Today, I want to look at using eigen-information to determine rankings of players in a strongly connected tournament – this will provide a framework for handling question 2 of the current HW assignment. For our purposes, we'll want the *largest* eigenvalue and corresponding eigenvector in this problem. The theory behind what we're doing today is based on the Perron-Frobenius theorem (for those of you who have encountered this result.)

Remember that Wolfram Alpha provides eigenvalues in *descending* order of their modulus (largest in absolute value is listed first) so for problem 1, you'll want to reverse that order (e.g., λ_2 in the HW exercise is λ_6 of the Wolfram Alpha output).

Example of a tournament (strongly connected) where we want to rank the players:

Suppose we have five players 1,2,3,4,5 where i defeats j if

- $j = i + 1 \bmod 5$, or otherwise
- if $i > j$

Let's write down the adjacency matrix A where $A_{ij} = 1$ if i defeats j and $A_{ij} = 0$ otherwise:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

How should we rank the players? We'll look at numbers of wins as a first pass to an algorithm. We can capture all of this information by computing

$$A\mathbf{1} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \\ 3 \end{bmatrix}.$$

Let's interpret the vector $\begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \\ 3 \end{bmatrix}$ as a crude "relative strength" vector. Let's compute $A(A\mathbf{1}) = (AA)\mathbf{1} =$ $A^2\mathbf{1}$:

$$A^2\mathbf{1} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1+3 \\ 1+1+3 \\ 1+1+2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \\ 5 \\ 4 \end{bmatrix}.$$

$$A^3\mathbf{1} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 4 \\ 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 6 \\ 7 \\ 7 \end{bmatrix}.$$

$$A^4\mathbf{1} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 6 \\ 7 \\ 7 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 9 \\ 13 \\ 12 \end{bmatrix}.$$

The "leading eigenvalue" is about $\lambda = 1.6593\ldots$ and the corresponding eigenvector is

$$\begin{bmatrix} 0.306564 \\ 0.508682 \\ 0.844057 \\ 1.09398 \\ 1 \end{bmatrix}$$

Notice that

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0.306564 \\ 0.508682 \\ 0.844057 \\ 1.09398 \\ 1 \end{bmatrix} = (1.6593\ldots) \begin{bmatrix} 0.306564 \\ 0.508682 \\ 0.844057 \\ 1.09398 \\ 1 \end{bmatrix}$$

It's not an accident that the outputs of $A^n\mathbf{1}$ as n gets large tend to a scalar multiple of this principal eigenvector.

We now see that this process produces the ranking 4,5,3,2,1 so player 4 wins the gold, etc.

The **spectral radius** of an $n \times n$ matrix is the largest magnitude among all eigenvalues.

A matrix M is **irreducible** if you replace all nonzero entries with 1s producing the matrix A, and then form the digraph whose adjacency matrix is A, resulting in a strongly connected digraph.

Notice that the adjacency matrix of a strongly connected digraph is irreducible. Thus, such a matrix for a strongly connected tournament is irreducible.

Theorem (Perron-Frobenius) If A is an irreducible non-negative $n \times n$ matrix with spectral radius r, then

1. $\lambda = r$ is an eigenvalue,
2. of multiplicity 1,
3. corresponding to an eigenspace of dimension 1,
4. spanned by a vector $\mathbf{w}$ whose entries are all positive,
5. and where no other eigenvector has all positive entries.

As a corollary, the Perron-Frobenius theorem guarantees that applying the above process to a strongly connected tournament is guaranteed to produce a ranking. I will neither confirm nor deny that this ranking precludes ties for strongly connected tournaments.

How do we apply this to non-strongly connected tournaments? Recall that the connected directed components of a tournament (strongly connected or not) form a structure called a chain. This allows you to rank the components first and then apply the above process to the sub-tournaments formed by each component.

Variants of this approach are used in practice, e.g., Google PageRank, NCAA computer rankings (Colley rankings, for instance). These are especially useful when the gaming structure is not a full round-robin tournament.