Question 0.0.1. *Kyle: What is the intuition behind the discretization of the box size and why is equivalent to the product definition* ($|B| = \prod_i |I_i|$).

Answer. *It will only hold in the case of a box. Recalling the definition of a box* $B = I_1 \times \ldots \times I_n$, *if* $I_i = [a_i, b_i]$ *we have*

$$|B| = \prod_i |I_i| = \prod_i (b_i - a_i).$$

Focusing on the single dimensional case (which generalizes via cross products), take $N \in \mathbb{N}_0$ *and note that* $k/N \in \mathbb{Z}/N$ *is in* I *iff* $a \leq \frac{k}{N} \leq b$, *which is the same as* $\lceil aN \rceil \leq k \leq \lfloor bN \rfloor$. *By simple counting it follows that*

$$\left| I \cap \frac{\mathbb{Z}}{N} \right| = \lfloor bN \rfloor - \lceil aN \rceil + 1.$$

This can be bounded to

$$(bN - 1) - (aN + 1) + 1 \leq \lfloor bN \rfloor - \lceil aN \rceil + 1 \leq bN - (aN - 1) + 1$$
$$N(b - a) - 1 \leq \lfloor bN \rfloor - \lceil aN \rceil + 1 \leq N(b - a) + 2$$

Therefore

$$(b - a) - \frac{1}{N} \leq \frac{1}{N} \cdot \left| I \cap \frac{\mathbb{Z}}{N} \right| \leq (b - a) + \frac{2}{N}$$

which in the limit gives the desired result $|I| = b - a = \lim_{n\to\infty} \frac{1}{N}\left|I \cap \frac{\mathbb{Z}}{N}\right|$.

Remark. Charlie: What about for an elementary set $E = \coprod_i B_i$ defining its "size" as a supremum over all disjoint decompositions? For example:

$$|E| = \sup \left\{ \sum_i^N |B_i| : B_i \cap B_j = \emptyset, B_i \in E \right\}.$$

This could possibly be extended to open sets as well? Open boxes with rational corners would provide a nice basis for $\mathbb{R}^d$.