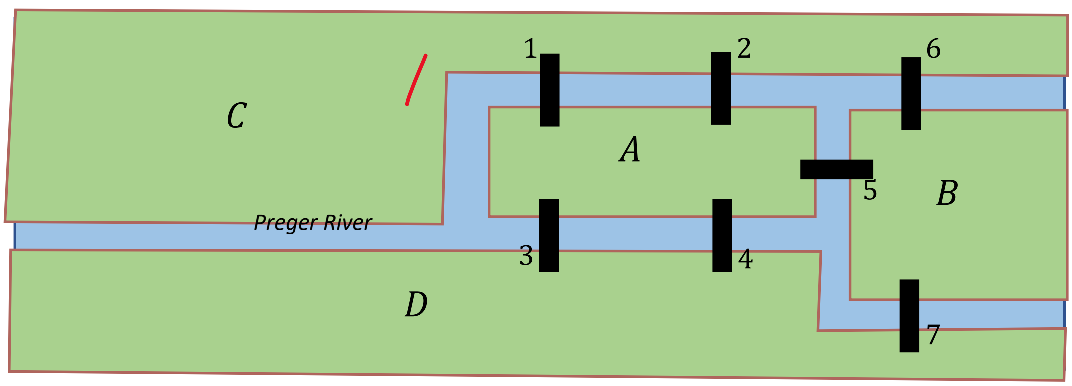
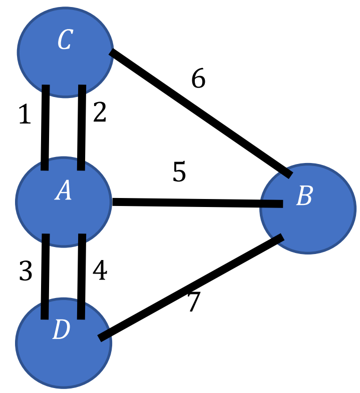
Traversals, i.e., walks that "cover" a graph in some way. For instance, we might want a walk that uses every edge exactly once, or where every vertex is used exactly once.

The famous "Bridges of Konigsberg" problem, solution is credited to Euler (pronounced "oiler"). This is a very early problem that was solved using graph theoretical techniques.





Schematic of Konigsberg

Konigsberg was split into four land masses A, B, C, D and the river was bridged seven times, denoted 1,2,3,4,5,6,7.

Can one traverse the seven bridges along a walk using each bridge exactly once, without getting your feet wet? No, according to Euler.

Euler's inspiration was to represent the land masses as vertices and the bridges as edges of a graph.

Question: Is there a trail (a trail is a walk that uses each edge at most once) that uses all of the edges?

We would call such a trail an Euler trail.

Proposition. There is no Euler trail for the Konigsberg graph.

Proof. Suppose an Euler trail exists for the Konigsberg graph. We would have a trail

$$W: v_0, e_1, v_1, e_2, v_2, \dots, e_7, v_7$$

where $v_i \in \{A, B, C, D\}$ and $e_j \in \{1,2,3,4,5,6,7\}$. For each interior vertex v_i, v_1 through v_6, there are two distinct edges incident with v_i. For the end vertices, there is one edge incident with it. So the degree of each of A, B, C, or D is twice the number of times it occurs as an interior vertex and once for each time it is an end vertex. If the degree of a vertex A, B, C, or D is odd, then it must be an end vertex. The contrapositive: If A, B, C, or D does not occur as an end vertex, its degree must be even in the graph. In the Konigsberg graph, the degrees of all four vertices are odd, namely 5,3,3,3. This implies all four vertices A, B, C, D are end vertices. But there are only two end vertices which is a contradiction. Hence, there is no Euler trail for the Konigsberg graph.

Convention. When discussing Eulerian graphs, loops contribute 2 to the degree of it endpoint.

Definition. An **Euler circuit** is an Euler trail that starts and ends at the same vertex. A graph with an Euler circuit is said to be **Eulerian**.

Definition. A vertex is an **odd vertex** if its degree is odd; an **even vertex** if its degree is even.

Theorem. If a graph is Eulerian, then all of its vertices have even degree.

Sketch of proof. Suppose G is Eulerian with odd vertices. There would have to be at least two odd vertices. These would have to be endpoints of any Euler trail. Therefore, such a trail would have two distinct endpoints and so no Euler circuit could exist.

Here is the converse:

Theorem. If G is connected and all of the vertices of a graph G are even vertices, then G is Eulerian.

This direction requires a bit of work.

Lemma. A finite graph with minimum degree at least 2 has a cycle as a subgraph.

Sketch of Proof. Let G be a graph with minimum degree at least 2. We form a walk W starting at any vertex v_0.

$$W: v_0, e_1, v_1, e_2, v_2$$

Since v_0 is not isolated, there is an edge e_1 not yet used in W where e_1 is incident with v_0 and its other endpoint v_1. If $v_1 = v_0$, then W is a one-cycle which we can use as our subgraph. If $v_1 \neq v_0$, then since v_1 has degree at least 2, there is an edge $e_2 \neq e_1$ incident with v_1; we assign v_2 to be the other endpoint of e_2. Do this until the first instance of a repeated vertex occurs; $v_i = v_j; i < j$. The walk

$$C: v_i, e_{i+1}, v_{i+1}, \dots, e_j, v_j$$

is a cycle.