Today, I want to start in on Section III.B. “Rooted Trees”

I did submit midterm grades – these are advisory only.

I plan on putting up Homework 6 on Friday. It’ll concentrate on trees.

Definition. A **rooted tree** $(\boldsymbol{T}, \boldsymbol{r})$ is a tree T with a distinguished vertex r called the **root**.

Convention. Given a rooted tree (T, r) , it is conventional to use r for the root vertex.

Common terminology, borrowed from genealogy:

Definitions. Given a rooted tree (T, r) and two <u>distinct</u> vertices u, w:

- If u is a vertex on the unique r, w-path, then u is an **ancestor** of w and w is a **descendant** of u.
- If u and w are adjacent as well, then u is <u>the</u> **parent** of w and w is <u>a</u> **child** of u. The root has no parents; the root has no ancestors.
- If u and w have the same parent, they are called **siblings**.
- If u is the parent of w’s parent, then u is the **grandparent** of w and w is a **grandchild** of u.
- (Less common). If u and w have a common grandparent, but not a common parent, then they are **cousins.**
- (Less common). If u is a sibling of w’s parent, then u is an **uncle (or aunt ; “pibling”?)** of w and w a **nephew (or niece; “nibling?”)** of u.

Observations.

- The root has no ancestors.
- The root is an ancestor of every non-root.
- Every non-root is a descendant of the root.
- The only non-child is the root.
- If T is nontrivial, then every non-parent is a leaf.
- It is possible for the root to have only one child, making it a vertex of degree 1, and so it would be a leaf by the established definition of “leaf”. Hence, “non-parent” is more precise than “leaf” in this context.

Lemma. For every edge e of a rooted tree, one endpoint of e is the parent of the other.

Proof. Let $e = uv$. If u is on the unique r, v-path, the path must end with “u, e, v” :

$$P\colon\ r, \ldots, u, e, v.$$

This implies u is the parent of v. If v is on the unique r, u-path, then v is the parent of u by the same reasoning.

If neither of these happen, then we can trace a u, v-walk W(not necessarily a path) without e by traversing the unique u, r-path followed by the unique r, v path. Deleting e must disconnect u and v because e is a bridge and deleting a bridge disconnects its endpoints. But W is left intact by deleting e and so u and v are not disconnected by deleting e. This is a contradiction.

Recall that $V - \{r\}$ means the set of non-root vertices.

Definition. Given a rooted tree (T, r), let the function $\text{child}: E \to V - \{r\}$ be the function where $\text{child}(e)$ is the endpoint of e that is the child of the other endpoint of e.

We will show that we can perfectly match the edges with the non-roots. This means the number of edges is equal to the number of non-roots.

Theorem. Given a rooted tree (T, r), the function $\text{child}: E \to V - \{r\}$ is a bijection, i.e., this function is both one-to-one and onto.

Proof. This function is onto ("surjective") because every non-root is a child so if v is a child with parent u, then uv is the edge joining them and $\text{child}(uv) = v$. No child is left behind. To show the function is one-to-one ("injective"), suppose $\text{child}(e) = \text{child}(f) = v$. Since there is a unique r, v-path, the last edge in this path must be e and it must also be f. This implies $e = f$. Hence, the function is one-to-one.

Corollary. For a rooted tree, the number of children equals the number of edges.