Question 1. We'll assume $n_k \geq 1$.

$$n = \sum_{j=1}^{\infty} n_j \,; \;\; \text{total degree} = 2n - 2 = \sum_{j=1}^{\infty} j n_j$$

$$n = n_1 + n_2 + n_3 + \cdots + n_k + \cdots$$
$$2n - 2 = n_1 + 2n_2 + 3n_3 + \cdots + kn_k + \cdots$$

We then have

$$2(n_1 + n_2 + n_3 + \cdots + n_k + \cdots) - 2 = n_1 + 2n_2 + 3n_3 + \cdots + kn_k + \cdots$$

Move the n_1 terms to the left; everything else to the right, and use $n_i \geq 0, n_k \geq 1$ to obtain an inequality.

The quantity $D(\mathbf{a}, \mathbf{b})$ being even means that $\mathbf{a}$ and $\mathbf{b}$ must be in the same "part" considering Q_n as a bipartite graph.

We're looking at n-ary trees:

Definition. A rooted tree (T, r) is said to be an $\boldsymbol{n}$**-ary tree** if every vertex has at most n children.

If $n = 2$, we have a **binary tree;** if $n = 3$, we have a **ternary tree**.

Definition. A vertex of an n-ary tree is said to be **saturated** if it has n children.

Definition. An n-ary tree is **full** if every parent is saturated.

Observations.

- A trivial rooted tree is an example of a full n-ary tree for any $n \geq 0$.
- If (T, r) is an n-ary tree, then it is also an $(n+1)$-ary tree. The word "full" is not being used in this observation.
- For a nontrivial full n-ary tree, the root has degree n, every other parent has degree $n+1$, and every nonparent has degree 1.

Theorem. Given a full n-ary tree (T, r), if there are k parents, then there are exactly $(n-1)k + 1$ non-parents.

Proof. Let ℓ be the number of non-parents. If T is trivial, then $k = 0$ and $\ell = 1 = (n-1)0 + 1$, as desired.

If T is nontrivial, the total degree of T is

$$\begin{aligned} n + (k-1)(n+1) + \ell &= n + 1 + (k-1)(n+1) + \ell - 1 \\ &= k(n+1) + \ell - 1. \end{aligned}$$

The number of edges is one less than the number of vertices; the number of vertices is $k + \ell$ and so the number of edges is $k + \ell - 1$.

By the handshaking lemma,

$$2(k + \ell - 1) = k(n+1) + \ell - 1$$

or

$$2k + 2\ell - 2 = kn + k + \ell - 1.$$

Solving this for ℓ gives us

$$\ell = (n-1)k + 1$$

as desired.

Corollary. In a full binary tree with k parents (these are often called "interior vertices" in the literature), there are $k + 1$ non-parents (usually called "leaves").

Corollary. In any full binary tree, the number of vertices must be odd.

Corollary. In any full n-ary tree, the number of vertices must be one more than a multiple of n.

Now, let's try to squeeze as many vertices onto level L as possible in an n-ary tree.

Theorem. For integers $n \geq 1$ and $L \geq 0$, an n-ary tree has at most n^L vertices at level L.

Proof. We proceed by induction on L, letting N_L be the maximum number of vertices at level L. For the base case, there is 1 vertex at level 0, namely the root. Hence, $N_0 = 1$.

For the inductive step, assume $N_L \leq n^L$. Since every vertex at level $L + 1$ is a child of a vertex at level L and every vertex at level L has at most n children, we have

$$N_{L+1} \leq n\, N_L \leq n \cdot n^L = n^{L+1},$$

completing the inductive step and the proof.