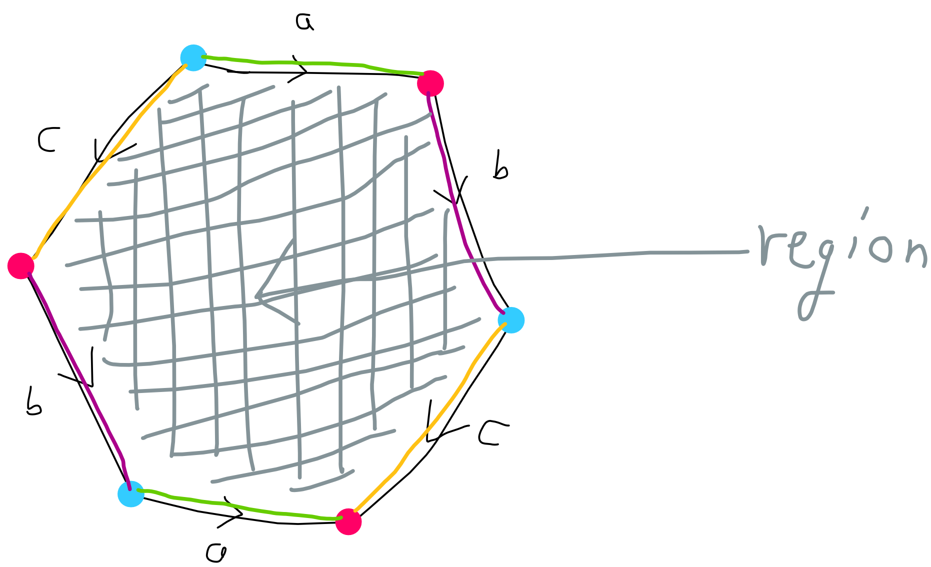


We claim that this diagram-coded by $abca^{-1}b^{-1}c^{-1}$ represents a 1-holed torus. Let's resolve what

$$n - m + r$$

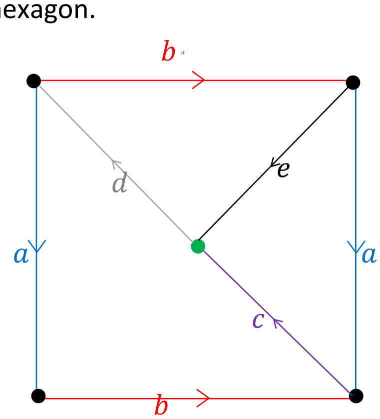
is going to be. We'll use $n = 2$ here, using the pink and blue points as my only vertices. We draw one edge for each letter used to label the figure. Here, there are three letters, so $m = 3$. By surrounding the interior of the hexagon, we form a single region. This graph satisfies $r = 1$.

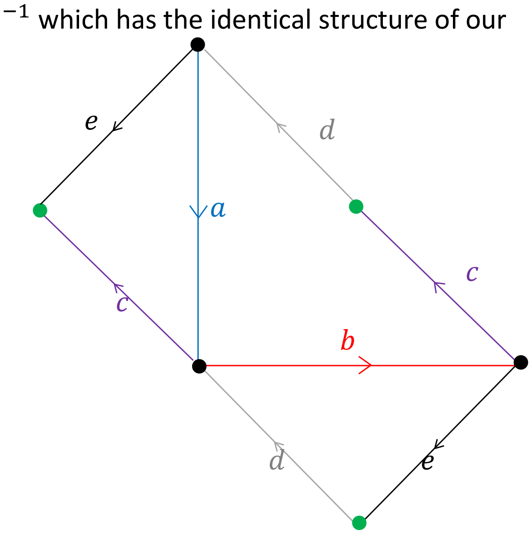
Hence,

$$n - m + r = 2 - 3 + 1 = 0 = 2 - 2(1); g = 1$$

this matches the right-hand side of Euler's formula for 1-holed tori.

Notice that traversing the right-hand figure in counter-clockwise direction starting at the far-right corner, we obtain the algebraic expression $cdec^{-1}d^{-1}e^{-1}$ which has the identical structure of our hexagon.





The Euler Characteristic for orientable surfaces is the right-hand side of this equation

$$n - m + r = 2 - 2g$$

where g (called the **genus**) corresponds to the "number of holes" in a g-holed torus.

Proposition. Each of the following algebraic expressions represents the n-holed torus:

$$a_1 b_1 a_1^{-1} b_1^{-1}\, a_2 b_2 a_2^{-1} b_2^{-1} \cdots a_n b_n a_n^{-1} b_n^{-1},$$

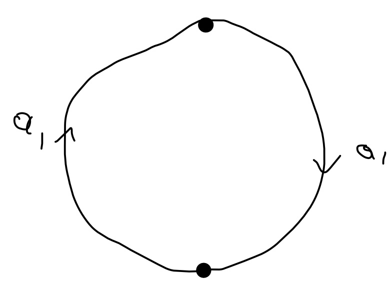
$$a_1 a_2 \cdots a_{2n} a_1^{-1} a_2^{-1} \cdots a_{2n}^{-1},$$
$$a_1 a_2 \cdots a_{2n} a_{2n+1} a_1^{-1} a_2^{-1} \cdots a_{2n}^{-1} a_{2n+1}^{-1}.$$

The first two have boundaries whose corners are all the same vertex and the last has a boundary whose corners represent two vertices.

Proposition. Every orientable surface is homeomorphic (continuously deformable) to an n-holed torus.

For non-orientable surfaces, we have the following:

Proposition. The algebraic expression $a_1 a_1 a_2 a_2 a_3 a_3 \ldots a_n a_n$ represents the boundary of a non-orientable surface. For instance, the projective plane is the $n = 1$ case.



Notice that we have $n = 1, m = 1, r = 1$ and so

$$n - m + r = 1,$$

which happens to be odd; if the answer is odd, your surface is non-orientable.

Proposition. Classification of Surfaces Theorem. Every closed and bounded surface embedded in $\mathbb{R}^3$ has a boundary that is algebraically some concatenation of strings of n-holed tori and non-orientable surfaces listed in the above propositions.

Here, we have $a_1 a_1 a_2 a_2$; $n = 1, m = 2, r = 1$ and so $n - m + r = 0$. This has the same Euler characteristic as a 1-holed torus, but it is *not* a 1-holed torus.