Today, I'll want to look at walks.

Definition. Given a graph $G = (V, E)$, a **walk W** is a finite sequence alternating between vertices and edges that begins and ends at vertices, i.e.,

$$W: v_0, e_1, v_1, e_2, v_2, \dots, v_{k-1}, e_k, v_k$$

and such that for all $i, 1 \le i \le k$, the endpoints of e_i are v_{i-1} and v_i; also, if $v_{i-1} = v_i$, then e_i is a loop.

- The graph G is said to **admit** the walk W.
- A **trivial walk** is a walk that has only one vertex and no edges.
- The vertices v_0 and v_k are the **starting point** and **ending point** of W. The vertices v_1 through v_{k-1} are **interior points** of W; it is allowed for v_0 or v_k to occur as interior points.

To illustrate this last point, let $e = uv$ be an edge:

$$u, e, v, e, u, e, v, e, u$$

is a walk in G.

- The integer k in the sequence $v_0, e_1, \dots, e_k, v_k$ is called the **length** of the walk. This is the number of edges in the walk.
- If we reverse the order of the items in a walk, we **reverse** the walk.
- Given two walks W, W':

  $$W: v_0, e_1, v_1, e_2, v_2, \dots, v_{k-1}, e_k, v_k$$
  $$W': v_k, e_{k+1}, v_{k+1}, e_{k+2}, v_{k+2}, \dots, v_{k+\ell-1}, e_{k+\ell}, v_{k+\ell}$$

  then the walk WW' is attained by **concatenating W and W':**

  $$WW': v_0, e_1, \dots, e_k, v_k, e_{k+1}, \dots, e_{k+\ell}, v_{k+\ell}.$$

- If G has no parallel edges, then it suffices to list the vertices in the walk.
- There are no *a priori* conditions or prohibitions about walks starting or ending at the same vertex, or repeating vertices.

A u, v-walk is a walk that starts at u and ends at v.

Definition. A **path** is a walk that does not repeat a vertex.

A nontrivial path is not allowed to start and end at the same vertex. A trivial walk is a path.

Definition. A **Hamilton path** is a path that includes every vertex of the graph.

Theorem. A path cannot repeat an edge.

Proof. Suppose a path P repeated an edge:

$$P\colon\ v_0, e_1, \dots, e_k, v_k$$

where $e_i = e_j$ with $i < j$. This would mean:

$$v_{i-1}, e_i, v_i \quad \text{and} \quad v_{j-1}, e_j, v_j$$

are subsequences of P. But then v_j is an endpoint of e_i, implying $v_j \in \{v_{i-1}, v_i\}$ and so a vertex would have to be repeated on P.

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Theorem. If a graph G admits a u, v-walk that repeats a vertex, then G admits a shorter u, v-walk.

Proof. Suppose G admits a u, v-walk W,

$$W\colon\ u = v_0, e_1, v_1, e_2, v_2, \dots, v_{k-1}, e_k, v_k = v$$

where a vertex is repeated, i.e., there exist indices $0 \le i < j \le k$ such that $v_i = v_j$. This means we can exclude the sub-sequence

$$e_{i+1}, v_{i+1}, e_{i+2}, v_{i+2}, \dots, e_{j-1}, v_{j-1}, e_j, v_j$$

from W, obtaining the walk

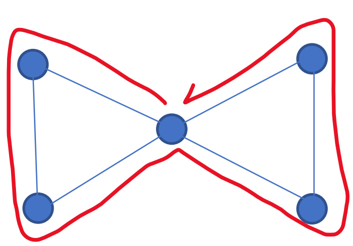
$$W'\colon u = v_0, \dots, v_i, e_{j+1}, v_{j+1}, \dots, v_k = v.$$

We note that the length of W is k and the length of W' is $k - (j - i) < k$, a lower number.

Corollary. If W is a shortest possible u, v-walk, then W is a path.

Definition. A **trail** is a walk that does not repeat an edge.

A trail can repeat a vertex.



Definition. An **Euler trail** is a trail that includes every edge of the graph.

Definition. A **cycle** is a u, u-trail of length at least one where no interior vertex is repeated.

A **closed walk** has the same starting and ending points.

Paths are trails, but not all trails are paths (think about the bowtie example.)

To get a handle on #3, suppose

$$W: v_0, e_1, v_1, \dots, v_i, e_{i+1}, v_{i+1}, \dots, v_{j-1}, e_j, v_j = v_i, \dots, e_k, v_k;$$

is a trail where $v_i = v_j$. You would be in your rights to consider only the subtrail

$$X: v_i, e_{i+1}, v_{i+1}, \dots, v_{j-1}, e_j, v_j = v_i.$$

Another hint for #3: Make i and j as close as possible, i.e., minimize $j - i$.

Foreshadowing: Given a graph G and two vertices $u, v \in V(G)$, the **distance** between u and v, written $d(u, v)$, is the length of a shortest u, v-walk (must be a path.) The length of a walk is the number of edges.

The walk u, ℓ, u where $\ell = uu$ is a loop is a length 1 walk. This is considered a cycle. The walk u, e_1, v, e_2, u is considered a cycle, if $e_1 \neq e_2$.

Two vertices u and v are **connected** if there exists a u, v-walk. They are **disconnected** otherwise.