

Today, I want to look at Euler's formula relating  $n, m, r$ , the numbers of vertices, edges, and regions for spherical (and hence, planar) graphs.

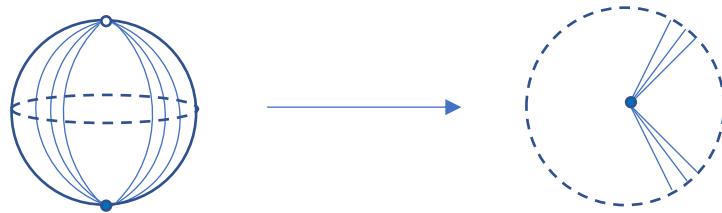
### Intuitive definitions.

A **region** is a set that is homeomorphic (continuously stretchable) with an open unit disk (filled in unit circle without the boundary).

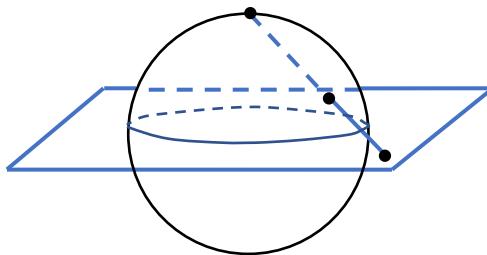
A graph is **spherical** if it can be drawn on the surface of a sphere without edges crossing.

A **punctured sphere** is a sphere without a single point (usually, one removes the “north pole” or “south pole.”)

**Observation.** The punctured sphere is a region.



**Observation.** The plane  $\mathbb{R}^2$  is homeomorphic to the punctured sphere, via **stereographic projection**.



**Proposition.** If a connected finite spherical graph  $G$  is drawn on the sphere  $S$  without edges crossing, then the set  $S - G$ , obtained by removing the vertices and edges from the sphere, is the disjoint union of a finite number of regions; this number is **invariant** and denoted by  $r$ .

**Proposition.** An edge is on the boundary of one or two regions. Furthermore, an edge is on the boundary of one region if and only if it is a bridge.

**Proposition.** A region bounded by non-bridges, exclusively, is bounded by the edges in a cycle.

### Observations.

- If  $G$  is a simple graph, then its “non-bridge exclusive” regions (covered by the third proposition) are bounded by cycles each with at least three edges.
- If  $G$  is simple and bipartite, then its “non-bridge exclusive” regions are bounded by cycles each with at least four edges.

**Corollary.** The graph  $T$  is a tree drawn on a sphere  $S$  without edges crossing if and only if  $S - T$  consists of a single region.

**Theorem (Euler's Formula).** For any connected spherical finite graph  $G$  drawn on a sphere without edges crossing,

$$n - m + r = 2.$$

*Proof.* We proceed by induction on  $r \geq 1$ , the number of regions. For the base case, suppose  $G$  is a graph where  $r = 1$ . By the above corollary,  $G$  is a tree. Hence,  $m = n - 1$  and

$$n - m + r = n - (n - 1) + 1 = 2,$$

establishing the base case.

For the inductive step, assume the result holds for  $r = k \geq 1$ , i.e., Euler's formula holds for any graph with exactly  $k$  regions. Let  $G$  be a spherical graph drawn with  $r = k + 1$  regions. Since  $r \geq 2$ ,  $G$  is not a tree and so it must have non-bridges. Let  $e$  be a non-bridge. As a non-bridge,  $e$  is on the boundary of two regions  $R_1, R_2$ . In the graph  $G' = G - e$ , these two regions are merged into one. More precisely,  $R_1 \cup R_2 \cup e$  is a single region for  $G - e$ . Let  $n', m', r'$  be the numbers of vertices, edges, and regions for  $G'$ . We observe:

$$n' = n, m' = m - 1, r' = r - 1.$$

This means

$$n = n', m = m' + 1, r = r' + 1$$

and so

$$\begin{aligned} n - m + r &= n' - (m' + 1) + (r' + 1) \\ &= n' - m' + r' \\ &= 2 \end{aligned}$$

because  $G'$  is a graph with  $r - 1 = k$  regions and so Euler's formula holds for  $G'$ . As a consequence, Euler's formula holds for  $G$  and this completes the inductive step and the proof.

As a physical example, consider the dodecahedron graph, with  $n = 20$  vertices,  $m = 30$  edges, and  $r = 12$  regions: Euler yields

$$n - m + r = 20 - 30 + 12 = 2$$

as advertised.

**Corollary.** If  $P$  is a convex polyhedron with  $n$  vertices,  $m$  edges, and  $r$  faces, then  $n - m + r = 2$ .

A shape is **convex** if for any two points on the shape, the line segment joining those points is entirely within the shape.