Today, I'll wrap up the "handshaking lemma":

Theorem. In any loopless graph, the total degree is twice the number of edges.

In symbols,

$$\sum_{v \in V} \deg(v) = 2|E|.$$

Then a few more remarks about degree (Section I.D) before moving on to subgraphs (Section I.E) and graph isomorphism (Section I.F). Graph Isomorphism is a deep subject, worthy of research considerations.

On Thursday morning 10:00 I promise I will hold office hours.

In section I.D.2, we look at bounds on degrees and regularity:

Notation. The **minimum degree** of a graph G is

$$\delta(G) = \min_{v \in V}\{\deg(v)\}.$$

The **maximum degree** of a graph G is

$$\Delta(G) = \max_{v \in V}\{\deg(v)\}.$$

If there is no upper bound on the set $\{\deg(v)\}$, then $\Delta(G) = \infty$.

Definition. The graph G is said to be r-**regular (we define this only for $\boldsymbol{r \geq 0, r \in \mathbb{Z}}$)** if any of the following equivalent conditions hold:

- For all $v \in V$, $\deg(v) = r$
- $\delta(G) = \Delta(G) = r$
- $\{\deg(v) : v \in V\} = \{r\}$; these are sets, not multisets

Technicality. A null graph is r-regular for every r. To disprove this, we would have to produce a vertex whose degree is not r. Since a null graph has no vertices, it has no vertex who degree fails to be r.

Observations.

- A 0-regular graph has only isolated vertices.
- A 1-regular graph is a collection of pairs of vertices, each pair joined by one edge. Notice that $\{1,1,1,1,1,1, \ldots, 1\} = \{1\}$ as sets.
- A finite 2-regular graph is a collection of cycles. Also, the infinite graph $P_{\mathbb{Z}}$ whose vertices are the integers and two vertices are joined if and only if they differ by exactly 1, is 2-regular.
- A 3-regular graph is called a **cubic** graphs. Notice that Q_3 is cubic, but Q_n is not cubic for any other n.
- The complete graph K_n is $(n-1)$-regular; every vertex is adjacent to exactly $n-1$ other vertices
- the cube graph Q_n is n-regular
- The complete bipartite graph $K_{n,n}$ is n-regular. If $a \neq b$, both positive, then $K_{a,b}$ is not regular.

- If G is a loopless d-regular graph with n vertices and m edges, then the handshaking lemma (the total degree is twice of edges) lets us conclude:
$$2m = nd.$$

In section I.D.3, we introduce "degree sequences":

Definition. Given a finite loopless graph G whose vertex set is $\{v_1, v_2, \dots, v_n\}$ where the indices of the vertices are given in descending order of their degrees, i.e., v_1 has maximum degree, v_2 is next, ..., until v_n has minimum degree. More precisely, if $i < j$, then $\deg(v_i) \geq \deg(v_j)$. Define $d_i = \deg(v_i)$. Then the **degree sequence** of G is

$$d_1, d_2, d_3, \dots, d_n.$$

Briefly, the degree sequence is a list of the degrees of the vertices in descending order.

Definition. For a non-null loopless finite graph, the **average degree** is the quantity

$$d_{\text{ave}} = \frac{1}{n} \sum_{i=1}^{n} d_i.$$

Observations.

- For an r-regular graph, $d_{\text{ave}} = r$.
- For finite graphs, $\delta(G) \leq d_{\text{ave}} \leq \Delta(G)$.

Suppose G is a graph without isolated vertices, and $d_{\text{ave}} < 2$. Then the minimum degree must be 1.