CPTS 553: Graph Theory
Assignment 6

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We know that $n = \sum_{j=1}^{\infty} n_j$ and total degree= $\sum_{j=1}^{\infty} jn_j$. We know that total degree is also $2n-2$. Therefore,

$$\sum_{j=1}^{\infty} jn_j = 2\sum_{j=1}^{\infty} n_j - 2$$

This implies,

$$2 = n_1 + \sum_{j=2}^{\infty} (2-j)n_j$$

Also,

$$2 = n_1 + (2-k)n_k + \sum_{j=2}^{k-1} (2-j)n_j + \sum_{j=k+1}^{\infty} (2-j)n_j$$

We can observe that n_j is always positive. And $(2-j)$ is always negative. So, the summation terms can be done away with by making it into an inequality as follows:

$$2 \leq (2-k)n_k + n_1$$

Or,

$$n_1 \geq 2 + (k-2)n_k$$

Now, the assumption is that $n_k \geq 1$.
For $k = 3$ and $n_k = 1$, we see that $n_1 \geq 3$ or $n_1 \geq k$.
When $k > 3$ and $n_k = 1$, $n_k \geq k$.
When $n_k > 1$, then $n_1 \geq kn_k + 2(1-n_k)$. But $2(1-n_k) < 0$ and $kn_k > k$. So, $n_1 \geq kn_k \geq k$. This proves that for any $n_k \geq 1$ and $k \geq 3$, $n_1 > k$.
In other words, for any $k \geq 3$, any tree with a vertex of degree k must have at least k leaves.

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2.1

A binary tree has minimum number of nodes when each non-leaf node has exactly 1 child (it is essentially a path graph). For that to happen, $n = H + 1$.

For a binary tree to have maximum number of nodes, each parent will have exactly 2 children. At every level k of the tree where $0 \leq k \leq H+1$, the corresponding number of nodes in that level is 2^k. So the total number of nodes $n = 2^0 + 2^1 + 2^3 + \cdots + 2^{H+1}$. This is a geometric series whose sum can be computed. This gives us $n = 2^{H+1} - 1$.

2.2

Given: Every parent has exactly two children.
If we consider all non-parents, then all of them are one of two siblings. So, the total number of non-parents is even.
If we consider all parents except the root, they are children themselves, and hence all of them are one of two siblings. So, the total number of parents excluding the root, is even.
The sum of non-parents and parents excluding the root comes out to $(n-1)$ which is even (sum of two even numbers). Since $(n-1)$ is even, n must be odd.

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3.1

H is the height of the tree. If $D(u,v) = 2H$, then that uv path has to pass through the root.

[This is because: Let the path from u to v with length $2H$ not pass through the root but some other vertex w. In that case, $D(u,w) = H - \delta_1$ and $D(w,v) = H - \delta_2$. This makes $D(u,v) = 2H - (\delta_1 + \delta_2)$. But we know that $D(u,v) = 2H$. Therefore, $\delta_1 + \delta_2 = 0$. This is only possible when w and r are the same vertex.]

Now, let u and v be parents having x and y as their respective children. Then, $D(x,y) = 2H+2$. Since the height of the tree is H, the longest path possible has a length $2H$. This is why x or y cannot exist. This contradicts our assumption that u or v are parents. Hence, it is proved that u and v are non-parents.

3.2

Proof of: $D(u,v) = level(u) + level(v) \implies r$ is on the unique u,v-path.

Let r not be on the unique u,v-path. We know that $level(x) = D(x,r)$. Therefore,
$D(u,v) = D(u,r) + D(v,r)$. This means $D(u,v)$ to be equal to the sum of the levels of u and v, the unique u,v-path can be decomposed into two paths of length $D(u,r)$ and $D(v,r)$. Let those two

paths be ux and xv for some vertex x. Now, without loss of generality if we consider just the ux path, then $D(u,x) = D(u,r)$. Since, the path is unique, $D(x,r) = 0$. This means that x is nothing but r. This contradicts our assumption that r is not on the unique u,v-path. Hence, it is proved that if $D(u,v) = level(u) + level(v)$ then r is on the unique u,v-path.

Proof of: r is on the unique u,v-path $\implies D(u,v) = level(u) + level(v)$.

Since r is on the unique u,v-path, $D(u,v) = D(u,r) + D(v,r)$. Now, we also know that $level(x) = D(x,r)$. Hence, $D(u,v) = level(u) + level(v)$. This proves that if r is on the unique u,v-path, then $D(u,v) = level(u) + level(v)$.

Combining the above two we can say that $D(u,v)$ is the sum of the levels of u and v **if and only if** r is on the unique u,v-path.

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Maximum number of components: Let all but one of the components in G be isolated. Also, let all the edges be present in the non-isolated component. To maximize the number of edges in a graph, we can consider the graph K_n which has $n(n-1)/2$ edges. The graph is question has 7 edges in total. Therefore, the minimum number of vertices in the non-isolated component is 5. We know $n = 14$. Therefore, there are 9 isolated vertices. This makes the maximum number of components to be 10.

Minimum number of components: If G is a forest, then $n = m + k$ where k is the number of components. If G is not a forest, then let us start removing edges from G one by one such that the edges removed are not bridges. There will come a time (after removing x many edges) when all the edges left are bridges [We note that the removal of those x edges from the graph hasn't increased the number of components since they weren't bridges]. At this point, the leftover graph G' is a forest with $m-x$ edges. So, $n = (m-r) + k$. Since the removal of x edges didn't change the number of components, the number of components in G and G' are same. Therefore, the minimum number of components in G is 7.

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For maximum height H, each non-leaf node will have exactly 1 child. Hence, $H = n-1$. Here, $n = 10^9$. Therefore, the maximum height of the tree is given by $H = 10^9 - 1$. Therefore, $H = 999999999$.

For minimum height H, the binary tree needs to have as many nodes as possible in the upper levels before populating the lower levels. For that to happen, $H = \lceil \log_2 n \rceil - 1$ (since this is a nontrivial tree). Here, $n = 10^9$. Therefore, $H = \lceil \log_2 10^9 \rceil - 1$. This makes $H = 29$.