CPTS 553: Graph Theory
Assignment 5

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1

1.1

We know that if a graph G is Eulerian then all of its vertices have even degree. From observation, we can see that all the vertices in the dodecahendron graph have a degree of 3. Hence, G is not Eulerian.

1.2

The Hamiltonian cycle is as follows: $0, 1, 2, 3, 8, 12, 7, 11, 6, 10, 5, 14, 15, 16, 17, 18, 19, 13, 9, 4, 0$. The presence of this cycle indicates that the given graph is Hamiltonian.

2

2.1

A 4-coloring of H:

- C1: 0,6
- C2: 5,2
- C3: 4,1
- C4: 3

2.2

Let there exist a 3-coloring of H with the colors: $C1, C2, C3$.
Let vertex '0' be colored with C1. Then vertex '5' can be colored with either C2 or C3. Without loss of generality, we can assign C2 to vertex '5' which forces vertex '4' (adjacent to '5') to be assigned C3.
Similarly, without loss of generality, we can assign vertices '1' and '2' with C3 and C2 respectively. Vertex '6' is adjacent to both '5' and '4'. Hence the only available choice for it is C1. Similarly, the only available choice for '3' is C1 (since it is adjacent to both '1' and '2'). But '6' and '1' are adjacent to each other as well making the assignment of C1 to both, invalid. Hence, we can say that a 3-coloring of H does not exist.

3

In $P_3 \times P_3$, we notice that there are 5 even vertices and 4 odd vertices. There are also, no edges which has 2 even vertices as endpoints.
Since there are more number of even vertices in the graph than there are odd vertices, a Hamiltonian path will need to start and end with even vertices.
A Hamiltonian cycle will be one where the start and end vertices of a Hamiltonian path are adjacent to each other. Here, that would mean the start and end vertices of the Hamiltonian cycle (if it exists) in $P_3 \times P_3$ where both are even, to be adjacent to each other. But as we have seen before from observation, there are no edges in $P_3 \times P_3$ such that its end vertices are both even. Hence, we can say that a Hamiltonian cycle doesn't exist. Therefore, $P_3 \times P_3$ is not Hamiltonian.

4

We know that for simple graph G such that e is an edge of G,

$$p_G(k) = p_{G-e}(k) - p_{G/e}(k)$$

Also, if G is C_n for $n \geq 4$, then $G - e$ is same as P_n and G/e is same as C_{n-1}. Therefore, for C_6,

$$p_{C_6}(k) = p_{P_6}(k) - p_{C_5}(k)$$

$$p_{C_5}(k) = p_{P_5}(k) - p_{C_4}(k)$$

$$p_{C_4}(k) = p_{P_4}(k) - p_{C_3}(k)$$

We know that $p_{C_3}(k) = k(k-1)(k-2)$ and $p_{P_n}(k) = k(k-1)^{n-1}$. Substituting them in, we get,

$$p_{C_6}(k) = k(k-1)^5 - (k(k-1)^4 - (k(k-1)^3 - k(k-1)(k-2)))$$

Factorizing the above, we get the following:

$$p_{C_6}(k) = k(k-1)((k-1)^4 - (k-1)^3 + (k-1)^2 - (k-2))$$

$$p_{C_6}(k) = k(k-1)((k-1)^3(k-2) + (k-1)^2 - (k-2))$$

$$p_{C_6}(k) = k(k-1)((k-1)^2k(k-2)-(k-2))$$
$$p_{C_6}(k) = k(k-1)(k-2)(k^3-2k^2+k-1)$$

From the above we see that $(k-2)$ is a factor of $p_{C_6}(k)$.