
Question 0.0.1. *Kyle: What is the intuition behind the discretization of the box size and why is equivalent to the product definition* ($|B| = \prod_i |I_i|$).

***Answer*.** *It will only hold in the case of a box. Recalling the definition of a box* $B = I_1 \times \ldots \times I_n$, *if* $I_i = [a_i, b_i]$ *we have*

$$|B| = \prod_i |I_i| = \prod_i (b_i - a_i).$$

Focusing on the single dimensional case (which generalizes via cross products), take $N \in \mathbb{N}_0$ *and note that* $k/N \in \mathbb{Z}/N$ *is in* I *iff* $a \leq \frac{k}{N} \leq b$, *which is the same as* $\lceil aN \rceil \leq k \leq \lfloor bN \rfloor$. *By simple counting it follows that*

$$\left| I \cap \frac{\mathbb{Z}}{N} \right| = \lfloor bN \rfloor - \lceil aN \rceil + 1.$$

This can be bounded to

$$\begin{aligned} (bN - 1) - (aN + 1) + 1 \leq \lfloor bN \rfloor - \lceil aN \rceil + 1 \leq bN - (aN - 1) + 1 \\ N(b-a) - 1 \leq \lfloor bN \rfloor - \lceil aN \rceil + 1 \leq N(b-a) + 2 \end{aligned}$$

Therefore

$$(b-a) - \frac{1}{N} \leq \frac{1}{N} \cdot \left| I \cap \frac{\mathbb{Z}}{N} \right| \leq (b-a) + \frac{2}{N}$$

which in the limit gives the desired result $|I| = b - a = \lim_{n\to\infty} \frac{1}{N}\left|I \cap \frac{\mathbb{Z}}{N}\right|$.

Remark. Charlie: What about for an elementary set $E = \coprod_i B_i$ defining its "size" as a supremum over all disjoint decompositions? For example:

$$|E| = \sup\left\{ \sum_i^N |B_i| : B_i \cap B_j = \emptyset, B_i \in E \right\}.$$

This could possibly be extended to open sets as well? Open boxes with rational corners would provide a nice basis for $\mathbb{R}^d$.

***Remark*.** *In addition to the previous proof, we found an alternative example via an explicit construction of a sequence of simple functions* f_n *that converge to* f. *From this point forward, we will assume that every binary expansion will be* *non-terminating as to ensure uniqueness of the expansions. Note this can be done since one can substitute the last* 1 *in a terminating sequence with* $0\overline{1}$*. Suppose that* $x \in [0,1]$ *has a binary expansion*

$$x = 0.b_1 b_2 b_3 \cdots .$$

Clearly the sequence of partial sums

$$f_n(x) = \sum_{j=1}^{n} 2b_j 3^{-j}$$

converges pointwise to f*. Thus if we can show they are simple functions, we obtain* f *is measurable and the previous answer's conclusion follows.*

Consider the case when b_1 *is* 1*. Then possible values* x *can take on are those satisfying*

$$\frac{1}{2} = 0.0111\cdots < x \le 0.111\cdots = 1.$$

Thus in the place of b_1 *in* f_n*, we can substitute* $\mathbb{1}_{(\frac{1}{2},1]}(x)$*. When* b_2 *is* 1*, then* b_1 *can be either* 0 *or* 1*, meaning the possible values for* x *must satisfy either of*

$$\frac{1}{4} = 0.00111\cdots < x \le 0.0111\cdots = \frac{1}{2}$$
$$\frac{3}{4} = 0.10111\cdots < x \le 0.1111\cdots = 1$$

Since these are disjoint intervals, we can again do a substitution of b_2 *in* f_n *for* $\mathbb{1}_{(\frac{1}{4},\frac{1}{2}]}(x) + \mathbb{1}_{(\frac{3}{4},1]}(x)$*. In general, the digit* b_i *can be written as a sum of disjoint indicator functions. More precisely, it is the sum over all indicator functions corresponding to all the possible bit flippings of* $0.b_1\cdots b_{i-1}$*, as those bits dictate where each interval starts, giving*

$$b_i = \sum_{k=1}^{2^{i-1}} \mathbb{1}_{(\frac{2k-1}{2^i},\frac{2k}{2^i}]}(x).$$

Thus we can write in total

$$f_n(x) = \sum_{j=1}^{n} \frac{2}{3^j} \cdot \left(\sum_{k=1}^{2^{j-1}} \mathbb{1}_{(\frac{2k-1}{2^i},\frac{2k}{2^i}]}(x) \right).$$

In this form it is then clear that each f_n *is a simple function, which was to be shown.*