We'll continue looking at adjacency matrices and then incidence matrices today.

Definition. If G has vertex set $V = \{v_1, v_2, \dots, v_n\}$, then its **adjacency matrix** is the $n \times n$ square matrix A whose entries are

$$A_{ij} = \text{the number of edges joining } v_i \text{ and } v_j.$$

Here, A_{ij} is the entry in the i^{th} row and j^{th} column.

Notice that this matrix A is **symmetric**, i.e., for all i and j, $A_{ij} = A_{ji}$.

For reference concerning **matrix multiplication**, if M is an $r \times k$ matrix and P is a $k \times c$ matrix, then MP is an $r \times c$ matrix with entries

$$(MP)_{ij} = \sum_{q=1}^{k} M_{iq} P_{qj}.$$

For reference concerning **matrix sums**, if M and P are matrices of the same shape, then

$$(M + P)_{ij} = M_{ij} + P_{ij}.$$

The "matrix power" A^p is defined recursively:

$$\begin{aligned} &A^0 = I, \text{the identity matrix;} \\ &A^1 = A; \\ &A^{p+1} = A^p A, \text{for } p \geq 1. \end{aligned}$$

Notation. Let $f_p(i, j)$ be the number of walks in G from v_i to v_j.

Theorem. Given a graph G with adjacency matrix A, for every $k \geq 0$,

$$f_k(i, j) = \left(A^k\right)_{ij}.$$

Proof. We use mathematical induction on k. The base cases are $k = 0$ and $k = 1$. When $k = 0$, we have

$$f_0(i, j) = \begin{cases} 1, i = j \\ 0, i \neq j \end{cases} = I_{ij}.$$

When $k = 1$, the result follows by the definition of the adjacency matrix A. We observe

$$f_1(i, j) = A_{ij} = (A^1)_{ij}.$$

For the inductive step, suppose the result holds for $k = p$. Consider counting the v_i, v_j-walks of length $p + 1$. Each of these walks can be constructed uniquely using the following steps:

1. Pick a vertex v_a
2. Construct a v_i, v_a-walk of length exactly p
3. Construct a v_a, v_j-walk of length exactly 1

The total number of our walks is

$$f_{p+1}(i,j) = \sum_{1\le a\le n} f_p(i,a)\ f_1(a,j) = \sum_{1\le a\le n} (A^p)_{ia}\ A_{aj} = (A^pA)_{ij} = (A^{p+1})_{ij},$$

which establishes the result.

This theorem holds for graphs with loops if each loop contributes 1 to the corresponding diagonal entry of the adjacency matrix.

For digraphs, the most common way to define adjacency matrices is to define A_{ij} to be the number of arcs that start at v_i and end at v_j. These adjacency matrices are generally not symmetric. The above theorem does hold for directed v_i, v_j-walks of length k.

Example. Let's look at C_5:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix};\ A^2 = \begin{bmatrix} 2 & 0 & 1 & 1 & 0 \\ 0 & 2 & 0 & 1 & 1 \\ 1 & 0 & 2 & 0 & 1 \\ 1 & 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 & 2 \end{bmatrix}$$

Using the adjacency matrix as a connectivity detector.

Corollary. If there exists a value p such that $(A^p)_{ij} > 0$, then vertices v_i and v_j are connected.

Corollary. If G has n vertices, then

$$(I + A + A^2 + \cdots + A^{n-1})_{ij} > 0$$

if and only if v_i and v_j are connected.

A very common and easily formed matrix: The degree matrix D (this D is not a digraph) has entries

$$D_{ij} = \begin{cases} \deg(v_i), & i = j \\ 0, & i \neq j \end{cases}$$

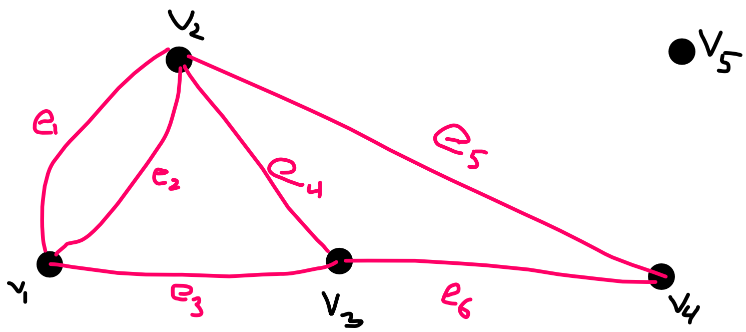
This matrix is a diagonal matrix, i.e., the only nonzero entries would be along the diagonal.

Incidence matrices.

Definition. Given a loopless graph G with vertex set $V = \{v_1, v_2, \dots, v_n\}$ and edge multiset $E = \{e_1, e_2, \dots, e_m\}$, its **incidence matrix** B is an $n \times m$ matrix whose entries are given by

$$B_{ij} = \begin{cases} 1, \text{ if } \ v_i \text{ is an endpoint of } e_j; \\ 0, \qquad\qquad\qquad \text{otherwise.} \end{cases}$$

The rows correspond to vertices while the columns correspond to edges.



For the graph above, its incidence matrix is

$$B = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Observations.

- Every column sums to 2.
- Every row sums to the degree of the corresponding vertex.
- The sum of all of the entries is the total degree.
- The sum of all of the entries is twice the number of edges.
- An all-zero row of the incidence matrix corresponds to an isolated vertex.
- Two graphs are isomorphic if and only if the incidence matrix of one graph can be obtained by permuting the columns and/or rows of the incidence matrix of the other.
- The matrix product BB^T yields the matrix $A + D$.