Remember the Class Motto: “We do this because it’s fun.”

Today I want to look at section I.C. “Constructing and Drawing Graphs” and start in on section I.D. “Degree”.

A usual practice when defining a graph is to specify the vertex set V and then some sort of rule that tells you exactly when two elements of V are joined by an edge.

Examples of this in action:

A **path graph** P_n has vertex set $V = \{1,2,3, \dots, n\}$ and two vertices u, v are joined by an edge if and only if $v = u - 1$? This doesn’t quite work: Suppose $u = 1, v = 2$? A better refinement would be

$$|u - v| = 1.$$

Alternatively, you could have said, “u and v are adjacent if and only if they differ by one.”

Alternatively, “u and v are adjacent if and only if $u - v = 1$ or $v - u = 1$.”

Convention. The path P_n has n vertices.

Also, the length of P_n is the number of edges.

A **cycle graph** C_n, $n \geq 1$, has vertex set $V = \{0,1,2, \dots, n-1\}$ and two vertices u, v are joined by an edge if and only if they differ by one or one of them is 0 and the other is $n - 1$.

On a standard clock, (treat 12 = 0), the arithmetic is mod 12.

Alternatively, “two vertices u, v are joined by an edge if and only if they differ by one modulo n.”

Example. C_4 has vertex set $V = \{0,1,2,3\}$ and edges $01,12,23,30$; here, we’re concatenating vertex names, so “30” should be interpreted as {3,0}. In mod 4, $3 + 1 = 0$, so $3 = -1 \bmod 4$. This is why 3 and 0 are considered to differ by 1 modulo 4. More precisely, $0 - 3 = 1 \bmod 4$.

Observe in modulo 4 arithmetic: 1+3=0 mod 4.

If we subtract 3 from both sides, we obtain 1=0-3 mod 4.

Often, C_3 is called a “triangle.” Sometimes, C_4 is called a “square”; C_5 are often called by their polygon names, pentagon, hexagon, etc.

Given $n \geq 1$, the **complete graph** K_n has vertex set $V = \{1,2,3, \dots, n\}$ and two vertices u, v are joined by an edge if and only if they are distinct. (Distinct means unequal.)

The number of edges of K_n:

1	|E|=0
2	|E|=1
3	|E|=3
4	|E|=6
5	|E|=10
6	|E|=15

Notice that the number of edges is the number of ways to choose two distinct vertices from $V = \{1,2,3,\dots,n\}$. This number is "n choose 2"; in symbols, $\binom{n}{2}$. The way to compute this is

$$\binom{n}{2} = \frac{n(n-1)}{2}.$$

General formula:

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}.$$

Another family of graphs: The n-cubes Q_n. The vertex set of Q_n is the set

$$V = \{(a_1, a_2, \dots, a_n):\ a_i \in \{0,1\}\}$$

The order of Q_n, i.e., the number of vertices is $|V| = 2^n$. The number of edges is left as an exercise.

For instance, the vertices of Q_3 are triples:

$$(0,0,0), \qquad (0,0,1), (0,1,0), (0,1,1), (1,0,0), (1,0,1), (1,1,0), (1,1,1).$$

Two vertices are joined an edge if and only if they differ by 1 in one coordinate and are equal in all other coordinates.

For instance, $(0,0,0)$ is adjacent to each of $(0,0,1), (1,0,0), (0,1,0)$.

If you draw all of the joinings in Q_3, you get a familiar picture of a cube.