Problem 1

Proof. Suppose towards contradiction that F is uniformly continuous. Take $\varepsilon = 1$. Then $\exists\delta > 0$ such that $\|x - y\| \le \delta \implies \|f(x) - f(y)\| \le 1$ for $x, y \in \mathbb{R}^2$. Take $x = (n, 0)$ and $y = (n + \frac{1}{n}, 0)$ with $n \in \mathbb{N}$ such that $\frac{1}{n} \le \delta$. Then

$$\|x - y\| = \frac{1}{n} \le \delta$$

and

$$\|f(x) - f(y)\| = \left\| n^2 - \left(n^2 + 2 + \frac{1}{n^2} \right) \right\| = \left\| 2 + \frac{1}{n^2} \right\| = 2 + \frac{1}{n^2} > 1 = \varepsilon.$$

Therefore F is not uniformly continuous on $\mathbb{R}^2$. ◇

Problem 2

Proof. Take $\varepsilon > 0$. Since F approximates f, for any $x \in E$

$$\|f(x) - F(x)\| \le \frac{\varepsilon}{3}.$$

Furthermore since F is uniformly continuous, there exists some $\delta > 0$ such that for $\|x - y\| \le \delta$

$$\|F(x) - F(y)\| \le \frac{\varepsilon}{3}$$

and thus by triangle inequality

$$\begin{aligned} \|f(x) - f(y)\| &\le \|f(x) - F(x)\| + \|F(x) - F(y)\| + \|F(y) - f(y)\| \\ &\le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ &= \varepsilon \end{aligned}$$

Therefore f is uniformly continuous on E ◇

Problem 3

Proof. For $x \in K \times L$, let $x_L := (x_1, \dots, x_n)$ and $x_R := (x_{n+1}, \dots, x_{n+p})$. Since K and L are compact, both are closed and bounded meaning there exists M_K and M_L such that $\|x\|_{\mathbb{R}^n} \leq M_K$ for $x \in K$ and $\|x\|_{\mathbb{R}^p} \leq M_L$ for $x \in L$. Therefore

$$\|x\|_{\mathbb{R}^{n+p}}^2 = \|x_K\|_{\mathbb{R}^n}^2 + \|x_K\|_{\mathbb{R}^p}^2 \leq (\|x_K\|_{\mathbb{R}^n} + \|x_L\|_{\mathbb{R}^p})^2 \leq (M_K + M_L)^2$$

meaning $\|x\|_{\mathbb{R}^{n+p}} \leq M_K + M_L$ for all $x \in K \times L$. Thus $K \times L$ is bounded.

Let $\left(x^{(k)}\right)$ be a sequence in $K \times L$ that converges to some $a \in \mathbb{R}^{n+p}$. Note then that each component must converge, meaning the sequences $\left(x_K^{(k)}\right)$ and $\left(x_L^{(k)}\right)$ both converge to a_K and a_L respectively. Since K and L are closed, $a_K \in K$ and $a_L \in L$. Therefore $a \in K \times L$ meaning $K \times L$ is closed. Since $K \times L$ is both closed and bounded, it is compact. $\diamond$

Problem 4

Proof. Suppose A is closed and B is compact. Consider some sequence $\left(x^{(k)}\right)$ in $A + B$ that converges to $x \in \mathbb{R}^n$. Note that $x^{(k)} = a^{(k)} + b^{(k)}$ for some $a^{(k)} \in A$ and $b^{(k)} \in B$ for all $k \in \mathbb{N}$. Since B is compact, there exists a subsequence $\left(b^{(k_j)}\right)$ that converges to some $b \in B$. Then since $a^{(k_j)} = x^{(k_j)} - b^{(k_j)}$ and both $\left(x^{(k_j)}\right)$ and $\left(b^{(k_j)}\right)$ are convergent sequences, $\left(a^{(k_j)}\right)$ converges to $x - b$. Since A is closed, $x - b \in A$. Thus $(x-b)+b = x \in A+B$, hence $A + B$ is closed. $\diamond$

Problem 5

Proof. First note that $\mathbb{S}^{n-1}$ is bounded since $\|x\| \leq 1$ for all $x \in \mathbb{S}^{n-1}$. Additionally, since $d(x) = \|x\|$ is continuous on $\mathbb{R}^n$, $\{1\}$ is closed in $\mathbb{R}$, and $\mathbb{S}^{n-1} = d^{-1}(\{1\})$, $\mathbb{S}^{n-1}$ is closed. Since $\mathbb{S}^{n-1}$ is both closed and bounded, it is compact.

Consider some $x \in \mathbb{R}^n \setminus 0$. Note that it can be written as $r\hat{x}$ where $r = \|x\| > 0$ and $\hat{x} = \frac{x}{\|x\|}$. Therefore by the homogeneity of f, $f(x) = f(r\hat{x}) = r^d f(\hat{x})$. Since $\|\hat{x}\| = 1$, $\hat{x} \in \mathbb{S}^{n-1}$. Furthermore f is continuous on $\mathbb{S}^{n-1}$ which is compact, thus f is bounded by some $M > 0$ on it. Hence

$$\|f(x)\| = \left\|r^d f(\hat{x})\right\| = \|x\|^d \|f(\hat{x})\| \leq M\|x\|^d$$

which was to be shown. ◇

Problem 6

Part I

Proof. Suppose g is continuous and let $\left(x^{(k)}\right)$ be a Cauchy sequence in $\mathbb{R}^n$. Since $\left(x^{(k)}\right)$ is Cauchy and $\mathbb{R}^n$ is complete, it must converge to some $a \in \mathbb{R}^n$. Take $\varepsilon > 0$. Since g is continuous at a, there exists $\delta > 0$ such that $\|x - a\| \leq \delta \implies \|g(x) - g(a)\| \leq \varepsilon$. By convergence of $\left(x^{(k)}\right)$ there exists $K \in \mathbb{N}$ such that $\left\|x^{(k)} - a\right\| \leq \delta$ for $k \geq K$. Therefore $\left\|g(x^{(k)}) - g(a)\right\| \leq \varepsilon$ for $k \geq K$, meaning the sequence $\left(g(x^{(k)})\right)$ converges to $g(a)$. Since convergent sequences are Cauchy, g takes Cauchy sequences to Cauchy sequences. ◇

Part II

Proof. Suppose f is uniformly continuous and let $\left(x^{(k)}\right)$ be a Cauchy sequence in D. Take $\varepsilon > 0$. Since f is uniformly continuous, $\exists \delta > 0$ such that $\|x - y\| \leq \delta \implies \|f(x) - f(y)\| \leq \varepsilon$. Since $\delta > 0$ and $\left(x^{(k)}\right)$ is Cauchy, there exists K such that for $m, n \geq K$, $\left\|x^{(n)} - x^{(m)}\right\| \leq \delta$. Therefore $\left\|f(x^{(n)}) - f(x^{(m)})\right\| \leq \varepsilon$ for $m, n \geq K$, hence $\left(f(x^{(k)})\right)$ is a Cauchy sequence. ◇

Problem 7

Proof. Let $F \subseteq \mathbb{R}^n$ be closed. Consider some $y \in \overline{f(F)}$. Then there is a sequence $\left(y^{(k)}\right)$ in $f(F)$ that converges to y. Let $K = \left\{y^{(k)} : k \in \mathbb{N}\right\} \cup \{y\}$. Let (G_α) be an open cover of K. Note then there is some G_α such that $y \in G_\alpha$. Thus there exists $r > 0$ such that $y \in B_r(y) \subseteq G_\alpha$. Since $\left(y^{(k)}\right)$ is convergent, there exists $K \in \mathbb{N}$ such that $\left\|y^{(k)} - y\right\| < r$, thus $y^{(k)} \in B_r(y) \subseteq G_\alpha$ for all $k > K$. For each $1 \leq k \leq K$, there is then some G_{α_k} such that $y^{(k)} \in G_{\alpha_k}$. Thus the finite subcover $G_{\alpha_1} \cup \ldots \cup G_{\alpha_K} \cup G_\alpha$ covers K.

Therefore K is compact meaning the preimage $f^{-1}(K)$ is also compact. Note that $y^{(k)} = f(x^{(k)})$ for some $x^{(k)} \in f^{-1}(K) \subseteq F$. Thus there is a convergent subsequence $\left(x^{(k_j)}\right)$ which converges to some $a \in F$. Therefore by continuity of f and uniqueness of limits

$$f(a) = \lim_{j \to \infty} f(x^{(k_j)}) = y.$$

Hence $y \in f(F)$ meaning $f(F)$ is closed. ◇