Today, I want to briefly look at subgraphs (Section I.E) and isomorphic graphs (Section I.F). I then want to get into graph operations (Section I.G).

Also, one way to approach question 3 is to let $V = \{1,2,3,\dots,2n\}$ and describe which pairs of vertices would get edges.

For instance, to describe a 1-regular graph with $2n$ vertices: For each $k \in \{1,2,3,\dots,n\}$ join vertices $2k - 1$ and $2k$ with an edge.

Subgraphs. (Section I.E)

Definition. Given graphs G, H with vertex sets V_G, V_H and edge multisets E_G, E_H, we say H is a **subgraph** of G when $V_H \subseteq V_G$ and $E_H \subseteq E_G$; recall that $E_H \subseteq E_G$ means the number of occurrences of an edge in E_H is less than or equal to the number of occurrences of that edge in E_G. Notation will be $H \leq G$.

Example: Given G, H with vertices $V_G = \{1,2,3,4,5\}$, $V_H = \{1,2,5\}$ and edges $E_G = \{12,23,34,45,15,25\}$, $E_H = \{12,25,15\}$. Then $H \leq G$. If $H \leq G$ but $H \neq G$, then H is a **proper subgraph** of G; you can write this as $H < G$.

Technically, if H is a null graph, then $H \leq G$ for any G.

Caution! It is important that the edges of H actually join vertices of H. It would not make sense to have $V_H = \{1,2,3\}$ and $E_H = \{12,25,15\}$, because now H is not a graph.
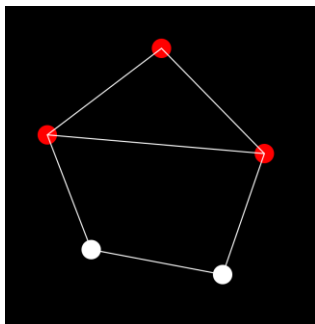
What if we change E_H to $\{12,15\}$? Yes, this is still a subgraph.

What if we change E_H to $\{12,24,15\}$? No, this is not a subgraph. There is an edge of H that is not an edge of G.

Induced subgraphs. Let $G = (V_G, E_G)$ be a graph and let $W \subseteq V_G$ be given. Then if H is the graph with vertex set W and edge multiset E_H where for every edge $e \in E_G$ with both endpoints in W,

$$\mu_{E_H}(e) = \mu_{E_G}(e),$$

then H is said to be **induced** by W. We'll write $G(W)$ for the subgraph of G induced by W.
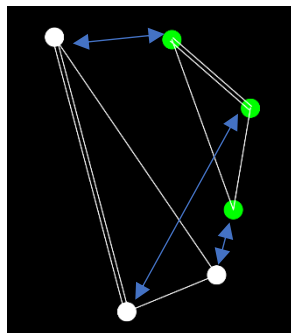

If G is the graph depicted above with the five vertices and six edges, and W is the set of red vertices, then $G(W)$, the subgraph of G induced by W, is the upper triangle.

Graph Isomorphism. Given two graphs, we want to clarify when their structures are the same.

Definition. Given two graphs G and G', a **graph isomorphism** is a function $\phi: V_G \to V_{G'}$ such that

- ϕ is one-to-one and onto, i.e., ϕ is a bijection.
- If $\phi(u) = u'$ and $\phi(v) = v'$, then the multiplicity of the edge uv in E_G is the same as the multiplicity of the edge $u'v'$ in $E_{G'}$.

Observations about isomorphic graphs G and G'.

- They have the same numbers of vertices and the same numbers of edges.
- They have the same degree sequences
- Either both are bipartite or both fail to be bipartite
- Either both are simple or both fail to be simple
- For any subgraph of G, there is an isomorphic subgraph of G'

Example. Let G, G' both have vertex set $\{0,1,2,3,4\}$ (Treat these as integers modulo 5)

The edges of G are $01, 12, 23, 34, 40$, the edges of G' are $02, 13, 24, 30, 41$

I claim these are isomorphic graphs:

$$\phi(0) = 0; \phi(1) = 2; \phi(2) = 4; \phi(3) = 1; \phi(4) = 3$$

$\phi(v) = 2v \bmod 5$.

A **graph isomorphism** is a vertex bijection that preserves all multiplicities of adjacencies.

Intuitively, if $G \cong G'$ (this is the notation for isomorphism, you can typeset this "\cong"), then you can "move the vertices of G to get a copy of G'."

Example. $Q_2 \cong K_{2,2} \cong C_4$