Problem 1

Part A

Let Y be the number of games played. Note that $Y = X + 5$ where X is a geometric distribution with probability $1 - p$. This is because we are guaranteed to play 5 games and then continually play until the first failure (which has probability $1-p$). Thus the expected number of games played is

$$\mathbb{E}[Y] = \mathbb{E}[X+5] = \mathbb{E}[X] + 5 = \frac{1}{1-p} + 5.$$

Part B

Let Y be the number of games lost. If X is a binomial distribution with probability $1 - p$ and trial count 5 then $Y = X + 1$. That is, the number of games lost is the number of games lost in the first 5 plays plus the loss that stops further play. Thus the expected number of games lost is

$$\mathbb{E}[Y] = \mathbb{E}[X+1] = \mathbb{E}[X] + 1 = n(1-p) + 1.$$

Problem 2

Part A

In order for the maximum number on a ball to be a given value x, the other two balls must have values less than x written on them. Therefore the probability of a given value being the maximum is the probability that two other smaller value balls are drawn. There are $\binom{x-1}{2}$ possible choices for two balls with value smaller than x and $\binom{20}{3}$ possible choices of three balls. Thus

$$f_X(x) = \frac{\binom{x-1}{2}}{\binom{20}{3}}.$$

Part B

Problem 3

Let M be the event that A gets more heads after $n+1$ flips than B after n flips, H_A be the event that A gets more heads than B after both do n flips, H_B the event B has more heads after n flips, and H_0 the event that both have the same number of heads. Since H_A, H_B, H_0 are all mutually exclusive and cover all cases, by the law of total probability, we have

$$\mathbb{P}[M] = \mathbb{P}[M \mid H_A]\mathbb{P}[H_A] + \mathbb{P}[M \mid H_B]\mathbb{P}[H_B] + \mathbb{P}[M \mid H_0]\mathbb{P}[H_0].$$

Note that

- $\mathbb{P}[M \mid H_A] = 1$ since A would already have more heads than B, thus the $n+1$ flip will not change the outcome that A has more
- $\mathbb{P}[M \mid H_B] = 0$ since A is at least 1 head flip behind B and thus cannot overtake B in terms of heads with an additional flip
- $\mathbb{P}[M \mid H_0] = \frac{1}{2}$ since A will have more heads than B if a heads is flipped which occurs with probability $\frac{1}{2}$
- $\mathbb{P}[H_A] = \mathbb{P}[H_B]$ since the coin is fair and thus the outcome of either A or B having more heads is symmetric
- $\mathbb{P}[H_0] = 1 - \mathbb{P}[H_A \cup H_B] = 1 - \mathbb{P}[H_A] - \mathbb{P}[H_B] = 1 - 2\mathbb{P}[H_A]$

Therefore in total we have

$$\begin{aligned}
\mathbb{P}[M] &= \mathbb{P}[M \mid H_A]\mathbb{P}[H_A] + \mathbb{P}[M \mid H_B]\mathbb{P}[H_B] + \mathbb{P}[M \mid H_0]\mathbb{P}[H_0] \\
&= 1 \cdot (\mathbb{P}[H_A]) + 0 \cdot (\mathbb{P}[H_B]) + \frac{1}{2} \cdot (1 - 2\mathbb{P}[H_A]) \\
&= \mathbb{P}[H_A] + \frac{1}{2} - \mathbb{P}[H_A] = \boxed{\frac{1}{2}}
\end{aligned}$$

Problem 4

First consider the scenario that the barrel is spun again before shooting. Since the barrel is being respun, the first attempt by the opponent has no effect as a new random selection is being made. Therefore the probability of living is $\frac{4}{6}$ since there are 4 empty slots of the 6 total chambers.

Now consider the scenario in which the barrel is not spun again before shooting. Then the probability of being shot is the probability that a given empty chamber is followed by the adjacent loaded chambers. Since there is only 1 empty chamber out of 4 that is followed by the loaded chambers, the probability of getting shot is $\frac{1}{4}$ meaning a $\frac{3}{4}$ probability of survival.

Problem 5

Part A

We can use the following result to re-express $A^{(n)}$.

Theorem.

There exists a semicircle containing the points $P_1, \ldots, P_n$ if and only if there is a semicircle with an endpoint as some P_i containing all the points.

Proof. The reverse implication is trivial. Assume that there exists a semicircle C containing all points P_i. Without loss of generality, take $P_1, \ldots, P_n$ in clockwise order. Since all points are contained in C, can rotate C clockwise until one of the endpoints is at P_1. Call this new semicircle C'. Note all points must be in C' since the point furthest around the circle clockwise, P_n, is contained in C and in front P_1, and thus moving C to C' will not cause P_n to no longer be contained. Thus since P_1 and P_n are in C' and all other points are between P_1 and P_n, all points are contained in C'. $\diamond$

Therefore we can identify the event that such a semicircle exists for a set of randomnly sampled points with the events that a semicircle starting at one the points contains all other points, giving

$$A^{(n)} = \bigcup_{i=1}^{n} A_i^{(n)}.$$

Part B

Using part A we get

$$\mathbb{P}\left[A^{(n)}\right] = \mathbb{P}\left[\bigcup_i A_i^{(n)}\right] = \sum_i \mathbb{P}\left[A_i^{(n)}\right] + O(1).$$

We will have exact equality without any other factors if we have pairwise mutually exclusivity between the $A_i^{(n)}$.

$$\mathbb{P}\left[A_i^{(n)}\right] = \left(\frac{1}{2}\right)^{n-1} = \frac{1}{2^{n-1}}$$

since the other $n-1$ uniformly selected points have $\frac{1}{2}$ probability to be in the semicircle each (due to the fact that the semicircle is half the circumference and the points are sample uniformly along the circumference). Therefore

$$\mathbb{P}\left[A^{(n)}\right] = \sum_{i=1}^{n} \frac{1}{2^{n-1}} = \frac{n}{2^{n-1}}.$$