Problem 1

Note that $F(x,y,z) = 0$ defines implicitly $z = z(x,y)$. Let $h : \mathbb{R}^2 \to \mathbb{R}^3$ where $h(x,y) = (x,y,z(x,y))$. Then $F(h(x,y)) = F(x,y,z(x,y)) = 0$ and $F'(x,y,z) = 0$. Note that

$$F'(h(x,y)) = F'(x,y,z) = \begin{bmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} \end{bmatrix} \qquad h'(x,y) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \end{bmatrix}$$

Thus by chain rule

$$\begin{aligned} (F \circ h)'(x,y) &= F'(h(x,y)) \cdot h'(x,y) \\ &= \begin{bmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \end{bmatrix} \\ &= \left[\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x}, \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial y} \right] \end{aligned}$$

Since $F'(x,y,z) = 0$, that means each component above must also be identically 0, giving

$$\begin{aligned} \frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0 &\implies \frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} \\ \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial y} = 0 &\implies \frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} \end{aligned}$$

Problem 2

Using the previous equations for $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ from (1) and differentiating with respect to x gives

$$\frac{\partial z}{\partial x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial z}{\partial x} = -\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}\right] = -\frac{\frac{\partial^2 F}{\partial x^2}\frac{\partial F}{\partial z} - \frac{\partial F}{\partial x}\frac{\partial^2 F}{\partial z \partial x}}{\left(\frac{\partial F}{\partial z}\right)^2}.$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial z}{\partial y} = -\frac{\mathrm{d}}{\mathrm{d}y}\left[\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}\right] = -\frac{\frac{\partial^2 F}{\partial x \partial y}\frac{\partial F}{\partial z} - \frac{\partial F}{\partial x}\frac{\partial^2 F}{\partial z \partial y}}{\left(\frac{\partial F}{\partial z}\right)^2}.$$

Problem 3

Let $u, v : \mathbb{R}^3 \to \mathbb{R}$ with $u(x,y,z) = xz$ and $v(x,y,z) = yz$. Then $w = F(u,v)$, meaning by the chain rule

$$\frac{\partial w}{\partial x} = \frac{\partial F}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial F}{\partial v} \cdot \frac{\partial v}{\partial x} = \frac{\partial F}{\partial u} \cdot z$$

$$\frac{\partial w}{\partial y} = \frac{\partial F}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial F}{\partial v} \cdot \frac{\partial v}{\partial y} = \frac{\partial F}{\partial v} \cdot z$$

$$\frac{\partial w}{\partial z} = \frac{\partial F}{\partial u} \cdot \frac{\partial u}{\partial z} + \frac{\partial F}{\partial v} \cdot \frac{\partial v}{\partial z} = \frac{\partial F}{\partial u} \cdot x + \frac{\partial F}{\partial v} \cdot y$$

Note then that

$$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = xz\frac{\partial F}{\partial u} + yz\frac{\partial F}{\partial v} = z\left(x\frac{\partial F}{\partial u} + y\frac{\partial F}{\partial v}\right) = z\frac{\partial w}{\partial z}$$

which was to be shown.

Problem 4

Let $u, v : \mathbb{R}^2 \to \mathbb{R}$ where $u(t_1, t_2) = \frac{t_1}{t_2}$ and $v(t_2, t_3) = \frac{t_2}{t_3}$. Note then that $g(t_1, t_2, t_3) = f(u(t_1, t_2), v(t_2, t_3))$. Therefore by the chain rule

$$\frac{\partial g}{\partial t_1} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial t_1} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial t_1} = \frac{\partial f}{\partial u} \cdot \frac{1}{t_2} + \frac{\partial f}{\partial v} \cdot (0) \qquad = \boxed{\frac{\partial f}{\partial u} \cdot \frac{1}{t_2}}$$

$$\frac{\partial g}{\partial t_2} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial t_2} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial t_2} = \frac{\partial f}{\partial u} \cdot \left(-\frac{t_1}{t_2^2}\right) + \frac{\partial f}{\partial v} \cdot \frac{1}{t_3} \quad = \boxed{\frac{\partial f}{\partial v} \cdot \frac{1}{t_3} - \frac{\partial f}{\partial u} \cdot \frac{t_1}{t_2^2}}$$

$$\frac{\partial g}{\partial t_3} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial t_3} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial t_3} = \frac{\partial f}{\partial u} \cdot (0) + \frac{\partial f}{\partial v} \cdot \left(-\frac{t_2}{t_3^2}\right) = \boxed{-\frac{\partial f}{\partial v} \cdot \frac{t_2}{t_3^2}}$$

Problem 5

Assume that $u \in C^2(\mathbb{R}^2; \mathbb{R})$ and consider $u(s,t)$. By chain rule

$$u_x = \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial x} + \frac{\partial u}{\partial t} \cdot \frac{\partial t}{\partial x} = 2t \cdot u_s + u_t$$
$$u_y = \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial y} + \frac{\partial u}{\partial t} \cdot \frac{\partial t}{\partial y} = u_s$$

Applying chain rule again gives

$$\begin{aligned}
u''_{xx} &= \frac{\partial}{\partial x}\left(2t \cdot u'_s + u'_t\right) \\
&= \frac{\partial}{\partial x}\left(2x \cdot u'_s\right) + \frac{\partial}{\partial x}\left(u'_t\right) \\
&= 2u'_s + 2x\left(\frac{\partial u'_s}{\partial s} \cdot \frac{\partial s}{\partial x} + \frac{\partial u'_s}{\partial t} \cdot \frac{\partial t}{\partial x}\right) + \left(\frac{\partial u'_t}{\partial s} \cdot \frac{\partial s}{\partial x} + \frac{\partial u'_t}{\partial t} \cdot \frac{\partial t}{\partial x}\right) \\
&= 2u'_s + 2t\left(2t \cdot u''_{ss} + u''_{st}\right) + \left(2t \cdot u''_{ts} + u''_{tt}\right) \\
&= 4t^2 \cdot u''_{ss} + 4t \cdot u''_{st} + u''_{tt} + 2u'_s
\end{aligned}$$

$$\begin{aligned}
u''_{yy} &= \frac{\partial}{\partial y}\left(u'_s\right) \\
&= \frac{\partial u'_s}{\partial s} \cdot \frac{\partial s}{\partial y} + \frac{\partial u'_s}{\partial t} \cdot \frac{\partial t}{\partial y} \\
&= u''_{ss}
\end{aligned}$$

$$\begin{aligned}
u''_{xy} &= \frac{\partial}{\partial x}\left(u'_s\right) \\
&= \frac{\partial u'_s}{\partial s} \cdot \frac{\partial s}{\partial x} + \frac{\partial u'_s}{\partial t} \cdot \frac{\partial t}{\partial x} \\
&= 2t \cdot u''_{ss} + u''_{st}
\end{aligned}$$

Thus substituting these into the original PDE gives

$$\begin{aligned} 4t^2 \cdot u''_{ss} + 4t \cdot u''_{st} + u''_{tt} + 2u'_s - 4t\big(2t \cdot u''_{ss} + u''_{st}\big) + 4t^2 u''_{ss} - 2u'_s &= y \\ (4t^2 - 8t^2 + 4t^2)u''_{ss} + (4t - 4t)u''_{st} + (2-2)u'_s + u''_{tt} &= s - t^2 \\ u''_{tt} &= s - t^2 \end{aligned}$$

This new form can be solved then by integrating twice with respect to t

$$u = \iint (s - t^2)\mathrm{d}^2 t = \int \left(st - \frac{t^3}{3} + C_1(s)\right)\mathrm{d}t = \frac{st^2}{2} - \frac{t^4}{12} + C_1(s)t + C_2(s)$$

where $C_1(s)$ and $C_2(s)$ are functions due to indefinite integration. Substituting x and y back in in gives a final solution of

$$u(x,y) = \frac{(x^2+y)x^2}{2} - \frac{x^4}{12} + C_1(x^2+y)x + C_2(x^2+y).$$

Problem 6

Let $z = \frac{x_2}{x_1}$. Then $f(x) = x_1 g(z) + h(z)$. Note that

$$\frac{\partial z}{\partial x_1} = -\frac{x_2}{x_1^2} \qquad \frac{\partial z}{\partial x_2} = \frac{1}{x_1}.$$

Applying chain rule once gives

$$\begin{aligned} f'_{x_1}(x) &= 1 \cdot g(z) + x_1 \cdot g'(z) \cdot \frac{\partial z}{\partial x_1} + h'(z) \cdot \frac{\partial z}{\partial x_1} &&= g(z) - \frac{x_2}{x_1} \cdot g'(z) - \frac{x_2}{x_1^2} \cdot h'(z) \\ f'_{x_2}(x) &= x_1 \cdot g'(z) \cdot \frac{\partial z}{\partial x_2} + h'(z) \cdot \frac{\partial z}{\partial x_2} &&= g'(z) + \frac{1}{x_1} \cdot h'(z) \end{aligned}$$

And thus applying once again

$$\begin{aligned} f''_{x_1x_1} &= g'(z) \cdot \frac{\partial z}{\partial x_1} + \frac{x_2}{x_1^2} \cdot g'(z) - \frac{x_2}{x_1} \cdot g''(z) \cdot \frac{\partial z}{\partial x_1} + \frac{2x_2}{x_1^3} \cdot h'(z) - \frac{x_2}{x_1^2} \cdot h''(z) \cdot \frac{\partial z}{\partial x_1} \\ &= \frac{x_2^2}{x_1^3} \cdot g''(z) + \frac{2x_2}{x_1^3} \cdot h'(z) + \frac{x_2^2}{x_1^4} \cdot h''(z) \end{aligned}$$

$$f''_{x_2x_2} = g''(z)\frac{\partial z}{\partial x_2} + \frac{1}{x_1} \cdot h''(z) \cdot \frac{\partial z}{\partial x_2}$$

$$= \frac{1}{x_1} \cdot g''(z) + \frac{1}{x_1^2} \cdot h''(z)$$

$$\begin{aligned} f''_{x_1 x_2} &= g''(z) \cdot \frac{\partial z}{\partial x_1} - \frac{1}{x_1^2} \cdot h'(z) + \frac{1}{x_1} \cdot h''(z) \cdot \frac{\partial z}{\partial x_1} \\ &= -\frac{x_2}{x_1^2} \cdot g''(z) - \frac{1}{x_1^2} h'(z) - \frac{x_2}{x_1^3} h''(z) \end{aligned}$$

Therefore

$$\begin{aligned} x_1^2 f''_{x_1 x_1} &= \frac{x_2^2}{x_1} \cdot g''(z) + \frac{2x_2}{x_1} \cdot h'(z) + \frac{x_2^2}{x_1^2} \cdot h''(z) \\ x_2^2 f''_{x_2 x_2} &= \frac{x_2^2}{x_1} \cdot g''(z) + \frac{x_2^2}{x_1^2} \cdot h''(z) \\ 2x_1 x_2 f''_{x_1 x_2} &= -\frac{2x_2^2}{x_1} \cdot g''(z) - \frac{2x_2}{x_1} h'(z) - \frac{2x_2^2}{x_1^2} h''(z) \end{aligned}$$

Matching terms when summing these gives

$$\begin{aligned} h'(z) &\implies \frac{2x_2}{x_1} - \frac{2x_2}{x_1} &&= 0 \\ h''(z) &\implies \frac{x_2^2}{x_1^2} + \frac{x_2^2}{x_1^2} - \frac{2x_2^2}{x_1^2} &&= 0 \\ g''(z) &\implies \frac{x_2^2}{x_1} + \frac{x_2^2}{x_1} - \frac{2x_2^2}{x_1} &&= 0 \end{aligned}$$

Therefore the sum is zero giving

$$x_1^2 f''_{x_1 x_1}(x) + 2x_1 x_2 f''_{x_1 x_2}(x) + x_2^2 f''_{x_2 x_2}(x) = 0, x \in U.$$

Problem 7

Proof. Take $a, b \in E$ such that they differ only in their first components. That is

$$a = (a_1, a_2, \ldots, a_n), b = (b_1, a_2, \ldots, a_n)$$

with $a_1 \neq b_1$. Let $\phi : [0,1] \to \mathbb{R}$ where $\phi(t) = f(b_1 t + (1-t)a_1, a_2, \ldots, a_n)$. Since E is convex, $(b_1 t + (1-t)a_1, a_2, \ldots, a_n) \in E$ for all $t \in [0,1]$. Since $\partial_{x_1} f$

exists and is continuous on all of E, it follows by the chain rule that $\phi'(t)$ exists and

$$\phi'(t) = \partial_{x_1} f(b_1 t + (1-t)a_1, a_2, \ldots, a_n) \cdot (b_1 - a_1) = 0 \cdot (b_1 - a_1) = 0.$$

Thus $\phi(t)$ must be a constant function meaning $\phi(0) = \phi(1)$ which gives $f(a) = f(b)$. Since a, b were arbitrary, it follows that taking any $c \in E$ gives

$$f(x_1, \ldots, x_n) = f(c, x_2, \ldots, x_n)$$

meaning f only depends on $x_2, \ldots, x_n$. $\diamond$