Chapter 1

Euclidean Space

1.1 Basic Structure

Def 1.1. Euclidean Space

Euclidean Space, denoted as $\mathbb{R}^n$, is the set of all n-tuples $x = (x_1, \ldots, x_n)$ with each $x_i \in \mathbb{R}$. x is called a **point** or a **vector**. Addition is defined for $\mathbb{R}^n$ for $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ as

$$x + y = (x_1 + y_1, \ldots, x_n + y_n).$$

Scalar multiplication is defined for $\lambda \in \mathbb{R}$ as

$$\lambda x = (\lambda x_1, \ldots, \lambda x_n).$$

This definition of Euclidean space lends itself to a vector space structure where the underlying field is $\mathbb{R}$.

Theorem 1.1.

$\mathbb{R}^n$ is a vector space over $\mathbb{R}$.

The proof is ommitted as it follows from the fact that $\mathbb{R}$ is a vector space and its properties are preserved under component wise operations. We further endow $\mathbb{R}^n$ with a **scalar product**.

Def 1.2. Euclidean Scalar Product

The **scalar product** of two vectors $x, y \in \mathbb{R}^n$ is

$$x \cdot y = x_1 y_1 + \ldots + x_n y_n.$$

It can be checked that this defines an inner product over $\mathbb{R}^n$, and thus also gives a natural **Euclidean norm** defined simply as $|x| = \|x\| = \sqrt{x \cdot x}$.

Theorem 1.2. Cauchy-Schwarz Inequality

For all $x, y \in \mathbb{R}^n$, $|x \cdot y| \leq |x||y|$.

Proof. If $y = 0$, then the inequality follows trivially. Assume then that $y \neq 0$. Let $t \in \mathbb{R}$ and $z = x + ty$. Note that $z \cdot z = |z|^2 \geq 0$. Therefore

$$\begin{aligned} 0 \leq (x+ty) \cdot (x+ty) &= x \cdot x + 2t(x \cdot y) + t^2(y \cdot y) \\ &= |x|^2 + 2t(x \cdot y) + t^2|y|^2 \\ &= |x|^2 + \left(|y|t + \frac{x \cdot y}{|y|}\right)^2 - \frac{(x \cdot y)^2}{|y|^2} \end{aligned}$$

Since t was arbitrary, taking t to be

$$t = -\frac{x \cdot y}{|y|^2}$$

gives

$$0 \leq |x|^2 - \frac{(x \cdot y)^2}{|y|^2} \implies (x \cdot y)^2 \leq |x|^2|y|^2.$$

Rooting both sides gives the desired result. $\diamond$

Corollary 1.1. Triangle Inequality

For any $x, y \in \mathbb{R}^n$, $|x + y| \leq |x| + |y|$.

Proof. Note that

$$\begin{aligned} |x+y|^2 &= (x+y)\cdot(x+y) \\ &= |x|^2 + 2x\cdot y + |y|^2 \\ &\leq |x|^2 + 2|x||y| + |y|^2 \qquad (\star) \\ &= (|x|+|y|)^2 \end{aligned}$$

where $(\star)$ follows from Cauchy Schwarz. Taking the root of both sides gives the desired result. ◇

Def 1.3. Euclidean distance

The **distance** between $x, y \in \mathbb{R}^n$ is denoted as $d(x,y) := |x-y|$.

Theorem 1.3.

$d(x,y)$ defines a metric on $\mathbb{R}^n$ in the sense that for all $x, y, z \in \mathbb{R}^n$

i) $d(x,y) \geq 0$ and $d(x,y) = 0$ iff $x = y$

ii) $d(x,y) = d(y,x)$

iii) $d(x,z) \leq d(x,y) + d(y,z)$

Proof. Both (i) and (ii) follow from the properties of a norm on a vector space. For (iii), note that

$$d(x,z) = |x-z| = |(x-y)+(y-z)| \leq |x-y| + |y-z| = d(x,y) + d(y,z)$$

which was to be shown. ◇

Because $d(x,y)$ is a metric, it is called the **Euclidean metric** and $\mathbb{R}^n$ equipped with d is called a **metric space**.

1.2 Topology of $\mathbb{R}^n$

Def 1.4. Open Ball

Let $r > 0$ and $a \in \mathbb{R}^n$. Then the **open ball** centered at a or radius r is the set

$$B_r(a) = \{x \in \mathbb{R}^n | d(x, a) < r\}.$$

Def 1.5. Open Set

A set $G \subseteq \mathbb{R}^n$ is **open** if for every $a \in G$, $\exists r > 0$ such that $B_r(a) \subseteq G$.

Theorem 1.4.

Open balls are open sets.

Proof. Let $b \in B_r(a)$. That is $|b - a| < r$. Take $\rho = r - |a - b| \geq 0$ and consider some $x \in B_\rho(b)$. Then $|x - b| < \rho = r - |a - b|$ and

$$|x - a| \leq |x - b| + |b - a| < r - |a - b| + |b - a| = r.$$

Therefore $x \in B_r(a)$, meaning $B_\rho(b) \subseteq B_r(a)$. Hence $B_r(a)$ is open. $\diamond$

Theorem 1.5. $\mathbb{R}^n$ is a topology

The following hold in $\mathbb{R}^n$

i) Let $(G_\alpha)_{\alpha \in J}$ be a collection of open sets. Then $\bigcup_{\alpha \in J} G_\alpha$ is open.

ii) Let $(G_\alpha)_{\alpha \in J}$ be a *finite* collection of open sets. Then $\bigcap_{\alpha \in J} G_\alpha$ is open.

Proof.

i) Let $x \in \bigcup_{\alpha \in J} G_\alpha$. Then there is some G_α such that $x \in G_\alpha$. This set must be open, thus there is some $r > 0$ such that $B_r(x) \subseteq G_\alpha$. But note that $G_\alpha \subseteq \bigcup_{\alpha \in J} G_\alpha$. Thus the union is open.

ii) If $\bigcap_{\alpha \in J} G_\alpha = \varnothing$, then trivially the intersection is open. Assume then that $x \in \bigcap_{\alpha \in J} G_\alpha \neq \varnothing$. Then $x \in G_\alpha$ for all $\alpha \in J$. Thus there is a collection of radii r_α such that $B_{r_\alpha}(x) \subseteq G_\alpha$. Taking $r = \min_{\alpha\in J} r_\alpha$, the ball $B_r(x) \subseteq B_\alpha(x) \subseteq G_\alpha$ for all $\alpha \in J$. Thus the intersection is open.

◇

Remark. The intersection of an infinite collection of open sets is not neccesarily open. Consider the family of open intervals in $\mathbb{R}$ of the form

$$J_n = \left(-\frac{1}{n}, \frac{1}{n}\right).$$

Note that $\bigcap J_n = \{0\}$ which is not open.

Def 1.6. Neighborhood

Let $a \in \mathbb{R}^n$. A **neighborhood** of a is an open set $G \subseteq \mathbb{R}^n$ such that $a \in G$. Often the term *nbhd* is used as a shorthand.

Remark. If G is a nbhd of a, then $\exists r > 0$ such that $B_r(a) \subseteq G$.

Def 1.7. Interior

The **interior** of a set $A \subseteq \mathbb{R}^n$ is defined as

$$\text{int}(A) := \{x \in \mathbb{R}^n : x \text{ has a nbhd } G \subseteq A\}.$$

Example.

i) $\text{int}([a,b)) = (a,b)$ since any nbhd of a will contain points outside of the interval.

ii) Let $A = \{(x,y) \in \mathbb{R}^2 : x,y \geq 0\}$. Then $\text{int}(A) = \{(x,y) \in \mathbb{R}^2 : x,y > 0\}$ as any point along the axes fail by the same reasoning as above.

iii) $\text{int}(\mathbb{Q}) = \varnothing$ because there will always be an irrational x in any ball based around a rational number.

Theorem 1.6.

For any $A \subseteq \mathbb{R}^n$

i) $\text{int}(A)$ is open

ii) $\text{int}(A)$ is the largest open set contained in A

Proof. Let $x \in \text{int}(A)$. Then there is some nbhd G such that $G \subseteq A$. Let $y \in G$. Since G is open, G is a nbhd of y as well hence $y \in \text{int}(A)$. Therefore $G \subseteq \text{int}(A)$ meaning $\text{int}(A)$ is open. ◇

Def 1.8. Closed set

A set $F \subseteq \mathbb{R}^n$ is **closed** if its complement F^c is open.

Example.

i) Both $\varnothing$ and $\mathbb{R}^n$ are closed

ii) $[a, b]$ is closed for all $a \neq b$

iii) $[a, \infty)$ is closed since $[a, \infty)^c = (-\infty, a)$ which is open

Theorem 1.7.

For every $a \in \mathbb{R}^n$ and $r > 0$, the closed ball $B_r[a] = \{x \in \mathbb{R}^n : |x - a| \leq r\}$ is closed in $\mathbb{R}^n$.

Proof. If $B_r[a]^c = \{x \in \mathbb{R}^n : |x - a| > r\}$ is open, then the desired result is achieved. Let $x \in B_r[a]^c$. Since $|x - a| > r$, then $\exists \rho > 0$ such that $|x - a| = r + \rho$. Take $y \in B_\rho(x)$. Then

$$\begin{aligned} |x - a| \leq |x - y| + |y - a| &\implies |y - a| \geq |x - a| - |x - y| \\ &\implies |y - a| > |x - a| - \rho = r \end{aligned}$$

Therefore $y \in Br[a]^c$, meaning $B_r[a]$ is open. ◇

Def 1.9. Cluster Point

Let $A \subseteq \mathbb{R}^n$. Then $x \in \mathbb{R}^n$ is a **cluster point** of A if every nbhd of x intersects A. Equivalently, x is a cluster point of A iff for every $r > 0$, $B_r(x) \cap A \neq \varnothing$.

Remark. Any point $x \in A$ is a cluster point since $x \in B_r(x)$ for any $r > 0$ and hence $\varnothing \neq \{x\} \subseteq B_r(x) \cap A$. However, it need be that a cluster point is an element of A.

Example.

i) Consider $A_1 = \left\{\frac{1}{n} : n = 1, 2, 3, \dots\right\} \subseteq \mathbb{R}$. The point 0 is a cluster point since for any $r > 0$, $\exists n \geq 1$ such that $\frac{1}{n} < r$. However $0 \notin A_1$

ii) Consider $A_2 = \left\{(x, y) \in \mathbb{R}^2 : x, y > 0\right\}$. The set of all cluster points is $\left\{(x, y) \in \mathbb{R}^2 : x, y \geq 0\right\}$

Def 1.10. Closure

The set of all cluster points for a set $A \subseteq \mathbb{R}^n$ is the **closure** of A, denoted as $\overline{A}$.

For Example 3, the closure of A_1 is $\overline{A_1} = A_1 \cup \{0\}$ and the closure of A_2 is $\left\{(x, y) \in \mathbb{R}^2 : x, y \geq 0\right\}$. These sets are both closed, a fact which holds in general.

Theorem 1.8. Properties of Closure

Let $A \subseteq \mathbb{R}^n$. Then

i) $\overline{A}^c = \operatorname{int}(A^c)$

ii) $\overline{A}$ is closed

iii) $\overline{A}$ is the smallest closed set containing A

iv) F is closed if and only if $F = \overline{F}$

Proof.

i) Let $x \in \overline{A}^c$. Then x is not a cluster point. Therefore there is some nbhd G of x such that $G \cap A = \varnothing$. Thus $G \subseteq A^c$, hence $x \in \text{int}(A^c)$. Let $x \in \text{int}(A^c)$. Then there is some nbhd H such that $H \subseteq A^c$. Therefore $H \cap A = \varnothing$ meaning x is not a cluster point of A. Thus $x \notin \overline{A} \implies x \in \overline{A}^c$.

ii) From (i), the complement of the closure of a set is the interior of a set. The interior of a set is always open, thus the closure of a set is closed.

iii) Let $F \subseteq \mathbb{R}^n$ such that F is closed and $A \subseteq F$. Note that $A^c \supseteq F^c$ and that F^c is open. Furthermore $\text{int}(A^c)$ is the largest open set contained in A^c, therefore $F^c \subseteq \text{int}(A^c)$. Taking the complement and applying (i) gives $F \supseteq (\text{int}(A^c))^c = \overline{A}$.

iv) Assume that F is closed. Since trivially $F \subseteq F$, by (iii) it follows $\overline{F} \subseteq F$. By definition, $F \subseteq \overline{F}$. Thus $F = \overline{F}$. Assume that $F = \overline{F}$. By (ii), $\overline{F}$ is closed and therefore F is closed.

$\diamond$

Theorem 1.9. Closed Set Families

The following hold in $\mathbb{R}^n$

i) Let $(F_\alpha)_{\alpha \in J}$ be a collection of closed sets. Then $\bigcap_{\alpha \in J} F_\alpha$ is closed

ii) Let $(F_\alpha)_{\alpha \in J}$ be a finite collection of closed sets. Then $\bigcup_{\alpha \in J} F_\alpha$ is closed

Proof.

i) Note that by De'Morgan's,

$$\left(\bigcap_{\alpha \in J} F_\alpha\right)^c = \bigcup_{\alpha \in J} F_\alpha^c.$$

Since every F_α^c is open, and the union of a family of open sets is open, then the complement of the intersection is open. Hence the intersection is closed.

ii) The same application of De'Morgan's gives the desired result.

$\diamond$

Remark. Consider the family of closed sets $F_n = \left[-1 + \frac{1}{n}, 1 - \frac{1}{n}\right]$. Note that

$$\bigcup_{n=1}^{\infty} F_n = (-1, 1)$$

hence the infinite union of closed sets is not necessarily closed.

Def 1.11. Boundary

For a set $A \subseteq \mathbb{R}^n$, the **boundary** of A is $\partial A := \overline{A} \cap \overline{A^c}$. Equivalently, the boundary is $\partial A = \overline{A} \setminus \operatorname{int}(A)$.

Example. For an open ball $B_r(a) \subseteq \mathbb{R}^n$, its boundary is

$$\begin{aligned}
\partial B_r(a) &= \overline{B_r(a)} \setminus \operatorname{int}(B_r(a)) \\
&= \{x \in \mathbb{R}^n : |x - a| \leq r\} \setminus \{x \in \mathbb{R}^n : |x - a| < r\} \\
&= \{x \in \mathbb{R}^n : |x - a| = r\}
\end{aligned}$$

Chapter 2

Sequences

Def 2.1. Sequence

A **sequence** in $\mathbb{R}^n$ is a map $f : \mathbb{N} \to \mathbb{R}^n$ where

$$f(k) \coloneqq x^{(k)} = (x_1^{(k)}, \dots, x_n^{(k)}) \in \mathbb{R}^n$$

and is denoted by $\left\{x^{(k)}\right\}$ or $\left(x^{(k)}\right)_{k=1}^{\infty}$.

Def 2.2. Convergence

Let $\left\{x^{(k)}\right\}$ be a sequence in $\mathbb{R}^n$. Then $x^{(k)}$ **converges** to a point $a \in \mathbb{R}^n$ if

$$\lim_{k\to\infty} \left|x^{(k)} - a\right| = 0.$$

Equivalently

1. For all $\varepsilon > 0$, $\exists K \in \mathbb{N}$ such that for all $k \geq K$,

$$|x^{(k)} - a| \leq \varepsilon$$

2. For every nbhd V of a, there is some K such that $x^{(k)} \in V$ for all $k \geq K$

Theorem 2.1.

A sequence $\left\{x^{(k)}\right\}$ converges to $a \in \mathbb{R}^n$ iff for every $1 \leq j \leq n$, the sequence $\left\{x_j^{(k)}\right\}$ converges to a_j.

Proof. For $y \in \mathbb{R}^n$, note that

$$|y_j| \leq \|y\| \leq \sum_{i=1}^{n} |y_i|.$$

Therefore

$$0 \leq \left|x_j^{(k)} - a_j\right| \leq \left\|x^{(k)} - a\right\| \leq \sum_{j=1}^{n} \left|x_j^{(k)} - a_j\right|.$$

Assuming the forward direction, it follows that $\left\|x^{(k)} - a\right\| \to 0$ thus by the squeeze lemma $\left|x_j^{(k)} - a_j\right| \to 0$. Assuming the reverse direction, it follows that $\sum_{j=1}^{n} \left|x_j^{(k)} - a_j\right| \to 0$ which again by squeeze lemme means $\left\|x^{(k)} - a\right\| \to 0$, which was to be shown. ◇

Theorem 2.2. Cluster Point $\Leftrightarrow$ Limit Point

Let $A \subseteq \mathbb{R}^n$ and $x \in \mathbb{R}^n$. Then $x \in \overline{A}$ iff there exists a sequence $\left\{x^{(k)}\right\}$ in A that converges to x.

Proof.

$\Leftarrow$) Suppose such a sequence exists. Then for every nbhd V of x, there is some K such that $x^{(k)} \in V$ for all $k \geq K$. Since $x^{(k)} \in A$ for all k, then it follows $A \cap V \neq \varnothing$, thus $x \in \overline{A}$.

$\Rightarrow$) Suppose $x \in \overline{A}$. Then for any $k \geq 1$, $B_{k^{-1}}(x) \cap A \neq \varnothing$. Therefore for each k, pick some $x^{(k)} \in B_{k^{-1}}(x) \cap A$. Then

$$\left\|x^{(k)} - x\right\| < \frac{1}{k} \to 0$$

thus $\left\{x^{(k)}\right\}$ is such a sequence.

◇

Def 2.3. Bounded Sequence

A sequence $\{x^{(k)}\}$ in $\mathbb{R}^n$ is **bounded** if there exists $M \geq 0$ such that $\|x^{(k)}\| \leq M$ for all $k \geq 1$.

Def 2.4. Subsequence

Let $\left(x^{(k)}\right)_{k=1}^{\infty}$ be a sequence in $\mathbb{R}^n$ and $\phi : \mathbb{N} \to \mathbb{N}$ be a strictly increasing function. Then the sequence $\left(x^{(\phi(l))}\right)_{l=1}^{\infty}$ is a **subsequence** of the original sequence. The subindex will be denoted simply as $k_l \equiv \phi(l)$.

Theorem 2.3.

For any subsequence, $k_l \geq l$ for all $l \geq 1$.

Proof. Proceed with induction. Note that $k_1 \geq 1$ for any subsequence, thus the base case holds. Assume for some fixed l that $k_l \geq l$. Since the associated ϕ is strictly increasing

$$k_{l+1} > k_l \geq l \implies k_{l+1} \geq l + 1$$

which was to be shown. ◇

Theorem 2.4. Bolzano-Weierstrass

Every bounded sequence in $\mathbb{R}^n$ has a convergent subequence.

Proof. Let $\{x^{(k)}\}$ be a bounded sequence in $\mathbb{R}^n$. Note that $\left|x_j^{(k)}\right| \leq \|x^{(k)}\|$ for all $1 \leq j \leq n$ and $k \geq 1$. Therefore each element wise sequence is bounded. Thus by Bolzano-Weierstrass in $\mathbb{R}$, the first component has a convergent subsequence with index k_{j_1}. The second component under this index must also be bounded, thus Bolzano-Weiestrass applies to it as well to get another index k_{j_2}. This can be continued until an index k_{j_n}

is reached. It is guaranteed by its construction that every component of $x^{(k_{j_n})}$ converges. Thus the subsequence $\left\{x^{k_{j_n}}\right\}$ converges. $\diamond$

Def 2.5. Cauchy Sequence

A sequence $\left\{x^{(k)}\right\}$ in $\mathbb{R}^n$ is **Cauchy** if $\forall \varepsilon > 0$, there exists $K \in \mathbb{N}$ such that

$$\left\|x^{(k)} - x^{(l)}\right\| \leq \varepsilon \quad l, k \geq K.$$

Theorem 2.5.

Cauchy sequences in $\mathbb{R}^n$ are bounded.

Proof. Let $\left\{x^{(k)}\right\}$ be a Cauchy sequence in $\mathbb{R}^n$. Take $\varepsilon = 1$ and $l = k$. Then as in the definition, for all $k \geq K$,

$$\left\|x^{(k)}\right\| \leq \left\|x^{(k)} - x^{(K)}\right\| + \left\|x^{(K)}\right\| \leq \left\|x^{(K)}\right\| + 1.$$

Take $M = \max\left\{\left\|x^{(1)}\right\|, \ldots, \left\|x^{(K-1)}\right\|, \left\|x^{(K)} + 1\right\|\right\}$. This is well defined since K is finite, and bounds every element. $\diamond$

Theorem 2.6. Completeness

A sequence converges iff it is Cauchy.

Proof. Let $\left\{x^{(k)}\right\}$ be a sequence in $\mathbb{R}^n$.

$\Rightarrow$) Assume $\left\{x^{(k)}\right\}$ converges with limit $a \in \mathbb{R}^n$. Then for all $\varepsilon > 0$, there exists $K \in \mathbb{N}$ such that $\left\|x^{(k)} - a\right\| \leq \frac{\varepsilon}{2}$ for $k \geq 2$. Now consider $k, l \geq K$. Note then that

$$\left\|x^{(k)} - x^{(l)}\right\| \leq \left\|x^{(k)} - a\right\| + \left\|x^{(l)} - a\right\| \leq 2 \cdot \frac{\varepsilon}{2} = \varepsilon.$$

Thus $\left\{x^{(k)}\right\}$ is Cauchy.

$\Leftarrow$) Assume $\left\{x^{(k)}\right\}$ is Cauchy. Then it is bounded and thus by Bolzano-Weierstrass, it has a convergent subsequence $\left\{x^{(k_j)}\right\}$ with some limit

$a \in \mathbb{R}^n$. Take $\varepsilon > 0$. Then there exists $J \in \mathbb{N}$ such that for all $j \geq J$

$$\left\| x^{(k_j)} - a \right\| \leq \frac{\varepsilon}{2}.$$

Since $\left\{x^{(k)}\right\}$ is Cauchy, there exists $K \in \mathbb{N}$ such that for all $k, l \geq K$

$$\left\| x^{(k)} - x^{(l)} \right\| \leq \frac{\varepsilon}{2}.$$

Take $N = \max\{K, J\}$. Note that $k_j > j$ and so if $k \geq N$ and $j \geq N$

$$\left\| x^{(k)} - a \right\| \leq \left\| x^{(k)} - x^{(k_j)} \right\| + \left\| x^{(x_j)} - a \right\| \leq 2 \cdot \frac{\varepsilon}{2} = \varepsilon.$$

Thus $\left\{x^{(k)}\right\}$ converges.

$\diamond$

Chapter 3

Functions

Def 3.1. Function Terminology

Consider a function $f : D \to \mathbb{R}^p$ where $D \subseteq \mathbb{R}^n$. The **domain** of f is D and the **range** of f is $f(D) := \{f(x) : x \in D\}$.

Example. A function $L : \mathbb{R}^n \to \mathbb{R}$ is a *linear function* if it is of the form

$$L(x) = c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

where $c_i \in \mathbb{R}$. These functions are *linear* in their arguments, meaning $L(ax + by) = aL(x) + bL(y)$ for $a, b \in \mathbb{R}$.

Example. A function $Q : \mathbb{R}^n \to \mathbb{R}$ is a *quadratic form* if it is of the form

$$Q(x) = \sum_{i \leq j,k \leq n} c_{jk}x_jx_k$$

where $c_{jk} \in \mathbb{R}$. For example $\|\cdot\|$ is a quadratic form.

Example. A function's domain need not be all of $\mathbb{R}^n$. Consider $f : D \to \mathbb{R}^2$ where $D = \left\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 4, (x, y) \neq (0, 0)\right\}$ and

$$f(x, y) = \left(\sqrt{4 - x^2 - y^2}, \log\sqrt{x^2 + y^2}\right).$$

This function is well defined on D.

Def 3.2. Limit Point

Let $A \subseteq \mathbb{R}^n$. Then $x \in \mathbb{R}^n$ is a **limit point** of A if for all $r > 0$, $(B_r(x) \setminus \{x\}) \cap A \neq \emptyset$. The set of all limits points of A is denoted as A'.

Remark. If x is a limit point of A, then x is a cluster point. That is, $x \in \overline{A}$. However, the converse is not true.

- Consider $A = B_1(0) \cup P$ for some $P \notin B_1[0]$. Note that $\overline{A} = B_1[0] \cup P$ but $A' = B_1[0]$.
- Consider $A = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$. Note that $\overline{A} = A \cup \{0\}$ but $A' = \{0\}$.

Def 3.3. Limit

Let $D \subseteq \mathbb{R}^n$, $f : D \to \mathbb{R}^n$, and $a \in \mathbb{R}^n$ be a limit point of D. Then $f(x)$ has a **limit** to $b \in \mathbb{R}^p$ when x tends to a if for all $\varepsilon > 0$, $\exists \delta > 0$ such that

$$0 < |x - a| \leq \delta \implies |f(x) - b| \leq \varepsilon.$$

This limit is denoted as $\lim_{x \to a} f(x) = b$.

Remark. Limit points must be used when defining limits to ensure that $0 < |x - a| < \delta$ is a non empty set.

Theorem 3.1.

Let $D \subseteq \mathbb{R}^n$, $f : D \to \mathbb{R}^p$ and $a \in \mathbb{R}^n$ be a limt point of D. Then

$$\lim_{x \to a} = b \Leftrightarrow f\left(x^{(k)}\right) \to b, k \to \infty$$

for every $x^{(k)} \to a$ and $x^{(k)} \neq a$.

Corollary 3.1.

Using the same setup as above and letting $f_j : \mathbb{R}^n \to \mathbb{R}$ and b_j be the j^{th} components, then

$$\lim_{x \to a} f(x) = b \Leftrightarrow \lim_{x \to a} f_j\left(x^{(k)}\right) = b_j, 1 \leq j \leq n.$$

Corollary 3.2.

Let $f : D \to \mathbb{R}^p$ and $p : D \to \mathbb{R}^p$ with $D \subseteq \mathbb{R}^n$ and $a \in \mathbb{R}^n$ be a limit point of D. If $f(x) \to b$ and $p(x) \to d$ as $x \to a$ then

1. $f(x) + p(x) \to b + d$
2. $\lambda f(x) \to \lambda b$ for all $\lambda \in \mathbb{R}$
3. If $p = 1$, then $f(x)p(x) \to bd$ and $\frac{f(x)}{p(x)} \to \frac{b}{d}$ if $d \neq 0$.

3.1 Continuity

Def 3.4. Continuity

Let $D \subseteq \mathbb{R}^n$ and $a \in D$. A map $f : D \to \mathbb{R}^p$ is **continuous at a** if for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|x - a\| \leq \delta \implies \|f(x) - f(a)\| \leq \varepsilon.$$

If f is continuous at all $x \in D$, then f is **continuous on D**.

Remark. If a is a limit point, then f is continuous iff $\lim_{x \to a} f(x) = f(a)$.

Theorem 3.2. Sequential Continuity

Let $f : D \to \mathbb{R}^p$, $a \in D$. Then f is continuous at a iff for every sequence $x^{(k)}$ in D that converges to a, $\lim_{k \to \infty} f\left(x^{(k)}\right) = f(a)$.

Corollary 3.3.

$f : D \to \mathbb{R}^p$ is continuous at $a \in D$ iff f_j is continuous at a for all $1 \leq j \leq n$.

Corollary 3.4.

If $f, p : D \to \mathbb{R}^p$ are continuous at $a \in D$, then $f + p$, fp are continuous and if $p \neq 0$ then $\frac{f}{p}$ is continuous.

Example. Consider the map $f : \mathbb{R}^2 \to \mathbb{R}$ where

$$f(x) = \begin{cases} \frac{x_1 x_2}{x_1^2 + x_2^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Consider the continuity of f at $x = 0$. Note that $f(x_1, 0) \to 0$ as $x_1 \to 0$ and $f(x_1, x_1) \to \frac{1}{2}$ as $x_1 \to 0$. Thus $\lim_{x \to 0} f(x)$ does not exist, and thus f is not continuous at 0.

Example. Let $A \subseteq \mathbb{R}^n$ and $x \in \mathbb{R}^n$. Define the distance x from A as $d(x, A) = \inf_{a \in A} \|x - a\|$. Then $d(\cdot, A) : \mathbb{R}^n \to \mathbb{R}$ is continuous.

Proof. Let $x_0 \in \mathbb{R}^n$. Note for any $a \in A$ that $d(x_0, A) \leq \|x_0 - a\|$. Let $y \in \mathbb{R}^n$. Note then that

$$\begin{aligned} d(x_0, A) &\leq \|x_0 - a\| \\ &\leq \|x_0 - y\| + \|y - a\| \\ &\leq \|x_0 - y\| + d(y, A) \end{aligned}$$

Therefore $d(x_0, A) - d(y, A) \leq \|x_0 - y\|$. By symmetry, x_0 and y can be swapped and so $d(x_0, A) - d(y, A) \leq \|x_0 - y\|$. Thus $|d(x_0, A) - d(y, A)| \leq \|x_0 - y\|$. Let $\varepsilon > 0$ and $\delta = \varepsilon$. Then for $\|x_0 - y\| \leq \delta$, it follows that

$$|d(x_0, A) - d(y, A)| \leq \|x_0 - y\| \leq \delta = \varepsilon.$$

Therefore $d(\cdot, A)$ is continuous. ◇

Def 3.5. Preimage

Let $f : \mathbb{R}^n \to \mathbb{R}^p$ and $G \subseteq \mathbb{R}^p$. The **preimage** of G under f is $f^{-1}(G) = \{x \in \mathbb{R}^n : f(x) \in G\}$.

Theorem 3.3. Topological Continuity

A function $f : \mathbb{R}^n \to \mathbb{R}^p$ is continuous iff $f^{-1}(G)$ is open (closed) in $\mathbb{R}^n$ for every open (closed) set in $\mathbb{R}^p$.

Lemma 3.1.

For any map $f : A \to B$ and set $F \subseteq B$, $f^{-1}(F^c) = f^{-1}(F)^c$.

Proof. Note that

$$\begin{aligned} x \in f^{-1}(F^c) &\Leftrightarrow f(x) \in F^c \\ &\Leftrightarrow f(x) \notin F \\ &\Leftrightarrow x \notin f^{-1}(F) \\ &\Leftrightarrow x \in f^{-1}(F)^c \end{aligned}$$

which was to be shown. ◇

Proof of Theorem 3.3. We consider the open case first. Suppose G is open. Let $a \in f^{-1}(G)$. Then $f(a) \in G$, thus $\exists \varepsilon > 0$ such that $B_\varepsilon(f(a)) \subseteq G$. That is,

$$\|y - f(a)\| < \varepsilon \implies y \in G. \qquad (\star)$$

Since f is continuous at a, $\exists \delta > 0$ such that $\forall x \in \mathbb{R}^n, \|x - a\| < \delta \implies \|f(x) - f(a)\| < \varepsilon$. Consider $x \in B_\delta(a)$. Then $\|x - a\| < \delta$, thus by continuity of f and $(\star)$, $f(x) \in G$ meaning $x \in f^{-1}(G)$. Thus $B_\delta(a) \subseteq f^{(-1)}(G)$, meaning $f^{-1}(G)$ is open.

Suppose that $f^{-1}(G)$ is open for all open $G \subseteq \mathbb{R}^p$. Take $a \in \mathbb{R}^n$, $\varepsilon > 0$ and let $G = B_\varepsilon(f(a))$. Note that G is open. Suppose $a \in f^{-1}(G)$.

Then $\exists\delta > 0$ such that $B_\delta(a) \subseteq f^{-1}(G)$. Therfore $x \in B_\delta(a) \implies x \in f^{-1}(x) \implies f(x) \in G$. Equivalently,

$$\|x - a\| < \delta \implies \|f(x) - f(a)\| < \varepsilon.$$

Thus f is continuous.

To prove the closed case, suppose f is continuous. Take $F \subseteq \mathbb{R}^p$ closed. Then F^c is open in $\mathbb{R}^p$. Thus $f^{-1}(F^c)$ is open. By lemma 3.1, $f^{-1}(F^c) = f^{-1}(F)^c$, thus $f^{-1}(F)^c$ is open. Therefore $f^{-1}(C)$ is closed. The reverse direction follows by a similar argument as above. ◇

Remark. It is not true generally that a continuous map takes open sets to open sets, nor closed set into closed sets. The zero map $f : \mathbb{R} \to \mathbb{R} : x \mapsto 0$ is continuous, but $f((a, b)) = \{0\}$ which is closed. The map $f : \mathbb{R} \to \mathbb{R} : x \mapsto \frac{x^2}{x^2+1}$ is continuous as well, but $f(\mathbb{R}) = [0, 1)$, meaning both an open and closed set is mapped to a set that is neither open or closed.

3.2 Compactness and Uniform Continuity

Def 3.6. Sequential Compactness

A set $K \subseteq \mathbb{R}^n$ is **sequentially compact** if every sequence $(x^{(k)})$ in K has a convergent subsequence that converges to a point in K.

Example. The closed ball $B_r[a] \subseteq \mathbb{R}^n$ is compact. Let $(x^{(k)})$ be a sequence in $B_r[a]$. Note that $\left\|x^{(k)}\right\| \leq \left\|x^{(k)} - a\right\| + \|a\| \leq r + \|a\|$. Thus $(x^{(k)})$ is bounded. Therefore by Bolzano-Weierstrass there exists a subsequence $(x^{(k_j)})$ in K that converges to some point $x \in \mathbb{R}^n$. Since the norm is continuous,

$$\lim_{j\to\infty} \left\|x^{(k_j)} - a\right\| \leq r \implies \|x - a\| \leq r.$$

Thus $x \in B_r[a]$.

Theorem 3.4.

Let $f : \mathbb{R}^n \to \mathbb{R}^p$ be continuous and $K \subseteq \mathbb{R}^n$ be compact. Then $f(K)$ is also compact.

Proof. Let $(y^{(k)})$ be a sequence in $f(K)$. Therefore there exists a sequence $(x^{(k)})$ in K where $f(x^{(k)}) = y^{(k)}$. Since K is compact, there exists a subsequence (x^{k_j}) that converges to a point $a \in K$. Since f is continuous, then $f(x^{(k_j)}) = y^{(k_j)} \to f(a)$ as $j \to \infty$. Thus the subseqeunce $(y^{(k_j)})$ converges to $f(a) \in f(K)$, hence $f(K)$ is compact. $\diamond$

Def 3.7. Bounded Set

A set $A \subseteq \mathbb{R}^n$ is **bounded** if there exists $M > 0$ such that

$$\|a\| \leq M, \forall a \in A.$$

Theorem 3.5. Compactness $\Leftrightarrow$ Closed and Bounded

Let $K \subseteq \mathbb{R}^n$. Then K is compact iff K is closed and bounded.

Proof.

$\Leftarrow$) Suppose K is closed and bounded. Let $\left(x^{(k)}\right)$ be a sequence of elements in K. Since K is bounded, there exists $M > 0$ such that $\|a\| \leq M$ for all $a \in K$. Therefore $\left\|x^{(k)}\right\| \leq M$ for all $k \geq 0$, thus $\left(x^{(k)}\right)$ is bounded. By Bolzano-Weiestrass, there then exists a subsequence $\left(x^{(k_j)}\right)$ that converges to a point $x \in \mathbb{R}^n$. Since K is closed, $a \in K$. Therefore K is compact. $\diamond$

$\Rightarrow$) Suppose K is compact. Let $a \in \overline{K}$. Then there exists a sequence $\left(x^{(k)}\right)$ of elements in K that converges to a. Since K is compact, there exists a subsequence in K that converges to some $\tilde{a} \in K$. But by the uniqueness of the limit, $a = \tilde{a} \in K$. Therefore $\overline{K} \subseteq K \implies K = \overline{K}$ meaning K is closed. Suppose towards contradiction that K is *not bounded*. Then for any $l \in \mathbb{N}$, there exists $x^{(l)} \in K$ such that $\left\|x^{(l)}\right\| > l$. K is compact therefore there is a subsequence of these terms $\left(x^{(l_j)}\right)$ that converges to some $a \in K$. Since $\left(x^{(k)}\right)$ is convergent, it is bounded. On the other hand, $\left\|x^{(l_j)}\right\| > l_j \geq j$ which means $\left\|x^{(l_j)}\right\| \to \infty$ as $j \to \infty$, a contradiction. Therefore K must be bounded. $\diamond$

Remark. For a general metric space, it is only true in general that K is compact implies K is closed and bounded.

Remark. Let $f : \mathbb{R}^n \to \mathbb{R}^p$ be continuous and $K \subseteq \mathbb{R}^p$ be compact. Then $f^{-1}(K)$ is closed in $\mathbb{R}^n$. However, it need not be compact. For example, consider $f : \mathbb{R} \to \mathbb{R}^2$ where $f(t) = (\cos(t), \sin(t))$. Clearly f is continuous, and $f(\mathbb{R}) = S^1$. However, this means that S^1 which is a compact set under the preimage maps to $\mathbb{R}$, which is not bounded.

Theorem 3.6.

Let $K \subseteq \mathbb{R}^n$ be a compact non-empty set and $f : K \to \mathbb{R}$ be continuous. Then f is bounded and achieves its supremum and infimum. That is $\exists a, b \in K$ such that

$$\sup_{x \in K} f(x) = f(a) \qquad \inf_{x \in K} f(x) = f(b).$$

Proof. Since f is continuous, $f(K)$ is compact and therefore bounded. Hence f is bounded. Note that $f(K) \neq \varnothing$ is bounded. Thus there exists $\sup f(K) = L$. By definition of the supremum, $\forall \varepsilon > 0, \exists x \in K$ such that $L - \varepsilon < f(x) < L$. Take $\varepsilon = \frac{1}{k}$ for $k \in \mathbb{N}$. Then there exists an $x^{(k)}$ for each k such that $L - \frac{1}{k} < f(x^{(k)}) < L$. As $k \to \infty$, it follows $f(x^{(k)}) \to L$. Since $f(x^{(k)})$ is a sequence in $f(K)$ and $f(K)$ is compact and thus closed, $\exists a \in K$ such that $f(a) = L$. A similar argument can be applied to the infimum. $\diamond$

Def 3.8. Uniform Continuity

Let $f : D \to \mathbb{R}^p$ where $D \subseteq \mathbb{R}^n$. Then f is **uniformly continuous** on D if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in D$

$$\|x - y\| \le \delta \implies \|f(x) - f(y)\| \le \varepsilon.$$

Example. Consider the distance function $d(x, A)$ for some $A \subseteq \mathbb{R}^n$. Then the function $d(\cdot, A)$ is uniformly continuous. Consider $\varepsilon > 0$ and take $\delta = \varepsilon$. Take $x_0, y \in \mathbb{R}^n$ such that $\|x_0 - y\| \le \delta = \varepsilon$. Then

$$|d(y, A) - d(x_0, A)| \le \|y - x_0\| \le \varepsilon$$

follows from a previous example. Thus the distance function is uniformly continuous.

Theorem 3.7.

Let $K \subseteq \mathbb{R}^n$ be compact and $f : K \to \mathbb{R}^n$ be continuous. Then f is uniformly continuous on K.

Proof. Suppose towards contradiction that f is not uniformly continuous. Then $\exists \varepsilon > 0$ such that $\forall \delta > 0$, there exists $x, y \in K$ where $\|x - y\| < \delta$ while $\|f(x) - f(y)\| > \varepsilon$. Letting $\delta_k = \frac{1}{k}$ for $k \in \mathbb{N}$, there is then corresponding $x^{(k)}$ and $y^{(k)}$ such that $\left\|x^{(k)} - y^{(k)}\right\| \le \delta_k$ while $\left\|f(x^{(k)}) - f(y^{(k)})\right\|$. By compactness of K, there exists a subsequence $\left(x^{(k_j)}\right)$ that converges to some $x \in K$. Then

$$0 \le \left\|y^{(k_j)} - x\right\| \le \underbrace{\left\|y^{(k_j)} - x^{(k_j)}\right\|}_{\le \frac{1}{k_j} \le \frac{1}{j}} + \left\|x^{(k_j)} - x\right\|.$$

In the limit as $j \to \infty$, the upper bound goes to 0. Thus $\left\|y^{(k_j)} - x\right\|$ goes to 0, hence $\left(y^{(k_j)}\right)$ converges to x. Since f is continuous at x and y, $f(x^{(k_j)}) \to f(x)$ and $f(y^{(k_j)} \to f(y)$ as $j \to \infty$. Thus

$$\left\|f(x^{(k_j)}) - f(y^{(k_j)})\right\| \le \left\|f(x^{(k_j)} - f(x)\right\| + \left\|f(x) - f(y^{(k_j)}\right\|$$

which goes to 0 as $j \to \infty$, a contradiction. Thus f is uniformly continuous. $\diamond$

Def 3.9. Open Cover

Let $A \subseteq \mathbb{R}^n$. An **open cover** of A is a collection of open sets (G_α) in $\mathbb{R}^n$ such that $A \subseteq \bigcup G_\alpha$.

Def 3.10. Topological Compactness

A set $K \subseteq \mathbb{R}^n$ is **topologically compact** if every open cover of K has a finite subcover. In other words, for any open cover (G_α) of K, there are $\{\alpha_1, \ldots, \alpha_n\}$ indices with $n < \infty$ such that $K \subseteq G_{\alpha_1} \cup \ldots \cup G_{\alpha_n}$.

Example. The set $I = (0,1) \subseteq \mathbb{R}$ is not topologically compact. Consider the candidate open cover $\bigcup_{x \in (0,1)} \left(\frac{x}{2}, \frac{x+1}{2}\right)$. Let $x \in (0,1)$. Note that

$$\begin{aligned} x > 0 &\implies 2x > x &&\implies x > \frac{x}{2} \\ x < 1 &\implies 2x < x + 1 &&\implies x < \frac{x+1}{2} \end{aligned}$$

Thus it is an open cover. Assume then there exists a finite subcover

$$\left(\frac{x_1}{2}, \frac{x_1+1}{2}\right) \cup \ldots \cup \left(\frac{x_n}{2}, \frac{x_n+1}{2}\right)$$

for $x_1, \ldots, x_n \in (0,1)$. Take $x \in \min\{x_1, \ldots, x_n\} > 0$ and $0 < y < \frac{x}{2}$. Then $y \in (0,1)$ but is not in the subcover. Hence I cannot be topologically compact.

3.2.1 Compactness Equivalence

The goal of this section is to prove the following theorem.

Theorem 3.8. Sequential $\Leftrightarrow$ Topological Compactness

A set $K \subseteq \mathbb{R}^n$ is topologically compact iff K is sequentially compact.

The approach will be to use the result being close and bounded is equivalent to sequential compactness as a bridge. That is, show that topological compactness is equivalent to being closed and bounded, and thus sequentially compact as well.

Lemma 3.2.

Let $K \subseteq \mathbb{R}^n$ be (topologically) compact and $F \subseteq K$ be closed in $\mathbb{R}^n$. Then F is also (topologically) compact.

Proof. Let (G_α) be an open cover of F. Note then that $K \subseteq F^c \cup \bigcup_\alpha G_\alpha$. Since F is closed, F^c is open and thus this is an open cover of K. Since K is topologically compact, there then exists $\alpha_1, \ldots, a_n$ finite such that $K \subseteq G_{\alpha_1} \cup \ldots G_{\alpha_n} \cup F^c$. Since $F \subseteq K$, this is a finite cover of F as well. Hence F is compact. ◇

Theorem 3.9. Heine-Borel

Let $K \subseteq \mathbb{R}^n$. Then K is (topologically) compact iff K is closed and bounded.

The following definition and lemma will be pivotal in proving Heine-Borel. If for every compact set K a closed cube Q can be chosen such that $K \subseteq Q$, then by the previous lemma if Q is compact then so is K. Thus the reverse direction of Heine-Borel follows from the compactness of *cubes*.

Def 3.11. Closed Cube

A set $Q \subset \mathbb{R}^n$ is a **closed cube** if there exists closed and bounded intervals $I_1, \ldots, I_n$ in $\mathbb{R}$ such that $Q = I_1 \times \ldots \times I_n$.

Lemma 3.3. Cubes are Compact

Let Q be a closed cube in $\mathbb{R}^n$. Then Q is (topologically) compact.

Lemma 3.4.

Let (I_n) be a sequence of closed bounded intervals in $\mathbb{R}$ such that $I_n \supseteq I_{n+1}$. Then $\bigcap I_n \neq \varnothing$.

Proof. Denote $I_n = [a_n, b_n]$. Note that the set of left endpoints $M = \{a_n : n \in \mathbb{N}\}$ is bounded above by b_1. Let $x = \sup \mathbb{R}$. Note then that

$$a_n \leq a_{n+m} \leq b_{n+m} \leq b_m, \quad \forall n, m \in \mathbb{N}.$$

Thus b_m is an upper bound of M for all $m \geq 1$, meaning $a_m \leq x \leq b_m$ for all $m \geq 1$. Therefore $x \in \bigcap I_n$. ◇

Lemma 3.5.

Let (Q_j) be a sequence of closed cubes in $\mathbb{R}^n$ such that $Q_j \supseteq Q_{j+1}$. Then $\bigcap Q_j \neq \varnothing$.

Proof. Write each Q_j as $I_{1,j} \times \ldots \times I_{n,j}$. Then each $I_{k,j}$ are closed and bounded intervals such that $I_{k,j} \supseteq I_{k+1,j}$. Thus by Lemma 3.4, $\exists y_k \in \mathbb{R}$ for each $1 \leq k \leq n$ such that $y_k \in \bigcap_j I_{k,j}$. Thus the point $y = (y_1, \ldots, y_n) \in \bigcap_j Q_j$. ◇

Proof of Lemma 3.3. Write $Q = [a_1, b_1] \times \ldots \times [a_n \times b_n]$. Suppose towards contradiction that Q is not (topologically) compact. Then there exists an open cover (G_α) of Q that has no finite subcover. Divide Q into 4 subcubes Q_j^1 for $1 \leq j \leq 4$. Since no finite subcover exists for Q, then there is some Q_i^1 that does not have a finite subcovering. Denote $\tilde{Q}_1 = Q$ and $\tilde{Q}_2 = Q_i^1$. The same division and selection process can be applied to $\tilde{Q}_2$ to get some $\tilde{Q}_3$. Continuing gives a sequence $(\tilde{Q}_j)$ such that $\tilde{Q}_j \supseteq \tilde{Q}_{j+1}$, $\tilde{Q}_j$ has no finite subcovering, and

$$\operatorname{diam}(\tilde{Q}_j) \coloneqq \sup_{x,y \in \tilde{Q}_j} \|x - y\| \leq \frac{\operatorname{diam}(Q)}{2^{j-1}}.$$

for all $j \in \mathbb{N}$. By Lemma 3.5, there is some $y \in \bigcap \tilde{Q}_j$. Since (G_α) is an open cover of Q, there is some G_α with $y \in G_\alpha$. Let $r > 0$ such that

$B_r(y) \subseteq Q_\alpha$. Note then if j is taken large enough such that $\frac{\operatorname{diam}(Q)}{2^{j-1}} < r$, then if $x \in \tilde{Q}_j$

$$\|x - y\| \leq \operatorname{diam}(\tilde{Q}_j) \leq \frac{C}{2^{j-1}} < r.$$

Thus $\tilde{Q}_j$ is covered by the single open set G_α, a contradiction. Therefore Q is compact. $\diamond$

Proof of Theorem 3.9.

$\Leftarrow$) Assume K is closed and bounded. Since K is bounded, it is possible to choose a closed cube Q such that $K \subseteq Q$. Since Q is compact by Lemma 3.3 and K is closed, by Lemma 3.2 it follows K is also (topologically) compact.

$\Rightarrow$) Assume K is (topologically) compact.

$\diamond$