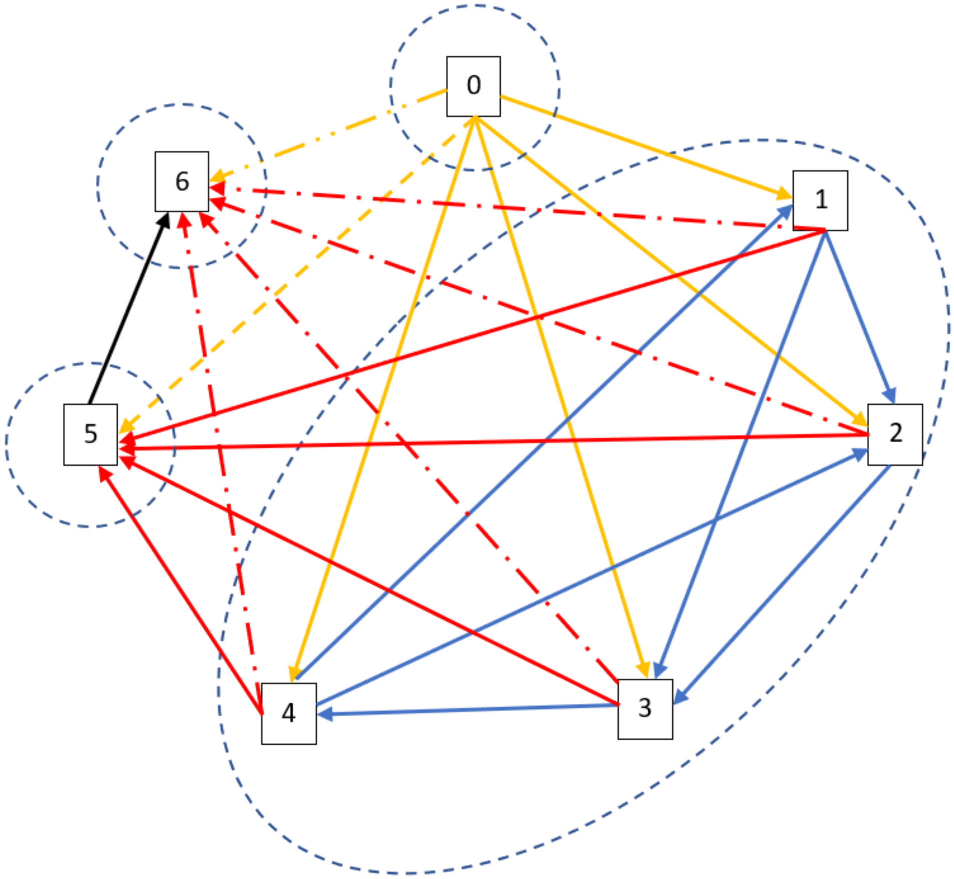
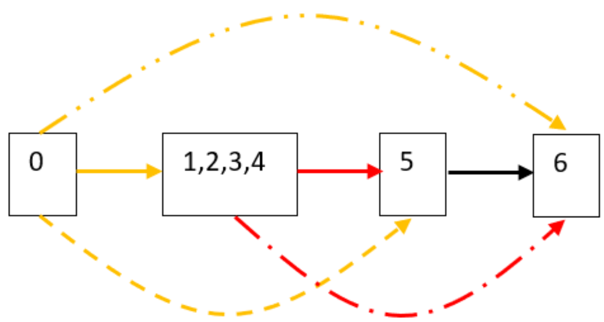
Example of tournament with directed cycles.



Chain of directed components:



Transitioning into linear algebra portion.

Largest topic involves matrices.

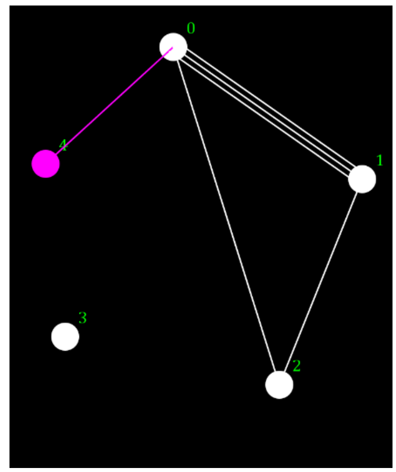
Adjacency matrices.

Definition. Given an undirected graph G (with no loops), we form its **adjacency matrix** A, an $n \times n$ matrix whose entries are indexed on the vertices. Let's suppose $V = \{v_1, v_2, \dots, v_n\}$ is the vertex set. The entries of A are given by

$$A_{ij} = \text{the number of edges joining } v_i \text{ to } v_j.$$

Here, A_{ij} represents the entry in row i and column j. The rows of a matrix are horizontal while the columns of a matrix are vertical.

In the example drawn, we'll index the vertices $\{v_0, v_1, v_2, v_3, v_4\}$:



$$A = \begin{bmatrix} 0 & 3 & 1 & 0 & 1 \\ 3 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

A^T

Notice that v_3 (represented by the fourth row and column of this matrix) has all zero entries in its row and column.

Recall that given a matrix M, its transpose is written M^T and has the property that

$$(M^T)_{ij} = M_{ji}.$$

Observations.

- The adjacency matrix is symmetric. In linear algebraic terms, $A = A^T$.
- If you sum the entries in the row or column corresponding to v_i, you get the degree of v_i.
- The entry A_{ij} represents the number of walks of length 1 from v_i to v_j.
- All of the entries of nonnegative.
- The diagonal entries are zero, by definition.

Recall that if you have two $n \times n$ matrices A and B, their matrix product AB is also $n \times n$. The entries of AB are defined by the following formula:

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}.$$

Here, think of this as computing "dot products" of rows of A with columns of B; the entry $(AB)_{ij}$ is the dot product of the i^{th} row of A and the j^{th} column of B. It turns out that this matrix product has the effect of composing the underlying linear transformations.

Proposition. The number of walks of length k from v_i to v_j is

$$\left(A^k\right)_{ij}.$$

Preliminary cases: When $k = 1$, this is the definition of the adjacency matrix $A = A^1$.

When $k = 0$, we have $A^0 = I$, the identity matrix. To make this fit our proposition, let's examine

$$(A^0)_{ij} = I_{ij} = \begin{cases} 1, \text{if } i = j \\ 0, \text{if } i \neq j \end{cases}$$

As an example, the 5×5 identity matrix looks like

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

How many walks of length 0 are there from v_i to v_i? There is exactly 1, the trivial walk.

How many walks of length 0 are there from v_i to v_j with $i \neq j$? There are 0.

Going back to the example matrix,

$$A = \begin{bmatrix} 0 & 3 & 1 & 0 & 1 \\ 3 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A^2 = AA = \begin{bmatrix} 11 & & & & \\ & & & & \\ & & & & \end{bmatrix}$$