I’ve set up Zoom meetings on Monday at 11:00 a.m. and Tuesday at 10:00. Recall that the final exam for this course was scheduled for Tuesday at 10:00, so that’s the “official” time to demonstrate Graph Theorist Sketchpads. Also, tomorrow’s office hours start at 10:00 as well. I’ll see what demand there is for meetings later next week as time progresses.

<u>There is no final exam for this course.</u>

Laplacians of cycles.

For instance, let’s look at C_4:

$$L = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}$$

Notice that this matrix has quite a few structural features. The feature worthy of attention is that

$$L_{r,c} = L_{r+i,c+i};\ \text{addition mod } 4$$

Notice, for instance, that $L_{1,2} = L_{4,1}$. Here, $r = 1, c = 2, i = 3$. Matrices with this feature are called “circulant” matrices.

In general, a 4×4 circulant matrix looks like, where $a, b, c, d \in \mathbb{R}$,

$$\begin{aligned} M &= \begin{bmatrix} a & b & c & d \\ d & a & b & c \\ c & d & a & b \\ b & c & d & a \end{bmatrix} \\ &= a\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} + b\begin{bmatrix} & 1 & & \\ & & 1 & \\ & & & 1 \\ 1 & & & \end{bmatrix} + c\begin{bmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{bmatrix} + d\begin{bmatrix} & & & 1 \\ 1 & & & \\ & 1 & & \\ & & 1 & \end{bmatrix} \\ &= aI + bK + cK^2 + dK^3 \end{aligned}$$

The “b” matrix plays a critical role here. Let’s give that matrix a name, calling it K.

Let’s compute $K\begin{bmatrix} 1 \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} & 1 & & \\ & & 1 & \\ & & & 1 \\ 1 & & & \end{bmatrix}\begin{bmatrix} 1 \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$. Notice that

$$\begin{bmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{bmatrix}\begin{bmatrix} 1 \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y \\ z \\ 1 \\ x \end{bmatrix} = K^2\begin{bmatrix} 1 \\ x \\ y \\ z \end{bmatrix}.$$

Suppose $\mathbf{v}$ is an eigenvector for K with eigenvalue λ, i.e., $K\mathbf{v} = \lambda\mathbf{v}$. What is

$$K^2\mathbf{v} = K(K\mathbf{v}) = K(\lambda\mathbf{v}) = \lambda(K\mathbf{v}) = \lambda^2\mathbf{v}; K^p\mathbf{v} = \lambda^p\mathbf{v}.$$

Moral of the story: $\mathbf{v}$ would then be an eigenvector for any power of K.

Moreover,

$$\begin{aligned} M\mathbf{v} &= (aI + bK + cK^2 + dK^3)\mathbf{v} \\ &= aI\mathbf{v} + bK\mathbf{v} + cK^2\mathbf{v} + dK^3\mathbf{v} \\ &= a\mathbf{v} + b\lambda\mathbf{v} + c\lambda^2\mathbf{v} + d\lambda^3\mathbf{v} \\ &= (a + b\lambda + c\lambda^2 + d\lambda^3)\mathbf{v} \end{aligned}$$

So, if $M = p(K)$ and $\mathbf{v}$ is an eigenvector of K with eigenvalue λ, then $\mathbf{v}$ is an eigenvector of $M = p(K)$ with eigenvalue $p(\lambda)$. Thus, we can analyze M by analyzing the easier matrix K.

Suppose $\mathbf{v} = \begin{bmatrix} 1 \\ x \\ y \\ z \end{bmatrix}$ is an eigenvector for K with eigenvalue λ. Then from the equation

$$K\begin{bmatrix} 1 \\ x \\ y \\ z \end{bmatrix} = \lambda\begin{bmatrix} 1 \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \lambda \\ \lambda x \\ \lambda y \\ \lambda z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

we obtain

$$\begin{aligned} &x = \lambda; y = \lambda x; z = \lambda y; 1 = \lambda z. \\ &y = \lambda^2; z = \lambda^3; 1 = \lambda^4 \end{aligned}$$

Hence, λ is a solution to the equation $\lambda^4 = 1$. The four possible solutions are $\lambda \in \{1, -1, i, -i\}$.

This tells us what the eigenvectors of K look like:

$$\lambda = 1, \mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix};\ \lambda = -1, \mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix};$$

$$\lambda = i, \mathbf{v} = \begin{bmatrix} 1 \\ i \\ -1 \\ -i \end{bmatrix};\ \lambda = -i, \mathbf{v} = \begin{bmatrix} 1 \\ -i \\ -1 \\ i \end{bmatrix}$$

These are also the eigenvectors of M. If $M = p(K)$, then the eigenvalues are $p(1), p(-1), p(i), p(-i)$.

Let's look at the Laplacian of C_4 in this light:

$$L = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix} = 2I - 1K + 0K^2 - 1K^3$$

$$p(K) = 2I - K - K^3$$

Eigenvalues:

$$\begin{aligned} &p(\lambda) = 2 - \lambda - \lambda^3 \\ &p(1) = 0; p(-1) = 4; p(i) = 2 - i - i^3 = 2 = p(-i). \end{aligned}$$

The eigenvalues of the Laplacian matrix L of C_4 are $\lambda = \{0,2,2,4\}$.

Notice that the eigenvectors for $\lambda = 2$ contain imaginary numbers:

$$\lambda = 2, \mathbf{v} = \begin{bmatrix} 1 \\ i \\ -1 \\ -i \end{bmatrix}; \ \lambda = 2, \mathbf{v} = \begin{bmatrix} 1 \\ -i \\ -1 \\ i \end{bmatrix}$$

How can we obtain vectors with real entries that correspond to $\lambda = 2$? Notice that their sum is the vector

$$\begin{bmatrix} 2 \\ 0 \\ -2 \\ 0 \end{bmatrix}.$$

Multiply both by i and subtract:

$$i\begin{bmatrix} 1 \\ i \\ -1 \\ -i \end{bmatrix} - i\begin{bmatrix} 1 \\ -i \\ -1 \\ i \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 0 \\ 2 \end{bmatrix}.$$

On Friday, I want to look at C_7. And then try to "prove" the Cantor-Schroeder-Bernstein theorem.