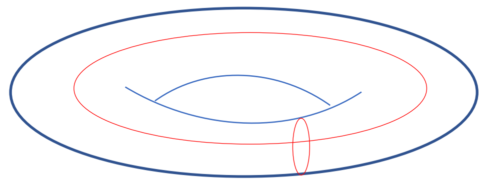
Today, I want to look at the "total edge count" (similar to total degree) and use this to prove that K_5 and $K_{3,3}$ are not planar graphs.

Then I want to look briefly and intuitively at drawing graphs on other surfaces, like the torus:

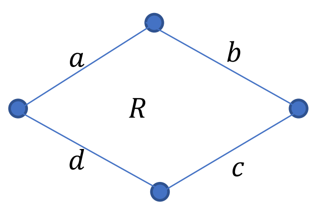


A full formal treatment of this topic would be a semester-long course in algebraic topology. One entertainment: Google "two-holed torus" and it'll come up with a You Tube video that's fascinating to watch.

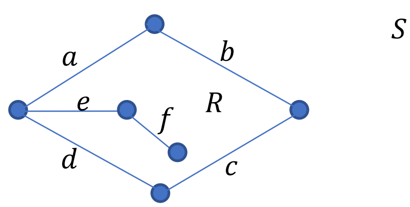
Consider a graph G drawn on some surface without edges crossing so that the surface is broken into regions.

Definition. Given a region R, the **edge count** of R is the number of edges forming the boundary of R, counting bridges twice. Equivalently, this is the number of edges in a shortest walk that traverses around R, starting and ending at the same vertex, using all the edges in the boundary of R.

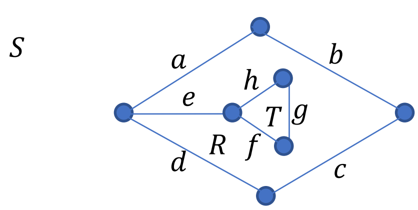
Examples. For a region bounded by a cycle, the walk we would use, listing only the edges would be "dcba"; the number of edges is four, so the edge count for this region would be 4.



More complicated example: Walk would be abcdeffe, so the edge count for the region R is 8. For the region S, the walk is abcd, and the edge count for S is 4.



One last example: The walk for R is abcdefghe, for a total of 9 edges. The edge count for R is 9.



Definition. The **total edge count** for a graph drawn on some surface so that the surface is broken into regions is the sum of the edge counts for the regions.

For extra reading, look up "dual graphs" some time to see how to turn the regions into vertices of what is known as the dual graph.

Observations.

- Each bridge contributes 2 to the edge count of its region, each non-bridge contributes 1 to the edge counts of two regions. Hence, every edge contributes exactly 2 to the total edge count.
- The total edge count, denoted $\varepsilon = 2m$, is twice the number of edges. Hence, ε must be even.

Theorem. The graph K_5 is not planar.

Proof. Suppose K_5 is planar and suppose G is a plane drawing of K_5. The graph G has $n = 5$ vertices and $m = 10$ edges, and since G drawn in the plane, it has r regions where

$$n - m + r = 2.$$

Solving this for r, G must have $r = 7$ regions. Each region must have an edge count of at least 3, because G is a simple graph. This means that the total edge count must be at least $\varepsilon \geq 7 \cdot 3 = 21$.

But then

$$20 = 2m = \varepsilon \geq 7 \cdot 3 = 21,$$

implying $20 \geq 21$ which contradicts $20 < 21$. Hence, K_5 is not planar.

Theorem. The graph $K_{3,3}$ is not planar.

Proof. Suppose $K_{3,3}$ is planar and suppose G is a plane drawing of $K_{3,3}$. The graph G has $n = 6$ vertices and $m = 9$ edges. Since G is drawn in the plane, it has r regions where

$$n - m + r = 2.$$

Solving this for r gives us $r = 5$. We note that G is bipartite, so each region must be bounded by at four edges (G has no triangles.) This means that the total edge count must be at least $\varepsilon \geq 5 \cdot 4 = 20$. But then

$$18 = 2m = \varepsilon \geq 5 \cdot 4 = 20,$$

implying $18 \geq 20$ which contradicts $18 < 20$. Hence, $K_{3,3}$ is not planar.

Kuratowski's Theorem. Any nonplanar graph has a subgraph that is either a subdivision of K_5 or a subdivision of $K_{3,3}$.

Definition. A planar graph is **triangulated** if all of its regions are triangles, i.e., each region is bounded by exactly three edges.

Definition. If R is a region bounded by more than three edges, then a new edge through R that that joins non-consecutive vertices of the boundary of R is called a **chord**.

Proposition. Any region bounded by $k \geq 4$ distinct edges can be dissected into triangles by adding chords with none of the chords crossing.

Corollary. If G is planar and simple with no bridges, then G can be triangulated by adding chords.

Theorem. For any triangulated planar graph, $m = 3n - 6$.

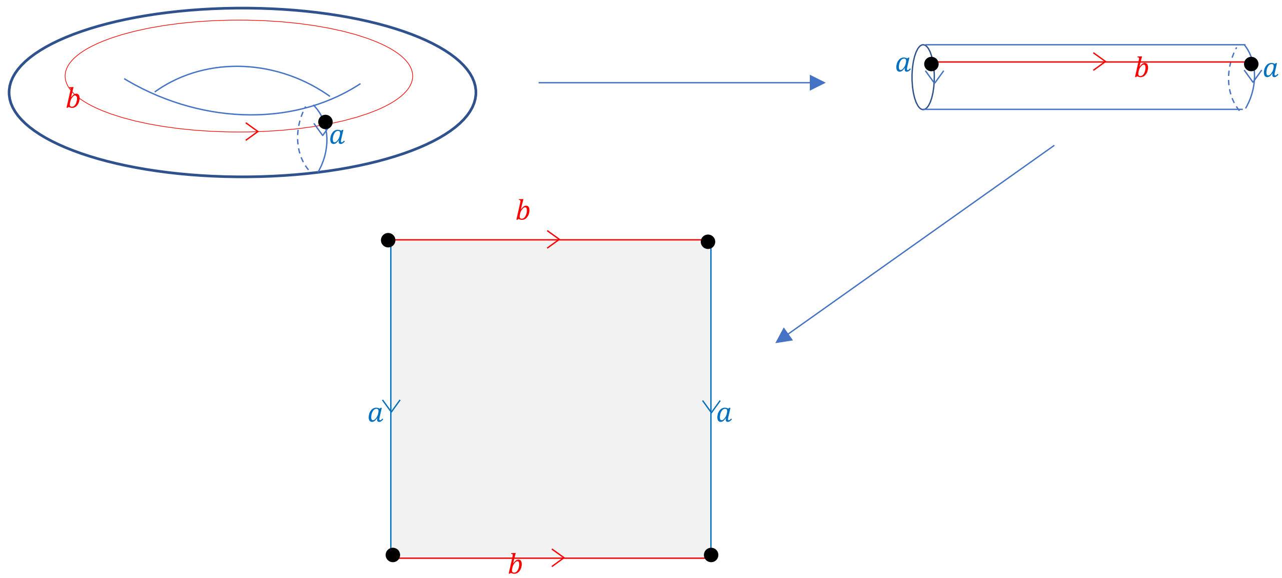
Proof. Let G be triangulated planar with r regions. Then the total edge count is $\varepsilon = 3r = 2m$. By Euler's formula,

$$\begin{aligned} n - m + r &= 2 \\ 3n - 3m + 3r &= 6 \\ 3n - 3m + 2m &= 6 \\ 3n - m &= 6 \\ m &= 3n - 6 \end{aligned}$$

Corollary. For any planar graph without bridges or parallel edges or loops, $m \leq 3n - 6$.

Corollary. The graph K_5 is nonplanar.

Proof. We have $m = 10 > 9 = 3(5) - 6 = 3n - 6$. Hence, for K_5, $m > 3n - 6$.



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