Homework observations:

Question 1:

Observation. If G is bipartite and $W: v_0, e_1, v_1, e_2, v_2, \dots, v_{k-1}, e_k, v_k$ is a walk in G, then all of the even-indexed vertices are in one part and all of the odd-indexed vertices are in the other part.

Observation. If G is bipartite and connected, then for any vertex $v \in V$, the two sets

$$V_{\text{even}} = \{w \in V: D(v,w) \text{ is even}\} \text{ and } V_{\text{odd}} = \{w \in V: D(v,w) \text{ is odd}\}$$

forms a bipartition for G.

Observation. Suppose G is bipartite and connected. Two vertices in the same part iff they are an even distance from each other.

Question 2. The cycles 12341 and 14321 should be listed as different cycles, even though they contain the same vertices and edges. Recall that a cycle is an alternating sequence of vertices and edges, so different sequences are different cycles.

Question 4. One approach is to let C be a cycle that contains both u and v. Suppose C does not contain e_1. Then C is still a cycle in $G - e_1$. Now consider the effect of deleting e_2 as well. Can u and v suddenly become disconnected by deleting one edge of the cycle C?

Notice that e_2 is a bridge of the graph $G - e_1$. This means e_2 is not on C and so u, v are still connected after removing e_1 and e_2.

Theorem. If G is connected and $e = st$ is a bridge, then in the graph $G - e$, for every vertex $v \in V$, either v is connected to s or v is connected to t.

Proof. Consider a v, s-path P in G,

$$P: v = v_0, e_1, v_1, \dots, v_{k-1}, e_k, v_k = s.$$

If e is not on P, then P is a path in $G - e$ and v is connected to s in $G - e$. If e is on P, it must be that $e = e_k$ and $v_{k-1} = t$. Then

$$Q: v = v_0, e_1, v_1, \dots, v_{k-2}, e_{k-1}, v_{k-1} = t$$

is a v, t-path in $G - e$. Hence, v is connected to t in $G - e$.

Informally, if G is connected, then every vertex is still connected to one of the endpoints of a removed bridge.

Corollary. Removing a bridge increases the number of components by exactly one.

Theorem. Given a graph G and an edge e of G, $e = st$ is on a cycle if and only if e is not a bridge.

Proof. For the forward direction, let

$$C: s = v_0, e_1, v_1, \ldots, v_{k-1}, e_k, v_k = t, e, s$$

be a cycle containing e. If e were a bridge, then its endpoints would become disconnected after deleting e. But, we still have an s, t-walk after removing e:

$$W: s = v_0, e_1, v_1, \ldots, v_{k-1}, e_k, v_k = t.$$

Hence, the endpoints of e remain connected and so e is not a bridge.

For the reverse direction, assume $e = st$ is not a bridge. Then there exists an s, t-path not containing e:

$$P: s = v_0, e_1, v_1, \ldots, v_{k-1}, e_k, v_k = t.$$

But then

$$C: s = v_0, e_1, v_1, \ldots, v_{k-1}, e_k, v_k = t, e, s$$

is a cycle containing e, as desired.

Question to ponder: What happens to a connected graph after you delete all of the non-bridges?

Section II.B. 4. “Cut” vertices and blocks.

What happens to the numbers of components when one deletes vertices? The answer can be highly variable.

Definition. Given a graph G, a vertex v is called a **cut vertex** if $G - v$ has more components than G.

Deleting a cut vertex is not guaranteed to increase the number of components by exactly one.

Definitions.

- A graph is **separable** if it is not connected or it has at least one cut vertex. Otherwise, the graph is **inseparable.**
- If $H \leq G$ is inseparable and there is no other inseparable subgraph K where $H < K \leq G$, then H is called a **block** of G.

Observations.

- An inseparable component is a block.

Definition. A **cactus** is a graph such that every edge is on at most one cycle.

A **Dutch windmill** is a cactus where all of the cycles share a single common vertex.

Question. Is a tree (connected graph with no cycles) a cactus?

In a tree, every edge is on zero cycles. Hence, every edge is on at most one cycle. Hence, trees are cacti.