
Ex 1.11

Proof. We will prove $1 \Leftrightarrow 2 \Leftrightarrow 3$.

$1 \Leftrightarrow 2$) Suppose E is Jordan measurable and take $\varepsilon > 0$. By the definition of the inner and outer Jordan measure, it follows that there exists elementary sets A and B such that $A \subset E$ and $E \subset B$ as well as

$$m(A) + \frac{\varepsilon}{2} \leq m_*^J(E) \qquad\qquad m_J^*(E) \leq m(B) - \frac{\varepsilon}{2}.$$

Subtracting the first inequality from the second gives

$$m(B) - \frac{\varepsilon}{2} - m(A) - \frac{\varepsilon}{2} \leq m_J^*(E) - m_*^J(E) = 0 \implies m(B) - m(A) \leq \varepsilon.$$

Since $B \setminus A$ and A are disjoint and $(B \setminus A) \cup A = A$,

$$\begin{aligned} m(A) &= m((B \setminus A) \cup A) \\ &= m(B \setminus A) + m(A) \implies m(B \setminus A) = m(B) - m(A) \end{aligned}$$

Therefore $m(B \setminus A) \leq \varepsilon$.

Suppose then (ii) holds.

$2 \Leftrightarrow 3$) Suppose that (iii) holds and take $\varepsilon > 0$. Since $m_J^*(A\Delta E) \leq \varepsilon$, then there exists an elementary set B such that $A\Delta E \subset B$ and $m_J^*(A\Delta E) \leq m(B) + \varepsilon$.

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Ex. 1.14

Proof.

i) Take $\varepsilon > 0$. Since B is closed and bounded, it is compact meaning f is uniformly continuous over B. Therefore $\exists \delta > 0$ such that $|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$. Partition B into a disjoint set of closed boxes Q_i whose diameters are smaller than δ. Associate then the set of points x_i with Q_i where $x_i \in Q_i$. Note then that $x \in Q_i$ gives $|x - x_i| < \delta \implies |f(x) - f(x_i)| < \varepsilon$. Thus we have

$$\{(x, f(x)) : x \in Q_i\} \subset Q_i \times [f(x_i) - \varepsilon, f(x_i) + \varepsilon].$$

Since all the Q_i cover B, it follows

$$G(f) \subset \bigcup_i Q_i \times [f(x_i) - \varepsilon, f(x_i) + \varepsilon].$$

The measure of the union can be bounded above by $2\varepsilon \cdot M$ where M is the total size of all the Q_i. The total size of all the Q_i is constant since it is simply the size of B. Therefore we have

$$0 \leq m_*^J(G(f)) \leq m_J^*(G(f)) \leq 2\varepsilon \cdot M.$$

Since ε is arbitrary, it follows that $m(G(f)) = 0$.

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Ex 1.18

Proof.

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Ex 1.25

Proof.

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Ex 1.26

Proof.

i) Since $E \subset \overline{E}$ it follows $m_J^*(E) \leq m_J^*(\overline{E})$, thus it suffices to show the reverse inequality. Take $\varepsilon > 0$. Then there exists boxes $B_1, \ldots, B_n$ such that $E \subset \bigcup B_i$ and $\sum |B_i| \leq m_J^*(E) + \varepsilon$. Note that $|\overline{B_i}| = |B_i|$ and

$$\overline{E} \subset \overline{\bigcup_i B_i} = \bigcup_i \overline{B_i}.$$

Therefore $m_J^*(\overline{E}) \leq \sum |\overline{B_i}| = \sum |B_i| \leq m_J^*(E) + \varepsilon$, giving $m_J^*(\overline{E}) \leq m_J^*(E) + \varepsilon$. Since ε was arbitrary, we have equality.

ii) We use a similar argument as above. Clearly $m_*^J(\mathring{E}) \leq m_*^J(E)$. Take $\varepsilon > 0$. Then there exists boxes $B_1, \ldots, B_n$ such that $\bigcup B_i \subset E$ with $m_*^J(E) + \varepsilon < \sum |B_i|$.

iii) Suppose E is Jordan measurable. Then $m_J^*(E) = m_*^J(E)$. Since $\mathring{E} \subset \overline{E}$, $m_*^J(\mathring{E}) \leq$

iv) Since $\overline{[0,1]^2 \setminus \mathbb{Q}^2} = [0,1]^2 = \overline{[0,1]^2 \cap \mathbb{Q}^2}$, it follows $m_J^*([0,1]^2 \setminus \mathbb{Q}^2) = m_J^*([0,1]^2 \cap \mathbb{Q}^2) = m_J^*([0,1]^2) = 1$. But $\operatorname{int}([0,1]^2 \setminus) = \varnothing = \operatorname{int}([0,1]^2 \cap \mathbb{Q}^2)$, thus $m_*^J([0,1]^2 \setminus \mathbb{Q}) = m_*^J([0,1]^2 \cap \mathbb{Q}^2) = m_*^J(\varnothing) = 0$. Therefore neither $[0,1]^2 \setminus \mathbb{Q}^2$ or $[0,1]^2 \cap \mathbb{Q}^2$ are Jordan measurable.

$\diamond$