Assignment 4 is on Blackboard. Also, I put up a rundown on the Project Assignment on Blackboard.

Section II.A.5. Even/Odd cycles and Bipartite graphs.

Definition. A cycle C_n is **even** if n is even and **odd** if n is odd.

Lemma. Odd cycles are not bipartite. Even cycles are bipartite.

Lemma. Every subgraph of a bipartite graph is bipartite.

(Suppose the vertex set V of G is bipartitioned into $U \sqcup W$. How does this lead to a bipartition of the vertex set $V' \subseteq V$ of a subgraph $G' \leq G$? We can write $U' = U \cap V'$ and $W' = W \cap V'$ leading to $V' = U' \sqcup W'$.)

Corollary (of these lemmas). If G is a bipartite graph, then G contains no odd cycles as subgraphs.

Corollary. If G has an odd cycle then G is not bipartite.

(If p then q) has as its **logical inverse** (If not p then not q).

(If p then q) has as its **converse** (If q then p).

(If p then q) has as its **contrapositive** (If not q then not p).

Theorem. If G has no odd cycles as subgraphs, then G is bipartite.

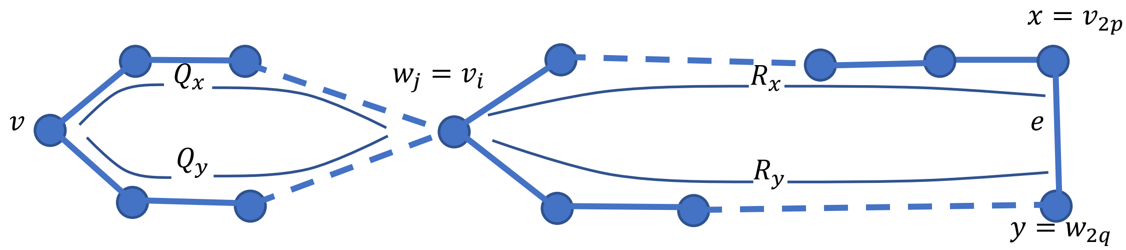
(This is the logical "inverse" of the preceding corollary.)

Sketch of Proof. Let's assume G is connected; if it isn't then apply this result to each connected "component." We assume that G has no odd cycles as subgraphs. We want to show that G is bipartite, i.e., that we can partition $V = V_0 \sqcup V_1$ so that every edge has an endpoint in V_0 and an endpoint in V_1. Choose a vertex v of G. We define V_0 and V_1 as follows:

$$V_0 = \{w \in V : D(v, w) \text{ is even}\}$$
$$V_1 = \{w \in V : D(v, w) \text{ is odd}\}$$

It is immediate that $V = V_0 \sqcup V_1$. We have to check the edge endpoint condition. Let $e = xy$ be any edge. Suppose $x \in V_0$ and $y \in V_0$. Then there are even-length shortest v, x- and v, y- paths

$$P_x : v = v_0, e_1, v_1, \dots, v_{2p-1}, e_{2p}, v_{2p} = x$$
$$P_y : w = w_0, f_1, w_1, \dots, w_{2q-1}, f_{2q}, w_{2q} = y$$



Here, $P_x = Q_x R_x$ and $P_y = Q_y R_y$.

Suppose w_j is the highest-indexed vertex on P_y that is also a vertex v_i on P_x. As a result of this definition, none of the vertices v_k or w_ℓ are equal if $k > i$ or $\ell > j$. Hence, we have a cycle formed by following R_x, then the edge xy, then R_y in reverse. We claim this cycle is an odd cycle. The length of R_x is $2p - i$, the length of R_y is $2q - j$. The fact that all of these paths are shortest paths leads to the conclusion that $i = j$. The length the cycle is then

$$2p - i + 2q - j + 1 = 2p + 2q - 2i + 1$$

which is an odd number. This is a contradiction. Therefore, it is impossible for x and y to both be elements of V_0. A similar argument prevents x and y from belonging to V_1 simultaneously. The only other possibility is that that x and y belong to different V_i sets.

Theorem. G is bipartite if and only if G has no odd cycles.

Corollary. If G has no cycles as subgraphs, then G is bipartite.

A graph with no cycles is called **acyclic.** These graphs are also called **forests**. An **acyclic connected** graph is also called a **tree**. Conclusion: Trees are bipartite.

Since Pete (The Petersen graph) has pentagons, Pete is not bipartite.

Section II.B Connectedness and Connectivity

These words carry different meanings; connectedness is a binary property, i.e., a graph is either connected or it is not. Connectivity will be a measure of "how connected a graph is."

Definition. Given a graph G, two vertices u, v are **connected** if there exists a u, v-walk. They are **disconnected** otherwise.

We can write $u \sim v$ (as a relation) when this occurs.

Observations.

- (Reflexivity) Vertices are connected to themselves, i.e., for all vertices u, $u \sim u$.
- (Symmetry) If $u \sim v$, then $v \sim u$.
- (Transitivity) If $u \sim v$ and $v \sim w$, then $u \sim w$.

Theorem. Vertex connectivity is an equivalence relation over V.

Any relation that is reflexive, symmetric, and transitive is an equivalence relation.