Today, we'll continue looking at "Hamiltonian" graphs, i.e., graphs that have Hamilton cycles. Specifically, we'll look at some non-Hamiltonian graph examples.

Definition. Given a graph G, a **Hamilton cycle** is a cycle subgraph that contains every vertex of G.

Definition. Given a graph G, a **Hamilton path** is a path subgraph that contains every vertex of G.

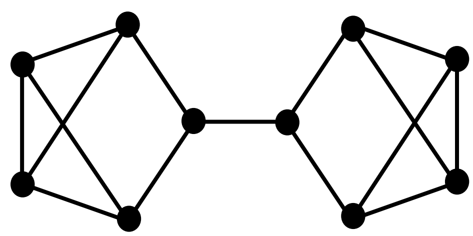
Definition. A graph G is **Hamiltonian** if it has a Hamilton cycle.

Last time, we saw that if G admits a Hamilton path, then $G \times K_2$ is Hamiltonian.

Intuitively, say the Hamilton path in G starts at u and ends at v. One traces the Hamilton path in G from u to v, then traverses the edge from v to v', the corresponding vertex in the copy of G, called G', then tracing the Hamilton path in G' in reverse (from v' to u') and finally traversing the edge from u' to u to close the Hamilton cycle in $G \times K_2$.

We mention in passing that being Hamiltonian is a more difficult property to assess in a graph than being Eulerian. The property of being Eulerian boils down to determining the degrees of the vertices, and judging whether the graph is connected (disregarding isolated vertices).

Question. Is the following graph Hamiltonian?

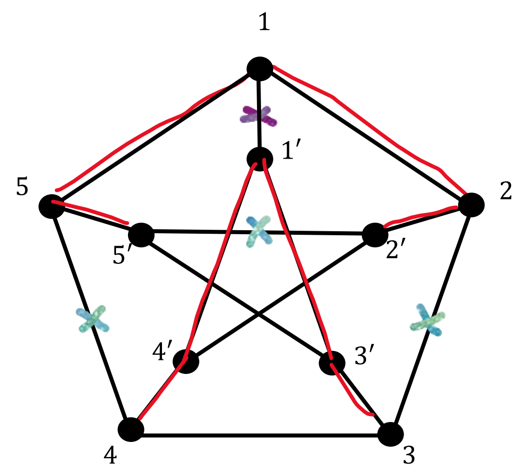


Proposition. If G has a bridge, then G is not Hamiltonian.

Proof. Suppose G has a bridge and is Hamiltonian. Let $e = uv$ be a bridge. From previous work, every u, v-walk must contain e. Any Hamilton cycle C would contain a u, v-walk and hence C would contain e. But cycles cannot contain bridges and so this leads to a contradiction. Therefore, G cannot both have a bridge and be Hamiltonian at the same time.

How about the converse? The converse is actually false. It is possible for a bridgeless graph to fail to be Hamiltonian.

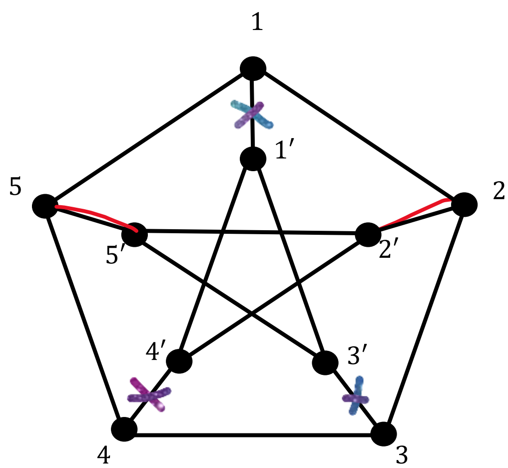
Example. The Petersen graph is not Hamiltonian. This is not immediately obvious.

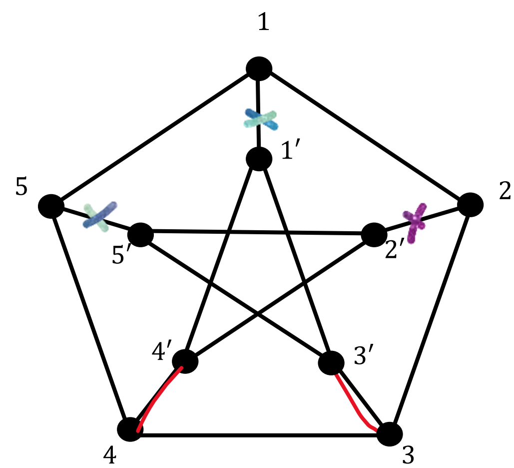


Sketch of proof. Suppose the Petersen graph is Hamiltonian. We focus on the edges 11’, 22’, 33’, 44’, 55’; let’s call these “radial edges.” Notice that any Hamilton cycle must use at least one radial edge and the number of radial edges used must be even. Hence, we must use 2 or 4 radial edges.

Case 1. We use four radial edges. This means a single radial edge is not used; without loss of generality, suppose edge 11’ is unused. This forces edges 12 and 15 to belong to our Hamilton cycle. Edges 1’3’ and 1’4’ must be used because two of the three edges incident on 1’ must be used. This then forces edges 23 and 45 not to be used. Notice that 34 cannot be used since it would close a 5-cycle (leaving us of a Hamilton cycle), but 34 must be used since otherwise vertices 3 and 4 would be degree-1 vertices in the cycle. We have a contradiction, finishing case 1, i.e., no Hamilton cycle can use four radial edges.

Case 2. We use two radial edges. There are two sub-cases to consider, depending on whether the radial edges differ by 1 or 2 in their indices. These are the sub-cases up to rotation in the graph.





Case 2a. We must use 12, 15, 45, 34, 23 in order for vertices 1, 3, 4 to have degree 2 in the Hamilton cycle. All of these edges close the outer pentagon.

Case 2b. Concentrating on the inner star, we're forced to use all five edges because every edge incident with 1', 2', or 5' must be used.

Corollary. Not every cubic bridgeless connected graph is Hamiltonian.

Proof. Pete (the Petersen graph) is a counterexample.

Section II.D. Graph coloring, specifically vertex coloring and edge coloring.

The main game is to assign as few colors as possible to the vertices so that no two adjacent vertices are assigned the same color. For bipartite graphs, two colors will suffice. We color the vertices in one part “red” and the vertices in the other part “blue.”