Ex 1.2

Proof. Let $E = B_1 \cup \ldots \cup B_n$ and $F = V_1 \cup \ldots \cup V_m$ where B_i, V_j are boxes in $\mathbb{R}^d$.

(i) By definition, E and F are the unions of finite boxes, and thus $E \cup F$ is the union of finite boxes meaning it is elementary.

(ii) Note that the intersection of two intervals is always an interval (assuming $\varnothing$ is also an interval). That is because if the intersection is non-empty and the endpoints of the intervals are a_1, b_1 and a_2, b_2, then the endpoints of their intersection are $\min(a_1, a_2)$ and $\max(b_1, b_2)$. Thus the intersection of two boxes is itself a box. Note

$$E \cap F = \bigcup_i \bigcup_j B_i \cap V_j$$

which is the union of the intersection of boxes, which themselves are boxes. Hence $E \cap F$ is elementary.

(iii)

(iv) Since $E \setminus F$ and $F \setminus E$ are elementary from (iii), then their union is also elementary from (i). Thus $E \Delta F$ is elementary.

(v) Note that $E + x = \bigcup_i (B_i + x)$ thus we only need to consider $B_i + x$. Suppose $B_i = [a_1, b_1] \times \ldots \times [a_d, b_d]$ (since the following argument works regardless of endpoint inclusion). Note then that

$$B_i + x = [a_1 + x_1, b_1 + x_1] \times \ldots \times [a_d + x_d, b_d + x_d]$$

which is still a box. Thus E is the union of a finite number of boxes and therefore elementary.

$\diamond$

Ex 1.7

Proof. Let $c = \tilde{m}([0, 1)^d)$. Note that finite additivity and non-negativity of $\tilde{m}$ give monotonicity because if $A \subset B$, then $\tilde{m}(B \setminus A) \geq 0$ and

$$\tilde{m}(B) = \tilde{m}(A \cup (B \setminus A)) \geq \tilde{m}(A) + \tilde{m}(B \setminus A) \geq \tilde{m}(A).$$

We can now extend the known value of $\tilde{m}$ to general boxes.

- Take $n \in \mathbb{N}$. Then $[0,1)^d$ can be subdivided into n^d cubes of side length $\frac{1}{n}$. By translation invariance, each of these cubes must have the same measure, thus $\tilde{m}\left([0,\frac{1}{n})^d\right) = \frac{c}{n^d}$.
- Let B be a box with rational endpoints. By translation invariance it can be assumed $B = \prod_{i=1}^{d}[0,r_i)$ where $r_i \in \mathbb{Q}$. For some $q \in \mathbb{N}$, we can then write $r_i = \frac{p_i}{q}$ for some $p_i \in \mathbb{Z}$. It is therefore possible to partition B into $p_1 p_2 \dots p_d$ boxes of side length $\frac{1}{q}$. It follows from the previous part then that
$$\tilde{m}(B) = p_1 \dots p_d \cdot \frac{c}{q^d} = c \cdot \prod_{i=1}^{d} r_i = cm(B).$$
- Now let B be a box with real endpoints. Again by translation invariance it can be assumed $B = \prod_{i=1}^{d}[0,t_i)$. Take $q \in \mathbb{N}$ and set $p_i = \lfloor qt_i \rfloor$. Define then
$$B_L = \prod_{i=1}^{d}\left[0,\frac{p_i}{q}\right) \qquad B_U = \prod_{i=1}^{d}\left[0,\frac{p_i+1}{q}\right).$$
Note that $B_L \subset B \subset B_U$ and
$$\frac{p_1 \dots p_d}{q} \le t_1 \dots t_d \le \frac{(p_1+1)\dots(p_d+1)}{q} \tag{$\star$}$$
which in the limit as $q \to \infty$ leads to both sides converging to $t_1 \dots t_d$. Monotonicity gives $\tilde{m}(B_L) \le \tilde{m}(B) \le \tilde{m}(B_U)$. From the previous part, we have
$$\tilde{m}(B_L) = c \cdot \frac{p_1 \dots p_d}{q} \qquad \tilde{m}(B_U) = c \cdot \frac{(p_1+1)\dots(p_d+1)}{q}.$$
Therefore
$$c \cdot \frac{p_1 \dots p_d}{q} \le \tilde{m}(B) \le c \cdot \frac{(p_1+1)\dots(p_d+1)}{q}.$$
In the limit as $q \to \infty$, it follows then in conjunction with ($\star$) that $\tilde{m}(B) = c \cdot t_1 \dots t_2 = cm(B)$.

Now consider an elementary set E. Then there is a disjoint decomposition of E into boxes $B_1 \cup \dots B_n$. By finite additivity of $\tilde{m}$, it follows

$$\tilde{m}(B) = \sum_i \tilde{B}_i = \sum_i c \cdot m(B_i) = c \sum_i m(B_i) = cm(E)$$

which was to be shown. ◇

Ex 1.11

Proof. We will prove $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 1$.

$1 \Rightarrow 2$) Suppose E is Jordan measurable and take $\varepsilon > 0$. By the definition of the inner and outer Jordan measure, it follows that there exists elementary sets A and B such that $A \subset E$ and $E \subset B$ as well as

$$m(A) + \frac{\varepsilon}{2} \leq m_*^J(E) \qquad\qquad m_J^*(E) \leq m(B) - \frac{\varepsilon}{2}.$$

Subtracting the first inequality from the second gives

$$m(B) - \frac{\varepsilon}{2} - m(A) - \frac{\varepsilon}{2} \leq m_J^*(E) - m_*^J(E) = 0 \implies m(B) - m(A) \leq \varepsilon.$$

Since $B \setminus A$ and A are disjoint and $(B \setminus A) \cup A = A$,

$$\begin{aligned} m(A) &= m((B \setminus A) \cup A) \\ &= m(B \setminus A) + m(A) \implies m(B \setminus A) = m(B) - m(A) \end{aligned}$$

Therefore $m(B \setminus A) \leq \varepsilon$.

$2 \Rightarrow 3$) Take $\varepsilon > 0$ and suppose there exists elementary sets $A \subset E \subset B$ such that $m(B \setminus A) \leq \varepsilon$. Note that $A \setminus E = \varnothing$, thus $A \Delta E = E \setminus A$. Since $E \setminus A \subset B$ and $(E \setminus A) \cap A = \varnothing$, $E \setminus A \subset B \setminus A$ and thus $A \Delta E \subset B \setminus A$. Therefore $m_J^*(A \Delta E) \leq m(B \setminus A) \leq \varepsilon$.

$3 \Rightarrow 1$) Take $\varepsilon > 0$ and suppose there exists and elementary set A such that $m_J^*(A \Delta E) \leq \varepsilon$. Then for any $\delta > 0$ there exists some elementary set C such that $A \Delta E \subset C$ and $m(C) \leq m_J^*(A \Delta E) + \delta \leq \varepsilon + \delta$. Note that $A \cup C \supset E$ and $A \setminus C \subset E$ are both elementary sets. Therefore we have

$$m_J^*(E) \leq m_J^*(A \cup C) = m(A \cup C) \leq m(A) + m(C) \leq m(A) + (\varepsilon + \delta)$$

and

$$m_*^J(E) \geq m_*^J(A \setminus C) = m(A \setminus C) \geq m(A) - m(C) \geq m(A) - (\varepsilon + \delta).$$

Putting these together gives

$$m(A) - (\varepsilon + \delta) \leq m_*^J(E) \leq m_J^*(E) \leq m(A) + (\varepsilon + \delta).$$

Since δ was arbitrary, we can reduce the inequality bounds to $m(A) \pm \varepsilon$. Furthermore since ε was arbitrary, $m_*^J(E) = m_J^*(E) = m(A)$ and thus E is measurable.

$\diamond$

Ex. 1.13

Proof.

(i) Take $\varepsilon > 0$. Then there exists elementary sets A_1, A_2, B_1, B_2 such that $A_1 \subset E_1 \subset B_1$, $A_2 \subset E_2 \subset B_2$ and $m(B_1 \setminus A_1), m(B_2 \setminus A_2) \leq \varepsilon$.

- Note that $A_1 \cup A_2 \subset E_1 \cup E_2 \subset B_1 \cup B_2$ and

$$\begin{aligned}(B_1 \cup B_2) \setminus (A_1 \cup A_2) &= (B_1 \setminus (A_1 \cup A_2)) \cup (B_2 \setminus (A_1 \cup A_2)) \\ &\subset (B_1 \setminus A_1) \cup (B_2 \setminus A_2)\end{aligned}$$

  Therefore $m((B_1 \cup B_2) \setminus (A_1 \cup A_2)) \leq m(B_1 \cup A_1) + m(B_2 \setminus A_2) \leq 2\varepsilon$. Since $A_1 \cup A_2$ and $B_1 \cup B_2$ are elementary, $E_1 \cup E_2$ is Jordan measurable.

- Note that $A_1 \cap A_2 \subset E_1 \cap E_2 \subset B_1 \cap B_2$ and

$$\begin{aligned}(B_1 \cap B_2) \setminus (A_1 \cap A_2) &= (B_1 \setminus (A_1 \cap A_2)) \cup (B_2 \setminus (A_1 \cap A_2)) \\ &\subset (B_1 \setminus A_1) \cup (B_2 \setminus A_2)\end{aligned}$$

-

- Since $E_1 \Delta E_2 = (E_1 \setminus E_2) \cup (E_2 \setminus E_1)$, $E_1 \Delta E_2$ is Jordan measurable by (i) and (iii).

(ii) Since the elementary measure is non-negative and $m(E_1) = m_*^J(E_1) = \sup\left\{m(A) : A \subset E_1, A \in \mathcal{E}(\mathbb{R}^d)\right\} \geq 0$

(iv) If $A \subset E_1$ is elementary, then $A \subset E_2$, thus the supremum over all inner elementary sets for E_1 is a subset of the inner elementary sets for E_2. Thus $m_*^J(E_1) \leq m_*^J(E_2)$, which gives $m(E_1) \leq m(E_2)$ since E_1 and E_2 are Jordan measurable.

(v) Pick $\varepsilon > 0$ and elementary sets $B_1 \supset E_1, B_2 \supset E_2$ such that $m(B_1) - m(E_1) \leq \varepsilon$ and $m(B_2) - m(E_2) \leq \varepsilon$. Since $E_1 \cup E_2 \subset B_1 \cup B_2$ by (iv) it follows

$$m(E_1 \cup E_2) \leq m(B_1 \cup B_2) \leq m(B_1) + m(B_2) \leq m(E_1) + m(E_2) + 2\varepsilon.$$

Since ε was arbitrary, it follows $m(E_1 \cup E_2) \leq m(E_1) + m(E_2)$.

(iii) Take $\varepsilon > 0$. Then there exists elementary sets $A_1 \subset E_1, A_2 \subset E_2$ such that $m(E_1) - m(A_1) \leq \varepsilon$ and $m(E_2) - m(A_2) \leq \varepsilon$. Since $E_1 \cap E_2 = \varnothing$, $A_1 \cap A_2 = \varnothing$ which means $m(A_1 \cup A_2) = m(A_1) + m(A_2)$. Note that $A_1 \cup A_2 \subset E_1 \cup E_2$, which by (iv) gives

$$m(E_1 \cup E_2) \geq m(A_1 \cup A_2) = m(A_1) + m(A_2) \geq m(E_1) + m(E_2) - 2\varepsilon.$$

Since ε was arbitrary, it follows $m(E_1 \cup E_2) \geq m(E_1) + m(E_2)$ which in conjunction with (v) gives $m(E_1 \cup E_2) = m(E_1) + m(E_2)$.

(vi) Take $\varepsilon > 0$. Then there exists elementary sets $A \subset E_1 \subset B$ such that $m(B \setminus A) \leq \varepsilon$. Note that $A + x \subset E_1 + x \subset B + x$. Since $m(\cdot)$ is translation invariant for elementary sets, it follows $m(A + x) = m(A)$ and $m(B + x) = m(B)$. Note then by (iv) that

$$m(A) \leq m(E_1) \leq m(B)$$

and by the definition of the outer/inner Jordan measure that

$$m(A) = m(A + x) \leq m_*^J(E_1 + x) \leq m_J^*(E_1 + x) \leq m(B + x) = m(B).$$

Since $m(B \setminus A) = m(B) - m(A) \leq \varepsilon$ we have both $|m(E_1) - m_J^*(E_1 + x)| \leq \varepsilon$ and $|m(E_1) - m_*^J(E_1 + x)|$. The choice of ε was arbitrary, hence $m_J^*(E_1 + x) = m_*^J(E_1 + x) = m(E)$.

◇

Ex. 1.14

Proof.

i) Take $\varepsilon > 0$. Since B is closed and bounded, it is compact meaning f is uniformly continuous over B. Therefore $\exists \delta > 0$ such that $|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$. Partition B into a disjoint set of closed boxes Q_i whose diameters are smaller than δ. Associate then the set of points x_i with Q_i where $x_i \in Q_i$. Note then that $x \in Q_i$ gives $|x - x_i| < \delta \implies |f(x) - f(x_i)| < \varepsilon$. Thus we have

$$\{(x, f(x)) : x \in Q_i\} \subset Q_i \times [f(x_i) - \varepsilon, f(x_i) + \varepsilon].$$

Since all the Q_i cover B, it follows

$$G(f) \subset \bigcup_i Q_i \times [f(x_i) - \varepsilon, f(x_i) + \varepsilon].$$

The measure of the union can be bounded above by $2\varepsilon \cdot M$ where M is the total size of all the Q_i. The total size of all the Q_i is constant since it is simply the size of B. Therefore we have

$$0 \leq m_*^J(G(f)) \leq m_J^*(G(f)) \leq 2\varepsilon \cdot M.$$

Since ε is arbitrary, it follows that $m(G(f)) = 0$.

ii) Take $\varepsilon > 0$. Partition B into the boxes Q_i as described above. Let $m_i = \inf_{x \in B_i} f(x)$ and $M_i = \sup_{x \in B_i} f(x)$, and define the elementary sets

$$L = \bigcup_i Q_i \times [0, m_i] \qquad U = \bigcup_i Q_i \times [0, M_i].$$

Note that $A \subset B(f) \subset B$ and that $0 \leq M_i - m_i \leq \varepsilon$. Therefore

$$m(U \setminus L) = m\left(\bigcup_i Q_i \times [m_i, M_i]\right) \leq \sum_i |Q_i|(M_i - m_i) \leq \varepsilon \sum_i |Q_i|.$$

Similar to above, the total size of all the Q_i is constant giving $m(U \setminus L) \leq \varepsilon \cdot M$, hence $B(f)$ is Jordan measurable.

◇

Ex 1.18

Proof.

$\diamond$

Ex 1.25

Proof.

$\diamond$

Ex 1.26

Proof.

i) Since $E \subset \overline{E}$ it follows $m_J^*(E) \leq m_J^*(\overline{E})$, thus it suffices to show the reverse inequality. Take $\varepsilon > 0$. Then there exists boxes $B_1, \dots, B_n$ such that $E \subset \bigcup B_i$ and $\sum |B_i| \leq m_J^*(E) + \varepsilon$. Note that $|\overline{B_i}| = |B_i|$ and

$$\overline{E} \subset \overline{\bigcup_i B_i} = \bigcup_i \overline{B_i}.$$

Therefore $m_J^*(\overline{E}) \leq \sum |\overline{B_i}| = \sum |B_i| \leq m_J^*(E) + \varepsilon$, giving $m_J^*(\overline{E}) \leq m_J^*(E) + \varepsilon$. Since ε was arbitrary, we have equality.

ii) We use a similar argument as above. Clearly $m_*^J(\mathring{E}) \leq m_*^J(E)$. Take $\varepsilon > 0$. Then there exists boxes $B_1, \dots, B_n$ such that $\bigcup B_i \subset E$ with $m_*^J(E) + \varepsilon < \sum |B_i|$.

iii) Suppose E is Jordan measurable. Then $m_J^*(E) = m_*^J(E)$. Since $\mathring{E} \subset \overline{E}$, $m_*^J(\mathring{E}) \leq$

iv) Since $\overline{[0,1]^2 \setminus \mathbb{Q}^2} = [0,1]^2 = \overline{[0,1]^2 \cap \mathbb{Q}^2}$, it follows $m_J^*([0,1]^2 \setminus \mathbb{Q}^2) = m_J^*([0,1]^2 \cap \mathbb{Q}^2) = m_J^*([0,1]^2) = 1$. But $\mathrm{int}([0,1]^2 \setminus) = \varnothing = \mathrm{int}([0,1]^2 \cap \mathbb{Q}^2)$, thus $m_*^J([0,1]^2 \setminus \mathbb{Q}) = m_*^J([0,1]^2 \cap \mathbb{Q}^2) = m_*^J(\varnothing) = 0$. Therefore neither $[0,1]^2 \setminus \mathbb{Q}^2$ or $[0,1]^2 \cap \mathbb{Q}^2$ are Jordan measurable.

$\diamond$