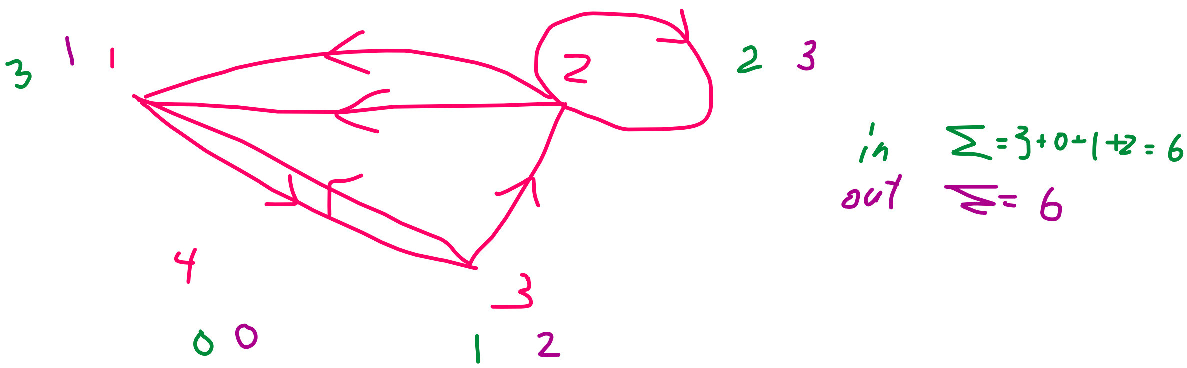
Today, we'll start looking at Section V, **directed graphs**, or **digraphs**.

We're discussing what happens if our "edges" become "one-way streets," i.e., if there is some orientation that causes the edges to "leave" vertices and "enter" vertices.

It's becoming common to rename "vertices" and "edges" (these are for undirected graphs) as "nodes" and "arcs" (these are the corresponding terms for directed graphs.)

Definition. A **directed graph D** is an ordered pair $D = (N, A)$ consisting of a set N of **nodes** and a multiset A of **arcs**, where each arc is an ordered pair of nodes.

In contrast, in an undirected graph, an edge is an unordered pair of (or a set of two) vertices.



Here,

$$N = \{1,2,3,4\}$$
$$A = \{(1,3), (2,1), (2,1), (2,2), (3,1), (3,2)\}$$

Notation. I'll drop the parentheses as a shorthand, renaming the arc $a = (u, v)$ as $a = uv$.

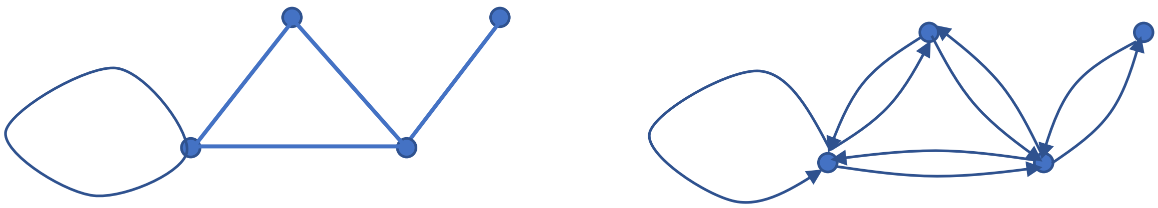
Definition. Two arcs $a = uw$ and $b = uw$ are said to be **parallel** while two arcs $a = uw$ and $b = wu$ are said to be **anti-parallel** or **opposite**.

Observation. Loops (arcs of the form $a = uu$) are their own opposite arcs.

Here, we have an undirected graph G and a directed graph D that is an **orientation** of G. You obtain an orientation of an undirected graph by choosing for each edge one of the two possible ordered pairs of its endpoints to be an arc.

We call G the **underlying graph** of D. You obtain the underlying graph by removing the arc orientations, turning directed pairs into undirected pairs.

Another common way to transform undirected graphs into directed graphs, maintaining the structure, is to replace every non-loop $e = xy$ with two opposite arcs $a = xy$ and $b = yx$; loops are replaced by themselves.



Terms such as subgraph and induced subgraph carry over directly to directed graphs, except to use the terms “node” and “arc” rather than “vertex” and “edge” in the definitions.

More terms that carry over: If $a = xy$, then x and y are **adjacent nodes**, the arc a is **incident** with x and y; x and y are still called **endpoints** or **end nodes** of a.

To capture the direction of arc $a = xy$ in terms of its endpoints, we have the following refinements:

Definitions. Each of the following is equivalent to $a = xy \in A$:

- x is the **tail** and y is the **head** of the arc a
- x **precedes** y
- y **follows** x
- a **goes from** x **to** y
- a goes **out of** x and **into** y

Moving on to the concept of “degree” in digraphs, we want to capture how often a node occurs as the tail of an arc and how often it occurs as the head of an arc.

Definitions.

- The **in-degree** of a node $u \in N$ is the number of arcs a having u as the head of the arc. This is denoted $d_{\text{in}}(u)$.
- The **out-degree** of a node $u \in N$ is the number of arcs a having u as the tail of the arc. This is denoted $d_{\text{out}}(u)$.
- The **total in-degree** is the sum of the in-degrees of all nodes.
- The **total out-degree** is the sum of the out-degrees of all nodes.
- The **minimum in-degree** is the minimum among all in-degrees of the nodes. This is denoted δ_{in}
- The **maximum in-degree** is the maximum among all in-degrees of the nodes. This is denoted Δ_{in}
- Similarly for minimum and maximum out-degrees.

There is a hand-shaking lemma for digraphs:

Observation. Every arc contributes exactly 1 to the total in-degree and the total out-degree. Therefore, these total degrees are both equal to the number of arcs; hence, they are equal to each other.