Recall that a trail is an **Euler trail** if every edge is used exactly once. A graph is **Eulerian** if it possesses an Euler circuit (recall that a **circuit** is a trail that begins and ends at the same vertex.)

We're talking about acyclic graphs (also known as "forests") and connected acyclic graphs (also known as "trees"). Notice that an individual tree is considered to be a forest.

To start, we'll list some results in Section III.A.2. (Path uniqueness in trees). Then moving to section III.A.3. (The Edge Formula) and section III.A.4. (Leaves).

Theorem. Given a tree T, for any two vertices u, v of T, there is a unique (exactly one) u, v-path.

Corollary. Given a forest F, for any two vertices u, v of F, there is at most one u, v-path.

Corollary. Given any two vertices u, v of a tree T, there exists a unique u, v-walk of length $D(u, v)$, namely, the unique u, v-path guaranteed by the theorem.

Corollary. Given three vertices u, v, w on a tree T, if $D(u, v) = D(u, w) + D(w, v)$, then w must be a vertex on the unique u, v-path guaranteed by the theorem.

Section III.A.3. The edge formula for forests.

Theorem III.A.3.a. If F is a forest with n vertices, m edges, and k components, then

$$n = m + k.$$

Proof. Let F be as hypothesized. This is a graph with exactly k components. Since F is a forest, it has no cycles. Therefore, every edge is a bridge (because it cannot lie on a cycle). We delete every edge one at a time. As we delete each edge, we increase the number of components by exactly one. Therefore, the effect of deleting all m of the edges is to increase the number of components by exactly m. The resulting graph has exactly $m + k$ components. But the result of deleting all m of the edges is also to isolate every vertex. Hence, we end up with a graph where every component is a single vertex. No vertices were deleted in this process, so our resulting graph still has n vertices. Hence, the number of vertices is equal to the number of components, or,

$$n = m + k,$$

as desired.

Corollary. If T is a tree with n vertices, then T has $n - 1$ edges.

Proof. This follows by applying the above theorem to the forest T that has $k = 1$ component.

A converse to our theorem:

Theorem. If G has n vertices, m edges, and k components where $n = m + k$, then G is a forest.

Proof. Delete every non-bridge from G one at a time; suppose there are j of them. Deleting a non-bridge has no effect on the number of components. This means that the resulting graph G' has n vertices, $m - j$ edges and k components. The graph G' has no cycles and so from the theorem,

$$m + k = n = m - j + k.$$

Hence, $j = 0$ and so there were no non-bridges to delete from G; hence, every edge is a bridge and so G has no cycles. Hence, G is a forest.

Corollary. If G is connected with n vertices and m edges, and $n = m + 1$, then G is a tree.

Theorem. Let G be a graph with n vertices and m edges. Any two of the following statements together imply the third statement:

1) G is acyclic (this is the same as saying G is a forest)
2) G is connected
3) $n = m + 1$

Furthermore, if any two of these hold, then all three hold and G is a tree.

If you have one of these and another fails, then the third fails.

Section III.A.4. Leaves

Definition. A **leaf** is a vertex of degree 1.

Observation. An isolated vertex is not a leaf.

Observation. A trivial tree (a single isolated vertex) has no leaves.

Theorem. Every nontrivial tree has at least two leaves.

Proof. Assume there are counterexamples. Let T be a counterexample with the smallest number of edges. This number of edges is at least one since T is nontrivial. If T has no leaves, then every vertex has degree at least 2. From earlier results, T must have a cycle. This is not allowed, so this case fails. Hence T has at least one leaf ℓ. If we delete the edge e incident with ℓ, then $T - e$ has two components. One of those components consists only of the vertex ℓ. Hence, $T - \ell$ is a tree. But then either $T - \ell$ is trivial, in which case T is isomorphic to K_2 and has two leaves, or $T - \ell$ is nontrivial.

Pick up here on Monday!!!