Laplacian matrices:

Recall that the Laplacian matrix L is defined by

$$L = D - A$$

where D is the diagonal matrix containing the degrees of the vertices and A is the adjacency matrix.

Also, if N is the incidence matrix of the graph, then

$$L = NN^T$$

Also, we showed that if N_e is the incidence matrix of a single edge e using the same vertices, then

$$L = \sum_{e \in E} (N_e N_e^T).$$

Now, consider an edge $e = ij$ and its Laplacian $L_e = N_e N_e^T$:

$$M = L_e = N_e N_e^T = \begin{bmatrix} 1 & & -1 & & \\ & & & & \\ -1 & & 1 & & \\ & & & & \\ & & & & \end{bmatrix}$$

For a single edge, you'll have 1's in entries M_{ii} and M_{jj} and you'll have -1 in entries M_{ij} and M_{ji} with zeros everywhere else.

Let $\mathbf{x} = [x_1, x_2, \ldots, x_n]^T$. What is

$$\begin{aligned}
\mathbf{x}^T L_e \mathbf{x} &= [x_1, x_2, \ldots, x_n] \begin{bmatrix} 1 & & -1 & & \\ & & & & \\ -1 & & 1 & & \\ & & & & \\ & & & & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \\
&= [x_1, x_2, \ldots, x_n] \begin{bmatrix} \vdots \\ x_i - x_j \\ \vdots \\ x_j - x_i \\ \vdots \end{bmatrix} \\
&= x_i(x_i - x_j) + x_j(x_j - x_i) \\
&= (x_i - x_j)^2
\end{aligned}$$

Therefore, for all $\mathbf{x} \in \mathbb{R}^n$

$$\mathbf{x}^T L\mathbf{x} = \mathbf{x}^T \left(\sum_{\substack{e \in E \\ e=ij}} (N_e N_e^T) \right) \mathbf{x} = \sum_{\substack{e \in E \\ e=ij}} (\mathbf{x}^T N_e N_e^T \mathbf{x}) = \sum_{\substack{e \in E \\ e=ij}} \left(x_i - x_j\right)^2$$

Consequences:

- $\mathbf{x}^T L\mathbf{x} \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$
- Hence, L is positive semi-definite
- Since L is symmetric, the eigenvalues are real numbers and there are n of them
- From being positive semi-definite, eigenvalues are nonnegative real numbers and so we can order them:

$$0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_n$$

When is $\lambda = 0$ an eigenvalue of L with eigenvector $\mathbf{x}$, i.e., when is

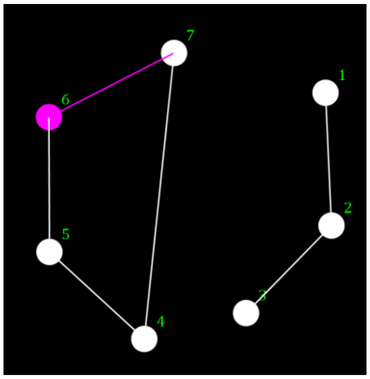
$$0 = \mathbf{x}^T L\mathbf{x} = \sum_{\substack{e \in E \\ e=ij}} \left(x_i - x_j\right)^2 \text{ ?}$$

For every edge $e = ij$, we must have $\left(x_i - x_j\right)^2 = 0$, forcing $x_i = x_j$. Hence, the endpoints of every edge must be assigned the same value if we use the entries of $\mathbf{x}$ as vertex labels.

Corollary. If $\mathbf{x} = \alpha\mathbf{1}$, then this happens.

If G has more than one component, then $0 = \mathbf{x}^T L\mathbf{x}$ is satisfied by setting the x_i equal in each component, independently. As a consequence, the multiplicity of $\lambda = 0$ is equal to the number of components of G.

In the example graph



The eigenvectors of $\lambda = 0$ of the Laplacian matrix take the form

$$a\begin{bmatrix}1\\1\\1\\0\\0\\0\\0\end{bmatrix} + b\begin{bmatrix}0\\0\\0\\1\\1\\1\\1\end{bmatrix}$$

and so the eigenspace would be two-dimensional, and $\lambda = 0$ would have multiplicity 2.

Let's try to draw the graph with the following Laplacian:

$$L = \begin{bmatrix} 2 & & & -1 & -1 & \\ & 3 & -1 & & -1 & -1 \\ & -1 & 4 & -1 & -1 & -1 \\ -1 & & -1 & 3 & & -1 \\ -1 & -1 & -1 & & 3 & \\ & -1 & -1 & -1 & & 3 \end{bmatrix}$$

Eigenvalues:

$$0 \leq \lambda_2 = 1.697 \leq \lambda_3 = 2.382 \leq \lambda_4 = 4.000 \leq 4.618 \leq 5.303$$

Corresponding eigenvectors:

$$\mathbf{x} = [0.781, -0.391, -0.237, 0.118, 0.118, -0.391]$$
$$\mathbf{y} = [0.000, -0.372, 0.000, 0.602, -0.602, 0.372]$$

$$\mathbf{z} = [0.408, 0.408, -0.408, -0.408, -0.408, 0.408]$$