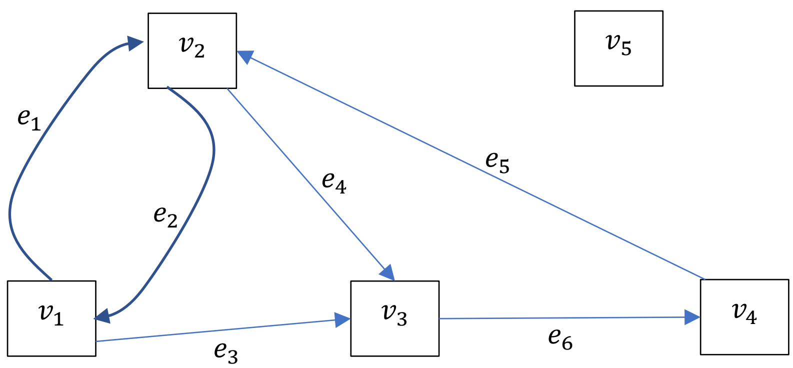
Today, I want to look at incidence matrices of digraphs, leading to **Laplacian** (why this adjective?) matrices.



After break, we'll look at eigenvalues and eigenvectors, so brace yourselves.

In the above example, we'll call G' will be the above digraph and G will be the underlying graph (without the orientations.)

In this example above,

$$B' = \begin{bmatrix} 1 & -1 & 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix};$$

$$L = B'\,B'^T = \begin{bmatrix} 3 & -2 & -1 & 0 & 0 \\ -2 & 4 & -1 & -1 & 0 \\ -1 & -1 & 3 & -1 & 0 \\ 0 & -1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

For digraphs, the incidence matrix B' will have entries

$$B'_{ij} = \begin{cases} 1, \text{if arc } e_j \text{ leaves node } v_i \\ -1, \text{if arc } e_j \text{ enters node } v_i \\ 0, \quad \text{otherwise} \end{cases}$$

Observations about B' incidence matrices.

- Every column sums to zero
- The row sums are $d_{\text{out}}(v_i) - d_{\text{in}}(v_i)$ where v_i is the node corresponding to that row
- The dot-product of a row with itself is the degree of the corresponding node, treated as a vertex of the underlying graph.
- The dot-product of rows i and j with $i \neq j$ is negative the number of arcs joining nodes v_i and v_j.

No matter what the orientation of G' is,

$$B'\,B'^T = D - A = L$$

where D is the degree matrix and A is the adjacency matrix of the underlying graph G.

Definition. Given a graph G with no loops, the matrix $L = D - A$ is its corresponding **Laplacian matrix**.

You can derive L by computing $B'B'^T$ where B' is the incidence matrix for <u>any</u> orientation to produce a digraph G'.

Looking at the example

$$L = B'\,B'^T = \begin{bmatrix} 3 & -2 & -1 & 0 & 0 \\ -2 & 4 & -1 & -1 & 0 \\ -1 & -1 & 3 & -1 & 0 \\ 0 & -1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Consider the matrix-vector product

$$\begin{bmatrix} 3 & -2 & -1 & 0 & 0 \\ -2 & 4 & -1 & -1 & 0 \\ -1 & -1 & 3 & -1 & 0 \\ 0 & -1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 0\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Observations.

- The row sums of the Laplacian matrix are all zero; each row's diagonal element is the degree, i.e., the total number of edges incident with that vertex. Each of the other entries subtracts the number of edges between our row-corresponding vertex and the vertex corresponding to the column. This is very important, theoretically.
- Notice that the above observation boils down to the equation $L\,\mathbf{1} = \mathbf{0} = 0\,\mathbf{1}$, where $\mathbf{1}$ represents the all-ones vector and $\mathbf{0}$ represents the all-zero vector.
- The above observation tells us that the all-ones vector $\mathbf{1}$ is actually an eigenvector of the Laplacian matrix, with corresponding eigenvalue 0.
- The Laplacian matrix is not invertible (singular).
- From linear algebra, since L is symmetric, it will have a collection of eigenvectors that can serve as an orthogonal basis for $\mathbb{R}^n$.

Consider "breaking the incidence matrix B' into single columns":

$$
\begin{aligned}
B' &= \begin{bmatrix} 1 & -1 & 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \cdots + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\
&= N_1 + N_2 + \cdots + N_6
\end{aligned}
$$

$$
\begin{aligned}
B'B'^T &= (N_1 + N_2 + \cdots + N_6)(N_1 + N_2 + \cdots + N_6)^T \\
&= (N_1 + N_2 + \cdots + N_6)(N_1^T + N_2^T + \cdots + N_6^T) \\
&= N_1N_1^T + N_1N_2^T + \cdots + N_6N_6^T \\
&= N_1N_1^T + N_2N_2^T + N_3N_3^T + \cdots + N_6N_6^T
\end{aligned}
$$

Let's look at $N_1N_2^T$:

$$
\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = O, \text{the all zero matrix}
$$

More generally, if $i \neq j$, then $N_iN_j^T = O$.

The upshot is that L, the Laplacian matrix, is the sum of the Laplacians of the individual arcs.

As a consequence, we will show that $L = D - A$ satisfies the property for any vector $\mathbf{x}$,

$$\mathbf{x}^T L \mathbf{x} \geq 0.$$

This is the property called "positive semi-definite." It turns out that L is positive semi-definite.

Consequence to all of this nonsense: The multiplicity of $\lambda = 0$, the zero eigenvalue, of L is the number of components of G.