

Today, I want to look at Euler's formula relating n, m, r , the numbers of vertices, edges, and regions for spherical (and hence, planar) graphs.

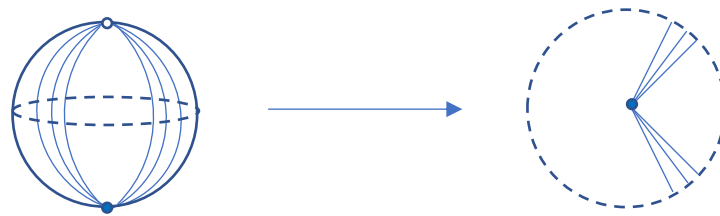
Intuitive definitions.

A **region** is a set that is homeomorphic (continuously stretchable) with an open unit disk (filled in unit circle without the boundary).

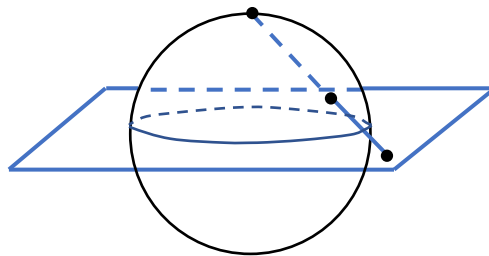
A graph is **spherical** if it can be drawn on the surface of a sphere without edges crossing.

A **punctured sphere** is a sphere without a single point (usually, one removes the "north pole" or "south pole.")

Observation. The punctured sphere is a region.



Observation. The plane \mathbb{R}^2 is homeomorphic to the punctured sphere, via **stereographic projection**.



Proposition. If a connected finite spherical graph G is drawn on the sphere S without edges crossing, then the set $S - G$, obtained by removing the vertices and edges from the sphere, is the disjoint union of a finite number of regions; this number is **invariant** and denoted by r .

Proposition. An edge is on the boundary of one or two regions. Furthermore, an edge is on the boundary of one region if and only if it is a bridge.

Proposition. A region bounded by non-bridges, exclusively, is bounded by the edges in a cycle.

Observations.

- If G is a simple graph, then its "non-bridge exclusive" regions (covered by the third proposition) are bounded by cycles each with at least three edges.
- If G is simple and bipartite, then its "non-bridge exclusive" regions are bounded by cycles each with at least four edges.

Corollary. The graph T is a tree drawn on a sphere S without edges crossing if and only if $S - T$ consists of a single region.

Theorem (Euler's Formula). For any connected spherical finite graph G drawn on a sphere without edges crossing,

$$n - m + r = 2.$$

Proof. We proceed by induction on $r \geq 1$, the number of regions. For the base case, suppose G is a graph where $r = 1$. By the above corollary, G is a tree. Hence, $m = n - 1$ and

$$n - m + r = n - (n - 1) + 1 = 2,$$

establishing the base case.

For the inductive step, assume the result holds for $r = k \geq 1$, i.e., Euler's formula holds for any graph with exactly k regions. Let G be a spherical graph drawn with $r = k + 1$ regions. Since $r \geq 2$, G is not a tree and so it must have non-bridges. Let e be a non-bridge. As a non-bridge, e is on the boundary of two regions R_1, R_2 . In the graph $G' = G - e$, these two regions are merged into one. More precisely, $R_1 \cup R_2 \cup e$ is a single region for $G - e$. Let n', m', r' be the numbers of vertices, edges, and regions for G' . We observe:

$$n' = n, m' = m - 1, r' = r - 1.$$

This means

$$n = n', m = m' + 1, r = r' + 1$$

and so

$$\begin{aligned} n - m + r &= n' - (m' + 1) + (r' + 1) \\ &= n' - m' + r' \\ &= 2 \end{aligned}$$

because G' is a graph with $r - 1 = k$ regions and so Euler's formula holds for G' . As a consequence, Euler's formula holds for G and this completes the inductive step and the proof.

As a physical example, consider the dodecahedron graph, with $n = 20$ vertices, $m = 30$ edges, and $r = 12$ regions: Euler yields

$$n - m + r = 20 - 30 + 12 = 2$$

as advertised.

Corollary. If P is a convex polyhedron with n vertices, m edges, and r faces, then $n - m + r = 2$.

A shape is **convex** if for any two points on the shape, the line segment joining those points is entirely within the shape.