Chapter 1

Review+

Definition 1.0.1 (Probability Space). A probability space is a triple $(\Omega, \mathcal{F}, P)$ such that

1. $\mathcal{F} \subseteq \mathcal{P}(\Omega)$
2. $\forall E \in \mathcal{F}, 0 \le P(E) \le 1$
3. $P(\Omega) = 1$
4. If $E_1, E_2, \ldots, E_n \in \mathcal{F}$ are pairwise disjoint then $P(\cup_i E_i) = \sum_i P(E_i)$

Ω is denote as the sample space, $\mathcal{F}$ the events, and P the probability function.

Definition 1.0.2 (Conditional Probability). For events $E, F \in \mathcal{F}$, the conditional probability is

$$\mathbb{P}[E|F] := \frac{\mathbb{P}[E \cap F]}{\mathbb{P}[F]}.$$

Definition 1.0.3 (Independence). Two events $E, F \in \mathcal{F}$ are **independent** if

$$\mathbb{P}[E|F] = \mathbb{P}[E] \Leftrightarrow \mathbb{P}[E \cap F] = \mathbb{P}[E]\mathbb{P}[F].$$

A collection of events $E_1, \ldots, E_n$ are **mutually independent** if

$$\forall I \subseteq [n], \mathbb{P}[\cap_{i \in I} E_i] = \prod_{i \in I} \mathbb{P}[E_i].$$

Theorem 1.0.1 (Law of Total Probability). Suppose that $F_1, \ldots, F_n \in \mathcal{F}$ are disjoint and $\Omega = \cup_i F_i$. Then for any other event $E \in \mathcal{F}$,

$$\mathbb{P}[E] = \sum_i \mathbb{P}[E|F_i] \cdot \mathbb{P}[F_i].$$

Example 1.0.1. Suppsoe that $S \subseteq [n]$ be a randomly chosen subset. We want to find $\mathbb{P}[|S| \text{ is even}]$. Take E_i to be the event that $i \in S$. Note that all E_i are mutually independent for $1 \le i \le n$. Take

$$F_{\overline{x}} = \bigcap_{i=1}^{n-1} E_i^{x_i}.$$

where $\overline{x}$ is a string of either complements or nothing. Note then that

$$\mathbb{P}[|S| \text{ is even}] = \sum_{\overline{x}} \mathbb{P}[|S| \text{ is even} \mid F_{\overline{x}}] \cdot \mathbb{P}[F_{\overline{x}}].$$

Note that $\mathbb{P}[|S| \text{ is even} \mid F_{\overline{x}}] = \frac{1}{2}$ since there are only two possible outcomes for adding the n^{th} element and only one makes $|S|$ even. Furthermore since all $F_{\overline{x}}$ across all strings $\overline{x}$ are disjoint and cover Ω, their total probability is 1. Hence

$$\mathbb{P}[|S| \text{ is even}] = \frac{1}{2} \cdot 1 = \frac{1}{2}.$$

Example 1.0.2. Suppose someone chooses $-\infty < x < y < \infty$ and each are placed into separate "envelopes". The goal is to determine a strategy for finding the envelope containg x with a greater than $\frac{1}{2}$ probability.

The strategy is as follows

1. Let $Z \sim N(0,1)$ and expose some outcome $z = Z$.
2. Pick a random evelope and open it. If the number in the envelope is $< z$ then guess that it is x, otherwise guess y.

Note that there are three outcomes or cases

1. $C_1 : z < x < y$
2. $C_2 : x < y < z$
3. $C_3 : x < z < y$

Thus by the law of total probability is

$$\begin{aligned}
\mathbb{P}[\text{win}] &= \sum_i \mathbb{P}[\text{win}|C_i]\mathbb{P}[C_i] \\
&= \frac{1}{2} \cdot \mathbb{P}[C_1] + \frac{1}{2} \cdot \mathbb{P}[C_2] + 1 \cdot \mathbb{P}[C_3] \\
&= \frac{1}{2}(\mathbb{P}[C_1] + \mathbb{P}[C_2] + \mathbb{P}[C_3]) + \frac{1}{2}\mathbb{P}[C_3] \\
&= \frac{1}{2} + \frac{1}{2}\mathbb{P}[C_3] > \frac{1}{2}
\end{aligned}$$

since any probability of a normal distribution lying in an interval is non zero.

Theorem 1.0.2. Let X be a random variable such that $\mathbb{P}[X < 0] = 0$. Then

$$\mathbb{E}[X] = \int_0^\infty \mathbb{P}[X \geq 0]\mathrm{d}x.$$

Proof. Note that

$$\begin{aligned}
\mathbb{E}[X] &= \int_0^\infty x f_X(x)\mathrm{d}x \\
&= \int_0^\infty f_X(x)\left(\int_0^x 1\mathrm{d}y\right)\mathrm{d}x \\
&= \int_0^\infty \int_y^\infty f_X(x)\mathrm{d}x\mathrm{d}y \\
&= \int_0^\infty \mathbb{P}[X \geq y]\mathrm{d}y \\
&= \int_0^\infty \mathbb{P}[X \geq x]\mathrm{d}x
\end{aligned}$$

which was to be shown. ■

1.1 Independence of Random Variables

Recall that the definition of independence of events.

Definition 1.1.1 (Independent Events). Two events E and F are independent if $\mathbb{P}[EF] = \mathbb{P}[E]\mathbb{P}[F]$.

How can this idea of independence be extended to random variables, say X and Y. Consider the example where X is the sum of two fair dice. Then the statement $X = 7$ is a shorthand for the the event

$$E = \{(i, j) : 1 \leq i, j \leq 6, i + j = 7\}.$$

All expressions of this form $X = k$ give disjoint events. Furthermore, we have $\Omega = \coprod_k$ "$X = k''$. Thus we can identify independence of random variables as independence of these partitioning events.

Definition 1.1.2 (Independent Random Variables). Two random variables X and Y are **independent** if $\forall x, y$ we have $\mathbb{P}[(X \leq x) \cap (Y \leq y)] = \mathbb{P}[X \leq x]\mathbb{P}[Y \leq y]$.