Exercise 6.12

Proof.

1. By the definition of the outer measure, for any $\varepsilon > 0$ there exists a cover $A_n \in \mathcal{B}^{\mathbb{N}}$ of E such that

$$\sum_{n\in\mathbb{N}} \mu_0(A_n) \le \mu^*(E) + \varepsilon.$$

Clearly A_n is a cover for A, meaning that

$$\mu^*(A) \le \sum_{n\in\mathbb{N}} \mu_0(A_n) \le \mu^*(E) + \varepsilon.$$

Since each $A_n \in \mathcal{B}$ and A is a countable union, it follows that $A \in \mathcal{B}_\sigma$.

2. Suppose E is μ^*-measurable and $\mu^*(E) < \infty$. Using part (a), let $B_n \in \mathcal{B}_\sigma^{\mathbb{N}}$ such that $E \subset B_n$ with $\mu^*(B_n) \le \mu^*(E) + \frac{1}{n}$. Take then $B = \bigcap_{n\in\mathbb{N}} B_n$. Note that both $E \subset B \subset B_n$ and $B \in \mathcal{B}_{\sigma\delta}$. Thus

$$\mu^*(E) \le \mu^*(B) \le \mu^*(E) + \frac{1}{n}$$

which in the limit $n \to \infty$ gives $\mu^*(B) = \mu^*(E)$. Since E is u^*-measurable, it follows that

$$\mu^*(E) = \mu^*(E \cap B) + \mu^*(E \setminus B) \implies \mu^*(E \setminus B) = \mu^*(E) - \mu^*(B).$$

But since $\mu^*(B) = \mu^*(E)$, it follows that $\mu^*(E \setminus B) = 0$.

Suppose then some $E \subset B \in \mathcal{B}_{\sigma\delta}$ where $\mu^*(E \setminus B) = 0$. Since $\mu^*(E \setminus B) = 0$, clearly the Caratheodory criterion holds for $E \setminus B$ and thus it is μ^*-measurable. Since the set of μ^*-measurable sets is a σ-algebra containing $\mathcal{B}$, it follows that B is μ^*-measurable and thus $(E \setminus B) \cup B = E$ is μ^*-measurable.

3. Note that the reverse direction did not require $\mu^*(E) < \infty$, so we only consider the forward direction. Suppose X is σ-finite. Then there exists $X_k \in \mathcal{B}^{\mathbb{N}}$ such that $\mu^*(X_k) < \infty$ and $X = \bigcup_{k\in\mathbb{N}} X_k$. Let $E_k = E \cap X_k$ and note that from part (a) that there exist $E_k \subset O_{k,n} \in \mathcal{B}_\sigma^{\mathbb{N}}$ with $\mu^*(O_{k,n}) \le \mu^*(E_k) + \frac{1}{n\cdot 2^k}$.

Take then $B_n = \bigcup_{k\in\mathbb{N}} O_{k,n}$ and note that $B_n \in \mathcal{B}_\sigma$ and

$$B_n \setminus E \subset \bigcup_{k\in\mathbb{N}} (O_{k,n} \setminus E_k).$$

Therefore by subadditivity it follows

$$\mu^*(B_n \setminus E) \le \sum_{k\in\mathbb{N}} \mu^*(O_{k,n} \setminus E) \le \sum_{k\in\mathbb{N}} \frac{1}{n \cdot 2^k} = \frac{1}{n}.$$

The exact same argument in (b) then works from here.

$\diamond$

Exercise 6.14

Proof.

1. Let $A_n \in \mathcal{A}^*$ be a cover of E. Since μ^* is an outer measure, it is subadditive and monotonic meaning

   $$\mu^*(E) \le \mu^*\left(\bigcup_{n\in\mathbb{N}} A_n\right) \le \sum_{n\in\mathbb{N}} \mu^*(A_n).$$

   Since μ^+ is the infimum over all such sums, it follows that $\mu^*(E) \le \mu^+(E)$.

   Suppose then there exists an $A \supset E$ with $\mu^*(A) = \mu^*(E)$. Since A covers E, by definition $\mu^+(E) \le \mu^*(A) = \mu^*(E)$. Combined with first inequality gives $\mu^+(E) = \mu^*(E)$.

   Suppose then that $\mu^*(E) = \mu^+(E)$.

   - Suppose $\mu^*(E) < \infty$. By the definition of μ^+ there then exists $A_n \in \mathcal{A}^*$ such that $E \subset A_n$ and

     $$\mu^*(A_n) \le \mu^+(E) + \frac{1}{n}.$$

     Take then $A = \bigcap_{n\in\mathbb{N}} A_n$ and note that both $A \in \mathcal{A}^*$ and $E \subset A$. In the limit as $n \longrightarrow \infty$ it then follows

     $$\mu^*(A) \le \mu^*(A_n) \le \mu^+(E) = \mu^*(E).$$

     Therefore $\mu^*(E) = \mu^*(A)$.

- Suppose $\mu^*(E) = \infty$. Since $\mu^*(E) \leq \mu^+(E)$, it follows $\mu^+(E) = \infty$. Take $A = X$. Clearly then $A \in \mathcal{A}^*$ and $E \subset A$. Thus by monotonicity $\infty = \mu^*(E) \leq \mu^*(A)$ meaning $\mu^*(A) = \mu^*(E) = \infty$

2. If μ^* is induced from a pre-measure over some algebra $\mathcal{B}$, then from problem 1 part (a), for any $\varepsilon > 0$ there is some $A \in \mathcal{A}_\sigma$ such that $E \subset A$ and $\mu^*(A) \leq \mu^*(E) + \varepsilon$. By Caratheodory's, every set in $\mathcal{B}$ is measurable and thus in $\mathcal{A}^*$. Since $\mathcal{A}^*$ is a σ-algebra, countable unions are contained in it and thus $\mathcal{B}_\sigma \subset \mathcal{A}^*$. Since then $A \in \mathcal{A}^*$ and covers E, it follows $\mu^+(E) \leq \mu^*(A) \leq \mu^*(E) + \varepsilon$. Letting $\varepsilon \to 0$ thus gives $\mu^+(E) \leq \mu^*(E)$, which combined with the initial result gives equality.

3. Define μ^* on 2^X where

$$\begin{aligned} \mu^*(\varnothing) &= 0 \\ \mu^*(\{0\}) &= 1 \\ \mu^*(\{1\}) &= 1 \\ \mu^*(\{0,1\}) &= 1.5 \end{aligned}$$

Clearly it is monotonic an subadditive, and $\mu^*(\varnothing) = 0$. Therefore μ^* is an outer measure on X. Note that

$$\mu^*(0,1) = 1.5 \neq 2 = 1 + 1 = \mu^*(\{0,1\} \cap \{0\}) + \mu^*(\{0,1\} \cap \{1\}).$$

Thus $\{0\}$ and $\{1\}$ are not measurable, leaving just $\varnothing$ and $\{0,1\}$ which are trivially measurable. Note that the only measurable set containing $\{0\}$ is $\{0,1\}$ and so $\mu^+(\{0\}) = \mu^*(\{0,1\}) = 1.5$, but

$$\mu^*(\{0\}) = 1 \neq 1.5 = \mu^+(\{0\}).$$

◇

Exercise 6.18

Proof.

1. First note that $(a, b] \cap \mathbb{Q}$ forms an elementary family.

iii) Note that $(a,b]^c = (b,\infty]$ and thus complements are maintained in $\mathbb{Q}$.

ii) Note that $(a_1,b_1] \cap (a_2,b_2] = (\max(a_1,a_2),\min(b_1,b_2)]$ and thus intersections are maintained in $\mathbb{Q}$

i) Note that $(a,b] \cap (a,b]^c = \varnothing$ and thus the empty set is in family

Since they form an elementary family, then $\mathcal{B}$ is an algebra since is the finite union of sets in the family.

2. Note that any singleton $\{q\} \subset \mathbb{Q}$ can be achieved by

$$\{q\} = \bigcap_{n\in\mathbb{N}} \left(q - \frac{1}{n}, q\right].$$

Since all the inner sets are contained in $\mathcal{B}$, it follows that $\{q\} \in \sigma(\mathcal{B})$ for all $q \in \mathbb{Q}$. But then that means that any $E \subset \mathbb{Q}$ can be made in $\sigma(\mathcal{B})$ since $\mathbb{Q}$ is countable. Thus $2^{\mathbb{Q}} \subset \sigma(\mathcal{B})$. By definition $\sigma(\mathcal{B}) \subset 2^{\mathbb{Q}}$, hence equality.

3. Consider the following measures on $2^{\mathbb{Q}}$:

$$\nu_1(E) = |E| \qquad\qquad \nu_2(E) = \begin{cases} 0 & E = \varnothing \\ \infty & E = \varnothing \end{cases}.$$

When restricted to $\mathcal{B}$, $\nu_1(E) = \infty$ since any set in $\mathcal{B}$ has countably many elements. Similarly, $\varnothing \notin \mathcal{B}$ so $\nu_2(E) = \infty$ for all E as well. Therefore both are equal to the premeasure on $\mathcal{B}$. However the measures are not the same on say $\{q\} \in 2^{\mathbb{Q}}$ since $\nu_1(\{q\}) = 1$ and $\nu_2(\{q\}) = \infty$.

$\diamond$

Exercise 6.19

Proof.

1. Let $D = A\Delta B$. Note that both $D \in \mathcal{A}$ and $D \cap E = \varnothing$ since

   - If $x \in A \cap E$, then $x \in B$ meaning $x \notin A \setminus B$

- If $x \in B \cap E$, then $x \in A$ meaning $x \notin B \setminus A$
- If $x \in E$ but $x \notin A$ or $x \notin B$, then $x \notin D$

Thus $E \subset D^c$, meaning $\mu(E) \leq \mu(D^c)$. Since μ is a finite measure, it follows that $\mu(D^c) = \mu(X) - \mu(D)$. Since $\mu(X) = \mu^*(X) = \mu^*(E) = \mu(E)$, then

$$\mu(X) \leq \mu(X) - \mu(D) \implies \mu(D) \leq 0.$$

Therefore $\mu(D) = 0$. Since $A \setminus B \subset D$ and $B \setminus A \subset D$, as well as $A = (A \cap B) \cup (A \setminus B)$ and $B = (B \cap A) \cup (B \setminus A)$, it follows by monotonicity and additivity

$$\begin{aligned}
\mu(A) &= \mu(A \cap B) + \mu(A \setminus B) \\
&= \mu(A \cap B) + 0 \\
&= \mu(B \cap A) + \mu(B \setminus A) \\
&= \mu(B)
\end{aligned}$$

2. It has been shown in prior exercises that a σ-algebra restricted to some set is still a σ-algebra, so all that needs to be shown is that ν is a measure on $\mathcal{A}_E$. Note that ν is well defined by part (a). Let $B_n \in \mathcal{A}_E^{\mathbb{N}}$ be pairwise disjoint. Then there exists $A_n \in \mathcal{A}$ such that $B_n = A_n \cap E$. Let then

$$C_n = A_n \setminus \bigcup_{k=1}^{n-1} A_k$$

and note that all the C_n are pairwise disjoint. Also note

$$C_n \cap E = (A_n \cap E) \setminus \bigcup_{k=1}^{n-1} (A_k \cap E) = B_n \setminus \bigcup_{k=1}^{n-1} B_k.$$

Since the B_k are disjoint, it follows then that $C_n \cap E = B_n$. Therefore

$$\bigcup_{n \in \mathbb{N}} B_n = \bigcup_{n \in \mathbb{N}} (C_n \cap E) = E \cap \bigcup_{n \in \mathbb{N}} C_n$$

meaning

$$\nu\left(\bigcup_{n \in \mathbb{N}} B_n\right) = \mu\left(\bigcup_{n \in \mathbb{N}} C_n\right) = \sum_{n \in \mathbb{N}} \mu(C_n) = \sum_{n \in \mathbb{N}} \nu(B_n).$$

Thus ν is countably additive and hence a measure.

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