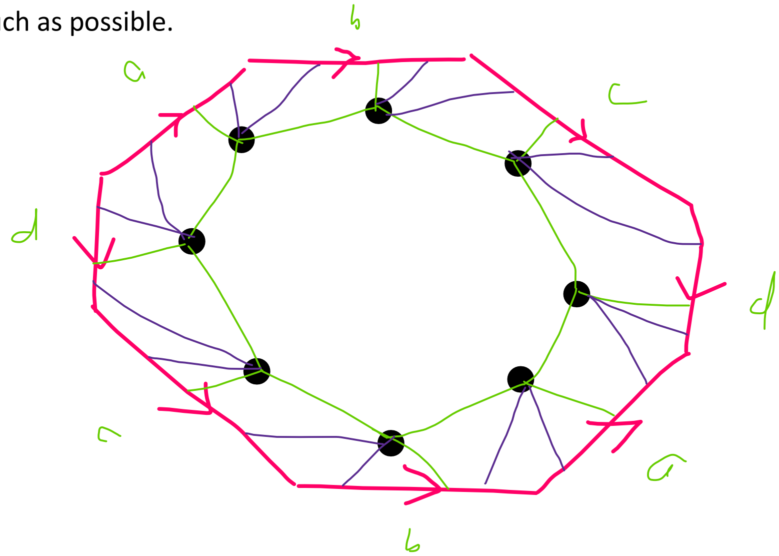
Today, we'll finish partially ordered sets and digraphs and then move on to linear algebra topics, starting with special matrices typically used in graph theory.

For the bonus question, I recommend drawing the boundary arcs using the recipe $abcda^{-1}b^{-1}c^{-1}d^{-1}$, try to avoid using the interior of the octagon until absolutely necessary. Finally, I recommend exploiting the symmetry as much as possible.



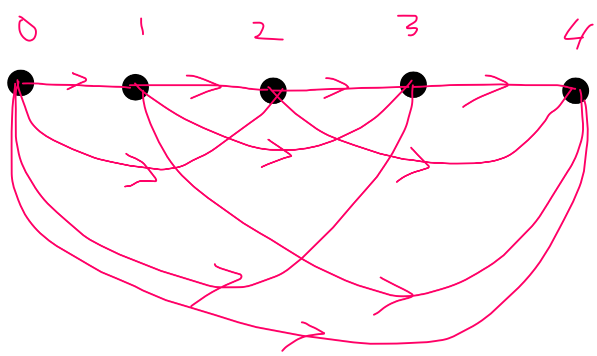
Recall that a **partial order** $\leq$ is a relation that is reflexive, anti-symmetric, and transitive.

Also, a **partially ordered set** (**poset**) is a set S equipped with a partial order $\leq$.

We can model a poset as a directed graph in the following manner:

Given a poset $S, \leq$, we define the directed graph D whose node set is S and where uv is an arc if and only if $u \leq v$ and $u \neq v$.

For instance, if $S = \{0,1,2,3,4\}$ and we use the standard less-than-or-equal-to partial order $\leq$, the digraph D looks like



Definition. A digraph that models a poset is called a **poset digraph**.

Observations. In a poset digraph,

- There are no loops.
- There is at most one arc joining any two given nodes.
- If $u \neq v$ and there is a directed u, v-path, then uv is an arc.

Definition. The sub-digraph induced by a directed u, v-path is called a **chain digraph**. This models all of the order statements implied by

$$u \leq w_1 \leq w_2 \leq w_3 \leq \cdots \leq w_{k-1} \leq v.$$

Observation. Chains are acyclic tournaments among the nodes that are involved.

Definition. If $S' \subseteq S$ has the property that every two elements of S' are comparable, then S' is called a **chain.**

Definition. If S itself is a chain (if every two elements are comparable), then S is **totally ordered under $\leq$.**

Definition. If $S' \subseteq S$ has the property that every two distinct elements of S' are incomparable, then S' is called an **antichain**. A sub-digraph induced by an antichain has no arcs.

For instance, the subset relation on a finite set $\{a, b, c, d\}$ has $\{\ \}, \{a\}, \{a, b\}, \{a, b, c\}, \{a, b, c, d\}$ as a chain and $\{a\}, \{b\}, \{c\}, \{d\}$ as an antichain.

Theorem. If D is a poset digraph, then D has no directed cycles.

Proof. Let C be a directed cycle in D. Suppose u and v are arbitrarily chosen nodes on C. Then there exists a directed u, v-path and a directed v, u-path. By transitivity, uv and vu are both arcs; hence, $u \leq v$ and $v \leq u$. This implies $u = v$ and so every node of C is equal. Hence, C is a loop. Since D has no loops, C does not exist.

Theorem. Let $D = (N, A)$ be a digraph without directed cycles. Let $\leq$ be the relation on N such that $u \leq v$ if there exists a u, v-directed path. Then N is a poset with partial order $\leq$.

Proof of transitivity. Suppose $u \leq v$ and $v \leq w$. These mean there exist u, v- and v, w- directed paths. Concatenating these paths results in u, w- directed walk. So there exist u, w-directed walks. The shortest such u, w-walk must be a u, w-directed path. Hence, $u \leq w$ by definition.

Definition. If $D = (N, A)$ is any digraph then its **transitive closure** $\widetilde{D}$ has the same node set N and for nodes u and w, if uw is an arc of D or there is a node v such that uv and vw are arcs of $\widetilde{D}$, then uw is an arc of $\widetilde{D}$. The effect is that $\widetilde{D}$ is the smallest supergraph that contains D and that is transitive.

Theorem. A digraph is a poset digraph if and only it is a transitively closed digraph and it has no directed cycles.

Let D be any digraph and let $C_1, C_2, C_3, \ldots, C_k$ be its directed components. We construct a digraph D' whose node set is $N' = \{C_1, C_2, C_3, \ldots, C_k\}$ and where for $i \neq j$, an arc is drawn from C_i to C_j if and only if there is a u, v-directed path for some $u \in C_i$ and $v \in C_j$.

Theorem. The digraph D' constructed above is a poset graph.

Suppose there was a directed cycle $C_1 \to C_2 \to C_3 \to C_1$. This would cause their union along with arcs between them to be one large directed component. This violates the conditions that each of C_1, C_2, C_3 is a directed component in its own right.

Corollary. If T is a tournament, then T' is a chain graph.

On Monday, expect to start reviewing linear algebra.