k=1}^{m} d_k \mathbb{1}_{G_{j,k}}.$$

Therefore

$$f + cg = \sum_{j=1}^{n} \sum_{k=1}^{m} (c_j + c \cdot d_k) \mathbb{1}_{G_{j,k}}$$

meaning

$$I_S(f + cg) = \sum_{j=1}^{n} \sum_{k=1}^{m} (c_j + c \cdot d_k) m(G_{j,k}).$$

But then this summation can be broken up into

$$\sum_{j=1}^{n}\sum_{k=1}^{m}(c_j + c\cdot d_k)m(G_{j,k}) = \left[\sum_{j=1}^{n}\sum_{k=1}^{m} c_j m(G_{j,k})\right] + c\cdot\left[\sum_{j=1}^{n}\sum_{k=1}^{m} d_k m(G_{j,k})\right]$$
$$= I_{\mathcal{S}}(f) + cI_{\mathcal{S}}(g)$$

Hence $I_{\mathcal{S}}(f+cg) = I_{\mathcal{S}}(f) + cI_{\mathcal{S}}(g)$.

2. Suppose f is finite a.e. and its support has finite measure. Then there exists some $X \subset \mathbb{R}^d$ such that $f(X^c) \subset [0,\infty)$ and $m(X) = 0$. Denote $S = [f \neq 0]$ and note $m(S) < \infty$. Note that f can be rewritten as

$$f = \sum_k \left[c_k \mathbb{1}_{X^c\cap E_k} + c_k \mathbb{1}_{X\cap E_k}\right].$$

Since $m(X\cap E_k) = 0$ for all k

$$I_{\mathcal{S}}(f) = \sum_k c_k m(X^c \cap E_k).$$

Whenever $m(X^c\cap E_k) \neq 0$, we have c_k finite. Denote then $C < \infty$ as the maximum over all such c_k. Since $X^c \cap E_k \subset S$, we have

$$I_{\mathcal{S}}(f) = \sum_k c_k m(X^c\cap E_k) \leq \sum_k Cm(S) < \infty.$$

Suppose now that $I_{\mathcal{S}}(f) < \infty$. Then

$$I_{\mathcal{S}}(f) = \sum_k c_k m(E_k) < \infty.$$

If f was not finite almost everywhere, then there would be a set A with $m(A) > 0$ such that $f(x) = \infty$ for $x \in A$. Let $A_E = \{E_k : A\cap E_k \neq \varnothing\}$ and note that A_E is non empty and its union is A. Therefore $\sum_{E_k\in A_E} c_k E_k = \infty$, meaning the integral of f is not finite, a contradiction. Assume then f is not supported on a set of finite measure. Then there must exist some $c = \min\{c_k : c_k > 0\}$. Note then that

$$c\sum_k m(E_k) < \sum_k c_k m(E_k)$$

but $\sum_k m(E_k) = m([f\neq 0]) = \infty$, hence $I_{\mathcal{S}}(f)$ is not finite, a contradiction.

3. If $f \overset{\text{a.e.}}{=} 0$, then there is some set $X \subset \mathbb{R}^d$ such that $f(X^c) = \{0\}$ and $m(X) = 0$. Note that f can be rewritten as

$$f = \sum_k \left[c_k \mathbb{1}_{X \cap E_k} + 0 \cdot \mathbb{1}_{X^c \cap E_k}\right].$$

Since $m(X \cap E_k) = 0$ for all k

$$I_{\mathcal{S}}(f) = \sum_k \left[c_k \cdot 0 + 0 \cdot m(X^c \cap E_k)\right] = 0.$$

4. If $f \overset{\text{a.e.}}{=} g$, then there is some set $X \subset \mathbb{R}^d$ such that $f(x) = g(x)$ for all $x \in X^c$ and $m(X) = 0$. Note that f and g can be rewritten as

$$f = \sum_k c_k \left(\mathbb{1}_{E_k \cap X} + \mathbb{1}_{E_k \cap X^c}\right) \qquad g = \sum_k d_k \left(\mathbb{1}_{F_k \cap X} + \mathbb{1}_{F_k} \cap X^c\right).$$

Since $m(E_k \cap X) = 0$ and $m(F_k \cap X) = 0$ for all k

$$I_{\mathcal{S}}(f) = \sum_k c_k m(E_k \cap X^c) \qquad I_{\mathcal{S}}(g) = \sum_k d_k m(F_k \cap X^c).$$

Note then that $f \cdot \mathbb{1}_{X^c} = g \cdot \mathbb{1}_{X^c}$ are simple and equal everywhere, and that $f \cdot \mathbb{1}_{X^c} = \sum_k c_k \mathbb{1}_{E_k \cap X^c}$ and $g \cdot \mathbb{1}_{X^c} = \sum_k d_k \mathbb{1}_{F_k \cap X^c}$. Therefore

$$I_{\mathcal{S}}(f) = I_{\mathcal{S}}(f \cdot \mathbb{1}_{X^c}) = I_{\mathcal{S}}(g \cdot \mathbb{1}_{X^c}) = I_{\mathcal{S}}(g).$$

5. If $f \leq g$ a.e., then there exists some set $X \subset \mathbb{R}^d$ such that $f(x) \leq g(x)$ for all $x \in X^c$ and $m(X) = 0$. Let $G_{j,k} = E_j \cap F_k$. Then

$$f = \sum_{j=1}^{n} \sum_{k=1}^{m} c_j \Big(\mathbb{1}_{G_{j,k} \cap X} + \mathbb{1}_{G_{j,k} \cap X^c}\Big) \qquad g = \sum_{j=1}^{n} \sum_{k=1}^{m} d_k \Big(\mathbb{1}_{G_{j,k} \cap X} + \mathbb{1}_{G_{j,k} \cap X^c}\Big).$$

Since $m(G_{j,k} \cap X) = 0$ for all j, k

$$I_{\mathcal{S}}(f) = \sum_{j=1}^{n} \sum_{k=1}^{m} c_j m(G_{j,k} \cap X^c) \qquad I_{\mathcal{S}}(g) = \sum_{j=1}^{n} \sum_{k=1}^{m} d_k m(G_{j,k} \cap X^c).$$

Note then for any $x \in G_{j,k} \cap X^c$ that $c_j = f(x) \leq g(x) = d_k$, thus $c_j m(G_{j,k} \cap X^c) \leq d_k m(G_{j,k} \cap X^c)$. Therefore

$$I_{\mathcal{S}}(f) = \sum_{j=1}^{n} \sum_{k=1}^{m} c_j m(G_{j,k} \cap X^c) \leq \sum_{j=1}^{n} \sum_{k=1}^{m} d_k m(G_{j,k} \cap X^c) = I_{\mathcal{S}}(g).$$

6. Note that $\mathbb{1}_E$ is a simple function with coefficient $c = 1$ since E is measurable, thus $I_{\mathcal{S}}(\mathbb{1}_E) = m(E)$.

Suppose there is a map $I : \mathcal{S}^+(\mathbb{R}^d) \to [0, \infty]$ satisfying all the above properties. Take $s \in \mathcal{S}^+(\mathbb{R}^d)$. Then s has the canonical representation

$$s = \sum_{y \in Y} y \mathbb{1}_{s^{-1}(y)}$$

where $Y = s(\mathbb{R}^d)$. Since s is non-negative, $y \geq 0$ for all $y \in Y$. Therefore by properties (i) and (vi)

$$I(s) = \sum_{y \in Y} y I(\mathbb{1}_{s^{-1}(y)}) = \sum_{y \in Y} y m(s^{-1}(y)).$$

But the right hand side is exactly $I_{\mathcal{S}}(s)$, thus $I_{\mathcal{S}}$ is the unique map from $\mathcal{S}^+(\mathbb{R}^d)$ to $[0, \infty]$ satisfying the above properties. $\diamond$

Exercise 3.14

Proof.

1. Suppose $f : \mathbb{R}^d \to [0, \infty]$ is continuous. Then for any relatively open set $U \subset [0, \infty)$, it follows from continuity that $f^{-1}(U)$ is also relatively open and thus measurable. Therefore f itself is measurable.

2. For any $s \in \mathcal{S}^+(\mathbb{R}^d)$, it is the pointwise limit of the sequence of simple functions $s_n = s$. Therefore it is measurable.

3. Take $\lambda \geq 0$ and let $g := \sup_n f_n$. Note that

$$\begin{aligned}
[g > \lambda] &= \left\{ x \in \mathbb{R}^d : \sup_n f_n(x) > \lambda \right\} \\
&= \left\{ x \in \mathbb{R}^d : \exists n \text{ s.t. } f_n(x) > \lambda \right\} \\
&= \bigcup_{n \geq 1} [f_n > \lambda]
\end{aligned}$$

Since each f_n is measurable, then $[f_n > \lambda]$ is measurable. Hence $[g > \lambda]$ is measurable as its the countable union of measurable sets, meaning g is measurable. Let $h := \inf_n f_n$. Similar to above,

$$[h < \lambda] = \bigcup_{n \geq 1} [f_n < \lambda]$$

which by the same argument implies h is measurable. Since then

$$\limsup_n f_n = \inf_{k \geq 1} \sup_{n \geq k} f_n \qquad \liminf_n f_n = \sup_{k \geq 1} \inf_{n \geq k} f_n$$

it follows from the measurability of supremum and infimum that both are measurable as well.

4. Suppose $f : \mathbb{R}^d \to [0, \infty]$ is equal a.e. to some non-negative measurable function s. Since s is measurable, there is some sequence of measurable simple functions s_n that converges to s pointwise. Clearly then this sequence s_n also converges to f a.e., thus f itself is measurable.

5. Let X denote the set on which $f_n \to f$ converges pointwise. Then restricting the sequence to $f_n|_X$ will converge everywhere to $f|_X$. Therefore

$$f|_X = \lim_{n \to \infty} f_n|_X = \limsup_{n \to \infty} f_n|_X.$$

Thus by (iii), $f|_X$ is measurable. Since f is equal a.e. to $f|_X$ (follows from $m(X^c) = 0$), which is measurable, then f is measurable by (iv).

6. Suppose $f : \mathbb{R}^d \to [0, \infty]$ is measurable and $\phi : [0, \infty] \to [0, \infty]$ is continuous. Let $U \subset [0, \infty]$ be a relatively open set. Note then that

$$(\phi \circ f)^{-1}(U) = f^{-1}(\phi^{-1}(U)).$$

Since ϕ is continuous, then $\phi^{-1}(U)$ is open, and since f is measurable then $f^{-1}(\phi^{-1}(U))$ is also measurable. Therefore $\phi \circ f$ is measurable.

7. If f and g are non-negative measurable functions, then there are sequences of non-negative simple functions f_n and g_n that converge pointwise to f and g. Note $f_n + g_n$ and $f_n \cdot g_n$ are still sequences of simple functions and by limit theorems $f + g$ and $f \cdot g$ are the pointwise limits of $f_n + g_n$ and $f_n \cdot g_n$ respectively. Therefore $f + g$ and $f \cdot g$ are measurable.

$\diamond$

Exercise 3.24

Proof.

1. If f is simple, then since $f \le f$, by definition $I_{\mathcal{S}}(f) \le \mathcal{L}_-[f]$. But for any simple function $g \le f$ we have $I_{\mathcal{S}}(g) \le I_{\mathcal{S}}(f)$, giving $\mathcal{L}_-[f] = I_{\mathcal{S}}(f)$.

2. Suppose $f \le g$ a.e. If $h \in \mathcal{S}^+ \le f$, then h can be changed to some $\tilde{h}$ that is 0 where $f > g$. Note then that $\tilde{h} \le g$ and $f > g$ on a null set, thus $I_{\mathcal{S}}(\tilde{h}) = I_{\mathcal{S}}(h)$. Therefore $I_{\mathcal{S}}(h) = I_{\mathcal{S}}(\tilde{h}) \le \mathcal{L}_-[g]$. Taking the supremum over both sides gives $\mathcal{L}_-[f] \le \mathcal{L}_-[g]$.

3. If $c = 0$, then clearly it holds. Suppose $c > 0$. Since for $h \in \mathcal{S}^+$ we have $I_{\mathcal{S}}(ch) = cI_{\mathcal{S}}(h)$, we have

$$\begin{aligned}
\mathcal{L}_-[cf] &= \sup_{cf \ge h \in \mathcal{S}^+} I_{\mathcal{S}}(h) \\
&= \sup_{f \ge h \in \mathcal{S}^+} I_{\mathcal{S}}(ch) \\
&= c \sup_{f \ge h \in \mathcal{S}^+} I_{\mathcal{S}}(h) \\
&= c\mathcal{L}_-[f]
\end{aligned}$$

4. If $f = g$ a.e., it follows $f \le g$ and $g \le f$ a.e. which in conjunction with (ii) gives the desired result.

5. Let $h_1 \le f$ and $h_2 \le g$ be simple. Then clearly $h_1 + h_2$ is simple and $h_1 + h_2 \le f + g$, thus

$$I_{\mathcal{S}}(h_1) + I_{\mathcal{S}}(h_2) = I_{\mathcal{S}}(h_1 + h_2) \le \mathcal{L}_-[f + g].$$

Taking the supremum over all h_1, h_2 on both sides gives then

$$\mathcal{L}_-[f] + \mathcal{L}_-[g] \le \mathcal{L}_-[f + g].$$

6. Let E be measurable. Note that $f = f\mathbb{1}_E + f\mathbb{1}_{E^c}$ as well as $f\mathbb{1}_E \le f$ and $f\mathbb{1}_{E^c} \le f$. Therefore by (v)

$$\mathcal{L}_-[f] \ge \mathcal{L}_-[f\mathbb{1}_E] + \mathcal{L}_-[f\mathbb{1}_{E^c}].$$

For any simple function $h \le f$, it can similarly be split into two simple function $h\mathbb{1}_E$ and $h\mathbb{1}_{E^c}$. Then

$$I_{\mathcal{S}}(h) = I_{\mathcal{S}}(h\mathbb{1}_E) + I_{\mathcal{S}}(h\mathbb{1}_{E^c}) \le \mathcal{L}_-[f\mathbb{1}_E] + \mathcal{L}_-[f\mathbb{1}_{E^c}].$$

Taking the supremum over all h combined with the previous inequality gives

$$\mathcal{L}_-[f] = \mathcal{L}_-[f\mathbb{1}_E] + \mathcal{L}_-[f\mathbb{1}_{E^c}].$$

7. Let $f_n = \min(f, n)$, and note that $f_n \le f_{n+1} \le \ldots \le f$. Therefore $\mathcal{L}_-[f_n]$ is an increasing sequence bounded above by $\mathcal{L}_-[f]$, so $\lim_{n\to\infty} \mathcal{L}_-[f_n]$ exists and is bounded above by $\mathcal{L}_-[f]$. Consider a simple function $h \le f$.

   - If h is ∞ on a set of positive measure E, the desired inequality is trivial as $\mathcal{L}_-[f] = \infty$, and for any $r > 0$ there is large enough N such if $n \ge N$ then $f_n(x) \ge r$ for all $x \in E$. Hence $\lim_{n\to\infty} \mathcal{L}_-[f_n] = \infty$ as well.
   - If h is ∞ on a null set, then we can simply set h to 0 on the null set and still get the same integral.

   We can assume then h is finite everywhere. There then exists sufficiently large N such that for $n \ge N$, $h \le f_n$. Therefore taking the limit and then supremum gives $\mathcal{L}_-[f] \le \lim_{n\to\infty} \mathcal{L}_-[f_n]$. Combined with the original inequality gives the desired result.

8. Let $B_n = \overline{B}_n(0)$. Note that $f\mathbb{1}_{B_n} \le f\mathbb{1}_{B_{n+1}} \le \ldots \le f$ for all n, thus the limit exists and $\lim_{n\to\infty} \mathcal{L}_-[f\mathbb{1}_{B_n}] \le \mathcal{L}_-[f]$. Let h be simple and $h \le f$. Then h can be written as

$$h = \sum_k c_k E_k.$$

   By continuity of the Lebesgue measure, it follows that $m(E_k) = \lim_{n\to\infty} E_k \cap B_n$, therefore

$$\lim_{n\to\infty} I_{\mathcal{S}}(h\mathbb{1}_{B_n}) = \lim_{n\to\infty} \sum_k c_k m(E_k \cap B_n) = \sum_k c_k m(E_k) = I_{\mathcal{S}}(h).$$

   Since $h\mathbb{1}_{B_n} \le f\mathbb{1}_{B_n}$, it follows in the limit that $I_{\mathcal{S}}(h) = \lim_{n\to\infty} \mathcal{L}_-[f\mathbb{1}_{B_n}]$. Taking the supremum over all h gives $\mathcal{L}_-[f] \le \lim_{n\to\infty} \mathcal{L}_-[f\mathbb{1}_{B_n}]$. This combined with the first inequality gives the desired result.

◇

Exercise 3.32

Proof. Let $g \in \mathcal{S}^+$ with the representation $g = \sum_k c_k E_k$.

1. Take $y_0 \in \mathbb{R}^d$. Note that

$$\mathbb{1}_{E_k}(x + y_0) = \mathbb{1}_{E_k - y_0}(x)$$

meaning $g(\cdot + y_0) = \sum_k c_k \mathbb{1}_{E_k - y_0}(x)$. Therefore

$$\begin{aligned} I_{\mathcal{S}}(g(\cdot + y_0)) &= \sum_k c_k m(E_k - y_0) \\ &= \sum_k c_k m(E_k) \\ &= I_{\mathcal{S}}(g) \end{aligned}$$

Since the integral of simple functions is translation invariant, so is the Lebesgue integral.

2. Since T is invertible and $Tx \in E_k$ iff $x \in T^{-1}(E_k)$, then

$$\mathbb{1}_{E_k}(Tx) = \mathbb{1}_{T^{-1}(E_k)}(x).$$

Therefore

$$\begin{aligned} I_{\mathcal{S}}(g \circ T) &= \sum_k c_k m(T^{-1}(E_k)) \\ &= \frac{1}{|\det T|} \cdot \sum_k c_k m(E_k) \\ &= \frac{1}{|\det T|} \cdot I_{\mathcal{S}}(g) \end{aligned}$$

Since T is a bijection and linear transformations are measurable, then

$$
\begin{aligned}
\mathcal{L}_-(f \circ T) &= \sup_{f\circ T \geq g \in \mathcal{S}^+} I_{\mathcal{S}}(g) \\
&= \sup_{f \geq g \in \mathcal{S}^+} I_{\mathcal{S}}(g \circ T) \\
&= \frac{1}{|\det T|} \cdot \sup_{f \geq g \in \mathcal{S}^+} \\
&= \frac{1}{|\det T|} \mathcal{L}_-(f)
\end{aligned}
$$

3. Let $P = [x_1, \ldots, x_n]$ be a partition of $[a, b]$. Then let

$$M_i = \sup_{x \in [x_i, x_{i+1}]} f(x).$$

Note that for any $g \in \mathcal{S}^+$ where $g \leq f$ that

$$x \in [x_i, x_{i+1}] \implies g(x) \leq M_i.$$

Thus we have

$$\int g(x)\mathrm{d}x \leq \sum_{i=1}^{n-1} M_i \cdot (x_{i+1} - x_i).$$

But the RHS is just the upper Darboux integral, so taking the infimum of both sides over all partitions of $[a, b]$ gives

$$\int_{\mathbb{R}} g(x)\mathrm{d}x \leq \int_a^b f(x)\mathrm{d}x$$

establishing the Riemann integral as an upper bound on the Lebesgue integral. Note for any partition that the lower Darboux integral gives a simple function. That is, if

$$m_i = \inf_{x \in [x_i, x_{i+1}]} f(x)$$

we get a simple function

$$f_P(x) = \sum_{i=1}^{n-1} m_i \mathbb{1}_{[x_i, x_{i+1}]}$$

Clearly then since f is Riemann integrable, the supremum of this over all partitions gives the Riemann integral of f. This can be rephrased as a supremum over all simple functions with this lower Darboux form, which combined with the Riemann integral being an upper bound gives us the desired result that

$$\int_{\mathbb{R}} f(x)\mathrm{d}x = \mathcal{R} - \int_a^b f(x)\mathrm{d}x.$$

$\diamond$

Exercise 3.34

Proof.

1. Suppose that $\int f(x)\mathrm{d}x < \infty$. Let $C = \int f(x)\mathrm{d}x$. Then for any $\lambda > 0$ we have from Markov's inequality

$$\lambda m([f \geq \lambda]) \leq C \implies m([f \geq \lambda]) \leq \frac{C}{\lambda}.$$

Note that $[f = \infty] = \bigcap_{n \geq 0}[f \geq n]$, and since $m([f \geq 0]) < \infty$, by continuity from above

$$m([f = \infty]) = \lim_{n \to \infty} \frac{C}{n} = 0.$$

Therefore f is finite almost everywhere. For a counterexample to the converse, consider

$$f(x) = \begin{cases} 0 & x \leq 0 \\ \frac{1}{x} & x > 0 \end{cases}.$$

Clearly f is finite everywhere, but the integral is infinite.

2. Suppose that f is zero almost everywhere. Then f is equal a.e. to the zero function which is simple, thus

$$\int f(x)\mathrm{d}x = I_{\mathcal{S}}(0) = 0.$$

Suppose then $\int f(x)\mathrm{d}x = 0$. Then for any $\lambda > 0$ we have from Markov's inequality

$$\lambda m([f \geq \lambda]) \leq 0 \implies m([f \geq \lambda]) = 0.$$

Since $[f > 0] = \bigcup_{n\in\mathbb{N}}[f \geq \frac{1}{n}]$, by subadditivity we have $m([f > 0]) = 0$. Therefore f is zero almost everywhere.

◇

Exercise 3.38

Proof. Since f and g are absolutely integrable, then

$$f = f_u + if_v \qquad g = g_u + ig_v$$

where f_u, f_v, g_u, g_v are real valued and absolutely integrable. Note then that by linearity of absolutely integrable real valued functions

$$\begin{aligned}
\int (f + cg) &= \int \mathrm{Re}[f + cg]\mathrm{d}x + i\int \mathrm{Im}[f + cg]\mathrm{d}x \\
&= \int (f_u + \mathrm{Re}[c]g_u)\mathrm{d}x + i\int (f_v + \mathrm{Im}[c]g_v)\mathrm{d}x \\
&= \left(\int f_u\mathrm{d}x + i\int f_v\mathrm{d}x\right) + \left(\mathrm{Re}[c]\int g_u\mathrm{d}x + i\,\mathrm{Im}[c]\int g_v\mathrm{d}x\right) \\
&= \int f\mathrm{d}x + c\int g\mathrm{d}x
\end{aligned}$$

Note as well that

$$\begin{aligned}
\overline{\int \overline{f}\mathrm{d}x} &= \overline{\int \mathrm{Re}\left[\overline{f}\right]\mathrm{d}x + i\int \mathrm{Im}[\overline{f}]\mathrm{d}x} \\
&= \overline{\int \mathrm{Re}\left[\overline{f}\right]\mathrm{d}x} - i\overline{\int \mathrm{Im}[\overline{f}]\mathrm{d}x} \\
&= \int f_u\mathrm{d}x - i\int (-f_v)\mathrm{d}x \\
&= \int f_u\mathrm{d}x + i\int f_v\mathrm{d}x \\
&= \int f\mathrm{d}x
\end{aligned}$$

which was to be shown. ◇

Exercise 3.50

Proof. Suppose that f is the pointwise a.e. limit of a sequence of continuous complex valued functions f_n. Each f_n is measurable since they are continuous. Therefore f is measurable since it is the pointwise a.e. limit of measurable functions.

Suppose then that f is measurable. Take $\varepsilon > 0$ and $n \in \mathbb{N}$ and consider $f\mathbb{1}_{B_n(0)}$. Since $f\mathbb{1}_{B_n(0)}$ has finite support, there is a set E with $m(E) \leq \varepsilon$ such that $f\mathbb{1}_{B_n(0)\setminus E}$ is bounded. Note then that $f\mathbb{1}_{B_n(0)\setminus E}$ is then $\mathcal{L}^1$, hence there is a continuous, compactly supported g_n such that $\|f - g_n\|_{L^1} \leq \varepsilon$. Let

$$A_n = \left\{ x \in B_n(0) : |f(x) - g_n(x)| \geq \frac{1}{n} \right\}.$$

Note then that

$$m(A_n) \leq m(A_n \setminus E) + m(E) \leq n \cdot \|f - g_n\|_{L^1} + m(E) = \varepsilon(n+1)$$

Therefore if $\varepsilon \leq \frac{1}{(n+1)\cdot 2^n}$ we have

$$m(A_n) \leq \frac{1}{2^n}.$$

Continuing this process for all $n \in \mathbb{N}$ gives a sequence of continuous function g_n. Consider the set of points where g_n does not converge to f

$$D = \left\{ x \in \mathbb{R}^d : \lim_{n\to\infty} g_n(x) \neq f(x) \right\}.$$

If $x \in D$, then there must exist some N and $\varepsilon_0 > 0$ such that $|f(x) - g_n(x)| > \varepsilon_0$ for all $n \geq N$. Taking N larger to where $\frac{1}{N} < \varepsilon_0$, we then have $x \in A_n$ for all $n \geq N$. Thus

$$D \subset \bigcup_{n\in\mathbb{N}} \bigcap_{k\geq n} A_k.$$

Note that $\bigcap_{k\geq n} A_k \subset A_n$ thus

$$m\left(\bigcap_{k\geq n} A_k \right) \leq m(A_n) \leq \frac{1}{2^n}.$$

By continuity from below, it then follows

$$m\left(\bigcup_{n\in\mathbb{N}} \bigcap_{k\geq n} A_k \right) = \lim_{n\to\infty} m(\bigcap_{k\geq n} A_k) \leq \lim_{n\to\infty} \frac{1}{2^n} = 0.$$

Thus $m(F) = 0$, meaning g_n converges to f pointwise almost everywhere. $\diamond$

Exercise 4.9

Proof.

1. Clearly if such a bijection exists the result holds. Suppose then that

$$\mathcal{B}((B_\alpha)_{\alpha\in I}) = \mathcal{B}((B'_\alpha)_{\alpha\in I'}).$$

   Since they are equal, it follows that for any B_α there is some $J \subset I'$ such that

$$B_\alpha = \bigcup_{\alpha'\in J} B'_{\alpha'}.$$

   By the same argument it follows that there exists for each $B'_{\alpha'}$ some $I_{\alpha'} \subset I$ such that

$$B'_{\alpha'} = \bigcup_{\alpha\in I_{\alpha'}} B_\alpha.$$

   Therefore

$$B_\alpha = \bigcup_{\alpha'\in J} \bigcup_{\beta\in I_{\alpha'}} B_\beta.$$

   If either $|J| > 1$ or $|I_{\alpha'}| > 1$, then B_α could not be an atom as it would be the union of at least two other atoms. Thus $|J| = 1$ and $|I_{\alpha'}| = 1$, meaning $B_\alpha = B'_{\alpha'}$ for some $\alpha' \in I'$. Define then $\phi(\alpha) = \alpha'$. This same argument applies to all B_α as well as all $B'_{\alpha'}$, hence $\phi : I \to I'$ is a bijection.

2. Since $\mathcal{B}$ is atomic and finite, there a finite number of atoms $A_1, \ldots, A_n$ that generate $\mathcal{B}$. Since $B \in \mathcal{B}$ can be expressed as the union of some subset of these atoms, and it is a binary choice to either include an atom or not in the union, there are 2^n possible B. Hence $|\mathcal{B}| = 2^n$.

   Note then that every finite Boolean algebra over X can be described as the atomic algebra generated by some finite partition of X. Any two partitions that are the same under relabelling will still preserve equality of the corresponding Boolean algebras under relabelling by part (i).

3. In the case of either the elementary or Jordan algebra, the non-empty atoms would have to be non-trivial intervals. However any non-trivial interval I can be split in half into two non-trivial intervals I_1 and I_2. Clearly I_1 and I_2 themselves are elementary and Jordan measurable, but cannot

be in generated algebra since I is an atom. Therefore neither the elementary or Jordan algebra's are atomic.

Suppose that the Lebesgue algebra (which is just the collection of all Lebesgue measurable sets) was atomic. Note that there must be an atom A such that $|A| \geq 2$. If not, then each atom would be a singleton and thus generate the discrete algebra, which does not equal the Lebesgue algebra. Note that A must be Lebesgue measurable, and since $\{x\}$ is also Lebesgue measurable, $A \setminus \{x\}$ is also Lebesgue measurable. But since $A \setminus \{x\} \subset A$, then A cannot be an atom, a contradiction. Therefore the Lebesgue algebra is not atomic

The same argument above holds for the null algebra since the Lebesgue measure of a singleton is 0. Hence the null algebra is not atomic.

4. Since $\emptyset$ and X must be contained all $\mathcal{B}_\alpha$ by definition, then $\emptyset, X \in \mathcal{B}$. Take $A, B \in \mathcal{B}$.

   - Since $A \in \mathcal{B}$, then $A \in \mathcal{B}_\alpha$ for all α, meaning $X \setminus A \in \mathcal{B}_\alpha$ for all α. Thus $X \setminus B \in \mathcal{B}$.
   - Since $A, B \in \mathcal{B}$, then $A, B \in \mathcal{B}_\alpha$ for all α, meaning $A \cup B \in \mathcal{B}_\alpha$ for all α. Thus $A \cup B \in \mathcal{B}$

   Therefore $\mathcal{B}$ is a boolean algebra. Since $\mathcal{B} \subset B_\alpha$ for all α, it follows that $\mathcal{B}$ is coarser than all B_α. Note that if C is a boolean algebra contained in all B_α, then any $A \in C$ is contained in all B_α. Hence $A \in \mathcal{B}$, meaning $C \subset \mathcal{B}$. Thus $\mathcal{B}$ is the finest Boolean algebra coarser than all $\mathcal{B}_\alpha$.

$\diamond$

Exercise 4.12

Proof.

1. Let $\mathcal{F}$ be the collection of all boxes in $\mathbb{R}^d$. Since the elementary algebra by definition contains all boxes since boxes are elementary sets, it follows that $b(\mathcal{F}) \subset \tilde{\mathcal{E}}$. Note that for any Boolean algebra $\mathcal{B}$ containing $\mathcal{F}$, that the collection of all finite unions of boxes and the complement of those

sets must be contained in it. But that is exactly the definition of $\tilde{\mathcal{E}}$, hence $\tilde{\mathcal{E}} \subset \mathcal{B}$. Therefore $\tilde{\mathcal{E}} \subset b(\mathcal{F})$, meaning $\tilde{\mathcal{E}} = b(\mathcal{F})$.

2. Let $\mathcal{F} = \{F_1, \dots, F_n\}$. For all n long binary string $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$, let

$$A_\varepsilon = \bigcap_{k=1}^{n} B_k, \qquad B_k = \begin{cases} F_k & \varepsilon_k = 1 \\ F_k^c & \varepsilon_k = 0 \end{cases}.$$

Note that the collection of all A_ε for every binary string partitions X.

- If $A_\varepsilon \neq A_{\varepsilon'}$, then $\varepsilon \neq \varepsilon'$ meaning some $\varepsilon_k \neq \varepsilon'_k$. Therefore $B_k \cap B'_k = \varnothing$ meaning $A_\varepsilon \cap A_{\varepsilon'} = \varnothing$.
- Let $x \in X$ and define $\varepsilon_k = 1$ if $x \in F_k$ and 0 otherwise. Then clearly $x \in A_\varepsilon$, meaning
$$X \subset \bigcup_{\varepsilon \in \{0,1\}^n} A_\varepsilon.$$
But clearly the reverse inclusion holds, hence equality holds.

Therefore we can construct the atomic algebra $\mathcal{A}$ from the atoms A_ε. Since we have 2^n atoms (since there are 2^n binary strings of length n), from the previous problem we have $|\mathcal{A}| \leq 2^{2^n}$. But clearly $b(\mathcal{F}) \subset \mathcal{A}$, hence $|b(\mathcal{F})| \leq 2^{2^n}$.

Consider $X = \{0,1\}^N$ and let $F_k = \{x \in X : x_k = 1\}$. Note that by the same construction above, $A_x = \{x\}$ for every $x \in \{0,1\}^N$. Since every Boolean algebra containing $\mathcal{F} = \{F_1, \dots, F_N\}$ must be closed under finite union, intersection, and complement it follows that every Boolean algebra contains A_x for all $x \in X$. But that means every singleton of X is contained in every Boolean algebra. Therefore every Boolean algebra must equal $\mathcal{P}(X)$, hence $|b(\mathcal{F})| = |\mathcal{F}| = 2^{|X|} = 2^{2^N}$.

3. Let $G = \bigcup_{n=0}^{\infty} \mathcal{F}_n$. Let $E, F \in G$.

- If $\mathcal{F} \neq \varnothing$, then take $A \in \mathcal{F}$. Note then that $A \cup A^c \in \mathcal{F}_\infty$ and thus $(A \cup A^c)^c \in \mathcal{F}_\in \subset G$. But $(A \cup A^c)^c = \varnothing$. Thus $\varnothing \in G$.
- Since E, F are in G, then WLOG $E \in \mathcal{F}_n$ and $F \in \mathcal{F}_m$ for some $n > m \geq 0$. Note that $\mathcal{F}_m \subset \mathcal{F}_n$. Therefore $F \in \mathcal{F}_n$, meaning $E \cup F \in \mathcal{F}_{n+1} \subset G$. Hence G is closed under union.

- Since E is in G, then $E \in \mathcal{F}_n$ for some $n \geq 0$. Thus by construction, $E^c \in \mathcal{F}_{n+1} \subset G$. Hence G is closed under complement.

Therefore G is a boolean algebra containing $\mathcal{F}$, hence $b(\mathcal{F}) \subset G$. Note that $\mathcal{F}_0 \subset b(\mathcal{F})$ and $\mathcal{F}_n \subset b(\mathcal{F}) \implies \mathcal{F}_{n+1} \subset b(\mathcal{F})$ since $b(\mathcal{F})$ is closed under finite intersection, union, and complement. Therefore by induction $\mathcal{F}_n \subset b(\mathcal{F})$ for all n and thus $G \subset b(\mathcal{F})$. Hence $G = b(\mathcal{F})$.

$\diamond$

Exercise 4.14

Proof.

1. Since atomic algebras are Boolean algebras, the first two properties follow by definition. Let $\mathcal{A}((A_\alpha)_{\alpha \in I})$ be an atomic algebra and $(E_n)_{n\in\mathbb{N}} \in \mathcal{A}^{\mathbb{N}}$. Note that for each E_n there is some $I_n \subset I$ such that $E_n = \bigcup_{\alpha \in I_n} A_\alpha$. Therefore if $I_E = \bigcup_{n\in\mathbb{N}} I_n$ then
$$\bigcup_{n\in\mathbb{N}} E_n = \bigcup_{\alpha \in I_E} A_\alpha.$$
Since $I_E \subset I$, then $\bigcup_{n\in\mathbb{N}} E_n \in \mathcal{A}$. Hence $\mathcal{A}$ is a σ-algebra.

   Since the trival, discrete, and dyadic algebras are atomic, then they are σ-algebras. Furthermore every finite algebra is also a σ-algebra since there are only finitely many sets in the algebra, hence any countable union of sets in the algebra can be reduced to a finite union which is closed.

2. Clearly $\mathcal{L}$ is a σ-algebra by its closure properties. Consider then $\mathcal{N}$. Since $\mathcal{N}$ is a Boolean algebra, all that is left to show is that it is closed under countable union. Let $(E_n)_{n\in\mathbb{N}} \subset \mathcal{N}^{\mathcal{N}}$. Consider then two cases

   - If there is at least one E_k such that $m(E_k^c) = 0$, then
$$\left(\bigcup_{n\in\mathbb{N}} E_n\right)^c = \bigcap_{n\in\mathbb{N}} E_n^c \subset E_k^c.$$
Therefore
$$m\left(\left(\bigcup_{n\in\mathbb{N}} E_n\right)^c\right) = 0 \implies \bigcup_{n\in\mathbb{N}} E_n \in \mathcal{N}.$$

- If $m(E_n) = 0$ for all n, then

$$m\left(\bigcup_{n\in\mathbb{N}} E_n\right) \leq \sum_{n\in\mathbb{N}} m(E_n) = 0 \implies \bigcup_{n\in\mathbb{N}} E_n \in \mathcal{N}.$$

In either case, we have closure under countable union, hence $\mathcal{N}$ is a σ-algebra.

The elementary and Jordan algebra's are not σ-algebras, since for any $r \in \mathbb{Q}$, $\{r\}$ is in both, but $\mathbb{Q}$ is not measurable in either and $\mathbb{Q} = \bigcup_{r\in\mathbb{Q}} \{r\}$ which is a countable union.

3. Consider the requirements of a σ-algebra:

   - Clearly $\varnothing \in \mathcal{A}_Y$ since $\varnothing \in \mathcal{A}$ and $Y \cap \varnothing = \varnothing$.
   - Let $E \cap Y \in \mathcal{A}_Y$ with $E \in \mathcal{A}$. Note then the complement of $E \cap Y$ in Y is

   $$Y \setminus (E \cap Y) = Y \cap (E^C \cup Y) = (Y \cap Y^c) \cup (E^c \cap Y) = E^c \cap Y.$$

   Since $E^c \in \mathcal{A}$, it follows the complement of $E^c \cap Y$ is Y is in $\mathcal{A}_Y$.
   - Let $(E_n \cap Y)_{n\in\mathbb{N}} \in \mathcal{A}_Y^{\mathbb{N}}$. Note that

   $$\bigcup_{n\in\mathbb{N}} (E_n \cap Y) = Y \cap \bigcup_{n\in\mathbb{N}} E_n.$$

   Since $\bigcup_{n\in\mathbb{N}} E_n \in \mathcal{A}$, it follows $\mathcal{A}_Y$ is closed under countable union.

   Therefore $\mathcal{A}_Y$ is a σ-algebra on Y.

4. Consider the requirements of a σ-algebra:

   - $\varnothing \in \mathcal{A}_\alpha$ for every α, thus $\varnothing \in \mathcal{A}$.
   - If $E \in \mathcal{A}$, then $E \in \mathcal{A}_\alpha$ for every α. Thus $E^c \in \mathcal{A}_\alpha$ for every α meaning $E^c \in \mathcal{A}$.
   - If $(E_n)_{n\in\mathbb{N}} \in \mathcal{A}^{\mathbb{N}}$, then for every n we have $E_n \in \mathcal{A}_\alpha$ for every α. Therefore

   $$\bigcup_{n\in\mathbb{N}} E_n \in \mathcal{A}_\alpha, \forall \alpha.$$

   Thus $\bigcup_{n\in\mathbb{N}} E_n \in \mathcal{A}$.

Therefore $\mathscr{A}$ is a σ-algebra.

$\diamond$

Exercise 4.20

Proof.

1. Consider first the case where E_1 is a box in $\mathbb{R}^{d_1}$. Define the property $P(B)$ where $B \subset \mathbb{R}^{d_2}$ by

$$P(B) \Leftrightarrow E_1 \times B \in \mathcal{B}(\mathbb{R}^{d_1+d_2}).$$

Let $\mathcal{F}$ be the family of all boxes in $\mathbb{R}^{d_2}$. Note then that

- $P(\varnothing)$ holds since $E_1 \times \varnothing = \varnothing$ is Borel
- $P(B)$ holds for every $B \in \mathcal{F}$ since the product of boxes is also a box, and all boxes are Borel.
- Suppose $P(B)$ holds. Then $E_1 \times B$ is Borel, hence $(E_1 \times B)^c$ is Borel. Note then that

$$E_1 \times B^c = (E_1 \times \mathbb{R}^{d_2}) \setminus (E_1 \times B) = (E_1 \times \mathbb{R}^{d_2}) \cap (E_1 \times B)^c.$$

  Since $E_1 \times \mathbb{R}^{d_2}$ is Borel, it follows that $E_1 \times B^c$ is Borel and hence $P(B^c)$ holds.
- Suppose $P(B_n)$ holds for every n. Then $E_1 \times B_n$ is Borel for every n, and

$$E_1 \times \bigcup_{n \in \mathbb{N}} B_n = \bigcup_{n \in \mathbb{N}} (E_1 \times B_n).$$

  Therefore $E_1 \times \bigcup_{n \in \mathbb{N}} B_n$ is a countable union of Borel sets, and thus Borel. Hence $P(\bigcup_{n \in \mathbb{N}} B_n)$ holds.

Therefore $P(B)$ holds for all $B \in \sigma(\mathcal{F}) = \mathcal{B}(\mathbb{R}^{d_2})$. If then E_2 is fixed to be Borel, the same argument applies over the family of boxes in $\mathbb{R}^{d_1}$, hence $E_1 \times E_2$ is Borel for all E_1 Borel and E_2 fixed. But this establishes the desired result.

2. First we prove that the preimage of Borel sets under a continuous function are Borel. Let $f : X \to Y$ be continuous and $\mathcal{B}_X$ and $\mathcal{B}_Y$ the Borel algebras of X and Y respectively. Let $\mathcal{F}$ be the family of all open sets in Y. Define the property $P(E)$ for $E \subset Y$ by

$$P(E) \Leftrightarrow f^{-1}(E) \in \mathcal{B}_X.$$

Note that

- $P(\varnothing)$ holds since $f^{-1}(\varnothing) = \varnothing \in \mathcal{B}_X$.
- If $U \subset Y$ is open, then $f^{-1}(U)$ is open in X and hence Borel. Therefore $P(U)$ holds for all $U \in \mathcal{F}$.
- Suppose that $P(E)$ holds. Then $f^{-1}(E)$ is Borel meaning $\left(f^{-1}(E)\right)^c$ is Borel. But $\left(f^{-1}(E)\right)^c = f^{-1}(E^c)$, thus $P(E^c)$ holds.
- Suppose $P(E_n)$ holds for all n. Then $f^{-1}(E_n)$ is Borel for all n, thus

$$f^{-1}\left(\bigcup_{n\in\mathbb{N}} E_n\right) = \bigcup_{n\in\mathbb{N}} f^{-1}(E_n)$$

  is a countable union of Borel sets, hence it is Borel. Therefore $P(\bigcup_{n\in\mathbb{N}} E_n)$ holds.

Therefore it follows $P(E)$ holds for all $E \in \sigma(\mathcal{F}) = \mathcal{B}_Y$. Hence the preimage of Borel sets under a continuous map are Borel.

Now take $E \in \mathcal{B}(\mathbb{R}^{d_1+d_2})$ and fix $x_1 \in \mathbb{R}^{d_1}$. Let

$$f_{x_1} : \mathbb{R}^{d_2} \to \mathbb{R}^{d_1+d_2} : y \mapsto (x_1, y).$$

Note that $E_{x_1} = f_{x_1}^{-1}(E)$ and that f is continuous. Thus from the previous lemma, it follows E_{x_1} is Borel. The same argument with the coordinates reversed establishes the desired result.

This result does not hold if Borel is substituted for Lebesgue. Let V be the Vital set and $x \in \mathbb{R}$. Then clearly $E = V \times \{x\}$ is a null set in $\mathbb{R}^2$, but $E_x = V$ is not Lebesgue measurable in $\mathbb{R}$.

3. Since every borel and null set are Lebesgue measurable, it follows $\mathcal{B} \cup \mathcal{N} \subset \mathcal{L}$, and thus $\sigma(\mathcal{B} \cup \mathcal{N}) \subset \sigma(\mathcal{L})$. Consider then some Lebesgue

measurable set $E \in \mathcal{L}$. Then there exists a F_σ set B and null set N such that $E = B \cup N \in \mathcal{B} \cup \mathcal{N}$. Therefore $\sigma(\mathcal{L}) = \mathcal{L} \subset \sigma(\mathcal{B} \cup \mathcal{N})$. Hence $\sigma(\mathcal{L}) = \sigma(\mathcal{B} \cup \mathcal{N})$.

$\diamond$

Exercise 4.31

Proof.

1. Let $(F_n)_{n\in\mathbb{N}} \in \mathcal{A}^{\mathbb{N}}$ where
$$F_1 = E_1, \qquad F_{n+1} = E_{n+1} \setminus \left(\bigcup_{k=1}^{n} E_n \right).$$
Note that each F_n are disjoint, $E_n \subset F_n$, and that
$$\bigcup_{n\in\mathbb{N}} F_n = \bigcup_{n\in\mathbb{N}} E_n.$$
Therefore
$$\mu\left(\bigcup_{n\in\mathbb{N}} E_n \right) = \mu\left(\bigcup_{n\in\mathbb{N}} F_n \right) = \sum_{n\in\mathbb{N}} \mu(F_n) \leq \sum_{n\in\mathbb{N}} \mu(E_n)$$
which was to be shown.

2. If $\mu(E_n) = \infty$ for some $n \in \mathbb{N}$, then clearly the result holds by montonocity. Suppose then that $\mu(E_n) < \infty$ for all n. Let $F_{n+1} = E_{n+1} \setminus E_n$ with $F_1 = E_1$. Note that F_n is a disjoint sequence, and that $\bigcup_{n\in\mathbb{N}} F_n = \bigcup_{n\in\mathbb{N}} E_n$. Therefore
$$\mu\left(\bigcup_{n\in\mathbb{N}} E_n \right) = \mu\left(\bigcup_{n\in\mathbb{N}} F_n \right) = \sum_{n\in\mathbb{N}} \mu(F_n).$$
Note that the partial sums of the right hand side for some N are
$$\begin{aligned}\sum_{n=1}^{N} \mu(F_n) &= \mu(E_1) + \mu(E_2 \setminus E_1) + \ldots + \mu(E_{N-1} \setminus E_{N-2}) + \mu(E_N \setminus E_{N-1}) \\ &= \mu(E_1) - \mu(E_1) + \mu(E_2) - \ldots - \mu(E_{N-2}) + \mu(E_{N-1}) - \mu(E_{N-1}) + \mu(E_N) \\ &= \mu(E_N)\end{aligned}$$

Therefore $\sum_{n\in\mathbb{N}} \mu(F_n) = \lim_{n\to\infty} \mu(E_n)$. Note also by monotonicty that $\mu(E_n) \leq \mu(E_{n+1})$, therefore $\lim_{n\to\infty} \mu(E_n) = \sup_{n\in\mathbb{N}} \mu(E_n)$. In total then

$$\mu\left(\bigcup_{n\in\mathbb{N}} E_n\right) = \lim_{n\to\infty} \mu(E_n) = \sup_{n\in\mathbb{N}} \mu(E_n).$$

3. Let E_k be the set such that $\mu(E_k) < \infty$ and $F = \bigcap_{n\in\mathbb{N}} E_n$. Note then that

$$E_k = F \dot\cup \left(\bigcup_{n=k}^{\infty} E_n \setminus E_{n+1}\right) \tag{$\star_1$}$$

since $E_n \setminus E_{n+1}$ doesn't contain E_{n+1} and $F \subset E_{n+1}$. Each $E_n \setminus E_{n+1}$ is disjoint since the sequence of E_n is non-increasing, therefore

$$\mu(E_k) = \mu(F) + \mu\left(\bigcup_{n=k}^{\infty} E_n \setminus E_{n+1}\right) = \mu(F) + \sum_{n=k}^{\infty} \mu(E_n \setminus E_{n+1})$$

By montonicity $\mu(F) \leq \mu(E_k) < \infty$, thus

$$\sum_{n=k}^{\infty} \mu(E_n \setminus E_{n+1}) = \mu(E_k) - \mu(F) < \infty. \tag{$\star_2$}$$

Note that $(\star_1)$ holds for any $n \geq k$, therefore if $n \geq k$

$$\mu(E_n) - \mu(F) = \sum_{j=n}^{\infty} \mu(E_n \setminus E_{n+1}).$$

The RHS is the tail of the sum in $(\star_2)$ which a convergent series. Therefore in the limit $n \longrightarrow \infty$, it goes to 0. Thus

$$\lim_{n\to\infty} \mu(E_n) - \mu(F) = 0 \implies \lim_{n\to\infty} \mu(E_n) = \mu(F) = \mu\left(\bigcap_{n\in\mathbb{N}} E_n\right).$$

Since $\mu(E_n) \geq \mu(E_{n+1})$ by monotonicity, $\lim_{n\to\infty} \mu(E_n) = \inf_{n\in\mathbb{N}} \mu(E_n)$. In total then

$$\mu\left(\bigcap_{n\in\mathbb{N}} E_n\right) = \lim_{n\to\infty} \mu(E_n) = \inf_{n\in\mathbb{N}} \mu(E_n).$$

In (iii), suppose the condition for one of the sets being finite is dropped. Consider then the non-increasing sequence $E_n = [n, \infty)$. Then $m(E_n) = \infty$ for all n, meaning

$$\lim_{n\to\infty} m(E_n) = \infty.$$

But then

$$m\left(\bigcap_{n=1}^{\infty} E_n\right) = m(\varnothing) = 0.$$

Hence the condition for eventual finiteness is needed. ◇

Exercise 4.31

Proof.

1. Take $x \in E$. From pointwise convergence it follows $\exists N \in \mathbb{N}$ such that $|\mathbb{1}_{E_n}(x) - \mathbb{1}_E(x)| < \frac{1}{2}$ for every $n \geq N$. But this means that $\mathbb{1}_{E_n}(x) = \mathbb{1}_E(x)$ since any indicator function only takes on the values $\{0, 1\}$. Therefore $x \in E_n$ for every $n \geq N$. Thus

   $$E \subset \bigcup_{N\in\mathbb{N}} \bigcap_{n\geq N} E_n \liminf_{n\to\infty} E_n =: L.$$

   Now take $x \in L$. Then there is some $N \in \mathbb{N}$ such that $x \in E_n$ for all $n \geq N$. Thus $\mathbb{1}_{E_n}(x) = 1$ for every $n \geq N$. It follows then $\mathbb{1}_E = \lim_{n\to\infty} \mathbb{1}_{E_n}(x) = 1$, meaning $x \in E$. Therefore we have the other subset inclusion giving $E = L$.

   Since every E_n is measurable, the countable intersection of them is also measurable. This collection of countable intersections are all measurable, thus the countable union of them is also measurable. Hence L is measurable, which means E is measurable.

2. By the same observations as above, if $x \notin E$, then $1_{E_n}(x) = 0$ for sufficiently large n, and hence $x \notin E_n$. Therefore $x \in E_n$ for only finitely many n and thus $x \notin \limsup_n E_n$. Hence $\limsup_n E_n \subset E$ meaning

   $$E = \liminf_{n\to\infty} E_n \subset \limsup_{n\to\infty} E_n \subset E \implies E = \limsup_{n\to\infty} E_n = \bigcup_{n\in N} \bigcap_{k\geq n} E_n.$$

Take then $C_n = \bigcap_{k \geq n} E_n$ and $D_n = \bigcup_{k \geq n}$. Note that we have

$$C_n \subset E_n \subset D_n \subset F.$$

Since $\mu(F) < \infty$, continuity from above and below can be applied to get

$$\mu(E) = \lim_{n\to\infty} \mu(C_n) \qquad \mu(E) = \lim_{n\to\infty} \mu(D_n).$$

But note that by montonicty $\mu(C_n) \leq \mu(E_n)$ and $\mu(D_n) \geq \mu(E_n)$, thus

$$\lim_{n\to\infty} \mu(C_n) \leq \liminf_{n\to\infty} \mu(E_n) \qquad \lim_{n\to\infty} \mu(D_n) \geq \limsup_{n\to\infty} \mu(E_n).$$

Therefore

$$\limsup_{n\to\infty} \mu(E_n) \leq \mu(E) \leq \liminf_{n\to\infty} \mu(E_n)$$

which gives $\lim_{n\to\infty} \mu(E_n) = \mu(E)$

3. Consider the Lebesgue algebra and measure. Let $E_n = n + [0,1]$. Note that $m(E_n) = 1$ for all n by translation invariance. However, for any $x \in \mathbb{R}$, there exists some $N \in \mathbb{N}$ such that $x < n$ for all $n > N$. Thus $x \notin E_n$ for $n > N$, and hence $\mathbb{1}_{E_n}(x) \neq 1$ for $n > N$. This means that $\lim_{n\to\infty} \mathbb{1}_{E_n} = \mathbb{1}_{\varnothing}$, and since $m(\varnothing) = 0 \neq 1$, (b) does not hold.

◇

Exercise 4.35

Proof. Define the following

$$\begin{aligned}
\mathcal{N} &= \{N \in \mathcal{A} : \mu(N) = 0\} \\
\mathcal{N}^* &= \left\{M \in 2^X : M \subset N \in \mathcal{N}\right\} \\
\overline{\mathcal{A}} &= \{A \cup M : A \in \mathcal{A}, M \in \mathcal{N}^*\}
\end{aligned}$$

Clearly $\varnothing$ and X are in $\overline{\mathcal{A}}$ since M can be taken as $\varnothing$ in both cases. Let then $E_n = A_n \cup M_n \in \overline{\mathcal{A}}$ with $M_n \subset N_n \in \mathcal{N}$. Note that

$$\bigcup_{n\in\mathbb{N}} E_n = \left(\bigcup_{n\in\mathbb{N}} A_n\right) \cup \left(\bigcup_{n\in\mathbb{N}} M_n\right).$$

Since $\mathcal{A}$ is a σ-algebra, $\bigcup A_n \in \mathcal{A}$. Furthermore $\mu(\bigcup N_n) = 0$ and since $\bigcup M_n \subset \bigcup N_n$, it follows $\bigcup E_n \in \overline{\mathcal{A}}$. Take now $E = A \cup M \in \overline{\mathcal{A}}$ and $M \subset N \in \mathcal{N}$. Note that

$$E^c = A^c \cap M^c = (A^c \cap (X \setminus N)) \cup (A^c \cap (N \setminus M)).$$

Since $A^c \in \mathcal{A}$ and $X \setminus N \in \mathcal{A}$, it follows $A^c \cap (X \setminus N) \in N$. Furthermore, $A^c \cap (N \setminus M) \subset N$ meaning $E^c \in \overline{\mathcal{A}}$. Hence $\overline{\mathcal{A}}$ is a σ-algebra.

Take $E = A \cup M \in \overline{\mathcal{A}}$ with and define $\overline{\mu}(E) := \mu(A)$. Clearly if $E \in \mathcal{A}$ ($M = \varnothing$) then $\overline{\mu}(E) = \mu(E)$. Suppose then that $A \cup M = B \cup M' \in \overline{\mathcal{A}}$ and $M \subset N \in \mathcal{N}$ and $M' \subset N' \in \mathcal{N}$. Note that

$$A \setminus B \subset (A \cup M) \setminus (B \cup M') \subset M \cup M'$$

which also holds for $B \setminus A$. Therefore $A \Delta B \subset M \cup M' \subset N \cup N'$. Since $\mu(N \cup N') = 0$, it follows $\mu(A \Delta B) = 0$ and thus $\mu(A) = \mu(B)$. Hence $\overline{\mu}$ is well-defined. Clearly $\overline{\mu}$ enjoys the measure properties of μ directly except for countable additivity. Let $E_n = A_n \cup M_n$ with $M_n \subset N_n \in \mathcal{N}$ pairwise disjoint. Note that

$$\bigcup_{n \in \mathbb{N}} E_n = \left(\bigcup_{n \in \mathbb{N}} A_n\right) \cup \left(\bigcup_{n \in \mathbb{N}} M_n\right).$$

Since E_n are pairwise disjoint, then so are the A_n meaning

$$\mu\left(\bigcup_{n \in \mathbb{N}} A_n\right) = \sum_{n \in \mathbb{N}} \mu(A_n).$$

Furthermore since $\bigcup M_n \subset \bigcup N_n$ and $\bigcup N_n$ is a null set,

$$\overline{\mu}\left(\bigcup_{n \in \mathbb{N}} E_n\right) = \mu\left(\bigcup_{n \in \mathbb{N}} A_n\right) = \sum_{n \in \mathbb{N}} \mu(A_n) = \sum_{n \in \mathbb{N}} \overline{\mu}(E_n).$$

Therefore $\overline{\mu}$ is a measure on $\overline{\mathcal{A}}$.

Completeness of $\overline{\mathcal{A}}$ under $\overline{\mu}$ follows directly by its construction and the fact that $\varnothing \in \mathcal{A}$, thus leaving the question of coarsity. Let $\mathcal{B} \supset \mathcal{A}$ be a complete σ-algebra with meausre ν that extends μ. Note that $\mathcal{N}^* \subset \mathcal{B}$ since for any $M \subset N \in \mathcal{N} \subset \mathcal{B}$, by completeness $\mu(N) = 0 \implies \mu(M) = 0$. Therefore $\mathcal{A} \cup \mathcal{N}^* \subset \mathcal{B}$, meaning it must contain $\sigma(A \cup \mathcal{N}^*)$ which is simply $\overline{\mathcal{A}}$. Thus $\overline{\mathcal{A}}$ is the coarset complete refinement of $\mathcal{A}$. $\diamond$

Exercise 4.41

Proof.

1. Let
$$\mathcal{C} = \{E \in \sigma(\mathcal{B}) : \forall \varepsilon > 0, \exists F \in \mathcal{B} \text{ s.t. } \mu(E\Delta F) < \varepsilon\}.$$

   Note that $\mathcal{B} \subset \mathcal{B}$ since $\mu(A\Delta A) = 0$ for $A \in \mathcal{B}$. We proceed to show $\mathcal{C}$ is a montone class.

   - Let $E_n \in \mathcal{C}^{\mathbb{N}}$ be non-decreasing and $E = \bigcup_{n\in\mathbb{N}} E_n$. Take $\varepsilon > 0$. For each n take $F_n \in \mathcal{B}$ such that
   $$\mu(E_n\Delta F_n) < \varepsilon \cdot 2^{-(n+1)}.$$
   Let $G_N = \bigcup_{k\leq N} F_k$ and note $G_N \in \mathcal{B}$ since it is a finite union. Note that we have
   $$E \setminus G_N \subset \left(\bigcup_{k\leq N} E_k \setminus G_N\right) \cup \left(\bigcup_{k>N} E_k\right) \subset \left(\bigcup_{k\leq N} E_k \setminus F_k\right) \cup \left(\bigcup_{k>N} E_k\right)$$
   and
   $$G_N \setminus E \subset \bigcup_{k\leq N} F_k \setminus E_k \subset \bigcup_{k\leq N} E_k\Delta F_k.$$
   Therefore
   $$\begin{aligned}\mu(E\Delta G_N) &= \mu((E \setminus G_N) \cup (G_N \setminus E))\\ &\leq \mu\left(\bigcup_{k>N} E_k\right) + \mu(\bigcup_{k\leq N} E_k\Delta F_k)\\ &\leq \mu(E \setminus E_N) + \sum_{k\leq N} \mu(E_k\Delta F_k)\end{aligned}$$
   By continuity from below, it follows $\mu(E \setminus E_N) \to 0$ as $N \to \infty$. Take N large enough such that $\mu(E \setminus E_N) \leq \frac{\varepsilon}{2}$. Note then by above
   $$\mu(E\Delta G_N) \leq \frac{\varepsilon}{2} + \sum_{k\leq N} \mu(E_k\Delta F_k) \leq \frac{\varepsilon}{2} + \sum_{k\in\mathbb{N}} \varepsilon \cdot 2^{-(n+1)} = \frac{e}{2} + \frac{\varepsilon}{2} = \varepsilon.$$
   Therefore $E \in \mathcal{C}$.

- Let $E_n \in C^{\mathbb{N}}$ be non-increasing and $E = \bigcap_{n\in\mathbb{N}} E_n$. Since $\mu(X) < \infty$, the sequence E_n^c and E^c match the above criteria, thus there is some $F \in \mathcal{B}$ such that $\mu(E^c \Delta F) < \varepsilon$. Note then that $\mu(E\Delta F^c) = \mu(E^c\Delta F) < \varepsilon$ which can be done again since $\mu(X) < \infty$. Thus $E \in C$ as $F^c \in \mathcal{B}$.

Therefore C is a monotone class that is generated by $\mathcal{B}$, thus $C = \sigma(\mathcal{B})$ meaning every set in $\sigma(\mathcal{B})$ has the desired property.

2. Take $\varepsilon > 0$ and $E \in \sigma(\mathcal{B})$ with finite measure. For any N,consider the restricted subalgebra $\sigma(\mathcal{B}) \cap B_N$. On this sub algebra, the previous result holds and thus there is some $F \in \mathcal{B}$ such that

$$\mu((E \cap B_N)\Delta F) < \frac{\varepsilon}{2}.$$

Since E has finite measure and $X = \bigcup_{n\in\mathbb{N}} B_n$, then

$$E = \bigcup_{E\cap B_n} .$$

Therefore by continuity from below it follows $\mu(E) = \lim_{n\to\infty} \mu(E \cap B_n)$. Therefore there exists N large enough such that $\mu(E \setminus B_N) < \frac{\varepsilon}{2}$. Note then that

$$E\Delta F \subset ((E \cap B_N)\Delta F) \cup (E \setminus B_N)$$

meaning

$$\mu(E\Delta F) \leq \mu((E \cap B_N)\Delta F) \leq \mu(E \setminus B_N) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

which was to be shown.

◇

Exercise 4.46

Proof. Suppose the backwards direction. Since each A_α is measurable, each $\mathbb{1}_{A_\alpha}$ is measurable and thus f is measurable since it is sum of measurable functions.

Suppose then the forwards direction. $f : \mathcal{A} \to \mathbb{C}$ being measurable is equivalent to its components being measurable, so we reduce to the case of $f : \mathcal{A} \to [0, \infty]$ since the resulting atomic sum representations can be added together. Consider some atom A_α and suppose towards contradiction that there are $x, y \in A_\alpha$ such that $f(x) \neq f(y)$. WLOG, suppose $f(x) < f(y)$. Let $B = [0, t)$ where $f(x) < t < f(y)$. Since f is measurable, it follows $f^{-1}(B) \in \mathcal{A}$. Consider then $A_\alpha \cap f^{-1}(B)$.

- Since $x \in A_\alpha$ and $f(x) \in B$, then $x \in A_\alpha \cap f^{-1}(B)$ meaning it is non-empty
- Since $f(y) \notin B$, it follows $y \notin A_\alpha \cap f^{-1}(B)$ meaning is a proper subset.

Therefore $A_\alpha \cap f^{-1}(B)$ is a proper non-empty subset of A_α. But since A_α and $f^{-1}(B)$ are measurable, $A_\alpha \cap f^{-1}(B)$ is measurable and smaller than A_α, a contradiction. Therefore f must be constant on every atom A_α with some value c_α. Since the partition of X is a disjoint partition, it then follows

$$f = \sum_{\alpha \in I} c_\alpha \mathbb{1}_{A_\alpha} \qquad \diamond$$

Exercise 4.47

Proof. First note that we can modify the f_n on a null set K to converge pointwise everywhere to f, thus we assume f_n converges pointwise everywhere to f and defer dealing with K. Since the f_n converge pointwise, for any $x \in X$ and $M \in \mathbb{N}$, there exists some $N \geq 0$ such that

$$|f_n(x) - f(x)| \leq \frac{1}{M}, \forall n \geq N.$$

Therefore if we take

$$E_{N,M} = \left\{ x \in X : |f_n(x) - f(x)| \leq \frac{1}{M}, n \geq N \right\}$$

then it follows for every M that $X = \bigcup_{N \in N} E_{N,M}$ and $\varnothing = \bigcap_{N \in \mathbb{N}} E^c_{N,M}$.

To see that each $E_{N,M}$ (and thus $E^c_{N,M}$) is measurable, let $g_n(x) = |f_n(x) - f(x)|$. Note that g_n is measurable since f_n is measurable and f is the limit of measurable functions and thus also measurable. Furthermore

$$E_{N,M} = \bigcap_{n \geq N} g_n^{-1}\left(\left[0, \frac{1}{M}\right]\right).$$

Since $E_{N,M}$ is the countable intersection of measurable sets (since preimage of measurable sets under a measurable function are measurable), $E_{N,M}$ is measurable.

Note that $E_{N,M}^c$ is decreasing in N, and since $\mu(X) < \infty$ then by continuity from above

$$\lim_{N\to\infty} \mu(E_{N,M}^c) = 0.$$

Note then for any $M \geq 1$, there exists some N_M such that

$$\mu(E_{N_M,M}^c) \leq \frac{\varepsilon}{2^M}$$

with $\varepsilon > 0$. Therefore if we take

$$A = K \cup \bigcup_{M\in\mathbb{N}} E_{N_M,M}^c$$

then by subadditivity $\mu(A) \leq \varepsilon$. Take $\delta > 0$ and M large enough such that $\frac{1}{M} < \delta$. Then for any $x \in X \setminus A$ we have $x \in E_{N_M,M}$ for all $n \geq N_M$, hence

$$|f_n(x) - f(x)| \leq \frac{1}{M} < \delta.$$

Since N_M only depends on M which itself only depends on δ, it follows f_n converges uniformly to f. $\diamond$

Exercise 4.54

Proof. Let $f = \sum_{i=j}^m c_j \mathbb{1}_{E_j}$ and $g = \sum_{k=1}^m d_k \mathbb{1}_{F_k}$ be simple functions.

1. If $f \leq g$ pointwise, then note on the refinement $A_{jk} = E_j \cap F_k$ that $c_j \leq d_k$. Therefore

$$I_{\mathcal{S}}(f,\mu) = \sum_{j,k} c_j \mu(A_{jk}) \leq \sum_{j,k} d_k \mu(A_{jk}) = I_{\mathcal{S}}(g,\mu).$$

2. Note the only values that $\mathbb{1}_E$ take are 0 and 1, and that $\mathbb{1}_E^{-1}(1) = E$. Therefore since $E \in \mathcal{A}$, we have measurability and

$$I_{\mathcal{S}}(\mathbb{1}_E,\mu) = 1 \cdot \mu(E) + 0 \cdot \mu(\mathbb{1}_E^{-1}(0)) = \mu(E).$$

3. Note that
$$cf = \sum_j (c \cdot c_j)\mathbb{1}_{E_j} = c \cdot \sum_j c_j \mathbb{1}_{E_j}.$$
Therefore
$$I_{\mathcal{S}}(cf, \mu) = \sum_j (c \cdot c_j)\mu(E_j) = c \cdot \sum_j c_j \mu(E_j) = c \cdot I_{\mathcal{S}}(f, \mu).$$

4. Note that on the refinement $A_{jk} = E_j \cap F_k$ that $f + g = (c_j + d_k)\mathbb{1}_{A_{jk}}$. Therefore
$$\begin{aligned} I_{\mathcal{S}}(f + g, \mu) = \sum_{j,k}(c_j + d_k)\mu(A_{jk}) &= \left(\sum_{j,k} c_j A_{jk}\right) + \left(\sum_{j,k} d_k A_{jk}\right) \\ &= I_{\mathcal{S}}(f, \mu) + I_{\mathcal{S}}(g, \mu) \end{aligned}$$

5. We have from definition 4.52 the finite σ-algebra $C = \sigma(\{f^{-1}(c_1), \ldots, f^{-1}(c_m)\})$. Note that $C \subset \mathcal{A} \subset \mathcal{B}$. Since $\nu|_{\mathcal{A}} = \mu$, it follows that $\nu|_C = \mu|_C$ and thus
$$I_{\mathcal{S}}(f, \nu) = I_{\mathcal{S}}(f, \mu|_{\mathcal{B}}) = I_{\mathcal{S}}(f, \mu).$$

6. Suppose $f \overset{\text{a.e.}}{=} g$ with respect to μ. Consider the refinement $A_{jk} = E_j \cap F_k$ and let $I = \{(j, k) : c_k \neq d_k\}$. Then
$$\begin{aligned} I_{\mathcal{S}}(f, \mu) &= \sum_{(i,j)\notin I} c_j \mu(A_{jk}) + \sum_{(i,j)\in I} c_j A_{jk} \\ I_{\mathcal{S}}(g, \mu) &= \sum_{(i,j)\notin I} d_k \mu(A_{jk}) + \sum_{(i,j)\in I} d_k A_{jk} \end{aligned}$$
Note that if $N = \bigcup_{(i,j)\in I} A_{jk}$ that $\mu(N) = 0$ since it is comprised of sets where $f \neq g$. Thus $(i, j) \in I$ means $\mu(A_{jk}) \leq \mu(N) = 0$. Therefore
$$I_{\mathcal{S}}(f, \mu) = \sum_{(i,j)\notin I} c_j \mu(A_{jk}) = \sum_{(i,j)\notin I} d_k \mu(A_{jk}) = I_{\mathcal{S}}(g, \mu).$$

7. Suppose that $I_{\mathcal{S}}(f, \mu) < \infty$. Then
$$c_i \mu(E_i) \leq \sum_j c_j \mu(E_j) = I_{\mathcal{S}}(f, \mu) < \infty$$

for all i. Note this implies that f cannot be ∞ on a set of positive measure, thus $f < \infty$ almost everywhere. Take $c = \min\left\{c_j : c_j > 0\right\}$. If c does not exist, then clearly the statement holds. Note then that

$$I_{\mathcal{S}}(f,\mu) = \sum_j c_j\mu(E_j) = \sum_{j,c_j>0} c_j\mu(E_j) \geq \sum_{j,c_j} c\mu(E_j) = c\cdot\mu(\mathrm{supp}(f)).$$

Therefore $\mu(\mathrm{supp}(f)) \leq \frac{1}{c}\cdot I_{\mathcal{S}}(f,\mu) < \infty$.

Suppose then that $f < \infty$ almost everywhere and $\mu(\mathrm{supp}(f)) < \infty$. Let

$$\begin{aligned} J_\infty &= \left\{j : c_j = \infty\right\} \\ J_+ &= \left\{j : 0 < c_j < \infty\right\} \\ J_0 &= \left\{j : c_j = 0\right\} \end{aligned}$$

Note that since $f < \infty$ almost everywhere, $c_j\mu(E_j) = 0$ for any $j \in J_\infty$ since $\mu(E_j) = 0$. If J_+ is empty, then clearly the result holds. Assume then J_+ is non-empty and let $c = \max_{j\in J_+} c_j < \infty$. Therefore

$$I_{\mathcal{S}}(f,\mu) = \left[\sum_{j\in J_\infty} c_j\mu(E_j)\right] + \left[\sum_{j\in J_0} c_j\mu(E_j)\right] + \left[\sum_{j\in J_+} c_j\mu(E_j)\right] = \sum_{j\in J_+} c_j\mu(E_j).$$

Since the E_j can be taken pairwise disjoint, we get

$$\sum_{j\in J_+} c_j\mu(E_j) \leq M \sum_{j\in J_+} \mu(E_j) = M\cdot\mu\left(\bigcup_{j\in J_+} E_j\right).$$

Note that $\bigcup_{j\in J_+} E_j \subset \mathrm{supp}(f)$, thus

$$I_{\mathcal{S}}(f,\mu) \leq M\cdot\mu\left(\bigcup_{j\in J_+} E_j\right) \leq M\cdot\mu(\mathrm{supp}(f)) < \infty.$$

8. This follows directly from (vi) since $I_{\mathcal{S}}(0,\mu) = 0$ always.

◇

Exercise 4.58

Proof.

(i) Suppose $f \overset{\text{a.e.}}{=} g$ with respect to μ. Let $N = \{x \in X : f(x) \neq g(x)\}$ and $\mu(N) = 0$. Take $h \leq f$ to be simple. Note that $\tilde{h} = h \cdot \mathbb{1}_{N^c}$ is also simple and that $\tilde{h} \leq g$. Since h and $\tilde{h}$ are equal μ-a.e., it follows $I_{\mathcal{S}}(h, \mu) = I_{\mathcal{S}}(\tilde{h}, \mu)$ as well. Therefore every simple function $h \leq f$ has some $\tilde{h} \leq g$ with the same integral, meaning

$$\sup_{h \leq f} I_{\mathcal{S}}(h, \mu) \leq \sup_{\tilde{h} \leq g} I_{\mathcal{S}}(\tilde{h}, g).$$

The roles of f and g can be reversed in the previous argument, turning the previous inequality into equality. Thus $\int_X f d\mu = \int_X g d\mu$.

(ii) This follows from the first part of the argument above by replacing $N = \{x \in X : f(x) > g(x)\}$.

(v) Since f is simple and $f \leq f$, it follows

$$I_{\mathcal{S}}(f, \mu) \leq \sup_{h \leq f} I_{\mathcal{S}}(h, \mu).$$

But by montonicity of simple functions $h \leq f$, $I_{\mathcal{S}}(h, \mu) \leq I_{\mathcal{S}}(f, \mu)$ giving the reverse inequality. Hence $\int_X f d\mu = I_{\mathcal{S}}(f, \mu)$.

(vi) Note that pointwise $\lambda \cdot \mathbb{1}_{[f \geq \lambda]} \leq f$. Integrating both sides and applying both (ii) and (v) we get

$$\lambda \cdot \mu([f \geq \lambda]) = \lambda \cdot I_{\mathcal{S}}(\mathbb{1}_{[f \geq \lambda]}, \mu) = I_{\mathcal{S}}(\lambda \cdot \mathbb{1}_{[f \geq \lambda]}, \mu) = \int_X \lambda \cdot \mathbb{1}_{[f \geq \lambda]} d\mu \leq \int_X f d\mu.$$

(viii) If $\int_X f d\mu = 0$, for each $n \in \mathbb{N}$ we have from (vi) that

$$\mu([f \geq \frac{1}{n}]) \leq n \cdot \int_X f d\mu = 0.$$

Therefore since the countable union of null sets is also a null set

$$\mu([f > 0]) = \mu\left(\bigcup_{n \in \mathbb{N}} [f \geq \frac{1}{n}]\right) = 0.$$

Thus f is 0 μ-a.e.

(iii) If $c = 0$ then the result clearly holds as $cf = 0$ and the only simple function bounded by 0 is the 0 function itself. If $c = \infty$, then $\int_X f d\mu > 0$ gives $c \int_X f d\mu = \int_X cf d\mu = \infty$ and $\int_X f d\mu = 0$ gives $cf = 0$ a.e. and the result holds. Suppose then that $0 < c < \infty$. Then the map $h \mapsto ch$ is a bijection between simple functions $h \leq f$ and simple functions $ch \leq cf$. Since $I_{\mathcal{S}}(ch, \mu) = cI_{\mathcal{S}}(h, \mu)$, it follows

$$\int_X cf d\mu = \sup_{h \leq cf} I_{\mathcal{S}}(h, \mu) = \sup_{h \leq f} I_{\mathcal{S}}(ch, \mu) = c \cdot \sup_{h \leq f} I_{\mathcal{S}}(h, \mu) = c \cdot \int_X f d\mu.$$

(iv) If h_1, h_2 are simple with $h_1 \leq f$ and $h_2 \leq g$ then $\tilde{h} = h_1 + h_2$ is also simple and $\tilde{h} \leq f + g$ as well as $h_1 + h_2 \leq \tilde{h}$. By linearity we have

$$I_{\mathcal{S}}(h_1 + h_2, \mu) = I_{\mathcal{S}}(h_1, \mu) + I_{\mathcal{S}}(h_2, \mu).$$

Note then that

$$I_{\mathcal{S}}(h_1, \mu) + I_{\mathcal{S}}(h_2, \mu) = I_{\mathcal{S}}(h_1 + h_2, \mu) \leq \sup_{\tilde{h} \leq f+g} I_{\mathcal{S}}(\tilde{h}, \mu) = \int_X (f+g) d\mu.$$

Thus taking the supremum over h_1 and h_2 we get

$$\int_X f d\mu + \int_X g d\mu \leq \int_X (f+g) d\mu.$$

(vii) Suppose that $\int_X f d\mu < \infty$. Let $E = \{x \in X : f(x) = \infty\}$. Note that for each $n \in \mathbb{N}$ that $n \cdot \mathbb{1}_E \leq f$. Therefore

$$n\mu(E) = I_{\mathcal{S}}(n \cdot \mathbb{1}_E, \mu) \leq \int_X f d\mu.$$

If $\mu(E) > 0$, then $\lim_{n\to\infty} n\mu(E) = \infty$, giving $\int_X f d\mu = \infty$, a contradiction. Therefore $\mu(E) = 0$, hence $f < \infty$ μ-a.e.

(ix) Let $f_n = \min(f, n)$. Note that each f_n is measurable and $\lim_{n\to\infty} f_n = f$ pointwise. Since $f_n \leq f_{n+1}$ and $f_n \leq f$ for all n, it follows

$$\lim_{n\to\infty} \int_X f_n d\mu \leq \int_X f d\mu.$$

Consider then a simple function $h \leq f$.

- If $h = \infty$ on a set of positive measure E, then we have from montonicity that $\int_X f\,d\mu = \infty$. Note also that for any $r > 0$ there then must be some N large such that for $n \geq N$ we have $f_n(x) \geq r$ for all $x \in E$. But this means that

$$r \cdot \mu(E) \leq \int_X f_N d\mu.$$

  Since r was arbitrary, this means that $\lim_{n\to\infty} \int_X f_n d\mu = \infty$.
- If $h = \infty$ on a null set, then we can simply set h to 0 on the null set and get the same integral.

We can assume then h is finite everywhere. Thus there exists some N large such that for $n \geq N$, $h \leq f_n$. Therefore taking the limit and then supremum gives

$$\int_X f\,d\mu \leq \lim_{n\to\infty} \int_X f_n d\mu.$$

(x) Let $f_n = f\mathbb{1}_{E_n}$ and $E = \bigcup_{n\in\mathbb{N}} E_n$. Note that $f_n \leq f\mathbb{1}_E$ for all n, therefore by montonicity

$$\lim_{n\to\infty} \int_X f_n d\mu \leq \int_X f\mathbb{1}_E d\mu.$$

(xi) Note that if h is a simple function on E such that $h \leq f$ on E, then extending h to $\tilde{h}$ that is 0 on E^c, we get $\tilde{h} \leq f\mathbb{1}_E$. Furthermore, if g is a simple function on X with $g \leq f\mathbb{1}_E$, then $\operatorname{supp}(g) = E$ and thus the restriction of $\tilde{g}$ of g to E is simple and $\tilde{g} \leq f$ on E. This establishes a bijection between simple functions $h \leq f$ on E and simple function $g \leq f\mathbb{1}_E$ on X. Since $I_{\mathcal{S}}(h, \mu|_E) = I_{\mathcal{S}}(\tilde{h}, \mu)$ and $I_{\mathcal{S}}(g, \mu) = I_{\mathcal{S}}(\tilde{g}, \mu|_E)$, we have

$$\int_X f\mathbb{1}_E d\mu = \sup_{g\leq f\mathbb{1}_E} I_{\mathcal{S}}(g, \mu) = \sup_{h\leq f \text{ on } E} I_{\mathcal{S}}(h, \mu|_E) = \int_E f\,d\mu|_E.$$

◇

Exercise 4.61

Proof.

1. To show $\phi_*\mu(E)$ is a measure, we check the axioms. Let $E \in \mathcal{B}$.
   - $\phi_*\mu(E) = \mu(\phi^{-1}(E)) \geq 0$ since μ is non-negative.
   - $\phi_*\mu(\varnothing) = \mu(\phi^{-1}(\varnothing)) = \mu(\varnothing) = 0$
   - Let $E_n \in \mathcal{B}^{\mathbb{N}}$ be pairwise disjoint. Note that the preimages $\phi^{-1}(E_n)$ are therefore also disjoint and
   $$\phi^{-1}\left(\bigcup_{n\in\mathbb{N}} E_n\right) = \bigcup_{n\in\mathbb{N}} \phi^{-1}(E_n).$$
   Thus
   $$\phi_*\mu\left(\bigcup_{n\in\mathbb{N}} E_n\right) = \mu\left(\bigcup_{n\in\mathbb{N}} \phi^{-1}(E_n)\right) = \sum_{n\in\mathbb{N}} \mu(\phi^{-1}(E_n)) = \sum_{n\in\mathbb{N}} \phi_*\mu(E_n).$$
   Hence countable additivity holds.

2. First consider some simple function $s = \sum_k c_k \mathbb{1}_{E_k}$ over Y. Note that $\phi^{-1} \circ \mathbb{1}_{E_k} = \mathbb{1}_{\phi^{-1}(E_k)}$, thus
   $$\int_Y s d(\phi_*\mu) = \sum_k c_k \phi_*\mu(E_k) = \sum_k c_k \mu(\phi^{-1}(E_k)) = \int_X s \circ \phi d\mu.$$
   Note if $s \leq f$ that both $s \circ \phi$ is simple and $s \circ \phi \leq f \circ \phi$. Therefore by the previous result
   $$\int_Y s d(\phi_*\mu) = \int_X s \circ \phi d\mu \leq \int_X f \circ \phi d\mu.$$
   Taking the supremum of the left side gives then
   $$\int_Y f d(\phi_*\mu) \leq \int_X f \circ \phi d\mu.$$
   Take then $t \leq f \circ \phi$ simple over X where $t = \sum_j d_j \mathbb{1}_{F_j}$. Note that for each j there is some $F_j = \phi^{-1}(B_j)$. Thus t can be rewritten as
   $$t = \sum_j d_j \mathbb{1}_{\phi^{-1}(B_j)} = \sum_j d_j \mathbb{1}_{B_j} \circ \phi = \left(\sum_j d_j \mathbb{1}_{B_j}\right) \circ \phi.$$

Therefore if $h = \sum_j d_j \mathbb{1}_{B_j}$, h is a simple function on Y and $t = h \circ \phi$. Since $t \leq f \circ \phi$, then $h \circ \phi \leq f \circ \phi$ and thus $h \leq f$ on $\phi(X)$. Setting $h = 0$ outside of $\phi(X)$ gives then $h \leq f$ everywhere on Y. Therefore

$$\int_X t d\mu = \int_X (h \circ \phi) d\mu = \int_Y h d(\phi_* \mu) \leq \int_Y f d(\phi_* \mu).$$

Taking the supremum over all t thus gives

$$\int_X (f \circ \phi) d\mu \leq \int_Y f d(\phi_* \mu).$$

If $T : \mathbb{R}^d \to \mathbb{R}^d$ is invertible, from previous HW we have for every Lebesgue measurable set E that

$$T_* m(E) = m(T^{-1}(E)) = \frac{1}{|\det T|} m(E).$$

Therefore $T_* m = \frac{1}{|\det T|} m$.

◇

Exercise 4.65

Proof. Suppose $f_n \to f$ uniformly and take $\varepsilon > 0$. Then there exists some N such that for all $n \geq N$ and $x \in X$

$$-\varepsilon < f_n(x) - f(x) < \varepsilon \implies f_n(x) - \varepsilon < f(x) < f_n(x) + \varepsilon.$$

Since uniform convergence implies pointwise, it follows f is measurable. Thus by montonicity of the integral we then have

$$\int_X (f_n - \varepsilon) d\mu < \int_X f d\mu < \int_X (f_n + \varepsilon) d\mu.$$

Since $\mu(X) < \infty$, we get

$$\int_X f_n d\mu - \frac{\varepsilon}{\mu(X)} < \int_X f_n d\mu + \frac{\varepsilon}{\mu(X)} \implies -\frac{\varepsilon}{\mu(X)} < \int_X f_n d\mu - \int_X f d\mu < \frac{\varepsilon}{\mu(X)}.$$

As $\varepsilon \to 0$, then $n \to \infty$ and the difference between the two goes to 0. Therefore

$$\lim_{n \to \infty} \int_X f_n d\mu = \int_X f d\mu.$$

◇

Exercise 4.73

Proof.

1. Note that
$$\limsup_{n\to\infty} E_n = \bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} = \{x \in X : x \in E_n \text{ for infinitely many } n\}.$$
Thus an equivalent result is to show that
$$\mu\left(\limsup_{n\to\infty} E_n\right) = 0.$$
Note that by montonicity and subadditivity
$$\mu(\limsup_{n\to\infty} E_n) = \mu\left(\bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} E_k\right) \leq \mu\left(\bigcup_{k\geq n} E_k\right) \leq \sum_{k\geq n} \mu(E_k)$$
for all $n \in \mathbb{N}$. Since $\sum_n \mu(E_n) < \infty$, then the tail of the sum goes to 0. Therefore
$$\mu(\limsup_{n\to\infty} E_n) \leq \sum_{k\geq n} \mu(E_k) \to 0 \ (k \to \infty).$$
Thus $\mu(\limsup_{n\to\infty} E_n) = 0$.

2. We will work in the interval $[0,1]$ equipped with the Lebesgue measure. Let H_n be the n^{th} harmonic number. Let E_n denote the segment with $m(E_n) = \frac{1}{n+1}$ starting at $x = H_n$ that "wraps" around the endpoints of $[0,1]$. That is an interval $[a,b]$ with $a < 1$ and $b > 1$ turns into $[a,1] \cup [0, b \bmod 1]$. Since $\lim_{n\to\infty} H_n = \infty$, it follows that the E_n will cover every point infinitely many times, however $\lim_{n\to\infty} \mu(E_n) = 0$.

◇

Exercise 4.81

Proof. Let $g_n = \min(f_n, f)$. Note that both $g_n \leq f$ everywhere and $g_n \to f$ as $n \to \infty$. Since f is $\mathcal{L}^1$, then DCT applies to g_n meaning
$$\lim_{n\to\infty} \int_X g_n d\mu = \int_X f d\mu.$$

Note that $f_n - f - |f - f_n| = 2 \cdot (f_n - g_n)$, therefore

$$\int_X f_n d\mu - \int_X f d\mu - \int_X |f - f_n| d\mu = 2 \cdot \left(\int_X f_n d\mu - \int_X g_n d\mu \right).$$

Since the RHS goes to 0 as $n \to \infty$, it follows that

$$\int_X f_n d\mu - \int_X f d\mu - \|f - f_n\|_{L_1(\mu)} \to 0 \quad (n \to \infty).$$

◇

Exercise 4.82

Proof. Consider the axioms of a measure

- Since g is non-negative, the integral of g over any set will also be non-negative. Thus $\mu_g(E) \geq 0$ for all $E \in \mathcal{A}$.
- Let $E, F \in \mathcal{A}$ with $E \subset F$. Note that $g \cdot \mathbb{1}_E \leq g \cdot \mathbb{1}_F$, therefore
$$\mu_g(E) = \int_E g d\mu = \int_X g \cdot \mathbb{1}_E d\mu \leq \int_X g \cdot \mathbb{1}_F d\mu = \int_F g d\mu = \mu_g(F).$$
- Let $(E_n) \in \mathcal{A}^{\mathbb{N}}$ be disjoint and $E = \bigcup_{n \in \mathbb{N}} E_n$. Note that the sequence $F_n = \bigcup_{k \leq n} E_n$ is an increasing sequence and $\lim_{n \to \infty} F_n = E$. Therefore
$$\lim_{n \to \infty} \int_X g \cdot \mathbb{1}_{F_n} d\mu = \int_X g \cdot \mathbb{1}_E d\mu = \mu_g(E).$$
Since $\mathbb{1}_{A \cup B} = \mathbb{1}_A + \mathbb{1}_B$ when A and B are disjoint, it follows by linearity that
$$\int_X g \cdot \mathbb{1}_{F_n} = \int_X g \cdot \left(\sum_{k \leq n} \mathbb{1}_{E_n} \right) d\mu = \sum_{k \leq n} \int_X g \cdot \mathbb{1}_{E_n} d\mu = \sum_{k \leq n} \int_{E_n} g d\mu = \sum_{k \leq n} \mu_g(E_n).$$
In total then
$$\mu_g(E) = \lim_{n \to \infty} \int_X g \cdot \mathbb{1}_{F_n} d\mu = \sum_{n \in \mathbb{N}} \mu_g(E_n)$$
hence μ_g is countably additive.

Thus μ_g is a measure. ◇

Exercise 5.22

Proof. Suppose that $f_n \to f$ in the L^1 norm. Note that by Markov's inequality that for some $\varepsilon > 0$

$$\mu([|f - f_n| \geq \varepsilon]) \leq \frac{1}{\varepsilon} \cdot \int |f - f_n| d\mu = \frac{1}{\varepsilon} \cdot \|f - f_n\|_{L^1}.$$

Therefore in the limit as $n \to \infty$, the RHS goes to zero by the assumption, meaning $f_n \to f$ in measure.

Suppose towards contradiction then that $f_n \to f$ in measure but $f_n \not\to f$ in L^1. Then there exists some $\varepsilon > 0$ and subsequence f_{n_j} such that

$$\left\|f - f_{n_j}\right\| \geq \varepsilon, \forall j.$$

Denote this subsequence as h_n. Since $f_n \to f$ in measure, it is also the case that $h_n \to f$ in measure as well. Since f_n is a dominated sequence, so is h_n meaning there is some $g \in \mathcal{L}^1$ such that both $|h_n| \leq g$ and $|f| \leq g$. Therefore by the triangle inequality, $|f - h_n| \leq 2g$ meaning $|f - h_n|$ is also a dominated sequence. Since $h_n \to f$ in measure, there is some subsequence $h_{n_k} \to f$ pointwise a.e. Therefore by the Dominated Convergence Theorem it follows that

$$\lim_{n \to \infty} \int |f - h_{n_k}| d\mu = 0 \Leftrightarrow \lim_{n \to \infty} \left\|f - h_{n_k}\right\|_{L^1} = 0.$$

But for a sufficiently large k it follows that $\left\|f - h_{n_k}\right\| < \varepsilon$, a contradiction. Therefore $f_n \to f$ in L^1. $\diamond$

Exercise 5.26

Proof. The forward direction is trivial as the result is a requirement of uniform integrability. Suppose then that

$$\sup_{n \in \mathbb{N}} \int_{[|f_n| \geq M]} |f_n| d\mu \to 0$$

as $M \to \infty$. One can then take M large enough such that

$$\sup_{n \in \mathbb{N}} \int_{[|f_n| \geq M]} \leq 1.$$

Since $\mu(X) < \infty$, for any n it then follows that

$$\int |f_n| d\mu = \int_{[|f_n| \geq M]} |f_n| d\mu + \int_{[|f_n| < M]} |f_n| d\mu \leq 1 + M \cdot \mu(X) < \infty.$$

Therefore the first condition for uniform integrability holds. Clearly the second holds as it is identical to the assumption, leaving the third to be shown. Note that for any n

$$\int_{[|f_n| \leq \delta]} |f_n| \leq \int_{[|f_n| \leq \delta]} \delta d\mu = \delta \cdot \mu(|f_n| \leq \delta]) \leq \delta \cdot \mu(X).$$

In the limit $\delta \to 0$, the integral goes to 0 and thus $(f_n)_{n \in \mathbb{N}}$ is uniformly integrable. $\diamond$

Exercise 5.27

Proof.

1. Denote $C = \sup_{n \in \mathbb{N}} \int |f_n|^p d\mu$. Note that $|f_n| \leq |f_n|^p + 1$ for any $p > 1$, therefore

   $$\sup_{n \in \mathbb{N}} \int |f_n| d\mu \leq \sup_{n \in \mathbb{N}} \int (|f_n|^p + 1) d\mu = \mu(X) + C < \infty.$$

   Therefore the first condition of uniform integrability holds. Consider $|f_n|$ when $|f_n| \geq M$. Note that for $p > 1$

   $$|f_n| = \frac{|f_n|^p}{|f_n|^{p-1}} \leq \frac{|f_n|^p}{M^{p-1}}.$$

   Integrating over both sides then gives

   $$\int_{[|f_n| \geq M]} |f_n| d\mu \leq \frac{1}{M^{p-1}} \cdot \int_{[|f_n| \geq M]} |f_n|^p d\mu \leq \frac{C}{M^{p-1}}$$

   which in the limit as $M \to \infty$ goes to 0. Therefore the second condition of uniform integrability holds. Take $\delta > 0$ and note that

   $$\int_{[|f_n| \leq \delta]} |f_n| d\mu \leq \int_{[|f_n| \leq \delta]} \delta d\mu = \delta \cdot \mu([|f_n| \leq \delta]) \leq \delta \cdot \mu(X).$$

   Since $\mu(X) < \infty$, it follows in the limit $\delta \to 0$ that the integral goes to 0 for all n. Thus $(f_n)_{n \in \mathbb{N}}$ is uniformly integrable.

2. Take $\varepsilon > 0$. By uniform integrability, there is some $\delta > 0$ and $M > 0$ such that
$$\sup_{n\in\mathbb{N}} \int_{[|f_n|\le\delta]} |f_n| d\mu \le \varepsilon \qquad \sup_{n\in\mathbb{N}} \int_{[|f_n|\ge M]} |f_n| \le \varepsilon$$
If $\mu(E) \le \varepsilon$, then note
$$\int_E |f_n| d\mu = \int_{E\cap[|f_n|\le\delta]} |f_n| d\mu + \int_{E\cap[\delta<|f_n|<M]} |f_n| d\mu + \int_{E\cap[|f_n|\ge M]} |f_n| d\mu$$
in which each term can be bounded to give
$$\int_E |f_n| d\mu \le \varepsilon + \mu(E)\cdot M + \varepsilon \le 2\varepsilon + M\cdot\varepsilon$$
which was to be shown.

3. The first condition for uniform integrability is satisfied by taking $C = \sup_{n\in\mathbb{N}} \|f_n\|_{L^1}$. Note that by Markov's inequality
$$\mu([|f_n| \ge M]) \le \frac{\|f_n\|_{L^1}}{M} \le \frac{C}{M}.$$
For any $\varepsilon > 0$ and the associated δ given by the assumption, taking M large enough such that $\frac{C}{M} \le \delta$ gives
$$\sup_{n\in\mathbb{N}} \int_{[|f_n|\ge M]} |f_n| d\mu \le \varepsilon$$
hence the second condition for uniform integrability holds. Note that for $\delta > 0$ and any n that
$$\int_{[|f_n|\le\delta]} |f_n| \le \int_{[|f_n|\le\delta]} \delta d\mu \le \delta\cdot\mu(X).$$
Since $\delta\cdot\mu(X) \longrightarrow 0$ as $\delta \longrightarrow 0$, the third condition for uniform integrability then holds.

4. Let $f_n = \mathbb{1}_{[n,n+1]}$. Note that
   - $\|f_n\|_{L^1} = 1$ for all n
   - $\int_{[|f_n|\ge M]} |f_n| d\mu = 0$ for $M > 1$ and all n
   - $\int_{[|f_n|\le\delta]} |f_n| d\mu = 0$ for $\delta < 1$ and all n

Thus f_n is uniformly integrable and converges pointwise almost everywhere to 0. However,

- It does not converge in L^1 since $\|f_n - 0\|_{L^1} = \|f_n\|_{L^1} = 1$
- It does not converge in measure since $\mu([f_n - 0] > \frac{1}{2}) = 1 \not\to 0$
- It does not converge almost uniformly since the set f_n differs from 0 will always have a measure of 1 and hence cant be arbitrarily small

◇

Exercise 5.31

Proof. Let $f := \sup_{n\in\mathbb{N}} f_n = \lim_{n\to\infty} f_n$. Since f_n is a non-decreasing sequence, it follows from the Monotone Convergence Theorem that

$$\int f d\mu = \lim_{n\to\infty} \int f_n d\mu = \sup_{n\in\mathbb{N}} \int f_n d\mu < \infty.$$

Therefore $f \in \mathcal{L}^1$. Since $f_n \le f$, it follows that $f - f_n \ge 0$. Thus

$$\|f - f_n\|_{L^1} = \int |f - f_n| d\mu = \int (f - f_n) d\mu = \int f d\mu - \int f_n d\mu.$$

In the limit the integrals of the RHS are equal, meaning $\lim_{n\to\infty} \|f - f_n\|_{L^1} = 0$. Hence $f_n \to f$ in the L^1 norm. ◇

Exercise 5.33

Proof. The forward direction follows from exercise 5.5. Suppose then that $f_n \to f$ pointwise a.e. and that f_n is dominated by some $g \in \mathcal{L}^1$. Take $\varepsilon > 0$ and define

$$X_k = \left\{ x \in X : g(x) \ge \frac{1}{k} \right\}.$$

By Markov's inequality it follows that

$$\mu(X_k) \le k \cdot \int g d\mu \le \infty$$

thus each of the X_k are finite. For each k, it follows then by Egorov's that there is a $B_k \subset X_k$ with $\mu(B_k) \le \frac{\varepsilon}{2^k}$ where $f_n \to f$ uniformly on $X_k \setminus B_k$. Let then $B = \bigcup B_k$ and note that $\mu(B) \le \varepsilon$. Take now $K > 0$ large such that $\frac{1}{K} \le \varepsilon$. If $x \notin X_K \cup E$, then $g(x) < \frac{1}{K} \le \varepsilon$. Note then that

$$|f_n(x) - f(x)| \le |f_n(x)| + |f(x)| \le 2g(x) \le 2\varepsilon.$$

By the construction of X_K, it follows that $f_n \to f$ uniformly on $X_K \setminus E$. Therefore $f_n \to f$ uniformly everywhere except on B which is arbitrarily small. ◇

Exercise 6.12

Proof.

1. By the definition of the outer measure, for any $\varepsilon > 0$ there exists a cover $A_n \in \mathcal{B}^{\mathbb{N}}$ of E such that

$$\sum_{n\in\mathbb{N}} \mu_0(A_n) \le \mu^*(E) + \varepsilon.$$

   Clearly A_n is a cover for A, meaning that

$$\mu^*(A) \le \sum_{n\in\mathbb{N}} \mu_0(A_n) \le \mu^*(E) + \varepsilon.$$

   Since each $A_n \in \mathcal{B}$ and A is a countable union, it follows that $A \in \mathcal{B}_\sigma$.

2. Suppose E is μ^*-measurable and $\mu^*(E) < \infty$. Using part (a), let $B_n \in \mathcal{B}_\sigma^{\mathbb{N}}$ such that $E \subset B_n$ with $\mu^*(B_n) \le \mu^*(E) + \frac{1}{n}$. Take then $B = \bigcap_{n\in\mathbb{N}} B_n$. Note that both $E \subset B \subset B_n$ and $B \in \mathcal{B}_{\sigma\delta}$. Thus

$$\mu^*(E) \le \mu^*(B) \le \mu^*(E) + \frac{1}{n}$$

   which in the limit $n \to \infty$ gives $\mu^*(B) = \mu^*(E)$. Since E is u^*-measurable, it follows that

$$\mu^*(E) = \mu^*(E \cap B) + \mu^*(E \setminus B) \implies \mu^*(E \setminus B) = \mu^*(E) - \mu^*(B).$$

   But since $\mu^*(B) = \mu^*(E)$, it follows that $\mu^*(E \setminus B) = 0$.

Suppose then some $E \subset B \in \mathcal{B}_{\sigma\delta}$ where $\mu^*(E \setminus B) = 0$. Since $\mu^*(E \setminus B) = 0$, clearly the Caratheodory criterion holds for $E \setminus B$ and thus it is μ^*-measurable. Since the set of μ^*-measurable sets is a σ-algebra containing $\mathcal{B}$, it follows that B is μ^*-measurable and thus $(E \setminus B) \cup B = E$ is μ^*-measurable.

3. Note that the reverse direction did not require $\mu^*(E) < \infty$, so we only consider the forward direction. Suppose X is σ-finite. Then there exists $X_k \in \mathcal{B}^{\mathbb{N}}$ such that $\mu^*(X_k) < \infty$ and $X = \bigcup_{k\in\mathbb{N}} X_k$. Let $E_k = E \cap X_k$ and note that from part (a) that there exist $E_k \subset O_{k,n} \in \mathcal{B}_\sigma^{\mathbb{N}}$ with $\mu^*(O_{k,n}) \leq \mu^*(E_k) + \frac{1}{n\cdot 2^k}$.

   Take then $B_n = \bigcup_{k\in\mathbb{N}} O_{k,n}$ and note that $B_n \in \mathcal{B}_\sigma$ and
   $$B_n \setminus E \subset \bigcup_{k\in\mathbb{N}} (O_{k,n} \setminus E_k).$$
   Therefore by subadditivity it follows
   $$\mu^*(B_n \setminus E) \leq \sum_{k\in\mathbb{N}} \mu^*(O_{k,n} \setminus E) \leq \sum_{k\in\mathbb{N}} \frac{1}{n\cdot 2^k} = \frac{1}{n}.$$
   The exact same argument in (b) then works from here.

◇

Exercise 6.14

Proof.

1. Let $A_n \in \mathcal{A}^*$ be a cover of E. Since μ^* is an outer measure, it is subadditive and monotonic meaning
   $$\mu^*(E) \leq \mu^*\left(\bigcup_{n\in\mathbb{N}} A_n\right) \leq \sum_{n\in\mathbb{N}} \mu^*(A_n).$$
   Since μ^+ is the infimum over all such sums, it follows that $\mu^*(E) \leq \mu^+(E)$.

   Suppose then there exists an $A \supset E$ with $\mu^*(A) = \mu^*(E)$. Since A covers E, by definition $\mu^+(E) \leq \mu^*(A) = \mu^*(E)$. Combined with first inequality gives $\mu^+(E) = \mu^*(E)$.

   Suppose then that $\mu^*(E) = \mu^+(E)$.

- Suppose $\mu^*(E) < \infty$. By the definition of μ^+ there then exists $A_n \in \mathcal{A}^*$ such that $E \subset A_n$ and
$$\mu^*(A_n) \leq \mu^+(E) + \frac{1}{n}.$$
Take then $A = \bigcap_{n\in\mathbb{N}} A_n$ and note that both $A \in \mathcal{A}^*$ and $E \subset A$. In the limit as $n \to \infty$ it then follows
$$\mu^*(A) \leq \mu^*(A_n) \leq \mu^+(E) = \mu^*(E).$$
Therefore $\mu^*(E) = \mu^*(A)$.
- Suppose $\mu^*(E) = \infty$. Since $\mu^*(E) \leq \mu^+(E)$, it follows $\mu^+(E) = \infty$. Take $A = X$. Clearly then $A \in \mathcal{A}^*$ and $E \subset A$. Thus by monotonicity $\infty = \mu^*(E) \leq \mu^*(A)$ meaning $\mu^*(A) = \mu^*(E) = \infty$

2. If μ^* is induced from a pre-measure over some algebra $\mathcal{B}$, then from problem 1 part (a), for any $\varepsilon > 0$ there is some $A \in \mathcal{A}_\sigma$ such that $E \subset A$ and $\mu^*(A) \leq \mu^*(E) + \varepsilon$. By Caratheodory's, every set in $\mathcal{B}$ is measurable and thus in $\mathcal{A}^*$. Since $\mathcal{A}^*$ is a σ-algebra, countable unions are contained in it and thus $\mathcal{B}_\sigma \subset \mathcal{A}^*$. Since then $A \in \mathcal{A}^*$ and covers E, it follows $\mu^+(E) \leq \mu^*(A) \leq \mu^*(E) + \varepsilon$. Letting $\varepsilon \to 0$ thus gives $\mu^+(E) \leq \mu^*(E)$, which combined with the initial result gives equality.

3. Define μ^* on 2^X where
$$\begin{aligned} \mu^*(\varnothing) &= 0 \\ \mu^*(\{0\}) &= 1 \\ \mu^*(\{1\}) &= 1 \\ \mu^*(\{0,1\}) &= 1.5 \end{aligned}$$
Clearly it is monotonic an subadditive, and $\mu^*(\varnothing) = 0$. Therefore μ^* is an outer measure on X. Note that
$$\mu^*(0,1) = 1.5 \neq 2 = 1 + 1 = \mu^*(\{0,1\} \cap \{0\}) + \mu^*(\{0,1\} \cap \{1\}).$$
Thus $\{0\}$ and $\{1\}$ are not measurable, leaving just $\varnothing$ and $\{0,1\}$ which are trivially measurable. Note that the only measurable set containing $\{0\}$ is $\{0,1\}$ and so $\mu^+(\{0\}) = \mu^*(\{0,1\}) = 1.5$, but
$$\mu^*(\{0\}) = 1 \neq 1.5 = \mu^+(\{0\}).$$

◇

Exercise 6.18

Proof.

1. First note that $(a, b] \cap \mathbb{Q}$ forms an elementary family.

   iii) Note that $(a, b]^c = (b, \infty]$ and thus complements are maintained in $\mathbb{Q}$.

   ii) Note that $(a_1, b_1] \cap (a_2, b_2] = (\max(a_1, a_2), \min(b_1, b_2)]$ and thus intersections are maintained in $\mathbb{Q}$

   i) Note that $(a, b] \cap (a, b]^c = \varnothing$ and thus the empty set is in family

   Since they form an elementary family, then $\mathcal{B}$ is an algebra since is the finite union of sets in the family.

2. Note that any singleton $\{q\} \subset \mathbb{Q}$ can be achieved by

$$\{q\} = \bigcap_{n \in \mathbb{N}} \left(q - \frac{1}{n}, q \right].$$

   Since all the inner sets are contained in $\mathcal{B}$, it follows that $\{q\} \in \sigma(\mathcal{B})$ for all $q \in \mathbb{Q}$. But then that means that any $E \subset \mathbb{Q}$ can be made in $\sigma(\mathcal{B})$ since $\mathbb{Q}$ is countable. Thus $2^{\mathbb{Q}} \subset \sigma(\mathcal{B})$. By definition $\sigma(\mathcal{B}) \subset 2^{\mathbb{Q}}$, hence equality.

3. Consider the following measures on $2^{\mathbb{Q}}$:

$$\nu_1(E) = |E| \qquad \qquad \nu_2(E) = \begin{cases} 0 & E = \varnothing \\ \infty & E = \varnothing \end{cases}.$$

   When restricted to $\mathcal{B}$, $\nu_1(E) = \infty$ since any set in $\mathcal{B}$ has countably many elements. Similarly, $\varnothing \notin \mathcal{B}$ so $\nu_2(E) = \infty$ for all E as well. Therefore both are equal to the premeasure on $\mathcal{B}$. However the measures are not the same on say $\{q\} \in 2^{\mathbb{Q}}$ since $\nu_1(\{q\}) = 1$ and $\nu_2(\{q\}) = \infty$.

◇

Exercise 6.19

Proof.

1. Let $D = A\Delta B$. Note that both $D \in \mathcal{A}$ and $D \cap E = \varnothing$ since

   - If $x \in A \cap E$, then $x \in B$ meaning $x \notin A \setminus B$
   - If $x \in B \cap E$, then $x \in A$ meaning $x \notin B \setminus A$
   - If $x \in E$ but $x \notin A$ or $x \notin B$, then $x \notin D$

   Thus $E \subset D^c$, meaning $\mu(E) \leq \mu(D^c)$. Since μ is a finite measure, it follows that $\mu(D^c) = \mu(X) - \mu(D)$. Since $\mu(X) = \mu^*(X) = \mu^*(E) = \mu(E)$, then
   $$\mu(X) \leq \mu(X) - \mu(D) \implies \mu(D) \leq 0.$$
   Therefore $\mu(D) = 0$. Since $A \setminus B \subset D$ and $B \setminus A \subset D$, as well as $A = (A \cap B) \cup (A \setminus B)$ and $B = (B \cap A) \cup (B \setminus A)$, it follows by monotonicity and additivity

   $$\begin{aligned}\mu(A) &= \mu(A \cap B) + \mu(A \setminus B)\\ &= \mu(A \cap B) + 0\\ &= \mu(B \cap A) + \mu(B \setminus A)\\ &= \mu(B)\end{aligned}$$

2. It has been shown in prior exercises that a σ-algebra restricted to some set is still a σ-algebra, so all that needs to be shown is that ν is a measure on $\mathcal{A}_E$. Note that ν is well defined by part (a). Let $B_n \in \mathcal{A}_E^{\mathbb{N}}$ be pairwise disjoint. Then there exists $A_n \in \mathcal{A}$ such that $B_n = A_n \cap E$. Let then
   $$C_n = A_n \setminus \bigcup_{k=1}^{n-1} A_k$$
   and note that all the C_n are pairwise disjoint. Also note
   $$C_n \cap E = (A_n \cap E) \setminus \bigcup_{k=1}^{n-1}(A_k \cap E) = B_n \setminus \bigcup_{k=1}^{n-1} B_k.$$

Since the B_k are disjoint, it follows then that $C_n \cap E = B_n$. Therefore

$$\bigcup_{n\in\mathbb{N}} B_n = \bigcup_{n\in\mathbb{N}} (C_n \cap E) = E \cap \bigcup_{n\in\mathbb{N}} C_n$$

meaning

$$\nu\left(\bigcup_{n\in\mathbb{N}} B_n\right) = \mu\left(\bigcup_{n\in\mathbb{N}} C_n\right) = \sum_{n\in\mathbb{N}} \mu(C_n) = \sum_{n\in\mathbb{N}} \nu(B_n).$$

Thus ν is countably additive and hence a measure.

$\diamond$

Exercise 6.29

Proof. Since each $\mathcal{A}_i$ generates themselves ($\sigma(\mathcal{A}_i) = \mathcal{A}_i$), then by proposition 6.22 it follows that

$$\mathcal{A}_1 \otimes \mathcal{A}_2 \otimes \mathcal{A}_3 = \sigma(\{A_1 \times A_2 \times A_3 : A_i \in \mathcal{A}_i\}).$$

But note that $\{A_1 \times A_2 : A_i \in \mathcal{A}_i\}$ generates $\mathcal{A}_1 \otimes \mathcal{A}_2$, so again by proposition 6.22

$$(\mathcal{A}_1 \otimes \mathcal{A}_2) \otimes \mathcal{A}_3 = \sigma(\{(A_1 \times A_2) \times A_3 : A_i \in \mathcal{A}_i\}).$$

Since $(A_1 \times A_2) \times A_3 = A_1 \times A_2 \times A_3$, equality of the two product algebras follows.

Suppose each of the μ_i are σ-finite. Then $X_i = \bigcup_{j\in\mathbb{N}} X_i^j$ where $\mu_i(X_i^j) < \infty$. Therefore $X_1 \times X_2 \times X_3 = \bigcup_{j,k,l\in\mathbb{N}} X_1^j \times X_2^k \times X_3^l$ and

$$(\mu_1 \times \mu_2 \times \mu_3)(X_1^j \times X_2^k \times X_3^l) = \mu_1(X_1^j)\mu_2(X_2^k)\mu_3(X_3^l) < \infty.$$

Therefore $\mu_1 \times \mu_2 \times \mu_3$ is σ-finite. Therefore by Theorem 6.10, it follows that $\mu_1 \times \mu_2 \times \mu_3$ is the unique measure on $\mathcal{A}_1 \times \mathcal{A}_2 \times \mathcal{A}_3$ in which

$$(\mu_1 \times \mu_2 \times \mu_3)(A_1 \times A_2 \times A_3) = \mu_1(A_1)\mu_2(A_2)\mu_3(A_3), \quad A_i \in \mathcal{A}_\rangle.$$

However since for any $A_i \in \mathcal{A}_i$ we have

$$((\mu_1 \times \mu_2) \times \mu_3)(A_1 \times A_2 \times A_3) = (\mu_1 \times \mu_2)(A_1 \times A_2)\mu_3(A_3) = \mu_1(A_1)\mu_2(A_2)\mu_3(A_3)$$

it follows $\mu_1 \times \mu_2 \times \mu_3 = (\mu_1 \times \mu_2) \times \mu_3$ by uniqueness. $\diamond$

Exercise 6.34

Proof.

Suppose towards contradiction that $(\mu_1 \times \mu_2)(D) < \infty$. Let $A_n \times B_n$ be a cover of D in $X_1 \times X_2$. Without loss of generality, it can be assumed that $m(A_0) = 0$ and $m(A_n) > 0$ for all $n \geq 1$, as well as $\mu_2(B_n) \geq 1$ for all n. Note that since $(\mu_1 \times \mu_2)(D) < \infty$, it follows

$$\sum_{n \geq 0} \mu_1(A_n)\mu_2(B_n) < \infty$$

which clearly means that $\mu_2(B_n) < \infty$ and thus the B_n are finite for all $n \geq 1$. Therefore letting

$$C = \bigcup_{n \geq 1} A_n \cap B_n$$

is a countable set and thus $m(C) = 0$. Since D is this special diagonal set, it follows that $x \in A_n \times B_n$ implies $x \in A_n$ and $x \in B_n$, thus

$$[0,1] \subset \bigcup_{n \geq 0} A_n \cap B_n = (A_0 \cap B_0) \cup C.$$

But then this means that

$$[0,1] \setminus C \subset A_0 \cap B_0 \implies A_0 \supset [0,1] \setminus C.$$

Since $m(C) = 0$, then by monotonicity it follows $m(A_0) \geq m([0,1]) = 1$, a contradiction. Therefore

$$\int \mathbb{1}_D d\mu_1 \times \mu_2 = (\mu_1 \times \mu_2)(D) = \infty.$$

$\diamond$