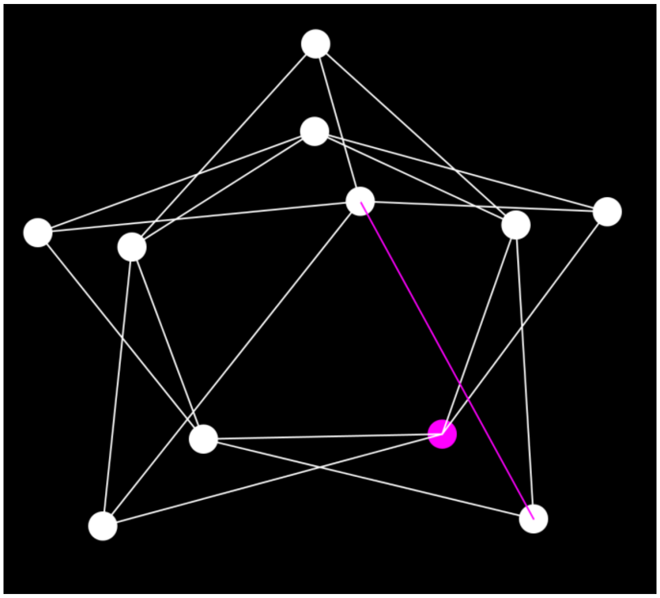
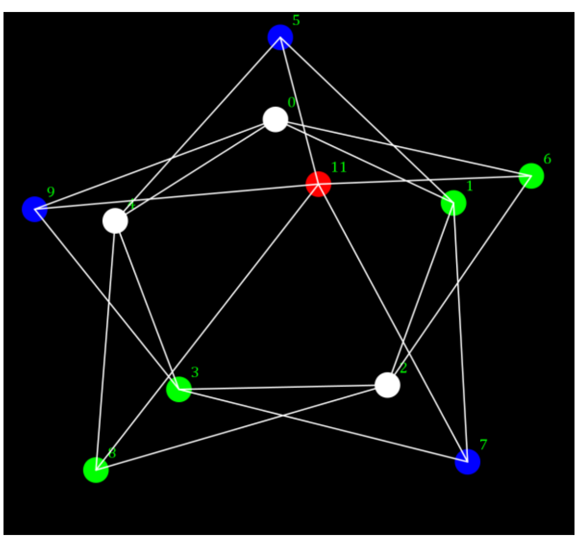


Claim: The above graph cannot be 3-colored.

Proof. Suppose it could be and let it be 3-colored. Then the center vertex has a color; call that color 0. The neighbors of that center vertex would be assigned up to two colors. Call those colors 1 and 2. Any "outer" vertex v that is assigned color 1 does not have neighbors that have been assigned color 1. Therefore, the inner vertex that corresponds to v may be re-assigned to have color 1 without causing two adjacent vertices to be assigned the same color. Similarly, any "outer" vertex w that is assigned color 2 does not have neighbors that have been assigned color 2. Therefore, the inner vertex that corresponds to w may be re-assigned to have color 2 without causing two adjacent vertices to be assigned the same color. We have just reassigned colors 1 and 2 to the pentagon successfully, i.e., without any of their neighbors being assigned the same color. Hence, the pentagon can be 2-colored. This is a contradiction since pentagons cannot be 2-colored. Therefore, this graph cannot be 3-colored.



Chromatic Polynomials (II.D.2)

Definition. Given a graph G, we define a function $p_G: \mathbb{Z} \to \mathbb{Z}$ called the **chromatic polynomial** of G. when $k \geq 0$, the value of $p_G(k)$ is the number of ways to k-color the vertices of G using colors from $\{1,2,3,\dots,k\}$.

Convention. If G is a null graph, then $p_G(k) = 1$, a constant function.

Warm-up example. Suppose G is a trivial graph (no edges, one vertex). Then

$$p_G(k) = k.$$

Warm-up example 2. Suppose G has two vertices v_1 and v_2 and no edges. Then

$$p_G(k) = k^2.$$

To see this, consider assigning colors using the following list of subtasks:

- Assign a color to v_1; there are k choices.
- Assign a color to v_2; there are k choices, regardless of the choice made in the previous subtask.

Since these subtasks are independent, we can multiply the numbers of choices, arriving at

$$p_G(k) = k^2.$$

Generalization. If G has n isolated vertices and no edges, then

$$p_G(k) = k^n.$$

Example. Suppose $G = P_n$, $n \geq 2$, a path graph with n vertices $v_1, v_2, \dots, v_n$. Assume edges are of the form $v_i v_{i+1}$.

Subtask analysis.

- Assign a color to v_1; there are k choices.
- Assign a color to v_2; there are $k-1$ choices, regardless of the choice made in the previous subtask.
- Assign a color to v_3; there are $k-1$ choices, regardless of the choices made in the previous subtasks.
- Assign a color to v_4; there are $k-1$ choices, regardless of the choices made in the previous subtasks.
- $\vdots$
- Assign a color to v_n; there are $k-1$ choices, regardless of the choices made in the previous subtasks.

These subtasks are independent (the magic phrase is "regardless of the choices made in the previous subtasks") and so

$$p_G(k) = k(k-1)^{n-1}.$$

Observation. If G is a path with $n \geq 2$, i.e. at least two vertices, $p_G(1) = 0$. This tells us that there are no 1-colorings of a path with $n \geq 2$.

Example. Now suppose $G = K_n$, the complete graph on n vertices, $v_1, v_2, v_3, \dots, v_n$.

Subtasks:

- Assign a color to v_1; there are k choices.
- Assign a color to v_2; there are $k-1$ choices, regardless of the choice made in the previous subtask.
- Assign a color to v_3; there are $k-2$ choices, regardless of the choice made in the previous subtask.
- Assign a color to v_4; there are $k-3$ choices, regardless of the choice made in the previous subtask.
- $\vdots$
- Assign a color to v_n; there are $k-(n-1)$ choices, regardless of the choice made in the previous subtask.

$$p_G(k) = k(k-1)(k-2)\cdots\big(k-(n-1)\big)$$

If $n \leq k$, then

$$\begin{aligned} p_G(k) &= k(k-1)(k-2)\cdots\big(k-(n-1)\big) \\ &= \frac{k(k-1)(k-2)\cdots\big(k-(n-1)\big)\cdot(k-n)\big(k-(n+1)\big)\cdots 1}{(k-n)\big(k-(n+1)\big)\cdots 1} \\ &= \frac{k!}{(k-n)!} \end{aligned}$$

If $n > k$, then $p_G(k) = 0$.