Feel free to ignore (leave unanswered) question 2B; there isn't really anything asked in that question.

If your software is already producing orthogonal eigenvectors, demonstrate that fact rather than going through the Gram-Schmidt process in the homework assignment.

Today, I want to look at how to count the number of spanning trees in a labeled graph, i.e., whose vertices are {1,2,3,...,n}.

This will be answered by looking at the determinant of a submatrix of the Laplacian; this result is called the Kirchhoff matrix-tree theorem.

Warm-up question: How many spanning trees (subgraph that is a tree that contains all of the vertices) does the labeled C_4 graph have? (Answer = 4, the number of ways of removing one edge to leave a tree. Such a tree, in this example, would be a path P_4)

How many spanning trees does the labeled K_4 have? This is the same as asking how many labeled trees with four vertices are there? It will turn out that there are $16 = 4^2$ of them. In general, there are n^{n-2} labeled trees on n vertices. This is a very famous result with numerous different-styled proofs.

Theorem (The Kirchhoff matrix-tree theorem). Let G be a labeled graph with Laplacian matrix L. For any vertex v_i, let L_i be the matrix produced by deleting row i and column i of L. Then the number of spanning trees of G is $\det(L_i)$.

Example. Consider the graph C_4. Its Laplacian is the matrix

$$L = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}$$

Quick observation: $\det(L) = 0$. This is true of any Laplacian since $\lambda = 0$ corresponds to the all-one eigenvector.

To apply Kirchhoff, we choose $i = 1$, deleting the first row and column, obtaining

$$L_1 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

$$\begin{aligned} \det(L_1) &= 2\det\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} - (-1)\det\begin{bmatrix} -1 & -1 \\ 0 & 2 \end{bmatrix} + 0\det\begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix} \\ &= 6 + (-2) + 0 \\ &= 4 \end{aligned}$$

Example. Consider the graph K_4. Its Laplacian is the matrix

$$L = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$$

The number of spanning trees is

$$\det \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}$$

We could certainly expand this, but let's operate a bit more generally and ask what the determinant of

$$M = aJ + (b-a)I = \begin{bmatrix} b & a & a & a & & a \\ a & b & a & a & & a \\ a & a & b & a & \cdots & a \\ a & a & a & b & & a \\ & & \vdots & & \ddots & \vdots \\ a & a & a & a & \cdots & b \end{bmatrix}$$

where J is the all-ones $n \times n$ matrix and I is the $n \times n$ identity matrix.

Observe that this matrix is symmetric and so it is diagonalizable. This means that it is similar to a diagonal matrix whose entries are its eigenvalues. The moral of this story is that the determinant of M equals the product of its eigenvalues, with multiplicities. So, it suffices to find the eigenvalues of M and their multiplicities. The trick is to decompose M

$$M = aJ + (b-a)I = \begin{bmatrix} a & a & a & a & & a \\ a & a & a & a & & a \\ a & a & a & a & \cdots & a \\ a & a & a & a & & a \\ & & \vdots & & \ddots & \vdots \\ a & a & a & a & \cdots & a \end{bmatrix} + (b-a)I$$

What are the eigenvalues of aJ? One of them has corresponding eigenvector $\begin{bmatrix} 1 \\ 1 \\ 1 \\ \cdots \\ 1 \end{bmatrix}$; the eigenvalue is $\lambda = na$.

$$\begin{bmatrix} a & a & a & a & & a \\ a & a & a & a & & a \\ a & a & a & a & \cdots & a \\ a & a & a & a & & a \\ & & \vdots & & \ddots & \vdots \\ a & a & a & a & \cdots & a \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} na \\ na \\ na \\ na \\ \vdots \\ na \end{bmatrix} = na \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

The matrix aJ is a rank-one matrix and so the value 0 will be an eigenvalue of multiplicity $n-1$ for this matrix. Thus, the eigenvalues of aJ are

$$na \text{ (multiplicity 1), and } 0 \text{ (multiplicity } n-1).$$

Adding $(b-a)I$ to aJ has the effect of increasing the eigenvalues by $b-a$. This means that the eigenvalues of M are

$$na + b - a \text{ (multiplicity 1), and } b - a \text{ (multiplicity } n-1).$$

Furthermore, the determinant of M is

$$\det(aJ + (b-a)I) = (na + b - a)(b-a)^{n-1}.$$

$$\det \begin{bmatrix} b & a & a & a & & a \\ a & b & a & a & & a \\ a & a & b & a & \cdots & a \\ a & a & a & b & & a \\ & & \vdots & & \ddots & \vdots \\ a & a & a & a & \cdots & b \end{bmatrix} = (na + b - a)(b-a)^{n-1}.$$

Example. If $n = 3, a = -1, b = 3$

$$\det \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix} = \big(3(-1) + 3 - (-1)\big)\big(3 - (-1)\big)^{3-1} = 1(4)^2 = 16,$$

as advertised.

More generally, given K_p and its $p \times p$ Laplacian L, we have L_1

$$L_1 = \begin{bmatrix} b & a & a & a & & a \\ a & b & a & a & & a \\ a & a & b & a & \cdots & a \\ a & a & a & b & & a \\ & & \vdots & & \ddots & \vdots \\ a & a & a & a & \cdots & b \end{bmatrix}; \ n = p-1, b = p-1, a = -1$$

$$\det(L_1) = \big(1 - p + p - 1 - (-1)\big)(b-a)^{n-1} = p^{p-2}$$

and so K_p has exactly p^{p-2} labeled spanning trees. Equivalently, there are exactly p^{p-2} labeled trees on p distinct vertices.