There is a sneaky way to handle Hamming distances using addition mod 2:

$3 = D(0010,1100)$; Notice that (coordinate-wise addition mod 2): 0010+1100=1110

This is also known as the bitwise XOR.

Starting exploration for homework question 1:

$D(\mathbf{a},\mathbf{b}) = k$ would mean $\mathbf{a}+\mathbf{b}$ has k ones

$D(\mathbf{b},\mathbf{c}) = \ell$ would mean $\mathbf{b}+\mathbf{c}$ has ℓ ones

$D(\mathbf{a},\mathbf{c}) = m$ would mean $\mathbf{a}+\mathbf{c}$ has m ones. Notice that $\mathbf{a}+\mathbf{c}$ has exactly as many ones as $(\mathbf{a}+\mathbf{b})+(\mathbf{b}+\mathbf{c})$

Connectedness.

Given a graph G, two vertices u, v are **connected** if there exists a u, v-walk in G.

It turns out that if we write $u{\sim}v$ for u is connected to v,

- (Reflexive) For any vertex u, $u{\sim}u$. (Use the trivial walk $W: u$)
- (Symmetric) For any two vertices u, v, if $u{\sim}v$, then $v{\sim}u$. (Reverse the u, v-walk to get a v, u-walk.)
- (Transitive) For any three vertices u, v, w, if $u{\sim}v$ and $v{\sim}w$, then $u{\sim}w$. (Concatenate the u, v-walk and the v, w-walk to get a u, w-walk)

The connectedness relation $\sim$ is an equivalence relation.

The set (**equivalence class**) of vertices $[u] = \{x : x{\sim}u\}$ is known as the "connectedness class" of u.

Definition. The **component** of a graph G that contains u is the <u>induced subgraph</u> $G([u])$. Intuitively, this is the largest "connected piece" that contains u.

Definition. A <u>non-null</u> graph G is **connected** if for any two vertices u and v of the graph, u and v are connected.

Components are graphs; these are the largest connected subgraphs that contain a given vertex of graphs.

Two separate components must be disconnected from one another.

Conventions and Observations.

- A graph is connected if and only if it has exactly one component.
- Conventionally, a null graph is not considered to be connected.
- Conventionally, a null graph has zero components.
- A trivial graph is connected because it has one component.
- For any two vertices $u, v \in V$, the following are equivalent:
    - u is connected to v, written $u{\sim}v$
    - $u \in [v]$

- $[u] \cap [v] \neq \emptyset$
- $[u] = [v]$
- u and v lie in the same component

Theorem II.B.2.a. For any two vertices u and v in the same component, every edge in any u, v-walk is an edge of the component containing u.

Theorem II.B.2.b. If $H \leq G$ and $u, v \in V(H)$ are in the same component of H, then they are in the same component of G.

Corollary II.B.2.c. If $H \leq G$ and $u, v \in V(H)$ are in different components of G, then they are in different components of H.

Bridges

Definition. Given a graph $G = (V, E)$, an edge $e \in E$ is called a **bridge** if there exist vertices u and v such that u and v are connected in G, but disconnected in $G - e$.

Intuitively, a bridge is an edge whose removal causes a connected pair of vertices to become disconnected.

Observation. Given a connected graph G and a bridge e of G, the vertices u and v are connected in $G - e$ if and only if there exists a u, v-walk in G that does not include the edge e.

Rephrasing this in contrapositive form:

Theorem II.B.3.a. Given a connected graph G and a bridge e of G, every u, v-walk contains the edge e in the graph G if and only if u and v are disconnected in $G - e$.

Theorem II.B.3.b. Given a connected graph G and a bridge e of G, deleting e disconnects its endpoints.

Corollary II.B.3.c. Loops are not bridges. One of a set of at least two parallel edges is not a bridge.

Theorem II.B.3.d. Given a connected graph G and a bridge $e = st$ of G, in the graph $G - e$, every vertex is connected to either s or t, but not both.

Phrasing in the book:

$$V = [s]_{G-e} \sqcup [t]_{G-e}.$$

Corollary II.B.3.e. Deleting a bridge from a connected graph results in a graph with exactly two components.

Theorem II.B.3.g. Given a graph G with finitely many components, the number of components increases by exactly 1 by deleting a bridge.