Theorem 2.

Let $K \subseteq \mathbb{R}^n$ be compact and $f : K \to \mathbb{R}^n$ be continuous. Then f is uniformly continuous on K.

Proof. Suppose towards contradiction that f is not uniformly continuous. Then $\exists \varepsilon > 0$ such that $\forall \delta > 0$, there exists $x, y \in K$ where $\|x - y\| < \delta$ while $\|f(x) - f(y)\| > \varepsilon$. Letting $\delta_k = \frac{1}{k}$ for $k \in \mathbb{N}$, there is then corresponding $x^{(k)}$ and $y^{(k)}$ such that $\left\|x^{(k)} - y^{(k)}\right\| \leq \delta_k$ while $\left\|f(x^{(k)}) - f(y^{(k)})\right\|$. By compactness of K, there exists a subsequence $\left(x^{(k_j)}\right)$ that converges to some $x \in K$. Then

$$0 \leq \left\|y^{(k_j)} - x\right\| \leq \underbrace{\left\|y^{(k_j)} - x^{(k_j)}\right\|}_{\leq \frac{1}{k_j} \leq \frac{1}{j}} + \left\|x^{(k_j)} - x\right\|.$$

In the limit as $j \to \infty$, the upper bound goes to 0. Thus $\left\|y^{(k_j)} - x\right\|$ goes to 0, hence $\left(y^{(k_j)}\right)$ converges to x. Since f is continuous at x and y, $f(x^{(k_j)}) \to f(x)$ and $f(y^{(k_j)} \to f(y)$ as $j \to \infty$. Thus

$$\left\|f(x^{(k_j)}) - f(y^{(k_j)})\right\| \leq \left\|f(x^{(k_j)} - f(x)\right\| + \left\|f(x) - f(y^{(k_j)}\right\|$$

which goes to 0 as $j \to \infty$, a contradiction. Thus f is uniformly continuous. ◇

Def. Open Cover

Let $A \subseteq \mathbb{R}^n$. An **open cover** of A is a collection of open sets (G_α) in $\mathbb{R}^n$ such that $A \subseteq \bigcup G_\alpha$.

Def. Topological Compactness

A set $K \subseteq \mathbb{R}^n$ is **topologically compact** if every open cover of K has a finite subcover. In other words, for any open cover (G_α) of K, there are $\{\alpha_1, \ldots, \alpha_n\}$ indices with $n < \infty$ such that $K \subseteq G_{\alpha_1} \cup \ldots \cup G_{\alpha_n}$.

Example 1. The set $I = (0,1) \subseteq \mathbb{R}$ is not topologically compact. Consider the candidate open cover $\bigcup_{x\in(0,1)} \left(\frac{x}{2}, \frac{x+1}{2}\right)$. Let $x \in (0,1)$. Note that

$$\begin{aligned} x > 0 &\implies 2x > x &&\implies x > \frac{x}{2} \\ x < 1 &\implies 2x < x+1 &&\implies x < \frac{x+1}{2} \end{aligned}$$

Thus it is an open cover. Assume then there exists a finite subcover

$$\left(\frac{x_1}{2}, \frac{x_1+1}{2}\right) \cup \ldots \cup \left(\frac{x_n}{2}, \frac{x_n+1}{2}\right)$$

for $x_1, \ldots, x_n \in (0,1)$. Take $x \in \min\{x_1, \ldots, x_n\} > 0$ and $0 < y < \frac{x}{2}$. Then $y \in (0,1)$ but not in the subcover. Hence I cannot be topologically compact.

0.0.1 Compactness Equivalence

The goal of this section is to prove the following theorem.

Theorem. Sequential $\Leftrightarrow$ Topological Compactness

A set $K \subseteq \mathbb{R}^n$ is topologically compact iff K is sequentially compact.

The approach will be to use the result being close and bounded is equivalent to sequential compactness as a bridge. That is, show that topological compactness is equivalent to being closed and bounded, and thus sequentially compact as well.

Lemma 2.

Let $K \subseteq \mathbb{R}^n$ be (topologically) compact and $F \subseteq K$ be closed in $\mathbb{R}^n$. Then F is also (topologically) compact.

Proof. Let (G_α) be an open cover of F. Note then that $K \subseteq F^c \cup \bigcup_\alpha G_\alpha$. Since F is closed, F^c is open and thus this is an open cover of K. Since K is topologically compact, there then exists $\alpha_1, \ldots, a_n$ finite such that

$K \subseteq G_{\alpha_1} \cup \ldots G_{\alpha_n} \cup F^c$. Since $F \subseteq K$, this is a finite cover of F as well. Hence F is compact. $\diamond$

Theorem. Heine-Borel

Let $K \subseteq \mathbb{R}^n$. Then K is (topologically) compact iff K is closed and bounded.

Def. Closed Cube

A set $Q \subset \mathbb{R}^n$ is a **closed cube** if there exists closed intervals $I_1, \ldots, I_n$ in $\mathbb{R}$ such that $Q = I_1 \times \ldots \times I_n$.

Proof.

$\Leftarrow$) Assume K is closed and bounded.

$\diamond$