Problem 1

Proof. Suppose towards contradiction that f has no critical points and E has interior points. Then there exists a point $a \in E$ such that $\exists r > 0$ where $B_r(a) \subseteq E$. Note that $a + te_k \in B_r(a)$ for $|t| < r$ since

$$|t| = |te_k| = |(a + te_k) - a| < r.$$

Therefore since $f(p) = 0$ for all $p \in B_r(a)$ and both a and $a + te_k$ are in $B_r(a)$ for sufficiently small t,

$$\partial_{x_k} f(a) = \lim_{t \to 0} \frac{f(a + te_k) - f(a)}{t} = 0.$$

Since each partial derivative of f at a is 0, then $\nabla f(a) = 0$. Hence a is a critical point, a contradiction. ◇

Problem 2

Proof. Take $x, h \in \mathbb{R}$ with $h > 0$. Since $f \in C^2(\mathbb{R}; \mathbb{R})$, by the single variable Taylors theorem there exists $\xi \in (x, x+h)$ such that

$$f(x+h) = f(x) + f'(x)h + \frac{1}{2} f''(\xi)h^2.$$

This can be rearranged to get

$$f'(x) = \frac{f(x+h) - f(x)}{h} - \frac{1}{2} f''(\xi)h.$$

Since $|f''(x)| \leq M$ for any x, it follows

$$f'(x) = \frac{f(x+h) - f(x)}{h} - \frac{M}{2} \cdot h.$$

Take $\varepsilon > 0$. Since $\lim_{x\to\infty} f(x) = 0$, then there exists $N \in \mathbb{N}$ such that for $x \geq N$, $|f(x)| \leq \varepsilon$. Thus if $h = \sqrt{\varepsilon}$, then for $x \geq N$

$$|f'(x)| = \left| \frac{f(x+h) - f(x)}{h} \right| + \frac{M}{2} \cdot h \leq \frac{2\varepsilon}{\sqrt{\varepsilon}} + \frac{M}{2} \cdot \sqrt{\varepsilon} = \left(2 + \frac{M}{2}\right)\sqrt{\varepsilon}.$$

Therefore since $2 + \frac{M}{2}$ is a constant, it follows that for $x \to \infty$ that $f'(x) = 0$. ◇

Problem 3

Proof. Suppose towards contradiction that $f(x) = f(x')$ for some $x, x' \in U$ where $x \neq x'$. Then there exists some $h \in \mathbb{R}^n \setminus \{0\}$ such that $x' = x + h$. Let $g : [0,1] \to \mathbb{R}$ where $g(t) = h \cdot f(x + th)$. Since U is convex, $x + th \in U$ for all $t \in [0,1]$ and thus g is well defined on U. Thus by the chain rule

$$g'(t) = h \cdot f'(x + th)h.$$

Since $h \cdot f'(x)h > 0$ for all $x \in U$, $g'(t) > 0$ for all $t \in [0,1]$. Therefore g is strictly increasing meaning $g(0) \neq g(1) \implies h \cdot f(x) \neq h \cdot f(x')$. But since $f(x) = f(x')$, $h \cdot f(x) = h \cdot f(x')$, a contradiction. Therefore $f(x) = f(x')$ only when $x = x'$, hence f is one-to-one. ◇

Problem 4

Proof. First note that for $c \in \mathbb{R} \setminus \{0\}$ that

$$\lim_{x \to 0} \frac{\arctan(cx)}{x} = c \lim_{h \to 0} \frac{\arctan(0 + ch) - \arctan(0)}{ch} = c \arctan'(0) = \frac{c}{1 + 0^2} = c$$

and that

$$\lim_{x \to 0} x \arctan\left(\frac{c}{x}\right) = 0$$

since $\arctan\left(\frac{c}{x}\right)$ is bounded. Consider $\partial^2_{x_2 x_1} f(0,0)$. Note that for $x_2 \neq 0$ that

$$\begin{aligned}
\partial_{x_1} f(0, x_2) &= \lim_{h \to 0} \frac{f(h, x_2) - f(0, x_2)}{h} \\
&= \lim_{h \to 0} \frac{f(h, x_2)}{h} \\
&= \lim_{h \to 0} h \arctan\left(\frac{x_2}{h}\right) - x_2^2 \frac{\arctan\left(\frac{h}{x_2}\right)}{h} \\
&= 0 - x_2^2 \cdot \frac{1}{x_2} = -x_2
\end{aligned}$$

Since $\partial_{x_1} f(0,0) = 0$ which is the same as above when $x_2 = 0$, $\partial^2_{x_2 x_1} f(0,0) = -1$. Now consider $\partial^2_{x_1 x_2} f(0,0)$. Note that for $x_1 \neq 0$ that

$$\begin{aligned}\partial_{x_2} f(x_1, 0) &= \lim_{h \to 0} \frac{f(x_1, h) - f(x_1, 0)}{h} \\ &= \lim_{h \to 0} \frac{f(x_1, h)}{h} \\ &= \lim_{h \to 0} x_1^2 \frac{\arctan\left(\frac{h}{x_1}\right)}{h} - h \arctan\left(\frac{x_1}{h}\right) \\ &= x_1^2 \cdot \frac{1}{x_1} - 0 = x_1\end{aligned}$$

Since $\partial_{x_2} f(0,0) = 0$ which is the same as above when $x_1 = 0$, $\partial^2_{x_1 x_2} f(0,0) = 1 \neq -1 = \partial^2_{x_2 x_1} f(0,0)$. ◇

Problem 5

Proof. Suppose f has a local extremum at a. Take $h \in \mathbb{R}^n \setminus \{0\}$ and note that f has a Taylor expansion about a where

$$f(a+h) = f(a) + L(h) + \frac{1}{2} Q(h) + R(h).$$

Since a is a local extremum, $L(h) = \nabla f(h) = 0$. Therefore

$$f(a+h) - f(a) = \frac{1}{2} Q(h) + R(h).$$

Suppose towards contradiction that a is a local minimum and $Q(h)$ is not positive semidefinite. Therefore there exists $v \in \mathbb{R}^n \setminus \{0\}$ such that $Q(v) < 0$. WLOG $|v| = 1$ since $Q\left(\frac{v}{|v|}\right) = \frac{1}{|v|^2} Q(v)$ and thus both have the same sign. Set $h = tv$ for t small enough that $a + h$ is in the open ball around a where a is local minimum. Therefore $f(a+h) \geq f(a)$. Note then that

$$f(a+tv) - f(a) = \frac{1}{2} Q(tv) + R(tv) = \frac{t^2}{2} Q(v) + R(h).$$

Thus dividing through by t^2 gives

$$\frac{f(a+tv) - f(a)}{t^2} = \frac{1}{2} Q(v) + \frac{R(h)}{t^2} = \frac{1}{2} Q(v) + \frac{R(h)}{|h|^2}.$$

Let $\frac{1}{2}Q(v) = -c$ for some $c > 0$. Since $\lim_{h\to 0} \frac{R(h)}{|h|^2} = 0$, then for t sufficiently small (take it to be smaller than the previous t selected)

$$\left|\frac{R(tv)}{t^2}\right| < \frac{c}{2}.$$

Therefore

$$\frac{f(a+tv)-f(a)}{t^2} < -c + \frac{c}{2} = -\frac{c}{2} < 0 \implies f(a+h) - f(a) < 0$$

meaning $f(a+tv) < f(a)$. Note that since t can be made arbitrarily small, in every open neighborhood of a, t can be chosen such that $a+tv$ is in this nbhd and $f(a+tv) < f(a)$, a contradiction. Therefore Q must be positive semidefinite. Proving the case in which a is a local maximum and Q is negative semidefinite is identical to above except

1. Q is not negative semidefinite so there exists v where $Q(v) > 0$.
2. $\frac{1}{2}Q(v) = c$ for some $c > 0$.
3. The final inequality is instead

   $$\frac{f(a+tv)-f(a)}{t^2} > c - \frac{c}{2} = \frac{c}{2} > 0$$

   giving $f(a+tv) - f(a) > 0 \implies f(a+tv) > f(a)$.
4. Since t can be chosen small enough such that $a+tv$ is in every open neighborhood of a, a cannot be a local maximum. Thus Q is negative semidefinite.

$\diamond$

Problem 6

To find possible critical points, consider the points where $\nabla f(x,y) = 0$. Note that

$$\nabla f(x,y) = \left[\frac{4y}{1+(xy)^2} - 2x, \frac{4x}{1+(xy)^2} - 2y\right].$$

Note that then $\nabla f(x,y) = 0$ iff

$$4y - 2x(1+(xy)^2) = 0 \implies 2y = x(1+(xy)^2)$$

and

$$4x - 2y(1+(xy)^2) = 0 \implies 2x = y(1+(xy)^2).$$

Subtracting these from each other gives

$$2(x-y) = (y-x)(1+(xy)^2).$$

Suppose $x \neq y$. Then $x - y \neq 0$ and so

$$2(x-y) = -(x-y)(1+(xy)^2) \implies -2 = 1+(xy)^2.$$

But $1+(xy)^2 \geq 1$ thus there is no real solution in this case. Thus $x = y$ and substituting gives

$$2x = x(1+x^4) \implies 2x = x + x^4 \implies x = x^4.$$

The solutions to this are $x \in \{0,-1,1\}$. Therefore the critical points of f are $(0,0), (-1,-1)$ and $(1,1)$. Since $f \in C^2(R^2;\mathbb{R})$, f has a Taylor expansion at each of these critical points (x,y) whose $Q(h) = Hf(x,y)h \cdot h$ where

$$Hf(x,y) = \begin{bmatrix} -\frac{8xy^3}{(1+(xy)^2)^2} - 2 & \frac{4}{1+(xy)^2} - \frac{8(xy)^2}{(1+(xy)^2)^2} \\ \frac{4}{1+(xy)^2} - \frac{8(xy)^2}{(1+(xy)^2)^2} & -\frac{8x^3y}{(1+(xy)^2)^2} - 2 \end{bmatrix}.$$

Therefore

- $Hf(0,0) = \begin{bmatrix} -2 & 4 \\ 4 & -2 \end{bmatrix}$. Take $h^+ = (-1,-1)$ and $h^- = (1,-1)$. Note that

  $$Hf(0,0)h^+ \cdot h^+ = (-1)(-2) + (-1)(-2) = 4 > 0$$

  and

  $$Hf(0,0)h^- \cdot h^- = (1)(-6) + (-1)(6) = -12 < 0.$$

  Therefore $Q(h)$ at $(0,0)$ is indefinite and thus $(0,0)$ is neither a local maximum or minimum.

- $Hf(1,1) = \begin{bmatrix} -4 & 0 \\ 0 & -4 \end{bmatrix}$. Note then that for any $h = (h_1, h_2) \neq 0$ that

$$Hf(1,1)h \cdot h = -4h_1^2 - 4h_2^2 < 0.$$

Therefore $Q(h)$ at $(1,1)$ is negative definite and thus $(1,1)$ is a local maximum.

- $Hf(-1,-1) = \begin{bmatrix} -4 & 0 \\ 0 & -4 \end{bmatrix}$, which by the same logic as above means $(-1,-1)$ is a local maximum.