I want to look at section I.G. today; this will finish “Chapter I” so we should begin discussing connectivity “Chapter II” shortly.

I have posted Assignment 3 on Blackboard.

Graph Operations (Section I.G.)

Operation #1: The **Complement** of a simple graph.

Given a simple graph $G = (V, E)$, its **complement** is the simple graph $\overline{G}$, typeset this as “\overbar<space>G<space>”, whose vertex set is also V and whose edge set is $\overline{E} = \{uv\colon\ uv \notin E\}$. Intuitively, u and v are adjacent in $\overline{G}$ if and only if u and v are not adjacent in G.

If G has n vertices and m edges, then $\overline{G}$ has n vertices and $\binom{n}{2} - m$ edges. Recall the notation for “n choose 2”:

$$\binom{n}{2} = \frac{n(n-1)}{2}.$$

Operations #2: The **union** and the **intersection**.

Given two graphs $G_1 = (V_1, E_1); G_2 = (V_2, E_2)$, their **union** is written $G_1 \cup G_2$ and it has vertex set $V = V_1 \cup V_2$ and edge multiset $E = E_1 \cup E_2$ where the union of multisets has the property that the multiplicity of $e \in E_1 \cup E_2$ is the maximum of the multiplicities of e in E_1 and in E_2.

Given two graphs $G_1 = (V_1, E_1); G_2 = (V_2, E_2)$, their **intersection** is written $G_1 \cap G_2$ and it has vertex set $V = V_1 \cap V_2$ and edge multiset $E = E_1 \cap E_2$ where the intersection of multisets has the property that the multiplicity of $e \in E_1 \cap E_2$ is the minimum of the multiplicities of e in E_1 and in E_2.

To perform the union and/or the intersection, you have to specify the actual vertex and edge sets of the graphs.

Operation #3: The **disjoint union**.

Given two graphs $G_1 = (V_1, E_1); G_2 = (V_2, E_2)$, their **disjoint union** is written $G_1 \sqcup G_2$ (typeset this symbol "\sqcup") and it has vertex set

$$V = \{(v, 1): v \in V_1\} \cup \{(w, 2): w \in V_2\}.$$

For each edge $e = uv \in G_i$, there is a corresponding edge in $G_1 \sqcup G_2$ joining vertices (u, i) and (v, i).

Intuitively, one draws copies of each graph in different colors and gathers all of them into one graph to form the disjoint union.

A technical distinction would be vertices of the form $(v, 3)$ vs. $\big((v, 1), 1\big)$.

Operation #4: The Cartesian Product

Given two graphs $G_1 = (V_1, E_1); G_2 = (V_2, E_2)$, we define their **Cartesian Product** $G_1 \times G_2$ as the graph whose vertex set is

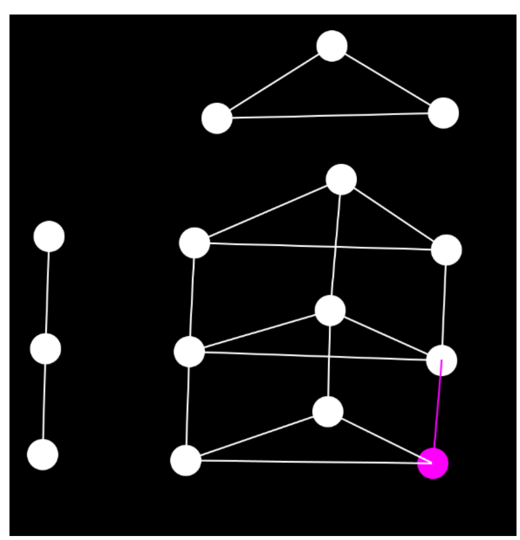
$$V = V_1 \times V_2 = \{(v_1, v_2):\ v_1 \in V_1, v_2 \in V_2\}$$

Two vertices (v_1, v_2) and (w_1, w_2) are adjacent if and only if either

- $v_1 = w_1$ and v_2 is adjacent to w_2
- $v_2 = w_2$ and v_1 is adjacent to w_1

Example: $G = C_3 \times P_3$

The vertices of C_3 are $\{0,1,2\}$; the vertices of P_3 are $\{a, b, c\}$. Vertices of G include $(0, a)$ up to $(2, c)$.



The natural generalization for the Cartesian product of $G_1, G_2, \ldots, G_k$ would be to use k-tuples for the vertices:

$$(v_1, v_2, \ldots, v_k)$$

Two vertices are adjacent if they equal in all but one coordinate and are adjacent in the remaining coordinate.

An example:

$$Q_n = K_2 \times \ldots \times K_2$$

where there are n copies of K_2.

An example: A “grid graph” is a Cartesian product of two paths.

Operation #5. Edge deletion.

Given a graph $G = (V, E)$ and an edge $e \in E$, the graph $G - e$ produced by **deleting the edge $\boldsymbol{e}$** has the same vertex set V and its edge set has the multiplicity of e reduced by 1.

Operation #6. Vertex deletion.

Given a graph $G = (V, E)$ and a vertex $v \in V$, the graph $G - v$ produced by **deleting the vertex $\boldsymbol{v}$** is the induced subgraph $G(V - \{v\})$. The vertex set becomes $V - \{v\}$. The edges that have v as an endpoint are removed from E.

If S is a set of vertices, then $G - S$ is the graph produced by deleting all of the vertices in S.