Problem 1

By integration note that

$$T_1(x,y) = \int \frac{\partial T_1}{\partial x}\mathrm{d}x = \int 3x^2y\mathrm{d}x = x^3y + C_1(y).$$

Since $\frac{\partial T}{\partial y} = x^3$, then if the previous expression is to hold

$$\frac{\partial}{\partial y}\Big(x^3y + C(y)\Big) = x^3 \implies C'(y) = 0.$$

Therefore $C_1(y) = c_1 \in \mathbb{R}$. Integrating again but this time for f_2 gives

$$T_2(x,y) = \int \frac{\partial T_2}{\partial x}\mathrm{d}x = \int y\mathrm{d}x = xy + C_2(y).$$

Since $\frac{\partial T}{\partial y} = x$, then if the previous expression is to hold

$$\frac{\partial}{\partial y}(xy + C_2(y)) = x \implies C_2'(y) = 0.$$

Therefore similarly $C_2(y) = c_2 \in \mathbb{R}$. Thus the transformation

$$T(x,y) = \Big(x^3y + c_1,\ xy + c_2\Big)$$

for constants $c_1, c_2 \in \mathbb{R}$ has the desired differential.

Problem 2

Yes. Let $F(x,y) = x^2 + y + \sin(xy)$. Since $F(0,0) = 0^2 + 0 + \sin(0) = 0$ and

$$F_y'(x,y) = 1 + x\cos(xy) \implies F_y'(0,0) = 1 \neq 0$$

then by inverse function theorem, there does exist $f(x)$ such that $y = f(x)$ and $F(x, f(x)) = 0$ in a neighborhood of $(0,0)$.

Problem 3

Yes. Let $F(x,y,z) = xy - z\log y + e^{xz} - 1$. Since $F(0,1,1) = 0$ and

$$F_y'(x,y,z) = x - \frac{z}{y} \implies F_y'(0,1,1) = 0 - \frac{1}{1} = -1 \neq 0$$

then by inverse function theorem, there does exist $g(x,z)$ such that $y = g(x,z)$ and $F(x, g(x,z), z) = 0$ in a neighborhood of $(0,1,1)$.

Problem 4

Proof. Let $r = |x|$. Since f only depends on $|x|$, then $f(x) = g(r)$ where $g : (0, \infty) \to \mathbb{R}$ is a C^2 function. Note that

$$\frac{\partial r}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\sqrt{x_1^2 + x_2^2 + x_3^2}\right) = 2 \cdot x_j \cdot \frac{1}{2\sqrt{x_1^2 + x_2^2 + x_3^2}} = \frac{x_j}{r}$$

and thus

$$\frac{\partial^2 r}{\partial x_j^2} = \frac{\partial}{\partial x_j}\left(\frac{x_j}{r}\right) = \frac{r - x_j \cdot \frac{x_j}{r}}{r^2} = \frac{r^2 - x_j^2}{r^3} = \frac{1}{r} - \frac{x_j^2}{r^3}.$$

Therefore by chain rule

$$\frac{\partial f}{\partial x_j} = g'(r) \cdot \frac{\partial r}{\partial x_j} = g'(r) \cdot \frac{x_j}{r}.$$

Applying the chain rule once again gives

$$\frac{\partial^2 f}{\partial x_j^2} = \frac{\partial}{\partial x_j}\left(g'(r) \cdot \frac{x_j}{r}\right) = g''(r) \cdot \left(\frac{x_j}{r}\right)^2 + g'(r) \cdot \left(\frac{1}{r} - \frac{x_j^2}{r^3}\right).$$

To get Δf, simply sum over all these terms to get

$$\Delta f = \sum_{k=1}^{3} \frac{\partial^2 f}{\partial x_j^2} = \frac{g''(r)}{r^2} \cdot \sum_{k=1}^{3} x_j^2 + g'(r) \cdot \left(\frac{3}{r} - \frac{1}{r^3} \sum_{k=1}^{3} x_j^2\right).$$

Since $x_1^2 + x_2^2 + x_3^2 = r^2$ and $\Delta f = 0$, we obtain $\Delta f = g''(r) + g'(r) \cdot \frac{2}{r} = 0$. Let $q(r) = g'(r)$ and note that $q'(r) + q(r) \cdot \frac{2}{r} = 0$. Therefore

$$q'(r) + q(r) \cdot \frac{2}{r} = 0 \implies r^2 q'(r) + 2r \cdot q(r) = 0 \implies \frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 \cdot q(r)\right) = 0.$$

Thus integrating gives $r^2 \cdot q(r) = c_1$ or equivalently $g'(r) = \frac{c_1}{r^2}$. Integrating again gives $g(r) = -\frac{c_1}{r} + c_2$. Letting $a = -c_1$ and $b = c_2$ gives

$$f(x) = g(r) = g(|x|) = \frac{a}{|x|} + b.$$

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Problem 5

Proof. Let $t > 0$ and $x^{(1)}, x^{(2)} \in \mathbb{R}^n$ such that $\Phi_t(x^{(1)}) = \Phi_t(x^{(2)})$. Note then that

$$\begin{aligned} \Phi_t(x^{(1)}) &= \Phi_t(x^{(2)}) \\ x^{(1)} - tf(x^{(1)}) &= x^{(2)} - tf(x^{(2)}) \\ x^{(1)} - x^{(2)} &= t\left[f(x^{(1)}) - f(x^{(2)})\right] \end{aligned}$$

Thus $|x^{(1)} - x^{(2)}| = t|f(x^{(1)}) - f(x^{(2)})|$. Since $f \in C^1$, so is Φ_t.

Since K is compact, it is bounded and thus there exists $R > 0$ such that $|x| \leq R$ for every $x \in K$. Since Φ_t is C^1 on $\mathbb{R}^n$, it follows that Φ_t is C^1 on $B_{R+2}(0)$. Therefore since an open ball is convex, by MVT there exists $M > 0$ such that for $x, y \in B_{R+1}(0)$

$$|f(x) - f(y)| \leq M|x - y|.$$

Since $K \subseteq B_{R+1}(0)$, this holds for $x, y \in K$. Therefore if $x^{(1)}, x^{(2)} \in K$, then

$$|x^{(1)} - x^{(2)}| = t|f(x^{(1)}) - f(x^{(2)})| \leq tM|x^{(1)} - x^{(2)}|.$$

If $x^{(1)} \neq x^{(2)}$, then $|x^{(1)} - x^{(2)}| \neq 0$ and so dividing through gives $1 \leq tM \implies \frac{1}{M} \leq t$. But t can be taken to be $t < \frac{1}{M}$, meaning this cannot be the case. Therefore $x^{(1)} = x^{(2)}$, and hence Φ_t is injective.

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Problem 6

Proof. Note that $f(x) = 3x_1^4 - 4x_1^2x_2 + x_2^2$, therefore

$$\nabla f(x) = \begin{bmatrix} 12x_1^3 - 8x_1x_2 & 2x_2 - 4x_1^2 \end{bmatrix}.$$

Since $\nabla f(0) = (0,0)$, 0 is a critical point of f. Suppose towards contradiction that f has a local extremum at 0. If 0 is a local maximum, then $f(x) \leq f(0) = 0$ for any x in some open ball $B_r(0)$. Note that $f(0,t) = t^2$, and taking $0 < t < r$ gives $(0,t) \in B_r(0)$. But $t^2 > 0 = f(0)$, hence 0 cannot be a local maximum.

Suppose then 0 is a local minimum. Then $f(x) \geq f(0) = 0$ for any x in some open ball $B_r(0)$. Note that $f(t, 2t^2) = -t^4$, and taking t such that $0 < |(t, 2t^2)| < r$ gives $(t, 2t^2) \in B_r(0)$. But $-t^4 < 0 = f(0)$, hence 0 cannot be a local minimum.

In either case, 0 cannot be a local maximum or minimum and thus is not a local extremum of f. ◇

Problem 7

Proof.

i) Let $K \subseteq \mathbb{R}^n$ be compact and assume towards contradiction that f has infinitely many critical points in K. Let $S = \{x \in K : \nabla f(x) = 0\}$ and note that S is infinite. Therefore it is possible to pick a sequence $\left(x^{(k)}\right)$ in S with distinct terms. Since $\left(x^{(k)}\right)$ is also a sequence in K and K is closed and bounded, then by Bolzano-Weierstrass there is a subsequence $\left(x^{(k_j)}\right)$ that converges to some $x \in K$. Furthermore since f is C^2, then ∇f is continuous meaning

$$\lim_{j\to\infty} \nabla f(x^{(k_j)}) = 0 \implies \nabla f(x) = 0.$$

Thus x is a critical point of f and is non degenerate since f is a morse function. Since ∇f is C^1 and Hf is both the differential of ∇f and invertible at x (by non degeneracy), by the inverse function theorem there exists an open nbhd U of x and open nbhd V of $\nabla f(x)$ such that $\nabla f : U \to V$ is bijective. Since $\nabla f(x) = 0$ and f must be injective, then $\nabla f(a) \neq 0$ for any $a \in U \setminus \{x\}$. Note that $\left(x^{(k_j)}\right)$ converges to x so there must exist some $J \in \mathbb{N}$ such that for $j \geq J$, $x^{(k_j)} \in U$ and $x^{(k_j)} \neq x$. But $\nabla f(x^{(k_j)}) = 0$, a contradiction. Therefore f has finitely many critical points.

ii) Note that for $x \in \mathbb{R}^n$ that $|x|^2 = x_1^2 + \ldots + x_n^2$. Therefore

$$f(x) = 2\left(x_1^2 + \ldots + x_n^2\right) - \left(x_1^2 + \ldots + x_n^2\right)^2.$$

Thus the partial derivatives are of the form

$$\partial_{x_j} f(x) = 4x_j - 4x_j(x_1^2 + \ldots + x_n^2).$$

Since the gradient of f is simply the matrix of its partials, it is equal to 0 when all the partial are simultaneously equal to 0. It follows that

$$\partial_{x_j} f(x) = 0 \implies x_j = x_j(x_1^2 + \ldots + x_n^2)$$

which is only true when either $x_j = 0$, or $x_1^2 + \ldots + x_n^2 = 1$ or equivalently $|x| = 1$. Since this must hold for all $1 \leq j \leq n$, the gradient of f is zero when

$$x \in \{0\} \cup \{x \in \mathbb{R}^n : |x| = 1\}.$$

This set of points is infinite, and also contained entirely in the closed ball $B_2(0)$ which is compact. By the contrapositive of (i), it follows that f cannot be a Morse function.

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