The Zoom meetings for Sketchpad people are Monday and Tuesday are at 11:00-1:00 for Monday and 10:00-1:00 for Tuesday.

Theorem. (Cantor-Schroder-Bernstein). Given two sets A, B, if there exist injections (one-to-one but not onto) $f: A \to B$ and $g: B \to A$, then there exists a bijection (one-to-one correspondence; perfect matching) $h: A \to B$.

Sketch of Proof. Without loss of generality, let's suppose A and B are disjoint sets. If they're not disjoint, replace A with $A \times \{1\}$ and B with $B \times \{2\}$.

We construct a digraph D whose node set is $A \cup B$. For each $a \in A$ there is an arc going from a to $f(a)$ and for each $b \in B$, there is an arc going from b to $g(b)$.

Since f, g are injections, so different elements a_1, a_2 in A are sent to different elements $f(a_1)$ and $f(a_2)$ in B; likewise for b_1, b_2 in B. Consider the out-degree of each node. How many arcs have $a \in A$ or $b \in B$ at the tail? Answer 1; this is a consequence of f and g being functions.

For the in-degrees, we use the fact that f, g are injections. If $b \in B$ is the head of two or more arcs, then there must be two or more elements $a_1, a_2, \ldots \in A$ such that $f(a_1) = f(a_2) = \cdots = b$. This is not allowed since f is an injection. Therefore, the in-degree of b is at most 1. Similarly, using g, the in-degree of $a \in A$ is at most 1.

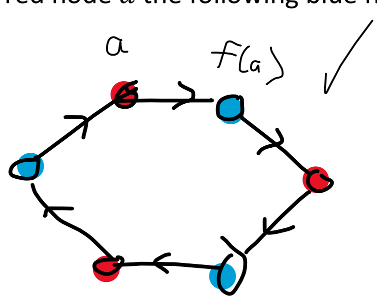
So the digraph D has maximum out-degree and maximum in-degree 1.

What sorts of connected (underlying graph) digraphs have this property?

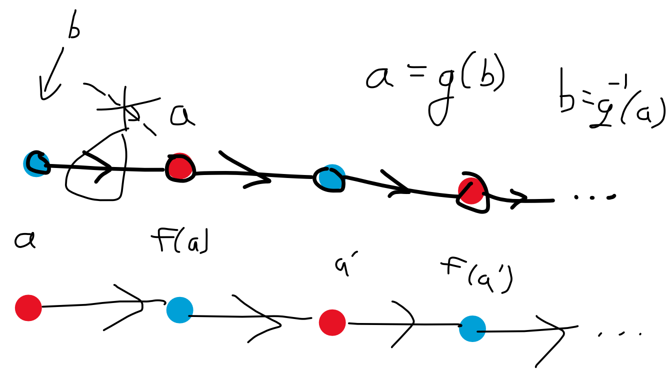
1) Directed cycles. Observing that the underlying graph is bipartite, these cycles must be even.
2) Directed path; noting that the out-degree of every node is 1, such a path can't stop. We might call such a directed graph a "directed ray", i.e., an infinitely long path with one endpoint.
3) A directed path in both directions, sort of a "directed line."

We use "red" for A and "blue" for B, it makes sense in most of our possibilities below to associate with each red node a the following blue node.

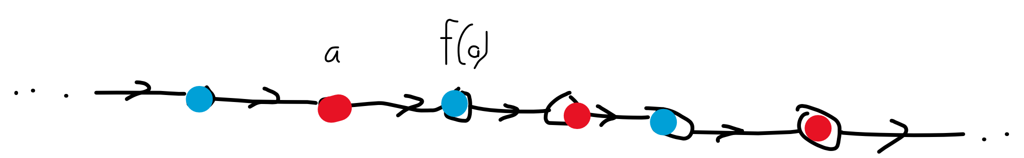
1)



2)



3)



Recall that the goal is to show that there exists a bijection $h: A \to B$. In cases 1, 2(bottom) and 3, we simply declare $h(a) = f(a)$.

For 2(top), there is a unique b such that $g(b) = a$; we define $h(a) = b = g^{-1}(a)$ in this case.

·This result is much more interesting when applied to infinite sets.

Application. Let $\mathbb{N} = \{1,2,3,4,5,6,7, \dots\}$ and $\mathbb{N} \times \mathbb{N} = \{(a, b):\ a, b \in \mathbb{N}\}$.

Consider

$$f: \mathbb{N} \to \mathbb{N} \times \mathbb{N}$$

given by $f(a) = (a, a)$; this is seen to be an injection by inspection.

For the harder direction, we need an injection $g: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. Here's one:

$$g(a, b) = 2^a\, 3^b.$$

For instance, $g(1,4) = 2^1\, 3^4 = 162$. What about $g(4,1) = 2^4\, 3^1 = 48$. More generally, suppose

$$g(a, b) = g(c, d).$$

Then $2^a 3^b = 2^c 3^d$. Since both sides equal the same positive integer, by the unique prime factorization property, we must have $a = c$ and $b = d$, forcing $(a, b) = (c, d)$. This shows that g is an injection.

Hence, from CSB, there exists a bijection $h: \mathbb{N} \to \mathbb{N} \times \mathbb{N}$.

"There are exactly as many ordered pairs of natural numbers as there are natural numbers."

Application. There is a bijection $h_2: \mathbb{N} \times \mathbb{N} \to \mathbb{Q}^+$. Use the fact that any $q \in \mathbb{Q}^+$ can be written in lowest terms $q = \frac{a}{b}$.

The easy direction: $g: \mathbb{Q}^+ \to \mathbb{N} \times \mathbb{N}$. Define $g\left(\frac{a}{b}\right) = (a, b)$. By inspection, this is injective.

For the other direction: $f: \mathbb{N} \times \mathbb{N} \to \mathbb{Q}^+$. Use $f(a, b) = 2^a 3^b$. Done.

Since there are bijections from $\mathbb{N} \to \mathbb{N} \times \mathbb{N}$ and from $\mathbb{N} \times \mathbb{N} \to \mathbb{Q}^+$, the conclusion is that there is a bijection from $\mathbb{N} \to \mathbb{Q}^+$.