Graph Theory Fall 2020

Assignment 6

Due at 5:00 pm on Friday, October 23

1. Show that for any $k \geq 3$, any tree with a vertex of degree k must have at least k leaves. The proof that uses summations of the result that a tree always has two leaves is probably easiest to adapt here. You will want to assume $n_k \geq 1$ in the summations

$$n = \sum_{j=1}^{\infty} n_j \,; \quad \text{total degree} = \sum_{j=1}^{\infty} j n_j.$$

2. Suppose T is a binary tree of height H with n vertices.
   A. As a function of H, what are the minimum and maximum possible values of n?
   B. Suppose every parent has exactly two children in T. Show that n must be odd.

3. Let (T, r) be a rooted tree.
   A. Show that if $D(u, v) = 2H$ where H is the height of T, then u and v are non-parents.
   B. Show that $D(u, v)$ is the sum of the levels of u and v if and only if r is on the unique u, v-path.

4. Suppose G is a simple graph (no loops; no parallel edges) with $n = 14$ vertices and $m = 7$ edges. What are the possible values for k, the number of components of G?

5. Suppose T is a binary tree with 10^9 vertices. What are the minimum and maximum possible values of H, the height of T?