From last time,

Definition. Given a graph G and color set $S = \{1,2,3,\dots,k\}$, the value of $p_G(k)$ is the number of ways to k-color the vertices of G. The function $p_G: \mathbb{Z}^{\geq 0} \to \mathbb{Z}^{\geq 0}$ is called the **chromatic polynomial** of G.

Observations.

- For $G = P_n$, a path with n vertices, $p_G(k) = k(k-1)^{n-1}$.
- For $G = K_n$, a complete graph, $p_G(k) = k(k-1)(k-2)\cdots\big(k-(n-1)\big) = \frac{k!}{(k-n)!}$.
- For G is n isolated vertices, $p_G(k) = k^n$.

Theorem. Let G be a simple graph and $e = uv$ be an edge of G. Then

$$p_G(k) = p_{G-e}(k) - p_{G/e}(k),$$

where $G - e$ is obtained by deleting the edge e from G and G/e is obtained by contracting the edge e; recall that contracting e is the same as identifying its endpoints u and v.

Proof. Consider a k-coloring of $G - e$.

Case 1. If u and v are colored using different colors, then we can insert the edge e to produce a k-coloring of G. This is reversible; we can remove the edge e from a k-colored version of G to produce a k-colored version of $G - e$ where u and v are colored differently.

Case 2. If u and v are coloring using the same colors in $G - e$, we can identify the vertices u and v to obtain a k-coloring of G/e.

This leads to the following:

$$p_{G-e}(k) = p_G(k) + p_{G/e}(k).$$

The result follows.

Corollary. For $n \geq 4$,

$$\begin{aligned} p_{C_n}(k) &= p_{P_n}(k) - p_{C_{n-1}}(k) \\ &= k(k-1)^{n-1} - p_{C_{n-1}}(k) \end{aligned}$$

Notice also that when $n = 3$,

$$p_{C_3}(k) = p_{K_3}(k) = k(k-1)(k-2) = k^3 - 3k^2 + 2k.$$

This is the initial condition for the recurrence in the corollary.

Notice that $p_{C_3}(2) = 0$ and so there are no 2-colorings of C_3. Hence, C_3 is not bipartite.

Consider $p_{C_6}(k)$. Will $k - 2$ be a factor of this polynomial?

Observation. The graph G is r-colorable if and only if $p_G(r) \neq 0$.

Observation. The graph G is not r-colorable if and only if $(k - r)$ is a factor of $p_G(k)$.

Theorem. For any simple graph G with n vertices, the function $p_G(k)$ is a polynomial of degree n in the variable k and this polynomial has leading coefficient equal to 1 (such polynomials are called **monic** polynomials.)

III. Trees and forests

Definition. A **forest** is a graph that has no cycles as subgraphs. Such a graph is also said to be **acyclic.**

Definition. A **tree** is an acyclic connected graph.

Observations.

- Every component of a forest is a tree.
- A trivial empty graph is a tree.
- A null graph is a forest but not a tree (because it has zero rather than one component and so it is not connected.)
- A path is a tree.
- Every edge in a forest is a bridge.
- The only complete graphs that are trees (or forests) are K_1 and K_2.
- Forests are bipartite because they have no cycles at all, so they have no odd cycles.
- If the complete bipartite graph $K_{n,m}$ is a tree, then $n = 1$ or $m = 1$ (with each positive).