PROPERTIES OF GRAPHS IN RELATION TO THEIR SPECTRA

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Abstract. We will explain basic concepts of spectral graph theory, which studies graph properties via the algebraic characteristics of its adjacency matrix or Laplacian matrix. It turns out that investigating eigenvalues and eigenvectors of a graph provides unexpected and interesting results. Methods of spectral graph theory can be used to examine large random graphs and help tackle many difficult combinatorial problems (in particular because of recently invented algorithms which are able to compute eigenvalues and eigenvectors even of huge graphs in relatively short time). This approach uses highly efficient tools from advanced linear algebra, matrix theory and geometry.
This paper will cover the following issues :

- obtaining *nice* embeddings of graphs using eigenvectors of their Laplacian matrices
- use of graph spectra in testing for isomorphism
- partitioning a graph into two equal pieces minimizing the number of edges between them (a problem arising for example in parallel computing) - Fiedler value and isoperimetric number
- spectra of large random graphs.

We will outline proofs of some presented theorems.

1. Introduction

First of all, let us recall some useful definitions and terminology. A graph $G = (V, E)$ is specified by a vertex set V and an edge set E. E is a set of pairs of vertices. Unless stated otherwise, graph G will be finite and undirected.
Without loss of generality let $V = \{1, 2, \ldots, n\}$.
The *adjacency matrix* of a graph G is a matrix $A_G = [a_{i,j}]$, defined as follows:

$$a_{i,j} = \begin{cases} 1 & \text{if } \{i,j\} \in E \\ 0 & \text{otherwise.} \end{cases}$$

The *Laplacian matrix* of a graph G is a matrix $L_G = [l_{i,j}]$, defined as follows:

$$l_{i,j} = \begin{cases} -1 & \text{if } \{i,j\} \in E \\ d_i & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

where $d_i \equiv deg(i)$ denotes a degree of vertex i of the graph G.
Note that the Laplacian matrix can be defined as the difference:

$$L_G = D_G - A_G,$$

where D_G denotes the diagonal matrix of dimension n and with vertex degrees on the diagonal, i.e. $D_G = diag\{d_i, i \in V\}$.

2. Properties of graphs in relation to their Laplacians

Now, let us redefine the Laplacian matrix.
Let $G_{1,2}$ be the graph on two vertices with one edge.

$$L_{G_{1,2}} := \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

As we see $x^T L_{G_{1,2}} x = (x_1 - x_2)^2$, where $x = (x_1, x_2)$.
For any graph $G_{u,v}$ we define:

$$L_{G_{u,w}}(i,j) = \begin{cases} 1 & \text{if } i = j \text{ and } i \in \{u, w\} \\ -1 & \text{if } i = u \text{ and } j = w, \text{ or } i = w \text{ and } j = u \\ 0 & \text{otherwise.} \end{cases}$$

For a graph $G = (V, E)$ we define the Laplacian matrix as follows:

$$L_G = \sum_{\{u,w\} \in E} L_{G_{u,w}}.$$

From this definition some properties of the Laplacian matrix follow easily:

Lemma 1. *Let L be the Laplacian matrix of dimension n and let $x \in \mathbb{R}^n$. Then*

$$x^T L x = \sum_{\{i,j\} \in E} (x_i - x_j)^2,$$

Corollary 2. *The Laplacian matrix of every graph is positive semidefinite (hence its eigenvalues are non-negative).*

Lemma 3. *Let L_G be the Laplacian matrix of a graph G. Then the eigenvector of eigenvalue* 0 *is* the all 1s *vector, that is* $(1, 1, ..., 1)$.

Lemma 4. *Let G be a connected graph, and let $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ be the eigenvalues of the Laplacian matrix L_G of the graph G. Then $\lambda_2 > 0$.*

Proof. Let x be an eigenvector of the matrix L_G of eigenvalue 0. Then

$$L_G x = 0,$$
$$x^T L_G x = \sum_{\{i,j\} \in E} (x_i - x_j)^2 = 0.$$

Thus, for each edge $\{i, j\} \in E$ (for each pair of vertices i and j connected by an edge), we have $x_i = x_j$. The graph G is connected which implies that x is some constant times the all 1s vector. Thus, the eigenspace of 0 has dimension 1.

□

This lemma led Fiedler to consider the magnitude of λ_2 as a measure of compactness (algebraic connectivity) of a graph (Fig. 1). Accordingly, we often call λ_2 the *Fiedler value* of a graph and eigenvector v_2 of λ_2 the *Fiedler vector*.

In practice it is pointless to consider unconnected graphs, because their spectra are just the union of the spectra of their connected components.

Corollary 5. *The multiplicity of* 0 *as an eigenvalue of the Laplacian matrix L_G of a graph G equals the number of connected components of G.*

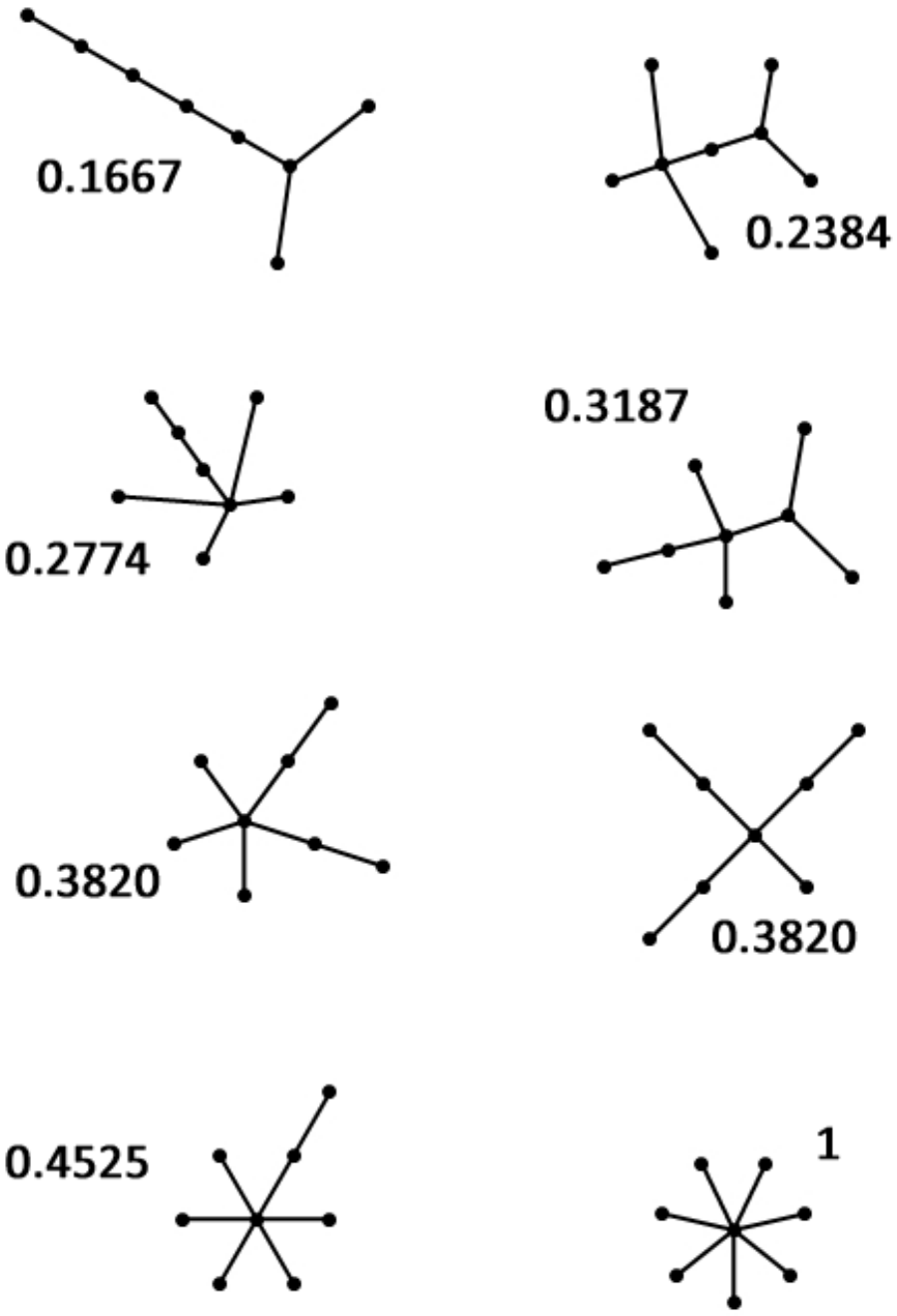

FIGURE 1. Relationship between the second Laplacian eigenvalue and a graph structure.

3. Graph drawing

Let us treat an eigenvector as a function of the vertices:

$$F_u : V \to \mathbb{R}$$

$$F_u(i) = u_i,$$

where $G = (V, E)$, $i \in V$, u - an eigenvector of the Laplacian matrix L_G.
So if we take two eigenvectors, we obtain two real numbers for each vertex of the graph. These two numbers can become coordinates of a vertex on the plane. It is probably not obvious but if we take two eigenvectors corresponding to two the smallest non-zero eigenvalues of the Laplacian matrix L_G, this *often* gives us a good picture of the graph with straight-line edges.

Example. $A_G = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$, $L_G = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix}$

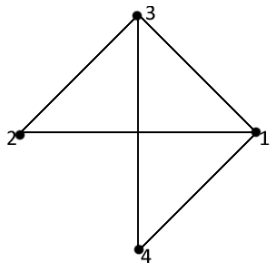

Figure 2. Graph G specified by A_G.

The spectrum of G is $\sigma(L_G) = \{0, 2, 4, 4\}$ and the eigenvectors are $v_1 = (1, 1, 1, 1)$, $v_2 = (0, -1, 0, 1)$, $v_3 = (-3, 1, 1, 1)$ and $v_4 = (1, -1, 1, -1)$.
So let $u = v_2$ i $w = v_3$ (two eigenvectors corresponding to two the smallest non-zero eigenvalues). Assign $(x_i, y_i) = (u(i), w(i))$, for $i = 1, 2, 3, 4$. We obtain a planar representation of G:

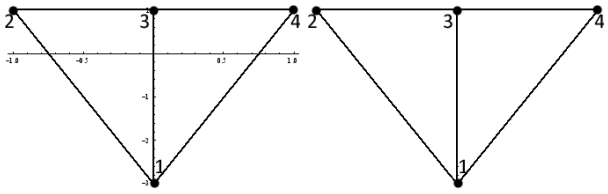

Figure 3. Spectral embedding of graph G.

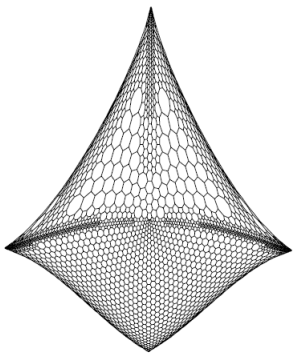

FIGURE 4. Drawing obtained from the Laplacian eigenvectors, $|V| = 4970$, $|E| = 7400$.

In order to obtain a two-dimensional representation of a graph in this way, a certain condition has to be satisfied: the smallest non-zero eigenvalue of the Laplacian matrix has to be of multiplicity 1 or 2. Otherwise, for example when its multiplicity is 3, we cannot reasonably choose just two vectors, but if we choose three vectors that span the eigenspace, we will get a representation of the graph in three dimensions.

We may wonder why the eigenvectors of the Laplacian matrix of graph give us such a good picture for planar graphs. Unfortunately, to our knowledge nobody has proven a satisfactory theorem about that yet.

4. Isomorphism and spectra of graphs

One of the oldest problems in graph theory is that of determining whether or not two graphs are isomorphic.

Recall that two graphs $G = (V, E)$ and $H = (V, F)$ are *isomorphic*, if there exist a permutation (re-labeling) $\varphi : V \to V$, such that $\{i, j\} \in E \Leftrightarrow \{\varphi(i), \varphi(j)\} \in F$.
As we know such problem for large graphs might be computationally hard. Let's try to use Laplacian matrix spectrum to compare and find isomorphic graphs.

Theorem 6. *Let $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ be the eigenvalues of the Laplacian matrix L_G of a graph G and $\eta_1 \leq \eta_2 \leq \ldots \leq \eta_n$ be the eigenvalues of the Laplacian matrix L_H of graph H. Then $\lambda_i \neq \eta_i$ implies that G and H are not isomorphic.*

To prove this theorem we should know some other theorems and lemmas:

Theorem 7. *Two graphs G and H are isomorphic if and only if there exist a permutation matrix P such that $L_G = PL_HP^T$.*

Lemma 8. *Two square matrices are* similar *if and only if they represent the same linear transformation.*

Thus, isomorphic graphs are represented by similar matrices.

Lemma 9. *Similar matrices have the same characteristic polynomial.*

Thus, similar matrices have the same spectrum.

Proof. 6 Isomorphic graphs are related by permutation of vertex labels. Label-permutation of matrices is a linear transformation. Thus, isomorphic graphs are represented by similar matrices and therefore have the same spectra.

□

However this method is imperfect because of *cospectral non-isomorphic graphs* - graphs that have identical eigenvalues but are not isomorphic. The characteristic polynomial distinguishes all the graphs, up to isomorphism, only of at most four vertices.

Because the spectral embeddings (section 3.) were uniquely determined up to rotation, we can try to use the eigenvectors to test the isomorphism.

It can be proved that:

Theorem 10. *Let*

(1) G and H be isomorphic graphs,
(2) $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ be the eigenvalues of the adjacency matrix A_G
(3) λ_i be isolated, that means $\lambda_{i-1} < \lambda_i < \lambda_{i+1}$
(4) v_i be an eigenvector corresponding to λ_i,
(5) u_i be the i-th eigenvector of the adjacency matrix A_H.

Then there exist a permutation $\varphi : \{1, 2, \ldots n\} \to \{1, 2, \ldots n\}$ such that $v_i(j) = u_i(\varphi(j))$.

So if we find a few eigenvectors that map each vertex to a distinct point, then we can use them to test for isomorphism. But unfortunately, there exist graphs for which the eigenvectors do not tell us anything about isomorphism, for example when an eigenvector has only two values (then all vertices have one coordinate being one of these two values).

5. Cutting graphs

Let λ_2 denote the smallest non-zero eigenvalue of the Laplacian which is often called *Fiedler value*. The table below presents values of λ_2 computed for some basic graphs (n is the number of vertices):

Graph	Fiedler value
Path	$\Theta(1/n^2)$
Grid	$\Theta(1/n)$
3d grid	$\Theta(n^{-2/3})$
Binary tree	$\Theta(1/n)$

Table 1. λ_2 for different types of graphs.

By a grid we understand a Cartesian product of two path-graphs (kd grid is its k-dimensional generalization).

One explanation of why the Fiedler value is distinguished among other eigenvalues is that it tells us something about cutting a graph. Cutting a graph means dividing its vertices into two disjoint sets S and $\overline{S}$. One is usually interested in minimizing the number of edges which have to be cut when two graphs emerge from one given graph. Moreover, we often want the sets S and $\overline{S}$ to be of roughly equal sizes. The quality telling us how good a cut is in these terms, is the ratio of the cut:

$$\varphi(S) := \frac{|E(S,\overline{S})|}{\min(|S|,|\overline{S}|)},$$

where $E(S,\overline{S})$ is the set of edges joining vertices lying on the opposite sides of the cut. The smaller the ratio, the better cut. The minimum ratio at which we can cut a graph is called *isoperimetric number*:

$$\varphi(G) := \min_{S\subset V} \varphi(S).$$

The relation between the isoperimetric number and Fiedler value is captured by *Cheegers's Inequality*:

Theorem 11 (Cheegers's Inequality)**.** *Let d_{max} denote the maximum degree of vertices in the graph. Then:*

$$\varphi(G) \geq \lambda_2 \geq \frac{\varphi^2(G)}{2d_{max}}.$$

This double inequality is quite informative for any graph G, because if λ_2 is small it tells us that it is possible to cut a graph without cutting too many edges (depending on d). However, if λ_2 is big, then every cut will cut many edges. The proof of Cheegers's Inequality can be found in [1].

Graph partitioning is a fundamental part of many algorithms. A problem of graph partition was in fact one of the inspirations for spectral graph theory. Consider for instance parallel computing. Here a graph can represent communication required between different subprograms and graph partitioning can be used to partition a computational task among parallel processors. But how can one actually obtain the optimal cut? For the time being some algorithms have been proposed. Unfortunately, many of them are computationally complex. For the algorithms based on computing the eigenvectors of a graph and other algorithms, the reader is referred to [1, 2].

The concept of isoperimetric number can be generalized for weighted graphs. Here the corresponding quality is called *conductance*. For partitioning the vertex set V into two sets S and $\overline{S}$ the conductance of the cut $(S,\overline{S})$ is defined as follows:

$$\Phi(S) := \frac{\sum_{u\in S, v\notin S} a_{u,v}}{\min\left(\sum_{w\in S} d_w, \sum_{w\notin S} d_w\right)},$$

where $a_{u,v}$ denotes the weight attributed to the edge $\{u,v\}$ and we set:

$$d_u := \sum_w a_{u,w}.$$

The *volume* of the set of vertices S we define by:

$$\mathrm{vol}(S) := \sum_{w\in S} d_w$$

and the volume of the set of edges F by:

$$\mathrm{vol}(F) := \sum_{\{u,v\}\in F} a_{u,v}.$$

Introducing one more definition

$$\partial(S) := \{\{u,v\} \in E : u \in S, v \in \overline{S}\},$$

allows us to rewrite the conductance of a set S in a simpler form:

$$\Phi(S) = \frac{\mathrm{vol}(\partial(S))}{\min(\mathrm{vol}(S), \mathrm{vol}(\overline{S}))}.$$

The conductance of a graph is given by

$$\Phi(G) := \min_{S\subset V} \Phi(S).$$

The symmetry property of set conductance $\Phi(S) = \Phi(\overline{S})$ arises immediately from its definition.

The problem of a *graph decomposition* is slightly different than that of just cutting a graph. Firstly, we should state what we mean by a decomposition. In decomposing a graph we want to divide a vertex set into sets of approximately the same volumes (so the partition would be balanced), cut as few edges as possible and keep high conductance of these newly obtained smaller graphs. Sometimes we have to sacrifice the balance of the cut for achieving high conductance of each subgraph. We define a $\Phi-$*decomposition* of a graph $G = (V, E)$ to be a partition of V into sets $V_1, V_2, ..., V_k$ such that for all i, the graph G_i induced by G on V_i satisfies $\Phi(G_i) \geq \varphi$. The induced graph $G_i = (V_i, E_i)$ has all the edges of G whose both ends lie in V_i. Let $\partial(V_1, V_2, ..., V_k)$ denote the boundary of the decomposition, which is the set of edges joining components V_i,

$$\partial(V_1, V_2, ..., V_k) := E \setminus \cup_i E_i.$$

Cutting V into sets S and $\overline{S}$ induces graphs $G(S)$ and $G(\overline{S})$. We will denote by $\mathrm{vol}_{\overline{S}}(T)$ and $\partial_{\overline{S}}(T)$ the volume and the boundary of the set $T \subset \overline{S}$ in the graph $G(\overline{S})$, respectively. The following two properties result easily from definitions:

Proposition 12. *Let $S \subset V$ and $T \subset \overline{S}$. Then*

$$vol_{\overline{S}}(T) \leq vol_V(T).$$

Proposition 13. *Let $S \subset V$ and $T \subset \overline{S}$. Then*

$$\partial(S \cup T) = \partial_{V\setminus T}(S) \cup \partial_{\overline{S}}(T) \subset (\partial_V(S) \cup \partial_{\overline{S}}(T))$$

and

$$vol(\partial(S \cup T)) = vol(\partial_{V\setminus T}(S)) + vol(\partial_{\overline{S}}(T)) \leq vol(\partial_V(S) + \partial_{\overline{S}}(T)).$$

Of course, a decomposition can be obtained by first cutting G, then cutting each of the two "halves" and continuing in this manner. The theorem below tells us that every graph has a good decomposition and when we cut a graph into two pieces and they are not balanced (in terms of volume), it is not necessary to keep cutting the larger part, so the procedure of recursively decomposing a graph will not last too long.

Theorem 14. *For any $\varphi \leq \Phi(G)$, let S be the set of the largest possible volume not greater than vol$(V)/2$ such that $\Phi(S) \leq \varphi$. If vol$(S) <$ vol$(V)/4$, then*

$$\Phi(G(\overline{S})) \geq \varphi/3.$$

Proof. Assume, contradicting the thesis, that $\Phi(G(\overline{S})) < \varphi/3$. Then there exists a set $R \subset \overline{S}$ such that

$$\frac{\mathrm{vol}(\partial_{\overline{S}}(R))}{\mathrm{vol}_{\overline{S}}(R)} < \Phi_{\overline{S}}(R) < \varphi/3.$$

Let T denote the set R or the set $\overline{S} \setminus R$ such that $\mathrm{vol}_V(T) \leq \mathrm{vol}_V(\overline{S})/2$. From proposition 12 we have

$$\frac{\mathrm{vol}(\partial_{\overline{S}}(T))}{\mathrm{vol}_V(T)} < \varphi/3. \tag{1}$$

We will obtain a contradiction by showing that there exist a set ($S \cup T$ or $\overline{S \cup T}$) whose volume is larger than vol(S) but still at most vol$(V)/2$ and which conductance is less than φ, so a different set would have been chosen instead of S. We will consider two cases.

When vol$(S \cup T) \leq$ vol$(V)/2$, we have

$$\begin{aligned}
\Phi(S \cup T) &= \frac{\mathrm{vol}(\partial(S \cup T))}{\mathrm{vol}(S \cup T)} \\
&\leq \frac{\mathrm{vol}(\partial(S)) + \mathrm{vol}(\partial_{\overline{S}}(T))}{\mathrm{vol}(S \cup T)} \\
&= \frac{\mathrm{vol}(\partial(S)) + \mathrm{vol}(\partial_{\overline{S}}(T))}{\mathrm{vol}(S) + \mathrm{vol}(T)} \\
&\leq \max\left(\frac{\mathrm{vol}(\partial(S))}{\mathrm{vol}(S)}, \frac{\mathrm{vol}(\partial_{\overline{S}}(T))}{\mathrm{vol}(T)}\right) \\
&\leq \max(\Phi(S), \varphi/3) \leq \varphi,
\end{aligned} \tag{2}$$

where we have made use of the inequality $\frac{a+b}{c+d} \leq \max(a/c, b/d)$ and of (1). Thus vol$(S) <$ vol$(S \cup T) \leq$ vol$(V)/2$ and $\Phi(S \cup T) < \varphi$ so the set S is not the most balanced set satisfying given assumptions as stated in the theorem. A contradiction.

When vol$(S \cup T) >$ vol$(V)/2$, taking into account that the set T satisfies vol$(T) \leq$ (vol(V) − vol$(S))/2$, we can upper-bound vol$(S \cup T)$:

$$\begin{aligned}
\mathrm{vol}(S \cup T) = \mathrm{vol}(S) + \mathrm{vol}(T) \leq \frac{\mathrm{vol}(V) - \mathrm{vol}(S)}{2} + \\
+ \mathrm{vol}(S) = \frac{\mathrm{vol}(V)}{2} + \frac{\mathrm{vol}(S)}{2} < (5/8)\mathrm{vol}(V),
\end{aligned} \tag{3}$$

since vol$(S) <$ vol$(V)/4$. Making use of (3) we obtain vol$(\overline{S \cup T}) \geq (3/8)vol(V)$ and

$$\begin{aligned}
\mathrm{vol}(\partial(S \cup T)) &\leq \mathrm{vol}(\partial(S)) + \mathrm{vol}(\partial_{\overline{S}}(T)) \\
&\leq \varphi\mathrm{vol}(S) + (\varphi/3)\mathrm{vol}(T) \\
&\leq \varphi\mathrm{vol}(S) + (\varphi/3)(\mathrm{vol}(V) - \mathrm{vol}(S))/2 \\
&\leq (5/6)\varphi\mathrm{vol}(S) + (1/6)\varphi(\mathrm{vol}(V)) \\
&< (3/8)\varphi\mathrm{vol}(V).
\end{aligned} \tag{4}$$

Hence,

$$\Phi(\overline{S \cup T}) = \frac{\mathrm{vol}(\partial(S \cup T))}{\mathrm{vol}(\overline{S \cup T})} < \frac{(3/8)\varphi \mathrm{vol}(V)}{(3/8)\mathrm{vol}(V)} = \varphi$$

and for the same reasons as in the previous case we have a contradiction. □

Theorem 14 allows us to formulate a convenient decomposition procedure. Given a graph G, start with finding the most balanced set S such that $\Phi(S) \leq \varphi$. Take this to be the first cut. If $\mathrm{vol}(S) < \mathrm{vol}(V)/4$, the theorem says that $\Phi(G(\overline{S})) \geq \varphi/3$, so the part $G(\overline{S})$ already has a relatively high conductance and decomposition may continue only on the set S. Otherwise, perform the recursive procedure on both sets. Since the depth of the recursion is at most

$$\log_{4/3} \mathrm{vol}(V)$$

and the number of edges cut at each stage of the recursion is at most $(\varphi/2)\mathrm{vol}(V)$, the total number of edges cut does not increase

$$\frac{\varphi \mathrm{vol}(V) \log_{4/3} \mathrm{vol}(V)}{2}.$$

If φ is assigned an appropriate value, then at most $|E|/2$ edges will be cut and each of the obtained graphs will have high conductance (at least $\varphi/3$).

One application of $\Phi(G)$ is examining the convergence of random walks on graphs. One can show that if $\Phi(G)$ is big then every random walk must converge quickly. This is explained in [1].

6. Spectra of large random graphs

There are usually two ways of generating fancy graphs: by algebraic construction or at random. However, those created at random should be really considered as a special family of graphs, especially if they are very large. When one wants to prove something about random graphs, it sometimes requires sophisticated tools of probability theory. Nevertheless, let us look how the spectra of random graphs behave.

Take a classical Erdös-Rényi random graph. Let it be $G(n, 1/2)$ - the one where each edge appears independently at random with uniform probability $1/2$. Figure 5 presents the probability density function of eigenvalues of its adjacency matrix for $n = 40$.

The density function was obtained by generating many such adjacency matrices, calculating their eigenvalues and then computing a histogram of values which they take. Additionally, the histograms of just the largest and second largest eigenvalues were added in. The distributions are very tightly concentrated. Note the appearance of the big "hump" in the limit - it turns out that the largest eigenvalue would be always isolated. It is connected with the fact that it is not smaller than an average vertex degree and it grows faster with the number of nodes or connectivity.

The shape of the density function can be analytically proved - for large n it follows *Wigner's Semi-circle law*. So-called Wigner semi-circle distribution, named after the physicist Eugene Wigner, is the probability distribution supported on the interval

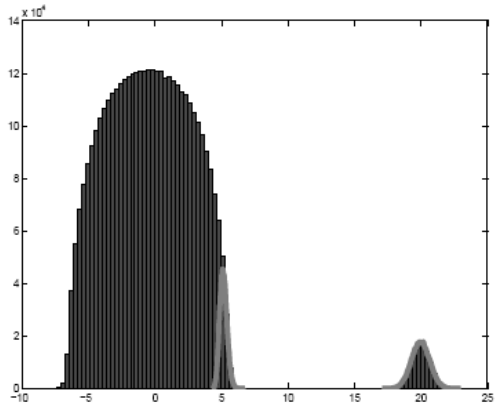

FIGURE 5. A histogram of eigenvalues of $G(40, 1/2)$.

$[-R, R]$ the graph of whose probability density function f is a semicircle of radius R centered at (0,0) and then suitably normalized (so it in fact becomes a semi-ellipse):

$$f(x) = \begin{cases} \frac{2}{\pi R^2}\sqrt{R^2 - x^2}, & \text{if } -R < x < R; \\ 0, & \text{else.} \end{cases}$$

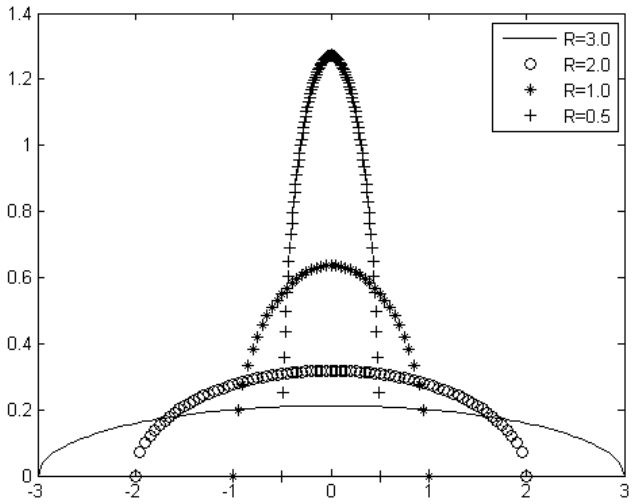

FIGURE 6. Wigner semicircle distribution

Wigner was dealing with advanced quantum physics and he noticed that some matrices emerging there have eigenvalues with this limiting distribution. One formulation of his famous theorem is the following:

Theorem 15 (Wigner's Semi-circle Law, 1958)**.** *For $1 \leq i \leq j \leq n$ let a_{ij} be real valued independent random variables satisfying:*

(1) The laws of distributions of $\{a_{ij}\}$ are symmetric;
(2) $\mathbb{E}[a_{ij}^2] = \frac{1}{4}$, $1 \leq i < j \leq n$, $\mathbb{E}[a_{ii}^2] \leq C$, $1 \leq i \leq n$;
(3) $\mathbb{E}[(a_{ij})^{2m}] \leq (Cm)^m$, for all $m \geq 1$,

where $C > 0$ is an absolute constant. For $i < j$ set $a_{ji} = a_{ij}$. Let A_n denote the n-dimensional random matrix with entries a_{ij}. Finally, denote by $W_n(x)$ the number of eigenvalues of A_n not larger than x, divided by n. Then

$$\lim_{n\to\infty} W_n(x\sqrt{n}) = W(x),$$

in distribution, where

$$W(x) = \begin{cases} 0, & \text{if } x \leq -1; \\ \frac{2}{\pi}\int_{-1}^{x} \sqrt{1-y^2}dy, & \text{if } -1 \leq x \leq 1; \\ 1, & \text{if } x \geq 1. \end{cases} \tag{5}$$

The proof of this theorem can be found in [3].

It should be noticed that for the adjacency matrix of $G(n,p)$ we have $\mathbb{E}[a_{ij}^2] = p$, $(1 \leq i < j \leq n)$, which is not always equal to $1/4$ as in the assumptions of Wigner's theorem. However, distribution of eigenvalues of $G(n,p)$-adjacency matrix also converges to the semicircle distribution, when properly rescaled [4]. Hence, the histogram of $G(n,p)$ eigenvalues looks like a semicircle.

Nevertheless, observe that the Semi-circle Law provides very limited information about the asymptotic behavior of any particular (for instance, the largest) eigenvalue. This problem was studied, among others, by the authors of [5]. They considered quite general model of random symmetric matrices. Here is their main result:

Theorem 16 (Alon, Krivelevich, Vu)**.** *For $1 \leq i \leq j \leq n$, let a_{ij} be independent, real random variables with absolute value at most 1. Define $a_{ij} = a_{ji}$ for all admissible i, j, and let A be the n-by-n matrix $(a_{ij})_{n\times n}$ with eigenvalues $\mu_1(A) \leq \mu_2(A) \leq ... \leq \mu_n(A)$. Then for every positive integer $1 \leq s \leq n$, the probability that $\mu_s(A)$ deviates from its median by more than t is at most $4e^{-t^2/32s^2}$. The same estimate holds for $\mu_{n-s+1}(A)$.*

Using theorem 16 it can be shown that the expectation and median for the distribution of eigenvalues of such matrices are very close [5]. This theorem explains why the distribution of a particular eigenvalue is so tightly concentrated around its expected value.

Wigner's Semi-circle Law is in the theory of random matrices almost as important as the Central Limit Theorem is in probability theory. However, it appears that spectra of "real world" graphs (for example the graph of the Internet with power law distribution of vertices degree) do not obey this semi-circle rule. More discussion on this topic the reader will find in [4].

Spectral density is also related to the appearance of so-called *Giant Connected Component* in a random graph ensemble (see [4] and references therein).

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