Assignment 2 is posted; it's due next Tuesday at 5:00 pm via Blackboard.

Today, we'll finish looking at families of graphs – Complete bipartite graphs. Quick discussion of bipartite graphs in general.

Then on to Section I.D, "degree."

The commands \cdots, \vdots, \ddots will produce $\cdots$, $\vdots$, $\ddots$

Definition. A **bipartite graph** is a graph G whose vertex set V can be partitioned into two subsets W, X such that every edge joins a vertex in W to a vertex in X.

Recall that "V is partitioned into W and X" means

$$V = W \cup X; \ \ \emptyset = W \cap X.$$

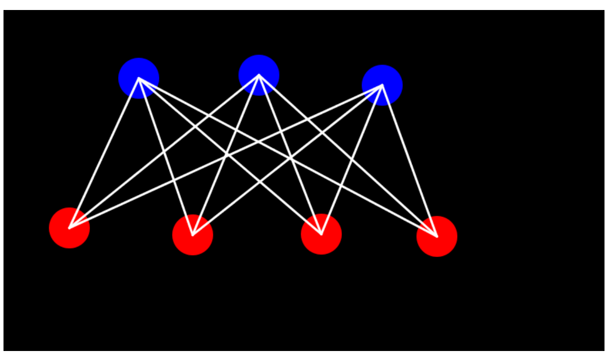
Definition. The **complete bipartite graph** $K_{p,q}$ has vertex set

$$V = \{w_1, w_2, \dots, w_p, x_1, x_2, \dots, x_q\};$$

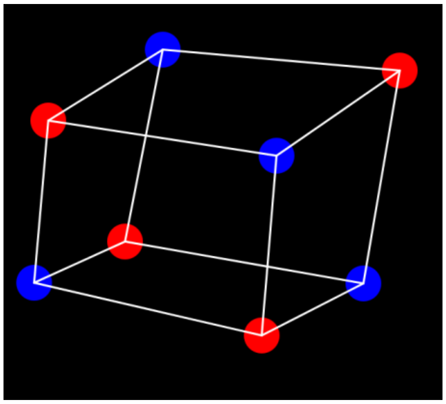
$$W = \{w_1, w_2, \dots, w_p\}; X = \{x_1, x_2, \dots, x_q\}; W \cap X = \emptyset$$

And there is an edge joining w_i to x_j for every $i, 1 \le i \le p$ and every $j, 1 \le j \le q$.

Example. The following is a depiction of $K_{3,4}$ as well as a depiction of $K_{4,3}$:



The following graph is bipartite, but not complete bipartite:



Conjecture. C_7 is not bipartite.

Sketch of proof. The vertices of $C_7 = \{0,1,2,3,4,5,6\}$ and two vertices are joined if and only if they differ by 1 mod 7; particularly, 06 is an edge. Suppose C_7 is bipartite with vertex parts W_0 and W_1. Let W_0 be the part that contains 0. Then W_1 has to contain 1. <u>We observe, in general, that W_0 contains the even vertices and W_1 contains the odd vertices.</u> (A formal proof of this underlined statement is by mathematical induction.) But this means $6 \in W_0$ but then there is an edge, namely 06, that joins vertices in the same part. That contradicts C_7 being bipartite.

Section I.D. Vertex Degree

Let $G = (V, E)$ be a finite graph without loops.

Definition. For any vertex $v \in V$, the **degree of v**, denoted $\deg(v)$, is the number of edges incident with v.

Observation. If G is a simple graph, then

$$\deg(v) = |N(v)|,$$

the number of neighbors of v. Recall that the absolute value symbols denote the number of elements of the set of neighbors of v, denoted $N(v)$.

Terminology.

- A vertex of degree 0 is called an **isolated** vertex.
- A vertex of degree 1 is often called a **leaf** or an **end vertex.** The edge that is incident with that vertex is called a **pendant edge**.

Observation. If G has no loops, then for $v \in V$,

$$\deg(v) = \sum_{u \in V} \mu_E(uv),$$

recalling that $\mu_E(uv)$ is the number of occurrences of uv in the edge multiset E.

Also, if $V = \{v_1, v_2, v_3, \dots, v_n\}$,

$$2|E| = \sum_{1 \le i < j \le n} \mu_E(v_i v_j) + \sum_{1 \le j < i \le n} \mu_E(v_i v_j) = \sum_{j=1}^{n} \sum_{i=1}^{n} \mu_E(v_i v_j) = \sum_{j=1}^{n} \left(\deg(v_j)\right)$$

or, the famous result:

"Twice the number of edges is equal to the total degree."

Corollary. The total degree of a loopless finite graph is even.

Corollary. In any loopless finite graph, the number of vertices with odd degree must be even.