I did realign the first question (the one where you have to draw K_8 on a 2-holed torus) as extra credit (5 points). There is a new link to hand this question in as extra credit, so if you've already turned in the current assignment, feel free to resubmit it to this new link.

On the question where all of the boundary arcs are oriented in the same direction, the number of corner points may or may not be 1 or 2.

We'll continue with directed walks and directed cycles today, then on to weakly- and strongly-connected digraphs; probably getting into "tournaments," which are digraphs whose underlying graphs are complete graphs.

We're not that far away from linear algebra – good topics for review include: matrices and vectors (220-level material should suffice) and look at determinants, eigenvectors and eigenvalues.

Definition. Given a digraph $D = (N, A)$, a **directed walk** is a finite sequence alternating between nodes and arcs, beginning and ending at nodes, i.e.,

$$W: v_0, a_1, v_1, a_2, v_2, \dots, v_{n-1}, a_n, v_n$$

such that for each k, $a_k = v_{k-1}v_k$. If $v_{k-1} = v_k$, then a_k is a loop joining v_k to itself.

We'll still call n (the number of arcs) the **length** of W. If $n = 0$, you have a trivial directed walk.

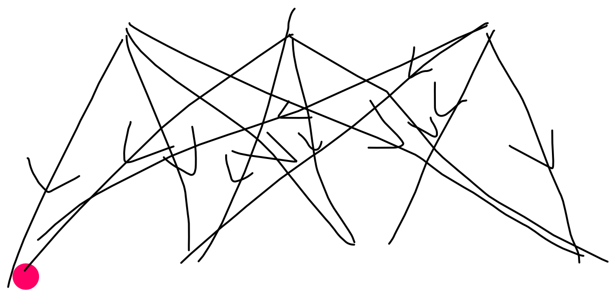
Definition. A **directed path** is a directed walk where all nodes are distinct.

Definition. A **directed cycle** is a directed walk with $n \geq 1$ such that the only repeated node $v_0 = v_n$; all other nodes are distinct from each other and the endpoints.

Theorem (Roy (1967) and Gallai (1968), independently). Given a digraph D with underlying graph G, there exists a directed path of length $\chi(G) - 1$, where $\chi(G)$ is the chromatic number, i.e., the minimum number of colors needed to color the vertices of G with no two adjacent vertices assigned the same color.

Suppose D (this will be called a "tournament") has K_n as its underlying graph, noting that $\chi(K_n) = n$. Roy and Gallai's theorem says there exists a directed path of length $\chi(K_n) - 1 = n - 1$. Such a path would use n of the nodes in K_n; hence, D has a Hamilton directed path.

For the other extreme, suppose the underlying graph G is bipartite with parts X and Y. We can orient the arcs from X to Y (or from Y to X) so that the longest directed path in the digraph D so obtained has length 1.



Connectedness in digraphs.

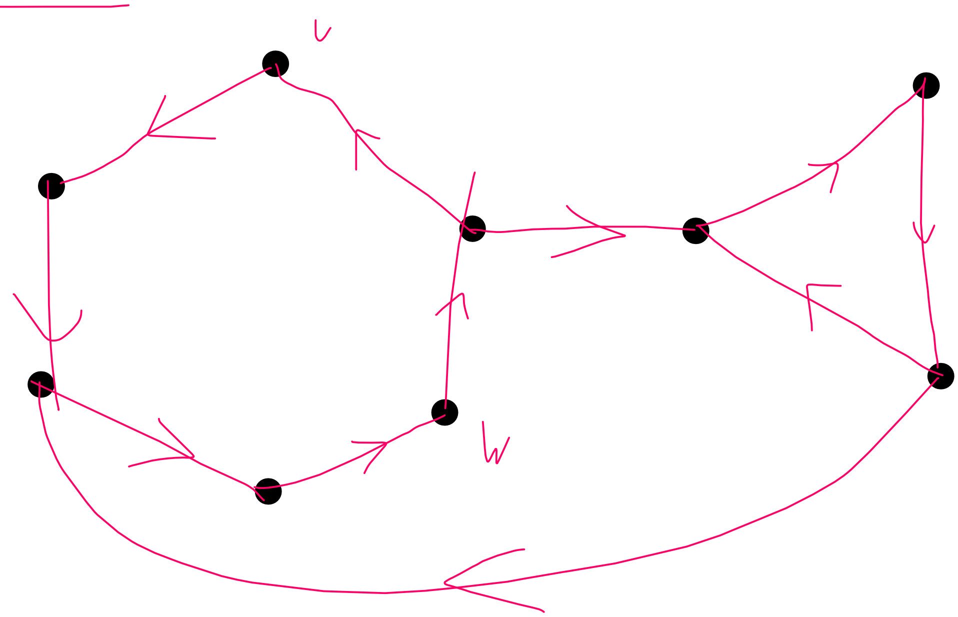
If D is a digraph whose underlying graph G is disconnected, then D is disconnected.

At this point, we consider digraphs with connected underlying graphs. Such a digraph is guaranteed to be either "weakly connected" or "strongly connected."

(Strongly connected implies weakly connected.)

Definition. A digraph D is **weakly connected** if its underlying graph is connected.

Definition. A digraph D is **strongly connected** if for any two nodes v, w, there exists a directed path from v to w (you could say a directed v, w-path).



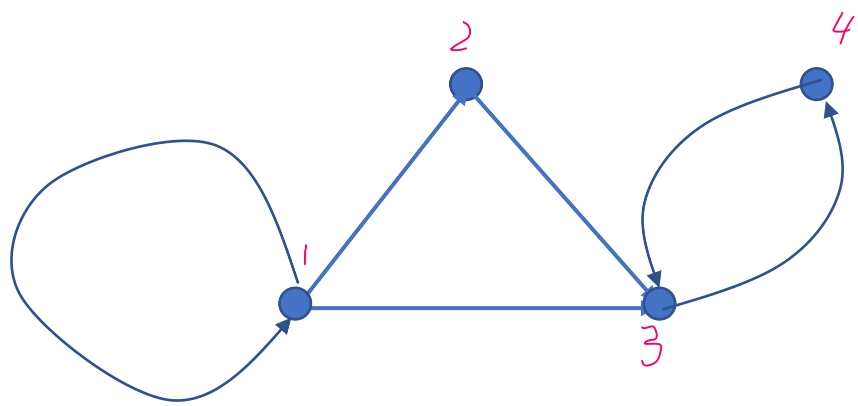
Directed components.

Definition. Given a digraph D, for any two vertices v and w, we say $v \sim w$ if there exist both a v, w-directed path and a w, v-directed path.

One can show that $\sim$ is an equivalence relation; that means it is reflexive, it is symmetric, and it is transitive.

Definition. Let $N' = [u]$ be an equivalence class of nodes under $\sim$ above. This is the set of nodes that are related to u under $\sim$.

Definition. The induced sub-digraph $D[N']$ is the **directed component** of D that contains u.



$$[1] = \{1\}; [2] = \{2\}; [3] = \{3,4\}$$

The directed components: