Today, we'll look at "vertex coloring." This will generalize "bipartite graphs"; a 2-colorable graph will be the same as a bipartite graph.

Once I finish with vertex coloring, I want to head into chapter 3, trees.

If you're doing a "Graph Theorist's Sketchpad" application, I've put a document (The third document in the project assignment) in the project assignment that outlines some of the features I'm hoping to see.

Formal definition for vertex coloring:

Definition. Given a graph G, a function $c: V \to \{1,2,3, \dots, k\}$ is said to be a k-**vertex coloring** of G if whenever u and v are adjacent, $c(u) \neq c(v)$.

The literature on colorings assigned to vertices is significantly more extensive than that for edges, so the word "vertex" is often omitted. Thus, a "k-coloring" is really a k-vertex coloring.

Notice that if a 2-coloring of G exists, then G is bipartite. The parts would be the sets of vertices assigned to each of the two colors. More generally, a graph is k-**partite** iff a k-coloring exists.

Definition. A graph is k-**colorable** if there exists a k-coloring of it.

Observations.

- For $n \geq 3$, if n is odd, then C_n is 3-colorable, but not 2-colorable.
- If n is even, then C_n is 2-colorable.
- The only graphs that are 1-colorable are those where every vertex is isolated.
- Notice that if G has a loop, there is no k-coloring of G for any k.
- Parallel edges have no effect on vertex colorings.

The usual game that's played at this point is to find the minimum value of k such that G has a k-coloring.

Definition. The quantity $\min\{k: G \text{ has a } k\text{-coloring}\}$ is called the **chromatic number** of G. This is denoted $\chi(G)$. This symbol χ is the Greek letter "chi" and is the first letter in the word "chromatic" written in Greek.

Observations and conventions

- For $n \geq 1, \chi(C_{2n}) = 2;\ \chi(C_{2n+1}) = 3$
- Convention: If G has a loop, then $\chi(G) = \infty$
- Convention: If G is a null graph, then $\chi(G) = 0$
- A graph is k-colorable if and only if it is k-partite
- For any nonempty bipartite graph G, $\chi(G) = 2$
- For any completely disconnected graph G with at least one vertex, $\chi(G) = 1$
- For any $n \geq 1$, $\chi(K_n) = n$
- If G has K_n as a subgraph, then $\chi(G) \geq n$.
- For any $a, b \geq 1$, $\chi\left(K_{a,b}\right) = 2$

Recall that $\Delta(G)$ is the maximum among all degrees of its vertices.

Recall that $\delta(G)$ is the minimum among all degrees.

Theorem. For any loopless finite graph G, $\chi(G) \leq \Delta(G) + 1$.

Proof. Let $p = \Delta(G) + 1$ and let $S = \{1,2,3, \dots, p\}$, the set of p colors. Consider the following algorithm:

1. Start with any assignment of colors (not necessarily a p-coloring) from S to the vertices.
2. Assign s to be the number of vertices adjacent to other vertices with the same color.
3. If $s = 0$, STOP; the assignment of colors is a p-coloring.
4. If $s > 0$, find a vertex v that is adjacent to another vertex with the same color.
5. Let S' be the set of colors used for vertices adjacent to v. We note that v is adjacent to at most $\Delta(G) = p - 1$ vertices. Therefore, $|S'| < p$.
6. Pick a color in $S - S'$ and assign v to have that new color. The vertex v is no longer adjacent to vertices with the same color.
7. Return to step 2. Note that the value of s will be reduced by at least 1 when we return to step 2.

Since the value of s can never go below 0 and it must reduce at each iteration of this process, this process must stop.

Theorem. For every $n \geq 1$, there exist triangle-free graphs G such that $\chi(G) \geq n$.