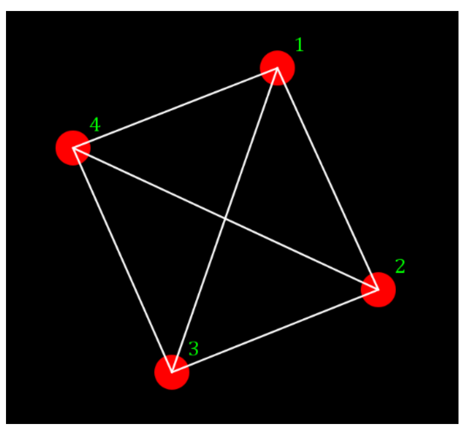
Graph Theory Fall 2020

Assignment 4

Due at 5:00 pm on Friday, September 25

1. For the cube graph Q_n, the distance $D(\mathbf{a}, \mathbf{b})$ between two vertices
$$\mathbf{a} = (a_1, a_2, \dots, a_n) \text{ and } \mathbf{b} = (b_1, b_2, \dots, b_n)$$
is called the "Hamming distance." This is the number of positions where $\mathbf{a}$ and $\mathbf{b}$ differ. For instance, the Hamming distance between $(0,0,1,0)$ and $(1,1,0,0)$ is 3 because these two vertices differ in three positions. In each of the parts A,B below, $D(\mathbf{x}, \mathbf{y})$ is the Hamming distance in Q_n:
   A. Show that if $D(\mathbf{a}, \mathbf{b})$ and $D(\mathbf{b}, \mathbf{c})$ have the same parity (i.e., are both even or are both odd), then $D(\mathbf{a}, \mathbf{c})$ must be even.
   B. Show that if $D(\mathbf{a}, \mathbf{b})$ and $D(\mathbf{b}, \mathbf{c})$ have different parity, then $D(\mathbf{a}, \mathbf{c})$ must be odd.

2. Consider K_4 as drawn and labeled below:



Since this graph is simple, we can specify a walk by listing only the vertices. For instance, C: 1,2,3,4,1 is a 4-cycle; this can be abbreviated as "12341". List all of the cycles (abbreviated style is fine) that start and end at vertex 1 in this drawing of K_4.

3. For $1 \leq m \leq 11$, let G_m be the graph with vertex set
$$V = \{0,1,2,3,4,5,6,7,8,9,10,11\}$$
and where vertices u and w are adjacent iff $w - u = m$ modulo 12 or $u - w = m$ modulo 12. We observe that $G_1 = C_{12}$, a twelve-cycle.
   A. For what values of m is G_m connected?
   B. What are the possible numbers of components of G_m?

4. Let G be a graph and let e_1 and e_2 be edges. Show that if deleting e_1 and e_2 disconnects vertices u and v, then any cycle containing both u and v must contain both e_1 and e_2. One approach: You could apply to the graphs $G - e_1$ and $G - e_2$ the fact that if e is a bridge whose removal disconnects u and v, then any path connecting u and v must contain e.