Problem 1

Proof.

i) Clearly $B \subseteq A \cup B$. Let $x \in A \cup B$. If $x \in B$, then trivially $x \in B$. Consider the case when $x \notin B$. Then $x \in A$. Since $A \subset B$, it follows $x \in B$. Therefore $A \cup B \subseteq B \implies A \cup B = B$.

ii) Clearly $A \cap B \subseteq A$. Let $x \in A$. Since $A \subset B$, $x \in B$ meaning $x \in A \cap B$. Therefore $A \subseteq A \cap B \implies A \cap B = A$. $\diamond$

Problem 2

a) $(2,1,-3) + P = (0,2,4) \implies P = (0,2,4) - (2,1,-3) = (-2,1,7)$

b) $$(1,-1,4) + 2P = 3P + (2,0,5) \implies P = (1,-1,4) - (2,0,5) = (-1,-1,-1)$$

Problem 3

Adding the equations gives

$$\begin{aligned} 3P + Q &= (1,0,1,-4) \\ P - Q &= (2,1,2,3) \end{aligned} \implies 4P = (3,1,3,-1) \implies P = \left(\frac{3}{4}, \frac{1}{4}, \frac{3}{4}, -\frac{1}{4}\right).$$

which when plugged into the second equation

$$Q = P - (2,1,2,3) = \left(-\frac{5}{4}, -\frac{3}{4}, -\frac{5}{4}, -\frac{13}{4}\right).$$

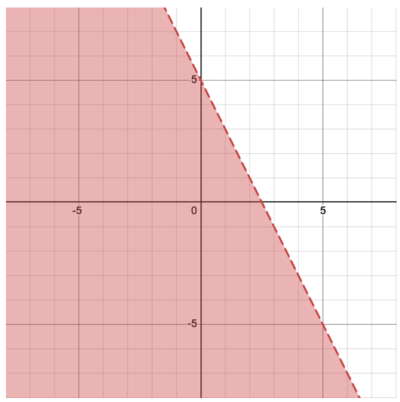
Problem 4

Yes. Choosing $p = (4,5,-3)$ gives $p \cdot A = 4 + 5 - 9 = 0$ and $p \cdot B = 8 - 5 - 3 = 0$.

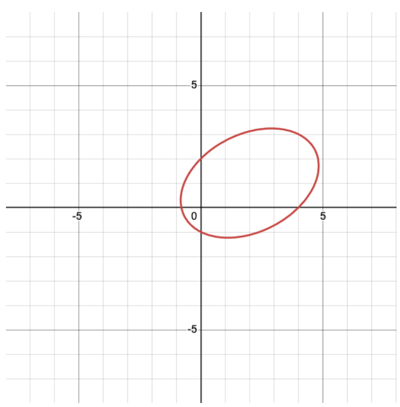
Problem 5

Let $p = (x, y)$.

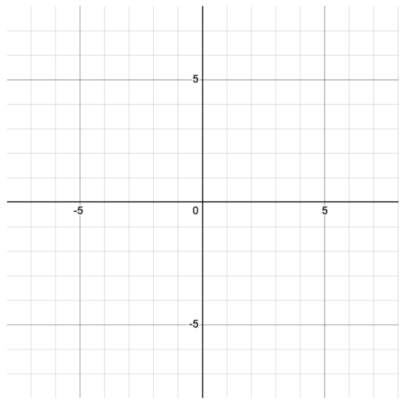
a) $$\begin{aligned} |p| < |p - A| &\implies x^2 + y^2 < (x-4)^2 + (y-2)^2 \\ &\implies 0 < -8x + 16 - 4y + 4 \\ &\implies y < -2x + 5 \end{aligned}$$



b) The statement is the same as saying the distance from $(0,0)$ to p to A is contant, which by definition is an ellipse with foci $(0,0)$ and A and constant distance 6. Note this is a non empty set of points since $p = (0,2)$ works.



c) No such points in the plane satisfy this. The smallest possible sum is achieved when p is on the line between $(0,0)$ and A (derived from the equations similarity to an ellipse but the point can be on the interior), but this gives a total distance of $|A| = 2\sqrt{5} > 4$. Thus the graph would be empty



Problem 6

Proof. We proceed with induction on n. Consider the base case $n = 1$. Trivially $|p_1| \le |p_1|$. Fix $n \in \mathbb{N}$ and assume that the statements holds. Consider the $n+1$ case. Note that if $s = p_{n+1} + p_n$ that

$$\begin{aligned} |p_1 + \ldots + p_n + p_{n+1}| &= |p_1 + \ldots + p_{n-1} + s| \\ &\le |p_1| + \ldots + |p_{n-1}| + |s| \qquad (\star) \\ &\le |p_1| + \ldots + |p_{n-1}| + |p_n| + |p_{n+1}| \end{aligned}$$

where $(\star)$ follows from the induction hypothesis and the last line from the triangle inequality applied to $|s|$. Thus the $n+1$ case holds, hence the statement holds for all n. ◇

Problem 7

Proof. By triangle inequality

$$|p| = |(p-q) + q| \le |p-q| + |q| \implies |p-q| \ge |p| - |q|$$

which was to be shown. ◇

Problem 8

Proof.

a) Note that since $|u|, |v|, |w| > 0$, $(|u| + |v| + |w|)^2 \geq |u|^2 + |v|^2 + |w|^2$ meaning

$$|p|^2 = u^2 + v^2 + w^2 = |u|^2 + |v|^2 + |w|^2 \leq (|u| + |v| + |w|)^2.$$

Rooting both sides then gives $|p| \leq |u| + |v| + |w|$.

b) Since $|p|^2 = u^2 + v^2 + w^2$ and $u^2, v^2, w^2 \geq 0$

$$\begin{aligned} |p|^2 \geq u^2 &= |u|^2 \implies |u| \leq |p| \\ |p|^2 \geq v^2 &= |v|^2 \implies |v| \leq |p| \\ |p|^2 \geq w^2 &= |w|^2 \implies |w| \leq |p| \end{aligned}$$

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Problem 9

Intersections are Convex

Proof. Let $A, B \subseteq \mathbb{R}^n$ be convex and $x, y \in A \cap B$. Then $x, y \in A$ and $x, y \in B$. Since both sets are convex, for all $\lambda \in (0, 1)$

$$\lambda x + (1 - \lambda)y \in A$$

$$\lambda x + (1 - \lambda)y \in B$$

Therefore $\lambda x + (1 - \lambda)y \in A \cap B$, hence $A \cap B$ is convex. ◇

Unions arent always Convex

Proof. Note that any singleton $\{x\} \subseteq \mathbb{R}^n$ is convex since $\lambda x + (1 - \lambda)x = x \in \{x\}$. However, for $x \neq y$, $\{x\} \cup \{y\}$ cannot be convex, otherwise $\frac{1}{2}(x + y) \in \{x, y\}$ which would imply $x = y$. ◇