Problem 1

Proof. Let $K = \{k+1, \dots, 2k+1\}$.

Part A

Assume towards contradiction that A is not sum free. Then there exists some b_i, b_j, b_k such that $b_i + b_j = b_k$. Since $b_i, b_j, b_k < \max\{|b_i| : 1 \le i \le n\}$ and $p > 2\max\{|b_i| : 1 \le 1 \le n\}$, $b_i + b_j < p$ thus

$$b_i + b_j \equiv b_k \pmod{p} \implies xb_i + xb_j \equiv xb_k \pmod{p}.$$

But this means that $d_i + d_j = d_k$, which are elements of a sum free subset. This is a contradiction, hence A must be sum free.

Part B

Let D_i be the indicator random variable that $d_i \in K$. Then we have

$$\mathbb{E}[|\{d_1, \dots, d_n\} \cap K|] = \mathbb{E}\left[\sum_{i=1}^{n} D_i\right] = \sum_{i=1}^{n} \mathbb{E}[D_i].$$

Since $x \neq 0$ and $b_i \neq 0$, $d_i \neq 0$ and hence the number of total possibilities for d_i is $p-1$. Therefore since x is chosen uniformly at random

$$\mathbb{P}[d_i \in K] = \frac{(2k+2) - (k+1)}{p-1} = \frac{k+1}{3k+1}$$

Thus we have that the desired expectation equals

$$\sum_{i=1}^{n} \mathbb{E}[D_i] = \sum_{i=1}^{n} \frac{k+1}{3k+1} = n\left(\frac{k+1}{3k+1}\right).$$

Since $\frac{k+1}{3k+1} > \frac{1}{3}$ for all $k \ge 0$,

$$\mathbb{E}[|\{d_1, \dots, d_n\} \cap K|] > \frac{n}{3}.$$

Part C

Since the expectation is greater than $\frac{n}{3}$, it follows that

$$\mathbb{P}\left[|\{d_1, \ldots, d_n\} \cap K| > \frac{n}{3}\right] > 0.$$

Therefore there must exist some x such that

$$|\{d_1, \ldots, d_n\} \cap K| > \frac{n}{3}.$$

Part D

Take A to be the collection of associated b_i to x such that $d_i \in K$. Then by (a) this set is A sum free and by (c) this set has size greater than $\frac{n}{3}$. $\diamond$

Problem 2

Proof. Let $\varepsilon_1, \ldots, \varepsilon_n$ be independently randomnly chosen weights such that $\mathbb{P}[\varepsilon_i = -1] = \mathbb{P}[\varepsilon_i = 1] = \frac{1}{2}$. If $\langle\ ,\ \rangle$ is the Euclidean inner product, then $\|\cdot\|^2 = \langle \cdot, \cdot \rangle$ and thus by linearity of the inner product

$$\begin{aligned}
\left\|\sum_i \varepsilon_i v_i\right\|^2 &= \left\langle \left(\sum_i \varepsilon_i v_i\right), \left(\sum_j \varepsilon_j v_j\right)\right\rangle \\
&= \sum_i \left\langle \varepsilon_i v_i, \sum_j \varepsilon_j v_j \right\rangle \\
&= \sum_i \sum_j \left\langle \varepsilon_i v_i, \varepsilon_j v_j \right\rangle \\
&= \sum_i \sum_j \varepsilon_i \varepsilon_j \left\langle v_i, v_j \right\rangle .
\end{aligned}$$

Note that since $\mathbb{E}\left[\varepsilon_i^2\right] = 1$ and for $i \neq j$, $\mathbb{E}\left[\varepsilon_i \varepsilon_j\right] = \mathbb{E}[\varepsilon_i]\mathbb{E}\left[\varepsilon_j\right] = 0$,

$$\mathbb{E}\left[\varepsilon_i \varepsilon_j\right] = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}.$$

Therefore by linearity of expectation, we have

$$\begin{aligned}\mathbb{E}\left[\left\|\sum_i \varepsilon_i v_i\right\|^2\right] &= \mathbb{E}\left[\sum_i \sum_j \varepsilon_i \varepsilon_j \left\langle v_i, v_j\right\rangle\right] \\ &= \sum_i \sum_j \mathbb{E}\left[\varepsilon_i \varepsilon_j\right] \left\langle v_i, v_j\right\rangle \\ &= \sum_i \langle v_i, v_i\rangle \\ &= n\end{aligned}$$

since all of the v_i are unit vectors. Therefore since the expected value of the norm squared is n, then the probability that there exists an assignment of the ε_i that makes the norm squared less than or equal to n is non zero. Since this probability is non zero, there then must exist such an assignment. Thus

$$\left\|\sum_i \varepsilon_i v_i\right\|^2 \leq n \implies \left\|\sum_i \varepsilon_i v_i\right\| \leq \sqrt{n}$$

which was to be shown. ◇

Problem 3

Proof. Trivially, if $n < 3$ then K_n has no possible rainbow triangles and $\binom{n}{3}\frac{2}{9} < 1$. Assume then that $n \geq 3$. Consider a random assignment of a color to each edge $e \in K_n$ such that each color is equally likely. Let $X_{i,j,k}$ be the indicator random variable that the triangle comprised of edges i, j, k is rainbow. Then the expected number of rainbow triangles is

$$\mathbb{E}\left[\sum_{i \neq j \neq k} X_{i,j,k}\right] = \sum_{i \neq j \neq k} \mathbb{E}\left[X_{i,j,k}\right] = \binom{n}{3} \mathbb{E}\left[X_{1,2,3}\right]$$

since each $X_{i,j,k}$ is identically distributed hence we only need to find the expectation of a single one, and there are $\binom{n}{3}$ edges since K_n is complete.

The expectation is simply the probability a given triangle is rainbow. There are $3^3 = 27$ possible colorings of the edges of a triangle, and there are $3 \cdot 2 \cdot 1 = 6$ colorings in which all edges are different colors. Since every coloring is equally likely, this means

$$\mathbb{E}\left[\sum_{i \neq j \neq k} X_{i,j,k}\right] = \binom{n}{3}\mathbb{E}\left[X_{1,2,3}\right] = \binom{n}{3}\frac{6}{27} = \binom{n}{3}\frac{2}{9}.$$

Therefore the probability that a random edge coloring of K_n contains more than $\binom{n}{3}\frac{2}{9}$ rainbow triangles is non zero, hence such a random edge coloring must exist. ◇

Problem 4

Proof. Let $S = \{1, \ldots, n\}$ and $I = (c, c+1)$ where $c \in \mathbb{R}$. Given a vector $\varepsilon \in \{0,1\}^n$, we can associate a subset $E \subseteq S$ in which $i \in E$ iff $\varepsilon_i = 1$. Note then that the sum

$$\sum_{i=1}^{n} \varepsilon_i a_i = \sum_{i \in E} a_i.$$

Assume towards contradiction we have a family of such subsets such that they satisfy the unit interval summation property, but there is two subsets E_1 and E_2 such that $E_1 \neq E_2$ and $E_1 \subset E_2$. Note that

$$c < \sum_{i \in E_1} a_i < c + 1.$$

Since $E_1 \subset E_2$,

$$\sum_{i \in E_2} a_i = \sum_{i \in E_1} a_i + \sum_{i \in E_2 \setminus E_1} a_i.$$

Furthermore, there must be at least one element a_i in E_2 that isnt in E_1. Combined with the fact that all $a_i \geq 1$,

$$\sum_{i \in E_2 \setminus E_1} a_i \geq 1.$$

Thus

$$\sum_{i\in E_2} a_i = \sum_{i\in E_1} a_i + \sum_{i\in E_2\setminus E_1} a_i \geq 1 + \sum_{i\in E_1} a_i > c + 1.$$

But this means E_2 can not be in the family since it does not lay in I, a contradiction. Therefore the family of E_i is a Sperner Family and hence has at most $\binom{n}{\frac{n}{2}} = \binom{n}{\lfloor\frac{n}{2}\rfloor}$ members. By the initial equivalence, this means that a family of ε vectors is also subject to the same upper bound. $\diamond$

Problem 5

Proof. Let $\tilde{X} = X - a \cdot \mathbb{1}_{X\geq a}$. Note that $\tilde{X} \geq 0$, since when $X < a$ then $\tilde{X} = X$ and when $X \geq a$, then $\tilde{X} = X - a \geq a - a = 0$. Furthermore, by linearity of expectation $\mathbb{E}\left[\tilde{X}\right] = \mathbb{E}[X] - a\mathbb{P}[X \geq a]$. When $\mathbb{E}\left[\tilde{X}\right] = 0$, we get equality of the Markov inequality. Since $\tilde{X}$ is non-negative, the expectation being 0 means that $\mathbb{P}\left[\tilde{X} = 0\right] = 1$. The only two possibilities for X that satisfy this are either $\mathbb{P}[X = 0] = 1$ or $\mathbb{P}[X = a] = 1$. $\diamond$