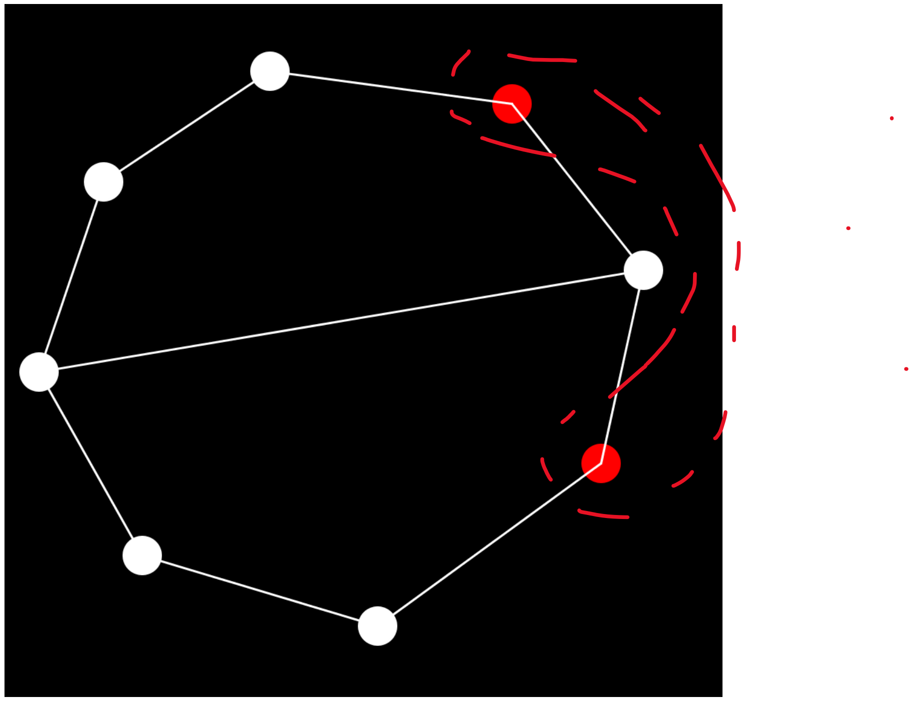
Some advice on question 5 of the current HW assignment: Watch carefully what the number of edges in your graphs will be as you perform these various operations. and use this info to determine total degrees.

On assignment 2, question 1, the minimum number of edges of a simple graph with $n \geq 1$ vertices is zero.



Recall that a **walk** W is an alternating sequence of vertices and edges

$$W: v_0, e_1, v_1, e_2, v_2, \dots, v_{k-1}, e_k, v_k$$

where the endpoints of edge e_i are v_{i-1} and v_i. If $v_{i-1} = v_i$, then e_i must be loop. If the walk starts at u and ends at v, it can be referred to as a u, v-walk.

If you want to specify that a vertex is repeated, you'd say something like, "there exists $i < j$ such that $v_i = v_j$."

A trivial walk consists of a single vertex and no edges: $W: v_0$.

The value k is called the **length** of the walk (length = number of edges used in the walk, including repetitions.)

Also recall that a **path** is walk that does not repeat a vertex.

If there exists a u, v-walk in a graph G, then there exists a u, v-path in G. The shortest (minimum length) u, v-walk is guaranteed to be a path.

A walk

$$u, e, v, e, u, e, v, e, u, e, v$$

that repeats vertices can always be shortened into a path.

Definition. If a u, v-walk W exists, we say that u and v are **connected** by W. Two vertices are said to be connected if there exists a walk connecting them.

Terminology. A **trail** is a walk that does not repeat an edge.

A **closed walk** is a walk that starts and ends at the same vertex (and has at least one edge.) Some authors will say that a trivial walk is closed, other will say that it's not.

A **circuit** is a closed trail with at least one edge.

A **cycle** (must have at least one edge) is a circuit where the only occurrence of a repeated vertex is the start and the end.

Cycles look like:

$$C:\ v_0, e_1, v_1, e_2, v_2, \dots, v_{k-1}, e_k, v_k = v_0$$

and all v_i are distinct except for $v_0 = v_k$. Such a cycle is called a k-cycle. Since cycles are trails, no edge is repeated.

A 1-cycle would require that the edge be a loop. $C: v, e, v$ with e being a loop with its endpoints both equal to v.

Observation. If we are considering a 2-cycle $C: v_0, e_1, v_1, e_2, v_2 = v_0$, we require $e_1 \neq e_2$. In shorter words, a K_2 graph is not a 2-cycle.

Observation. If your graph is simple, then there will be no 1- or 2-cycles.

Distances between vertices (Section II.A.3.)

Given two vertices u, v in a graph G, we'd like some sort of measure as to "how far apart they are," some distance between them.

Definition. The **distance** between two vertices u, v in a graph G, denote $D(u, v)$, is the length of a shortest u, v-path (also shortest u, v-walk), recalling that "length = edges." If there is no u, v-path (this means u and v are disconnected), then we declare $D(u, v) = \infty$.

It turns out that the distance defined here is called an "extended metric" because of the presence of ∞ in the definition.

Observations.

- For any vertex v, $D(v, v) = 0$ because of the trivial v, v-path existing.
- If $u \neq v$, then $D(u, v) > 0$ (you'd have to traverse at least one edge to walk from u to v.)
- For any two vertices u and v, $D(u, v) = D(v, u)$.
- For any three vertices u, v, w, $D(u, w) \leq D(u, v) + D(v, w)$. This is called the **triangle inequality.**

Theorem II.A.3.a. If P is a shortest u, v-path and x is a vertex on P, then the u, x-subpath along P is a shortest u, x-path and the x, v-subpath along P is a shortest x, v-path.

Proof. Let P' be the u, x-subpath along P and Q' be the remaining x, v-subpath along P. This means $P = P'Q'$. If P' is not shortest among u, x-subpaths, then there is some shorter u, x-subpath P''. But then $P''Q'$ would be a u, v-walk shorter than P. This can't happen. Similarly for Q'.