Exercise 5.22

Proof. Suppose that $f_n \to f$ in the L^1 norm. Note that by Markov's inequality that for some $\varepsilon > 0$

$$\mu([|f - f_n| \geq \varepsilon]) \leq \frac{1}{\varepsilon} \cdot \int |f - f_n| d\mu = \frac{1}{\varepsilon} \cdot \|f - f_n\|_{L^1}.$$

Therefore in the limit as $n \to \infty$, the RHS goes to zero by the assumption, meaning $f_n \to f$ in measure.

Suppose towards contradiction then that $f_n \to f$ in measure but $f_n \not\to f$ in L^1. Then there exists some $\varepsilon > 0$ and subsequence f_{n_j} such that

$$\left\|f - f_{n_j}\right\| \geq \varepsilon, \forall j.$$

Denote this subsequence as h_n. Since $f_n \to f$ in measure, it is also the case that $h_n \to f$ in measure as well. Since f_n is a dominated sequence, so is h_n meaning there is some $g \in \mathcal{L}^1$ such that both $|h_n| \leq g$ and $|f| \leq g$. Therefore by the triangle inequality, $|f - h_n| \leq 2g$ meaning $|f - h_n|$ is also a dominated sequence. Since $h_n \to f$ in measure, there is some subsequence $h_{n_k} \to f$ pointwise a.e. Therefore by the Dominated Convergence Theorem it follows that

$$\lim_{n\to\infty} \int |f - h_{n_k}| d\mu = 0 \Leftrightarrow \lim_{n\to\infty} \left\|f - h_{n_k}\right\|_{L^1} = 0.$$

But for a sufficiently large k it follows that $\left\|f - h_{n_k}\right\| < \varepsilon$, a contradiction. Therefore $f_n \to f$ in L^1. ◇

Exercise 5.26

Proof. The forward direction is trivial as the result is a requirement of uniform integrability. Suppose then that

$$\sup_{n\in\mathbb{N}} \int_{[|f_n| \geq M]} |f_n| d\mu \to 0$$

as $M \to \infty$. One can then take M large enough such that

$$\sup_{n\in\mathbb{N}} \int_{[|f_n| \geq M]} \leq 1.$$

Since $\mu(X) < \infty$, for any n it then follows that

$$\int |f_n| d\mu = \int_{[|f_n| \geq M]} |f_n| d\mu + \int_{[|f_n| < M]} |f_n| d\mu \leq 1 + M \cdot \mu(X) < \infty.$$

Therefore the first condition for uniform integrability holds. Clearly the second holds as it is identical to the assumption, leaving the third to be shown. Note that for any n

$$\int_{[|f_n| \leq \delta]} |f_n| \leq \int_{[|f_n| \leq \delta]} \delta d\mu = \delta \cdot \mu(|f_n| \leq \delta]) \leq \delta \cdot \mu(X).$$

In the limit $\delta \to 0$, the integral goes to 0 and thus $(f_n)_{n \in \mathbb{N}}$ is uniformly integrable. $\diamond$

Exercise 5.27

Proof.

1. Denote $C = \sup_{n \in \mathbb{N}} \int |f_n|^p d\mu$. Note that $|f_n| \leq |f_n|^p + 1$ for any $p > 1$, therefore

   $$\sup_{n \in \mathbb{N}} \int |f_n| d\mu \leq \sup_{n \in \mathbb{N}} \int (|f_n|^p + 1) d\mu = \mu(X) + C < \infty.$$

   Therefore the first condition of uniform integrability holds. Consider $|f_n|$ when $|f_n| \geq M$. Note that for $p > 1$

   $$|f_n| = \frac{|f_n|^p}{|f_n|^{p-1}} \leq \frac{|f_n|^p}{M^{p-1}}.$$

   Integrating over both sides then gives

   $$\int_{[|f_n| \geq M]} |f_n| d\mu \leq \frac{1}{M^{p-1}} \cdot \int_{[|f_n| \geq M]} |f_n|^p d\mu \leq \frac{C}{M^{p-1}}$$

   which in the limit as $M \to \infty$ goes to 0. Therefore the second condition of uniform integrability holds. Take $\delta > 0$ and note that

   $$\int_{[|f_n| \leq \delta]} |f_n| d\mu \leq \int_{[|f_n| \leq \delta]} \delta d\mu = \delta \cdot \mu([|f_n| \leq \delta]) \leq \delta \cdot \mu(X).$$

   Since $\mu(X) < \infty$, it follows in the limit $\delta \to 0$ that the integral goes to 0 for all n. Thus $(f_n)_{n \in \mathbb{N}}$ is uniformly integrable.

2. Take $\varepsilon > 0$. By uniform integrability, there is some $\delta > 0$ and $M > 0$ such that
$$\sup_{n\in\mathbb{N}} \int_{[|f_n|\leq\delta]} |f_n| d\mu \leq \varepsilon \qquad \sup_{n\in\mathbb{N}} \int_{[|f_n|\geq M]} |f_n| \leq \varepsilon$$
If $\mu(E) \leq \varepsilon$, then note
$$\int_E |f_n| d\mu = \int_{E\cap[|f_n|\leq\delta]} |f_n| d\mu + \int_{E\cap[\delta<|f_n|<M]} |f_n| d\mu + \int_{E\cap[|f_n|\geq M]} |f_n| d\mu$$
in which each term can be bounded to give
$$\int_E |f_n| d\mu \leq \varepsilon + \mu(E)\cdot M + \varepsilon \leq 2\varepsilon + M\cdot\varepsilon$$
which was to be shown.

3. The first condition for uniform integrability is satisfied by taking $C = \sup_{n\in\mathbb{N}} \|f_n\|_{L^1}$. Note that by Markov's inequality
$$\mu([|f_n| \geq M]) \leq \frac{\|f_n\|_{L^1}}{M} \leq \frac{C}{M}.$$
For any $\varepsilon > 0$ and the associated δ given by the assumption, taking M large enough such that $\frac{C}{M} \leq \delta$ gives
$$\sup_{n\in\mathbb{N}} \int_{[|f_n|\geq M]} |f_n| d\mu \leq \varepsilon$$
hence the second condition for uniform integrability holds. Note that for $\delta > 0$ and any n that
$$\int_{[|f_n|\leq\delta]} |f_n| \leq \int_{[|f_n|\leq\delta]} \delta d\mu \leq \delta\cdot\mu(X).$$
Since $\delta\cdot\mu(X) \longrightarrow 0$ as $\delta \longrightarrow 0$, the third condition for uniform integrability then holds.

4. Let $f_n = \mathbb{1}_{[n,n+1]}$. Note that
   - $\|f_n\|_{L^1} = 1$ for all n
   - $\int_{[|f_n|\geq M]} |f_n| d\mu = 0$ for $M > 1$ and all n
   - $\int_{[|f_n|\leq\delta]} |f_n| d\mu = 0$ for $\delta < 1$ and all n

Thus f_n is uniformly integrable and converges pointwise almost everywhere to 0. However,

- It does not converge in L^1 since $\|f_n - 0\|_{L^1} = \|f_n\|_{L^1} = 1$
- It does not converge in measure since $\mu([f_n - 0] > \frac{1}{2}) = 1 \not\to 0$
- It does not converge almost uniformly since the set f_n differs from 0 will always have a measure of 1 and hence cant be arbitrarily small

$\diamond$

Exercise 5.31

Proof. Let $f := \sup_{n\in\mathbb{N}} f_n = \lim_{n\to\infty} f_n$. Since f_n is a non-decreasing sequence, it follows from the Monotone Convergence Theorem that

$$\int f d\mu = \lim_{n\to\infty} \int f_n d\mu = \sup_{n\in\mathbb{N}} \int f_n d\mu < \infty.$$

Therefore $f \in \mathcal{L}^1$. Since $f_n \leq f$, it follows that $f - f_n \geq 0$. Thus

$$\|f - f_n\|_{L^1} = \int |f - f_n| d\mu = \int (f - f_n) d\mu = \int f d\mu - \int f_n d\mu.$$

In the limit the integrals of the RHS are equal, meaning $\lim_{n\to\infty} \|f - f_n\|_{L^1} = 0$. Hence $f_n \to f$ in the L^1 norm. $\diamond$

Exercise 5.33

Proof. The forward direction follows from exercise 5.5. Suppose then that $f_n \to f$ pointwise a.e. and that f_n is dominated by some $g \in \mathcal{L}^1$. Take $\varepsilon > 0$ and define

$$X_k = \left\{ x \in X : g(x) \geq \frac{1}{k} \right\}.$$

By Markov's inequality it follows that

$$\mu(X_k) \leq k \cdot \int g d\mu \leq \infty$$

thus each of the X_k are finite. For each k, it follows then by Egorov's that there is a $B_k \subset X_k$ with $\mu(B_k) \leq \frac{\varepsilon}{2^k}$ where $f_n \to f$ uniformly on $X_k \setminus B_k$. Let then $B = \bigcup B_k$ and note that $\mu(B) \leq \varepsilon$. Take now $K > 0$ large such that $\frac{1}{K} \leq \varepsilon$. If $x \notin X_K \cup E$, then $g(x) < \frac{1}{K} \leq \varepsilon$. Note then that

$$|f_n(x) - f(x)| \leq |f_n(x)| + |f(x)| \leq 2g(x) \leq 2\varepsilon.$$

By the construction of X_K, it follows that $f_n \to f$ uniformly on $X_K \setminus E$. Therefore $f_n \to f$ uniformly everywhere except on B which is arbitrarily small. $\diamond$