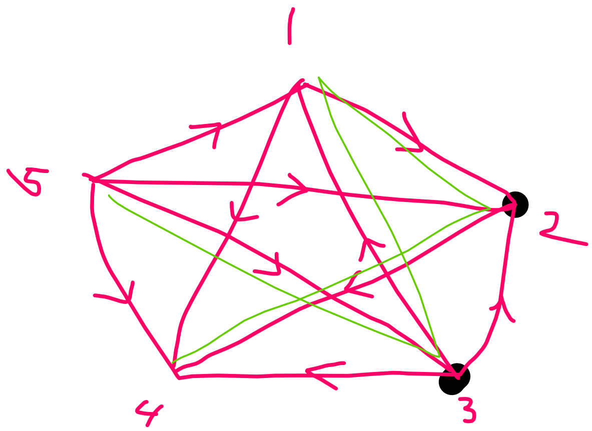
Today, we'll look at tournaments and partially ordered sets (posets).

Remember that Wednesday, Nov. 11 is a holiday.



Definition. A **tournament on n nodes** is an orientation of K_n.

Tournaments (the digraph definition) can be used to model the results of round-robin tournaments among n players. An arc $a = ij$ represents player i defeating player j.

Every tournament has a Hamilton directed path, i.e., a directed path that contains all of the nodes. In the example above, the Hamilton directed path is 5, 3, 1, 2, 4.

At one extreme for tournaments are those that have no directed cycles.

It turns out that for each K_n, there is only one tournament up to isomorphic that has no directed cycles.

Theorem. All tournaments on n nodes that have no directed cycles are isomorphic.

Proof. Let T be a tournament on n nodes with no directed cycles and let

$$P: v_1, v_2, v_3, \dots, v_n$$

be a Hamilton directed path in T. This path is guaranteed to exist from previous work. Suppose $a = v_i v_j$ is an arc of T such that $i > j$. Then

$$C:\ v_j, v_{j+1}, \dots, v_i, v_j$$

would be a directed cycle in T which is not allowed. This means that every arc $a = v_i v_j$ is oriented with $i < j$.

Now suppose T' is another tournament on n nodes with no directed cycles and let

$$P': v'_1, v'_2, v'_3, \dots, v'_n$$

be a Hamilton directed path in T'. By the exact same argument as before, every arc $a' = v'_i v'_j$ must be oriented with $i < j$. To show T and T' are isomorphic, we exhibit an isomorphism $\phi: N(T) \to N(T')$

between their node sets that preserves the directions of the arcs. Notice that $\phi(v_i) = v_i{}'$ preserves the direction of every arc. Also, ϕ is clearly bijective, and so it is an isomorphism.

Given an acyclic tournament, ranking the players is immediate, using the unique Hamilton path.

If your tournament has directed cycles, then rankings are not as straightforward.

If your tournament is strongly connected, then something weird happens.

Theorem (Moon 1966). In any strongly connected tournament with $n \geq 3$ nodes and any k with $3 \leq k \leq n$, every node is on a directed k-cycle.

Here are the lemmas that would lead to this theorem:

Lemma. Every node of a strongly connected tournament with $n \geq 3$ nodes lies on a directed triangle.

This would be the base case where $k = 3$ to prove Moon's theorem using mathematical induction.

Lemma. In every strongly connected tournament on $n \geq 3$ nodes, given k with $3 \leq k \leq n - 1$, if the node u lies on a directed k-cycle, then the node u lies on a directed $(k + 1)$-cycle.

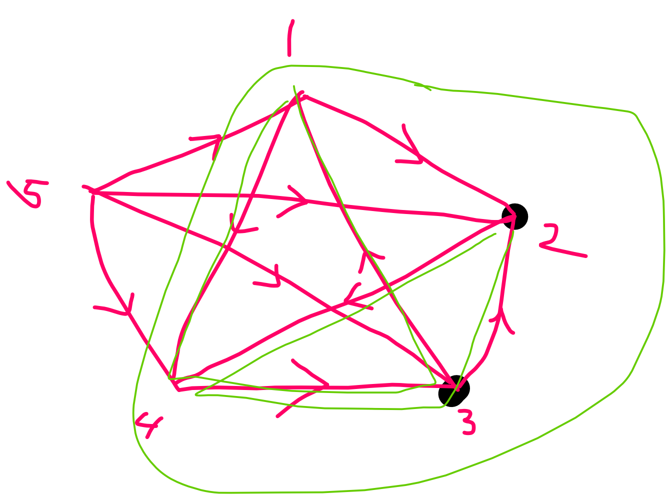
This would form the inductive step for the proof of Moon's theorem.

Corollary. Any strongly connected tournament has a directed Hamilton cycle.

Apply Moon's theorem in the case $k = n$.

Corollary. If the tournament T has a directed Hamilton cycle, then Moon's theorem applies to T.

Acyclic tournaments and strongly connected tournaments are extreme cases of tournament behavior. We want to look "in between" now, at tournaments that have directed cycles, but not Hamilton directed cycles.



Partially ordered sets.

Definition. A **partial order** on a set S is a relation $\sim$ such that

- (Reflexivity) For all $a \in S$, $a \sim a$.
- (Anti-symmetry) For all $a, b \in S$, if $a \sim b$ and $b \sim a$, then $a = b$.
- (Transitivity) For all $a, b, c \in S$, if $a \sim b$ and $b \sim c$, then $a \sim c$.

The classical examples are the relations $\geq$ and $\leq$ on $\mathbb{R}$.

Anti-symmetry is equivalent to saying at most one of $a \sim b$ or $b \sim a$ holds. We note that if $a \neq b$, there is no guarantee that either $a \sim b$ or $b \sim a$.

Definition. Two elements are **comparable** if $a \sim b$ or $b \sim a$ hold. They are **incomparable** if neither statement holds.

Example. Consider the subset relation $\subseteq$.

Given a set A, $A \subseteq A$ is true.

Given two sets A, B, if $A \subseteq B$ and $B \subseteq A$, then $A = B$

Given three sets A, B, C, if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

On the other hand, consider $\{1,2,3\}$ vs. $\{1,4,5\}$. This is an example of two incomparable objects.