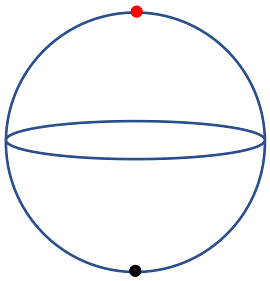
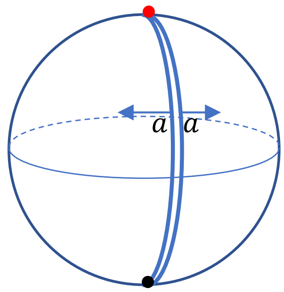
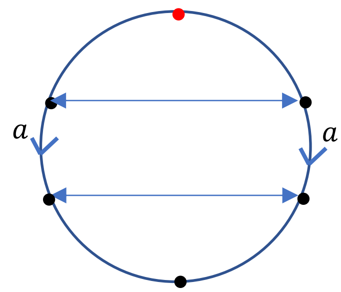
Today, I want to look at more interesting surfaces embedded in $\mathbb{R}^3$ (the set of triples of real numbers, i.e., "standard three-dimensional space"). We'll look at "surgical techniques" in an effort to better categorize these surfaces, via symbolic algebra.

I anticipate putting together Assignment 7 for release on Monday, Nov. 2 and due on Monday, Nov. 9.







Suppose we start with a sphere ("north pole" = red point, "south pole" = black point.)

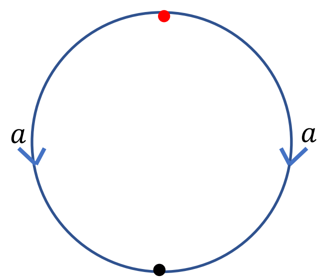
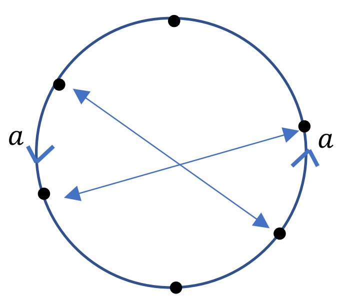


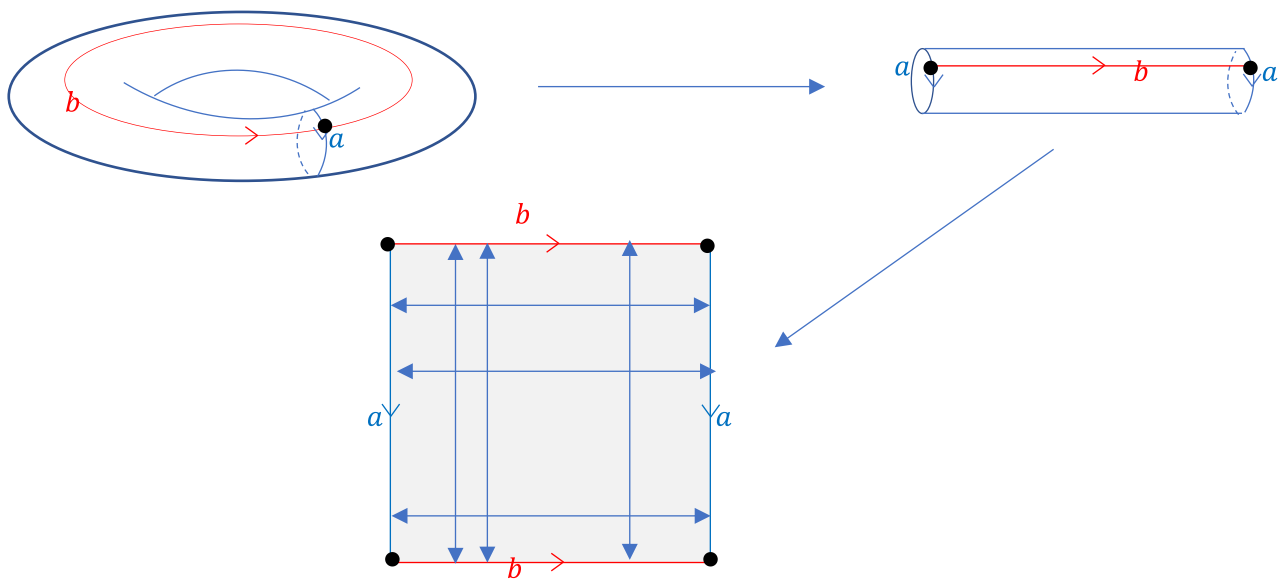
Diagram representing a sphere. Algebraically, if we traverse the boundary of this diagram, keeping track of which direction we follow each arc, we obtain an algebraic expression. Here, our algebraic expression would be aa^{-1}. It turns out that one could use $a^{-1}a$ for this surface as well.

In contrast, if we identify opposite points in this diagram, we get the following diagram:



One observation in this new diagram: The north and south poles are identified here; they are actually the same point on this surface. This surface is not a sphere. This is called a "projective plane." This surface is non-orientable. An algebraic expression for the projective plane is aa, obtained by traversing in counterclockwise direction. Notice that we could just as well have $a^{-1}a^{-1}$, by traversing in a clockwise direction.

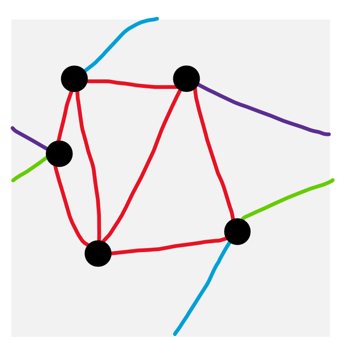
Now consider "surgery" on a torus:



Here, after slicing the torus along "a" and then "b", we obtain the representation that is the "piece of paper". Notice that the corners of this paper represent the same point of the torus. An algebraic representation: Starting at the upper left and proceeding counterclockwise, we obtain the string $aba^{-1}b^{-1}$. There are others: $ba^{-1}b^{-1}a$, etc.

Anyone who has played the video game Asteroids has played on a torus. Any video game with "double wraparound" is played on a torus.

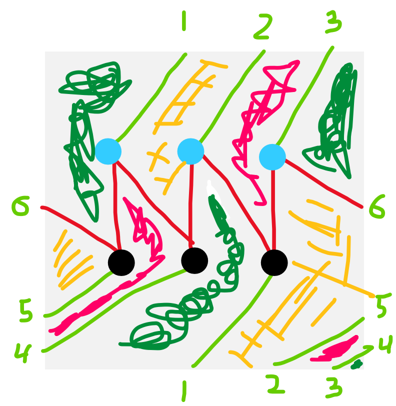
Drawing a graph on a torus:



No edges cross. This is a drawing of K_5 on a torus without edges crossing. In fact, one can draw K_7 on a torus without edges crossing. Here, $n = 5, m = 10, r = 5$: Euler:

$$n - m + r = 5 - 10 + 5 = 0.$$

Here is $K_{3,3}$:



What does Euler say? We have $n = 6, m = 9, r = 3$. From Euler:

$$n - m + r = 6 - 9 + 3 = 0.$$

It turns out that if G is drawn on a torus S such that $S - G$ is a set of regions, then

$$n - m + r = 0.$$

If we draw a tree on a torus,

$$n - (n - 1) + r = 0$$

$$r + 1 = 0$$

$$r = -1$$

and this doesn't make sense. The problem is that $S - G$ is not a region when G is a tree.