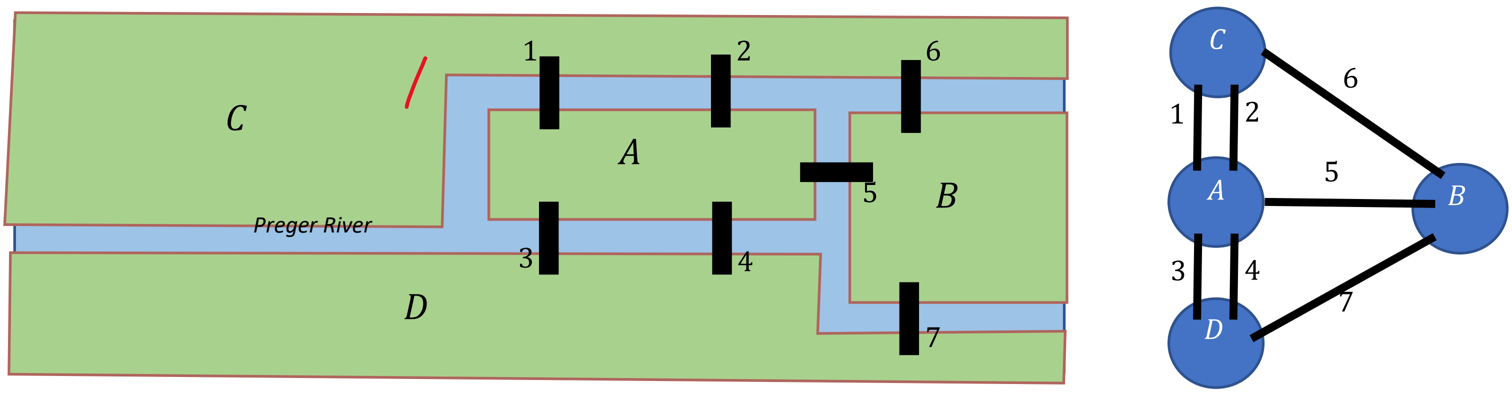
Traversals, i.e., walks that "cover" a graph in some way. For instance, we might want a walk that uses every edge exactly once, or where every vertex is used exactly once.

The famous "Bridges of Konigsberg" problem, solution is credited to Euler (pronounced "oiler"). This is a very early problem that was solved using graph theoretical techniques.



Schematic of Konigsberg

Konigsberg was split into four land masses A, B, C, D and the river was bridged seven times, denoted 1,2,3,4,5,6,7.

Can one traverse the seven bridges along a walk using each bridge exactly once, without getting your feet wet? No, according to Euler.

Euler's inspiration was to represent the land masses as vertices and the bridges as edges of a graph.

Question: Is there a trail (a trail is a walk that uses each edge at most once) that uses all of the edges?

We would call such a trail an Euler trail.

Proposition. There is no Euler trail for the Konigsberg graph.

Proof. Suppose an Euler trail exists for the Konigsberg graph. We would have a trail

$$W: v_0, e_1, v_1, e_2, v_2, \dots, e_7, v_7$$

where $v_i \in \{A, B, C, D\}$ and $e_j \in \{1,2,3,4,5,6,7\}$. For each interior vertex v_i, v_1 through v_6, there are two distinct edges incident with v_i. For the end vertices, there is one edge incident with it. So the degree of each of A, B, C, or D is twice the number of times it occurs as an interior vertex and once for each time it is an end vertex. If the degree of a vertex A, B, C, or D is odd, then it must be an end vertex. The contrapositive: If A, B, C, or D does not occur as an end vertex, its degree must be even in the graph. In the Konigsberg graph, the degrees of all four vertices are odd, namely 5,3,3,3. This implies all four vertices A, B, C, D are end vertices. But there are only two end vertices which is a contradiction. Hence, there is no Euler trail for the Konigsberg graph.

Convention. When discussing Eulerian graphs, loops contribute 2 to the degree of it endpoint.

Definition. An **Euler circuit** is an Euler trail that starts and ends at the same vertex. A graph with an Euler circuit is said to be **Eulerian**.

Picking up from Friday's lecture; we're starting Monday here.

I'm holding off on assignment #5 until we get a few more concepts looked at. This should give everyone a chance to get caught up on other aspects of their lives.

Recall that an **Euler circuit** is a circuit that uses every edge exactly once. We're looking at **Eulerian graphs**, i.e., those graphs that have Euler circuits.

Definition. A vertex is an **odd vertex** if its degree is odd; an **even vertex** if its degree is even.

Theorem. If a graph is Eulerian, then all of its vertices have even degree.

Sketch of proof. Suppose G is Eulerian with odd vertices. There would have to be at least two odd vertices. These would have to be distinct endpoints of any Euler trail. Therefore, such a trail would have two distinct endpoints and so no Euler circuit could exist.

Here is the converse:

Theorem. If G is connected and all of the vertices of a graph G are even vertices, then G is Eulerian.

This direction requires a bit of work.

Lemma. A <u>finite</u> graph with minimum degree at least 2 has a cycle as a subgraph.

Sketch of Proof. Let G be a graph with minimum degree at least 2. We form a walk W starting at any vertex v_0.

$$W: v_0, e_1, v_1, e_2, v_2$$

Since v_0 is not isolated, there is an edge e_1 not yet used in W where e_1 is incident with v_0 and its other endpoint v_1. If $v_1 = v_0$, then W is a one-cycle which we can use as our subgraph. If $v_1 \neq v_0$, then since v_1 has degree at least 2, there is an edge $e_2 \neq e_1$ incident with v_1; we assign v_2 to be the other endpoint of e_2. Do this until the first instance of a repeated vertex occurs; $v_i = v_j; i < j$. The walk

$$C: v_i, e_{i+1}, v_{i+1}, \dots, e_j, v_j$$

is a cycle.

Theorem. If G is connected and all of the vertices of a graph G are even vertices, then G is Eulerian.

Proof. Suppose counterexamples exist. Let G be a counterexample with the minimum possible number of edges; call this number of edges m. First, note that if $m = 0$, then G is trivial, and the trivial walk is an Euler circuit. Therefore, $m > 0$. The graph G must be connected and all vertices must be even. Therefore, the minimum degree of G must be at least 2. Therefore, by the lemma, G has a subgraph C that is a cycle:

$$C: v_0, e_1, v_1, e_2, \dots, v_{k-1}, e_k, v_k = v_0.$$

Let $H = G - E(C)$, the graph obtained by deleting the edges of C. Notice that every vertex of H is an even vertex. There is no guarantee that H is connected, but we will list the components of H:

$$H_1, H_2, \dots, H_p$$

in the order they are first encountered as we traverse the vertices in C. Every component H_i has fewer edges than G, is connected, and every vertex in H_i is an even vertex. Therefore, each H_i is not a counterexample to the theorem. This means that H_i has an Euler circuit:

$$C_i: w_{i0}, e_{i1}, w_{i1}, \dots, w_{i,q-1}, e_{i,q}, w_{i,q} = w_{i0}$$

where w_{i0} is the first vertex v_j in the cycle C contained in H_i. In the cycle C, replace v_j with the circuit C_i. This recipe yields an Euler circuit for G. This contradicts G being a counterexample. Hence, counterexamples do not exist and so the result follows.

Corollary. A connected graph is Eulerian if and only if all of its vertices have even degree.

Corollary. For all positive integers n, the graph Q_n is Eulerian if and only if n is even.

Corollary. A connected graph has an Euler walk that begins and ends at different vertices if and only if it has exactly two odd vertices.

Sketch of proof. Add an edge between the two odd vertices. The resulting graph is connected with all even vertices and is hence Eulerian. Removing that edge from an Eulerian circuit starting at one of the two vertices results in an Euler walk.

Moral of the story (Eulerian properties are easy): It is easy to tell whether a walk can be constructed that covers the edges of a graph exactly once.

Section II.C.2. Hamiltonian graphs

The game now is to ask whether we can manufacture walks that cover the vertices of a graph exactly once. Here, a cycle is allowed if the only repetition is the start and end vertex.

Definition. A **Hamilton path** in a graph G is a path subgraph that contains every vertex.

Definition. A **Hamilton cycle** in a graph G is a cycle subgraph that contains every vertex. A graph is **Hamiltonian** if it has a Hamilton cycle.

Observation. If G has a Hamilton cycle, then it has a Hamilton path. Simply remove any edge from the cycle.

Theorem. If G is nontrivial and has a Hamilton path, then $G \times K_2$ has a Hamilton cycle.

Proof. Let $P: v_1, e_2, v_2, \dots, v_{n-1}, e_n, v_n$ be a Hamilton path for G and let w_1, f, w_2 be the Hamiltonian path for K_2. Recall that the vertex set of $G \times K_2$ is

$$\{(v_i, w_j):\ 1 \le i \le n; 1 \le j \le 2\}$$

Let $u_{ij} = (v_i, w_j)$. The edges of $G \times K_2$: e_{ij} joins $u_{i-1,j}$ to u_{ij} and f' will join u_{i1} to u_{i2}. Then

$$C:\ u_{11}, e_{21}, u_{21}, e_{31}, \dots, e_{n1}, u_{n1}, f_n', u_{n2}, e_{n2}, \dots, u_{32}, e_{32}, u_{22}, e_{22}, u_{12}, f_1', u_{11}$$

is a Hamilton cycle for $G \times K_2$.